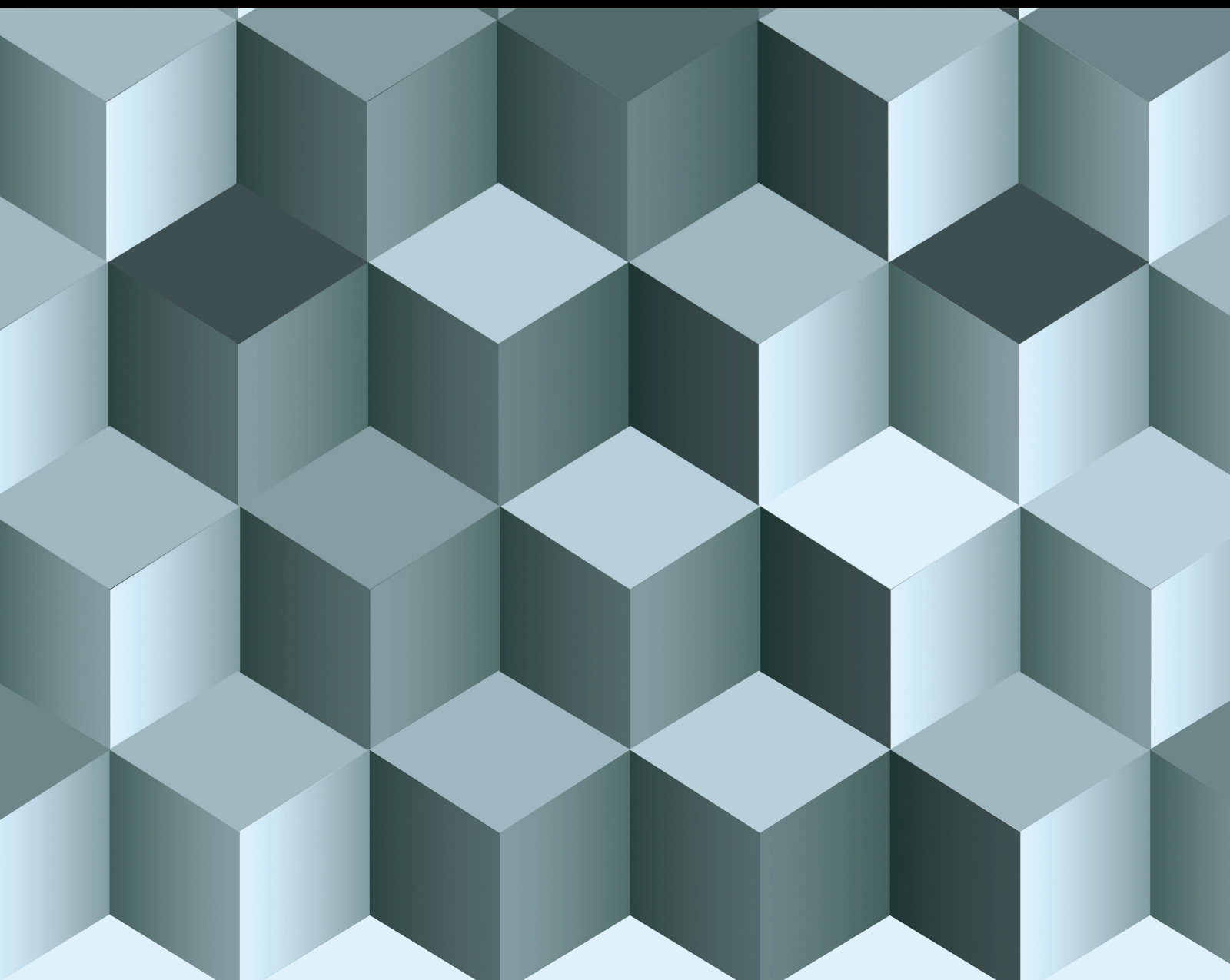


Fixed Point Theory and Applications for Function Spaces 2021

Lead Guest Editor: Nawab Hussain

Guest Editors: Hüseyin Işık and Mohamed Aziz Taoudi





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Research Article

On Multiple Positive Solutions for Singular Fractional Boundary Value Problems with Riemann-Stieltjes Integrals

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In this paper, the existence result of at least two positive solutions is obtained for a nonlinear Riemann-Liouville fractional differential equation subject to nonlocal boundary conditions, where fractional derivatives and Riemann-Stieltjes integrals are involved. The nonlinearity possesses singularities on both its time and space variables. The discussion is based on the fixed point index theory on cones.

1. Introduction

We consider the existence of at least two positive solutions for the following nonlinear fractional differential equation with integral boundary value conditions

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, & D_{0+}^{\beta_0} u(1) = \sum_{i=1}^m \int_0^1 D_{0+}^{\beta_i} u(t) dH_i(t), \end{cases} \quad (1)$$

where $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n, m \in \mathbb{N}$, $n \geq 3$, $\beta_i \in \mathbb{R}$ for all $i = 0, 1, \dots, m$, $0 \leq \beta_1 < \beta_2 < \dots < \beta_m \leq \beta_0 < \alpha - 1$, $\beta_0 \geq 1$, $f \in C((0, 1) \times (0, +\infty), [0, +\infty))$. D_{0+}^{μ} represents the Riemann-Liouville derivative of order μ (for $\mu = \alpha, \beta_0, \beta_1, \dots, \beta_m$). The integrals involved in the boundary conditions are Riemann-Stieltjes integrals; here, $H_i (i = 1, 2, \dots, m)$ are functions of bounded variation. For more general integral conditions, see [1] and the references therein. The nonlinearity f permits singularities at both $t = 0, 1$ and $u = 0$.

In recent years, there has been a gradual increase in the investigation of fractional differential equations and systems of fractional differential equations with nonlocal boundary value conditions owing to their better descriptions in very important phenomena in science and technology than in integers. For fractional calculus and its applications in non-

local problems, see monographs [2–6] and papers [7–25] and the references therein. Very recently, by means of the fixed point theory, principal characteristic value, and fixed point theorems together with height functions, Tudorache et al. [7, 9] investigated the existence of one, two, or three positive solutions for BVP (1). Existence results can be found in [15–19] for the system of fractional differential equations with boundary conditions related to BVP (1).

As is well known, Riemann-Stieltjes integral boundary conditions are more general, and they include many special cases such as two-point, three-point, and other classical integral conditions or a combination of them. As a consequence, boundary conditions in BVP (1) are generalizations of those adopted in the literature [10–14], which can be listed below:

$$\begin{aligned} u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad D_{0+}^{\beta_0} u(1) &= \sum_{i=1}^m a_i D_{0+}^{\beta_i} u(1), \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad D_{0+}^{\beta_0} u(1) &= \lambda \int_0^{\eta} D_{0+}^{\beta_0} h(t) u(t) dt, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad D_{0+}^{\beta_0} u(1) &= \int_0^1 D_{0+}^{\beta_0} u(t) dH(t), \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad D_{0+}^{\beta_0} u(1) &= \lambda \int_0^{\eta} D_{0+}^{\beta_0} u(t) dH(t). \end{aligned} \quad (2)$$

We make an effort in this paper to investigate the existence of multiple positive solutions for BVP (1). By a positive solution of BVP (1), we mean a function $u \in C[0, 1]$ satisfying BVP (1) with $u(t) > 0$ for all $t \in (0, 1]$. This paper admits the following features. Firstly, compared with [16–18], the nonlinearity f in this paper possesses singularities not only on the time but also on the space variables. Secondly, compared with [7–9], super linear conditions on the nonlinearity at 0 and ∞ are imposed to obtain the existence of at least two positive solutions. Thirdly, conditions given in this paper are shown to be easy to verify by an example. The main tools employed in this paper are the cone theory and fixed point index theorems on cones.

2. Preliminaries and Several Lemmas

First, we introduce some useful lemmas from [7, 9] which will be used in the latter. For notational convenience, denote

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_0)} - \sum_{i=1}^m \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_i)} \int_0^1 s^{\alpha - \beta_i - 1} dH_i(s). \quad (3)$$

Consider the fractional differential equation

$$\begin{cases} D_{0+}^\alpha u(t) + x(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, & D_{0+}^{\beta_0} u(1) = \sum_{i=1}^m \int_0^1 D_{0+}^{\beta_i} u(t) dH_i(t), \end{cases} \quad (4)$$

where $x \in C(0, 1) \cap L^1(0, 1)$.

Lemma 1 (see [7, 9]). *If $\Delta \neq 0$, then the unique solution $u \in C[0, 1]$ of problems (4) is given by*

$$u(t) = \int_0^1 \mathcal{G}(t, s) x(s) ds, \quad t \in [0, 1], \quad (5)$$

where

$$\mathcal{G}(t, s) = g_1(t, s) + \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m \int_0^1 g_{2i}(\tau, s) dH_i(\tau), \quad (6)$$

$$g_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} (1-s)^{\alpha-\beta_0-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1} (1-s)^{\alpha-\beta_0-1}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (7)$$

$$g_{2i}(t, s) = \frac{1}{\Gamma(\alpha - \beta_i)} \begin{cases} t^{\alpha-\beta_i-1} (1-s)^{\alpha-\beta_0-1} - (t-s)^{\alpha-\beta_i-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-\beta_i-1} (1-s)^{\alpha-\beta_0-1}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (8)$$

for all $(t, s) \in [0, 1] \times [0, 1]$, $i = 1, 2, \dots, m$.

Lemma 2 (see [7, 9]). *We suppose that $\Delta > 0$. Then, the Green function \mathcal{G} given by (6) is a continuous function on $[0, 1] \times [0, 1]$ and satisfies the following inequalities:*

(i) $\mathcal{G}(t, s) \leq \mathcal{J}(s)$ for all $t, s \in [0, 1]$, where

$$\begin{aligned} \mathcal{J}(s) &= h_1(s) + \frac{1}{\Delta} \sum_{i=1}^m \int_0^1 g_{2i}(\tau, s) dH_i(\tau), \quad s \in [0, 1], \\ h_1(s) &= \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-\beta_0-1} \left(1 - (1-s)^{\beta_0} \right), \quad s \in [0, 1] \end{aligned} \quad (9)$$

(ii) $\mathcal{G}(t, s) \geq t^{\alpha-1} \mathcal{J}(s)$ for all $t, s \in [0, 1]$

(iii) $\mathcal{G}(t, s) \leq \sigma t^{\alpha-1}$ for all $t, s \in [0, 1]$, where

$$\sigma = \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^1 \tau^{\alpha-\beta_i-1} dH_i(\tau). \quad (10)$$

We make the following assumptions:

(H₁) $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n, m \in \mathbb{N}$, $n \geq 3$, $\beta_i \in \mathbb{R}$ for all $i = 0, 1, \dots, m$, $0 \leq \beta_1 < \beta_2 < \dots < \beta_m \leq \beta_0 < \alpha - 1$, $\beta_0 \geq 1$, $H_i : [0, 1] \rightarrow \mathbb{R}$ ($i = 1, 2, \dots, m$) are nondecreasing functions and $\Delta > 0$.

(H₂) $f \in C((0, 1) \times (0, +\infty), [0, +\infty))$.

(H₃) There exist $a, b \in C((0, 1), [0, +\infty)) \cap L^1[0, 1]$, $g \in C((0, +\infty), [0, +\infty))$ such that

$$f(t, u) \leq a(t)g(u) + b(t), \quad \forall t \in (0, 1), u \in (0, +\infty),$$

$$\widetilde{a}_{pq}^* = \int_0^1 a(t)g_{pq}(t)dt < +\infty, \quad \widetilde{b}^* = \int_0^1 b(t)dt < +\infty, \quad (11)$$

for any $q \geq p > 0$, where

$$g_{pq}(t) = \max \{g(u) : t^{\alpha-1}p \leq u \leq q\}. \quad (12)$$

(H₄) There exists $c \in C((0, 1), [0, +\infty))$ such that

$$\frac{f(t, u)}{c(t)u} \longrightarrow +\infty \text{ as } u \longrightarrow +\infty, \quad (13)$$

uniformly for $t \in (0, 1)$, and

$$c^* = \int_0^1 c(t)dt < +\infty. \quad (14)$$

(H₅) There exists $d \in C((0, 1), [0, +\infty))$ such that

$$\frac{f(t, u)}{d(t)} \longrightarrow +\infty \text{ as } u \longrightarrow 0^+, \quad (15)$$

uniformly for $t \in (0, 1)$, and

$$d^* = \int_0^1 d(t)dt < +\infty. \quad (16)$$

In addition, considering the boundedness of $\mathcal{J}(t)$, $t \in [0, 1]$, it is easy to know that

$$\begin{aligned} a_{pq}^* &= \int_0^1 \mathcal{J}(t)a(t)g_{pq}(t)dt < +\infty, \quad b^* = \int_0^1 \mathcal{J}(t)b(t)dt < +\infty, \\ a_p^* &= \int_0^1 \mathcal{J}(t)a(t)g_{pp}(t)dt < +\infty, \end{aligned} \quad (17)$$

where

$$g_{pp}(t) = \max \{g(u) : t^{\alpha-1}p \leq u \leq p\}, \quad \text{for all } p > 0. \quad (18)$$

Let $E = C[0, 1]$ be the traditional Banach space of all continuous functions defined on $[0, 1]$ with the maximum norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ and P the cone

$$P = \{u \in E : u(t) \geq t^{\alpha-1}\|u\|, t \in [0, 1]\}. \quad (19)$$

Denote $P_{pq} = \{u \in P : p \leq \|u\| \leq q\}$, $P_r = \{u \in P : \|u\| \leq r\}$, $\partial P_r = \{u \in P : \|u\| = r\}$ for $q > p > 0, r > 0$.

Define the operator T as follows:

$$(Tu)(t) = \int_0^1 \mathcal{G}(t, s)f(s, u(s))ds, \quad 0 \leq t \leq 1. \quad (20)$$

Clearly, $T : P \setminus \{0\} \rightarrow C[0, 1]$.

Lemma 3. Suppose that (H_1) – (H_3) hold; then, for any $q > p > 0$, $T : P_{pq} \rightarrow P$ is completely continuous.

Proof. For any $u \in P_{pq}$, we have $p \leq \|u\| \leq q$. It follows from the definition of cone P that

$$t^{\alpha-1}p \leq u(t) \leq q, \quad \forall t \in [0, 1]. \quad (21)$$

By (H_2) , (H_3) , and Lemma 2, we get that

$$f(t, u(t)) \leq a(t)g_{pq}(t) + b(t), \quad \forall t \in (0, 1), u \in P_{pq}, \quad (22)$$

$$\begin{aligned} (Tu)(t) &= \int_0^1 \mathcal{G}(t, s)f(s, u(s))ds \leq \int_0^1 \mathcal{J}(s)f(s, u(s)) \\ &\quad \cdot ds \leq \int_0^1 \mathcal{J}(s) \left[a(s)g_{pq}(s) + b(s) \right] \\ &\quad \cdot ds = a_{pq}^* + b^*, \quad \forall t \in [0, 1], \end{aligned} \quad (23)$$

which means that T is well defined. For any $t \in [0, 1]$, we have from (23) that

$$(Tu)(t) = \int_0^1 \mathcal{G}(t, s)f(s, u(s))ds \leq \int_0^1 \mathcal{J}(s)f(s, u(s))ds. \quad (24)$$

Therefore,

$$\|Tu\| \leq \int_0^1 \mathcal{J}(s)f(s, u(s))ds. \quad (25)$$

On the other hand, it follows from Lemma 2 and (25) that

$$(Tu)(t) = \int_0^1 \mathcal{G}(t, s)f(s, u(s))ds \geq t^{\alpha-1} \int_0^1 \mathcal{J}(s)f(s, u(s))ds \geq t^{\alpha-1}\|Tu\|, \quad \forall t \in [0, 1]. \quad (26)$$

Thus, we have proven that T maps P_{pq} into P .

In the following, we are in the position to show that T is completely continuous. First, we prove that T is continuous. For $u_n, \bar{u} \in P_{pq}$ with $\|u_n - \bar{u}\| \rightarrow 0$ ($n \rightarrow \infty$), we have $\lim_{n \rightarrow \infty} u_n(t) = \bar{u}(t)$, $t \in [0, 1]$. By (H_1) , we know

$$\lim_{n \rightarrow \infty} f(t, u_n(t)) = f(t, \bar{u}(t)), \quad 0 < t < 1. \quad (27)$$

Similar to (22), for $u_n, \bar{u} \in P_{pq}$, we have

$$f(t, u_n(t)) \leq a(t)g_{pq}(t) + b(t), \quad f(t, \bar{u}(t)) \leq a(t)g_{pq}(t) + b(t), \quad \forall t \in (0, 1). \quad (28)$$

Thus,

$$|\mathcal{G}(t, s)f(s, u_n(s)) - \mathcal{G}(t, s)f(s, \bar{u}(s))| \leq 2\mathcal{J}(s) \left[a(s)g_{pq}(s) + b(s) \right] = \sigma(s) \in L^1[0, 1]. \quad (29)$$

It follows from (27), (29), (H_3) , and the Lebesgue-dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \|Tu_n - T\bar{u}\| \leq \lim_{n \rightarrow \infty} \max_{t \in [0, 1]} \left| \int_0^1 \mathcal{G}(t, s)f(s, u_n(s)) - \mathcal{G}(t, s)f(s, \bar{u}(s))ds \right| = 0, \quad (30)$$

which means that T is continuous.

Next, we will show that T is a compact operator. Let V be a bounded set in P_{pq} . For any $u \in V$, we have $p \leq \|u\| \leq q$. Similar to (23), we know

$$(Tu)(t) \leq a_{pq}^* + b^*, \quad \forall t \in [0, 1], u \in P_{pq}, \quad (31)$$

which means that $T(V)$ is bounded uniformly. In the following, we shall prove that $T(V)$ is equicontinuous. To this end, we estimate $(Tu)'$ for $u \in V$.

$$\begin{aligned}
\|(Tu)'(t)\| &= \int_0^1 \frac{\partial \mathcal{G}(t,s)}{\partial t} f(s, u(s)) ds \\
&= \int_0^1 \left(\frac{\partial g_1(t,s)}{\partial t} + \frac{(\alpha-1)t^{\alpha-2}}{\Delta} \sum_{i=1}^m \int_0^1 g_{2i}(\tau,s) dH_i(\tau) \right) \\
&\quad \cdot f(s, u(s)) ds \leq (\alpha-1) \int_0^1 \left[\frac{1}{\Gamma(\alpha)} \left(t^{\alpha-2}(1-s)^{\alpha-\beta_0-1} + (t-s)^{\alpha-2} \right) \right. \\
&\quad \left. + \frac{t^{\alpha-2}}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} \tau^{\alpha-\beta_i-1} (1-s)^{\alpha-\beta_0-1} dH_i(\tau) \right] f(s, u(s)) \\
&\quad \cdot ds \leq (\alpha-1) \int_0^1 \left(\frac{2}{\Gamma(\alpha)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} (H_i(1) - H_i(0)) \right) \\
&\quad \cdot (a(s)g_{pq}(s) + b(s)) ds \leq (\alpha-1) \\
&\quad \cdot \left(\frac{2}{\Gamma(\alpha)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} (H_i(1) - H_i(0)) \right) \\
&\quad \cdot (\widetilde{a_{pq}^*} + \widetilde{b^*}) \stackrel{\Delta}{=} \mathfrak{F}, \quad \forall t \in [0, 1].
\end{aligned} \tag{32}$$

Thus, for any $0 \leq t_1 \leq t_2 \leq 1$ and $u \in V$, one has

$$|Tu(t_1) - Tu(t_2)| = \left| \int_{t_1}^{t_2} (T_n u)'(s) ds \right| \leq \mathfrak{F} |t_1 - t_2|. \tag{33}$$

Thus, $T(V)$ is equicontinuous. It follows from the Arzelà-Ascoli theorem that $T(V)$ is relatively compact, and then, T is a compact operator. Hence, $T : P_{pq} \longrightarrow P$ is completely continuous. \square

Lemma 4 (see [26]). *Let E be a Banach space and $P \subset E$ a cone in E . Assume that $T : P_r \longrightarrow P$ is a compact map such that $Tu \neq u$ for $u \in \partial P_r$,*

(i) *If $\|u\| \leq \|Tu\|$, $\forall u \in \partial P_r$, then*

$$i(T, P_r, P) = 0 \tag{34}$$

(ii) *If $\|u\| \geq \|Tu\|$, $\forall u \in \partial P_r$, then*

$$i(T, P_r, P) = 1. \tag{35}$$

3. Main Result

Theorem 5. *Assume that (H_1) – (H_5) hold. In addition, there exists $R_0 > 0$ such that*

$$a_{R_0}^* + b^* < R_0. \tag{36}$$

Then, the BVP (1) has at least two positive solutions u^ and u^{**} with $0 < \|u^*\| < R_0 < \|u^{**}\|$.*

Proof. It follows from Lemma 3 that for any $q > p > 0$, the operator $T : P_{pq} \longrightarrow P$ is completely continuous. In the following, we shall prove that T has two different fixed points u^* and u^{**} in P satisfying $0 < \|u^*\| < R_0 < \|u^{**}\|$.

Choose $\theta \in (0, 1/2)$. We know from (H_4) that there exists $r_1 > 0$ such that

$$f(t, u) \geq \theta^{1-\alpha} \left(\int_{\theta}^{1-\theta} \mathcal{G}(1/2, s) c(s) ds \right)^{-1} c(t) u, \quad \forall t \in (0, 1), u \geq r_1. \tag{37}$$

Let

$$R_1 > \max \{ \theta^{1-\alpha} r_1, R_0 \}. \tag{38}$$

For $u \in \partial P_{R_1}$, we have, by the construction of cone P , that

$$u(t) \geq t^{\alpha-1} R_1 \geq \theta^{\alpha-1} R_1 > r_1, \quad \forall t \in [\theta, 1 - \theta]. \tag{39}$$

It follows from (37) to (39) that

$$\begin{aligned}
(Tu)\left(\frac{1}{2}\right) &= \int_0^1 \mathcal{G}\left(\frac{1}{2}, s\right) f(s, u(s)) ds > \theta^{1-\alpha} \left(\int_{\theta}^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) c(s) ds \right)^{-1} \\
&\quad \cdot \int_{\theta}^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) c(s) u(s) ds \geq \theta^{1-\alpha} \left(\int_{\theta}^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) c(s) ds \right)^{-1} \\
&\quad \cdot \int_{\theta}^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) c(s) ds \cdot \theta^{\alpha-1} R_1 = R_1.
\end{aligned} \tag{40}$$

Thus,

$$\|Tu\| = \max_{t \in [0, 1]} \|(Tu)(t)\| \geq \left\| (Tu)\left(\frac{1}{2}\right) \right\| > R_1 = \|u\|, \quad \forall u \in \partial P_{R_1}. \tag{41}$$

Hence, by Lemma 4,

$$i(T, P_{R_1}, P) = 0. \tag{42}$$

By condition (H_4) , there exists $r_2 > 0$ such that

$$f(t, u) \geq \left(\int_{\theta}^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) d(s) ds \right)^{-1} d(t) R_0, \quad \forall t \in (0, 1), 0 < u < r_2. \tag{43}$$

Choose

$$0 < R_2 < \min \{ r_2, R_0 \}. \tag{44}$$

For $u \in P_{R_2}$, we have

$$0 < R_2 t^{\alpha-1} \leq u(t) \leq \|u\| = R_2 < r_2, \quad \forall t \in (0, 1). \tag{45}$$

Consequently, we have from (43) to (45) and (H₅) that

$$\begin{aligned} (Tu)\left(\frac{1}{2}\right) &= \int_0^1 \mathcal{G}\left(\frac{1}{2}, s\right) f(s, u(s)) ds \geq \left(\int_\theta^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) d(s) ds\right)^{-1} \\ &\quad \cdot \int_\theta^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) d(s) R_0 ds \geq \left(\int_\theta^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) d(s) ds\right)^{-1} \\ &\quad \cdot \int_\theta^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) d(s) ds \cdot R_0 = R_0 > R_2. \end{aligned} \quad (46)$$

Thus,

$$\|Tu\| = \max_{t \in [0,1]} \|(Tu)(t)\| \geq \left\| (Tu)\left(\frac{1}{2}\right) \right\| > R_2 = \|u\|, \quad \forall u \in P_{R_2}. \quad (47)$$

As a consequence, we get

$$i(T, P_{R_2}, P) = 0. \quad (48)$$

On the other hand, for $u \in \partial P_{R_0}$, by (H₃), Lemma 2, and (36), we get

$$\begin{aligned} (Tu)(t) &= \int_0^1 \mathcal{G}(t, s) f(s, u(s)) ds \leq \int_0^1 \mathcal{J}(s) f(s, u(s)) ds \leq \int_0^1 \mathcal{J}(s) \\ &\quad \cdot [a(s) g_{R_0 R_0}(s) + b(s)] ds \leq a_{R_0}^* + b^* < R_0, \quad \forall t \in [0, 1], \end{aligned} \quad (49)$$

i.e.,

$$\|Tu\| \leq \|u\|, \quad u \in \partial P_{R_0}. \quad (50)$$

Then, Lemma 4 guarantees that

$$i(T, P_{R_0}, P) = 1. \quad (51)$$

It follows from (42), (48), (51) and the additivity of the fixed point index that

$$\begin{aligned} i\left(T, P_{R_1} \setminus \overset{\circ}{P}_{R_0}, P\right) &= -1, \\ i\left(T, P_{R_0} \setminus \overset{\circ}{P}_{R_2}, P\right) &= 1. \end{aligned} \quad (52)$$

Hence, T has two distinct fixed points u^* and u^{**} belonging to $P_{R_0} \setminus \overset{\circ}{P}_{R_2}$ and $P_{R_1} \setminus \overset{\circ}{P}_{R_0}$, respectively, with $0 < R_2 < \|u^*\| < R_0 < \|u^{**}\| \leq R_1$. \square

4. An Example

Example 1. Consider the following fractional differential equations with nonlocal boundary value problems

$$\begin{cases} D_{0+}^{11/3} u(t) + \frac{1}{75\sqrt[5]{t^2(1-t)^4}} (u^3 + u^{-1/7}) + \frac{1}{25\sqrt[7]{t(1-t)^3}} = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & D_{0+}^{13/6} u(1) = \int_0^1 D_{0+}^{2/3} u(t) dt + D_{0+}^{5/3} u\left(\frac{1}{2}\right). \end{cases} \quad (53)$$

Conclusion: BVP (53) has at least two positive solutions u^* and u^{**} with $0 < \|u^*\| < 5 < \|u^{**}\|$.

Proof. In this problem, $\alpha = 11/3$, $n = 4$, $m = 2$, $\beta_0 = 13/6$, $\beta_1 = 2/3$, $\beta_2 = 5/3$, $H_1(t) = t$ for all $t \in [0, 1]$, $H_2(t) = \{0$ for $t \in [0, 1/2]$; 1 for $t \in [1/2, 1]\}$. By simple computation, we have $\Delta = 1.852483495372207 > 0$. It is clear that (H₁) and (H₂) are satisfied. Furthermore,

$$h_1(s) = \frac{1}{\Gamma(11/3)} (1-s)^{1/2} (1-(1-s)^{13/6}), \quad s \in [0, 1], \quad (54)$$

$$g_{21}(t, s) = \frac{1}{\Gamma(3)} \begin{cases} t^2(1-s)^{1/2} - (t-s)^2, & 0 \leq s \leq t \leq 1, \\ t^2(1-s)^{1/2}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (55)$$

$$g_{22}(t, s) = \frac{1}{\Gamma(2)} \begin{cases} t(1-s)^{1/2} - (t-s), & 0 \leq s \leq t \leq 1, \\ t(1-s)^{1/2}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (56)$$

$$\mathcal{J}(s) = \begin{cases} h_1(s) + \frac{1}{\Delta} \left\{ \frac{1}{\Gamma(4)} (1-s)^{1/2} - \frac{1}{\Gamma(4)} (1-s)^3 + \frac{1}{\Gamma(2)} \left[\frac{1}{2} \cdot (1-s)^{1/2} - \left(\frac{1}{2} - s \right) \right] \right\}, & 0 \leq s \leq \frac{1}{2}, \\ h_1(s) + \frac{1}{\Delta} \left\{ \frac{1}{\Gamma(4)} (1-s)^{1/2} - \frac{1}{\Gamma(4)} (1-s)^3 + \frac{1}{\Gamma(2)} \cdot \frac{1}{2} \cdot (1-s)^{1/2}, \right. & \left. \frac{1}{2} < s \leq 1. \right\} \end{cases} \quad (57)$$

For any $r > 0$, (H₃) holds for $a(t) = 1/75 \sqrt[5]{t^2(1-t)^4}$, $g(u) = u^3 + u^{-1/7}$, $b(t) = 1/25 \sqrt[7]{t(1-t)^3}$, and

$$\begin{aligned} \widetilde{a}_{pq}^* &= \int_0^1 a(t) g_{pq}(t) dt = \int_0^1 \frac{1}{75\sqrt[5]{t^2(1-t)^4}} \left(q^3 + \frac{1}{\sqrt[7]{t^{8/3}p}} \right) \\ &\quad \cdot dt = \frac{1}{75} \left(\int_0^1 t^{-2/5} (1-t)^{-4/5} dt \cdot q^3 + \int_0^1 t^{-82/105} (1-t)^{-4/5} dt \cdot p^{-1/7} \right) \\ &= \frac{1}{75} \left[B\left(\frac{3}{5}, \frac{1}{5}\right) q^3 + B\left(\frac{23}{105}, \frac{1}{5}\right) p^{-1/7} \right] < +\infty. \end{aligned} \quad (58)$$

$\tilde{b}^* = \int_0^1 (1/25 \sqrt[4]{t(1-t)^3}) dt = (1/25) B(3/4, 1/4) \approx 0.17771 5317526335$. Thus, (H₃) is verified. Obviously, (H₄) and (H₅) are valid for $c(t) = d(t) = 1/75 \sqrt[5]{t^2(1-t)^4}$, $c^* = d^* \approx 0.078296677374705$.

Next, we focus on checking (36). Take $R_0 = 5$. By (57), we know that

$$\begin{aligned}
 b^* &= \int_0^1 \mathcal{J}(s)b(s)ds \leq \int_0^1 \left\{ h_1(s) + \frac{1}{\Delta} \left[\frac{1}{\Gamma(4)}(1-s)^{1/2} + \frac{1}{\Gamma(3)}(1-s)^{1/2} \right] \right\} \\
 &\quad \cdot \frac{1}{25\sqrt[4]{s(1-s)^3}} ds \leq \int_0^1 \left\{ \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta} \left[\frac{1}{\Gamma(4)}(1-s)^{1/2} + \frac{1}{\Gamma(3)}(1-s)^{1/2} \right] \right\} \\
 &\quad \cdot \frac{1}{25\sqrt[4]{s(1-s)^3}} ds \leq \frac{1}{25} \cdot \left[\frac{1}{\Gamma(11/3)} \cdot \int_0^1 s^{-1/4}(1-s)^{-3/4} ds + \frac{1}{\Delta} \left(\frac{1}{\Gamma(4)} + \frac{1}{\Gamma(3)} \right) \right. \\
 &\quad \cdot \int_0^1 s^{-1/4}(1-s)^{-1/4} ds \Big] = \frac{1}{25} \cdot \left[\frac{1}{\Gamma(11/3)} B\left(\frac{3}{4}, \frac{1}{4}\right) + \frac{1}{1.852483495372207} \right. \\
 &\quad \cdot \left(\frac{1}{\Gamma(4)} + \frac{1}{\Gamma(3)} \right) \cdot B\left(\frac{3}{4}, \frac{3}{4}\right) \Big] \approx \frac{1}{25} \left[0.249239737672306 \cdot 4.442882938158366 \right. \\
 &\quad \left. + 0.539815875552012 \cdot \left(\frac{1}{6} + \frac{1}{2} \right) \cdot 1.694426169587958 \right] \approx 0.068685136355131, \\
 a_{R_0}^* &= \int_0^1 \mathcal{J}(s)a(s)g_{R_0} ds \leq \frac{1}{75} \int_0^1 \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta} \left(\frac{1}{\Gamma(4)} + \frac{1}{\Gamma(3)} \right) (1-s)^{1/2} \right] \\
 &\quad \cdot \frac{1}{\sqrt[5]{s^2(1-s)^4}} \left(R_0^3 + \frac{1}{\sqrt[7]{s^{8/3}R_0}} \right) ds \leq \frac{1}{75} \\
 &\quad \cdot \left\{ \frac{1}{\Gamma(11/3)} \cdot \left[B\left(\frac{3}{5}, \frac{1}{5}\right) \cdot R_0^3 + B\left(\frac{23}{105}, \frac{1}{5}\right) \cdot R_0^{-1/7} \right] \right. \\
 &\quad \left. + \frac{1}{1.852483495372207} \cdot \left(\frac{1}{\Gamma(4)} + \frac{1}{\Gamma(3)} \right) \right. \\
 &\quad \cdot \left[B\left(\frac{3}{5}, \frac{7}{10}\right) \cdot R_0^3 + B\left(\frac{23}{105}, \frac{7}{10}\right) \cdot R_0^{-1/7} \right] \Big\} \approx \frac{1}{75} \\
 &\quad \cdot \left[0.249239737672306 \cdot (5.872250803102905 \cdot 5^3 + 9.049460301355431 \cdot 5^{-1/7}) \right. \\
 &\quad \left. + 0.539815875552012 \cdot \left(\frac{1}{6} + \frac{1}{2} \right) \cdot (2.153890871161322 \cdot 5^3 \right. \\
 &\quad \left. + 5.136337054837259 \cdot 5^{-1/7}) \right] \approx 3.774703981617563.
 \end{aligned} \tag{59}$$

Hence,

$$\begin{aligned}
 a_{R_0}^* + b^* &= 3.774703981617563 + 0.068685136355131 \\
 &= 3.843389117972694 < 5 = R_0,
 \end{aligned} \tag{60}$$

which implies that (36) holds. Consequently, our conclusion follows from Theorem 5. \square

5. Conclusions

In this paper, we focus on the existence and multiplicity of positive solutions for a class of a higher-order Riemann-Liouville fractional differential equation with Riemann-Stieltjes integrals. The nonlinearity possesses singularities on both its time and space variables. By means of the fixed point index theory on cones, the existence result of at least two positive solutions is obtained. Conditions imposed on the nonlinearity are shown to be easy to verify by an example.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

H^β -Hausdorff Functions and Common Fixed Points of Multivalued Operators in a b -Metric Space and Their Applications

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H^β -Hausdorff functions for $\beta \in [0, 1]$ are introduced, and common fixed-point theorems for a pair of multivalued operators satisfying generalized contraction conditions are proven in a b -metric space. Our results are proper extensions and new variants of many contraction conditions existing in literature. In order to demonstrate applications of our result, we have proven an existence theorem for a unique common multivalued fractal of a pair of iterated multifunction systems and also an existence theorem for a common solution of a pair of Volterra-type integral equations.

1. Introduction

In the last few decades, a wide range of extensions, generalizations, and applications of the infamous Banach contraction principle came into existence. In the sequel, Bakhtin [1] initiated the idea of a b -metric space followed by Czerwik [2], in which the author by weakening the triangular inequality formally defined a b -metric space and proved the Banach contraction principle in a b -metric space. Some examples and other details of a b -metric space can be found in Kirk and Shahzad [3] whereas a wide range of generalized fixed-point theorems in a b -metric space can be found in [4–7]. On the other hand, the study of a metric function on the set of closed and bounded subsets of a metric space was initiated by Pompeiu in [8] and then continued by Hausdorff [9]. Such a metric function is referred to as the Hausdorff-Pompeiu metric. Banach's contraction principle was extended to a multivalued function in a metric space by Nadler [10] and in a b -metric space by Czerwik [2] using the Hausdorff-Pompeiu metric H . Further generalized results of multivalued contractions can be found in ([11–14]). Czerwik's contraction was also generalized in many directions to name a few: q -quasi-contraction [15],

Hardy-Rogers contraction [16], weak quasi-contraction [17], Ćirić contraction [18], etc. More results on multivalued contraction mappings in a b -metric space can be found in [19–23]. Very recently, Debnath [24] proved the set-valued Meir-Keeler-type as well as Geraghty- and Edelstein-type fixed-point theorems in a b -metric space whereas Altun et al. [25] and Kumar and Luambano [26] proved fixed-point results for multivalued F -contraction mappings in complete metric space and partial metric space, respectively. In [27], the authors introduced the concept of H^β -Hausdorff-Pompeiu b -metric for some $0 \leq \beta \leq 1$ and proved fixed-point theorems for multivalued mappings belonging to various classes of multivalued H^β -contractions in a b -metric space. Applications of fixed-point results in dealing with solutions of nonlinear problems arising in engineering and science are an important area in present-day research. Fruitful applications of fixed-point problems in solution of various types of integral equations, fractional differential equations, and optimization problems can be found in [28–32]. Barnsley [33] introduced the idea of data interpolation using the fractal methodology of iterated function systems. Nowadays, fractal functions constitute a method of approximation of nondifferentiable mappings, providing

suitable tools for the description of irregular signals (see [34–39]). The aim of this work is to prove common fixed-point theorems for a pair of multivalued mappings in a b -metric space using H^β -Hausdorff-Pompeiu b -metric and thereby extend and introduce new variants of various fixed-point results for multivalued mappings existing in literature. We have provided two applications of our main results: one to prove the existence of a unique common multivalued fractal of a pair of iterated multifunction system defined on a b -metric space and the second to prove the existence of a common solution of a pair of Volterra-type nonlinear integral equations.

2. Preliminaries

In this section, we provide some preliminary definitions, lemmas, and propositions required in our main results.

Definition 1 (see [1]). Let X be a nonempty set and $d_s : X \times X \rightarrow [0, \infty)$ satisfy the following:

- (1) $d_s(i, j) = 0$ if and only if $i = j$ for all $i, j \in X$
- (2) $d_s(i, j) = d_s(j, i)$ for all $i, j \in X$
- (3) There exists a real number $s \geq 1$ such that $d_s(i, j) \leq s[d_s(i, \ell) + d_s(\ell, j)]$ for all $i, j, \ell \in X$

Then, d_s is a b -metric on X and (X, d_s) is a b -metric space with coefficient s .

Let $CB^{d_s}(X)$ be the collection of all nonempty closed and bounded subsets of a b -metric space (X, d_s) . For $A, B \in CB^{d_s}(X)$, define $d_s(x, A) = \inf \{d_s(x, a) : a \in A\}$, $\delta_{d_s}(A, B) = \sup_{a \in A} d_s(a, B)$, and $H_{d_s}(A, B) = \max \{\delta_{d_s}(A, B), \delta_{d_s}(B, A)\}$. Czerwik [2] has shown that H_{d_s} is a b -metric in the set $CB^{d_s}(X)$ and is called the Hausdorff-Pompeiu b -metric induced by d_s . In [27], the authors introduced the function $H^\beta(A, B) = \max \{\beta \delta_{d_s}(A, B) + (1 - \beta) \delta_{d_s}(B, A), \beta \delta_{d_s}(B, A) + (1 - \beta) \delta_{d_s}(A, B)\}$ for some $\beta \in [0, 1]$ and showed that H^β is a b -metric for the set $CB^{d_s}(X)$. They called this function the H^β -Hausdorff-Pompeiu b -metric induced by the b -metric d_s . Note that for $\beta = 0$ or 1 , H^β is the Hausdorff-Pompeiu metric H_{d_s} .

Proposition 2 (see [27]). For any $x, y \in X$, $H^\beta(\{x\}, \{y\}) = d_s(x, y)$.

Definition 3 (see [18]). The b -metric d_s is $*$ -continuous if and only if for any $A \in CB^{d_s}(X)$ and sequence $\{x_n\}$ in (X, d_s) with $\lim_{n \rightarrow \infty} x_n = x$, we have $\lim_{n \rightarrow \infty} d_s(x_n, A) = d_s(x, A)$.

Proposition 4 (see [19]). For any $A \subseteq X$,

$$a \in \bar{A} \Leftrightarrow d_s(a, A) = 0. \quad (1)$$

Lemma 5 (see [18]). Let $\{x_n\}$ be a sequence in (X, d_s) . If there exists $\lambda \in [0, 1)$ such that $d_s(x_n, x_{n+1}) \leq \lambda d_s(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

The following lemma follows immediately from the above lemma.

Lemma 6. If for some $\lambda, \epsilon \in [0, 1)$, with $\lambda < \epsilon$, $d_s(x_n, x_{n+1}) \leq \lambda d_s(x_{n-1}, x_n) + \epsilon^n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

3. Main Results

We introduce pairwise H^β -Hausdorff functions as follows:

Definition 7. Let $S, T : X \rightarrow CB^{d_s}(X)$. For any $i \in X$, $j \in T$ (or Si) and any $\epsilon > 0$, if there exist $\ell \in Sj$ (or Tj) such that

$$d_s(j, \ell) \leq H^\beta(Ti, Sj) + \epsilon \text{ or respectively } d_s(j, \ell) \leq H^\beta(Si, Tj) + \epsilon, \quad (2)$$

then we say that T and S are pairwise H^β -Hausdorff functions.

For $S = T$, we get the following.

Definition 8. For any $i \in X, j \in Ti$ and any $\epsilon > 0$ if there exist $\ell \in Tj$ such that

$$d_s(j, \ell) \leq H^\beta(Ti, Tj) + \epsilon, \quad (3)$$

then we say that T is a H^β -Hausdorff function.

Remark 9.

- (i) For $\beta = 1$, $T : X \rightarrow CB(X)$ is always a H^β -Hausdorff function
- (ii) If for any $0 \leq \beta_1 \leq 1$, the function $T : X \rightarrow CB(X)$ is a H^{β_1} -Hausdorff function, then for any $0 \leq \beta_1 \leq \beta_2 \leq 1$, the function $T : X \rightarrow CB(X)$ is a H^{β_2} -Hausdorff function

Example 10. Let $X = [0, 33/48] \cup \{1\}$,

$$d_s(i, j) = |i - j|^2 \text{ for all } i, j \in X. \quad (4)$$

and $S, T : X \rightarrow CB(X)$ be as follows:

$$S(t) = \begin{cases} \left\{ \frac{t}{4} \right\}, & \text{for } t \in \left(0, \frac{33}{48}\right], \\ \left\{ \frac{33}{48}, 1 \right\}, & \text{for } t \in \{0, 1\}, \end{cases}$$

$$T(t) = \begin{cases} \left\{ \frac{t}{2} \right\}, & \text{for } t \in \left(0, \frac{33}{48}\right], \\ \left\{ \frac{1}{3}, \frac{33}{48}, 1 \right\}, & \text{for } t \in \{0, 1\}. \end{cases} \quad (5)$$

We will show that the functions S and T satisfy (2). We will consider the values of t in X as follows:

- (i) $t \in (0, 33/48]$. In this case, S_t and T_j are singleton sets and so (2) is obviously true
- (ii) $t = 0$. $S_t = \{33/48, 1\}$. If $j = 33/48$, $J = \{33/96\}$, then we have $\ell = 33/96$ and $d_s(j, \ell) = 1089/9216$, $\delta_s(S_t, T_j) = 3969/9216$, $\delta_s(T_j, S_t) = 1089/9216$, and $H^{3/4}(S_t, T_j) = 3249/9216$. Thus, (2) is true for all $\epsilon > 0$. If $j = 1$, $T_j = \{1/3, 33/48, 1\}$, then inequality (2) holds with $\ell = 1$
- (iii) $t = 1$. $S_t = \{33/48, 1\}$, and the result follows in the same way as in (ii) above.
- (iv) $t = 0$. $T_t = \{1/3, 33/48, 1\}$. If $j = 1/3$, $S_j = \{1/12\}$, then we have $\ell = 1/12$ and $d_s(j, \ell) = 9/144$, $\delta_s(T_t, S_j) = 121/144$, $\delta_s(S_j, T_t) = 9/144$, and $H^{3/4}(S_t, T_j) = 93/144$. Thus, (2) is true for all $\epsilon > 0$. If $j = 33/48$, $S_j = \{33/192\}$, then we take $\ell = 33/192$ and then $d_s(j, \ell) = 1089/4096$, $\delta_s(T_t, S_j) = 2809/4096$, $\delta_s(S_j, T_t) = 961/36864$, and $H^{3/4}(T_t, S_j) = 19201/36864$. Thus, (2) is true for all $\epsilon > 0$. If $j = 1$, $S_j = \{33/48, 1\}$, inequality (2) holds with $\ell = 1$

Thus, S and T are pairwise H^β -Hausdorff functions for $\beta = 3/4$. However, S and T are not pairwise H^β -Hausdorff functions for $\beta = 1/2$, as we see that inequality (2) is not satisfied for $i = 0$, $T_t = \{1/3, 33/48, 1\}$, and $j = 33/48$. In fact, S and T are not pairwise H^β -Hausdorff functions for $34/95 < \beta < 61/95$.

We now present our main result as follows:

Theorem 11. Let (X, d_s) be a complete b -metric space with constant $s \geq 1$, d_s be $*$ -continuous, and $T, S : X \rightarrow CB^{d_s}(X)$ be multivalued pairwise H^β -Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:

$$H^\beta(T_t, S_j) \leq \alpha_1 d_s(t, T_t) + \alpha_2 d_s(j, S_j) + \alpha_3 d_s(t, S_j) + \alpha_4 d_s(j, T_t) + \alpha_5 \left(\frac{d_s(t, S_j) + d_s(j, T_t)}{2} \right) + \alpha_6 \frac{d_s(t, T_t) d_s(j, S_j)}{1 + d_s(t, j)} + \alpha_7 d_s(t, j), \quad (6)$$

for all $t, j \in X$ and some $\alpha_k \geq 0$, $k = 1, 2, \dots, 7$, with $\alpha_1 + \alpha_2 + s\alpha_5 + \alpha_6 + \alpha_7 + \max\{2s\alpha_3, 2s\alpha_4\} < 1$, $s(\alpha_1 + \alpha_4 + (\alpha_5/2)) < \beta$, and $s(\alpha_2 + \alpha_3 + (\alpha_5/2)) < \beta$. Then, S and T have a common fixed point.

Proof. Let $t_0 \in X$, $t_1 \in T t_0$, and $0 < \epsilon < 1$. By (2), there exist $t_2 \in S t_1$, such that $d_s(t_1, t_2) \leq H^\beta(T t_0, S t_1) + \epsilon$. By (2) again, there exist $t_3 \in T t_2$, such that $d_s(t_2, t_3) \leq H^\beta(S t_1, T t_2) + \epsilon^2$.

Continuing these ways, we construct the sequence $\langle t_n \rangle$ such that

$$t_{2n+1} \in T t_{2n}, t_{2n+2} \in S t_{2n+1},$$

$$d_s(t_{2n+1}, t_{2n+2}) \leq H^\beta(T t_{2n}, S t_{2n+1}) + \epsilon^{2n+1}, \quad (7)$$

$$d_s(t_{2n+2}, t_{2n+3}) \leq H^\beta(S t_{2n+1}, T t_{2n+2}) + \epsilon^{2n+2}.$$

Now,

$$d_s(t_{2n+1}, t_{2n+2}) \leq H^\beta(T t_{2n}, S t_{2n+1}) + \epsilon^{2n+1}$$

$$\leq \alpha_1 d_s(t_{2n}, T t_{2n}) + \alpha_2 d_s(t_{2n+1}, S t_{2n+1})$$

$$+ \alpha_3 d_s(t_{2n}, S t_{2n+1}) + \alpha_4 d_s(t_{2n+1}, T t_{2n})$$

$$+ \alpha_5 \left[\frac{d_s(t_{2n}, S t_{2n+1}) + d_s(t_{2n+1}, T t_{2n})}{2} \right]$$

$$+ \alpha_6 \left[\frac{d_s(t_{2n}, T t_{2n}) * d_s(t_{2n+1}, S t_{2n+1})}{1 + d_s(t_{2n}, t_{2n+1})} \right]$$

$$+ \alpha_7 d_s(t_{2n}, t_{2n+1}) + \epsilon^{2n+1}$$

$$\leq \alpha_1 d_s(t_{2n}, t_{2n+1}) + \alpha_2 d_s(t_{2n+1}, t_{2n+2})$$

$$+ \alpha_3 d_s(t_{2n}, t_{2n+2}) + \alpha_4 (0)$$

$$+ \alpha_5 \left[\frac{d_s(t_{2n}, t_{2n+2}) + 0}{2} \right]$$

$$+ \alpha_6 \left[\frac{d_s(t_{2n}, t_{2n+1}) * d_s(t_{2n+1}, t_{2n+2})}{1 + d_s(t_{2n}, t_{2n+1})} \right]$$

$$+ \alpha_7 d_s(t_{2n}, t_{2n+1}) + \epsilon^{2n+1}$$

$$\leq \alpha_1 d_s(t_{2n}, t_{2n+1}) + \alpha_2 d_s(t_{2n+1}, t_{2n+2})$$

$$+ \alpha_3 [s(d_s(t_{2n}, t_{2n+1}) + d_s(t_{2n+1}, t_{2n+2}))]$$

$$+ \alpha_5 s \left[\frac{d_s(t_{2n}, t_{2n+1}) + d_s(t_{2n+1}, t_{2n+2})}{2} \right]$$

$$+ \alpha_6 d_s(t_{2n+1}, t_{2n+2}) + \alpha_7 d_s(t_{2n}, t_{2n+1}) + \epsilon^{2n+1}.$$

Therefore,

$$d_s(t_{2n+1}, t_{2n+2}) \leq \frac{\alpha_1 + s\alpha_3 + (s\alpha_5/2) + \alpha_7}{1 - \alpha_2 - s\alpha_3 - (s\alpha_5/2) - \alpha_6} d_s(t_{2n}, t_{2n+1}) + \epsilon^{2n+1}. \quad (9)$$

Again,

$$d_s(t_{2n+2}, t_{2n+3}) \leq H^\beta(S t_{2n+1}, T t_{2n+2}) + \epsilon^{2n+2} \leq \alpha \max\{d_s(t_{2n+1}, t_{2n+2}),$$

$$\cdot d_s(t_{2n+2}, T t_{2n+2}), d_s(t_{2n+1}, S t_{2n+1}),$$

$$\cdot d_s(t_{2n+2}, S t_{2n+1}), d_s(t_{2n+1}, T t_{2n+2})\}$$

$$+ L \min\{d_s(t_{2n+2}, S t_{2n+1}), d_s(t_{2n+1}, T t_{2n+2})\} + \epsilon^{2n+2} \quad (10)$$

$$\leq \alpha \max\{d_s(t_{2n+1}, t_{2n+2}), d_s(t_{2n+2}, t_{2n+3}),$$

$$\cdot d_s(t_{2n+1}, t_{2n+2}), d_s(t_{2n+2}, t_{2n+2}), d_s(t_{2n+1}, t_{2n+3})\}$$

$$+ L \min\{d_s(t_{2n+2}, t_{2n+2}), d_s(t_{2n+1}, t_{2n+3})\} + \epsilon^{2n+2},$$

$$\begin{aligned} & \text{or } d_s(t_{2n+2}, t_{2n+3}) \\ & \leq \frac{\alpha_2 + s\alpha_4 + (s\alpha_5/2) + \alpha_7}{1 - \alpha_1 - s\alpha_4 - (s\alpha_5/2) - \alpha_6} d_s(t_{2n+1}, t_{2n+3}) + \epsilon^{2n+2}. \end{aligned} \quad (11)$$

Thus, we have

$$d_s(t_n, t_{n+1}) \leq \lambda d_s(t_{n-1}, t_n) + \epsilon^n, \quad (12)$$

where $\lambda = \max \{ (\alpha_1 + s\alpha_3 + (s\alpha_5/2) + \alpha_7/1 - \alpha_2 - s\alpha_3 - (s\alpha_5/2) - \alpha_6), (\alpha_2 + s\alpha_4 + (s\alpha_5/2) + \alpha_7/1 - \alpha_1 - s\alpha_4 - (s\alpha_5/2) - \alpha_6) \} < 1$.

By Lemma 6, the sequence $\langle t_n \rangle$ is a Cauchy sequence. Since (X, d_s) is complete, there exists $\bar{h} \in X$ such that the Cauchy sequence $\langle t_n \rangle$ is convergent to \bar{h} . We will show that $\bar{h} \in T\bar{h} \cap S\bar{h}$. By the definition of H^β , we have

$$\begin{aligned} & \beta\delta_s(Si_{2n+1}, T\bar{h}) + (1 - \beta)\delta_s(T\bar{h}, Si_{2n+1}) \\ & \leq H^\beta(Si_{2n+1}, T\bar{h}) \leq \alpha_1 d_s(\bar{h}, T\bar{h}) + \alpha_2 d_s(t_{2n+1}, Si_{2n+1}) \\ & \quad + \alpha_3 d_s(\bar{h}, Si_{2n+1}) + \alpha_4 d_s(t_{2n+1}, T\bar{h}) \\ & \quad + \alpha_5 \left[\frac{d_s(\bar{h}, Si_{2n+1}) + d_s(t_{2n+1}, T\bar{h})}{2} \right] \\ & \quad + \alpha_6 \frac{d_s(\bar{h}, T\bar{h}) * d_s(t_{2n+1}, Si_{2n+1})}{1 + d_s(\bar{h}, t_{2n+1})} \\ & \quad + \alpha_7 d_s(\bar{h}, t_{2n+1}) + \epsilon^{2n+1} \\ & \leq \alpha_1 d_s(\bar{h}, T\bar{h}) + \alpha_2 d_s(t_{2n+1}, t_{2n+2}) \\ & \quad + \alpha_3 d_s(\bar{h}, t_{2n+2}) + \alpha_4 d_s(t_{2n+1}, T\bar{h}) \\ & \quad + \alpha_5 \left[\frac{d_s(\bar{h}, t_{2n+2}) + d_s(t_{2n+1}, T\bar{h})}{2} \right] \\ & \quad + \alpha_6 \frac{d_s(\bar{h}, T\bar{h}) d_s(t_{2n+1}, t_{2n+2})}{1 + d_s(\bar{h}, t_{2n+1})} \\ & \quad + \alpha_7 d_s(\bar{h}, t_{2n+1}) + \epsilon^{2n+1}. \end{aligned} \quad (13)$$

□

It follows that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta\delta_s(Si_{2n+1}, T\bar{h}) + (1 - \beta)\delta_s(T\bar{h}, Si_{2n+1}) \\ & \leq \lim [\alpha_1 d_s(\bar{h}, T\bar{h}) + \alpha_2 d_s(t_{2n+1}, t_{2n+2}) \\ & \quad + \alpha_3 d_s(\bar{h}, t_{2n+2}) + \alpha_4 d_s(t_{2n+1}, T\bar{h}) \\ & \quad + \alpha_5 \left[\frac{d_s(\bar{h}, t_{2n+1}) + d_s(t_{2n+1}, T\bar{h})}{2} \right] \\ & \quad + \alpha_6 \frac{d_s(\bar{h}, T\bar{h}) d_s(t_{2n+1}, t_{2n+2})}{1 + d_s(\bar{h}, t_{2n+1})} + \alpha_7 d_s(\bar{h}, t_{2n+1})] \\ & \leq \alpha_1 d_s(\bar{h}, T\bar{h}) + \alpha_4 d_s(\bar{h}, T\bar{h}) + \alpha_5 \frac{d_s(\bar{h}, T\bar{h})}{2} \\ & \leq \left(\alpha_1 + \alpha_4 + \frac{\alpha_5}{2} \right) d_s(\bar{h}, T\bar{h}). \end{aligned} \quad (14)$$

Since

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta\delta_s(Si_{2n+1}, T\bar{h}) + \lim_{n \rightarrow \infty} (1 - \beta)\delta_s(T\bar{h}, Si_{2n+1}) \\ & \leq \lim_{n \rightarrow \infty} \beta\delta_s(Si_{2n+1}, T\bar{h}) + (1 - \beta)\delta_s(T\bar{h}, Si_{2n+1}), \end{aligned} \quad (15)$$

we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta\delta_s(Si_{2n+1}, T\bar{h}) + \lim_{n \rightarrow \infty} (1 - \beta)\delta_s(T\bar{h}, Si_{2n+1}) \\ & \leq \left(\alpha_1 + \alpha_4 + \frac{\alpha_5}{2} \right) d_s(\bar{h}, T\bar{h}). \end{aligned} \quad (16)$$

This implies

$$\lim_{n \rightarrow \infty} \beta\delta_s(Si_{2n+1}, T\bar{h}) \leq \left(\alpha_1 + \alpha_4 + \frac{\alpha_5}{2} \right) d_s(\bar{h}, T\bar{h}). \quad (17)$$

Again, we have

$$\begin{aligned} & \beta\delta_s(Ti_{2n}, S\bar{h}) + (1 - \beta)\delta_s(S\bar{h}, Ti_{2n}) \\ & \leq H^\beta(Ti_{2n}, S\bar{h}) \leq \alpha_1 d_s(t_{2n}, Ti_{2n}) + \alpha_2 d_s(\bar{h}, S\bar{h}) \\ & \quad + \alpha_3 d_s(t_{2n}, S\bar{h}) + \alpha_4 d_s(\bar{h}, Ti_{2n}) \\ & \quad + \alpha_5 \left[\frac{d_s(\bar{h}, Ti_{2n}) + d_s(t_{2n}, S\bar{h})}{2} \right] \\ & \quad + \alpha_6 \left[\frac{d_s(\bar{h}, S\bar{h}) d_s(t_{2n}, Ti_{2n})}{1 + d_s(\bar{h}, t_{2n})} \right] + \alpha_7 d_s(\bar{h}, t_{2n}) + \epsilon^{2n+1} \\ & \leq \alpha_1 d_s(t_{2n}, t_{2n+1}) + \alpha_2 d_s(\bar{h}, S\bar{h}) + \alpha_3 d_s(t_{2n}, S\bar{h}) \\ & \quad + \alpha_4 d_s(\bar{h}, t_{2n+1}) + \alpha_5 \left[\frac{d_s(\bar{h}, t_{2n+1}) + d_s(t_{2n}, S\bar{h})}{2} \right] \\ & \quad + \alpha_6 \frac{d_s(\bar{h}, S\bar{h}) d_s(t_{2n}, t_{2n+1})}{1 + d_s(\bar{h}, t_{2n})} + \alpha_7 d_s(\bar{h}, t_{2n}) + \epsilon^{2n+1}. \end{aligned} \quad (18)$$

It follows that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta\delta_s(Ti_{2n}, S\bar{h}) + (1 - \beta)\delta_s(S\bar{h}, Ti_{2n}) \\ & \leq \lim \left[\alpha_1 d_s(t_{2n}, t_{2n+1}) + \alpha_2 d_s(\bar{h}, S\bar{h}) \right. \\ & \quad + \alpha_3 d_s(t_{2n}, S\bar{h}) + \alpha_4 d_s(\bar{h}, t_{2n+1}) \\ & \quad + \alpha_5 \left[\frac{d_s(\bar{h}, t_{2n+1}) + d_s(t_{2n}, S\bar{h})}{2} \right] \\ & \quad + \alpha_6 \frac{d_s(\bar{h}, S\bar{h}) d_s(t_{2n}, t_{2n+1})}{1 + d_s(\bar{h}, t_{2n})} + \alpha_7 d_s(\bar{h}, t_{2n}) \left. \right] \\ & \leq \alpha_2 d_s(\bar{h}, S\bar{h}) + \alpha_3 d_s(\bar{h}, S\bar{h}) + \alpha_5 \frac{d_s(\bar{h}, S\bar{h})}{2} \\ & \leq \left(\alpha_2 + \alpha_3 + \frac{\alpha_5}{2} \right) d_s(\bar{h}, S\bar{h}). \end{aligned} \quad (19)$$

Since

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta\delta_s(Ti_{2n}, S\bar{h}) + \lim_{n \rightarrow \infty} (1 - \beta)\delta_s(S\bar{h}, Ti_{2n}) \\ & \leq \lim_{n \rightarrow \infty} \beta\delta_s(Ti_{2n}, S\bar{h}) + (1 - \beta)\delta_s(S\bar{h}, Ti_{2n}), \end{aligned} \quad (20)$$

we have

$$\lim_{n \rightarrow \infty} \beta \delta_s(Ti_{2n}, Sh) + \lim_{n \rightarrow \infty} (1 - \beta) \delta_s(Sh, Ti_{2n}) \leq \left(\alpha_2 + \alpha_3 + \frac{\alpha_5}{2} \right) d_s(h, Sh). \quad (21)$$

This implies

$$\lim_{n \rightarrow \infty} \beta \delta_s(Ti_{2n}, Sh) \leq \left(\alpha_2 + \alpha_3 + \frac{\alpha_5}{2} \right) d_s(h, Sh). \quad (22)$$

Now

$$\begin{aligned} d_s(h, Th) &\leq s[d_s(h, i_{2n+2}) + \delta_s(Si_{2n+1}, Th)], \\ d_s(h, Sh) &\leq s[d_s(h, i_{2n+1}) + \delta_s(Ti_{2n}, Sh)]. \end{aligned} \quad (23)$$

Using (17) and (22) in the above two inequalities, we get

$$\begin{aligned} d_s(h, Th) &\leq s \lim_{n \rightarrow \infty} d_s(h, i_{2n+2}) + s \lim_{n \rightarrow \infty} \delta_s(Si_{2n+1}, Th) \\ &\leq \frac{s(\alpha_1 + \alpha_4 + (\alpha_5/2))}{\beta} d_s(h, Th), \\ d_s(h, Sh) &\leq s \lim_{n \rightarrow \infty} d_s(h, i_{2n+1}) + s \lim_{n \rightarrow \infty} \delta_s(Ti_{2n}, Sh) \\ &\leq \frac{s(\alpha_2 + \alpha_3 + (\alpha_5/2))}{\beta} d_s(h, Sh). \end{aligned} \quad (24)$$

This gives $d_s(h, Th) = 0$ and $d_s(h, Sh) = 0$. Since T and S are closed, we have $h \in T$ and $h \in S$.

Our next result provides an extension and new variants of Ćirić's quasi-contraction [15] and multivalued weak quasi-contraction [17], for a pair of multivalued mappings in a b -metric space.

Theorem 12. Let (X, d_s) be a complete b -metric space with constant $s \geq 1$, d_s be $*$ -continuous, and $T, S : X \rightarrow CB^{d_s}(X)$ be multivalued pairwise H^β -Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:

$$H^\beta(Ti, Sj) \leq \alpha \max \{d_s(i, j), d(i, Ti), d_s(j, Sj), d_s(i, Sj), d_s(j, Ti)\} + L \min \{d_s(i, Sj), d_s(j, Ti)\}, \quad (25)$$

for all $i, j \in X$, some $\alpha \geq 0$ with $0 \leq s\alpha < 1/2$ and $L \geq 0$. Then, S and T have a common fixed point.

Proof. Proceeding as in the proof of Theorem 11, for some $i_0 \in X$, $i_1 \in Ti_0$, and $0 < \epsilon < 1$, we construct the sequence $\langle i_n \rangle$ satisfying (7). Then, we have

$$\begin{aligned} d_s(i_{2n+1}, i_{2n+2}) &\leq H^\beta(Ti_{2n}, Si_{2n+1}) + \epsilon^{2n+1} \\ &\leq \alpha \max \{d_s(i_{2n}, i_{2n+1}), d_s(i_{2n}, Ti_{2n}), d_s(i_{2n+1}, Si_{2n+1}), \\ &\quad \cdot d_s(i_{2n}, Si_{2n+1}), d_s(i_{2n+1}, Ti_{2n})\} \\ &\quad + L \min \{d_s(i_{2n}, Si_{2n+1}), d_s(i_{2n+1}, Ti_{2n})\} + \epsilon^{2n+1} \\ &\leq \alpha \max \{d_s(i_{2n}, i_{2n+1}), d_s(i_{2n}, i_{2n+1}), d_s(i_{2n+1}, i_{2n+2}), \\ &\quad \cdot d_s(i_{2n}, i_{2n+2}), d_s(i_{2n+1}, i_{2n+1})\} \\ &\quad + L \min \{d_s(i_{2n}, i_{2n+2}), d_s(i_{2n+1}, i_{2n+1})\} + \epsilon^{2n+1}. \end{aligned} \quad (26)$$

Therefore,

$$d_s(i_{2n+1}, i_{2n+2}) \leq \frac{s\alpha}{1 - s\alpha} d_s(i_{2n}, i_{2n+1}) + \epsilon^{2n+1}. \quad (27)$$

Again,

$$\begin{aligned} d_s(i_{2n+2}, i_{2n+3}) &\leq H^\beta(Si_{2n+1}, Ti_{2n+2}) + \epsilon^{2n+2} \\ &\leq \alpha \max \{d_s(i_{2n+1}, i_{2n+2}), d_s(i_{2n+2}, Ti_{2n+2}), \\ &\quad \cdot d_s(i_{2n+1}, Si_{2n+1}), d_s(i_{2n+2}, Si_{2n+1}), d_s(i_{2n+1}, Ti_{2n+2})\} \\ &\quad + L \min \{d_s(i_{2n+2}, Si_{2n+1}), d_s(i_{2n+1}, Ti_{2n+2})\} + \epsilon^{2n+2} \\ &\leq \alpha \max \{d_s(i_{2n+1}, i_{2n+2}), d_s(i_{2n+2}, i_{2n+3}), \\ &\quad \cdot d_s(i_{2n+1}, i_{2n+2}), d_s(i_{2n+2}, i_{2n+2}), d_s(i_{2n+1}, i_{2n+3})\} \\ &\quad + L \min \{d_s(i_{2n+2}, i_{2n+2}), d_s(i_{2n+1}, i_{2n+3})\} + \epsilon^{2n+2}, \end{aligned} \quad (28)$$

and we get

$$d_s(i_{2n+2}, i_{2n+3}) \leq \frac{s\alpha}{1 - s\alpha} d_s(i_{2n+1}, i_{2n+2}) + \epsilon^{2n+2}. \quad (29)$$

Thus, we have

$$d_s(i_n, i_{n+1}) \leq \lambda d_s(i_{n-1}, i_n) + \epsilon^n, \quad (30)$$

where $\lambda = (s\alpha/(1 - s\alpha)) < 1$. \square

By Lemma 6, the sequence $\langle i_n \rangle$ is a Cauchy sequence. Since (X, d_s) is complete, there exists $h \in X$ such that the Cauchy sequence $\langle i_n \rangle$ is convergent to h . We will show that $h \in Th \cap Sh$. By the definition of H^β , we have

$$\begin{aligned} \beta \delta_s(Si_{2n+1}, Th) + (1 - \beta) \delta_s(Th, Si_{2n+1}) &\leq H^\beta(Si_{2n+1}, Th) \leq \alpha \max \{d_s(i_{2n+1}, h), d_s(h, Th), \\ &\quad \cdot d_s(i_{2n+1}, Si_{2n+1}), d_s(h, Si_{2n+1}), d_s(i_{2n+1}, Th)\} \\ &\quad + L \min \{d_s(h, Si_{2n+1}), d_s(i_{2n+1}, Th)\} + \epsilon^{2n+2} \\ &\leq \alpha \max \{d_s(i_{2n+1}, h), d_s(h, Th), d_s(i_{2n+1}, i_{2n+2}), \\ &\quad \cdot d_s(h, i_{2n+2}), d_s(i_{2n+1}, Th)\} \\ &\quad + L \min \{d_s(h, i_{2n+2}), d_s(i_{2n+1}, Th)\} + \epsilon^{2n+2}. \end{aligned} \quad (31)$$

It follows that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta s \delta_s(Si_{2n+1}, Th) + (1 - \beta) \delta_s(Th, Si_{2n+1}) \\ & \leq \lim [\alpha \max \{d_s(i_{2n+1}, \bar{h}), d_s(\bar{h}, Th), \\ & \quad \cdot d_s(i_{2n+1}, i_{2n+2}), d_s(\bar{h}, i_{2n+2}), d_s(i_{2n+1}, Th)\} \\ & \quad + L \min \{d_s(\bar{h}, i_{2n+2}), d_s(i_{2n+1}, Th)\} \epsilon^{2n+2}] \\ & \leq \alpha d_s(\bar{h}, Th). \end{aligned} \quad (32)$$

Since

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta s \delta_s(Si_{2n+1}, Th) + \lim_{n \rightarrow \infty} (1 - \beta) \delta_s(Th, Si_{2n+1}) \\ & \leq \lim_{n \rightarrow \infty} \beta s \delta_s(Si_{2n+1}, Th) + (1 - \beta) \delta_s(Th, Si_{2n+1}), \end{aligned} \quad (33)$$

we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta s \delta_s(Si_{2n+1}, Th) + \lim_{n \rightarrow \infty} (1 - \beta) \delta_s(Th, Si_{2n+1}) \\ & \leq \alpha d_s(\bar{h}, Th). \end{aligned} \quad (34)$$

This implies

$$\lim_{n \rightarrow \infty} \beta s \delta_s(Si_{2n+1}, Th) \leq \alpha d_s(\bar{h}, Th). \quad (35)$$

Again, we have

$$\begin{aligned} & \beta s \delta_s(Ti_{2n}, Sh) + (1 - \beta) \delta_s(Sh, Ti_{2n}) \\ & \leq H^\beta(Ti_{2n}, Sh) \leq \alpha \max \{d_s(i_{2n}, \bar{h}), d_s(i_{2n}, Ti_{2n}), \\ & \quad \cdot d_s(\bar{h}, Sh), d_s(i_{2n}, Sh), d_s(\bar{h}, Ti_{2n})\} \\ & \quad + L \min \{d_s(i_{2n}, Sh), d_s(\bar{h}, Ti_{2n})\} + \epsilon^{2n+1} \\ & \leq \alpha \max \{d_s(i_{2n}, \bar{h}), d_s(i_{2n}, i_{2n+1}), \\ & \quad \cdot d_s(\bar{h}, Sh), d_s(i_{2n}, Sh), d_s(\bar{h}, i_{2n+1})\} \\ & \quad + L \min \{d_s(i_{2n}, Sh), d_s(\bar{h}, i_{2n+1})\} + \epsilon^{2n+1}. \end{aligned} \quad (36)$$

It follows that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta s \delta_s(Ti_{2n}, Sh) + (1 - \beta) \delta_s(Sh, Ti_{2n}) \\ & \leq \lim [\alpha \max \{d_s(i_{2n}, \bar{h}), d_s(i_{2n}, i_{2n+1}), \\ & \quad \cdot d_s(\bar{h}, Sh), d_s(i_{2n}, Sh), d_s(\bar{h}, i_{2n+1})\} \\ & \quad + L \min \{d_s(i_{2n}, Sh), d_s(\bar{h}, i_{2n+1})\} + \epsilon^{2n+1}] \\ & \leq \alpha d_s(\bar{h}, Sh). \end{aligned} \quad (37)$$

Since

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta s \delta_s(Ti_{2n}, Sh) + \lim_{n \rightarrow \infty} (1 - \beta) \delta_s(Sh, Ti_{2n}) \\ & \leq \lim_{n \rightarrow \infty} \beta s \delta_s(Ti_{2n}, Sh) + (1 - \beta) \delta_s(Sh, Ti_{2n}), \end{aligned} \quad (38)$$

we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \beta s \delta_s(Ti_{2n}, Sh) + \lim_{n \rightarrow \infty} (1 - \beta) \delta_s(Sh, Ti_{2n}) \leq \alpha d_s(\bar{h}, Sh). \end{aligned} \quad (39)$$

This implies

$$\lim_{n \rightarrow \infty} \beta s \delta_s(Ti_{2n}, Sh) \leq \alpha d_s(\bar{h}, Sh). \quad (40)$$

Now,

$$\begin{aligned} d_s(\bar{h}, Th) & \leq s[d_s(\bar{h}, i_{2n+2}) + \delta_s(Si_{2n+1}, Th)], \\ d_s(\bar{h}, Sh) & \leq s[d_s(\bar{h}, i_{2n+1}) + \delta_s(Ti_{2n}, Sh)]. \end{aligned} \quad (41)$$

Using (35) and (40) in the above two inequalities, we get

$$\begin{aligned} d_s(\bar{h}, Th) & \leq s \lim_{n \rightarrow \infty} d_s(\bar{h}, i_{2n+2}) + s \lim_{n \rightarrow \infty} \delta_s(Si_{2n+1}, Th) \\ & \leq \frac{s\alpha}{\beta} d_s(\bar{h}, Th), \\ d_s(\bar{h}, Sh) & \leq s \lim_{n \rightarrow \infty} d_s(\bar{h}, i_{2n+1}) + s \lim_{n \rightarrow \infty} \delta_s(Ti_{2n}, Sh) \\ & \leq \frac{s\alpha}{\beta} d_s(\bar{h}, Th). \end{aligned} \quad (42)$$

Since $s\alpha < 1/2$ and $1/2 \leq \beta \leq 1$, we get $d_s(\bar{h}, Th) = 0$ and $d_s(\bar{h}, Sh) = 0$. As T and S are closed, we have $\bar{h} \in T$ and $\bar{h} \in S$.

Applying the same technique as in the proof of Theorem 12, we can prove the following extension and new variant of Ciric's contraction for a pair of multivalued mappings in a b -metric space.

Theorem 13. Let (X, d_s) be a complete b -metric space with constant $s \geq 1$, d_s be $*$ -continuous, and $T, S : X \longrightarrow CB^d_s(X)$ be multivalued pairwise H^β -Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:

$$\begin{aligned} H^\beta(Ti, Sj) & \leq \alpha \max \left\{ d_s(i, j), d_s(i, Ti), d_s(j, Sj), \right. \\ & \quad \left. \cdot \frac{d_s(i, Sj) + d_s(j, Ti)}{2} \right\}, \end{aligned} \quad (43)$$

for all $i, j \in X$ and some $\alpha \geq 0$ with $s\alpha < \beta$. Then, S and T have a common fixed point.

For $S = T$ in Theorem 11, we get the following result:

Corollary 14. Let (X, d_s) be a complete b -metric space with constant $s \geq 1$, d_s be $*$ -continuous, and $T : X \longrightarrow CB^d_s(X)$ be a multivalued H^β -Hausdorff function for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:

$$\begin{aligned} H^\beta(Ti, Tj) & \leq \alpha_1 d_s(i, Ti) + \alpha_2 d_s(j, Tj) + \alpha_3 d_s(i, Tj) \\ & \quad + \alpha_4 d_s(j, Ti) + \alpha_5 \left(\frac{d_s(i, Tj) + d_s(j, Ti)}{2} \right) \\ & \quad + \alpha_6 \frac{d_s(i, Ti) d_s(j, Tj)}{1 + d_s(i, j)} + \alpha_7 d_s(i, j), \end{aligned} \quad (44)$$

for all $i, j \in X$ and some $\alpha_k \geq 0$, $1 \leq k \leq 7$, with $\alpha_1 + \alpha_2 + s\alpha_5 + \alpha_6 + \alpha_7 + \max\{2s\alpha_3, 2s\alpha_4\} < 1$, $s(\alpha_1 + \alpha_4 + (\alpha_5/2)) < \beta$, and $s(\alpha_2 + \alpha_3 + (\alpha_5/2)) < \beta$. Then, T has a fixed point.

Example 15. Let $X = [0, 5/12] \cup \{2\}$, $d_s(i, j) = |i - j|^2$ for all $i, j \in X$, and $S, T : X \rightarrow CB(X)$ be as follows:

$$\begin{aligned} S(i) &= \begin{cases} \left\{\frac{i}{4}\right\}, & \text{for } i \in \left[0, \frac{5}{12}\right], \\ \left\{0, \frac{1}{3}, 2\right\}, & \text{for } i = 2, \end{cases} \\ T(i) &= \begin{cases} \left\{\frac{i}{4}\right\}, & \text{for } i \in \left[0, \frac{5}{12}\right], \\ \left\{0, \frac{5}{12}, 2\right\}, & \text{for } i = 2. \end{cases} \end{aligned} \quad (45)$$

We will show that the functions S and T satisfy contraction condition (6) for $\beta = 1/2$.

Case 1. $i, j \in [0, 5/12]$. By Proposition 2, we have

$$\begin{aligned} H^{1/2}(S_i, T_j) &= H^{1/2}\left(\left\{\frac{i}{4}\right\}, \left\{\frac{j}{4}\right\}\right) = d_s\left(\frac{i}{4}, \frac{j}{4}\right) = \left|\frac{i}{4} - \frac{j}{4}\right|^2 \\ &\leq \alpha_1 |i - j|^2, \quad \text{for any } \alpha_7 \geq \frac{1}{16} = \alpha_7 d_x(i, j). \end{aligned} \quad (46)$$

Case 2. $i \in [0, 5/12], j = 2$. We have $d_s(i, j) = |2 - i|^2$. The minimum value of $d_s(i, j)$ for $i \in [0, 5/12]$ is $361/144$.

$$\begin{aligned} \delta_s(S_i, T_j) &= \delta_s\left(\left\{\frac{i}{4}\right\}, \left\{0, \frac{5}{12}, 2\right\}\right) = \frac{i^2}{16}, \\ \delta_s(T_j, S_i) &= \delta_s\left(\left\{0, \frac{5}{12}, 1\right\}, \left\{\frac{i}{4}\right\}\right) = \left(2 - \frac{i}{4}\right)^2, \\ H^{1/2}(S_i, T_j) &= \frac{1}{2} \left(\frac{i^2}{16} + \left(2 - \frac{i}{4}\right)^2 \right). \end{aligned} \quad (47)$$

The maximum value of $H^{1/2}(S_i, T_j)$ for $i \in [0, 5/12]$ is 2 (at $i = 0$). Thus, $H^{1/2}(S_i, T_j) \leq \alpha_7 d_s(i, j)$ for any $\alpha_7 \geq 288/361$.

Case 3. $i = 2, j \in [0, 5/12]$. We have $d_s(i, j) = |2 - j|^2$. The minimum value of $d_s(i, j)$ for $j \in [0, 5/12]$ is $361/144$. $\delta_s(S_i, T_j) = \delta_s(\{0, 5/12, 2\}, \{j/4\}) = (2 - (j/4))^2$.

$$\begin{aligned} \delta_s(T_j, S_i) &= \delta_s\left(\left\{\frac{j}{4}\right\}, \left\{0, \frac{5}{12}, 1\right\}\right) = \frac{j^2}{16}, \\ H^{1/2}(S_i, T_j) &= \frac{1}{2} \left(\frac{j^2}{16} + \left(2 - \frac{j}{4}\right)^2 \right). \end{aligned} \quad (48)$$

The maximum value of $H^{1/2}(S_i, T_j)$ for $j \in [0, 5/12]$ is 2 (at $j = 0$). Thus, $H^{1/2}(S_i, T_j) \leq \alpha_7 d_s(i, j)$ for any $\alpha_7 \geq 288/361$.

Thus, S and T satisfy contraction condition (6) for $\beta = 1/2$, $288/361 \leq \alpha_7 < 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$. Simple calculations show that S and T are pairwise H^β -Hausdorff functions. All conditions of Theorem 11 are satisfied, and 0 is a common fixed point of S and T . However, we see that at $i = 0, j = 2$, S and T do not satisfy contraction condition (6) for $\beta = 1$ and so do not satisfy Nadler's contraction and Czerwik's contraction.

Remark 16. In Example 15, simple calculations show that S and T do not satisfy contraction condition (6) for $62/100 < \beta \leq 1$. However, in view of Remark 9 (i), there may exist functions S and T which satisfy contraction condition (6) for $\beta = 1$ but may not satisfy for $\beta < 1$. Thus, for $\beta = 1$, Theorem 11 is an extension of Nadler's contraction [10], Czerwik's contraction [2], and many of their generalizations. For $\beta < 1$, Theorem 11 provides new variants of Nadler's contraction [10], Czerwik's contraction [2], and many of their generalizations.

Example 17. Let $X = \{0, 1/4, 1\}$,

$$d_s(i, j) = |i - j|^2 \quad \text{for all } i, j \in X. \quad (49)$$

and $T : X \rightarrow CB(X)$ be as follows:

$$T(x) = \begin{cases} \{0\}, & \text{for } i \in \left\{0, \frac{1}{4}\right\}, \\ \{0, 1\}, & \text{for } i = 1. \end{cases} \quad (50)$$

We will show that T satisfies (44) with $\beta \in (7/16, 9/16)$.

For if $i, j \in \{0, 1/4\}$, then the result is clear. Suppose $i \in \{0, 1/4\}$ and $j = 1$. Then, $\delta_{d_s}(T_i, T_1) = 0$ and $\delta_{d_s}(T_1, T_i) = 1$ so that $H^\beta(T_i, T_1) = \max\{\beta, 1 - \beta\}$. Also, we have $d_s(i, 1) = 1$ or $9/16$.

If $\beta \in (7/16, 1/2]$, then $H^\beta(T_i, T_1) = 1 - \beta$. Now $1 - \beta \in [8/16, 9/16]$. So $1 - \beta = 16/9(1 - \beta)9/16$ and $1 - \beta < (16/9)(1 - \beta)1$, that is, $1 - \beta \leq (16/9)(1 - \beta)d_s(i, 1)$. Thus, we have $H^\beta(T_i, T_1) = 1 - \beta \leq kd_s(i, 1)$, where $k = 16/9(1 - \beta) < 1$.

Similarly, if $\beta \in [1/2, 9/16)$, we get $H^\beta(T_i, T_1) = \beta \leq kd_s(i, 1)$, where $k = 16/9\beta < 1$.

However, for $i = 1/4$ and $j = 1$, we have

$$\begin{aligned} H\left(T\left(\frac{1}{4}\right), T(1)\right) &= \max\left\{\delta_{d_s}\left(T\left(\frac{1}{4}\right), T_1\right), \delta_{d_s}\left(T_1, T\left(\frac{1}{4}\right)\right)\right\} \\ &= 1 \text{ and } d_s\left(\frac{1}{4}, 1\right) = \frac{9}{16}. \end{aligned} \quad (51)$$

We see that T does not satisfy condition (2.2) of [24] and condition (2.1) of [26]. Thus, Theorem 2.2 of Debnath [24]

and Theorem 2.3 of Kumar and Luambano [26] are not applicable.

Remark 18 (an open question). Obtain the version of results in fixed points in the sense of Debnath [24], Kumar and Luambano [26], and Altun et al. [25] for two or more mappings using H^β -Hausdorff-Pompeiu b -metric, which will give extension and new variants of the respective results and will also generalize Corollary 19.

By taking different values of α_k in Theorem 11, we get the following extension and new variants of well-known contraction principles:

For $\alpha_k = 0$, $k = 1, 2, 3, 4, 5, 6$, we have the following.

Corollary 19 (Nadler's and Czerwik's contraction). *Let (X, d_s) be a complete b -metric space with constant $s \geq 1$ and $T, S : X \rightarrow CB^{d_s}(X)$ be multivalued pairwise H^β -Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:*

$$H^\beta(Ti, Sj) \leq \alpha_d(i, j), \quad (52)$$

for all $i, j \in X$ and $0 \leq \alpha < 1$. Then, S and T have a common fixed point.

For $\alpha_k = 0$, $k = 3, 4, 5, 6, 7$, we have the following.

Corollary 20 (Kannan's contraction). *Let (X, d_s) be a complete b -metric space with constant $s \geq 1$ and $T, S : X \rightarrow CB^{d_s}(X)$ be multivalued pairwise H^β -Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:*

$$H^\beta(Ti, Sj) \leq \alpha_1 d_s(i, Ti) + \alpha_2 d_s(j, Sj), \quad (53)$$

for all $i, j \in X$ and $0 \leq \alpha_1 + \alpha_2 < 1$. Then, S and T have a common fixed point.

For $\alpha_k = 0$, $k = 1, 2, 5, 6, 7$, we have the following.

Corollary 21 (Chattarjee contraction). *Let (X, d_s) be a complete b -metric space with constant $s \geq 1$ and $T, S : X \rightarrow CB^{d_s}(X)$ be multivalued pairwise H^β -Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:*

$$H^\beta(Ti, Sj) \leq \alpha_3 d_s(i, Si) + \alpha_4 d_s(j, Ti), \quad (54)$$

for all $i, j \in X$ and $\max \{s\alpha_3, s\alpha_4\} < 1/2$. Then, S and T have a common fixed point.

For $\alpha_k = 0$, $k = 5, 6$, we have the following.

Corollary 22 (Hardy-Rogers contraction). *Let (X, d_s) be a complete b -metric space with constant $s \geq 1$ and $T, S : X \rightarrow CB^{d_s}(X)$ be multivalued pairwise H^β -Hausdorff func-*

tions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:

$$H^\beta(Ti, Sj) \leq \alpha_1 d_s(i, Ti) + \alpha_2 d_s(j, Sj) + \alpha_3 d_s(i, Si) + \alpha_4 d_s(j, Ti) + \alpha_7 d_s(i, j), \quad (55)$$

for all $i, j \in X$ and $\alpha_1 + \alpha_2 + \alpha_7 + \max \{2s\alpha_3, 2s\alpha_4\} < 1$, $s(\alpha_1 + \alpha_4) < \beta$, and $s(\alpha_2 + \alpha_3) < \beta$. Then, S and T have a common fixed point.

Remark 23. Corollary 19 is an extension and new variant of the results of Nadler [10] and Czerwik [2], Corollaries 20 and 21 are extended and new variants of the set-valued versions of the Kannan contraction and Chatterjee contraction, respectively, whereas Corollary 22 is an extended and new variant of the result of Mirmostafae [16].

If $T, S : X \rightarrow X$ are single-valued mappings and then by Proposition 2, $H^\beta(Ti, Sj) = d_s(Ti, Sj)$ for all $1/2 \leq \beta \leq 1$. So taking $\beta = 1$ in Theorem 11, we get the following results for single-valued mappings.

Corollary 24. *Let (X, d_s) be a complete b -metric space with constant $s \geq 1$ and $T, S : X \rightarrow X$ be single-valued mappings satisfying the following condition:*

$$d_s(Ti, Sj) \leq \alpha_1 d_s(i, Ti) + \alpha_2 d_s(j, Sj) + \alpha_3 d_s(i, Si) + \alpha_4 d_s(j, Ti) + \alpha_5 \left(\frac{d_s(i, Sj) + d_s(j, Ti)}{2} \right) + \alpha_6 \frac{d_s(i, Ti) d_s(j, Sj)}{1 + d_s(i, j)} + \alpha_7 d_s(i, j), \quad (56)$$

for all $i, j \in X$ and $\alpha_1 + \alpha_2 + s\alpha_5 + \alpha_6 + \alpha_7 + \max \{2s\alpha_3, 2s\alpha_4\} < 1$, $s(\alpha_1 + \alpha_4 + (s\alpha_5/2)) < 1$, and $s(\alpha_2 + \alpha_3 + (s\alpha_5/2)) < 1$. Then, S and T have a common fixed point.

Remark 25. Corollary 24 is an extension and b -metric version of the result of Wong [40].

4. Applications

In this section, we provide two applications of our results.

4.1. Application to Multivalued Fractals. In this section inspiring from some recent works in [20, 41, 42], we will apply our result to prove the existence of a unique common multivalued fractal for a pair of iterated multifunction systems. Let $P_i, Q_i : X \rightarrow CB^{d_s}(X)$, $i = 1, 2, \dots, n$, be upper semicontinuous mappings. Then, $P = (P_1, P_2, \dots, P_n)$ and $Q = (Q_1, Q_2, \dots, Q_n)$ form a pair of iterated multifunction systems defined on the b -metric space (X, d_s) . The extended multifractal operators generated by the iterated multifunction systems $P = (P_1, P_2, \dots, P_n)$ and $Q = (Q_1, Q_2, \dots, Q_n)$ are the operators $T_P, T_Q : CB^{d_s}(X) \rightarrow CB^{d_s}(X)$ defined by $T_P(Y) = \bigcup_{i=1}^n P_i(Y)$ and $T_Q(Y) = \bigcup_{i=1}^n Q_i(Y)$, respectively. A common fixed point of T_P and T_Q is called the common multivalued fractal of the iterated multifunction systems $P = (P_1, P_2, \dots, P_n)$ and $Q = (Q_1, Q_2, \dots, Q_n)$.

Theorem 26. Let $P_i, Q_i : X \longrightarrow CB^d_i(X)$, $i = 1, 2, \dots, n$, be upper semicontinuous mappings satisfying the following condition:

For $i = 1, 2, \dots, n$, there exist $\beta \in [1/2, 1]$ and $a_i, e_i \in (0, 1)$, $a_i + 2se_i < 1$, such that for all $x, y \in X$,

$$H^\beta(P_i x, Q_i y) \leq a_i \cdot d_s(x, y) + e_i [d_s(x, Q_i y) + d_s(y, P_i x)]. \quad (57)$$

Then,

(i) For all $U_1, U_2 \in CB(X)$, $H^\beta(T_P(U_1), T_Q(U_2)) \leq a$.
 $H^\beta(U_1, U_2) + b \cdot H^\beta(U_1, T_P(U_1)) + c \cdot H^\beta(U_2, T_Q(U_2))$
 $+ e [H^\beta(U_1, T_Q(U_2)) + H^\beta(U_2, T_P(U_1))]$

(v) The pair of systems $P = (P_1, P_2, \dots, P_n)$ and $Q = (Q_1, Q_2, \dots, Q_n)$ has a unique common multivalued fractal

Proof. Suppose condition (57) holds. Then, for $U_1, U_2 \in CB(X)$, we have

$$\begin{aligned} & \beta \delta(P_i(U_1), Q_i(U_2)) + (1 - \beta) \delta(Q_i(U_2), P_i(U_1)) \\ &= \beta \sup_{x \in U_1} \left(\inf_{y \in U_2} H^\beta(P_i(x), Q_i(y)) \right) \\ & \quad + (1 - \beta) \sup_{y \in U_2} \left(\inf_{x \in U_1} H^\beta(P_i(x), Q_i(y)) \right) \\ & \leq \beta \sup_{x \in U_1} \left(\inf_{y \in U_2} \{a_i \cdot d_s(x, y) + e_i [d_s(x, Q_i y) \right. \\ & \quad \left. + d_s(y, P_i x)]\} \right) + (1 - \beta) \sup_{y \in U_2} \left(\inf_{x \in U_1} \{a_i \cdot d_s(x, y) \right. \\ & \quad \left. + e_i [d_s(x, Q_i y) + d_s(y, P_i x)]\} \right) \\ &= a_i \cdot H^\beta(U_1, U_2) + e_i \left[H^\beta(U_1, Q_i(U_2)) \right. \\ & \quad \left. + H^\beta(U_2, P_i(U_1)) \right]. \end{aligned} \quad (58)$$

Similarly, we get

$$\begin{aligned} & \beta \delta(Q_i(U_2), P_i(U_1)) + (1 - \beta) \delta(P_i(U_1), Q_i(U_2)) \\ & \leq a_i \cdot H^\beta(U_2, U_1) + e_i \left[H^\beta(U_2, P_i(U_1)) + H^\beta(U_1, Q_i(U_2)) \right]. \end{aligned} \quad (59)$$

Then, we have

$$\begin{aligned} & H^\beta(P_i(U_1), Q_i(U_2)) \\ & \leq a_i \cdot H^\beta(U_1, U_2) + e_i \left[H^\beta(U_2, P_i(U_1)) \right. \\ & \quad \left. + H^\beta(U_1, Q_i(U_2)) \right] \quad i = 1, 2, \dots, n \\ & \leq a \cdot H^\beta(U_2, U_1) + e \left[H^\beta(U_2, P_i(U_1)) \right. \\ & \quad \left. + H^\beta(U_1, Q_i(U_2)) \right], \end{aligned} \quad (60)$$

where $a = \max \{a_1, a_2, \dots, a_n\}$ and $e = \max \{e_1, e_2, \dots, e_n\}$. Note that

$$\begin{aligned} & H^\beta \left(\bigcup_{i=1}^n P_i(U_1), \bigcup_{i=1}^n Q_i(U_2) \right) \\ & \leq \max \left\{ H^\beta(P_1(U_1), Q_1(U_2)), \right. \\ & \quad \left. \cdot H^\beta(P_2(U_1), Q_2(U_2)), \dots, H^\beta(P_n(U_1), Q_n(U_2)) \right\}, \end{aligned} \quad (61)$$

and so

$$\begin{aligned} H^\beta(T_P(U_1), T_Q(U_2)) & \leq a \cdot H^\beta(U_1, U_2) + e \left[H^\beta(U_1, T_Q(U_2)) \right. \\ & \quad \left. + H^\beta(U_2, T_P(U_1)) \right]. \end{aligned} \quad (62)$$

Thus, $T_P, T_Q : CB(X) \longrightarrow CB(X)$ satisfies the conditions of Corollary 24 in the metric space $\{CB(X), H^\beta\}$ and hence has a common fixed point U^* in $CB(X)$, which in turn is the unique common multivalued fractal of the pair of iterated multifunction systems $P = (P_1, P_2, \dots, P_n)$ and $Q = (Q_1, Q_2, \dots, Q_n)$. \square

Remark 27. Since $H^\beta(A, B) \leq H(A, B)$, Theorem 26 is a proper improvement and generalization of Theorem 3.4 of [20], Theorem 3.1 of [41], and Theorem 3.8 of [42].

4.2. Application to the Integral Equation. In this section, motivated by the applications given in [28–30] and [31], we establish the sufficient conditions for the existence of a common solution of a pair of nonlinear Volterra-type integral equations.

For some real numbers a, b with $0 \leq a < b$ and $I = [a, b]$, let $X = C(I, \mathbb{R})$ be the Banach space of real continuous functions defined on I equipped with a norm given by $\|t\| = \max_{t \in I} |t(t)|$. For some $p \geq 1$, define a b -metric d_s on X by

$$d_s(t, j) = \max_{t \in I} |t(t) - j(t)|^p, \quad \text{for all } t, j \in X. \quad (63)$$

Then, $(X, d_s, 2^{p-1})$ is a complete b -metric space. Consider the following pair of Volterra-type integral equations:

$$\begin{cases} t(t) = q(t) + \int_a^{\mu(t)} \mathcal{P}(t, s) \mathcal{K}_1(t, s, t(s)) ds + \int_a^{\sigma(t)} \mathcal{Q}(t, s) \mathcal{K}_2(t, s, j(s)) ds, \\ j(t) = q(t) + \int_a^{\mu(t)} \mathcal{P}(t, s) \mathcal{K}_2(t, s, j(s)) ds + \int_a^{\sigma(t)} \mathcal{Q}(t, s) \mathcal{K}_1(t, s, j(s)) ds, \end{cases} \quad (64)$$

for all $t, s \in I = [a, b] \subseteq \mathbb{R}$, $|\lambda| > 0$, $\mathcal{K}_{i=1,2} : I \times I \times X \longrightarrow \mathbb{R}$, and $q : I \longrightarrow \mathbb{R}$, and $\mathcal{P}, \mathcal{Q} : I \times I \longrightarrow \mathbb{R}$ are continuous functions and $\mu, \sigma : I \longrightarrow I$.

Suppose $T, S : X \longrightarrow X$ is self-mappings defined by

$$\begin{cases} Tt(t) = q(t) + \int_a^{\mu(t)} \mathcal{P}(t, s) \mathcal{K}_1(t, s, t(s)) ds + \int_a^{\sigma(t)} \mathcal{Q}(t, s) \mathcal{K}_2(t, s, j(s)) ds, \\ Sj(t) = q(t) + \int_a^{\mu(t)} \mathcal{P}(t, s) \mathcal{K}_2(t, s, j(s)) ds + \int_a^{\sigma(t)} \mathcal{Q}(t, s) \mathcal{K}_1(t, s, j(s)) ds, \end{cases} \quad (65)$$

for all $t, j \in X$, where $t \in I$. It is obvious that $\tilde{h}(t)$ is a solution of (64) if and only if it has a common fixed point of T and S .

Theorem 28. Suppose that the following hypotheses hold:

(H₁) $T(X)$ and $S(X)$ are closed in X

(H₂) There exist nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ with $\alpha_1 + \alpha_2 + 2^p \max\{\alpha_3, \alpha_4\} + 2^{p-1}\alpha_5 + \alpha_6 + \alpha_7 < 1$ such that

$$|\mathcal{K}_1(t, s, t(s)) - \mathcal{K}_2(t, s, j(s))|^p \leq N(T, S, p, t), \quad (66)$$

where

$$\begin{aligned} N(T, S, p, t) &= \alpha_1 |t(t) - Tt(t)|^p + \alpha_2 |j(t) - Sj(t)|^p \\ &\quad + \alpha_3 |t(t) - Sj(t)|^p + \alpha_4 |j(t) - Tt(t)|^p \\ &\quad + \alpha_7 |t(t) - j(t)|^p \\ &\quad + \alpha_5 \left(\frac{|t(t) - Sj(t)|^p + |j(t) - Tt(t)|^p}{2} \right) \\ &\quad + \alpha_6 \frac{|t(t) - Tt(t)|^p |j(t) - Sj(t)|^p}{1 + |t(t) - j(t)|^p}. \end{aligned} \quad (67)$$

$$(H_3) \int_a^{\mu(t)} \mathcal{P}(t, s) ds + \int_a^{\sigma(t)} \mathcal{Q}(t, s) ds \leq 1/2^{p-1}$$

Then, the system (64) of integral equations has unique common solutions in X .

Proof. Using (H₂) and (H₃), we have

$$\begin{aligned} d_s(Tt, Sj) &= \max_{t \in I} |Tt(t) - Sj(t)|^p \leq \max_{t \in I} \left| \int_a^{\mu(t)} \mathcal{P}(t, s) \mathcal{K}_1(t, s, t(s)) ds \right. \\ &\quad + \int_a^{\sigma(t)} \mathcal{Q}(t, s) \mathcal{K}_2(t, s, j(s)) ds - \int_a^{\mu(t)} \mathcal{P}(t, s) \mathcal{K}_2(t, s, j(s)) ds \\ &\quad \left. - \int_a^{\sigma(t)} \mathcal{Q}(t, s) \mathcal{K}_1(t, s, j(s)) ds \right|^p \\ &\leq \max_{t \in I} 2^{p-1} \left\{ \left| \int_a^{\mu(t)} \mathcal{P}(t, s) \mathcal{K}_1(t, s, t(s)) ds \right. \right. \\ &\quad \left. - \int_a^{\mu(t)} \mathcal{P}(t, s) \mathcal{K}_2(t, s, j(s)) ds \right|^p + \left| \int_a^{\sigma(t)} \mathcal{Q}(t, s) \mathcal{K}_2(t, s, j(s)) ds \right. \\ &\quad \left. - \int_a^{\sigma(t)} \mathcal{Q}(t, s) \mathcal{K}_1(t, s, j(s)) ds \right|^p \Big\} \end{aligned}$$

$$\begin{aligned} &\leq \max_{t \in I} 2^{p-1} \left\{ \left| \int_a^{\mu(t)} \mathcal{P}(t, s) (\mathcal{K}_1(t, s, t(s)) - \mathcal{K}_2(t, s, j(s))) ds \right|^p \right. \\ &\quad \left. + \left| \int_a^{\sigma(t)} \mathcal{Q}(t, s) (\mathcal{K}_2(t, s, j(s)) - \mathcal{K}_1(t, s, j(s))) ds \right|^p \right\} \\ &\leq \max_{t \in I} 2^{p-1} \left\{ \int_a^{\mu(t)} |\mathcal{P}(t, s)|^p |\mathcal{K}_1(t, s, t(s)) - \mathcal{K}_2(t, s, j(s))|^p ds \right. \\ &\quad \left. + \int_a^{\sigma(t)} |\mathcal{Q}(t, s)|^p |\mathcal{K}_2(t, s, j(s)) - \mathcal{K}_1(t, s, j(s))|^p ds \right\} \\ &\leq \max_{t \in I} 2^{p-1} \left\{ \int_a^{\mu(t)} |\mathcal{P}(t, s)|^p N(T, S, p, t) ds \right. \\ &\quad \left. + \int_a^{\sigma(t)} |\mathcal{Q}(t, s)|^p N(T, S, p, t) ds \right\} \\ &\leq \max_{t \in I} 2^{p-1} N(T, S, p, t) \left\{ \int_a^{\mu(t)} |\mathcal{P}(t, s)|^p ds + \int_a^{\sigma(t)} |\mathcal{Q}(t, s)|^p ds \right\} \\ &\leq \max_{t \in I} N(T, S, p, t) \leq \alpha_1 d_s(t, Tt) + \alpha_2 d_s(j, Sj) \\ &\quad + \alpha_3 d_s(t, Sj) + \alpha_4 d_s(j, Tt) + \alpha_5 \left(\frac{d_s(t, Sj) + d_s(j, Tt)}{2} \right) \\ &\quad + \alpha_6 \frac{d_s(t, Tt) d_s(j, Sj)}{1 + d_s(t, j)} + \alpha_7 d_s(t, j). \end{aligned} \quad (68)$$

Thus, conditions of Theorem 11 are satisfied. Theorem 11 therefore ensures a common fixed point of T and S , which in turn is a common solution of the pair of integral equations (64). \square

Remark 29. Taking $\mathcal{Q}(t, s) = 0$, $\mathcal{P}(t, s) = 1$, $q(t) = 0$, $\mu(t) = t$ and $a = 0$ in (64), we get the Volterra-type integral equations considered in Rasham et al. [31] and Alshoraify et al. [30].

Remark 30. Taking $\mathcal{Q}(t, s) = 0$, $\mu(t) = 1$ and $a = 0$ in (64), we get the Fredholm-type integral equations (III.3) considered in Shoaib et al. [29].

Remark 31. Taking $\mathcal{Q}(t, s) = 0$, $\mathcal{P}(t, s) = 1$ and $\mu(t) = b$ in (64), we get the Fredholm-type integral equations (III.1) considered in Shoaib et al. [29].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Fixed Points and Continuity Conditions of Generalized b -Quasicontractions

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In this research, we first check that the abstract cone b -metric is discontinuous in the case of normal cone by a counter example. We obtain several meaningful results about generalized b -quasicontraction and Ćirić-type b -quasicontraction in cone b -metric spaces over Banach algebras, weakening certain important conditions of the spaces and the mappings. Meanwhile, several valid examples are given to demonstrate the new notions and fixed point results, when the existing theorems in the literature are not applicable.

1. Introduction and Preliminaries

Since the concept of cone metric space was reintroduced by Huang and Zhang [1], a large number of fixed point results were gained in such spaces. In 2009, the authors in [2] defined quasicontraction in cone metric space with a normal cone. Subsequently, by removing the normality of the cone, Kadelburg et al. [3] established a fixed point theorem with a quasicontractive constant $k \in (0, 1/2)$ ([3], Theorem 2.2). Sequentially, Gajić and Rakočević [4] showed the result holds when $k \in [0, 1)$ in the same spaces. In 2011, the notion of cone b -metric space was given by Hussian and Shah [5], which generalized b -metric space and cone metric space. Afterwards, Huang and Xu [6] gave several fixed point results of different classes contraction in this spaces. In 2013, the scholars in [7] claimed the cone metric space over Banach algebra while the Banach space E is substituted with the Banach algebra \mathcal{A} . In their paper, the most important work was to verify that the fixed point conclusions in cone metric spaces over Banach algebras were not equivalent to those in metric spaces by a nontrivial example. In [8], the authors redefined cone b -metric space over Banach algebra. They obtained some fixed points of contractions in such spaces which were not equivalent to the corresponding work in b -metric spaces. Later on,

numerous interesting fixed point theorems in these spaces were promoted to be studied by many scholars, see [9–19] and their references, but most of the results were established under the completeness of the spaces and some even required the continuity of b -metric (while cone metric and metric are continuous) (see [1–4, 6–8, 18–29]).

In 2018, Aleksić et al. [20] proved that b -metric is discontinuous in general by some examples, which is a generalization of metric. However, the fixed point conclusions of b -quasicontraction in b -metric spaces were also discussed under continuous b -metric and complete b -metric spaces. In order to improve these too strong conditions, we prove that the abstract cone b -metric is discontinuous even with a normal cone. Furthermore, we gain some fixed point results in cone b -metric spaces over Banach algebras when the cone b -metric is discontinuous. Some other conditions are weakened by giving several new concepts, such as T -orbital completeness, orbital continuity, and orbital compactness in these spaces. Our work develops and broadens some significant well-known theorems in the literature [8, 11, 20, 23, 28, 30]. Furthermore, some nontrivial examples are provided to demonstrate that the new concepts and main theorems in this paper are genuine developments and generalizations of some existing ones in the literature.

Now, we start our paper with some preliminary definitions in the literature.

Suppose \mathcal{A} is a real Banach algebra and P is a cone over Banach algebra \mathcal{A} with $\text{int } P \neq \emptyset$, the notation \leq expresses the partial ordering in terms of P . For the definitions of Banach algebras and cones, the readers may refer to [24, 31].

Definition 1. (see [5, 8]). Suppose X is a nonempty set and $s \geq 1$ is a constant, the mapping $d : X \times X \longrightarrow \mathcal{A}$ is said to be a cone b -metric if

(d1) $\theta \leq d(\bar{\omega}, \eta)$ for all $\bar{\omega}, \eta \in X$ and $d(\bar{\omega}, \eta) = \theta$ if and only if $\bar{\omega} = \eta$

(d2) $d(\bar{\omega}, \eta) = d(\eta, \bar{\omega})$ for all $\bar{\omega}, \eta \in X$

(d3) $d(\bar{\omega}, \eta) \leq s[d(\bar{\omega}, \zeta) + d(\zeta, \eta)]$ for all $\bar{\omega}, \eta, \zeta \in X$

The pair (X, d) is called a cone b -metric space over Banach algebra \mathcal{A} .

Definition 2. (see [8]). Suppose (X, d) is a cone b -metric space over Banach algebra \mathcal{A} , $\bar{\omega} \in X$, and $\{\bar{\omega}_n\}$ is a sequence in X , we say

(i) $\{\bar{\omega}_n\}$ converges to x if for each $c \in \mathcal{A}$ with $c \gg \theta$, there is an integer $N \geq 1$ such that $d(\bar{\omega}_n, \bar{\omega}) \ll c$ for all $n > N$

(ii) $\{\bar{\omega}_n\}$ is a Cauchy sequence if for each $c \in \mathcal{A}$ with $c \gg \theta$, there is an integer $N \geq 1$ such that $d(\bar{\omega}_n, \bar{\omega}_m) \ll c$ for all $n, m > N$

(iii) (X, d) is complete if each Cauchy sequence in X is convergent

It is significant to note that different from the usual metric and cone metric with a normal cone, cone b -metric is generally discontinuous even with a normal cone. Let us show an example.

Example 3. Take $\mathcal{A} = \mathbb{R}^2$ with a norm $\|\bar{\omega}_1, \bar{\omega}_2\| = |\bar{\omega}_1| + |\bar{\omega}_2|$. For any $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2)$ and $\eta = (\eta_1, \eta_2)$ in \mathcal{A} , set the multiplication as

$$\bar{\omega}\eta = (\bar{\omega}_1, \bar{\omega}_2)(\eta_1, \eta_2) = (\bar{\omega}_1\eta_1, \bar{\omega}_1\eta_2 + \bar{\omega}_2\eta_1). \quad (1)$$

Let $P = \{(\bar{\omega}_1, \bar{\omega}_2) \in \mathbb{R}^2 : \bar{\omega}_1, \bar{\omega}_2 \geq 0\}$. Then, \mathcal{A} is a real Banach algebra owing the unit element $e = (1, 0)$, and the cone $P \subseteq \mathcal{A}$ is normal. Set $X = \mathbb{N} \cup \{\infty\} \times \mathbb{N} \cup \{\infty\}$ and $d : X \times X \longrightarrow \mathcal{A}$ be defined as

$$d((\bar{\omega}_1, \bar{\omega}_2), (\eta_1, \eta_2)) = \begin{cases} (0, 0), & \bar{\omega}_1 = \eta_1, \bar{\omega}_2 = \eta_2; \\ \left(\left| \frac{1}{\bar{\omega}_1} - \frac{1}{\eta_1} \right|, \left| \frac{1}{\bar{\omega}_2} - \frac{1}{\eta_2} \right| \right), & \text{one of } (\bar{\omega}_1, \bar{\omega}_2) \text{ and } (\eta_1, \eta_2) \text{ is odd and the other is odd or } \infty; \\ (5, 5), & \text{one of } (\bar{\omega}_1, \bar{\omega}_2) \text{ and } (\eta_1, \eta_2) \text{ is even and the other is even or } \infty; \\ (2, 2), & \text{otherwise.} \end{cases} \quad (2)$$

In the above definition, $(\bar{\omega}_1, \bar{\omega}_2)$ is odd if both $\bar{\omega}_1$ and $\bar{\omega}_2$ are odd; $(\bar{\omega}_1, \bar{\omega}_2)$ is ∞ if both $\bar{\omega}_1$ and $\bar{\omega}_2$ are ∞ . We can check that (X, d) is a cone b -metric space over Banach algebra \mathcal{A} where $s = 5/2$.

Let $\zeta_m = (4m - 1, 4m + 1)$, $m \in \mathbb{N}^+$. We have

$$\begin{aligned} d(\zeta_m, \infty) &= d((4m - 1, 4m + 1), (\infty, \infty)) \\ &= \left(\left| \frac{1}{4m - 1} - \frac{1}{\infty} \right|, \left| \frac{1}{4m + 1} - \frac{1}{\infty} \right| \right) \\ &= \left(\frac{1}{4m - 1}, \frac{1}{4m + 1} \right) \longrightarrow (0, 0), \end{aligned} \quad (3)$$

which indicates $\zeta_m \longrightarrow \infty$ but

$$\begin{aligned} d(\zeta_m, 2) &= d((4m - 1, 4m + 1), (2, 2)) \\ &= (2, 2) \rightarrow (5, 5) \\ &= d((\infty, \infty), (2, 2)) = d(\infty, 2). \end{aligned} \quad (4)$$

So, we have showed that the cone b -metric is discontinuous in the case of normal cone.

Definition 4. (see [25]). Suppose P is a solid cone, $P \subseteq \mathcal{A}$. A sequence $\{\sigma_n\} \subset P$ is a c -sequence if for any $c \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $\sigma_n \ll c$ for all $n \geq n_0$.

Lemma 5. (see [8]). Suppose P is a solid cone, $P \subseteq \mathcal{A}$. If $\alpha, \beta \in P$, $\{\bar{\omega}_n\}$, and $\{\eta_n\}$ are c -sequences in \mathcal{A} , then $\{\alpha\bar{\omega}_n + \beta\eta_n\}$ is a c -sequence in \mathcal{A} .

Lemma 6. (see [31]). Suppose \mathcal{A} is a Banach algebra with a unit e and $\bar{\omega} \in \mathcal{A}$, if the spectral radius $\rho(\bar{\omega})$ of $\bar{\omega}$ satisfies

$$\rho(\bar{\omega}) = \lim_{n \rightarrow \infty} \|\bar{\omega}^n\|^{1/n} = \inf_{n \geq 1} \|\bar{\omega}^n\|^{1/n} < 1, \quad (5)$$

then $e - \bar{\omega}$ is invertible. Actually, $(e - \bar{\omega})^{-1} = \sum_{i=0}^{\infty} \bar{\omega}^i$.

Throughout this paper, we always suppose (X, d) is a cone b -metric space over Banach algebra \mathcal{A} with a unit e and $s \geq 1$.

2. Orbital Completeness

In this section, we give several fixed point theorems of generalized b -quasicontraction in orbitally complete cone b -metric spaces over Banach algebras. The cone is neither regular nor normal. The cone b -metric and the self mapping are not required continuous. At first, encouraged by the concepts of orbital continuity, Φ -orbital completeness [32], and k -continuity [21] in usual metric space, we provide the analogous concepts in cone b -metric space over Banach algebra \mathcal{A} , which are important in our proof.

Definition 7. Suppose (X, d) is a cone b -metric space over Banach algebra \mathcal{A} and $\Phi : X \rightarrow X$, take $\omega \in X$ and $O_\Phi(\omega) = \{\omega, \Phi\omega, \Phi^2\omega, \Phi^3\omega, \dots\}$, namely, the orbit of ω under Φ .

The mapping Φ is orbitally continuous at an element $\zeta \in X$ if for any sequence $\{\omega_n\} \subset O_\Phi(\omega)$ (for all $\omega \in X$), $\omega_n \rightarrow \zeta$ as $n \rightarrow \infty$ implies $\Phi\omega_n \rightarrow \Phi\zeta$ as $n \rightarrow \infty$. Note that each continuous mapping is orbitally continuous, but not the converse.

The mapping Φ is k -continuous for $k = 1, 2, \dots$, if $\Phi^{k-1}\omega_n \rightarrow \zeta$ implies $\Phi^k\omega_n \rightarrow \Phi\zeta$ ($n \rightarrow \infty$). It is clear that Φ is 1-continuous if and only if it is continuous, and k -continuity implies $(k+1)$ -continuity for any $k = 1, 2, \dots$ but not the converse. Furthermore, continuity of Φ^k and k -continuity of Φ are independent when $k > 1$. See the following examples.

Example 8. Suppose $\mathcal{A} = C_{\mathbb{R}}^1[0, 1] \times C_{\mathbb{R}}^1[0, 1]$ with a norm,

$$\|\omega_1, \omega_2\| = \|\omega_1\|_{\infty} + \|\omega_2\|_{\infty} + \|\omega_1'\|_{\infty} + \|\omega_2'\|_{\infty}. \quad (6)$$

For any $\omega = (\omega_1, \omega_2)$ and $\eta = (\eta_1, \eta_2)$ in \mathcal{A} , set the multiplication as

$$\omega\eta = (\omega_1, \omega_2)(\eta_1, \eta_2) = (\omega_1\eta_1, \omega_1\eta_2 + \omega_2\eta_1). \quad (7)$$

Let $P = \{(\omega_1(z), \omega_2(z)) \in \mathcal{A} : \omega_1(z) \geq 0, \omega_2(z) \geq 0, z \in [0, 1]\}$. It follows that there is a unit element $e = (1, 0)$ in the real Banach algebra \mathcal{A} . Let $X = [0, 4] \times [0, 4]$ and define $d : X \times X \rightarrow \mathcal{A}$ by

$$d((\omega_1, \omega_2), (\eta_1, \eta_2))(z) = (|\omega_1 - \eta_1|^2 \exp(z), |\omega_2 - \eta_2|^2 \exp(z)), \quad (8)$$

for any $\omega = (\omega_1, \omega_2), \eta = (\eta_1, \eta_2) \in X$. It is obvious that (X, d) is complete and $s = 2$. Suppose $\Phi : X \rightarrow X$ is defined as

$$\Phi(\omega_1, \omega_2) = \begin{cases} \left(\sin \frac{\omega_1}{2}, \sin \frac{\omega_2}{3} \right), & (\omega_1, \omega_2) \in [0, 2] \times [0, 2]; \\ \left(\log \left(1 + \frac{\omega_1}{2} \right), \log \left(1 + \frac{\omega_2}{3} \right) \right), & \text{otherwise.} \end{cases} \quad (9)$$

For any $\zeta_0 = (\omega_1, \omega_2) \in X$, if $\zeta_n = \Phi\zeta_{n-1}, n \in \mathbb{N}$, then $\zeta_n \rightarrow \theta$ implies $\Phi\zeta_n \rightarrow \Phi\theta = \theta$ while $\theta = (0, 0)$. Clearly, Φ is an orbitally continuous mapping rather than continuous.

Moreover, notice that Φ is 2-continuous but not continuous, that is, 2-continuity of Φ does not imply continuity of Φ^2 . Furthermore, for each integer $k \geq 2$, Φ^k is discontinuous while Φ is k -continuous. This indicates that k -continuity of Φ does not imply continuity of Φ^k in usual situation.

In [12], we have given the following definitions of T -orbital completeness.

Definition 9. The space (X, d) is named Φ -orbitally complete, if each Cauchy sequence included in $O_\Phi(\omega)$ for some $\omega \in X$ converges in X . Each complete space (X, d) is Φ -orbitally complete for any Φ but not the converse.

For being convenient, we give the notion of generalized b -quasicontraction in (X, d) . The mapping $\Phi : X \rightarrow X$ is named a generalized b -quasicontraction if there exists $r \in P$ with $\rho(r) < 1/s$, and one has

$$d(\Phi\omega, \Phi\eta) \leq ru(\omega, \eta), \quad (\omega, \eta \in X), \quad (10)$$

where

$$u(\omega, \eta) \in \{d(\omega, \eta), d(\omega, \Phi\omega), d(\eta, \Phi\eta), d(\omega, \Phi\eta), d(\eta, \Phi\omega)\}. \quad (11)$$

When $s = 1$, we call it a generalized quasicontraction in cone metric space over Banach algebra. Before showing our main results, we give an important lemma without the assumptions of completeness and normality.

Lemma 10. Assume the mapping $\Phi : X \rightarrow X$ is a generalized b -quasicontraction in the space (X, d) , for each $\omega_0 \in X$, let $\omega_n = \Phi\omega_{n-1}$. Then, for any integers $i, j \geq 1$, it holds that

$$d(\omega_i, \omega_j) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1). \quad (12)$$

Proof. At first, we show that for any $n \geq 1$ and $1 \leq i \leq n$,

$$d(\omega_i, \omega_n) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1). \quad (13)$$

If $n = 1$, the result is trivial. Assume $n = 2, i = 1$, then

$$d(\omega_1, \omega_2) = d(\Phi\omega_0, \Phi\omega_1) \leq ru(\omega_0, \omega_1), \quad (14)$$

where

$$u(\omega_0, \omega_1) \in \{d(\omega_0, \omega_1), d(\omega_0, \Phi\omega_1), d(\omega_1, \omega_2), d(\omega_0, \omega_2), d(\omega_1, \omega_1)\}. \quad (15)$$

Obviously, $u(\omega_0, \omega_1) \neq d(\omega_1, \omega_2)$ and $u(\omega_0, \omega_1) \neq d(\omega_1, \omega_1)$; otherwise, there is a contradiction.

If $u(\omega_0, \omega_1) = d(\omega_0, \omega_1)$, then

$$d(\omega_1, \omega_2) \leq rd(\omega_0, \omega_1) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1). \quad (16)$$

If $u(\omega_0, \omega_1) = d(\omega_0, \omega_2)$, then

$$d(\omega_1, \omega_2) \leq rd(\omega_0, \omega_2) \leq sr[d(\omega_0, \omega_1) + d(\omega_1, \omega_2)], \quad (17)$$

which implies that $d(\omega_1, \omega_2) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1)$. Thus, (13) is true. Now, we suppose for all integers $n \geq 2$ and $1 \leq i \leq n$,

$$d(\omega_i, \omega_n) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1). \quad (18)$$

We have to prove

$$d(\omega_i, \omega_{n+1}) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1), \quad (19)$$

for $n + 1 \geq 2, 1 \leq i \leq n + 1$. Now, we prove

$$d(\omega_1, \omega_{n+1}) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1). \quad (20)$$

By (10), we obtain

$$d(\omega_1, \omega_{n+1}) = d(\Phi\omega_0, \Phi\omega_n) \leq ru(\omega_0, \omega_n), \quad (21)$$

where

$$u(\omega_0, \omega_n) \in \{d(\omega_0, \omega_n), d(\omega_0, \omega_1), d(\omega_n, \omega_{n+1}), d(\omega_0, \omega_{n+1}), d(\omega_n, \omega_1)\}. \quad (22)$$

If $u(\omega_0, \omega_n) = d(\omega_0, \omega_n)$, then

$$\begin{aligned} d(\omega_1, \omega_{n+1}) &\leq rd(\omega_0, \omega_n) \\ &\leq sr[d(\omega_0, \omega_1) + d(\omega_1, \omega_n)] \\ &= sr[d(\omega_0, \omega_1) + sr(e - sr)^{-1}d(\omega_0, \omega_1)] \\ &= sr\left(e + \sum_{i=1}^{+\infty} (sr)^i\right)d(\omega_0, \omega_1) \\ &= sr(e - sr)^{-1}d(\omega_0, \omega_1). \end{aligned} \quad (23)$$

If $u(\omega_0, \omega_n) = d(\omega_0, \omega_1)$, then $d(\omega_1, \omega_{n+1}) \leq rd(\omega_0, \omega_1) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1)$.

If $u(\omega_0, \omega_n) = d(\omega_0, \omega_{n+1})$, then

$$d(\omega_1, \omega_{n+1}) \leq rd(\omega_0, \omega_{n+1}) \leq sr[d(\omega_0, \omega_1) + d(\omega_1, \omega_{n+1})], \quad (24)$$

which yields $(e - sr)d(\omega_1, \omega_{n+1}) \leq sr d(\omega_0, \omega_1)$. That is, $d(\omega_1, \omega_{n+1}) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1)$.

If $u(\omega_0, \omega_n) = d(\omega_1, \omega_n)$, then

$$d(\omega_1, \omega_{n+1}) \leq rd(\omega_1, \omega_n) \leq r \cdot sr(e - sr)^{-1}d(\omega_0, \omega_1) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1). \quad (25)$$

At last, we only check that (20) is true when $u(\omega_0, \omega_n) = d(\omega_n, \omega_{n+1})$. That is,

$$d(\omega_1, \omega_{n+1}) \leq rd(\omega_n, \omega_{n+1}) = rd(\Phi\omega_{n-1}, \Phi\omega_n) \leq r^2u(\omega_{n-1}, \omega_n), \quad (26)$$

where

$$u(\omega_{n-1}, \omega_n) \in \{d(\omega_{n-1}, \omega_n), d(\omega_{n-1}, \omega_n), d(\omega_n, \omega_{n+1}), d(\omega_{n-1}, \omega_{n+1}), d(\omega_n, \omega_n)\}. \quad (27)$$

Clearly, $u(\omega_{n-1}, \omega_n) \neq d(\omega_n, \omega_{n+1})$ and $u(\omega_{n-1}, \omega_n) \neq d(\omega_n, \omega_n)$. If $u(\omega_{n-1}, \omega_n) = d(\omega_{n-1}, \omega_n)$, then

$$\begin{aligned} d(\omega_1, \omega_{n+1}) &\leq r^2d(\omega_{n-1}, \omega_n) \\ &\leq r^2 \cdot sr(e - sr)^{-1}d(\omega_0, \omega_1) \\ &\leq sr(e - sr)^{-1}d(\omega_0, \omega_1). \end{aligned} \quad (28)$$

If $u(\omega_{n-1}, \omega_n) = d(\omega_{n-1}, \omega_{n+1})$, then

$$d(\omega_1, \omega_{n+1}) \leq rd(\omega_n, \omega_{n+1}) \leq r^2d(\omega_{n-1}, \omega_{n+1}) \leq r^3u(\omega_{n-2}, \omega_n), \quad (29)$$

where

$$u(\omega_{n-2}, \omega_n) \in \{d(\omega_{n-2}, \omega_n), d(\omega_{n-2}, \omega_{n-1}), d(\omega_n, \omega_{n+1}), d(\omega_{n-2}, \omega_{n+1}), d(\omega_{n-1}, \omega_n)\}. \quad (30)$$

Similarly, we also have $u(\omega_{n-2}, \omega_n) \neq d(\omega_n, \omega_{n+1})$. If $u(\omega_{n-2}, \omega_n)$ equals to one of $d(\omega_{n-2}, \omega_n), d(\omega_{n-2}, \omega_{n-1})$ and $d(\omega_{n-1}, \omega_n)$, then by the assumption (18), we have

$$d(\omega_1, \omega_{n+1}) \leq r^3 \cdot sr(e - sr)^{-1}d(\omega_0, \omega_1) \leq sr(e - sr)^{-1}d(\omega_0, \omega_1). \quad (31)$$

It remains to check (20) when $u(\omega_{n-1}, \omega_n) = d(\omega_{n-2}, \omega_{n+1})$; that is,

$$d(\omega_1, \omega_{n+1}) \leq rd(\omega_n, \omega_{n+1}) \leq r^2d(\omega_{n-1}, \omega_{n+1}) \leq r^3d(\omega_{n-2}, \omega_{n+1}). \quad (32)$$

By a similar analysis, we can deduce that

$$\begin{aligned} d(\omega_1, \omega_{n+1}) &\leq rd(\omega_n, \omega_{n+1}) \leq r^2d(\omega_{n-1}, \omega_{n+1}) \leq \dots \\ &\leq r^{n-1}d(\omega_2, \omega_{n+1}) \leq r^n u(\omega_1, \omega_n), \end{aligned} \quad (33)$$

where

$$u(\omega_1, \omega_n) \in \{d(\omega_1, \omega_n), d(\omega_1, \omega_2), d(\omega_n, \omega_{n+1}), d(\omega_1, \omega_{n+1}), d(\omega_n, \omega_2)\}. \quad (34)$$

Since $u(\omega_1, \omega_n) \neq d(\omega_n, \omega_{n+1})$ and $u(\omega_1, \omega_n) \neq d(\omega_1, \omega_{n+1})$, we know $u(\omega_1, \omega_n)$ equals to one of $d(\omega_1, \omega_n), d(\omega_1, \omega_2)$ and $d(\omega_n, \omega_2)$. Therefore, we finally obtain

$$\begin{aligned} d(\omega_1, \omega_{n+1}) &\leq r^n d(\omega_1, \omega_n) \leq r^n \cdot sr(e - sr)^{-1}d(\omega_0, \omega_1) \\ &\leq sr(e - sr)^{-1}d(\omega_0, \omega_1) \end{aligned} \quad (35)$$

or

$$d(\omega_1, \omega_{n+1}) \leq r^n d(\omega_1, \omega_2) \leq r^n \cdot sr(e - sr)^{-1} d(\omega_0, \omega_1) \leq sr(e - sr)^{-1} d(\omega_0, \omega_1) \quad (36)$$

or

$$d(\omega_1, \omega_{n+1}) \leq r^n d(\omega_n, \omega_2) \leq r^n \cdot sr(e - sr)^{-1} d(\omega_0, \omega_1) \leq sr(e - sr)^{-1} d(\omega_0, \omega_1). \quad (37)$$

Hence, (20) is always true. Moreover, by (18), (20), and (33), we know

$$d(\omega_n, \omega_{n+1}) \leq r^{n-1} u(\omega_1, \omega_n) \leq r^{n-1} \cdot sr(e - sr)^{-1} d(\omega_0, \omega_1) \leq sr(e - sr)^{-1} d(\omega_0, \omega_1). \quad (38)$$

For $2 \leq i \leq n+1$, we have

$$d(\omega_i, \omega_{n+1}) = d(\Phi\omega_{i-1}, \Phi\omega_n) \leq ru(\omega_{i-1}, \omega_n), \quad (39)$$

while

$$u(\omega_{i-1}, \omega_n) \in \{d(\omega_{i-1}, \omega_n), d(\omega_{i-1}, \omega_i), d(\omega_n, \omega_{n+1}), d(\omega_{i-1}, \omega_{n+1}), d(\omega_n, \omega_i)\}. \quad (40)$$

If $u(\omega_{i-1}, \omega_n)$ equals to one of $d(\omega_{i-1}, \omega_n), d(\omega_{i-1}, \omega_i), d(\omega_n, \omega_i)$ and $d(\omega_n, \omega_{n+1})$, then by (18) and (38),

$$d(\omega_i, \omega_{n+1}) \leq r \cdot sr(e - sr)^{-1} d(\omega_0, \omega_1) \leq sr(e - sr)^{-1} d(\omega_0, \omega_1). \quad (41)$$

If $u(\omega_{i-1}, \omega_n) = d(\omega_{i-1}, \omega_{n+1})$, by (18) and (20), (33), and (38), we conclude that

$$\begin{aligned} d(\omega_i, \omega_{n+1}) &\leq rd(\omega_{i-1}, \omega_{n+1}) \leq r^{i-1} u(\omega_1, \omega_n) \\ &\leq r^{i-1} \cdot sr(e - sr)^{-1} d(\omega_0, \omega_1) \\ &\leq sr(e - sr)^{-1} d(\omega_0, \omega_1). \end{aligned} \quad (42)$$

Therefore, (19) is true. The proof is finished. \square

Now, we present and prove our main results without requiring the cone to be normal or d to be continuous.

Theorem 11. Suppose $\Phi : X \rightarrow X$ is a generalized b -quasi-contraction mapping in the Φ -orbitally complete space (X, d) , if $\rho(r) < 1/s$, then the mapping Φ possesses one and only one fixed point $\zeta \in X$, and the sequence $\{\Phi^n \omega_0\}$ converges to ζ for each $\omega_0 \in X$.

Proof. For each $\omega_0 \in X$, take $\omega_n = \Phi\omega_{n-1}$. If there is some $n \in \mathbb{N}$ such that $\omega_n = \omega_{n+1} = \Phi\omega_n$, then ω_n is the fixed point. Hence, we assume $\omega_n \neq \omega_{n+1}$ for all $n \in \mathbb{N}$. Let us show that $\{\omega_n\}$ is a Cauchy sequence. For any $n > m > 1$, write $E(m, n) = \{d(\omega_i, \omega_j) : m \leq i < j \leq n\}$. From the concept of general-

ized b -quasi-contraction, for any $u \in E(m, n)$, there exists $v \in E(m-1, n)$ satisfying $u \leq rv$. Thus, it follows that

$$d(\omega_m, \omega_n) \leq ru_{m-1} \leq r^2 u_{m-2} \leq \dots \leq r^{m-1} u_1 \leq r^{m-1} \cdot sr(e - sr)^{-1} d(\omega_0, \omega_1), \quad (43)$$

where $u_{m-1} \in E(m-1, n), u_{m-2} \in E(m-2, n), \dots, u_1 \in E(1, n)$. The last inequality is obtained by Lemma 10. Since $\|sr^m(e - sr)^{-1} d(\omega_0, \omega_1)\| \rightarrow 0$ as $m \rightarrow \infty$ ($\|r^m\| \rightarrow 0$ as $m \rightarrow \infty$). Therefore, for any $c \in \mathcal{A}$ with $c \gg \theta$, there exists an integer $N \geq 1$ satisfying

$$d(\omega_m, \omega_n) \leq sr^m(e - sr)^{-1} d(\omega_0, \omega_1) \ll c, \quad (n > m \geq N). \quad (44)$$

That is, $\{\omega_n\}$ is a Cauchy sequence. By the Φ -orbital completeness of (X, d) , we have $\zeta \in X$ such that $\omega_n \rightarrow \zeta$ as $n \rightarrow \infty$. Next, we check $\Phi\zeta = \zeta$. By (10), we see

$$d(\omega_{n+1}, \Phi\zeta) = d(\Phi\omega_n, \Phi\zeta) \leq ru(\omega_n, \zeta), \quad (45)$$

where

$$u(\omega_n, \zeta) \in \{d(\omega_n, \zeta), d(\omega_n, \omega_{n+1}), d(\zeta, \Phi\zeta), d(\omega_n, \Phi\zeta), d(\zeta, \Phi\omega_n)\}. \quad (46)$$

There are the following three cases:

Case 1. If $u(\omega_n, \zeta)$ equals to one of $d(\omega_n, \zeta), d(\omega_n, \omega_{n+1})$ and $d(\zeta, \Phi\omega_n)$, then $d(\omega_{n+1}, \Phi\zeta)$ is a c -sequence.

Case 2. If $u(\omega_n, \zeta) = d(\zeta, \Phi\zeta)$, we get

$$d(\omega_{n+1}, \Phi\zeta) \leq rd(\zeta, \Phi\zeta) \leq sr[d(\zeta, \omega_{n+1}) + d(\omega_{n+1}, \Phi\zeta)], \quad (47)$$

that is, $d(\omega_{n+1}, \Phi\zeta) \leq sr(e - sr)^{-1} d(\omega_{n+1}, \zeta)$. Thus, $d(\omega_{n+1}, \Phi\zeta)$ is a c -sequence.

Case 3. If $u(\omega_n, \zeta) = d(\omega_n, \Phi\zeta)$, we gain

$$d(\omega_{n+1}, \Phi\zeta) \leq rd(\omega_n, \Phi\zeta) \leq sr[d(\omega_n, \omega_{n+1}) + d(\omega_{n+1}, \Phi\zeta)], \quad (48)$$

that is, $d(\omega_{n+1}, \Phi\zeta) \leq sr(e - sr)^{-1} d(\omega_n, \omega_{n+1})$. Then, $d(\omega_{n+1}, \Phi\zeta)$ is a c -sequence.

In summary, $d(\omega_{n+1}, \Phi\zeta)$ is always a c -sequence. This gives $\omega_n \rightarrow \Phi\zeta$ as $n \rightarrow \infty$. According to the uniqueness of the limit, we know $\zeta = \Phi\zeta$.

It remains to prove the uniqueness of ζ . We assume there exists another fixed point $\hat{\zeta}$ such that $\Phi\hat{\zeta} = \hat{\zeta}$, and then

$$d(\zeta, \hat{\zeta}) = d(\Phi\zeta, \Phi\hat{\zeta}) \leq ru(\zeta, \hat{\zeta}), \quad (49)$$

where

$$u(\zeta, \hat{\zeta}) \in \left\{ d(\zeta, \hat{\zeta}), d(\zeta, \Phi\hat{\zeta}), d(\hat{\zeta}, \Phi\hat{\zeta}), d(\zeta, \Phi\hat{\zeta}), d(\hat{\zeta}, \Phi\hat{\zeta}) \right\} \\ = \left\{ d(\zeta, \hat{\zeta}), \theta \right\}. \quad (50)$$

It is a contradiction. In conclusion, the fixed point ζ is unique, and the sequence $\{\Phi^n \omega_0\}$ converges to ζ . The proof is finished. \square

Taking $s = 1$, we obtain the fixed point results in cone metric spaces over Banach algebras.

Corollary 12. Suppose $T : X \longrightarrow X$ is a generalized quasicontraction mapping in Φ -orbitally complete cone metric space over Banach algebra \mathcal{A} with a unit e , if $\rho(r) < 1$, then the mapping Φ possesses one and only one fixed point $\zeta \in X$, and the sequence $\{\Phi^n \omega_0\}$ converges to ζ for every $\omega_0 \in X$.

Remark 13. Theorem 11 greatly improves Theorem 3.1 in [20], while Theorem 3.1 in [20] depends strongly on the continuity of b -metric. It also generalizes the condition $sh < 1$ (i.e. $h < 1/s$) of Theorem 2.13 in [23] to $\rho(r) < 1/s$. The assumption of completeness in Theorem 3.1 of [11] and Theorem 2.13 in [23] is relaxed by Φ -orbital completeness. Corollary 12 mainly improves and generalizes Theorem 9 in [28], while the results rely on the conditions that the cone is normal and the d is continuous.

Turning to the next theorem, we show that another type of b -quasicontraction in the space (X, d) has a unique fixed point when $1/s < \rho(r) < 1$. Before giving the related result, we require an important lemma in [26].

Lemma 14. Suppose $\{\omega_n\}$ is a sequence in (X, d) satisfying

$$d(\omega_n, \omega_{n+1}) \leq rd(\omega_{n-1}, \omega_n), \quad (51)$$

for some $r \in P$ with $\rho(r) < 1$ and $n \in \mathbb{N}$. Then, $\{\omega_n\}$ is a Cauchy sequence in (X, d) .

Theorem 15. Suppose the space (X, d) is Φ -orbitally complete. Assume the mapping $\Phi : X \longrightarrow X$ satisfies

$$d(\Phi\omega, \Phi\eta) \leq ru(\omega, \eta), \quad (52)$$

for all $\omega, \eta \in X$ and $\rho(r) \in [(1/s), 1)$, where

$$u(\omega, \eta) \in \left\{ d(\omega, \eta), d(\omega, \Phi\omega), d(\eta, \Phi\eta), \frac{d(\omega, \Phi\eta)}{2s}, \frac{d(\eta, \Phi\omega)}{2s} \right\}. \quad (53)$$

If Φ is k -continuous for some $k \geq 1$ or orbitally continuous, then the mapping Φ possesses one and only one fixed point $\zeta \in X$, and the sequence $\{\Phi^n \omega_0\}$ converges to ζ for every $\omega_0 \in X$.

Proof. From Theorem 11, we obtain a sequence $\{\omega_n\}$ by $\omega_{n+1} = \Phi\omega_n$ and suppose $\omega_n \neq \omega_{n+1}$ for all $n \in \mathbb{N}$ and $n \geq 0$. In view of (52), we see that

$$d(\omega_n, \omega_{n+1}) = d(\Phi\omega_{n-1}, \Phi\omega_n) \leq ru(\omega_{n-1}, \omega_n), \quad (54)$$

where

$$u(\omega_{n-1}, \omega_n) \in \left\{ d(\omega_{n-1}, \omega_n), d(\omega_{n-1}, \omega_n), d(\omega_n, \omega_{n+1}), \frac{d(\omega_{n-1}, \omega_{n+1})}{2s}, \frac{d(\omega_n, \omega_n)}{2s} \right\} \\ = \left\{ d(\omega_{n-1}, \omega_n), d(\omega_n, \omega_{n+1}), \frac{d(\omega_{n-1}, \omega_{n+1})}{2s}, \theta \right\}. \quad (55)$$

We immediately get $u(\omega_{n-1}, \omega_n) \neq d(\omega_n, \omega_{n+1})$ and $u(\omega_{n-1}, \omega_n) \neq \theta$. If $u(\omega_{n-1}, \omega_n) = d(\omega_{n-1}, \omega_n)$, then

$$d(\omega_n, \omega_{n+1}) \leq rd(\omega_{n-1}, \omega_n). \quad (56)$$

If $u(\omega_{n-1}, \omega_n) = d(\omega_{n-1}, \omega_{n+1})/2s$, then

$$d(\omega_n, \omega_{n+1}) \leq r \cdot \frac{s[d(\omega_{n-1}, \omega_n) + d(\omega_n, \omega_{n+1})]}{2s} \\ = \frac{r[d(\omega_{n-1}, \omega_n) + d(\omega_n, \omega_{n+1})]}{2}, \quad (57)$$

which gives

$$d(\omega_n, \omega_{n+1}) \leq r(2e - r)^{-1}d(\omega_{n-1}, \omega_n). \quad (58)$$

In fact,

$$\rho(r(2e - r)^{-1}) \leq \rho(r)\rho((2e - r)^{-1}) \leq \frac{\rho(r)}{2 - \rho(r)} < 1. \quad (59)$$

So, the conditions of Lemma 14 are satisfied for each case. By Lemma 14, $\{\omega_n\}$ is a Cauchy sequence. In view of Φ -orbital completeness of (X, d) , we have $\zeta \in X$ such that $\omega_n \longrightarrow \zeta$.

We are now in a position to show that $\Phi\zeta = \zeta$. If Φ is k -continuous, then $\Phi^k \omega_n \longrightarrow \Phi\zeta$ by k -continuity of Φ and $\Phi^{k-1} \omega_n \longrightarrow \zeta$ as $n \longrightarrow \infty$. According to the uniqueness of the limit, we have $\Phi\zeta = \zeta$.

If Φ is orbitally continuous, then $\Phi\omega_n \longrightarrow \Phi\zeta$ due to the fact that $\omega_n \longrightarrow \zeta$. This yields $\zeta = \Phi\zeta$.

Uniqueness of the fixed point follows immediately from (52). \square

Once we replace the set (53) by the following set (61), then the result is true without any continuity of the mapping Φ .

Theorem 16. Suppose (X, d) is the same as in Theorem 15, assume the mapping $\Phi : X \longrightarrow X$ that satisfies

$$d(\Phi\omega, \Phi\eta) \leq ru(\omega, \eta), \quad (60)$$

for all $\omega, \eta \in X$ and $\rho(r) \in [(1/s), 1)$, where

$$u(\bar{\omega}, \eta) \in \left\{ d(\bar{\omega}, \eta), d(\bar{\omega}, \Phi\bar{\omega}), \frac{d(\eta, \Phi\eta)}{s}, \frac{d(\bar{\omega}, \Phi\eta)}{2s}, d(\eta, \Phi\bar{\omega}) \right\}. \quad (61)$$

Then, the mapping Φ possesses one and only one fixed point $\zeta \in X$, and the sequence $\{\Phi^n \bar{\omega}_0\}$ converges to ζ for every $\bar{\omega}_0 \in X$.

Proof. The proof is analogous to Theorem 15. We at first gain a sequence $\bar{\omega}_n = \Phi \bar{\omega}_{n-1} = \Phi^n \bar{\omega}_0$, $n \geq 1$ and suppose that $\bar{\omega}_n \neq \bar{\omega}_{n+1}$, $\forall n \in \mathbb{N}$. By an analogous analysis with Theorem 15, $\bar{\omega}_n \rightarrow \zeta$ for some $\zeta \in X$. We proceed to show that $\zeta = \Phi\zeta$. By (60), we see that

$$d(\bar{\omega}_n, \Phi\zeta) = d(\Phi\bar{\omega}_{n-1}, \Phi\zeta) \leq ru(\bar{\omega}_{n-1}, \zeta), \quad (62)$$

where

$$u(\bar{\omega}_{n-1}, \zeta) \in \left\{ d(\bar{\omega}_{n-1}, \zeta), d(\bar{\omega}_{n-1}, \bar{\omega}_n), \frac{d(\zeta, \Phi\zeta)}{s}, \frac{d(\bar{\omega}_{n-1}, \Phi\zeta)}{2s}, d(\zeta, \bar{\omega}_n) \right\}. \quad (63)$$

There are the following three cases.

Case 1. If $u(\bar{\omega}_{n-1}, \zeta)$ equals to one of $d(\bar{\omega}_{n-1}, \zeta)$, $d(\bar{\omega}_{n-1}, \bar{\omega}_n)$ and $d(\zeta, \bar{\omega}_n)$, then $\{d(\bar{\omega}_n, \Phi\zeta)\}$ is a c -sequence.

Case 2. If $u(\bar{\omega}_{n-1}, \zeta) = d(\zeta, \Phi\zeta)/s$, we have

$$d(\bar{\omega}_n, \Phi\zeta) \leq \frac{rd(\zeta, \Phi\zeta)}{s} \leq \frac{sr[d(\zeta, \bar{\omega}_n) + d(\bar{\omega}_n, \Phi\zeta)]}{s}, \quad (64)$$

that is, $d(\bar{\omega}_n, \Phi\zeta) \leq r(e-r)^{-1}d(\bar{\omega}_n, \zeta)$. Thus, $\{d(\bar{\omega}_n, \Phi\zeta)\}$ is a c -sequence.

Case 3. If $u(\bar{\omega}_{n-1}, \zeta) = d(\bar{\omega}_{n-1}, \Phi\zeta)/2s$, we have

$$d(\bar{\omega}_n, \Phi\zeta) \leq \frac{rd(\bar{\omega}_{n-1}, \Phi\zeta)}{2s} \leq \frac{sr[d(\bar{\omega}_{n-1}, \bar{\omega}_n) + d(\bar{\omega}_n, \Phi\zeta)]}{2s}, \quad (65)$$

that is, $d(\bar{\omega}_n, \Phi\zeta) \leq r(2e-r)^{-1}d(\bar{\omega}_{n-1}, \bar{\omega}_n)$. Thus, $\{d(\bar{\omega}_n, \Phi\zeta)\}$ is a c -sequence.

In summary, we always deduce that $\{d(\bar{\omega}_n, \Phi\zeta)\}$ is a c -sequence. This gives $\bar{\omega}_n \rightarrow \Phi\zeta$ as $n \rightarrow \infty$. Since the limit is unique, we know $\zeta = \Phi\zeta$. The remaining proof is analogous to Theorem 11. \square

Remark 17. In Theorem 15, we complement and perfect Theorem 11 in [8], which obtained the conclusions under the condition $\rho(r) < 1/s$ in complete spaces (X, d) . Moreover, the conditions in Theorem 15 are much weaker than Theorem 2.1 in [20], since we obtain the results by k -continuity for some $k \geq 1$ or orbital continuity of the mapping, without appealing the continuity of cone b -metric or the mapping Φ .

According to the proof of Theorem 16 and the symmetry of the cone b -metric d , we see at once that (61) can be replaced by

$$u(\bar{\omega}, \eta) \in \left\{ d(\bar{\omega}, \eta), \frac{d(\bar{\omega}, \Phi\bar{\omega})}{s}, d(\eta, \Phi\eta), d(\bar{\omega}, \Phi\eta), \frac{d(\eta, \Phi\bar{\omega})}{2s} \right\}, \quad (66)$$

for all $\bar{\omega}, \eta \in X$.

Example 18. Set $\mathcal{A} = C_{\mathbb{R}}^1[0, 2]$ with a norm $\|\bar{\omega}\| = \|\bar{\omega}\|_{\infty} + \|\bar{\omega}'\|_{\infty}$, for any $\bar{\omega} \in \mathcal{A}$. The multiplication in \mathcal{A} is taken as the pointwise multiplication. We conclude that \mathcal{A} is a Banach algebra owing the unit element $e = 1$. Let $X = [0, 2]$. For all $\bar{\omega}, \eta \in X$, define

$$d(\bar{\omega}, \eta)(z) = \begin{cases} |\bar{\omega} - \eta|^2 \psi, & \bar{\omega}, \eta \in [0, 1]; \\ 2|\bar{\omega} - \eta|^2 \psi, & \text{otherwise,} \end{cases} \quad (67)$$

where $\psi \in P = \{\varphi(z) \in \mathcal{A} : \varphi(z) \geq 0, z \in [0, 2]\}$. Note that the cone P is nonnormal, and the cone b -metric d is discontinuous. Indeed, let $\bar{\omega}_n = 1 + (1/n)$, $n \in \mathbb{N}$, and then

$$d(\bar{\omega}_n, 1)(z) = d\left(1 + \frac{1}{n}, 1\right)(z) = 2\left|1 + \frac{1}{n} - 1\right|^2 \psi \rightarrow 0. \quad (68)$$

So, $\bar{\omega}_n \rightarrow 1$. However,

$$\begin{aligned} d\left(\bar{\omega}_n, \frac{1}{2}\right)(z) &= d\left(1 + \frac{1}{n}, \frac{1}{2}\right)(z) = 2\left|1 + \frac{1}{n} - \frac{1}{2}\right|^2 \psi \rightarrow \frac{1}{2}\psi, \\ d\left(1, \frac{1}{2}\right)(z) &= \left|1 - \frac{1}{2}\right|^2 \psi = \frac{1}{4}\psi, \end{aligned} \quad (69)$$

that is,

$$d\left(\bar{\omega}_n, \frac{1}{2}\right)(z) \not\rightarrow d\left(1, \frac{1}{2}\right)(z). \quad (70)$$

Thus, the cone b -metric d is discontinuous.

The mapping $\Phi : X \rightarrow X$ is defined as

$$\Phi\bar{\omega} = \begin{cases} \frac{\bar{\omega}}{3}, & \bar{\omega} \in [0, 1], \\ 0, & \bar{\omega} \in (1, 2). \end{cases} \quad (71)$$

It suffices to show that Φ is orbitally continuous rather than continuous (one also can check that Φ is k -continuous for each integer $k \geq 2$ but Φ^k is discontinuous for each $k \geq 1$). In addition, (X, d) is Φ -orbitally complete but not complete cone b -metric space over Banach algebra \mathcal{A} with the coefficient $s = 4$. In fact, for $\bar{\omega}_n = 2 - 1/n$, $n \in \mathbb{N}$, we get

$$d(\bar{\omega}_n, \bar{\omega}_m) = 2\left|\frac{1}{n} - \frac{1}{m}\right|^2 \psi \rightarrow 0, n, m \rightarrow \infty, \quad (72)$$

but there exists no $\bar{\omega} \in X$ satisfying $d(\bar{\omega}_n, \bar{\omega}) \rightarrow 0$. Hence, (X, d) is not complete. Take $r(z) = (z/4) + 1/2$. We can calculate that $\rho(r) = 3/4 \in [(1/s), 1)$. Now, there are the following three cases to verify the inequality (52).

For all $\bar{\omega}, \eta \in [0, 1]$, we get $d(\Phi\bar{\omega}, \Phi\eta)(z) = |\bar{\omega}/3 - \eta/3|^2\psi$ and

$$u(\bar{\omega}, \eta)(z) \in \left\{ |\bar{\omega} - \eta|^2\psi, \left| \bar{\omega} - \frac{\bar{\omega}}{3} \right|^2\psi, \left| \eta - \frac{\eta}{3} \right|^2\psi, \frac{1}{8} \left| \bar{\omega} - \frac{\eta}{3} \right|^2\psi, \frac{1}{8} \left| \eta - \frac{\bar{\omega}}{3} \right|^2\psi \right\}. \quad (73)$$

Then, (52) holds by taking $u(\bar{\omega}, \eta)(z) = |\bar{\omega} - \eta|^2\psi$.

For all $\bar{\omega} \in [0, 1], \eta \in (1, 2)$, we gain $d(\Phi\bar{\omega}, \Phi\eta)(z) = |\bar{\omega}/3 - 0|^2\psi = (\bar{\omega}^2/9)\psi$ and

$$\begin{aligned} u(\bar{\omega}, \eta)(z) &\in \left\{ 2|\bar{\omega} - \eta|^2\psi, \left| \bar{\omega} - \frac{\bar{\omega}}{3} \right|^2\psi, 2|\eta - 0|^2\psi, \frac{1}{8}|\bar{\omega} - 0|^2\psi, \frac{1}{8} \cdot 2 \left| \eta - \frac{\bar{\omega}}{3} \right|^2\psi \right\} \\ &= \left\{ 2|\bar{\omega} - \eta|^2\psi, \frac{4\bar{\omega}^2}{9}\psi, 2\eta^2\psi, \frac{\bar{\omega}^2}{8}\psi, \frac{1}{4} \left| \eta - \frac{\bar{\omega}}{3} \right|^2\psi \right\}. \end{aligned} \quad (74)$$

Then, (52) holds by taking $u(\bar{\omega}, \eta)(z) = (4\bar{\omega}^2/9)\psi$.

For all $\bar{\omega}, \eta \in (1, 2)$, then $\Phi\bar{\omega} = \Phi\eta = 0$. We observe that $d(\Phi\bar{\omega}, \Phi\eta)(z) = 0$ and

$$\begin{aligned} u(\bar{\omega}, \eta)(z) &\in \left\{ 2|\bar{\omega} - \eta|^2\psi, 2|\bar{\omega} - 0|^2\psi, 2|\eta - 0|^2\psi, \frac{1}{8} \cdot 2|\bar{\omega} - 0|^2\psi, \frac{1}{8} \cdot 2|\eta - 0|^2\psi \right\} \\ &= \left\{ 2|\bar{\omega} - \eta|^2\psi, 2\bar{\omega}^2\psi, 2\eta^2\psi, \frac{\bar{\omega}^2}{4}\psi, \frac{\eta^2}{4}\psi \right\}. \end{aligned} \quad (75)$$

So, (52) holds trivially. In the same manner, we can prove that (52) is true for all $\bar{\omega} \in (1, 2), \eta \in [0, 1]$. Therefore, Φ possesses a unique fixed point $0 \in X$, and the sequence $\{\Phi^n \bar{\omega}\}$ converges to 0 for each $\bar{\omega} \in X$ by Theorem 15.

Furthermore, there is no $r \in P$ with $\rho(r) \in [0, 1)$ satisfy $(\bar{\omega}^2/9)\psi \leq 2r|\bar{\omega} - \eta|^2\psi$ for

$$\bar{\omega} = 1 \in [0, 1], \eta = 1 + \frac{1}{9} \in (1, 2) \text{ and } \psi(z) > 0, \quad (76)$$

in Case 2, which means that Φ is not a Banach-type b -contraction. The work from b -metric space, cone b -metric space, or cone b -metric space over Banach algebra which requires completeness, continuity, or Banach-type b -contraction is not applicable here (see [6, 8, 11, 18–20, 23, 26]). Due to the continuity of metric and cone metric, the corresponding theorems from such spaces (see [1–4, 7, 22, 24, 28]) cannot be used in this example, either.

3. Orbital Compactness

Garai et al. [29] and Haokip and Goswami [33] defined Φ -orbitally compact metric spaces and Φ -orbitally compact b -metric spaces, respectively, which extend sequentially compact metric (b -metric) spaces. Now, the similar definition of Φ -orbital compactness and a fixed point theorem of

Ćirić-type b -quasicontraction in cone b -metric spaces over Banach algebras is showed.

Definition 19. The mapping $\Phi : X \rightarrow X$ is named a Ćirić-type b -quasicontraction in (X, d) , if for all $\bar{\omega}, \eta \in X$ with $\bar{\omega} \neq \eta$, we have

$$d(\Phi\bar{\omega}, \Phi\eta) < u(\bar{\omega}, \eta), \quad (77)$$

where

$$u(\bar{\omega}, \eta) \in \left\{ d(\bar{\omega}, \eta), d(\bar{\omega}, \Phi\bar{\omega}), d(\eta, \Phi\eta), \frac{d(\bar{\omega}, \Phi\eta) + d(\eta, \Phi\bar{\omega})}{2s} \right\}. \quad (78)$$

Definition 20. Suppose the mapping $\Phi : X \rightarrow X$, the set X is Φ -orbitally compact, if each sequence in $O_\Phi(\bar{\omega})$ has a convergent subsequence for all $\bar{\omega} \in X$. Clearly, every Φ -orbitally compact cone b -metric space over Banach algebra does not need to be complete.

Example 21. Suppose the Banach algebra \mathcal{A} and cone P are the same as in Example 8, take $X = [0, 2) \times [0, 2)$ and $\Phi : X \rightarrow X$ be given by $\Phi(\bar{\omega}_1, \bar{\omega}_2) = ((\bar{\omega}_1/3), (\bar{\omega}_2/5))$. Then, X is Φ -orbitally compact rather than complete.

In the last theorem, suppose that (X, d) owes a regular cone P with $d(\bar{\omega}, \eta) \in \text{int } P$, where $\bar{\omega}, \eta \in X$ and $\bar{\omega} \neq \eta$, the cone b -metric d is continuous.

Theorem 22. Suppose (X, d) is Φ -orbitally compact, if $\Phi : X \rightarrow X$ is a Ćirić-type b -quasicontraction and orbitally continuous, then the mapping Φ possesses one and only one fixed point $\zeta \in X$.

Proof. For any $\bar{\omega}_0 \in X$, set $\bar{\omega}_n = \Phi\bar{\omega}_{n-1} = T^n\bar{\omega}_0, n \geq 1$. Note that $\bar{\omega}_n \neq \bar{\omega}_{n+1}$ for all $n \in \mathbb{N}$. In fact, if $\bar{\omega}_n = \bar{\omega}_{n+1} = \Phi\bar{\omega}_n$ for some $n \in \mathbb{N}$, then $\bar{\omega}_n$ is a fixed point of Φ . Let $l_n = d(\bar{\omega}_n, \bar{\omega}_{n+1})$ for every $n \in \mathbb{N}$. From (77), we know

$$l_n = d(\bar{\omega}_n, \bar{\omega}_{n+1}) = d(\Phi\bar{\omega}_{n-1}, \Phi\bar{\omega}_n) < u(\bar{\omega}_{n-1}, \bar{\omega}_n), \quad (79)$$

where

$$u(\bar{\omega}_{n-1}, \bar{\omega}_n) \in \left\{ d(\bar{\omega}_{n-1}, \bar{\omega}_n), d(\bar{\omega}_{n-1}, \bar{\omega}_n), d(\bar{\omega}_n, \bar{\omega}_{n+1}), \frac{d(\bar{\omega}_{n-1}, \bar{\omega}_{n+1}) + d(\bar{\omega}_n, \bar{\omega}_n)}{2s} \right\}. \quad (80)$$

Indeed, $u(\bar{\omega}_{n-1}, \bar{\omega}_n) \neq d(\bar{\omega}_n, \bar{\omega}_{n+1})$; so, it remains the following two cases.

Case 1. When $u(\bar{\omega}_{n-1}, \bar{\omega}_n) = d(\bar{\omega}_{n-1}, \bar{\omega}_n)$, then $d(\bar{\omega}_n, \bar{\omega}_{n+1}) < d(\bar{\omega}_{n-1}, \bar{\omega}_n)$.

Case 2. When $u(\bar{\omega}_{n-1}, \bar{\omega}_n) = d(\bar{\omega}_{n-1}, \bar{\omega}_{n+1}) + d(\bar{\omega}_n, \bar{\omega}_n)/2s$, then

$$\begin{aligned} d(\bar{\omega}_n, \bar{\omega}_{n+1}) &< \frac{d(\bar{\omega}_{n-1}, \bar{\omega}_{n+1}) + d(\bar{\omega}_n, \bar{\omega}_n)}{2s} \\ &\leq \frac{s[d(\bar{\omega}_{n-1}, \bar{\omega}_n) + d(\bar{\omega}_n, \bar{\omega}_{n+1})]}{2s}, \end{aligned} \quad (81)$$

which means $d(\bar{\omega}_n, \bar{\omega}_{n+1}) < d(\bar{\omega}_{n-1}, \bar{\omega}_n)$. Thus, by repeating the above process, we deduce that

$$\theta < l_n = d(\bar{\omega}_n, \bar{\omega}_{n+1}) < l_{n-1} < \cdots < l_0 = d(\bar{\omega}_1, \bar{\omega}_0). \quad (82)$$

By the regularity of the cone, there exists $c_0 \in \mathcal{A}$, $c_0 \geq \theta$ such that $l_n \rightarrow c_0$ as $n \rightarrow \infty$. Because X is Φ -orbitally compact, we have a convergent subsequence $\{\bar{\omega}_{n_i}\}$ of $\{\bar{\omega}_n\}$ and a point $\zeta \in X$ satisfying $\bar{\omega}_{n_i} \rightarrow \zeta$ as $n \rightarrow \infty$, according to orbital continuity of Φ , $\Phi\bar{\omega}_{n_i} \rightarrow \Phi\zeta$.

When $c_0 > \theta$, we gain

$$\theta < c_0 = \lim_{i \rightarrow \infty} d(\bar{\omega}_{n_i}, \Phi\bar{\omega}_{n_i}) = d(\zeta, \Phi\zeta). \quad (83)$$

Moreover, since the cone is regular, we see that

$$\theta < c_0 = \lim_{i \rightarrow \infty} l_{n_i} = \lim_{i \rightarrow \infty} d(\Phi\bar{\omega}_{n_i}, \Phi^2\bar{\omega}_{n_i}) = d(\Phi\zeta, \Phi^2\zeta) < u(\zeta, \Phi\zeta), \quad (84)$$

where

$$u(\zeta, \Phi\zeta) \in \left\{ d(\zeta, \Phi\zeta), d(\zeta, \Phi\zeta), d(\Phi\zeta, \Phi^2\zeta), \frac{d(\zeta, \Phi^2\zeta) + d(\Phi\zeta, \Phi\zeta)}{2s} \right\}. \quad (85)$$

It is evidence to get

$$c_0 = d(\Phi\zeta, \Phi^2\zeta) < d(\zeta, \Phi\zeta) = c_0, \quad (86)$$

a contradiction. So, $c_0 = \theta$ and $\zeta = \Phi\zeta$, namely, ζ is a fixed point.

Finally, let us prove that the fixed point is unique by (77). Otherwise, if there is another fixed point η , then

$$d(\eta, \zeta) = d(\Phi\eta, \Phi\zeta) < u(\eta, \zeta), \quad (87)$$

where

$$\begin{aligned} u(\eta, \zeta) &\in \left\{ d(\eta, \zeta), d(\eta, \Phi\eta), d(\zeta, \Phi\zeta), \frac{d(\eta, \Phi\zeta) + d(\zeta, \Phi\eta)}{2s} \right\} \\ &= \left\{ d(\eta, \zeta), \theta, \frac{d(\eta, \Phi\zeta) + d(\zeta, \Phi\eta)}{2s} \right\}. \end{aligned} \quad (88)$$

If $u(\eta, \zeta) = d(\eta, \zeta)$ or $u(\eta, \zeta) = \theta$, then this is a contradiction. If $u(\eta, \zeta) = [d(\eta, \Phi\zeta) + d(\zeta, \Phi\eta)]/2s$, then

$$\begin{aligned} d(\eta, \zeta) &< \frac{d(\eta, \Phi\zeta) + d(\zeta, \Phi\eta)}{2s} \\ &\leq \frac{s[d(\eta, \zeta) + d(\zeta, \Phi\zeta)] + s[d(\zeta, \eta) + d(\eta, \Phi\eta)]}{2s} = d(\eta, \zeta), \end{aligned} \quad (89)$$

a contradiction too. The result follows. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Endpoints of Generalized Contractions in \mathcal{F} -Metric Spaces with Application to Integral Equations

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The purpose of this article is to introduce locally α - ζ -multivalued contraction and rational Ćirić type α - ζ -multivalued contraction in the context of \mathcal{F} -metric spaces and prove some endpoint results. We provide a nontrivial example to show the authenticity of our main result. Our results generalize some well-known results of literature. We also present some endpoint results in both graphic \mathcal{F} -metric spaces and ordered \mathcal{F} -metric spaces. As an application of our main result, we investigate the solution of an integral equation.

1. Introduction

In 2010, Amini-Harandi [1] showed that a multivalued mapping has a unique endpoint if and only if this multivalued mapping has the approximate endpoint property. Hussain et al. [2] established some approximate endpoints of the multivalued almost I-contractions in complete metric spaces. Later on, Moradi and Khojasteh [3] proved a result for generalized weak contractive multifunctions.

On the other hand, Samet et al. [4] introduced the notion of α -admissibility and α - ζ -contraction in 2012. Asl et al. [5] extended this notion of α -admissibility to α^* -admissibility and proved some results for multivalued mappings. In 2015, Mohammadi and Rezapour [6] improved the α -admissibility concept and obtained endpoint of α - ζ -multivalued contraction. Later on, Choudhury et al. [7] used the notion of α -admissibility and proved end point results of multivalued mappings without continuity. Very recently, Isik et al. [8] proved endpoint results for α - ζ -contraction in the newly introduced space of Jleli and Samet [9] which is named as \mathcal{F} -metric space (\mathcal{F} -MS). In this article, we give locally α - ζ -multivalued contraction and rational Ćirić type α - ζ -multivalued contraction in the framework of \mathcal{F} -metric space and generalized the main result of Isik et al. [8].

2. Preliminaries

Let $\mathcal{M} = \emptyset$ and $\mathcal{T} : \mathcal{M} \rightarrow 2^{\mathcal{M}}$ (nonempty subsets of \mathcal{M}) be a multivalued mapping. A point $\sigma \in \mathcal{M}$ is professed to be an endpoint (fixed point) of \mathcal{T} if $\mathcal{T}\sigma = \{\sigma\}$ ($\sigma \in \mathcal{T}\sigma$). Now, let (\mathcal{M}, d) be a metric space, then \mathcal{T} is said to satisfy the approximate fixed point property if

$$\inf_{\sigma \in \mathcal{M}} \sup_{y \in \mathcal{T}\sigma} d(\sigma, y) = 0. \quad (1)$$

Let $\mathcal{CB}(\mathcal{M})$ represents the set of all nonempty, closed, and bounded subsets of \mathcal{M} . The Hausdorff metric \mathcal{H} is defined on $\mathcal{CB}(\mathcal{M})$ as follows:

$$\mathcal{H}(A, B) = \max \left\{ \sup_{\sigma \in A} d(\sigma, B), \sup_{y \in B} d(y, A) \right\}. \quad (2)$$

In 2012, Samet et al. [4] used the following set Ψ of non-decreasing functions $\zeta : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\sum_{n=1}^{\infty} \zeta^n(t) < \infty, \text{ for all } t > 0, \quad (3)$$

and introduced α - ζ -contraction. Clearly, $\zeta(t) < t$ for all $t > 0$ ([30]).

Samet et al. [4] also initiated the concept of α -admissibility of a single valued mapping in this way.

Definition 1 (see [4]). Let $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ and let $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$, then \mathcal{T} is said to be α -admissible if $\forall \sigma, y \in \mathcal{M}$, $\alpha(\sigma, y) \geq 1$ implies $\alpha(\mathcal{T}\sigma, \mathcal{T}y) \geq 1$.

They gave the following property of \mathcal{M} that is \mathcal{M} is α -regular, if for each sequence $\{\sigma_n\}$ in \mathcal{M} with $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$, and $\sigma_n \longrightarrow \sigma$, then $\alpha(\sigma_n, \sigma) \geq 1$, $\forall n$.

In 2013, Asl et al. [5] extended this concept to multivalued mapping and gave the notion of α^* -admissibility as follows.

Definition 2 (see [5]). Let $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ and let $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$, then \mathcal{T} is said to be α^* -admissible if for all $\sigma, y \in \mathcal{M}$, $\alpha(\sigma, y) \geq 1$ implies $\alpha^*(\mathcal{T}\sigma, \mathcal{T}y) \geq 1$, where $\alpha^*(A, B) = \inf \{\alpha(a, b) : a \in A, b \in B\}$, for all $A, B \in \mathcal{CB}(\mathcal{M})$.

In 2015, Mohammadi and Rezapour [6] extended the above notion in this way.

Definition 3 (see [6]). Let $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ and $\mathcal{T} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$, then \mathcal{T} is α -admissible provided that for all $\sigma \in \mathcal{M}$ and $y \in \mathcal{T}\sigma$ with $\alpha(\sigma, y) \geq 1$, then $\alpha(y, z) \geq 1$, for all $z \in \mathcal{T}y$.

They proved endpoint results for α - ζ -multivalued contraction by using the following property.

A multivalued mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is said to satisfy the property (\mathcal{BS}) , if for all $\sigma \in \mathcal{M}$, there exists $y \in \mathcal{T}\sigma$ such that $\mathcal{H}(\mathcal{T}\sigma, \mathcal{T}y) = \sup_{b \in \mathcal{T}y} d(y, b)$. Isik et al. [8] used the property (\mathcal{BS}) of Mohammadi and Rezapour [6] to prove their results, that is, for each sequence $\{\sigma_n\}$ with

$$d(\sigma_n, \mathcal{T}\sigma_n) \leq d(\sigma_n, \sigma_{n+1}) + \zeta(d(\sigma_n, \sigma_{n+1})), \quad (4)$$

for all n and $\sigma_n \longrightarrow \sigma$, then $d(\sigma_n, \mathcal{T}\sigma_n) \leq d(\sigma_n, \sigma) + \zeta(d(\sigma_n, \sigma))$, for all $n \geq N$.

For more details in this direction, we refer the readers (see [10–14]).

Recently, Jleli and Samet [9] introduced an interesting generalization of metric space which is called \mathcal{F} -metric space (\mathcal{F} -MS) as follows.

Let \mathcal{F} be the class of $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $f(\sigma_1) < f(\sigma_2)$, for $\{\sigma_n\} \subseteq \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \sigma_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} f(\sigma_n) = -\infty$.

Definition 4 (see [9]). Let $\mathcal{M} \neq \emptyset$, and let $d_{\mathcal{F}} : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$. Suppose that there exists $f \in \mathcal{F}$ and $\alpha \in [0, +\infty)$ such that

$$\begin{aligned} (D_1) d_{\mathcal{F}}(\sigma, y) = 0 &\Leftrightarrow \sigma = y, \text{ for all } (\sigma, y) \in \mathcal{M} \times \mathcal{M} \\ (D_2) d_{\mathcal{F}}(\sigma, y) &= d_{\mathcal{F}}(y, \sigma), \text{ for all } (\sigma, y) \in \mathcal{M} \times \mathcal{M} \end{aligned}$$

(D_3) for every $(\sigma, y) \in \mathcal{M} \times \mathcal{M}$, for every $N \in \mathbb{N}$, $N \geq 2$ and for every $(\sigma_i)_{i=1}^N \subset \mathcal{M}$ with $(u_1, u_N) = (\sigma, y)$, we have

$$d_{\mathcal{F}}(\sigma, y) > 0 \Rightarrow f(d_{\mathcal{F}}(\sigma, y)) \leq f\left(\sum_{i=1}^{N-1} d_{\mathcal{F}}(u_i, u_{i+1})\right) + \alpha \quad (5)$$

Then, $(\mathcal{M}, d_{\mathcal{F}})$ is called an \mathcal{F} -MS.

Theorem 5 (see [9]). Let $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -MS and let $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$. Suppose that these assertions hold:

- (i) $(\mathcal{M}, d_{\mathcal{F}})$ is \mathcal{F} -complete
- (ii) there exists $k \in (0, 1)$ such that

$$d_{\mathcal{F}}(\mathcal{T}(\sigma), \mathcal{T}(y)) \leq kd_{\mathcal{F}}(\sigma, y) \quad (6)$$

Then, there exists $\sigma^* \in \mathcal{M}$ such that $\mathcal{T}\sigma^* = \sigma^*$ which is unique.

Hussain and Kanwal [15] utilized an \mathcal{F} -metric space and generalized the above result by considering the notion of α - ζ -contraction to prove a fixed point theorem. Many researchers (see [16–18]) worked in this newly generalized space.

Very recently, Isik et al. [8] introduced the notion of Hausdorff metric $\mathcal{H}_{\mathcal{F}}(\cdot, \cdot)$ on $\mathcal{CB}(\mathcal{M})$ influence by \mathcal{F} -metric $d_{\mathcal{F}}$ as follows:

$$\mathcal{H}_{\mathcal{F}}(A, B) = \max \left\{ \sup_{\sigma \in A} d_{\mathcal{F}}(\sigma, B), \sup_{y \in B} d_{\mathcal{F}}(y, A) \right\}, \quad (7)$$

for all $A, B \in \mathcal{CB}(\mathcal{M})$, where $d_{\mathcal{F}}(\sigma, B) = \inf_{y \in B} d_{\mathcal{F}}(\sigma, y)$ and obtained endpoint results for α - ζ -multivalued contraction in this way.

Theorem 6. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -MS and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ be an α -admissible mapping which satisfies the property (\mathcal{BS}) . Suppose there exists $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ and $\zeta \in \Psi$ such that

$$\alpha(\sigma, y) \geq 1 \Rightarrow \mathcal{H}_{\mathcal{F}}(\mathcal{T}(\sigma), \mathcal{T}(y)) \leq \zeta(d_{\mathcal{F}}(\sigma, y)). \quad (8)$$

Also, suppose that these assertions hold:

- (i) $(\mathcal{M}, d_{\mathcal{F}})$ is \mathcal{F} -complete
- (ii) $\alpha(\sigma_0, \sigma_1) \geq 1$ for an $\sigma_0 \in \mathcal{M}$ and $\sigma_1 \in \mathcal{T}(\sigma_0)$
- (iii) \mathcal{M} is α -regular

Then, \mathcal{T} has an endpoint.

3. Main Results

Definition 7. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -MS. A mapping $\mathcal{T} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$ is called a locally α - ζ -multivalued contraction if there exists $\zeta \in \Psi$ and $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ such that

$$\alpha(\sigma, \gamma) \geq 1 \Rightarrow \mathcal{H}_{\mathcal{F}}(\mathcal{T}(\sigma), \mathcal{T}(\gamma)) \leq \zeta(d_{\mathcal{F}}(\sigma, \gamma)), \quad (9)$$

for $\sigma, \gamma \in B(\bar{\sigma}_0, r)$.

Now, we state our main result regarding the existence of the endpoint of an α - ζ -multivalued contraction on the closed ball $B(\bar{\sigma}_0, r)$ which is very advantageous in the perception that it needs the contractiveness of the multivalued mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ only on the closed ball instead of the whole space.

Theorem 8. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -MS and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ be an α -admissible, locally α - ζ -multivalued contraction such that \mathcal{T} satisfies the property (\mathcal{BS}) and for $\sigma_0 \in \mathcal{M}$, there exists $\sigma_1 \in \mathcal{T}\sigma_0$ such that

$$\zeta^i(d(\sigma_0, \sigma_1)) < r, \quad (10)$$

for all $n = 0, 1, 2, \dots$ and $r > 0$. Also, suppose that the following assertions hold:

- (i) $(\mathcal{M}, d_{\mathcal{F}})$ is \mathcal{F} -complete
- (ii) $\alpha(\sigma_0, \sigma_1) \geq 1$ for an $\sigma_0 \in \mathcal{M}$ and $\sigma_1 \in \mathcal{T}(\sigma_0)$
- (iii) \mathcal{M} is α -regular

Then, \mathcal{T} has an endpoint.

Proof. Choose $\sigma_0 \in \mathcal{M}$ and $\sigma_1 \in \mathcal{T}\sigma_0$ such that $\alpha(\sigma_0, \sigma_1) \geq 1$. It follows directly from (10); we have

$$d(\sigma_0, \sigma_1) < r, \quad (11)$$

which implies that

$$\sigma_1 \in B(\bar{\sigma}_0, r). \quad (12)$$

□

It follows from (10) that

$$\alpha(\sigma_0, \sigma_1) \geq 1 \Rightarrow H(\mathcal{T}\sigma_0, \mathcal{T}\sigma_1) \leq \zeta(d(\sigma_0, \sigma_1)). \quad (13)$$

Since \mathcal{T} satisfies the property (\mathcal{BS}) , so $\exists \sigma_2 \in \mathcal{T}\sigma_1$ such that $\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_1, \mathcal{T}\sigma_2) = \sup_{b \in \mathcal{T}\sigma_2} d_{\mathcal{F}}(\sigma_2, b)$. Now, from (13), we have

$$\begin{aligned} d(\sigma_1, \sigma_2) &\leq \sup_{b \in \mathcal{T}\sigma_1} d_{\mathcal{F}}(\sigma_1, b) = \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_0, \mathcal{T}\sigma_1) \\ &\leq \zeta(d(\sigma_0, \sigma_1)) < r. \end{aligned} \quad (14)$$

This implies that

$$\sigma_2 \in \overline{B(\sigma_0, r)}. \quad (15)$$

Since \mathcal{T} is α -admissible, $\alpha(\sigma_1, \sigma_2) \geq 1$, so t follows from (9) that

$$\alpha(\sigma_1, \sigma_2) \geq 1 \Rightarrow H(\mathcal{T}\sigma_1, \mathcal{T}\sigma_2) \leq \zeta(d(\sigma_1, \sigma_2)). \quad (16)$$

Continuing this process, we obtain a sequence $\{\sigma_n\}$ in $B(\bar{\sigma}_0, r)$ such that $\sigma_{n+1} \in \mathcal{T}\sigma_n$, $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$ and $\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma_{n+1}) = \sup_{b \in \mathcal{T}\sigma_{n+1}} d_{\mathcal{F}}(\sigma_{n+1}, b)$, for all n . If $\sigma_n = \sigma_{n+1}$ for some $n \in \mathbb{N}$, then we get that $\mathcal{H}_{\mathcal{F}}(\{\sigma_{n+1}\}, \mathcal{T}\sigma_{n+1}) = \sup_{b \in \mathcal{T}\sigma_{n+1}} d_{\mathcal{F}}(\sigma_{n+1}, b) = \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma_{n+1}) = 0$. It implies that σ_{n+1} is an endpoint. Hence, we suppose that $\sigma_n \neq \sigma_{n+1}$, for all $n \in \mathbb{N}$.

Now, since $\alpha(\sigma_{n-1}, \sigma_n) \geq 1$, so

$$\begin{aligned} d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}) &\leq \sup_{b \in \mathcal{T}\sigma_n} d_{\mathcal{F}}(\sigma_n, b) = \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma_n) \\ &\leq \zeta(d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n)) \leq \zeta^2(d_{\mathcal{F}}(\sigma_{n-2}, \sigma_{n-1})) \\ &\leq \dots \leq \zeta^n(d_{\mathcal{F}}(\sigma_0, \sigma_1)), \end{aligned} \quad (17)$$

for all $n \geq 0$. Assume that $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied and fix $\varepsilon > 0$. By (F_2) , $\exists \delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\varepsilon) - \alpha. \quad (18)$$

Suppose that $N \in \mathbb{N}$ be such that $0 < \sum_{i \geq N} \zeta^{i-1}(d_{\mathcal{F}}(\sigma_1, \sigma_2)) < \delta$. Hence, by (17), (18) and (\mathcal{F}_1) , we have

$$\begin{aligned} f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\sigma_i, \sigma_{i+1})\right) &\leq f\left(\sum_{i=n}^{m-1} \zeta^{i-1}(d_{\mathcal{F}}(\sigma_1, \sigma_2))\right) \\ &\leq f\left(\sum_{i \geq N} \zeta^{i-1}(d_{\mathcal{F}}(\sigma_1, \sigma_2))\right) \\ &< f(\varepsilon) - \alpha, \end{aligned} \quad (19)$$

for $m > n \geq N$. Using (\mathfrak{D}_3) and (19), we obtain that $d_{\mathcal{F}}(\sigma_n, \sigma_m) > 0$ where $m > n \geq N$ which implies that

$$f(d_{\mathcal{F}}(\sigma_n, \sigma_m)) \leq f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\sigma_i, \sigma_{i+1})\right) + \alpha < f(\varepsilon), \quad (20)$$

which implies by (\mathcal{F}_1) that $d_{\mathcal{F}}(\sigma_n, \sigma_m) < \varepsilon$, for all $m > n \geq N$. This proves that $\{\sigma_n\}$ is \mathcal{F} -Cauchy. Because of \mathcal{F} -completeness of \mathcal{M} , there exists $\sigma^* \in B(\bar{\sigma}_0, r)$ such that $\sigma_n \longrightarrow \sigma^*$. We shall prove that σ^* is an endpoint of \mathcal{T} . We assume on the contrary that $\mathcal{T}\sigma^* \neq \{\sigma^*\}$. Then $\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \mathcal{T}\sigma^*) > 0$. Since \mathcal{M} is locally α -regular, so $\alpha(\sigma_n, \sigma^*) \geq 1$, for all $n \in \mathbb{N}$. Then, by (9) and (\mathcal{F}_1) , we have

$$\begin{aligned} f(\mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n)) &= f(\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma_n)) \\ &\leq f(\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma^*)) \\ &\quad + \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma^*) + \alpha \\ &\leq f(\zeta(d_{\mathcal{F}}(\sigma_{n-1}, \sigma^*))) \\ &\quad + \zeta(d_{\mathcal{F}}(\sigma_n, \sigma^*)) \\ &\quad + \alpha \longrightarrow -\infty, \end{aligned} \quad (21)$$

as $n \longrightarrow \infty$. Thus,

$$\lim_{n \longrightarrow \infty} \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) = 0. \quad (22)$$

On the other side,

$$\begin{aligned}
 & f(\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \mathcal{T}\sigma^*)) \\
 & \leq f(\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \{\sigma_n\}) + \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) \\
 & \quad + \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma^*)) + \alpha \\
 & \leq f(d(\sigma^*, \sigma_n) + \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) \\
 & \quad + \zeta(d_{\mathcal{F}}(\sigma_n, \sigma^*))) \longrightarrow -\infty,
 \end{aligned} \tag{23}$$

as $n \longrightarrow \infty$, that is a contradiction. Hence, $\{\sigma^*\} = \mathcal{T}\sigma^*$.

Definition 9. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -MS. A mapping $\mathcal{T} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$ is called a rational Ćirić type α - ζ -multivalued contraction if there exists two functions $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ and $\zeta \in \Psi$ such that

$$\alpha(\sigma, \gamma) \mathcal{H}_{\mathcal{F}}(\mathcal{T}(\sigma), \mathcal{T}(\gamma)) \leq \zeta(R_{\mathcal{F}}(\sigma, \gamma)), \tag{24}$$

for $(\sigma, \gamma) \in \mathcal{M} \times \mathcal{M}$, where

$$R_{\mathcal{F}}(\sigma, \gamma) = \max \left\{ d_{\mathcal{F}}(\sigma, \gamma), \frac{d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) d_{\mathcal{F}}(\gamma, \mathcal{T}\gamma)}{1 + d_{\mathcal{F}}(\sigma, \gamma)} \right\}. \tag{25}$$

Theorem 10. Suppose that $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -MS and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ be an α -admissible and rational Ćirić type α - ζ -multivalued contraction such that \mathcal{T} satisfies the property (\mathcal{BS}) . Also, suppose that these conditions hold:

- (i) $(\mathcal{M}, d_{\mathcal{F}})$ is \mathcal{F} -complete
- (ii) $\alpha(\sigma_0, \sigma_1) \geq 1$ for an $\sigma_0 \in \mathcal{M}$ and $\sigma_1 \in \mathcal{T}(\sigma_0)$;
- (iii) \mathcal{T} is continuous

Then, \mathcal{T} has an endpoint.

Proof. Choose $\sigma_0 \in \mathcal{M}$ and $\sigma_1 \in \mathcal{T}\sigma_0$ such that $\alpha(\sigma_0, \sigma_1) \geq 1$. Since \mathcal{T} satisfies the property (\mathcal{BS}) , there exists $\sigma_2 \in \mathcal{T}\sigma_1$ such that

$$\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_1, \mathcal{T}\sigma_2) = \sup_{b \in \mathcal{T}\sigma_2} d_{\mathcal{F}}(\sigma_2, b). \tag{26}$$

□

Since \mathcal{T} is α -admissible, $\alpha(\sigma_1, \sigma_2) \geq 1$. Continuing this process, we obtain a sequence $\{\sigma_n\}$ such that $\sigma_{n+1} \in \mathcal{T}\sigma_n$, $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$ and

$$\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma_{n+1}) = \sup_{b \in \mathcal{T}\sigma_{n+1}} d_{\mathcal{F}}(\sigma_{n+1}, b), \tag{27}$$

for all n . If $\sigma_n = \sigma_{n+1}$ for some $n \in \mathbb{N}$, then we get that

$$\begin{aligned}
 \mathcal{H}_{\mathcal{F}}(\{\sigma_{n+1}\}, \mathcal{T}\sigma_{n+1}) &= \sup_{b \in \mathcal{T}\sigma_{n+1}} d_{\mathcal{F}}(\sigma_{n+1}, b) \\
 &= \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma_{n+1}) = 0.
 \end{aligned} \tag{28}$$

It implies that σ_{n+1} is an endpoint. Hence, we suppose that $\sigma_n \neq \sigma_{n+1}$, for all $n \in \mathbb{N}$.

Note that

$$\begin{aligned}
 d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}) &\leq \sup_{b \in \mathcal{T}\sigma_n} d_{\mathcal{F}}(\sigma_n, b) = \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma_n) \\
 &\leq \alpha(\sigma_{n-1}, \sigma_n) \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma_n) \leq \zeta(R_{\mathcal{F}}(\sigma_{n-1}, \sigma_n)) \\
 &= \zeta \left(\max \left\{ d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n), \frac{d_{\mathcal{F}}(\sigma_{n-1}, \mathcal{T}\sigma_{n-1}) d_{\mathcal{F}}(\sigma_n, \mathcal{T}\sigma_n)}{1 + d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n)} \right\} \right) \\
 &\leq \zeta \left(\max \left\{ d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n), \frac{d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n) d_{\mathcal{F}}(\sigma_n, \sigma_{n+1})}{1 + d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n)} \right\} \right) \\
 &\leq \zeta(\max \{ d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n), d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}) \}),
 \end{aligned} \tag{29}$$

for all $n \geq 2$. If $\max \{ d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n), d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}) \} = d_{\mathcal{F}}(\sigma_n, \sigma_{n+1})$, then

$$d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}) \leq \zeta(d_{\mathcal{F}}(\sigma_n, \sigma_{n+1})) < d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}), \tag{30}$$

which is a contradiction. So, we have

$$\max \{ d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n), d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}) \} = d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n), \tag{31}$$

which implies

$$d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}) \leq \zeta(d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n)). \tag{32}$$

Continuing in this way, we obtain that

$$\begin{aligned}
 d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}) &\leq \zeta(d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n)) \leq \zeta^2(d_{\mathcal{F}}(\sigma_{n-2}, \sigma_{n-1})) \\
 &\leq \zeta^3(d_{\mathcal{F}}(\sigma_{n-3}, \sigma_{n-2})) \leq \dots \leq \zeta^n(d_{\mathcal{F}}(\sigma_0, \sigma_1)),
 \end{aligned} \tag{33}$$

for all $n \geq 2$ which yields that

$$\sum_{i=n}^{m-1} d_{\mathcal{F}}(\sigma_i, \sigma_{i+1}) \leq \sum_{i=n}^{m-1} \zeta^i(d_{\mathcal{F}}(\sigma_0, \sigma_1)), \tag{34}$$

for $m > n \geq 2$. Suppose that $\varepsilon > 0$ be arbitrary. Next, let $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ be such that $(d_{\mathcal{F}_3})$ is satisfied. By (\mathcal{F}_2) , there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\varepsilon) - \alpha. \tag{35}$$

Suppose that $N \in \mathbb{N}$ be such that $\sum_{i \geq N} \zeta^i(d_{\mathcal{F}}(\sigma_1, \sigma_2)) < \delta$. Hence, by (24), (35) and (\mathcal{F}_1) , we have

$$\begin{aligned}
 f \left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\sigma_i, \sigma_{i+1}) \right) &\leq f \left(\sum_{i=n}^{m-1} \zeta^i(d_{\mathcal{F}}(\sigma_0, \sigma_1)) \right) \\
 &\leq f \left(\sum_{i \geq N} \zeta^i(d_{\mathcal{F}}(\sigma_0, \sigma_1)) \right) \\
 &< f(\varepsilon) - \alpha,
 \end{aligned} \tag{36}$$

for $m > n \geq N$. Using (\mathfrak{D}_3) and (36), we obtain that $d_{\mathcal{F}}(\sigma_n, \sigma_m) > 0$ where $m > n \geq N$ which implies that

$$f(d_{\mathcal{F}}(\sigma_n, \sigma_m)) \leq f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\sigma_i, \sigma_{i+1})\right) + \alpha < f(\varepsilon), \quad (37)$$

which implies by (\mathcal{F}_1) that $d_{\mathcal{F}}(\sigma_n, \sigma_m) < \varepsilon$, for all $m > n \geq N$. This proves that $\{\sigma_n\}$ is \mathcal{F} -Cauchy. As \mathcal{M} is \mathcal{F} -complete, so $\exists \sigma^* \in \mathcal{M}$ such that $\sigma_n \rightarrow \sigma^*$. We shall prove that σ^* is an endpoint of \mathcal{T} . We assume on contrary that $\mathcal{T}\sigma^* \neq \{\sigma^*\}$. Then, $\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \mathcal{T}\sigma^*) > 0$. Now, we have

$$\begin{aligned} f(\mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n)) \\ = f(\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma_n)) \\ \leq f(\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma^*) + \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma^*)) + \alpha. \end{aligned} \quad (38)$$

Note that we used the property (\mathcal{BS}) in the above inequality. Taking the limit in both sides of the above inequality and using continuity assumption of \mathcal{T} , we get $\lim_{n \rightarrow \infty} f(\mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n)) = -\infty$ which implies that $\lim_{n \rightarrow \infty} \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) = 0$. Hence,

$$\begin{aligned} f(\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \mathcal{T}\sigma^*)) \\ \leq f(\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \{\sigma_n\}) + \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) \\ + \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma^*)) + \alpha \\ \leq f(d(\sigma^*, \sigma_n) + \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) \\ + \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma^*)) \rightarrow -\infty, \end{aligned} \quad (39)$$

as $n \rightarrow \infty$, that is a contradiction. Hence, $\{\sigma^*\} = \mathcal{T}\sigma^*$.

Example 1. Consider the set $\mathcal{M} = \{1, 2, 3\}$. Suppose that the mapping $d_{\mathcal{F}} : \mathcal{M} \times \mathcal{M} \rightarrow [0, +\infty)$ be given by

$$\begin{aligned} d_{\mathcal{F}}(1, 1) &= d_{\mathcal{F}}(2, 2) = d_{\mathcal{F}}(3, 3) = 0, \\ d_{\mathcal{F}}(1, 2) &= d_{\mathcal{F}}(2, 1) = \frac{1}{2}, \\ d_{\mathcal{F}}(2, 3) &= d_{\mathcal{F}}(3, 2) = \frac{2}{3}, \\ d_{\mathcal{F}}(1, 3) &= d_{\mathcal{F}}(3, 1) = \frac{4}{3}. \end{aligned} \quad (40)$$

So, $(\mathcal{M}, d_{\mathcal{F}})$ is an \mathcal{F} -metric on \mathcal{M} with $f(t) = \ln(\sqrt{t})$ and $\alpha = \ln \sqrt{7/6}$. Now, define $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{CB}(\mathcal{M})$ by $\mathcal{T}(1) = \mathcal{T}(2) = \{1\}$ and $\mathcal{T}(3) = \{1, 2\}$. Taking $\zeta(t) = (3/4)t$, we have

$$\begin{aligned} \mathcal{H}_{\mathcal{F}}(\mathcal{T}(1), \mathcal{T}(2)) &= 0, \\ \mathcal{H}_{\mathcal{F}}(\mathcal{T}(1), \mathcal{T}(3)) &= d_{\mathcal{F}}(1, 2) = \frac{1}{2} \leq \frac{3}{4} = \frac{3}{4} R_{\mathcal{F}}(1, 3), \end{aligned} \quad (41)$$

where

$$\begin{aligned} R_{\mathcal{F}}(1, 3) &= \max \left\{ d_{\mathcal{F}}(1, 3), \frac{d_{\mathcal{F}}(1, \mathcal{T}(1))d_{\mathcal{F}}(3, \mathcal{T}(3))}{1 + d_{\mathcal{F}}(1, 3)} \right\} \\ &= \max \left\{ d_{\mathcal{F}}(1, 3), \frac{d_{\mathcal{F}}(1, 1)d_{\mathcal{F}}(3, \{1, 2\})}{1 + d_{\mathcal{F}}(1, 3)} \right\}, \\ \mathcal{H}_{\mathcal{F}}(\mathcal{T}(2), \mathcal{T}(3)) &= d_{\mathcal{F}}(1, 2) = \frac{1}{2} \leq \frac{3}{4} = \frac{3}{4} R_{\mathcal{F}}(2, 3), \end{aligned} \quad (42)$$

where

$$\begin{aligned} R_{\mathcal{F}}(2, 3) &= \max \left\{ d_{\mathcal{F}}(2, 3), \frac{d_{\mathcal{F}}(2, \mathcal{T}(2))d_{\mathcal{F}}(3, \mathcal{T}(3))}{1 + d_{\mathcal{F}}(2, 3)} \right\} \\ &= \max \left\{ d_{\mathcal{F}}(2, 3), \frac{d_{\mathcal{F}}(2, 1)d_{\mathcal{F}}(3, \{1, 2\})}{1 + d_{\mathcal{F}}(2, 3)} \right\}. \end{aligned} \quad (43)$$

Therefore,

$$\alpha(\sigma, \gamma) \mathcal{H}_{\mathcal{F}}(\mathcal{T}(\sigma), \mathcal{T}(\gamma)) \leq \zeta(R_{\mathcal{F}}(\sigma, \gamma)), \quad (44)$$

where

$$R_{\mathcal{F}}(\sigma, \gamma) = \max \left\{ d_{\mathcal{F}}(\sigma, \gamma), \frac{d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma)d_{\mathcal{F}}(\gamma, \mathcal{T}\gamma)}{1 + d_{\mathcal{F}}(\sigma, \gamma)} \right\}, \quad (45)$$

for all $\sigma, \gamma \in \mathcal{M}$. Taking $\alpha(\sigma, \gamma) = 1$ for all $\sigma, \gamma \in \mathcal{M}$, \mathcal{T} satisfies all of the conditions of Theorem 10 and so \mathcal{T} has an endpoint. Here, $\mathcal{T}(1) = \{1\}$.

4. Endpoint Theorem in Graphic \mathcal{F} -Metric Spaces

In the present section, we will discuss the existence of endpoints on an \mathcal{F} -MS equipped with a graph G , i.e., $(\mathcal{F}$ -GMS).

Jachymski [19] has obtained an extension of Banach's contraction principle in metric space equipped with a graph G . Afterwards, Dinevari and Frigon [20] proved his results for multivalued mappings. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -MS. A set $\{(\sigma, \sigma) : \sigma \in \mathcal{M}\}$ is said to be a *diagonal* of $\mathcal{M} \times \mathcal{M}$, and represented by Γ . Let G be a graph such that the set $\mathfrak{V}(\mathfrak{G}) = \mathcal{M}$, that is, the set of its vertices and the set $\mathfrak{E}(\mathfrak{G})$ of its edges consists of all loops, i.e., $\Gamma \subseteq \mathfrak{E}(\mathfrak{G})$.

Definition 11. [21] Let $\mathcal{M} = \emptyset$ equipped with a graph G and $\mathcal{T} : \mathcal{M} \rightarrow 2^{\mathcal{M}}$. The mapping \mathcal{T} is said to preserves edges weakly if, for all $\sigma \in \mathcal{M}$ and $\gamma \in \mathcal{T}\sigma$ with $(\sigma, \gamma) \in \mathfrak{E}(\mathfrak{G})$, we get $(\gamma, z) \in \mathfrak{E}(\mathfrak{G}), \forall z \in \mathcal{T}\gamma$.

We give the following definition from [21] which is required in our proof.

Definition 12. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -GMS.

The \mathcal{F} -GMS \mathcal{M} is called $\mathfrak{G}(\mathfrak{G})$ -complete if every Cauchy sequence $\{\sigma_n\}$ in \mathcal{M} with $(\sigma_n, \sigma_{n+1}) \in \mathfrak{G}(\mathfrak{G})$, for all $n \in \mathbb{N}$ converges in \mathcal{M} .

Definition 13. A mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is called a $\mathfrak{G}(\mathfrak{G})$ -continuous mapping if, for any $\sigma \in \mathcal{M}$ and any sequence $\{\sigma_n\}$ with $\lim_{n \rightarrow \infty} d_{\mathcal{F}}(\sigma_n, \sigma) = 0$ and $(\sigma_n, \sigma_{n+1}) \in \mathfrak{G}(\mathfrak{G})$ for all $n \in \mathbb{N}$, we have

$$\lim_{n \rightarrow \infty} \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma) = 0. \quad (46)$$

Definition 14. A multivalued mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is called a rational Ćirić type $(\mathfrak{G}(\mathfrak{G}), \zeta)$ -contraction multivalued mapping if there exist a function $\zeta \in \Psi$ such that

$$\sigma, \mathfrak{y} \in \mathcal{M}, (\sigma, \mathfrak{y}) \in \mathfrak{G}(\mathfrak{G}) \Rightarrow \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma, \mathcal{T}\mathfrak{y}) \leq \zeta(R_{\mathcal{F}}(\sigma, \mathfrak{y})), \quad (47)$$

where $R_{\mathcal{F}}(\sigma, \mathfrak{y}) = \max \{d_{\mathcal{F}}(\sigma, \mathfrak{y}), (d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma)d_{\mathcal{F}}(\mathfrak{y}, \mathcal{T}\mathfrak{y})) / (1 + d_{\mathcal{F}}(\sigma, \mathfrak{y}))\}$.

Theorem 15. Suppose that $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -GMS and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ be a rational Ćirić type $(\mathfrak{G}(\mathfrak{G}), \zeta)$ -multivalued contraction. Suppose that the following conditions hold:

- (S₁) $(\mathcal{M}, d_{\mathcal{F}})$ is an $\mathfrak{G}(\mathfrak{G})$ -complete \mathcal{F} -GMS
 - (S₂) \mathcal{T} preserves edges weakly
 - (S₃) there exist σ_0 and $\sigma_1 \in \mathcal{T}\sigma_0$ such that $(\sigma_0, \sigma_1) \in \mathfrak{G}(\mathfrak{G})$
 - (S₄) \mathcal{T} is an $\mathfrak{G}(\mathfrak{G})$ -continuous multivalued mapping
- Then, \mathcal{T} has an endpoint point in \mathcal{M} .

Proof. This result can be obtain from Theorem 10 if we define a mapping $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ by $\alpha(\sigma, \mathfrak{y}) = 1$, if $(\sigma, \mathfrak{y}) \in \mathfrak{G}(\mathfrak{G})$ and $\alpha(\sigma, \mathfrak{y}) = 0$, otherwise. \square

5. Endpoint Theorem in Ordered \mathcal{F} -Metric Spaces

In 2004, Ran and Reurings [22] gave the idea of ordered metric space (OMS) by combing classical metric space (\mathcal{M}, d) and partial order \circ on \mathcal{M} . Fixed point results in OMS have many applications in integral and differential equations and other fields of mathematical analysis (see [23, 24]). In this section, we will consider $(\mathcal{F}$ -OMS), i.e., $(\mathcal{M}, d_{\mathcal{F}}^{\circ})$ where $(\mathcal{M}, d_{\mathcal{F}})$ is an \mathcal{F} -MS and \circ is a partial order on \mathcal{M} and we will derive some new results from Theorems 8 and 10. Remember that $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ is nondecreasing if $\forall \sigma, \mathfrak{y} \in \mathcal{M}, \sigma^{\circ} \mathfrak{y} \Rightarrow \mathcal{T}(\sigma)^{\circ} \mathcal{T}(\mathfrak{y})$.

Here, we state the following notion motivated from [25].

Definition 16. Let $\mathcal{M} = \emptyset$ with partial order \circ on \mathcal{M} and $\mathcal{T} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$. Then, \mathcal{T} is said to be weakly increasing if, for all $\sigma \in \mathcal{M}$ and $\mathfrak{y} \in \mathcal{T}\sigma$ with $\sigma^{\circ} \mathfrak{y}$, we get that $\mathfrak{y}^{\circ} z$, for all $z \in \mathcal{T}\mathfrak{y}$.

Definition 17. Let $(\mathcal{M}, d_{\mathcal{F}}^{\circ})$ be an \mathcal{F} -OMS.

The \mathcal{F} -OMS \mathcal{M} is called \circ -complete if every Cauchy sequence $\{\sigma_n\}$ in \mathcal{M} with $\sigma_n^{\circ} \sigma_{n+1}$, for all $n \in \mathbb{N}$ converges in \mathcal{M} .

Definition 18. A mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is said to be a \circ -continuous mapping if, for any $\sigma \in \mathcal{M}$ and any sequence $\{\sigma_n\}$ with $\lim_{n \rightarrow \infty} d_{\mathcal{F}}(\sigma_n, \sigma) = 0$ and $\sigma_n^{\circ} \sigma_{n+1}$, for all $n \in \mathbb{N}$, we get

$$\lim_{n \rightarrow \infty} \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma) = 0. \quad (48)$$

Motivated from [8], we define the notion of an ordered rational Ćirić' type ζ -multivalued contraction in an \mathcal{F} -OMS.

Definition 19. A multivalued $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is called an ordered rational Ćirić' type ζ -multivalued contraction if there exists $\zeta \in \Psi$ such that

$$\sigma, \mathfrak{y} \in \mathcal{M}, \sigma^{\circ} \mathfrak{y} \Rightarrow \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma, \mathcal{T}\mathfrak{y}) \leq \zeta(R_{\mathcal{F}}(\sigma, \mathfrak{y})), \quad (49)$$

where $R_{\mathcal{F}}(\sigma, \mathfrak{y}) = \max \{d_{\mathcal{F}}(\sigma, \mathfrak{y}), (d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma)d_{\mathcal{F}}(\mathfrak{y}, \mathcal{T}\mathfrak{y})) / (1 + d_{\mathcal{F}}(\sigma, \mathfrak{y}))\}$.

Theorem 20. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an \mathcal{F} -OMS \circ and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ be an ordered rational Ćirić type ζ -multivalued contraction. Assume that these hold:

- (S₁) $(\mathcal{M}, d_{\mathcal{F}})$ is an \circ -complete \mathcal{F} -OMS
 - (S₂) \mathcal{T} is weakly increasing
 - (S₃) there exist σ_0 and $\sigma_1 \in \mathcal{T}\sigma_0$ such that $\sigma_0^{\circ} \sigma_1$
 - (S₄) \mathcal{T} is an \circ -continuous multivalued mapping
- Then, \mathcal{T} has an endpoint point in \mathcal{M} .

Proof. This result can be obtained from Theorem 10 if we define a mapping $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ by $\alpha(\sigma, \mathfrak{y}) = 1$, if $\sigma^{\circ} \mathfrak{y}$, and $\alpha(\sigma, \mathfrak{y}) = 0$, otherwise. \square

6. Suzuki Type Endpoint Results in \mathcal{F} -MS

In 2008, Suzuki [26] obtained a fixed point result as generalization of the Banach fixed point theorem. In this section, we derive endpoint results for rational Suzuki type ζ -multivalued contraction in \mathcal{F} -MS as consequence of our result.

Corollary 21. Let $(\mathcal{M}, d_{\mathcal{F}})$ be a complete \mathcal{F} -MS, $\zeta \in \Psi$ and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ such that $d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) \leq d_{\mathcal{F}}(\sigma, \mathfrak{y}) + \zeta(d_{\mathcal{F}}(\sigma, \mathfrak{y}))$ implies

$$\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma, \mathcal{T}\mathfrak{y}) \leq \zeta(R_{\mathcal{F}}(\sigma, \mathfrak{y})), \quad (50)$$

where $R_{\mathcal{F}}(\sigma, \mathfrak{y}) = \max \{d_{\mathcal{F}}(\sigma, \mathfrak{y}), (d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma)d_{\mathcal{F}}(\mathfrak{y}, \mathcal{T}\mathfrak{y})) / (1 + d_{\mathcal{F}}(\sigma, \mathfrak{y}))\}$, for all $\sigma, \mathfrak{y} \in \mathcal{M}$ and \mathcal{T} satisfies the property (BS). If \mathcal{T} is continuous, then \mathcal{T} has an endpoint.

Proof. Define $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ by

$$\alpha(\sigma, y) = \begin{cases} 1, & d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) \leq d_{\mathcal{F}}(\sigma, y) + \zeta(d_{\mathcal{F}}(\sigma, y)), \\ 0, & \text{otherwise.} \end{cases} \quad (51)$$

□

It is easy to check that \mathcal{T} is α -admissible. Also, for every $\sigma_0 \in \mathcal{M}$ and $\sigma_1 \in \mathcal{T}\sigma_0$, we have $d_{\mathcal{F}}(\sigma_0, \mathcal{T}\sigma_0) \leq d_{\mathcal{F}}(\sigma_0, \sigma_1) \leq d_{\mathcal{F}}(\sigma_0, \sigma_1) + \zeta(d_{\mathcal{F}}(\sigma_0, \sigma_1))$. Hence, $\alpha(\sigma_0, \sigma_1) = 1$. It is very simple to check that

$$\alpha(\sigma, y) \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma, \mathcal{T}y) \leq \zeta(R(\sigma, y)), \quad (52)$$

where $R_{\mathcal{F}}(\sigma, y) = \max \{d_{\mathcal{F}}(\sigma, y), (d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma)d_{\mathcal{F}}(y, \mathcal{T}y))/(1 + d_{\mathcal{F}}(\sigma, y))\}$, for all $\sigma, y \in \mathcal{M}$. Therefore, by Theorem 10, \mathcal{T} has an endpoint.

Corollary 22. Suppose that $(\mathcal{M}, d_{\mathcal{F}})$ be a complete \mathcal{F} -MS, $r \in [0, 1)$ and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ such that $1/(1+r)d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) \leq d_{\mathcal{F}}(\sigma, y)$ implies that

$$\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma, \mathcal{T}y) \leq rR_{\mathcal{F}}(\sigma, y), \quad (53)$$

where $R_{\mathcal{F}}(\sigma, y) = \max \{d_{\mathcal{F}}(\sigma, y), (d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma)d_{\mathcal{F}}(y, \mathcal{T}y))/(1 + d_{\mathcal{F}}(\sigma, y))\}$, for all $\sigma, y \in \mathcal{M}$ and \mathcal{T} enjoys property (\mathcal{BS}) . If \mathcal{T} is continuous, then \mathcal{T} has an endpoint.

7. Application to Nonlinear Integral Equations

Let $CB(\mathbb{R})$ represents the set of all nonempty closed and bounded subsets of \mathbb{R} and $\mathfrak{B} = C(I, \mathbb{R})$ be the space of all real-valued continuous functions on $[0, 1]$. Clearly, \mathfrak{B} equipped with the \mathcal{F} -metric $d_{\mathcal{F}} : \mathfrak{B} \times \mathfrak{B} \longrightarrow [0, +\infty)$ given by

$$d_{\mathcal{F}}(\sigma, y) = \begin{cases} e^{\|\sigma - y\|}, & \text{if } \sigma = y, \\ 0, & \text{otherwise,} \end{cases} \quad (54)$$

where

$$\|\sigma - y\| = \sup_{t \in I} |\sigma(t) - y(t)|, \quad (55)$$

is a \mathcal{F} -complete \mathcal{F} -metric space (see [15]).

Now, we consider the integral equation

$$\sigma(t) = {}^t_0 K(t, s, \sigma(s)) d_{\mathcal{F}} s + g(t), \quad (56)$$

$t \in I$, where $\sigma \in \mathfrak{B}, K : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow CB(\mathbb{R})$ and $g : [0, 1] \longrightarrow \mathbb{R}$ is continuous.

Theorem 23. Suppose that these conditions hold:

- (i) for all $\sigma \in \mathfrak{B}, K : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow CB(\mathbb{R})$ is such that $K(t, s, \sigma(s))$ is continuous in $[0, 1] \times [0, 1]$
- (ii) there exists $\mathfrak{F} : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ which is continuous that satisfy the property $\inf_{t \in I} \int_0^t \mathfrak{F}(t, s) ds = \tau > 0$

such that for any $\sigma, h \in \mathfrak{B}$ and each $k_{\sigma}(t, s) \in K_{\sigma}(t, s, \sigma(s))$, there exists $k_h(t, s) \in K_{\sigma}(t, s, h(s))$ such that

$$\begin{aligned} & |k_{\sigma}(t, s) - k_h(t, s)| \\ & \leq \max \{ |\sigma(s) - h(s)|, (|\sigma(s) - k_{\sigma}(t, s)| \\ & \quad \cdot |h(s) - k_h(t, s)|)/(1 + |\sigma(s) - h(s)|) \} - \mathfrak{F}(t, s) \end{aligned} \quad (57)$$

for all $t, s \in [0, 1]$.

Then, the integral equation (56) has at least one solution in \mathfrak{B} .

Proof. Suppose that multivalued mapping $\mathcal{T} : \mathfrak{B} \longrightarrow CB(\mathfrak{B})$ defined by

$$\mathcal{T}\sigma = \left\{ \omega \in \mathfrak{B} : \omega(t) \in g(t) + \int_0^t K(t, s, \sigma(s)) d_{\mathcal{F}} s, t \in [a, b] \right\}, \quad (58)$$

for all $\sigma \in \mathfrak{B}$. Evidently, each endpoint of \mathcal{T} is a solution of (56). □

Next, consider the set-valued operator $K_{\sigma}(t, s) : [0, 1] \times [0, 1] \longrightarrow CB(\mathbb{R})$, defined by

$$K_{\sigma}(t, s) = K(t, s, \sigma(s)). \quad (59)$$

Then, by Michael's selection theorem, $\exists k_{\sigma}(t, s) : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ such that $k_{\sigma}(t, s) \in K_{\sigma}(t, s)$ for each $t, s \in [0, 1]$. This implies that $g(t) + \int_0^t k_{\rho}(t, s) d_{\mathcal{F}} s \in T\sigma$. Hence, $T\sigma = \emptyset$. Next, we prove that the multivalued function \mathcal{T} satisfies all the conditions of Theorem 10. Let $\sigma, \mathfrak{h} \in \mathfrak{B}$ and $\rho(t) \in \mathcal{T}\sigma$. Then, $\exists k_{\sigma}(t, s) \in K_{\sigma}(t, s)$ for each $t, s \in [0, 1]$ such that

$$\rho(t) = g(t) + \int_0^t k_{\sigma}(t, s) d_{\mathcal{F}} s, \quad (60)$$

for $t \in [0, 1]$. On the other side, by assumption (ii), $\exists k_h(t, s) \in K_{\sigma}(t, s)$ such that (57) holds. Now, by taking

$$\omega(t) = g(t) + \int_0^t k_h(t, s) d_{\mathcal{F}} s, \quad (61)$$

we get

$$\omega(t) = g(t) + \int_0^t K(t, s, h(s)) d_{\mathcal{F}} s = \mathcal{T}h, \quad (62)$$

for $t \in [0, 1]$,

$$\begin{aligned}
 d_{\mathcal{F}}(\rho, \omega) &= e^{\|\rho - \omega\|} \leq e^{\sup_{t \in [0,1]} \left| \int_0^t k_{\sigma}(t,s) ds - \int_0^t k_{\eta}(t,s) ds \right|} \\
 &\leq e^{\sup_{t \in [0,1]} \int_0^t |k_{\sigma}(t,s) - k_{\eta}(t,s)| ds} \\
 &\leq e^{\sup_{t \in [0,1]} \int_0^t \max \{ |\sigma(s) - \eta(s)|, (|\sigma(s) - k_{\sigma}(t,s)| |h(s) - k_{\eta}(t,s)|) / (1 + |\sigma(s) - h(s)|) \} - \mathfrak{A}(t,s) ds} \\
 &= e^{\sup_{t \in [0,1]} \int_0^t \max \{ |\sigma(s) - h(s)|, (|\sigma(s) - k_{\sigma}(t,s)| |h(s) - k_{\eta}(t,s)|) / (1 + |\sigma(s) - h(s)|) \} ds - \int_0^t \mathfrak{A}(t,s) ds} \\
 &\leq e^{\max \{ \|\sigma(s) - h(s)\|, (|\sigma(s) - k_{\sigma}(t,s)| \|h(s) - k_{\eta}(t,s)\|) / (1 + \|\sigma(s) - h(s)\|) \} - \inf_{t \in [0,1]} \int_0^t \mathfrak{A}(t,s) ds} \\
 &\leq e^{\max \{ \|\sigma(s) - h(s)\|, (|\sigma(s) - k_{\sigma}(t,s)| \|h(s) - k_{\eta}(t,s)\|) / (1 + \|\sigma(s) - h(s)\|) \} - \tau} \\
 &\leq e^{\max \{ \|\sigma(s) - h(s)\|, (|\sigma(s) - k_{\sigma}(t,s)| \|h(s) - k_{\eta}(t,s)\|) / (1 + \|\sigma(s) - h(s)\|) \}} \cdot e^{-\tau} \\
 &= \zeta(R_{\mathcal{F}}(\sigma, \eta)),
 \end{aligned} \tag{63}$$

where $\zeta(t) = e^{-\tau} t$. By interchanging the roles of σ and η , we get that

$$\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma, \mathcal{T}\eta) \leq \zeta(R_{\mathcal{F}}(\sigma, \eta)), \tag{64}$$

where $R_{\mathcal{F}}(\sigma, \eta) = \max \{ d_{\mathcal{F}}(\sigma, \eta), (d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) d_{\mathcal{F}}(\eta, \mathcal{T}\eta)) / (1 + d_{\mathcal{F}}(\sigma, \eta)) \}$, for all $\sigma, \eta \in \mathfrak{B}$. Taking $\alpha(\sigma, \eta) = 1$, for all $\sigma, \eta \in \mathfrak{B}$, all of the conditions of Theorem 10 are satisfied, and thus, \mathcal{T} has an endpoint, which is a solution of integral equation (56).

Data Availability

No such data were used for this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On System of Mixed Fractional Hybrid Differential Equations

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In this article, we find the necessary conditions for the existence and uniqueness of solutions to a system of hybrid equations that contain mixed fractional derivatives (Caputo and Riemann-Liouville). We also verify the stability of these solutions using the Ulam-Hyers (U-H) technique. Finally, this study ends with applied examples that show how to proceed and verify the conditions of our theoretical results.

1. Introduction

Although the concept of fractional calculus was established 300 years ago, interest in this type of derivative appeared for a short period. So that it is no secret to anyone that the most important use of fractional derivatives is to find analytical solutions to differential equations if possible, or by using numerical analysis methods to find an approximation to these solutions. In this study, we will focus on the idea of studying theories that investigate the existence of a solution to a system of hybrid fractional equations that contain mixed fractional derivatives with boundary conditions attached to them.

As mentioned before, fractional calculus as a concept is not very recent. It is worth mentioning here the great names who have given a lot to this science, such as A.V. Letnikov, J. Hadamard, J. Liouville, B. Riemann M., and Caputo L. worked in this field. These names must be mentioned by way of example. To get acquainted with some of the names of scientists who have made great contributions to fractional calculus in the modern world, we ask the reader to look at [1].

Fractional derivatives have played a very important role in mathematical modeling in many diverse applied sciences, see [2, 3]. For example, the authors in [4] employed the fractional derivative of the Psi-Caputo type in modeling the

logistic population equation, through which they were able to show that the model with the fractional derivative led to a better approximation of the variables than the classical model. In addition, the authors in [5] employed the fractional derivative of the Psi-Caputo type and used the kernel Rayleigh, to improve the model again in modeling the logistic population equation.

As a final example, the authors in [6] employed the fractional derivatives of the Caputo and Caputo-Fabrizio type by modeling the equation that gives the relationship between atmospheric pressure and altitude, and they were also able to show that the fractional equation gave less error in estimating atmospheric pressure at a certain altitude. There are many scientific papers in the literature that prove the superiority of fractional derivatives over classical ones.

There are a large number of manuscripts published in the literature that investigate the issue of the existence of a solution to fractional differential equations, whether they are sequential equations of type or nonsequential equations [7–14].

In 2012, the authors in [7] studied a nonlinear three-point boundary value problem of sequential fractional differential equations. Green's function of the associated problem involving the classical gamma function is obtained. Existence results are obtained using Banach's contraction

mapping principle and Krasnoselskii's fixed point theorem.

$$\begin{cases} {}^C D^q (D + \lambda) \chi(\tau) = w(\tau, \chi(\tau)), \tau \in [0, 1], q \in (1, 2], \\ \chi(0) = 0, \chi'(0) = 0, \chi(1) = \delta \chi(\eta), \eta \in (0, 1). \end{cases} \quad (1)$$

Here, D is the ordinary derivative, $\psi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$, δ is a real number such that $\delta \neq ((\lambda + e^{-\lambda} - 1) / (\lambda\eta + e^{-\lambda\eta} - 1))$.

In 2019, Ahmad et al. [15] developed the existence theory for a new kind of nonlocal three-point boundary value problems for differential equations involving both Caputo and Riemann–Liouville fractional derivatives. The existence of solutions for the multivalued problem concerning the upper semicontinuous and Lipschitz cases is proved by applying nonlinear alternative for Kakutani maps and Covitz and Nadler fixed point theorem.

$$\begin{cases} \left({}^C D_{1-}^q {}^{RL} D_{0+}^p \right) \omega(\tau) = \vartheta(\tau, \omega(\tau)), 1 < q \leq 2, 0 < p \leq 1, \\ \omega(0) = \omega'(0) = 0, \omega(1) = \delta \omega(\zeta), \end{cases} \quad (2)$$

where $\varphi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, $\delta \in \mathbb{R}$, $\zeta \in (0, 1)$.

It is known that fractional calculus and FDEs are used in different fields such as physics, signal and image processing, control theory, robotics, economics, biology, and metallurgy, see for example [16, 17] and references therein. On the other hand, recently, many researchers have paid much attention to hybrid differential equations of fractional order. This is because of the development and new advanced applications of fractional calculus. The fractional hybrid modeling is of great significance in different engineering fields, and it can be a unique idea for future combined research between various applied sciences, for example, see [18] in which fractional hybrid modeling of a thermostat is simulated, for some recent results on hybrid.

For FDEs, we refer to [19, 20]. Freshly, some authors have studied different characteristics of hybrid FDEs including the existence of solutions, see for some detail [21–29], and some go further and studied Hyers–Ulam stability for FDEs by different mathematical theories, see for some detail [26].

Zhao et al. [29] investigated the existence result for the fractional hybrid differential equations with Riemann–Liouville fractional derivatives given by

$$\begin{cases} \left({}^{RL} D_{0+}^r \right) \left(\frac{x(t)}{f(t, x(t))} \right) = g(t, x(t)), t \in [0, T], r \in (0, 1), \\ x(0) = 0, \end{cases} \quad (3)$$

where ${}^{RL} D^r$ is Riemann–Liouville fractional derivative, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$, $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are assumed to be continuous.

Hilal and Kajouni [30] studied the Caputo hybrid BVP of the form

$$\begin{cases} \left({}^C D_{0+}^r \right) \left(\frac{x(t)}{f(t, x(t))} \right) = g(t, x(t)), t \in [0, L], r \in (0, 1), \\ a_1 \frac{x(0)}{f(0, x(0))} + a_2 \frac{x(L)}{f(L, x(L))} = d, \end{cases} \quad (4)$$

in which $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$, $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are assumed to be continuous and $a_1 + a_2 \neq 0$.

In [31], the authors have considered the following coupled hybrid system. A new generalization of Darbo's theorem associated with measures of noncompactness is the main tool in their approach:

$$\begin{cases} \left({}^C D^q \right) \left(\frac{\chi(\tau)}{h(\tau, \chi(\tau), \vartheta(\tau))} \right) = \psi(\tau, \chi(\tau), \vartheta(\tau)), \tau \in [0, 1], 0 < q \leq 1, \\ \left({}^{RL} D^p \right) \left(\frac{\vartheta(t)}{\lambda(\tau, \chi(\tau), \vartheta(\tau))} \right) = \varphi(\tau, \chi(\tau), \vartheta(\tau)), 1 < p \leq 2, \end{cases} \quad (5)$$

supplemented with nonlocal hybrid boundary conditions.

Inspired by the aforementioned studies, the following sequential hybrid BVP is considered for investigating the existence of the solution and for the stability of its solution via the U–H sense

$$\begin{cases} \left({}^C D_{1-}^q {}^{RL} D_{0+}^r \right) \left(\frac{\chi(\tau)}{h(\tau, \chi(\tau), \vartheta(\tau))} \right) = \psi(\tau, \chi(\tau), \vartheta(\tau)), \tau \in [0, 1], 1 < q \leq 2, 0 < r \leq 1, \\ \left({}^C D_{1-}^q {}^{RL} D_{0+}^p \right) \left(\frac{\vartheta(t)}{\lambda(\tau, \chi(\tau), \vartheta(\tau))} \right) = \varphi(\tau, \chi(\tau), \vartheta(\tau)), 0 < p \leq 1, \\ \chi(0) = \chi'(0) = 0, & \chi(1) = \delta \chi(\zeta), \delta \in \mathbb{R}, \zeta \in (0, 1), \\ \vartheta(0) = \vartheta'(0) = 0, & \vartheta(1) = \varepsilon \vartheta(\xi), \varepsilon \in \mathbb{R}, \xi \in (0, 1). \end{cases} \quad (6)$$

After this introductory section of this work, the manuscript is organized as the following hierarchical structure: Section 2 delivers the basic elements of fractional calculus definitions, Section 3 introduces the main results of the work, Section 4 introduces the (U-H) stability result for our problem, and the last section is arranged for a numerical example to support the theoretical results.

2. Preliminaries

In this part, we present some basic elements and definitions needed to find solutions to the main mathematical problem presented in this study.

Definition 1 (see [3]). The Riemann-Liouville (RL) fractional integral is defined by

$$\left({}^{RL}I_{0+}^{\delta}\vartheta\right)(\omega):=\frac{1}{\Gamma(\delta)}\int_0^{\omega}(\omega-t)^{\delta-1}\vartheta(t)dt, \omega>0, \operatorname{Re}(\delta)>0. \quad (7)$$

Definition 2 (see [3]). The Caputo fractional derivative of order ν of a function $\vartheta: \mathbb{R}^+ \rightarrow \mathbb{R}$ is given by

$$\left({}^CD_{0+}^{\nu}\vartheta\right)(\tau)=\int_0^{\tau}\frac{(\tau-z)^{p-\nu-1}\vartheta^{(p)}(z)}{\Gamma(p-\nu)}dz, p-1<\nu<p, p=[\nu]+1. \quad (8)$$

Theorem 3 (see [3], Banach's contraction mapping principle). Let (S, d) be a complete metric space; $H: S \rightarrow S$ is a contraction then

- (i) H has a unique fixed point $s \in S$; that, is $H(s) = s$
- (ii) $\forall s_0 \in S$, we have $\lim_{n \rightarrow \infty} H^n(u_0) = u$

Theorem 4 (see [3], nonlinear alternative of Leray-Schauder type). Assume that V is an open subset of a Banach space U , $0 \in V$, and $F: \bar{V} \rightarrow U$ be a contraction such that $F(\bar{V})$ is bounded then either

- (i) F has a fixed point in \bar{V} , or
- (ii) $\exists \mu \in (0, 1)$ and $v \in \partial V$ such that $v = \mu F(v)$ holds

Theorem 5 (see [2], Arzela-Ascoli theorem). $F \subset C(U, \mathbb{R})$ is compact if and only if it is closed, bounded, and equicontinuous.

3. Main Results

Lemma 6. If $h \in C([0, 1], \mathbb{R})$, and

$$\begin{cases} \left({}^CD_{1-}^q {}^{RL}D_{0+}^r\right)\left(\frac{\chi(\tau)}{\hbar(\tau, \chi(\tau), \vartheta(\tau))}\right) = w(\tau) & , \tau \in [0, 1], 1 < q \leq 2, 0 < r \leq 1, \\ \chi(0) = \chi'(0) = 0, & \chi(1) = \delta\chi(\zeta), \delta \in \mathbb{R}, \zeta \in (0, 1), \end{cases} \quad (9)$$

then the solution to the problem mentioned above is given by

$$\begin{aligned} \chi(\tau) = & \hbar(\tau, \chi(\tau), \vartheta(\tau)) \\ & \times \left(\frac{1}{\Gamma(r)} \int_0^{\tau} (\tau-z)^{r-1} I_{1-}^q w(z) dz + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})\Gamma(r)} \right. \\ & \left. \cdot \left[\delta \int_0^{\zeta} (\zeta-z)^{r-1} I_{1-}^q w(z) dz - \int_0^1 (1-z)^{r-1} I_{1-}^q w(z) dz \right] \right). \end{aligned} \quad (10)$$

Proof. Taking ${}^{RL}I_{1-}^q$ to $({}^CD_{1-}^q {}^{RL}D_{0+}^r)(\chi(\tau)/\hbar(\tau, \chi(\tau), \vartheta(\tau)))$

$= w(\tau)$, then take ${}^{RL}I_{0+}^r$ to the resulting equation, we get

$$\begin{aligned} \frac{\chi(\tau)}{\hbar(\tau, \chi(\tau), \vartheta(\tau))} = & {}^{RL}I_{0+}^r \left({}^{RL}I_{1-}^q w(\tau) + a_0 + a_1 t \right) \\ & + a_2 \tau^{r-1} = {}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(\tau) + a_0 \frac{\tau^r}{\Gamma(r+1)} \\ & + a_1 \frac{\tau^{r+1}}{\Gamma(r+2)} + a_2 \tau^{r-1}. \end{aligned} \quad (11)$$

Substitution of $\chi(0) = 0$ and $\chi'(0) = 0$ in Equation (11) gives $a_2 = 0$ and $a_0 = 0$, respectively, and consequently,

Equation (6) becomes

$$\frac{\chi(\tau)}{\hbar(\tau, \chi(\tau), \vartheta(\tau))} = {}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(\tau) + a_1 \frac{\tau^{r+1}}{\Gamma(r+2)}. \quad (12)$$

Use of the condition $\chi(1) = \delta\chi(\zeta)$ in Equation (12) yields

$$a_1 = \frac{\Gamma(r+2)}{1 - \delta\zeta^{r+1}} \left(\delta {}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(\zeta) - {}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(1) \right). \quad (13)$$

Inserting a_1 in Equation (12) gives

$$\begin{aligned} \frac{\chi(\tau)}{\hbar(\tau, \chi(\tau), \vartheta(\tau))} &= {}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(\tau) + \frac{\Gamma(r+2)}{1 - \delta\zeta^{r+1}} \\ &\cdot \left(\delta {}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(\zeta) - {}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(1) \right) \frac{\tau^{r+1}}{\Gamma(r+2)}. \end{aligned} \quad (14)$$

Alternatively, we have

$$\begin{aligned} \chi(\tau) &= \hbar(\tau, \chi(\tau), \vartheta(\tau)) \\ &\times \left({}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(\tau) + \frac{\Gamma(r+2)}{1 - \delta\zeta^{r+1}} \left(\delta {}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(\zeta) \right. \right. \\ &\left. \left. - {}^{RL}I_{0+}^r {}^{RL}I_{1-}^q w(1) \right) \frac{\tau^{r+1}}{\Gamma(r+2)} \right), \tau \in [0, 1]. \end{aligned} \quad (15)$$

Equation (15) is equivalent to Equation (10), which makes the proof done. \square

Denote the Banach space by $C = C[0, 1]$ with the norm $\|h\| = \sup_{0 \leq t \leq 1} |h(t)|$. Then, the product space $(C \times C, \|(\chi, \vartheta)\|)$ with the norm $\|(\chi, \vartheta)\| = \|\chi\| + \|\vartheta\|$, $\forall (\chi, \vartheta) \in C \times C$ is indeed a Banach space too. We define an operator $\Upsilon : C \times C \longrightarrow C \times C$ as

$$\Upsilon(\chi, \vartheta)(\tau) = \begin{pmatrix} \Upsilon_1(\chi, \vartheta)(\tau) \\ \Upsilon_2(\chi, \vartheta)(\tau) \end{pmatrix}, \quad (16)$$

where

$$\begin{aligned} \Upsilon_1(\chi, \vartheta)(\tau) &= \hbar(\tau, \chi(\tau), \vartheta(\tau)) \\ &\times \left(\frac{1}{\Gamma(r)} \int_0^\tau (\tau - z)^{r-1} {}^{RL}I_{1-}^q \psi(z, \chi(z), \vartheta(z)) dz \right. \\ &+ \frac{\tau^{r+1}}{(1 - \delta\zeta^{r+1})\Gamma(r)} \times \left[\delta \int_0^\zeta (\zeta - z)^{r-1} {}^{RL}I_{1-}^q \psi(z, \chi(z), \vartheta(z)) dz \right. \\ &\left. \left. - \int_0^1 (1 - z)^{r-1} {}^{RL}I_{1-}^q \psi(z, \chi(z), \vartheta(z)) dz \right] \right), \Upsilon_2(\chi, \vartheta)(\tau) \\ &= \lambda(\tau, \chi(\tau), \vartheta(\tau)) \times \left(\frac{1}{\Gamma(p)} \int_0^\tau (\tau - z)^{p-1} {}^{RL}I_{1-}^q \varphi(z, \chi(z), \vartheta(z)) dz \right. \\ &+ \frac{\tau^{p+1}}{(1 - \varepsilon\zeta^{p+1})\Gamma(p)} \times \left[\varepsilon \int_0^\zeta (\zeta - z)^{p-1} {}^{RL}I_{1-}^q \varphi(z, \chi(z), \vartheta(z)) dz \right. \\ &\left. \left. - \int_0^1 (1 - z)^{p-1} {}^{RL}I_{1-}^q \varphi(z, \chi(z), \vartheta(z)) dz \right] \right). \end{aligned} \quad (17)$$

To construct the necessary conditions for the results of uniqueness and existence of the problem (6), let us consider the following hypotheses.

(C1) Let the functions f and g are assumed to be continuous and bounded; that is, $\exists \lambda_f, \lambda_g > 0$ such that

$$|\hbar(\tau, \chi, \vartheta)| \leq \lambda_h, \text{ and } |\lambda(\tau, \chi, \vartheta)| \leq \lambda_\lambda, \forall (\tau, \chi, \vartheta) \in [0, 1] \times \mathbb{R}^2. \quad (18)$$

(C2) Let the functions ψ and φ are assumed to be continuous, and $\exists v_i, \ell_i > 0, (i = 1, 2)$ such that

$$\begin{aligned} |\psi(\tau, \chi_1, \vartheta_1) - \psi(\tau, \chi_2, \vartheta_2)| &\leq v_1 |\chi_1 - \chi_2| + v_2 |\vartheta_1 - \vartheta_2|, \\ |\varphi(\tau, \chi_1, \vartheta_1) - \varphi(\tau, \chi_2, \vartheta_2)| &\leq \ell_1 |\chi_1 - \chi_2| \\ &+ \ell_2 |\vartheta_1 - \vartheta_2|, \forall \tau \in [0, 1], \chi_i, \vartheta_i \in \mathbb{R}, (i = 1, 2). \end{aligned} \quad (19)$$

(C3) There is positive constants ω_0, θ_0 , and $\omega_i, \theta_i \geq 0 (i = 1, 2)$ such that

$$|\psi(\tau, \chi, \vartheta)| \leq \omega_0 + \omega_1 |\chi| + \omega_2 |\vartheta|, \quad (20)$$

$$|\varphi(\tau, \chi, \vartheta)| \leq \theta_0 + \theta_1 |\chi| + \theta_2 |\vartheta|, \forall \tau \in [0, 1], \chi_i, \vartheta_i \in \mathbb{R}, (i = 1, 2). \quad (21)$$

(C4) Let $S \subset C \times C$ be a bounded set, then $\exists \kappa_i > 0, (i = 1, 2)$ such that $|\psi(\tau, \chi(\tau), \vartheta(\tau))| \leq \kappa_1$, and

$$|\varphi(\tau, \chi(\tau), \vartheta(\tau))| \leq \kappa_2, \forall (\chi, \vartheta) \in S. \quad (22)$$

Observe that

$$\begin{aligned}
 & \frac{1}{\Gamma(r)\Gamma(q)} \int_0^\tau (\tau-z)^{r-1} \int_s^1 (u-z)^{q-1} du dz \\
 &= \int_0^\tau \frac{(\tau-z)^{r-1}}{\Gamma(r)} \int_z^1 \frac{(u-z)^{q-1}}{\Gamma(q)} du dz \\
 &= \int_0^\tau \frac{(\tau-z)^{r-1}}{\Gamma(r)} \frac{(u-z)^q}{\Gamma(q+1)} \Big|_{u=z}^{u=1} dz \\
 &= \int_0^\tau \frac{(\tau-z)^{r-1}}{\Gamma(r)} \frac{(u-z)^q}{\Gamma(q+1)} \Big|_{u=z}^{u=1} dz = \int_0^\tau \frac{(\tau-z)^{r-1}}{\Gamma(r)} \frac{(1-z)^q}{\Gamma(q+1)} dz \\
 &\leq \int_0^\tau \frac{(\tau-z)^{r-1}}{\Gamma(r)} \frac{1}{\Gamma(q+1)} dz, ((1-s)^q \leq 1, 1 < q \leq 2) \\
 &= \frac{\tau^r}{\Gamma(r+1)\Gamma(q+1)}.
 \end{aligned} \tag{23}$$

To facilitate the calculations below, let us say

$$\begin{aligned}
 \Lambda_1 &= \sup_{0 \leq \tau \leq 1} \left\{ \left\{ \frac{1}{\Gamma(r)\Gamma(q)} \int_0^\tau (\tau-z)^{r-1} \int_z^1 (u-z)^{q-1} du dz \right. \right. \\
 &\quad + \frac{\tau^{r+1}}{|1-\delta\zeta^{r+1}|\Gamma(r)\Gamma(q)} \times \left[|\delta| \int_0^\zeta (\zeta-z)^{r-1} \int_z^1 (u-z)^{q-1} du dz \right. \\
 &\quad \left. \left. - \int_0^1 (1-z)^{r-1} \int_z^1 (u-z)^{q-1} du dz \right] \right\} \\
 &\leq \frac{1}{\Gamma(q+1)\Gamma(r+1)} \left[1 + \frac{|\delta|\zeta^r}{|1-\delta\zeta^{r+1}|} \right], \\
 \Lambda_2 &= \sup_{0 \leq t \leq 1} \left\{ \frac{1}{\Gamma(p)\Gamma(q)} \int_0^\tau (\tau-z)^{p-1} \int_z^1 (u-z)^{q-1} du dz \right. \\
 &\quad + \frac{\tau^{p+1}}{\Gamma(p)|1-\varepsilon\zeta^{p+1}|\Gamma(q)} \times \left[|\varepsilon| \int_0^\zeta (\zeta-z)^{p-1} \int_z^1 (u-z)^{q-1} du dz \right. \\
 &\quad \left. \left. - \int_0^1 (1-z)^{p-1} \int_z^1 (u-z)^{q-1} du dz \right] \right\} \\
 &\leq \frac{1}{\Gamma(p+1)\Gamma(q+1)} \left[1 + \frac{|\varepsilon|\zeta^p}{|1-\varepsilon\zeta^{p+1}|} \right].
 \end{aligned} \tag{24}$$

Theorem 7. If both (C1) and (C2) are satisfied, and assume that $[\lambda_h \Lambda_1(v_1 + v_2) + \lambda_\lambda \Lambda_2(\ell_1 + \ell_2)] < 1$. Then, the system in Equation (6) has a unique solution.

Proof. Define a closed ball $\overline{\mathfrak{B}}_\gamma = \{(\chi, \vartheta) \in C \times C : \|(\chi, \vartheta)\| \leq \gamma\}$ with $\gamma \geq (\lambda_h \Lambda_1 N_\psi + \lambda_\lambda \Lambda_2 N_\varphi) / (1 - (\lambda_h \Lambda_1(v_1 + v_2) + \lambda_\lambda \Lambda_2(\ell_1 + \ell_2)))$, where $N_\psi = \sup_{0 \leq \tau \leq T} |\psi(\tau, 0, 0)|$, $N_\varphi = \sup_{0 \leq \tau \leq T} |\varphi(\tau, 0, 0)|$.

Observe that $|\psi(\tau, \chi, \vartheta)| = |\psi(\tau, \chi, \vartheta) - \psi(\tau, 0, 0) + \psi(\tau, 0, 0)| \leq v_1 \|\chi\| + v_2 \|\vartheta\| + N_\psi \leq (v_1 + v_2)\gamma + N_\psi$.

First, we show that $\mathfrak{B}_\gamma \subset \overline{\mathfrak{B}}_\gamma$. For any $(\chi, \vartheta) \in \mathfrak{B}_\gamma$, $\tau \in [0, 1]$, we have

$$\begin{aligned}
 |Y_1(\chi, \vartheta)(\tau)| &= \left| h(\tau, \chi(\tau), \vartheta(\tau)) \times \left(\frac{1}{\Gamma(r)} \int_0^\tau (\tau-z)^{r-1} I_{1-}^q \psi(z, \chi(z), \vartheta(z)) dz \right. \right. \\
 &\quad + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})\Gamma(r)} \times \left[\delta \int_0^\zeta (\zeta-z)^{r-1} I_{1-}^q \psi(z, \chi(z), \vartheta(z)) dz \right. \\
 &\quad \left. \left. \cdot \int_0^1 (1-z)^{r-1} I_{1-}^q \psi(z, \chi(z), \vartheta(z)) dz \right] \right| \\
 &\leq \lambda_h \sup_{0 \leq \tau \leq 1} \left\{ \frac{1}{\Gamma(r)\Gamma(q)} \int_0^\tau (\tau-z)^{r-1} \right. \\
 &\quad \cdot \int_z^1 (u-z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz + \frac{\tau^{r+1}}{|1-\delta\zeta^{r+1}|\Gamma(r)\Gamma(q)} \\
 &\quad \cdot \left[|\delta| \int_0^\zeta (\zeta-z)^{r-1} \int_z^1 (u-z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz \right. \\
 &\quad \left. \left. - \int_0^1 (1-z)^{r-1} \int_z^1 (u-z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz \right] \right\} \\
 &\leq \lambda_h [(v_1 + v_2)\gamma + N_\psi] \sup_{0 \leq \tau \leq 1} \left\{ \int_0^\tau (\tau-z)^{r-1} \int_z^1 (u-z)^{q-1} du dz \right. \\
 &\quad + \frac{\tau^{r+1}}{|1-\delta\zeta^{r+1}|\Gamma(r)\Gamma(q)} \times \left[|\delta| \int_0^\zeta (\zeta-z)^{r-1} \int_z^1 (u-z)^{q-1} du dz \right. \\
 &\quad \left. \left. - \int_0^1 (1-z)^{r-1} \int_z^1 (u-z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz \right] \right\} \\
 &\leq \lambda_h \Lambda_1 [(v_1 + v_2)\gamma + N_\psi],
 \end{aligned} \tag{25}$$

similar to what was done above, we get

$$\|Y_2(\chi, \vartheta)\| \leq \lambda_\lambda \Lambda_2 [(\ell_1 + \ell_2)\gamma + N_\varphi]. \tag{26}$$

From Equation (25) and Equation (26), we deduce that $\|Y(\chi, \vartheta)\| \leq \gamma$.

Next, for $(\chi_1, \vartheta_1), (\chi_2, \vartheta_2) \in C \times C$, $\forall \tau \in [0, 1]$, we have

$$\begin{aligned}
 |Y_1(\chi_1, \vartheta_1)(\tau) - Y_1(\chi_2, \vartheta_2)(\tau)| &\leq \lambda_h \sup_{0 \leq \tau \leq 1} \left\{ \frac{1}{\Gamma(r)\Gamma(q)} \int_0^\tau (\tau-z)^{r-1} \int_z^1 (u-z)^{q-1} |\psi(z, \chi_1(z), \vartheta_1(z)) \right. \\
 &\quad \left. - \psi(z, \chi_2(z), \vartheta_2(z))| du dz + \frac{\tau^{r+1}}{|1-\delta\zeta^{r+1}|\Gamma(r)\Gamma(q)} \right. \\
 &\quad \times \left[|\delta| \int_0^\zeta (\zeta-z)^{r-1} \int_z^1 (u-z)^{q-1} |\psi(z, \chi_1(z), \vartheta_1(z)) \right. \\
 &\quad \left. - \psi(z, \chi_2(z), \vartheta_2(z))| du dz - \int_0^1 (1-z)^{r-1} \right. \\
 &\quad \left. \times \int_z^1 (u-z)^{q-1} |\psi(z, \chi_1(z), \vartheta_1(z)) - \psi(z, \chi_2(z), \vartheta_2(z))| du dz \right] \Big\} \\
 &\leq \lambda_h (v_1 \|\chi_1 - \chi_2\| + v_2 \|\vartheta_1 - \vartheta_2\|) \sup_{0 \leq \tau \leq 1} \left\{ \frac{1}{\Gamma(r)\Gamma(q)} \int_0^\tau (\tau-z)^{r-1} \right. \\
 &\quad \times \int_z^1 (u-z)^{q-1} du dz + \frac{\tau^{r+1}}{|1-\delta\zeta^{r+1}|\Gamma(r)\Gamma(q)} \times \left[|\delta| \int_0^\zeta (\zeta-z)^{r-1} \right. \\
 &\quad \times \int_z^1 (u-z)^{q-1} du dz - \int_0^1 (1-z)^{r-1} \int_z^1 (u-z)^{q-1} du dz \Big\} \\
 &\leq \lambda_h \Lambda_1 (v_1 \|\chi_1 - \chi_2\| + v_2 \|\vartheta_1 - \vartheta_2\|) \leq \lambda_h \Lambda_1 (v_1 + v_2) (\|\chi_1 - \chi_2\| + \|\vartheta_1 - \vartheta_2\|).
 \end{aligned} \tag{27}$$

Similarly, we can find

$$\|\Upsilon_2(\chi_1, \vartheta_1) - \Upsilon_2(\chi_2, \vartheta_2)\| \leq \lambda_{\tilde{\lambda}} \Lambda_2 (\ell_1 + \ell_2) (\|\chi_1 - \chi_2\| + \|\vartheta_1 - \vartheta_2\|). \quad (28)$$

Combining Equation (27) and Equation (28) yields

$$\begin{aligned} \|\Upsilon(\chi_1, \vartheta_1) - \Upsilon(\chi_2, \vartheta_2)\| &\leq [\lambda_h \Lambda_1 (v_1 + v_2) + \lambda_{\tilde{\lambda}} \Lambda_2 (\ell_1 + \ell_2)] \\ &\quad \times (\|\chi_1 - \chi_2\| + \|\vartheta_1 - \vartheta_2\|). \end{aligned} \quad (29)$$

Equation (29) becomes $\|\Upsilon(\chi_1, \vartheta_1) - \Upsilon(\chi_2, \vartheta_2)\| \leq (\|\chi_1 - \chi_2\| + \|\vartheta_1 - \vartheta_2\|)$. That is, Υ is a contraction; consequently, Banach fixed point theorem applies; thus, the uniqueness of solutions for Equation (6) holds on $[0, 1]$. \square

Theorem 8. *If (C1), (C3), and (C4) are satisfied, and if $(\lambda_h \Lambda_1 \omega_1 + \lambda_{\tilde{\lambda}} \Lambda_2 \theta_1) < 1$ and $(\lambda_h \Lambda_1 \omega_2 + \lambda_{\tilde{\lambda}} \Lambda_2 \theta_2) < 1$, then Equation (6) has at least one solution.*

Proof. In the first step, we verify that the operator $\Upsilon : C \times C \longrightarrow C \times C$ is completely continuous; obviously, the operator is continuous as a result that \hbar , $\tilde{\lambda}$, ψ , and φ are all assumed to be continuous.

With the aid of (C4), $\forall (\chi, \vartheta) \in S$, we have

$$\begin{aligned} |\Upsilon_1(\chi, \vartheta)(\tau)| &\leq \lambda_h \sup_{0 \leq \tau \leq 1} \left\{ \frac{1}{\Gamma(r)\Gamma(q)} \int_0^\tau (\tau - z)^{r-1} \right. \\ &\quad \cdot \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz \\ &\quad + \frac{\tau^{r+1}}{|1 - \delta \zeta^{r+1}| \Gamma(r)\Gamma(q)} \times \left[|\delta| \int_0^\zeta (\zeta - z)^{r-1} \right. \\ &\quad \cdot \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz \\ &\quad \left. \left. - \int_0^1 (1 - z)^{r-1} \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz \right] \right\} \\ &\leq \lambda_h \Lambda_1 \kappa_1. \end{aligned} \quad (30)$$

Similarly,

$$\|\Upsilon_2(\chi, \vartheta)\| \leq \lambda_{\tilde{\lambda}} \Lambda_2 \kappa_2. \quad (31)$$

Combining the inequalities (30) and (31) yields $\|\Upsilon(\chi, \vartheta)\| \leq \lambda_h \Lambda_1 \kappa_1 + \lambda_{\tilde{\lambda}} \Lambda_2 \kappa_2$, implying that Υ is uniformly bounded.

Next, to verify the equicontinuity for the operator Υ , we let $\tau_1, \tau_2 \in [0, 1]$, $(\tau_1 < \tau_2)$ then

$$\begin{aligned} |\Upsilon_1(\chi, \vartheta)(\tau_2) - \Upsilon_1(\chi, \vartheta)(\tau_1)| &\leq \lambda_h \sup_{0 \leq \tau \leq 1} \left\{ \frac{1}{\Gamma(r)\Gamma(q)} \int_0^{\tau_1} ((\tau_1 - z)^{r-1} - (\tau_2 - z)^{r-1}) \right. \\ &\quad \times \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz \\ &\quad - \frac{1}{\Gamma(r)\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - z)^{r-1} \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz \\ &\quad + \frac{|\tau_2^{r+1} - \tau_1^{r+1}|}{|1 - \delta \zeta^{r+1}| \Gamma(r)\Gamma(q)} \times \left[|\delta| \int_0^\zeta (\zeta - z)^{r-1} \right. \\ &\quad \times \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz - \int_0^1 (1 - z)^{r-1} \\ &\quad \left. \times \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| du dz \right] \Big\}, \end{aligned} \quad (32)$$

$$\begin{aligned} |\Upsilon_1(\chi, \vartheta)(\tau_2) - \Upsilon_1(\chi, \vartheta)(\tau_1)| &\leq \lambda_h \kappa_1 \\ &\times \left| \frac{1}{\Gamma(r)\Gamma(q)} \int_0^{\tau_1} ((\tau_1 - z)^{r-1} - (\tau_2 - z)^{r-1}) \int_z^1 (u - z)^{q-1} du dz \right. \\ &\quad \left. - \frac{1}{\Gamma(r)\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - z)^{r-1} \int_z^1 (u - z)^{q-1} du dz \right| \\ &\quad + \frac{|\tau_2^{r+1} - \tau_1^{r+1}|}{|1 - \delta \zeta^{r+1}| \Gamma(r)\Gamma(q)} \times \left[|\delta| \int_0^\zeta (\zeta - z)^{r-1} \int_z^1 (u - z)^{q-1} du dz \right. \\ &\quad \left. - \int_0^1 (1 - z)^{r-1} \int_z^1 (u - z)^{q-1} du dz \right], \end{aligned} \quad (33)$$

$$\begin{aligned} |\Upsilon_2(\chi, \vartheta)(\tau_2) - \Upsilon_2(\chi, \vartheta)(\tau_1)| &\leq \lambda_{\tilde{\lambda}} \kappa_2 \\ &\times \left| \frac{1}{\Gamma(p)\Gamma(q)} \int_0^{\tau_1} ((\tau_1 - z)^{p-1} - (\tau_2 - z)^{p-1}) \int_s^1 (u - s)^{q-1} du ds \right. \\ &\quad \left. - \frac{1}{\Gamma(p)\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{p-1} \int_s^1 (u - s)^{q-1} du ds \right| \\ &\quad + \frac{|\tau_2^{p+1} - \tau_1^{p+1}|}{|1 - \varepsilon \xi^{p+1}| \Gamma(p)\Gamma(q)} \times \left[|\varepsilon| \int_0^\xi (\xi - z)^{p-1} \int_z^1 (u - z)^{q-1} du dz \right. \\ &\quad \left. - \int_0^1 (1 - z)^{p-1} \int_z^1 (u - z)^{q-1} du dz \right]. \end{aligned} \quad (34)$$

The R.H.S for both inequalities (33) and (34) tend to zero as $\tau_1 \longrightarrow \tau_2$, and they are both independent on (χ, ϑ) . So, operator $\Upsilon(\chi, \vartheta)$ is equicontinuous and yields; $\Upsilon(\chi, \vartheta)$ is completely continuous.

Finally, we establish the bounded set given by $\Omega = \{(x, y) \in C \times C | (x, y) = \beta \mathfrak{F}(x, y), \beta \in [0, 1]\}$; then, $\forall \tau \in [0, 1]$; the equation $(\chi, \vartheta) = \beta \Upsilon(\chi, \vartheta)$ gives

$$\chi(\tau) = \beta \Upsilon_1(\chi, \vartheta)(\tau), \quad \vartheta(\tau) = \beta \Upsilon_2(\chi, \vartheta)(\tau). \quad (35)$$

Using the hypothesis (C3), we get

$$\begin{aligned}\|\chi\| &\leq \lambda_h \Lambda_1 (\omega_0 + \omega_1 \|\chi\| + \omega_2 \|\vartheta\|), \\ \|\vartheta\| &\leq \lambda_{\bar{h}} \Lambda_2 (\theta_0 + \theta_1 \|\chi\| + \theta_2 \|\vartheta\|).\end{aligned}\quad (36)$$

Consequently, we have

$$\begin{aligned}\|\chi\| + \|\vartheta\| &\leq (\lambda_h \Lambda_1 \omega_0 + \lambda_{\bar{h}} \Lambda_2 \theta_0) + (\lambda_h \Lambda_1 \omega_1 + \lambda_{\bar{h}} \Lambda_2 \theta_1) \|\chi\| \\ &\quad + (\lambda_h \Lambda_1 \omega_2 + \lambda_{\bar{h}} \Lambda_2 \theta_2) \|\vartheta\|.\end{aligned}\quad (37)$$

Inequality (37) can be written as follows:

$$\|(\chi, \vartheta)\| \leq \frac{(\lambda_h \Lambda_1 \omega_0 + \lambda_{\bar{h}} \Lambda_2 \theta_0)}{\Lambda_0}, \quad (38)$$

where $\Lambda_0 = \min \{1 - (\lambda_h \Lambda_1 \omega_1 + \lambda_{\bar{h}} \Lambda_2 \theta_1), 1 - (\lambda_h \Lambda_1 \omega_2 + \lambda_{\bar{h}} \Lambda_2 \theta_2)\}$.

Inequality (38) shows that Ω is bounded. Hence, *Leray-Schauder alternative* applies, implying the existence of the solution for Equation (6). \square

4. Stability

In this part, we address the issue of stability of solutions to the system of equations defined by Equation (6) via U-H definition.

Definition 9. The system of the coupled sequential fractional differential BVPs Equation (6) is stable in U-H sense if a real number $c = \max(c_1, c_2) > 0$ exists so that, for any $\varepsilon = \max(\varepsilon_1, \varepsilon_2) > 0$ and for any $(\bar{\chi}, \bar{\vartheta}) \in C \times C$ satisfying

$$\begin{aligned}\left| \left({}^C D_{1-}^{q, RL} D_{0+}^r \right) \left(\frac{\bar{\chi}(\tau)}{h(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))} \right) - \psi(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \right| &< \varepsilon_1, \tau \in [0, 1], \\ \left| \left({}^C D_{1-}^{q, RL} D_{0+}^p \right) \left(\frac{\bar{\vartheta}(\tau)}{\bar{h}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))} \right) - \varphi(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \right| &< \varepsilon_2,\end{aligned}\quad (39)$$

there exists a unique solution $(\chi, \vartheta) \in C \times C$ of (6) with

$$\|(\chi, \vartheta) - (\bar{\chi}, \bar{\vartheta})\| < c\varepsilon. \quad (40)$$

It is clear that $(\bar{\chi}, \bar{\vartheta}) \in C \times C$ satisfies the inequalities (39) if there exists a function $(h_1, h_2) \in C \times C$ (which depends on $(\bar{\chi}, \bar{\vartheta})$), such that

$$(i) \quad |h_1(\tau)| < \varepsilon_1 \text{ and } |h_2(\tau)| < \varepsilon_2, \tau \in [0, 1]$$

$$(ii) \quad \text{For } \tau \in [0, 1]$$

$$\begin{cases} \left({}^C D_{1-}^{q, RL} D_{0+}^r \right) \left(\frac{\bar{\chi}(\tau)}{h(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))} \right) = \psi(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) + h_1(\tau), \\ \left({}^C D_{1-}^{q, RL} D_{0+}^p \right) \left(\frac{\bar{\vartheta}(\tau)}{\bar{h}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))} \right) = \varphi(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) + h_2(\tau). \end{cases} \quad (41)$$

Theorem 10. Suppose that (C2) is fulfilled. Moreover

$$\begin{aligned}\lambda_h \Lambda_1 (v_1 + v_2) &< 1, \lambda_{\bar{h}} \Lambda_2 (\ell_1 + \ell_2) < 1, \\ \Delta &= (1 - \lambda_h \Lambda_1 (v_1 + v_2))(1 - \lambda_{\bar{h}} \Lambda_2 (\ell_1 + \ell_2)) \\ &\quad - \lambda_h \Lambda_1 (v_1 + v_2) \lambda_{\bar{h}} \Lambda_2 (\ell_1 + \ell_2) > 0.\end{aligned}\quad (42)$$

Then, the system of coupled sequential fractional differential BVPs (6) is U-H stable.

Proof. Assume that for $\varepsilon_1, \varepsilon_2 > 0$ a couple $(\bar{\chi}, \bar{\vartheta}) \in C \times C$ satisfies the inequalities (39). Introduce the following operator

$$\begin{aligned}K_1(\tau; h) &= \frac{1}{\Gamma(r)} \int_0^\tau (\tau - z)^{r-1} {}^C D_{1-}^q I_{1-}^q h(z) dz, \\ K_2(\tau; h) &= \frac{1}{\Gamma(p)} \int_0^\tau (\tau - u)^{p-1} {}^C D_{1-}^q I_{1-}^q h(u) du.\end{aligned}\quad (43)$$

Then

$$\begin{aligned}\bar{\chi}(\tau) &= h(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) (K_1(\tau; \psi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) \\ &\quad + \frac{\tau^{r+1}}{(1 - \delta \zeta^{r+1})} [\delta K_1(\zeta; \psi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) \\ &\quad - K_1(1; \psi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))]) + h(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \\ &\quad \cdot \left(K_1(\tau; h_1) + \frac{\tau^{r+1}}{(1 - \delta \zeta^{r+1})} [\delta K_1(\zeta; h_1) - K_1(1; h_1)] \right),\end{aligned}\quad (44)$$

$$\begin{aligned}\bar{\vartheta}(\tau) &= \bar{h}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) (K_2(\tau; \varphi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) \\ &\quad + \frac{\tau^{r+1}}{(1 - \delta \zeta^{r+1})} [\delta K_2(\zeta; \varphi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) - K_2(1; \varphi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))]) \\ &\quad + \bar{h}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \left(K_2(\tau; h_2) + \frac{\tau^{r+1}}{(1 - \delta \zeta^{r+1})} [\delta K_2(\zeta; h_2) - K_2(1; h_2)] \right).\end{aligned}\quad (45)$$

From Equation (44) and Equation (45), we obtain

$$\begin{aligned} & \bar{\chi}(\tau) - \bar{h}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \left(K_1 \left(\tau; \psi \left(\bar{\chi} \left(\frac{\bar{A}}{\bar{n}} \right), \bar{\vartheta} \left(\frac{\bar{A}}{\bar{n}} \right) \right) \right) \right. \\ & \left. + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} \left[\delta K_1 \left(\zeta; \psi \left(\bar{\chi} \left(\frac{\bar{A}}{\bar{n}} \right), \bar{\vartheta} \left(\frac{\bar{A}}{\bar{n}} \right) \right) \right) - K_1 \left(1; \psi \left(\bar{\chi} \left(\frac{\bar{A}}{\bar{n}} \right), \bar{\vartheta} \left(\frac{\bar{A}}{\bar{n}} \right) \right) \right) \right] \right) \\ & \leq \bar{h}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \left(K_1(\tau; h_1) + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_1(\zeta; h_1) - K_1(1; h_1)] \right), \quad (46) \end{aligned}$$

$$\begin{aligned} & \bar{\vartheta}(\tau) - \bar{\lambda}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \left(K_2 \left(\tau; \varphi \left(\bar{\chi} \left(\frac{\bar{A}}{\bar{n}} \right), \bar{\vartheta} \left(\frac{\bar{A}}{\bar{n}} \right) \right) \right) \right. \\ & \left. + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} \left[\delta K_2 \left(\zeta; \varphi \left(\bar{\chi} \left(\frac{\bar{A}}{\bar{n}} \right), \bar{\vartheta} \left(\frac{\bar{A}}{\bar{n}} \right) \right) \right) - K_2 \left(1; \varphi \left(\bar{\chi} \left(\frac{\bar{A}}{\bar{n}} \right), \bar{\vartheta} \left(\frac{\bar{A}}{\bar{n}} \right) \right) \right) \right] \right) \\ & \leq \bar{\lambda}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \left(K_2(\tau; h_2) + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_2(\zeta; h_2) - K_2(1; h_2)] \right) \quad (47) \end{aligned}$$

From Equation (27) and Equation (28), we obtain

$$\begin{aligned} & |\bar{\chi}(\tau) - \bar{h}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) (K_1(\tau; \psi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) \\ & + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_1(\zeta; \psi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) - K_1(1; \psi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))])| \\ & \leq \lambda_h \Lambda_1 \|h_1\| \leq \lambda_h \Lambda_1 \varepsilon_1, \quad (48) \end{aligned}$$

$$\begin{aligned} & |\bar{\vartheta}(\tau) - \bar{\lambda}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) (K_2(\tau; \varphi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) \\ & + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_2(\zeta; \varphi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) - K_2(1; \varphi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))])| \\ & \leq \lambda_\lambda \Lambda_2 \|h_2\| \leq \lambda_\lambda \Lambda_2 \varepsilon_2. \quad (49) \end{aligned}$$

Let $(\chi, \vartheta) \in C \times C$ be a solution of Equation (6). Thanks to Lemma 6, it is equivalent to the following integral equations:

$$\begin{aligned} \chi(\tau) &= \bar{h}(\tau, \chi(\tau), \vartheta(\tau)) (K_1(\tau; \psi(\cdot, \chi(\cdot), \vartheta(\cdot))) \\ & + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_1(\zeta; \psi(\cdot, \chi(\cdot), \vartheta(\cdot))) - K_1(1; \psi(\cdot, \chi(\cdot), \vartheta(\cdot))]) \\ & = \bar{\lambda}(\tau, \chi(\tau), \vartheta(\tau)) (K_2(t; \varphi(\cdot, \chi(\cdot), \vartheta(\cdot))) \\ & + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_2(\zeta; \varphi(\cdot, \chi(\cdot), \vartheta(\cdot))) - K_2(1; \varphi(\cdot, \chi(\cdot), \vartheta(\cdot))]) \end{aligned} \quad (50)$$

By the same arguments in Theorem 7, we get

$$\begin{aligned} |\chi(\tau) - \bar{\chi}(\tau)| &= |\Upsilon_1(\chi, \vartheta)(\tau) - \Upsilon_1(\bar{\chi}, \bar{\vartheta})(\tau) - \bar{h}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \\ & \cdot \left(K_1(\tau; h_1) + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_1(\zeta; h_1) - K_1(1; h_1)] \right)| \\ & \leq |\Upsilon_1(\chi, \vartheta)(\tau) - \Upsilon_1(\bar{\chi}, \bar{\vartheta})(\tau)| + |\bar{h}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \\ & \cdot \left(K_1(\tau; h_1) + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_1(\zeta; h_1) - K_1(1; h_1)] \right)| \\ & \leq \lambda_h \Lambda_1 (v_1 + v_2) (\|\chi - \bar{\chi}\| + \|\vartheta - \bar{\vartheta}\|) + \lambda_h \Lambda_1 \varepsilon_1, \quad (51) \end{aligned}$$

$$\begin{aligned} |\vartheta(\tau) - \bar{\vartheta}(\tau)| &= |\Upsilon_2(\chi, \vartheta)(\tau) - \Upsilon_2(\bar{\chi}, \bar{\vartheta})(\tau) - \bar{\lambda}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \\ & \cdot \left(K_2(\tau; h_2) + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_2(\zeta; h_2) - K_2(1; h_2)] \right)| \\ & \leq |\Upsilon_2(\chi, \vartheta)(\tau) - \Upsilon_2(\bar{\chi}, \bar{\vartheta})(\tau)| + |\bar{\lambda}(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \\ & \cdot \left(K_2(\tau; h_2) + \frac{\tau^{r+1}}{(1-\delta\zeta^{r+1})} [\delta K_2(\zeta; h_2) - K_2(1; h_2)] \right)| \\ & \leq \lambda_\lambda \Lambda_2 (\ell_1 + \ell_2) (\|\chi - \bar{\chi}\| + \|\vartheta - \bar{\vartheta}\|) + \lambda_\lambda \Lambda_2 \varepsilon_2. \quad (52) \end{aligned}$$

It follows that

$$\begin{aligned} \|\chi - \bar{\chi}\| &\leq \lambda_h \Lambda_1 (v_1 + v_2) (\|\chi - \bar{\chi}\| + \|\vartheta - \bar{\vartheta}\|) \leq \lambda_h \Lambda_1 \varepsilon_1, \\ \|\vartheta - \bar{\vartheta}\| &\leq \lambda_\lambda \Lambda_2 (\ell_1 + \ell_2) (\|\chi - \bar{\chi}\| + \|\vartheta - \bar{\vartheta}\|) \leq \lambda_\lambda \Lambda_2 \varepsilon_2. \quad (53) \end{aligned}$$

Representing these inequalities as matrices, we get

$$\begin{pmatrix} 1 - \lambda_h \Lambda_1 (v_1 + v_2) & -\lambda_h \Lambda_1 (v_1 + v_2) \\ 1 - \lambda_\lambda \Lambda_2 (\ell_1 + \ell_2) & -\lambda_\lambda \Lambda_2 (\ell_1 + \ell_2) \end{pmatrix} \begin{pmatrix} \|\chi - \bar{\chi}\| \\ \|\vartheta - \bar{\vartheta}\| \end{pmatrix} \leq \begin{pmatrix} \lambda_h \Lambda_1 \varepsilon_1 \\ \lambda_\lambda \Lambda_2 \varepsilon_2 \end{pmatrix}. \quad (54)$$

Solving the above inequality, we get

$$\begin{aligned} \|\chi - \bar{\chi}\| &\leq \frac{1 - \lambda_h \Lambda_1 (v_1 + v_2)}{\Delta} \lambda_h \Lambda_1 \varepsilon_1 + \frac{\lambda_h \Lambda_1 (v_1 + v_2)}{\Delta} \lambda_\lambda \Lambda_2 \varepsilon_2, \\ \|\vartheta - \bar{\vartheta}\| &\leq \frac{\lambda_\lambda \Lambda_2 (\ell_1 + \ell_2)}{\Delta} \lambda_h \Lambda_1 \varepsilon_1 + \frac{1 - \lambda_\lambda \Lambda_2 (\ell_1 + \ell_2)}{\Delta} \lambda_\lambda \Lambda_2 \varepsilon_2, \quad (55) \end{aligned}$$

where $\Delta = (1 - \lambda_h \Lambda_1 (v_1 + v_2))(1 - \lambda_\lambda \Lambda_2 (\ell_1 + \ell_2)) - \lambda_h \Lambda_1 (v_1 + v_2) \lambda_\lambda \Lambda_2 (\ell_1 + \ell_2) \neq 0$.

Thus

$$\begin{aligned} \|\chi - \bar{\chi}\| + \|\vartheta - \bar{\vartheta}\| &\leq \left(\frac{1 - \lambda_h \Lambda_1 (v_1 + v_2)}{\Delta} + \frac{\lambda_h \Lambda_1 (v_1 + v_2)}{\Delta} \right) \lambda_h \Lambda_1 \varepsilon_1 \\ &+ \left(\frac{1 - \lambda_\lambda \Lambda_2 (\ell_1 + \ell_2)}{\Delta} + \frac{\lambda_\lambda \Lambda_1 (v_1 + v_2)}{\Delta} \right) \lambda_\lambda \Lambda_2 \varepsilon_2. \quad (56) \end{aligned}$$

For $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$ and

$$c = \frac{(1 - \lambda_h \Lambda_1(v_1 + v_2) + \lambda_{\bar{\lambda}} \Lambda_2(\ell_1 + \ell_2)) \lambda_h \Lambda_{11} + (1 - \lambda_{\bar{\lambda}} \Lambda_2(\ell_1 + \ell_2) + \lambda_h \Lambda_1(v_1 + v_2)) \lambda_{\bar{\lambda}} \Lambda_2}{\Delta}, \quad (57)$$

we get

$$\|(\chi, \bar{\chi}) - (\vartheta, \bar{\vartheta})\| \leq \|\chi - \bar{\chi}\| + \|\vartheta - \bar{\vartheta}\| \leq c\varepsilon. \quad (58)$$

Therefore, with the aid of Definition 9, the solution of the problem Equation (6) is U–H stable. \square

5. Example

In this part, we present an applied example to support the theoretical results we reached in the previous part, consider the following system:

$$\begin{cases} \left({}^C D_{1-}^{7/4} {}^{RL} D_{0+}^{1/4} \right) \left(\frac{\chi(\tau)}{1/2 |\sin \chi(\tau)| + 7/5} \right) = 3e^\tau + \frac{1}{10\sqrt{\tau^3 + 15}} \frac{|\chi|}{1 + |\chi|} + \frac{1}{40} \tan^{-1} \vartheta, \tau \in [0, 1], \\ \left({}^C D_{1-}^{7/4} {}^{RL} D_{0+}^{3/4} \right) \left(\frac{\vartheta(\tau)}{1/3 |\cos \vartheta(\tau)| + 1} \right) = 2e^{-3\tau} \sin \tau + \frac{1}{20} (\tan^{-1} \vartheta + \tan^{-1} \chi), \\ \chi(0) = \chi'(0) = 0, \quad \chi(1) = \chi\left(\frac{1}{2}\right), \\ \vartheta(0) = \vartheta'(0) = 0, \quad \vartheta(1) = \vartheta\left(\frac{1}{3}\right). \end{cases} \quad (59)$$

Here,

$$\begin{aligned} 1 &= \delta = \varepsilon, \\ p &= \frac{7}{4}, q = \frac{5}{4}, h(\tau, \chi, \vartheta) = \frac{1}{2} |\sin \chi(\tau)| + \frac{7}{5}, \lambda(\tau, \chi, \vartheta) = \frac{1}{3} |\cos \vartheta(\tau)| + 1, \\ \psi(\tau, \chi(\tau), \vartheta(\tau)) &= 3e^\tau + \frac{1}{10\sqrt{\tau^3 + 15}} \frac{|\chi|}{1 + |\chi|} + \frac{1}{40} \tan^{-1} \vartheta, \varphi(\tau, \chi(\tau), \vartheta(\tau)) \\ &= 2e^{-3\tau} \sin \tau + \frac{1}{20} (\tan^{-1} \vartheta + \tan^{-1} \chi). \end{aligned} \quad (60)$$

Observe that

$$\begin{aligned} |\psi(\tau, \chi_1, \vartheta_1) - \psi(\tau, \chi_2, \vartheta_2)| &\leq \frac{1}{40} |\chi_2 - \chi_1| + \frac{1}{40} |\vartheta_2 - \vartheta_1|, \\ |\varphi(\tau, \chi_1, \vartheta_1) - \varphi(\tau, \chi_2, \vartheta_2)| &\leq \frac{1}{20} |\chi_2 - \chi_1| + \frac{1}{20} |\vartheta_2 - \vartheta_1|, \\ [\lambda_h \Lambda_1(v_1 + v_2) + \lambda_{\bar{\lambda}} \Lambda_2(\ell_1 + \ell_2)] &\leq 0.330291 < 1. \end{aligned} \quad (61)$$

Thus, the boundary value problem Equation (59) satisfies all the conditions of Theorem 7; consequently, the uniqueness of solution of Equation (59) is satisfied on $[0, 1]$.

In order to explain Theorem 7, it is clear that (C1) is satisfied as follows:

$$\begin{aligned} |h(\tau, \chi, \vartheta)| &\leq \frac{1}{2} |\sin \chi(\tau)| + \frac{7}{5} \leq 2 = \lambda_h, \\ |\lambda(\tau, \chi, \vartheta)| &\leq \frac{1}{3} |\cos \vartheta(\tau)| + 1 \leq \frac{3}{2} = \lambda_{\bar{\lambda}}, \end{aligned} \quad (62)$$

Also, one can easily show that (C3) holds, taking into account that $\tau \in [0, 1]$, then

$$\begin{aligned} |\psi(\tau, \chi, \vartheta)| &= \left| 3e^\tau + \frac{1}{10\sqrt{\tau^3 + 15}} \frac{|\chi|}{1 + |\chi|} + \frac{1}{40} \tan^{-1} \vartheta \right| \\ &\leq 3e + \frac{1}{40} |\chi| + \frac{1}{40} |\vartheta|, \\ |\varphi(\tau, \chi, \vartheta)| &= \left| 2e^{-3\tau} \sin \tau + \frac{1}{20} (\tan^{-1} \vartheta + \tan^{-1} \chi) \right| \\ &\leq 2 + \frac{1}{20} |\chi| + \frac{1}{20} |\vartheta|. \end{aligned} \quad (63)$$

Also, (C4) satisfied with

$$|\psi(\tau, \chi, \vartheta)| \leq 3e + \frac{1 + 2\pi}{40}, |\varphi(\tau, \chi, \vartheta)| \leq 2 + \frac{\pi}{5}. \quad (64)$$

Finally, easy calculations with the data above give $(\lambda_h \Lambda_1 \omega_1 + \lambda_{\bar{\lambda}} \Lambda_2 \theta_1) = 0.203275 < 1$ and $(\lambda_h \Lambda_1 \omega_2 + \lambda_{\bar{\lambda}} \Lambda_2 \theta_2) =$

$0.203 < 1$; all conditions of Theorem 8 hold; that is, the problem (59) has at least one solution in $[0, 1]$.

6. Conclusion

We have studied a coupled hybrid FDEs consisting of mixed fractional derivatives such as Caputo and Riemann-Liouville fractional derivatives and nonlocal boundary conditions. Existence/uniqueness results are established via a nonlinear alternative of the Leray-Schauder and Banach fixed point theorem. We also studied the Ulam-Hyers stability of these couple of hybrid FDEs. The obtained result is well illustrated by a numerical example. The result obtained in this paper is new and significantly contributes to the existing literature on the topic.

One possible direction in which to extend the results of this paper is toward different kinds of mixed fractional differential and mixed conformable fractional differential systems of higher order. Another challenge is to find out if similar results can be derived in the case of constant/variable delays in linear/nonlinear terms.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

M.A and N. M contributed to each part of this work equally and read and approved the final version of the manuscript.

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Research Article

On the 3D Incompressible Boussinesq Equations in a Class of Variant Spherical Coordinates

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This paper investigates the global stabilizing effects of the geometry of the domain at which the flow locates and of the geometric structure of the solution to the incompressible flows by studying the three-dimensional (3D) incompressible, viscosity, and diffusivity Boussinesq system in spherical coordinates. We establish the global existence and uniqueness of the smooth solution to the Cauchy problem for a full 3D incompressible Boussinesq system in a class of variant spherical coordinates for a class of smooth large initial data. We also construct one class of nonempty bounded domains in the three-dimensional space \mathbb{R}^3 , in which the initial boundary value problem for the full 3D Boussinesq system in a class of variant spherical coordinates with a class of large smooth initial data with swirl has a unique global strong or smooth solution with exponential decay rate in time.

1. Introduction and Main Results

In this paper, we consider the Cauchy problem for the three-dimensional (3D) incompressible Boussinesq ($\nu, \mu > 0$) equations

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} + \rho \mathbf{e}_3, & \mathbf{x} \in \mathbb{R}^3, \quad t > 0, \\ \rho_t + \mathbf{u} \cdot \nabla \rho = \mu \Delta \rho, & \mathbf{x} \in \mathbb{R}^3, \quad t > 0, \\ \operatorname{div} \mathbf{u} = 0, & \mathbf{x} \in \mathbb{R}^3, \quad t > 0, \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \rho(0, \mathbf{x}) = \rho_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3, \end{cases} \quad (1)$$

and the initial boundary value problem for the 3D incompressible Boussinesq ($\nu, \mu > 0$) equations in the bounded domain

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} + \rho \mathbf{e}_3, & \mathbf{x} \in \Omega, \quad t > 0, \\ \rho_t + \mathbf{u} \cdot \nabla \rho = \mu \Delta \rho, & \mathbf{x} \in \Omega, \quad t > 0, \\ \operatorname{div} \mathbf{u} = 0, & \mathbf{x} \in \Omega, \quad t > 0, \\ \mathbf{u} = 0, \rho = 0, & \mathbf{x} \in \partial\Omega, \quad t > 0, \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \rho(0, \mathbf{x}) = \rho_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \quad (2)$$

respectively. Here, $\mathbf{x} = (x_1, x_2, x_3)$; the unknowns $\mathbf{u} = (u^1(t, \mathbf{x}), u^2(t, \mathbf{x}), u^3(t, \mathbf{x}))^T$ denote the fluid velocity vector field; $P = P(t, \mathbf{x})$ is the scalar pressure and $\rho = \rho(t, \mathbf{x})$ is the scalar density; ν, μ are viscosity and thermal diffusivity, respectively; \mathbf{n} is the unit outer normal vector of bounded domain Ω ; $\mathbf{e}_3 = (0, 0, 1)^T$ is the unit vector in the vertical direction; and \mathbf{u}_0 and ρ_0 are the given initial velocity and initial density, respectively, with $\operatorname{div} \mathbf{u}_0 = 0$. It should be noted that, if $\rho \equiv 0$, (1) comes back to the classical 3D incompressible Navier-Stokes equations.

It is well known that the 3D incompressible Navier-Stokes equations have at least one global weak solution with the finite energy [1, 2]. However, the issue of the regularity and uniqueness for the global weak solution is still a challenging open problem in the field of mathematical fluid dynamics [3–8].

Recently, motivated by the studies on the axisymmetric flow (see [6–11] and the references therein), the helical flow (see [12] and the references therein), and the 3D incompressible Euler and the SQG (surface quasigeostrophic) equations [13–15], we investigate further the global

dynamical stabilizing effects of the geometry of the domain at which the flow locates and of the geometry structure of the solution to the 3D incompressible Navier-Stokes equations. As an example, we study the 3D incompressible Navier-Stokes and Euler equations in the spherical coordinate system, see S. Wang and Y.X. Wang [16], where the existence and uniqueness of the global strong solution of the 3D incompressible Navier-Stokes and Euler equations in the spherical coordinates are obtained for a class of large smooth initial data with swirl or without swirl.

As stated in the beginning, the present paper is focused on the Boussinesq system, which plays an important role in the atmospheric and oceanographic sciences [11, 17–20]. Considering the 2D standard Boussinesq equations with the viscosity or diffusive coefficient, Hou and Li [21] and Chae [22] obtain the global well-posedness results similar to the 2D incompressible Navier-Stokes equations [6]. On global regularity on the smooth solution for the 2D Boussinesq system, see also, e.g., [23–25] and the references therein. On the other hand, comparing with the magnitude of research conducted on the Boussinesq equations on Euclidean domains, the qualitative behaviour of the model on Riemannian manifolds has been investigated relatively little, see [26], in which the convergence of the average of weak solutions of the 3D equations to a 2D problem is proved by Saito, and see [27], in which the nondegenerate and partially degenerate Boussinesq equations on a closed surface are studied by Li et al..

The global well-posedness for a 3D axisymmetric Boussinesq system without swirl and with partial viscosity or thermal diffusivity in the system of cylindrical coordinates is obtained by Abidi et al. in [28], Hmidi and Keraani in [29], and Hmidi et al. [30, 31], respectively. For the general 3D Boussinesq system, there exist some results on the local well-posedness problem, partial regularity, or the global regularity with respect to small initial data; see [32–37], etc.

In this paper, we further investigate the global stabilizing effects of the geometry of the domain and the solution to the three-dimensional incompressible flows by studying the 3D incompressible axisymmetric Boussinesq system in the system of a class of variant spherical coordinates.

Let the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad (3)$$

be a real orthogonal matrix, i.e., $\mathbf{A}^T \mathbf{A} = \mathbf{I}$, where \mathbf{I} is an identity matrix and \mathbf{A}^T is a transpose of the matrix \mathbf{A} . For the given

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \in \mathbb{R}^3, \quad (4)$$

and the constant $a > 0$, introduce a class of variant spherical coordinates (r, θ, φ) defined as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + a \mathbf{A} \begin{pmatrix} r \sin \theta \cos \varphi \\ r \sin \theta \sin \varphi \\ r \cos \theta \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^3. \quad (5)$$

Because the matrix \mathbf{A} is an orthogonal one, we have

$$\begin{aligned} r &= \frac{1}{a} \sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x}) + \zeta^2(\mathbf{x})} \\ &= \frac{1}{a} \sqrt{(x_1 - \alpha_1)^2 + (x_2 - \alpha_2)^2 + (x_3 - \alpha_3)^2} \geq 0, \\ 0 \leq \theta &= \arctan \frac{\sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})}}{\zeta(\mathbf{x})} \leq \pi, \quad 0 \leq \varphi = \arctan \frac{\eta(\mathbf{x})}{\xi(\mathbf{x})} < 2\pi, \end{aligned} \quad (6)$$

where

$$\begin{cases} \xi(\mathbf{x}) = a_{11}(x_1 - \alpha_1) + a_{21}(x_2 - \alpha_2) + a_{31}(x_3 - \alpha_3), \\ \eta(\mathbf{x}) = a_{12}(x_1 - \alpha_1) + a_{22}(x_2 - \alpha_2) + a_{32}(x_3 - \alpha_3), \\ \zeta(\mathbf{x}) = a_{13}(x_1 - \alpha_1) + a_{23}(x_2 - \alpha_2) + a_{33}(x_3 - \alpha_3). \end{cases} \quad (7)$$

Note that, for variant spherical coordinates (r, θ, φ) , the r coordinate is spherical symmetric in \mathbb{R}^3 , but the θ coordinate and φ coordinate are not axisymmetric with respect to the Cartesian coordinates $\mathbf{x} \in \mathbb{R}^3$ except that $\mathbf{A} = \mathbf{I}$. Denote

$$\begin{aligned} \mathbf{e}_r &= \mathbf{A} \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix} = \frac{1}{\sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x}) + \zeta^2(\mathbf{x})}} \begin{pmatrix} x_1 - \alpha_1 \\ x_2 - \alpha_2 \\ x_3 - \alpha_3 \end{pmatrix}, \\ \mathbf{e}_\theta &= \mathbf{A} \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{pmatrix} \\ &= \frac{\begin{pmatrix} a_{11}\xi^2(\mathbf{x}) + a_{12}\xi(\mathbf{x})\eta(\mathbf{x}) - a_{13}(\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})) \\ a_{21}\xi^2(\mathbf{x}) + a_{22}\xi(\mathbf{x})\eta(\mathbf{x}) - a_{23}(\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})) \\ a_{31}\xi^2(\mathbf{x}) + a_{32}\xi(\mathbf{x})\eta(\mathbf{x}) - a_{33}(\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})) \end{pmatrix}}{\sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x}) + \zeta^2(\mathbf{x})} \sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})}}, \end{aligned}$$

$$\begin{aligned} \mathbf{e}_\varphi &= \mathbf{A} \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{\xi^2(\mathbf{x}) + \eta^2(\mathbf{x})}} \begin{pmatrix} -a_{11}\xi(\mathbf{x}) + a_{12}\eta(\mathbf{x}) \\ -a_{21}\xi(\mathbf{x}) + a_{22}\eta(\mathbf{x}) \\ -a_{31}\xi(\mathbf{x}) + a_{32}\eta(\mathbf{x}) \end{pmatrix}. \end{aligned} \quad (8)$$

Also, denote the special bounded domain $\tilde{\Omega}$ described by variant spherical coordinates by

$$\begin{aligned} \tilde{\Omega} &= \{(x_1, x_2, x_3) \\ &= (\alpha_1 + aa_{11}r \sin \theta \cos \varphi + aa_{12}r \sin \theta \sin \varphi \\ &\quad + aa_{13}r \cos \theta, \alpha_2 + aa_{21}r \sin \theta \cos \varphi + aa_{22}r \sin \theta \sin \varphi \\ &\quad + aa_{23}r \cos \theta, \alpha_3 + aa_{31}r \sin \theta \cos \varphi + aa_{32}r \sin \theta \sin \varphi \\ &\quad + aa_{33}r \cos \theta) \\ &\in \mathbb{R}^3 : 0 < r_0 \leq r \leq R_0 < \infty, 0 < \theta_0 \leq \theta \leq \theta_1 < \pi, 0 \leq \varphi < 2\pi\}, \end{aligned} \quad (9)$$

where $r_0, R_0, \theta_0, \theta_1$ are given fixed positive constants. Here, we give an explicit example for the domain

$$\begin{aligned} \tilde{\Omega} &= \left\{ (x_1, x_2, x_3) \right. \\ &= \left(\sqrt{2}r \sin \theta \sin \varphi, -r \sin \theta \cos \varphi + r \cos \theta, r \sin \theta \cos \varphi \right. \\ &\quad \left. + r \cos \theta \right) \\ &\in \mathbb{R}^3 : 1 \leq r \leq 10, \frac{\pi}{8} \leq \theta \leq \frac{3\pi}{4}, 0 \leq \varphi < 2\pi \}, \end{aligned} \quad (10)$$

by taking

$$\begin{aligned} a &= \sqrt{2}, \alpha = 0, \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \\ r_0 &= 1, R_0 = 10, \theta_0 = \frac{\pi}{8}, \theta_1 = \frac{3\pi}{4}. \end{aligned} \quad (11)$$

Now, we consider the 3D incompressible Boussinesq equations (1) and (2) with the form

$$\begin{aligned} \mathbf{u}(t, \mathbf{x}) &= u^r(t, r, \theta)\mathbf{e}_r + u^\theta(t, r, \theta)\mathbf{e}_\theta + u^\varphi(t, r, \theta)\mathbf{e}_\varphi, \\ P(t, \mathbf{x}) &= P(t, r, \theta), \rho(t, \mathbf{x}) = \rho(t, r, \theta), \end{aligned} \quad (12)$$

with

$$\mathbf{u}_0(\mathbf{x}) = u_0^r(t, r, \theta)\mathbf{e}_r + u_0^\theta(t, r, \theta)\mathbf{e}_\theta + u_0^\varphi(t, r, \theta)\mathbf{e}_\varphi, \rho_0(\mathbf{x}) = \rho_0(r, \theta). \quad (13)$$

When the matrix \mathbf{A} is an orthogonal matrix, the gradient operator ∇ and Laplacian Δ have the expression

$$\begin{aligned} \nabla &= \mathbf{e}_r \frac{1}{a} \partial_r + \frac{1}{ar} \mathbf{e}_\theta \partial_\theta + \frac{1}{ar \sin \theta} \mathbf{e}_\varphi \partial_\varphi, \\ \Delta &= \frac{1}{a^2} \left(\partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi^2 \right), \end{aligned} \quad (14)$$

respectively.

Then, one can derive the evolution equations for $(u^r, u^\theta, u^\varphi, \rho)(t, r, \theta)$ for 3D incompressible Boussinesq equations as follows:

$$\begin{cases} \partial_t u^r + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) u^r + \frac{1}{a} \partial_r P = \nu \left[\left(\tilde{\Delta} - \frac{2}{a^2 r^2} \right) u^r - \frac{2 \cos \theta}{a^2 r^2 \sin \theta} u^\theta - \frac{2}{a^2 r^2} \partial_\theta u^\theta \right] + \frac{(u^\theta)^2 + (u^\varphi)^2}{ar} + \rho \cos \theta, \\ \partial_t u^\theta + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) u^\theta + \frac{1}{ar} \partial_\theta P = \nu \left[\left(\tilde{\Delta} - \frac{1}{a^2 r^2 \sin^2 \theta} \right) u^\theta + \frac{2}{a^2 r^2} \partial_\theta u^r \right] - \frac{u^r u^\theta}{ar} + \frac{\cos \theta}{ar \sin \theta} (u^\varphi)^2 - \rho \sin \theta, \\ \partial_t u^\varphi + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) u^\varphi = \nu \left(\tilde{\Delta} - \frac{1}{a^2 r^2 \sin^2 \theta} \right) u^\varphi - \frac{u^r u^\varphi}{ar} - \frac{\cos \theta}{ar \sin \theta} u^\theta u^\varphi, \\ \partial_t \rho + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \rho = \nu \tilde{\Delta} \rho, \\ \partial_r u^r + \frac{2}{r} u^r + \frac{1}{r} \partial_\theta u^\theta + \frac{\cos \theta}{r \sin \theta} u^\theta = 0, \end{cases} \quad (15)$$

where $\tilde{\mathbf{u}} = u^r \mathbf{e}_r + u^\theta \mathbf{e}_\theta$, and

$$\tilde{\nabla} = \mathbf{e}_r \partial_r + \mathbf{e}_\theta \frac{1}{r} \partial_\theta, \tilde{\Delta} = \partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta. \quad (16)$$

Note that equations (15) completely determine the evolution of the 3D Boussinesq equations in a class of variant spherical coordinates once the initial conditions and/or the boundary value conditions are given. Also, the 3D incompressible Boussinesq system in a class of variant spherical coordinates is completely different from the one in cylindrical coordinates because of the complexity of the last equation in system (15) and of Laplace operator $\tilde{\Delta}$ given by (16).

We take the initial condition for system (15) as follows:

$$(u^r, u^\theta, u^\varphi, \rho)(t=0, r, \theta) = (u_0^r, u_0^\theta, u_0^\varphi)(r, \theta). \quad (17)$$

Moreover, the boundary condition $\mathbf{u}|_{\partial\Omega} = 0, t \geq 0$ is equivalent to the following condition:

$$(u^r, u^\theta, u^\varphi, \rho)|_{\partial\Omega} = 0, t \geq 0. \quad (18)$$

It is easy to know, by direct computation, that the vorticity $\omega = \nabla \times \mathbf{u}$ can be expressed as

$$\omega(t, \mathbf{x}) = \omega^r(t, r, \theta) \mathbf{e}_r + \omega^\theta(t, r, \theta) \mathbf{e}_\theta + \omega^\varphi(t, r, \theta) \mathbf{e}_\varphi, \quad (19)$$

with the initial vorticity

$$\omega_0 = \omega(0, \mathbf{x}) = \omega_0^r(r, \theta) \mathbf{e}_r + \omega_0^\theta(r, \theta) \mathbf{e}_\theta + \omega_0^\varphi(r, \theta) \mathbf{e}_\varphi, \quad (20)$$

where

$$\begin{aligned} \omega^r &= \frac{1}{ar \sin \theta} \partial_\theta (\sin \theta u^\varphi), \omega^\theta = -\frac{1}{ar} \partial_r (ru^\varphi), \\ \omega^\varphi &= \frac{1}{a} \left(\partial_r u^\theta + \frac{u^\theta}{r} - \frac{\partial_\theta u^r}{r} \right). \end{aligned} \quad (21)$$

It is clear that

$$\operatorname{div} \omega = \frac{1}{a} \left(\partial_r \omega^r + \frac{2}{r} \omega^r + \frac{1}{r} \partial_\theta \omega^\theta + \frac{\cos \theta}{r \sin \theta} \omega^\theta \right) \equiv 0. \quad (22)$$

In addition, we can obtain the equation of ω^φ from (15) as

$$\begin{aligned} \partial_t \omega^\varphi + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \omega^\varphi &= v \left(\tilde{\Delta} - \frac{1}{a^2 r^2 \sin^2 \theta} \right) \omega^\varphi + \frac{u^r \omega^\varphi}{ar} + \frac{\cos \theta}{ar \sin \theta} u^\theta \omega^\varphi \\ &\quad + \left(\frac{\cos \theta}{ar \sin \theta} \partial_r - \frac{1}{ar^2} \partial_\theta \right) |u^\varphi|^2 \\ &\quad - \left(\sin \theta \partial_r \rho + \cos \theta \frac{1}{r} \partial_\theta \rho \right). \end{aligned} \quad (23)$$

We now state our main results as follows:

Theorem 1 (the case of 3D incompressible Boussinesq equations in \mathbb{R}^3 without swirl in the sense of spherical coordinates). Assume that $v > 0$ and $\mu > 0$. Let $(\mathbf{u}_0, \rho_0)(t, \mathbf{x})$ be given by (13) with $u_0^\varphi = 0$. Let $\omega_0^\varphi = \partial_r u_0^\theta + u_0^\theta/r - \partial_\theta u_0^r/r$. If $(\mathbf{u}_0, \rho_0) \in H^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$ and $\omega_0^\varphi/r \sin \theta \in L^2(\mathbb{R}^3)$, then the Cauchy problems (15) and (17) have a unique global strong solution $(u^r, u^\theta, u^\varphi, P, \rho)(t, r, \theta, \varphi)$ with $u^\varphi \equiv 0$ satisfying $\mathbf{u} \in L^\infty(0, +\infty; H^1(\mathbb{R}^3))$, given by (12). Moreover, assume that $\mathbf{u}_0(\mathbf{x}) = u_0^r(r, \theta) \mathbf{e}_r + u_0^\theta(r, \theta) \mathbf{e}_\theta$ is smooth with $u_0^r(0, \theta) = u_0^\theta(0, \theta) = \rho_0(0, \theta)|_{\theta=0, \pi} = 0$, and furthermore, with some compatibility conditions for the initial data with respect to $\theta = 0, \pi$ and $r = 0$, then the Cauchy problem (1) to the 3D incompressible Boussinesq equations has a unique global smooth solution in time.

Theorem 2 (the exponential decay rate in time and the global strong solution of 3D incompressible Boussinesq equations in the special bounded domain of \mathbb{R}^3 with swirl in the sense of spherical coordinates). Assume that $v > 0$ and $\mu > 0$. Let $\Omega = \tilde{\Omega} \subset \mathbb{R}^3$ in (2), given by (9). Let $(\mathbf{u}_0, \rho_0)(t, \mathbf{x})$ be given by (13) with $u_0^\varphi \neq 0$. If $(\mathbf{u}_0, \rho_0) \in H^2(\Omega)$ with $\operatorname{div} \mathbf{u}_0 = 0$ and $(\mathbf{u}_0, \rho_0)|_{\partial\Omega} = 0$, then the initial-boundary value problems (15), (17), and (18) to the incompressible Boussinesq equation (2) have a unique global strong solution $(u^r, u^\theta, u^\varphi, P, \rho)(t, r, \theta, \varphi)$ satisfying $\partial_t^i(\mathbf{u}, \rho) \in L^\infty(0, +\infty; H^{1-i}(\Omega))$, $i = 0, 1$, given by (12), and the exponential decay rate in time

$$\|(\mathbf{u}, \rho)(t, \cdot)\|_{H^1(\Omega)}^2 + \|(\mathbf{u}_t, \rho_t)(t, \cdot)\|_{L^2(\Omega)}^2 \leq Ce^{-\alpha t}, \quad 0 \leq t \leq +\infty, \quad (24)$$

for some constants $C = C(\Omega, v, \mu, \|(\mathbf{u}_0, \rho_0)\|_{H^2(\Omega)}) > 0$ and $\alpha = \alpha(\Omega, v, \mu) > 0$, independent of $t : 0 \leq t \leq \infty$. Moreover, any Leray-Hopf-type global weak solution (\mathbf{u}, ρ, P) , given by (12), to the initial-boundary value problem (2) is globally smooth in $(0, T] \times \Omega_1$ for any $0 \leq T \leq \infty$ and any smooth domain $\Omega_1 \subset \subset \Omega \subset \mathbb{R}^3$.

Remark 3. The assumptions $v > 0$ and $\mu > 0$ are key in the proofs of Theorems 1 and 2. The key point of the proof of Theorem 1 is to establish the a priori estimate on the quantity $\omega^\varphi/r \sin \theta - (1/2v)\rho$ and then to use the special geometry

structure (12) of the solutions $(\mathbf{u}, \rho)(t, \mathbf{x})$, which guarantees that there exist some kinds of cancelation regimes so that we can deal with the vortex stretching term $\omega \cdot \nabla \mathbf{u}$ in the vorticity equation for ω . The present method used in this paper cannot be extended to the case of $\nu = 0$ or $\mu = 0$. The global well-posedness problem on the 3D incompressible Boussinesq system with partial viscosity or diffusivity and without swirl in spherical coordinates is complex because each component of the velocity field in spherical coordinates in the Boussinesq system given by the classical Biot-Savart law is very complex, which will be discussed in the future. The classical Biot-Savart law expresses the velocity field that transports the vorticity in terms of the vorticity itself; see [38] and the references therein. The assumption in Theorem 2 on the domain $\Omega = \tilde{\Omega}$ with the special geometry structure given by (9) is key for one to prove our global regularity for the strong solution and global interior regularity for the smooth solution in time for 3D Boussinesq equations with large smooth initial data, which yields to one inequality of Ladyzhenskaya's type (see [3] and Lemma 6 for details), close to a two-dimensional case, for the function $(\mathbf{u}, \rho)(t, \mathbf{x})$ having the special geometry structure (12) for $\mathbf{x} \in \tilde{\Omega} \subset \mathbb{R}^3$. Also, if we replace the domain $\tilde{\Omega}$ in Theorem 2 by one smooth domain $\Omega_2 \subset \mathbb{R}^3$ satisfying that there exists one positive constant $\epsilon_1, \epsilon_2, \epsilon_3$, and ϵ_4 such that $\Omega_2 \subset \Omega_{\epsilon} \subset \mathbb{R}^3$ with

$$\begin{aligned} \Omega_{\epsilon} &= \{(x_1, x_2, x_3) \\ &= (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) \\ &\in \mathbb{R}^3 : 0 < \epsilon_1 \leq r \leq \epsilon_2 < \infty, 0 < \epsilon_3 \leq \theta \leq \epsilon_4 < \pi, 0 \leq \varphi < 2\pi\}, \end{aligned} \quad (25)$$

then the global strong solution obtained in Theorem 2 is also smooth in $(0, \infty) \times \Omega_2$.

Remark 4. The axisymmetric flow makes the 3D flow close to the 2D flow; that is, all velocity components (radial, angular (or swirl) and x_3 component) as well as the pressure are independent of the angular variable in the cylindrical coordinates. As a kind of fluid with special geometry structure, we know that the 1D parabolic Hausdorff measure of the set of possible singular points to the suitable weak solutions of the incompressible Navier-Stokes or Boussinesq system is zero; see [7, 8, 39] for details. This implies that the incompressible axisymmetric Navier-Stokes or Boussinesq equations cannot develop finite time singularities away from the symmetry axis. Based on this fact, it is not clear whether the potential finite-time-blow-up set for 3D incompressible Boussinesq equations in spherical coordinates is only one point set, where the flow is a special variant of axisymmetric, i.e., spherically symmetric, in \mathbb{R}^3 . This is the main motivation of the current paper.

The rest of this paper is organized as follows. In Section 2, we introduce some technical lemmas used for the proof of the main theorems. In Section 3, we prove Theorems 1 and 2.

2. Preliminaries

In this section, we provide some lemmas used for the proof of the main theorems.

Lemma 5. (see [40]). *Let $\mathbf{u} \in W^{1,p}(\mathbb{R}^3)$ be a velocity field with its divergence free and vorticity ω ; then, the inequality*

$$\|\nabla \mathbf{u}\|_{L^p} \leq C(p) \|\omega\|_{L^p}, \quad (26)$$

holds for any $p \in (1, \infty)$, where the constant $C(p)$ depends only on p .

Lemma 6 (see [3]). *Let $D \subseteq \mathbb{R}^2$; then, there exists a constant $C(D)$ such that, for any $f \in H_0^1(D)$,*

$$\|f\|_{L^4(D)} \leq C(D) \|f\|_{L^2(D)}^{1/2} \|\nabla f\|_{L^2(D)}^{1/2}. \quad (27)$$

Lemma 7 (see [41]). *Suppose that the initial data $(\mathbf{u}_0, \rho_0) \in H^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$ in (1); then, any Leray-Hopf weak solution \mathbf{u} of 3D incompressible Boussinesq equation (1) is also a smooth solution in $(0, T] \times \mathbb{R}^3$ if there holds that*

$$u \in L^p(0, T; L^q(\mathbb{R}^3)), \quad (28)$$

in which p and q satisfy the conditions

$$\frac{2}{p} + \frac{3}{q} \leq 1 \quad \text{with } 3 < q < \infty, 2 < p \leq \infty. \quad (29)$$

Lemma 8 (see [42]). *Suppose that Ω is smooth and the initial data (\mathbf{u}_0, ρ_0) in (2) satisfies $(\mathbf{u}_0, \rho_0) \in H^2(\Omega)$ with $\operatorname{div} \mathbf{u}_0 = 0$ and $(\mathbf{u}_0, \rho_0)|_{\partial\Omega} = 0$; then, any Leray-Hopf weak solution \mathbf{u} of 3D incompressible Boussinesq equation (2) is also a smooth solution in $(0, T] \times \Omega$ if there holds that*

$$u \in L^p(0, T; L^q(\Omega)), \quad (30)$$

in which p and q satisfy the conditions

$$\frac{2}{p} + \frac{3}{q} \leq 1 \quad \text{with } 3 < q < \infty, 2 \leq p \leq \infty. \quad (31)$$

3. Proof of Main Results

In this section, we give the proofs of Theorems 1 and 2.

Proof of Theorem 1. From (1), for any $T > 0$, we have the energy inequality

$$\begin{aligned} &\sup_{0 \leq t \leq T} \left(\|\mathbf{u}\|_{L^2(\mathbb{R}^3)}^2 + \|\rho\|_{L^2(\mathbb{R}^3)}^2 \right) \\ &\quad + 2\nu \int_0^T \|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^3)}^2 dt + 2\mu \int_0^T \|\nabla \rho\|_{L^2(\mathbb{R}^3)}^2 dt \\ &\leq C \left(T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|\rho_0\|_{L^2(\mathbb{R}^3)} \right). \end{aligned} \quad (32)$$

By the existence and uniqueness of the local smooth solution to the Cauchy problem (1) for the 3D Boussinesq equations, it is easy to get that $u^\theta \equiv 0$ for the case of no swirl initial data $u_0^\theta \equiv 0$. In this kind of case of no swirl, the velocity and vorticity satisfy the following special form:

$$\mathbf{u}(t, \mathbf{x}) = u^r(t, r, \theta)\mathbf{e}_r + u^\theta(t, r, \theta)\mathbf{e}_\theta, \omega(t, \mathbf{x}) = \omega^\theta(t, r, \theta)\mathbf{e}_\phi, \quad (33)$$

and hence, equation (23) for ω^θ is simplified as

$$\begin{aligned} \partial_t \omega^\theta + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \omega^\theta \\ = \nu \left(\tilde{\Delta} - \frac{1}{r^2 \sin^2 \theta} \right) \omega^\theta + \frac{u^r \omega^\theta}{r} + \frac{\cos \theta}{r \sin \theta} u^\theta \omega^\theta \\ - \left(\sin \theta \partial_r \rho + \cos \theta \frac{1}{r} \partial_\theta \rho \right). \end{aligned} \quad (34)$$

Multiplying (34) by $r^2 \sin^2 \theta$ and then letting $r = 0$, $\theta = 0$, or $\theta = \pi$ and by the existence and uniqueness of the local smooth solution to the Cauchy problem (1) or (15)–(17) for the 3D Boussinesq equations, it is easy to see that

$$\omega^\theta(t, 0, \theta) = \omega^\theta(t, r, 0) = \omega^\theta(t, r, \pi) = 0. \quad (35)$$

Similarly, we have

$$\left. \frac{\omega^\theta(t, r, \theta)}{r \sin \theta} \right|_{r=0} = \left. \frac{\omega^\theta(t, r, \theta)}{r \sin \theta} \right|_{\theta=0, \pi} = 0. \quad (36)$$

Taking $\omega^\theta(t, r, \theta) = g(t, r, \theta)r \sin \theta$, i.e., $g(t, r, \theta) = \omega^\theta(t, r, \theta)/r \sin \theta$, satisfying $g(t, 0, \theta) = g(t, r, 0) = g(t, r, \pi) = 0$, then we have

$$\begin{aligned} (\tilde{\mathbf{u}} \cdot \tilde{\nabla})(gr \sin \theta) &= \left(u^r \partial_r + \frac{u^\theta}{r} \partial_\theta \right) (gr \sin \theta) \\ &= u^r \sin \theta (g + r \partial_r g) + u^\theta (g \cos \theta + \sin \theta \partial_\theta g) \\ &= r \sin \theta (\tilde{\mathbf{u}} \cdot \tilde{\nabla})g + u^r g \sin \theta + u^\theta g \cos \theta, \end{aligned} \quad (37)$$

$$\begin{aligned} \left(\tilde{\Delta} - \frac{1}{r^2 \sin^2 \theta} \right) (gr \sin \theta) \\ = \left(\partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta - \frac{1}{r^2 \sin^2 \theta} \right) (gr \sin \theta) \\ = \sin \theta \left[\partial_r^2 (rg) + \frac{2}{r} \partial_r (rg) \right] + \frac{1}{r} \partial_\theta^2 (g \sin \theta) \\ + \frac{\cos \theta}{r \sin \theta} \partial_\theta (g \sin \theta) - \frac{g}{r \sin \theta} \\ = \sin \theta \left(r \partial_r^2 g + 4 \partial_r g + \frac{2}{r} g \right) \\ + \frac{1}{r} \left(3 \cos \theta \partial_\theta g - g \sin \theta + \sin \theta \partial_\theta^2 g + \frac{\cos^2 \theta}{\sin \theta} g \right) \\ - \frac{g}{r \sin \theta} \end{aligned}$$

$$\begin{aligned} &= r \sin \theta \left(\partial_r^2 + \frac{4}{r} \partial_r \right) g + \frac{2 \sin \theta}{r} g \\ &\quad + r \sin \theta \left(\frac{1}{r^2} \partial_\theta^2 g + \frac{3 \cos \theta}{r^2 \sin \theta} \partial_\theta g \right) \\ &\quad + \left(\frac{\cos^2 \theta}{r \sin \theta} - \frac{\sin \theta}{r} \right) g - \frac{g}{r \sin \theta} \\ &= r \sin \theta \left(\partial_r^2 + \frac{4}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \frac{3 \cos \theta}{r^2 \sin \theta} \partial_\theta \right) g \\ &\quad + \left(\frac{2 \sin \theta}{r} + \frac{1 - 2 \sin^2 \theta}{r \sin \theta} - \frac{1}{r \sin \theta} \right) g \\ &= r \sin \theta \left(\tilde{\Delta} + \frac{2}{r} \partial_r + \frac{2 \cos \theta}{r^2 \sin \theta} \partial_\theta \right) g. \end{aligned} \quad (38)$$

Now putting $\omega^\theta(t, r, \theta) = g(t, r, \theta)r \sin \theta$ into (34) and using (37)–(38), we obtain the following equation for $g(t, r, \theta)$:

$$\begin{aligned} \partial_t g + (\tilde{\mathbf{u}} \cdot \tilde{\nabla})g - \nu \tilde{\Delta} g &= 2\nu \left(\frac{1}{r} \partial_r + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta \right) g \\ &\quad - \frac{1}{r \sin \theta} \left(\sin \theta \partial_r \rho + \cos \theta \frac{1}{r} \partial_\theta \rho \right). \end{aligned} \quad (39)$$

To deal with the more singular second term in the right-hand side of (39), we decompose g into $g = G + (1/2\nu)\rho$; then, $G(t, r, \theta) = g(t, r, \theta) - (1/2\nu)\rho(t, r, \theta)$ satisfies

$$G(t, 0, \theta) = G(t, r, 0) = G(t, r, \pi) = 0, \quad (40)$$

and the following equation

$$\begin{aligned} \partial_t \left(G + \frac{1}{2\nu} \rho \right) + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \left(G + \frac{1}{2\nu} \rho \right) - \nu \tilde{\Delta} \left(G + \frac{1}{2\nu} \rho \right) \\ = 2\nu \left(\frac{1}{r} \partial_r + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta \right) \left(G + \frac{1}{2\nu} \rho \right) \\ - \frac{1}{r \sin \theta} \left(\sin \theta \partial_r \rho + \cos \theta \frac{1}{r} \partial_\theta \rho \right), \end{aligned} \quad (41)$$

which implies that

$$\partial_t G + (\tilde{\mathbf{u}} \cdot \tilde{\nabla})G - \nu \tilde{\Delta} G = 2\nu \left(\frac{1}{r} \partial_r + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta \right) G - \frac{\mu - \nu}{2\nu} \tilde{\Delta} \rho. \quad (42)$$

Multiplying equation (42) by G and integrating the resulting equation on \mathbb{R}^3 , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|G\|_{L^2(\mathbb{R}^3)}^2 + \nu \|\tilde{\nabla} G\|_{L^2(\mathbb{R}^3)}^2 \\ = \nu \left(\int_{\mathbb{R}^3} \frac{1}{r} \partial_r |G|^2 d\mathbf{x} + \int_{\mathbb{R}^3} \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta |G|^2 d\mathbf{x} \right) \\ - \frac{\mu - \nu}{2\nu} \int_{\mathbb{R}^3} G \tilde{\Delta} \rho d\mathbf{x} = I_1 + I_2, \end{aligned} \quad (43)$$

where I_1 and I_2 are defined by and can be estimated as follows:

$$\begin{aligned}
I_1 &= \nu \left(\int_{\mathbb{R}^3} \frac{1}{r} \partial_r |G|^2 d\mathbf{x} + \int_{\mathbb{R}^3} \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta |G|^2 d\mathbf{x} \right) \\
&= \nu \left(\int_0^{2\pi} \int_0^\pi \int_0^\infty \left(\frac{1}{r} \partial_r |G|^2 \right) r^2 \sin \theta dr d\theta d\varphi \right. \\
&\quad \left. + \int_0^{2\pi} \int_0^\pi \int_0^\infty \left(\frac{\cos \theta}{r^2 \sin \theta} \partial_\theta |G|^2 \right) r^2 \sin \theta dr d\theta d\varphi \right) \\
&= \nu \left(\int_0^{2\pi} \int_0^\pi \int_0^\infty r \sin \theta \partial_r |G|^2 dr d\theta d\varphi \right. \\
&\quad \left. + \int_0^{2\pi} \int_0^\pi \int_0^\infty \cos \theta \partial_\theta |G|^2 dr d\theta d\varphi \right) \\
&= -\nu \left(\int_0^{2\pi} \int_0^\pi \int_0^\infty |G|^2 \sin \theta (\partial_r r) dr d\theta d\varphi \right. \\
&\quad \left. + \int_0^{2\pi} \int_0^\pi \int_0^\infty |G|^2 \partial_\theta (\cos \theta) dr d\theta d\varphi \right) \equiv 0,
\end{aligned} \tag{44}$$

$$\begin{aligned}
I_2 &= -\frac{\mu - \nu}{2\nu} \int_{\mathbb{R}^3} G \tilde{\Delta} \rho d\mathbf{x} = \frac{\mu - \nu}{2\nu} \int_{\mathbb{R}^3} \tilde{\nabla} G \cdot \tilde{\nabla} \rho d\mathbf{x} \\
&\leq \frac{\nu}{2} \|\tilde{\nabla} G\|_{L^2(\mathbb{R}^3)}^2 + C \|\tilde{\nabla} \rho\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned} \tag{45}$$

Putting (44) and (45) into (43), we get

$$\frac{1}{2} \frac{d}{dt} \|G\|_{L^2(\mathbb{R}^3)}^2 + \nu \|\tilde{\nabla} G\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{\nu}{2} \|\tilde{\nabla} G\|_{L^2(\mathbb{R}^3)}^2 + C \|\tilde{\nabla} \rho\|_{L^2(\mathbb{R}^3)}^2, \tag{46}$$

which, together with (32), yields to the following estimate for $G(t, r, \theta)$:

$$\begin{aligned}
\|G(t, r, \theta)\|_{L^2(\mathbb{R}^3)} &\leq C \left(T, \|G(0, r, \theta)\|_{L^2(\mathbb{R}^3)}, \|u_0\|_{L^2(\mathbb{R}^3)}, \|\rho_0\|_{L^2(\mathbb{R}^3)} \right).
\end{aligned} \tag{47}$$

Thus, we have

$$\begin{aligned}
\|g(t, r, \theta)\|_{L^2(\mathbb{R}^3)} &= \left\| G + \frac{1}{2\nu} \rho \right\|_{L^2(\mathbb{R}^3)} \\
&\leq \|G\|_{L^2(\mathbb{R}^3)} + C \|\rho\|_{L^2(\mathbb{R}^3)} \\
&\leq C \left(T, \left\| \frac{\omega_0^\varphi(r, \theta)}{r \sin \theta} \right\|_{L^2(\mathbb{R}^3)}, \|u_0\|_{L^2(\mathbb{R}^3)}, \|\rho_0\|_{L^2(\mathbb{R}^3)} \right).
\end{aligned} \tag{48}$$

Next, we obtain the estimate for the vorticity $\omega = \omega^\varphi(t, r, \theta)\mathbf{e}_\varphi$, given by (33) in the case of no swirl for the

3D incompressible Boussinesq equation in the spherical coordinate system.

It is known that the vorticity equation for the vorticity $\omega = \nabla \times \mathbf{u}$ for the 3D incompressible Boussinesq equation is the following:

$$\partial_t \omega + (\mathbf{u} \cdot \nabla) \omega - \nu \Delta \omega = \omega \cdot \nabla \mathbf{u} - \left(\sin \theta \partial_r \rho + \cos \theta \frac{1}{r} \partial_\theta \rho \right) \mathbf{e}_\varphi. \tag{49}$$

Multiplying equation (49) by ω and integrating the resulting equation on \mathbb{R}^3 , we have, for any $T > 0$, $0 \leq t \leq T$,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \nu \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} (\omega \cdot \nabla) \mathbf{u} \cdot \omega d\mathbf{x} - \int_{\mathbb{R}^3} \omega^\varphi \left(\sin \theta \partial_r \rho + \cos \theta \frac{1}{r} \partial_\theta \rho \right) d\mathbf{x} \\
&= J_1 + J_2,
\end{aligned} \tag{50}$$

where $J_1 = \int_{\mathbb{R}^3} (\omega \cdot \nabla) \mathbf{u} \cdot \omega d\mathbf{x}$ and $J_2 = -\int_{\mathbb{R}^3} \omega^\varphi (\sin \theta \partial_r \rho + \cos \theta (1/r) \partial_\theta \rho) d\mathbf{x}$ can be estimated as follows by using the special structure (33) of the velocity \mathbf{u} and the vorticity ω . Using (33), with the help of the Hölder inequality, Gagliardo-Nirenberg inequality, and Young inequality, we have

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}^3} \left[\omega^\varphi \mathbf{e}_\varphi \cdot \left(\mathbf{e}_r \partial_r + \mathbf{e}_\theta \frac{1}{r} \partial_\theta + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \partial_\varphi \right) \right] \\
&\quad \cdot \left(u^r \mathbf{e}_r + u^\theta \mathbf{e}_\theta \right) \cdot (\omega^\varphi \mathbf{e}_\varphi) d\mathbf{x} \\
&= \int_{\mathbb{R}^3} \left[\frac{\omega^\varphi}{r \sin \theta} \partial_\varphi \left(u^r \mathbf{e}_r + u^\theta \mathbf{e}_\theta \right) \right] \cdot (\omega^\varphi \mathbf{e}_\varphi) d\mathbf{x} \\
&= \int_{\mathbb{R}^3} \left[\frac{\omega^\varphi}{r \sin \theta} \left(u^r \sin \theta + u^\theta \cos \theta \right) \mathbf{e}_\varphi \right] \cdot (\omega^\varphi \mathbf{e}_\varphi) d\mathbf{x} \\
&= \int_{\mathbb{R}^3} \frac{1}{r} u^r \omega^\varphi \omega^\varphi d\mathbf{x} + \int_{\mathbb{R}^3} \frac{\cos \theta}{r \sin \theta} u^\theta \omega^\varphi \omega^\varphi d\mathbf{x} \\
&= \int_{\mathbb{R}^3} u^r g \omega^\varphi \sin \theta d\mathbf{x} + \int_{\mathbb{R}^3} u^\theta g \omega^\varphi \cos \theta d\mathbf{x} \\
&\leq \int_{\mathbb{R}^3} |u^r g \omega^\varphi| d\mathbf{x} + \int_{\mathbb{R}^3} |u^\theta g \omega^\varphi| d\mathbf{x} \\
&\leq \left(\|u^r\|_{L^3(\mathbb{R}^3)} + \|u^\theta\|_{L^3(\mathbb{R}^3)} \right) \|g\|_{L^2(\mathbb{R}^3)} \|\omega\|_{L^6(\mathbb{R}^3)} \\
&\leq C \|u\|_{L^2(\mathbb{R}^3)}^{1/2} \|\nabla u\|_{L^2(\mathbb{R}^3)}^{1/2} \|g\|_{L^2(\mathbb{R}^3)} \|\nabla \omega\|_{L^2(\mathbb{R}^3)} \\
&\leq C \|u\|_{H^1(\mathbb{R}^3)}^2 + \frac{\nu}{2} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2,
\end{aligned} \tag{51}$$

$$J_2 \leq \int_{\mathbb{R}^3} |\omega^\varphi| \left(|\partial_r \rho| + \left| \frac{1}{r} \partial_\theta \rho \right| \right) d\mathbf{x} \leq \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \|\tilde{\nabla} \rho\|_{L^2(\mathbb{R}^3)}^2. \tag{52}$$

Putting (51) and (52) into (50), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{\nu}{2} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2 &\leq C \|\mathbf{u}\|_{H^1(\mathbb{R}^3)}^2 + \|\omega\|_{L^2(\mathbb{R}^3)}^2 \\ &\quad + \|\tilde{\nabla} \rho\|_{L^2(\mathbb{R}^3)}^2, \end{aligned} \quad (53)$$

which, by applying Gronwall's inequality and by using (32), yields to, for any $T > 0$,

$$\begin{aligned} \|\omega(t, \cdot)\|_{L^2(\mathbb{R}^3)} &\leq C \left(T, \|\mathbf{u}_0\|_{H^1(\mathbb{R}^3)}, \left\| \frac{\omega_0^\varphi}{r \sin \theta} \right\|_{L^2(\mathbb{R}^3)}, \|\rho_0\|_{L^2(\mathbb{R}^3)} \right), \quad 0 \leq t \leq T, \end{aligned} \quad (54)$$

$$\begin{aligned} \int_0^t \|\nabla \omega(s, \cdot)\|_{L^2(\mathbb{R}^3)}^2 ds &\leq C \left(T, \|\mathbf{u}_0\|_{H^1(\mathbb{R}^3)}, \left\| \frac{\omega_0^\varphi}{r \sin \theta} \right\|_{L^2(\mathbb{R}^3)}, \|\rho_0\|_{L^2(\mathbb{R}^3)} \right), \quad 0 \leq t \leq T. \end{aligned} \quad (55)$$

Using Lemma 5, we get from (54) that, for any $0 \leq T \leq \infty$,

$$\nabla \mathbf{u} \in L^\infty(0, T; L^2(\mathbb{R}^3)), \quad (56)$$

and hence, by Sobolev's imbedding theorem, we have, for any $0 \leq T \leq \infty$,

$$\mathbf{u} \in L^\infty(0, T; L^6(\mathbb{R}^3)). \quad (57)$$

Now, the desired regularity estimate for the 3D incompressible Boussinesq equation (1) is obtained; hence, by applying Lemma 7, we obtain the results stated in Theorem 1.

The proof of Theorem 1 is complete. \square

Proof of Theorem 2. We take $\Omega = \tilde{\Omega}$ in Theorem 2, where $\tilde{\Omega}$ is given by (9) having one special geometry structure. Also, \mathbf{u} in Theorem 2 is given by (12), which satisfies that $|\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2 = (u^r)^2 + (u^\theta)^2 + (u^\varphi)^2$ by using the orthogonality of three spherical coordinate unit vectors. Firstly, for the system (2), we have the following basic energy estimates, for $0 \leq t \leq +\infty$,

$$\begin{aligned} \frac{d}{dt} \int_\Omega |\mathbf{u}(t, \cdot)|^2 d\mathbf{x} + 2\nu \int_\Omega |\nabla \mathbf{u}(t, \cdot)|^2 d\mathbf{x} &\leq \delta \int_\Omega |\mathbf{u}(t, \cdot)|^2 d\mathbf{x} + C(\delta) \int_\Omega |\rho(t, \cdot)|^2 d\mathbf{x}, \end{aligned} \quad (58)$$

$$\frac{d}{dt} \int_\Omega |\rho(t, \cdot)|^2 d\mathbf{x} + 2\mu \int_\Omega |\nabla \rho(t, \cdot)|^2 d\mathbf{x} = 0, \quad (59)$$

for some constant $C(\delta) > 0$ and any $\delta > 0$, which, together with Poincaré's inequality for \mathbf{u} and ρ , yield the energy estimate, for $0 \leq t \leq +\infty$,

$$\begin{aligned} &\left(\|\mathbf{u}(t, \cdot)\|_{L^2(\Omega)}^2 + \|\rho(t, \cdot)\|_{L^2(\Omega)}^2 \right) \\ &\quad + 2\nu \int_0^t \|\nabla \mathbf{u}(t, \cdot)\|_{L^2(\Omega)}^2 dt + 2\mu \int_0^t \|\nabla \rho(t, \cdot)\|_{L^2(\Omega)}^2 dt \\ &\leq C \left(\Omega, \nu, \mu, \|\mathbf{u}_0\|_{L^2(\Omega)}, \|\rho_0\|_{L^2(\Omega)} \right), \end{aligned} \quad (60)$$

$$\|(\mathbf{u}, \rho)(t, \cdot)\|_{L^2(\Omega)}^2 \leq C \left(\Omega, \nu, \mu, \|\mathbf{u}_0\|_{L^2(\Omega)}, \|\rho_0\|_{L^2(\Omega)} \right) e^{-\alpha t}, \quad (61)$$

for some constants $C = C(\Omega, \nu, \mu, \|\mathbf{u}_0\|_{L^2(\Omega)}, \|\rho_0\|_{L^2(\Omega)}) > 0$ and $\alpha = \alpha(\Omega, \nu, \mu) > 0$.

Next, we give the estimates of $\|\partial_t(\mathbf{u}, \rho)(t, \cdot)\|_{L^2(\Omega)}^2$.

Differentiating (2) with respect to t , one gets

$$\begin{cases} \mathbf{u}_{tt} + \mathbf{u}_t \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}_t + \nabla P_t = \nu \Delta \mathbf{u}_t + \rho_t \mathbf{e}_3, \\ \rho_{tt} + \mathbf{u} \cdot \nabla \rho_t + \mathbf{u}_t \cdot \nabla \rho = \mu \Delta \rho_t, \\ \operatorname{div} \mathbf{u}_t = 0, \\ \mathbf{u}_t|_{\partial\Omega} = 0, \rho_t|_{\partial\Omega} = 0, \\ \mathbf{u}_t(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \rho_t(0, \mathbf{x}) = \rho_0(\mathbf{x}), \end{cases} \quad (62)$$

where \mathbf{v}_0 and ρ_0 satisfy, by using (2), that

$$\mathbf{v}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + \nabla P_0 = \nu \Delta \mathbf{u}_0 + \rho_0 \mathbf{e}_3, \operatorname{div} \mathbf{v}_0 = 0, \quad (63)$$

$$\rho_0 + (\mathbf{u}_0 \cdot \nabla) \rho_0 = \mu \Delta \rho_0. \quad (64)$$

It is easy to get that

$$\|\mathbf{v}_0\|_{L^2(\Omega)} \leq C \left(\|\mathbf{u}_0\|_{H^2(\Omega)}, \|\rho_0\|_{H^2(\Omega)} \right), \quad (65)$$

$$\|\rho_0\|_{L^2(\Omega)} \leq C \left(\|\mathbf{u}_0\|_{H^2(\Omega)}, \|\rho_0\|_{H^2(\Omega)} \right). \quad (66)$$

In fact, multiplying (64) by \mathbf{v}_0 and integrating the resulting equation on Ω , applying the Hölder inequality, Gagliardo-Nirenberg inequality, and Young inequality, we have

$$\begin{aligned} \|\mathbf{v}_0\|_{L^2(\Omega)}^2 &= - \int_\Omega (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 \cdot \mathbf{v}_0 d\mathbf{x} + \nu \int_\Omega \Delta \mathbf{u}_0 \cdot \mathbf{v}_0 d\mathbf{x} \\ &\quad + \int_\Omega \rho_0 \mathbf{e}_3 \cdot \mathbf{v}_0 d\mathbf{x} \\ &\leq C \|\mathbf{u}_0\|_{L^3(\Omega)} \|\nabla \mathbf{u}_0\|_{L^6(\Omega)} \|\mathbf{v}_0\|_{L^2(\Omega)} \\ &\quad + C \|\Delta \mathbf{u}_0\|_{L^2(\Omega)} \|\mathbf{v}_0\|_{L^2(\Omega)} + \|\rho_0\|_{L^2(\Omega)} \|\mathbf{v}_0\|_{L^2(\Omega)} \\ &\leq C \left(\|\mathbf{u}_0\|_{H^2(\Omega)} + \|\rho_0\|_{L^2(\Omega)} \right) \|\mathbf{v}_0\|_{L^2(\Omega)}, \end{aligned} \quad (67)$$

which implies (65). Similarly, we have (66).

Multiplying the first equation in (62) by \mathbf{u}_t and integrating the resulting equation on Ω , with the help of the Hölder inequality, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_t\|_{L^2(\Omega)}^2 + \nu \|\nabla \mathbf{u}_t\|_{L^2(\Omega)}^2 \\ &= \int_{\Omega} \rho_t \mathbf{e}_3 \cdot \mathbf{u}_t d\mathbf{x} - \int_{\Omega} (\mathbf{u}_t \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_t d\mathbf{x} \leq C(\delta) \|\rho_t\|_{L^2(\Omega)}^2 \\ & \quad + \delta \|\mathbf{u}_t\|_{L^2(\Omega)}^2 + C \|\mathbf{u}_t\|_{L^4(\Omega)}^2 \|\nabla \mathbf{u}\|_{L^2(\Omega)}. \end{aligned} \quad (68)$$

Multiplying the second equation in (62) by ρ_t and integrating the resulting equation on Ω , with the help of the Hölder inequality, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\rho_t\|_{L^2(\Omega)}^2 + \mu \|\nabla \rho_t\|_{L^2(\Omega)}^2 \\ &= - \int_{\Omega} (\mathbf{u}_t \cdot \nabla) \rho \cdot \rho_t d\mathbf{x} \leq C \|\mathbf{u}_t\|_{L^4(\Omega)} \|\nabla \rho\|_{L^2(\Omega)} \|\rho_t\|_{L^4(\Omega)}. \end{aligned} \quad (69)$$

In the following, we use the special geometry structure (9) of the domain $\Omega = \tilde{\Omega} \subset \mathbb{R}^3$ and the special geometry structure (12) of the functions $(\mathbf{u}, \rho)(t, \mathbf{x})$ in spherical coordinates in \mathbb{R}^3 to obtain the following inequality for $(\mathbf{u}_t, \rho_t)(t, \mathbf{x})$ defined in $[0, t) \times \tilde{\Omega} \subset [0, \infty) \times \mathbb{R}^3$ with $(\mathbf{u}, \rho)|_{\partial\Omega} = 0$: there exists a constant $C = C(\Omega) > 0$ such that

$$\|\mathbf{u}_t(t, \cdot)\|_{L^4(\Omega)} \leq C(\Omega) \|\mathbf{u}_t(t, \cdot)\|_{L^2(\Omega)}^{1/2} \|\nabla \mathbf{u}_t(t, \cdot)\|_{L^2(\Omega)}^{1/2}, \quad (70)$$

$$\|\rho_t(t, \cdot)\|_{L^4(\Omega)} \leq C(\Omega) \|\rho_t(t, \cdot)\|_{L^2(\Omega)}^{1/2} \|\nabla \rho_t(t, \cdot)\|_{L^2(\Omega)}^{1/2}, \quad (71)$$

where

$$\nabla = \mathbf{e}_r \partial_r + \frac{1}{r} \mathbf{e}_{\theta} \partial_{\theta} + \frac{1}{r \sin \theta} \mathbf{e}_{\varphi} \partial_{\varphi}. \quad (72)$$

We note that the inequalities (70) are the same as in the two-dimensional case, which are, in general, not true for the general functions $\mathbf{u}(t, \mathbf{x})$ or $\rho(t, \mathbf{x})$ when $\mathbf{x} \in \Omega$ if Ω is the general bounded domain of \mathbb{R}^3 . However, the equalities (70) are true under the assumption of Theorem 2 because of the special geometry structures (9) and (12) for the domain Ω and the functions $(\mathbf{u}, \rho)(t, \mathbf{x})$, especially $\rho(t, \mathbf{x}) = \rho(t, r, \theta)$ independent of φ and $\mathbf{u}(t, \mathbf{x}) = \mathbf{u}(t, r, \theta, \varphi)$ depending upon φ only by the three orthogonal unit vectors in spherical coordinates (the combination of the functions $\cos \varphi$ and $\sin \varphi$). In fact, the inequalities (70) and (71) has been proven by S. Wang and Y.X. Wang in [16] for the domain $\Omega \subset \mathbb{R}^3$ having the special geometry structure (9) and for the function $\mathbf{u}(t, \mathbf{x})$, $\mathbf{x} \in \Omega \subset \mathbb{R}^3$, with $\mathbf{u}(t, \mathbf{x})|_{\partial\Omega} = 0$ having the special geometry structure (12).

For completeness, we give the proof of the inequalities (70) and (71). Taking $f(t, r, \theta) = \rho_t r^{1/2} \sin^{1/4} \theta$ in Lemma 6

with the domain $D = (r_0, r_1) \times (\theta_0, \theta_1) \subset \mathbb{R}^2$ and $\rho(t, r, \theta)|_{\partial D} = 0$, we get

$$\begin{aligned} \|\rho_t\|_{L^4(\Omega)}^4 &= \int_{\Omega} |\rho_t|^4(t, r, \theta) d\mathbf{x} \\ &= \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} (\rho_t)^4(t, r, \theta) r^2 \sin \theta dr d\theta d\varphi \\ &= 2\pi \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} (\rho_t r^{1/2} \sin^{1/4} \theta)^4 dr d\theta \\ &\leq C \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} (|\rho_t| r^{1/2} \sin^{1/4} \theta)^2 dr d\theta \\ &\quad \cdot \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} |\nabla_{r,\theta} (\rho_t r^{1/2} \sin^{1/4} \theta)|^2 dr d\theta, \end{aligned} \quad (73)$$

where $C = C(\Omega)$ is a constant depending upon the domain $\Omega = \tilde{\Omega}$, and $\nabla_{r,\theta} = (\partial_r, \partial_{\theta})$. It is clear that

$$\begin{aligned} & \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} (|\rho_t| r^{1/2} \sin^{1/4} \theta)^2 dr d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} \frac{1}{r \sqrt{\sin \theta}} |\rho_t|^2 r^2 \sin \theta dr d\theta d\varphi \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} \frac{1}{r_0 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} |\rho_t|^2 r^2 \sin \theta dr d\theta d\varphi \\ &= \frac{1}{2\pi r_0 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \|\rho_t\|_{L^2(\Omega)}^2, \end{aligned} \quad (74)$$

$$\begin{aligned} & \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} |\nabla_{r,\theta} (\rho_t r^{1/2} \sin^{1/4} \theta)|^2 dr d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} \left(|\partial_r (\rho_t r^{1/2} \sin^{1/4} \theta)|^2 \right. \\ & \quad \left. + |\partial_{\theta} (\rho_t r^{1/2} \sin^{1/4} \theta)|^2 \right) dr d\theta d\varphi \\ &\leq \frac{1}{\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} \left[r \sin^{1/2} \theta (|\partial_r \rho_t|^2 + |\partial_{\theta} \rho_t|^2) \right. \\ & \quad \left. + \left(\frac{1}{4r} \sin^{1/2} \theta + \frac{\cos^2 \theta}{16} r \sin^{-3/2} \theta \right) |\rho_t|^2 \right] dr d\theta d\varphi \\ &= K_1 + K_2, \end{aligned} \quad (75)$$

where

$$\begin{aligned} K_1 &= \frac{1}{\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} \left(\frac{1}{r \sqrt{\sin \theta}} |\partial_r \rho_t|^2 + \frac{r}{\sqrt{\sin \theta}} \left| \frac{1}{r} \partial_{\theta} \rho_t \right|^2 \right) \\ &\quad \cdot r^2 \sin \theta dr d\theta d\varphi \end{aligned}$$

$$\begin{aligned}
& \leq \frac{1}{\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} \left(\frac{1}{r_0 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} |\partial_r \rho_t|^2 \right. \\
& \quad \left. + \frac{R_0}{\sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \left| \frac{1}{r} \partial_\theta \rho_t \right|^2 \right) r^2 \sin \theta dr d\theta d\varphi \\
& \leq \frac{1}{\pi \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \max \left\{ \frac{1}{r_0}, R_0 \right\} \\
& \quad \cdot \int_\Omega \left(|\partial_r \rho_t|^2 + \left| \frac{1}{r} \partial_\theta \rho_t \right|^2 \right) dx \\
& \leq \frac{1}{\pi \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \max \left\{ \frac{1}{r_0}, R_0 \right\} \\
& \quad \cdot \int_\Omega \left(|\partial_r \rho_t|^2 + \left| \frac{1}{r} \partial_\theta \rho_t \right|^2 + \left| \frac{1}{r \sin \theta} \partial_\varphi \rho_t \right|^2 \right) dx \\
& = \frac{1}{\pi \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \max \left\{ \frac{1}{r_0}, R_0 \right\} \|\nabla \rho_t\|_{L^2(\Omega)}^2,
\end{aligned} \tag{76}$$

$$\begin{aligned}
K_2 &= \frac{1}{\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} \left(\frac{1}{4r^3 \sqrt{\sin \theta}} + \frac{1}{16r \sin^2 \theta \sqrt{\sin \theta}} \right) \\
& \quad \cdot |\rho_t|^2 r^2 \sin \theta dr d\theta d\varphi \\
& \leq \frac{1}{\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} \left(\frac{1}{4r_0^3 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \right. \\
& \quad \left. + \frac{1}{16r_0 \left(\min_{\theta \in (\theta_0, \theta_1)} \sin \theta \right)^2 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \right) \\
& \quad \cdot |\rho_t|^2 r^2 \sin \theta dr d\theta d\varphi \\
& \leq \frac{1}{\pi} \left(\frac{1}{4r_0^3 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \right. \\
& \quad \left. + \frac{1}{16r_0 \left(\min_{\theta \in (\theta_0, \theta_1)} \sin \theta \right)^2 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \right) \int_\Omega |\rho_t|^2 dx \\
& \leq \frac{1}{\pi} \left(\frac{1}{4r_0^3 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \right. \\
& \quad \left. + \frac{1}{16r_0 \left(\min_{\theta \in (\theta_0, \theta_1)} \sin \theta \right)^2 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \right) \|\rho_t\|_{L^2(\Omega)}^2
\end{aligned}$$

$$\begin{aligned}
& \leq C(\Omega) \left(\frac{1}{4r_0^3 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \right. \\
& \quad \left. + \frac{1}{16r_0 \left(\min_{\theta \in (\theta_0, \theta_1)} \sin \theta \right)^2 \sqrt{\min_{\theta \in (\theta_0, \theta_1)} \sin \theta}} \right) \|\nabla \rho_t\|_{L^2(\Omega)}^2,
\end{aligned} \tag{77}$$

with the help of Poincaré's inequality and the fact that $|\nabla \rho(r, \theta, \varphi, t)|^2 = |\partial_r \rho|^2 + |\partial_\theta \rho/r|^2 + |\partial_\varphi \rho/r \sin \theta|^2$.

Combining (75) together with (76)–(77), we have

$$\int_{\theta_0}^{\theta_1} \int_{r_0}^{R_0} |\nabla_r(\rho_t r^{1/2} \sin^{1/4} \theta)|^2 dr d\theta \leq C(\Omega) \|\nabla \rho_t\|_{L^2(\Omega)}^2. \tag{78}$$

Thus, putting (74) and (78) into (73), we get (70) and (71).

Combining (68), (69), and (70), with the help of the Young inequality, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|\mathbf{u}_t\|_{L^2(\Omega)}^2 + M_1 \|\rho_t\|_{L^2(\Omega)}^2 \right) + \nu \|\nabla \mathbf{u}_t\|_{L^2(\Omega)}^2 + M_1 \mu \|\nabla \rho_t\|_{L^2(\Omega)}^2 \\
& \leq C(\delta) \|\rho_t\|_{L^2(\Omega)}^2 + \delta \|\mathbf{u}_t\|_{L^2(\Omega)}^2 + C \|\mathbf{u}_t\|_{L^4(\Omega)}^2 \|\nabla \mathbf{u}\|_{L^2(\Omega)} \\
& \quad + C \|\mathbf{u}_t\|_{L^4(\Omega)} \|\nabla \rho\|_{L^2(\Omega)} \|\rho_t\|_{L^4(\Omega)} \\
& \leq C(\delta) \|\rho_t\|_{L^2(\Omega)}^2 + \delta \|\mathbf{u}_t\|_{L^2(\Omega)}^2 \\
& \quad + C \|\mathbf{u}_t\|_{L^2(\Omega)} \|\nabla \mathbf{u}_t\|_{L^2(\Omega)} \|\nabla \mathbf{u}\|_{L^2(\Omega)} \\
& \quad + C \left(\|\mathbf{u}_t\|_{L^2(\Omega)}^{1/2} \|\nabla \mathbf{u}_t\|_{L^2(\Omega)}^{1/2} \|\nabla \rho\|_{L^2(\Omega)}^{1/2} \right) \\
& \quad \cdot \left(\|\rho_t\|_{L^2(\Omega)}^{1/2} \|\nabla \rho_t\|_{L^2(\Omega)}^{1/2} \|\nabla \rho\|_{L^2(\Omega)}^{1/2} \right) \\
& \leq C(\delta) \|\rho_t\|_{L^2(\Omega)}^2 + \delta \|\mathbf{u}_t\|_{L^2(\Omega)}^2 + C \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 \|\mathbf{u}_t\|_{L^2(\Omega)}^2 \\
& \quad + C \|\nabla \rho\|_{L^2(\Omega)}^2 \|\mathbf{u}_t\|_{L^2(\Omega)}^2 + C \|\nabla \rho\|_{L^2(\Omega)}^2 \|\rho_t\|_{L^2(\Omega)}^2 \\
& \quad + \frac{\nu}{2} \|\nabla \mathbf{u}_t\|_{L^2(\Omega)}^2 + \frac{\mu}{2} \|\nabla \rho_t\|_{L^2(\Omega)}^2
\end{aligned} \tag{79}$$

for any $\delta > 0$, for suitably large constant $M_1 > 0$ and for some constants $C(\delta) > 0$ and $C = C(M_1, \Omega, \nu, \mu) > 0$, which yields, by applying Gronwall's inequality and Poincaré's inequality, using the estimate (60), to the decay exponentially in time

$$\begin{aligned}
& \|\mathbf{u}_t(t, \cdot)\|_{L^2(\Omega)} + \|\rho_t(t, \cdot)\|_{L^2(\Omega)} \\
& \leq C \left(\|\mathbf{u}_0\|_{H^2(\Omega)} + \|\rho_0\|_{H^2(\Omega)} \right) e^{-at}, \quad 0 \leq t \leq +\infty.
\end{aligned} \tag{80}$$

Finally, we obtain the estimate $L^\infty([0, T]; \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2)$ for \mathbf{u} and for any $T > 0$.

Multiplying (2) by \mathbf{u} and integrating the resulting equation on Ω , and by using the estimates (61) and (80), we have,

for $0 \leq t \leq +\infty$,

$$\begin{aligned} \|\nabla \mathbf{u}(t, \cdot)\|_{L^2(\Omega)}^2 &= \frac{1}{\nu} \int_{\Omega} \rho \mathbf{e}_3 \cdot \mathbf{u} d\mathbf{x} - \frac{1}{\nu} \int_{\Omega} \mathbf{u} \cdot \mathbf{u}_t d\mathbf{x} \\ &\leq C(\nu) \left(\|\rho\|_{L^2(\Omega)}^2 + \|\mathbf{u}\|_{L^2(\Omega)}^2 + \|\mathbf{u}_t\|_{L^2(\Omega)} \|\mathbf{u}\|_{L^2(\Omega)} \right) \\ &\leq C \left(\|\mathbf{u}_0\|_{H^2(\Omega)}, \|\rho_0\|_{L^2(\Omega)} \right) e^{-\alpha t}. \end{aligned} \quad (81)$$

Combining (60) and (81) together, we have

$$\mathbf{u} \in L^\infty(0, T; H_0^1(\Omega)), \quad (82)$$

which gives, by Sobolev's imbedding theorem, that

$$\mathbf{u} \in L^\infty(0, T; L^6(\Omega)). \quad (83)$$

Thus, we can obtain the desired global regularity estimate for strong solution (\mathbf{u}, ρ) and the global smooth interior regularity for the solution (\mathbf{u}, ρ) by choosing the suitable cutoff function with compact subset of the domain Ω , and we can conclude the regularity results on Theorem 2 by using Lemma 8.

Also, it is easy from (59) to get that

$$\begin{aligned} \|\nabla \rho(t, \cdot)\|_{L^2(\Omega)}^2 &\leq C \|\rho(t, \cdot)\|_{L^2(\Omega)} \|\rho_t(t, \cdot)\|_{L^2(\Omega)} \\ &\leq C e^{-\alpha t}, \quad 0 \leq t \leq +\infty. \end{aligned} \quad (84)$$

Thus, the decay rate (24) in Theorem 2 can be obtained from the estimates (61) and (80)–(84).

The proof of Theorem 2 is complete. \square

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Fixed Point Results on Closed Ball in Convex Rectangular b – Metric Spaces and Applications

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In this paper, the concept of convex rectangular b – metric spaces is introduced as a generalization of both convex metric spaces and rectangular b – metric spaces. The purpose of this study is to indicate a way of generalizing Mann’s iteration algorithm and a series of fixed point results in rectangular b – metric spaces. Furthermore, certain examples are given to support the results. We also study well posedness of fixed point problems of some mappings in convex rectangular b – metric spaces, and an application to the dynamic programming is entrusted to manifest the viability of the obtained results. Our results extend comparable results in the existing literature.

1. Introduction and Preliminaries

It is well known that fixed point theory has become an important field of mathematics due to its high degree of unity and wide application. No doubt that the most significant fundamental result of this theory is Banach contraction principle [1] which was published in 1922. Banach contraction principle proposes for the first time to use Picard iteration to approximate a fixed point, which not only proves the existence of the fixed point but also proves the uniqueness of the fixed point. Later in 1968, Kannan [2] studied a new type of contractive mapping. Since then, there have been many results related to mappings satisfy various types of contractive inequality, see for example [3–9].

In 2000, Branciari [10] developed the notion of a rectangular space as a generalization of normal metric space via substituting the triangle inequality with the quadrilateral inequality and extended Banach contraction principle to this space. Successively, George et al. [11] introduced the notion of a rectangular b – metric space as a generalization of rectangular metric space and they also proved some fixed point results for contractive mappings. The concept of a convex structure and a convex metric space was introduced by

Takahashi [12]. Later, Goebel and Kirk [13] studied some iterative processes for nonexpansive mappings in a hyperbolic metric space, and in 1988, Ding [14] found fixed points of quasicontraction mappings in convex metric spaces by Ishikawa’s iteration scheme. However, iterative methods have received vast investigation for finding fixed points of nonexpansive mappings, see [15–17]. Particularly, in the process of the research on some fixed point problem, one of the most famous fixed point method is the Mann iteration [18, 19] as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad (1)$$

for some suitably chosen scalars $\alpha_n \in [0, 1]$. Due to [20], Mann iterative sequence $\{x_n\}$ converges weakly to a fixed point of T if the sequence $\{\alpha_n\} \in [0, 1]$ satisfies following conditions: $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$.

Very recently, Chen et al. [21] introduced the notion of a convex b – metric space and extend Mann’s algorithms directly to b – metric spaces. After that, Asif et al. [22] investigate fixed point of single-valued Hardy-Roger’s type F – contraction globally as well as locally in a convex b – metric space. Along the line, Chen et al. [23] introduce the concept

of a convex graphical rectangular b – metric space and prove some fixed point theorems in this space. New some fixed point results on a closed ball can be seen in [22, 24–26].

In this work, we firstly introduce the concept of the convex rectangular b – metric spaces which is a combination of properties of rectangular b – metric spaces and convex metric spaces. However, we prove some fixed point theorems using generalized Mann's iteration algorithm and show concrete examples supporting our main results. In addition, we claim that fixed point problem is well posed and as an application, we apply our main results to solve the dynamic programming problem.

Some fundamental definitions related to our work are given below:

Definition 1. (See [11]). Let X be a nonempty set and the mapping $d : X \times X \longrightarrow [0, \infty)$ satisfy

- $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$
- $d(x, y) = d(y, x)$ for all $x, y \in X$
- (3) There exists a real number s such that $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$

Then, d is called a rectangular b – metric on X , and (X, d) is called a rectangular b – metric space (R_bMS) with coefficient $s \geq 1$.

Remark 2. Note that every metric space is a rectangular metric space (RMS) (see [11]), and every RMS is a R_bMS with coefficient $s = 1$.

Definition 3. (See [11]). Let (X, d) be a R_bMS , $\{x_n\}$ be a sequence in X and $x \in X$. Then,

- (a) The sequence $\{x_n\}$ is said to be convergent in X to x , if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$, and this fact is represented by $\lim_{n \rightarrow \infty} x_n = x$
- (b) The sequence $\{x_n\}$ is said to be a Cauchy sequence in X if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_0$, and this fact is represented by $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$

X is said to be a complete R_bMS if every Cauchy sequence in X converges to some $x \in X$

Definition 4. (See [12]). Let (X, d) be a metric space and $I = [0, 1]$. A continuous function $w : X \times X \times [0, 1] \longrightarrow X$ is said to be a convex structure on X if for each $x, y \in X$ and $\alpha \in I$,

$$d(u, w(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y), \quad (2)$$

for all $u \in X$. A metric space (X, d) with a convex structure w is called a convex metric space.

2. Main Results

In this section, we introduce a generalization of both convex metric spaces and rectangular b – metric spaces, which we call convex rectangular b – metric spaces. We also establish some fixed point theorems arising from this metric space.

Definition 5. Let (X, d) be a R_bMS with constant $s \geq 1$. If a mapping $w : X \times X \times [0, 1] \longrightarrow X$ satisfies

$$d(u, w(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y), \quad (3)$$

for all $x, y, u \in X$ and $\alpha \in [0, 1]$, then (X, d, w) is said to be a convex rectangular b – metric space (CR_bMS).

Definition 6. Let (X, d, w) be a CR_bMS and $T : X \longrightarrow X$ be a mapping. Let $\{x_n\}$ be the sequence generated by Mann's iterative procedure involving the mapping T , as follows:

$$x_{n+1} = w(x_n, Tx_n; \alpha_n), \quad (4)$$

where $\alpha_n \in [0, 1]$ and $x_0 \in X$ are the initial value.

Definition 7. If $s = 1$ in Definition 5, we call the resultant space to be a convex rectangular metric space ($CRMS$), which is, indeed, the RMS with a convex structure w .

Next, we see some specific examples of CR_bMS .

Example 8. Let $X = \mathbb{R}$. For any $x, y \in X$, we define the metric $d : X \times X \longrightarrow [0, +\infty)$ by $d(x, y) = |x - y|^r$ and $r \geq 1$. Notice that, for any $a, b, c \in [0, +\infty)$ and $1 \leq r < \infty$, then the convex of the function $f(x) = x^r (x > 0)$ implies that

$$\left(\frac{a + b + c}{3}\right)^r \leq \frac{a^r + b^r + c^r}{3} \quad (a, b, c > 0). \quad (5)$$

Then, for any distinct points $u, v \in X \setminus \{x, y\}$, we have

$$\begin{aligned} d(x, y) &= |x - y|^r = |x - u + u - v + v - y|^r \\ &\leq [|x - u| + |u - v| + |v - y|]^r \\ &\leq 3^{r-1} [|x - u|^r + |u - v|^r + |v - y|^r] \\ &= 3^{r-1} [d(x, u) + d(u, v) + d(v, y)]. \end{aligned} \quad (6)$$

Hence, (X, d) is a R_bMS with $s = 3^{r-1}$. For any $x, y \in X$, let $w : X \times X \times \{1/2\} \longrightarrow X$ be a mapping defined by

$$w(x, y; \alpha) = \alpha x + (1 - \alpha)y, \quad \alpha = \frac{1}{2}. \quad (7)$$

Now, we verify that w satisfies inequality (3). In fact, for any $x, y, u \in X$, we can see that

$$\begin{aligned} d(u, w(x, y; \alpha)) &= |u - [\alpha x + (1 - \alpha)y]|^r \\ &\leq 2^{r-1} [\alpha^r |u - x|^r + (1 - \alpha)^r |u - y|^r] \\ &= \alpha d(u, x) + (1 - \alpha)d(u, y). \end{aligned} \quad (8)$$

Therefore, (X, d, w) is a CR_bMS with $s = 3^{r-1}$. Note that (X, d, w) is not a metric space as follows:

$$d(1, 3) = 2^2 > d(1, 2) + d(2, 3) = 2, \quad (9)$$

for we take $r = 2$. Moreover, (X, d, w) is a CRMS when we let $r = 1$, and it shows that CR_bMS reduces to a CRMS for $s = 1$.

Example 9. Let $X = \mathbb{R}$. For any $x, y \in X$, we define the metric $d : X \times X \rightarrow [0, +\infty)$ by $d(x, y) = |x - y|^2$. From Example 8, it follows that (X, d) is a R_bMS with $s = 3$. For any $x, y \in X$, let $w : X \times X \times [0, 1] \rightarrow X$ be a mapping defined by

$$w(x, y; \alpha) = \alpha x + (1 - \alpha)y. \quad (10)$$

For any $x, y, u \in X$, we obtain that

$$\begin{aligned} d(u, w(x, y; \alpha)) &= |u - [\alpha x + (1 - \alpha)y]|^2 \\ &\leq [\alpha|u - x| + (1 - \alpha)|u - y|]^2 \\ &= \alpha^2 d(u, x) + (1 - \alpha)^2 d(u, y) \\ &\quad + 2\alpha(1 - \alpha)|u - x||u - y| \\ &\leq \alpha^2 d(u, x) + (1 - \alpha)^2 d(u, y) \\ &\quad + 2\alpha(1 - \alpha) \frac{|u - x|^2 + |u - y|^2}{2} \\ &= \alpha^2 d(u, x) + (1 - \alpha)^2 d(u, y) \\ &\quad + \alpha(1 - \alpha)\{d(u, x) + d(u, y)\} \\ &= \alpha d(u, x) + (1 - \alpha)d(u, y). \end{aligned} \quad (11)$$

Therefore, (X, d, w) is a CR_bMS with $s = 3$, but not a CRMS.

Example 10. Let $X = [0, 2]$, $d : X \times X \rightarrow [0, +\infty)$, such that $d(x, y) = d(y, x)$ and

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2a & \text{if } x, y \in [0, 1], \\ \frac{1}{2}a & \text{otherwise,} \end{cases} \quad (12)$$

where $a > 0$ is a constant. Then, (X, d) is a R_bMS with coefficient $s = 4/3$. The mapping $w : X \times X \times [0, 1] \rightarrow X$ is defined by $w(x, y; \alpha) = 2 - \alpha xy$, $\alpha = 1/4$, and then

$$d(u, w(x, y; \alpha)) \leq \frac{1}{4}d(u, x) + \frac{3}{4}d(u, y). \quad (13)$$

So, (X, d, w) is a CR_bMS with coefficient $s = 4/3$, but not a CRMS.

Definition 11. Let (X, d, w) be a CR_bMS with constant $s \geq 1$, x_0 is some element in X , and $\varepsilon > 0$, and then the set $B_\varepsilon[x_0] = \{x \in X : d(x_0, x) \leq \varepsilon\}$ is called a closed ball in X .

In the paper [3], George et al. proved Banach contraction principle in complete R_bMS by means of Picard iteration.

Now, we will show Banach contraction principle for complete CR_bMS using generalized Mann's iteration algorithm.

Theorem 12. Let (X, d, w) be a complete CR_bMS with constant $s \geq 1$ and $T : X \rightarrow X$ be a mapping satisfying

$$d(Tx, Ty) \leq \beta d(x, y), \quad (14)$$

for all $x, y \in X$, where $\beta \in [0, 1)$. Let the sequence $\{x_n\}$ generated by the Mann iterative process and $x_0 \in X$ such that $d(x_0, Tx_0) = M < \infty$. If $\beta \in [0, (1/2s^2)]$ and $\alpha_n \in [0, (sr/2s^3 + s - 2)]$ (r is an arbitrary positive real number and $r < 1$), then T has a unique fixed point in X . Moreover, the sequence $\{x_n\} \subseteq B_\varepsilon[x_0]$ and $x_n \rightarrow x^* \in B_\varepsilon[x_0]$ as $n \rightarrow \infty$, if the following inequality holds:

$$d(x_0, Tx_0) \leq \beta(1 - s\beta)\varepsilon. \quad (15)$$

Proof. Without loss of generality, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Indeed, If $x_n = x_{n+1}$, then $x_n = w(x_n, Tx_n; \alpha_n)$. We conclude that

$$d(x_n, Tx_n) = d(w(x_n, Tx_n; \alpha_n), Tx_n) \leq \alpha_n d(x_n, Tx_n), \quad (16)$$

and it shows $d(x_n, Tx_n) = 0$; then, x_n is a fixed point of T , and the proof is finished. It follows from Definition 5 and Definition 6,

$$d(x_n, x_{n+1}) = d(x_n, w(x_n, Tx_n; \alpha_n)) \leq (1 - \alpha_n)d(x_n, Tx_n). \quad (17)$$

□

Now, we consider the following two cases:

Case 13. If $x_n \neq Tx_{n-1}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, Tx_n) &= d(w(x_{n-1}, Tx_{n-1}; \alpha_{n-1}), Tx_n) \\ &\leq \alpha_{n-1}d(x_{n-1}, Tx_n) + (1 - \alpha_{n-1})d(Tx_{n-1}, Tx_n) \\ &\leq \alpha_{n-1}d(x_{n-1}, Tx_n) + (1 - \alpha_{n-1})\beta d(x_{n-1}, x_n) \\ &\leq s\alpha_{n-1}[d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n)] \\ &\quad + (1 - \alpha_{n-1})^2\beta d(x_{n-1}, Tx_{n-1}) \\ &\leq s\alpha_{n-1}[(1 - \alpha_{n-1})d(x_{n-1}, Tx_{n-1}) \\ &\quad + \alpha_{n-1}d(x_{n-1}, Tx_{n-1}) + (1 - \alpha_{n-1})\beta d(x_{n-1}, Tx_{n-1})] \\ &\quad + (1 - \alpha_{n-1})^2\beta d(x_{n-1}, Tx_{n-1}) \\ &\leq [s\alpha_{n-1}(1 + (1 - \alpha_{n-1})\beta) \\ &\quad + (1 - \alpha_{n-1})^2\beta]d(x_{n-1}, Tx_{n-1}). \end{aligned} \quad (18)$$

Let $\lambda_{n-1} = s\alpha_{n-1}[1 + (1 - \alpha_{n-1})\beta] + (1 - \alpha_{n-1})^2\beta$, with the assumption $0 \leq \beta \leq 1/2s^2$ and $0 \leq \alpha_n \leq sr/2s^3 + s - 2$, and we obtain that

$$\begin{aligned}
\lambda_{n-1} &\leq s\alpha_{n-1} \left(1 + (1 - \alpha_{n-1}) \frac{1}{2s^2} \right) + (1 - \alpha_{n-1})^2 \frac{1}{2s^2} \\
&= s\alpha_{n-1} + \frac{\alpha_{n-1}}{2s} - \frac{\alpha_{n-1}^2}{2s} + \frac{1}{2s^2} - \frac{\alpha_{n-1}}{s^2} + \frac{\alpha_{n-1}^2}{2s^2} \\
&= \left(\frac{1}{2s^2} - \frac{1}{2s} \right) \alpha_{n-1}^2 + \left(s + \frac{1}{2s} - \frac{1}{s^2} \right) \alpha_{n-1} + \frac{1}{2s^2} \quad (19) \\
&\leq \left(s + \frac{1}{2s} - \frac{1}{s^2} \right) \alpha_{n-1} + \frac{1}{2s^2} = \left(\frac{2s^3 + s - 2}{2s^2} \right) \\
&\quad \times \frac{sr}{2s^3 + s - 2} + \frac{1}{2s} \leq \frac{r+1}{2s} \leq \frac{r+1}{2}.
\end{aligned}$$

Hence,

$$d(x_n, Tx_n) \leq \lambda_{n-1} d(x_{n-1}, Tx_{n-1}) \leq \frac{r+1}{2} d(x_{n-1}, Tx_{n-1}). \quad (20)$$

Case 14. If $x_n = Tx_{n-1}$ for some $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_n, Tx_n) &= d(Tx_{n-1}, Tx_n) \leq \beta d(x_{n-1}, x_n) \\
&\leq \beta(1 - \alpha_n) d(x_{n-1}, Tx_{n-1}) \\
&\leq \frac{1}{2s^2} d(x_{n-1}, Tx_{n-1}). \quad (21)
\end{aligned}$$

Denote that $\lambda = r + 1/2 < 1$, and it follows from (20) and (21) that

$$d(x_n, Tx_n) \leq \lambda d(x_{n-1}, Tx_{n-1}), \text{ for all } n \in \mathbb{N}, \quad (22)$$

which implies that $\{d(x_n, Tx_n)\}$ is a decreasing sequence of nonnegative reals. Hence, there exists $\gamma \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \gamma. \quad (23)$$

We will show that $\gamma = 0$. Suppose that $\gamma > 0$, letting $n \rightarrow \infty$ in inequality (22), we obtain

$$\gamma \leq \lambda \gamma, \quad (24)$$

a contradiction. Hence, we get that $\gamma = 0$. Furthermore, we have

$$d(x_n, x_{n+1}) \leq (1 - \alpha_n) d(x_n, Tx_n), \quad (25)$$

which shows that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Also, we can assume $x_n \neq x_{n+p}$ for any $p > 1$. Indeed, if $x_n = x_{n+p}$, then using the inequality (21), we have

$$d(x_n, Tx_n) = d(x_{n+p}, Tx_{n+p}) \leq \lambda^{p-1} d(x_n, Tx_n), \quad (26)$$

in which shows that $d(x_n, Tx_n) = 0$ and $x_n = Tx_n$, and then x_n is a fixed point, and the proof is finished. Next, we shall prove $\lim_{n \rightarrow \infty} d(x_n, x_{n+2}) = 0$ for all $n \in \mathbb{N}$. In order to do it, we will consider the following two cases:

Case 15. If $x_{n+2} \neq Tx_n$ for all $n \in \mathbb{N}$, then we have

$$\begin{aligned}
d(x_n, x_{n+2}) &\leq s[d(x_n, Tx_n) + d(Tx_n, Tx_{n+2}) + d(Tx_{n+2}, x_{n+2})] \\
&\leq s[d(x_n, Tx_n) + \beta d(x_n, x_{n+2}) + d(Tx_{n+2}, x_{n+2})], \quad (27)
\end{aligned}$$

which establishes that

$$d(x_n, x_{n+2}) \leq \frac{s}{1 - s\beta} [d(x_n, Tx_n) + d(x_{n+2}, Tx_{n+2})] \left(\text{as } 0 \leq \beta < \frac{1}{2s^2} \right). \quad (28)$$

Case 16. If there exist some $n \in \mathbb{N}$ such that $x_{n+2} = Tx_n$, then

$$d(x_n, x_{n+2}) \leq d(x_n, Tx_n). \quad (29)$$

It follows from (28) and (29) that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+2}) = 0. \quad (30)$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence by contradiction. Assume that there exists $\varepsilon_0 > 0$ and the subsequences $\{x_{\theta(k)}\}$ and $\{x_{\eta(k)}\}$ of x_n such for $\theta(k) > \eta(k) > k$ with $d(x_{\theta(k)}, x_{\eta(k)}) \geq \varepsilon_0$, $d(x_{\theta(k)-1}, x_{\eta(k)}) < \varepsilon_0$. On the one hand,

$$\begin{aligned}
\varepsilon_0 \leq d(x_{\theta(k)}, x_{\eta(k)}) &\leq s \left[d(x_{\theta(k)}, x_{\eta(k)+1}) \right. \\
&\quad \left. + d(x_{\eta(k)+1}, x_{\eta(k)+2}) + d(x_{\eta(k)+2}, x_{\eta(k)}) \right], \quad (31)
\end{aligned}$$

taking the limit superior in above inequality as $k \rightarrow \infty$, and we conclude

$$\frac{\varepsilon_0}{s} \leq \limsup_{k \rightarrow \infty} d(x_{\theta(k)}, x_{\eta(k)+1}). \quad (32)$$

On the other hand, let $x_{\eta(k)} \neq x_{\eta(k)+2} \neq x_{\theta(k)-1} \neq x_{\eta(k)+1}$ and $Tx_{\eta(k)} \neq x_{\eta(k)} \neq Tx_{\theta(k)-1} \neq x_{\eta(k)+1}$, and we have

$$\begin{aligned}
d(x_{\theta(k)}, x_{\eta(k)+1}) &= d(w(x_{\theta(k)-1}, Tx_{\theta(k)-1}; \alpha_{\theta(k)-1}), x_{\eta(k)+1}) \\
&\leq \alpha_{\theta(k)-1} d(x_{\theta(k)-1}, x_{\eta(k)+1}) \\
&\quad + (1 - \alpha_{\theta(k)-1}) d(Tx_{\theta(k)-1}, x_{\eta(k)+1}) \\
&\leq \alpha_{\theta(k)-1} s \left[d(x_{\theta(k)-1}, x_{\eta(k)}) \right. \\
&\quad \left. + d(x_{\eta(k)}, x_{\eta(k)+2}) + d(x_{\eta(k)+2}, x_{\eta(k)+1}) \right] \\
&\quad + (1 - \alpha_{\theta(k)-1}) s \left[d(Tx_{\theta(k)-1}, Tx_{\eta(k)}) \right. \\
&\quad \left. + d(Tx_{\eta(k)}, x_{\eta(k)}) + d(x_{\eta(k)}, x_{\eta(k)+1}) \right] \\
&\leq (\alpha_{\theta(k)-1} s + (1 - \alpha_{\theta(k)-1}) s \beta) d(x_{\theta(k)-1}, x_{\eta(k)}) \\
&\quad + \alpha_{\theta(k)-1} s \left[d(x_{\eta(k)}, x_{\eta(k)+2}) + d(x_{\eta(k)+2}, x_{\eta(k)+1}) \right] \\
&\quad + (1 - \alpha_{\theta(k)-1}) s \left[d(Tx_{\eta(k)}, x_{\eta(k)}) + d(x_{\eta(k)}, x_{\eta(k)+1}) \right], \quad (33)
\end{aligned}$$

by taking the limit superior on both sides of above the inequality as $k \rightarrow \infty$, and we get

$$\begin{aligned}
 \frac{\varepsilon_0}{s} &\leq \limsup_{k \rightarrow \infty} d(x_{\theta(k)}, x_{\eta(k)+1}) \\
 &\leq \left(\alpha_{\theta(k)-1} s + \left(1 - \alpha_{\theta(k)-1} \right) s \beta \right) \varepsilon_0 \\
 &\leq \left(\alpha_{\theta(k)-1} s + \frac{1}{2s} \left(1 - \alpha_{\theta(k)-1} \right) \right) \varepsilon_0 \\
 &\leq \left(\alpha_{\theta(k)-1} s + \frac{1}{2s} - \frac{\alpha_{\theta(k)-1}}{2s} \right) \varepsilon_0 \\
 &\leq \left(\frac{1}{2s^2 - 1} \times \left(s - \frac{1}{2s} \right) + \frac{1}{2s} \right) \varepsilon_0 \\
 &= \left(\frac{r+1}{2} \right) \frac{\varepsilon_0}{s} < \frac{1}{s} \varepsilon_0,
 \end{aligned} \tag{34}$$

a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in X . Since the space (X, d, w) is complete, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x^*$. We shall show that x^* is a fixed point of T . Applying the rectangular inequality, we obtain that

$$\begin{aligned}
 d(x^*, Tx^*) &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*)] \\
 &\leq sd(x^*, x_n) + sd(x_n, x_{n+1}) \\
 &\quad + s[\alpha_n d(x_n, Tx^*) + (1 - \alpha_n) d(Tx_n, Tx^*)] \\
 &\leq sd(x^*, x_n) + sd(x_n, x_{n+1}) \\
 &\quad + s^2 \alpha_n [d(x_n, x_{n+1}) + d(x_{n+1}, x^*) + d(x^*, Tx^*)] \\
 &\quad + s(1 - \alpha_n) \beta d(x_n, x^*),
 \end{aligned} \tag{35}$$

since $s^2 \alpha_n < 1$, and then

$$\begin{aligned}
 d(x^*, Tx^*) &\leq \frac{1}{1 - s^2 \alpha_n} \{ sd(x^*, x_n) + sd(x_n, x_{n+1}) \\
 &\quad + s^2 \alpha_n [d(x_n, x_{n+1}) + d(x_{n+1}, x^*)] \\
 &\quad + s(1 - \alpha_n) \beta d(x_n, x^*) \},
 \end{aligned} \tag{36}$$

letting $n \rightarrow \infty$, and we deduce $d(x^*, Tx^*) = 0$ which implies $Tx^* = x^*$. Thus, x^* is a fixed point of T . Suppose that $x^*, y^* \in X$ are two distinct fixed points of T , that is, $Tx^* = x^*$ and $Ty^* = y^*$. Then,

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \beta d(x^*, y^*), \tag{37}$$

which is a contradiction. Therefore, we must have $d(x^*, y^*) = 0$, i.e., $x^* = y^*$. Thus, T has a unique fixed point. Next, we proceed to show that the sequence $\{x_n\} \subseteq B_\varepsilon[x_0]$. In order to complete it, we will use mathematical induction. Thanks to Definition 5 and Definition 6, we obtain

$$\begin{aligned}
 d(x_0, x_1) &= d(x_0, w(x_0, Tx_0; \alpha_0)) \\
 &\leq (1 - \alpha_0) d(x_0, Tx_0) \\
 &\leq (1 - \alpha_0) \beta (1 - s\beta) \varepsilon < \varepsilon,
 \end{aligned} \tag{38}$$

which implies $d(x_0, x_1) < \varepsilon$; therefore, $x_1 \in B_\varepsilon[x_0]$. Suppose $x_2, x_3, \dots, x_m \in B_\varepsilon[x_0]$, observe from above proof, we get $d(x_n, Tx_n) \leq \lambda^n d(x_0, Tx_0)$ for all $n \in \mathbb{N}$. It is easy to see that $\beta(1 - s\beta) \leq 1/4s$. Now, we can assume that $x_{m+1} \neq x_m$. If $Tx_0 \neq Tx_m \neq x_0 \neq x_{m+1}$, then

$$\begin{aligned}
 d(x_0, x_{m+1}) &\leq s[d(x_0, Tx_0) + d(Tx_0, Tx_m) + d(Tx_m, x_{m+1})] \\
 &\leq s[\beta(1 - s\beta)\varepsilon + \beta d(x_0, x_m) + \alpha_m d(Tx_m, x_m)] \\
 &\leq s[\beta(1 - s\beta)\varepsilon + \beta\varepsilon + \alpha_m \lambda^m \beta(1 - s\beta)\varepsilon] \\
 &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
 \end{aligned} \tag{39}$$

We also need to distinguish the following four cases:

Case 17. If $x_0 = Tx_0$, then we have

$$\begin{aligned}
 d(x_0, x_{m+1}) &= d(x_0, w(x_m, Tx_m; \alpha_m)) \\
 &\leq \alpha_m d(x_0, x_m) + (1 - \alpha_m) d(Tx_0, Tx_m) \\
 &\leq \alpha_m \varepsilon + (1 - \alpha_m) \beta \varepsilon < \varepsilon.
 \end{aligned} \tag{40}$$

Case 18. if $x_0 = Tx_m$, then we have

$$\begin{aligned}
 d(x_0, x_{m+1}) &= d(Tx_m, w(x_m, Tx_m; \alpha_m)) \\
 &\leq \alpha_m d(Tx_m, x_m) \\
 &\leq \alpha_m \lambda^m d(Tx_0, x_0) \\
 &\leq \alpha_m \lambda^m \beta (1 - s\beta) \varepsilon < \varepsilon.
 \end{aligned} \tag{41}$$

Case 19. if $x_{m+1} = Tx_0$, then we have

$$d(x_0, x_{m+1}) = d(x_0, Tx_0) \leq \beta(1 - \beta) \varepsilon < \varepsilon. \tag{42}$$

Case 20. if $x_{m+1} = Tx_m$, then we have

$$\begin{aligned}
 d(x_0, x_{m+1}) &= d(x_0, w(x_m, Tx_m; \alpha_m)) \\
 &\leq \alpha_m d(x_0, x_m) + (1 - \alpha_m) d(x_0, Tx_m) \\
 &= \alpha_m d(x_0, x_m) + (1 - \alpha_m) d(x_0, x_{m+1}),
 \end{aligned} \tag{43}$$

which implies

$$d(x_0, x_{m+1}) \leq d(x_0, x_m) \leq \varepsilon. \tag{44}$$

Finally, by above cases, we prove that $d(x_0, x_{m+1}) \leq \varepsilon$ which show that $x_{m+1} \in B_\varepsilon[x_0]$. Hence, by induction $x_n \in B_\varepsilon[x_0]$, therefore, we conclude that $x_n \in B_\varepsilon[x_0]$ for all $n \in \mathbb{N}$. As every closed ball in a complete metric space is complete, so $x_n \rightarrow x^* \in B_\varepsilon[x_0]$, as $n \rightarrow \infty$.

The following example illustrates the above theorem.

Example 21. Let $X = \mathbb{R}^+ \cup \{0\}$ and $Tx = x/5$ for all $x \in X$. For any $x, y \in X$, we define $d : X \times X \rightarrow [0, +\infty)$ by $d(x, y) = (x - y)^2$. The mapping $w : X \times X \times [0, 1] \rightarrow X$ is defined by

$$w(x, y; \alpha) = \alpha x + (1 - \alpha)y, x, y \in X. \tag{45}$$

Set $x_{n+1} = w(x_n, Tx_n; \alpha_n)$ and $\alpha_n = 1/2s^2 + 2$. If $\beta = 1/2s^2 + 1$, then $x_n \in B_\varepsilon[x_0]$ and T have a unique fixed point in $B_\varepsilon[x_0]$.

Proof. It is easy to see that (X, d) is a CR_bMS with $s = 3$. In addition, for any $x, y, u \in X$, we have

$$\begin{aligned} d(u, w(x, y; \alpha_n)) &= [\alpha_n(u - x) + (1 - \alpha_n)(u - y)]^2 \\ &\leq \alpha_n^2(u - x)^2 + (1 - \alpha_n)^2(u - y)^2 \\ &\quad + 2\alpha_n(1 - \alpha_n)(u - x)(u - y) \\ &\leq \alpha_n^2(u - x)^2 + (1 - \alpha_n)^2(u - y)^2 \\ &\quad + \alpha_n(1 - \alpha_n)((u - x)^2 + (u - y)^2) \\ &\leq \alpha_n(u - x)^2 + (1 - \alpha_n)(u - y)^2. \end{aligned} \quad (46)$$

So, (X, d, w) is a CR_bMS with $s = 3$. It is not difficult to see that T satisfies

$$d(Tx, Ty) = \frac{1}{25}d(x, y) \leq \beta d(x, y), \quad (47)$$

for $\beta = 1/18$. According to $x_{n+1} = w(x_n, Tx_n; \alpha_n)$, we have $x_{n+1} = 1/20x_n + 19/20Tx_n$, since $Tx = x/5$, and we obtain

$$x_{n+1} = \frac{1}{20}x_n + \frac{19}{20} \times \frac{1}{5}x_n, \quad (48)$$

that is, $x_{n+1} = 6/25x_n$, then

$$x_n = \frac{6}{25}x_{n-1}, x_{n-1} = \frac{6}{25}x_{n-2}, \dots, x_1 = \frac{6}{25}x_0, \quad (49)$$

And we obtain

$$x_n = \left(\frac{6}{25}\right)^n x_0, Tx_n = \frac{1}{5} \times \left(\frac{6}{25}\right)^n x_0, \quad (50)$$

while $n \rightarrow \infty$, getting $x_n \rightarrow 0 \in X$ and $Tx_n \rightarrow 0 \in X$. Hence, 0 is a fixed point of T in X . Suppose $x^*, y^* \in X$ are two distinct fixed points of T , then we have $d(x^*, y^*) = d(Tx^*, Ty^*) = 1/25d(x^*, y^*)$ which shows that $d(x^*, y^*) = 0$, that is, $x^* = y^*$. Thus, T has a unique fixed point in X . Let $\varepsilon = x_0^2/\beta(1 - s\beta) > 0$, then $\beta(1 - s\beta)\varepsilon = x_0^2 \geq d(x_0, Tx_0) = 16/25x_0^2$. For all $n \in \mathbb{N}$, we obtain $d(x_0, x_n) = (x_0 - (6/25)^n x_0)^2 < x_0^2 < \varepsilon$, and this means that the sequence $\{x_n\} \subseteq B_\varepsilon[x_0]$. Furthermore, $d(x_0, 0) = x_0^2 < \varepsilon$, that is, $0 \in B_\varepsilon[x_0]$. \square

Now, we prove the Kannan type fixed point theorem for a complete CR_bMS , which extends the results in the paper [3], replacing Picard's iteration algorithm by Mann's iteration algorithm.

Theorem 22. Let (X, d, w) be a CR_bMS with constant $s \geq 1$ and the mapping $T : X \rightarrow X$ be defined by

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \quad (51)$$

for all $x, y \in X$, and $k \in [0, 1/2)$. Let the sequence $\{x_n\}$ generated by the Mann iterative process and $x_0 \in X$ such that $d(x_0, Tx_0) = M < \infty$. If $k \in [0, (1/3s)]$ and $\alpha_n \in [0, (1/us^2)]$ (u is an arbitrary real number and $u > 5$), then T has a unique fixed point in X . Moreover, the sequence $\{x_n\} \subseteq B_\varepsilon[x_0]$ and $x_n \rightarrow x^* \in B_\varepsilon[x_0]$ as $n \rightarrow \infty$, if the following inequality holds:

$$d(x_0, Tx_0) \leq k(1 - sk)\varepsilon. \quad (52)$$

Proof. Without loss of generality, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Indeed, If $x_n = x_{n+1}$, that is, $x_n = w(x_n, Tx_n; \alpha_n)$. Then, we have

$$d(x_n, Tx_n) = d(w(x_n, Tx_n; \alpha_n), Tx_n) \leq \alpha_n d(x_n, Tx_n), \quad (53)$$

and it shows that $d(x_n, Tx_n) = 0$ and $x_n = Tx_n$, which means that x_n is a fixed point of T , and the proof is finished. Thanks to Definition 5 and Definition 6, we have

$$d(x_n, x_{n+1}) = d(x_n, w(x_n, Tx_n; \alpha_n)) \leq (1 - \alpha_n)d(x_n, Tx_n). \quad (54)$$

\square

Now, we have the following two cases:

Case 23. If $x_n \neq Tx_{n-1}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, Tx_n) &= d(w(x_{n-1}, Tx_{n-1}; \alpha_{n-1}), Tx_n) \\ &\leq \alpha_{n-1}d(x_{n-1}, Tx_n) + (1 - \alpha_{n-1})d(Tx_{n-1}, Tx_n) \\ &\leq \alpha_{n-1}[d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n)] \\ &\quad + (1 - \alpha_{n-1})d(Tx_{n-1}, Tx_n) \\ &\leq \alpha_{n-1}[(1 - \alpha_{n-1})d(x_{n-1}, Tx_{n-1}) + \alpha_{n-1}d(x_{n-1}, Tx_{n-1}) \\ &\quad + kd(x_{n-1}, Tx_{n-1}) + kd(x_n, Tx_n)] \\ &\quad + (1 - \alpha_{n-1})k[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\leq [\alpha_{n-1} + \alpha_{n-1}k + (1 - \alpha_{n-1})k]d(x_{n-1}, Tx_{n-1}) \\ &\quad + [\alpha_{n-1}k + (1 - \alpha_{n-1})k]d(x_n, Tx_n), \end{aligned} \quad (55)$$

which establishes that

$$\begin{aligned} [1 - (1 - \alpha_{n-1})k - \alpha_{n-1}k]d(x_n, Tx_n) \\ \leq [\alpha_{n-1} + \alpha_{n-1}k + (1 - \alpha_{n-1})k]d(x_{n-1}, Tx_{n-1}). \end{aligned} \quad (56)$$

Notice that $(1 - \alpha_{n-1})k + \alpha_{n-1}k < 1$, then we have

$$d(x_n, Tx_n) \leq \frac{\alpha_{n-1} + \alpha_{n-1}k + (1 - \alpha_{n-1})k}{1 - (1 - \alpha_{n-1})k - \alpha_{n-1}k} d(x_{n-1}, Tx_{n-1}). \quad (57)$$

Since $u > 5$, we conclude that

$$\frac{s\alpha_{n-1} + s\alpha_{n-1}k + (1 - \alpha_{n-1})k}{1 - s\alpha_{n-1}k - (1 - \alpha_{n-1})k} \leq \frac{(1/us) + (1/3us^2) + (1/3s)}{1 - 1/3us^2 - 1/3s} \leq \frac{u+4}{2u-1} < 1; \quad (58)$$

Case 24. If $x_n = Tx_{n-1}$ for some $n \in \mathbb{N}$, then

$$d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n) \leq k[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)], \quad (59)$$

and this implies that

$$(1 - k)d(x_n, Tx_n) \leq kd(x_{n-1}, Tx_{n-1}). \quad (60)$$

Since $0 \leq k \leq 1/3s$, then we get

$$d(x_n, Tx_n) \leq \frac{k}{1-k} d(x_{n-1}, Tx_{n-1}). \quad (61)$$

Noticing that

$$\frac{k}{1-k} \leq \frac{1}{2} \leq \frac{u+4}{2u-1}. \quad (62)$$

Let $\lambda_u = u + 4/2u - 1$, it is clear that $\lambda_u < 1$, and for any $n \in \mathbb{N}$, we obtain the following inequality:

$$d(x_n, Tx_n) \leq \lambda_u d(x_{n-1}, Tx_{n-1}), \text{ for all } n \in \mathbb{N}, \quad (63)$$

and it implies that $\{d(x_n, Tx_n)\}$ is a decreasing sequence of nonnegative reals. Hence, there exists $\gamma \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \gamma. \quad (64)$$

We will show that $\gamma = 0$. Suppose that $\gamma > 0$. Letting $n \rightarrow \infty$ in inequality (63), we obtain

$$\gamma \leq \lambda_u \gamma, \quad (65)$$

a contradiction. Hence, we get that $\gamma = 0$. Moreover, we have

$$d(x_n, x_{n+1}) \leq (1 - \alpha_n)d(x_n, Tx_n), \quad (66)$$

which shows that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Next, we shall prove that $\lim_{n \rightarrow \infty} d(x_n, x_{n+2}) = 0$. In order to do it, we will consider the following two cases:

Case 25. if $x_{n+1} \neq Tx_n$ for all $n \in \mathbb{N}$, then we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &\leq d(x_n, w(x_{n+1}, Tx_{n+1}; \alpha_{n+1})) \\ &\leq \alpha_{n+1}d(x_n, x_{n+1}) + (1 - \alpha_{n+1})d(x_n, Tx_{n+1}) \\ &\leq \alpha_{n+1}d(x_n, x_{n+1}) + (1 - \alpha_{n+1})s[d(x_n, Tx_n) \\ &\quad + d(Tx_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1})] \\ &\leq \alpha_{n+1}d(x_n, x_{n+1}) + (1 - \alpha_{n+1})s[d(x_n, Tx_n) \\ &\quad + \alpha_n d(Tx_n, x_n) + d(x_{n+1}, Tx_{n+1})]. \end{aligned} \quad (67)$$

Hence,

$$\begin{aligned} d(x_n, x_{n+2}) &\leq \alpha_{n+1}d(x_n, x_{n+1}) \\ &\quad + (1 - \alpha_{n+1})s[(1 + \alpha_n)d(Tx_n, x_n) + d(x_{n+1}, Tx_{n+1})]. \end{aligned} \quad (68)$$

Case 26. If there exist some $n \in \mathbb{N}$ such that $x_{n+1} = Tx_n$, then we get

$$\begin{aligned} d(x_n, x_{n+2}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+3}) + d(x_{n+3}, x_{n+2})] \\ &= s[d(x_n, x_{n+1}) + d(x_{n+1}, w(x_{n+2}, Tx_{n+2}; \alpha_{n+2})) \\ &\quad + d(x_{n+3}, x_{n+2})] \\ &\leq s[d(x_n, x_{n+1}) + \alpha_{n+2}d(x_{n+1}, x_{n+2}) \\ &\quad + (1 - \alpha_{n+2})d(x_{n+1}, Tx_{n+2}) + (1 - \alpha_{n+2})d(Tx_{n+2}, x_{n+2})] \\ &\leq s[d(x_n, x_{n+1}) + \alpha_{n+2}d(x_{n+1}, x_{n+2}) \\ &\quad + (1 - \alpha_{n+2})kd(x_n, Tx_n) + (1 - \alpha_{n+2})kd(x_{n+2}, Tx_{n+2}) \\ &\quad + (1 - \alpha_{n+2})d(Tx_{n+2}, x_{n+2})]. \end{aligned} \quad (69)$$

Hence,

$$\begin{aligned} d(x_n, x_{n+2}) &\leq s[d(x_n, x_{n+1}) + \alpha_{n+2}d(x_{n+1}, x_{n+2}) \\ &\quad + (1 - \alpha_{n+2})kd(x_n, Tx_n) + (1 - \alpha_{n+2})kd(x_{n+2}, Tx_{n+2}) \\ &\quad + (1 - \alpha_{n+2})d(Tx_{n+2}, x_{n+2})]. \end{aligned} \quad (70)$$

It follows from (68) and (70) that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+2}) = 0. \quad (71)$$

Next, we will claim that $\{x_n\}$ is a Cauchy sequence by contradiction. Assume there exists $\varepsilon_0 > 0$ and the subsequences $\{x_{\theta(k)}\}$ and $\{x_{\eta(k)}\}$ of $\{x_n\}$ such for $\theta(k) > \eta(k) > k$ with $d(x_{\theta(k)}, x_{\eta(k)}) \geq \varepsilon_0$, $d(x_{\theta(k)-1}, x_{\eta(k)}) < \varepsilon_0$. On the one hand,

$$\begin{aligned} \varepsilon_0 \leq d(x_{\theta(k)}, x_{\eta(k)}) &\leq s \left[d(x_{\theta(k)}, x_{\eta(k)+1}) \right. \\ &\quad \left. + d(x_{\eta(k)+1}, x_{\eta(k)+2}) + d(x_{\eta(k)+2}, x_{\eta(k)}) \right], \end{aligned} \quad (72)$$

taking the limit superior in above inequality as $k \rightarrow \infty$, and we get

$$\frac{\varepsilon_0}{s} \leq \limsup_{k \rightarrow \infty} d(x_{\theta(k)}, x_{\eta(k)+1}). \quad (73)$$

On the other hand, let $x_{\eta(k)} \neq x_{\eta(k)+2} \neq x_{\theta(k)-1} \neq x_{\eta(k)+1}$ and $Tx_{\eta(k)} \neq x_{\eta(k)} \neq Tx_{\theta(k)-1} \neq x_{\eta(k)+1}$, and we have

$$\begin{aligned} d(x_{\theta(k)}, x_{\eta(k)+1}) &= d(w(x_{\theta(k)-1}, Tx_{\theta(k)-1}; \alpha_{\theta(k)-1}), x_{\eta(k)+1}) \\ &\leq \alpha_{\theta(k)-1} d(x_{\theta(k)-1}, x_{\eta(k)+1}) \\ &\quad + (1 - \alpha_{\theta(k)-1}) d(Tx_{\theta(k)-1}, x_{\eta(k)+1}) \\ &\leq \alpha_{\theta(k)-1} s [d(x_{\theta(k)-1}, x_{\eta(k)}) \\ &\quad + d(x_{\eta(k)}, x_{\eta(k)+2}) + d(x_{\eta(k)+2}, x_{\eta(k)+1})] \\ &\quad + (1 - \alpha_{\theta(k)-1}) s [d(Tx_{\theta(k)-1}, Tx_{\eta(k)}) \\ &\quad + d(Tx_{\eta(k)}, x_{\eta(k)}) + d(x_{\eta(k)}, x_{\eta(k)+1})] \\ &\leq \alpha_{\theta(k)-1} s [d(x_{\theta(k)-1}, x_{\eta(k)}) \\ &\quad + d(x_{\eta(k)}, x_{\eta(k)+2}) + d(x_{\eta(k)+2}, x_{\eta(k)+1})] \\ &\quad + (1 - \alpha_{\theta(k)-1}) s [kd(x_{\theta(k)-1}, Tx_{\theta(k)-1}) \\ &\quad + kd(x_{\eta(k)}, Tx_{\eta(k)}) \\ &\quad + d(Tx_{\eta(k)}, x_{\eta(k)}) + d(x_{\eta(k)}, x_{\eta(k)+1})]. \end{aligned} \quad (74)$$

We obtain

$$\frac{\varepsilon_0}{s} \leq \limsup_{k \rightarrow \infty} d(x_{\theta(k)}, x_{\eta(k)+1}) \leq \frac{1}{us} \varepsilon_0 < \frac{1}{s} \varepsilon_0, \quad (75)$$

a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in X . Since the space (X, d, w) is complete, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x^*$. We shall show that x^* is a fixed point of T . Applying the rectangular inequality, we obtain that

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &\leq sd(x^*, x_n) + sd(x_n, x_{n+1}) \\ &\quad + s[\alpha_n d(x_n, Tx^*) + (1 - \alpha_n) d(Tx_n, Tx^*)] \\ &\leq sd(x^*, x_n) + sd(x_n, x_{n+1}) \\ &\quad + s^2 \alpha_n [d(x_n, x_{n+1}) + d(x_{n+1}, x^*) + d(x^*, Tx^*)] \\ &\quad + s(1 - \alpha_n) \{kd(x_n, Tx_n) + kd(x^*, Tx^*)\}, \end{aligned} \quad (76)$$

since $s^2 \alpha_n + s(1 - \alpha_n)k < 1$, and then

$$\begin{aligned} d(x^*, Tx^*) &\leq \frac{1}{1 - s^2 \alpha_n - s(1 - \alpha_n)k} \{sd(x^*, x_n) + sd(x_n, x_{n+1}) \\ &\quad + s^2 \alpha_n [d(x_n, x_{n+1}) + d(x_{n+1}, x^*)] \\ &\quad + s(1 - \alpha_n)kd(x_n, Tx_n)\}, \end{aligned} \quad (77)$$

letting $n \rightarrow \infty$, and we deduce $d(x^*, Tx^*) = 0$ which implies $Tx^* = x^*$. Thus, x^* is a fixed point of T . Suppose that $x^*, y^* \in X$ are two distinct fixed points of T , that is, $Tx^* = x^*, Ty^* = y^*$. Then,

$$0 < d(x^*, y^*) = d(Tx^*, Ty^*) \leq k[d(x^*, Tx^*) + d(y^*, Ty^*)] = 0, \quad (78)$$

which is a contradiction. Therefore, we must have $d(x^*, y^*) = 0$, that is, $x^* = y^*$. Thus, T has a unique fixed point. Finally, we will prove the iteration sequence $\{x_n\} \subseteq B_\varepsilon[x_0]$. In order to complete it, we will use mathematical induction. Choose $x_0 \in X$, and we have

$$\begin{aligned} d(x_0, x_1) &= d(x_0, w(x_0, Tx_0; \alpha_0)) \\ &\leq (1 - \alpha_0)d(x_0, Tx_0) \\ &\leq (1 - \alpha_0)\beta(1 - s\beta)\varepsilon < \varepsilon, \end{aligned} \quad (79)$$

which implies $d(x_0, x_1) < \varepsilon$; therefore, $x_1 \in B_\varepsilon[x_0]$. Suppose $x_2, x_3, \dots, x_m \in B_\varepsilon[x_0]$. It is easy to see that $s[k(1 - sk) < 2/9]$. Without loss of generality, we can assume that $x_{m+1} \neq x_m$. If $Tx_0 \neq Tx_m \neq x_0 \neq x_{m+1}$, then

$$\begin{aligned} d(x_0, x_{m+1}) &\leq s[d(x_0, Tx_0) + d(Tx_0, Tx_m) + d(Tx_m, x_{m+1})] \\ &\leq s[k(1 - sk)\varepsilon + k^2(1 - sk)\varepsilon + k\lambda_u^m d(x_0, Tx_0) \\ &\quad + \alpha_n \lambda_u^m d(Tx_0, x_0)] \\ &\leq s[k(1 - sk)\varepsilon + k^2(1 - sk)\varepsilon + \lambda_u^m k^2(1 - sk) \\ &\quad + \alpha_m \lambda^m k(1 - sk)\varepsilon] \\ &\leq \frac{2\varepsilon}{9} + \frac{2\varepsilon}{27} + \frac{2\varepsilon}{27} + \frac{2\varepsilon}{9} < \varepsilon. \end{aligned} \quad (80)$$

We also need to distinguish the following four cases:

Case 27. If $x_0 = Tx_0$, then we have

$$\begin{aligned} d(x_0, x_{m+1}) &= d(x_0, w(x_m, Tx_m; \alpha_m)) \\ &\leq \alpha_m d(x_0, x_m) + (1 - \alpha_m) d(Tx_0, Tx_m) \\ &\leq \alpha_m \varepsilon + (1 - \alpha_m)k[d(x_0, Tx_0) + d(x_m, Tx_m)] \\ &\leq \alpha_m \varepsilon + 2k(1 - sk)(1 - \alpha_m)k\varepsilon < \varepsilon. \end{aligned} \quad (81)$$

Case 28. If $x_0 = Tx_m$, then we have

$$\begin{aligned} d(x_0, x_{m+1}) &= d(Tx_m, w(x_m, Tx_m; \alpha_m)) \leq \alpha_m d(Tx_m, x_m) \\ &\leq \alpha_m \lambda_u^m d(Tx_0, x_0) \leq \alpha_m \lambda_u^m k(1 - sk)\varepsilon < \varepsilon. \end{aligned} \quad (82)$$

Case 29. If $x_{m+1} = Tx_0$, then we have

$$d(x_0, x_{m+1}) = d(x_0, Tx_0) \leq k(1 - sk)\varepsilon < \varepsilon. \quad (83)$$

Case 30. If $x_{m+1} = Tx_m$, then we have

$$\begin{aligned} d(x_0, x_{m+1}) &= d(x_0, w(x_m, Tx_m; \alpha_m)) \\ &\leq \alpha_m d(x_0, x_m) + (1 - \alpha_m) d(x_0, Tx_m) \\ &= \alpha_m d(x_0, x_m) + (1 - \alpha_m) d(x_0, x_{m+1}), \end{aligned} \quad (84)$$

which implies

$$d(x_0, x_{m+1}) \leq d(x_0, x_m) \leq \varepsilon. \quad (85)$$

Finally, by above cases, we prove that $d(x_0, x_{m+1}) \leq \varepsilon$, which show that $x_{m+1} \in B_\varepsilon[x_0]$. Hence, by induction $x_n \in B_\varepsilon[x_0]$. Therefore, we conclude that $x_n \in B_\varepsilon[x_0]$ for all $n \in \mathbb{N}$. As every closed ball in a complete metric space is complete, so $x^* \in B_\varepsilon[x_0]$, as $n \rightarrow \infty$.

Next, we give the following example to illustrate above theorem.

Example 31. Let $X = \mathbb{R}^+ \cup \{0\}$ and the mapping $T : X \rightarrow X$ such that

$$Tx = \begin{cases} 0, & \text{if } x \in [0, \sqrt{2}), \\ \frac{1}{2x}, & \text{if } x \in [\sqrt{2}, +\infty), \end{cases} \quad (86)$$

for any $x, y \in X$. Let us define the metric $d : X \times X \rightarrow X$ by the formula $d(x, y) = (x - y)^2$ as well as the mapping $w : X \times X \times [0, 1] \rightarrow X$ by the formula $w(x, y; \alpha) = \alpha x + (1 - \alpha)y$. Choose $x_0 \geq 0$ to be the initial value and $x_{n+1} = w(x_n, Tx_n; \alpha_n)$, where $\alpha_n = 1/49$. If $k = 1/9$, then $x_n \in B_\varepsilon[x_0]$, and T has a unique fixed point in $B_\varepsilon[x_0]$.

Proof. It is easy to see that (X, d, w) is a CR_bMS with $s = 3$. We claim that T satisfies inequality

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \quad (87)$$

for any $x, y \in X$. Next, we will consider the four cases:

- (a) If $x, y \in [0, \sqrt{2})$, then it is easy to see that inequality (87) holds

- (b) If $x \in [0, \sqrt{2})$ and $y \in [\sqrt{2}, +\infty)$, then

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{9} [d(x, Tx) + d(y, Ty)] \\ &= \left(\frac{1}{2y}\right)^2 - \frac{1}{9} \left[x^2 + \left(y - \frac{1}{2y}\right)^2 \right] \\ &\leq \left(\frac{1}{2y}\right)^2 - \frac{1}{9} \left(y - \frac{1}{2y}\right)^2 \leq 0, \end{aligned} \quad (88)$$

which implies that

$$d(Tx, Ty) \leq \frac{1}{9} [d(x, Tx) + d(y, Ty)], \quad (89)$$

holds for any $x \in [0, \sqrt{2})$ and $y \in [\sqrt{2}, +\infty)$.

- (c) If $x \in [\sqrt{2}, +\infty)$ and $y \in [0, \sqrt{2})$, then, similarly to case (b), we can also get that inequality (87) holds

- (d) If $x, y \in [\sqrt{2}, +\infty)$, then

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{9} [d(x, Tx) + d(y, Ty)] \\ &= \frac{1}{4} \left(x - \frac{1}{x}\right)^2 - \frac{1}{9} \left[\left(x - \frac{1}{2x}\right)^2 + \left(y - \frac{1}{2y}\right)^2\right] \\ &= \frac{1}{4} \left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{2}{xy}\right) - \frac{1}{9} \left(x^2 + y^2 + \frac{1}{4x^2} + \frac{1}{4y^2} - 2\right) \\ &= \frac{8}{36} \left(\frac{1}{x^2} + \frac{1}{y^2}\right) - \frac{1}{9} \left(x^2 + y^2 + \frac{1}{2xy} - 2\right) \\ &\leq \frac{8}{36} \left(\frac{1}{2} + \frac{1}{2}\right) - \frac{1}{9} \left(2 + 2 + \frac{1}{4} - 2\right) \\ &= \frac{8}{36} - \frac{9}{36} < 0, \end{aligned} \quad (90)$$

which shows that

$$d(Tx, Ty) < \frac{1}{9} [d(x, Tx) + d(y, Ty)] \quad (91)$$

holds for all $x, y \in [\sqrt{2}, +\infty)$.

Summarizing, inequality (87) holds for all $x, y \in X$. Next, we will show that T has a unique fixed point in X . In order to do it, we will consider the following two cases:

- (i) If $x_0 < \sqrt{2}$, then

$$\begin{aligned}
Tx_0 &= 0, \\
x_1 &= \frac{1}{49}x_0 + \frac{48}{49}Tx_0 = \frac{1}{49}x_0, Tx_1 = 0, \\
x_2 &= \frac{1}{49}x_1 + \frac{48}{49}Tx_1 = \left(\frac{1}{49}\right)^2 x_0, Tx_2 = 0, \\
&\dots \\
x_n &= \frac{1}{49}x_{n-1} + \frac{48}{49}Tx_{n-1} = \left(\frac{1}{49}\right)^n x_0.
\end{aligned} \tag{92}$$

Obviously, $x_n \rightarrow 0$ as $n \rightarrow \infty$,

(ii) If $x_0 \geq \sqrt{2}$, then

$$\begin{aligned}
Tx_0 &= \frac{1}{2x_0}, \\
x_1 &= \frac{1}{49}x_0 + \frac{48}{49}Tx_0, \\
\frac{x_1}{x_0} &= \frac{1}{49} + \frac{48}{49} \times \frac{1}{2x_0^2} \leq \frac{13}{49}.
\end{aligned} \tag{93}$$

If $0 \leq x_1 < \sqrt{2}$, then $Tx_1 = 0$. From the case (i), it follows that $x_n \rightarrow 0$ as $n \rightarrow \infty$. If $x_1 \geq \sqrt{2}$, then $x_2/x_1 = 1/49 + 48/(49 \times (1/2x_1^2)) \leq 13/49$. From the above procedure, without loss of generality, we can assume that $x_{n-1} \geq \sqrt{2}$. Then, we obtain

$$\begin{aligned}
\frac{x_n}{x_{n-1}} &= \frac{1}{49} + \frac{48}{49} \times \frac{1}{2x_{n-1}^2} \leq \frac{13}{49}, \\
\frac{x_n}{x_0} &= \frac{x_1}{x_0} \times \frac{x_2}{x_1} \times \dots \times \frac{x_n}{x_{n-1}} \leq \left(\frac{13}{49}\right)^n,
\end{aligned} \tag{94}$$

which implies that $x_n \leq (13/49)^n x_0$.

Hence, $\lim_{n \rightarrow \infty} x_n = 0$, where 0 is a fixed point of T . Actually, 0 is a unique fixed point of T in \mathbb{R} . Indeed, suppose that $y^* \in [\sqrt{2}, +\infty)$ is a fixed point of T , then $Ty^* = y^*$, that is, $y^* = Ty^* = 1/2y^*$, which implies $y^* = \sqrt{2}/2 < \sqrt{2}$, a contradiction. Thus, T has a unique fixed point in \mathbb{R} . Let $\varepsilon = x_0^2/k(1 - sk) > 0$, then $k(1 - sk)\varepsilon = x_0^2 \geq d(x_0, Tx_0)$. For all $n \in \mathbb{N}$, from above proof, we can obtain $x_n \leq (13/49)^n x_0$, then $d(x_0, x_n) = (x_0 - (13/49)^n x_0)^2 < x_0^2 < \varepsilon$, and this means that the sequence $\{x_n\} \subseteq B_\varepsilon[x_0]$. Furthermore, $d(x_0, 0) = x_0^2 < \varepsilon$, that is, $0 \in B_\varepsilon[x_0]$, and the proof is finished. \square

The concept of well posedness is very important in many fields of mathematics and has evoked much interest to several researchers [27–29].

Definition 32. (see [26]). Let (X, d) be a metric space and T be a self-map. The fixed point problem of T is said to be well posed if

T has a unique fixed point $x^* \in X$

(2) For any sequence $\{y_n\}$ in X with $\lim_{n \rightarrow \infty} d(y_n, Ty_n) = 0$, we have $\lim_{n \rightarrow \infty} d(y_n, x^*) = 0$

We next study the well posedness of the fixed point problem of T in complete CR_bMS .

Theorem 33. Let (X, d, w) be a CR_bMS with constant $s \geq 1$ and all the hypotheses of Theorem 12 hold. If the constant $0 < \alpha < 1$, then fixed point problem of T is well posed.

Proof. Let x^* is a unique fixed point of T and assume $\{y_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} d(y_n, Ty_n) = 0$. Because of uniqueness of the fixed point of T , for all $n \in \mathbb{N}$, we can assume that $y_n \neq Ty_n$. If $y_n = w(y_n, Ty_n; \alpha)$ for some $\alpha \in (0, 1)$, $n \in \mathbb{N}$, then

$$\begin{aligned}
d(y_n, x^*) &= d(w(y_n, Ty_n; \alpha), x^*) \\
&\leq \alpha d(y_n, x^*) + (1 - \alpha)d(Ty_n, x^*) \\
&\leq (\alpha + (1 - \alpha)\beta)d(y_n, x^*),
\end{aligned} \tag{95}$$

since $\alpha + (1 - \alpha)\beta < 1$, and we get $d(y_n, x^*) = 0$. Due to $\alpha > 0$, it is not difficult to see that $Ty_n \neq w(y_n, Ty_n; \alpha)$, indeed, if not,

$$d(y_n, Ty_n) = d(y_n, w(y_n, Ty_n; \alpha)) \leq (1 - \alpha)d(y_n, Ty_n), \tag{96}$$

a contradiction. Therefore, let us assume that $y_n \neq Ty_n \neq w(y_n, Ty_n; \alpha)$, and then

$$\begin{aligned}
d(y_n, x^*) &\leq s[d(y_n, w(y_n, Ty_n; \alpha)) + d(w(y_n, Ty_n; \alpha), Ty_n) + d(Ty_n, x^*)] \\
&\leq s(1 - \alpha)d(y_n, Ty_n) + sad(y_n, Ty_n) + s\beta d(y_n, x^*),
\end{aligned} \tag{97}$$

combining with $1 - s\beta > 0$, and we obtain

$$d(y_n, x^*) \leq \frac{s}{1 - s\beta} d(y_n, Ty_n), \tag{98}$$

which implies $\lim_{n \rightarrow \infty} d(y_n, x^*) = 0$, which completes the proof. \square

Theorem 34. Let (X, d, w) be a CR_bMS with constant $s \geq 1$ and all the hypotheses of Theorem 22 hold. If the constant $0 < \alpha < 1$, then fixed point problem of T is well posed.

Proof. Let x^* be a unique fixed point of T and a sequence y_n in sequence in X such that $\lim_{n \rightarrow \infty} d(y_n, Ty_n) = 0$. Without loss of generality, let $y_n \neq x^*$, for all $n \in \mathbb{N}$. By the help of uniqueness of the fixed point of T , then we have $y_n \neq Ty_n$. If $y_n = w(y_n, Ty_n; \alpha)$ for some $\alpha \in (0, 1)$, $n \in \mathbb{N}$, then

$$\begin{aligned}
d(y_n, x^*) &= d(w(y_n, Ty_n; \alpha), x^*) \\
&\leq \alpha d(y_n, x^*) + (1 - \alpha)d(Ty_n, x^*) \\
&\leq \alpha d(y_n, x^*) + (1 - \alpha)k\{d(y_n, Ty_n) + d(x^*, Tx^*)\} \\
&\leq \alpha d(y_n, x^*) + (1 - \alpha)kd(y_n, Ty_n).
\end{aligned} \tag{99}$$

Hence,

$$d(y_n, x^*) \leq kd(y_n, Ty_n), \tag{100}$$

and we conclude that $\lim_{n \rightarrow \infty} d(y_n, x^*) = 0$. Due to $\alpha > 0$, it is not difficult to see that $Ty_n \neq w(y_n, Ty_n; \alpha)$, indeed, if not,

$$d(y_n, Ty_n) = d(y_n, w(y_n, Ty_n; \alpha)) \leq (1 - \alpha)d(y_n, Ty_n), \tag{101}$$

a contradiction. Therefore, let us assume that $y_n \neq Ty_n \neq w(y_n, Ty_n; \alpha)$, and then

$$\begin{aligned}
d(y_n, x^*) &\leq s[d(y_n, w(y_n, Ty_n; \alpha)) + d(w(y_n, Ty_n; \alpha), Ty_n) + d(Ty_n, x^*)] \\
&\leq s(1 - \alpha)d(y_n, Ty_n) + sad(y_n, Ty_n) + sk\{d(y_n, Ty_n) + d(x^*, Tx^*)\},
\end{aligned} \tag{102}$$

combining with $1 - s\beta > 0$, and we obtain

$$d(y_n, x^*) \leq (s + sk)d(y_n, Ty_n), \tag{103}$$

which implies $\lim_{n \rightarrow \infty} d(y_n, x^*) = 0$, which completes the proof. \square

3. Applications

In this section, we will apply our result to solving the following functional equation arising in dynamic programming:

$$p(u) = \sup_{v \in B} \{f(u, v) + G(u, v, p(\varphi(u, v)))\}, \tag{104}$$

for all $u \in A$, where $f : A \times B \rightarrow R$, $\varphi : A \times B \rightarrow A$, and $G : A \times B \times R \rightarrow R$. We assume that C and D are Banach spaces, $A \subseteq C$ is a state space, and $B \subseteq D$ is a decision space. Precisely, see also [30, 31]. Let $X = R(A)$ denote the set of all bounded real-valued functions on A and the norm $\|\cdot\|$ defined as $\|x\| = \sup_{u \in A} |x(u)|$ for all $x \in X$.

Clearly, $(X, \|\cdot\|)$ is a Banach space. Moreover, we can define a rectangular b -metric d by

$$d(x, y) = \sup_{u \in A} |x(u) - y(u)|^2, \tag{105}$$

for all $x, y \in X$. Since $(X, \|\cdot\|)$ is complete, we deduce that (X, d) is a complete rectangular b -metric space with $s = 3$. In order to show the existence of a solution of equation (104), we consider the operator $T : X \rightarrow X$ of the form

$$T(x)(u) = \sup_{v \in B} \{f(u, v) + G(u, v, x(\varphi(u, v)))\}, \tag{106}$$

for all $u \in A$ and $x \in X$. We will prove the following theorem.

Theorem 35. Let $T : X \rightarrow X$ be given by (106). Suppose that the following hypotheses hold:

(A1) $f : A \times B \rightarrow R$ and $G : A \times B \times R \rightarrow R$ are bounded functions;

(A2) There exists $a > 0$, for all $u \in A$, $v \in B$ and $x, y \in X$, such that

$$|G_1(u, v, x(u)) - G_2(u, v, y(u))| \leq a|x(u) - y(u)|. \tag{107}$$

Then, the functional equation (104) has a bounded solution.

Proof. Obviously, T is well defined, since f and G are bounded. That is, $Tx \in X$ and operator T are well defined. Then, from (A2), we have

$$\begin{aligned}
|Tx(u) - Ty(u)|^2 &= \left| \sup_{v \in B} \{f(u, v) + G(u, v, x(\varphi(u, v)))\} \right. \\
&\quad \left. - \sup_{v \in B} \{f(u, v) + G(u, v, y(\varphi(u, v)))\} \right|^2 \\
&\leq \sup_{v \in B} |G(u, v, x(\varphi(u, v))) - G(u, v, y(\varphi(u, v)))|^2 \\
&\leq a^2 \sup_{v \in B} |x(u) - y(u)|^2.
\end{aligned} \tag{108}$$

Let $0 \leq a \leq 1/3\sqrt{2}$; thus, all the conditions of Theorem 12 are fulfilled, and there exists a fixed point $x^* \in X$ of T such that $Tx^* = x^*$. In other words,

$$x^*(u) = \sup_{v \in B} \{f(u, v) + G(u, v, x^*(\varphi(u, v)))\}, \tag{109}$$

for all $u \in A$. This completes the proof. \square

Example 36. Consider the functional equation

$$x(u) = \sup_{v \in [0, 1]} \left\{ \sin(u + v) + \ln \left(1 + uv + \frac{1}{6}x(uv) \right) \right\} \tag{110}$$

for $u \in [0, 2]$. We let $A = [0, 2]$, $B = [0, 1]$. $f : A \times B \rightarrow R$ is defined by $f(u, v) = \sin(u + v)$, $\varphi : A \times B \rightarrow A$ is defined by $\varphi(u, v) = uv$, and

$$G : A \times B \times R \rightarrow R \tag{111}$$

is defined by $G(u, v, x) = \ln(1 + uv + 1/6x)$ for $x \in X$. It is

not difficult to see that f and G are bounded functions. Moreover,

$$\begin{aligned}
 & |G(u, v, x(\varphi(u, v))) - G(u, v, y(\varphi(u, v)))|^2 \\
 &= |\ln(1 + uv + x) - \ln(1 + uv + y)|^2 \\
 &= \left| \ln \frac{1 + uv + 1/6x + 1/6y - 1/6y}{1 + uv + 1/6y} \right|^2 \\
 &= \left| \ln \left(1 + \frac{1/6x - 1/6y}{1 + uv + 1/6y} \right) \right|^2 \\
 &\leq \left| \ln \left(1 + \frac{1}{6}x - \frac{1}{6}y \right) \right|^2 \\
 &\leq \frac{1}{36} |x - y|^2.
 \end{aligned} \tag{112}$$

Thus, all the conditions of Theorem 35 are fulfilled. Hence, functional equation (110) has a solution $x^*(u) \in R(A)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Research Article

Solving an Integral Equation via Orthogonal Branciari Metric Spaces

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In this article, we present an orthogonal L-contraction mapping concepts and prove a fixed point theorem on orthogonal complete Branciari metric spaces. As an application, we apply our major results to solving integral equations.

1. Introduction

The general metric concept was introduced by Branciari [1] in 2000 and which is known as the Branciari metric. Later, many authors were interested to the Branciari metric space for extending the results of Branciari b-metric spaces (see [2–7]). The $\tilde{\Theta}$ -contraction concept was introduced by Jleli and Samet [8] in 2014. It is based on some fixed point results [9, 10]. An orthogonality concept in metric spaces was introduced by Gordji et al. [11, 12]. Several authors proved the fixed point results in the generalized orthogonal metric space of Branciari metric spaces (BMS) [13–17]. The L-contraction concept was introduced by Cho [17] in 2018. In this article, we present the new concepts of L-contractive orthogonal mapping and prove fixed point theorems in an orthogonal complete Branciari metric space (OCBMS). We also give an example to our

current results for using the integral equation solved, respectively.

2. Preliminaries

The basic definitions and results are required in the next section as follows.

Definition 1 (see [1]). Let P be a non-empty set and $\mathfrak{S} : P \times P \rightarrow \mathbb{R}_+$ a mapping such that for all $\mathfrak{S}_1, \mathfrak{S}_2 \in P$ and all $\mathfrak{S}_3 \neq \mathfrak{S}_4 \in P/\{\mathfrak{S}_1, \mathfrak{S}_2\}$:

(BM1) $\mathfrak{S}(\mathfrak{S}_1, \mathfrak{S}_2) = 0$, iff $\mathfrak{S}_1 = \mathfrak{S}_2$

(BM2) $\mathfrak{S}(\mathfrak{S}_1, \mathfrak{S}_2) = \mathfrak{S}(\mathfrak{S}_2, \mathfrak{S}_1)$

(BM3) $\mathfrak{S}(\mathfrak{S}_1, \mathfrak{S}_2) \leq \mathfrak{S}(\mathfrak{S}_1, \mathfrak{S}_3) + \mathfrak{S}(\mathfrak{S}_3, \mathfrak{S}_4) + \mathfrak{S}(\mathfrak{S}_4, \mathfrak{S}_2)$.

The metric \mathfrak{S} is called a Branciari metric, and the pair (P, \mathfrak{S}) is called a BMS.

Definition 2 (see [1]). Let (P, \mathfrak{S}) be a BMS. A self-map $H : P \longrightarrow P$ is called $\ddot{\Theta}$ -contraction if there exist $\ddot{\Theta} \in \Gamma_{1,2,3}$ and $\nu \in (0, 1)$ such that $\forall \mathfrak{S}_1, \mathfrak{S}_2 \in P$:

$$\mathfrak{S}(\mathcal{H}\mathfrak{S}_1, \mathcal{H}\mathfrak{S}_2) > 0 \Rightarrow \ddot{\Theta}(\mathfrak{S}(\mathcal{H}\mathfrak{S}_1, \mathcal{H}\mathfrak{S}_2)) \leq \left[\ddot{\Theta}(\mathfrak{S}(\mathfrak{S}_1, \mathfrak{S}_2)) \right]^\nu, \quad (1)$$

where $\Gamma_{1,2,3}$ is the family of all functions $\ddot{\Theta} : (0, \infty) \longrightarrow (0, \infty)$ which satisfy the following conditions:

- ($\ddot{\Theta}_1$) $\ddot{\Theta}$ is increasing
- ($\ddot{\Theta}_2$) For each sequence $\{\alpha_i\} \subset (0, \infty)$, $\lim_{i \rightarrow \infty} \ddot{\Theta}(\alpha_i) = 1 \Leftrightarrow \lim_{i \rightarrow \infty} \alpha_i = 0^+$.
- ($\ddot{\Theta}_3$) $\ddot{\Theta}$ is continuous.

Remark 3. We know that every $\ddot{\Theta}$ -contraction mapping is continuous.

The following notes are subsequently adopted:

- (1) $\Gamma_{1,2,3}$ is the class of all functions $\ddot{\Theta}$ which satisfy $[\ddot{\Theta}_1 - \ddot{\Theta}_3]$

Definition 4 (see [17]). Let (P, \mathfrak{S}) be a BMS. A mapping $H : P \longrightarrow P$ is called L -contraction with respect to $\varsigma \in L$ if there exists $\ddot{\Theta} \in \Gamma_{1,2,3}$ such that (for all $\mathfrak{S}_1, \mathfrak{S}_2 \in P$):

$$\mathfrak{S}(\mathcal{H}\mathfrak{S}_1, \mathcal{H}\mathfrak{S}_2) > 0 \Rightarrow \varsigma \left[\ddot{\Theta}(\mathfrak{S}(\mathcal{H}\mathfrak{S}_1, \mathcal{H}\mathfrak{S}_2)), \ddot{\Theta}(\mathfrak{S}(\mathfrak{S}_1, \mathfrak{S}_2)) \right] \geq 1, \quad (2)$$

where L is the class of all functions $\varsigma : [1, \infty) \times [1, \infty) \longrightarrow \mathbb{R}$ in which the following conditions are satisfied (ς_1^*):

- (ς_1^*) $\varsigma(1, 1) = 1$
- (ς_2^*) $\varsigma(\rho, \rho_1) < (\rho_1/\rho)$, for all $\rho, \rho_1 > 1$
- (ς_3^*) If $\{\rho_i\}$ and $\{\rho_{1i}\}$ are two sequence in $(1, \infty)$ with $\rho_i < \rho_{1i}$, such that $\lim_{i \rightarrow \infty} \rho_i = \lim_{i \rightarrow \infty} \rho_{1i} > 1$, then $\limsup_{i \rightarrow \infty} \varsigma(\rho_i, \rho_{1i}) < 1$.

Example 1 (see [17]). Let $\varsigma_\nu, \varsigma_\psi : [1, \infty) \times [1, \infty) \longrightarrow \mathbb{R}$ be two functions defined by

- (a) $\varsigma_\nu(\rho, \rho_1) = \rho_1^\nu / \rho$, for all $\rho, \rho_1 \geq 1$, where $\nu \in (0, 1)$
- (b) $\varsigma_\psi(\rho, \rho_1) = \rho_1 / \rho \psi(\rho_1)$, for all $\rho, \rho_1 \geq 1$, where $\psi : [1, \infty) \longrightarrow [1, \infty)$ is a lower semicontinuous and increasing function with $\psi^{-1}(\{1\}) = 1$

Then, $\varsigma_\nu, \varsigma_\psi \in L$.

Cho [17] proved the following theorem.

Theorem 5 (see [17]). Let (P, \mathfrak{S}) be a complete BMS and $H : P \longrightarrow P$ an L -contraction mapping. Then, H has a unique fixed point.

Remark 6. Let $\{\mathbf{a}_i\}, \{\mathbf{b}_i\}, \{\mathbf{c}_i\}$ are sequences of \mathbb{R}_+ such that $\lim_{i \rightarrow \infty} \mathbf{a}_i = \mathbf{a}$, $\lim_{i \rightarrow \infty} \mathbf{b}_i = \mathbf{b}$ and $\lim_{i \rightarrow \infty} \mathbf{c}_i = \mathbf{c}$. Then,

$$\begin{aligned} \lim_{i \rightarrow \infty} \max \{\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i\} &= \max \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \\ \lim_{i \rightarrow \infty} \min \{\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i\} &= \min \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}. \end{aligned} \quad (3)$$

Lemma 7 (see [7]). Let $\{\mathfrak{S}_{1i}\}$ be a Cauchy sequence in a BMS (P, \mathfrak{S}) such that $\lim_{i \rightarrow \infty} \mathfrak{S}(\mathfrak{S}_{1i}, \mathfrak{S}_1) = 0$, for some $\mathfrak{S}_1 \in P$. Then, $\lim_{i \rightarrow \infty} \mathfrak{S}(\mathfrak{S}_{1i}, \mathfrak{S}_2) = \mathfrak{S}(\mathfrak{S}_1, \mathfrak{S}_2)$, for all $\mathfrak{S}_2 \in P$. In particular, $\{\mathfrak{S}_{1i}\}$ diverge to \mathfrak{S}_2 if $\mathfrak{S}_2 \neq \mathfrak{S}_1$.

Definition 8 (see [11]). Let $P \neq \emptyset$ and $\nabla \subseteq P \times P$ be a binary relation. If ∇ satisfies the following condition:

$$\exists \mathfrak{S}_{10} \in P : (\forall \mathfrak{S}_1 \in P, \mathfrak{S}_1 \nabla \mathfrak{S}_{10}) \text{ or } (\forall \mathfrak{S}_1 \in P, \mathfrak{S}_{10} \nabla \mathfrak{S}_1), \quad (4)$$

then it is called an orthogonal set. We denote this O-set by (P, ∇) .

Example 2. Let $P = \mathbb{Z}$ and define $\mathfrak{S}_2 \nabla \mathfrak{S}_1$ if there exists $\nu \in \mathbb{Z}$ such that $\mathfrak{S}_2 = \nu \mathfrak{S}_1$. It is easy to see that $0 \nabla \mathfrak{S}_1$ for all $\mathfrak{S}_1 \in \mathbb{Z}$. Hence (P, ∇) is an O-set.

Example 3 (see [11]). A wheel graph \mathscr{W}_i is a graph (see, for example, Figure 1) with i vertices for each $i \geq 4$, a single vertex connect to all vertex to all vertices of an $(i-1)$ -cycle. Let P be the set of all vertices of \mathscr{W}_i for each $i \geq 4$. Define $\mathfrak{S}_1 \nabla \mathfrak{S}_2$ if there is a connection from \mathfrak{S}_1 to \mathfrak{S}_2 . Then, (P, ∇) is an O-set.

Definition 9 (see [11]). Let (P, ∇) be an O-set. A sequence $\{\mathfrak{S}_{1i}\}$ is called an orthogonal sequence (shortly, O-sequence) if

$$(\forall i \in \mathbb{N}, \mathfrak{S}_{1i} \nabla \mathfrak{S}_{1i+1}) \text{ or } (\forall i \in \mathbb{N}, \mathfrak{S}_{1i+1} \nabla \mathfrak{S}_{1i}). \quad (5)$$

Definition 10 (see [11]). The triplet $(P, \nabla, \mathfrak{S})$ is called an orthogonal metric space if (P, ∇) is an O-set and (P, \mathfrak{S}) is a metric space.

Definition 11 (see [11]). Let $(P, \nabla, \mathfrak{S})$ be an orthogonal metric space. Then, a mapping $H : P \longrightarrow P$ is said to be orthogonally continuous in $\mathfrak{S}_1 \in P$ if for each O-sequence $\{\mathfrak{S}_{1i}\}$ in P with $\mathfrak{S}_{1i} \longrightarrow \mathfrak{S}_1$ as $i \longrightarrow \infty$, we have $H(\mathfrak{S}_{1i}) \longrightarrow H(\mathfrak{S}_1)$ as $i \longrightarrow \infty$. Also, H is said to be ∇ -continuous on P if H is ∇ -continuous in each $\mathfrak{S}_1 \in P$.

Definition 12 (see [11]). Let $(P, \nabla, \mathfrak{S})$ be an orthogonal metric space. Then, P is said to be an orthogonally complete, if every Cauchy O-sequence is convergent.

Definition 13 (see [11]). Let (P, ∇) be an O-set. A mapping $H : P \longrightarrow P$ is said to be ∇ -preserving if $\mathcal{H}\mathfrak{S}_1 \nabla \mathcal{H}\mathfrak{S}_2$ whenever $\mathfrak{S}_1 \nabla \mathfrak{S}_2$. Also, $H : P \longrightarrow P$ is said to be weakly ∇ -preserving if $H(\mathfrak{S}_1) \nabla H(\mathfrak{S}_2)$ or $H(\mathfrak{S}_2) \nabla H(\mathfrak{S}_1)$ whenever $\mathfrak{S}_1 \nabla \mathfrak{S}_2$ for all $\mathfrak{S}_1, \mathfrak{S}_2 \in P$.

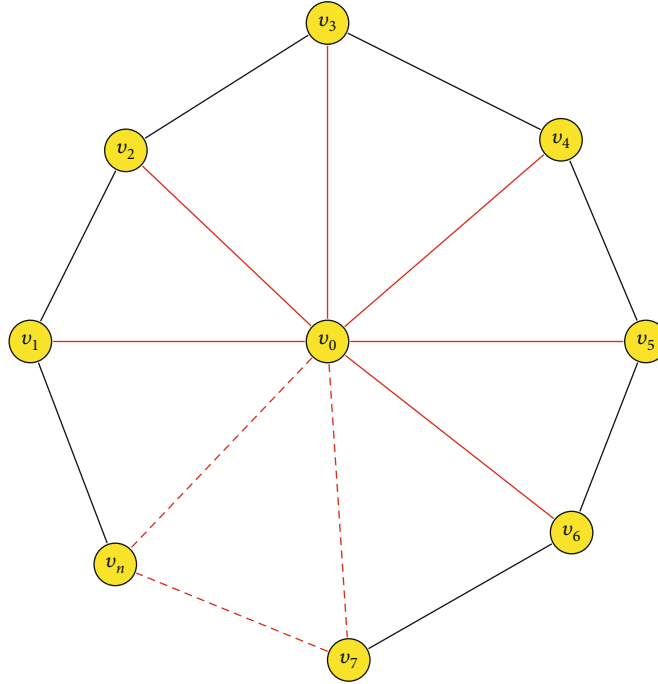


FIGURE 1: An image of a wheel graph.

3. Major Results

In this section, we present the generalized orthogonal L-contraction notion.

Definition 14. Let (P, ∇) be an O-set and $\mathfrak{E} : P \times P \longrightarrow \mathbb{R}_+$ a mapping such that for all $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \mathfrak{I}_4 \in P$ and all $\mathfrak{I}_3 \neq \mathfrak{I}_4 \in P/\{\mathfrak{I}_1, \mathfrak{I}_2\}$:

(OBM1) $\mathfrak{E}(\mathfrak{I}_1, \mathfrak{I}_2) = 0$, if and only if $\mathfrak{I}_1 = \mathfrak{I}_2$

(OBM2) $\mathfrak{E}(\mathfrak{I}_1, \mathfrak{I}_2) = \mathfrak{E}(\mathfrak{I}_2, \mathfrak{I}_1)$

(OBM3) $\mathfrak{E}(\mathfrak{I}_1, \mathfrak{I}_2) \leq \mathfrak{E}(\mathfrak{I}_2, \mathfrak{I}_3) + \mathfrak{E}(\mathfrak{I}_3, \mathfrak{I}_4) + \mathfrak{E}(\mathfrak{I}_4, \mathfrak{I}_1)$ for all $\mathfrak{I}_1 \nabla \mathfrak{I}_2, \mathfrak{I}_1 \nabla \mathfrak{I}_3, \mathfrak{I}_3 \nabla \mathfrak{I}_4, \mathfrak{I}_4 \nabla \mathfrak{I}_1$.

The metric \mathfrak{E} is an orthogonal Branciari metric (shortly OBM), and the pair $(P, \nabla, \mathfrak{E})$ is an orthogonal BMS (shortly OCBMS).

Definition 15. Let $(P, \nabla, \mathfrak{E})$ be a OCBMS and $H : P \longrightarrow P$. Then, H is said to be generalized orthogonal L-contraction with respect to $\varsigma \in L$ if there exist $\ddot{\Theta} \in \Gamma_{1,2,3}$ such that

$$\begin{aligned} \forall \mathfrak{I}_1, \mathfrak{I}_2 \in P \text{ with } \mathfrak{I}_1 \nabla \mathfrak{I}_2 \mathfrak{E}(H\mathfrak{I}_1, H\mathfrak{I}_2) \\ > 0 \Rightarrow \varsigma \left(\ddot{\Theta}(\mathfrak{E}(H\mathfrak{I}_1, H\mathfrak{I}_2)), \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_1, \mathfrak{I}_2)) \right) \geq 1. \end{aligned} \quad (6)$$

Theorem 16. Let $(P, \nabla, \mathfrak{E})$ be a complete OCBMS with an orthogonal element \mathfrak{I}_2 and a self-mapping $H : P \longrightarrow P$. Suppose that there exist $\varsigma \in L$ and $l > 0$ such that the following conditions hold:

- (i) H is ∇ -preserving;
- (ii) H is generalized orthogonal L-contraction mapping;

(iii) H is ∇ -continuous.

Then, H has a unique fixed point.

Proof. Since (P, ∇) is an O-set,

$$\exists \mathfrak{I}_2 \in P : (\forall \mathfrak{I}_1 \in P, \mathfrak{I}_1 \nabla \mathfrak{I}_2) \text{ or } (\forall \mathfrak{I}_1 \in P, \mathfrak{I}_2 \nabla \mathfrak{I}_1). \quad (7)$$

It follows that $\mathfrak{I}_2 \nabla H\mathfrak{I}_2$ or $H\mathfrak{I}_2 \nabla \mathfrak{I}_2$. Let

$$\begin{aligned} \mathfrak{I}_{11} = H\mathfrak{I}_{10}, \mathfrak{I}_{12} = H\mathfrak{I}_{11} = H^2\mathfrak{I}_{10}, \dots, \mathfrak{I}_{1i+1} \\ = H\mathfrak{I}_{1i} = H^{i+1}\mathfrak{I}_{10}, \end{aligned} \quad (8)$$

for all $i \in \mathbb{N} \cup \{0\}$. If $\mathfrak{I}_{1i_0} = \mathfrak{I}_{1i_0+1}$ for any $i_0 \in \mathbb{N} \cup \{0\}$, then it is clear that \mathfrak{I}_{1i_0} is a fixed point of H . Now, we consider $\mathfrak{I}_{1i_0} \neq \mathfrak{I}_{1i_0+1}$ for all $i_0 \in \mathbb{N} \cup \{0\}$. Since H is ∇ -preserving, we have

$$\mathfrak{I}_{1i_0} \nabla \mathfrak{I}_{1i_0+1} \text{ or } \mathfrak{I}_{1i_0+1} \nabla \mathfrak{I}_{1i_0}, \quad (9)$$

for all $i_0 \in \mathbb{N} \cup \{0\}$. This implies $\{\mathfrak{I}_{1i}\}$ is an O-sequence. Using contractive Condition (6) and (ς_2^*) , we have

$$\begin{aligned} 1 &\leq \varsigma \left[\ddot{\Theta}(\mathfrak{E}(H\mathfrak{I}_{1n-1}, H\mathfrak{I}_{1i}), \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1i-1}, \mathfrak{I}_{1i}))) \right] \\ &= \varsigma \left[\ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1i}, \mathfrak{I}_{1i+1})), \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1i-1}, \mathfrak{I}_{1i})) \right] \\ &< \frac{\ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1i-1}, \mathfrak{I}_{1i}))}{\ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1i}, \mathfrak{I}_{1i+1}))}, \end{aligned} \quad (10)$$

which implies that

$$\ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1t}, \mathfrak{I}_{1t+1})) < \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1t-1}, \mathfrak{I}_{1t})) \forall t \in \mathbb{N}. \quad (11)$$

Hence, inequality (11) becomes (in view of $(\ddot{\Theta}_1)$) that

$$\mathfrak{E}(\mathfrak{I}_{1t}, \mathfrak{I}_{1t+1}) < \mathfrak{E}(\mathfrak{I}_{1t-1}, \mathfrak{I}_{1t}), \forall t \in \mathbb{N}. \quad (12)$$

Therefore, the sequence $\{\mathfrak{E}(\mathfrak{I}_{1t-1}, \mathfrak{I}_{1t})\}$ is non-increasing and bounded below by 0. Then, $\ell \geq 0$ such that $\lim_{t \rightarrow \infty} \mathfrak{E}(\mathfrak{I}_{1t-1}, \mathfrak{I}_{1t}) = \ell$. We can claim that $\ell \neq 0$, then

$$\lim_{t \rightarrow \infty} \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1t-1}, \mathfrak{I}_{1t})) > 1. \quad (13)$$

Setting $\rho_t = \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1t}, \mathfrak{I}_{1t+1}))$ and $\rho_{1t} = \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1t-1}, \mathfrak{I}_{1t}))$. In view of (11), (13), and $(\ddot{\Theta}_3)$, we have $\lim_{t \rightarrow \infty} \rho_t = \lim_{t \rightarrow \infty} \rho_{1t} > 1$ and $\rho_t < \rho_{1t}$, for all $t \in \mathbb{N}$. Therefore, applying the condition (ζ_3^*) , we deduce

$$1 \leq \limsup_{t \rightarrow \infty} \varsigma(\rho_t, \rho_{1t}) < 1, \quad (14)$$

which is a contradiction, and therefore

$$\lim_{t \rightarrow \infty} \mathfrak{E}(\mathfrak{I}_{1t-1}, \mathfrak{I}_{1t}) = 0. \quad (15)$$

Now, we consider $\mathfrak{I}_{1j} = \mathfrak{I}_{1t}$, for some $j > t$. Then, also $\mathfrak{I}_{1j+1} = \mathfrak{I}_{1t+1}$. Using (11), we get

$$\begin{aligned} \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1j}, \mathfrak{I}_{1j+1})) &< \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1j-1}, \mathfrak{I}_{1j})) \\ &< \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1j-2}, \mathfrak{I}_{1j-1})) \\ &< \dots < \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1t}, \mathfrak{I}_{1t+1})) \\ &= \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1j}, \mathfrak{I}_{1j+1})), \end{aligned} \quad (16)$$

which is a contradiction. Hence, we conclude that $\mathfrak{I}_{1j} \neq \mathfrak{I}_{1t}$, $\forall t \neq j$. \square

Next, we show that $\{\mathfrak{I}_{1t}\}$ is a Cauchy sequence in $(P, \nabla, \mathfrak{E})$. On the contrary, assume that it is not Cauchy, then there exists an $\epsilon > 0$ for which we can find two subsequences $\{\mathfrak{I}_{1_{j_v}}\}$ and $\{\mathfrak{I}_{1_{t_v}}\}$ of $\{\mathfrak{I}_{1t}\}$ such that $t_v > j_v \geq v$, for all $v \in \mathbb{N}$ and

$$\mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v}}) \geq \epsilon. \quad (17)$$

Suppose that i_v is the least integer exceeding j_v satisfying inequality (17). Then,

$$\mathfrak{E}(\mathfrak{I}_{1_{t_v}}, \mathfrak{I}_{1_{t_v-1}}) < \epsilon. \quad (18)$$

Using (17), (18), and the triangular inequality, we get

$$\begin{aligned} \epsilon &< \mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v}}) \\ &\leq \mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v-2}}) + \mathfrak{E}(\mathfrak{I}_{1_{t_v-2}}, \mathfrak{I}_{1_{t_v-1}}) + \mathfrak{E}(\mathfrak{I}_{1_{t_v-1}}, \mathfrak{I}_{1_{t_v}}) \\ &< \epsilon + \mathfrak{E}(\mathfrak{I}_{1_{t_v-2}}, \mathfrak{I}_{1_{t_v-1}}) + \mathfrak{E}(\mathfrak{I}_{1_{t_v-1}}, \mathfrak{I}_{1_{t_v}}). \end{aligned} \quad (19)$$

As $v \rightarrow \infty$,

$$\lim_{v \rightarrow \infty} \mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v}}) = \epsilon. \quad (20)$$

Employing the triangular inequality once again, we get

$$\begin{aligned} \mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v}}) &\leq \mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{j_v-1}}) + \mathfrak{E}(\mathfrak{I}_{1_{j_v-1}}, \mathfrak{I}_{1_{t_v-1}}) \\ &\quad + \mathfrak{E}(\mathfrak{I}_{1_{t_v-1}}, \mathfrak{I}_{1_{t_v}}) \\ &\leq 2\mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{j_v-1}}) + \mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v}}) \\ &\quad + 2\mathfrak{E}(\mathfrak{I}_{1_{t_v-1}}, \mathfrak{I}_{1_{t_v}}). \end{aligned} \quad (21)$$

On letting $v \rightarrow \infty$ and using (15) as well as (20) we get

$$\lim_{v \rightarrow \infty} \mathfrak{E}(\mathfrak{I}_{1_{j_v-1}}, \mathfrak{I}_{1_{t_v-1}}) = \epsilon. \quad (22)$$

Now, using (6) and (ζ_2^*) , we obtain

$$\begin{aligned} 1 &\leq \varsigma[\ddot{\Theta}(\mathfrak{E}(\mathfrak{H}\mathfrak{I}_{1_{j_v-1}}, \mathfrak{H}\mathfrak{I}_{1_{t_v-1}})), \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1_{j_v-1}}, \mathfrak{I}_{1_{t_v-1}}))] \\ &= [\ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v}})), \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1_{j_v-1}}, \mathfrak{I}_{1_{t_v-1}}))] \\ &< \frac{\ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1_{j_v-1}}, \mathfrak{I}_{1_{t_v-1}}))}{\ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v}}))}. \end{aligned} \quad (23)$$

Consequently, we deduce that

$$\ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v}})) < \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1_{j_v-1}}, \mathfrak{I}_{1_{t_v-1}})), \forall v \in \mathbb{N}. \quad (24)$$

Let $\rho_v = \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1_{j_v}}, \mathfrak{I}_{1_{t_v}}))$ and $\rho_{1v} = \ddot{\Theta}(\mathfrak{E}(\mathfrak{I}_{1_{j_v-1}}, \mathfrak{I}_{1_{t_v-1}}))$. Then, in view of Remark 6 and (24), we have $\lim_{v \rightarrow \infty} \rho_v = \lim_{v \rightarrow \infty} \rho_{1v} > 1$ and $\rho_v < \rho_{1v}$, $\forall v \in \mathbb{N}$. So, on using ζ_3^* , we obtain

$$1 \leq \limsup_{v \rightarrow \infty} \varsigma(\rho_v, \rho_{1v}) < 1, \quad (25)$$

which is a reductio ad absurdum. Therefore, $\{\mathfrak{F}_{1_i}\}$ must be a Cauchy sequence in $(P, \nabla, \mathfrak{E})$. Since $(P, \nabla, \mathfrak{E})$ is a complete, then there exists $v_1 \in P$ such that $\lim_{i \rightarrow \infty} \mathfrak{F}_{1_i} = v_1$, then,

$$\lim_{i \rightarrow \infty} \mathfrak{E}(\mathfrak{F}_{1_i}, v_1) = 0. \quad (26)$$

As H is continuous, then we get that (due to (26))

$$\lim_{i \rightarrow \infty} \mathfrak{E}(\mathfrak{F}_{1_{i+1}}, Hv_1) = \lim_{i \rightarrow \infty} \mathfrak{E}(H\mathfrak{F}_{1_i}, Hv_1) = 0. \quad (27)$$

Using Lemma 7, we have $v_1 = Hv_1$ that is, v_1 is a fixed point of H . On the contrary, assume that there are two fixed points such that $\mathfrak{E}(v_1, \mathfrak{F}_3) = \mathfrak{E}(Hv_1, H\mathfrak{F}_3) > 0$. From (6), since H is preserving, $\forall Hv_1 \nabla H\mathfrak{F}_3$ we have

$$\begin{aligned} 1 &\leq \varsigma \left[\ddot{\Theta}(\mathfrak{E}(Hv_1, H\mathfrak{F}_3)), \ddot{\Theta}(\mathfrak{E}(v_1, \mathfrak{F}_3)) \right] \\ &= \varsigma \left[\ddot{\Theta}(\mathfrak{E}(v_1, \mathfrak{F}_3)), \ddot{\Theta}(\mathfrak{E}(v_1, \mathfrak{F}_3)) \right] \\ &< \frac{\ddot{\Theta}(\mathfrak{E}(v_1, \mathfrak{F}_3))}{\ddot{\Theta}(\mathfrak{E}(v_1, \mathfrak{F}_3))}. \end{aligned} \quad (28)$$

This implies that

$$\ddot{\Theta}(\mathfrak{E}(v_1, \mathfrak{F}_3)) < \ddot{\Theta}(\mathfrak{E}(v_1, \mathfrak{F}_3)), \quad (29)$$

which is a reductio ad absurdum. Then, H has a unique fixed point.

Example 4. Let $P = E \cup G$, where $E = [0, 2]$ and $G = \{(1/i) : n = 2, 3, 4, 5\}$. Define the binary relation ∇ on P by $\mathfrak{F}_1 \nabla \mathfrak{F}_2$ if $\mathfrak{F}_1, \mathfrak{F}_2 \geq 0$. Define a mapping $\mathfrak{E} : P \times P \rightarrow [0, \infty)$ defined by $\mathfrak{E}(\mathfrak{F}_1, \mathfrak{F}_2) = |\mathfrak{F}_1 - \mathfrak{F}_2|$, for all $\mathfrak{F}_1, \mathfrak{F}_2 \in P$.

It is easy to see that $(P, \nabla, \mathfrak{E})$ is an orthogonal complete BMS. Let $H : P \rightarrow P$ be defined as $H\mathfrak{F}_1 = \mathfrak{F}_1/6$ for all $\mathfrak{F}_1 \in P$. Clearly, H is an orthogonal preserving and orthogonal continuous. Observe that H is an L -contraction with respect to $\varsigma : [1, \infty) \times [1, \infty) \rightarrow \mathbb{R}$, where

$$\varsigma_v(\rho, \rho_1) = \frac{\rho_1^v}{\rho}, \forall \rho, \rho_1 \in [1, \infty), v \in (0, 1), \quad (30)$$

and $\ddot{\Theta} : (0, \infty) \rightarrow (1, \infty)$ such that $\ddot{\Theta}(\rho) = e^\rho, \forall \rho > 0$.

Let $\mathfrak{F}_1, \mathfrak{F}_2 \in P$; then,

$$\begin{aligned} &\varsigma \left[\ddot{\Theta}(\mathfrak{E}(H\mathfrak{F}_1, H\mathfrak{F}_2)), \ddot{\Theta}(\mathfrak{E}(\mathfrak{F}_1, \mathfrak{F}_2)) \right] \\ &= \frac{\left[\ddot{\Theta}(\mathfrak{E}(\mathfrak{F}_1, \mathfrak{F}_2)) \right]^v}{\ddot{\Theta}(\mathfrak{E}(H\mathfrak{F}_1, H\mathfrak{F}_2))} = \frac{e^{v|\mathfrak{F}_1 - \mathfrak{F}_2|}}{e^{|\mathfrak{F}_1 - \mathfrak{F}_2|/6}} \geq 1. \end{aligned} \quad (31)$$

Hence, all the hypotheses of Theorem 16 are satisfied, and $\mathfrak{F}_1 = 0$ is the unique fixed point of H .

4. Applications

As an application of Theorem 16, we find an existence and uniqueness of the solution of the following integral equation:

$$\mathfrak{F}_1(\rho) = g(\rho) + \int_0^a j(\rho, v_1) f(v_1, \mathfrak{F}_1(v_1)) dv_1, \quad \rho \in [0, a], a > 0. \quad (32)$$

Let $\mathcal{U} = \mathcal{C}([0, a], \mathbb{R})$ be the set of real continuous functions defined on $[0, a]$ and the mapping $H : \mathcal{U} \rightarrow \mathcal{U}$ defined by

$$H(\mathfrak{F}_1(\rho)) = g(\rho) + \int_0^a j(\rho, v_1) f(v_1, \mathfrak{F}_1(v_1)) dv_1, \quad \rho \in [0, a]. \quad (33)$$

Obviously, $\mathfrak{F}_1(\rho)$ is a solution of integral Equation (32) iff $\mathfrak{F}_1(\rho)$ is a fixed point of H .

Theorem 17. Suppose that

- (R1) The mappings $j : [0, a] \times \mathbb{R} \rightarrow [0, \infty)$, $f : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$, and $g : [0, a] \rightarrow \mathbb{R}$ are continuous functions;
- (R2) there exist $\tau > 0$ and $v \in (0, 1)$ such that

$$|f(v_1, \mathfrak{F}_1(v_1)) - f(v_1, \mathfrak{F}_2(v_1))| \leq v |\mathfrak{F}_1(v_1) - \mathfrak{F}_2(v_1)|; \quad (34)$$

$$(R3) \int_0^a j(\rho, v_1) dv_1 \leq 1.$$

Then, the integral Equation (32) has a unique solution in \mathcal{U} .

Proof. Define the orthogonality relation ∇ on \mathcal{U} by

$$\begin{aligned} \mathfrak{F}_1 \nabla \mathfrak{F}_2 &\Leftrightarrow \mathfrak{F}_1(\rho) \mathfrak{F}_2(\rho) \geq \mathfrak{F}_1(\rho) \text{ or } \mathfrak{F}_1(\rho) \mathfrak{F}_2(\rho) \\ &\geq \mathfrak{F}_2(\rho), \forall \rho \in [0, a]. \end{aligned} \quad (35)$$

Define $\mathfrak{E} : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ given by

$$\mathfrak{E}(\mathfrak{F}_1, \mathfrak{F}_2) = \sup_{\rho \in [0, a]} |\mathfrak{F}_1(\rho) - \mathfrak{F}_2(\rho)|, \quad (36)$$

for all $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{U}$. It is easy to see that $(\mathcal{U}, \nabla, \mathfrak{E})$ is complete orthogonal BMS. For each $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{U}$ with $\mathfrak{F}_1 \nabla \mathfrak{F}_2$ and $\rho \in [0, a]$, we have

$$H(\mathfrak{F}_1(\rho)) = g(\rho) + \int_0^a j(\rho, v_1) f(v_1, \mathfrak{F}_1(v_1)) dv_1 \geq 1. \quad (37)$$

Accordingly, $[(H\mathfrak{F}_1)(\rho)][(H\mathfrak{F}_2)(\rho)] \geq (H\mathfrak{F}_2)(\rho)$ and so $(H\mathfrak{F}_1)(\rho) \nabla (H\mathfrak{F}_2)(\rho)$. Then, H is ∇ -preserving. Let \mathfrak{F}_1 ,

$\mathfrak{F}_2 \in \mathcal{U}$ with $\mathfrak{F}_1 \nabla \mathfrak{F}_2$. Suppose that $H(\mathfrak{F}_1) \neq H(\mathfrak{F}_2)$. For each $\rho \in [0, a]$, we have

$$\begin{aligned} & |H\mathfrak{F}_1(\rho) - H\mathfrak{F}_2(\rho)| \\ &= \left| \int_0^a j(\rho, v_1) [\mathfrak{f}(v_1, \mathfrak{F}_1(v_1)) - \mathfrak{f}(v_1, \mathfrak{F}_2(v_1))] \mathfrak{d}u \right| \\ &\leq \int_0^a j(\rho, v_1) v |\mathfrak{F}_1(v_1) - \mathfrak{F}_2(v_1)| \mathfrak{d}u \\ &\leq v \int_0^a j(\rho, v_1) \mathfrak{d}u \sup_{u \in [0, a]} |\mathfrak{F}_1(u) - \mathfrak{F}_2(u)| \leq v \mathfrak{E}(\mathfrak{F}_1, \mathfrak{F}_2). \end{aligned} \quad (38)$$

Thus,

$$\mathfrak{E}(H\mathfrak{F}_1, H\mathfrak{F}_2) \leq v \mathfrak{E}(\mathfrak{F}_1, \mathfrak{F}_2), \quad (39)$$

which implies that

$$e^{\mathfrak{E}(H\mathfrak{F}_1, H\mathfrak{F}_2)} \leq e^{v \mathfrak{E}(\mathfrak{F}_1, \mathfrak{F}_2)}, \quad (40)$$

for each $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{U}$. We consider L-simulation mapping $\varsigma : [1, \infty) \times [1, \infty) \longrightarrow \mathbb{R}$, where

$$\varsigma_v(\rho, \rho_1) = \frac{\rho_1^v}{\rho}, \forall \rho, \rho_1 \in [1, \infty), v \in (0, 1), \quad (41)$$

and $\ddot{\Theta} : (0, \infty) \longrightarrow (1, \infty)$ such that $\ddot{\Theta}(\rho) = e^\rho, \forall \rho > 0$. Then,

$$\varsigma \left[\ddot{\Theta}(\mathfrak{E}(H\mathfrak{F}_1, H\mathfrak{F}_2)), \ddot{\Theta}(\mathfrak{E}(\mathfrak{F}_1, \mathfrak{F}_2)) \right] \geq 1. \quad (42)$$

Hence, all the conditions of Theorem 16 are fulfilled. Therefore, the integral equation has a unique solution. \square

5. Conclusion

In this article, we proved the fixed point theorems for orthogonal L-contraction mapping on orthogonal complete BMS.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

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Research Article

On Coincidence Theorem in Intuitionistic Fuzzy b -Metric Spaces with Application

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The concept of intuitionistic fuzzy b -metric spaces (shortly, IFbMS) has been introduced and studied to generalize both the notion of intuitionistic fuzzy metric spaces and fuzzy b -metric spaces. The existence of coincident point and common fixed point for two self-mappings has been established. In order to show the strength of these results, some interesting examples are established as well. Our results generalize many previous results existing in literature. Some nontrivial examples are furnished as well as an application is created to give the strength of our main result.

1. Introduction

The advancement and the rich progress of fixed-point theorems in metric spaces (and in its various generalizations) have significant theoretic and useful applications. These developments in the last three decades were fabulous. Most of them used Banach's contraction theorem [1] in their reference result. Many problems in engineering and research can be solved by confining nonlinear equations to similar fixed-point cases. A fixed point $Fx = x$ can be proved for an operator sum $Gx = 0$, where F is a self-mapping in some relevant discipline. Fixed-point theory has various key modes for addressing difficulties arising from a variety of mathematical inspection offshoots, such as split feasibility concerns, supporting problems, equilibrium problems, and matching and selection issues. The theory of fixed points is a fascinating and energising field of study. This idea has already been identified as an over-the-top attempt to pack nonlinear analysis into a small amount of time.

In 1965, Zadeh [2] initiated the concept of FSs which is used to characterize/manipulate information and data having nonstatistical uncertainties. The objectives of the theory of FS are to offer rational and set hypothetical tools to deal with such problems where errors and degree of uncertainties are

present. Later, the idea of intuitionistic-FSs was given by Atanassov [3] in 1986. This set theory not only defines the degree of membership but also degree of nonmembership which is a more generalized form of FS-theory. Many authors apply this concept in different fields of mathematics. Gulzar et al. (e.g., [4–6]) have applied this theory in groups and its characteristics. Akber [7] used this idea to define intuitionistic fuzzy mappings and obtained common fixed points for such types of mappings. Since the notion of distance function have a central part in approximation theory, therefore, FSs further have been applied to the classical notion of metric. In this direction, some authors [8–10] suggested the generalization of metric spaces to the fuzzy situations.

In 1975, the concept of FMS was offered by Kramosil and Michalek [11], which was further developed by George and Veeramani [12] in 1994, in order to build a Hausdorff topology using fuzzy metric. Rehman and Aydi [13] constructed results in fuzzy cone metric space. Bakhtin [14] first proposed the concept of b -metric. b -metric spaces are a broader category than metric spaces. Later, Javed et al. [15] established results on orthogonal partial b -metric spaces. Nădăban [16] proposed the notion of fuzzy b -metric space in 2016, which is a generalization of FMSs. Javed et al. [17] worked on finding fixed-point

results in fuzzy b -metric spaces. Shazia et al. [18] established fixed points for various contractions in fuzzy strong b -metric spaces. Park [19] defined IFMSs as a refinement of FMSs by using theory of intuitionistic-FSs and continuous t -norm and continuous t -conorm in the year 2004. Jungck [20] enhanced Banach's theorem in 1976 by analyzing at coincidence and common fixed points of commuting mappings. Jungck [21] established the concept of compatible maps for a pair of self-mappings, as well as the existence of shared fixed points, in 1986. Turkoglu et al. [22] expanded Jungck's common fixed-point theorems in IFMSs in a paper published in 2006. Jungck and Rhoades [23] introduced weakly compatible mappings in 2006. Weakly compatible mappings are more generic since any pair of compatible mappings is weakly compatible, but not the other way around. Altun et al. [24,25] recently established excellent results on the best proximity points in 2021.

Grabiec [26] defined the completion of FMS in 1988. We proved the existence of the coincidence theorem and the common fixed-point theorem in IFbMS in this work. The structure of the paper is as follows.

After the preliminaries, in Section 3, the notion of IFbMSs has been defined, and this concept is clarified with the help of comprehensible examples. The conceptual definitions of convergent sequence, Cauchy sequence, and topology induced by an IFbMS are presented as well. In Section 4, we formulate and prove our main results regarding coincidence points and common fixed point of weakly compatible mappings in IFbMS and establish some nontrivial examples to justify the validity of our results. Section 5 consists of an application of our main result.

2. Preliminaries

For the reader's convenience, some definitions and results are recalled.

Definition 2.1 (see [27]). Let ζ be an arbitrary nonempty set and $s \geq 1$ be a given real number. A function $\varpi: \zeta \times \zeta \rightarrow [0, \infty)$ is a b -metric on ζ if, for all $\omega, v, z \in \zeta$, the following conditions are satisfied:

- (b_1) $\varpi(\omega, v) = 0 \Leftrightarrow \omega = v$
- (b_2) $\varpi(\omega, v) = \varpi(v, \omega)$
- (b_3) $\varpi(\omega, z) \leq s[\varpi(\omega, v) + \varpi(v, z)]$

The triple (ζ, ϖ, s) will be called b -metric space.

Example 1 (see [28]). The space k_i ($0 < i < 1$),

$$k_i = \left\{ (\omega_j) \subset R : \sum_{j=1}^{\infty} |\omega_j|^i < \infty \right\}, \quad (1)$$

with a function $\varpi: k_i \times k_i \rightarrow R$,

$$\varpi(\omega, v) = \left(\sum_{j=1}^{\infty} |\omega_j - v_j|^i \right)^{1/i}, \quad (2)$$

is a b -metric space, where $\omega = (\omega_j)$ and $v = (v_j) \in k_i$. By some calculation, we obtain

$$\varpi(\omega, z) \leq 2^{1/i} [\varpi(\omega, v) + \varpi(v, z)]. \quad (3)$$

Here, $s = 2^{1/i} > 1$.

Example 2 (see [28]). The space G_i ($0 < i < 1$), of all real functions $\omega(t), t \in [0, 1]$ such that $\int_0^1 |\omega(t)|^i dt < \infty$, is b -metric space if we take

$$\varpi(\omega, v) = \left[\int_0^1 |\omega(t) - v(t)|^i dt \right]^{1/i}, \quad (4)$$

for each $\omega, v \in G_i$.

Definition 2.2 (see [2]). Consider a nonempty set ζ . A mapping from ζ to $[0, 1]$ is called the fuzzy set. If K is a FS and $\eta \in \zeta$, then the function value $K(\eta)$ is called the grade of membership of η in K .

Definition 2.3 (see [3]). For a nonempty set ζ an intuitionistic-FS is defined as

$$A = \{ \omega \in \zeta : \langle \mu_A(\omega), \nu_A(\omega) \rangle \}, \quad (5)$$

where $\mu_A: \zeta \rightarrow [0, 1]$ is called the degree of membership and $\nu_A: \zeta \rightarrow [0, 1]$ is called the degree of nonmembership of every ω to the set A such that $\mu_A(\omega) + \nu_A(\omega) \in [0, 1]$, for all $\omega \in \zeta$.

Definition 2.4 (see [29]). A binary operation $\odot: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is known as continuous- t -norm if the following axioms are satisfied:

- (1) Associativity and commutativity properties are satisfied by \odot
- (2) \odot is a continuous function
- (3) $\lambda \odot 1 = \lambda, \forall \lambda \in [0, 1]$
- (4) If $\lambda \leq k$ and $\epsilon \leq l$ with $\lambda, \epsilon, k, l \in [0, 1]$, then $\lambda \odot \epsilon \leq k \odot l$

Example

- (1) $\lambda \odot_1 \epsilon = \min(\lambda, \epsilon)$
- (2) $\lambda \odot_2 \epsilon = \lambda \cdot \epsilon$
- (3) $\lambda \odot_3 \epsilon = \max(\lambda + \epsilon - 1, 0)$

Definition 2.5 (see [29]). A binary operation $\circ: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is known as continuous- t -conorm if the following axioms are satisfied:

- (1) Associativity and commutativity properties are satisfied by \circ
- (2) \circ is continuous function
- (3) $\lambda \circ 0 = \lambda, \forall \lambda \in [0, 1]$
- (4) $\lambda \circ \epsilon \leq k \circ l$, whenever $\lambda \leq k$ and $\epsilon \leq l$ for all $\lambda, \epsilon, k, l \in [0, 1]$

Example

- (1) $\lambda \circ_1 \epsilon = \min(\lambda + \epsilon, 1)$
- (2) $\lambda \circ_2 \epsilon = \lambda + \epsilon - \lambda \epsilon$
- (3) $\lambda \circ_3 \epsilon = \max(\lambda, \epsilon)$

Definition 2.6 (see [16]). Let ζ be a nonempty set. Let $s \in R(\text{set of reals})$, $s \geq 1$, and \odot be a continuous-t-norm. A FS Φ on $\zeta \times \zeta \times [0, \infty)$ is called fuzzy b -metric if, for all $\omega, v, z \in \zeta$, the following conditions hold:

- (bM1) $\Phi(\omega, v, 0) = 0$
- (bM2) $\Phi(\omega, v, t) = 1, \forall t \geq 0 \Leftrightarrow S\omega = v$
- (bM3) $\Phi(\omega, v, t) = \Phi(v, \omega, t), \forall t \geq 0$
- (bM4) $\Phi(\omega, z, s(t + \theta)) \geq \Phi(\omega, v, t) \odot \Phi(v, z, \theta), \forall t, \theta \geq 0$
- (bM5) $\lim_{t \rightarrow \infty} \Phi(\omega, v, t) = 1$ and $\Phi(\omega, v, \cdot): [0, \infty) \rightarrow [0, 1]$ is left continuous

The quadruple (ζ, Φ, \odot, s) is known as fuzzy b -metric space.

3. Intuitionistic Fuzzy b -Metric Spaces and Coincidence Point Results

Definition 3.1. A 6-tuple $(\zeta, \Phi, \varphi, \odot, \circ, s)$ is said to be IFbMS if ζ is an arbitrary set, $s \geq 1$ is a given real number, \odot is a continuous-t-norm, \circ is a continuous-t-conorm, Φ and φ are FSs on $\zeta^2 \times [0, \infty)$ satisfying the following conditions. For all $\omega, v, z \in \zeta$,

- (a) $\Phi(\omega, v, t) + \varphi(\omega, v, t) \leq 1$
- (b) $\Phi(\omega, v, 0) = 0$
- (c) $\Phi(\omega, v, t) = 1, \forall t > 0$ iff $\omega = v$
- (d) $\Phi(\omega, v, t) = \Phi(v, \omega, t), \forall t > 0$
- (e) $\Phi(\omega, z, s(t + \theta)) \geq \Phi(\omega, v, t) \odot \Phi(v, z, \theta), \forall t, \theta > 0$
- (f) $\Phi(\omega, v, \cdot): [0, \infty) \rightarrow [0, 1]$ is left continuous and $\lim_{t \rightarrow \infty} \Phi(\omega, v, t) = 1$
- (g) $\varphi(\omega, v, 0) = 1$
- (h) $\varphi(\omega, v, t) = 0, \forall t > 0$ iff $\omega = v$
- (i) $\varphi(\omega, v, t) = \varphi(v, \omega, t), \forall t > 0$
- (j) $\varphi(\omega, z, s(t + \theta)) \leq \varphi(\omega, v, t) \circ \varphi(v, z, \theta), \forall t, \theta > 0$
- (k) $\lim_{t \rightarrow \infty} \varphi(\omega, v, t) = 0$ and $\varphi(\omega, v, \cdot): [0, \infty) \rightarrow [0, 1]$ is right continuous

Here, $\Phi(\omega, v, t)$ and $\varphi(\omega, v, t)$ denote the degree of nearness and the degree of nonnearness between ω and v with respect to t , respectively.

Example 3.2.1. Let $(\zeta, \bar{\omega}, s)$ be a b -metric space and $a = \min(a, b), a \circ b = \max(a, b) \forall a, b \in [0, 1]$, and let $\Phi_{\bar{\omega}}, \varphi_{\bar{\omega}}$ be FSs on $\zeta^2 \times [0, \infty)$, defined as follows:

$$\Phi_{\bar{\omega}}(\omega, v, t) = \begin{cases} \frac{t}{t + \bar{\omega}(\omega, v)}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases} \quad (6)$$

$$\varphi_{\bar{\omega}}(\omega, v, t) = \begin{cases} \frac{\bar{\omega}(\omega, v)}{t + \bar{\omega}(\omega, v)}, & \text{if } t > 0, \\ 1, & \text{if } t = 0. \end{cases}$$

We check only axioms (e) and (j) of Definition 3.1 because verification of the other conditions is standard. Let $\omega, v, z \in \zeta$ and $t, u > 0$.

Without restraining the generality, we assume that

$$\begin{aligned} \Phi_{\bar{\omega}}(\omega, v, t) &\leq \Phi_{\bar{\omega}}(v, z, u), \\ \varphi_{\bar{\omega}}(\omega, v, t) &\geq \varphi_{\bar{\omega}}(v, z, u). \end{aligned} \quad (7)$$

Thus,

$$\begin{aligned} \frac{t}{t + \bar{\omega}(\omega, v)} &\leq \frac{u}{u + \bar{\omega}(v, z)}, \\ \frac{\bar{\omega}(\omega, v)}{t + \bar{\omega}(\omega, v)} &\geq \frac{\bar{\omega}(v, z)}{u + \bar{\omega}(v, z)}, \end{aligned} \quad (8)$$

i.e.,

$$t\bar{\omega}(v, z) \leq u\bar{\omega}(\omega, v). \quad (9)$$

On the contrary,

$$\begin{aligned} \Phi_{\bar{\omega}}(\omega, z, s(t + u)) &= \frac{s(t + u)}{s(t + u) + \bar{\omega}(\omega, z)} \\ &\geq \frac{s(t + u)}{s(t + u) + s[\bar{\omega}(\omega, v) + \bar{\omega}(v, z)]} \\ &= \frac{t + u}{t + u + \bar{\omega}(\omega, v) + \bar{\omega}(v, z)}. \end{aligned} \quad (10)$$

Also,

$$\begin{aligned} \varphi_{\bar{\omega}}(\omega, z, s(t + u)) &= \frac{\bar{\omega}(\omega, z)}{s(t + u) + \bar{\omega}(\omega, z)} \\ &\leq \frac{s[\bar{\omega}(\omega, v) + \bar{\omega}(v, z)]}{s(t + u) + s[\bar{\omega}(\omega, v) + \bar{\omega}(v, z)]} \\ &= \frac{\bar{\omega}(\omega, v) + \bar{\omega}(v, z)}{t + u + \bar{\omega}(\omega, v) + \bar{\omega}(v, z)}, \\ \frac{\bar{\omega}(\omega, v) + \bar{\omega}(v, z)}{t + u + \bar{\omega}(\omega, v) + \bar{\omega}(v, z)} &\leq \frac{\bar{\omega}(\omega, v)}{t + \bar{\omega}(\omega, v)}. \end{aligned} \quad (11)$$

Hence, we will obtain that

$$\begin{aligned} \Phi_{\bar{\omega}}(\omega, z, s(t + u)) &\geq \Phi_{\bar{\omega}}(\omega, v, t) = \Phi_{\bar{\omega}}(\omega, v, t) \odot \Phi_{\bar{\omega}}(v, z, u), \\ \varphi_{\bar{\omega}}(\omega, z, s(t + u)) &\leq \varphi_{\bar{\omega}}(\omega, v, t) = \varphi_{\bar{\omega}}(\omega, v, t) \circ \varphi_{\bar{\omega}}(v, z, u), \end{aligned} \quad (12)$$

which had to be verified. We remark that

$$\begin{aligned} \frac{t + u}{t + u + \bar{\omega}(\omega, v) + \bar{\omega}(v, z)} &\geq \frac{t}{t + \bar{\omega}(\omega, v)} \\ \Leftrightarrow t^2 + ut + t\bar{\omega}(\omega, v) + u\bar{\omega}(\omega, v) &\geq t^2 + ut + t\bar{\omega}(\omega, v) + t\bar{\omega}(v, z) \\ \Leftrightarrow u\bar{\omega}(\omega, v) &\geq t\bar{\omega}(v, z), \end{aligned} \quad (13)$$

which is true.

Also,

$$\begin{aligned} \frac{\varpi(\omega, v) + \varpi(v, z)}{v + u + \varpi(\omega, v) + \varpi(v, z)} &\leq \frac{\varpi(\omega, v)}{v + \varpi(\omega, v)} \\ \Leftrightarrow v\varpi(\omega, v) + v\varpi(v, z) + \varpi(\omega, v)\varpi(v, z) + (\varpi(\omega, v))^2 \\ &\leq v\varpi(\omega, v) + u\varpi(\omega, v) + \varpi(v, z)\varpi(v, z) + (\varpi(\omega, v))^2 \\ \Leftrightarrow v\varpi(v, z) &\leq u\varpi(\omega, v), \end{aligned} \quad (14)$$

which is true.

Hence, $(\zeta, \Phi, \varphi, \odot, \circ, s)$ is (standard) IFbMS.

Example 3.2.2. Let (ζ, ϖ, s) be a b -metric space and $a = \min(a, b)$ and $a^\circ b = \max(a, b) \forall a, b \in [0, 1]$, and let Φ and φ be FSSs on $\zeta^2 \times [0, \infty)$, defined as follows:

$$\begin{aligned} \Phi(\omega, v, t) &= \begin{cases} \left(\exp\left(\frac{\varpi(\omega, v)}{t}\right) \right)^{-1}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases} \\ \varphi\varpi(\omega, v, t) &= \begin{cases} \frac{\exp(\varpi(\omega, v)/t) - 1}{\exp(\varpi(\omega, v)/t)} & \text{if } t > 0, \\ 1, & \text{if } t = 0. \end{cases} \end{aligned} \quad (15)$$

Then, $(\zeta, \Phi, \varphi, \odot, \circ, s)$ is an IFbMS.

Definition 3.3. Let $s \geq 1$ be a given real number. A function $f: R \rightarrow R$ will be called s -nondecreasing if $t < u$ implies that $f(t) \leq f(su)$ and f is called s -nonincreasing if $t < u$ implies that $f(t) \geq f(su)$.

Proposition 3.4. In an IFbMS $(\zeta, \Phi, \varphi, \odot, \circ, s)$, $\Phi(\omega, v, \cdot): [0, \infty) \rightarrow [0, 1]$ is s -nondecreasing and $\varphi(\omega, v, \cdot): [0, \infty) \rightarrow [0, 1]$ is s -nonincreasing, for all $\omega, v \in \zeta$.

Proof. For $0 < t < u$, we have

$$\begin{aligned} \Phi(\omega, v, su) &= \Phi(\omega, v, s(u - t + t)) \\ &\geq \Phi(\omega, \omega, u - t) \odot \Phi(\omega, v, t) \\ &= 1 \odot \Phi(\omega, v, t) = \Phi(\omega, v, t). \end{aligned} \quad (16)$$

Also,

$$\begin{aligned} \varphi(\omega, v, su) &= \varphi(\omega, v, s(u - t + t)) \\ &\leq \varphi(\omega, \omega, u - t) \circ \varphi(\omega, v, t) \\ &= 0^\circ \varphi(\omega, v, t) = \varphi(\omega, v, t). \end{aligned} \quad (17)$$

□

Definition 3.5. Let $(\zeta, \Phi, \varphi, \odot, \circ, s)$ be an IFbMS.

- (a) Any sequence ω_n in ζ is said to be convergent if there exists $\omega \in \zeta$ such that $\lim_{n \rightarrow \infty} \Phi(\omega_n, \omega, t) = 1$ and $\lim_{n \rightarrow \infty} \varphi(\omega_n, \omega, t) = 0, \forall t > 0$. ω is called the limit of the sequence ω_n , and it is written as $\lim_{n \rightarrow \infty} \omega_n = \omega$, or $\omega_n \rightarrow \omega$.

- (b) Any sequence ω_n in $(\zeta, \Phi, \odot, \circ, s)$ is said to be a Cauchy sequence if, for every ϵ in $(0, 1)$, there is $n_0 \in \varphi$ such that $\Phi(\omega_n, \omega_m, t) > 1 - \epsilon$ and $\varphi(\omega_n, \omega_m, t) < \epsilon$, for all $m, n \geq n_0$ and $t > 0$.

- (c) ζ is said to be complete if every Cauchy sequence in ζ is convergent in ζ .

In 2012, Tirado [30] proved that (standard) FMS is complete. It can be easily checked that (standard) IFbMS is also complete.

Definition 3.6. Let $(\zeta, \Phi, \varphi, \odot, \circ, s)$ be an IFbMS. An open ball $B(\omega, r, t)$ with center $\omega \in \zeta$ and radius $r, 0 < r < 1$, and $t > 0$ is defined as $B(\omega, r, t) = \{v \in \zeta: \Phi(\omega, v, t) > 1 - r, \varphi(\omega, v, t) < r\}$.

Definition 3.7. Let $(\zeta, \Phi, \varphi, \odot, \circ, s)$ be an IFbMS and A be a subset of ζ . A is said to be open if, for each $\omega \in A$, there is an open ball $B(\omega, r, t)$ contained in A .

Result: let $(\zeta, \Phi, \varphi, \odot, \circ, s)$ be an IFbMS. Define $\tau_{\Phi, \varphi}$ as $\tau_{\Phi, \varphi} = \{A \in P(\zeta): \omega \in A \text{ if } \exists t > 0 \text{ and } r \in (0, 1): B(\omega, r, t) \subset A\}$, then $\tau_{\Phi, \varphi}$ is a **topology** on ζ , where $P(\zeta)$ is the power set of ζ .

4. Coincidence and Common Fixed-Point theorems

This section concerns with the constructing and proving of coincidence theorem and common fixed-point theorem in IFbMS. Many useful results existing in literature are presented here as corollaries of our results.

Definition 4.1. Let ζ be a nonempty set and $\Pi, \sigma: \zeta \rightarrow \zeta$ be two mappings on ζ .

- (i) A point $\omega \in \zeta$ is called a coincidence point of Π and σ if $\Pi(\omega) = \sigma(\omega)$
- (ii) A point $v \in \zeta$ is called point of coincidence of Π and σ if there exists $\omega \in \zeta$ such that $v = \Pi(\omega) = \sigma(\omega)$
- (iii) A point $z \in \zeta$ is known as common fixed point of Π and σ if $z = \Pi(z) = \sigma(z)$

Definition 4.2. Two self-maps $\Pi, \sigma: \zeta \rightarrow \zeta$ are said to be weakly compatible if $\Pi\sigma(\omega) = \sigma\Pi(\omega)$ when $\Pi(\omega) = \sigma(\omega)$.

Theorem 4.1. Let ζ be a nonempty set and $(Y, \Phi, \varphi, \odot, \circ, s)$ be an IFbMS and $\Pi, \sigma: \zeta \rightarrow Y$ be mappings satisfying the following conditions:

- (1) $\sigma(\zeta) \subseteq \Pi(\zeta); \setminus$
- (2) There is $k, 0 \leq k < 1$, such that, for all $\omega, v \in \zeta$,

$$\begin{aligned} \Phi(\sigma(\omega), \sigma(v), kt) &\geq \Phi(\Pi(\omega), \Pi(v), t), \\ \Pi\varphi(\sigma(\omega), \sigma(v), kt) &\leq \varphi(\Pi(\omega), \Pi(v), t). \end{aligned} \quad (18)$$

If $\Pi(\zeta)$ or $\sigma(\zeta)$ is complete, then there exists a point $z \in \zeta$ such that $\Pi(z) = \sigma(z)$. Moreover, Π and σ have a unique point of coincidence.

Proof. Let $\omega_0 \in \zeta$. By (1), we can find $\omega_1 \in \zeta$ such that $\Pi(\omega_1) = \sigma(\omega_0)$.

For $k = 0$,

$$\begin{aligned} \Phi(\sigma(\omega_0), \sigma(\omega_1), 0t) &\geq \Phi(\Pi(\omega_0), \Pi(\omega_1), t), \\ \varphi(\sigma(\omega_0), \sigma(\omega_1), 0t) &\leq \varphi(\Pi(\omega_0), \Pi(\omega_1), t), \\ \Rightarrow \Phi(\sigma(\omega_0), \sigma(\omega_1), 0t) &= 1, \\ \varphi(\sigma(\omega_0), \sigma(\omega_1), 0t) &= 0. \end{aligned} \quad (19)$$

Hence,

$$\begin{aligned} \sigma(\omega_0) &= \sigma(\omega_1) \\ \Rightarrow \Pi(\omega_1) &= \sigma(\omega_1). \end{aligned} \quad (20)$$

This implies that ω_1 is the coincidence point of Π and σ .

For $k \neq 0$, by induction, we can define a sequence $\{\omega_n\}$ in ζ such that $\Pi(\omega_n) = \sigma(\omega_{n-1})$:

$$\begin{aligned} \Phi(\Pi(\omega_n), \Pi(\omega_{n+1}), t) &= \Phi(\sigma(\omega_{n-1}), \sigma(\omega_n), t) \\ &\geq \Phi(\Pi(\omega_{n-1}), \Pi(\omega_n), t/k) \\ &\geq \dots \\ &\geq \Phi(\Pi(\omega_0), \Pi(\omega_1), t/k^n). \end{aligned} \quad (21)$$

Clearly, $1 \geq \Phi(\Pi(\omega_n), \Pi(\omega_{n+1}), t) \geq \Phi(\Pi(\omega_0), \Pi(\omega_1), t/k^n) \rightarrow 1$, when $n \rightarrow \infty$.

Thus,

$$\lim_{n \rightarrow \infty} \Phi(\Pi(\omega_n), \Pi(\omega_{n+1}), t) = 1.$$

And

$$\begin{aligned} \varphi(\Pi(\omega_n), \Pi(\omega_{n+1}), t) &= \varphi(\sigma(\omega_{n-1}), \sigma(\omega_n), t) \\ &\leq \varphi(\Pi(\omega_{n-1}), \Pi(\omega_n), t/k) \\ &\leq \dots \\ &\leq \varphi(\Pi(\omega_0), \Pi(\omega_1), t/k^n). \end{aligned} \quad (22)$$

Clearly, $0 \leq \varphi(\Pi(\omega_n), \Pi(\omega_{n+1}), t) \leq \varphi(\Pi(\omega_0), \Pi(\omega_1), t/k^n) \rightarrow 0$, when $n \rightarrow \infty$.

Thus, $\lim_{n \rightarrow \infty} \varphi(\Pi(\omega_n), \Pi(\omega_{n+1}), t) = 0$.

Let $\tau_n(t) = \Phi(\Pi(\omega_n), \Pi(\omega_{n+1}), t)$ and $\mu_n(t) = \varphi(\Pi(\omega_n), \Pi(\omega_{n+1}), t)$, for all $n \in N \cup \{0\}$, $t > 0$.

Clearly, $\lim_{n \rightarrow \infty} \tau_n(t) = 1$ and $\lim_{n \rightarrow \infty} \mu_n(t) = 0$.

To show that $\Pi(\omega_n)$ is a Cauchy sequence, suppose it is not; then, there exists $0 < \varepsilon < 1$ and two sequences $p(\eta)$ and $q(\eta)$ such that, for every $\eta \in N \cup \{0\}$, $t > 0$, $p(\eta) > q(\eta) \geq \eta$, $\Phi(\Pi(\omega_{p(\eta)}), \Pi(\omega_{q(\eta)}), t) \leq 1 - \varepsilon$, and $\varphi(\Pi(\omega_{p(\eta)}), \Pi(\omega_{q(\eta)}), t) \geq \varepsilon$.

Then, $\Phi(\Pi(\omega_{p(\eta)-1}), \Pi(\omega_{q(\eta)-1}), t) > 1 - \varepsilon$, $\Phi(\Pi(\omega_{p(\eta)-1}), \Pi(\omega_{q(\eta)}), t) > 1 - \varepsilon$, and $\varphi(\Pi(\omega_{p(\eta)-1}), \Pi(\omega_{q(\eta)-1}), t) < \varepsilon$, $\varphi(\Pi(\omega_{p(\eta)-1}), \Pi(\omega_{q(\eta)}), t) < \varepsilon$.

Now,

$$\begin{aligned} 1 - \varepsilon &\geq \Phi(\Pi(\omega_{p(\eta)}), \Pi(\omega_{q(\eta)}), t) \\ &\geq \Phi(\Pi(\omega_{p(\eta)-1}), \Pi(\omega_{p(\eta)}), t/2s) \odot \Phi(\Pi(\omega_{p(\eta)-1}), \Pi(\omega_{q(\eta)}), t/2s) \\ &> \tau_{p(\eta)-1}(t/2s) \odot (1 - \varepsilon), \\ \varepsilon &\leq \varphi(\Pi(\omega_{p(\eta)}), \Pi(\omega_{q(\eta)}), t) \\ &\leq \varphi(\Pi(\omega_{p(\eta)-1}), \Pi(\omega_{p(\eta)}), t/2s) \circ \varphi(\Pi(\omega_{p(\eta)-1}), \Pi(\omega_{q(\eta)}), t/2s) \\ &< \mu_{p(\eta)-1}(t/2s) \circ \varepsilon. \end{aligned} \quad (23)$$

Since $\tau_{p(\eta)-1}(t/2s) \rightarrow 1$ as $\eta \rightarrow \infty$ and $\mu_{p(\eta)-1}(t/2s) \rightarrow 0$ as $\eta \rightarrow \infty$ for every t , supposing that $\eta \rightarrow \infty$, we have $1 - \varepsilon \geq \Phi(\Pi(\omega_{p(\eta)}), \Pi(\omega_{q(\eta)}), t) > 1 - \varepsilon$, $\varepsilon \leq \varphi(\Pi(\omega_{p(\eta)}), \Pi(\omega_{q(\eta)}), t) < \varepsilon$.

Clearly, this leads to the contradiction.

Hence, $\Pi(\omega_n)$ is a Cauchy sequence in $\Pi(\zeta)$. \square

Case I: suppose that $\Pi(\zeta)$ is complete; then, there exists a point $v \in \Pi(\zeta)$ such that $\lim_{n \rightarrow \infty} \Pi(\omega_n) = v$.

This implies that there exists $z \in \zeta$ such that $v = \Pi(z)$.

Now,

$$\begin{aligned} \Phi(\Pi(z), \sigma(z), t) &\geq \Phi(\Pi(z), \Pi(\omega_n), t/2s) \odot \Phi(\Pi(\omega_n), \sigma(z), t/2s) \\ &= \Phi(\Pi(z), \Pi(\omega_n), t/2s) \odot \Phi(\sigma(\omega_{n-1}), \sigma(z), t/2s) \geq \Phi(\Pi(z), \Pi(\omega_n), t/2s) \odot \Phi(\Pi(\omega_{n-1}), \Pi(z), t/2sk) \\ &\geq 1 \odot 1 = 1, \text{ as } n \rightarrow \infty, \\ \varphi(\Pi(z), \sigma(z), t) &\leq \varphi(\Pi(z), \Pi(\omega_n), t/2s) \circ \varphi(\Pi(\omega_n), \sigma(z), t/2s) \\ &= \varphi(\Pi(z), \Pi(\omega_n), t/2s) \circ \varphi(\sigma(\omega_{n-1}), \sigma(z), t/2s) \leq \varphi(\Pi(z), \Pi(\omega_n), t/2s) \circ \varphi(\Pi(\omega_{n-1}), \Pi(z), t/2sk) \\ &\leq 0 \circ 0 = 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (24)$$

By (c) and (h) of Definition 3.1, it follows that $\Pi(z) = \sigma(z)$.

Hence, z is a coincidence point and v is the point of coincidence of Π and σ .

Case II : suppose that $\sigma(\zeta)$ is complete; then, there exists a point $v \in \sigma(\zeta)$ such that $\lim_{n \rightarrow \infty} \Pi(\omega_n) = v$.

However, $\sigma(\zeta) \subseteq \Pi(\zeta)$; this implies that $v \in \Pi(\zeta)$, so there exists $z \in \zeta$ such that $v = \Pi(z)$.

Next onward, proof is the same as in case I.

Now, we show that the point of coincidence of Π and σ is unique.

Let v_1 be another point of coincidence of Π and σ . Then, $v_1 = \Pi(z_1) = \sigma(z_1)$ for some z_1 in ζ :

$$\begin{aligned} 1 &\geq \Phi(v, v_1, t) = \Phi(\sigma(z), \sigma(z_1), t) \\ &\geq \Phi(\Pi(z), \Pi(z_1), t/k) = \Phi(v, v_1, t/k) \\ &\geq \dots \geq \Phi(v, v_1, t/k^n). \end{aligned} \quad (25)$$

Also,

$$\begin{aligned} 0 &\leq \varphi(v, v_1, t) = \varphi(\sigma(z), \sigma(z_1), t) \\ &\leq \varphi(\Pi(z), \Pi(z_1), t/k) = \varphi(v, v_1, t/k) \\ &\leq \dots \leq \varphi(v, v_1, t/k^n). \end{aligned} \quad (26)$$

Thus, by (II) and (k) of Definition 3.1, $\lim_{n \rightarrow \infty} \Phi(v, v_1, t/k^n) = 1$ and $\lim_{n \rightarrow \infty} \varphi(v, v_1, t/k^n) = 0$.

It follows that $1 \geq \Phi(v, v_1, t) \geq 1$ and $0 \leq \varphi(v, v_1, t) \leq 0$, which implies that $v = v_1$ by (c) and (h) of Definition 3.1. This completes the proof.

Note: the uniqueness of the coincidence point will be sure when Π or σ is one-one.

The following result gives common fixed point of Π and σ with the assumption of weakly compatibility.

Theorem 4.2. Let $(\zeta, \Phi, \varphi, \odot, \circ, s)$ be a complete IFbMS and $\Pi, \sigma: \zeta \rightarrow \zeta$ be mappings satisfying the following conditions:

- (1) $\sigma(\zeta) \subseteq \Pi(\zeta)$.
- (2) There is $k, 0 \leq k < 1$, such that, for all $\omega, v \in \zeta$,

$$\begin{aligned} \Phi(\sigma(\omega), \sigma(v), kt) &\geq \Phi(\Pi(\omega), \Pi(v), t) \\ \varphi(\sigma(\omega), \sigma(v), kt) &\leq \varphi(\Pi(\omega), \Pi(v), t). \end{aligned} \quad (27)$$

- (3) Π and σ are weakly compatible.

Then, Π and σ have a unique-common fixed point in ζ .

Proof. By the above theorem, there is a unique point of coincidence of Π and σ in ζ . That is, we can get z, v in ζ such that $v = \Pi(z) = \sigma(z)$.

Since $v = \Pi(z)$ and Π and σ are weakly compatible, so $\sigma(v) = \sigma(\Pi(z)) = \Pi(\sigma(z)) = \Pi(v)$.

Let $u = \Pi(v) = \sigma(v)$; then, u is a point of coincidence of Π and σ . Since the point of coincidence is unique, this implies that $u = v \Rightarrow v = \Pi(v) = \sigma(v)$.

Hence, v is unique-common fixed point of Π and σ . This completes the proof. \square

Corollary 1. Let $(\zeta, \Phi, \varphi, \odot, \circ)$ be a complete IFMS and $\Pi, \sigma: \zeta \rightarrow \zeta$ be mappings satisfying the following conditions:

- (1) $\sigma(\zeta) \subseteq \Pi(\zeta)$.
- (2) There is $k, 0 \leq k < 1$, such that, for all $\omega, v \in \zeta$,

$$\begin{aligned} \Phi(\sigma(\omega), \sigma(v), kt) &\geq \Phi(\Pi(\omega), \Pi(v), t), \\ \varphi(\sigma(\omega), \sigma(v), kt) &\leq \varphi(\Pi(\omega), \Pi(v), t). \end{aligned} \quad (28)$$

- (3) Π and σ are weakly compatible.

Then, Π and σ have unique common fixed point in ζ .

Proof. By putting $s = 1$ in Theorem 4.2, we get the required result. \square

Corollary 2. Let (ζ, Φ, \odot) be a complete fuzzy b-metric space and $\Pi, \sigma: \zeta \rightarrow \zeta$ be mappings satisfying the following conditions:

- (1) $\sigma(\zeta) \subseteq \Pi(\zeta)$.
- (2) There exist $k \in [0, 1)$ such that $\forall \omega, v \in \zeta$,

$$\Phi(\sigma(\omega), \sigma(v), kt) \geq \Phi(\Pi(\omega), \Pi(v), t). \quad (29)$$

- (3) Π and σ are weakly compatible.

Then, Π and σ have unique-common fixed point in ζ .

Proof. By putting $\varphi = O$ (i.e., φ is a zero function) in Theorem 4.2, we get the required result. \square

Corollary 3 (see [31]). Let (ζ, Φ, \odot) be a complete FMS and $\Pi, \sigma: \zeta \rightarrow \zeta$ be mappings satisfying the following conditions:

- (1) $\sigma(\zeta) \subseteq \Pi(\zeta)$.
- (2) There exist $k \in [0, 1)$ such that $\forall \omega, v \in \zeta$,

$$\Phi(\sigma(\omega), \sigma(v), kt) \geq \Phi(\Pi(\omega), \Pi(v), t). \quad (30)$$

- (3) Π and σ are weakly compatible.

Then, Π and σ have unique-common fixed point in ζ .

Proof. By putting $\varphi = O$ (i.e., φ is a zero function) and $s = 1$ in Theorem 4.2, we get the required result. \square

Example. Let $\zeta = \mathbb{R}$ and $\Pi: \zeta \rightarrow \zeta$ be a self-map on ζ defined as $\Pi(\omega) = 3\omega, \forall \omega \in \zeta$.

Define $\Phi, \varphi: \zeta^2 \times [0, \infty) \rightarrow [0, 1]$ as

$$\begin{aligned} \Phi(\omega, v, t) &= \begin{cases} \frac{t}{t + |\omega - v|}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases} \\ \varphi(\omega, v, t) &= \begin{cases} \frac{|\omega - v|}{t + |\omega - v|}, & \text{if } t > 0, \\ 1, & \text{if } t = 0. \end{cases} \end{aligned} \quad (31)$$

By [30] and Example 3.2.1, it is clear that $(\zeta, \Phi, \varphi, \odot, \circ, s)$ is complete IFbMS, where $a = \min(a, b)$, $a \circ b = \max(a, b)$, and $\forall a, b \in [0, 1]$. Note that conventional Banach's contraction principle fails to find the fixed point of Π as Π is not a contraction.

Now, define $\sigma: \zeta \longrightarrow \zeta$ as $\sigma(\omega) = 2\omega, \forall \omega \in \zeta$.

It is evident that $\sigma(\zeta) \subseteq \Pi(\zeta)$ and Π and σ are weakly compatible. Then,

$$\begin{aligned} \Phi(\sigma(\omega), \sigma(v), 2t/3) &= \frac{2t/3}{2t/3 + |2\omega - 2v|} \\ &= \frac{t/3}{t/3 + |\omega - v|} \\ &\geq \frac{t}{t + |3\omega - 3v|}, \\ &= \Phi(\Pi(\omega), \Pi(v), t) \\ \varphi(\sigma(\omega), \sigma(v), 2t/3) &= \frac{|2\omega - 2v|}{2t/3 + |2\omega - 2v|} \\ &= \frac{|\omega - v|}{t/3 + |\omega - v|} \\ &\leq \frac{|3\omega - 3v|}{t + |3\omega - 3v|} \\ &= \varphi(\Pi(\omega), \Pi(v), t). \end{aligned} \quad (32)$$

Thus, all the conditions of Theorem 4.2 are satisfied for $k = 2/3$; hence, Π and σ have a unique-common fixed point: $0 = \Pi(0) = \sigma(0)$.

5. Application

Now, as an application of coincidence theorem, we give the following theorem.

Theorem. Let $F, G: R \times I \longrightarrow R$ and $f: R \longrightarrow R$ be continuous mappings such that

$$G(\omega, u) = F(\omega, u) + f(\omega), \quad (33)$$

where $I = \{u \in R: a \leq u \leq b, a, b \in R\}$.

Let $C(I)$ be the collection of all continuous functions defined from I into R . Suppose that, for each $\omega \in C(I)$, there exists $v \in C(I)$, such that $(fv)(u) = G(\omega(u), u)$ and $\{f\omega: \omega \in C(I)\}$ is complete. If there exists a number $k \in [0, 1)$ such that, for all $\omega_1, \omega_2 \in C(I)$ and $u \in I$,

$$|G(\omega_1(u), u) - G(\omega_2(u), u)| \leq k|f(\omega_1(u)) - f(\omega_2(u))|, \quad (34)$$

then the equation,

$$F(\omega, u) = 0, \quad (35)$$

defines a continuous function ω in terms of u .

Proof. Let $\zeta = Y = C(I)$.

Define $\Phi, \varphi: \zeta^2 \times [0, \infty) \longrightarrow [0, 1]$ as

$$\begin{aligned} \Phi_{\omega}(\omega, v, t) &= \begin{cases} \frac{t}{t + \max_{u \in I} |\omega(u) - v(u)|}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases} \\ \varphi_{\omega}(\omega, v, t) &= \begin{cases} \frac{\max_{u \in I} |\omega(u) - v(u)|}{t + \max_{u \in I} |\omega(u) - v(u)|}, & \text{if } t > 0, \\ 1, & \text{if } t = 0. \end{cases} \end{aligned} \quad (36)$$

Define mapping $\sigma: \zeta \longrightarrow \zeta$ as follows:

$$\sigma(\omega(u)) = G(\omega(u), u). \quad (37)$$

Then, by assumption, $f(\zeta) = \{f\omega: \omega \in \zeta\}$ is complete. Let $\omega^{\odot} \in \sigma(\zeta)$; then, $\omega^{\odot} = \sigma\omega$ for $\omega \in \zeta$ and $\omega^{\odot}(u) = \sigma\omega(u) = G(\omega(u), u)$.

By assumptions, there exists $v \in \zeta$ such that $\sigma\omega(u) = G(\omega(u), u) = fv(u)$.

Hence, $\sigma(\zeta) \subseteq f(\zeta)$.

Since

$$\begin{aligned} |(\sigma\omega)(u) - (\sigma v)(u)| &= |G(\omega(u), u) - G(v(u), u)| \\ &\leq k|(f\omega)(u) - (fv)(u)| \\ &\leq k(\max_{u \in I} |(f\omega)(u) - (fv)(u)|), \end{aligned} \quad (38)$$

it further implies that

$$\begin{aligned} \max_{u \in I} |(\sigma\omega)(u) - (\sigma v)(u)| &\leq k(\max_{u \in I} |(f\omega)(u) - (fv)(u)|) \\ \Rightarrow \frac{\max_{u \in I} |(\sigma\omega)(u) - (\sigma v)(u)|}{kt} &\leq \frac{(\max_{u \in I} |(f\omega)(u) - (fv)(u)|)}{t} \\ \Rightarrow \frac{kt}{\max_{u \in I} |(\sigma\omega)(u) - (\sigma v)(u)|} &\geq \frac{t}{(\max_{u \in I} |(f\omega)(u) - (fv)(u)|)} \\ \Rightarrow \frac{kt}{kt + (\max_{u \in I} |(\sigma\omega)(u) - (\sigma v)(u)|)} &\geq \frac{t}{t + (\max_{u \in I} |(f\omega)(u) - (fv)(u)|)} \Rightarrow \Phi(\sigma\omega, \sigma v, kt) \geq \Phi(f\omega, fv, t). \end{aligned} \quad (39)$$

Also, inequality (39) implies that

$$\begin{aligned} & \frac{\max_{u \in I} |(\sigma\omega)(u) - (\sigma v)(u)|}{kt} \leq \frac{(\max_{u \in I} |(f\omega)(u) - (fv)(u)|)}{t} \\ \Rightarrow & \frac{\max_{u \in I} |(\sigma\omega)(u) - (\sigma v)(u)|}{kt + \max_{u \in I} |(\sigma\omega)(u) - (\sigma v)(u)|} \leq \frac{(\max_{u \in I} |(f\omega)(u) - (fv)(u)|)}{t + (\max_{u \in I} |(f\omega)(u) - (fv)(u)|)} \quad (40) \\ \Rightarrow & \varphi(\sigma\omega, \sigma v, kt) \leq \varphi(f\omega, fv, t). \end{aligned}$$

Hence, all the conditions of theorem (4.1) are satisfied to obtain a continuous function $z: I \longrightarrow R$ such that $\sigma z = fz$. Then,

$$G(z(u), u) - f(z(u)) = 0, \quad (41)$$

where z will be a solution of the equation $F(z, u) = 0$. \square

Remark. If we consider an implicit form $F(\omega, u) = 10\omega^5(u-1) + u$, then, by the assumptions $G(\omega, u) = 10\omega^5(u-1) + u + 90\omega^5$ and $f(\omega(u)) = 90\omega^5$ in Theorem 4.3, we can easily obtain the explicit representation as $\omega = \sqrt[5]{5}u/10(1-u)$.

For a nontrivial example, consider the implicit equation,

$$u + \sin(8\omega^5 u) - \omega^5 = 0, \quad (42)$$

in the space $C([-1/9, t1/9])$. Let

$$\begin{aligned} F(\omega, u) &= u + \sin(8\omega^5 u) - \omega^5, \\ f(\omega) &= 5\omega^5 - 5, \end{aligned} \quad (43)$$

where $F: R \times ([-1/9, t1/9]) \longrightarrow R$ and $f: R \longrightarrow R$. Then, let $G(\omega, u) = u + \sin(8\omega^5 u) + 4\omega^5 - 5$. Define $\sigma: C([-1/9, t1/9]) \longrightarrow C([-1/9, t1/9])$ as

$$\sigma(\omega(u)) = G(\omega(u), u) = u + \sin(8\omega^5(u)u) + 4\omega^5(u) - 5. \quad (44)$$

Here, $f(\omega) = 5\omega^5 - 5$ implies that $f(R) = R$. Now,

$$\begin{aligned} |\sigma\omega_1 - \sigma\omega_2| &= |G(\omega_1, u) - G(\omega_2, u)| = |u + \sin(8\omega_1^5 u) + 4\omega_1^5 - 5 - u - \sin(8\omega_2^5 u) - 4\omega_2^5 + 5| \\ &\leq |\sin(8\omega_1^5 u) - \sin(8\omega_2^5 u) + 4\omega_1^5 - 4\omega_2^5| \\ &\leq |\sin(8\omega_1^5 u) - \sin(8\omega_2^5 u)| + 4|\omega_1^5 - \omega_2^5| \\ &\leq 8|u||\omega_1^5 - \omega_2^5| + 4|\omega_1^5 - \omega_2^5| \\ &\leq \frac{44}{45}|5\omega_1^5 - 5 - 5\omega_2^5 + 5|. \end{aligned} \quad (45)$$

Hence, all the conditions of Theorem 4.3 are satisfied. To apply Theorem 4.1, choose an initial guess $\omega_0(u) = 0$; then,

$$\sigma(\omega_0(u)) = G(\omega_0(u), u) = u - 5 = f(\omega_1(u)) = 5\omega_1^5 - 5. \quad (46)$$

This implies that $\omega_1(u) = \sqrt[5]{5}u/5$. So,

$$\begin{aligned}
\sigma(\omega_1(u)) &= G(\omega_1(u), u) = u + \sin(8\omega_1^5 u) + 4\omega_1^5 - 5 \\
&= u + \sin\left(8\frac{u^2}{5}\right) + 4\left(\frac{u}{5}\right) - 5, \\
f(\omega_2) &= u + \sin\left(8\frac{u^2}{5}\right) + 4\left(\frac{u}{5}\right) - 5, \\
5\omega_2^5(u) &= u + \sin\left(8\frac{u^2}{5}\right) + 4\left(\frac{u}{5}\right), \\
\Rightarrow \omega_2(u) &= \sqrt[5]{\frac{\sin(8u^2/5) + 9(u/5)}{5}}.
\end{aligned}
\tag{47}$$

Now,

$$\begin{aligned}
\sigma(\omega_2(u)) &= G(\omega_2(u), u) = u + \sin(8\omega_2^5 u) + 4\omega_2^5 - 5, \\
f(\omega_3) &= u + \sin 8\left(\frac{u \sin(8(u^2/5)) + 9(u^2/5)}{5}\right) + 4\left(\frac{\sin 8(u^2/5) + 9(u/5)}{5}\right) - 5, \\
\Rightarrow \omega_3 &= \sqrt[5]{\frac{u + \sin 8\left(\frac{u \sin(8(u^2/5)) + 9(u^2/5)}{5}\right) + 4\left(\frac{\sin 8(u^2/5) + 9(u/5)}{5}\right)}{5}},
\end{aligned}
\tag{48}$$

is an approximation of the explicit form of $F(\omega, u)$.

It is worthwhile to point out here that the application given in the above remark is not found in the literature even as an application of the following corollary of Theorem 4.1 regarding metric spaces.

Corollary. Let ζ be a nonempty set and (Y, ω) be a b -metric space and $f, \sigma: \zeta \rightarrow Y$ be mappings satisfying the following conditions:

- (1) $\sigma(\zeta) \subseteq f(\zeta)$.
- (2) There exist $k \in [0, 1)$ such that $\forall \omega, v \in \zeta$:

$$\omega(\sigma(\omega), \sigma(v)) \leq k\omega(f(\omega), f(v)). \tag{49}$$

If $f(\zeta)$ or $\sigma(\zeta)$ is complete, then there exist a point $z \in \zeta$ such that $f(z) = \sigma(z)$. Moreover, f and σ have a unique point of coincidence.

6. Conclusion

Metric spaces play a vital role in functional analysis and its related concepts. Modern and latest developments are due to fuzzy theory, fuzzy logic, and its vast applications in almost all fields of research. This motivated us to define metric-type spaces in fuzzy version. We called this IFbMSs. Moreover, interesting nontrivial examples are created as well. These spaces generalize fuzzy

b -metric spaces and IFMSs which are already generalized forms of classic metric spaces. Since fixed-point techniques have a lot of wonderful applications in science and technology, so in this research article, we intended to put our efforts in obtaining coincidence points and common fixed points in IFbMS. In such a way, many useful, present, and conventional results are presented as consequences of our results. Furthermore, as an application we have proved an implicit function theorem with the help of our main result. This activity will definitely motivate researchers to do further work in these spaces and fixed-point theory.

Abbreviations

FS: Fuzzy set
FMS: Fuzzy-metric-space
IFMS: Intuitionistic-fuzzy-metricspace
IFbMS: Intuitionistic-fuzzy-b-metric-space

Data Availability

The data used to support the finding of the study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this work.

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Research Article

A Self-Adaptive Extragradient Algorithm for Solving Quasimonotone Variational Inequalities

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This article aims to research iterative schemes for searching a solution of a quasimonotone variational inequality in a Hilbert space. For solving this quasimonotone variational inequality, we propose an iterative procedure which combines a self-adaptive rule and the extragradient algorithm. We demonstrate that the procedure weakly converges to the solution of the investigated quasimonotone variational inequality provided the considered operator satisfies several additional conditions.

1. Introduction

Variational inequality emerged in 1964 arising from the study of mechanics has many applications in engineering, economics, operations research, etc. ([1–4]). Variational inequality theory acts as a tool for solving many problems, such as equilibrium problems ([5, 6]), optimization problems ([7–9]), fixed point problems ([10–12]), and split problems ([13–17]). There are numerous iterative schemes for solving variational inequalities in the existing results; see [18–25]. Next, we briefly review several valuable iterative methods.

Throughout, suppose that \mathcal{H} is a Hilbert space. The symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the inner and norm of \mathcal{H} , respectively. Let $\mathcal{D} \subset \mathcal{H}$ be a convex and closed set. For an operator $\phi : \mathcal{D} \rightarrow \mathcal{H}$, the variational inequality aims to seek a point $v \in \mathcal{D}$ satisfying

$$\langle \phi(v), v - \hat{v} \rangle \leq 0, \forall \hat{v} \in \mathcal{D}. \quad (1)$$

We use $S(\phi, \mathcal{D})$ to indicate the set of solutions of (1).

A valuable algorithm for solving (1) is the projection algorithm ([26, 27]) which generates a procedure as follows:

$$s_0 \in \mathcal{D}, s_{n+1} = \text{proj}_{\mathcal{D}}(s_n - \rho\phi(s_n)), \text{ for all } n \geq 0, \quad (2)$$

where $\text{proj}_{\mathcal{D}} : \mathcal{H} \rightarrow \mathcal{D}$ stands for the orthogonal projection and $\rho > 0$ means the step-size.

When ϕ is strongly monotone, strongly pseudomonotone, or inverse strongly monotone, iterative scheme (2) is convergent ([28, 29]).

Another powerful method is extragradient method studied by Korpelevich [30] which generates a procedure starting from an initial point $s_0 \in \mathcal{D}$:

$$\begin{cases} t_n = \text{proj}_{\mathcal{D}}(s_n - \rho_n\phi(s_n)), \\ s_{n+1} = \text{proj}_{\mathcal{D}}(s_n - \rho_n\phi(t_n)), \text{ for all } n \geq 0. \end{cases} \quad (3)$$

Thereafter, (3) has been discussed extensively for solving (1); see, e.g., [30–34]. The main reason why the extragradient method attracts so much attention is that extragradient method can be used to find a solution of plain monotone operators. In fact, extragradient algorithm can be used to

solve (1) if ϕ is pseudomonotone and sequentially weakly continuous ([35–37]).

Very recently, iterative methods for solving quasimonotone variational inequality have been investigated in the literature [24, 38, 39]. Especially, Salahuddin [40] utilized (3) for solving a Lipschitz quasimonotone variational inequality and achieve the following result.

Conclusion 1 ([40]). Assume that the operator ϕ satisfies (i) quasimonotone on \mathcal{H} ; (ii) sequentially weakly continuous on \mathcal{D} ; and (iii) Lipschitz continuous on \mathcal{D} . Suppose that $S(\phi, \mathcal{D}) \neq \emptyset$ and $\rho_n \in [a, b] \subset (0, (1/L))$, $\forall n \geq 0$. Then, $\{s_n\}$ obtained from (3) weakly converges to $\hat{z} \in S(\phi, \mathcal{D})$.

In this article, we further utilize extragradient method (3) for solving quasimonotone variational inequality (1). For this task, we will make use of an auxiliary tool regarding the following dual variational inequality which is to find $v \in \mathcal{D}$ satisfying:

$$\langle \phi(\hat{v}), v - \hat{v} \rangle \leq 0, \forall v \in \mathcal{D}. \quad (4)$$

We use $S_d(\phi, \mathcal{D})$ to indicate the set of solutions of (4).

Notice that $S_d(\phi, \mathcal{D})$ is closed convex. At the same time, we have $S_d(\phi, \mathcal{D}) \subset S(\phi, \mathcal{D})$ when ϕ is continuous and \mathcal{D} is convex. However, to acquire the convergence of the constructed sequence, one has to add the following extra condition

$$S(\phi, \mathcal{D}) \subset S_d(\phi, \mathcal{D}), \quad (5)$$

which implies that

$$\langle \phi(z), z - z^\dagger \rangle \geq 0, \forall z^\dagger \in S(\phi, \mathcal{D}) \text{ and } z \in \mathcal{D}. \quad (6)$$

Note that the above condition (5) holds if ϕ is pseudomonotone. However, this condition (5) is not satisfied when ϕ is quasimonotone. Further, self-adaptive rule was applied for solving variational inequality problems, see [41–45]. In this paper, for solving quasimonotone variational inequality (1), we propose an iterative procedure which combines a self-adaptive method and extragradient method (3) without using condition (5). We show that the suggested iterative procedure is weakly convergent. Our result extends the above theorem (1) at two aspects: on the one hand “sequential weak continuity” imposed on ϕ can be replaced by a more general restriction and on the other hand a self-adaptive technique is used to relax Lipschitz condition of ϕ .

2. Notions and Lemmas

Throughout, suppose that \mathcal{H} is a Hilbert space and $\emptyset \neq \mathcal{D} \subset \mathcal{H}$ is convex and closed. A map $\phi : \mathcal{D} \rightarrow \mathcal{H}$ is called

(1) Monotone if

$$\langle \phi(q) - \phi(\hat{q}), q - \hat{q} \rangle \geq 0, \forall q, \hat{q} \in \mathcal{D}. \quad (7)$$

(2) Pseudomonotone if

$$\langle \phi(\hat{q}), q - \hat{q} \rangle \geq 0 \implies \langle \phi(q), q - \hat{q} \rangle \geq 0, \forall q, \hat{q} \in \mathcal{D}. \quad (8)$$

(3) Quasimonotone if

$$\langle \phi(\hat{q}), q - \hat{v} \rangle > 0 \implies \langle \phi(q), q - \hat{q} \rangle \geq 0, \forall q, \hat{q} \in \mathcal{D}. \quad (9)$$

By the above definition, we can deduce that if ϕ is pseudomonotone, then ϕ must be quasimonotone. However, the reverse conclusion may fail.

A map $\phi : \mathcal{D} \rightarrow \mathcal{H}$ is called Lipschitz continuous if

$$\|\phi(q) - \phi(\hat{q})\| \leq \tau \|q - \hat{q}\|, \forall q, \hat{q} \in \mathcal{D}, \quad (10)$$

where τ is some positive constant. In this case, we call ϕ τ -Lipschitz. ϕ is called nonexpansive provided $\tau = 1$.

An orthogonal projection from \mathcal{H} onto \mathcal{D} , denoted by $proj_{\mathcal{D}}$ fulfills

$$v \in \mathcal{H}, \|v - proj_{\mathcal{D}}(v)\| \leq \|\hat{v} - v\|, \forall \hat{v} \in \mathcal{D}. \quad (11)$$

$proj_{\mathcal{D}}$ possesses the following characteristic inequality:

$$v \in \mathcal{H}, \langle v - proj_{\mathcal{D}}(v), \hat{v} - proj_{\mathcal{D}}(v) \rangle \leq 0, \forall \hat{v} \in \mathcal{D}. \quad (12)$$

3. Algorithms and Convergence Results

First, we declare several related conditions. Suppose that \mathcal{H} is a Hilbert space and $\emptyset \neq \mathcal{D} \subset \mathcal{H}$ is convex and closed. Suppose that the involved operator ϕ satisfies three restrictions:

(t1) $\phi : \mathcal{H} \rightarrow \mathcal{H}$ is a quasimonotone operator

(t2) ϕ is μ -Lipschitz on \mathcal{D}

(t3) If $\lim_{n \rightarrow +\infty} \|\phi(s_n)\| = 0$ with $\{s_n\}$ being a sequence in \mathcal{H} and $s_n \rightharpoonup s^\ddagger$, then $\phi(s^\ddagger) = 0$

In the sequel, assume that $S_d(\phi, \mathcal{D}) \neq \emptyset$ and the set $\{t^\dagger \in \mathcal{D} : \phi(t^\dagger) = 0\} \setminus S_d(\phi, \mathcal{D})$ is finite.

Suppose that ω, ζ and $\hat{\zeta}$ are three constants in the open interval $(0, 1)$. Suppose that $\{\tau_n\}$ is a sequence in $(0, 2)$ satisfying $0 < \liminf_{n \rightarrow \infty} \tau_n \leq \limsup_{n \rightarrow \infty} \tau_n < 2$.

Next, we state our scheme for solving (1).

Algorithm 2. Select a fixed point u_0 in \mathcal{D} . Set $n = 0$.

Step 1. Assume u_n is presented. Compute

$$w_n = proj_{\mathcal{D}}[u_n - \omega \zeta_n \phi(u_n)], \quad (13)$$

where $\zeta_n = \max \{1, \zeta, \zeta^2, \dots\}$ fulfills

$$\zeta_n \|\phi(w_n) - \phi(u_n)\| \leq \frac{1 - \hat{\zeta}}{\omega} \|w_n - u_n\|. \quad (14)$$

Step 2. Compute $\widehat{w}_n = \text{proj}_{\mathcal{D}}[u_n - \phi(u_n)]$. (2a) If $\widehat{w}_n = u_n$, then stop. (2b) If $\widehat{w}_n \neq u_n$, then calculate

$$v_n = \frac{\widehat{v}_n}{\|\widehat{v}_n\|^2} \|w_n - u_n\|^2, \quad (15)$$

where $\widehat{v}_n = u_n - w_n + \omega\zeta_n\phi(w_n)$ and compute

$$u_{n+1} = \text{proj}_{\mathcal{D}}(u_n - \widehat{\zeta}\tau_n v_n). \quad (16)$$

Let $n := n + 1$ and return to Step 1.

Conclusion 3. Inequality (14) is well-defined. Moreover, $0 < (1 - \widehat{\zeta})\zeta/\omega\mu < \zeta_n \leq 1 - \widehat{\zeta}/\omega\mu$ or $\zeta_n = 1$.

Proof. Since ϕ is μ -Lipschitz, we obtain

$$\|\phi(\text{proj}_{\mathcal{D}}[u_n - \omega\zeta_n\phi(u_n)]) - \phi(u_n)\| \leq \mu\|\text{proj}_{\mathcal{D}}[u_n - \omega\zeta_n\phi(u_n)] - u_n\|, \quad (17)$$

which equals to

$$\begin{aligned} & \frac{1 - \widehat{\zeta}}{\mu\omega} \|\phi(\text{proj}_{\mathcal{D}}[u_n - \omega\zeta_n\phi(u_n)]) - \phi(u_n)\| \\ & \leq \frac{1 - \widehat{\zeta}}{\omega} \|\text{proj}_{\mathcal{D}}[u_n - \omega\zeta_n\phi(u_n)] - u_n\|. \end{aligned} \quad (18)$$

This implies that (14) holds for all $\zeta_n \leq 1 - \widehat{\zeta}/\mu\omega$. It is obviously that

$$\omega \frac{\zeta_n}{\zeta} \mu \|w_n - u_n\| \geq \omega \frac{\zeta_n}{\zeta} \|\phi(w_n) - \phi(u_n)\| > (1 - \widehat{\zeta}) \|w_n - u_n\|, \quad (19)$$

which implies that $\zeta_n > (1 - \widehat{\zeta})\zeta/\omega\mu > 0$. \square

Conclusion 4. (i) If $\widehat{w}_n = u_n$, then $u_n \in S(\phi, \mathcal{D})$. (ii) If $\widehat{w}_n \neq u_n$, then $\|\widehat{v}_n\| > 0$ and (15) is well-defined.

Proof.

(i) If $\widehat{w}_n = u_n$, that is $u_n = \text{proj}_{\mathcal{D}}[u_n - \phi(u_n)]$, by virtue of (12), we acquire

$$\langle u_n - [u_n - \phi(u_n)], v - u_n \rangle \geq 0, \forall v \in \mathcal{D}, \quad (20)$$

which results in that $u_n \in S(\phi, \mathcal{D})$.

(ii) Take $x^* \in S_d(\phi, \mathcal{D})$. Notice that

$$\begin{aligned} \langle \widehat{v}_n, u_n - x^* \rangle &= \langle u_n - w_n + \omega\zeta_n\phi(w_n), u_n - x^* \rangle \\ &= \langle u_n - w_n - \omega\zeta_n\phi(u_n), u_n - x^* \rangle \\ &\quad + \omega\zeta_n \langle \phi(u_n), u_n - x^* \rangle \\ &\quad + \omega\zeta_n \langle \phi(w_n), u_n - w_n \rangle \\ &\quad + \omega\zeta_n \langle \phi(w_n), w_n - x^* \rangle. \end{aligned} \quad (21)$$

As a result of $x^* \in S_d(\phi, \mathcal{D})$ and $u_n \in \mathcal{D}$, we have

$$\langle \phi(u_n), u_n - x^* \rangle \geq 0. \quad (22)$$

As the same as (22), we obtain

$$\langle \phi(w_n), w_n - x^* \rangle \geq 0, \quad (23)$$

due to $w_n \in \mathcal{D}$.

In view of (21)-(23), we receive

$$\begin{aligned} \langle \widehat{v}_n, u_n - x^* \rangle &\geq \langle u_n - w_n - \omega\zeta_n\phi(u_n), u_n - x^* \rangle \\ &\quad - \omega\zeta_n \langle \phi(w_n), w_n - u_n \rangle \\ &= \langle w_n - u_n + \omega\zeta_n(\phi(u_n) - \phi(w_n)), w_n - u_n \rangle \\ &\quad + \langle w_n - u_n + \omega\zeta_n\phi(u_n), x^* - w_n \rangle \\ &= \|w_n - u_n\|^2 - \omega\zeta_n \langle \phi(w_n) - \phi(u_n), w_n - u_n \rangle \\ &\quad + \langle w_n - u_n + \omega\zeta_n\phi(u_n), x^* - w_n \rangle. \end{aligned} \quad (24)$$

Owing to $\zeta_n > (1 - \widehat{\zeta})\zeta/\omega\mu > 0$, from (14), we get

$$\begin{aligned} \langle \phi(w_n) - \phi(u_n), w_n - u_n \rangle &\leq \|\phi(w_n) - \phi(u_n)\| \|w_n - u_n\| \\ &\leq \frac{1 - \widehat{\zeta}}{\omega\zeta_n} \|u_n - w_n\|^2. \end{aligned} \quad (25)$$

Using (12) of $\text{proj}_{\mathcal{D}}$ and (14), we acquire

$$\langle u_n - w_n - \omega\zeta_n\phi(u_n), w_n - x^* \rangle \geq 0. \quad (26)$$

In the light of (24), (25) and (26), we achieve

$$\langle \widehat{v}_n, u_n - x^* \rangle \geq \widehat{\zeta} \|w_n - u_n\|^2. \quad (27)$$

If $\widehat{w}_n \neq u_n$, then $w_n \neq u_n$. Otherwise, by virtue of (12), $\langle u_n - [u_n - \omega\zeta_n\phi(u_n)], v - u_n \rangle \geq 0, \forall v \in \mathcal{D}$ which results in that $u_n \in S(\phi, \mathcal{D})$ and hence $\widehat{w}_n = u_n$. This leads to a contradiction. So, $\|w_n - u_n\| > 0$. It follows from (27) that $\langle \widehat{v}_n, u_n - x^* \rangle > 0$ which yields that $\|\widehat{v}_n\| > 0$. Therefore, (15) is well-defined.

In this position, we prove a main theorem. \square

Theorem 5. $\{u_n\}$ defined by Algorithm 2 weakly converges to an element in $S(\phi, \mathcal{D})$.

Proof. Let $x^* \in S_d(\phi, \mathcal{D})$. Since $\text{proj}_{\mathcal{D}}$ is nonexpansive and $x^* \in \mathcal{D}$, from (16) and (21), we have

$$\begin{aligned} \|u_{n+1} - x^*\|^2 &= \|\text{proj}_{\mathcal{D}}(u_n - \widehat{\zeta}\tau_n v_n) - \text{proj}_{\mathcal{D}}(x^*)\|^2 \leq \|u_n - x^* - \widehat{\zeta}\tau_n v_n\|^2 \\ &= (\widehat{\zeta}\tau_n)^2 \|v_n\|^2 - 2\widehat{\zeta}\tau_n \langle v_n, u_n - x^* \rangle + \|u_n - x^*\|^2 \\ &= \|u_n - x^*\|^2 - 2\widehat{\zeta}\tau_n \frac{\|w_n - u_n\|^2}{\|\widehat{v}_n\|^2} \langle \widehat{v}_n, u_n - x^* \rangle \\ &\quad + (\widehat{\zeta}\tau_n)^2 \frac{\|w_n - u_n\|^4}{\|\widehat{v}_n\|^2}. \end{aligned} \quad (28)$$

Substituting (27) into (28) to derive

$$\|u_{n+1} - x^*\|^2 \leq \|u_n - x^*\|^2 - \widehat{\zeta}^2(2 - \tau_n)\tau_n \frac{\|w_n - u_n\|^4}{\|\widehat{v}_n\|^2}. \quad (29)$$

Noting that $0 < \underline{\lim}_{n \rightarrow \infty} \tau_n \leq \overline{\lim}_{n \rightarrow \infty} \tau_n < 2$, by (29), we acquire

$$\|u_{n+1} - x^*\| \leq \|u_n - x^*\|, \quad (30)$$

which leads to that $\lim_{n \rightarrow \infty} \|u_n - x^*\|$ exists. Then, $\{u_n\}$ is bounded and so is $\{\phi(u_n)\}$. From (13), we have

$$\|w_n - u_n\| = \|\text{proj}_{\mathcal{D}}[u_n - \omega\zeta_n \phi(u_n)] - \text{proj}_{\mathcal{D}}[u_n]\| \leq \omega\zeta_n \|\phi(u_n)\|. \quad (31)$$

Hence, $\{w_n\}$ and $\{\phi(w_n)\}$ are bounded. \square

Taking into account (29), we gain

$$\widehat{\zeta}^2(2 - \tau_n)\tau_n \frac{\|w_n - u_n\|^4}{\|\widehat{v}_n\|^2} \leq -\|u_{n+1} - x^*\|^2 + \|u_n - x^*\|^2. \quad (32)$$

This leads to

$$\frac{\|w_n - u_n\|^2}{\|\widehat{v}_n\|} \longrightarrow 0. \quad (33)$$

Thanks to the boundedness of $\widehat{v}_n = u_n - w_n + \omega\zeta_n \phi(w_n)$, by (33), we have

$$\lim_{n \rightarrow \infty} \|w_n - u_n\| = 0. \quad (34)$$

With the help of the Lipschitz continuity of ϕ , from (34), we deduce

$$\lim_{n \rightarrow \infty} \|\phi(w_n) - \phi(u_n)\| = 0. \quad (35)$$

Based on (15) and (16), we derive

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|\text{proj}_{\mathcal{D}}(u_n - \widehat{\zeta}\tau_n v_n) - \text{proj}_{\mathcal{D}}(u_n)\| \leq \widehat{\zeta}\tau_n \|v_n\| \\ &= \widehat{\zeta}\tau_n \frac{\|w_n - u_n\|^2}{\|\widehat{v}_n\|}. \end{aligned} \quad (36)$$

In the light of (33) and (36), we achieve

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0. \quad (37)$$

By (12) and (13), we deduce

$$\langle u_n - \omega\zeta_n \phi(u_n) - w_n, z - w_n \rangle \leq 0, \forall z \in \mathcal{D}. \quad (38)$$

So,

$$\frac{1}{\omega\zeta_n} \langle u_n - w_n, z - w_n \rangle + \langle \phi(u_n), w_n - u_n \rangle \leq \langle \phi(u_n), z - u_n \rangle, \forall z \in \mathcal{D}. \quad (39)$$

Observe that $\{u_n\}$, $\{w_n\}$, and $\{\phi(u_n)\}$ are bounded. According to (34) and (39), we have

$$\underline{\lim}_{n \rightarrow +\infty} \langle \phi(u_n), z - u_n \rangle \geq 0, \forall z \in \mathcal{D}. \quad (40)$$

Owing to $\{u_n\}$ is bounded, there is $\{u_{n_i}\} \subset \{u_n\}$ fulfilling $u_{n_i} \rightarrow b^\dagger \in \mathcal{D}$ as $i \rightarrow +\infty$. Taking into account (40), we attain

$$\underline{\lim}_{i \rightarrow +\infty} \langle \phi(u_{n_i}), z - u_{n_i} \rangle \geq 0, \forall z \in \mathcal{D}. \quad (41)$$

If $\underline{\lim}_{i \rightarrow +\infty} \|\phi(u_{n_i})\| = 0$, by $u_{n_i} \rightarrow b^\dagger$ and ϕ verifying (t1), we get that $\phi(b^\dagger) = 0$. Then, $b^\dagger \in S(\phi, \mathcal{D})$.

Now, we assume that $\underline{\lim}_{i \rightarrow +\infty} \|\phi(u_{n_i})\| > 0$. Then, there is an integer $m > 0$ fulfilling $\|\phi(u_{n_i})\| > 0$ for all $i \geq m$. By virtue of (41), we attain

$$\underline{\lim}_{i \rightarrow +\infty} \left\langle \frac{\phi(u_{n_i})}{\|\phi(u_{n_i})\|}, z - u_{n_i} \right\rangle \geq 0, \forall z \in \mathcal{D}. \quad (42)$$

Let $\{\varepsilon_k\}$ be a real number sequence fulfilling $\varepsilon_k > 0$, $\varepsilon_{k+1} < \varepsilon_k$ and $\varepsilon_k \rightarrow 0$ as $k \rightarrow +\infty$. Based on (42), there is $\{n_{i_k}\}$ of $\{n_i\}$ fulfilling $n_{i_k} \geq m(k \geq 0)$ and

$$\left\langle \frac{\phi(u_{n_{i_k}})}{\|\phi(u_{n_{i_k}})\|}, z - u_{n_{i_k}} \right\rangle + \varepsilon_k > 0, \forall k \geq 0, \forall z \in \mathcal{D}, \quad (43)$$

which results in that

$$\left\langle \phi(u_{n_{i_k}}), z - u_{n_{i_k}} \right\rangle + \varepsilon_k \|\phi(u_{n_{i_k}})\| > 0, \forall k \geq 0, \forall z \in \mathcal{D}. \quad (44)$$

Put $b_k = \phi(u_{n_{i_k}})/\|\phi(u_{n_{i_k}})\|^2$ for all $k \geq 0$. It is easily seen

that $\langle \phi(u_{n_k}), b_k \rangle = 1$ for all $k \geq 0$. With the help of (44), we achieve

$$\left\langle \phi(u_{n_k}), \varepsilon_k b_k \left\| \phi(u_{n_k}) \right\| + z - u_{n_k} \right\rangle > 0, \forall k \geq 0, \forall z \in \mathcal{D}. \quad (45)$$

Owing to (45) and using the quasimonotonicity of ϕ , we acquire

$$\left\langle \phi(z + \varepsilon_k b_k \left\| \phi(u_{n_k}) \right\|), \varepsilon_k b_k \left\| \phi(u_{n_k}) \right\| + z - u_{n_k} \right\rangle \geq 0, \forall k \geq 0, \forall z \in \mathcal{D}. \quad (46)$$

As a result of Lipschitz continuity of ϕ and $\lim_{k \rightarrow +\infty} \varepsilon_k \left\| \phi(u_{n_k}) \right\| = \lim_{k \rightarrow +\infty} \varepsilon_k = 0$, we deduce $\phi(z + \varepsilon_k b_k \left\| \phi(u_{n_k}) \right\|) \rightarrow \phi(z)$ as $k \rightarrow +\infty$. In (46), letting $k \rightarrow +\infty$, we receive

$$\left\langle \phi(z), z - b^\dagger \right\rangle \geq 0, \forall z \in \mathcal{D}. \quad (47)$$

Thus, $b^\dagger \in S_d(\phi, \mathcal{D})$. Therefore, $\omega_w(u_n) \subset (\{t^\dagger \in \mathcal{D} : \phi(t^\dagger) = 0\} \cup S_d(\phi, \mathcal{D})) \subset S(\phi, \mathcal{D})$. Next, we prove $\{u_n\}$ has no more than one weak cluster point in $S_d(\phi, \mathcal{D})$. Suppose that $b^\dagger \in S_d(\phi, \mathcal{D})$ and $c^\ddagger \in S_d(\phi, \mathcal{D})$ are two weak cluster points of $\{u_n\}$. Then, there exist two subsequences $\{u_{n_i}\} \subset \{u_n\}$ and $\{u_{n_j}\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup b^\dagger$ and $u_{n_j} \rightharpoonup c^\ddagger$.

It is obviously that

$$2\langle u_n, b^\dagger - c^\ddagger \rangle = \|u_n - c^\ddagger\|^2 - \|u_n - b^\dagger\|^2 + \|b^\dagger\|^2 - \|c^\ddagger\|^2, \forall n \geq 0. \quad (48)$$

Letting $n \rightarrow \infty$ on both sides of (48), we have that $\lim_{n \rightarrow +\infty} \langle u_n, b^\dagger - c^\ddagger \rangle$ exists, denoted by \hat{a} . Therefore,

$$\lim_{i \rightarrow +\infty} \langle u_{n_i}, b^\dagger - c^\ddagger \rangle = \hat{a} = \lim_{j \rightarrow +\infty} \langle u_{n_j}, b^\dagger - c^\ddagger \rangle \quad (49)$$

Note that $u_{n_i} \rightharpoonup b^\dagger$ and $u_{n_j} \rightharpoonup c^\ddagger$. By (45), we obtain

$$\langle b^\dagger, b^\dagger - c^\ddagger \rangle = \hat{a} = \langle b^\dagger - c^\ddagger, c^\ddagger \rangle, \quad (50)$$

which yields that $c^\ddagger = b^\dagger$. So, $\{u_n\}$ has no more than one weak cluster point in $S_d(\phi, \mathcal{D})$. Since the set $\{t^\dagger \in \mathcal{D} : \phi(t^\dagger) = 0\} \setminus S_d(\phi, \mathcal{D})$ is finite, we deduce that $\{u_n\}$ has only finite weak cluster points in $S(\phi, \mathcal{D})$. Let w_1, w_2, \dots, w_p be p unequal weak cluster points of $\{u_n\}$ in $S(\phi, \mathcal{D})$. Let $\Gamma = \{1, 2, \dots, p\}$ and

$$\alpha = \min \{ \|w_r - w_s\|/4, r, s \in \Gamma, r \neq s \}. \quad (51)$$

For $w_r, r \in \Gamma$, there is $\{u_{n_i}^r\}$ of $\{u_n\}$ fulfilling $u_{n_i}^r \rightharpoonup w_r$ when $i \rightarrow +\infty$. Hence,

$$\lim_{i \rightarrow +\infty} \left\langle u_{n_i}^r, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle = \left\langle w_r, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle, \forall s \in \Gamma. \quad (52)$$

For $\forall s \neq r$, we have

$$\begin{aligned} \langle w_r, (w_r - w_s)/\|w_r - w_s\| \rangle &= (\|w_r\|^2 - \|w_s\|^2)/(2\|w_r - w_s\|) \\ &+ \|w_r - w_s\|/2 > \alpha + (\|w_r\|^2 - \|w_s\|^2)/(2\|w_r - w_s\|). \end{aligned} \quad (53)$$

Thanks to (52) and (53), there exists an integer $N_i^r > 0$ satisfying for all $i \geq N_i^r$,

$$u_{n_i}^r \in \left\{ \hat{u} : \left\langle \hat{u}, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle > \alpha + \frac{\|w_r\|^2 - \|w_s\|^2}{2\|w_r - w_s\|} \right\}, s \in \Gamma, s \neq r. \quad (54)$$

Let

$$\Omega_r = \bigcap_{s=1, s \neq r}^p \left\{ \hat{u} : \left\langle \hat{u}, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle > \alpha + \frac{\|w_r\|^2 - \|w_s\|^2}{2\|w_r - w_s\|} \right\}. \quad (55)$$

Combining (54) with (55), we obtain $u_{n_i}^r \in \Omega_r, \forall i \geq \max \{N_i^r, r \in \Gamma\}$.

Next we demonstrate that if n is large enough, $u_n \in \cup_{r=1}^p \Omega_r$. Suppose that there is $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \notin \cup_{r=1}^p \Omega_r$. According to the boundedness of $\{u_{n_j}\}$, there exists a subsequence of $\{u_{n_j}\}$, without loss of generality, still denoted by $\{u_{n_j}\}$, which converges weakly to \hat{v} . Hence $u_{n_j} \notin \Omega_r$ for any $r \in \Gamma$. Then, there is a subsequence $\{u_{n_{j_s}}\}$ of $\{u_{n_j}\}$ such that $\forall s \geq 0$:

$$\begin{aligned} u_{n_{j_s}} &\notin \left\{ \hat{u} : \langle \hat{u}, (w_r - w_s)/\|w_r - w_s\| \rangle \right. \\ &\quad \left. > (\|w_r\|^2 - \|w_s\|^2)/(2\|w_r - w_s\|) + \alpha, s \in \Gamma, s \neq r \right\}. \end{aligned} \quad (56)$$

So,

$$\begin{aligned} \hat{v} &\notin \left\{ \hat{u} : \langle \hat{u}, (w_r - w_s)/\|w_r - w_s\| \rangle \right. \\ &\quad \left. > (\|w_r\|^2 - \|w_s\|^2)/(2\|w_r - w_s\|) + \alpha, s \in \Gamma, s \neq r \right\}, \end{aligned} \quad (57)$$

which results in that $\hat{v} \neq w_r (\forall r \in \Gamma)$. It leads to a contradiction. Thus, there is a large enough integer \tilde{N} such that $u_n \in \cup_{r=1}^p \Omega_r$ for all $n \geq \tilde{N}$.

Finally, we show that $\omega_w(u_n)$ is singleton in $S(\phi, \mathcal{D})$. Suppose that $p \geq 2$. Taking into account (37), there is $\tilde{N} \geq \tilde{N}$ fulfilling $\|u_{n+1} - u_n\| < \alpha$ when $n \geq \tilde{N}$. Hence, there is $m \geq \tilde{N}$ fulfilling $u_m \in \Omega_r$ and $u_{m+1} \in \Omega_s$, where $r, s \in \Gamma$ and $p \geq 2$, that is

$$\begin{aligned}
u_m \in \Omega_r &= \bigcap_{s=1, s \neq r}^p \{ \hat{u} : \langle \hat{u}, (w_r - w_s) / \|w_r - w_s\| \rangle \\
&> \alpha + (\|w_r\|^2 - \|w_s\|^2) / (2\|w_r - w_s\|) \} \\
u_{m+1} \in \Omega_s &= \bigcap_{r=1, r \neq s}^p \{ \hat{u} : \langle \hat{u}, (w_s - w_r) / \|w_s - w_r\| \rangle \\
&> \alpha + (\|w_s\|^2 - \|w_r\|^2) / (2\|w_s - w_r\|) \}.
\end{aligned} \tag{58}$$

Thus, we have

$$\langle u_m, (w_r - w_s) / \|w_r - w_s\| \rangle > \alpha + (\|w_r\|^2 - \|w_s\|^2) / (2\|w_r - w_s\|) \tag{59}$$

$$\langle u_{m+1}, (w_s - w_r) / \|w_s - w_r\| \rangle > \alpha + (\|w_s\|^2 - \|w_r\|^2) / (2\|w_s - w_r\|). \tag{60}$$

Thanks to (59) and (60), we acquire

$$\left\langle u_m - u_{m+1}, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle > 2\alpha. \tag{61}$$

Note that

$$\|u_{m+1} - u_n\| < \alpha. \tag{62}$$

By virtue of (61) and (62), we receive

$$2\alpha < \left\langle u_m - u_{m+1}, \frac{w_r - w_s}{\|w_r - w_s\|} \right\rangle \leq \|u_n - u_{m+1}\| < \alpha, \tag{63}$$

which is impossible. Then, $\omega_w(u_n)$ is singleton in $S(\phi, \mathcal{D})$. So, $\{u_n\}$ weakly converges to an element in $S(\phi, \mathcal{D})$.

Remark 6. A map $\phi : \mathcal{H} \longrightarrow \mathcal{H}$ is called weakly sequentially continuous, if $z_n \rightharpoonup \tilde{z} \Rightarrow \phi(z_n) \rightharpoonup \phi(\tilde{z})$, where $\{z_n\}$ is any sequence in \mathcal{H} .

To solve (1), many existing results have imposed the above “sequential weak-to-weak continuity” condition on ϕ ; see, [35, 36, 40]. We can check if ϕ satisfies sequential weak continuity and then ϕ satisfies condition (t3).

4. Conclusions

The main purpose of this paper is to investigate iterative algorithms for solving variational inequality (1). A powerful method to solve (1) is extragradient method (3) introduced by Korpelevich [30] where the involved operator ϕ is pseudomonotone monotone. Based on the corresponding result of Salahuddin [40], we further apply extragradient method (3) to solve quasimonotone variational inequality (1).

We propose an iterative algorithm (Algorithm 2) which combines a self-adaptive rule and the extragradient algorithm. In general, in order to show $\omega(u_n)$ belongs to the solution set, ϕ should be sequentially weakly continuous. In this paper, we replace these conditions by a weaker condition (t3). We demonstrate that the procedure weakly

converges to the solution of the investigated quasimonotone variational inequality under several additional conditions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

A New Iterative Algorithm for General Variational Inequality Problem with Applications

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This study aims at investigation of a generalized variational inequality problem. We initiate a new iterative algorithm and examine its convergence analysis. Using this newly proposed iterative method, we estimate the common solution of generalized variational inequality problem and fixed points of a nonexpansive mapping. A numerical example is illustrated to verify our existence result. Further, we demonstrate that the considered iterative algorithm converges with faster rate than normal S-iterative scheme. Furthermore, we apply our proposed iterative algorithm to estimate the solution of a convex minimization problem and a split feasibility problem.

1. Introduction

All through this study, we presume that \mathcal{H} is a real Hilbert space equipped with norm $\|\cdot\|$ induced by inner product $\langle \cdot, \cdot \rangle$. Let \mathcal{C} be a nonempty closed convex subset of \mathcal{H} and $f, g: \mathcal{C} \rightarrow \mathcal{H}$ be nonlinear mappings. The generalized nonlinear variational inequality is to locate a point $x^* \in \mathcal{H}$ such that

$$\langle f(x^*), g(x^*) - g(y^*) \rangle \geq 0, \quad \forall y^* \in \mathcal{C}, g(x^*), g(y^*) \in \mathcal{H}, \quad (1)$$

which was introduced by Noor [18]. We denote the set of solutions of (1) by $\text{Sol}(\mathcal{C}, f, g)$.

If $g = I$, then generalized nonlinear variational inequality (1) reduces to the classical variational inequality studied by Stampacchia [23], which is to allocate a point $x^* \in \mathcal{H}$, such that

$$\langle f(x^*), y^* - x^* \rangle \geq 0, \quad \forall y^* \in \mathcal{H}. \quad (2)$$

If $\mathcal{C}^* = \{x^* \in \mathcal{H}: \langle x^*, y^* \rangle \geq 0, \forall y^* \in \mathcal{C}\}$ is a dual cone of a convex cone \mathcal{C} , then generalized nonlinear variational inequality (1) coincides to generalized nonlinear complementarity problem which is to locate a point $x^* \in \mathcal{H}$ such that

$$\begin{aligned} \langle f(x^*), g(x^*) \rangle &= 0, \\ g(x^*) &\in \mathcal{C}, f(x^*) \in \mathcal{C}^*. \end{aligned} \quad (3)$$

It is worthy to adduce that variational inequalities which are unconventional and remarkable augmentation of variational principles provide well organized unified framework for figuring out a wide range of nonlinear problems arising in optimization, economics, physics, engineering science, operations research, and control theory, for example, [2, 8, 15, 20, 21, 24, 26, 33] and references cited therein.

Next, we recall the following definitions of a nonlinear mapping $f: \mathcal{C} \subset \mathcal{H} \rightarrow \mathcal{H}$.

Definition 1. The mapping $f: \mathcal{C} \longrightarrow \mathcal{H}$ is said to be

- (i) a -inverse strongly monotone or cocoercive if there exists a constant $a > 0$, such that

$$\langle f(x) - f(y), x - y \rangle \geq a \|f(x) - f(y)\|^2, \quad \forall x, y \in \mathcal{C}, \quad (4)$$

- (ii) L -Lipschitz continuous if there exists a constant $L > 0$, such that

$$\|f(x) - f(y)\| \leq L \|x - y\|, \quad \forall x, y \in \mathcal{C}. \quad (5)$$

For $L = 1$, f is nonexpansive, and if $0 < L < 1$, then f is a contraction. Note that a -inverse strongly monotone mapping is $1/a$ -Lipschitz continuous.

It is customary to mention that variational inequalities, variational inclusions, and related optimization problems can be posed as fixed-point problems. This unusual formulation plays a dominant role in studying variational inequalities and nonlinear problems by employing fixed-point iterative methods.

Lemma 1. Let $p_{\mathcal{C}}: \mathcal{H} \longrightarrow \mathcal{C}$ be a projection mapping of \mathcal{H} onto \mathcal{C} . For a given $q \in \mathcal{H}$, $p \in \mathcal{C}$ satisfies the inequality

$$\langle p - q, r - p \rangle \geq 0, \quad \forall r \in \mathcal{C} \text{ if and only if } p = p_{\mathcal{C}}(q). \quad (6)$$

Note that the projection mapping $p_{\mathcal{C}}$ is nonexpansive [16]. For more details on projection mapping $p_{\mathcal{C}}$, we refer to [12]. By utilizing Lemma 1, the generalized nonlinear variational inequality (1) can be designed as a fixed-point problem as follows:

Lemma 2 (see [17]). Let $p_{\mathcal{C}}: \mathcal{H} \longrightarrow \mathcal{C}$ be a projection mapping. For any $\rho > 0$, $x^* \in \mathcal{H}$, $g(x^*) \in \mathcal{C}$ solves the generalized nonlinear variational inequality (1) if and only if

$$g(x^*) = p_{\mathcal{C}}[g(x^*) - \rho f(x^*)]. \quad (7)$$

Relation (7) can be rescripted as

$$x^* = x^* - g(x^*) + p_{\mathcal{C}}[g - \rho f](x^*). \quad (8)$$

Let T be a nonexpansive mapping and $F(T)$ denotes the set of fixed points of T . If $x^* \in F(T) \cap \text{Sol}(\mathcal{H}, f, g)$, then

$$\begin{aligned} x^* &= T(x^*) = x^* - g(x^*) + p_{\mathcal{C}}[g - \rho f](x^*) \\ &= T\{x^* - g(x^*) + p_{\mathcal{C}}[g(x^*) - \rho f(x^*)]\}, \quad \rho > 0. \end{aligned} \quad (9)$$

It is significant to achieve better rate of convergence if two or more iterative algorithms converge to the same point for a given problem. We recall the following concepts which are versatile tools to find finer convergence rate for different iterative methods.

Definition 2 (see [3]). Let $\{p_n\}$ and $\{q_n\}$ be two real sequences converging to p and q , respectively. Suppose that $\lim_{n \rightarrow \infty} \|p_n - p\|/\|q_n - q\| = l$ exists. Then,

- (i) $\{p_n\}$ converges faster than $\{q_n\}$ if $l = 0$
(ii) $\{p_n\}$ and $\{q_n\}$ converges with identical rates if $0 < l < \infty$

Definition 3 (See [3]). Let $\{p_n\}$ and $\{q_n\}$ be two real sequences converging to the same fixed point t . If $\{u_n\}$ and $\{v_n\}$ are two sequences of positive real numbers converging to 0 such that $\|p_n - t\| \leq u_n$ and $\|q_n - t\| \leq v_n$ for all $n \in \mathbb{N}$. Then, $\{p_n\}$ converges to t faster than $\{q_n\}$ if $\{u_n\}$ converges faster than $\{v_n\}$.

Lemma 3 (see [4]). Let $\{\phi_n\}$ and $\{\psi_n\}$ be nonnegative sequences of real numbers satisfying

$$\phi_{n+1} \leq \sigma \phi_n + \psi_n, \quad \forall n \in \mathbb{N}, \quad (10)$$

where $\sigma \in (0, 1)$ and $\lim_{n \rightarrow \infty} \psi_n = 0$. Then $\lim_{n \rightarrow \infty} \phi_n = 0$.

Lemma 4 (see [31]). Let $\{\phi_n\}$, $\{\varphi_n\}$, and $\{\psi_n\}$ be nonnegative sequences of real numbers satisfying

$$\phi_{n+1} \leq (1 - \varphi_n)\phi_n + \psi_n, \quad \forall n \in \mathbb{N}, \quad (11)$$

where $\varphi_n \in (0, 1)$, $\sum_{n=1}^{\infty} \varphi_n = \infty$, and $\psi_n = o(\varphi_n)$. Then, $\lim_{n \rightarrow \infty} \phi_n = 0$.

Mann, Ishikawa, and Halpern iterative methods are fundamental tools for solving fixed-point problems of nonexpansive mappings. In recent past, a number of fixed point iterative methods have been constructed and implemented to solve various classes of nonlinear mappings [2, 9, 10, 19, 22, 25, 28–30, 34]. Agarwal and others [1] introduced the S-iteration method which converges faster than some well-known iterative algorithms such as Mann, Ishikawa, and Picard for contraction as well as nonexpansive mappings. Due to the super convergence rate, it attracted number of researchers to study fixed-point problems, minimization problems, variational inclusions, variational inequalities, and alternate points problems in different settings. In [18], Noor utilized formulation (9) to propose following iterative algorithm:

$$\begin{cases} u_0 \in \mathcal{C}, \\ u_{n+1} = (1 - a_n)u_n + a_n T\{u_n - g(u_n) + p_{\mathcal{C}}[g(u_n) - \rho f(u_n)]\}, \end{cases} \quad (12)$$

where $\{a_n\}$ is a sequence in $[0, 1]$. The author proved strong convergence of the proposed iterative algorithm. Furthermore, it is customary that the normal S-iterative algorithm converges faster than the Mann and Picard iterative algorithm. Owing to its uncomplicated nature and faster convergence rate, Gursay and others [14] investigated the following normal S-iterative algorithm to examine (1) as follows:

$$\begin{cases} p_0 \in \mathcal{C}, \\ p_{n+1} = T\{q_n - g(q_n) + p_{\mathcal{C}}[g(q_n) - \rho f(q_n)]\}, \\ q_n = (1 - \xi_n)p_n + \xi_n T\{p_n - g(p_n) + p_{\mathcal{C}}[g(p_n) - \rho f(p_n)]\}, \xi_n \in [0, 1]. \end{cases} \quad (13)$$

Recently, Ullah and Arshad [27] introduced a more efficient iterative algorithm called the M -iterative method for Suzuki's generalized nonexpansive mappings as follows:

$$\begin{cases} u_1 \in \mathcal{C}, \\ w_n = (1 - a_n)u_n + a_n T u_n, \\ v_n = T w_n, \\ u_{n+1} = T v_n, \end{cases} \quad (14)$$

where $\{a_n\}$ is a sequence in $(0, 1)$. They analyzed convergence and showed that their iterative procedure converges faster than the Picard S [13] and S-iteration process [1]. In recent work, Garodia and Uddin [11] developed a new iterative algorithm for Suzuki's generalized nonexpansive mappings as follows:

$$\begin{cases} u_1 \in \mathcal{C}, \\ w_n = T u_n, \\ v_n = T((1 - a_n)w_n + a_n T w_n), \\ u_{n+1} = T v_n, \end{cases} \quad (15)$$

where $\{a_n\}$ is a sequence in $(0, 1)$. The authors approximated fixed-points and inspected the convergence. Also, they proved that the posed iterative method converges with faster rate than that of the M -iterative method.

Stimulated by the work discussed in above-mentioned references, in this study, we investigate algorithm (15) to estimate the common solution of fixed points of a non-expansive mapping T and the generalized nonlinear variational inequality (1) as follows:

$$\begin{cases} u_1 \in \mathcal{C}, \\ w_n = T\{u_n - g(u_n) + p_{\mathcal{C}}[g(u_n) - \rho f(u_n)]\}, \\ r_n = (1 - a_n)w_n + a_n T\{w_n - g(w_n) + p_{\mathcal{C}}[g(w_n) - \rho f(w_n)]\}, \\ v_n = T\{r_n - g(r_n) + p_{\mathcal{C}}[g(r_n) - \rho f(r_n)]\}, \\ u_{n+1} = T\{v_n - g(v_n) + p_{\mathcal{C}}[g(v_n) - \rho f(v_n)]\}, \end{cases} \quad (16)$$

where $\{a_n\}$ is a sequence in $(0, 1)$ satisfying certain assumptions. We analyze strong convergence of our proposed iterative algorithm (16) under some mild assumptions. We also pose a modified form of our iterative algorithm (16) to investigate convex optimization and split feasibility problems. Theoretical findings are validated by an illustrative numerical example. Our existence and convergence results can be seen as generalizations and prevalent of some known results.

2. Convergence Results

Theorem 1. Let $f, g: \mathcal{C} \longrightarrow \mathcal{H}$ be a_1, a_2 -inverse strongly monotone mappings, respectively, and $T: \mathcal{H} \longrightarrow \mathcal{C}$ be a nonexpansive mapping such that $F(T) \cap \text{Sol}(\mathcal{C}, f, g) \neq \emptyset$. Suppose that the assumption

$$|\rho - a_1| < a_1(1 - Y), \quad (17)$$

holds, where $Y = 2|a_2 - 1/a_2|$. Then, the iterative sequence $\{u_n\}$ defined by (16) converges strongly to $u^* \in F(T) \cap \text{Sol}(\mathcal{C}, f, g)$ with the following error estimates:

$$\|u_{n+1} - u^*\| \leq \zeta^{3(n+1)} \|u_0 - u^*\| \prod_{k=0}^n [1 - a_k(1 - \zeta)], \quad \forall n \in \mathbb{N}, \quad (18)$$

where

$$\zeta = 2 \left| \frac{a_2 - 1}{a_2} \right| + \left| \frac{a_1 - \rho}{a_1} \right|. \quad (19)$$

Proof. Note that

$$u^* = T\{u^* - g(u^*) + p_{\mathcal{C}}[g(u^*) - \rho f(u^*)]\}. \quad (20)$$

Since f being a_1 -inverse strongly monotone is $1/a_1$ -Lipschitz continuous mapping, T and $p_{\mathcal{C}}$ are the nonexpansive mappings. Then, from (16 and 20), we obtain

$$\begin{aligned} \|w_n - u^*\| &= \|T\{u_n - g(u_n) + p_{\mathcal{C}}[g(u_n) - \rho f(u_n)]\} \\ &\quad - T\{u^* - g(u^*) + p_{\mathcal{C}}[g(u^*) - \rho f(u^*)]\}\| \\ &\leq 2\|u_n - u^* - (g(u_n) - g(u^*))\| \\ &\quad + \|u_n - u^* - \rho(f(u_n) - f(u^*))\|. \end{aligned} \quad (21)$$

Since f is a_1 -inverse strongly monotone mapping, then we have

$$\begin{aligned} &\|u_n - u^* - \rho(f(u_n) - f(u^*))\|^2 \\ &= \|u_n - u^*\|^2 + \rho^2 \|f(u_n) - f(u^*)\|^2 \\ &\quad - 2\rho \langle u_n - u^*, f(u_n) - f(u^*) \rangle \\ &\leq \|u_n - u^*\|^2 + \frac{\rho^2}{a_1^2} \|u_n - u^*\|^2 - 2\rho a_1 \|f(u_n) - f(u^*)\|^2 \\ &\leq \left(\frac{a_1 - \rho}{a_1} \right)^2 \|u_n - u^*\|^2. \end{aligned} \quad (22)$$

Also, g is a_2 -inverse strongly monotone mapping; then we have

$$\begin{aligned} &\|u_n - u^* - (g(u_n) - g(u^*))\|^2 \\ &= \|u_n - u^*\|^2 + \|g(u_n) - g(u^*)\|^2 \\ &\quad - 2\langle u_n - u^*, g(u_n) - g(u^*) \rangle \\ &\leq \|u_n - u^*\|^2 + \frac{1}{a_2^2} \|u_n - u^*\|^2 - 2a_2 \|g(u_n) - g(u^*)\|^2 \\ &\leq \left(\frac{a_2 - 1}{a_2} \right)^2 \|u_n - u^*\|^2. \end{aligned} \quad (23)$$

Thus, from (21) to (23), we have

$$\|w_n - u^*\| \leq \left(2 \left| \frac{a_2 - 1}{a_2} \right| + \left| \frac{a_1 - \rho}{a_1} \right| \right) \|u_n - u^*\| = \zeta \|u_n - u^*\|, \quad (24)$$

where ζ is defined by (19), and from (17), we have $\zeta < 1$. Again, following the same steps (21)–(24) and from (16), we obtain

$$\|v_n - u^*\| \leq \zeta \|r_n - u^*\|. \quad (25)$$

Next, we estimate

$$\begin{aligned} \|r_n - u^*\| &= \|(1 - a_n)w_n + a_n T\{w_n - g(w_n) \\ &\quad + p_{\mathcal{E}}[g(w_n) - \rho f(w_n)]\} - u^*\| \\ &\leq (1 - a_n)\|w_n - u^*\| + a_n \zeta \|w_n - u^*\| \\ &= 1 - a_n(1 - \zeta)\|w_n - u^*\|, \end{aligned} \quad (26)$$

which amounts to say

$$\begin{aligned} \|v_n - u^*\| &\leq \zeta [1 - a_n(1 - \zeta)]\|w_n - u^*\|, \\ \|u_{n+1} - u^*\| &= \|T\{v_n - g(v_n) + p_{\mathcal{E}}[g(v_n) - \rho f(v_n)]\} \\ &\quad - T\{u^* - g(u^*) + p_{\mathcal{E}}[g(u^*) - \rho f(u^*)]\}\| \\ &\leq \zeta \|v_n - u^*\| \leq \zeta^3 [1 - a_n(1 - \zeta)]\|u_n - u^*\|. \end{aligned} \quad (27)$$

Since, $1 - a_n(1 - \zeta) < 1$. Therefore, we get $\|u_{n+1} - u^*\| \leq \zeta^3 \|u_n - u^*\|$, $\forall n \in \mathbb{N}$. By repeating the process in this fashion, we obtain

$$\|u_{n+1} - u^*\| \leq \zeta^{3(n+1)} \|u_0 - u^*\|, \quad \forall n \in \mathbb{N}, \quad (28)$$

which gives that $\lim_{n \rightarrow \infty} \|u_n - u^*\| = 0$. \square

Now, we exemplify the existence of solution.

Example 1. Let $\mathcal{H} = \mathbb{R}$, $\mathcal{E} = [1, 2]$ be equipped with norm $\|u\| = |u|$ and inner product $\langle u, v \rangle = u \cdot v$. Let $f, g, T: [1, 2] \rightarrow \mathbb{R}$ be defined by

$$f(u) = u^2, g(u) = \frac{u^3}{4} + \frac{3}{4}, T(u) = \frac{u^2 + u^3}{16} + \frac{7}{8}. \quad (29)$$

Then, for all $u, v \in \mathcal{E}$, observe that

$$\begin{aligned} \langle f(u) - f(v), u - v \rangle &= (u - v)^2(u + v) \geq 2|u - v|^2, \\ \langle g(u) - g(v), u - v \rangle &= \frac{1}{4}(u - v)^2(u^2 + uv + v^2) \geq \frac{3}{4}|u - v|^2, \\ |T(u) - T(v)| &= \frac{1}{16}|u - v|u^2 + uv + v^2 + u + v| \leq |u - v|. \end{aligned} \quad (30)$$

Then, f and g are 2 and 3/4-inverse strongly monotone mapping, respectively, and T is nonexpansive mapping. One

can easily verify that $u^* = 1 \in \mathcal{E}$ is the unique fixed point of T . Also,

$$\langle f(u^*), g(v) - g(u^*) \rangle = \frac{v^3 - 1}{4} \geq 0, \quad \text{for all } v \in \mathcal{E}. \quad (31)$$

Thus, we have $u^* = 1 \in F(T) \cap \text{Sol}(\mathcal{E}, f, g)$.

Theorem 2. Let \mathcal{H} be a real Hilbert space and \mathcal{E} be a nonempty closed convex subset of \mathcal{H} . Let f, g, T , and ζ be same as defined in Theorem 1. Let $\{p_n\}$ and $\{u_n\}$ be the sequences defined by (13) and (16), respectively. Suppose that (17) holds and $F(T) \cap (\mathcal{E}, f, g) \neq \emptyset$. Then, the following statements hold:

(i) If $\{(1 + \zeta^3)/\xi_n\}$ is bounded and $\sum_{n=0}^{\infty} a_n = \infty$, then the sequence $\{u_n - p_n\}$ converges strongly to 0 with following error estimates:

$$\begin{aligned} \|u_{n+1} - p_{n+1}\| &\leq [1 - \xi_n(1 - \zeta)]\|u_n - p_n\| \\ &\quad + (1 + \zeta^3)\|u_n - u^*\|, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (32)$$

$\{p_n\}$ converges strongly to $u^* \in F(T) \cap \text{Sol}(\mathcal{E}, f, g)$.

(ii) If $\{p_n\}$ converges strongly to $u^* \in F(T) \cap \text{Sol}(\mathcal{E}, f, g)$, then $\{p_n - u_n\}$ converges strongly to 0 with following error estimates:

$$\begin{aligned} \|p_{n+1} - u_{n+1}\| &\leq \zeta^3 \|p_n - u_n\| \\ &\quad + (1 + \zeta^3)\|p_n - u^*\|, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (33)$$

Proof

(i) It follows from Theorem 1 that $\lim_{n \rightarrow \infty} \|u_n - u^*\| = 0$. Next, we prove that $\lim_{n \rightarrow \infty} \|p_n - u^*\| = 0$. Following (13) and (16) and the steps as in (21)–(24), we obtain

$$\begin{aligned} \|u_{n+1} - p_{n+1}\| &= \|T\{v_n - g(v_n) + p_{\mathcal{E}}[g(v_n) - \rho f(v_n)]\} \\ &\quad - T\{q_n - g(q_n) + p_{\mathcal{E}}[g(q_n) - \rho f(q_n)]\}\| \\ &\leq \zeta \|v_n - q_n\|, \end{aligned} \quad (34)$$

where ζ is same as in (19). Again, utilizing (13), (16), and (34), we have

$$\begin{aligned}
\|u_{n+1} - p_{n+1}\| &\leq \zeta \|v_n - (1 - \xi_n)p_n - \xi_n T\{p_n - g(p_n) + p_{\mathcal{C}}[g(p_n) - \rho f(p_n)]\}\| \\
&\leq \zeta \|v_n - u^*\| + (1 - \xi_n)\zeta \|p_n - u^*\| + \xi_n \zeta \|T\{p_n - g(p_n) \\
&\quad + p_{\mathcal{C}}[g(p_n) - \rho f(p_n)]\} \\
&\quad - T\{u^* - g(u^*) + p_{\mathcal{C}}[g(u^*) - \rho f(u^*)]\}\| \\
&\leq \zeta \|v_n - u^*\| + (1 - \xi_n)\zeta \|p_n - u^*\| + \xi_n \zeta^2 \|p_n - u^*\| \\
&= \zeta [\|v_n - u^*\| + 1 - \xi_n(1 - \zeta)\|p_n - u^*\|] \\
&\leq \zeta [\zeta^2(1 - a_n(1 - \zeta))\|u_n - u^*\| + 1 - \xi_n(1 - \zeta)\|p_n - u^*\|] \\
&= \zeta [\zeta^2(1 - a_n(1 - \zeta))\|u_n - u^*\| + 1 - \xi_n(1 - \zeta)\|u_n - u^*\| \\
&\quad + 1 - \xi_n(1 - \zeta)\|u_n - p_n\|] \\
&\leq \zeta [1 - \xi_n(1 - \zeta)]\|u_n - p_n\| + (1 + \zeta^3)\max\{1 - a_n(1 - \zeta), \\
&\quad 1 - \xi_n(1 - \zeta)\}\|u_n - u^*\| \\
&\leq [1 - \xi_n(1 - \zeta)]\|u_n - p_n\| + (1 + \zeta^3)\|u_n - u^*\|.
\end{aligned} \tag{35}$$

Let $\phi_n = \|u_n - p_n\|$, $\varphi_n = \xi_n(1 - \zeta)$, $\psi_n = (1 + \zeta^3)\|u_n - u^*\|$, and $\delta_n = \|u_n - u^*\|$, $\forall n \in \mathbb{N}$. It follows from assumption of the theorem that $\{(1 + \zeta^3)/\xi_n\}$ is bounded; therefore, $\{(1 + \zeta^3)/\xi_n(1 - \zeta)\}$ is also bounded. Then, there exists a constant $M > 0$, such that $|(1 + \zeta^3)/\xi_n(1 - \zeta)| < M$, $\forall n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\{(1 + \zeta^3)/\xi_n(1 - \zeta)\}$ is bounded, therefore, $\{(1 + \zeta^3)/\xi_n(1 - \zeta)\delta_n\} \rightarrow 0$ as $n \rightarrow \infty$, i.e., $\lim_{n \rightarrow \infty} (\psi_n/\varphi_n) = 0$, which amounts to say that

$\psi_n = o(\varphi_n)$. Thus, all the assumptions of Lemma 4 are fulfilled. Hence, $\lim_{n \rightarrow \infty} \|u_n - p_n\| = 0$ and $\|p_n - u^*\| \leq \|u_n - p_n\| + \|u_n - u^*\|$. Thus, we have $\lim_{n \rightarrow \infty} \|p_n - u^*\| = 0$.

(ii) Next, we estimate that $\{p_n - u_n\} \rightarrow 0$. Since $\{p_n\}$ converges to $u^* \in F(T) \cap \text{Sol}(\mathcal{C}, f, g)$, then following the same arguments as in (34) and (35), we obtain

$$\begin{aligned}
\|p_{n+1} - u_{n+1}\| &\leq \zeta [1 - \xi_n(1 - \zeta)]\|p_n - u^*\| + \zeta [\zeta^2(1 - a_n(1 - \zeta))\|u_n - u^*\| \\
&\leq \zeta^3 [1 - a_n(1 - \zeta)]\|p_n - u_n\| + \zeta^3 [1 - a_n(1 - \zeta)]\|p_n - u^*\| \\
&\quad + \zeta [1 - \xi_n(1 - \zeta)]\|p_n - u^*\| \\
&\leq \zeta^3 [1 - a_n(1 - \zeta)]\|p_n - u_n\| + \zeta^3 \|p_n - u^*\| + \|p_n - u^*\| \\
&\leq \zeta^3 \|p_n - u_n\| + (1 + \zeta^3)\|p_n - u^*\|.
\end{aligned} \tag{36}$$

Let $\phi'_n = \|p_n - u_n\|$, $\psi'_n = (1 + \zeta^3)\|p_n - u^*\|$, $\forall n \in \mathbb{N}$. By the assumption $\{p_n\}$ converges to u^* and utilizing the fact that $(1 + \zeta^3)$ is bounded, we obtain that $\psi'_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, all the assumptions of Lemma 3 are fulfilled. Hence, $\lim_{n \rightarrow \infty} \|p_n - u_n\| = 0$. Also, we know that $\|u_n - u^*\| \leq \|p_n - u_n\| + \|p_n - u^*\|$, $\forall n \in \mathbb{N}$. Thus, $\lim_{n \rightarrow \infty} \|u_n - u^*\| = 0$. Hence, $\{p_n - u_n\} \rightarrow 0$ as $n \rightarrow \infty$. \square

Theorem 3. Let \mathcal{H} be a real Hilbert space and \mathcal{C} be a closed convex subset of \mathcal{H} . Suppose f, g, T , and ζ are identical as in Theorem 1. Let $\{p_n\}$ and $\{u_n\}$ be sequences defined by (13) and (16), respectively. Suppose that assumption (17) holds and $F(T) \cap \text{Sol}(\mathcal{C}, f, g) \neq \emptyset$. If $p_0 = u_0$, then $\{u_n\}$ converges faster than $\{p_n\}$ to u^* , such that $u^* \in F(T) \cap \text{Sol}(\mathcal{C}, f, g)$.

Proof. It follows from (27) that

$$\|u_{n+1} - u^*\| \leq \zeta^3 [1 - a_n(1 - \zeta)]\|u_n - u^*\|. \tag{37}$$

Since $\{a_n\}$ is a sequence in $(0, 1)$, we can choose a constant $a \in \mathbb{R}$, such that $0 < a \leq a_n < 1$, $\forall n \in \mathbb{N}$. Then,

$$\|u_{n+1} - u^*\| \leq \zeta^3 [1 - a(1 - \zeta)]\|u_n - u^*\|. \tag{38}$$

By repeating the process, we obtain

$$\|u_{n+1} - u^*\| \leq \zeta^{3(n+1)} [1 - a(1 - \zeta)]^{n+1} \|u_0 - u^*\|, \quad \forall n \in \mathbb{N}. \tag{39}$$

Also, it follows from (13) that

$$\begin{aligned}
\|p_{n+1} - u^*\| &= \|T\{q_n - g(q_n) + p_{\mathcal{C}}[g(q_n) - \rho f(q_n)]\} - T\{u^* - g(u^*) + p_{\mathcal{C}}[g(u^*) - \rho f(u^*)]\}\| \\
&\leq \|q_n - u^* - (g(q_n) - g(u^*))\| + \|g(q_n) - g(u^*) - \rho(f(q_n) - f(u^*))\| \\
&\leq 2\|q_n - u^* - (g(q_n) - g(u^*))\| + \|q_n - u^* - \rho(f(q_n) - f(u^*))\|.
\end{aligned} \tag{40}$$

By following the arguments as discussed from (21) to (24), we have

$$\|p_{n+1} - u^*\| \leq \zeta \|q_n - u^*\|. \tag{41}$$

Also,

$$\begin{aligned}
\|q_n - u^*\| &= \|(1 - \xi_n)p_n + \xi_n T\{p_n - g(p_n) + p_{\mathcal{C}}[g(p_n) - \rho f(p_n)]\} - u^*\| \\
&\leq (1 - \xi_n)\|p_n - u^*\| + 2\xi_n\|p_n - u^* - (g(p_n) - g(u^*))\| \\
&\quad + \xi_n\|p_n - u^* - \rho(f(p_n) - f(u^*))\| \\
&\leq (1 - \xi_n)\|p_n - u^*\| + \xi_n\zeta\|p_n - u^*\| \\
&= [1 - \xi_n(1 - \zeta)]\|p_n - u^*\|.
\end{aligned} \tag{42}$$

By combining (41) and (42), we get

$$\|p_{n+1} - u^*\| \leq \zeta [1 - \xi_n(1 - \zeta)]\|p_n - u^*\|. \tag{43}$$

Since $\{\xi_n\}$ is a sequence in $[0, 1]$, we can choose a constant $\xi \in \mathbb{R}$, such that $0 < \xi \leq \xi_n < 1, \forall n \in \mathbb{N}$. Then,

$$\|p_{n+1} - u^*\| \leq \zeta [1 - \xi(1 - \zeta)]\|p_n - u^*\|. \tag{44}$$

Thus, by repeating the process, we obtain

$$\|p_{n+1} - u^*\| \leq \zeta^{(n+1)} [1 - \xi(1 - \zeta)]^{n+1} \|p_0 - u^*\|, \quad \forall n \in \mathbb{N}, \tag{45}$$

Set $a_n = \zeta^{3(n+1)} [1 - a(1 - \zeta)]^{n+1} \|u_0 - u^*\|$ and $b_n = \zeta^{(n+1)} [1 - \xi(1 - \zeta)]^{n+1} \|p_0 - u^*\|$; then,

$$A_n = \frac{a_n}{b_n} = \frac{\zeta^{3(n+1)} [1 - a(1 - \zeta)]^{n+1} \|u_0 - u^*\|}{\zeta^{(n+1)} [1 - \xi(1 - \zeta)]^{n+1} \|p_0 - u^*\|} \tag{46}$$

$$\longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence, $\{u_n\}$ converges faster than $\{p_n\}$. \square

3. Applications

3.1. Convex Minimization Problem. Now, we solve convex minimization problem as an application of Theorem 1.

Let \mathcal{C} be a closed convex subset of a real Hilbert space \mathcal{H} , $p_{\mathcal{C}}: \mathcal{H} \rightarrow \mathcal{C}$ be a projection, and $F: \mathcal{C} \rightarrow \mathbb{R}$ be a convex, Frechet differentiable mapping. We consider the following convex minimization problem:

$$\min_{u^* \in \mathcal{C}} F(u^*). \tag{47}$$

Clearly, $u^* \in \mathcal{C}$ is a solution of $p_{\mathcal{C}}(I - \rho \nabla F)$ if and only if

$$\langle \nabla F(u^*), u - u^* \rangle \geq 0, \quad \forall u \in \mathcal{C}. \tag{48}$$

More precisely, $u^* \in \mathcal{C}$ solves problem (47) if and only if u^* is a fixed point of the projection mapping $p_{\mathcal{C}}(I - \rho \nabla F)$, i.e.,

$$u^* = p_{\mathcal{C}}[u^* - \rho \nabla F(u^*)], \tag{49}$$

where ∇F is the gradient of mapping F . This formulation is known as gradient projection, which plays a key role in solving problem (47). So far, several iterative methods have been employed to solve minimization problems [7, 26, 32]. By considering $f := \nabla F$ and assuming $T = g = I$, the identity mapping, we propose the following modified gradient projection algorithm for solving $p_{\mathcal{C}}(I - \rho \nabla F)$ as follows:

$$\begin{cases} u_1 \in \mathcal{C}, \\ w_n = p_{\mathcal{C}}[u_n - \rho \nabla F(u_n)], \\ r_n = (1 - a_n)w_n + a_n p_{\mathcal{C}}[w_n - \rho \nabla F(w_n)], \\ v_n = p_{\mathcal{C}}[r_n - \rho \nabla F(r_n)], \\ u_{n+1} = p_{\mathcal{C}}[v_n - \rho \nabla F(v_n)], \end{cases} \tag{50}$$

where $\{a_n\}$ is a sequence in $(0, 1)$. Now, we approximate the proposed algorithm (50) to estimate the solution of (47).

Theorem 4. Let \mathcal{C} be a nonempty closed convex subset of real Hilbert space \mathcal{H} . Let $F: \mathcal{C} \rightarrow \mathbb{R}$ be a convex, Frechet differentiable mapping, and ∇F is α -inverse strongly monotone mapping. Suppose that the convex minimization problem (47) has a solution and condition (17) holds. Then, the sequence $\{u_n\}$ generated by (50) converges strongly to u^* which solves convex minimization problem (47) with the following error estimates:

$$\|u_{n+1} - u^*\| \leq \zeta^{3(n+1)} \prod_{k=0}^n [1 - a_k(1 - \zeta)] \|u_0 - u^*\|, \quad \forall n \in \mathbb{N}, \quad (51)$$

where

$$\zeta = \left| \frac{a_1 - \rho}{a_1} \right|. \quad (52)$$

Proof. The desired conclusion is accomplished by taking $f = \nabla F$ and $T, g = I$ in Theorem 1. \square

Example 2. Let $\mathcal{H} = L^2[0, 1] = \left\{ \mathcal{G}: [0, 1] \rightarrow \mathbb{R}: \int_0^1 \mathcal{G}^2(u) du < \infty \right\}$. Then, $(\mathcal{H}, \|\cdot\|_2)$ is a Hilbert space given by

$$\|\mathcal{G}(u)\|_2^2 = \langle \mathcal{G}(u), \mathcal{G}(u) \rangle = \int_0^1 \mathcal{G}^2(u) du. \quad (53)$$

Consider a closed convex subset $\mathcal{C} = \{\mathcal{G} \in L^2[0, 1]: \|\mathcal{G}(u)\|_2^2 \leq 1\}$ of \mathcal{H} . Define $F: \mathcal{C} \rightarrow \mathbb{R}$ by $F(\mathcal{G}) = \|\mathcal{G}(u)\|_2^2$. Then, $\mathcal{G}(u) = 0$ is a unique minimum of a convex function f , and f is the Frechet differentiable at \mathcal{G} . The gradient $\nabla F: \mathcal{C} \rightarrow \mathcal{H}$ is evaluated as $\nabla F(\mathcal{G}) = 2\mathcal{G}$. Then, for all $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{C}$, we get

$$\begin{aligned} \langle \nabla F(\mathcal{G}_1) - \nabla F(\mathcal{G}_2), \mathcal{G}_1 - \mathcal{G}_2 \rangle &= \int_0^1 (2\mathcal{G}_1(u) - 2\mathcal{G}_2(u))(\mathcal{G}_1 - \mathcal{G}_2) du \\ &= 2 \int_0^1 (\mathcal{G}_1(u) - \mathcal{G}_2(u))^2 du \\ &\geq -\frac{1}{4} \int_0^1 (2\mathcal{G}_1(u) - 2\mathcal{G}_2(u))^2 du \\ &= -\frac{1}{4} \|\nabla F(\mathcal{G}_1) - \nabla F(\mathcal{G}_2)\|_2^2, \end{aligned} \quad (54)$$

i.e., ∇F is $1/4$ inverse strongly monotone. Also, $\zeta < 1$ for $\rho = 1/4$. Thus, all the assumptions of Theorem 4 are satisfied, and for $a_n = 1/n + 1$, the sequence $\{u_n\}$ generated by (50) is given as

$$\begin{cases} u_0 \in \mathcal{C}, \\ w_n = p_{\mathcal{C}} \left[\frac{1}{2} u_n \right], \\ r_n = \left(1 - \frac{1}{n+1} \right) w_n + \frac{1}{n+1} p_{\mathcal{C}} \left[\frac{1}{2} w_n \right], \\ v_n = p_{\mathcal{C}} \left[\frac{1}{2} r_n \right], \\ u_{n+1} = p_{\mathcal{C}} \left[\frac{1}{2} v_n \right], \end{cases} \quad (55)$$

where $p_{\mathcal{C}} = \begin{cases} \mathcal{G}, & \mathcal{G} \in \mathcal{C}, \\ \mathcal{G}/\|\mathcal{G}\|, & \mathcal{G} \notin \mathcal{C}. \end{cases}$ Then, the sequence $\{u_n\}$ generated by (50) converges to 0 function.

3.2. Split Feasibility Problem. This subsection is devoted to utilization of Theorem 1 to examine a split feasibility problem (SFP). Let \mathcal{C}_1 and \mathcal{C}_2 be nonempty closed convex subsets of real Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , respectively. Let $\mathcal{A}: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator. The SFP is to locate a point u^* , such that

$$u^* \in \mathcal{C}_1: \mathcal{A}u^* \in \mathcal{C}_2. \quad (56)$$

Let Γ denotes the solution set of SFP (56); then,

$$\Gamma = \{u^* \in \mathcal{C}_1: \mathcal{A}u^* \in \mathcal{C}_2\} = \mathcal{C}_1 \cap \mathcal{A}^{-1}\mathcal{C}_2. \quad (57)$$

A class of inverse problems has been solved by using SFP, for example, [6]. In [32], Xu established the relationship between SFP (56) and the fixed point of problem $p_{\mathcal{C}_1}[I - \rho \mathcal{A}^*(I - p_{\mathcal{C}_2})\mathcal{A}]$. More precisely, for $\rho > 0$, $u^* \in \mathcal{C}_1$ solves SFP (56) if and only if $p_{\mathcal{C}_1}[I - \rho \mathcal{A}^*(I - p_{\mathcal{C}_2})\mathcal{A}](u^*) = u^*$. Byrne [5] posed the following iterative algorithm for solving SFP (56) as follows:

$$u_{n+1} = p_{\mathcal{C}_1}[I - \rho \mathcal{A}^*(I - p_{\mathcal{C}_2})\mathcal{A}](u_n), \quad \forall n \geq 0, \quad (58)$$

where $0 < \rho < 2/\|\mathcal{A}\|^2$, \mathcal{A}^* is the adjoint of operator \mathcal{A} , and $p_{\mathcal{C}_1}$ and $p_{\mathcal{C}_2}$ are the projections onto \mathcal{C}_1 and \mathcal{C}_2 , respectively.

Note that the operator $p_{\mathcal{C}_1}[I - \rho \mathcal{A}^*(I - p_{\mathcal{C}_2})\mathcal{A}]$ with $0 < \rho < 2/\|\mathcal{A}\|^2$ is nonexpansive. Now, we propose following iterative algorithm to solve SFP (56):

$$\begin{cases} u_1 \in \mathcal{C}_1, \\ w_n = p_{\mathcal{C}_1}[I - \rho \mathcal{A}^*(I - p_{\mathcal{C}_2})\mathcal{A}](u_n), \\ r_n = (1 - a_n)w_n + a_n p_{\mathcal{C}_1}[I - \rho \mathcal{A}^*(I - p_{\mathcal{C}_2})\mathcal{A}](w_n), \\ v_n = p_{\mathcal{C}_1}[I - \rho \mathcal{A}^*(I - p_{\mathcal{C}_2})\mathcal{A}](r_n), \\ u_{n+1} = p_{\mathcal{C}_1}[I - \rho \mathcal{A}^*(I - p_{\mathcal{C}_2})\mathcal{A}](v_n), \end{cases} \quad (59)$$

where $\{a_n\}$ is a sequence in $(0, 1)$ and $0 < \rho < 2/\|\mathcal{A}\|^2$.

Theorem 5. Suppose that $\Gamma \neq \emptyset$ and condition (17) holds. Then, the sequence $\{u_n\}$ initiated in (59) converges weakly to u^* , which solves SFP (56) with following error estimates:

$$\|u_{n+1} - u^*\| \leq \zeta^{3(n+1)} \prod_{k=0}^n [1 - a_k(1 - \zeta)] \|u_0 - u^*\|, \quad \forall n \in \mathbb{N}, \quad (60)$$

where

$$\zeta = \left| \frac{a_1 - \rho}{a_1} \right|. \quad (61)$$

Proof. The desired conclusion follows by taking $\nabla F = \mathcal{A}^*(I - p_{\mathcal{C}_2})\mathcal{A}$ and $T, g = I$ in Theorem 1. \square

4. Conclusion

In this study, a new iterative algorithm (16) has been proposed and employed to explore convergence analysis. Using this newly constructed iterative procedure, a common solution of the generalized variational inequality problem and fixed points of nonexpansive mapping is investigated, and theoretical findings are verified by a numerical example. Furthermore, we have shown that our iteration algorithm converges faster than the normal S-iteration process for contraction mapping. Finally, we applied our newly constructed iterative algorithm to investigate the convex optimization problem and split feasibility problem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Fixed Point Property of Variable Exponent Cesàro Complex Function Space of Formal Power Series under Premodular

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We have defined the variable exponent of the Cesàro complex function space of formal power series. We have constructed the prequasi-ideal generated by s -numbers and this new space of complex functions. We present some topological and geometric structures of this class of ideal. The existence of Caristi's fixed point is examined. Some geometric properties related to the fixed point theory are presented. Finally, real-world examples and applications show solutions to some nonlinear difference equations.

1. Introduction

Since the publishing of the book [1] on the Banach fixed point theorem, several mathematicians have studied possible extensions to the Banach fixed point theorem. The nonlinear analysis relies heavily on the Banach contraction principle, a powerful nonlinear analysis tool. The variable exponent Lebesgue spaces $L_{(r)}$ contain Nakano sequence spaces. Variable exponent spaces were thought to offer adequate frameworks for the mathematical components of several issues. Standard Lebesgue spaces were inadequate throughout the second half of the twentieth century. Since these spaces and their effects have become a well-known and efficient instrument for solving a range of problems, they have become a flourishing topic of research, with ramifications that extend into a wide variety [2] of mathematical disciplines. The study of variable exponent Lebesgue spaces $L_{(r)}$ received additional impetus from the mathematical description of non-Newtonian fluid hydrodynamics [3, 4]. Non-Newtonian fluids, also known as electrorheological fluids, have various applications ranging from military science to civil engineering and orthopedics. Guo and Zhu [5] investigated a class of stochastic Volterra-Levin equations with Poisson jumps. Mao et al. [6] were concerned with neutral

stochastic functional differential equations driven by pure jumps (NSFDEwPJs). They proved the existence and uniqueness of the solution to NSFDEwPJs whose coefficients satisfy the local Lipschitz condition and established the p th exponential estimations and almost surely asymptotic estimations of the solution for NSFDEwJs. Yang and Zhu [7] concerned with a class of stochastic neutral functional differential equations of Sobolev type with Poisson jumps. The mapping ideal theory is well regarded in functional analysis. Using s -numbers is an essential technique. Pietsch [8–11] developed and studied the theory of s -numbers of linear bound mappings between Banach spaces. He offered and explained some topological and geometric structures of the quasi ideals of ℓ_p -type mappings. Then, Constantin [12] generalized the class of ℓ_p -type mappings to the class of ces_p -type mappings. Makarov and Faried [13] showed some inclusion relations of ℓ_p -type mappings. As a generalization of ℓ_p -type mappings, Stolz mappings and mappings' ideal were examined by Tita [14, 15]. In [16], Maji and Srivastava studied the class $A_p^{(s)}$ of s -type ces_p mappings using s -number sequence and Cesàro sequence spaces and they introduced a new class $A_{p,q}^{(s)}$ of s -type $\text{ces}(p, q)$ mappings by weighted ces_p with $1 < p < \infty$. In [17], the class of s -type $Z(u, v; \ell_p)$

mappings was defined and some of their properties were explained. Yaying et al. [18] defined and studied χ_r^η , whose its r -Cesàro matrix in ℓ_η , with $r \in (0, 1]$ and $1 < \eta < \infty$. They explained the quasi-Banach ideal of type χ_r^η , with $r \in (0, 1]$ and $1 < \eta < \infty$. Kannan [19] gave an example of a class of mappings with the same fixed point actions as contractions, though that fails to be continuous. The only attempt to describe Kannan operators in modular vector spaces was once made in Reference [20]. Bakery and Mohamed [21] investigated the concept of a prequasinorm on Nakano sequence space with a variable exponent in the range $(0; 1]$. They discussed the adequate circumstances for it to generate prequasi-Banach and closed space when endowed with a definite prequasinorm and the Fatou property of various prequasinorms on it. Additionally, they established a fixed point for Kannan prequasinorm contraction mappings on it and the prequasi-Banach mappings' ideal generated from s -numbers belonging to this sequence space. Also, in [22], they found some fixed points results of Kannan nonexpan-

sive mappings on generalized Cesàro backward difference sequence space of the nonabsolute type. The set of nonnegative integers, real, and complex numbers will be denoted by \mathcal{N} , \mathfrak{R} , and \mathbb{C} , respectively. By $\mathfrak{R}^{\mathcal{N}}$ and $\mathfrak{R}_+^{\mathcal{N}}$, we denote the space of real and positive real sequences. By ℓ_∞ and ℓ_r , we denote the spaces of bounded and r -absolutely summable sequences of \mathfrak{R} .

Lemma 1 (see [23]). *Suppose $\tau_q > 0$ and $y_q \in \mathfrak{R}$ for all $q \in \mathcal{N}$, then*

$$|y_q + z_q|^{\tau_q} \leq 2^{K-1} \left(|y_q|^{\tau_q} + |z_q|^{\tau_q} \right), \quad (1)$$

where $K = \max \{1, \sup_q \tau_q\}$.

If $\tau = (\tau_a) \in \mathfrak{R}_+^{\mathcal{N}}$ and $\tau_a \geq 1$, for all $a \in \mathcal{N}$, the variable exponent Cesàro complex function space is denoted by

$$\mathfrak{C}_{\tau(\cdot)} = \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \hat{f}_v y^v \text{ and } h(\mu f) < \infty, \text{ for some } \mu > 0 \right\}, \text{ when } h(f) = \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^{\infty} |\hat{f}_k|}{a+1} \right)^{\tau_a}. \quad (2)$$

For more information on formal power series spaces and their behaviors, see [24–27]. Many fixed point theorems in a particular space work by either expanding the self-mapping acting on it or expanding the space itself. In this paper, we have introduced the concept of premodular special spaces of formal power series, which are important extensions of the concept of modular spaces. We have built large spaces of solutions to many nonlinear summable and difference equations. It is the first attempt to examine the fixed point theory and Caristi's fixed point in certain premodular special spaces of formal power series. The purpose of this study is arranged, as follows: In Section 2, we present and study the space $(\mathfrak{C}_{\tau(\cdot)})_h$ equipped with a definite function h . In Section 3, we suggest a generalization of Caristi's fixed point theorem. In Section 4, the mapping ideals formed by s -numbers and this function space are constructed, and their geometric and topological properties are presented. Specifically, we explore, in Section 5, some geometric properties connected with fixed point theory in $(\mathfrak{C}_{\tau(\cdot)})_h$. Finally, in Section 6, we discuss several applications of solutions to summable equations and illustrate our findings with some instances.

2. Some Properties of $\mathfrak{C}_{\tau(\cdot)}$

In this section, we investigate sufficient setups of $\mathfrak{C}_{\tau(\cdot)}$ equipped with definite function h to be prequasiclosed and Banach (ssfps). We also present the Fatou property of various h on $\mathfrak{C}_{\tau(\cdot)}$.

Theorem 2. *If $(\tau_a) \in \ell_\infty$ and $\tau_a > 1$, for all $a \in \mathcal{N}$, then*

$$\mathfrak{C}_{\tau(\cdot)} = \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \hat{f}_v y^v \text{ and } h(\mu f) < \infty, \text{ for any } \mu > 0 \right\}. \quad (3)$$

Proof.

$$\begin{aligned} \mathfrak{C}_{\tau(\cdot)} &= \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \hat{f}_v y^v \text{ and } h(\mu f) < \infty, \text{ for some } \mu > 0 \right\} \\ &= \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \hat{f}_v y^v, \inf |\mu|^{\tau_a} \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^a |\hat{f}_k|}{a+1} \right)^{\tau_a} \right. \\ &\quad \left. \leq \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^a |\mu \hat{f}_k|}{a+1} \right)^{\tau_a} < \infty, \text{ for some } \mu > 0 \right\} \\ &= \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \hat{f}_v y^v, \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^a |\hat{f}_k|}{a+1} \right)^{\tau_a} < \infty \right\} \\ &= \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \hat{f}_v y^v \text{ and } h(\mu f) < \infty, \text{ for any } \mu > 0 \right\}. \end{aligned} \quad (4)$$

Let us indicate ϑ , the zero function of \mathfrak{H} and the space of

finite formal power series by \mathfrak{F} , i.e., when $f \in \mathfrak{F}$, then there is $k \in \mathcal{N}$ so that $f(y) = \sum_{a=0}^k \hat{f}_a y^a$. Nakano [28] introduced the concept of modular vector spaces. \square

Definition 3. Suppose \mathcal{H} is a vector space. A function $h : \mathcal{H} \rightarrow [0, \infty)$ is said to be modular, if the next conditions hold

- (a) If $g \in \mathcal{H}$, then $h(g) \geq 0$ and $g = \vartheta \iff h(g) = 0$
- (b) $h(\eta g) = h(g)$ holds, for all $g \in \mathcal{H}$ and $|\eta| = 1$
- (c) The inequality $h(\alpha g + (1 - \alpha)f) \leq h(g) + h(f)$ satisfies, for all $g, f \in \mathcal{H}$ and $\alpha \in [0, 1]$

Definition 4 (see [29]). The space $\mathcal{H} = \{f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{a=0}^{\infty} \hat{f}_a y^a\}$ is said to be a special space of formal power series (or in short ssfps), if it verifies the following settings:

- (1) $e^{(p)} \in \mathcal{H}$, for every $p \in \mathcal{N}$, where $e^{(p)}(y) = \sum_{a=0}^{\infty} e_a^{(p)} y^a = y^p$
- (2) For all $g \in \mathcal{H}$ and $|\hat{f}_a| \leq |\hat{g}_a|$, for every $a \in \mathcal{N}$, then $f \in \mathcal{H}$
- (3) If $g \in \mathcal{H}$ then $g_{[\cdot]} \in \mathcal{H}$, where $g_{[\cdot]}(y) = \sum_{p=0}^{\infty} \widehat{g[p/2]} y^p$ and $[p/2]$ indicates the integral part of $p/2$

Definition 5 (see [29]). A subspace \mathcal{H}_h of the ssfps is said to be a premodular ssfps, if there is a function $h : \mathcal{H} \rightarrow [0, \infty)$ verifies the following conditions:

- (i) If $g \in \mathcal{H}$, then $h(g) \geq 0$ and $g = \vartheta \iff h(g) = 0$
- (ii) When $f \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, then there are $Q \geq 1$ such that $h(\lambda f) \leq |\lambda| Q h(f)$
- (iii) Suppose $f, g \in \mathcal{H}$, then there are $P \geq 1$ such that $h(f + g) \leq P(h(f) + h(g))$
- (iv) Suppose $|\hat{f}_b| \leq |\hat{g}_b|$, for all $b \in \mathcal{N}$, then $h(f) + h(g)$
- (v) There are $P_0 \geq 1$ such that $h(f) \leq h(f_{[\cdot]}) \leq P_0 h(f)$
- (vi) The closure of $\mathfrak{F} = \mathcal{H}_h$
- (vii) There are $\xi > 0$ so that $h(\lambda e^{(0)}) \geq \xi |\lambda| h(e^{(0)})$, where $\lambda \in \mathbb{C}$

Clearly, the concept of premodular vector spaces is more general than modular vector spaces, an example of premodular vector space but not modular vector space.

Example 1. The function $h(f) = \sum_{q=0}^{\infty} (\sum_{p=0}^q |\hat{f}_p|/(q+1))^{(2q+3)/(q+4)}$ is a premodular (not a modular) on the vector space $\mathfrak{C}((2q+3)/(q+4))_{q=0}^{\infty}$. As for every $f, g \in \mathfrak{C}((2q+3)/(q+4))_{q=0}^{\infty}$, one has

$$h\left(\frac{f+g}{2}\right) = \sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\hat{f}_p + \hat{g}_p|/2}{q+1} \right)^{2q+3/q+4} \leq \frac{2}{\sqrt[4]{8}} (h(f) + h(g)), \quad (5)$$

an example of premodular vector space and modular vector space.

Example 2. The function $h(f) = \inf \{\alpha > 0 : \sum_{q=0}^{\infty} (\sum_{p=0}^q |\hat{f}_p|/\alpha/(q+1))^{(2q+3)/(q+4)} \leq 1\}$ is a premodular (modular) on the vector space $\mathfrak{C}(((2q+3)/(q+2))_{q=0}^{\infty})$.

Definition 6 (see [29]). A subspace \mathcal{H}_h of the ssfps is said to be a prequasinormed ssfps, if there is a function $h : \mathcal{H} \rightarrow [0, \infty)$ verifies the following conditions:

- (i) If $g \in \mathcal{H}$, then the $h(g) \geq 0$ and $g = \vartheta \iff h(g) = 0$
- (ii) When $f \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, then there are $Q \geq 1$ such that $h(\lambda f) \leq |\lambda| Q h(f)$
- (iii) Suppose $f, g \in \mathcal{H}$ then there are $P \geq 1$ such that $h(f + g) \leq P(h(f) + h(g))$

Recall that \mathcal{H}_h is said to be a prequasi-Banach ssfps, when \mathcal{H}_h is complete.

Theorem 7 (see [30]). All premodular ssfps \mathcal{H}_h is a prequasinormed ssfps.

Theorem 8 (see [30]). All quasinormed (ssfps) is a prequasinormed (ssfps).

Definition 9.

- (a) The function h on $\mathfrak{C}_{\tau(\cdot)}$ is said to be h -convex, if

$$h(\alpha f + (1 - \alpha)g) \leq \alpha h(f) + (1 - \alpha)h(g), \quad (6)$$

for every $\alpha \in [0, 1]$ and $f, g \in \mathfrak{C}_{\tau(\cdot)}$

- (b) $\{g_q\}_{q \in \mathcal{N}} \subseteq (\mathfrak{C}_{\tau(\cdot)})_h$ is h -convergent to $g \in (\mathfrak{C}_{\tau(\cdot)})_h$, if and only if, $\lim_{q \rightarrow \infty} h(g_q - g) = 0$. When the h -limit exists, then it is unique
- (c) $\{g_q\}_{q \in \mathcal{N}} \subseteq (\mathfrak{C}_{\tau(\cdot)})_h$ is h -Cauchy, if $\lim_{q, r \rightarrow \infty} h(g_q - g_r) = 0$
- (d) $\Gamma \subset (\mathfrak{C}_{\tau(\cdot)})_h$ is h -closed, when for all h -converges, $\{g_1\}_{a \in \mathcal{N}} \subset \Gamma$ to g , then $g \in \Gamma$
- (e) $\Gamma \subset (\mathfrak{C}_{\tau(\cdot)})_h$ is h -bounded, if $\delta_h(\Gamma) = \sup \{h(f - g) : f, g \in \Gamma\} < \infty$
- (f) The h -ball of radius $\varepsilon \geq 0$ and center f , for every $f \in (\mathfrak{C}_{\tau(\cdot)})_h$, is described as

$$B_h(f, \varepsilon) = \left\{ g \in (\mathfrak{G}_{\tau(\cdot)})_h : h(f - g) \leq \varepsilon \right\} \quad (7)$$

- (g) A prequasinorm h on $\mathfrak{G}_{\tau(\cdot)}$ holds the Fatou property, if for every sequence $\{g^q\} \subseteq (\mathfrak{G}_{\tau(\cdot)})_h$ under $\lim_{q \rightarrow \infty} h(g^q - g) = 0$ and all $f \in (\mathfrak{G}_{\tau(\cdot)})_h$, one has $h(f - g) \leq \sup_r \inf_{q \geq r} h(f - g^q)$

Recall that the Fatou property explains the h -closedness of the h -balls. We will mark the space of all increasing sequences of real numbers by \mathbf{I} .

Theorem 10. $(\mathfrak{G}_{\tau(\cdot)})_h$, where $h(f) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K}$ for all $f \in \mathfrak{G}_{\tau(\cdot)}$, is a premodular (ssfps), when $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$.

Proof. Evidently, $h(f) \geq 0$ and $h(f) = 0 \iff f = 0$.

Let $f, g \in \mathfrak{G}_{\tau(\cdot)}$. One has $(f + g)(y) = \sum_{v=0}^{\infty} (\widehat{f_v} + \widehat{g_v}) y^v \in \mathbb{C}$ with

$$\begin{aligned} h(f + g) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p} + \widehat{g_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{g_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &= h(f) + h(g) < \infty. \end{aligned} \quad (8)$$

As $\alpha f \in \mathfrak{G}_{\tau(\cdot)}$, hence from conditions (1-i) and (1-ii), one has $\mathfrak{G}_{\tau(\cdot)}$ is linear. Also $e^{(r)} \in \mathfrak{G}_{\tau(\cdot)}$, for all $r \in \mathcal{N}$, since

$$h(e^{(r)}) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{e_p^{(r)}}|}{q+1} \right)^{\tau_q} \right]^{1/K} = \left[\sum_{r=0}^{\infty} \left(\frac{1}{r+1} \right)^{\tau_0} \right]^{1/K} < \infty. \quad (9)$$

There is $Q = \max \{1, \sup_q |\alpha|^{(\tau_q/K)-1}\} \geq 1$ with $h(\alpha f) \leq Q |\alpha| h(f)$ for all $f \in \mathfrak{G}_{\tau(\cdot)}$ and $\alpha \in \mathbb{C}$

Assume $|f_q| \leq |g_q|$, for all $q \in \mathcal{N}$ and $g \in \mathfrak{G}_{\tau(\cdot)}$. One finds

$$h(f) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{g_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} = h(g) < \infty, \quad (10)$$

then $f \in \mathfrak{G}_{\tau(\cdot)}$.

Obviously, from (58).

Let $(f_q) \in \mathfrak{G}_{\tau(\cdot)}$, we get

$$\begin{aligned} h((f_{[p/2]})) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_{[p/2]}}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{2q} |\widehat{f_{[p/2]}}|}{2q+1} \right)^{\tau_{2q}} + \sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{2q+1} |\widehat{f_{[p/2]}}|}{2q+1} \right)^{\tau_{2q+1}} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \left(\frac{|\widehat{f_q}| + 2 \sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} + \sum_{q=0}^{\infty} \left(\frac{2 \sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \left(\frac{3 \sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} + \sum_{q=0}^{\infty} \left(\frac{2 \sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq (3^K + 2^K)^{1/K} \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &= (3^K + 2^K)^{1/K} h((Y_q)), \end{aligned} \quad (11)$$

then $(f_{[p/2]}) \in \mathfrak{G}_{\tau(\cdot)}$.

From (59), we obtain $P_0 = (3^K + 2^K)^{1/K} \geq 1$.

Evidently the closure of $\mathfrak{F} = \mathfrak{G}_{\tau(\cdot)}$.

There is $0 < \sigma \leq \sup_q |\alpha|^{(\tau_q/K)-1}$, for $\alpha \neq 0$ or $\sigma > 0$, for $\alpha = 0$ with $h(\alpha e^{(0)}) \geq \sigma |\alpha| h(e^{(0)})$. \square

Theorem 11. If $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{G}_{\tau(\cdot)})_h$ is a prequasi-Banach (ssfps), where

$$h(f) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K}, \quad (12)$$

for every $f \in \mathfrak{G}_{\tau(\cdot)}$.

Proof. According to Theorems 10 and 7, the space $(\mathfrak{G}_{\tau(\cdot)})_h$ is a prequasinormed (ssfps). Assume $f^l = (f_q^l)_{q=0}^{\infty}$ is a Cauchy sequence in $(\mathfrak{G}_{\tau(\cdot)})_h$, hence for every $\varepsilon \in (0, 1)$, one has $l_0 \in \mathcal{N}$ such that for all $l, m \geq l_0$, one gets

$$h(f^l - f^m) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p^l} - \widehat{f_p^m}|}{q+1} \right)^{\tau_q} \right]^{1/K} < \varepsilon. \quad (13)$$

□

This implies $|\widehat{f_q^l} - \widehat{f_q^m}| < \varepsilon$. Hence, $(\widehat{f_q^m})$ is a Cauchy sequence in \mathbb{C} , for constant $q \in \mathcal{N}$, which implies $\lim_{m \rightarrow \infty} |\widehat{f_q^m} - \widehat{f_q^0}| = 0$, for constant $q \in \mathcal{N}$. Hence, $h(f^l - f^0) < \varepsilon$, for every $l \geq l_0$. Since $h(f^0) = h(f^0 - f^l + f^l) \leq h(f^l - f^0) + h(f^l) < \infty$. So, $f^0 \in \mathfrak{G}_{\tau(\cdot)}$.

Theorem 12. Suppose $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{G}_{\tau(\cdot)})_h$ is a pre-quasi closed (ssfps), where

$$h(f) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K}, \quad (14)$$

for every $f \in \mathfrak{G}_{\tau(\cdot)}$.

Proof. According to Theorems 10 and 7, the space $(\mathfrak{G}_{\tau(\cdot)})_h$ is a prequasinormed (ssfps). Assume $f^l = (f_q^l)_{q=0}^{\infty} \in (\mathfrak{G}_{\tau(\cdot)})_h$ and $\lim_{l \rightarrow \infty} h(f^l - f^0) = 0$, then for all, $\varepsilon \in (0, 1)$, there is $l_0 \in \mathcal{N}$ such that for all $l \geq l_0$, we obtain

$$\varepsilon > h(f^l - f^0) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p^l} - \widehat{f_p^0}|}{q+1} \right)^{\tau_q} \right]^{1/K}, \quad (15)$$

which implies $|\widehat{f_q^l} - \widehat{f_q^0}| < \varepsilon$, as \mathbb{C} is a complete space.

Therefore, $(\widehat{f_q^l})$ is a convergent sequence in \mathbb{C} , for fixed $q \in \mathcal{N}$. So $\lim_{l \rightarrow \infty} |\widehat{f_q^l} - \widehat{f_q^0}| = 0$, for fixed $q \in \mathcal{N}$. Since, $h(f^0) \leq h(f^l - f^0) + h(f^l) < \infty$. So, $f^0 \in \mathfrak{G}_{\tau(\cdot)}$. □

Theorem 13. The function

$$h(f) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K}, \quad (16)$$

holds the Fatou property, when $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, for all $f \in \mathfrak{G}_{\tau(\cdot)}$.

Proof. Let $\{g^r\} \subseteq (\mathfrak{G}_{\tau(\cdot)})_h$ such that $\lim_{r \rightarrow \infty} h(g^r - g) = 0$. Since $(\mathfrak{G}_{\tau(\cdot)})_h$ is a pre-quasi closed space, one has $g \in (\mathfrak{G}_{\tau(\cdot)})_h$. For all $f \in (\mathfrak{G}_{\tau(\cdot)})_h$, one gets

$$\begin{aligned} h(f - g) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p} - \widehat{g_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p} - \widehat{g_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{g_p^r} - \widehat{g_p}|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \sup_m \inf_{r \geq m} h(f - g^r). \end{aligned} \quad (17)$$

□

Theorem 14. The function

$$h(f) = \sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q}, \quad (18)$$

does not hold the Fatou property, for all $f \in \mathfrak{G}_{\tau(\cdot)}$, when $(\tau_q) \in \ell_{\infty}$ and $\tau_q > 1$, for all $q \in \mathcal{N}$.

Proof. Let $\{g^r\} \subseteq (\mathfrak{G}_{\tau(\cdot)})_h$ so that $\lim_{r \rightarrow \infty} h(g^r - g) = 0$. Since $(\mathfrak{G}_{\tau(\cdot)})_h$ is a pre-quasi closed space, one gets $g \in (\mathfrak{G}_{\tau(\cdot)})_h$. For every $f \in \mathfrak{G}_{\tau(\cdot)}$, we obtain

$$\begin{aligned} h(f - g) &= \sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p} - \widehat{g_p}|}{q+1} \right)^{\tau_q} \\ &\leq 2^{K-1} \left(\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p} - \widehat{g_p}|}{q+1} \right)^{\tau_q} + \sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{g_p^r} - \widehat{g_p}|}{q+1} \right)^{\tau_q} \right) \\ &\leq 2^{K-1} \sup_m \inf_{r \geq m} h(f - g^r). \end{aligned} \quad (19)$$

□

Example 3. For $(\tau_q) \in [1, \infty)^{\mathcal{N}}$, the function

$$h(f) = \inf \left\{ \alpha > 0 : \sum_{q \in \mathcal{N}} \left(\frac{\sum_{p=0}^q |\widehat{f_p}/\alpha|}{q+1} \right)^{\tau_q} \leq 1 \right\}, \quad (20)$$

is a norm on $\mathfrak{G}_{\tau(\cdot)}$.

Example 4. The function

$$h(f) = \sqrt[3]{\sum_{q \in \mathcal{N}} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{3q+2/q+1}}, \quad (21)$$

is a prequasinorm (not a quasinorm) on $\mathfrak{C}(((3q+2)/(q+1))_{q=0}^\infty)$.

Example 5. The function

$$h(f) = \sum_{q \in \mathcal{N}} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{3q+2/q+1}, \quad (22)$$

is a prequasinorm (not a quasinorm) on $\mathfrak{C}(((3q+2)/(q+1))_{q=0}^\infty)$.

Example 6. The function

$$h(f) = \sqrt[d]{\sum_{q \in \mathcal{N}} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^d}, \quad (23)$$

is a prequasinorm, quasinorm, and not a norm on \mathfrak{C}_d , for $0 < d < 1$.

3. Caristi's Fixed Point Theorem in $(\mathfrak{C}_{\tau(\cdot)})_h$

In this section, the existence of Caristi's fixed point in $(\mathfrak{C}_{\tau(\cdot)})_h$ is presented according to Farkas [31], where

$$h(f) = \left[\sum_{q \in \mathcal{N}} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K}, \quad (24)$$

for all $f \in \mathfrak{C}_{\tau(\cdot)}$.

Definition 15. The function $\Psi_1 : (\mathfrak{C}_{\tau(\cdot)})_h \longrightarrow (-\infty, \infty]$ is said to be lower semicontinuous at $G^{(0)} \in (\mathfrak{C}_{\tau(\cdot)})_h$ if $\liminf_{G \rightarrow G^{(0)}} \Psi_1(G) = \Psi_1(G^{(0)})$, where $\liminf_{G \rightarrow G^{(0)}} \Psi_1(G) = \sup_{V \in \mathcal{V}(G^{(0)})} \inf_{G \in V} \Psi_1(G)$, where $V(G^{(0)})$ is a neighborhood system of $G^{(0)}$.

Definition 16. The function $\Psi_1 : (\mathfrak{C}_{\tau(\cdot)})_h \longrightarrow (-\infty, \infty]$ is said to be proper, when

$$\mathcal{D}(\Psi_1) = \left\{ G \in (\mathfrak{C}_{\tau(\cdot)})_h : \Psi_1(G) < \infty \right\} \neq \emptyset. \quad (25)$$

Theorem 17. Suppose $\Xi \neq \emptyset$ and Ξ is a h -closed subset of $(\mathfrak{C}_{\tau(\cdot)})_h$, and $\Psi_1 : \Xi \longrightarrow (-\infty, \infty]$ is a proper, h -lower semicontinuous function with $\inf_{G \in \Xi} \Psi_1(G) > -\infty$. If $\gamma > 0$, $\{\omega_q\} \subset (0, \infty)$, and $G^{(0)} \in \Xi$ so that $\Psi_1(G^{(0)}) \leq \inf_{G \in \Xi} \Psi_1(G) + \gamma$. One gets $\{G^{(q)}\} \in \Xi$ which h -converges to some $G^{(\gamma)}$, and

$$(i) \ h(G^{(\gamma)} - G^{(q)}) \leq \gamma/2^q \omega_0, \text{ with } q \in \mathcal{N}$$

$$\Psi_1(G^{(\gamma)}) + \sum_{q=0}^\infty \omega_q h(G^{(\gamma)} - G^{(q)}) \leq \Psi_1(G^{(0)}) \quad (26)$$

(ii) when $G \neq G^{(\gamma)}$, then

$$\Psi_1(G^{(\gamma)}) + \sum_{q=0}^\infty \omega_q h(G^{(\gamma)} - G^{(q)}) < \Psi_1(G) + \sum_{q=0}^\infty \omega_q h(G - G^{(q)}) \quad (27)$$

Proof. If $S(G^{(0)}) = \{G \in \Xi : \Psi_1(G) + \omega_0 h(G - G^{(0)}) \leq \Psi_1(G^{(0)})\}$. Since $G^{(0)} \in S(G^{(0)})$, then $S(G^{(0)}) \neq \emptyset$. As Ψ_1 is h -lower semicontinuous, h holds the Fatou property and Ξ is h -closed, then $S(G^{(0)})$ is h -closed. Take $G^{(1)} \in S(G^{(0)})$ and

$$\Psi_1(G^{(1)}) + \omega_0 h(G^{(1)} - G^{(0)}) \leq \inf_{G \in S(G^{(0)})} \left\{ \Psi_1(G) + \omega_0 h(G - G^{(0)}) \right\} + \frac{\gamma \omega_1}{2\omega_0}. \quad (28)$$

Choose

$$\begin{aligned} S(G^{(1)}) &= \left\{ G \in S(G^{(0)}) : \Psi_1(G) + \sum_{j=0}^1 \omega_j h(G - G^{(j)}) \right. \\ &\quad \left. \leq \Psi_1(G^{(1)}) + \omega_0 h(G^{(1)} - G^{(j)}) \right\}. \end{aligned} \quad (29)$$

As $S(G^{(0)})$, we get $S(G^{(1)}) \neq \emptyset$ and h -closed. Suppose that one has built $\{G^{(0)}, G^{(1)}, G^{(2)}, \dots, G^{(q)}\}$ and $\{S(G^{(0)}), S(G^{(1)}), S(G^{(2)}), \dots, S(G^{(q)})\}$. Next, take $G^{(q+1)} \in S(G^{(q)})$ and

$$\begin{aligned} \Psi_1(G^{(q+1)}) + \sum_{j=0}^q \omega_j h(G^{(q+1)} - G^{(j)}) \\ \leq \inf_{G \in S(G^{(q)})} \left\{ \Psi_1(G) + \sum_{j=0}^q \omega_j h(G - G^{(j)}) \right\} + \frac{\gamma \omega_{q+1}}{2^q \omega_0}. \end{aligned} \quad (30)$$

Let

$$\begin{aligned} S(G^{(q+1)}) &:= \left\{ G \in S(G^{(q)}) : \Psi_1(G) + \sum_{j=0}^{q+1} \omega_j h(G - G^{(j)}) \right. \\ &\quad \left. \leq \Psi_1(G^{(q+1)}) + \sum_{j=0}^q \omega_j h(G^{(q+1)} - G^{(j)}) \right\}, \end{aligned} \quad (31)$$

hence we form by induction, the sequences $\{G^{(q)}\}$ and $\{S(G^{(q)})\}$. Fix $q \in \mathcal{N}$. Suppose $W \in S(G^{(q)})$. One obtains

$$\Psi_1(W) + \sum_{j=0}^q \omega_j h(W - G^{(j)}) \leq \Psi_1(G^{(q)}) + \sum_{j=0}^{q-1} \omega_j h(G^{(q)} - G^{(j)}), \quad (32)$$

then

$$\begin{aligned} \omega_q h(W - G^{(q)}) &\leq \Psi_1(G^{(q)}) + \sum_{j=0}^{q-1} \omega_j h(G^{(q)} - G^{(j)}) \\ &\quad - \left[\Psi_1(W) + \sum_{j=0}^{q-1} \omega_j h(W - G^{(j)}) \right] \\ &\leq \Psi_1(G^{(q)}) + \sum_{j=0}^{q-1} \omega_j h(G^{(q)} - G^{(j)}) - \inf_{G \in S(G^{(q-1)})} \\ &\quad \cdot \left[\Psi_1(G) + \sum_{j=0}^{q-1} \omega_j h(G - G^{(j)}) \right] \leq \frac{\gamma \omega_q}{2^q \omega_0}. \end{aligned} \quad (33)$$

As $\{S(G^{(q)})\}$ is decreasing with $G^{(q)} \in S(G^{(q)})$, for all $q \in \mathcal{N}$, one gets

$$h(G^{(q+p)} - G^{(q)}) \leq \frac{\gamma}{2^q \omega_0}, \quad (34)$$

with $q, p \in \mathcal{N}$. This implies $\{G^{(q)}\}$ is h -Cauchy. Since $(\mathfrak{C}_{\tau(\cdot)})_h$ is h -Banach space; hence, $\{G^{(q)}\}$ has h -limits $G^{(\gamma)}$ and $\cap_{q \in \mathcal{N}} S(G^{(q)}) = \{G^{(\gamma)}\}$. Since $G^{(q+1)} \in S(G^{(q)})$, we can see

$$\Psi_1(G^{(q+1)}) + \sum_{j=0}^q \omega_j h(G^{(q+1)} - G^{(j)}) \leq \Psi_1(G^{(q)}) + \sum_{j=0}^{q-1} \omega_j h(G^{(q)} - G^{(j)}), \quad (35)$$

hence, $\{\Psi_1(G^{(q)}) + \sum_{j=0}^{q-1} \omega_j h(G^{(q+1)} - G^{(j)})\}$ is decreasing. After, let $G \neq G^{(\gamma)}$. One gets $m \in \mathcal{N}$ with $G \in S(G^{(q)})$, with $q \geq m$, i.e.,

$$\Psi_1(G^{(q)}) + \sum_{j=0}^{q-1} \omega_j h(G^{(q)} - G^{(j)}) < \Psi_1(G) + \sum_{j=0}^q \omega_j h(G - G^{(j)}). \quad (36)$$

Since $G^{(\gamma)} \in S(G^{(q)})$, with $q \geq m$, we get

$$\begin{aligned} \Psi_1(G^{(\gamma)}) + \sum_{j=0}^q \omega_j h(G^{(\gamma)} - G^{(j)}) \\ \leq \Psi_1(G^{(q)}) + \sum_{j=0}^{q-1} \omega_j h(G^{(q)} - G^{(j)}) \\ \leq \Psi_1(G^{(m)}) + \sum_{j=0}^{m-1} \omega_j h(G^{(m)} - G^{(j)}). \end{aligned} \quad (37)$$

Put $q \longrightarrow \infty$ in the previous inequality, then

$$\begin{aligned} \Psi_1(G^{(\gamma)}) + \sum_{j=0}^{\infty} \omega_j h(G^{(\gamma)} - G^{(j)}) \\ \leq \Psi_1(x_m) + \sum_{j=0}^{m-1} \omega_j h(G^m - G^{(j)}) \\ < \Psi_1(G) + \sum_{j=0}^m \omega_j h(G - G^{(j)}) \\ \leq \Psi_1(G) + \sum_{j=0}^{\infty} \omega_j h(G - G^{(j)}). \end{aligned} \quad (38)$$

This gives

$$\Psi_1(G^{(\gamma)}) + \sum_{q=0}^{\infty} \omega_q h(G^{(\gamma)} - G^{(q)}) < \Psi_1(G) + \sum_{q=0}^{\infty} \omega_q h(G - G^{(q)}). \quad (39)$$

□

Theorem 18. Suppose $\Xi \neq \emptyset$ and Ξ is a h -closed subset of $(\mathfrak{C}_{\tau(\cdot)})_h$. By taking $\gamma > 0$ and $\{\omega_n\}$ and $0 < \omega = \sum_{n=0}^{\infty} \omega_n < \infty$. If $H : \Xi \longrightarrow \Xi$ is a mapping and there is a function $\Psi_1 : \Xi \longrightarrow (-\infty, \infty]$ holds a proper and h -lower semicontinuous with $\inf_{G \in \Xi} \Psi_1(G) > -\infty$ and

- (1) $h(H(G) - Y) - h(G - Y) \leq h(H(G) - Y)$, for any $G, Y \in \Xi$
- (2) $h(H(G) - G) \leq \Psi_1(G) - \Psi_1(H(G))$, with $G \in \Xi$

Then, H has a fixed point in Ξ .

Proof. As $0 < \omega = \sum_{n=0}^{\infty} \omega_n < \infty$, one has $\Psi_2 := \omega \Psi_1$ is also proper, h -lower semicontinuous and bounded from below. If $G \in \Xi$, one gets

$$\omega h(H(G) - G) \leq \Psi_2(G) - \Psi_2(H(G)). \quad (40)$$

As $\inf_{G \in \Xi} \Psi_2(G) > -\infty$, one obtains $G^{(0)} \in \Xi$ with $\Psi_2(G^{(0)}) < \inf_{G \in \Xi} \Psi_2(G) + \gamma$. From Theorem 17, there is $\{G^{(q)}\}$ which h -converges to some $G^{(\gamma)} \in \Xi$, and

$$\Psi_2(G^{(\gamma)}) + \sum_{q=0}^{\infty} \omega_q h(G^{(\gamma)} - G^{(q)}) < \Psi_2(G) + \sum_{q=0}^{\infty} \omega_q h(G - G^{(q)}), \quad (41)$$

for every $G \neq G^{(\gamma)}$. Assume that $H(G^{(\gamma)}) \neq G^{(\gamma)}$, we have

$$\begin{aligned} & \Psi_2(G^{(\gamma)}) + \sum_{q=0}^{\infty} \omega_q h(G^{(\gamma)} - G^{(q)}) \\ & < \Psi_2(H(G^{(\gamma)})) + \sum_{q=0}^{\infty} \omega_q h(H(G^{(\gamma)}) - G^{(q)}), \end{aligned} \quad (42)$$

then

$$\begin{aligned} & \Psi_2(G^{(\gamma)}) - \Psi_2(H(G^{(\gamma)})) \\ & < \sum_{q=0}^{\infty} \omega_q h(H(G^{(\gamma)}) - G^{(q)}) - \sum_{q=0}^{\infty} \omega_q h(G^{(\gamma)} - G^{(q)}) \\ & = \sum_{q=0}^{\infty} \omega_q (h(H(G^{(\gamma)}) - G^{(q)}) - h(G^{(\gamma)} - G^{(q)})). \end{aligned} \quad (43)$$

From condition (40), then

$$\begin{aligned} \Psi_2(G^{(\gamma)}) - \Psi_2(H(G^{(\gamma)})) & < \sum_{q=0}^{\infty} \omega_q h(H(G^{(\gamma)}) - G^{(q)}) \\ & = \omega h(H(G^{(\gamma)}) - G^{(\gamma)}). \end{aligned} \quad (44)$$

The inequality (40) implies that

$$\begin{aligned} \omega h(H(G^{(\gamma)}) - G^{(\gamma)}) & \leq \Psi_2(G^{(\gamma)}) - \Psi_2(H(G^{(\gamma)})) \\ & < \omega h(H(G^{(\gamma)}) - G^{(\gamma)}). \end{aligned} \quad (45)$$

This is a contradiction, hence $H(G^{(\gamma)}) = G^{(\gamma)}$. \square

4. Structure of Mappings' Ideal

The structure of the mappings' ideal by $(\mathfrak{G}_{\tau(\cdot)})_h$, where

$$h(f) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K}, \quad (46)$$

for all $f \in \mathfrak{G}_{\tau(\cdot)}$, and s -numbers have been explained. We study enough setups on $(\mathfrak{G}_{\tau(\cdot)})_h$ such that the class $^*(\mathfrak{G}_{\tau(\cdot)})_h$ is complete and closed. We investigate enough setups (not necessary) on $(\mathfrak{G}_{\tau(\cdot)})_h$ such that the closure of $F = {}^{*\alpha}(\mathfrak{G}_{\tau(\cdot)})_h$. This gives a negative answer of Rhoades' [32] open problem about the linearity of s -type $(\mathfrak{G}_{\tau(\cdot)})_h$ spaces. We explain enough setups on $(\mathfrak{G}_{\tau(\cdot)})_h$ such that $^*(\mathfrak{G}_{\tau(\cdot)})_h$ is strictly contained for different powers, ${}^{*\alpha}(\mathfrak{G}_{\tau(\cdot)})_h$ is the minimum, the class $^*(\mathfrak{G}_{\tau(\cdot)})_h$ is simple, and $(^*(\mathfrak{G}_{\tau(\cdot)})_h)^\lambda = ^*(\mathfrak{G}_{\tau(\cdot)})_h$.

We denote the space of all bounded, finite rank linear mappings from an infinite-dimensional Banach space Δ into an infinite-dimensional Banach space Λ by $\mathcal{L}(\Delta, \Lambda)$, and $\mathbf{F}(\Delta, \Lambda)$ and when $\Delta = \Lambda$, we inscribe $\mathcal{L}(\Delta)$ and $\mathbf{F}(\Delta)$. The

space of approximable and compact-bounded linear mappings from a Banach space Δ into a Banach space Λ will be indicated by $\mathcal{Y}(\Delta, \Lambda)$ and $\mathcal{L}_c(\Delta, \Lambda)$, and if $\Delta = \Lambda$, we mark $\mathcal{Y}(\Delta)$ and $\mathcal{L}_c(\Delta)$, respectively.

Definition 19 (see [33]). An s -number function is a mapping $s : \mathcal{L}(\Delta, \Lambda) \longrightarrow \mathfrak{R}^{+\mathcal{N}}$ that sorts every $V \in \mathcal{L}(\Delta, \Lambda)$ unique sequence $(s_d(V))_{d=0}^{\infty}$ validates the following settings:

- (a) $\|V\| = s_0(V) \geq s_1(V) \geq s_2(V) \geq \dots \geq 0$, for all $V \in \mathcal{L}(\Delta, \Lambda)$
- (b) $s_{l+d-1}(V_1 + V_2) \leq s_l(V_1) + s_d(V_2)$, for all $V_1, V_2 \in \mathcal{L}(\Delta, \Lambda)$ and $l, d \in \mathcal{N}$
- (c) $s_d(VYW) \leq \|V\| s_d(Y) \|W\|$, for all $W \in \mathcal{L}(\Delta_0, \Delta)$, $Y \in \mathcal{L}(\Delta, \Lambda)$, and $V \in \mathcal{L}(\Lambda, \Lambda_0)$, where Δ_0 and Λ_0 are arbitrary Banach spaces
- (d) when $V \in \mathcal{L}(\Delta, \Lambda)$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$
- (e) suppose $\text{rank}(V) \leq d$, then $s_d(V) = 0$, for each $V \in \mathcal{L}(\Delta, \Lambda)$
- (f) $s_{l \geq q}(I_q) = 0$ or $s_{l < q}(I_q) = 1$, where I_q denotes the unit map on the q -dimensional Hilbert space ℓ_2^q

Some examples of s -numbers are as follows:

- (1) The q th Kolmogorov number, described by $d_q(X)$, is marked by

$$d_q(X) = \inf_{\dim J \leq q} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\|. \quad (47)$$

- (2) The q th approximation number, described by $\alpha_q(X)$, is marked by

$$\alpha_q(X) = \inf \{ \|X - Y\| : Y \in \mathcal{L}(\Delta, \Lambda) \text{ and } \text{rank}(Y) \leq q \}. \quad (48)$$

Definition 20 (see [10]). Assume \mathcal{L} is the class of all bounded linear mappings within any two arbitrary Banach spaces. A subclass \mathcal{U} of \mathcal{L} is said to be a mappings' ideal, when all $\mathcal{U}(\Delta, \Lambda) = \mathcal{U} \cap \mathcal{L}(\Delta, \Lambda)$ verifies the following conditions:

- (i) $I_\Gamma \in \mathcal{U}$, where Γ marks Banach space of one dimension
- (ii) The space $\mathcal{U}(\Delta, \Lambda)$ is linear over \mathfrak{R}
- (iii) If $W \in \mathcal{L}(\Delta_0, \Delta)$, $X \in \mathcal{U}(\Delta, \Lambda)$, and $Y \in \mathcal{L}(\Lambda, \Lambda_0)$ then, $YXW \in \mathcal{U}(\Delta_0, \Lambda_0)$

Notations 21 (see [30]).

$$\mathfrak{H}_{\mathcal{H}} := \{\mathfrak{H}_{\mathcal{H}}\}$$

$(\Delta, \Lambda)\}$, where $\mathfrak{H}_{\mathcal{H}}(\Delta, \Lambda) := \{V \in \mathcal{L}(\Delta, \Lambda) : f_s \in \mathcal{H}, \text{ where } f_s(y) = \sum_{n=0}^{\infty} s_n(V)y^n\}$, $\mathfrak{H}_{\mathcal{H}}^{\alpha} := \{\mathfrak{H}_{\mathcal{H}}^{\alpha}(\Delta, \Lambda)\}$, where $\mathfrak{H}_{\mathcal{H}}^{\alpha}(\Delta, \Lambda) := \{V \in \mathcal{L}(\Delta, \Lambda) : f_{\alpha} \in \mathcal{H}, \text{ where } f_{\alpha}(y) = \sum_{n=0}^{\infty} \alpha_n(V)y^n\}$, $\mathfrak{H}_{\mathcal{H}}^d := \{\mathfrak{H}_{\mathcal{H}}^d(\Delta, \Lambda)\}$, where $\mathfrak{H}_{\mathcal{H}}^d(\Delta, \Lambda) := \{V \in \mathcal{L}(\Delta, \Lambda) : f_d \in \mathcal{H}, \text{ where } f_d(y) = \sum_{n=0}^{\infty} d_n(V)y^n\}$. **Theorem 22.** (see [29]). Suppose \mathcal{H} is a (ssfps), then $\mathfrak{H}_{\mathcal{H}}$ is mappings' ideal.

According to Theorems 10 and 22, one concludes the next theorem.

Theorem 23. Suppose $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, then ${}^*(\mathfrak{G}_{\tau(\cdot)})_h$ is a mappings' ideal.

Definition 24 (see [34]). A function $H \in [0, \infty)^{\mathcal{U}}$ is said to be a pre-quasi norm on the ideal \mathcal{U} , if it verifies the following setups:

- (1) Let $V \in \mathcal{U}(\Delta, \Lambda)$, $H(V) \geq 0$, and $H(V) = 0$, if and only if, $V = 0$
- (2) we have $Q \geq 1$ so as to $H(\alpha V) \leq D|\alpha|H(V)$, for every $V \in \mathcal{U}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$
- (3) we have $P \geq 1$ so that $H(V_1 + V_2) \leq P[H(V_1) + H(V_2)]$, for each $V_1, V_2 \in \mathcal{U}(\Delta, \Lambda)$
- (4) we have $\sigma \geq 1$ when $V \in \mathcal{L}(\Delta_0, \Delta)$, $X \in \mathcal{U}(\Delta, \Lambda)$, and $Y \in \mathcal{L}(\Lambda, \Lambda_0)$ then $H(YXV) \leq \sigma\|Y\|H(X)\|V\|$

Theorem 25 (see [35]). Every quasinorm on the ideal \mathcal{U} is a prequasinorm on the same ideal.

Theorem 26. If $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, then H is a pre-quasinorm on ${}^*(\mathfrak{G}_{\tau(\cdot)})_h$, so that $H(Z) = h(f_s)$, where $f_s \in (\mathfrak{G}_{\tau(\cdot)})_h$ and $f_s(y) = \sum_{n=0}^{\infty} s_n(Z)y^n$.

Proof.

- (1) When $X \in {}^*(\mathfrak{G}_{\tau(\cdot)})_h(\Delta, \Lambda)$, $H(X) = h(f_s) \geq 0$, and $H(X) = h(f_s) = 0$, if and only if, $s_n(X) = 0$, for all $n \in \mathcal{N}$; if and only if, $X = 0$
- (2) There is $Q \geq 1$ with $H(\varepsilon X) \leq h(\varepsilon f_s) \leq Q\|\varepsilon\|H(X)$ for every $X \in {}^*(\mathfrak{G}_{\tau(\cdot)})_h(\Delta, \Lambda)$ and $\varepsilon \in \mathbb{C}$
- (3) One has $PP_0 \geq 1$ so that for $X_1, X_2 \in {}^*(\mathfrak{G}_{\tau(\cdot)})_h(\Delta, \Lambda)$; hence, there are $f_1, f_2 \in (\mathfrak{G}_{\tau(\cdot)})_h$ with $f_1(y) = \sum_{n=0}^{\infty} s_n(X_1)y^n$ and $f_2(y) = \sum_{n=0}^{\infty} s_n(X_2)y^n$. Therefore, for $g_s(y) = \sum_{n=0}^{\infty} s_n(X_1 + X_2)y^n$, we have $KK_0 \geq 1$ so that

$$\begin{aligned} H(X_1 + X_2) &= h(g_s) \leq h\left((f_1)_s + (f_2)_s\right) \\ &\leq P\left(h(f_1)_s + h(f_2)_s\right) \\ &\leq PP_0(H(X_1) + H(X_2)) \end{aligned} \quad (50)$$

- (4) We have $Q \geq 1$ if $X \in \mathcal{L}(\Delta_0, \Delta)$, $Y \in {}^*(\mathfrak{G}_{\tau(\cdot)})_h(\Delta, \Lambda)$, and $Z \in \mathcal{L}(\Lambda, \Lambda_0)$; hence, there is $f_s \in (\mathfrak{G}_{\tau(\cdot)})_h$ with $f_s(y) = \sum_{n=0}^{\infty} s_n(Y)y^n$. Then, for $g_s(y) = \sum_{n=0}^{\infty} s_n(ZYX)y^n$, one has

$$H(ZYX) = h(g_s) \leq h(\|X\|\|Z\|f_s) \leq Q\|X\|H(Y)\|Z\|. \quad (51)$$

□

Theorem 27. Suppose $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$ one has $({}^*(\mathfrak{G}_{\tau(\cdot)})_h, H)$ is a prequasi-Banach mappings' ideal.

Proof. Suppose $(V_a)_{a \in \mathcal{N}}$ is a Cauchy sequence in ${}^*(\mathfrak{G}_{\tau(\cdot)})_h(\Delta, \Lambda)$. As $\mathcal{L}(\Delta, \Lambda) \supseteq \mathcal{S}_{(\mathfrak{G}_{\tau(\cdot)})_h}(\Delta, \Lambda)$, hence, there is $f_s^a \in (\mathfrak{G}_{\tau(\cdot)})_h$ with $f_s^a(y) = \sum_{n=0}^{\infty} s_n(V_a)y^n$ for every $a \in \mathcal{N}$, then

$$\begin{aligned} H(V_r - V_a) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_p(V_r - V_a)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\geq \inf_q \|V_r - V_a\|^{\tau_q/K} \left[\sum_{q=0}^{\infty} \left(\frac{1}{q+1} \right)^{\tau_q} \right]^{1/K}, \end{aligned} \quad (52)$$

hence, $(V_a)_{a \in \mathcal{N}}$ is a Cauchy sequence in $\mathcal{L}(\Delta, \Lambda)$ is a Banach space, so there exists $V \in \mathcal{L}(\Delta, \Lambda)$ so that $\lim_{a \rightarrow \infty} \|V_a - V\| = 0$ and since $f_s^a \in (\mathfrak{G}_{\tau(\cdot)})_h$, for all $a \in \mathcal{N}$ and $(\mathfrak{G}_{\tau(\cdot)})_h$ is a premodular (ssfps), hence, one can see

$$\begin{aligned} H(V) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_p(V)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_{[p/2]}(V - V_a)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_{[p/2]}(V_a)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \|V_a - V\|^{\tau_q} \right]^{1/K} (3^K + 2^K)^{1/K} \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_p(V_a)}{q+1} \right)^{\tau_q} \right]^{1/K} < \varepsilon. \end{aligned} \quad (53)$$

We obtain $f_s^a \in (\mathfrak{G}_{\tau(\cdot)})_h$, hence $V \in {}^*(\mathfrak{G}_{\tau(\cdot)})_h(\Delta, \Lambda)$. □

Theorem 28. If $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, one has $({}^*(\mathfrak{G}_{\tau(\cdot)})_h, H)$ is a prequasiclosed mappings' ideal.

Proof. Suppose $V_a \in {}^*(\mathfrak{C}_{\tau(\cdot)})_h(\Delta, \Lambda)$, for all $a \in \mathcal{N}$ and $\lim_{a \rightarrow \infty} H\|V_a - V\| = 0$, hence, there is $f_s^a \in (\mathfrak{C}_{\tau(\cdot)})_h$ with $f_s^a(y) = \sum_{n=0}^{\infty} s_n(V_a)y^n$, for all $a \in \mathcal{N}$, there is $\varsigma > 0$ and as $\mathcal{L}(\Delta, \Lambda) \supseteq S_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$, one has

$$\begin{aligned} H(V_a - V) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_p(V_a - V)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \inf_q \|V_a - V\|^{\tau_q/K} \left[\sum_{q=0}^{\infty} \left(\frac{1}{q+1} \right)^{\tau_q} \right]^{1/K}. \end{aligned} \quad (54)$$

□

So $(V_a)_{a \in \mathcal{N}}$ is convergent in $\mathcal{L}(\Delta, \Lambda)$, i.e., $\lim_{a \rightarrow \infty} \|V_a - V\| = 0$ and since $f_s^a(\mathfrak{C}_{\tau(\cdot)})_h$, for all $a \in \mathcal{N}$ and $(\mathfrak{C}_{\tau(\cdot)})_h$ is a premodular (ssfps), hence, one can see

$$\begin{aligned} H(V) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_p(V)}{q+1} \right)^{\tau_q} \right]^{1/K} \leq \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_{[p/2]}(V - V_a)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_{[p/2]}(V_a)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} (\|V_a - V\|)^{\tau_q} \right]^{1/K} \\ &\quad + (3^K + 2^K)^{1/K} \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_p(V_a)}{q+1} \right)^{\tau_q} \right]^{1/K} < \varepsilon. \end{aligned} \quad (55)$$

We obtain $f_s \in (\mathfrak{C}_{\tau(\cdot)})_h$, hence $V \in \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$.

Definition 29. A prequasinorm H on the ideal $\mathfrak{H}_{\mathcal{H}_h}$ verifies the Fatou property if for every $\{T_q\}_{q \in \mathcal{N}} \subseteq \mathfrak{H}_{\mathcal{H}_h}(\Delta, \Lambda)$ so that $\lim_{q \rightarrow \infty} H(T_q - T) = 0$ and $M \in \mathfrak{H}_{\mathcal{H}_h}(\Delta, \Lambda)$, one gets

$$H(M - T) \leq \sup_q \inf_{j \geq q} H(M - T_j). \quad (56)$$

Theorem 30. Suppose $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}, H)$ does not verify the Fatou property.

Proof. Assume $\{T_q\}_{q \in \mathcal{N}} \subseteq \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$ with $\lim_{q \rightarrow \infty} H(T_q - T) = 0$. Since $\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}$ is a prequasiclosed ideal, then $T \in \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$. So for every $M \in \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$, one has

$$\begin{aligned} H(M - T) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_p(M - T)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_{[p/2]}(M - T_j)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_{[p/2]}(T_j - T)}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq (3^K + 2^K)^{1/K} \sup_r \inf_{j \geq r} \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q s_p(M - T_j)}{q+1} \right)^{\tau_q} \right]^{1/K}. \end{aligned} \quad (57)$$

□

Theorem 31. $\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}^{\alpha}(\Delta, \Lambda)$ = the closure of $\mathbf{F}(\Delta, \Lambda)$, if $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$. But the converse is not necessarily true.

Proof. As $e^{(q)} \in (\mathfrak{C}_{\tau(\cdot)})_h$, for every $q \in \mathcal{N}$ and $(\mathfrak{C}_{\tau(\cdot)})_h$ is a linear space. Suppose $Z \in \mathbf{F}(\Delta, \Lambda)$ with $\text{rank}(Z) = m$, where $m \in \mathcal{N}$, hence $f_{\alpha} \in \mathfrak{F}$ with $f_{\alpha}(y) = \sum_{n=0}^{m-1} \alpha_n(Z)y^n$, one has $f_{\alpha} \in (\mathfrak{C}_{\tau(\cdot)})_h$. Therefore, the closure of $\mathbf{F}(\Delta, \Lambda) \subseteq \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}^{\alpha}(\Delta, \Lambda)$. Assume $Z \in \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}^{\alpha}(\Delta, \Lambda)$, we have $g_{\alpha} \in (\mathfrak{C}_{\tau(\cdot)})_h$. As $h(g_{\alpha}) < \infty$, assume $\rho \in (0, 1)$, then there is $q_0 \in \mathcal{N} - \{0\}$ with $h(g_{\alpha} - \sum_{n=0}^{q_0-1} e^{(n)}) < \rho/2^{K+3}\eta d$, for some $d \geq 1$, where $\eta = \max\{1, \sum_{q=q_0}^{\infty} (1/q+1)^{\tau_q}\}$. Since $(\alpha_q(Z))$ is decreasing, we have

$$\begin{aligned} \sum_{q=q_0+1}^{2q_0} \left(\frac{\sum_{p=0}^q \alpha_{2q_0}(Z)}{q+1} \right)^{\tau_q} &\leq \sum_{q=q_0+1}^{2q_0} \left(\frac{\sum_{p=0}^q \alpha_p(Z)}{q+1} \right)^{\tau_q} \\ &\leq \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^q \alpha_p(Z)}{q+1} \right)^{\tau_q} < \frac{\rho}{2^{K+3}\eta d}. \end{aligned} \quad (58)$$

Hence, there is $Y \in \mathbf{F}_{2q_0}(\Delta, \Lambda)$ so that $\text{rank}(Y) \leq 2q_0$ and

$$\sum_{q=2q_0+1}^{3q_0} \left(\frac{\sum_{p=0}^q \|Z - Y\|}{q+1} \right)^{\tau_q} \leq \sum_{q=q_0+1}^{2q_0} \left(\frac{\sum_{p=0}^q \|Z - Y\|}{q+1} \right)^{\tau_q} < \frac{\rho}{2^{K+3}\eta d}, \quad (59)$$

since $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, we have

$$\sup_{q=q_0}^{\infty} (2q_0 \|Z - Y\|)^{\tau_q} < \frac{\rho}{2^{2K+2}\eta}. \quad (60)$$

Therefore, one has

$$\sum_{q=0}^{q_0} (\|Z - Y\|)^{\tau_q} < \frac{\rho}{2^{K+3}\eta d}. \quad (61)$$

As $Z - Y \in \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}^{\alpha}(\Delta, \Lambda)$, hence $g_{\alpha} \in (\mathfrak{C}_{\tau(\cdot)})_h$, where g_{α}

$(y) := \sum_{n=0}^{\infty} \alpha_n(Z-Y)y^n$. In view of inequalities (58)–(61), one has

$$\begin{aligned}
 d(Z, Y) &= h(g_a) = \sum_{q=0}^{3q_0-1} \left(\frac{\sum_{p=0}^q \alpha_p(Z-Y)}{q+1} \right)^{\tau_q} + \sum_{q=3q_0}^{\infty} \left(\frac{\sum_{p=0}^q \alpha_p(Z-Y)}{q+1} \right)^{\tau_q} \\
 &\leq \sum_{q=0}^{3q_0} \left(\frac{\sum_{p=0}^q \|Z-Y\|}{q+1} \right)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{q+2q_0} \alpha_p(Z-Y)}{q+2q_0+1} \right)^{\tau_{q+2q_0}} \\
 &\leq \sum_{q=0}^{3q_0} (\|Z-Y\|)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{q+2q_0} \alpha_p(Z-Y)}{q+1} \right)^{\tau_q} \\
 &\leq 3 \sum_{q=0}^{q_0} (\|Z-Y\|)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{2q_0-1} \alpha_p(Z-Y) + \sum_{p=2q_0}^{q+2q_0} \alpha_p(Z-Y)}{q+1} \right)^{\tau_q} \\
 &\leq 3 \sum_{q=0}^{q_0} (\|Z-Y\|)^{\tau_q} + 2^{K-1} \cdot \left[\sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{2q_0-1} \alpha_p(Z-Y)}{q+1} \right)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=2q_0}^{q+2q_0} \alpha_p(Z-Y)}{q+1} \right)^{\tau_q} \right] \\
 &\leq 3 \sum_{q=0}^{q_0} (\|Z-Y\|)^{\tau_q} + 2^{K-1} \cdot \left[\sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{2q_0-1} \alpha_p(Z-Y)}{q+1} \right)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=2q_0}^{q+2q_0} \alpha_p(Z-Y)}{q+1} \right)^{\tau_q} \right] \\
 &\leq 3 \sum_{q=0}^{q_0} (\|Z-Y\|)^{\tau_q} + 2^{K-1} \left[\eta \sup_{q=q_0} (2q_0 \|Z-Y\|)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^q \alpha_p(Z)}{q+1} \right)^{\tau_q} \right]
 \end{aligned} \quad (62)$$

□

Therefore, $\mathfrak{H}_{(\mathfrak{G}_{\tau(\cdot)})_h}^{\alpha}(\Delta, \Lambda) \subseteq$ the closure of $\mathbf{F}(\Delta, \Lambda)$. Contrarily, one has a counter example as $I_2 \in \mathfrak{H}_{(\mathfrak{G}_{((0,0,2,2,2,\dots))}_h)}^{\alpha}(\Delta, \Lambda)$, but $\tau_0 > 1$ is not verified.

Theorem 32. Suppose $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $1 < \tau_x^{(1)} < \tau_x^{(2)}$, for all $x \in \mathcal{N}$, hence

$$\mathfrak{H}_{(\mathfrak{G}_{(\tau_x^{(1)})})_h}(\Delta, \Lambda) \mathfrak{H}_{(\mathfrak{G}_{(\tau_x^{(2)})})_h}(\Delta, \Lambda) \subsetneq \mathcal{L}(\Delta, \Lambda). \quad (63)$$

Proof. Let $Z \in \mathfrak{H}_{(\mathfrak{G}_{((\tau_x^{(1)})})_h)}(\Delta, \Lambda)$, hence $(g_s) \in (\mathfrak{G}_{((\tau_x^{(1)})})_h})$, where $g_s(y) := \sum_{n=0}^{\infty} s_n(Z) y^n$. One gets

$$\sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^x s_p(Z)}{x+1} \right)^{\tau_x^{(2)}} < \sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^x s_p(Z)}{x+1} \right)^{\tau_x^{(1)}} < \infty, \quad (64)$$

then $(g_s) \in (\mathfrak{G}_{((\tau_x^{(2)})})_h})$ this implies $Z \in \mathfrak{H}_{(\mathfrak{G}_{((\tau_x^{(2)})})_h)}(\Delta, \Lambda)$.

After, if we choose $(s_x(Z))_{x=0}^{\infty}$ with $\sum_{p=0}^x s_p(Z) = (x+1)^{1-(1/\tau_x^{(1)})}$, we have $Z \in \mathcal{L}(\Delta, \Lambda)$ such that

$$\sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^x s_p(Z)}{x+1} \right)^{\tau_x^{(1)}} = \sum_{x=0}^{\infty} \frac{1}{x+1} = \infty, \quad (65)$$

$$\sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^x s_p(Z)}{x+1} \right)^{\tau_x^{(2)}} \leq \sum_{x=0}^{\infty} \left(\frac{1}{x+1} \right)^{\tau_x^{(2)}/\tau_x^{(1)}} < \infty.$$

□

Then, $Z \notin \mathfrak{H}_{(\mathfrak{G}_{((\tau_x^{(1)})})_h)}(\Delta, \Lambda)$ and $Z \in \mathfrak{H}_{(\mathfrak{G}_{((\tau_x^{(2)})})_h)}(\Delta, \Lambda)$.

Clearly, $\mathfrak{H}_{(\mathfrak{G}_{((\tau_x^{(2)})})_h)}(\Delta, \Lambda) \subset \mathcal{L}(\Delta, \Lambda)$. Next, if we put $(s_x(Z))_{x=0}^{\infty}$ with $\sum_{p=0}^x s_p(Z) = (x+1)^{1-(1/\tau_x^{(2)})}$. We have $Z \in \mathcal{L}(\Delta, \Lambda)$ such that $Z \notin \mathfrak{H}_{(\mathfrak{G}_{((\tau_x^{(2)})})_h)}(\Delta, \Lambda)$.

Theorem 33. Assume $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, hence $\mathfrak{H}_{(\mathfrak{G}_{\tau(\cdot)})_h}^{\alpha}$ is minimum.

Proof. Let $\mathfrak{H}_{(\mathfrak{G}_{\tau(\cdot)})_h}^{\alpha}(\Delta, \Lambda) = \mathcal{L}(\Delta, \Lambda)$, one has $\eta > 0$ so that $H(Z) \leq \eta \|Z\|$, where

$$H(Z) = \sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q \alpha_p(Z)}{q+1} \right)^{\tau_q}, \quad (66)$$

for all $Z \in \mathcal{L}(\Delta, \Lambda)$. According to Dvoretzky's theorem [36], with $r \in \mathcal{N}$, we get quotient spaces Δ/Y_r and subspaces M_r of Λ which can be transformed onto ℓ_2^r by isomorphisms V_r and X_r with $\|V_r\| \|V_r^{-1}\| \leq 2$ and $\|X_r\| \|X_r^{-1}\| \leq 2$. If I_r is the identity map on ℓ_2^r , T_r is the quotient map from Δ onto Δ/Y_r and J_r is the natural embedding map from M_r into Λ . □

Assume m_q is the Bernstein numbers [9], then

$$\begin{aligned}
 1 &= m_q(I_r) = m_q(X_r X_r^{-1} I_r V_r V_r^{-1}) \leq \|X_r\| m_q(X_r^{-1} I_r V_r) \|V_r^{-1}\| \\
 &= \|X_r\| m_q(J_r X_r^{-1} I_r V_r) \|V_r^{-1}\| \leq \|X_r\| d_q(J_r X_r^{-1} I_r V_r) \|V_r^{-1}\| \\
 &= \|X_r\| d_q(J_r X_r^{-1} I_r V_r T_r) \|V_r^{-1}\| \leq \|X_r\| \alpha_q(J_r X_r^{-1} I_r V_r T_r) \|V_r^{-1}\|,
 \end{aligned} \quad (67)$$

for $0 \leq q \leq r$. Then, we have

$$1 \leq (\|X_r\| \|V_r^{-1}\|)^{\tau_q} \left(\frac{\sum_{p=0}^q \alpha_p(J_r X_r^{-1} I_r V_r T_r)}{q+1} \right)^{\tau_q}. \quad (68)$$

So, there are $q \geq 1$, we obtain

$$\begin{aligned}
\sum_{q=0}^r 1 &\leq \mathfrak{Q} \|X_r\| \|V_r^{-1}\| \sum_{q=0}^r \left(\frac{\sum_{p=0}^q \alpha_p (J_r X_r^{-1} I_r V_r T_r)}{q+1} \right)^{\tau_q} \Rightarrow \sum_{q=0}^r 1 \\
&\leq \mathfrak{Q} \|X_r\| \|V_r^{-1}\| H(J_r X_r^{-1} I_r V_r T_r) \Rightarrow \sum_{q=0}^r 1 \\
&\leq \mathfrak{Q} \eta \|X_r\| \|V_r^{-1}\| \|J_r X_r^{-1} I_r V_r T_r\| \Rightarrow \sum_{q=0}^r 1 \\
&\leq \mathfrak{Q} \eta \|X_r\| \|V_r^{-1}\| \|J_r X_r^{-1}\| \|I_r\| \|V_r T_r\| \\
&= \mathfrak{Q} \eta \|X_r\| \|V_r^{-1}\| \|X_r^{-1}\| \|I_r\| \|V_r\| \leq 4\mathfrak{Q} \eta.
\end{aligned} \tag{69}$$

So there is a contradiction, if $r \rightarrow \infty$. Therefore, Δ and Λ both cannot be infinite dimensional if $\mathfrak{H}_{\mathfrak{C}_{\tau(\cdot)}}^{\alpha}(\Delta, \Lambda) = \mathcal{L}(\Delta, \Lambda)$.

As with the previous theorem, we can easily prove the following theorem.

Theorem 34. If $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, hence $\mathfrak{H}_{\mathfrak{C}_{\tau(\cdot)}}^d$ is minimum.

Lemma 35 (see [10]). If $B \in \mathcal{L}(\Delta, \Lambda)$ and $B \notin \Upsilon(\Delta, \Lambda)$, then $D \in \mathcal{L}(\Delta)$ and $M \in \mathcal{L}(\Lambda)$ with $MBDI_b = I_b$, with $b \in \mathcal{N}$.

Theorem 36 (see [10]). In general, we have

$$\mathbf{F}(\Delta) \subsetneq \Upsilon(\Delta) \subsetneq \mathcal{L}_c(\Delta) \subsetneq \mathcal{L}(\Delta). \tag{70}$$

Theorem 37. Let $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $1 < \tau_x^{(1)} < \tau_x^{(2)}$, for all $x \in \mathcal{N}$, hence

$$\begin{aligned}
&\mathcal{L}\left(\mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(2)}})}(\Delta, \Lambda), \mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(1)}})}(\Delta, \Lambda)\right) \\
&= \Upsilon\left(\mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(2)}})}(\Delta, \Lambda), \mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(1)}})}(\Delta, \Lambda)\right).
\end{aligned} \tag{71}$$

Proof. Assume $X \in \mathcal{L}(\mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(2)}})}(\Delta, \Lambda), \mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(1)}})}(\Delta, \Lambda))$ and $X \notin \Upsilon(\mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(2)}})}(\Delta, \Lambda), \mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(1)}})}(\Delta, \Lambda))$. By using Lemma 35, we have $Y \in \mathcal{L}(\mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(2)}})}(\Delta, \Lambda))$ and $Z \in \mathcal{L}(\mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(1)}})}(\Delta, \Lambda))$ so that $ZXYI_b = I_b$, hence with $b \in \mathcal{N}$, one has

$$\begin{aligned}
\|I_b\|_{\mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(1)}})}(\Delta, \Lambda)} &= \sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^x s_p(I_b)}{x+1} \right)^{\tau_x^{(1)}} \\
&\leq \|ZXY\| \|I_b\|_{\mathfrak{H}_{(\mathfrak{C}_{\tau_x^{(2)}})}(\Delta, \Lambda)} \\
&\leq \sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^x s_p(I_b)}{x+1} \right)^{\tau_x^{(2)}}.
\end{aligned} \tag{72}$$

□

This fails Theorem 32. So, $X \in \Upsilon(\mathfrak{H}_{(\mathfrak{C}_{(\tau_x^{(2)}))})_h}(\Delta, \Lambda), \mathfrak{H}_{(\mathfrak{C}_{(\tau_x^{(1)}))})_h}(\Delta, \Lambda))$.

Corollary 38. Assume $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $1 < \tau_x^{(1)} < \tau_x^{(2)}$, for all $x \in \mathcal{N}$, hence,

$$\begin{aligned}
&\mathcal{L}\left(\mathfrak{H}_{(\mathfrak{C}_{(\tau_x^{(2)}))})_h}(\Delta, \Lambda), \mathfrak{H}_{(\mathfrak{C}_{(\tau_x^{(1)}))})_h}(\Delta, \Lambda)\right) \\
&= \mathcal{L}_c\left(\mathfrak{H}_{(\mathfrak{C}_{(\tau_x^{(2)}))})_h}(\Delta, \Lambda), \mathfrak{H}_{(\mathfrak{C}_{(\tau_x^{(1)}))})_h}(\Delta, \Lambda)\right).
\end{aligned} \tag{73}$$

Proof. Evidently, as $\Upsilon \subset \mathcal{L}_c$. □

Definition 39 (see [10]). A Banach space Δ is said to be simple, if there is a unique nontrivial closed ideal in $\mathcal{L}(\Delta)$.

Theorem 40. Let $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, hence $\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}$ is simple.

Proof. Let $X \in \mathcal{L}_c(\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda))$ and $X \notin \Upsilon(\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda))$. From Lemma 35, there exist $Y, Z \in \mathcal{L}(\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda))$ with $ZXYI_b = I_b$, which gives that $I_{\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)} \in \mathcal{L}_c(\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda))$. Then, $\mathcal{L}(\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)) = \mathcal{L}_c(\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda))$; hence, $\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}$ is simple Banach space. □

Notations 41.

$$(\mathfrak{H}_{\mathcal{H}})^{\lambda} := \left\{ (\mathfrak{H}_{\mathcal{H}})^{\lambda}(\Delta, \Lambda); \Delta \text{ and } \Lambda \text{ are Banach Spaces} \right\}, \text{ where} \tag{74}$$

$$(\mathfrak{H}_{\mathcal{H}})^{\lambda}(\Delta, \Lambda) := \{X \in \mathcal{L}(\Delta, \Lambda): f_{\lambda} \in \mathcal{H}_h, \text{ where } f_{\lambda}(y) = \sum_{n=0}^{\infty} \lambda_n(T)y^n \text{ and } \|X - \lambda_x(X)I\| = 0, \text{ for every } x \in \mathcal{N}\}.$$

Theorem 42. Assume $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, hence,

$$(\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h})^{\lambda}(\Delta, \Lambda) = \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda). \tag{75}$$

Proof. Let $X \in (\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h})^{\lambda}(\Delta, \Lambda)$, hence $f_{\lambda} \in (\mathfrak{C}_{\tau(\cdot)})_h$, where $f_{\lambda}(y) = \sum_{n=0}^{\infty} \lambda_n(T)y^n$ and $\|X - \lambda_x(X)I\| = 0$, with $x \in \mathcal{N}$. We have $X = \lambda_x(X)I$, for all $x \in \mathcal{N}$, so

$$s_x(X) = s_x(\lambda_x(X)I) = |\lambda_x(X)|, \tag{76}$$

with $x \in \mathcal{N}$. One gets $f_s \in (\mathfrak{C}_{\tau(\cdot)})_h$; hence, $X \in \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$. Next, suppose $X \in \mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$. Hence, $f_s \in (\mathfrak{C}_{\tau(\cdot)})_h$. One gets

$$\sum_{x=0}^{\infty} (s_x(X))^{\tau_x} \leq \sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^x s_p(X)}{x+1} \right)^{\tau_x} < \infty. \tag{77}$$

□

Then, $\lim_{x \rightarrow \infty} s_x(X) = 0$. If $\|X - s_x(X)I\|^{-1}$ exists, with $x \in \mathcal{N}$. Then, $\|X - s_x(X)I\|^{-1}$ exists and bounded, for all $x \in \mathcal{N}$. So, $\lim_{x \rightarrow \infty} \|X - s_x(X)I\|^{-1} = \|X\|^{-1}$ exists and bounded. Since $(\mathfrak{P}_{(\mathfrak{G}_{\tau(\cdot)})_h}, H)$ is a pre-quasi mappings' ideal, one has

$$I = XX^{-1} \in \mathfrak{P}_{(\mathfrak{G}_{\tau(\cdot)})_h}(\Delta, \Lambda) \Rightarrow g_s \in \mathfrak{G}_{\tau(\cdot)} \Rightarrow \lim_{x \rightarrow \infty} s_x(I) = 0, \quad (78)$$

where $g_s(y) = \sum_{n=0}^{\infty} s_n(I)y^n$. This gives a contradiction, as $\lim_{x \rightarrow \infty} s_x(I) = 1$. Therefore, $\|X - s_x(X)I\| = 0$, with $x \in \mathcal{N}$, which explains $X \in (\mathfrak{P}_{(\mathfrak{G}_{\tau(\cdot)})_h})^\lambda(\Delta, \Lambda)$.

5. Nonexpansive Mappings on $(\mathfrak{G}_{\tau(\cdot)})_h$

In this section, we have presented some geometric properties connected with the fixed point theory in $(\mathfrak{G}_{\tau(\cdot)})_h$.

In the next part of this section, we will use the function h as

$$h(f) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f_p}|}{q+1} \right)^{\tau_q} \right]^{1/K}, \quad (79)$$

for all $f \in \mathfrak{G}_{\tau(\cdot)}$.

Definition 43 (see [37]). A sequence $\{g_p\} \subseteq \mathcal{H}_h$, is said to be ε -separated sequence for some $\varepsilon > 0$, if

$$\text{sep}(g_p) = \inf \left\{ h(g_p - g_q) : p \neq q \right\} > \varepsilon. \quad (80)$$

Definition 44. [37]. If $k \geq 2$ is an integer, a Banach space \mathcal{H}_h is said to be k -nearly uniformly convex (k -NUC) when for all $\varepsilon > 0$ one has $\delta \in (0, 1)$ so that for every sequence $\{g_p\} \subseteq B(\mathcal{H}_h)$, with $\text{sep}(g_p) \geq \varepsilon$, we have $p_1, p_2, p_3, \dots, p_k \in \mathcal{N}$.

Such that

$$h\left(\frac{g_{p_1} + g_{p_2} + g_{p_3} + \dots + g_{p_k}}{k}\right) < 1 - \delta. \quad (81)$$

Definition 45 [38]. A function h is said to hold the δ_2 -condition ($h \in \delta_2$), if for any $\varepsilon > 0$, there exists a constant $k \geq 2$ and $a > 0$ such that,

$$h(2g) \leq kh(g) + \varepsilon \text{ for each } g \in \mathcal{H}_h, \text{ with } h(g) \leq a. \quad (82)$$

If h satisfies the δ_2 -condition for any $a > 0$ with $k \geq 2$ depending on a , we say that h satisfies the strong δ_2 -condition ($\rho \in \delta_2^s$).

Theorem 46 ((see [38]), Lemma 2.1). Suppose $h \in \delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$ one has $\delta > 0$ with $|h(f+g) - h(f)| < \varepsilon, g \in \mathcal{H}_h$, with $h(f) \leq L$ and $h(g) \leq \delta$.

Theorem 47. Pick an $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then for any $L > 0$ and $\varepsilon > 0$ one has $\delta > 0$ with $|h(f+g) - h(f)| < \varepsilon$, for every $f, g \in (\mathfrak{G}_{\tau(\cdot)})_h$, so that $h(f) \leq L$ and $h(g) \leq \delta$.

Proof. Since $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $h \in \delta_2^s$. According to Theorem 46, the proof follows.

We denote $S(\mathcal{H}_h)$ and $B(\mathcal{H}_h)$ for the unit sphere and the unit ball of \mathcal{H}_h , respectively. \square

Theorem 48. Suppose $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then is k -NUC, for any integer $k \geq 2$.

Proof. Assume $\varepsilon \in (0, 1)$ and $\{f_n\} \subseteq B((\mathfrak{G}_{\tau(\cdot)})_h)$, where $f_n(y) = \sum_{i=0}^{\infty} \widehat{f_n(i)} y^i$ so that $\text{sep}(f_n) \geq \varepsilon$. For all $m \in \mathcal{N}$, suppose $f_n^m(y) = \sum_{i=0}^{\infty} \widehat{f_n^m(i)} y^i$, where $(\widehat{f_n^m(i)})_{i=0}^{\infty} = (0, 0, 0, \dots, \widehat{f_n(m)}, \widehat{f_n(m+1)}, \dots)$. As for all $i \in \mathcal{N}$, $(\widehat{f_n(i)})_{n=0}^{\infty} \in \ell_\infty$, from the diagonal method, one has a subsequence (f_{n_j}) of (f_n) with $(\widehat{f_{n_j}(i)})$ converges for all $i \in \mathcal{N}$, $0 \leq i \leq m$. One obtains an increasing sequence of positive integers (t_m) so that $\text{sep}((f_{n_j})_{j > t_m}) \geq \varepsilon$. Therefore, one has a sequence of positive integers $(r_m)_{m=0}^{\infty}$ with $r_0 < r_1 < r_2 < \dots$, so that

$$h^K(f_{r_m}^m) \geq \frac{\varepsilon}{2}, \quad (83)$$

for all $m \in \mathcal{N}$. For constant integer $k \geq 2$, assume $\varepsilon_1 = ((k^{p_0-1} - 1)/(k - 1)k^{p_0})(\varepsilon/4)$ from Theorem 47, one gets $\delta > 0$ with

$$|h^K(f+g) - h^K(f)| < \varepsilon_1. \quad (84)$$

\square

If $h^K(g) \leq \delta$. As $h^K(f_n) \leq 1$, for every $n \in \mathcal{N}$, one has positive integers $m_i (i = 0, 1, 2, \dots, k-2)$ with $m_0 < m_1 < m_2 < \dots < m_{k-2}$ with $h^K(f_{m_i}^{m_i}) \leq \delta$. Define $m_{k-1} = m_{k-2} + 1$. From inequality (83), one can see $h(f_{r_{m_k}}^{m_k}) \geq \varepsilon/2$. Suppose $p_i = i$ for $0 \leq i \leq k-2$ and $p_{k-1} = r_{m_{k-1}}$. According to inequalities (83), (84), and convexity of $J_n(u) = |u|^{\tau_n}$ for every $n \in \mathcal{N}$, one has

$$\begin{aligned} & h^K\left(\frac{f_{p_0} + f_{p_1} + f_{p_2} + \dots + f_{p_{k-1}}}{k}\right) \\ &= \sum_{n=0}^{\infty} \left(\frac{\sum_{i=0}^n |\widehat{f_{p_2}(i)} + \widehat{f_{p_3}(i)} + \dots + \widehat{f_{p_{k-1}}(i)}|/k}{n+1} \right)^{\tau_n} \\ &= \sum_{n=0}^{m_1-1} \left(\frac{\sum_{i=0}^n |\widehat{f_{p_2}(i)} + \widehat{f_{p_3}(i)} + \dots + \widehat{f_{p_{k-1}}(i)}|/k}{n+1} \right)^{\tau_n} \\ &+ \sum_{n=m_1}^{\infty} \left(\frac{\sum_{i=0}^n |\widehat{f_{p_2}(i)} + \widehat{f_{p_3}(i)} + \dots + \widehat{f_{p_{k-1}}(i)}|/k}{n+1} \right)^{\tau_n} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{m_1-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_2}(i)} + f_{p_3}(i) + \dots + \widehat{f_{p_{k-1}}(i)} \right| / k}{n+1} \right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{\infty} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_2}(i)} + f_{p_3}(i) + \dots + \widehat{f_{p_{k-1}}(i)} \right| / k}{n+1} \right)^{\tau_n} + \varepsilon_1 \\
&\leq \sum_{n=0}^{m_1-1} \frac{1}{k} \sum_{j=0}^{k-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_j}(i)} \right|}{n+1} \right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{m_2-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_2}(i)} + f_{p_3}(i) + \dots + \widehat{f_{p_{k-1}}(i)} \right| / k}{n+1} \right)^{\tau_n} \\
&\quad + \sum_{n=m_2}^{\infty} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_2}(i)} + f_{p_3}(i) + \dots + \widehat{f_{p_{k-1}}(i)} \right| / k}{n+1} \right)^{\tau_n} + \varepsilon_1 \\
&\leq \sum_{n=0}^{m_1-1} \frac{1}{k} \sum_{j=0}^{k-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_j}(i)} \right|}{n+1} \right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{m_2-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_2}(i)} + f_{p_3}(i) + \dots + \widehat{f_{p_{k-1}}(i)} \right| / k}{n+1} \right)^{\tau_n} \\
&\quad + \sum_{n=m_2}^{\infty} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_2}(i)} + f_{p_3}(i) + \dots + \widehat{f_{p_{k-1}}(i)} \right| / k}{n+1} \right)^{\tau_n} + 2\varepsilon_1 \\
&\leq \sum_{n=0}^{m_1-1} \frac{1}{k} \sum_{j=0}^{k-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_j}(i)} \right|}{n+1} \right)^{\tau_n} + \sum_{n=m_1}^{m_2-1} \frac{1}{k} \sum_{j=1}^{k-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_j}(i)} \right|}{n+1} \right)^{\tau_n} \\
&\quad + \sum_{n=m_2}^{m_3-1} \frac{1}{k} \sum_{j=2}^{k-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_j}(i)} \right|}{n+1} \right)^{\tau_n} + \dots + \sum_{n=m_{k-1}}^{m_k-1} \frac{1}{k} \sum_{j=k-2}^{k-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_j}(i)} \right|}{n+1} \right)^{\tau_n} \\
&\quad + \sum_{n=m_k}^{\infty} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_k}(i)} \right| / k}{n+1} \right)^{\tau_n} + (k-1)\varepsilon_1 \\
&\leq \frac{h^k(f_{p_0} + f_{p_1} + f_{p_2} + \dots + f_{p_{k-2}})}{k} + \frac{1}{k} \sum_{n=0}^{m_k-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_k}(i)} \right|}{n+1} \right)^{\tau_n} \\
&\quad + \sum_{n=m_k}^{\infty} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_k}(i)} \right| / k}{n+1} \right)^{\tau_n} + (k-1)\varepsilon_1 \\
&\leq \frac{k-1}{k} + \frac{1}{k} \sum_{n=0}^{m_k-1} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_k}(i)} \right|}{n+1} \right)^{\tau_n} \\
&\quad + \frac{1}{k^{\tau_0}} \sum_{n=m_k}^{\infty} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_k}(i)} \right|}{n+1} \right)^{\tau_n} + (k-1)\varepsilon_1 \\
&\leq 1 - \frac{1}{k} + \frac{1}{k} \left(1 - \sum_{n=m_k}^{\infty} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_k}(i)} \right|}{n+1} \right)^{\tau_n} \right)
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{k^{\tau_0}} \sum_{n=m_k}^{\infty} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_k}(i)} \right|}{n+1} \right)^{\tau_n} + (k-1)\varepsilon_1 \\
&= 1 + (k-1)\varepsilon_1 - \left(\frac{k^{\tau_0-1} - 1}{k^{\tau_0}} \right) \sum_{n=m_k}^{\infty} \left(\frac{\sum_{i=0}^n \left| \widehat{f_{p_k}(i)} \right|}{n+1} \right)^{\tau_n} \\
&\leq 1 + (k-1)\varepsilon_1 - \left(\frac{k^{\tau_0-1} - 1}{k^{\tau_0}} \right) \frac{\varepsilon}{2} = 1 - \left(\frac{k^{\tau_0-1} - 1}{k^{\tau_0}} \right) \frac{\varepsilon}{4}. \quad (85)
\end{aligned}$$

So, $(\mathfrak{C}_{\tau(\cdot)})_h$ is k -NUC.

Recall that k -NUC implies reflexivity.

Definition 49 (see [39]). A Banach space \mathcal{H}_h holds the uniform Opial property, if for all $\varepsilon > 0$ one has $\gamma > 0$ so that for every weakly null sequence $\{f_n\} \subset S(\mathcal{H}_h)$ and $f \in \mathcal{H}_h$ so that $h(f) \geq \varepsilon$, then

$$1 + \gamma \leq \liminf_{n \rightarrow \infty} h(f_n + f). \quad (86)$$

Definition 50 (see [40]). For a bounded subset $E \subset \mathcal{H}_h$, the set-measure of noncompactness defined by

$$\alpha(E) = \inf \{ \xi > 0 : E \text{ can be covered by finitely many sets of diameter } \leq \xi \}. \quad (87)$$

Definition 51 (see [41, 42]). The ball-measure of noncompactness is defined by

$$\beta(E) = \inf \{ \xi > 0 : E \text{ can be covered by finitely many balls of diameter } \leq \xi \}. \quad (88)$$

Definition 52 (see [43]). For a subset $E \subset \mathcal{H}_h$ is said to be α -minimal if $\alpha(C) = \alpha(E)$, for any infinite subset C of E .

Definition 53 (see [43]). The packing rate of a Banach space \mathcal{H}_h is denoted by $\gamma(\mathcal{H}_h)$, and the formula defines it

$$\gamma(\mathcal{H}_h) = \frac{\delta(\mathcal{H}_h)}{\sigma(\mathcal{H}_h)}, \quad (89)$$

where $\delta(\mathcal{H}_h)$ and $\sigma(\mathcal{H}_h)$ are defined as the supremum and the infimum, respectively, of the set

$$\left\{ \frac{\beta(E)}{\alpha(E)} : E \subset \mathcal{H}_h, E \text{ is } \alpha\text{-minimal}, \alpha(E) > 0 \right\}. \quad (90)$$

Definition 54 (see [41]). The function Δ is said to be the modulus of noncompact convexity, if for every $\xi > 0$ define

$$\Delta(\xi) = \inf \left\{ 1 - \inf_{f \in E} h(f) : E \text{ is a closed convex subset of } B(\mathcal{H}_h) \text{ with } \beta(E) \geq \xi \right\}. \quad (91)$$

Definition 55 (see [39]). A Banach space \mathcal{H}_h is said to be hold property (L), when $\lim_{\varepsilon \rightarrow 1^-} \Delta(\varepsilon) = 1$.

Definition 56. An operator $V : \mathcal{H}_h \longrightarrow \mathcal{H}_h$ is said to be a h -contraction, if one gets $\alpha \in [0, 1)$ with $h(Vg - Vf) \leq \alpha h(g - f)$, for all $g, f \in \mathcal{H}_h$. The operator V is said to be h -non-expansive, when $\alpha = 1$. An element $g \in \mathcal{H}_h$ is said to be a fixed point of V , when $V(g) = g$.

Theorem 57 (see [39]).

(1) Suppose a Banach space \mathcal{H}_h holds property (L), then it has the fixed point property, i.e., for every nonexpansive self-mapping of a nonempty, closed, bounded, convex subset has a fixed point

(2) A Banach space \mathcal{H}_h holds property (L), if and only if, it is reflexive and has the uniform Opial property

Theorem 58. Suppose $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{C}_{\tau(\cdot)})_h$ has the uniform Opial property.

Proof. Let $\varepsilon > 0$ one finds a positive number $\varepsilon_0 \in (0, \varepsilon)$ with

$$1 + \frac{\varepsilon^K}{2} > (1 + \varepsilon_0)^K. \quad (92)$$

If $f \in (\mathfrak{C}_{\tau(\cdot)})_h$ and $h(f) \geq \varepsilon$, one has $n_1 \in \mathcal{N}$ with

$$\sum_{n=n_1+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^n |\widehat{f(i)}| \right)^{\tau_n} < \left(\frac{\varepsilon_0}{4} \right)^K. \quad (93)$$

Therefore, one gets

$$h \left(\sum_{i=n_1+1}^{\infty} \widehat{f(i)} e^{(i)} \right) < \frac{\varepsilon_0}{4} < \frac{\varepsilon}{4}. \quad (94)$$

Also, one has

$$\begin{aligned} \varepsilon^K &\leq \sum_{n=0}^{n_1} \left(\frac{1}{n+1} \sum_{i=0}^n |\widehat{f(i)}| \right)^{\tau_n} + \sum_{n=n_1+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^n |\widehat{f(i)}| \right)^{\tau_n} \\ &< \sum_{n=0}^{n_1} \left(\frac{1}{n+1} \sum_{i=0}^n |\widehat{f(i)}| \right)^{\tau_n} + \left(\frac{\varepsilon_0}{4} \right)^K \\ &< \sum_{n=0}^{n_1} \left(\frac{1}{n+1} \sum_{i=0}^n |\widehat{f(i)}| \right)^{\tau_n} + \frac{\varepsilon^K}{4}, \end{aligned} \quad (95)$$

if

$$\frac{3\varepsilon^K}{4} \leq \sum_{n=0}^{n_1} \left(\frac{1}{n+1} \sum_{i=0}^n |\widehat{f(i)}| \right)^{\tau_n}. \quad (96)$$

For any weakly null sequence $\{f_m\} \subset S((\mathfrak{C}_{\tau(\cdot)})_h)$, in virtue of $\widehat{f_m(i)} \longrightarrow 0$ for $i = 0, 1, 2, \dots$, one has $m_0 \in \mathcal{N}$ with

$$h \left(\sum_{i=0}^{n_1} \widehat{f_m(i)} e^{(i)} \right) < \frac{\varepsilon_0}{4}, \quad (97)$$

for $m > m_0$. One can see

$$\begin{aligned} h(f_m + f) &= h \left(\sum_{i=0}^{n_1} (\widehat{f_m(i)} + \widehat{f(i)}) e^{(i)} + \sum_{i=n_1+1}^{\infty} (\widehat{f_m(i)} + \widehat{f(i)}) e^{(i)} \right) \\ &\geq h \left(\sum_{i=0}^{n_1} \widehat{f(i)} e^{(i)} + \sum_{i=n_1+1}^{\infty} \widehat{f_m(i)} e^{(i)} \right) \\ &\quad - h \left(\sum_{i=0}^{n_1} \widehat{f_m(i)} e^{(i)} \right) - h \left(\sum_{i=n_1+1}^{\infty} \widehat{f(i)} e^{(i)} \right) \\ &\geq h \left(\sum_{i=0}^{n_1} \widehat{f(i)} e^{(i)} + \sum_{i=n_1+1}^{\infty} \widehat{f_m(i)} e^{(i)} \right) - \frac{\varepsilon_0}{2}, \end{aligned} \quad (98)$$

if $m > m_0$. For $a := \sum_{i=0}^{n_1} |\widehat{f(i)}|$, one obtains

$$\begin{aligned} h^K \left(\sum_{i=0}^{n_1} \widehat{f(i)} e^{(i)} + \sum_{i=n_1+1}^{\infty} \widehat{f_m(i)} e^{(i)} \right) &= \sum_{n=0}^{n_1} \left(\frac{1}{n+1} \sum_{i=0}^n |\widehat{f(i)}| \right)^{\tau_n} \\ &\quad + \sum_{n=n_1+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^n (a + |\widehat{f_m(i)}|) \right)^{\tau_n} \\ &\geq \sum_{n=0}^{n_1} \left(\frac{1}{n+1} \sum_{i=0}^n |\widehat{f(i)}| \right)^{\tau_n} + \sum_{n=n_1+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^n |\widehat{f_m(i)}| \right)^{\tau_n} \\ &\geq \frac{3\varepsilon^K}{4} + \left(1 - \frac{\varepsilon^K}{4} \right) = 1 + \frac{\varepsilon^K}{2} > (1 + \varepsilon_0)^K. \end{aligned} \quad (99)$$

Combining this with the previous inequality, one has

$$\begin{aligned} h(f_m + f) &\geq h \left(\sum_{i=0}^{n_1} \widehat{f(i)} e^{(i)} + \sum_{i=n_1+1}^{\infty} \widehat{f_m(i)} e^{(i)} \right) - \frac{\varepsilon_0}{2} \\ &\geq 1 + \varepsilon_0 - \frac{\varepsilon_0}{2} = 1 + \frac{\varepsilon_0}{2}. \end{aligned} \quad (100)$$

□

Therefore, the space $(\mathfrak{C}_{\tau(\cdot)})_h$ has the uniform Opial property.

From Theorem 58 and the reflexivity of the space $(\mathfrak{C}_{\tau(\cdot)})_h$, by applying Theorem 47, we get the following.

Corollary 59. If $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{C}_{\tau(\cdot)})_h$ has the property (L) and the fixed point property.

Definition 60. \mathcal{H}_h holds the h -normal structure property, if and only if, for every nonempty h -bounded, h -convex, and h -closed subset Γ of \mathcal{H}_h not decreased to one point, one has $f \in \Gamma$ with

$$\sup_{g \in \Gamma} h(f - g) < \delta_h(\Gamma) := \sup \{h(f - g) : f, g \in \Gamma\} < \infty. \quad (101)$$

Definition 61 (see [44]). The weakly convergent sequence coefficient of a Banach space \mathcal{H}_h , denoted by $\text{WCS}(\mathcal{H}_h)$, is defined as follows:

$$\begin{aligned} \text{WCS}(\mathcal{H}_h) &= \inf \{A(\{f_n\}) : \{f_n\}_{n=1}^\infty \subset S(\mathcal{H}_h), A(\{f_n\}) \\ &= A_1(\{f_n\}), f_n \xrightarrow{w} 0\}, \end{aligned} \quad (102)$$

where

$$\begin{aligned} A(\{f_n\}) &= \limsup_{n \rightarrow \infty} \left\{ h(f_i - f_j) : i, j \geq n, i \neq j \right\}, \\ A_1(\{f_n\}) &= \liminf_{n \rightarrow \infty} \left\{ h(f_i - f_j) : i, j \geq n, i \neq j \right\}. \end{aligned} \quad (103)$$

Theorem 62 (see [45]). A reflexive Banach space \mathcal{H}_h such that $\text{WCS}(\mathcal{H}_h) > 1$ has the normal structure property.

Theorem 63. If $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{E}_{\tau(\cdot)})_h$ holds the h -normal structure property.

Proof. Take any $\varepsilon > 0$ and an asymptotic equidistant sequence $\{f_n\} \subset S((\mathfrak{E}_{\tau(\cdot)})_h)$ with $f_n \xrightarrow{w} 0$ and let $v_1 = f_1$. One has $i_1 \in \mathcal{N}$ with

$$h\left(\sum_{i=i_1+1}^\infty \widehat{v_1(i)} e^{(i)}\right) < \varepsilon. \quad (104)$$

As $f_n \xrightarrow{w} 0$ coordinate-wise, one gets $n_2 \in \mathcal{N}$ with

$$h\left(\sum_{i=1}^{i_1} \widehat{f_n(i)} e^{(i)}\right) < \varepsilon. \quad (105)$$

For $n \geq n_2$, put $v_2 = f_{n_2}$, one gets $i_2 > i_1$ with

$$h\left(\sum_{i=i_2+1}^\infty \widehat{v_2(i)} e^{(i)}\right) < \varepsilon. \quad (106)$$

As $f_n(i) \xrightarrow{w} 0$ coordinate-wise, one obtains $n_3 \in \mathcal{N}$ with

$$h\left(\sum_{i=1}^{i_2} \widehat{f_n(i)} e^{(i)}\right) < \varepsilon. \quad (107)$$

For $n \geq n_3$. By induction, one has a subsequence $\{v_n\}$ of $\{f_n\}$ with

$$h\left(\sum_{i=i_n+1}^\infty \widehat{v_n(i)} e^{(i)}\right) < \varepsilon, h\left(\sum_{i=1}^{i_n} \widehat{v_{n+1}(i)} e^{(i)}\right) < \varepsilon. \quad (108)$$

Take

$$z_n = \sum_{i=i_{n-1}+1}^{i_n} \widehat{v_n(i)} e^{(i)}, \quad (109)$$

for $n = 2, 3, \dots$. So,

$$\begin{aligned} 1 \geq h(z_n) &= h\left(\sum_{i=1}^\infty \widehat{v_n(i)} e^{(i)} - \sum_{i=1}^{i_n-1} \widehat{v_n(i)} e^{(i)} - \sum_{i=i_n+1}^\infty \widehat{v_n(i)} e^{(i)}\right) \\ &\geq h\left(\sum_{i=1}^\infty \widehat{v_n(i)} e^{(i)}\right) - h\left(\sum_{i=1}^{i_n-1} \widehat{v_n(i)} e^{(i)}\right) - h\left(\sum_{i=i_n+1}^\infty \widehat{v_n(i)} e^{(i)}\right) > 1 - 2\varepsilon. \end{aligned} \quad (110)$$

For every $n, m \in \mathcal{N}$ so that $n \neq m$, one can see

$$\begin{aligned} h(v_n - v_m) &= h\left(\sum_{i=1}^\infty \widehat{v_n(i)} e^{(i)} - \sum_{i=1}^\infty \widehat{v_m(i)} e^{(i)}\right) \\ &\geq h\left(\sum_{i=i_{n-1}+1}^{i_n} \widehat{v_n(i)} e^{(i)} - \sum_{i=i_{m-1}+1}^{i_m} \widehat{v_m(i)} e^{(i)}\right) \\ &\quad - h\left(\sum_{i=1}^{i_n-1} \widehat{v_n(i)} e^{(i)}\right) - h\left(\sum_{i=i_n+1}^\infty \widehat{v_n(i)} e^{(i)}\right) \\ &\quad - h\left(\sum_{i=1}^{i_m-1} \widehat{v_m(i)} e^{(i)}\right) - h\left(\sum_{i=i_m+1}^\infty \widehat{v_m(i)} e^{(i)}\right) \\ &\geq h(z_n - z_m) - 4\varepsilon, \end{aligned} \quad (111)$$

which gives $A(\{f_n\}) = A(\{v_n\}) \geq A(\{z_n\}) - 4\varepsilon$. Take $u_n = z_n / \|z_n\|$, for $n = 2, 3, \dots$. Then,

$$u_n \in S\left((\mathfrak{E}_{\tau(\cdot)})_h\right); \quad (112)$$

$$A(\{f_n\}) \geq 1 - \varepsilon A(\{u_n\}) - 4\varepsilon. \quad (113)$$

On the other hand,

$$h(v_n - v_m) \leq h(z_n - z_m) + 4\varepsilon \leq h(u_n - u_m) + 4\varepsilon, \quad (114)$$

for any $n, m \in \mathcal{N}$ with $n \neq m$. Therefore,

$$A(\{u_n\}) \geq A(\{f_n\}) - 4\varepsilon. \quad (115)$$

By the arbitrariness of $\varepsilon > 0$, we have from the relations (112), (113), and (115) that

$$\text{WCS}\left((\mathfrak{E}_{\tau(\cdot)})_h\right) = \inf \{A(\{u_n\})\}, \quad (116)$$

such that

$$\begin{aligned}
u_n &= \sum_{i=i_{n-1}+1}^{i_n} \widehat{u_n(i)} e^{(i)} \in S\left(\left(\mathfrak{E}_{\tau(\cdot)}\right)_h\right), 0 \\
&= i_0 < i_1 < \dots, u_n \xrightarrow{w} 0 \text{ and } \{u_n\} \text{ is asymptotic equidistant.}
\end{aligned} \tag{117}$$

Take $m \in \mathcal{N}$ large enough such that

$$\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k}\right)^{\tau_k} < \varepsilon, \tag{118}$$

where $b := \sum_{i=i_{n-1}+1}^{i_n} |u_n(i)|$. One gets for

$$\begin{aligned}
h^K(u_n - u_m) &= \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k} \sum_{i=1}^k |\widehat{u_n(i)}|\right)^{\tau_k} \\
&\quad + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k} \left(b + \sum_{i=1}^k |\widehat{u_m(i)}|\right)\right)^{\tau_k} \\
&\geq \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k} \sum_{i=1}^k |\widehat{u_n(i)}|\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^k |\widehat{u_m(i)}|\right)^{\tau_k} \\
&= \sum_{k=i_{n-1}+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^k |\widehat{u_n(i)}|\right)^{\tau_k} - \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k}\right)^{\tau_k} \\
&\quad + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^k |\widehat{u_m(i)}|\right)^{\tau_k} > 1 - \varepsilon + 1 = 2 - \varepsilon,
\end{aligned} \tag{119}$$

that is $A_n(\{u_n\}) \geq (2 - \varepsilon)^{1/K}$. Note that

$$\begin{aligned}
&\left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k} \left(b + \sum_{i=1}^k |\widehat{u_m(i)}|\right)\right)^{\tau_k} \right]^{1/K} \\
&\leq \left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k}\right)^{\tau_k} \right]^{1/K} + \left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^k |\widehat{u_m(i)}|\right)^{\tau_k} \right]^{1/K} < \varepsilon^{1/K} + 1.
\end{aligned} \tag{120}$$

Therefore,

$$\begin{aligned}
h^K(u_n - u_m) &= \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k} \sum_{i=1}^k |\widehat{u_n(i)}|\right)^{\tau_k} \\
&\quad + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k} \left(b + \sum_{i=1}^k |\widehat{u_m(i)}|\right)\right)^{\tau_k} \\
&\leq \sum_{k=i_{n-1}+1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^k |\widehat{u_m(i)}|\right)^{\tau_k} \\
&\quad + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k} \left(b + \sum_{i=1}^k |\widehat{u_m(i)}|\right)\right)^{\tau_k} \\
&\leq 1 + (1 + \varepsilon^{1/K})^K,
\end{aligned} \tag{121}$$

with $n, m \in \mathcal{N}$ and $n \neq m$. Therefore, $A_n(\{u_n\}) \leq$

$(1 + (1 + \varepsilon^{1/K})^K)^{1/K}$ and, by the arbitrariness of $\varepsilon > 0$, one has $\text{WCS}((\mathfrak{E}_{\tau(\cdot)})_h) = 2^{1/K}$. From Theorems 48 and 62, then, the function space $(\mathfrak{E}_{\tau(\cdot)})_h$ has the h -normal structure property. \square

Theorem 64 (see [46]). *If \mathcal{H}_h is reflexive Banach space with the uniform Opial property, one has $\gamma(\mathcal{H}_h) = 2/\text{WCS}(\mathcal{H}_h)$.*

Theorem 65. *If $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$, then $\gamma((\mathfrak{E}_{\tau(\cdot)})_h) = 2^{1-(1/K)}$.*

Proof. Since $(\mathfrak{E}_{\tau(\cdot)})_h$ is reflexive Banach space with the uniform Opial property, one obtains

$$\gamma((\mathfrak{E}_{\tau(\cdot)})_h) = \frac{2}{\text{WCS}((\mathfrak{E}_{\tau(\cdot)})_h)} = 2^{1-1/K}. \tag{122}$$

\square

Theorem 66. *If $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$ and $\mathfrak{B} : (\mathfrak{E}_{\tau(\cdot)})_h \rightarrow (\mathfrak{E}_{\tau(\cdot)})_h$ is h -contraction mapping, where $h(f) = [\sum_{q=0}^{\infty} (\sum_{p=0}^q |\widehat{f_p}|/(q+1))^{\tau_q}]^{1/K}$, for every $f \in \mathfrak{E}_{\tau(\cdot)}$, then \mathfrak{B} has a unique fixed point.*

Proof. Let the setups be satisfied. For every $f \in (\mathfrak{E}_{\tau(\cdot)})_h$, then $\mathfrak{B}^p f \in (\mathfrak{E}_{\tau(\cdot)})_h$. As \mathfrak{B} is a h -contraction mapping, one gets

$$\begin{aligned}
h(\mathfrak{B}^{p+1}f - \mathfrak{B}^p f) &\leq \alpha h(\mathfrak{B}^p f - \mathfrak{B}^{p-1}f) \\
&\leq \alpha^2 h(\mathfrak{B}^{p-1}f - \mathfrak{B}^{p-2}f) \leq \dots \leq \alpha^p h(\mathfrak{B}f - f).
\end{aligned} \tag{123}$$

So, for all $p, q \in \mathcal{N}$ so that $q > p$, one has

$$h(\mathfrak{B}^q f - \mathfrak{B}^p f) \leq \alpha^p h(\mathfrak{B}^{q-p} f - f). \tag{124}$$

Therefore, $\{\mathfrak{B}^p f\}$ is a Cauchy sequence in $(\mathfrak{E}_{\tau(\cdot)})_h$. Since the space $(\mathfrak{E}_{\tau(\cdot)})_h$ is prequasi-Banach (ssfps). One gets $g \in (\mathfrak{E}_{\tau(\cdot)})_h$ with $\lim_{p \rightarrow \infty} \mathfrak{B}^p f = g$, to prove that $\mathfrak{B}g = g$. According to Theorem 13, h verifies the Fatou property; one can see

$$h(\mathfrak{B}g - g) \leq \sup_i \inf_{p \geq i} h(\mathfrak{B}^{p+1}f - \mathfrak{B}^p f) \leq \sup_i \inf_{p \geq i} \alpha^p h(\mathfrak{B}f - f) = 0, \tag{125}$$

so $\mathfrak{B}g = g$. Then, g is a fixed point of \mathfrak{B} . To prove that the fixed point is unique, let us have two different fixed points $f, g \in (\mathfrak{E}_{\tau(\cdot)})_h$ of \mathfrak{B} . One obtains

$$h(f - g) \leq h(\mathfrak{B}f - \mathfrak{B}g) \leq \alpha h(f - g). \tag{126}$$

So, $f = g$. \square

Example 7. Assume

$$V : \left(\mathfrak{C} \left(\left(\frac{2q+3}{q+2} \right)_{q=0}^{\infty} \right) \right)_h \longrightarrow \left(\mathfrak{C} \left(\left(\frac{2q+3}{q+2} \right)_{q=0}^{\infty} \right) \right)_h, \quad (127)$$

where

$$h(g) = \sqrt{\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{g}_p|}{q+1} \right)^{2q+3/q+2}}, \quad (128)$$

for every $g \in \mathfrak{C}(((2q+3)/(q+2))_{q=0}^{\infty})$ and $V(g) = g/4$.

Since for all $f_1, f_2 \in (\mathfrak{C}(((2q+3)/(q+2))_{q=0}^{\infty}))_h$, one gets

$$h(Vf_1 - Vf_2) = h\left(\frac{f_1}{4} - \frac{f_2}{4}\right) \leq \frac{1}{\sqrt[3]{64}}(h(f_1 - f_2)). \quad (129)$$

So V is h -contraction. Assume $V : \Gamma \longrightarrow \Gamma$ with $V(g) = g/4$, where

$$\Gamma = \left\{ f \in \left(\mathfrak{C} \left(\left(\frac{2q+3}{q+2} \right)_{q=0}^{\infty} \right) \right)_h : \widehat{f}_0 = \widehat{f}_1 = 0 \right\}. \quad (130)$$

Since V is h -contraction. So, it is h -nonexpansive. By Corollary 59, V holds a fixed point ϑ in Γ .

6. Applications to Nonlinear Summable Equations

Numerous authors, for example in [47], have examined nonlinear summable equations such as (132). This section is dedicated to locating a solution to (132) in $(\mathfrak{C}_{\tau(\cdot)})_h$, where the conditions $(\tau_q)_{q \in \mathcal{N}} \in \ell_{\infty} \cap \mathbf{I}$ with $\tau_0 > 1$ are satisfied and

$$h(f) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^q |\widehat{f}_p|}{q+1} \right)^{\tau_q} \right]^{1/K}, \quad (131)$$

for every $f \in \mathfrak{C}_{\tau(\cdot)}$. Take a look at the equations that are summable:

$$\widehat{g}_a = \widehat{r}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m), \quad (132)$$

and assume $W : (\mathfrak{C}_{\tau(\cdot)})_h \longrightarrow (\mathfrak{C}_{\tau(\cdot)})_h$ defined by

$$(W(g))(z) = \sum_{a=0}^{\infty} \left(\widehat{r}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m) \right) z^a. \quad (133)$$

Theorem 67. *The summable equations (132) have only one solution in $(\mathfrak{C}_{\tau(\cdot)})_h$ if $A : \mathcal{N}^2 \longrightarrow \mathbb{C}, f : \mathcal{N} \times \mathbb{C} \longrightarrow \mathbb{C}$,*

$\widehat{r} : \mathcal{N} \longrightarrow \mathbb{C}, \widehat{t} : \mathcal{N} \longrightarrow \mathbb{C}$, assume there is $\kappa \in \mathbb{C}$ so that $\sup_q |\kappa|^{\tau_q/K} \in [0, 1)$ and for every $a \in \mathcal{N}$, we have

$$\left| \sum_{m=0}^{\infty} A(a, m) \left(f(m, \widehat{g}_m) - f(m, \widehat{t}_m) \right) \right| \leq |\kappa| |\widehat{g}_a - \widehat{t}_a|. \quad (134)$$

Proof. Let the setups be verified. Consider the mapping $W : (\mathfrak{C}_{\tau(\cdot)})_h \longrightarrow (\mathfrak{C}_{\tau(\cdot)})_h$ defined by (133). We have

$$\begin{aligned} h(Wg - Wt) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{Wg}_a - \widehat{Wt}_a|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\sum_{m=0}^{\infty} A(a, m) [f(m, \widehat{g}_m) - f(m, \widehat{t}_m)]|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \sup_q |\kappa|^{\tau_q/K} \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{g}_a - \widehat{t}_a|}{q+1} \right)^{\tau_q} \right]^{1/K}. \end{aligned} \quad (135)$$

□

According to Theorem 66, one obtains a unique solution of equation (132) in $(\mathfrak{C}_{\tau(\cdot)})_h$.

Example 1. Assume the function space $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$, where

$$h(f) = \sqrt[3]{\sum_{a=0}^{\infty} \left(\frac{\sum_{b=0}^a |\widehat{f}_b|}{a+1} \right)^{3a+2/a+1}}, \quad (136)$$

for all $f \in \mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty})$.

$$\widehat{g}_a = 5^{-(2a+3i)} + \sum_{m=0}^{\infty} (-1)^{ai+3m} \left(\frac{\cos |\widehat{g}_a|}{\sinh |\widehat{g}_a| + \sin ma + 1} \right)^q, \quad (137)$$

where $q > 0$, $i^2 = -1$ and let $W : (\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h \longrightarrow (\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$ defined by

$$(W(g))(z) = \sum_{a=0}^{\infty} \left(5^{-(2a+3i)} + \sum_{m=0}^{\infty} (-1)^{ai+3m} \left(\frac{\cos |\widehat{g}_a|}{\sinh |\widehat{g}_a| + \sin ma + 1} \right)^q \right) z^a. \quad (138)$$

It is easy to see that

$$\left| \sum_{m=0}^{\infty} (-1)^{ai} \left(\frac{\cos |\widehat{g}_a|}{\sinh |\widehat{g}_a| + \sin ma + 1} \right)^q ((-1)^{3m} - (-1)^{3m}) \right| \leq \frac{1}{3} |\widehat{g}_a - \widehat{t}_a|. \quad (139)$$

By Theorem 67, the summable equations (137) have one solution in $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$.

Example 2. Given the function space $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$, where

$$h(f) = \sqrt{\sum_{a=0}^{\infty} \left(\frac{\sum_{b=0}^a |\widehat{f}_b|}{a+1} \right)^{2a+3/a+2}}, \quad (140)$$

for all $f \in \mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty})$. Consider the summable equations (137) with $a \geq 2$ and let $W : \mathfrak{E} \rightarrow \mathfrak{E}$, where $\mathfrak{E} = \{f \in (\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h : \widehat{f}_0 = \widehat{f}_1 = 0\}$, defined by

$$(W(f))(z) = \sum_{a=2}^{\infty} \left(5^{-(2a+3i)} + \sum_{m=0}^{\infty} (-1)^{ai+3m} \left(\frac{\cos |\widehat{f}_a|}{\sinh |\widehat{f}_a| + \sin ma + 1} \right)^q \right) z^a. \quad (141)$$

Clearly, \mathfrak{E} is a nonempty, h -convex, h -closed, and h -bounded subset of $(\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h$. It is easy to see that

$$\left| \sum_{m=0}^{\infty} (-1)^{ai} \left(\frac{\cos |\widehat{g}_a|}{\sinh |\widehat{g}_a| + \sin ma + 1} \right)^q \left((-1)^{3m} - (-1)^{3m} \right) \right| \leq \frac{1}{9} |\widehat{g}_a - \widehat{t}_a|. \quad (142)$$

By Theorem 67 and Corollary 59, the summable equations (137) with $a \geq 2$ have a solution in \mathfrak{E} .

Example 3. Assume the function space $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$, where

$$h(g) = \sqrt[3]{\sum_{a=0}^{\infty} \left(\frac{\sum_{b=0}^a |\widehat{g}_b|}{a+1} \right)^{3a+2/a+1}}, \quad (143)$$

for all $g \in \mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty})$.

Consider the non-linear difference equations,

$$\widehat{g}_a = e^{-(2a+3i)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1) \cosh(3mi-a) \cos^p |\widehat{g}_{a-2}|}{\sinh^q |\widehat{g}_{a-1}| + \sin ma + 1}, \quad (144)$$

where $\widehat{g}_{-2}, \widehat{g}_{-1}, p, q > 0$, $i^2 = -1$ and let $W : (\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h \rightarrow (\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$ defined by

$$(W(g))(z) = \sum_{a=0}^{\infty} \left(e^{-(2a+3i)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1) \cosh(3mi-a) \cos^p |\widehat{g}_{a-2}|}{\sinh^q |\widehat{g}_{a-1}| + \sin ma + 1} \right) z^a. \quad (145)$$

It is easy to see that

$$\left| \sum_{m=0}^{\infty} \frac{\cosh(3mi-a) \cos^p |\widehat{g}_{a-2}|}{\sinh^q |\widehat{g}_{a-1}| + \sin ma + 1} (\tan(2m+1) - \tan(2m+1)) \right| \leq \frac{1}{5} |\widehat{g}_a - \widehat{t}_a|. \quad (146)$$

By Theorem 67, the nonlinear difference equations (144) have one solution in $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$.

Example 4. Given the function space $(\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h$, where

$h(g) = \sqrt{\sum_{a=0}^{\infty} (\sum_{b=0}^a |\widehat{g}_b|/(a+1))^{(2a+3)/(a+2)}}$, for all $g \in \mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty})$. Consider the non-linear difference equations (144) with $a \geq 1$ and let $W : \mathfrak{E} \rightarrow \mathfrak{E}$, where $\mathfrak{E} = \{g \in (\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h : \widehat{g}_0 = 0\}$, defined by

$$(W(g))(z) = \sum_{a=1}^{\infty} \left(e^{-(2a+3i)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1) \cosh(3mi-a) \cos^p |\widehat{g}_{a-2}|}{\sinh^q |\widehat{g}_{a-1}| + \sin ma + 1} \right) z^a. \quad (147)$$

Clearly, \mathfrak{E} is a nonempty, h -convex, h -closed, and h -bounded subset of $(\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h$. It is easy to see that

$$\left| \sum_{m=0}^{\infty} \frac{\cosh(3mi-a) \cos^p |\widehat{g}_{a-2}|}{\sinh^q |\widehat{g}_{a-1}| + \sin ma + 1} (\tan(2m+1) - \tan(2m+1)) \right| \leq \frac{1}{5} |\widehat{g}_a - \widehat{t}_a|. \quad (148)$$

By Theorem 67 and Corollary 59, the nonlinear difference equations (144) with $a \geq 1$ have a solution in \mathfrak{E} .

Example 5. The summable equations (132) have a solution in $(\mathfrak{C}_{\tau(\cdot)})_h$ if

$$K \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{r}_a - \widehat{g}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m)|}{q+1} \right)^{\tau_q} \right]^{1/K} \leq \ln \frac{\sum_{q=0}^{\infty} ((\sum_{a=0}^q |\widehat{r}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m)|)/(q+1))^{\tau_q}}{\sum_{q=0}^{\infty} (\sum_{a=0}^q |\widehat{g}_a|/(q+1))^{\tau_q}}. \quad (149)$$

Evidently, we have

$$\begin{aligned} h(Wg - g) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{r}_a - \widehat{g}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m)|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \frac{1}{K} \ln \frac{\sum_{q=0}^{\infty} ((\sum_{a=0}^q |\widehat{r}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m)|) / (q+1))^{\tau_q}}{\sum_{q=0}^{\infty} (\sum_{a=0}^q |\widehat{g}_a| / (q+1))^{\tau_q}} \\ &= \ln(h(Wg)) - \ln(h(g)). \end{aligned} \quad (150)$$

By Theorem 18, one gets a solution of equation (132) in $(\mathfrak{C}_{\tau(\cdot)})_h$.

Example 6. The summable equations (132) have a solution in $(\mathfrak{C}_{\tau(\cdot)})_h$, if

$$\begin{aligned} &\left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{r}_a - \widehat{g}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m)|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{g}_a|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\quad - \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{r}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m)|}{q+1} \right)^{\tau_q} \right]^{1/K}. \end{aligned} \quad (151)$$

Clearly, we have

$$\begin{aligned} h(Wg - g) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{r}_a - \widehat{g}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m)|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{g}_a|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &\quad - \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\widehat{r}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m)|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &= h(g) - h(Wg). \end{aligned} \quad (152)$$

By Theorem 18, one gets a solution of equation (132) in $(\mathfrak{C}_{\tau(\cdot)})_h$.

Assume Ω is the set of all closed and bounded intervals on the real line \mathfrak{R} . For $t = [t_1, t_2]$ and $g = [g_1, g_2]$ in Ω , suppose

$$t \leq g \text{ if and only if } t_1 \leq g_1 \text{ and } t_2 \leq g_2. \quad (153)$$

Define a metric ρ on Ω by

$$\rho(t, g) = \max \{|t_1 - g_1|, |t_2 - g_2|\}. \quad (154)$$

Matloka [48] showed that ρ is a metric on Ω , and (Ω, ρ) is a complete metric space.

Definition 68. A fuzzy number g is a fuzzy subset of \mathfrak{R} , i.e., a mapping $g : \mathfrak{R} \rightarrow [0, 1]$ which verifies the following four settings:

- (a) g is fuzzy convex, i.e., for $x, y \in \mathfrak{R}$ and $\alpha \in [0, 1]$, $g(\alpha x + (1 - \alpha)y) \geq \min \{g(x), g(y)\}$
- (b) g is normal, i.e., there is $y_0 \in \mathfrak{R}$ such that $g(y_0) = 1$
- (c) g is an upper semicontinuous, i.e., for all $\alpha > 0$, $g^{-1}([0, \alpha])$ for all $x \in [0, 1]$ is open in the usual topology of \mathfrak{R}
- (d) The closure of $g^0 = \{y \in \mathfrak{R} : g(y) > 0\}$ is compact

Recall that the β -level set of a fuzzy real number g , $0 < \beta < 1$ indicated by g^β is defined as

$$g^\beta = \{y \in \mathfrak{R} : g(y) \geq \beta\}. \quad (155)$$

The set of every upper semicontinuous, normal, convex fuzzy number, and is compact and is denoted by $\mathfrak{R}([0, 1])$. The set \mathfrak{R} can be embedded in $\mathfrak{R}([0, 1])$, if we define $r \in \mathfrak{R}([0, 1])$ by

$$\bar{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r. \end{cases} \quad (156)$$

Consider the summable equations of fuzzy reals (132) and assume $W : (\mathfrak{C}_{\tau(\cdot)})_h \rightarrow (\mathfrak{C}_{\tau(\cdot)})_h$ defined by

$$(W(g))(z) = \sum_{a=0}^{\infty} \bar{\rho} \left(\widehat{r}_a + \sum_{m=0}^{\infty} A(a, m) f(m, \widehat{g}_m), \bar{0} \right) z^a, \quad (157)$$

where $\bar{\rho} : \mathfrak{R}[0, 1] \times \mathfrak{R}[0, 1] \rightarrow \mathfrak{R}^+ \cup \{0\}$ is defined by $\bar{\rho}(t, g) = \sup_{0 \leq \beta \leq 1} \rho(t^\beta, g^\beta)$. For more details about the fuzzy numbers and their properties, see Zadeh [49].

Theorem 69. The summable equations (132) have an unique solution in $(\mathfrak{C}_{\tau(\cdot)})_h$ if $A : \mathcal{N}^2 \rightarrow \mathfrak{R}^+$, $f : \mathcal{N} \times \mathfrak{R}^+[0, 1] \rightarrow \mathfrak{R}^+[0, 1]$, $\widehat{r}_a : \mathcal{N} \rightarrow \mathfrak{R}^+[0, 1]$, $\widehat{t}_a : \mathcal{N} \rightarrow \mathfrak{R}^+[0, 1]$, assume there is $\kappa \in \mathbb{C}$ so that $\sup_q |\kappa|^{\tau_q/K} \in [0, 1]$ and for every $a \in \mathcal{N}$, we have

$$\left| \sum_{m=0}^{\infty} A(a, m) \left(f(m, \widehat{g}_m) - f(m, \widehat{t}_m) \right) \right| \leq |\kappa| |\widehat{g}_a - \widehat{t}_a|. \quad (158)$$

Proof. Let the setups be verified. Consider the mapping $W : (\mathfrak{C}_{\tau(\cdot)})_h \longrightarrow (\mathfrak{C}_{\tau(\cdot)})_h$ defined by (157). We have

$$\begin{aligned} h(Wg - Wt) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |(\widehat{Wg})_a - (\widehat{Wt})_a|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\bar{\rho}(\widehat{r}_a + \sum_{m=0}^{\infty} A(a, m)f(m, \widehat{g}_m), \bar{0}) - \bar{\rho}(\widehat{r}_a + \sum_{m=0}^{\infty} A(a, m)f(m, \widehat{t}_m), \bar{0})|}{q+1} \right)^{\tau_q} \right]^{1/K} \\ &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\sum_{m=0}^{\infty} A(a, m)[f(m, \widehat{g}_m) - f(m, \widehat{t}_m)]|}{q+1} \right)^{\tau_q} \right]^{1/K} \leq \sup_q |\kappa|^{\tau_q/K} \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{a=0}^q |\bar{\rho}(\widehat{g}_a, \bar{0}) - \bar{\rho}(\widehat{t}_a, \bar{0})|}{q+1} \right)^{\tau_q} \right]^{1/K} = \sup_q |\kappa|^{\tau_q/K} h(g - t). \end{aligned} \quad (159)$$

□

According to Theorem 66, one obtains a unique solution of equation (132) in $(\mathfrak{C}_{\tau(\cdot)})_h$.

7. Conclusion

We discuss in this paper some topological and geometric structure of $(\mathfrak{C}_{\tau(\cdot)})_h$, the existence of Caristi's fixed point in it, of the class $\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h}$, and of the class $(\mathfrak{H}_{(\mathfrak{C}_{\tau(\cdot)})_h})^\lambda$. Moreover, some geometric properties related to the fixed point theory in $(\mathfrak{C}_{\tau(\cdot)})_h$ are introduced. Finally, we investigate several solutions applications to summable equations and illustrate our findings with some instances. This article has several advantages for researchers, such as studying the fixed points of any contraction mapping on this prequasispace, which is a generalization of the quasinormed spaces, a new general space of solutions for many difference equations, examining the eigenvalue problem in these new settings, and noting that the closed mappings' ideals are certain to play an important function in the principle of Banach lattices, hence since many fixed point theorems in a particular space work by either expanding the self-mapping acting on it or expanding the space itself, as future work, we can enlarge the space $(\mathfrak{C}_{\tau(\cdot)})_h$ by q -analogue or generalize the self-mapping acting on it.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Caristi's Fixed Point Theorem in Cone Metric Space

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In this paper, we provide a short, comprehensive, and brief proof for Caristi-Kirk fixed point result for single and set-valued mappings in cone metric spaces. In addition, we partially addressed an open problem in which Caristi-Kirk fixed point result in cone metric spaces reduces to a classical result in metric spaces and provided a brief justification for a partial positive answer to this open problem using Caristi-Kirk fixed point theorem on uniform space. The proofs given to Caristi-Kirk's result vary and use different techniques.

1. Introduction and Preliminaries

Caristi-Kirk's fixed point theorem in [1] states that if X is a complete metric space and φ is a lower semicontinuous mapping from X into the nonnegative real numbers, then any mapping $T : X \rightarrow X$ satisfying

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \quad (x \in X) \quad (1)$$

has a fixed point.

Several researchers generalized the Caristi-Kirk's fixed point theorem in various directions, for details see [2–9].

Angelov [10] provided an extension of the Caristi-Kirk theorem to T_2 -separated uniform spaces, the uniform space X is known as T_2 -separated if each convergent sequence in X has a unique limit. As we know that every uniform space is generated by a family of pseudometrics $\{d_a(x, y) : a \in A\}$, where A is an indexing set. Also, a sequence $(x_n) \in X$ is known as a Cauchy sequence, if for each $a \in A$, we have $\lim_{n, m \rightarrow +\infty} d_a(x_n, x_m) = 0$, and a sequence $(x_n) \in X$ is convergent and converges to $x \in X$, if for each $a \in A$, we have

$\lim_{n \rightarrow +\infty} d_a(x_n, x) = 0$. Thus, a uniform space X is called complete if every Cauchy sequence is convergent in X .

In this regard, Angelov [10] generalizes the Caristi-Kirk fixed point theorem on uniform space, which stated as:

Theorem 1 [10]. *Let X be a T_2 -separated complete uniform space which is generated by a family of pseudometrics $\{d_a : a \in A\}$, where A is an indexing set. Let $T : X \rightarrow X$ be a mapping and $\{\psi_a\}$ be a family of lower semicontinuous functionals. Suppose that the following inequality holds for each $a \in A$,*

$$d_{j(a)}(x, T(x)) \leq \psi_a(x) - \psi_a(T(x)), \quad (2)$$

where $x \in X$ and $j : A \rightarrow A$ is a surjective mapping. Then, T has a fixed point in X .

The following theorem is a Banach fixed point theorem on uniform space, which stated as:

Theorem 2 [11]. *Let X be a T_2 -separated complete uniform space which is generated by a family of pseudometrics $\{d_a$*

$: a \in A\}$, where A is an indexing set. Suppose that $T : X \rightarrow X$ is a mapping which is satisfying

$$d_a(T(x), T(y)) \leq k_a d_a(x, y), \quad (3)$$

for each $x, y \in X$ and $a \in A$. Then, T has a unique fixed point in X .

In 2007, Huang et al. [12] introduced the concept of cone metric space and proved some well-known fixed point results. The authors extended fixed point results proved for cone metric spaces which was just a simple reformulation of classical results presented in metric spaces. The obtained results are generalizations from classical results to cone metric spaces, for details see [13–16].

In this paper, we aim to reformulate Caristi-Kirk's fixed point theorem for single and set-valued mappings in cone metric space and obtained a detailed answer to a question posed by Khamsi and Wojciechowski in ([17], Theorem 3-1) "whether the vectorial version and the classical version of Caristi-Kirk's fixed point theorem are equivalent." To address this particular answer, we defined a uniform space by considering cone metric space, and then, we addressed Theorem 1 in cone metric spaces and uniform spaces. Our proof is shorter, comprehensive, and easier than proof provided until now and our results generalize the existing results due to Khamsi and Wojciechowski in [17].

2. Cone Metric Version of Caristi-Kirk's Theorem

Suppose that P is a nonempty closed convex cone of a real Banach space E such that $P \neq \{\theta\}$, where θ is the null vector, $P \cap -P = \{\theta\}$ and $\text{int}P \neq \emptyset$.

In addition, P induces a partial order \leq on E which is defined as $x \leq y$ if and only if $y - x \in P$ and we write $x \ll y$ if and only if $y - x \in \text{int}P$.

A convex subset $B \subset P$ is a base of P if $\theta \in \bar{B}$ and $P = \bigcup_{t \geq 0} tB$ and E^* is the topological dual space of E and $P^* = \{\psi \in E^* : \psi(x) \geq 0, \forall x \in P\}$ is known as dual cone of P . The dual cone P^* of a cone P in a Banach space E has a weak*-compact base B^* . A set $A \subset E$ is called bounded from above (below) if there exists $z \in E$ such that for all $a \in A$, $a \leq z$ ($z \leq a$). A cone is called regular if every nondecreasing (decreasing) sequence which is bounded from above (below) is convergent in norm. The cone P is called normal if there is a number $K > 0$, such that we have for all $x, y \in E$,

$$\theta \leq x \leq y \implies \|x\| \leq K\|y\|. \quad (4)$$

The least positive number satisfying this inequality is called a normal constant of P .

The following lemma will be used in proving our main results.

Lemma 3 [18, 19]. *The weak*-compact base B^* satisfies:*

- (1) any element $x \in P$ if and only if $\psi(x) \geq 0$, for all $\psi \in B^*$

- (2) any element $x \in \text{int}P$ if and only if $\psi(x) > 0$, for all $\psi \in B^*$

Definition 4 [12]. Let X be a nonempty set. Consider a mapping $d : X \times X \rightarrow E$ is satisfied as follows:

- (1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$
- (3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$

Then, d is called a cone metric on X and (X, d) is named a cone metric space.

Definition 5 [12]. Let (x_n) be a sequence in a cone metric space (X, d) and some $x \in X$. A sequence (x_n) is as follows:

- (1) a d -Cauchy sequence if for every $\theta \ll \varepsilon \in E$, there exists $N \in \mathbb{N}$, such that $d(x_m, x_n) \ll \varepsilon$, for all $m, n \geq N$
- (2) a d -convergent and d -converges to $x \in X$ if for every $\theta \ll \varepsilon \in E$, there exists $N \in \mathbb{N}$, such that $d(x_n, x) \ll \varepsilon$, for all $n \geq N$, which is denoted as $x_n \rightarrow x$

Definition 6 [12]. A cone metric space (X, d) is d -complete if every d -Cauchy sequence is d -convergent in (X, d) .

Definition 7 [20]. Let (X, d) be a cone metric space. A mapping $\varphi : X \rightarrow E$ is considered as a cone lower semicontinuous mapping at $x \in X$ if for any $\theta \ll \varepsilon \in E$, there exists a natural number $N_\varepsilon \in \mathbb{N}$ such that

$$\varphi(x) \leq \varphi(x_n) + \varepsilon, \quad (5)$$

for all $n > N_\varepsilon$, where (x_n) is a sequence in X and $x_n \rightarrow x$. If $E = \mathbb{R}$, then $P = \mathbb{R}^{\geq 0}$, (X, d) is a metric space and $\varphi : X \rightarrow \mathbb{R}$, so φ is a lower semicontinuous mapping at $x \in X$, if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for any $n > N_\varepsilon$, we have

$$\varphi(x) \leq \varphi(x_n) + \varepsilon, \quad (6)$$

where (x_n) is a convergent sequence and converges to x in a metric space (X, d) .

The following theorem is a cone metric version of the Caristi-Kirk's theorem with the some extra normal cone condition.

Theorem 8 [21]. *Let (X, d) be a cone metric space with normal and regular cone of a Banach space $(E, \|\cdot\|)$ such that $\lim_{m, n \rightarrow \infty} \|d(x_m, x_n)\| = 0$ implies $\lim_{n \rightarrow \infty} \|d(x_n, x)\| = 0$ for some $x \in X$. Also, $\varphi : X \rightarrow P$ satisfies $\varphi(x) \preceq \liminf_n \varphi(x_n)$, for every*

$\lim_{n \rightarrow \infty} \|d(x_n, x)\| = 0$. Suppose that the mapping $T : X \rightarrow X$ satisfying the following condition:

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \quad (7)$$

for all $x \in X$. Then, T has a fixed point.

Since then, some studies have focused on extending and improving the cone metric version of the Caristi-Kirk's fixed point theorem in many ways. In [16, 17], authors proved the Caristi-Kirk's fixed point theorem but the authors supposed that the cone is normal which is a strict condition and researchers did not accept it as a good condition. Further, the results are proved for regular and normal cone in [21].

In this paper, none of these conditions was considered for the cone. Now, we will omit the stronger conditions, the normality, and regularity of the cone in our main results and we will prove this result under the weaker condition as compared to the result proved in the literature under strict conditions.

3. Main Results

The following lemmas are handy tools that are used in the sequel.

Lemma 9. Let (X, d) be a cone metric space and $\psi \in B^*$. Also suppose that X is a uniform space which is generated by a family of pseudometrics $\{\psi \circ d : \psi \in B^*\}$. Then, X is T_2 -separated.

Proof. On contrary suppose that the sequence (x_n) has two different limits, i.e., $\lim x_n = x$ and $\lim x_n = y$ in the uniform space X . Then, according to definition, for each pseudometric $\psi \circ d$, we have $\lim_{n \rightarrow \infty} \psi(d(x_n, x)) = 0$ and $\lim_{n \rightarrow \infty} \psi(d(x_n, y)) = 0$. In addition, by the third property of the cone metric, we have $d(x, y) \leq d(x_n, x) + d(x_n, y)$, (by lemma 3) we have $\psi(d(x, y)) \leq \psi(d(x_n, x)) + \psi(d(x_n, y))$, for each $\psi \in B^*$. When $n \rightarrow \infty$, we have that for each $\psi \in B^*$, $\psi(d(x, y)) = 0$. Thus, $\|d(x, y)\| = \sup_{\|\psi\|=1} |\psi(d(x, y))| = 0$. Thus, $d(x, y) = \theta$, i.e., $x = y$. \square

Lemma 10. Let (X, d) be a cone metric space and $\psi \in B^*$. Also suppose that X is a uniform space which is generated by a family of pseudometrics $\{\psi \circ d : \psi \in B^*\}$. Then, X is a complete uniform space if and only if (X, d) is a d -complete cone metric space.

Proof. First, we suppose that X is a complete uniform space. Let (x_n) be a d -Cauchy sequence in the cone metric space (X, d) . Then, for each $\theta \ll \varepsilon/k$, where $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for each $m, n > N$, we have $d(x_m, x_n) \ll \varepsilon/k$. Using Lemma 3 part (2), for each $\psi \in B^*$, we have $\psi(\varepsilon)/k > 0$ and

$$\psi(d(x_m, x_n)) < \frac{\psi(\varepsilon)}{k}. \quad (8)$$

For any $\varepsilon > 0$ and $\psi \in B^*$, there is a $k \in \mathbb{N}$ such that $\psi(\varepsilon)/k < \varepsilon$. For $m, n > N$, inequality (8) implies that $\psi(d(x_m, x_n)) < \varepsilon$. Using the definition, (x_n) is a Cauchy sequence in a complete uniform space. Therefore, (x_n) is convergent and converges to x which belongs to the uniform space X . Then, for each $\varepsilon > 0$ and $\psi \in B^*$, there exists $N \in \mathbb{N}$, such that for each $n > N$,

$$\psi(d(x_n, x)) < \varepsilon. \quad (9)$$

Now, we demonstrate that the sequence (x_n) is d -convergent. On the contrary, suppose that there is some $\theta \ll \varepsilon$, such that for each $N \in \mathbb{N}$ there is $n > N$ such that $d(x_n, x) \geq \varepsilon$. From lemma 3 part (8), for each $\psi \in B^*$, $\psi(\varepsilon) > 0$, and $\psi(d(x_n, x)) > \psi(\varepsilon)$, which is a contradiction, if $\varepsilon = \psi(\varepsilon)$.

On contrary suppose that (X, d) is a d -complete space. Let (x_n) be a Cauchy sequence in a uniform space X , for each $\varepsilon > 0$ and $\psi \in B^*$, there is $N \in \mathbb{N}$, such that for each $m, n > N$,

$$\psi(d(x_m, x_n)) < \varepsilon. \quad (10)$$

Now, we show that (x_n) is a d -Cauchy sequence. On the contrary, suppose that (x_n) is not a d -Cauchy sequence. Then, there is $\theta \ll \varepsilon$ such that for each $N \in \mathbb{N}$, there are $m, n > N$ such that $d(x_m, x_n) \geq \varepsilon$. Thus, by using lemma 3 for each $\psi \in B^*$, we have $\psi(\varepsilon) > 0$ and $\psi(d(x_m, x_n)) \geq \psi(\varepsilon) > 0$, which is a contradiction, as $\varepsilon = \psi(\varepsilon)$. Therefore, (x_n) is a d -Cauchy sequence, and accordingly, it is a d -convergent and converges to some x (from definition 5) for each $\theta \ll \varepsilon/k$ where $k \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for each $n > N$, we have $d(x_n, x) \ll \varepsilon/k$, (using lemma 3) for each $\psi \in B^*$, we have $\psi(\varepsilon)/k > 0$ and

$$\psi(d(x_n, x)) < \frac{\psi(\varepsilon)}{k}. \quad (11)$$

For each $\varepsilon > 0$ and $\psi \in B^*$, there is $k \in \mathbb{N}$ such that $\psi(\varepsilon)/k < \varepsilon$ thus (11) implies that for $n > N$, we have

$$\psi(d(x_n, x)) < \frac{\psi(\varepsilon)}{k} < \varepsilon. \quad (12)$$

Thus, (x_n) is convergent to x in the uniform space X . \square

Lemma 11. Let (X, d) be a cone metric space and $x \in X$. Then, $d(x, \cdot) : X \rightarrow E$ is a cone lower semicontinuous mapping.

Proof. Let $\theta \ll \varepsilon \in E$ and (y_n) be a sequence in X such that $y_n \rightarrow y \in X$. There exists $N \in \mathbb{N}$ such that $d(y_n, y) \ll \varepsilon$ for all $n \geq N$. Then, $d(x, y) \ll d(x, y_n) + d(y_n, y) \ll d(x, y_n) + \varepsilon$, for all $n \geq N$. Thus, $d(x, \cdot)$ is a cone lower semicontinuous mapping. \square

Lemma 12. Let (X, d) be a cone metric space, $\varphi : X \rightarrow E$ be a cone lower semicontinuous mapping and $\psi \in B^*$. Then, $\varphi \circ \psi : X \rightarrow \mathbb{R}$ is a lower semicontinuous function.

Proof. Let $\theta \ll \varepsilon \in E$ be fixed. For any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\psi(\varepsilon/m) < \varepsilon$ (ψ is linear) and φ is cone lower semicontinuous and $x_n \rightarrow x$. Thus, there exists $N \in \mathbb{N}$ such that

$$\varphi(x) \leq \varphi(x_n) + \frac{1}{m} \varepsilon, \quad (13)$$

for all $n \geq N$, and so

$$\psi(\varphi(x)) \leq \psi(\varphi(x_n)) + \psi\left(\frac{1}{m} \varepsilon\right) \leq \psi(\varphi(x_n)) + \varepsilon, \quad (14)$$

for all $n \geq N$. This relation indicates the lower semicontinuity of $\psi \circ \varphi$. \square

As is shown in [14], all fixed point results in cone metric spaces obtained recently, in which the assumption that the underlying cone is normal and with the nonempty interior is present, can be reduced to the corresponding results in metric spaces. On the other hand, when we deal with non-normal cones, this is not possible.

Theorem 13 is a cone metric version of Caristi-Kirk's theorem without extra conditions normality and regularity which are always put in cone metric theorems, so our results are original. To prove this theorem, we show that the cone metric space is uniform too; then, it will be proved by applying Theorem 1. We know that a T_2 -separated uniform space is metrizable if its uniformity can be defined by a countable family of pseudometrics. Indeed, such uniformity can be defined by a single pseudometric, which is necessarily a metric. This implies that a cone metric version of Caristi-Kirk's theorem may be derived from the classical one if B^* which is defined in Section 2 is countable. This is a partial answer to the open question mentioned before.

Theorem 13. *Let (X, d) be a d -complete cone metric space and $\varphi : X \rightarrow P$ be a cone lower semicontinuous mapping. Suppose that the self-mapping $T : X \rightarrow X$ satisfying the following condition:*

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \quad (15)$$

for all $x \in X$. Then, T has a fixed point.

Proof. We provide the conditions of Theorem 1 to conclude that T has a fixed point. It is easily shown that $\{\psi \circ d : \psi \in B^*\}$ is a family of pseudometrics on X , and X will be a uniform space with the topology generated by these pseudometrics. By Lemma 9, the uniform space X is T_2 -separated. Using lemma 10, X is a d -complete cone metric space, since X is a complete uniform space. By lemma 12, $\psi \circ \varphi$ is a lower semicontinuous mapping. Further, lemma 3 and assumption

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x)) \quad (16)$$

imply that for each $\psi \in B^*$, $\psi(d(x, T(x))) \leq \psi(\varphi(x)) - \psi(\varphi(T(x)))$. By considering j as an identity mapping, all assumptions are satisfied. \square

In Theorem 13, the regularity of the cone, which is an essential condition in [21] is omitted. So, our theorem is a real generalization of Theorem 8.

For example, we cannot even conclude from Theorem 8 that the identity mapping has a fixed point but it is possible by Theorem 8. The following example is presented in the support of the theorem 13.

Example 14. Consider the Banach space $\ell_\infty(R)$ with its cone $P = \{(x_n) \in \ell_\infty(R) : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$. It is not difficult to see that $\ell_\infty(R)$ is complete and P is normal with nonempty interior. Let B be a subset of $\ell_\infty(R)$ consisting of all (x_n) which are nondecreasing and converging to 1 with $1/2 \leq x_n \leq 1$, for all $n \in \mathbb{N}$. Define $d : B \times B \rightarrow P$ as, $d((x_n), (y_n)) = (|x_1 - y_1|, \dots, |x_n - y_n|, \dots)$, for every $(x_n), (y_n) \in B$. It is not hard to check that (B, d) is a d -complete space. Now, define the mapping $T : B \rightarrow B$ by $T((x_n)) = (x_n)$ and $\varphi : B \rightarrow P$ is the inclusion mapping. It is clear that φ is cone lower semicontinuous and T satisfies $d(x, T(x)) \leq \varphi(x) - \varphi(T(x))$, since $d(x, Tx) = d(x, x) = \theta$. But P is not regular because the sequence (a_n) that $a_n = (\underbrace{1, \dots, 1}_n, 0, 0, \dots)$, for each $n \in \mathbb{N}$, is

nondecreasing and bounded from above but it is not convergent.

Thus, one of the conditions of Theorem 8 is not satisfied, although $T(x) = x$, for all $x \in B$. But Theorem 13 implies that T has a fixed point.

Remark 15. In example 14, one of conditions of Theorem 8 is not satisfied, as P is not regular although $T(x) = x$, for all $x \in B$. In the example 14, all the conditions of the Theorem 13 are satisfied, and hence, the underlying mapping T has a fixed point. This shows that Theorem 13 is a real and proper generalization of Theorem 8.

The following example shows that the cone lower semicontinuity of φ is essential in Theorem 13 and may not be dropped.

Example 16. Let $X \subseteq \ell_\infty(R)$ be a family of the sequences $\bar{0} = \{0, 0, \dots, 0, \dots\}$, $\tilde{1} = \{1, 1, \dots, 1, \dots\}$, $\bar{1} = \{1/2, 1/3, \dots, (1/m), \dots\}$, $\bar{2} = \{(1/2)^2, (1/3)^2, \dots, (1/m)^2, \dots\}$, $\bar{n} = \{(1/2)^n, (1/3)^n, \dots, (1/m)^n, \dots\}$ and P is defined as same as the cone defined in Example 14 and the cone metric is

$$d(\bar{x}, \bar{y}) = \left\{ \left| \frac{1}{2^{x+1}} - \frac{1}{2^{y+1}} \right|, \left| \frac{1}{3^{x+1}} - \frac{1}{3^{y+1}} \right|, \dots, \left| \frac{1}{m^{x+1}} - \frac{1}{m^{y+1}} \right|, \dots \right\}, \quad (17)$$

for every $\bar{x}, \bar{y} \in X$. Define the mapping $T : X \rightarrow X$ and $\varphi : X \rightarrow P$ in the following way:

$$T(\bar{n}) = \bar{n} + 1, T(\bar{0}) = \tilde{1}, T(\tilde{1}) = \bar{0}, \quad (18)$$

and $\varphi = T$. Obviously $d(\bar{x}, T(\bar{x})) = \varphi(\bar{x}) - \varphi(T(\bar{x}))$ but φ is not cone lower semicontinuous map because $\lim \bar{n} = \bar{0}$ does

imply $\bar{1} = \varphi(\bar{0}) \neq \liminf \varphi(\bar{n}) = \bar{0}$, and therefore, one of conditions of Theorem 13 is not satisfied. It is clear that T has no fixed point because $T(\bar{n}) = n + 1 \neq \bar{n}$.

In the next theorem, we give a short proof for a set-valued version of Caristi-Kirk's fixed point theorem in cone metric space. An element $x \in X$ is considered as a fixed point of set-valued mapping $f : X \rightarrow X$ if $x \in f(x)$.

Theorem 17. *Let (X, d) be a d -complete cone metric space, $\varphi : X \rightarrow P$ be a cone lower semicontinuous mapping and there exists $y \in f(x)$ for a set-valued mapping $f : X \rightarrow X$ such that*

$$d(x, y) \leq \varphi(x) - \varphi(y), \quad (19)$$

for each $x \in X$. Then, f has a fixed point.

Proof. By assumption, for each $x \in X$, the set $\{y \in f(x) : d(x, y) \leq \varphi(x) - \varphi(y)\}$ is nonempty. Using the axiom of choice, there is a single-valued mapping $T : X \rightarrow X$ such that $d(x, T(x)) \leq \varphi(x) - \varphi(T(x))$, for each $x \in X$. Theorem 13 is applied for T to find a fixed point x (say) of T . Since $T(x) \in f(x)$, we have $x \in f(x)$. \square

Additionally, Khamsi (2010) proved the Theorem 18, which is the cone metric version of the Banach fixed point theorem. In Theorem 19, we improve it by removing the stronger condition of "normal cone."

Theorem 18 [16]. *Let (X, d) be a d -complete cone metric space over the Banach space $(E, \|\cdot\|)$ with the cone P which is normal. Suppose that for some $0 < \alpha < 1$, the mapping $T : X \rightarrow X$ satisfies*

$$\|d(T(x), T(y))\| \leq \alpha \|d(x, y)\|, \quad (20)$$

for all $x, y \in X$. Then, T has a unique fixed point.

Theorem 19. *Let (X, d) be a d -complete cone metric space and for some $0 < \alpha < 1$, the mapping $T : X \rightarrow X$ satisfying*

$$d(T(x), T(y)) \leq \alpha d(x, y), \quad (21)$$

for all $x, y \in X$. Then, T has a unique fixed point.

Proof. We provide the conditions of Theorem 2 to conclude that T has a fixed point. We know that $\{\psi \circ d : \psi \in B^*\}$ is a family of pseudometrics on X , and X will be a uniform space with the topology generated by these pseudometrics. By lemma 9, this uniform space is T_2 -separated. Using lemma 10, X is a complete uniform space since (X, d) is a d -complete cone metric space. In addition, lemma 3 and assumption

$$d(T(x), T(y)) \leq \alpha d(x, y), \quad (22)$$

imply that for each $\psi \in B^*$, $\psi(d(T(x), T(y))) \leq \alpha \psi(d(x, y))$. Thus, Theorem 2 implies that T has a fixed point. \square

Remark 20. It is worth noting that Theorem 19 is a generalization of the Theorem 18. We used cone metric space with a nonnormal cone in our main results. Therefore, our theorems are the strict generalizations of the results which are proved in [16, 17, 21].

Theorem 21. *Let (X, d) be a d -complete cone metric space, and $f : B \rightarrow B$ be a set-valued mapping that for each $x, y \in X$ and $z \in f(x)$, there exists $w \in f(y)$ such that*

$$d(z, w) \leq \alpha d(x, y). \quad (23)$$

Then, f has a fixed point.

Proof. It is direct consequence of the Theorem 19. \square

In this article, we provided a brief proof for the Caristi-Kirk's fixed point result for single and set-valued mappings in cone metric spaces. Also, we partially addressed an open problem in which Caristi-Kirk's fixed point resulted in cone metric spaces. We improved the already existing results on Caristi-Kirk's fixed point in cone metric spaces by improving and removing the extra and strict conditions on the underlying spaces and mappings as well. Further, we provided a brief justification as a partial positive answer to this open problem using Caristi-Kirk's fixed point theorem on uniform space. We further provided a short proof in the cone metric version of the Banach fixed point theorem by using a short and comprehensive approach.

Data Availability

No data were used.

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

Self-Adaptive Predictor-Corrector Approach for General Variational Inequalities Using a Fixed-Point Formulation

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A literature review revealed that the general variational inequalities, fixed-point problems, and Winner–Hopf equations are equivalent. In this study, general variational inequality and fixed-point problem are considered. We introduced a new iterative method based on a self-adaptive predictor-corrector approach for finding a solution to the GVI. Adaptations in the fixed-point formulation and self-adaptive techniques have been used to predict a novel iterative approach. Convergence analyses of the suggested algorithm are demonstrated. Moreover, numerical analysis shows that we establish the new best method for solving general variational inequality which performs better than the previous one. Furthermore, it is known that GVI consisted of several classes including variational inequalities and related optimization problems, and results obtained in this study continue to hold for these problems.

1. Introduction

Applied mathematics has adorned the most promising and panoramic field referred to as variational inequality theory. This theory is a powerful unifying methodology for the study of equilibrium problems and provides us algorithms with accompanying convergence analysis for computational purposes. Therefore, in recent few years, various branches of mathematical and engineering sciences can be transformed in the framework of variational inequalities such as electronics, heat transportation, elasticity, optimization, network analysis, water resources, game theory, equilibrium problems in economics, mechanics, and traffic analysis; see [1–10]. Such remarkable development claims the most simple and unidirectional models of linear and nonlinear techniques. The idea of variational inequalities was first originated by Stampacchia [11]. Related to the variational inequalities, we have the concept of the Wiener–Hopf

equations and general variational inequalities which were introduced by Noor [12] and Shi [13] in conjunction with variational inequalities from different points of view.

Several conventional improvements approach to establish the solutions for open, moving boundary value problems and asymmetric obstacle, unilateral, even-order, and odd-order problems utilizing general variational inequalities, see [13–18]. Equivalent effects of general variational inequalities and fixed-point problems utilizing the projection techniques in recent days are an active research field, see [8, 19–21]. Quantitative knowledge of pseudocontractive and nonlinear monotone (accretive) operators combined with Lipschitz-type conditions is vital to prove the convergence of fixed-point iterative procedures. The phenomena of variational inequalities have a significant contribution to solving the Wiener–Hopf equations. Salient features of Wiener–Hopf equations and optimization problems in the presence of variational inequalities are addressed by Shi, see [12, 20–23].

Variants of projection methods such as Wiener–Hopf equation techniques, auxiliary principle scheme, decomposition, and dynamical systems are advanced for solving various kinds of variational inequalities and other related optimization problems, see [8, 17–19, 22–26]. A detailed study by Lions and Stampacchia revealed the utilization of such tools for finding the detailed solution of variational inequalities was in consumption a long time ago, see [27, 28]. The primary objective for employing these abilities is to keep the variational inequality and the fixed-point problem similar through projection. Based on this formula, many projection methods for resolving variational inequalities can be developed. This approach has been critical. Convergence of the projection method has a drawback; it required a strongly monotone and Lipschitz continuous operator, which has limited much application. Therefore, innovative methods or modifications in the projection method are required to diversify the field.

Publications such as [23, 29–36] comprised of extra gradient-type methods which delimited the projection phenomenon by taking additional forward steps, and projection at each iteration is considered according to the double projection. These methods are a predictor-corrector tool and have been suggested to quantify variational inequalities and their special cases. We improve the recent best results for GVI by introducing innovative iterative methods.

The self-adaptive predictor-corrector approach is the primary goal of this research, which modifies the fixed point by incorporating a generalized residue vector that includes general variational inequalities. For the convergence of the method, we require only pseudomonotonicity, which is considered a weaker condition than monotonicity. It is comprehended that the proposed model is simple and robust. We can see from the numerical results that the proposed technique is both rapid and easy to implement, as demonstrated by an example.

2. Preliminaries

We denote Hilbert space H with the norm and inner product by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. A convex set is represented by M in H , and let $A, \phi: H \rightarrow H$ be considered the nonlinear operators. For finding $\alpha \in H, \phi(\alpha) \in M$, such that

$$\langle A\alpha, \phi(\beta) - \phi(\alpha) \rangle \geq 0, \quad \text{for all } \beta \in H: \phi(\beta) \in M. \quad (1)$$

Problem (1) is called the general variational inequalities (GVI), considered by Noor in 1988. We have observed that a large number of problems in pure and applied mathematics related to physical sciences, engineering, equilibrium, moving, nonsymmetric, unified, obstacle, and contact can be discussed and studied via inequalities (1), see [18, 19, 22].

For $\phi \equiv I$ (take identity operator), problem (1) reduces to finding $\alpha \in M$, such that

$$\langle A\alpha, \beta - \alpha \rangle \geq 0, \quad \forall \beta \in M. \quad (2)$$

Problem (2) is defined by the original variational inequalities introduced by Stampacchia, see [11].

The following concept and known results are required to approach the main algorithms.

Definition 1. Let $A: H \rightarrow H$ be called the operator and also ϕ -pseudomonotone if $\langle A\alpha, \phi(\beta) - \phi(\alpha) \rangle \geq 0$ provides $\langle A\beta, \phi(\beta) - \phi(\alpha) \rangle \geq 0, \forall \alpha, \beta \in H$.

It is considered [22, 24] that monotonicity implies pseudomonotonicity, but the converse does not exist.

Lemma 1. For $z \in H, \alpha \in M$ holds for the inequality:

$$\langle \alpha - z, \beta - \alpha \rangle \geq 0, \quad \forall \beta \in M, \quad (3)$$

if and only if

$$\alpha = P_M z, \quad (4)$$

where P_M is called the projection of H onto the convex set M .

It is also known that P_M is called the projection operator and also nonexpansive which satisfies the following inequality:

$$\|P_M z - \alpha\| \leq \|z - \alpha\| - \|z - P_M z\|. \quad (5)$$

Lemma 2. α is a solution of the given GVI (1) if and only if $\alpha \in M$ satisfies the relation

$$\phi(\alpha) = P_M [\phi(\alpha) - \rho A\alpha], \quad (6)$$

where $\rho \geq 0$ is taken as the constant and P_M is considered the projection operator H onto M .

Residue vector $R_1(\alpha)$ is defined by

$$R_1(\alpha) := \phi(\alpha) - P_M [\phi(\alpha) - \rho A\alpha]. \quad (7)$$

From Lemma 1, we can see that α satisfy (1) if and only if α is a zero point of the function:

$$R_1(\alpha) := 0. \quad (8)$$

For the GVI (1), we consider the problem for the Wiener–Hopf equations. Let $Q_M = I - P_M$, where I is the identity operator and P_M is projection operation. For the operators $A, \phi: H \rightarrow H$, and ϕ^{-1} exists; then, for finding $z \in H$, we have

$$\rho A\phi^{-1}P_M z + Q_M z = 0, \quad (9)$$

where (9) is the general Wiener–Hopf equation (GWHE), investigated by Noor [18]. We have seen that the Wiener–Hopf equations are considered and used to establish various efficient and powerful iterative schemes.

Lemma 3. The function $\alpha \in H: \phi(\alpha) \in M$ satisfies inequalities (1) if and only if $z \in H$ satisfies equation (9), provided

$$\phi(\alpha) = P_M z, \quad (10)$$

$$z = \phi(\alpha) - \rho A\alpha. \quad (11)$$

Lemma 3 provides that the GVI (1) is equivalent to GWHE (9). This fixed-point formulation was considered by Noor [24] to establish various iterative schemes for solving the GVI and other optimization theory and related problems.

This useful scheme has been considered to make and establish a self-adaptive method for solving the GVI (1).

By using (7) and (10) and (11), the GWHE (9) can be modified in the form:

$$\begin{aligned} 0 &= \phi(\alpha) - P_M[\phi(\alpha) - \rho A\alpha] - \rho A\alpha + \rho A\phi^{-1}P_M[\phi(\alpha) - \rho A\alpha], \\ &= R_1(\alpha) - \rho A\alpha + \rho A\phi^{-1}P_M[\phi(\alpha) - \rho A\alpha]. \end{aligned} \quad (12)$$

For $\omega \in [0, 1]$, (9) can be mentioned as

$$\phi(\alpha) = P_M[\phi(\alpha) - \omega d_1(\alpha)], \quad (13)$$

where

$$d_1(\alpha) = R_1(\alpha) - \rho A\alpha + \rho A\phi^{-1}P_M[\phi(\alpha) - \rho A\alpha]. \quad (14)$$

This equivalent modification has been considered by Noor [31] for solving the general variational inequalities (GVI).

Algorithm 1.

Step 0. We set the parameters as follows. For $\alpha_o \in H$, set $n = 0$, and take $\delta_0, \delta \in (0, 1), \epsilon > 0, \gamma \in [1, 2), \mu \in (0, 1)$, and $\rho > 0$.

Step 1. If $R_1(\alpha_n) < \epsilon$, then we terminate; otherwise, take $\rho_n = \rho\mu^{m_n}$, where m_n finds the smallest nonnegative integer that satisfies the inequality $\rho_n \langle A(\alpha_n) - A\phi^{-1}P_M[\phi(\alpha_n) - \rho_n A(\alpha_n)], R_1(\alpha_n) \rangle \leq \delta \|R_1(\alpha_n)\|^2$.

Step 2. Compute $d_1(\alpha_n) = R_1(\alpha_n) - \rho_n A(\alpha_n) + \rho_n A\phi^{-1}P_M[\phi(\alpha_n) - \rho_n A(\alpha_n)]$ and $\omega_n = ((1 - \delta) R_1(\alpha_n)^2) / d_1(\alpha_n)^2$.

Step 3. Find the next iteration, $\phi(\alpha_{n+1}) = \phi(\alpha_n) - \omega_n d_1(\alpha_n)$.

Step 4. If $\rho_n A(\alpha_n) - A\phi^{-1}P_M[\phi(\alpha_n) - \rho_n A(\alpha_n)], R_1(\alpha_n) \leq \delta_0 R_1(\alpha_n)^2$, then again take $\rho = \rho_n / \mu$, else $\rho = \rho_n$. Consider $n = n + 1$, and start iteration from step 1.

3. Main Results

We suggest predictor-corrector techniques for updating the scheme to find the solution of the GVI (1):

$$\begin{aligned} \phi(y) &= P_M[\phi(\alpha) - \rho A\alpha], \\ \phi(\alpha) &= P_M[\phi(y) - \rho A y]. \end{aligned} \quad (15)$$

Here, we suggest the residue vector involving projection by the relation:

$$\begin{aligned} R(\alpha) &= \phi(\alpha) - P_M[\phi(y) - \rho A(y)] = \phi(\alpha) - \phi(y) \\ &= \phi(\alpha) - P_M[P_M[\phi(\alpha) - \rho A\alpha] \\ &\quad - \rho A\phi^{-1}P_M[\phi(\alpha) - \rho A\alpha]]. \end{aligned} \quad (16)$$

It is clear that $\alpha \in H$, and $\phi(\alpha) \in M$ is a solution of the GVI (1) if and only if $\alpha \in H$, and $\phi(\alpha) \in M$ is satisfied with the residue vector:

$$R(\alpha) = 0. \quad (17)$$

Since the convex set is defined by M , then, for all $\eta \in [0, 1]$, using (16), we have

$$\begin{aligned} \phi(\alpha) &= P_M[\phi(\alpha) - \omega\{\eta R(\alpha) + \rho A\phi^{-1}(\phi(\alpha) - \eta R(\alpha))\}], \\ &= P_M[\phi(\alpha) - \omega d(\alpha)]. \end{aligned} \quad (18)$$

where

$$d(\alpha) = \eta R(\alpha) + \rho A\phi^{-1}(\phi(\alpha) - \eta R(\alpha)). \quad (19)$$

We now analyze and recommend the following predictor-corrector scheme for finding the GVI (1).

Algorithm 2.

Step 0. For parameters $\epsilon > 0, \rho > 0, \delta_0, \delta \in (0, 1), \gamma \in [0, 1], \mu \in (0, 1)$, and $\alpha_o \in H$, we start from $n = 0, \eta \in (0, 1)$.

Step 1. We again take $\rho_n = \rho$. If $\|R(\alpha_n)\| < \epsilon$, then computation stops; otherwise, we continue and consider $\rho_n = \rho\mu^{m_n}$ and find the smallest nonnegative integer m_n , which satisfies the inequality $\rho_n \langle A(\alpha_n) - A\phi^{-1}(\phi(\alpha_n) - \eta_n R(\alpha_n)), R(\alpha_n) \rangle \leq \|\sigma R(\alpha_n)\|^2, \sigma \in [0, 1]$.

Step 2. Calculate the next iterate:

$$\phi(\alpha_{n+1}) = P_M[\phi(\alpha_n) - \omega_n d(\alpha_n)], \quad (20)$$

where

$$d(\alpha_n) = R(\alpha_n) + \rho_n A\phi^{-1}(\phi(\alpha_n) - \eta_n R(\alpha_n)), \quad (21)$$

$$\phi(\alpha_{n+1}) = P_M[\phi(\alpha_n) - \omega_n d(\alpha_n)],$$

$$\omega_n = \frac{\langle R(\alpha_n), D(\alpha_n) \rangle}{\|d(\alpha_n)\|^2}. \quad (22)$$

Step 3. If

$$\begin{aligned} \rho_n \langle A(\alpha_n) - A\phi^{-1}(\phi(\alpha_n) - \eta_n R(\alpha_n)), R(\alpha_n) \rangle \\ \leq \sigma R(\alpha_n)^2, \text{ and } \sigma \in [0, 1], \end{aligned} \quad (23)$$

then again we set $\rho = \rho_n / \mu$, else by setting $\rho = \rho_n$. Take $n = n + 1$, and repeat step 1.

Here, ω_n is taken as corrector step size which contains the GWHE (9).

We consider the convergence of the main established results, which is the main target of this research.

Theorem 1. If α^* is a solution of inequality(1) and the operator $A: H \longrightarrow H$ is ϕ -pseudomonotone, then

$$\langle \phi(\alpha) - \phi(\alpha^*), d(\alpha) \rangle \geq (\eta - \sigma) \|R(\alpha)^2\|, \quad \forall \alpha \in H. \quad (24)$$

Proof. Let $\alpha^* \in H$ be a solution of GVI (1). Then,

$$\langle A\alpha^*, \phi(\beta) - \phi(\alpha^*) \rangle \geq 0, \quad \forall \phi(\beta) \in M, \quad (25)$$

since T is ϕ -pseudomonotone. Taking $\phi(\beta) = \phi(\alpha) - \eta R(\alpha)$ in (25), we have

$$\langle A\phi^{-1}(\phi(\alpha) - \eta R(\alpha)), \phi(\alpha) - \eta R(\alpha) - \phi(\alpha^*) \rangle \geq 0. \quad (26)$$

This implies that

$$\langle A\phi^{-1}(\phi(\alpha) - \eta R(\alpha)), \phi(\alpha) - \phi(\alpha^*) \rangle \geq \eta \langle A\phi^{-1}(\phi(\alpha) - \eta R(\alpha)), R(\alpha) \rangle. \quad (27)$$

Taking $z = \phi(\alpha) - \rho A\alpha$, $\alpha = P_M[\phi(y) - \rho A\alpha]$, and $\beta = \phi(\alpha^*)$ in (3), we obtain

$$\begin{aligned} 0 &\leq \langle P_M[\phi(y) - \rho A\alpha] - \phi(\alpha) + \rho A\alpha, \phi(\alpha^*) - P_M[\phi(y) - \rho A\alpha] \rangle \\ &= \langle \rho A\alpha - [\phi(\alpha) - P_M[\phi(y) - \rho A\alpha]], \phi(\alpha^*) - \phi(\alpha) + \phi(\alpha) - P_M[\phi(y) - \rho A\alpha] \rangle \\ &= \langle \rho A\alpha - R(\alpha), \phi(\alpha^*) - \phi(\alpha) + R(\alpha) \rangle. \end{aligned} \quad (28)$$

Using (3.1),

$$\begin{aligned} &= \langle R(\alpha) - \rho A\alpha, \phi(\alpha) - \phi(\alpha^*) - R(\alpha) \rangle \\ &= \langle R(\alpha), \phi(\alpha) - \phi(\alpha^*) \rangle - \langle \rho A\alpha, \phi(\alpha) - \phi(\alpha^*) \rangle - \langle R(\alpha) - \rho A\alpha, R(\alpha) \rangle \\ &\leq \langle R(\alpha), \phi(\alpha) - \phi(\alpha^*) \rangle - \langle R(\alpha) - \rho A\alpha, R(\alpha) \rangle, \end{aligned} \quad (29)$$

from which, we have

$$\langle R(\alpha), \phi(\alpha) - \phi(\alpha^*) \rangle \geq \langle R(\alpha) - \rho A\alpha, R(\alpha) \rangle. \quad (30)$$

Adding (27) and (30), we obtain

$$\langle \phi(\alpha) - \phi(\alpha^*), A\phi^{-1}(\phi(\alpha) - \eta R(\alpha)) + R(\alpha) \rangle \geq \eta \langle R(\alpha) - \rho A\alpha + A\phi^{-1}(\phi(\alpha) - \eta R(\alpha)), R(\alpha) \rangle. \quad (31)$$

Using (19), (21), (23), (27), and (31), we have

$$\begin{aligned} &\langle \phi(\alpha) - \phi(\alpha^*), d(\alpha) \rangle \geq \eta \langle R(\alpha), D(\alpha) \rangle \\ &= \eta \langle R(\alpha), R(\alpha) \rangle - \eta \rho \langle R(\alpha), A\alpha - A\phi^{-1}(\phi(\alpha) - \eta R(\alpha)) \rangle \\ &= \eta \|R(\alpha)^2\| - \eta \rho \langle R(\alpha), A\alpha - A\phi^{-1}(\phi(\alpha) - \eta R(\alpha)) \rangle \\ &\geq (\eta - \sigma) \|R(\alpha)^2\|, \end{aligned} \quad (32)$$

which is the desired result. \square

Theorem 2. If $\alpha^* \in H$ is a solution of GVI (1) and α_{n+1} is the approximate solution found from Algorithm 2, then

$$\phi(\alpha_{n+1}) - \phi(\alpha^*) \leq \phi(\alpha_n) - \phi(\alpha^*) - \frac{(\eta_n - \sigma)R(\alpha_n)^4}{d(\alpha_n)^2}. \quad (33)$$

Proof. From (20), (21), (22), and (32), we have

$$\begin{aligned}
 \phi(\alpha_{n+1}) - \phi(\alpha^*)^2 &= P_M[\phi(\alpha_n) - \omega_n d(\alpha_n)] - \phi(\alpha^*)^2 \\
 &\leq \phi(\alpha_n) - \omega_n d(\alpha_n) - \phi(\alpha^*)^2 \\
 &\leq \phi(\alpha_n) - \phi(\alpha^*)^2 - 2\omega_n(\eta_n - \sigma)R(\alpha_n)^2 + \omega_n^2 d(\alpha_n)^2 \\
 &= \phi(\alpha_n) - \phi(\alpha^*)^2 - 2(\eta_n - \sigma) \frac{R(\alpha_n), D(\alpha_n)}{d(\alpha_n)^2} R(\alpha_n)^2 + \frac{[R(\alpha_n), D(\alpha_n)]^2}{d(\alpha_n)^4} d(\alpha_n)^2 \\
 &\leq \phi(\alpha_n) - \phi(\alpha^*)^2 - 2(\eta_n - \sigma) \frac{R(\alpha_n)^4}{d(\alpha_n)^2} + (\eta_n - \sigma) \frac{R(\alpha_n)^4}{d(\alpha_n)^2} \\
 &= \phi(\alpha_n) - \phi(\alpha^*)^2 - (\eta_n - \sigma) \frac{2R(\alpha_n)^4}{d(\alpha_n)^2},
 \end{aligned} \tag{34}$$

which is the required result. \square

Theorem 3. Let α_{n+1} be the approximated solution obtained from Algorithm 2 and $\alpha^* \in M$ be a solution of GVI (1). If H is a finite-dimensional space, then $\lim_{n \rightarrow \infty} \alpha_n = \alpha^*$.

Proof. Let $\alpha^* \in M$ be a solution of GVI (1). From (34), we get that the sequence $\{\alpha_n\}$ is bounded; we have

$$(\eta_n - \sigma) \frac{R(\alpha_n)^4}{d(\alpha_n)^2} \leq \phi(\alpha_n) - \phi(\alpha^*)^2, \tag{35}$$

which shows both expressions are going to be zero when $n \rightarrow \infty$ such as

$$\lim_{n \rightarrow \infty} R(\alpha_n) = 0 \tag{36}$$

and

$$\lim_{n \rightarrow \infty} \eta_n = 0, \tag{37}$$

which implies (36) holds. Let α^* be taken as the cluster point of $\{\alpha_n\}$, and consider the subsequence $\{\alpha_{n_i}\}$ of the sequence $\{\alpha_n\}$ converge to point α^* . We know continuity of R holds; we have

$$R(\alpha^*) = \lim_{n \rightarrow \infty} R(\alpha_n) = 0, \tag{38}$$

which provides that α^* is a solution of GVI (1) by Theorem 3 and

$$\phi(\alpha_{n+1}) - \phi(\alpha^*)^2 \leq \phi(\alpha_n) - \phi(\alpha^*)^2. \tag{39}$$

Thus, the sequence $\{\alpha_n\}$ converges exactly one cluster point and the consequences, and we obtain

$$\lim_{n \rightarrow \infty} \phi(\alpha_n) = \phi(\alpha^*). \tag{40}$$

Since ϕ is injective, it gives that $\lim_{n \rightarrow \infty} \alpha_n = \alpha^* \in H$, which satisfies the GVI (1).

Suppose that (37) holds and $\lim_{n \rightarrow \infty} \eta_n = 0$. If (32) does not hold, then, by taking the value of η_n , we obtain

$$\sigma R(\alpha_n) \leq \eta_n \rho_n A(\alpha_n) - A\phi^{-1}(\phi(\alpha_n) - \eta_n R(\alpha_n)), R(\alpha_n). \tag{41}$$

Let α^* be the cluster point of $\{\alpha_n\}$ and let $\{\alpha_{n_i}\}$ be the subsequence $\{\alpha_{n_i}\}$ converge to α^* . We apply the limit in (41); then,

$$\sigma R(\alpha^*)^2 \leq 0, \tag{42}$$

which gives $R(\alpha^*) = 0$, that is, $\alpha^* \in H$ is a solution of inequality (1), and by Lemma 1, inequality (41) holds. By repeating the same process and arguments, we approach that $\lim_{n \rightarrow \infty} \alpha_n = \alpha^*$, the desired result. \square

4. Numerical Example

Problem 1. This problem is relevant to inequality (1), with $\phi(\alpha) = A\alpha + q$ and $A\alpha = \alpha$, where

$$A = \begin{bmatrix} 4 & -2 & 0 & \dots & 0 & 0 \\ 1 & 4 & -2 & \dots & 0 & 0 \\ 0 & 1 & 4 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & -2 \\ 0 & 0 & 0 & \dots & 1 & 4 \end{bmatrix}, \tag{43}$$

$$q = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}.$$

We set the following domain for the considered problem: $\mathbf{M} = \{\alpha \in (R^n/0) \leq \alpha_i \leq 1, \text{ for } i = 1, 2, 3, \dots, n\}$. Results for

TABLE 1: Algorithm 1 (numerical results).

| Parameters | $\rho = 7, \delta = 0.02,$ $\mu = 0.7$ | $\rho = 7, \delta = 0.02,$ $\mu = 0.8$ | $\rho = 7, \delta = 0.02,$ $\mu = 0.9$ |
|------------|---|---|---|
| Iterations | 14 | 22 | 47 |

TABLE 2: Algorithm 2 (numerical results).

| Parameters | $\rho = 7, \delta = 0.02,$ $\mu = 0.7$ | $\rho = 7, \delta = 0.02,$ $\mu = 0.8$ | $\rho = 7, \delta = 0.02,$ $\mu = 0.9$ |
|------------|---|---|---|
| Iterations | 12 | 19 | 42 |

TABLE 3: Algorithm 2 (numerical results).

| Parameters | $\rho = 5, \delta = 0.3,$ $\mu = 0.6$ | $\rho = 7, \delta = 0.2,$ $\mu = 0.4$ | $\rho = 7, \delta = 0.05,$ $\mu = 0.7$ |
|------------|--|--|---|
| Iterations | 4 | 3 | 11 |

TABLE 4: Algorithm 2 (numerical results).

| Parameters | $\rho = 4, \delta = 0.2,$ $\mu = 0.8$ | $\rho = 4, \delta = 0.1,$ $\mu = 0.8$ | $\rho = 4, \delta = 0.04,$ $\mu = 0.8$ |
|------------|--|--|---|
| Iterations | 9 | 12 | 16 |

Algorithm 1 are mentioned in Table 1. Tables 2 and 3 represent the outcomes of Algorithm 2 with the initial point $\alpha^0 = -A^{-1}q$ for the order $n = 100$ of the generated matrix. For all output, we consider $\mu, \delta \in (0, 1), \gamma \in [1, 2]$, and $\rho > 0$. The iterative process will stop when we have $r(\alpha_n, \rho_n) \leq 10^{-7}$.

From Tables 1 and 2, we observe with the change of parameters that the number of iterations also varies. Table 2 gives the results for the newly established method (Algorithm 2). From the output, we observe that the newly established method converges more quickly than Algorithm 1 for solving the main GVI.

From Tables 3 and 4, we see that, in the new iterative scheme, the number varies (iterations) by changing the parameters δ, ρ , and μ . By changing the parameters accordingly, we can reduce the number of iterations.

5. Conclusion

For this study, the predictor-corrector self-adaptive method has been applied and considered to find the solution of the GVI. We used pseudomonotone of the operator, which is considered as a weaker condition than monotonicity. We also proved the convergence analysis, which is the main motivation of this paper. It has been analyzed that the new technique is more efficient than the already proved methods. The efficiency of the method has been illustrated through an example. Comparison is provided with other known methods. The numerical results reflect the output of our newly established algorithms well for the considered problem.

Data Availability

All data and information used for implementation are available within the article.

Conflicts of Interest

All authors declare no conflicts of interest in this paper.

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Research Article

Modelling to Engineering Data: Using a New Two-Parameter Lifetime Model

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A novel two-parameter continuous lifespan model is developed, based on a truncated Fréchet produced family of distributions known as the truncated Fréchet inverted Lindley distribution. It includes a thorough discussion of statistical features such as the quantile function, moments, order statistics, incomplete moments, and Lorenz and Bonferroni curves. The greatest likelihood approach for estimating population parameters is described. Finally, one real-world data set to application is utilized to demonstrate the new distribution's utility. The data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20 mm.

1. Introduction

Adding parameter(s) to baseline distributions is a traditional approach for generating families of probability distributions. These families have the capacity to increase the desirable aspects of probability distributions as well as extract additional information from a variety of data sets, which may be used in a variety of fields such as engineering, economics, biology, and environmental sciences. Another generator utilizes the shortened random variable. In this context, significant research on the truncated (T)-G families is the T Fréchet-G [1], T Weibull-G [2], Type II T Fréchet-G (TIITFG) [3], T Burr X-G [4], T Lomax-G [5], T power Lo-G (TPLoG) [6], TX family of distributions [7], T log-logistic-G [8], generalized odd Weibull-G [9], Topp-Leone-G [10], transmuted odd Fréchet-G [11] and truncated Cauchy power [12].

Aldahlan [3] proposed the TIITFG family with the following cumulative distribution function (cdf):

$$F(z; b, \varphi) = 1 - ee^{-(1-G(z; \varphi))^{-b}}, \quad (1)$$

and the density function (pdf)

$$f(z; b, \varphi) = \text{beg}(z; \varphi)(1 - G(z; \varphi))^{-b-1} e^{-(1-G(z; \varphi))^{-b}}. \quad (2)$$

The following exponential series is used to generate the expansion of pdf (2):

$$e^{-cz} = \sum_{j=0}^{\infty} \frac{(-1)^j c^j}{j!} z^j, \quad c > 0. \quad (3)$$

Regarding the existing binomial series can be used,

$$(1 - Z)^{-c} = \sum_{i=0}^{\infty} \binom{c+i-1}{i} Z^i, \quad c > 0, \text{ and } |Z| < 1. \quad (4)$$

Employing (3) and (4) in (2), then the pdf of TIITFG, where b is real, is

$$f(z; b, \varphi) = \sum_{i=0}^{\infty} \eta_i g(z; \varphi) G(z; \varphi)^{i+1}, \quad (5)$$

where

$$\eta_i = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} b e (-1)^i \binom{b(j+1)+i}{i}. \quad (6)$$

The quantile function $Q_{(u)}$ of X is given by

$$Q_{(u)} = G^{-1} \left\{ 1 - \left[\ln \left(\frac{e}{1-u} \right) \right]^{(-1/b)} \right\}. \quad (7)$$

Sharma et al.[13] investigated the inverted Lindley (ILi) distribution, which has the following pdf and cdf, respectively

$$g(z; \theta) = \frac{\theta^2}{1+\theta} \left(\frac{1+z}{z^3} \right) e^{(-\theta/z)}, \quad z > 0, \theta > 0, \quad (8)$$

$$G(z; \theta) = \left(1 + \frac{\theta}{(1+\theta)z} \right) e^{(-\theta/z)}, \quad z > 0, \theta > 0. \quad (9)$$

Many statisticians have generalized the ILi distribution in recent years, such as [14] who studied the extended ILi distribution, [15] who proposed the generalized ILi distribution, [16] who proposed the power ILi distribution, a new extension of ILi proposed by [17, 18] who studied weighted ILi

distribution, [19] who studied alpha power transformed ILi (APTILi) distribution, and [20] who studied extended exponentiated ILi distribution, and logarithmic ILi model was studied by [21] and generalized Marshall Olkin ILi by [22].

The main goal of this article is to present the Type II truncated Fréchet inverted Lindley (TIIFILi) distribution, a novel two-parameter life-time model. The new model is very flexible. The pdf can be symmetric, right skewness, and unimodal. Also, the hrf can be unimodal, increasing, and J-shaped. Investigate some of its statistical features to discuss the statistical inference of the TIIFILi model and to give leading fits than some known models with favourable results for the TIIFILi model.

The new model is extremely adaptable, and we may obtain the cdf and pdf by adding (8) and (9), as shown in (1) and (2).

$$F(z; b, \theta) = 1 - e e^{-\left(1 - (1 + (\theta/(1+\theta)z))e^{(-\theta/z)}\right)^{-b}}, \quad z > 0, b, \theta > 0, \quad (10)$$

$$f(z; b, \theta) = \frac{\theta^2 b e}{1+\theta} \left(\frac{1+z}{z^3} \right) e^{(-\theta/z)} \left(1 - \left(1 + \frac{\theta}{(1+\theta)z} \right) e^{(-\theta/z)} \right)^{-b-1} e^{-\left(1 - (1 + (\theta/(1+\theta)z))e^{(-\theta/z)}\right)^{-b}}. \quad (11)$$

Using the three equations (3)–(5), we can rewrite (11) as follows:

$$f(z; b, \theta) = \sum_{i=0}^{\infty} \eta_i \frac{\theta^2}{1+\theta} \left(\frac{1+z}{z^3} \right) e^{(-\theta(z+1)/z)} \left(1 + \frac{\theta}{(1+\theta)z} \right)^j. \quad (12)$$

We may rewrite the above equation using the binomial expansion as

$$f(z; b, \theta) = \sum_{k=0}^{\infty} \mathbb{C}_k \left(\frac{1+z}{z^{k+3}} \right) e^{(-\theta(z+1)/z)}, \quad (13)$$

where

$$\mathbb{C}_k = \sum_{i=0}^{\infty} \eta_i \binom{j}{k} \frac{\theta^{k+2}}{(1+\theta)^{k+1}}. \quad (14)$$

The TIIFILi distribution function, the hazard rate function (hrf), the inverted hazard rate function, and the cumulative hazard rating function are given when a random variable Z follows the TIIFILi model,

$$R(z; b, \theta) = e e^{-\left(1 - \left(1 + \frac{\theta}{(1+\theta)z}\right) e^{(-\theta/z)}\right)^{-b}}, \quad (15)$$

$$h(z; b, \theta) = \frac{\theta^2 b}{1+\theta} \left(\frac{1+z}{z^3} \right) e^{(-\theta/z)} \left(1 - \left(1 + \frac{\theta}{(1+\theta)z} \right) e^{(-\theta/z)} \right)^{-b-1},$$

$$\tau(z; b, \theta) = \frac{\left(\theta^2 b e / (1+\theta) \right) \left((1+z)/z^3 \right) e^{(-\theta/z)} \left(1 - (1 + (\theta/(1+\theta)z)) e^{(-\theta/z)} \right)^{-b-1} e^{-\left(1 - (1 + (\theta/(1+\theta)z)) e^{(-\theta/z)}\right)^{-b}}}{1 - e e^{-\left(1 - (1 + (\theta/(1+\theta)z)) e^{(-\theta/z)}\right)^{-b}}}, \quad (16)$$

and

$$H(z; b, \theta) = \frac{1}{1 - \left(1 - (1 + (\theta/(1 + \theta)z))e^{(-\theta/z)}\right)^{-b}}. \quad (17)$$

Figures 1 and 2 show the pdf and hrf plots of the TIIFILi distribution.

Figure 1 demonstrates how the pdf might be unimodal and tilted to the right. Figure 2 depicts various potential hrf forms, including monotone increasing, up-side-down, and J-shaped.

The remainder of this article is arranged as follows: Section 2 investigates distribution's mathematical characteristics of the proposed model. Section 3 covers the estimate of distribution parameters using the maximum likelihood method of estimation. Section 4 provides actual data

applications to illustrate the potential of the new distribution, and Section 5 ends with remarks.

2. Mathematical Properties

In this section, we will study some statistical properties such as the quantile function, median, order statistics, moments, moment generating function, incomplete moments, and Lorenz and Bonferroni curves.

The quantile function

$$\left(1 + \frac{\theta}{(1 + \theta)Q(u)}\right)e^{(-\theta/Q(u))} = 1 - \left[\ln\left(\frac{e}{1 - u}\right)\right]^{(-1/b)}, \quad 0 < u < 1. \quad (18)$$

By multiplying (9) both sides by $-(1 + \theta)e^{-(1+\theta)}$,

$$-\left(1 + \theta + \frac{\theta}{Q(u)}\right)e^{-(1+\theta+(-\theta/Q(u)))} = -(1 + \theta)e^{-(1+\theta)}\left(1 - \left[\ln\left(\frac{e}{1 - u}\right)\right]^{(-1/b)}\right). \quad (19)$$

Then,

$$Q(u) = -\left[1 + \frac{1}{\theta} + \frac{1}{\theta}W_{-1}\left(-(1 + \theta)e^{-(1+\theta)}\left(1 - \left[\ln\left(\frac{e}{1 - u}\right)\right]^{(-1/b)}\right)\right)\right]^{-1}. \quad (20)$$

Corollary 1. If $Z \sim TIIFILi$, the median M of Z is given by

$$Q(u) = -\left[1 + \frac{1}{\theta} + \frac{1}{\theta}W_{-1}\left(-(1 + \theta)e^{-(1+\theta)}\left(1 - [\ln(2e)]^{(-1/b)}\right)\right)\right]^{-1}. \quad (21)$$

Assuming $Z_1 < Z_2 < \dots < Z_n$ is an order sample from TIIFILi population, the pdf of the i^{th} ordered statistics is given as

$$f(z_{i:n}) = \frac{n!}{(i-1)!(n-i)!} f(z; b, \theta) F(z; b, \theta)^{i-1} (1 - F(z; b, \theta))^{n-i}. \quad (22)$$

Substituting (10) and (11), and applying general binomial series expansion, (13) becomes

$$f(z_{i:n}) = \frac{n! \theta^2 b e^{n-i+1} e^{(-\theta/z)}}{(i-1)!(n-i)!(1+\theta)} \left(\frac{1+z}{z^3}\right) \left(1 - \left(1 + \frac{\theta}{(1+\theta)z}\right)e^{(-\theta/z)}\right)^{-b-1} e^{-(n-i+1)(1-(1+\theta/(1+\theta)z))e^{(-\theta/z)}}^{-b} \left(1 - e e^{-(1-(1+\theta/(1+\theta)z))e^{(-\theta/z)}}^{-b}\right)^{i-1}. \quad (23)$$

We can get the first and last order statistics at $i = 1$ and $i = n$, respectively, as follows:

$$f(z_{1:n}) = \frac{n \theta^2 b e^n}{(1+\theta)} \left(\frac{1+z}{z^3}\right) e^{(-\theta/z)} \left(1 - \left(1 + \frac{\theta}{(1+\theta)z}\right)e^{(-\theta/z)}\right)^{-b-1} e^{-n(1-(1+\theta/(1+\theta)z))e^{(-\theta/z)}}^{-b}, \quad (24)$$

and

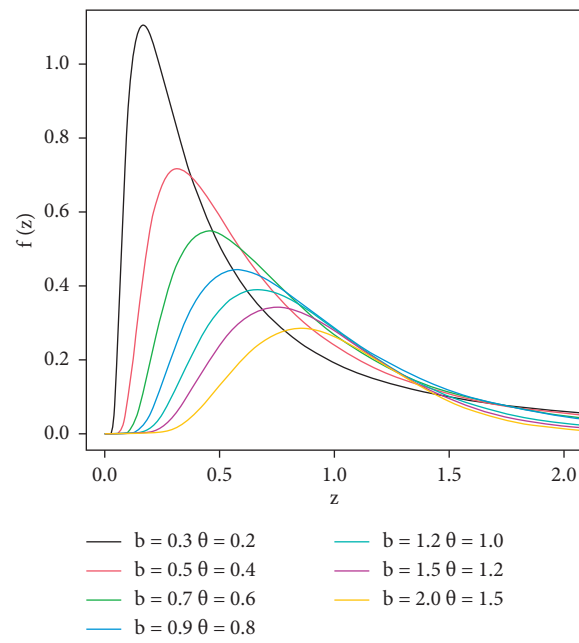


FIGURE 1: The pdf of the TIIFILi model for various parameter values.

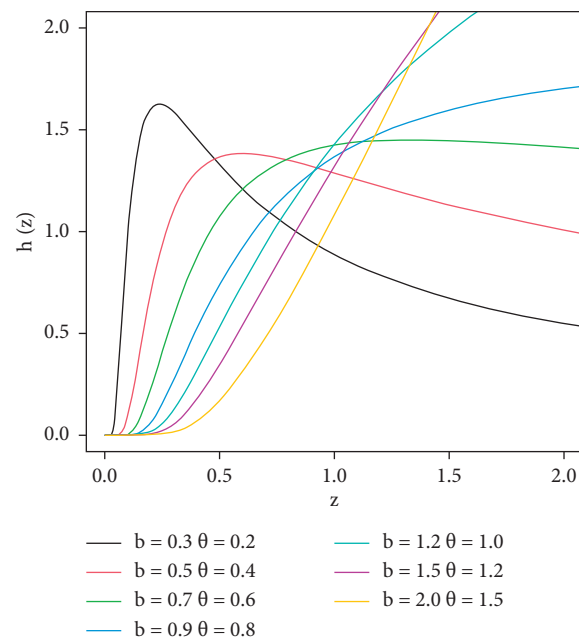


FIGURE 2: The cdf of the TIIFILi model for various parameter values.

$$f(z_{n:n}) = \frac{n\theta^2 b e}{(1+\theta)} \left(\frac{1+z}{z^3} \right) e^{(-\theta/z)} \left(1 - \left(1 + \frac{\theta}{(1+\theta)z} \right) e^{(-\theta/z)} \right)^{-b-1} e^{-\left(1 - (1+(\theta/(1+\theta)z))e^{(-\theta/z)}\right)^{-b}} \left(1 - e e^{-\left(1 - (1+(\theta/(1+\theta)z))e^{(-\theta/z)}\right)^{-b}} \right)^{n-1}. \quad (25)$$

The r^{th} moments of TIITFILI distribution are defined as follows:

$$\begin{aligned} E(Z^r) &= \int_0^\infty z^r f(z; b, \theta) dz \\ &= \sum_{k=0}^\infty C_k \int_0^\infty z^r \left(\frac{1+z}{z^{k+3}} \right) e^{(-\theta(j+1)/z)} dz \\ &= \sum_{k=0}^\infty C_k \int_0^\infty (z^{r-k-3} + z^{r-k-2}) e^{(-\theta(j+1)/z)} dz. \end{aligned} \quad (26)$$

Letting $y = (\theta(j+1)/z)$, $z = (\theta(j+1)/y)$, $dz = (-\theta(j+1)/y^2)dy$, and simplifying further, then

$$E(Z^r) = \sum_{k=0}^\infty C_k \int_0^\infty \left(\left(\frac{\theta(j+1)}{y} \right)^{r-k-3} + \left(\frac{\theta(j+1)}{y} \right)^{r-k-2} \right) e^{-y} \frac{\theta(j+1)}{y^2} dy, \quad (27)$$

$$\begin{aligned} E(Z^r) &= \sum_{k=0}^\infty C_k (\theta(j+1))^{r-k-2} \int_0^\infty (y^{k-r+1} + \theta(j+1)y^{k-r}) e^{-y} dy, \\ &= \sum_{k=0}^\infty C_k (\theta(j+1))^{r-k-2} (\Gamma(k-r+2) + \theta(j+1)\Gamma(k-r+1)). \end{aligned} \quad (28)$$

The moment generating function of TIITFILI model can be calculated by

$$M_Z(t) = \sum_{r=0}^\infty \frac{t^r}{r!} E(Z^r) = \sum_{r,k=0}^\infty \frac{t^r}{r!} C_k (\theta(j+1))^{r-k-2} (\Gamma(k-r+2) + \theta(j+1)\Gamma(k-r+1)), \quad r < k+2. \quad (29)$$

The incomplete moments, for example, $\omega_s(t)$ are provided by

$$\omega_s(t) = \int_0^t z^s f(z; b, \theta) dz. \quad (30)$$

Using (10), then, $\omega_s(t)$ can be taken, the next formula

$$\omega_s(t) = \sum_{k=0}^\infty C_k \int_0^t z^s \left(\frac{1+z}{z^{k+3}} \right) e^{(-\theta(j+1)/z)} dz. \quad (31)$$

The lower incomplete gamma function is then used to produce

$$\omega_s(t) = \sum_{k=0}^\infty C_k (\theta(j+1))^{s-k-2} \left(\gamma(k-s+2, \theta(j+1)t^{-1}) + \theta(j+1)\gamma(k-s+1, \theta(j+1)t^{-1}) \right), \quad s < k+2. \quad (32)$$

TABLE 1: Analysis results for the first data.

| Model | MLEs (SErs) | $\bar{O}1$ | $\bar{O}2$ | $\bar{O}3$ | $\bar{O}4$ | $\bar{O}5$ |
|----------|--|------------|------------|------------|------------|------------|
| TIITFiLi | $\hat{b} = 0.254(0.0396)$ $\hat{\theta} = 2.625(0.804)$ | 162.44 | 328.88 | 327.835 | 329.325 | 329.777 |
| APTEE | $\hat{\alpha} = 0.161(0.282)$ $\hat{\beta} = 2.01 \times 10^{-4}(0.024)$ $\hat{\gamma} = 0.011(0.022)$ | 176.631 | 359.262 | 357.694 | 360.186 | 360.607 |
| APTLi | $\hat{\alpha} = 0.1(0.1037)$ $\hat{\gamma} = 0.011(0.024)$ | 183.415 | 370.83 | 369.784 | 371.274 | 371.727 |
| PLi | $\hat{\beta} = 1.525(0.155)$ $\hat{\theta} = 2.63 \times 10^{-3}(2.058 \times 10^{-3})$ | 195.999 | 395.999 | 394.953 | 396.443 | 396.895 |

In this case, $\nu(s, t)$ is the lower incomplete gamma function.

The Lorenz and Bonferroni curves are given by

$$\begin{aligned}
 L_F(z) &= \frac{\int_0^t z f(z; b, \theta) dz}{E(Z)} = \frac{\sum_{k=0}^{\infty} C_k (\theta(j+1))^{-k-1} (\nu(k+1, \theta(j+1)t^{-1}) + \theta(j+1)\nu(k, \theta(j+1)t^{-1}))}{\sum_{k=0}^{\infty} C_k (\theta(j+1))^{-k-1} (\Gamma(k+1) + \theta(j+1)\Gamma(k))}, \\
 B_F(z) &= \frac{\int_0^t z f(z; b, \theta) dz}{E(z)F(z; b, \theta)} = \frac{L_F(z)}{F(z; b, \theta)} \\
 &= \frac{\sum_{k=0}^{\infty} C_k (\theta(j+1))^{-k-1} (\nu(k+1, \theta(j+1)t^{-1}) + \theta(j+1)\nu(k, \theta(j+1)t^{-1}))}{\left(1 - e^{-\left(1 - (1 + \theta/(1 + \theta)z)\right)e^{(-\theta/z)}}\right)^b \left(\sum_{k=0}^{\infty} C_k (\theta(j+1))^{-k-1} (\Gamma(k+1) + \theta(j+1)\Gamma(k))\right)}.
 \end{aligned} \tag{33}$$

3. Maximum Likelihood Estimation

Assume that Z_1, Z_2, \dots, Z_n is a random sample of size n from a population with TIITFiLi pdf and that the log-likelihood function is provided by

$$\text{Log}L = n + n \log(b) + 2n \log(\theta) - n \log(1 + \theta) + \sum_{i=1}^n \log\left(\frac{1 + z_i}{z_i^3}\right) - \theta \sum_{i=1}^n \frac{1}{z_i} - (b + 1) \sum_{i=1}^n \log(N_i) - \sum_{i=1}^n (N_i)^{-b}. \tag{34}$$

The score functions which correspond to equating the first-order partial derivative of the last equation to zero is given by

$$\frac{\partial \text{Log}L}{\partial b} = \frac{n}{b} - \sum_{i=1}^n \log(N_i) + \sum_{i=1}^n (N_i)^{-b} \ln(N_i), \tag{35}$$

$$\frac{\partial \text{Log}L}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{1 + \theta} - \sum_{i=1}^n \frac{1}{z_i} - (b + 1) \sum_{i=1}^n \frac{M_i}{N_i} + b \sum_{i=1}^n \frac{M_i}{N_i^{b+1}}, \tag{36}$$

where $N_i = 1 - (1 + (\theta/(1 + \theta)z_i))e^{(-\theta/z_i)}$, $M_i = (\partial N_i / \partial \theta)$ are the solutions, say \hat{b} and $\hat{\theta}$. The maximum likelihood estimators of the TIITFiLi distribution correspond to the scoring functions. The score functions, on the other hand, are nonlinear functions; the numerical values of the maximum likelihood estimates may be derived using the Newton Raphson iterative optimisation technique.

4. Modelling to Data Sets

To describe the performance of the TIITFiLi model in reality, actual data sets are explored. The whole first set of data comes from [23]. The outcomes of the fits are compared in data set to the power Li (PLi) by [24], alpha power transformed Li (APTLi) by [25], and APT extended exponential (APTEE) by [26] models.

Statistics measures such as minus log-likelihood ($\bar{O}1$), Akaike information criterion (IC) ($\bar{O}2$), Bayesian IC ($\bar{O}3$), corrected AIC ($\bar{O}4$), and Hannan-Quinn IC ($\bar{O}5$) are obtained. Several criteria are used to compare the TIITFiLi model's performance against those of other models.

The maximum likelihood estimates (MLEs), standard errors (SErs) of parameters, and the above statistics measures for the both data sets are given in Table 1. Figures 3–6 provide further information.

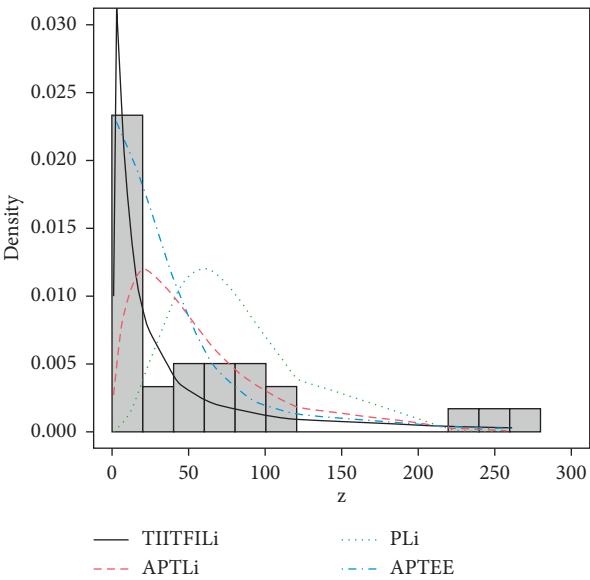


FIGURE 3: Estimated pdf of the TIITFILI and other competing models for first data.

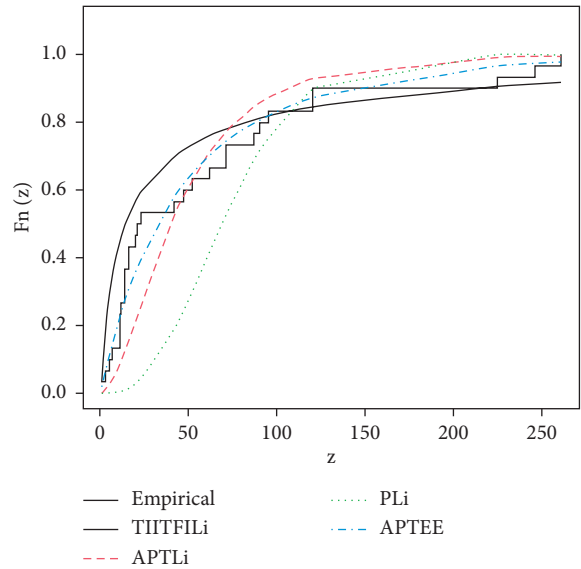


FIGURE 4: Estimated cdf of the TIITFILI and other competing models for the data.

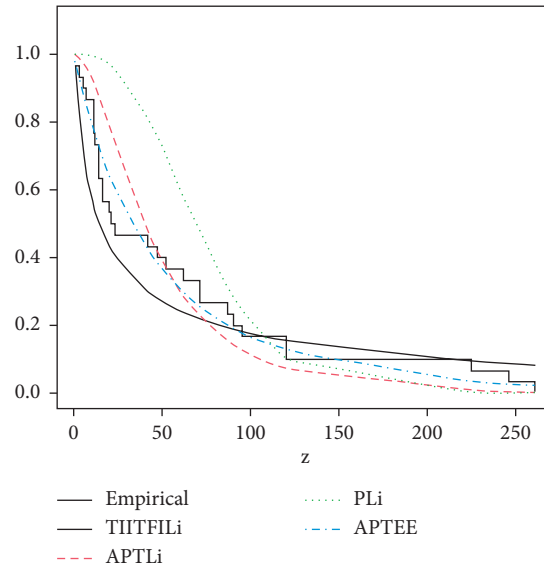


FIGURE 5: Estimated sf of the TIITFiLi and other competing models for the data.

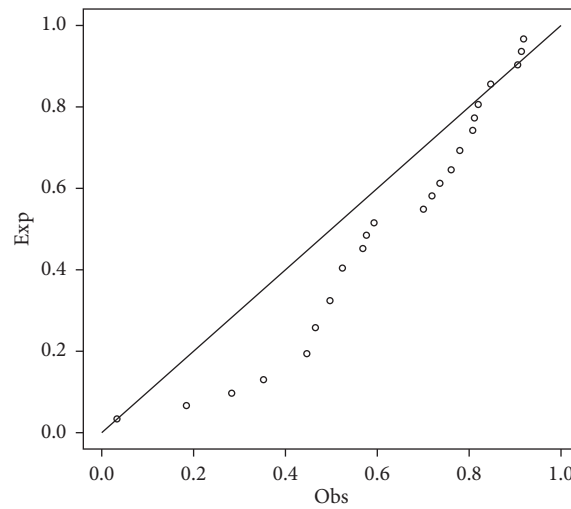


FIGURE 6: PP plots of the TIITFiLi for the data.

Table 1 reveals that the TIITFiLi model is best suitable than the APTLi, APTEE, and PLi models. Figures 3 to 6 attempt to estimate pdfs, cdfs, sfs, and pp plots for the fitted models. We infer that now the TIITFiLi model fits the data set better.

5. Conclusion

This research suggested a novel two-parameter truncated Fréchet inverted Lindley model (TIITFiLi) distribution for modelling engineering data and other applications. The TIITFiLi model generalizes and extends the inverted Lindley distribution. The TIITFiLi distribution's hazard rate might be increasing, unimodal, and J-shaped. Mathematical properties of the new model such as ordinary moments, incomplete moments, and the quantile function, order statistics, are discussed. The maximum likelihood approach

is used to estimate the parameters of the new distribution. The novel distribution's value and potential are proven by comparing its fit to a real-world data set to those of existing distributions. According to the goodness-of-fit statistics, the new distribution fits better than the other competing distributions.

Abbreviations

| | |
|---------|--|
| T: | Truncated |
| TIITFG: | Type II truncated Fréchet-G |
| TPLoG: | Truncated power Lomax-G |
| ILi: | Inverted Lindley |
| APTiLi: | Alpha power transformed inverted Lindley |
| hrf: | The hazard rate function |
| cdf: | Cumulative distribution function |
| pdf: | Density function |

MLEs: Maximum likelihood estimates
 SEs: Standard errors
 APTEE: Alpha power transformed extended exponential
 PLi: Power Lindley
 APTLi: Alpha power transformed Lindley.

Data Availability

Please contact the relevant author if you would like to acquire the numerical data set used to perform the study presented in the paper.

Conflicts of Interest

The author declares no conflicts of interest.

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Research Article

Some New Aspects in the Intuitionistic Fuzzy and Neutrosophic Fixed Point Theory

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In this manuscript, we use the concepts of continuous t -norms and continuous t -conorms to introduce some definitions, in which intuitionistic fuzzy rectangular metric spaces, intuitionistic fuzzy rectangular metric-like spaces, intuitionistic fuzzy rectangular b -metric spaces, intuitionistic fuzzy rectangular b -metric-like spaces, neutrosophic rectangular metric spaces, neutrosophic rectangular metric-like spaces, neutrosophic rectangular b -metric spaces, and neutrosophic rectangular b -metric-like spaces are included. Continuous t -norms and continuous t -conorms are used to generalize the probability distribution of triangular inequalities in metric space axioms. Nontrivial examples, some fixed point results, and an application to the integral equation are imparted in this manuscript.

1. Introduction

Fuzzy set (FS) presented by Zadeh [1] is a useful tool for those situations in which the data are imprecise and the idea of degree of membership is involved in FS theory. Intuitionistic fuzzy sets (IFSs) introduced by Atanassov [2] are the generalization of the FS, in which degrees of membership and nonmembership are involved. Smarandache [3] presented the idea of neutrosophic sets (NSs) that are the generalization of the IFS, in which degrees of membership, nonmembership, and uncertainty are involved.

By combining the concepts of FS and metric spaces, fuzzy metric spaces (FMSs) were presented by Kramosil and Michalek [4]. Kaleva and Seikkala [5] coined FMS in which they defined a distance between two points to be a non-negative fuzzy number, and Garbiec [6] presented the fuzzy interpretation of the Banach contraction principle in the FMS. Park [7] presented the intuitionistic fuzzy metric space (IFMS), in which he used George and Veeramani's [8] approach of applying continuous t -norm (CTN) and continuous t -conorm (CTCN) to the FMS. Kirişçi and Şimşek [9] presented the notion of neutrosophic metric space (NMS), in which they used the idea of NS and probabilistic

metric spaces. FMS deals with membership functions, and IFMS deals with membership and nonmembership functions. NMS is a generalization of the IFMS that deals with membership, nonmembership, and inconsistent functions. Altun et al. [10] and Aslantas et al. [11] proved some interesting results for cyclic p -contractions and KW-type nonlinear contractions. Al-Omeri et al. [12, 13] proved several neutrosophic fixed point results and generalized theorems in the sense of neutrosophic cone metric spaces.

Javed et al. [14] presented the idea of fuzzy b -metric-like spaces (FBMLSs) and proved several fixed point results. Mehmood et al. [15] presented the concept of fuzzy rectangular b -metric spaces (FRBMSs) and proved the Banach contraction principle in the sense of FRBMS. For some necessary definitions and related fixed point results, see [16–19].

In this manuscript, we generalized the concepts used in [14, 15]. The main objectives of this manuscript are as follows:

- (i) To present different notions in the intuitionistic fuzzy and neutrosophic fixed point theory
- (ii) To prove certain fixed point theorems

- (iii) To enhance the existing literature of the FMS and fuzzy fixed point theory

This study is organized with some basic notions of FRBMS, FBMLS, IFMS, and NMS. The notions of intuitionistic fuzzy rectangular metric spaces (IFRMSs), intuitionistic fuzzy rectangular metric-like spaces (IFRMLSs), intuitionistic fuzzy rectangular b -metric spaces (IFRBMSs), intuitionistic fuzzy rectangular b -metric-like spaces (IFRBMLSs), neutrosophic rectangular metric spaces (NRMSs), neutrosophic rectangular metric-like spaces (NRMLSs), neutrosophic rectangular b -metric spaces (NRBMSs), and neutrosophic rectangular b -metric-like spaces (NRBMLSs) are discussed in detail, and several fixed point results, nontrivial examples, and an application to the integral equation are imparted. At the end, conclusion is given for the examined results.

2. Preliminaries

In this section, some basic definitions are imparted that are helpful to understand the main section.

Definition 1 (see [7]). A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a CTN if it meets the following assertions:

- C1. $\zeta * b = b * \zeta$, $(\forall) \zeta, b \in [0, 1]$
- C2. $*$ is continuous
- C3. $\zeta * 1 = \zeta$, $(\forall) \zeta \in [0, 1]$
- C4. $(\zeta * b) * c = \zeta * (b * c)$, $(\forall) \zeta, b, c \in [0, 1]$
- C5. If $\zeta \leq c$ and $b \leq \sigma$, with $\zeta, b, c, \sigma \in [0, 1]$, then $\zeta * b \leq c * \sigma$

Definition 2 (see [7]). A binary operation \circ : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a CTCN if it meets the following assertions:

- T1. $\zeta \circ b = b \circ \zeta$, for all $\zeta, b \in [0, 1]$
- T2. \circ is continuous
- T3. $\zeta \circ 0 = 0$, for all $\zeta \in [0, 1]$
- T4. $(\zeta \circ b) \circ c = \zeta \circ (b \circ c)$, for all $\zeta, b, c \in [0, 1]$
- T5. If $\zeta \leq c$ and $b \leq \sigma$, with $\zeta, b, c, \sigma \in [0, 1]$, then $\zeta \circ b \leq c \circ \sigma$

Definition 3 (see [19]). Let a set $E \neq \emptyset$ and $\vartheta \in E$. A NSG in E is categorized by a truth membership function $\mathfrak{B}_G(\vartheta)$, an indeterminacy membership function $\mathfrak{D}_G(\vartheta)$, and a falsity membership function $\mathfrak{Q}_G(\vartheta)$. The functions $\mathfrak{B}_G(\vartheta)$, $\mathfrak{D}_G(\vartheta)$, and $\mathfrak{Q}_G(\vartheta)$ are real standard or nonstandard subsets of $]0^-, 1^+[$; that is, $\mathfrak{B}_G(\vartheta): E \rightarrow]0^-, 1^+[$, $\mathfrak{D}_G(\vartheta): E \rightarrow]0^-, 1^+[$ and $\mathfrak{Q}_G(\vartheta): E \rightarrow]0^-, 1^+[$. So,

$$0^- \leq \sup \mathfrak{B}_G(\vartheta) + \sup \mathfrak{D}_G(\vartheta) + \sup \mathfrak{Q}_G(\vartheta) \leq 3^+. \quad (1)$$

Definition 4 (see [14]). Let E be a nonempty set. A triplet $(E, F_b, *)$ is called a FBMLS if $*$ is a CTN and F_b is a FS on

$E \times E \times (0, \infty)$ and fulfills the following assertions for all $\vartheta, \delta, g \in E$ and $\tau, z > 0$:

- A1. $F_b(\vartheta, \delta, \tau) > 0$
- A2. $F_b(\vartheta, \delta, \tau) = 1$; then, $\vartheta = \delta$
- A3. $F_b(\vartheta, \delta, \tau) = F_b(\delta, \vartheta, \tau)$
- A4. $F_b(\vartheta, g, b(\tau + z)) \geq F_b(\vartheta, \delta, \tau) * F_b(\delta, g, z)$
- A5. $F_b(\vartheta, \delta, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} F_b(\vartheta, \delta, \tau) = 1$

Definition 5 (see [15]). Let E be a nonempty set. A triplet $(E, R_b, *)$ is called a FRMS if $*$ is a CTN and R_b is a FS on $E \times E \times [0, \infty)$ and fulfills the following assertions for all $\vartheta, \delta, g \in E$ and $\tau, z, w > 0$:

- (1) $R_b(\vartheta, \delta, 0) = 0$
- (2) $R_b(\vartheta, \delta, \tau) = 1$ if and only if $\vartheta = \delta$
- (3) $R_b(\vartheta, \delta, \tau) = R_b(\delta, \vartheta, \tau)$
- (4) $R_b(\vartheta, g, \tau + z + w) \geq R_b(\vartheta, \delta, \tau) * R_b(\delta, u, z) + R_b(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- (5) $R_b(\vartheta, \delta, \cdot): (0, \infty) \rightarrow [0, 1]$ is left continuous, and $\lim_{\tau \rightarrow \infty} R_b(\vartheta, \delta, \tau) = 1$

Definition 6 (see [7]). Take $E \neq \emptyset$. Let $*$ be a CTN, \circ be a CTCN, and F, V be FSs on $E \times E \times (0, \infty)$. If $(E, F, V, *, \circ)$ verifies the following assertions for all $\vartheta, \delta \in E$ and $z, \tau > 0$,

- F1. $F(\vartheta, \delta, \tau) + V(\vartheta, \delta, \tau) \leq 1$
 - F2. $F(\vartheta, \delta, \tau) > 0$
 - F3. $F(\vartheta, \delta, \tau) = 1 \iff \vartheta = \delta$
 - F4. $F(\vartheta, \delta, \tau) = F(\delta, \vartheta, \tau)$
 - F5. $F(\vartheta, g, \tau + z) \geq F(\vartheta, \delta, \tau) * F(\delta, g, z)$
 - F6. $F(\vartheta, \delta, z): (0, \infty) \rightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} F(\vartheta, \delta, \tau) = 1$ for all $\tau > 0$
 - F7. $V(\vartheta, \delta, \tau) > 0$
 - F8. $V(\vartheta, \delta, \tau) = 0 \iff z\vartheta = \delta$
 - F9. $V(\vartheta, \delta, \tau) = V(\delta, \vartheta, \tau)$
 - F10. $V(\vartheta, g, \tau + z) \leq V(\vartheta, \delta, \tau) \circ V(\delta, g, z)$
 - F11. $V(\vartheta, \delta, z): (0, \infty) \rightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} V(\vartheta, \delta, \tau) = 0$ for all $\tau > 0$
- Then, $(E, F, V, *, \circ)$ is an IFMS

Definition 7 (see [8]). Let $E \neq \emptyset$, $*$ be a CTN, and \circ be a CTCN. L, W , and Q are NSs on $E \times E \times (0, \infty)$ which are said to be a neutrosophic metric on E if for all $\vartheta, \delta, g \in E$, the following circumstances fulfill:

- S1. $L(\vartheta, \delta, \tau) + W(\vartheta, \delta, \tau) + Q(\vartheta, \delta, \tau) \leq 3$ for all $\tau \in \mathbb{R}^+$
- S2. $L(\vartheta, \delta, \tau) > 0$ for all $\tau > 0$
- S3. $L(\vartheta, \delta, \tau) = 1$ for all $\tau > 0$ if and only if $\vartheta = \delta$
- S4. $L(\vartheta, \delta, \tau) = L(\delta, \vartheta, \tau)$ for all $\tau > 0$
- S5. $L(\vartheta, g, \tau + z) \geq L(\vartheta, \delta, \tau) * L(\delta, g, z)$ for all $\tau, z > 0$

S6. $L(\vartheta, \delta, z): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} L(\vartheta, \delta, \tau) = 1$ for all $\tau > 0$

S7. $W(\vartheta, \delta, \tau) < 1$ for all $\tau > 0$

S8. $W(\vartheta, \delta, \tau) = 0$ for all $\tau > 0$ if and only if $\vartheta = \delta$

S9. $W(\vartheta, \delta, \tau) = W(\delta, \vartheta, \tau)$ for all $\tau > 0$

S10. $W(\vartheta, g, \tau + z) \leq W(\vartheta, \delta, \tau) \odot W(\delta, g, z)$ for all $\tau, z > 0$

S11. $W(\vartheta, \delta, z): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} W(\vartheta, \delta, \tau) = 0$ for all $\tau > 0$

S12. $Q(\vartheta, \delta, \tau) < 1$ for all $\tau > 0$

S13. $Q(\vartheta, \delta, \tau) = 0$ for all $\tau > 0$ if and only if $\vartheta = \delta$

S14. $Q(\vartheta, \delta, \tau) = Q(\delta, \vartheta, \tau)$ for all $\tau > 0$

S15. $Q(\vartheta, g, \tau + z) \leq Q(\vartheta, \delta, \tau) \odot Q(\delta, g, z)$ for all $\tau, z > 0$

S16. $Q(\vartheta, \delta, z): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} Q(\vartheta, \delta, \tau) = 0$ for all $\tau > 0$

Then, $(E, L, W, Q, *, \odot)$ is called a NMS

3. Main Results

In this section, we present some new notions as generalizations of intuitionistic fuzzy and neutrosophic metric spaces; also, some fixed point results are proved.

Definition 8. Let E be a nonempty set. A five-tuple $(E, \mathfrak{B}_i, \mathfrak{D}_i, *, \odot)$ is called an IFRMS if $*$ is a CTN, \odot is a CTCN, and \mathfrak{B}_i and \mathfrak{D}_i are two FSs on $E \times E \times [0, \infty)$ which fulfill the following assertions for all $\vartheta, \delta, g \in E$ and $\tau, z, w > 0$:

R1. $\mathfrak{B}_i(\vartheta, \delta, \tau) + \mathfrak{D}_i(\vartheta, \delta, \tau) \leq 1$

R2. $\mathfrak{B}_i(\vartheta, \delta, 0) = 0$

R3. $\mathfrak{B}_i(\vartheta, \delta, \tau) = 1$ if and only if $\vartheta = \delta$

R4. $\mathfrak{B}_i(\vartheta, \delta, \tau) = \mathfrak{B}_i(\delta, \vartheta, \tau)$

R5. $\mathfrak{B}_i(\vartheta, g, \tau + z + w) \geq \mathfrak{B}_i(\vartheta, \delta, \tau) * \mathfrak{B}_i(\delta, u, z) + \mathfrak{B}_i(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$

R6. $\mathfrak{B}_i(\vartheta, \delta, .): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{B}_i(\vartheta, \delta, \tau) = 1$

R7. $\mathfrak{D}_i(\vartheta, \delta, 0) = 1$

R8. $\mathfrak{D}_i(\vartheta, \delta, \tau) = 0$ if and only if $\vartheta = \delta$

R9. $\mathfrak{D}_i(\vartheta, \delta, \tau) = \mathfrak{D}_i(\delta, \vartheta, \tau)$

R10. $\mathfrak{D}_i(\vartheta, g, \tau + z + w) \leq \mathfrak{D}_i(\vartheta, \delta, \tau) \odot \mathfrak{D}_i(\delta, u, z) + \mathfrak{D}_i(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$

R11. $\mathfrak{D}_i(\vartheta, \delta, .): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{D}_i(\vartheta, \delta, \tau) = 0$

Example 1. Let (E, d) be a rectangular metric space, define $\mathfrak{B}_i, \mathfrak{D}_i: E \times E \times [0, \infty) \longrightarrow [0, 1]$ by

$$\begin{aligned} \mathfrak{B}_i(\vartheta, \delta, \tau) &= \frac{\tau}{\tau + d(\vartheta, \delta)}, \\ \mathfrak{D}_i(\vartheta, \delta, \tau) &= 1 - \frac{\tau}{\tau + d(\vartheta, \delta)} \text{ for all } \vartheta, \delta \in E \text{ and } \tau > 0, \end{aligned} \quad (2)$$

and let $*$ be a CTN and \odot be a CTCN on E . Then, it is easy to see that $(E, \mathfrak{B}_i, \mathfrak{D}_i, *, \odot)$ is an IRFMS.

Definition 9. Let E be a nonempty set. A five-tuple $(E, \mathfrak{B}_b, \mathfrak{D}_b, *, \odot)$ is called an IFRBMS if there is $b \geq 1$, $*$ is a CTN, \odot is a CTCN, and \mathfrak{B}_b and \mathfrak{D}_b are two FSs on $E \times E \times [0, \infty)$ verifying the following assertions for all $\vartheta, \delta, g \in E$ and $\tau, z, w > 0$:

I. $\mathfrak{B}_b(\vartheta, \delta, \tau) + \mathfrak{D}_b(\vartheta, \delta, \tau) \leq 1$

II. $\mathfrak{B}_b(\vartheta, \delta, 0) = 0$

III. $\mathfrak{B}_b(\vartheta, \delta, \tau) = 1$ if and only if $\vartheta = \delta$

IV. $\mathfrak{B}_b(\vartheta, \delta, \tau) = \mathfrak{B}_b(\delta, \vartheta, \tau)$

V. $\mathfrak{B}_b(\vartheta, g, b(\tau + z + w)) \geq \mathfrak{B}_b(\vartheta, \delta, \tau) * \mathfrak{B}_b(\delta, u, z) + \mathfrak{B}_b(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$

VI. $\mathfrak{B}_b(\vartheta, \delta, .): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{B}_b(\vartheta, \delta, \tau) = 1$

VII. $\mathfrak{D}_b(\vartheta, \delta, 0) = 1$

VIII. $\mathfrak{D}_b(\vartheta, \delta, \tau) = 0$ if and only if $\vartheta = \delta$

IX. $\mathfrak{D}_b(\vartheta, \delta, \tau) = \mathfrak{D}_b(\delta, \vartheta, \tau)$

X. $\mathfrak{D}_b(\vartheta, g, b(\tau + z + w)) \leq \mathfrak{D}_b(\vartheta, \delta, \tau) \odot \mathfrak{D}_b(\delta, u, z) + \mathfrak{D}_b(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$

XI. $\mathfrak{D}_b(\vartheta, \delta, .): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{D}_b(\vartheta, \delta, \tau) = 0$

Example 2. Let (E, d) be a rectangular b -metric space, and define $\mathfrak{B}_b, \mathfrak{D}_b: E \times E \times [0, \infty) \longrightarrow [0, 1]$ by

$$\mathfrak{B}_b(\vartheta, \delta, \tau) = \frac{\tau}{\tau + d(\vartheta, \delta)}, \quad (3)$$

$$\mathfrak{D}_b(\vartheta, \delta, \tau) = \frac{d(\vartheta, \delta)}{\tau + d(\vartheta, \delta)} \text{ for all } \vartheta, \delta \in E \text{ and } \tau > 0,$$

with CTN $\zeta * b = \min\{\zeta, b\}$ and CTCN $\zeta \odot b = \max\{\zeta, b\}$. Then, it is easy to see that $(E, \mathfrak{B}_b, \mathfrak{D}_b, *, \odot)$ is an IFRBMS.

Example 3. Let (E, d) be a rectangular b -metric space, and define $\mathfrak{B}_b, \mathfrak{D}_b: E \times E \times [0, \infty) \longrightarrow [0, 1]$ by

$$\begin{aligned} \mathfrak{B}_b(\vartheta, \delta, \tau) &= e^{-d(\vartheta, \delta)/\tau}, \\ \mathfrak{D}_b(\vartheta, \delta, \tau) &= 1 - e^{-d(\vartheta, \delta)/\tau} \text{ for all } \vartheta, \delta \in E \text{ and } \tau > 0, \end{aligned} \quad (4)$$

with CTN $\zeta * b = \min\{\zeta, b\}$ and CTCN $\zeta \odot b = \max\{\zeta, b\}$. Then, it is easy to see that $(E, \mathfrak{B}_b, \mathfrak{D}_b, *, \odot)$ is an IFRBMS.

Remark 1. The above Examples 2 and 3 are also an IFRBMS with CTN $\zeta * b = \zeta \blacklozenge b$ and CTCN $\zeta \odot b = \max\{\zeta, b\}$.

Remark 2. Every IFRMS is an IFRBMS, but the converse may not be true.

Definition 10. Let E be a nonempty set. A five-tuple $(E, \mathfrak{B}_l, \mathfrak{D}_l, *, \odot)$ is called an IFRBMLS if there is $b \geq 1$, $*$ is a CTN, \odot is a CTCN, and \mathfrak{B}_l and \mathfrak{D}_l are two FSs on $E \times E \times [0, \infty)$ fulfilling the following assertions for all $\vartheta, \delta, g \in E$ and $\tau, z, w > 0$:

- A. $\mathfrak{B}_I(\vartheta, \delta, \tau) + \mathfrak{D}_I(\vartheta, \delta, \tau) \leq 1$
- B. $\mathfrak{B}_I(\vartheta, \delta, 0) = 0$
- C. $\mathfrak{B}_I(\vartheta, \delta, \tau) = 1$ implies $\vartheta = \delta$
- D. $\mathfrak{B}_I(\vartheta, \delta, \tau) = \mathfrak{B}_I(\delta, \vartheta, \tau)$
- E. $\mathfrak{B}_I(\vartheta, g, b(\tau + z + w)) \geq \mathfrak{B}_I(\vartheta, \delta, \tau) * \mathfrak{B}_I(\delta, u, z) + \mathfrak{B}_I(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- F. $\mathfrak{B}_I(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{B}_I(\vartheta, \delta, \tau) = 1$
- G. $\mathfrak{D}_I(\vartheta, \delta, 0) = 1$
- H. $\mathfrak{D}_I(\vartheta, \delta, \tau) = 0$ implies if $\vartheta = \delta$
- I. $\mathfrak{D}_I(\vartheta, \delta, \tau) = \mathfrak{D}_I(\delta, \vartheta, \tau)$
- J. $\mathfrak{D}_I(\vartheta, g, b(\tau + z + w)) \leq \mathfrak{D}_I(\vartheta, \delta, \tau) \circ \mathfrak{D}_I(\delta, u, z) + \mathfrak{D}_I(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- K. $\mathfrak{D}_I(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{D}_I(\vartheta, \delta, \tau) = 0$

Definition 11. In the above Definition 10, if we take $b = 1$, then it becomes an IFRMLS.

Example 4. Define $\mathfrak{B}_I, \mathfrak{D}_I: E \times E \times [0, \infty) \longrightarrow [0, 1]$ by

$$\mathfrak{B}_I(\vartheta, \delta, \tau) = \frac{\tau}{\tau + \max\{\vartheta, \delta\}^p}, \quad (5)$$

$$\mathfrak{D}_I(\vartheta, \delta, \tau) = \frac{\max\{\vartheta, \delta\}^p}{\tau + \max\{\vartheta, \delta\}^p} \text{ for all } \vartheta, \delta \in E \text{ and } \tau > 0,$$

with CTN $\zeta * b = \min\{\zeta, b\}$ and CTCN $\zeta \circ b = \max\{\zeta, b\}$. Then, it is easy to see that $(E, \mathfrak{B}_I, \mathfrak{D}_I, *, \circ)$ is an IFRBMLS, and if we take $p = 1$, then it becomes an IFRMLS.

Example 5. Define $\mathfrak{B}_I, \mathfrak{D}_I: E \times E \times [0, \infty) \longrightarrow [0, 1]$ by

$$\begin{aligned} \mathfrak{B}_I(\vartheta, \delta, \tau) &= e^{-\max\{\vartheta, \delta\}^p/\tau}, \\ \mathfrak{D}_I(\vartheta, \delta, \tau) &= 1 - e^{-\max\{\vartheta, \delta\}^p/\tau} \text{ for all } \vartheta, \delta \in E, p \geq 1, \text{ and } \tau > 0, \end{aligned} \quad (6)$$

with CTN $\zeta * b = \min\{\zeta, b\}$ and CTCN $\zeta \circ b = \max\{\zeta, b\}$. Then, it is easy to see that $(E, \mathfrak{B}_I, \mathfrak{D}_I, *, \circ)$ is an IFRBMLS.

Remark 3. The above Examples 4 and 5 are also an IFRBMLS with CTN $\zeta * b = \zeta \diamond b$ and CTCN $\zeta \circ b = \max\{\zeta, b\}$.

Remark 4. In an IFRBMLS, the self-distance may not be equal to 1 and 0.

For this, consider the above Example 5; then, we have

$$\begin{aligned} \mathfrak{B}_I(\vartheta, \vartheta, \tau) &= e^{-\max\{\vartheta, \vartheta\}^p/\tau} = e^{-\vartheta^p/\tau} \neq 1, \\ \mathfrak{D}_I(\vartheta, \vartheta, \tau) &= 1 - e^{-\max\{\vartheta, \vartheta\}^p/\tau} = 1 - e^{-\vartheta^p/\tau} \neq 0. \end{aligned} \quad (7)$$

Remark 5. Every IFRBMS is an IFRBMLS, but the converse may not be true.

Remark 6. IFRBMLS may not be continuous.

Example 6. Let $E = [0, \infty)$, $\mathfrak{B}_I(\vartheta, \delta, \tau) = e^{-d(\vartheta, \delta)/\tau}$, and $\mathfrak{D}_I(\vartheta, \delta, \tau) = 1 - e^{-d(\vartheta, \delta)/\tau}$ for all $\vartheta, \delta \in E$, $\tau > 0$, and

$$d(\vartheta, \delta) = \begin{cases} 0, & \text{if } \vartheta = \delta, \\ 2(\vartheta + \delta)^2, & \text{if } \vartheta, \delta \in [0, 1], \\ \frac{1}{2}(\vartheta + \delta)^2, & \text{otherwise.} \end{cases} \quad (8)$$

If we define CTN by $\zeta * b = \zeta \diamond b$ and CTCN by $\zeta \circ b = \max\{\zeta, b\}$, then $(E, \mathfrak{B}_I, \mathfrak{D}_I, *, \circ)$ is an IFRBMLS. Now, to illustrate continuity, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{B}_I\left(0, 1 - \frac{1}{n}, \tau\right) &= \lim_{n \rightarrow \infty} e^{-2(1-(1/n))^2/\tau} \\ &= e^{-2/\tau} = \mathfrak{B}_I(0, 1, \tau), \end{aligned} \quad (9)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{D}_I\left(0, 1 - \frac{1}{n}, \tau\right) &= 1 - \lim_{n \rightarrow \infty} e^{-2(1-(1/n))^2/\tau} \\ &= 1 - e^{-2/\tau} = \mathfrak{D}_I(0, 1, \tau). \end{aligned}$$

However,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{B}_I\left(1, 1 - \frac{1}{n}, \tau\right) &= \lim_{n \rightarrow \infty} e^{-2(2-(1/n))^2/\tau} \\ &= e^{-8/\tau} \neq 1 = \mathfrak{B}_I(1, 1, \tau), \end{aligned} \quad (10)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{D}_I\left(1, 1 - \frac{1}{n}, \tau\right) &= 1 - \lim_{n \rightarrow \infty} e^{-2(2-(1/n))^2/\tau} \\ &= 1 - e^{-8/\tau} \neq 0 = \mathfrak{D}_I(1, 1, \tau). \end{aligned}$$

Hence, $(E, \mathfrak{B}_I, \mathfrak{D}_I, \mathcal{Q}, *, \circ)$ is not continuous.

Definition 12. Let E be a nonempty set. A six-tuple $(E, \mathfrak{B}_e, \mathfrak{D}_e, \mathcal{Q}_e, *, \circ)$ is called a NRMS if $*$ is a CTN, \circ is a CTCN, and $\mathfrak{B}_e, \mathfrak{D}_e$, and \mathcal{Q}_e are three NSs on $E \times E \times [0, \infty)$ fulfilling the following assertions for all $\vartheta, \delta, g \in E$ and $\tau, z, w > 0$:

- (i) $\mathfrak{B}_e(\vartheta, \delta, \tau) + \mathfrak{D}_e(\vartheta, \delta, \tau) + \mathcal{Q}_e(\vartheta, \delta, \tau) \leq 3$
- (ii) $\mathfrak{B}_e(\vartheta, \delta, 0) = 0$
- (iii) $\mathfrak{B}_e(\vartheta, \delta, \tau) = 1$ if and only if $\vartheta = \delta$
- (iv) $\mathfrak{B}_e(\vartheta, \delta, \tau) = \mathfrak{B}_e(\delta, \vartheta, \tau)$
- (v) $\mathfrak{B}_e(\vartheta, g, \tau + z + w) \geq \mathfrak{B}_e(\vartheta, \delta, \tau) * \mathfrak{B}_e(\delta, u, z) + \mathfrak{B}_e(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- (vi) $\mathfrak{B}_e(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{B}_e(\vartheta, \delta, \tau) = 1$
- (vii) $\mathfrak{D}_e(\vartheta, \delta, 0) = 1$
- (viii) $\mathfrak{D}_e(\vartheta, \delta, \tau) = 0$ if and only if $\vartheta = \delta$
- (ix) $\mathfrak{D}_e(\vartheta, \delta, \tau) = \mathfrak{D}_e(\delta, \vartheta, \tau)$
- (x) $\mathfrak{D}_e(\vartheta, g, \tau + z + w) \leq \mathfrak{D}_e(\vartheta, \delta, \tau) \circ \mathfrak{D}_e(\delta, u, z) + \mathfrak{D}_e(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- (xi) $\mathfrak{D}_e(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{D}_e(\vartheta, \delta, \tau) = 0$

- (xii) $\mathcal{Q}_e(\vartheta, \delta, 0) = 1$
- (xiii) $\mathcal{Q}_e(\vartheta, \delta, \tau) = 0$ if and only if $\vartheta = \delta$
- (xiv) $\mathcal{Q}_e(\vartheta, \delta, \tau) = \mathcal{Q}_e(\delta, \vartheta, \tau)$
- (xv) $\mathcal{Q}_e(\vartheta, g, \tau + z + w) \leq \mathcal{Q}_e(\vartheta, \delta, \tau) \mathcal{Q}_e(\delta, u, z) + \mathcal{Q}_e(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- (xvi) $\mathcal{Q}_e(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathcal{Q}_e(\vartheta, \delta, \tau) = 0$

Example 7. Let (E, d) be a rectangular metric space, and define $\mathfrak{B}_e, \mathfrak{D}_e, \mathcal{Q}_e: E \times E \times [0, \infty) \longrightarrow [0, 1]$ by

$$\begin{aligned}\mathfrak{B}_e(\vartheta, \delta, \tau) &= \frac{\tau}{\tau + d(\vartheta, \delta)}, \\ \mathfrak{D}_e(\vartheta, \delta, \tau) &= 1 - \frac{\tau}{\tau + d(\vartheta, \delta)}, \\ \mathcal{Q}_e(\vartheta, \delta, \tau) &= \frac{d(\vartheta, \delta)}{\tau},\end{aligned}\quad (11)$$

for all $\vartheta, \delta \in E$ and $\tau > 0$, with CTN $\zeta * b = \min\{\zeta, b\}$ and CTCN $\zeta \circ b = \max\{\zeta, b\}$. Then, it is easy to see that $(E, \mathfrak{B}_e, \mathfrak{D}_e, \mathcal{Q}_e, *, \circ)$ is a NRMS.

Definition 13. Let E be a nonempty set. A six-tuple $(E, \mathfrak{B}, \mathfrak{D}, \mathcal{Q}, *, \circ)$ is called a NRBMS if there is $b \geq 1$, $*$ is a CTN, \circ is a CTCN, and $\mathfrak{B}, \mathfrak{D}$, and \mathcal{Q} are three NSs on $E \times E \times [0, \infty)$ fulfilling the following assertions for all $\vartheta, \delta, g \in E$ and $\tau, z, w > 0$:

- (a) $\mathfrak{B}(\vartheta, \delta, \tau) + \mathfrak{D}(\vartheta, \delta, \tau) + \mathcal{Q}(\vartheta, \delta, \tau) \leq 3$
- (b) $\mathfrak{B}(\vartheta, \delta, 0) = 0$
- (c) $\mathfrak{B}(\vartheta, \delta, \tau) = 1$ if and only if $\vartheta = \delta$
- (d) $\mathfrak{B}(\vartheta, \delta, \tau) = \mathfrak{B}(\delta, \vartheta, \tau)$
- (e) $\mathfrak{B}(\vartheta, g, b(\tau + z + w)) \geq \mathfrak{B}(\vartheta, \delta, \tau) * \mathfrak{B}(\delta, u, z) + \mathfrak{B}(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- (f) $\mathfrak{B}(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{B}(\vartheta, \delta, \tau) = 1$
- (g) $\mathfrak{D}(\vartheta, \delta, 0) = 1$
- (h) $\mathfrak{D}(\vartheta, \delta, \tau) = 0$ if and only if $\vartheta = \delta$
- (i) $\mathfrak{D}(\vartheta, \delta, \tau) = \mathfrak{D}(\delta, \vartheta, \tau)$
- (j) $\mathfrak{D}(\vartheta, g, b(\tau + z + w)) \leq \mathfrak{D}(\vartheta, \delta, \tau) \circ \mathfrak{D}(\delta, u, z) + \mathfrak{D}(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- (k) $\mathfrak{D}(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{D}(\vartheta, \delta, \tau) = 0$
- (l) $\mathcal{Q}(\vartheta, \delta, 0) = 1$
- (m) $\mathcal{Q}(\vartheta, \delta, \tau) = 0$ if and only if $\vartheta = \delta$
- (n) $\mathcal{Q}(\vartheta, \delta, \tau) = \mathcal{Q}(\delta, \vartheta, \tau)$
- (o) $\mathcal{Q}(\vartheta, g, b(\tau + z + w)) \leq \mathcal{Q}(\vartheta, \delta, \tau) \circ \mathcal{Q}(\delta, u, z) + \mathcal{Q}(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- (p) $\mathcal{Q}(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathcal{Q}(\vartheta, \delta, \tau) = 0$

Example 8. Let (E, d) be a rectangular b -metric space, and define $\mathfrak{B}, \mathfrak{D}, \mathcal{Q}: E \times E \times [0, \infty) \longrightarrow [0, 1]$ by

$$\begin{aligned}\mathfrak{B}(\vartheta, \delta, \tau) &= \frac{\tau}{\tau + d(\vartheta, \delta)}, \\ \mathfrak{D}(\vartheta, \delta, \tau) &= 1 - \frac{\tau}{\tau + d(\vartheta, \delta)}, \\ \mathcal{Q}(\vartheta, \delta, \tau) &= \frac{d(\vartheta, \delta)}{\tau},\end{aligned}\quad (12)$$

for all $\vartheta, \delta \in E$ and $\tau > 0$, with CTN $\zeta * b = \min\{\zeta, b\}$ and CTCN $\zeta \circ b = \max\{\zeta, b\}$. Then, it is easy to see that $(E, \mathfrak{B}, \mathfrak{D}, \mathcal{Q}, *, \circ)$ is a NRBMS.

Remark 7. The above Example 6 is also a NRBMS with CTN $\zeta * b = \zeta \diamond b$ and CTCN $\zeta \circ b = \max\{\zeta, b\}$.

Remark 8. Every NRMS is a NRBMS, but the converse may not be true.

Definition 14. Let $(E, \mathfrak{B}, \mathfrak{D}, \mathcal{Q}, *, \circ)$ be a NRBMS, and assume $\{\vartheta_n\}$ to be a sequence in E . Then,

- (i) $\{\vartheta_n\}$ is named to be a convergent sequence if there exists $\vartheta \in E$ such that

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathfrak{B}(\vartheta_n, \vartheta, \tau) &= 1, \\ \lim_{n \rightarrow \infty} \mathfrak{D}(\vartheta_n, \vartheta, \tau) &= 0, \\ \lim_{n \rightarrow \infty} \mathcal{Q}(\vartheta_n, \vartheta, \tau) &= 0 \text{ for all } \tau > 0.\end{aligned}\quad (13)$$

- (ii) $\{\vartheta_n\}$ is named to be a Cauchy sequence if

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathfrak{B}(\vartheta_n, \vartheta_{n+q}, \tau) &= 1, \\ \lim_{n \rightarrow \infty} \mathfrak{D}(\vartheta_n, \vartheta_{n+q}, \tau) &= 0, \\ \lim_{n \rightarrow \infty} \mathcal{Q}(\vartheta_n, \vartheta_{n+q}, \tau) &= 0.\end{aligned}\quad (14)$$

- (iii) If every Cauchy sequence is convergent in E , then $(E, \mathfrak{B}, \mathfrak{D}, \mathcal{Q}, *, \circ)$ is said to be a complete NRBMS.

Definition 15. Let E be a nonempty set. A six-tuple $(E, \mathfrak{B}_h, \mathfrak{D}_h, \mathcal{Q}_h, *, \circ)$ is called a NRBMLS if there is $b \geq 1$, $*$ is a CTN, \circ is a CTCN, and $\mathfrak{B}_h, \mathfrak{D}_h$, and \mathcal{Q}_h are three NSs on $E \times E \times [0, \infty)$ fulfilling the following assertions for all $\vartheta, \delta, g \in E$ and $\tau, z, w > 0$:

- L1. $\mathfrak{B}_h(\vartheta, \delta, \tau) + \mathfrak{D}_h(\vartheta, \delta, \tau) + \mathcal{Q}_h(\vartheta, \delta, \tau) \leq 3$
- L2. $\mathfrak{B}_h(\vartheta, \delta, 0) = 0$
- L3. $\mathfrak{B}_h(\vartheta, \delta, \tau) = 1$ implies if $\vartheta = \delta$
- L4. $\mathfrak{B}_h(\vartheta, \delta, \tau) = \mathfrak{B}_h(\delta, \vartheta, \tau)$
- L5. $\mathfrak{B}_h(\vartheta, g, b(\tau + z + w)) \geq \mathfrak{B}_h(\vartheta, \delta, \tau) * \mathfrak{B}_h(\delta, u, z) + \mathfrak{B}_h(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
- L6. $\mathfrak{B}_h(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{B}_h(\vartheta, \delta, \tau) = 1$
- L7. $\mathfrak{D}_h(\vartheta, \delta, 0) = 1$
- L8. $\mathfrak{D}_h(\vartheta, \delta, \tau) = 0$ implies $\vartheta = \delta$

- L9. $\mathfrak{D}_h(\vartheta, \delta, \tau) = \mathfrak{D}_h(\delta, \vartheta, \tau)$
 L10. $\mathfrak{D}_h(\vartheta, g, b(\tau + z + w)) \leq \mathfrak{D}_h(\vartheta, \delta, \tau) \odot \mathfrak{D}_h(\delta, u, z) + \mathfrak{D}_h(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
 L11. $\mathfrak{D}_h(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathfrak{D}_h(\vartheta, \delta, \tau) = 0$
 L12. $\mathcal{Q}_h(\vartheta, \delta, 0) = 1$
 L13. $\mathcal{Q}_h(\vartheta, \delta, \tau) = 0$ implies $\vartheta = \delta$
 L14. $\mathcal{Q}_h(\vartheta, \delta, \tau) = \mathcal{Q}_h(\delta, \vartheta, \tau)$
 L15. $\mathcal{Q}_h(\vartheta, g, b(\tau + z + w)) \leq \mathcal{Q}_h(\vartheta, \delta, \tau) \odot \mathcal{Q}_h(\delta, u, z) + \mathcal{Q}_h(u, g, w)$ for all distinct $\delta, u \in E \setminus \{\vartheta, g\}$
 L16. $\mathcal{Q}_h(\vartheta, \delta, \cdot): (0, \infty) \longrightarrow [0, 1]$ is continuous, and $\lim_{\tau \rightarrow \infty} \mathcal{Q}_h(\vartheta, \delta, \tau) = 0$

Definition 16. In the above definition, if we take $b = 1$, then it becomes a NRMLS.

Example 9. Define $\mathfrak{B}_h, \mathfrak{D}_h, \mathcal{Q}_h: E \times E \times [0, \infty) \longrightarrow [0, 1]$ by

$$\begin{aligned}\mathfrak{B}_h(\vartheta, \delta, \tau) &= \frac{\tau}{\tau + (\vartheta + \delta)^p}, \\ \mathfrak{D}_h(\vartheta, \delta, \tau) &= 1 - \frac{\tau}{\tau + (\vartheta + \delta)^p}, \\ \mathcal{Q}_h(\vartheta, \delta, \tau) &= \frac{(\vartheta + \delta)^p}{\tau},\end{aligned}\quad (15)$$

for all $\vartheta, \delta \in E$, $p \geq 1$, and $\tau > 0$, with CTN $\zeta * b = \min\{\zeta, b\}$ and CTCN $\zeta \odot b = \max\{\zeta, b\}$. Then, it is easy to see that $(E, \mathfrak{B}_h, \mathfrak{D}_h, \mathcal{Q}_h, *, \odot)$ is a NRBMLS, and if we take $p = 1$, then it becomes a NRMLS.

Remark 9. The above Example 9 is also a NRBMLS with CTN $\zeta * b = \zeta \diamond b$ and CTCN $\zeta \odot b = \max\{\zeta, b\}$.

Remark 10. Every NRBMS is a NRBMLS, but the converse may not be true.

Remark 11. From Remark 4 and Example 9, it is clear that, in the NRBMLS, the self-distances $\mathfrak{B}_h(\vartheta, \vartheta, \tau) \neq 1$, $\mathfrak{D}_h(\vartheta, \vartheta, \tau) \neq 0$, and $\mathcal{Q}_h(\vartheta, \vartheta, \tau) \neq 0$.

Remark 12. It is clear from Example 6 that the NRBMLS may not be continuous.

Definition 17. Let $(E, \mathfrak{B}_h, \mathfrak{D}_h, \mathcal{Q}_h, *, \odot)$ be a NRBMLS, and assume $\{\vartheta_n\}$ to be a sequence in E . Then,

- (i) $\{\vartheta_n\}$ is named to be a convergent sequence if there exists $\vartheta \in E$ such that

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathfrak{B}_h(\vartheta_n, \vartheta, \tau) &= \mathfrak{B}_h(\vartheta, \vartheta, \tau), \\ \lim_{n \rightarrow \infty} \mathfrak{D}_h(\vartheta_n, \vartheta, \tau) &= \mathfrak{D}_h(\vartheta, \vartheta, \tau), \\ \lim_{n \rightarrow \infty} \mathcal{Q}_h(\vartheta_n, \vartheta, \tau) &= \mathcal{Q}_h(\vartheta, \vartheta, \tau) \text{ for all } \tau > 0.\end{aligned}\quad (16)$$

- (ii) $\{\vartheta_n\}$ is named to be a Cauchy sequence if

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathfrak{B}_h(\vartheta_n, \vartheta_{n+q}, \tau), \\ \lim_{n \rightarrow \infty} \mathfrak{D}_h(\vartheta_n, \vartheta_{n+q}, \tau), \\ \lim_{n \rightarrow \infty} \mathcal{Q}_h(\vartheta_n, \vartheta_{n+q}, \tau),\end{aligned}\quad (17)$$

exist and are finite for all $\tau > 0$.

- (iii) If every Cauchy sequence is convergent in E , then $(E, \mathfrak{B}_h, \mathfrak{D}_h, \mathcal{Q}_h, *, \odot)$ is said to be a complete NRBMLS such that

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathfrak{B}_h(\vartheta_n, \vartheta, \tau) &= \mathfrak{B}_h(\vartheta, \vartheta, \tau) = \lim_{n \rightarrow \infty} \mathfrak{B}_h(\vartheta_n, \vartheta_{n+q}, \tau), \\ \lim_{n \rightarrow \infty} \mathfrak{D}_h(\vartheta_n, \vartheta, \tau) &= \mathfrak{D}_h(\vartheta, \vartheta, \tau) = \lim_{n \rightarrow \infty} \mathfrak{D}_h(\vartheta_n, \vartheta_{n+q}, \tau), \\ \lim_{n \rightarrow \infty} \mathcal{Q}_h(\vartheta_n, \vartheta, \tau) &= \mathcal{Q}_h(\vartheta, \vartheta, \tau) = \lim_{n \rightarrow \infty} \mathcal{Q}_h(\vartheta_n, \vartheta_{n+q}, \tau),\end{aligned}\quad (18)$$

for all $\tau > 0$ and $q \geq 1$.

Definition 18. Let $(E, \mathfrak{B}_h, \mathfrak{D}_h, \mathcal{Q}_h, *, \odot)$ be a NRBMLS. For, $\vartheta \in E$, $\theta \in (0, 1)$, and $\tau > 0$, we define the open ball as

$$\begin{aligned}B(\vartheta, x, \tau) &= \{\delta \in E: \mathfrak{B}_h(\vartheta, \delta, \tau) > 1 - x, \mathfrak{D}_h(\vartheta, \delta, \tau) < x, \\ &\quad \mathcal{Q}_h(\vartheta, \delta, \tau) < x\} \\ &\quad (\text{center } \vartheta, \text{ radius } x \text{ with respect to } \tau).\end{aligned}\quad (19)$$

Lemma 1. Let $(E, \mathfrak{B}, \mathfrak{D}, \mathcal{Q}, *, \odot)$ be a NRBMS and

$$\begin{aligned}\mathfrak{B}(\vartheta, \delta, k\tau) &\geq \mathfrak{B}(\vartheta, \delta, \tau), \\ \mathfrak{D}(\vartheta, \delta, k\tau) &\leq \mathfrak{D}(\vartheta, \delta, \tau), \\ \mathcal{Q}(\vartheta, \delta, k\tau) &\leq \mathcal{Q}(\vartheta, \delta, \tau),\end{aligned}\quad (20)$$

for all $\vartheta, \delta \in E$, $0 < k < 1$, and $\tau > 0$; then, $\vartheta = \delta$.

Proof. It is immediate from (f), (k), and (p). \square

Theorem 1 (Banach contraction theorem in neutrosophic rectangular b -metric spaces). Let $(E, \mathfrak{B}, \mathfrak{D}, \mathcal{Q}, *, \odot)$ be a NRBMS with $b \geq 1$ such that

$$\begin{aligned}\lim_{\tau \rightarrow \infty} \mathfrak{B}(\vartheta, \delta, \tau) &= 1, \\ \lim_{\tau \rightarrow \infty} \mathfrak{D}(\vartheta, \delta, \tau) &= 0, \\ \lim_{\tau \rightarrow \infty} \mathcal{Q}(\vartheta, \delta, \tau) &= 0 \text{ for all } \vartheta, \delta \in E.\end{aligned}\quad (21)$$

Let $\Psi: E \longrightarrow E$ be a mapping satisfying

$$\begin{aligned}\mathfrak{B}(\Psi\vartheta, \Psi\delta, k\tau) &\geq \mathfrak{B}(\vartheta, \delta, \tau), \\ \mathfrak{D}(\Psi\vartheta, \Psi\delta, k\tau) &\leq \mathfrak{D}(\vartheta, \delta, \tau), \\ \mathcal{Q}(\Psi\vartheta, \Psi\delta, k\tau) &\leq \mathcal{Q}(\vartheta, \delta, \tau),\end{aligned}\quad (22)$$

for all $\vartheta, \delta \in E$ and $k \in [0, 1/b)$. Then, Ψ has a unique fixed point.

Proof. Fix an arbitrary point $\zeta_0 \in E$, and for $n = 0, 1, 2, \dots$, start an iterative process $\zeta_{n+1} = \Psi\zeta_n$. Successively applying inequality (22), we get for all $n, \tau > 0$,

$$\begin{aligned}\mathfrak{B}(\zeta_n, \zeta_{n+1}, \tau) &\geq \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{k^n}\right), \\ \mathfrak{D}(\zeta_n, \zeta_{n+1}, \tau) &\leq \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{k^n}\right), \\ \mathcal{Q}(\zeta_n, \zeta_{n+1}, \tau) &\leq \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{k^n}\right).\end{aligned}\quad (23)$$

Since $(E, \mathfrak{B}, \mathfrak{D}, \mathcal{Q}, *, \circ)$ is a NRBMS for the sequence $\{\zeta_n\}$, write $\tau = (\tau/3) + (\tau/3) + (\tau/3)$ and use the rectangular inequalities given in (e), (j), and (o) on $\mathfrak{B}(\zeta_n, \zeta_{n+p}, \tau)$, $\mathfrak{D}(\zeta_n, \zeta_{n+p}, \tau)$, and $\mathcal{Q}(\zeta_n, \zeta_{n+p}, \tau)$ in the following cases. \square

Case 1. If p is odd, then $p = 2m + 1$ where $m \in \{1, 2, 3, \dots\}$, and we have

$$\begin{aligned}\mathfrak{B}(\zeta_n, \zeta_{n+2m+1}, \tau) &\geq \mathfrak{B}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+2}, \zeta_{n+2m+1}, \frac{\tau}{3b}\right) \\ &\geq \mathfrak{B}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\ &\quad * \mathfrak{B}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) * \mathfrak{B}\left(\zeta_{n+4}, \zeta_{n+2m+1}, \frac{\tau}{(3b)^2}\right) \\ &\geq \mathfrak{B}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\ &\quad * \mathfrak{B}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) * \mathfrak{B}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3}\right) * \dots * \mathfrak{B}\left(\zeta_{n+2m}, \zeta_{n+2m+1}, \frac{\tau}{(3b)^m}\right), \\ \mathfrak{D}(\zeta_n, \zeta_{n+2m+1}, \tau) &\leq \mathfrak{D}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+2}, \zeta_{n+2m+1}, \frac{\tau}{3b}\right) \\ &\leq \mathfrak{D}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\ &\quad \circ \mathfrak{D}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) \circ \mathfrak{D}\left(\zeta_{n+4}, \zeta_{n+2m+1}, \frac{\tau}{(3b)^2}\right) \\ &\leq \mathfrak{D}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\ &\quad \circ \mathfrak{D}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) \circ \mathfrak{D}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3}\right) \circ \dots \circ \mathfrak{D}\left(\zeta_{n+2m}, \zeta_{n+2m+1}, \frac{\tau}{(3b)^m}\right), \\ \mathcal{Q}(\zeta_n, \zeta_{n+2m+1}, \tau) &\leq \mathcal{Q}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+2}, \zeta_{n+2m+1}, \frac{\tau}{3b}\right) \\ &\leq \mathcal{Q}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\ &\quad \circ \mathcal{Q}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) \circ \mathcal{Q}\left(\zeta_{n+4}, \zeta_{n+2m+1}, \frac{\tau}{(3b)^2}\right) \\ &\leq \mathcal{Q}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\ &\quad \circ \mathcal{Q}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) \circ \mathcal{Q}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3}\right) \circ \dots \circ \mathcal{Q}\left(\zeta_{n+2m}, \zeta_{n+2m+1}, \frac{\tau}{(3b)^m}\right).\end{aligned}\quad (24)$$

Using (23) in the above inequalities, we deduce

$$\begin{aligned}
\mathfrak{B}(\zeta_n, \zeta_{n+2m+1}, \tau) &\geq \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^{n+1}}\right) * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+2}}\right) \\
&\quad * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+3}}\right) * \mathfrak{B}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3 k^{n+4}}\right) * \cdots * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^m k^{n+m}}\right) \\
&\geq \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)k^n}\right) * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^n}\right) \\
&\quad * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^{n+1}}\right) * \mathfrak{B}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3bk)^3 k^{n+1}}\right) * \cdots * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^m k^{n+m}}\right), \\
\mathfrak{D}(\zeta_n, \zeta_{n+2m+1}, \tau) &\leq \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^{n+1}}\right) \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+2}}\right) \\
&\quad \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+3}}\right) \circ \mathfrak{D}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3 k^{n+4}}\right) \circ \cdots \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^m k^{n+m}}\right) \\
&\leq \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)k^n}\right) \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^n}\right) \\
&\quad \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^{n+1}}\right) \circ \mathfrak{D}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3bk)^3 k^{n+1}}\right) \circ \cdots \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^m k^{n+m}}\right), \\
\mathcal{Q}(\zeta_n, \zeta_{n+2m+1}, \tau) &\leq \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^{n+1}}\right) \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+2}}\right) \\
&\quad \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+3}}\right) \circ \mathcal{Q}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3 k^{n+4}}\right) \circ \cdots \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^m k^{n+m}}\right) \\
&\leq \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)k^n}\right) \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^n}\right) \\
&\quad \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^{n+1}}\right) \circ \mathcal{Q}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3bk)^3 k^{n+1}}\right) \circ \cdots \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^m k^{n+m}}\right).
\end{aligned} \tag{25}$$

Case 2. If p is even, then $p = 2m; m \in \{1, 2, 3, \dots\}$; then, we have

$$\begin{aligned}
\mathfrak{B}(\zeta_n, \zeta_{n+2m}, \tau) &\geq \mathfrak{B}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+2}, \zeta_{n+2m}, \frac{\tau}{3b}\right) \\
&\geq \mathfrak{B}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\
&\quad * \mathfrak{B}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) * \mathfrak{B}\left(\zeta_{n+4}, \zeta_{n+2m}, \frac{\tau}{(3b)^2}\right) \\
&\geq \mathfrak{B}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\
&\quad * \mathfrak{B}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) * \mathfrak{B}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3}\right) * \cdots * \mathfrak{B}\left(\zeta_{n+2m-4}, \zeta_{n+2m-3}, \frac{\tau}{(3b)^{m-1}}\right) \\
&\quad * \mathfrak{B}\left(\zeta_{n+2m-3}, \zeta_{n+2m-2}, \frac{\tau}{(3b)^{m-1}}\right) * \mathfrak{B}\left(\zeta_{n+2m-2}, \zeta_{n+2m}, \frac{\tau}{(3b)^{m-1}}\right),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{D}(\zeta_n, \zeta_{n+2m}, \tau) &\leq \mathfrak{D}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+2}, \zeta_{n+2m}, \frac{\tau}{3b}\right) \\
&\leq \mathfrak{D}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\
&\quad \circ \mathfrak{D}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) \circ \mathfrak{D}\left(\zeta_{n+4}, \zeta_{n+2m}, \frac{\tau}{(3b)^2}\right) \\
&\leq \mathfrak{D}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\
&\quad \circ \mathfrak{D}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) \circ \mathfrak{D}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3}\right) \circ \cdots \circ \mathfrak{D}\left(\zeta_{n+2m-4}, \zeta_{n+2m-3}, \frac{\tau}{(3b)^{m-1}}\right) \\
&\quad \circ \mathfrak{D}\left(\zeta_{n+2m-3}, \zeta_{n+2m-2}, \frac{\tau}{(3b)^{m-1}}\right) \circ \mathfrak{D}\left(\zeta_{n+2m-2}, \zeta_{n+2m}, \frac{\tau}{(3b)^{m-1}}\right), \\
\mathcal{Q}(\zeta_n, \zeta_{n+2m}, \tau) &\leq \mathcal{Q}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+2}, \zeta_{n+2m}, \frac{\tau}{3b}\right) \\
&\leq \mathcal{Q}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\
&\quad \circ \mathcal{Q}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) \circ \mathcal{Q}\left(\zeta_{n+4}, \zeta_{n+2m}, \frac{\tau}{(3b)^2}\right) \\
&\leq \mathcal{Q}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+1}, \zeta_{n+2}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+2}, \zeta_{n+3}, \frac{\tau}{(3b)^2}\right) \\
&\quad \circ \mathcal{Q}\left(\zeta_{n+3}, \zeta_{n+4}, \frac{\tau}{(3b)^2}\right) \circ \mathcal{Q}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3}\right) \circ \cdots \circ \mathcal{Q}\left(\zeta_{n+2m-4}, \zeta_{n+2m-3}, \frac{\tau}{(3b)^{m-1}}\right) \\
&\quad \circ \mathcal{Q}\left(\zeta_{n+2m-3}, \zeta_{n+2m-2}, \frac{\tau}{(3b)^{m-1}}\right) \circ \mathcal{Q}\left(\zeta_{n+2m-2}, \zeta_{n+2m}, \frac{\tau}{(3b)^{m-1}}\right).
\end{aligned}$$

(26)

Using (23) in the above inequalities, we deduce

$$\begin{aligned}
\mathfrak{B}(\zeta_n, \zeta_{n+2m}, \tau) &\geq \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^{n+1}}\right) * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+2}}\right) \\
&\quad * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+3}}\right) * \mathfrak{B}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3 k^{n+4}}\right) * \cdots * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^{m-1} k^{n+2m-2}}\right) \\
&\geq \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)k^n}\right) * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^n}\right) \\
&\quad * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^{n+1}}\right) * \mathfrak{B}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3bk)^3 k^{n+1}}\right) * \cdots * \mathfrak{B}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^{m-1} k^{n+m-1}}\right),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{D}(\zeta_n, \zeta_{n+2m}, \tau) &\leq \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^{n+1}}\right) \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+2}}\right) \\
&\quad \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+3}}\right) \circ \mathfrak{D}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3 k^{n+4}}\right) \circ \cdots \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^{m-1} k^{n+2m-2}}\right) \\
&\leq \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)k^n}\right) \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^n}\right) \\
&\quad \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^{n+1}}\right) \circ \mathfrak{D}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3bk)^3 k^{n+1}}\right) \circ \cdots \circ \mathfrak{D}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^{m-1} k^{n+m-1}}\right), \\
\mathcal{Q}(\zeta_n, \zeta_{n+2m}, \tau) &\leq \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^{n+1}}\right) \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+2}}\right) \\
&\quad \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^2 k^{n+3}}\right) \circ \mathcal{Q}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3b)^3 k^{n+4}}\right) \circ \cdots \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3b)^{m-1} k^{n+2m-2}}\right) \\
&\leq \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{3bk^n}\right) \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)k^n}\right) \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^n}\right) \\
&\quad \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^2 k^{n+1}}\right) \circ \mathcal{Q}\left(\zeta_{n+4}, \zeta_{n+5}, \frac{\tau}{(3bk)^3 k^{n+1}}\right) \circ \cdots \circ \mathcal{Q}\left(\zeta_0, \zeta_1, \frac{\tau}{(3bk)^{m-1} k^{n+m-1}}\right).
\end{aligned} \tag{27}$$

Therefore, from Cases 1 and 2 and together with (21), it follows that, for all $p \in \{1, 2, 3, \dots\}$, we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathfrak{B}(\zeta_n, \zeta_{n+p}, \tau) &= 1 * 1 * \cdots * 1 = 1, \\
\lim_{n \rightarrow \infty} \mathfrak{D}(\zeta_n, \zeta_{n+p}, \tau) &= 0 \circ 0 \circ \cdots \circ 0 = 0, \\
\lim_{n \rightarrow \infty} \mathfrak{D}(\zeta_n, \zeta_{n+p}, \tau) &= 0 \circ 0 \circ \cdots \circ 0 = 0.
\end{aligned} \tag{28}$$

Hence, $\{\zeta_n\}$ is a Cauchy sequence. Since $(E, \mathfrak{B}, \mathfrak{D}, \mathcal{Q}, *, \circ)$ is a complete NRBMS, there exists $u \in E$ such that $\lim_{n \rightarrow \infty} \zeta_n = u$.

Now, we examine that u is a fixed point of Ψ .

$$\begin{aligned}
\mathfrak{B}(u, \Psi u, \tau) &\geq \mathfrak{B}\left(u, \zeta_n, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n+1}, \Psi u, \frac{\tau}{3b}\right) \\
&\geq \mathfrak{B}\left(u, \zeta_n, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\Psi \zeta_{n-1}, \Psi \zeta_n, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\Psi \zeta_n, \Psi u, \frac{\tau}{3b}\right) \\
&\geq \mathfrak{B}\left(u, \zeta_n, \frac{\tau}{3b}\right) * \mathfrak{B}\left(\zeta_{n-1}, \zeta_n, \frac{\tau}{3bk}\right) * \mathfrak{B}\left(\zeta_n, u, \frac{\tau}{3bk}\right) \\
&\longrightarrow 1 * 1 * 1 = 1 \text{ as } n \longrightarrow \infty, \\
\mathfrak{D}(u, \Psi u, \tau) &\leq \mathfrak{D}\left(u, \zeta_n, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n+1}, \Psi u, \frac{\tau}{3b}\right) \\
&\leq \mathfrak{D}\left(u, \zeta_n, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\Psi \zeta_{n-1}, \Psi \zeta_n, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\Psi \zeta_n, \Psi u, \frac{\tau}{3b}\right) \\
&\leq \mathfrak{D}\left(u, \zeta_n, \frac{\tau}{3b}\right) \circ \mathfrak{D}\left(\zeta_{n-1}, \zeta_n, \frac{\tau}{3bk}\right) \circ \mathfrak{D}\left(\zeta_n, u, \frac{\tau}{3bk}\right) \\
&\longrightarrow 0 \circ 0 \circ 0 = 0 \text{ as } n \longrightarrow \infty,
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}(u, \Psi u, \tau) &\leq \mathcal{Q}\left(u, \zeta_n, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_n, \zeta_{n+1}, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n+1}, \Psi u, \frac{\tau}{3b}\right) \\
&\leq \mathcal{Q}\left(u, \zeta_n, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\Psi \zeta_{n-1}, \Psi \zeta_n, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\Psi \zeta_n, \Psi u, \frac{\tau}{3b}\right) \\
&\leq \mathcal{Q}\left(u, \zeta_n, \frac{\tau}{3b}\right) \circ \mathcal{Q}\left(\zeta_{n-1}, \zeta_n, \frac{\tau}{3bk}\right) \circ \mathcal{Q}\left(\zeta_n, u, \frac{\tau}{3bk}\right) \\
&\longrightarrow 0 \circ 0 \circ 0 = 0 \text{ as } n \longrightarrow \infty,
\end{aligned} \tag{29}$$

which shows that u is a fixed point of Ψ .

Now, we show the uniqueness.

Assume v is another fixed point of Ψ for some $v \in E$; then,

$$\begin{aligned}
\mathfrak{B}(v, u, \tau) &= \mathfrak{B}(\Psi v, \Psi u, \tau) \geq \mathfrak{B}\left(v, u, \frac{\tau}{k}\right) = \mathfrak{B}\left(\Psi v, \Psi u, \frac{\tau}{k}\right) \\
&\geq \mathfrak{B}\left(v, u, \frac{\tau}{k^2}\right) \geq \cdots \geq \mathfrak{B}\left(v, u, \frac{\tau}{k^n}\right) \longrightarrow 1 \text{ as } n \longrightarrow \infty, \\
\mathfrak{D}(v, u, \tau) &= \mathfrak{D}(\Psi v, \Psi u, \tau) \leq \mathfrak{D}\left(v, u, \frac{\tau}{k}\right) = \mathfrak{D}\left(\Psi v, \Psi u, \frac{\tau}{k}\right) \\
&\leq \mathfrak{D}\left(v, u, \frac{\tau}{k^2}\right) \leq \cdots \leq \mathfrak{D}\left(v, u, \frac{\tau}{k^n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \\
\mathcal{Q}(v, u, \tau) &= \mathcal{Q}(\Psi v, \Psi u, \tau) \leq \mathcal{Q}\left(v, u, \frac{\tau}{k}\right) = \mathcal{Q}\left(\Psi v, \Psi u, \frac{\tau}{k}\right) \\
&\leq \mathcal{Q}\left(v, u, \frac{\tau}{k^2}\right) \leq \cdots \leq \mathcal{Q}\left(v, u, \frac{\tau}{k^n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \tag{30}$$

Thus, $u = v$. Hence, the fixed point is unique.

Example 10. Let $E = [0, 1]$, and define $\mathfrak{B}, \mathfrak{D}, \mathcal{Q}: E \times E \times [0, \infty) \longrightarrow [0, 1]$ by

Remark 13. In Theorem 1, if we take $b = 1$, then it will become a Banach contraction theorem in the sense of NRMS.

$$\begin{aligned}
\mathfrak{B}(\vartheta, \delta, \tau) &= \frac{\tau}{\tau + |\vartheta - \delta|^2}, \\
\mathfrak{D}(\vartheta, \delta, \tau) &= 1 - \frac{\tau}{\tau + |\vartheta - \delta|^2}, \\
\mathcal{Q}(\vartheta, \delta, \tau) &= \frac{|\vartheta - \delta|^2}{\tau},
\end{aligned} \tag{31}$$

for all $\vartheta, \delta \in E$ and $\tau > 0$, with CTN $\zeta * b = \zeta \cdot b$ and CTCN $\zeta \circ b = \max\{\zeta, b\}$. Then, it is easy to see that $(E, \mathfrak{B}, \mathfrak{D}, * \circ)$ is a complete NRBMS.

Define $\Psi: E \longrightarrow E$ by $\Psi(\vartheta) = 1 - 2^{-\vartheta}/3$. Then,

$$\begin{aligned}
 \mathfrak{B}(\Psi\vartheta, \Psi\delta, k\tau) &= \mathfrak{B}\left(\frac{1-2^{-\vartheta}}{3}, \frac{1-2^{-\delta}}{3}, k\tau\right) = \frac{k\tau}{k\tau + \left|(1-2^{-\vartheta}/3) - (1-2^{-\delta}/3)\right|^2} \\
 &= \frac{9k\tau}{9k\tau + |2^{-\vartheta} - 2^{-\delta}|^2} \geq \frac{9k\tau}{9k\tau + |\vartheta - \delta|^2} \geq \frac{\tau}{\tau + |\vartheta - \delta|^2} = \mathfrak{B}(\vartheta, \delta, \tau), \\
 \mathfrak{D}(\Psi\vartheta, \Psi\delta, k\tau) &= \mathfrak{D}\left(\frac{1-2^{-\vartheta}}{3}, \frac{1-2^{-\delta}}{3}, k\tau\right) = 1 - \frac{k\tau}{k\tau + \left|(1-2^{-\vartheta}/3) - (1-2^{-\delta}/3)\right|^2} \\
 &= 1 - \frac{9k\tau}{9k\tau + |2^{-\vartheta} - 2^{-\delta}|^2} \leq 1 - \frac{9k\tau}{9k\tau + |\vartheta - \delta|^2} \leq 1 - \frac{\tau}{\tau + |\vartheta - \delta|^2} = \mathfrak{D}(\vartheta, \delta, \tau), \\
 \mathcal{Q}(\Psi\vartheta, \Psi\delta, k\tau) &= \mathcal{Q}\left(\frac{1-2^{-\vartheta}}{3}, \frac{1-2^{-\delta}}{3}, k\tau\right) = \frac{\left|(1-2^{-\vartheta}/3) - (1-2^{-\delta}/3)\right|^2}{k\tau} \\
 &= \frac{|2^{-\vartheta} - 2^{-\delta}|^2}{9k\tau} \leq \frac{|\vartheta - \delta|^2}{9k\tau} \leq \frac{|\vartheta - \delta|^2}{\tau} = \mathcal{Q}(\vartheta, \delta, \tau),
 \end{aligned} \tag{32}$$

for all $\vartheta, \delta \in E$, where $k \in [1/2, 1)$. Thus, all the conditions of Theorem 1 are satisfied, and hence, 0 is a unique fixed point of Ψ .

Corollary 1. Let $(E, \mathfrak{B}_b, \mathfrak{D}_b, *, \circ)$ be a IFRBMS with $b \geq 1$ such that

$$\begin{aligned}
 \lim_{\tau \rightarrow \infty} \mathfrak{B}_b(\vartheta, \delta, \tau) &= 1, \\
 \lim_{\tau \rightarrow \infty} \mathfrak{D}_b(\vartheta, \delta, \tau) &= 0, \text{ for all } \vartheta, \delta \in E.
 \end{aligned} \tag{33}$$

Let $\Psi: E \longrightarrow E$ be a mapping satisfying

$$\begin{aligned}
 \mathfrak{B}_b(\Psi\vartheta, \Psi\delta, k\tau) &\geq \mathfrak{B}_b(\vartheta, \delta, \tau), \\
 \mathfrak{D}_b(\Psi\vartheta, \Psi\delta, k\tau) &\leq \mathfrak{D}_b(\vartheta, \delta, \tau),
 \end{aligned} \tag{34}$$

for all $\vartheta, \delta \in E$ and $k \in [0, 1/b)$. Then, Ψ has a unique fixed point.

Proof. It is clear from Theorem 1. \square

Theorem 2. Let $(E, \mathfrak{B}_h, \mathfrak{D}_h, \mathcal{Q}_h, *, \circ)$ be a NRBMLS with $b \geq 1$ such that

$$\begin{aligned}
 \lim_{\tau \rightarrow \infty} \mathfrak{B}_h(\vartheta, \delta, \tau) &= 1, \\
 \lim_{\tau \rightarrow \infty} \mathfrak{D}_h(\vartheta, \delta, \tau) &= 0, \\
 \lim_{\tau \rightarrow \infty} \mathcal{Q}_h(\vartheta, \delta, \tau) &= 0 \text{ for all } \vartheta, \delta \in E.
 \end{aligned} \tag{35}$$

Let $\Psi: E \longrightarrow E$ be a mapping satisfying

$$\begin{aligned}
 \mathfrak{B}_h(\Psi\vartheta, \Psi\delta, k\tau) &\geq \mathfrak{B}_h(\vartheta, \delta, \tau), \\
 \mathfrak{D}_h(\Psi\vartheta, \Psi\delta, k\tau) &\leq \mathfrak{D}_h(\vartheta, \delta, \tau), \\
 \mathcal{Q}_h(\Psi\vartheta, \Psi\delta, k\tau) &\leq \mathcal{Q}_h(\vartheta, \delta, \tau),
 \end{aligned} \tag{36}$$

for all $\vartheta, \delta \in E$ and $k \in [0, 1/b)$. Then, Ψ has a unique fixed point.

Proof. It is easy to show on the lines of Theorems 1 and 2 in [14]. \square

4. Application

In this section, we present an application to the integral equation of Theorem 1. In particular, we show the existence of the solution of an integral equation of the form

$$\vartheta(j) = g(j) + \int_0^j F(j, r, \vartheta(r)) dr, \tag{37}$$

for all $j \in [0, l]$ where $l > 0$. Let $C([0, l], \mathbb{R})$ be the space of all continuous functions defined on $[0, l]$ with CTN $\zeta * b = \zeta.b$ and CTCN $\zeta \circ b = \max\{\zeta, b\}$ for all $\zeta, b \in [0, 1]$, and define a complete NRBMS by

$$\begin{aligned}
 \mathfrak{B}(\vartheta, \delta, \tau) &= \sup_{j \in [0, l]} \frac{\tau}{\tau + |\vartheta(j) - \delta(j)|^2}, \\
 \mathfrak{D}(\vartheta, \delta, \tau) &= \sup_{j \in [0, l]} \frac{|\vartheta(j) - \delta(j)|^2}{\tau + |\vartheta(j) - \delta(j)|^2}, \\
 \mathcal{Q}(\vartheta, \delta, \tau) &= \sup_{j \in [0, l]} \frac{|\vartheta(j) - \delta(j)|^2}{\tau} \text{ for all } \vartheta, \delta \in C([0, l], \mathbb{R}) \text{ and } \tau > 0.
 \end{aligned} \tag{38}$$

Theorem 3. Let $\Psi: C([0, l], \mathbb{R}) \longrightarrow C([0, l], \mathbb{R})$ be the integral operator given by

$$\Psi(\vartheta(j)) = g(j) + \int_0^j F(j, r, \vartheta(r)) dr, \quad g \in C([0, l], \mathbb{R}), \quad (39)$$

where $F \in C([0, l] \times [0, l] \times \mathbb{R}, \mathbb{R})$ satisfies the following conditions:

- (i) There exists $f: [0, l] \times [0, l] \longrightarrow [0, +\infty]$ such that, for all $r, j \in [0, l]$, $f(j, r) \in L^1([0, l], \mathbb{R})$ and for all $\vartheta, \delta \in C([0, l], \mathbb{R})$, we have

$$|F(j, r, \vartheta(r)) - F(j, r, \delta(r))|^2 \leq f^2(j, r) |\vartheta(r) - \delta(r)|^2. \quad (40)$$

(ii) Also,

$$\sup_{j \in [0, l]} \int_0^j f^2(j, r) dr \leq k < 1. \quad (41)$$

Then, the integral equation has the solution $\vartheta_* \in C([0, l], \mathbb{R})$.

Proof. For all $\vartheta, \delta \in C([0, l], \mathbb{R})$, we have

$$\begin{aligned} \mathfrak{B}(\Psi)(\vartheta(j), \Psi(\delta(j)), k\tau) &= \sup_{j \in [0, l]} \frac{k\tau}{k\tau + |\Psi(\vartheta(j)) - \Psi(\delta(j))|^2} \\ &\geq \sup_{j \in [0, l]} \frac{k\tau}{k\tau + \int_0^j |F(j, r, \vartheta(r)) - F(j, r, \delta(r))|^2 dr} \\ &\geq \sup_{j \in [0, l]} \frac{k\tau}{k\tau + \int_0^j f^2(j, r) |\vartheta(r) - \delta(r)|^2 dr} \\ &\geq \frac{k\tau}{k\tau + |\vartheta(r) - \delta(r)|^2 \sup_{j \in [0, l]} \int_0^j f^2(j, r) dr} \\ &\geq \frac{k\tau}{k\tau + |\vartheta(r) - \delta(r)|^2} \geq \frac{\tau}{\tau + |\vartheta(r) - \delta(r)|^2} = \mathfrak{B}(\vartheta, \delta, \tau), \\ \mathfrak{D}(\Psi)(\vartheta(j), \Psi(\delta(j)), k\tau) &= \sup_{j \in [0, l]} \frac{|\Psi(\vartheta(j)) - \Psi(\delta(j))|^2}{k\tau + |\Psi(\vartheta(j)) - \Psi(\delta(j))|^2} \\ &\leq \sup_{j \in [0, l]} \frac{\int_0^j |F(j, r, \vartheta(r)) - F(j, r, \delta(r))|^2 dr}{k\tau + \int_0^j |F(j, r, \vartheta(r)) - F(j, r, \delta(r))|^2 dr} \\ &\leq \sup_{j \in [0, l]} \frac{\int_0^j f^2(j, r) |\vartheta(r) - \delta(r)|^2 dr}{k\tau + \int_0^j f^2(j, r) |\vartheta(r) - \delta(r)|^2 dr} \\ &\leq \frac{|\vartheta(r) - \delta(r)|^2 \sup_{j \in [0, l]} \int_0^j f^2(j, r) dr}{k\tau + |\vartheta(r) - \delta(r)|^2 \sup_{j \in [0, l]} \int_0^j f^2(j, r) dr} \\ &\leq \frac{|\vartheta(r) - \delta(r)|^2}{k\tau + |\vartheta(r) - \delta(r)|^2} \leq \frac{|\vartheta(r) - \delta(r)|^2}{\tau + |\vartheta(r) - \delta(r)|^2} = \mathfrak{D}(\vartheta, \delta, \tau), \end{aligned}$$

$$\begin{aligned}
\mathcal{Q}(\Psi)(\vartheta(j), \Psi(\delta(j)), k\tau) &= \sup_{j \in [0, I]} \frac{|\Psi(\vartheta(j)) - \Psi(\delta(j))|^2}{k\tau} \\
&\leq \sup_{j \in [0, I]} \frac{\int_0^j |F(j, r, \vartheta(r)) - F(j, r, \delta(r))|^2 dr}{k\tau} \\
&\leq \sup_{j \in [0, I]} \frac{\int_0^j f^2(j, r) |\vartheta(r) - \delta(r)|^2 dr}{k\tau} \\
&\leq \frac{|\vartheta(r) - \delta(r)|^2 \sup_{j \in [0, I]} \int_0^j f^2(j, r) dr}{k\tau} \\
&\leq \frac{|\vartheta(r) - \delta(r)|^2}{k\tau} \leq \frac{|\vartheta(r) - \delta(r)|^2}{\tau} = \mathcal{Q}(\vartheta, \delta, \tau).
\end{aligned} \tag{42}$$

Hence, ϑ_* is a fixed point of Ψ , which is the solution of integral equation (37). \square

5. Conclusion

The aim of this study is to present the notions of intuitionistic fuzzy rectangular metric spaces, intuitionistic fuzzy rectangular metric-like spaces, intuitionistic fuzzy rectangular b -metric spaces, intuitionistic fuzzy rectangular b -metric-like spaces, neutrosophic rectangular metric spaces, neutrosophic rectangular metric-like spaces, neutrosophic rectangular b -metric spaces, and neutrosophic rectangular b -metric-like spaces and prove the Banach contraction theorem in these spaces, and nontrivial examples and an application to the integral equation are also given to support our results. Due to a diverse range of applications of the metric fixed point theory in mathematics, science, and economics, it is researched widely. Different types of fixed point results for single- and multivalued mappings can be proven in the sense of the above-defined notions in this manuscript. Also, presented notions can be extended in different mathematical structures, i.e., intuitionistic fuzzy controlled rectangular metric spaces, intuitionistic fuzzy triple controlled rectangular metric spaces, neutrosophic extended rectangular metric spaces, etc.

Data Availability

No such data were used for this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

BVP for Hadamard Sequential Fractional Hybrid Differential Inclusions

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The study is concerned with the Hadamard sequential fractional hybrid differential inclusions with two-point hybrid integral boundary conditions. With the help of the Dhage fixed-point theorem for the product of two operators and the Covitz-Nadler fixed-point theorem in the case of fractional inclusions, we obtain the existence results of solutions for Hadamard sequential fractional hybrid differential inclusions. Finally, two examples are presented to illustrate the main results.

1. Introduction

Nowadays, with the increasing demand of researchers for the study of natural phenomena, the use of fractional differential operators and fractional differential equations become an effective means to achieve this goal. Compared with integer order operators, fractional operators, which can simulate natural phenomena better, are a class of operators developed in recent years. This kind of operator has been expanded and widely used in modeling real-world phenomena such as biomathematics, electrical circuits, medicine, disease transmission, and control [1–6]. Also, some studies in the biological models with fractional-order derivative have been conducted in recent years [7–9]. In the past year, fractional differential operators and fractional differential equations have been used in modeling the spread of some viruses, such as Zika virus and mumps virus [10, 11]. All of these have enabled researchers to discover the structure of fractional boundary value problems (BVP) and the hereditary nature of their solutions from various aspects. In this regard, many researchers investigated advanced fractional-order models and related theoretical results and qualitative behaviors of such fractional-order boundary value problems, see [12–20] and the references therein.

There have been appeared different versions of fractional operators during these years. Much of the work on fractional

differential equations only involves either Riemann-Liouville derivative or Caputo derivative [21–29]. Guo et al. ([30, 31]) discussed the existence and Hyers–Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$ and the existence and Hyers–Ulam stability of the almost periodic solution to the fractional differential equation with impulse and fractional Brownian motion under nonlocal condition. Ma et al. [32] investigated the existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay in Hilbert space.

However, there is another concept of fractional derivative in the literature which was introduced by Hadamard in 1892 [33]. This derivative is known as Hadamard fractional derivative and differs from aforementioned derivatives in the sense that the kernel of the integral in its definition contains logarithmic function of arbitrary exponent. Many researchers have studied and obtained some results on the existence of solutions of Hadamard fractional differential equations in recent years. Yang ([34, 35]) studied the extremal solutions for a coupled system of nonlinear Hadamard fractional differential equations with Cauchy initial value conditions and the existence and nonexistence of positive solutions for the eigenvalue problems of nonlinear Hadamard fractional differential equations with

p-Laplacian operator. Tomar et al. [36] established certain generalized Hermite-Hadamard inequalities for generalized convex functions via local fractional integral.

In 1993, Miller and Ross also defined another type of fractional derivative called sequential derivative, which is a combination of the existing derivative operators. From then on, the attention of some researchers was attracted to finding

a connection between the Hadamard fractional derivative and the sequential fractional derivative [37–40]. In [41], by using the topological degree theory and Leray–Schauder fixed-point theory, Rezapour and Etemad studied the existence of solutions for the following Caputo–Hadamard fractional boundary value problem via mixed multiorde integroderivative conditions:

$$\begin{cases} \left({}^{\text{CH}}D_{1+}^{\alpha} + \lambda {}^{\text{CH}}D_{1+}^{\zeta} \right) u(t) = f(t, u(t)), & t \in [1, M], \\ u(1) = 0, \\ \mu_1 {}^{\text{CH}}D_{1+}^{\gamma_1} u(M) + {}^{\text{CH}}D_{1+}^{\gamma_2} u(\eta) = \delta_1, \\ \mu_2 {}^H I_{1+}^{q_1} u(M) + {}^H I_{1+}^{q_2} u(\eta) = \delta_2, \end{cases} \quad (1)$$

where $\lambda, \mu_1, \mu_2, \in (0, 1]$, $\gamma_1, \gamma_2 \in (0, \zeta - \alpha)$ with $2 < \alpha < \zeta < 3$, $q_1, q_2 > 0$, $\delta_1, \delta_2 \in R$. The symbol ${}^{\text{CH}}D_{1+}^{\xi}$ points out the Caputo–Hadamard fractional derivative of order $\xi \in \{\alpha, \zeta, \gamma_1, \gamma_2\}$, with the notation ${}^H I_{1+}^q$ standing for the Hadamard fractional integrals of order $q \in \{q_1, q_2\}$. The function f formulated by $f: [1, M] \times R \rightarrow R$ is assumed to be continuous on $[1, M] \times R$ with respect to its both components.

As a generalization of fractional boundary value problems, hybrid differential problems with different kinds of boundary conditions have received a lot of attention in recent years [42–44]. The research in this field started from Dhage and Lakshmikantham in 2010 [45]. There is a new

concept of differential equation in the literature which was introduced by Dhage and Lakshmikantham. They described this novel differential equation as a hybrid differential equation and investigated the extremal solutions of this new BVP by using some useful fundamental differential inequalities [45]. So far, there are few studies about the existence and various properties of solutions for hybrid boundary value problems of fractional order. In [46], by using a fixed-point theorem due to Dhage, the authors developed some existence theorem for Hadamard-type fractional hybrid differential inclusions problem:

$$\begin{cases} {}^H D^{\alpha} \left(\frac{x(t)}{f(t, x(t))} \right) \in F(t, x(t)), & t \in (1, e), \alpha \in (1, 2], \\ x(1) = x(e) = 0, \end{cases} \quad (2)$$

where ${}^H D^{\alpha}$ is the Hadamard fractional derivative, $f \in C([1, e] \times R, R \setminus \{0\})$, $F: [1, e] \times R \rightarrow \mathcal{P}(R)$ is a multivalued map, and $\mathcal{P}(R)$ is the family of all nonempty subsets of R . In [47], by using a hybrid fixed-point theorem

of Schaefer type for a sum of three operators due to Dhage, the authors investigated the existence of solutions for the nonlocal fractional BVP of hybrid inclusion problem given by

$$\begin{cases} {}^C D_{0+}^{\varrho} \left[\frac{w(s) - \sum_{j=1}^m {}^R I_{0+}^{\beta_j} h_j(s, w(s))}{f(s, w(s))} \right] \in F(s, w(s)), & s \in J := [0, 1], \\ w(0) = \mu(w), w(1) = \alpha, \end{cases} \quad (3)$$

where $\mu: C(J, R) \rightarrow R$, $\alpha \in R$, $\varrho \in (1, 2]$, ${}^C D_{0+}^{\varrho}$ is the Caputo derivative, and ${}^R I_{0+}^{\beta_j}$ is the Riemann–Liouville integral of order $\varphi > 0$, such that $\varphi \in \{\beta_1, \beta_2, \dots, \beta_m\}$. In [48], by

using the well-known Dhage fixed-point theorems for single-valued and set-valued maps, Baleanu and Etemad studied a new fractional hybrid model of thermostat in

which the thermostat controls an amount of heat based on the temperature detected by sensors. This hybrid differential inclusions of Caputo type are illustrated by

$$\left\{ \begin{array}{l} -{}^C D_{0+}^{\varrho} \left[\frac{x(s)}{\varphi(s, w(s))} \right] \in \Phi(s, x(s)), \quad \varrho \in (1, 2], s \in [0, 1], \\ D \left[\frac{x(s)}{\varphi(s, x(s))} \right] \Big|_{s=0} = 0, \\ \lambda {}^C D_{0+}^{\varrho-1} \left[\frac{x(s)}{\varphi(s, x(s))} \right] \Big|_{s=1} + \left[\frac{x(s)}{\varphi(s, x(s))} \right] \Big|_{s=\eta} = 0, \end{array} \right. \quad (4)$$

where $\lambda > 0, \eta \in [0, 1], \varrho - 1 \in (0, 1], D = d/ds, {}^C D_{0+}^{\alpha}$ is the Caputo derivative of fractional order $\alpha \in \{\varrho, \varrho - 1\}$, the function $\Phi: [0, 1] \times R \longrightarrow \mathcal{P}(R)$ is a multivalued map, and

$\varphi \in C([0, 1] \times R, R \setminus \{0\})$. In [49], the authors investigated the following fractional three-point hybrid problem:

$$\left\{ \begin{array}{l} {}^C D_{0+}^{\varrho} \left(\frac{w(s)}{g(s, w(s))} \right) = G(s, w(s)), \quad s \in [0, 1], \\ w(0) = 0, \\ \left[\frac{w(s)}{g(s, w(s))} \right] \Big|_{s=0} + {}^R I_{0+}^{\varrho^*} \left[\frac{w(s)}{g(s, w(s))} \right] \Big|_{s=\eta} = 0, \\ \left[\frac{w(s)}{g(s, w(s))} \right] \Big|_{s=0} + {}^R I_{0+}^{\varrho^*} \left[\frac{w(s)}{g(s, w(s))} \right] \Big|_{s=1} = 0, \end{array} \right. \quad (5)$$

where $\varrho \in (2, 3], \varrho^* > 0, \eta \in (0, 1)$. The function $G: [0, 1] \times R \longrightarrow R$ is continuous, and $g \in C([0, 1] \times R, R \setminus \{0\})$. In [50], by using various novel analytical techniques based on $\alpha - \psi$ -contractive mappings, endpoints, and the fixed points

of the product operators, the authors investigated a new category of the sequential hybrid inclusion problem with three-point integroderivative boundary conditions:

$$\left\{ \begin{array}{l} p_1 ({}^C D_{0+}^{\varrho} + p_2 {}^C D_{0+}^{\varrho-1}) \left[\frac{w(s)}{\zeta(s, w(s), {}^R I_{0+}^{\gamma} w(s))} \right] \in \mathcal{S}(s, w(s)), \quad s \in [0, 1], \\ \left[\frac{w(s)}{\zeta(s, w(s), {}^R I_{0+}^{\gamma} w(s))} \right] \Big|_{s=0} = 0, \\ {}^C D_{0+}^1 \left[\frac{w(s)}{\zeta(s, w(s), {}^R I_{0+}^{\gamma} w(s))} \right] \Big|_{s=0} + {}^C D_{0+}^2 \left[\frac{w(s)}{\zeta(s, w(s), {}^R I_{0+}^{\gamma} w(s))} \right] \Big|_{s=0} = 0, \\ \left[\frac{w(s)}{\zeta(s, w(s), {}^R I_{0+}^{\gamma} w(s))} \right] \Big|_{s=1} + {}^R I_{0+}^{\xi} \left[\frac{w(s)}{\zeta(s, w(s), {}^R I_{0+}^{\gamma} w(s))} \right] \Big|_{s=p} = 0, \end{array} \right. \quad (6)$$

where $\varrho \in (2, 3]$, $p \in (0, 1)$, $p_1, p_2, \gamma, \xi > 0$, ${}^C D_{0+}^{(\cdot)}$ and ${}^R I_{0+}^{(\cdot)}$ denote the Caputo-fractional derivative and the Riemann-Liouville fractional integral, respectively. Note that ${}^C D_{0+}^1 = d/ds$ and ${}^C D_{0+}^2 = d^2/ds^2$. The nonzero continuous real-valued function ζ is supposed to be defined on $[0, 1] \times R \times R$.

$\mathcal{S}: [0, 1] \times R \longrightarrow \mathcal{P}(R)$ is a set-valued map equipped via some properties.

Motivated by these problems, in this study, we will study the following Hadamard sequential fractional hybrid differential inclusion with two-point hybrid Hadamard integroboundary conditions:

$$\begin{cases} \left({}^H D^\alpha + \lambda {}^H D^{\alpha-1} \right) \left(\frac{x(t)}{\rho(t, x(t), {}^H I^p x(t))} \right) \in G(t, x(t), {}^H I^p x(t)), & t \in (1, e), \\ \alpha_1 \left(\frac{x(\xi)}{\rho(\xi, x(\xi), {}^H I^p x(\xi))} \right) = \alpha_2 {}^H I^r \left(\frac{x(e)}{\rho(e, x(e), {}^H I^p x(e))} \right), \\ \beta_1 \left(\frac{x(e)}{\rho(e, x(e), {}^H I^p x(e))} \right) = \beta_2 {}^H I^r \left(\frac{x(\xi)}{\rho(\xi, x(\xi), {}^H I^p x(\xi))} \right), \end{cases} \quad (7)$$

where $\alpha \in (1, 2]$, $\xi \in (1, e)$, $p \in (0, 1)$, $\lambda, r > 0$, $\alpha_i, \beta_i \in R$, $i = 1, 2$, ${}^H D^{(\cdot)}$ and ${}^H I^{(\cdot)}$ denote the Hadamard fractional derivative and the Hadamard fractional integral, respectively. The nonzero continuous real-valued function ρ is supposed to be defined $[1, e] \times R \times R$, and $G: [1, e] \times R \times R \longrightarrow \mathcal{P}(R)$ is a set-valued map equipped via some properties.

The Hadamard sequential fractional hybrid differential inclusion BVP (7) is modeled with respect to the generalized operators with kernels, including logarithmic functions. In other words, the presented formulation for the given Hadamard sequential fractional hybrid differential inclusion BVP (7) involves two different derivatives in the format of the Hadamard. The supposed abstract fractional hybrid differential inclusion problem (7) with given hybrid integral boundary conditions can describe some mathematical models of real and physical processes in which some parameters are often adjusted to suitable situations. The value of these parameters can change the effects of fractional

derivatives and integrals. Moreover, we express that such a Hadamard sequential fractional hybrid differential inclusion BVP is new and enriches the literature on boundary value problems for nonlinear Hadamard fractional differential inclusions. In this way, with the help of Dhage fixed-point theorem and Covitz-Nadler fixed-point theorem in the case of multivalued mapping, we try to find the existence criteria of solutions for the proposed problem (7).

The rest of this study is organized as follows. In Section 2, some preliminary facts that we need in the sequel are given. In Section 3, the existence results of solution for system (7) are discussed. In Section 4, two examples are given to prove validity of the results we obtained.

2. Preliminaries

Definition 1 (see [2]). The Hadamard derivative of fractional-order α for a function $f: [1, \infty) \longrightarrow R$ is defined as

$${}^H D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(\log \frac{t}{s} \right)^{n-\alpha-1} \frac{f(s)}{s} ds, \quad n-1 < \alpha < n, n = [\alpha] + 1, \quad (8)$$

provided the right side is pointwise defined on $[1, \infty)$, where $\Gamma(\cdot)$ is the gamma function and $\log(\cdot) = \log_e(\cdot)$.

Definition 2 (see [2]). The Hadamard fractional integral of order β for a function g is defined as

$${}^H I^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta-1} \frac{g(s)}{s} ds, \quad \beta > 0, \quad (9)$$

provided the integral exists.

Definition 3 (see [15]). Let $w: [1, +\infty) \longrightarrow R$ is a sufficiently smooth function; then, the sequential fractional derivative is defined by

$${}^H D^\varrho w(s) = ({}^H D^{\varrho_1} {}^H D^{\varrho_2} \dots {}^H D^{\varrho_n}) w(s), \quad (10)$$

where $\varrho = (\varrho_1, \varrho_2, \dots, \varrho_n)$ is a multiindex.

Lemma 1. For any $h \in C([1, e], R)$. A function $x \in AC([1, e], R)$ is a solution of the Hadamard sequential fractional hybrid differential equations:

$$\left({}^H D^\alpha + \lambda {}^H D^{\alpha-1}\right) \left(\frac{x(t)}{\rho(t, x(t), {}^H I^p x(t))} \right) = h(t), \quad t \in [1, e], \alpha \in (1, 2], \quad (11)$$

supplemented with the boundary conditions in (7) if and only if it satisfies the following integral equation:

$$\begin{aligned} x(t) = & \rho(t, x(t), {}^H I^p x(t)) \left(\frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} h(s) ds \right] \right. \right. \\ & \left. \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} h(s) ds \right] \right\} \right. \\ & \left. + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \right. \\ & \cdot \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} h(s) ds \right] \right. \\ & \left. - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} h(s) ds \right] \right\} \\ & \left. + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} h(s) ds \right), \end{aligned} \quad (12)$$

where

$$\Delta = A_1 B_2 - A_2 B_1 \neq 0,$$

$$A_1 = \alpha_1 \xi^{-\lambda} - \frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} ds, \quad (13)$$

$$\begin{aligned} A_2 &= \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1} (\log s)^{\alpha-2} ds - \frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1} (\log \tau)^{\alpha-2} d\tau \right) ds, \\ B_1 &= \beta_1 e^{-\lambda} - \frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} ds, \\ B_2 &= \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1} (\log s)^{\alpha-2} ds - \frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1} (\log \tau)^{\alpha-2} d\tau \right) ds. \end{aligned} \quad (14)$$

Proof. As argued in [2], the solution of Hadamard differential equation in (11) can be written as

$$\begin{aligned} x(t) = & \rho(t, x(t), {}^H I^p x(t)) \left(c_0 t^{-\lambda} + c_1 t^{-\lambda} \right. \\ & \left. \cdot \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} h(s) ds \right), \end{aligned} \quad (15)$$

where c_i , ($i = 0, 1$) are the unknown arbitrary constants. Making use of the integral boundary conditions given by (7) in (15), we obtain

$$\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}, \quad (16)$$

where A_i and B_i ($i = 1, 2$) are, respectively, given by (13) and (14), and

$$J_1 = \frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} h(s) ds, \quad (17)$$

$$J_2 = \frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} h(s) ds. \quad (18)$$

Solving (16) for c_0 and c_1 and using notation (13), we find that

$$c_0 = \frac{1}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} h(s) ds \right] \right. \\ \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} h(s) ds \right] \right\}, \quad (19)$$

$$c_1 = \frac{1}{\Delta} \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} h(s) ds \right] \right. \\ \left. - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} h(s) ds \right] \right\}. \quad (20)$$

Substituting the values of c_0 and c_1 in (15), we get the desired solution (12). This completes the proof.

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. \square

Definition 4 (see [51]). A multivalued map $G: X \longrightarrow \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$.

Definition 5 (see [51]). The multivalued map G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ (i.e., $\sup_{x \in B} \{\sup\{|y| : y \in G(x)\}\} < \infty$).

Definition 6 (see [51]). A multivalued map G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X , and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood \mathcal{N}_0 of x_0 , such that $G(\mathcal{N}_0) \subseteq N$.

Definition 7 (see [51]). A multivalued map G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$.

Definition 8 (see [51]). A multivalued map G has a fixed point if there is $x \in X$, such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\text{Fix } G$.

Definition 9 (see [51]). A multivalued map $G: [0, 1] \longrightarrow \mathcal{P}_d(R)$ is said to be measurable if for every $y \in R$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}, \quad (21)$$

is measurable.

Lemma 2 (see [51]). If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, that is, $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, and $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

Let $C([1, e], R)$ denote a Banach space of continuous functions from $[1, e]$ into R with the norm $\|x\| = \sup_{t \in [1, e]} |x(t)|$. Let $L^p([1, e], R)$ be the Banach space of measurable functions $x: [1, e] \longrightarrow R$ which are p -th Lebesgue integrable and normed by $\|x\|_{L^p} = (\int_1^e |x(t)|^p dt)^{1/p}$.

Definition 10 (see [51]). A collection of selections of multivalued map G at point $x \in C[1, e]$ is defined by

$$S_{G,x} = \{v(s) \in L^1([1, e]) : v(s) \in \cdot G(t, x(t), {}^H I^p x(t)) \text{ for a.e. } t \in [1, e]\}. \quad (22)$$

Definition 11 (see [51]). A multivalued map $G: [1, e] \times R^2 \longrightarrow \mathcal{P}(R)$ is said to be Caratheodory if

- (i) $t \longmapsto G(t, x, y)$ is measurable for each $x, y \in R$
- (ii) $(x, y) \longmapsto G(t, x, y)$ is upper semicontinuous for almost all $t \in [1, e]$

Definition 12 (see [37]). A function $x \in AC([1, e], R)$ is called a solution of problem (7) if there exists a function $v \in L^1([1, e], R)$ with $v(t) \in G(t, x(t), {}^H I^p x(t))$, a.e. on $[1, e]$, such that

$$\left\{ \begin{array}{l} \left({}^H D^\alpha + \lambda {}^H D^{\alpha-1} \right) \left(\frac{x(t)}{\rho(t, x(t), {}^H I^p x(t))} \right) = v(t), \quad \text{a.e. } t \in (1, e), \\ \alpha_1 \left(\frac{x(\xi)}{\rho(\xi, x(\xi), {}^H I^p x(\xi))} \right) = \alpha_2 {}^H I^r \left(\frac{x(e)}{\rho(e, x(e), {}^H I^p x(e))} \right), \\ \beta_1 \left(\frac{x(e)}{\rho(e, x(e), {}^H I^p x(e))} \right) = \beta_2 {}^H I^r \left(\frac{x(\xi)}{\rho(\xi, x(\xi), {}^H I^p x(\xi))} \right). \end{array} \right. \quad (23)$$

Lemma 3 (see [52]). Let X be a Banach space. Let $G: [1, e] \times \mathbb{R}^2 \longrightarrow \mathcal{P}_{cp,cv}(X)$ be an L^1 -Caratheodory multivalued map,

and let Θ be a linear continuous mapping from $L^1([1, e], X)$ to $C([1, e], X)$. Then, the operator

$$\Theta \circ S_G: C([1, e], X) \longrightarrow \mathcal{P}_{cp,cv}(C([1, e], X)), x \longmapsto (\theta^\circ S_G)(x) = \Theta(S_{G,x}), \quad (24)$$

is a closed graph operator in $C([1, e], X) \times C([1, e], X)$.

Lemma 4 (see [53]). Let X be a Banach algebra and $A: X \longrightarrow X$ be a single-valued and $B: X \longrightarrow \mathcal{P}_{cp,cv}(X)$ be a multivalued operator satisfying the following:

- (i) A is single-valued Lipschitz with a Lipschitz constant k
- (ii) B is compact and upper semicontinuous operator
- (iii) $2MK < 1$, where $M = \|B(X)\|$

Then, either

- (i) The operator inclusion $x \in Ax \cup Bx$ has a solution or
- (ii) The set $\mathcal{E} = \{u \in X: \mu u \in Au \cup Bu, \mu > 1\}$ is unbounded.

Definition 13 (see [51]). A multivalued map $G: X \longrightarrow \mathcal{P}(X)$ is said to be a contraction mapping if there is a constant $0 < \lambda < 1$, such that

$$H_d(G(x), G(y)) \leq \lambda \|x - y\|_X, \quad (25)$$

for every $x, y \in X$, where H_d is the Hausdorff metric.

Lemma 5 (see [54]). Let (X, d) be a complete metric space. If $N: X \longrightarrow \mathcal{P}_d(X)$ is a contraction, then $\text{Fix } N \neq \emptyset$.

3. Main Results

In this section, we will study the existence results of solutions for problem (7). First of all, we fix our terminology.

Let $X = C([1, e], \mathbb{R})$ denote the space equipped with the norm $\|x\| = \sup_{t \in [1, e]} |x(t)|$. Observe that $(X, \|\cdot\|)$ is a Banach space, and $(X, \|\cdot\|)$ with multiplication given by $(x \cdot x')(s) = x(s)x'(s)$ is a Banach algebra.

Now, we enlist the assumptions that we need in the sequel.

(H_1) The function $\rho: [1, e] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ is continuous, and there exists a bounded function Ψ , with bound $\|\Psi\|$, such that $\Psi(t) > 0$, a.e. $t \in [1, e]$, and

$$\begin{aligned} & |\rho(t, x_1, y_1) - \rho(t, x_2, y_2)| \\ & \leq \Psi(t)(|x_1 - x_2| + |y_1 - y_2|), \end{aligned} \quad (26)$$

a.e. $t \in [1, e]$, $\forall x_1, y_1, x_2, y_2 \in \mathbb{R}$.

(H_2) $G: [1, e] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{R}$ is Caratheodory and has nonempty compact and convex values

(H_3) There exists a constant $p_1 \in (0, p)$ and a function $g \in L^{1/p_1}([1, e], \mathbb{R}^+)$, such that

$$\|G(t, x, y)\| = \sup\{|v|: v \in G(t, x, y)\} \leq g(t), \quad (27)$$

For all $x, y \in \mathbb{R}$ and for a.e. $t \in [1, e]$.

(H_4) There exists a positive real number \mathcal{R} , such that

$$\mathcal{R} > \frac{\rho_0 M_1 \|g\|_{L^{1/p_1}}}{1 - M_0 M_1 \|g\|_{L^{1/p_1}}}, \quad (28)$$

where $M_0 M_1 \|g\|_{L^{1/p_1}} < 1/2$, $\rho_0 = \sup_{t \in [1, e]} |\rho(t, 0, 0)|$.

(H_5) There exists a continuous nondecreasing, subhomogeneous function $\Phi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ (that is, $\Phi(\mu x) \leq \mu \Phi(x)$ for all $\mu \geq 1$ and $x \in \mathbb{R}^+$) and a function $\varsigma \in L^{1/p_1}([1, e], \mathbb{R}^+)$, such that

$$\begin{aligned} \|G(t, x, y)\|_{\mathcal{R}} &:= \sup\{|y|: y \in G(t, x, y)\} \\ &\leq \varsigma(t) \Phi(|x| + |y|), \end{aligned} \quad (29)$$

For each $(t, x, y) \in [1, e] \times \mathbb{R}^2$.

(H_6) There exists a constant $r > 0$, such that

$$r > \frac{\rho_0 M_1 (1 + 1/\Gamma(p+1)) \|\varsigma\|_{L^{1/p_1}} \Phi(r)}{1 - M_0 M_1 (1 + 1/\Gamma(p+1)) \|\varsigma\|_{L^{1/p_1}} \Phi(r)}, \quad (30)$$

where

$$M_0 M_1 \left(1 + \frac{1}{\Gamma(p+1)} \right) \|\varsigma\|_{L^{1/p_1}} \Phi(r) < \frac{1}{2}, \quad \rho_0 = \sup_{t \in [1, e]} |\rho(t, 0, 0)|. \quad (31)$$

(H_7) $G: [1, e] \times R^2 \longrightarrow \mathcal{P}_{c_p}(R)$ satisfying the condition $G(\cdot, x, y): [1, e] \longrightarrow \mathcal{P}_{c_p}(R)$ is measurable for each $(x, y) \in R^2$

(H_8) $H_d(G(t, x, y), G(t, \bar{x}, \bar{y})) \leq \zeta(t)(|x - \bar{x}| + |y - \bar{y}|)$ for a.e. $t \in [1, e]$ and for all $x, y, \bar{x}, \bar{y} \in R$ with $\zeta \in L^{1/p_1}([1, e], R^+)$ and $d(0, G(t, 0, 0)) \leq \zeta(t)$ for a.e. $t \in [1, e]$.

(H_9) The function $\rho: [1, e] \times R \times R \longrightarrow R \setminus \{0\}$ is continuous, and there exists a function $\eta \in C([1, e], R^+)$, such that

$$|\rho(t, x, y)| \leq \eta(t), \quad \forall (t, x, y) \in [1, e] \times R^2. \quad (32)$$

In assumption (H_8), H_d is the Hausdorff metric, where d is the Euclidean metric in R defined by $d(x, y) = |x - y|$ for $x, y \in R$.

Furthermore, we set the notations:

$$\begin{aligned} M_1 &= \frac{1}{|\Delta|(\alpha-1)} \left\{ \frac{|\alpha_2|[(\alpha-1)|B_2| + |B_1|]}{\Gamma(\alpha-1)\Gamma(r+1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \right. \\ &\quad + \frac{|\beta_2|[(\alpha-1)|A_2| + |A_1|](\log \xi)^{(1+a)(1-p_1)+r+1}}{\Gamma(\alpha-1)\Gamma(r+1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \\ &\quad + \frac{|\beta_1|[(\alpha-1)|A_2| + |A_1|]}{\Gamma(\alpha-1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} + \frac{|\alpha_1|[(\alpha-1)|B_2| + |B_1|](\log \xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \\ &\quad \left. + \frac{|\Delta|(\alpha-1)}{\Gamma(\alpha-1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \right\}, \\ M_2 &= \frac{1}{|\Delta|} \left[\frac{|\alpha_2||B_2|}{\Gamma(\alpha-1)\Gamma(r+1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} + \frac{|\beta_2||A_2|(\log \xi)^{(1+a)(1-p_1)+r+1}}{\Gamma(\alpha-1)\Gamma(r+1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \right. \\ &\quad \left. + \frac{|\alpha_1||B_2|(\log \xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} + \frac{|\beta_1||A_2|}{\Gamma(\alpha-1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \right], \\ M_3 &= \frac{1}{|\Delta|(\alpha-1)} \left[\frac{|\alpha_2||B_1|}{\Gamma(\alpha-1)\Gamma(r+1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} + \frac{|\beta_2||A_1|(\log \xi)^{(1+a)(1-p_1)+r+1}}{\Gamma(\alpha-1)\Gamma(r+1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \right. \\ &\quad \left. + \frac{|\alpha_1||B_1|(\log \xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} + \frac{|\beta_1||A_1|}{\Gamma(\alpha-1)[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \right], \\ \|\Psi\| &= \sup_{t \in [1, e]} |\Psi(t)|, \quad \|\eta\| = \sup_{t \in [1, e]} |\eta(t)|, \quad a = \frac{\alpha-2}{1-p_1}, \quad \|\zeta\|_{L^{1/p_1}} = \left(\int_1^e |\zeta(t)|^{1/p_1} dt \right)^{p_1}. \end{aligned} \quad (33)$$

Theorem 1. Let the hypotheses $(H_1)-(H_4)$ be satisfied. Then, inclusion problem (7) has at least one mild solution on $C([1, e], R)$.

Proof. Consider the operator $\mathcal{N}: X \longrightarrow \mathcal{P}(X)$ defined by

$$\begin{aligned} \mathcal{N}x(t) = & \left\{ w \in C([1, e], R): w(t) = \rho(t, x(t), {}^H I^p x(t)) \left(\frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \right. \right. \\ & \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \Big] \\ & - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \Big\} \\ & + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \\ & \cdot \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right. \\ & - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \Big\} \\ & \left. + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v(s) ds \right\}, v \in S_{G,x}, \end{aligned} \quad (34)$$

and we define two operators $\mathcal{A}: X \longrightarrow X$ by

$$\mathcal{A}x(t) = \rho(t, x(t), {}^H I^p x(t)), \quad t \in [1, e], \quad (35)$$

and $\mathcal{B}: X \longrightarrow \mathcal{P}(X)$ by

$$\begin{aligned} \mathcal{B}x(t) = & \left\{ w \in C([1, e], R): w(t) = \frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \right. \right. \right. \\ & \cdot \left. \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \Big\} \\ & + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \\ & \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \Big] \\ & - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \Big\} \\ & \left. + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v(s) ds, \quad v \in S_{G,x} \right\}. \end{aligned} \quad (36)$$

Observe that $\mathcal{N}(x) = \mathcal{A}x\mathcal{B}x$. We will show that the operators \mathcal{A} and \mathcal{B} satisfy all the conditions of Lemma 4. For the sake of convenience, we split the proof into several steps. \square

Step 1. \mathcal{A} is a Lipschitz on X , that is, (i) of Lemma 4 holds.

Let $x, y \in X$. By H_1 , we have

$$\begin{aligned}
 |\mathcal{A}x(t) - \mathcal{A}y(t)| &= \left| \rho(t, x(t), {}^H I^p x(t)) - \rho(t, y(t), {}^H I^p y(t)) \right| \\
 &\leq |\Psi(t)| \left(|x(t) - y(t)| + \left| {}^H I^p x(t) - {}^H I^p y(t) \right| \right) \\
 &\leq |\Psi(t)| \left(|x(t) - y(t)| + \frac{1}{\Gamma(p)} \int_1^t \left(\log \frac{t}{s} \right)^{p-1} \frac{1}{s} |x(s) - y(s)| ds \right) \\
 &\leq |\Psi| \left(\|x - y\| + \frac{\|x - y\|}{\Gamma(p)} \int_1^t (\log t - \log s)^{p-1} d \log s \right) \\
 &\leq |\Psi| \left(\|x - y\| + \frac{\|x - y\|}{\Gamma(p)} \cdot \frac{(\log t)^p}{p} \right) \\
 &\leq |\Psi| \left(\|x - y\| + \frac{1}{\Gamma(1+p)} \|x - y\| \right) \\
 &\leq |\Psi| \left(1 + \frac{1}{\Gamma(1+p)} \right) \|x - y\|.
 \end{aligned} \tag{37}$$

Therefore,

$$\begin{aligned}
 \|\mathcal{A}x - \mathcal{A}y\| &= \sup_{t \in [1, e]} |\mathcal{A}x(t) - \mathcal{A}y(t)| \\
 &\leq \|\Psi\| \left(1 + \frac{1}{\Gamma(1+p)} \right) \|x - y\|,
 \end{aligned} \tag{38}$$

for all $x, y \in X$. So, \mathcal{A} is a Lipschitz on X with Lipschitz constant $M_0 = \|\Psi\| (1 + 1/\Gamma(1+p))$.

Step 2. The multivalued operator \mathcal{B} is compact and upper semicontinuous on X , that is, (ii) of Lemma 4 holds.

First, we show that \mathcal{B} has convex values. Let $w_1, w_2 \in \mathcal{B}x$, and then, there are $v_1, v_2 \in S_{G,x}$, such that

$$\begin{aligned}
 w_i &= \frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_i(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_i(s) ds \right] \right. \\
 &\quad \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_i(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_i(s) ds \right] \right\} \\
 &\quad + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_i(\tau) d\tau \right) ds \right. \right. \\
 &\quad \left. \left. - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_i(s) ds \right] - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_i(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_i(s) ds \right] \right\} \\
 &\quad + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v_i(s) ds,
 \end{aligned} \tag{39}$$

$i = 1, 2, t \in [1, e]$. For any $\theta \in [0, 1]$, we have

$$\begin{aligned}
 & \theta w_1(t) + (1 - \theta)w_2(t) \\
 &= \frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} (\theta v_1(\tau) + (1 - \theta)v_2(\tau)) d\tau \right) ds \right. \right. \\
 & \quad \left. \left. - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} (\theta v_1(s) + (1 - \theta)v_2(s)) ds \right] \right. \\
 & \quad \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} (\theta v_1(\tau) + (1 - \theta)v_2(\tau)) d\tau \right) ds \right. \right. \\
 & \quad \left. \left. - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} (\theta v_1(s) + (1 - \theta)v_2(s)) ds \right] \right\} \\
 & \quad + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \\
 & \quad \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} (\theta v_1(\tau) + (1 - \theta)v_2(\tau)) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} (\theta v_1(s) + (1 - \theta)v_2(s)) ds \Big] \\
 & \quad - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} (\theta v_1(\tau) + (1 - \theta)v_2(\tau)) d\tau \right) ds \right. \\
 & \quad \left. \left. - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} (\theta v_1(s) + (1 - \theta)v_2(s)) ds \right] \right\} \\
 & \quad + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} (\theta v_1(s) + (1 - \theta)v_2(s)) ds,
 \end{aligned} \tag{40}$$

where $\bar{v}(t) = (\theta v_1(t) + (1 - \theta)v_2(t)) \in G(t, x(t), {}^H I^q x(t))$ for all $t \in [1, e]$. Hence, $\theta u_1(t) + (1 - \theta)u_2(t) \in \mathcal{B}x$, and consequently, $\mathcal{B}x$ is convex for each $x \in X$. As a result, \mathcal{B} defines a multivalued operator $\mathcal{B}: X \longrightarrow \mathcal{P}_{cv}(X)$.

Next, we show that \mathcal{B} maps bounded sets into bounded sets in X . To see this, let Q be a bounded set in X , and then, there exists a real number $r > 0$, such that $\|x\| \leq r, \forall x \in Q$. Now, for each $h \in \mathcal{B}x$, there exist $v \in S_{G,x}$, such that

$$\begin{aligned}
 h(t) &= \frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right. \\
 & \quad \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right\} \\
 & \quad + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds \right. \right. \\
 & \quad \left. \left. - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right\} \\
 & \quad + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v(s) ds.
 \end{aligned} \tag{41}$$

Then, for each $t \in [1, e]$, using (H_2) , we have

$$\begin{aligned}
|h(t)| &\leq \frac{t^{-\lambda}}{|\Delta|} \left\{ |B_2| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |\nu(\tau)| d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1} \times {}^H I^{\alpha-1} |\nu(s)| ds \right] \right. \\
&\quad + |A_2| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |\nu(\tau)| d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} |\nu(s)| ds \left. \right] \Bigg\} \\
&\quad + \frac{1}{|\Delta|} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \\
&\quad \times \left\{ |A_1| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |\nu(\tau)| d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e \int_1^e s^{\lambda-1H} I^{\alpha-1} |\nu(s)| ds \right] \right. \\
&\quad + |B_1| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |\nu(\tau)| d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} |\nu(s)| ds \left. \right] \Bigg\} \\
&\quad + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} |\nu(s)| ds \\
&\leq \frac{t^{-\lambda}}{|\Delta|} \left\{ |B_2| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} g(\tau) d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1} \times {}^H I^{\alpha-1} g(s) ds \right] \right. \\
&\quad + |A_2| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} g(\tau) d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} g(s) ds \left. \right] \Bigg\} \\
&\quad + \frac{1}{|\Delta|} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \\
&\quad \times \left\{ |A_1| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} g(\tau) d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e \int_1^e s^{\lambda-1H} I^{\alpha-1} g(s) ds \right] \right. \\
&\quad + |B_1| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} g(\tau) d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} g(s) ds \left. \right] \Bigg\} \\
&\quad + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} g(s) ds \\
&\leq \frac{\|g\|_{L^{p_1/1}}}{|\Delta|} \left\{ |B_2| \left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-1} \left(\int_1^s \tau^{-1} \left(\int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\alpha-2/1-p_1} \sigma^{-1/1-p_1} d\sigma \right)^{1-p_1} d\tau \right) ds \right. \right. \\
&\quad + \frac{|\alpha_1|}{\Gamma(\alpha-1)} \int_1^\xi s^{-1} \left(\int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \tau^{-1/1-p_1} d\tau \right)^{1-p_1} ds \left. \right] \\
&\quad + |A_2| \left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-1} \left(\int_1^s \tau^{-1} \left(\int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\alpha-2/1-p_1} \sigma^{-1/1-p_1} d\sigma \right)^{1-p_1} d\tau \right) ds \right. \\
&\quad + \frac{|\beta_1|}{\Gamma(\alpha-1)} \int_1^e s^{-1} \left(\int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \tau^{-1/1-p_1} d\tau \right)^{1-p_1} ds \left. \right] \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\|g\|_{L^{1/p_1}}}{|\Delta|} \left(\int_1^t s^{-1} (\log s)^{\alpha-2} ds \right) \left\{ |A_1| \left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-1} \right. \right. \\
& \times \left(\int_1^s \tau^{-1} \left(\int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\alpha-2/1-p_1} \sigma^{-1/1-p_1} d\sigma \right)^{1-p_1} d\tau \right) ds + \frac{|\beta_1|}{\Gamma(\alpha-1)} \int_1^e s^{-1} \left(\int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \tau^{-1/1-p_1} d\tau \right)^{1-p_1} ds \Big] \\
& + |B_1| \left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-1} \left(\int_1^s \tau^{-1} \left(\int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\alpha-2/1-p_1} \sigma^{-1/1-p_1} d\sigma \right)^{1-p_1} d\tau \right) ds \right. \\
& \left. + \frac{|\alpha_1|}{\Gamma(\alpha-1)} \int_1^\xi s^{-1} \left(\int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \tau^{-1/1-p_1} d\tau \right)^{1-p_1} ds \right] \Big\} \\
& + \frac{\|g\|_{L^{1/p_1}}}{\Gamma(\alpha-1)} \int_1^t s^{-1} \left(\int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \tau^{-1/1-p_1} d\tau \right)^{1-p_1} ds \\
& \leq \frac{\|g\|_{L^{1/p_1}}}{|\Delta|} \left\{ |B_2| \left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-1} \left(\int_1^s \tau^{-1} \left(\int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\alpha-2/1-p_1} \frac{1}{\sigma} d\sigma \right)^{1-p_1} d\tau \right) ds \right. \right. \\
& + \frac{|\alpha_1|}{\Gamma(\alpha-1)} \int_1^{e\xi} s^{-1} \left(\int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \frac{1}{\tau} d\tau \right)^{1-p_1} ds \Big] \\
& + |A_2| \left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-1} \times \left(\int_1^s \tau^{-1} \left(\int_1^\tau \int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\alpha-2/1-p_1} \frac{1}{\sigma} d\sigma \right)^{1-p_1} d\tau \right) ds \right. \\
& \left. + \frac{|\beta_1|}{\Gamma(\alpha-1)} \int_1^e s^{-1} \left(\int_1^e \int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \frac{1}{\tau} d\tau \right)^{1-p_1} ds \right] \Big\} \\
& + \frac{\|g\|_{L^{1/p_1}}}{|\Delta|} \left(\int_1^t s^{-1} (\log s)^{\alpha-2} ds \right) \left\{ |A_1| \left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-1} \right. \right. \\
& \times \left(\int_1^s \tau^{-1} \left(\int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\alpha-2/1-p_1} \frac{1}{\sigma} d\sigma \right)^{1-p_1} d\tau \right) ds + \frac{|\beta_1|}{\Gamma(\alpha-1)} \int_1^e s^{-1} \left(\int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \frac{1}{\tau} d\tau \right)^{1-p_1} ds \Big] \\
& + |B_1| \left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-1} \left(\int_1^s \tau^{-1} \left(\int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\alpha-2/1-p_1} \frac{1}{\sigma} d\sigma \right)^{1-p_1} d\tau \right) ds \right. \\
& \left. + \frac{|\alpha_1|}{\Gamma(\alpha-1)} \int_1^\xi s^{-1} \left(\int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \frac{1}{\tau} d\tau \right)^{1-p_1} ds \right] + \frac{\|g\|_{L^{1/p_1}}}{\Gamma(\alpha-1)} \int_1^t s^{-1} \left(\int_1^s \left(\log \frac{s}{\tau} \right)^{\alpha-2/1-p_1} \frac{1}{\tau} d\tau \right)^{1-p_1} ds \\
& \leq \frac{\|g\|_{L^{1/p_1}}}{|\Delta|} \left\{ |B_2| \left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r) [(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-1} (\log s)^{(1+a)(1-p_1)+1} ds \right. \right. \\
& \left. + \frac{|\alpha_1| (\log \xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1) [(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1}]} \right]
\end{aligned}$$

$$\begin{aligned}
& + |A_2| \left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r) \left[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1} \right]} \times \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-1} (\log s)^{(1+a)(1-p_1)+1} ds \right. \\
& \left. + \frac{|\beta_1|}{\Gamma(\alpha-1) \left[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1} \right]} \right] \Bigg\} \\
& + \frac{\|g\|_{L^{1/p_1}}}{|\Delta|(\alpha-1)} \left\{ |A_1| \left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r) \left[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1} \right]} \right. \right. \\
& \times \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-1} (\log s)^{(1+a)(1-p_1)+1} ds + \frac{|\beta_1|}{\Gamma(\alpha-1) \left[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1} \right]} \Bigg] \\
& + |B_1| \left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r) \left[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1} \right]} \times \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-1} (\log s)^{(1+a)(1-p_1)+1} ds \right. \\
& \left. + \frac{|\alpha_1| (\log \xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1) \left[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1} \right]} \right] \Bigg\} \\
& + \frac{\|g\|_{L^{1/p_1}}}{\Gamma(\alpha-1) \left[(1+a)^{2-p_1}(1-p_1) + (1+a)^{1-p_1} \right]} \leq M_1 \|g\|_{L^{1/p_1}}, \tag{42}
\end{aligned}$$

hence,

$$\|h\| \leq M_1 \|g\|_{L^{1/p_1}}. \tag{43}$$

Therefore, $\mathcal{B}(Q)$ is uniformly bounded.

Next, we show that \mathcal{B} maps bounded sets into equi-continuous sets. For this purpose, we assume that Q be, as above, a bounded set and $h \in \mathcal{B}x$ for some $x \in Q$, and then, there exists a $v \in S_{G,x}$, such that

$$\begin{aligned}
h(t) = & \frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right. \\
& \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right\} \\
& + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \\
& \cdot \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right. \\
& \left. - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right\} \\
& + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v(s) ds. \tag{44}
\end{aligned}$$

Thus, for any $t_1, t_2 \in [1, e]$, $t_2 > t_1$, we have

$$\begin{aligned}
 & |h(t_2) - h(t_1)| \\
 & \leq \frac{M_2 \|g\|_{L^{1/p_1}}}{|\Delta|} |t_2^{-\lambda} - t_1^{-\lambda}| + \frac{M_3 \|g\|_{L^{1/p_1}}}{|\Delta|} \left| t_2^{-\lambda} \int_1^{t_2} s^{\lambda-1} (\log s)^{\alpha-2} ds - t_1^{-\lambda} \int_1^{t_1} s^{\lambda-1} (\log s)^{\alpha-2} ds \right| \\
 & \quad + \frac{\|g\|_{L^{1/p_1}}}{(1+a)^{1-p_1} \Gamma(\alpha-1)} \left| t_2^{-\lambda} \int_1^{t_2} s^{\lambda-1} (\log s)^{(1+a)(1-p_1)} ds - t_1^{-\lambda} \int_1^{t_1} s^{\lambda-1} (\log s)^{(1+a)(1-p_1)} ds \right| \\
 & \leq \frac{M_2 \|g\|_{L^{1/p_1}}}{|\Delta|} |t_2^{-\lambda} - t_1^{-\lambda}| + \frac{M_3 \|g\|_{L^{1/p_1}}}{|\Delta|} \left(|t_2^{-\lambda} - t_1^{-\lambda}| \cdot \left| \int_1^{t_2} \frac{(\log s)^{\alpha-2}}{s^{1-\lambda}} ds \right| + |t_1^{-\lambda}| \left| \int_{t_1}^{t_2} \frac{(\log s)^{\alpha-2}}{s^{1-\lambda}} ds \right| \right) \\
 & \quad + \frac{\|g\|_{L^{1/p_1}}}{(1+a)^{1-p_1} \Gamma(\alpha-1)} \left(|t_2^{-\lambda} - t_1^{-\lambda}| \cdot \left| \int_1^{t_2} \frac{(\log s)^{(1+a)(1-p_1)}}{s^{1-\lambda}} ds \right| + |t_1^{-\lambda}| \left| \int_{t_1}^{t_2} \frac{(\log s)^{(1+a)(1-p_1)}}{s^{1-\lambda}} ds \right| \right) \\
 & \longrightarrow 0,
 \end{aligned} \tag{45}$$

independent of $x \in Q$ as $t_1 - t_2 \longrightarrow 0$.

Therefore, $\mathcal{B}(Q)$ is an equicontinuous set in X . Now, an application of the Arzela-Ascoli theorem yields that $\mathcal{B}(Q)$ is relatively compact.

In our next step, we show that \mathcal{B} is upper semi-continuous. By Lemma 2, \mathcal{B} will be upper semicontinuous if

we prove that it has a closed graph. Let $x_n \longrightarrow x_*$, $h_n \in \mathcal{B}x_n$, and $h_n \longrightarrow h_*$. Then, we need to show that $h_* \in \mathcal{B}x_*$. Associated with $h_n \in \mathcal{B}x_n$, there exists $v_n \in S_{G, x_n}$, such that for each $t \in [1, e]$,

$$\begin{aligned}
 h_n(t) = & \frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_n(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_n(s) ds \right] \right. \\
 & \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_n(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_n(s) ds \right] \right\} \\
 & + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \\
 & \cdot \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_n(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_n(s) ds \right] \right. \\
 & \left. - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_n(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_n(s) ds \right] \right\} \\
 & + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v_n(s) ds.
 \end{aligned} \tag{46}$$

Thus, it suffices to show that there exists $v_* \in S_{G,x_*}$, such that for each $t \in [1, e]$,

$$\begin{aligned}
 h_*(t) = & \frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_*(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_*(s) ds \right] \right. \\
 & - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_*(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_*(s) ds \right] \Big\} \\
 & + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \\
 & \cdot \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_*(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_*(s) ds \right] \right. \\
 & - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_*(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_*(s) ds \right] \Big\} \\
 & + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v_*(s) ds.
 \end{aligned} \tag{47}$$

Let us consider the linear operator $\Theta: L^1([1, e], R) \rightarrow C([1, e], R)$ given by

$$\begin{aligned}
 v(t) & \longmapsto \Theta(v)(t) \\
 = & \frac{t^{-\lambda}}{\Delta} \left\{ B_2 \frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right. \\
 & - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \Big\} \\
 & + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \\
 & \cdot \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right. \\
 & - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \Big\} \\
 & + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v(s) ds.
 \end{aligned} \tag{48}$$

Notice that the operator Θ is continuous. Indeed, for $v_n, v_* \in L^1([1, e], R)$ with $v_n \rightarrow v_*$ in $L^1([1, e], R)$, we obtain

$$\begin{aligned}
& \|\Theta(v_n)(t) - \Theta(v_*)(t)\| \\
& \leq \frac{1}{|\Delta|} \left\| \left[B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} (v_n(\tau) - v_*(\tau)) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} (v_n(s) - v_*(s)) ds \right] \right. \right. \\
& \quad \left. \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} (v_n(\tau) - v_*(\tau)) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} (v_n(s) - v_*(s)) ds \right] \right] \right\| \\
& \quad + \frac{1}{|\Delta|(\alpha-1)} \left\| \left[A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} (v_n(\tau) - v_*(\tau)) d\tau \right) ds \right. \right. \right. \\
& \quad \left. \left. - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} (v_n(s) - v_*(s)) ds \right] \right. \\
& \quad \left. \left. - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} (v_n(\tau) - v_*(\tau)) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} (v_n(s) - v_*(s)) ds \right] \right] \right\| \\
& \quad + \left\| \int_1^t s^{-1H} I^{\alpha-1} (v_n(s) - v_*(s)) ds \right\|, \quad \forall t \in [1, e],
\end{aligned} \tag{49}$$

which implies that $\Theta(v_n) \rightarrow \Theta(v_*)$ in $C([1, e], R)$.

Thus, it follows by Lemma 3 that $\Theta^\circ S_G$ is a closed graph operator. Furthermore, we have $h_n(t) \in \Theta(S_{G, x_n})$. Since $x_n \rightarrow x_*$, therefore, we have

$$\begin{aligned}
h_*(t) &= \frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_*(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_*(s) ds \right] \right. \\
& \quad \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_*(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_*(s) ds \right] \right\} \\
& \quad + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \\
& \quad \left. \left. \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_*(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_*(s) ds \right] \right. \\
& \quad \left. \left. - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_*(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_*(s) ds \right] \right] \right\} \\
& \quad + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v_*(s) ds,
\end{aligned} \tag{50}$$

for some $v_* \in S_{G, x_*}$.

As a result, we have that the operator \mathcal{B} is compact and upper semicontinuous.

Step 3. Now, we show that $2MK < 1$, that is, (iii) of Lemma 4 holds.

This is obvious by (H_4) since we have

$$M = \|B(X)\| = \sup\{|\mathcal{B}x| : x \in X\} \leq M_1 \|\mathcal{G}\|_{L^{1/p_1}}, \quad (51)$$

and $K = M_0$.

Thus, all the conditions of Lemma 4 are satisfied, and a direct application of it yields that either conclusion (i) or

conclusion (ii) holds. We show that conclusion (ii) is not possible.

Supposed the conclusion (ii) is true. Let $u \in \mathcal{E}$ be arbitrary. Then, we have, for $\lambda > 1, \lambda u \in \mathcal{A}u\mathcal{B}u$, and then, there exists $v \in S_{G,x}$ such that

$$\begin{aligned} u(t) = & \lambda^{-1} \rho(t, x(t), {}^H I^p x(t)) \left(\frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \right. \\ & \cdot \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1p} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1p} v(s) ds \Big\} \\ & - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1p} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1p} v(s) ds \Big\} \right\} \\ & + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \\ & \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1p} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1p} v(s) ds \Big\} \\ & - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1p} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1p} v(s) ds \Big\} \right\} \\ & + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1p} v(s) ds, \end{aligned} \quad (52)$$

and so, for all $t \in [1, e]$, we have

$$\begin{aligned} |u(t)| \leq & \lambda^{-1} \left| \rho(t, u(t), {}^H I^p u(t)) \right| \\ & \times \left(\frac{t^{-\lambda}}{\Delta} \left\{ |B_2| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v(\tau)| d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \right] \right. \right. \\ & + |A_2| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v(\tau)| d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \Big\} \right\} \\ & + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ |A_1| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \\ & \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v(\tau)| d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \Big\} \\ & + |B_1| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v(\tau)| d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \Big\} \right\} \\ & + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \end{aligned}$$

$$\begin{aligned}
&\leq \left[|\rho(t, u(t), {}^H I^p u(t)) - \rho(t, 0, 0)| + |\rho(t, 0, 0)| \right] \\
&\quad \times \left(\frac{t^{-\lambda}}{\Delta} \left\{ |B_2| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v(\tau)| d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \right] \right. \right. \\
&\quad \left. \left. + |A_2| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v(\tau)| d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \right] \right\} \right. \\
&\quad \left. + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ |A_1| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \right. \\
&\quad \left. \left. \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v(\tau)| d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \right] \right. \right. \\
&\quad \left. \left. + |B_1| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v(\tau)| d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \right] \right\} \right. \\
&\quad \left. + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} |v(s)| ds \right) \leq M_1 \|g\|_{L^{1/p_1}} \left[\|\Psi\| \left(1 + \frac{1}{\Gamma(1+p)} \right) \|u\| + \rho_0 \right].
\end{aligned} \tag{53}$$

Therefore,

$$\begin{aligned}
\|u\| &= \sup_{t \in [1, e]} |u(t)| \\
&\leq M_1 \|g\|_{L^{1/p_1}} \left[\|\Psi\| \left(1 + \frac{1}{\Gamma(1+p)} \right) \|u\| + \rho_0 \right],
\end{aligned} \tag{54}$$

where we have put $\rho_0 = \sup_{t \in [1, e]} |\rho(t, 0, 0)|$. Then, with $\|u\| = \mathcal{R}$, we have

$$\mathcal{R} \leq \frac{\rho_0 M_1 \|g\|_{L^{1/p_1}}}{1 - M_0 M_1 \|g\|_{L^{1/p_1}}}. \tag{55}$$

This is contradictory. Thus, conclusion (ii) of Lemma 4 does not hold by (28). Therefore, the operator equation $\mathcal{A}x\mathcal{B}x$ and consequent problem (7) have a solution on $[1, e]$. This completes the proof.

Theorem 2. Suppose that the conditions (H_1) , (H_2) , (H_5) , and (H_6) hold. Then, inclusion problem (7) has at least one mild solution on $C([1, e], R)$.

Proof. The proof is similar to that of Theorem 1 and is omitted. \square

Theorem 3. Suppose that the conditions (H_1) , (H_3) , and (H_7) – (H_9) hold. If

$$\lambda_0 := M_1 \left(1 + \frac{1}{\Gamma(1+p)} \right) (\|g\|_{L^{1/p_1}} \|\Psi\| + \|\eta\| \|\zeta\|_{L^{1/p_1}}) < 1, \tag{56}$$

where M_1 , $\|\Psi\|$, $\|\eta\|$, $\|\zeta\|_{L^{1/p_1}}$ are given by (33); then, inclusion problem (7) has at least one mild solution on $C([1, e], R)$.

Proof. Observe that the set $S_{G,x}$ is nonempty for each $x \in C[1, e]$ by assumption (H_7) , and thus, G has a measurable selection. We now show that the operator $\mathcal{N}: C[1, e] \rightarrow \mathcal{P}(C[1, e])$ satisfies the assumptions of Lemma 5. To establish that $\mathcal{N}x \in \mathcal{P}_{cl}(C[1, e])$, for each $x \in C[1, e]$, let $\{w_n\}_{n \geq 1} \subset \mathcal{N}x$ be such that $w_n \rightarrow w$ as $n \rightarrow \infty$ in $C[1, e]$. Then, $w \in C[1, e]$, and there exists $v_n \in S_{G,x}$, such that for each $t \in [1, e]$, we have

$$\begin{aligned}
w_n(t) &= \rho(t, x(t), {}^H I^p x(t)) \left(\frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_n(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_n(s) ds \right] \right. \right. \\
&\quad \left. \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_n(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_n(s) ds \right] \right\} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \\
& \times \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_n(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_n(s) ds \right] \right. \\
& \left. - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_n(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_n(s) ds \right] \right\} \\
& + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v_n(s) ds,
\end{aligned} \tag{57}$$

with $v_n(t) \in G(t, x(t), {}^H I^q x(t))$, $t \in [1, e]$.

Since G has compact values, therefore, we can pass onto a subsequence (denoted in a same way) to obtain that v_n

converges to v in $L^1[1, e]$. Thus, $v \in S_{G,x}$, and for each $t \in [1, e]$, we have $w_n(t) \rightarrow w(t)$, where

$$\begin{aligned}
w(t) = & \rho(t, x(t), {}^H I^p x(t)) \left(\frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right. \right. \\
& \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right\} \\
& + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \\
& \times \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right. \\
& \left. - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v(s) ds \right] \right\} \\
& + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v(s) ds.
\end{aligned} \tag{58}$$

Hence, $w \in \mathcal{N}(x)$.

Next, we show that \mathcal{N} is a contraction, that is,

$$H_{d_1}(\mathcal{N}x, \mathcal{N}\bar{x}) \leq \lambda_0 \|x - \bar{x}\|, \quad \forall x, \bar{x} \in X, \tag{59}$$

where λ_0 is defined in (56), and d_1 is the metric induced by the norm $\|\cdot\|$ in $C[1, e]$.

For this, let $x, \bar{x} \in [1, e]$ and $w_1 \in \mathcal{N}x$. Then, there exists $v_1 \in S_{G,x}$, such that for all $t \in [1, e]$, we obtain

$$\begin{aligned}
w_1(t) = & \rho(t, x(t), {}^H I^p x(t)) \left(\frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \right. \\
& \cdot \left. \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_1(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_1(s) ds \right] \right. \\
& \left. - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_1(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_1(s) ds \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \\
& \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_1(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_1(s) ds \Big] \\
& - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_1(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_1(s) ds \right] \Big\} \\
& + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v_1(s) ds \Big).
\end{aligned} \tag{60}$$

By (H_7) , we have

$$\begin{aligned}
& H_{d_1} \left(G(t, x(t), {}^H I^p x(t)), G(t, \bar{x}(t), {}^H I^p \bar{x}(t)) \right) \\
& \leq \zeta(t) \left(|x(t) - \bar{x}(t)| + \left| {}^H I^p x(t) - {}^H I^p \bar{x}(t) \right| \right),
\end{aligned} \tag{61}$$

for a.e. $t \in [1, e]$, and then, there exists $\psi \in G(t, \bar{x}(t), {}^H D^q \bar{x}(t))$, such that

$$\begin{aligned}
& |v_1(t) - \psi| \leq \zeta(t) (|x(t) - \bar{x}(t)| \\
& + \left| {}^H I^p x(t) - {}^H I^p \bar{x}(t) \right|), \quad \text{a.e. } t \in [1, e].
\end{aligned} \tag{62}$$

We define $\tilde{U}: [1, e] \longrightarrow \mathcal{P}(R)$ by $\tilde{U}(t) = \{\psi \in R: |v_1(t) - \psi| \leq \zeta(t) (|x(t) - \bar{x}(t)| + \left| {}^H I^p x(t) - {}^H I^p \bar{x}(t) \right|)\}$. As the multivalued operator $V(t) = \tilde{U}(t) \cap G(t, \bar{x}(t), {}^H I^q \bar{x}(t))$ is measurable (proposition III.4, [55]), there exists a function $v_2(t)$ which is a measurable selection for $V(t)$. Hence, $v_2(t) \in G(t, \bar{x}(t), {}^H I^q \bar{x}(t))$ for a.e. $t \in [1, e]$ and

$$\begin{aligned}
& |v_1(t) - v_2(t)| \leq \zeta(t) (|x(t) - \bar{x}(t)| \\
& + \left| {}^H I^p x(t) - {}^H I^p \bar{x}(t) \right|), \quad \text{a.e. } t \in [1, e].
\end{aligned} \tag{63}$$

Let us define the function $w_2(t), t \in [1, e]$ by

$$\begin{aligned}
w_2(t) = & \rho(t, \bar{x}(t), {}^H I^p \bar{x}(t)) \left(\frac{t^{-\lambda}}{\Delta} \left\{ B_2 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \right. \\
& \cdot \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_2(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_2(s) ds \Big] \\
& - A_2 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_2(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_2(s) ds \right] \Big\} \\
& + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ A_1 \left[\frac{\beta_2}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \\
& \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_2(\tau) d\tau \right) ds - \beta_1 e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} v_2(s) ds \Big] \\
& - B_1 \left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} v_2(\tau) d\tau \right) ds - \alpha_1 \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} v_2(s) ds \right] \Big\} \\
& + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} v_2(s) ds \Big).
\end{aligned} \tag{64}$$

Then, we conclude that

$$\begin{aligned}
|w_1(t) - w_2(t)| &\leq \left| \rho(t, x(t), {}^H I^p x(t)) - \rho(t, \bar{x}(t), {}^H I^p \bar{x}(t)) \right| \\
&\quad \left(\frac{t^{-\lambda}}{\Delta} \left\{ |B_2| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v_1(\tau)| d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} |v_1(s)| ds \right] \right. \right. \\
&\quad \left. \left. + |A_2| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v_1(\tau)| d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} |v_1(s)| ds \right] \right\} \right. \\
&\quad \left. + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ |A_1| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \right. \\
&\quad \left. \left. \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v_1(\tau)| d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} |v_1(s)| ds \right] \right. \right. \\
&\quad \left. \left. + |B_1| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v_1(\tau)| d\tau \right) ds + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} |v_1(s)| ds \right] \right\} \right. \\
&\quad \left. + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} |v_1(s)| ds \right) \\
&\quad + \left| \rho(t, \bar{x}(t), {}^H I^p \bar{x}(t)) \right| \left(\frac{t^{-\lambda}}{\Delta} \left\{ |B_2| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v_1(\tau) - v_2(\tau)| d\tau \right) ds \right. \right. \right. \\
&\quad \left. \left. + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} |v_1(s) - v_2(s)| ds \right] \right. \right. \\
&\quad \left. \left. + |A_2| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v_1(\tau) - v_2(\tau)| d\tau \right) ds \right. \right. \right. \\
&\quad \left. \left. + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} |v_1(s) - v_2(s)| ds \right] \right\} \right. \\
&\quad \left. + \frac{1}{\Delta} \left(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right) \left\{ |A_1| \left[\frac{|\beta_2|}{\Gamma(r)} \int_1^\xi \left(\log \frac{\xi}{s} \right)^{r-1} s^{-\lambda-1} \right. \right. \right. \\
&\quad \left. \left. \times \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v_1(\tau) - v_2(\tau)| d\tau \right) ds + |\beta_1| e^{-\lambda} \int_1^e s^{\lambda-1H} I^{\alpha-1} |v_1(s) - v_2(s)| ds \right] \right. \right. \\
&\quad \left. \left. + |B_1| \left[\frac{|\alpha_2|}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s} \right)^{r-1} s^{-\lambda-1} \left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} |v_1(\tau) - v_2(\tau)| d\tau \right) ds \right. \right. \right. \\
&\quad \left. \left. + |\alpha_1| \xi^{-\lambda} \int_1^\xi s^{\lambda-1H} I^{\alpha-1} |v_1(s) - v_2(s)| ds \right] \right\} \right. \\
&\quad \left. + t^{-\lambda} \int_1^t s^{\lambda-1H} I^{\alpha-1} |v_1(s) - v_2(s)| ds \right) \\
&\leq M_1 \|g\|_{L^{1/p_1}} \left| \rho(t, x(t), {}^H I^p x(t)) - \rho(t, \bar{x}(t), {}^H I^p \bar{x}(t)) \right| + M_1 \|\zeta\|_{L^{1/p_1}} \left(1 + \frac{1}{\Gamma(1+p)} \right) \|x - \bar{x}\| \\
&\quad \times \left| \rho(t, \bar{x}(t), {}^H I^p \bar{x}(t)) \right| \\
&\leq M_1 \left(1 + \frac{1}{\Gamma(1+p)} \right) (\|g\|_{L^{1/p_1}} \|\Psi\| + \|\eta\| \|\zeta\|_{L^{1/p_1}}) \|x - \bar{x}\|, \quad \forall t \in [1, e],
\end{aligned} \tag{65}$$

which yield

$$\begin{aligned} \|w_1 - w_2\| &= \sup_{t \in [1, e]} |w_1(t) - w_2(t)| \\ &\leq M_1 \left(1 + \frac{1}{\Gamma(1+p)} \right) \\ &\quad \cdot (\|g\|_{L^{1/p_1}} \|\Psi\| + \|\eta\| \|\zeta\|_{L^{1/p_1}}) \|x - \bar{x}\| \\ &= \lambda_0 \|x - \bar{x}\|. \end{aligned} \quad (66)$$

By interchanging the roles of x and \bar{x} , we obtain a similar relation, and thus, we get

$$H_{d_1}(\mathcal{N}x, \mathcal{N}\bar{x}) \leq \lambda_0 \|x - \bar{x}\|. \quad (67)$$

In view of the condition $\lambda_0 < 1$ (given by (56)), it follows that \mathcal{N} is a contraction, and therefore, by Lemma 5, \mathcal{N} has a fixed point x , which is a solution of problem (7). This completes the proof. \square

4. Examples

(a) Consider the following equation:

$$\begin{cases} \left({}^H D^{3/2} + {}^H D^{1/2} \right) \left(\frac{x(t)}{e^{1-t}/460 \tan^{-1}(x(t) + {}^H I^{1/2} x(t) + \pi/4) + 2} \right) \in G(t, x(t), {}^H I^{1/2} x(t)), & t \in (1, e), \\ 3 \left(\frac{x(e^{1/2})}{e^{1-e^{1/2}}/460 \tan^{-1}(x(e^{1/2}) + {}^H I^{1/2} x(e^{1/2}) + \pi/4) + 2} \right) = {}^H I^{1/2} \left(\frac{x(e)}{e^{1-e}/460 \tan^{-1}(x(e) + {}^H I^{1/2} x(e) + \pi/4) + 2} \right), \\ 3 \left(\frac{x(e)}{e^{1-e}/460 \tan^{-1}(x(e) + {}^H I^{1/2} x(e) + \pi/4) + 2} \right) = {}^H I^{1/2} \left(\frac{x(e^{1/2})}{e^{1-e^{1/2}}/460 \tan^{-1}(x(e^{1/2}) + {}^H I^{1/2} x(e^{1/2}) + \pi/4) + 2} \right), \end{cases} \quad (68)$$

where $G: [1, e] \times R \times R \longrightarrow \mathcal{P}(R)$ is a multivalued map given by

$$t \longrightarrow G(t, x(t), {}^H I^{1/2} x(t)) = \left[\frac{|x|^3 + |{}^H I^{1/2} x|^3}{20(|x|^3 + |{}^H I^{1/2} x|^3 + 4)}, \frac{|\sin(x + {}^H I^{1/2} x)|}{9(|\sin(x + {}^H I^{1/2} x)| + 1)} + \frac{8}{9} \right]. \quad (69)$$

By condition (H_1) , $\Psi(t) = e^{1-t}/460$ with $\|\Psi\| = 1/460$. For $\tilde{g} \in G$, we have

$$|\tilde{g}| \leq \max \left(\frac{|x|^3 + |{}^H I^{1/2} x|^3}{20(|x|^3 + |{}^H I^{1/2} x|^3 + 4)}, \frac{|\sin(x + {}^H I^{1/2} x)|}{9(|\sin(x + {}^H I^{1/2} x)| + 1)} + \frac{8}{9} \right) \leq 1, \quad \forall x \in R, \quad (70)$$

$$\|G(t, x, y)\| = \sup\{|y|: y \in G(t, x, y)\} \leq 1 = g(t), \quad \forall x, y \in R.$$

Let $p_1 = 1/4$; then, $g(t) \in L^4([1, e], R^+)$. Using the given data, we find that $|A_1| \leq 1.8196, |A_2| \leq 4.2428, |B_1| \leq 1.1041,$

$|B_2| \leq 6, |\Delta| \leq 15.6018, M_0 \leq 0.0048, M_1 \leq 84.6585, \rho_0 = 1/460 + 2$. Furthermore,

$$M_0 M_1 \|g\|_{L^4} \leq 0.4485 < \frac{1}{2}, \quad (71)$$

and $\mathcal{R} > 160(e-1)^{1/4}(1/460+2) \geq \rho_0 M_1 \|g\|_{L^4}/(1-M_0 M_1 \|g\|_{L^4})$. Hence, all the conditions of Theorem 1 are satisfied, and accordingly, problem (68) has a solution on $[1, e]$.

(b) Let us consider the following inclusion problem:

$$\left\{ \begin{aligned} & \left({}^H D^{3/2} + {}^H D^{1/2} \right) \left(\frac{x(t)}{3/800e^{t-1} + 20(\sin(x(t)) + |{}^H I^{1/2} x(t)|/1 + |{}^H I^{1/2} x(t)|) + 1/10} \right) \in G(t, x(t), {}^H I^{1/2} x(t)), \quad t \in (1, e), \\ & 3 \left(\frac{x(e^{1/2})}{3/800e^{e^{1/2}-1} + 20(\sin(x(e^{1/2})) + |{}^H I^{1/2} x(e^{1/2})|/1 + |{}^H I^{1/2} x(e^{1/2})|) + 1/10} \right) \\ & = {}^H I^{1/2} \left(\frac{x(e)}{3/800e^{e-1} + 20(\sin(x(e)) + |{}^H I^{1/2} x(e)|/1 + |{}^H I^{1/2} x(e)|) + 1/10} \right), \\ & 3 \left(\frac{x(e)}{3/800e^{e-1} + 20(\sin(x(e)) + |{}^H I^{1/2} x(e)|/1 + |{}^H I^{1/2} x(e)|) + 1/10} \right) \\ & = {}^H I^{1/2} \left(\frac{x(e^{1/2})}{3/800e^{e^{1/2}-1} + 20(\sin(x(e^{1/2})) + |{}^H I^{1/2} x(e^{1/2})|/1 + |{}^H I^{1/2} x(e^{1/2})|) + 1/10} \right). \end{aligned} \right. \quad (72)$$

In order to demonstrate the application of Theorem 3, we consider

$$t \longrightarrow G(t, x(t), {}^H I^{1/2} x(t)) = \left[0, \frac{1}{512\sqrt[4]{t}} \left(\frac{|x(t)|}{12(8+|x(t)|)} + \frac{\tan^{-1}({}^H I^{1/2} x(t))}{1 + \tan^{-1}({}^H I^{1/2} x(t))} \right) + \frac{1}{300+t} \right]. \quad (73)$$

Clearly,

$$H_{d_1} \left(G(t, x, {}^H I^{1/2} x), G(t, \bar{x}, {}^H I^{1/2} \bar{x}) \right) \leq \frac{3}{512\sqrt[4]{t}} \|x - \bar{x}\|, \quad (74)$$

$$\|G(t, x, y)\| = \sup\{|v|: v \in G(t, x, y)\} \leq 1 = g(t), \quad \forall x, y \in R.$$

Letting $\zeta(t) = 3/512\sqrt[4]{t}$, it is easy to check that $d(0, G(t, 0, 0)) \leq \zeta(t)$ holds for almost $t \in [1, e]$ and that $\zeta(t) \in L^4([1, e], R^+)$ ($p_1 = 1/4$), $\|\zeta\|_{L^4} = 3/512$. From the following inequalities, we get $\eta(t) = 6/(800e^{t-1} + 20) + 1/10$ and $\|\eta\| = 88/820$:

$$|\rho(t, x, y)| \leq \frac{6}{800e^{t-1} + 20} + \frac{1}{10}, \quad (t, x, y) \in [1, e] \times R^2. \quad (75)$$

In addition, by condition (H_1) , we obtain $\Psi(t) = 3/(800e^{t-1} + 20)$ with $\|\Psi\| = 3/820$. Furthermore, using the given data, we find that $|A_1| \leq 1.8196$, $|A_2| \leq 4.2428$, $|B_1| \leq 1.1040$, $|B_2| \leq 6$, $|\Delta| \leq 15.6018$, $M_1 \leq 84.6585$. Furthermore,

$$M_1 \left(1 + \frac{1}{\Gamma(1+p)} \right) (\|g\|_{L^4} \|\Psi\| + \|\eta\| \|\zeta\|_{L^4}) \leq 0.8682 < 1. \quad (76)$$

Thus, all the conditions of Theorem 3 are satisfied. Hence, it follows by the conclusion of Theorem 3 that there exists a solution for problem (72) on $[1, e]$.

5. Conclusion

Nowadays, we need to study more natural phenomena to gain more abilities for modeling. Therefore, fractional calculus came into being, and today, their importance has become more and more apparent to researchers. In this way, it is necessary to design different and complicated modelings by utilizing the fractional differential problems. This is useful in making modern software which helps us to allow for more cost-free testing and less material consumption. In this work, we have developed the existence theory for a class of Hadamard sequential fractional hybrid differential inclusions equipped with two-point hybrid Hadamard integral boundary value conditions. The nonlinearities in the given problems implicitly depend on the unknown function together with its Hadamard fractional integral of order $p \in (0, 1)$. We apply fixed-point theorem due to Dhage and Covitz-Nadler fixed-point theorem to establish the desired results. Eventually, we give two numerical examples to support the applicability of our findings.

The work accomplished in this study is new and enriches the literature on boundary value problems for nonlinear Hadamard fractional differential inclusions. For future works, one can extend the given fractional boundary value problem to more general structures, such as finitely point multistrip integral boundary value conditions given by newly introduced generalized fractional operators with nonsingular kernels.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors read and approved the final manuscript.

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Research Article

Yosida Approximation Iterative Methods for Split Monotone Variational Inclusion Problems

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In this paper, we present two iterative algorithms involving Yosida approximation operators for split monotone variational inclusion problems (S_p MVIP). We prove the weak and strong convergence of the proposed iterative algorithms to the solution of S_p MVIP in real Hilbert spaces. Our algorithms are based on Yosida approximation operators of monotone mappings such that the step size does not require the precalculation of the operator norm. To show the reliability and accuracy of the proposed algorithms, a numerical example is also constructed.

1. Introduction

Variational inequality which was brought into existence by Hartman and Stampacchia [1] plays an important role as mathematical model in physics, economics, optimization, networking structural analysis, and medical images. In 1994, Censor and Elfving [2] first presented the split feasibility problems (in short, SFP) for modeling in medical image reconstruction. From the last two decades, SFP has been implemented widely in intensity-modulation therapy treatment planning and other branches of applied sciences (see, e.g., [3–5]). Censor et al. [6] combined the VIP and SFP and presented a new type of variational inequality problem called split variational inequality problem (in short, SVIP) as follows:

$$\text{Find } x^* \in C \text{ such that } x^* \in \text{VIP}(V_1; C) \text{ and } Ax^* \in \text{VIP}(V_2; Q), \quad (1)$$

where C and Q are closed, convex subsets of Hilbert spaces H_1 and H_2 , respectively, $A : H_1 \rightarrow H_2$ is a bounded linear operator, $V_1 : H_1 \rightarrow H_1$ and $V_2 : H_2 \rightarrow H_2$ are two operators, $\text{VIP}(V_1; C) = \{y \in C : \langle V_1(y), x - y \rangle \geq 0, \forall x \in C\}$ and $\text{VIP}(V_2; Q) = \{z \in Q : \langle g(z), x - z \rangle \geq 0, \forall x \in Q\}$.

Moudafi [7] generalized SVIP into split monotone variational inclusion problem (in short, S_p MVIP) as follows:

$$\text{Find } x^* \in H_1 \text{ such that } x^* \in \text{VI}(V_1, G_1; H_1) \text{ and } Ax^* \in \text{VI}(V_2, G_2; H_2), \quad (2)$$

where $G_1 : H_1 \rightarrow 2^{H_1}$ and $G_2 : H_2 \rightarrow 2^{H_2}$ are set-valued mappings on Hilbert spaces H_1 and H_2 , respectively, $\text{VI}(V_1, G_1; H_1) = \{y \in H_1 : 0 \in V_1(y) + G_1(y)\}$ and $\text{VI}(V_2, G_2; H_2) = \{z \in H_2 : 0 \in V_2(z) + G_2(z)\}$.

Moudafi [7] formulated the following iterative algorithm to find the solution of S_p MVIP. Let $\lambda > 0$, select an arbitrary

starting point $x_0 \in H_1$, and compute

$$x_{n+1} = U[x_n + \gamma A^*(W - I)Ax_n], \quad (3)$$

where A^* is an adjoint operator of A , $\gamma \in (0, 1/L)$ with L being a spectral radius of operator A^*A , $U = R_\lambda^{G_1}(I - \lambda V_1) = (I + \lambda G_1)^{-1}(I - \lambda V_1)$ and $W = R_\lambda^{G_2}(I - \lambda V_2) = (I + \lambda G_2)^{-1}(I - \lambda V_2)$.

Let $N_C(x) = \{z \in H_1 : \langle z, y - x \rangle \leq 0, \forall y \in C\}$ and $N_Q(x) = \{w \in H_2 : \langle w, y - x \rangle \leq 0, \forall y \in Q\}$ be normal cones to the closed and convex sets C and Q , respectively. If $G_1 = N_C$ and $G_2 = N_Q$, then S_p MVIP reduces to S_p VIP. If $V_1 = V_2 = 0$, then S_p MVIP reduces to the split variational inclusion problem (in short, S_p VIP) for set-valued maximal monotone mappings, introduced and studied by Byrne et al. [8]:

$$\text{Find } x^* \in H_1 \text{ such that } x^* \in \text{VI}(G_1; H_1) \text{ and } Ax^* \in \text{VI}(G_2; H_2), \quad (4)$$

where $\text{VI}(G_1; H_1) = \{y \in H_1 : 0 \in G_1(y)\}$ and $\text{VI}(G_2; H_2) = \{z \in H_2 : 0 \in G_2(z)\}$, G_1, G_2 are the same as in (2). We denote the solution set of S_p VIP by Δ . Moreover, Byrne et al. [8] presented the following iterative algorithm to find the solution of S_p VIP. Let $\lambda > 0$, and select a starting point $x_0 \in H_1$. Then, compute

$$x_{n+1} = R_\lambda^{G_1} [x_n + \gamma A^*(R_\lambda^{G_2} - I)Ax_n], \quad (5)$$

where A^* is the adjoint operator of A , $L = \|A^*A\|$, $\gamma \in (0, 2/L)$ and $R_\lambda^{G_1}, R_\lambda^{G_2}$ are the resolvents of monotone mappings G_1, G_2 , respectively. It can be easily seen that x^* solves S_p VIP if and only if $x^* = R_\lambda^{G_1}[x^* + \gamma A^*(I - R_\lambda^{G_2})Ax^*]$. Kazmi and Rizwi [9] proposed the following iterative method for approximating the common solutions of S_p VIP and fixed point problem of a nonexpansive mapping:

$$\begin{aligned} y_n &= R_\lambda^{G_1} [x_n + \gamma A^*(R_\lambda^{G_2} - I)Ax_n], \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Sy_n, \end{aligned} \quad (6)$$

where f is a contraction and S is nonexpansive mapping. Later, Sitthithakerngkiet et al. [10] studied the common solutions of S_p VIP and a fixed point of an infinite family of nonexpansive mappings and introduced the following iterative method:

$$\begin{aligned} y_n &= R_\lambda^{G_1} [x_n + \gamma A^*(R_\lambda^{G_2} - I)Ax_n], \\ x_{n+1} &= \alpha_n \xi u + \beta_n x_n + [(1 - \beta_n)I - \alpha_n D]W_n y_n, \quad \forall n \geq 1, \end{aligned} \quad (7)$$

where $u \in H_1$ is a given point and W_n is W -mapping which is generated by an infinite family of nonexpansive mappings. Similar results related to S_p VIP can be found in [11–17].

The common figure among the above-explained iterative methods is that they used the resolvent of associated monotone mappings; secondly, the step size depends on the operator norm $\|A^*A\|$. To avoid this obstacle, self-adaptive step size iterative algorithms have been introduced (see, for example, [18–24]). Lopez et al. [20] introduced a relaxed method for solving split feasibility problem with self-adaptive step size. Recently, Dilshad et al. [25] proposed two iterative algorithms to solve S_p VIP in which the precalculation of the operator norm $\|A^*A\|$ is not required. They studied the weak and strong convergence of the proposed methods to approximate the solution of S_p VIP with the step size $\gamma_n = (\|x_n - R_\lambda^{G_1}x_n\|^2 + \|A^*(I - R_\lambda^{G_2})Ax_n\|^2) / (\|x_n - R_\lambda^{G_1}x_n + A^*(I - R_\lambda^{G_2})Ax_n\|^2)$, which do not depend upon the precalculated operator norm.

The resolvent of a maximal monotone operator G is defined as $J_\lambda^G = (I + \lambda G)^{-1}$, where λ is a positive real number. A resolvent operator of maximal monotone operator is single valued and firmly nonexpansive. Due to the fact that the zeros of maximal monotone operator are the fixed point sets of resolvent operator, the resolvent associated with a set-valued maximal monotone operator plays an important role to find the zeros of monotone operators. Following Byrne's iterative method (5), which is mainly based on the resolvents of monotone mappings, many researchers introduced and studied various iterative methods for S_p VIP (see, for example, [7–9, 18, 25, 26] and references therein).

Yosida approximation operator for a monotone mapping G and parameter $\lambda > 0$ is defined as $J_\lambda^G = (1/\lambda)(I - R_\lambda^G)$. It is well known that set-valued monotone operator can be regularized into a single-valued monotone operator by the process known as the Yosida approximation. Yosida approximation is a tool to solve a variational inclusion problem using nonexpansive resolvent operator and has been used to solve various variational inclusions and system of variational inclusions in linear and nonlinear spaces (see, for example, [18, 25–30]).

Due to the fact that the zero of Yosida approximation operator associated with monotone operator G is the zero of inclusion problem $0 \in G(x)$ and inspired by the work of Moudafi, Byrne, Kazmi, and Dilshad et al., our motive is to propose two iterative methods to solve S_p MVIP. The rest of the paper is organized as follows.

The next section contains some fundamental results and preliminaries. In Section 3, we describe two iterative algorithms using Yosida approximation of monotone mappings G_1 and G_2 . Section 4 is devoted to the study of weak and strong convergence of the proposed iterative methods to the solution of S_p MVIP. In the last section, we give a numerical example in support of our main results and show the convergence of sequence obtained from the proposed algorithm to the solution of S_p MVIP.

2. Preliminaries

Let H be a real Hilbert space endowed with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. The strong and weak convergence of a sequence $\{x_n\}$ to x is denoted by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. The operator $T : H \rightarrow H$ is said to be a contraction if $\forall x, y \in H, \|T(x) - T(y)\| \leq \kappa \|x - y\|, \kappa \in (0, 1)$; if $\kappa = 1$, then T is called nonexpansive and firmly nonexpansive if $\forall x, y \in H, \|T(x) - T(y)\|^2 \leq \langle x - y, Tx - Ty \rangle$; T is called τ -inverse strongly monotone if there exists $\tau > 0$ such that $\langle T(x) - T(y), x - y \rangle \geq \tau \|T(x) - T(y)\|^2$.

For some $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \quad (8)$$

$P_C x$ is called the projection of x onto $C \subset H$, which satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (9)$$

Moreover, $P_C x$ is also characterized by the fact that

$$P_C x = z \Leftrightarrow \langle x - z, y - z \rangle \geq 0, \quad y \in C. \quad (10)$$

In Hilbert spaces, the following equality and inequality hold for all $x, y, z \in H, \alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma = 1$

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 &= \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta \|x - y\|^2 \\ &\quad - \beta \gamma \|y - z\|^2 - \gamma \alpha \|x - z\|^2, \end{aligned} \quad (11)$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \quad (12)$$

Let $G : H \rightarrow 2^H$ be a set-valued operator. The graph of G is defined by $\{(x, y) : y \in G(x)\}$, and inverse of G is denoted by $G^{-1} = \{(y, x) : y \in G(x)\}$. A set-valued mapping G is said to be monotone if $\langle u - v, x - y \rangle \geq 0$, for all $u \in G(x), v \in G(y)$. A monotone operator G is called a maximal monotone if there exists no other monotone operator such that its graph properly contains the graph of G .

Lemma 1 (see [31]). *If $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \beta_n) a_n + \delta_n, \quad n \geq 0, \quad (13)$$

where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in \mathbb{R} such that

$$(i) \sum_{n=1}^{\infty} \beta_n = \infty$$

$$(ii) \limsup_{n \rightarrow \infty} \delta_n / \beta_n \leq 0 \text{ or } \limsup_{n \rightarrow \infty} |\delta_n| < \infty$$

$$\text{then } \lim_{n \rightarrow \infty} a_n = 0.$$

Lemma 2 (see [32]). *Let H be a Hilbert space. A mapping $F : H \rightarrow H$ is τ -inverse strongly monotone if and only if $I - \tau F$ is firmly nonexpansive, for $\tau > 0$.*

Lemma 3 (see [33]). *Let H be a Hilbert space and $\{x_n\}$ be a bounded sequence in H . Assume there exists a nonempty subset $C \subset H$ satisfying the properties*

$$(i) \lim_{n \rightarrow \infty} \|x_n - z\| \text{ exists for every } z \in C$$

$$(ii) \omega_w(x_n) \subset C$$

Then, there exists $x^ \in C$ such that $\{x_n\}$ converges weakly to x^* .*

Lemma 4 (see [34]). *Let Γ_n be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence Γ_{n_k} of Γ_n such that $\Gamma_{n_k} < \Gamma_{n_k+1}$ for all $k \geq 0$. Also, consider the sequence of integers $\{\sigma(n)\}_{n \geq n_0}$ defined by*

$$\sigma(n) = \max \{k \leq n : \Gamma_k \leq \Gamma_{k+1}\}. \quad (14)$$

Then, $\{\sigma(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \rightarrow \infty} \sigma(n) = \infty$ and for all $n \geq n_0$,

$$\max \{\Gamma_{\sigma(n)}, \Gamma_{(n)}\} \leq \Gamma_{\sigma(n)+1}. \quad (15)$$

3. Yosida Approximation Iterative Methods

Let $V_1 : H_1 \rightarrow H_1, V_2 : H_2 \rightarrow H_2$ be single-valued monotone mappings and $G_1 : H_1 \rightarrow 2^{H_1}, G_2 : H_2 \rightarrow 2^{H_2}$ be set-valued mappings such that $V_1 + G_1 : H_1 \rightarrow 2^{H_1}$ and $V_2 + G_2 : H_2 \rightarrow 2^{H_2}$ are set-valued maximal monotone mappings; $R_{\lambda_1}^{V_1+G_1}, R_{\lambda_2}^{V_2+G_2}$ and $J_{\lambda_1}^{V_1+G_1}, J_{\lambda_2}^{V_2+G_2}$ are the resolvents and Yosida approximation operators of $V_1 + G_1$ and $V_2 + G_2$, respectively. We propose the following iterative methods to approximate the solution of $S_p\text{MVIP}$.

Algorithm 1. For an arbitrary x_0 , compute the $n + 1^{\text{th}}$ iteration as follows:

$$\begin{aligned} u_n &= x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n), \\ x_{n+1} &= u_n - \mu_n A^* J_{\lambda_2}^{V_2+G_2}(Au_n), \end{aligned} \quad (16)$$

where γ_n and μ_n are defined as

$$\gamma_n = \begin{cases} \frac{\tau_n \|J_{\lambda_1}^{V_1+G_1}(x_n)\|}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|}, & \text{if } \|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\| \neq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (17)$$

$$\mu_n = \begin{cases} \frac{\tau_n \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|}, & \text{if } \|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\| \neq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (18)$$

where $\lambda_1 > 0$, $\lambda_2 > 0$ and $\theta = \min \{2\lambda_1, 2\lambda_2\}$ such that $\tau_n \in (0, \theta)$.

$$\begin{aligned} \gamma_n &= \begin{cases} \frac{\tau_n \|J_{\lambda_1}^{V_1+G_1}(x_n)\|}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|}, & \text{if } \|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\| \neq 0, \\ 0, & \text{otherwise,} \end{cases} \\ \mu_n &= \begin{cases} \frac{\tau_n \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|}, & \text{if } \|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\| \neq 0, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (20)$$

where $\alpha_n, \beta_n \in (0, 1)$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\theta = \min \{2\lambda_1, 2\lambda_2\}$ such that $\tau_n \in (0, \theta)$.

4. Main Results

We assume that the problem $S_p\text{MVIP}$ is consistent and the solution set is denoted by Δ .

First, we prove following lemmas, which are used in the proof of our main results.

Lemma 5. Let $V_1 : H_1 \rightarrow H_1$ be single-valued monotone mappings and $G_1 : H_1 \rightarrow 2^{H_1}$ be set-valued mappings such that $V_1 + G_1 : H_1 \rightarrow 2^{H_1}$ be set-valued maximal monotone mapping. If $R_{\lambda_1}^{V_1+G_1}$ and $J_{\lambda_1}^{V_1+G_1}$ are the resolvent and Yosida approximation operators of $V_1 + G_1$, respectively, then for $\lambda_1 > 0$, following are equivalent:

- (i) $x^* \in H_1$ is the solution of $(V_1 + G_1)^{-1}(0)$
- (ii) $R_{\lambda_1}^{V_1+G_1}(x^*) = x^*$
- (iii) $J_{\lambda_1}^{V_1+G_1}(x^*) = 0$

Proof. The proof is trivial which is an immediate consequence of definitions of resolvent and Yosida approximation operator of maximal monotone mapping $V_1 + G_1$. \square

Algorithm 2. For an arbitrary x_0 , compute the $n + 1^{\text{th}}$ iteration as follows:

$$\begin{aligned} u_n &= x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n), \\ v_n &= u_n - \mu_n A^* J_{\lambda_2}^{V_2+G_2}(Au_n), \\ x_{n+1} &= (1 - \beta_n)u_n + \alpha_n(v_n - u_n). \end{aligned} \quad (19)$$

where γ_n and μ_n are defined as

Theorem 6. Let H_1, H_2 be real Hilbert spaces; $V_1 : H_1 \rightarrow H_1$, $V_2 : H_2 \rightarrow H_2$ be single-valued monotone mappings, $G_1 : H_1 \rightarrow 2^{H_1}$, $G_2 : H_2 \rightarrow 2^{H_2}$ be set-valued maximal monotone mappings such that $V_1 + G_1$ and $V_2 + G_2$ are maximal monotone, and $A : H_1 \rightarrow H_2$ be a bounded linear operator. Assume that $\theta = \min \{2\lambda_1, 2\lambda_2\}$ such that $\inf \tau_n(\theta - \tau_n) > 0$. Then, the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point $z \in \Delta$.

Proof. Let $z \in \Delta$. Since the Yosida approximation operator $J_{\lambda_1}^{V_1+G_1}$ is λ_1 -inverse strongly monotone, for $\lambda_1 > 0$, then by Algorithm 1 and (12), we have

$$\begin{aligned} \|u_n - z\|^2 &= \|x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n) - z\|^2 \\ &= \|x_n - z\|^2 + \gamma_n^2 \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^2 \\ &\quad - 2\gamma_n \langle J_{\lambda_1}^{V_1+G_1}(x_n), x_n - z \rangle \\ &\leq \|x_n - z\|^2 + \gamma_n^2 \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^2 \\ &\quad - 2\gamma_n \lambda_1 \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^2 = \|x_n - z\|^2 \\ &\quad + (\gamma_n^2 - 2\gamma_n \lambda_1) \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^2. \end{aligned} \quad (21)$$

Now, using (17), we estimate that

From (21) and (22), we get

(12), we estimate

$$\begin{aligned}
 (\gamma_n^2 - 2\gamma_n\lambda_1) \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^2 &= \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^2 \left[\frac{\tau_n^2 \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^2}{\left(\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|\right)^2} - \frac{2\tau_n\lambda_1 \|J_{\lambda_1}^{V_1+G_1}(x_n)\|}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|} \right] \\
 &= \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3 \left[\frac{\tau_n^2 \|J_{\lambda_1}^{V_1+G_1}(x_n)\| - 2\lambda_1\tau_n (\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|)}{\left(\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|\right)^2} \right] \\
 &\leq \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3 \left[\frac{(\tau_n^2 - 2\lambda_1\tau_n) (\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|)}{\left(\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|\right)^2} \right] \\
 &= \frac{(\tau_n^2 - 2\lambda_1\tau_n) \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|}.
 \end{aligned} \tag{22}$$

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 + \frac{(\tau_n^2 - 2\lambda_1\tau_n) \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|}. \tag{23}$$

Since $J_{\lambda_2}^{V_2+G_2}$ is λ_2 -inverse strongly monotone and using

$$\begin{aligned}
 \|x_{n+1} - z\|^2 &= \|u_n - \mu_n J_{\lambda_2}^{V_2+G_2}(Au_n) - z\|^2 \\
 &\leq \|u_n - z\|^2 + \mu_n^2 \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^2 \\
 &\quad - 2\mu_n \langle J_{\lambda_2}^{V_2+G_2}(Au_n), u_n - z \rangle \\
 &= \|u_n - z\|^2 + \mu_n^2 \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^2 \\
 &\quad - 2\mu_n \|J_{\lambda_1}^{V_1+G_1}(Au_n)\|^2 = \|u_n - z\|^2 \\
 &\quad + (\mu_n^2 - 2\mu_n\lambda_2) \|J_{\lambda_1}^{V_2+G_2}(Au_n)\|^2.
 \end{aligned} \tag{24}$$

By (18), it turns out that

$$\begin{aligned}
 (\mu_n^2 - 2\mu_n\lambda_2) \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^2 &= \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^2 \left[\frac{\tau_n^2 \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^2}{\left(\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|\right)^2} - \frac{2\tau_n\lambda_2 \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|} \right] \\
 &= \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3 \left[\frac{\tau_n^2 \|J_{\lambda_2}^{V_2+G_2}(Au_n)\| - 2\lambda_2\tau_n (\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|)}{\left(\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|\right)^2} \right] \\
 &= \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3 \left[\frac{(\tau_n^2 - 2\lambda_2\tau_n) (\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|)}{\left(\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|\right)^2} \right] \\
 &= \frac{(\tau_n^2 - 2\lambda_2\tau_n) \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|}.
 \end{aligned} \tag{25}$$

It follows from (24) and (25) that

$$\|x_{n+1} - z\|^2 \leq \|u_n - z\|^2 + \frac{(\tau_n^2 - 2\lambda_2 \tau_n) \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|}. \quad (26)$$

Combining (23) and (26), we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \|x_n - z\|^2 - \frac{\tau_n(2\lambda_1 - \tau_n) \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|} \\ &\quad - \frac{\tau_n(2\lambda_2 - \tau_n) \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|}, \end{aligned} \quad (27)$$

$$\leq \|x_n - z\|, \quad (28)$$

which implies that $\{x_n\}$ is Fejér monotone with respect to Δ and hence bounded, which assures that $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists for all $z \in \Delta$. Keeping in mind that $\theta = \min \{2\lambda_1, 2\lambda_2\}$, from (27), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \tau_n(\theta - \tau_n) &\left[\frac{\|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|} \right. \\ &\quad \left. + \frac{\|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|} \right] < \infty. \end{aligned} \quad (29)$$

Due to the assumption that $\inf \tau_n(\theta - \tau_n) > 0$ and the properties of convergent series, we conclude that

$$\lim_{n \rightarrow \infty} \|J_{\lambda_1}^{V_1+G_1}(x_n)\| = \lim_{n \rightarrow \infty} \|J_{\lambda_2}^{V_2+G_2}(Au_n)\| = 0. \quad (30)$$

Hence, there exist constants K_1 and K_2 such that

$$\|J_{\lambda_1}^{V_1+G_1}(x_n)\| \leq K_1, \quad \|J_{\lambda_2}^{V_2+G_2}(Au_n)\| \leq K_2. \quad (31)$$

By Algorithm 1 and (30), we get

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_n\| + \|u_n - x_n\| \leq K_1 \gamma_n + K_2 \mu_n \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (32)$$

Let $\{x^*\} \in \omega_w(x_n)$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges weakly to $\{x^*\}$, which implies that $\{x_{n_k}\}$ and $\{u_{n_k}\}$ also converge to $\{x^*\}$. Recall that $J_{\lambda_1}^{V_1+G_1}$ is λ_1 -inverse strongly monotone and $\{x_{n_k}\}$ converges to x^* , and using

(30), we get

$$\langle J_{\lambda_1}^{V_1+G_1}(x_{n_k}) - J_{\lambda_1}^{V_1+G_1}(x^*), x_{n_k} - x^* \rangle \geq \lambda_1 \|J_{\lambda_1}^{V_1+G_1}(x_{n_k}) - J_{\lambda_1}^{V_1+G_1}(x^*)\|^2. \quad (33)$$

Taking limit $k \rightarrow \infty$, we obtain $J_{\lambda_1}^{V_1+G_1}(x^*) = 0$.

Replacing $J_{\lambda_1}^{V_1+G_1}$ by $J_{\lambda_2}^{V_2+G_2}A$, x_{n_k} by Au_{n_k} with the same arguments, we get $J_{\lambda_2}^{V_2+G_2}A(x^*) = 0$. This completes the proof. \square

Theorem 7. Let H_1, H_2 be real Hilbert spaces; $V_1 : H_1 \rightarrow H_1$, $V_2 : H_2 \rightarrow H_2$ be single-valued monotone mappings, $G_1 : H_1 \rightarrow 2^{H_1}$, $G_2 : H_2 \rightarrow 2^{H_2}$ be set-valued maximal monotone mappings such that $V_1 + G_1$ and $V_2 + G_2$ are maximal monotone, and $A : H_1 \rightarrow H_2$ be a bounded linear operator. If $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$ and $\theta = \min \{2\lambda_1, 2\lambda_2\}$ such that $\tau_n \in (0, \theta)$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \beta_n &= 0, \\ \sum_{n=0}^{\infty} \beta_n &= \infty, \\ \lim_{n \rightarrow \infty} (1 - \alpha_n)\alpha_n &> 0, \\ \inf_n \tau_n(\theta - \tau_n) &> 0, \end{aligned} \quad (34)$$

then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $z = P_{\Delta}(0)$.

Proof. Let $z = P_{\Delta}(0)$; then, from (23) and (26) of the proof of Theorem 6, we have

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 + \frac{(\tau_n^2 - 2\lambda_1 \tau_n) \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|}, \quad (35)$$

$$\|v_n - z\|^2 \leq \|u_n - z\|^2 + \frac{(\tau_n^2 - 2\lambda_2 \tau_n) \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|}. \quad (36)$$

Since $\tau_n \leq \min \{2\lambda_1, 2\lambda_2\}$, we get $\|v_n - z\| \leq \|u_n - z\| \leq \|x_n - z\|$. From Algorithm 2, we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1 - \beta_n)u_n + \alpha_n(v_n - u_n) - z\| \\ &\leq (1 - \alpha_n - \beta_n)\|u_n - z\| + \|\alpha_n\|v_n - z\| + \beta_n\|u_n - z\| \\ &\leq (1 - \beta_n)\|x_n - z\| + \beta_n\|z\| \leq \max \{\|x_n - z\|, \|z\|\} \\ &\leq \dots \leq \max \{\|x_0 - z\|, \|z\|\}, \end{aligned} \quad (37)$$

which implies that the sequence $\{x_n\}$ is bounded and hence, the sequences $\{u_n\}, \{v_n\}, \{J_{\lambda_1}^{V_1+G_1}(u_n)\}$ and $\{A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\}$

are also bounded. Now,

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(1 - \beta_n)u_n + \alpha_n(v_n - u_n) - z\|^2 \\ &\leq (1 - \alpha_n - \beta_n)\|u_n - z\|^2 + \alpha_n\|v_n - z\|^2 \\ &\quad + \beta_n\|z\|^2 - \alpha_n(1 - \alpha_n - \beta_n)\|v_n - u_n\|^2. \end{aligned} \quad (38)$$

Combining (35), (36), and (38), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1 - \alpha_n - \beta_n) \left[\|x_n - z\|^2 + \frac{(\tau_n^2 - 2\lambda_1\tau_n) \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\|} \right] \\ &\quad + \alpha_n \left[\|u_n - z\|^2 + \frac{(\tau_n^2 - 2\lambda_2\tau_n) \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\|} \right] \\ &\quad + \beta_n\|z\|^2 - \alpha_n(1 - \alpha_n - \beta_n)\|v_n - u_n\|^2 \leq \|x_n - z\|^2 \\ &\quad + \beta_n(-\|x_n - z\|^2 + \|z\|^2) - \alpha_n(1 - \alpha_n - \beta_n)\|v_n - u_n\|^2 \\ &\quad - \frac{(1 - \alpha_n - \beta_n)(\tau_n^2 - 2\lambda_1\tau_n) \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\|} \\ &\quad - \frac{\alpha_n(\tau_n^2 - 2\lambda_2\tau_n) \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\|}. \end{aligned} \quad (39)$$

We discuss the two possible cases.

Case 1. If the sequence $\{\|x_n - z\|\}$ is nonincreasing, then there exists a number $k \geq 0$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$, for each $n \geq k$. Then, $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists and hence, $\lim_{n \rightarrow \infty} (\|x_{n+1} - z\| - \|x_n - z\|) = 0$. Thus, it follows from (39) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|v_n - u_n\| &= 0, \quad \lim_{n \rightarrow \infty} \|J_{\lambda_1}^{V_1+G_1}(x_n)\| = 0, \\ \lim_{n \rightarrow \infty} \|J_{\lambda_2}^{V_2+G_2}(Au_n)\| &= 0. \end{aligned} \quad (40)$$

From (40), we conclude that $\lim_{n \rightarrow \infty} \gamma_n = \lim_{n \rightarrow \infty} \mu_n = 0$ and $\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0$. We observe from Algorithm 2 that $x_{n+1} - u_n = \alpha_n(v_n - u_n) + \gamma_n u_n \rightarrow 0$; thus,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - u_n\| + \|u_n - x_n\| \leq \|v_n - u_n\| \\ &\quad + \gamma_n\|u_n\| + \|u_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (41)$$

This shows that the sequence $\{x_n\}$ is asymptotically regular. By Theorem 6, we have that $\omega_w(x_n) \subset \Delta$. Setting $z_n = (1 - \alpha_n)u_n + \alpha_nv_n$ and rewriting $x_{n+1} = (1 - \beta_n)z_n + \alpha_n\beta_n(v_n - u_n)$, we have

$$\begin{aligned} \|z_n - z\| &= \|(1 - \alpha_n)u_n + \alpha_nv_n - z\| \\ &\leq (1 - \alpha_n)\|u_n - z\| + \alpha_n\|v_n - z\| \leq \|x_n - z\|. \end{aligned} \quad (42)$$

From (42) and Algorithm 2, we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(1 - \beta_n)(z_n - z) + \beta_n(\alpha_n(v_n - u_n) - z)\|^2 \\ &\leq (1 - \beta_n)^2\|z_n - z\|^2 + 2\beta_n\langle \alpha_n(v_n - u_n) - z, x_{n+1} - z \rangle \\ &\leq (1 - \beta_n)\|x_n - z\|^2 + 2\beta_n\{\alpha_n\langle v_n - u_n, x_{n+1} - z \rangle \\ &\quad + \langle -z, x_{n+1} - z \rangle\}, \end{aligned} \quad (43)$$

or

$$a_{n+1} = (1 - \beta_n)a_n + b_n, \quad (44)$$

where $a_n = \|x_n - z\|$, $b_n = 2\beta_n\{\alpha_n\langle v_n - u_n, x_{n+1} - z \rangle + \langle -z, x_{n+1} - z \rangle\}$.

Since $\omega_w(x_n) \subset \Delta$ and $z = P_\Delta(0)$, then using (40), we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{b_n}{\beta_n} &= \limsup_{n \rightarrow \infty} \{2\alpha_n\langle v_n - u_n, x_{n+1} - z \rangle \\ &\quad + \langle -z, x_{n+1} - z \rangle\} = \limsup_{n \rightarrow \infty} \langle -z, x_{n+1} - z \rangle \leq 0. \end{aligned} \quad (45)$$

Thus, by Lemma 1, we obtain $x_n \rightarrow z$.

Case 2. If the sequence $\{\|x_n - z\|\}$ is not nonincreasing, we can select a subsequence $\{\|x_{n_k} - z\|\}$ of $\{\|x_n - z\|\}$ such that $\|x_{n_k} - z\| \leq \|x_n - z\|$ for all $k \in \mathbb{N}$. In this case, we define a subsequence of positive integers $\sigma(n) \rightarrow \infty$ with the properties

$$\begin{aligned} \|x_{\sigma(n)} - z\| &< \|x_{\sigma(n)+1} - z\|, \\ \max \left\{ \|x_{\sigma(n)} - z\|, \|x_n - z\| \right\} &\leq \|x_{\sigma(n)+1} - z\|. \end{aligned} \quad (46)$$

If $\|x_{n+1} - z\| > \|x_n - z\|$ for some $n \geq 0$, then it follows from (39) that

$$\begin{aligned} \alpha_n(1 - \alpha_n - \beta_n)\|v_n - u_n\|^2 &+ \frac{(1 - \alpha_n - \beta_n)(\tau_n^2 - 2\lambda_1\tau_n) \|J_{\lambda_1}^{V_1+G_1}(x_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\|} \\ &+ \frac{\alpha_n(\tau_n^2 - 2\lambda_2\tau_n) \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|^3}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\|} \leq \beta_n(\|z\|^2 - \|x_n - z\|^2). \end{aligned} \quad (47)$$

Replacing n by $\sigma(n)$ and taking limit $n \rightarrow \infty$, we get the following relation for the subsequences $\{x_{\sigma(n)}\}$, $\{u_{\sigma(n)}\}$, and $\{v_{\sigma(n)}\}$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \|v_{\sigma(n)} - u_{\sigma(n)}\| &= 0, \\ \lim_{n \rightarrow \infty} \|J_{\lambda_1}^{V_1+G_1}(x_{\sigma(n)})\| &= 0, \\ \lim_{n \rightarrow \infty} \|J_{\lambda_2}^{V_2+G_2}(Au_{\sigma(n)})\| &= 0 \end{aligned} \quad (48)$$

Thus, we have $\|x_{\sigma(n+1)} - x_{\sigma(n)}\| \rightarrow 0$, as $n \rightarrow \infty$ and $\omega_w(x_{\sigma(n)}) \subset \Delta$. It is remaining to show that $x_n \rightarrow z$.

Replacing n by $\sigma(n)$ in (47), using $\|x_{\sigma(n)} - z\| < \|x_{\sigma(n)+1} - z\|$ and boundedness of $\|x_n - z\|$, we have

$$\|x_{\sigma(n)} - z\|^2 \leq M \|v_{\sigma(n)} - u_{\sigma(n)}\| + 2 \langle -z, x_{\sigma(n)+1} - z \rangle. \quad (49)$$

Since $z = P_\Delta(0)$, $\omega(x_{\sigma(n)}) \subset \Delta$ with using $\|v_{\sigma(n)} - u_{\sigma(n)}\| \rightarrow 0$ and $\|x_{\sigma(n)+1} - x_{\sigma(n)}\| \rightarrow 0$, we have

$$\limsup_{n \rightarrow \infty} \langle -z, x_{\sigma(n)+1} - z \rangle = \limsup_{n \rightarrow \infty} \langle -z, x_{\sigma(n)} - z \rangle = \max_{r \in \omega_w(x_{\sigma(n)})} \langle -z, r - z \rangle \leq 0. \quad (50)$$

From (49) and (52), we conclude that $x_{\sigma(n)} \rightarrow z$ and

$$\|x_n - z\| \leq \|x_{\sigma(n)+1} - z\| \leq \|x_{\sigma(n)+1} - x_{\sigma(n)}\| + \|x_{\sigma(n)} - z\| \rightarrow 0, \quad (51)$$

that is, $x_n \rightarrow z$. This complete the proof. \square

For $\tau_n = 1$, we have the following result for the convergence of Algorithm 2.

Corollary 8. Let $H_1, H_2, V_1, V_2, G_1, G_2$, and A, A^* be the same as defined in Theorem 7. If $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0, 1)$ and assuming that $\lambda_1 > 1/2$ and $\lambda_2 > 1/2$ satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} \beta_n &= 0, \quad \sum_{n=0}^{\infty} \beta_n = \infty, \\ \lim_{n \rightarrow \infty} (1 - \alpha_n) \alpha_n &> 0, \end{aligned} \quad (52)$$

then the sequence $\{x_n\}$ generated by Algorithm 2 (with $\tau_n = 1$) converges strongly to $z = P_\Delta(0)$.

For $\beta_n = 0$, we have the following corollary for the convergence of Algorithm 2.

Corollary 9. Let $H_1, H_2, V_1, V_2, G_1, G_2$, and A, A^* be the same as defined in Theorem 7. If $\{\alpha_n\}$ is a sequence in $(0, 1)$ and assuming that $\theta = \min \{2\lambda_1, 2\lambda_2\}$ such that $\tau_n \in (0, \theta)$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} (1 - \alpha_n) \alpha_n &> 0, \\ \inf_n \tau_n (\theta - \tau_n) &> 0, \end{aligned} \quad (53)$$

then the sequence $\{x_n\}$ generated by the iterative method

$$\begin{aligned} u_n &= x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n), \\ v_n &= u_n - \mu_n A^* J_{\lambda_2}^{V_2+G_2}(Au_n), \\ x_{n+1} &= (1 - \alpha_n) u_n + \alpha_n v_n, \end{aligned} \quad (54)$$

where γ_n and μ_n are defined as in Algorithm 2 (with $\tau_n = 1$), converges strongly to $z \in \Delta$.

For $\tau_n = 1$ and $\beta_n = 0$, we have the following corollary for the convergence of Algorithm 2.

Corollary 10. Let $H_1, H_2, V_1, V_2, G_1, G_2$, and A, A^* be the same as defined in Theorem 7. If $\{\alpha_n\}$ be a sequence in $(0, 1)$ and assuming that

$$\begin{aligned} \lim_{n \rightarrow \infty} (1 - \alpha_n) \alpha_n &> 0, \\ \lambda_1 &> \frac{1}{2}, \\ \lambda_2 &> \frac{1}{2}, \end{aligned} \quad (55)$$

then the sequence $\{x_n\}$ generated by the iterative method

$$\begin{aligned} u_n &= x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n), \\ v_n &= u_n - \mu_n A^* J_{\lambda_2}^{V_2+G_2}(Au_n), \\ x_{n+1} &= (1 - \alpha_n) u_n + \alpha_n v_n, \end{aligned} \quad (56)$$

where γ_n and μ_n are defined in Algorithm 2 (with $\tau_n = 1$), converges strongly to $z \in \Delta$.

5. Numerical Example

Let $H_1 = H_2 = \mathbb{R}$ and $V_1 = V_2 = 0$; $G_1 : \mathbb{R} \rightarrow \mathbb{R}$, $G_2 : \mathbb{R} \rightarrow \mathbb{R}$ are defined as $G_1(x) = 2x + 3$ and $G_2(x) = 2(x + 1)$, respectively. One can easily check that G_1 and G_2 are monotone and the Yosida approximation operator of G_1 and G_2 for $\lambda_1 = \lambda_2 = 1$ is computed as

$$\begin{aligned} J_{\lambda_1}^{V_1+G_1}(x) &= \frac{2x+3}{3}, \\ J_{\lambda_2}^{V_2+G_2}(x) &= \frac{2x+2}{3}. \end{aligned} \quad (57)$$

Let $A : H_1 \rightarrow H_2$ be defined as $A(x) = 2x/3$, then, for $\tau_n = (2 - (e^{1/n}/2)) \in (0, 2)$, we compute the step size as

$$\begin{aligned} \gamma_n &= \frac{\tau_n \|J_{\lambda_1}^{V_1+G_1}(x_n)\|}{\|J_{\lambda_1}^{V_1+G_1}(x_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\|} = \left(2 - \frac{e^{1/n}}{2}\right) \frac{9}{13}, \\ \mu_n &= \frac{\tau_n \|J_{\lambda_2}^{V_2+G_2}(Au_n)\|}{\|J_{\lambda_1}^{V_1+G_1}(u_n)\| + \|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\|} = \left(2 - \frac{e^{1/n}}{2}\right) \frac{6}{13}. \end{aligned} \quad (58)$$

Then, for $\alpha_n = (2 - (e^{1/n}/3))$ and two different values of (β_n) (for example, $\beta_n = 1/(n+5)$ and $\beta_n = 1/(n+10)$) and for arbitrary x_0 (for example, $x_0 = -2$ and $x_0 = 0$), we compute the iterative sequences from Algorithm 2 as follows:

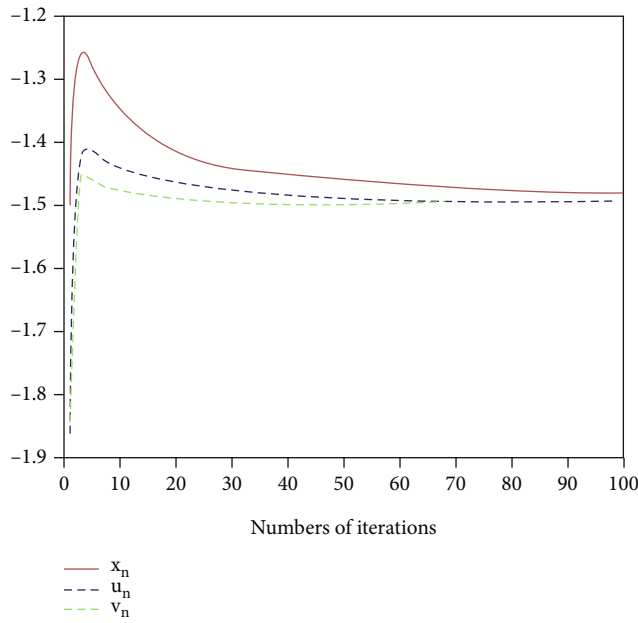


FIGURE 1: Convergence of iterative sequences $\{u_n\}$, $\{v_n\}$, and $\{x_n\}$ to $z = -1.5$ for $\beta_n = 1/n + 5$ and $x_0 = -2$.

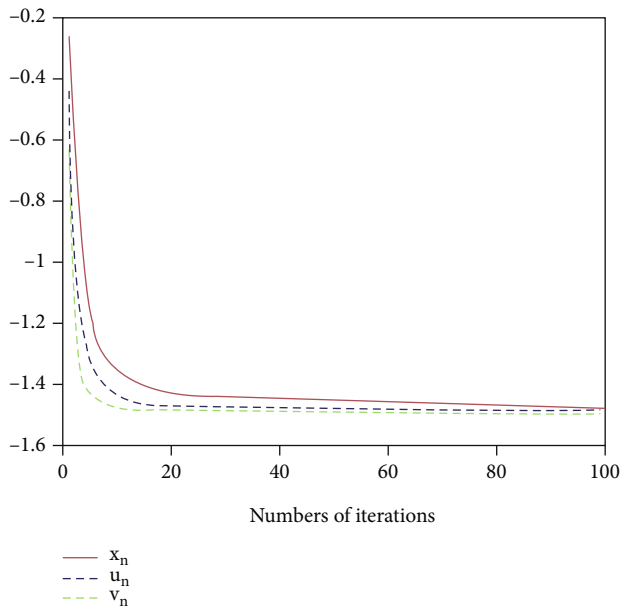


FIGURE 2: Convergence of iterative sequences $\{u_n\}$, $\{v_n\}$, and $\{x_n\}$ to $z = -1.5$ for $\beta_n = 1/n + 10$ and $x_0 = 0$.

$$\begin{aligned}
 u_n &= x_n - \left(2 - \frac{e^{1/n}}{2}\right) \frac{3}{13} (2x_n + 3), \\
 v_n &= u_n - \left(2 - \frac{e^{1/n}}{2}\right) \frac{8}{117} (2x_n + 3), \\
 x_{n+1} &= \left(1 - \frac{1}{n+5}\right) u_n + \left(2 - \frac{e^{1/n}}{3}\right) (v_n - u_n).
 \end{aligned} \tag{59}$$

In Figures 1 and 2, we show that the obtained sequences

$\{u_n\}$, $\{v_n\}$, and $\{x_n\}$ converge to $z = -(3/2)$ for randomly selected arbitrary values of $x_0 = -2$ and 0 .

6. Conclusions

We have proposed two iterative algorithms for S_p MVIP which are mainly based on the Yosida approximation operators. Since the zero of Yosida approximation of monotone mapping $V_1 + G_1$ is the solution of $(V_1 + G_1)^{-1}(0)$, we used the Yosida approximations of monotone mappings $V_1 + G_1$ and $V_2 + G_2$ to solve S_p MVIP. We proved the weak and strong convergence of the composed iterative algorithms to investigate the solution of S_p MVIP under some suitable assumptions such that the estimation of step size does not require any prior calculation of the operator norm $\|A^*A\|$. To show the accuracy and efficiency of our algorithms, we have present a numerical example and showed the convergence using different parameters.

Data Availability

We claim that this work is a theoretical result, and there is no available data source.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Common Fixed Point Theorems for Weakly Contractions in Rectangular b -Metric Spaces with Supportive Applications

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In this manuscript, two new classes of generalized weakly contractions are introduced and common fixed point results concerning the new contractions are proved in the context of rectangular b -metric spaces. Also, some examples are included to present the validity of our theorems. As an application, we provide the existence and uniqueness of solution of an integral equation.

1. Introduction

In the field of nonlinear analysis, Banach fixed point theorem, which is introduced by Banach [1], is a powerful and classical means to deal with problems on fixed points in metric spaces. It is widely used in many disciplines of mathematics and has been promoted in many aspects. One important extension is to extend the concept of metric spaces. b -metric spaces and rectangular metric spaces are regarded as two well-known generalizations of metric spaces.

As a extension of a metric space, b -metric space was firstly introduced by Czerwik [2], by modifying the third condition of metric function. In that paper, the author provided fixed point results for contraction conditions in this type space. Afterwards, some authors have obtained many excellent results concerning fixed point theory of a lot of new types of contractive mappings on b -metric spaces. Generalizing the results of Berinde [3], Zada et al. [4] obtained fixed point results for mappings with rational type and Pacurar [5] got fixed point theorems of φ -contractions. In [6], common fixed point results for weak φ -contraction mappings were proved in this type spaces by Aydi et al. In 2019, problems about periodic common fixed point were studied by Hussain et al. [7]. Recently, in [8], Gopal et al. explored the latest researches and developments on theory of fixed point in the framework of b -metric spaces. Younis

et al. [9] introduced new fixed point results for the underlying mappings in the framework of dislocated b -metric spaces. In [10], in b -metric-like spaces, the authors extended the concept of Kannan mappings in view of F -contraction. Lately, Younis et al. [11] presented the notion of graphical extended b -metric spaces and discussed the framework of an open ball in this new type space.

In 2000, by changing triangular inequality to quadrilateral inequality, more general inequality, Branciari [12] introduced the concept of rectangular metric spaces. Also, the author extended the Banach contraction mapping principle for this new context. Subsequently, a lot of fixed point theorems of various contractive conditions in rectangular metric spaces were obtained. Lakzian et al. [13] established fixed point theorems dealing with (ψ, ϕ) -weakly contraction conditions in this type space, which was ulteriorly extended by Erhan et al. in [14]. Bari and Vetro [15] got common fixed point results on given functions with (ψ, ϕ) -weakly contractive conditions. In [16], George and Rajagopalan studied problems of common fixed points of (ψ, ϕ) -contractive mappings. Lately, in complete rectangular metric spaces, Wang and Pei-Sheng [17] gave generalised θ -contraction mappings which can be regarded as generalized Suzuki-Berinde type θ -contraction mappings and provided conditions which ensured this type mapping possesses a unique fixed point. By the

help of C -functions, in [18], some fixed point results were established by Budhia et al. In graphical rectangular b -metric spaces, some errors from literature [19] were rectified by Younis et al. in [20].

Inspired by results of Czerwik [2] and Branciari [12], George et al. [21] extended b -metric space and rectangular metric space by introduced rectangular b -metric space. In that paper, the authors presented an analogue of Banach fixed point theorem and fixed point theorem of Kannan. After that, many researchers had solved problems of fixed point of new type of contractive mappings on this type space. Kadelburg and Radenovic [22] and Mitrovic [23] presented common fixed point theorems in this type space. In the setting of rectangular b -metric spaces, a Boyd-Wong type theorem was studied by Ding et al. in [24]. Sukprasert et al. [25] presented the concept of weak altering distance function and discussed fixed point result of a new generalized contractive mapping. Roshan et al. [26] gave some fixed point theorems concerning almost generalized weakly contractive mappings and rational type contractions. In [27], Mitrovic obtained an analogue of Banach contractive mapping principle and solved an open problem arose in [21].

Recently, Sunarsini et al. [28] introduced a new extension of metric space named as complex valued rectangular b -metric space and gave an example of Banach contractive mapping principle at linear equation system. In [29], common coupled fixed point theorems concerning generalised T -contraction conditions were studied by George and Reshma. Lately, in ordered partial rectangular b -metric spaces, Asim et al. [30] established some ordered-theoretic fixed point results of Geraghty-weak contractive mappings.

In 1997, by using the notion of weak contractive mappings, Alber et al. [31] extended Banach contraction mapping principle in Hilbert spaces. In [32], weak contraction principle was generalized to metric spaces by Rhoades. After that, many authors had generalised the weak contraction principle. For example, in [33], the authors obtained the fixed point results involving α - ψ contraction conditions and applied them to solve quadratic integral equations. In [34], Jamal et al. used (ψ, ϕ) -weak contraction to extend coincidence point theorems obtained in partially ordered b -metric spaces.

Set

$$\begin{aligned} \Psi &= \{\psi : [0, +\infty) \longrightarrow [0, +\infty) \text{ is a continuous and increasing function}\}, \\ \Phi &= \{\phi : [0, +\infty) \longrightarrow [0, +\infty) \text{ is a lower semi continuous and nondecreasing function and } \phi(t) = 0 \text{ if and only if } t = 0\}. \end{aligned} \quad (1)$$

Hao and Guan [35] proved common fixed point result dealing with a new class of generalized weakly contraction conditions in complete b -metric spaces as follows.

Theorem 1 (see [35]). *Let (\mathcal{E}, ρ) be a complete b -metric space with coefficient $s \geq 1$. Let $O, R : \mathcal{E} \longrightarrow \mathcal{E}$ be self-mappings such that R is injective and $O(\mathcal{E}) \subset R(\mathcal{E})$ where $R(\mathcal{E})$ is closed. Assume that $p \geq 2$ is a fixed number and $\varphi : \mathcal{E} \longrightarrow [0, +\infty)$ is lower semicontinuous. If there exist $\psi \in \Psi$ and $\phi \in \Phi$ satisfying*

$$\begin{aligned} &\psi(s^p[\rho(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta)]) \\ &\leq \psi(I(\xi, \eta, \rho, O, R, \varphi)) - \phi(J(\xi, \eta, \rho, O, R, \varphi)), \end{aligned} \quad (2)$$

where

$$\begin{aligned} I(\xi, \eta, \rho, O, R, \varphi) &= \max \left\{ \rho(R\xi, R\eta) + \varphi(R\xi) + \varphi(R\eta), \frac{1}{2} \{ \rho(O\xi, R\xi) + \varphi(O\xi) \right. \\ &\quad \left. + \varphi(R\xi) + \rho(O\eta, R\eta) + \varphi(O\eta) + \varphi(R\eta) \}, \frac{1}{2s} \{ \rho(O\xi, R\eta) \right. \\ &\quad \left. + \varphi(O\xi) + \varphi(R\eta) + \rho(O\eta, R\xi) + \varphi(O\eta) + \varphi(R\xi) \} \right\}, \\ J(\xi, \eta, \rho, O, R, \varphi) &= \max \{ \rho(R\xi, R\eta) + \varphi(R\xi) + \varphi(R\eta), \rho(O\eta, R\eta) \\ &\quad + \varphi(O\eta) + \varphi(R\eta) \}, \end{aligned} \quad (3)$$

then O and R possess a unique coincidence point in \mathcal{E} . Fur-

ther, if O and R are weakly compatible, then O and R have a unique common fixed point.

Continuing in the same direction, our aim is to give two new classes of generalized weakly contractions and establish some common fixed point theorems dealing with the new contractions in the setting of rectangular b -metric spaces. Moreover, we present some examples that elaborate the validity of our theorems. Also, as an application, we prove the existence of solution of an integral equation.

2. Preliminaries

First, we recall some definitions and lemmas as follows:

Definition 2 (see [2]). Let \mathcal{M} be a nonempty set and $s \geq 1$ be a constant. A function $\sigma : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ is said to be a b -metric iff

- (i) $\sigma(\xi, \eta) = 0$ iff $\xi = \eta$ for $\xi, \eta \in \mathcal{M}$
- (ii) $\sigma(\xi, \eta) = \sigma(\eta, \xi)$ for $\xi, \eta \in \mathcal{M}$
- (iii) there exists a real number $s \geq 1$ satisfying $\sigma(\xi, \eta) \leq s(\sigma(\xi, \vartheta) + \sigma(\eta, \vartheta))$ for $\xi, \eta, \vartheta \in \mathcal{M}$

Usually, we call (\mathcal{M}, σ) a b -metric space with coefficient $s \geq 1$.

Definition 3 (see [12]). Let \mathcal{M} be a nonempty set. A function $\rho : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ is said to be a rectangular metric iff

- (i) $\rho(\xi, \eta) = 0$ iff $\xi = \eta$ for $\xi, \eta \in \mathcal{M}$
- (ii) $\rho(\xi, \eta) = \rho(\eta, \xi)$ for $\xi, \eta \in \mathcal{M}$
- (iii) $\rho(\xi, \eta) \leq \rho(\xi, \kappa) + \rho(\kappa, \nu) + \rho(\nu, \eta)$ for $\xi, \eta \in \mathcal{M}$ and all different points $\kappa, \nu \in \mathcal{M} - \{\xi, \eta\}$

In general, we call (\mathcal{M}, ρ) a rectangular metric space.

Definition 4 (see [21]). Let \mathcal{M} be a nonempty set and $s \geq 1$ be a constant. A function $\mathfrak{q} : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ is said to be a rectangular b -metric iff

- (i) $\mathfrak{q}(\xi, \eta) = 0$ iff $\xi = \eta$ for $\xi, \eta \in \mathcal{M}$
- (ii) $\mathfrak{q}(\xi, \eta) = \mathfrak{q}(\eta, \xi)$ for $\xi, \eta \in \mathcal{M}$
- (iii) there exists a real number $s \geq 1$ satisfying $\mathfrak{q}(\xi, \eta) \leq s(\mathfrak{q}(\xi, \kappa) + \mathfrak{q}(\kappa, \nu) + \mathfrak{q}(\nu, \eta))$ for $\xi, \eta \in \mathcal{M}$ and all different points $\kappa, \nu \in \mathcal{M} - \{\xi, \eta\}$

As usual, we call $(\mathcal{M}, \mathfrak{q})$ a rectangular b -metric space with coefficient $s \geq 1$.

Remark 5. It is obvious that a rectangular metric function becomes a metric function when $\kappa = \nu$ and a rectangular b -metric function becomes a rectangular metric function when $s = 1$, whereas the converse of this statement may not be true (see [21], Examples 1.4 and 1.5).

Example 1. Let $\mathcal{M} = A \cup B$, where $A = \{1/2, 1/3, 1/4, 1/5\}$, $B = [1, 2]$. Define $\mathfrak{q} : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ with $\mathfrak{q}(\xi, \eta) = \mathfrak{q}(\eta, \xi)$ for $x, y \in X$ and

$$\left\{ \begin{array}{l} \mathfrak{q}\left(\frac{1}{2}, \frac{1}{3}\right) = \mathfrak{q}\left(\frac{1}{3}, \frac{1}{4}\right) = \mathfrak{q}\left(\frac{1}{4}, \frac{1}{5}\right) = 0.05, \\ \mathfrak{q}\left(\frac{1}{2}, \frac{1}{4}\right) = \mathfrak{q}\left(\frac{1}{3}, \frac{1}{5}\right) = 0.08, \\ \mathfrak{q}\left(\frac{1}{2}, \frac{1}{5}\right) = 0.6, \\ \mathfrak{q}(\xi, \eta) = |\xi - \eta|, \text{ otherwise.} \end{array} \right. \quad (4)$$

By calculation, we get $(\mathcal{M}, \mathfrak{q})$ is a rectangular b -metric space as $s = 4$, whereas we obtain the following results:

- (1) $(\mathcal{M}, \mathfrak{q})$ is not a metric space, as

$$\mathfrak{q}\left(\frac{1}{2}, \frac{1}{5}\right) = 0.6 > 0.13 = \mathfrak{q}\left(\frac{1}{2}, \frac{1}{3}\right) + \mathfrak{q}\left(\frac{1}{3}, \frac{1}{5}\right) \quad (5)$$

- (2) $(\mathcal{M}, \mathfrak{q})$ is not a rectangular metric space, as

$$\mathfrak{q}\left(\frac{1}{2}, \frac{1}{5}\right) = 0.6 > 0.15 = \mathfrak{q}\left(\frac{1}{2}, \frac{1}{3}\right) + \mathfrak{q}\left(\frac{1}{3}, \frac{1}{4}\right) + \mathfrak{q}\left(\frac{1}{4}, \frac{1}{5}\right) \quad (6)$$

- (3) $(\mathcal{M}, \mathfrak{q})$ is not a b -metric space with $s = 4$, as

$$\mathfrak{q}\left(\frac{1}{2}, \frac{1}{5}\right) = 0.6 > 0.52 = 4 \cdot \left(\mathfrak{q}\left(\frac{1}{2}, \frac{1}{3}\right) + \mathfrak{q}\left(\frac{1}{3}, \frac{1}{5}\right) \right) \quad (7)$$

Example 2. Assume $(\mathcal{M}, \mathfrak{q}^*)$ is a metric space. For $p \geq 2$, define $\mathfrak{q}(\xi, \eta) = (\mathfrak{q}^*(\xi, \eta))^p$. Then, $(\mathcal{M}, \mathfrak{q})$ is a rectangular b -metric space with parameter $s = 3^{p-1}$.

Proof. One can verify easily the conditions (i) and (ii) hold by definition of $\mathfrak{q}(\xi, \eta)$. In order to check (iii), we can infer from the following inequality:

$$(m + n + l)^p \leq 3^{p-1} (m^p + n^p + l^p), \text{ for any } m, n, l \geq 0 \text{ and } p \geq 2. \quad (8)$$

Then, for $\xi, \eta \in \mathcal{M}$ and all different points $\tau, \nu \in \mathcal{M} - \{\xi, \eta\}$, we have

$$\begin{aligned} \mathfrak{q}(\xi, \eta) &= (\mathfrak{q}^*(\xi, \eta))^p \leq (\mathfrak{q}^*(\xi, \tau) + \mathfrak{q}^*(\tau, \nu) + \mathfrak{q}^*(\nu, \eta))^p \\ &\leq 3^{p-1} (\mathfrak{q}(\xi, \tau) + \mathfrak{q}(\tau, \nu) + \mathfrak{q}(\nu, \eta)). \end{aligned} \quad (9)$$

That is, $(\mathcal{M}, \mathfrak{q})$ is a rectangular b -metric space when $s = 3^{p-1}$. \square

Definition 6 (see [21]). Let $(\mathcal{M}, \mathfrak{q})$ be a rectangular b -metric space with coefficient $s \geq 1$. A sequence $\{\xi_n\}$ in \mathcal{M} is called:

- (i) convergent sequence iff there is $\xi \in \mathcal{M}$ such that $\mathfrak{q}(\xi_n, \xi) \longrightarrow 0$ as $n \longrightarrow +\infty$
- (ii) a Cauchy sequence iff $\mathfrak{q}(\xi_n, \xi_m) \longrightarrow 0$ when $n, m \longrightarrow +\infty$

Furthermore, a rectangular b -metric space is called completeness iff every Cauchy sequence is convergent.

Remark 9. In rectangular b -metric spaces, one can show that the limit of a sequence may not unique and every convergent sequence in a rectangular b -metric space may not be a Cauchy sequence (see [21], Example 1.7).

Definition 8 (see [36]). Let O and R be two self-maps defined on a nonempty set \mathcal{M} . If $v = O\xi = R\xi$, for some $\xi \in \mathcal{M}$, then v is called the point of coincidence of O and R , where ξ is said to be the coincidence point of O and R . Let $C(O, R)$ represent the collection of all coincidence points of O and R .

Definition 9 (see [36]). Let O and R be two self-maps defined on a nonempty set \mathcal{M} . Then, O and R are called weakly compatible mappings when they commute at each coincidence point, i.e., $O\xi = R\xi \Rightarrow OR\xi = RO\xi$ for each $\xi \in C(O, R)$.

Lemma 10 (see [26]). Let $(\mathcal{M}, \mathcal{Q})$ be a rectangular b -metric space with parameter $s \geq 1$. Assume that $\{\xi_n\}$ and $\{\eta_n\}$ are convergent to ξ and η , respectively. Then, one can get

$$\frac{1}{s}\mathcal{Q}(\xi, \eta) \leq \liminf_{n \rightarrow +\infty} \mathcal{Q}(\xi_n, \eta_n) \leq \limsup_{n \rightarrow +\infty} \mathcal{Q}(\xi_n, \eta_n) \leq s\mathcal{Q}(\xi, \eta). \quad (10)$$

Moreover, if $\xi = \eta$, then we have $\lim_{n \rightarrow +\infty} \mathcal{Q}(\xi_n, \eta_n) = 0$. Further, for $\zeta \in \mathcal{M}$, we deduce

$$\frac{1}{s}\mathcal{Q}(\xi, \zeta) \leq \liminf_{n \rightarrow +\infty} \mathcal{Q}(\xi_n, \zeta) \leq \limsup_{n \rightarrow +\infty} \mathcal{Q}(\xi_n, \zeta) \leq s\mathcal{Q}(\xi, \zeta). \quad (11)$$

3. Main Results

In this section, a few of new common fixed point results on generalized weakly contractive conditions in a complete rectangular b -metric space will be presented. Moreover, two examples will be provided to prove the validity of our theorems.

Suppose $(\mathcal{M}, \mathcal{Q})$ is a rectangular b -metric space. A mapping $O : \mathcal{M} \rightarrow [0, +\infty)$ is named as a lower semicontinuous mapping if, for $\xi \in \mathcal{M}$ and $\{\xi_n\}$ is convergent to ξ , one get

$$O(\xi) \leq \liminf_{n \rightarrow +\infty} O(\xi_n). \quad (12)$$

Let Ω represent the set of all functions $\beta : \mathbb{R}_0^+ \rightarrow [0, 1]$. We shall consider the contractive conditions defined by the family Θ :

$$\Theta = \{\theta : [0, +\infty) \rightarrow [0, +\infty) \text{ is a continuous and increasing function, for all } \kappa > 0, \theta(\kappa) < \kappa \text{ and } \theta(\kappa) = 0 \text{ iff } \kappa = 0\}.$$

(13)

Lemma 11 (see [37]). Let $\theta : [0, +\infty) \rightarrow [0, +\infty)$ be a non-decreasing and upper semicontinuous mapping. Then, $\theta(x) < x$ for any $x > 0$ iff $\theta^n(x) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 12. Let (\mathcal{M}, ρ) be a rectangular b -metric space with coefficient $s \geq 1$. Let $\alpha : \mathcal{M} \times \mathcal{M} \rightarrow [0, +\infty)$ and $O, R : \mathcal{M} \rightarrow \mathcal{M}$ be given functions and $p \geq 2$ be a real number. A function $O : \mathcal{M} \rightarrow \mathcal{M}$ is called $R - \alpha_{s^p}$ -admissible function if, for all $\xi, \eta \in \mathcal{M}$, $\alpha(R\xi, R\eta) \geq s^p$ implies $\alpha(O\xi, O\eta) \geq s^p$.

Definition 13. Let $(\mathcal{M}, \mathcal{Q})$ be a rectangular b -metric space with coefficient $s \geq 1$. Let $\alpha : \mathcal{M} \times \mathcal{M} \rightarrow [0, +\infty)$ and O, R

$: \mathcal{M} \rightarrow \mathcal{M}$ be three given mappings. Suppose that $p \geq 2$ is a real number and $\varphi : \mathcal{M} \rightarrow [0, +\infty)$ is a lower semicontinuous function. A mapping O is called a generalized $(R - \alpha_{s^p}, \theta, \varphi)$ contractive mapping, if there exist $\theta \in \Theta, \beta \in \Omega$ and $L \geq 0, 1/s + L < 1$ satisfying

$$\begin{aligned} & \theta(\alpha(R\xi, R\eta)[\mathcal{Q}(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta)]) \\ & \leq \beta(\theta(h(\xi, \eta, \mathcal{Q}, O, R, \varphi)))\theta(h(\xi, \eta, \mathcal{Q}, O, R, \varphi)) \\ & + L\theta(q(\xi, \eta, \mathcal{Q}, O, R, \varphi)), \end{aligned} \quad (14)$$

for all $\xi, \eta \in \mathcal{M}$ with $\alpha(R\xi, R\eta) \geq s^p$ and $\mathcal{Q}(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta) \neq 0$, where

$$\begin{aligned} h(\xi, \eta, \mathcal{Q}, O, R, \varphi) = \max \bigg\{ & \mathcal{Q}(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta), \frac{\mathcal{Q}(O\eta, O\eta) + \varphi(O\eta) + \varphi(R\eta)}{1 + \mathcal{Q}(O\xi, R\xi) + \varphi(O\xi) + \varphi(R\xi)} \\ & \cdot \{\mathcal{Q}(R\xi, R\eta) + \varphi(R\xi) + \varphi(R\eta)\}, \frac{1}{2} \{\mathcal{Q}(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta) \\ & + \mathcal{Q}(R\xi, R\eta) + \varphi(R\xi) + \varphi(R\eta)\}, \\ & q(\xi, \eta, \mathcal{Q}, O, R, \varphi) = \frac{1}{2} \min \{\mathcal{Q}(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta), \mathcal{Q}(R\xi, R\eta) \\ & + \varphi(R\xi) + \varphi(R\eta)\}. \end{aligned} \quad (15)$$

Let $\alpha : \mathcal{M} \times \mathcal{M} \rightarrow [0, +\infty)$ be a mapping. Set

(A_{s^p}) If $\{\xi_n\}$ is a sequence in \mathcal{M} satisfying $R\xi_n \rightarrow R\xi$ as $n \rightarrow +\infty$, then there is a subsequence $\{R\xi_{n_k}\}$ of $\{R\xi_n\}$ with $\alpha(R\xi_{n_k}, R\xi) \geq s^p$ for $k \in \mathbb{N}$

(B_{s^p}) For $x, y \in C(O, R)$, one can get the condition of $\alpha(Rx, Ry) \geq s^p$ and $\alpha(Rx, Ry) \geq s^p$

Theorem 14. Let $(\mathcal{M}, \mathcal{Q})$ be a complete rectangular b -metric space with coefficient $s \geq 1$. Let $O, R : \mathcal{M} \rightarrow \mathcal{M}$ be given self-mappings satisfying $O(\mathcal{M}) \subset R(\mathcal{M})$ and $R(\mathcal{M})$ is closed. Assume that $\varphi : \mathcal{M} \rightarrow [0, +\infty)$ is a lower semicontinuous mapping and $\alpha : \mathcal{M} \times \mathcal{M} \rightarrow [0, +\infty)$. If

(i) O is $R - \alpha_{s^p}$ -admissible

(ii) O is generalized $(R - \alpha_{s^p}, \theta, \varphi)$ contractive

(iii) there is $\xi_0 \in \mathcal{M}$ satisfying $\alpha(R\xi_0, O\xi_0) \geq s^p$

(iv) properties (A_{s^p}) and (B_{s^p}) are fulfilled

(v) α satisfies transitive property, i.e., for $\xi, \eta, \zeta \in \mathcal{M}$

$$\alpha(\xi, \eta) \geq s^p \text{ and } \alpha(\eta, \zeta) \geq s^p \Rightarrow \alpha(\xi, \zeta) \geq s^p, \quad (16)$$

then O and R possess a unique point of coincidence. Furthermore, if O and R are weakly compatible, then O and R possess a unique common fixed point in \mathcal{M} .

Proof. It follows from condition (iii) that one can choose an $\xi_0 \in \mathcal{M}$ with $\alpha(R\xi_0, O\xi_0) \geq s^p$. Define sequences $\{\xi_n\}$ and $\{\eta_n\}$ in \mathcal{M} by $\eta_n = O\xi_n = R\xi_{n+1}$ for $n \in \mathbb{N}$. If $\eta_n = \eta_{n+1}$ for some n , then we deduce $\eta_n = \eta_{n+1} = O\xi_{n+1} = R\xi_{n+1}$ and O and R possess a point of coincidence. Next, we suppose that $\eta_n \neq \eta_{n+1}$ for $n \in \mathbb{N}$. In light of contraction condition (i), we

obtain

$$\begin{aligned}\alpha(R\xi_0, R\xi_1) &= \alpha(R\xi_0, O\xi_0) \geq s^p, \\ \alpha(R\xi_1, R\xi_2) &= \alpha(O\xi_0, O\xi_1) \geq s^p, \\ \alpha(R\xi_2, R\xi_3) &= \alpha(O\xi_1, O\xi_2) \geq s^p.\end{aligned}\quad (17)$$

Hence, for all $n \in \mathbb{N}$, we deduce $\alpha(R\xi_n, R\xi_{n+1}) = \alpha(\eta_{n-1}, \eta_n) \geq s^p$. Applying (14) with $\xi = \xi_n$ and $\eta = \xi_{n+1}$,

$$\begin{aligned}\theta(\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1})) &\leq \theta(s^p[\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1})]) \\ &\leq \theta(\alpha(R\xi_n, R\xi_{n+1})[\mathcal{Q}(O\xi_n, O\xi_{n+1}) + \varphi(O\xi_n) + \varphi(O\xi_{n+1})]) \\ &\leq \beta(\theta(h(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi))\theta(h(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi)) \\ &\quad + L\theta(q(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi))),\end{aligned}\quad (18)$$

where

$$\begin{aligned}h(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi) &= \max \left\{ \mathcal{Q}(O\xi_n, O\xi_{n+1}) + \varphi(O\xi_n) + \varphi(O\xi_{n+1}), \frac{\mathcal{Q}(O\xi_{n+1}, R\xi_{n+1}) + \varphi(O\xi_{n+1}) + \varphi(R\xi_{n+1})}{1 + \mathcal{Q}(O\xi_n, R\xi_n) + \varphi(O\xi_n) + \varphi(R\xi_n)} \right. \\ &\quad \cdot \{ \mathcal{Q}(R\xi_n, R\xi_{n+1}) + \varphi(R\xi_n) + \varphi(R\xi_{n+1}) \}, \frac{1}{2} \{ \mathcal{Q}(O\xi_n, O\xi_{n+1}) \\ &\quad \left. + \varphi(O\xi_n) + \varphi(O\xi_{n+1}) + \mathcal{Q}(R\xi_n, R\xi_{n+1}) + \varphi(R\xi_n) + \varphi(R\xi_{n+1}) \} \right\} \\ &= \max \left\{ \mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1}), \frac{\mathcal{Q}(\eta_{n+1}, \eta_n) + \varphi(\eta_{n+1}) + \varphi(\eta_n)}{1 + \mathcal{Q}(\eta_n, \eta_{n-1}) + \varphi(\eta_n) + \varphi(\eta_{n-1})} \right. \\ &\quad \cdot \{ \mathcal{Q}(\eta_{n-1}, \eta_n) + \varphi(\eta_{n-1}) + \varphi(\eta_n) \}, \frac{1}{2} \{ \mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) \\ &\quad \left. + \varphi(\eta_{n+1}) + \mathcal{Q}(\eta_{n-1}, \eta_n) + \varphi(\eta_{n-1}) + \varphi(\eta_n) \} \right\} \\ &\leq \max \{ \mathcal{Q}(\eta_{n-1}, \eta_n) + \varphi(\eta_{n-1}) + \varphi(\eta_n), \mathcal{Q}(\eta_{n+1}, \eta_n) + \varphi(\eta_{n+1}) + \varphi(\eta_n) \},\end{aligned}\quad (19)$$

$$\begin{aligned}q(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi) &= \frac{1}{2} \min \{ \mathcal{Q}(O\xi_n, O\xi_{n+1}) + \varphi(O\xi_n) \\ &\quad + \varphi(O\xi_{n+1}), \mathcal{Q}(R\xi_n, R\xi_{n+1}) + \varphi(R\xi_n) + \varphi(R\xi_{n+1}) \} \\ &= \frac{1}{2} \min \{ \mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1}), \mathcal{Q}(\eta_{n-1}, \eta_n) \\ &\quad + \varphi(\eta_{n-1}) + \varphi(\eta_n) \}.\end{aligned}\quad (20)$$

If we assume that $\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1}) > \mathcal{Q}(\eta_n, \eta_{n-1}) + \varphi(\eta_n) + \varphi(\eta_{n-1})$ for some $n \in \mathbb{N}$, according to (18), (19), and (20), we have

$$\begin{aligned}\theta(\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1})) &\leq \frac{1}{s} \theta(h(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi)) \\ &\quad + L\theta(q(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi)) \\ &\leq \frac{1}{s} \theta(\mathcal{Q}(\eta_{n+1}, \eta_n) + \varphi(\eta_{n+1}) + \varphi(\eta_n)) + L\theta(\mathcal{Q}(\eta_{n+1}, \eta_n) \\ &\quad + \varphi(\eta_{n+1}) + \varphi(\eta_n)) < \theta(\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1})),\end{aligned}\quad (21)$$

which is a contradiction. Thus,

$$\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1}) \leq \mathcal{Q}(\eta_n, \eta_{n-1}) + \varphi(\eta_n) + \varphi(\eta_{n-1}),\quad (22)$$

$$h(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi) \leq \mathcal{Q}(\eta_n, \eta_{n-1}) + \varphi(\eta_n) + \varphi(\eta_{n-1}),\quad (23)$$

$$q(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi) < \mathcal{Q}(\eta_n, \eta_{n-1}) + \varphi(\eta_n) + \varphi(\eta_{n-1}).\quad (24)$$

It follows from (22) that $\{\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1})\}$ is decreasing. It follows that there exists a real number $\gamma \geq 0$ satisfying

$$\lim_{n \rightarrow +\infty} (\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1})) = \gamma.\quad (25)$$

In view of (18), (23), and (24), one can obtain

$$\begin{aligned}\theta(\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1})) &\leq \beta(\theta(h(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi))\theta(h(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi)) \\ &\quad + L\theta(q(\xi_n, \xi_{n+1}, \mathcal{Q}, O, R, \varphi))) \\ &< \theta(\mathcal{Q}(\eta_n, \eta_{n-1}) + \varphi(\eta_n) + \varphi(\eta_{n-1})).\end{aligned}\quad (26)$$

If $\gamma > 0$, putting $n \rightarrow \infty$ in (26), we obtain

$$\dots\quad (27)$$

a contradiction. Hence,

$$\lim_{n \rightarrow +\infty} (\mathcal{Q}(\eta_n, \eta_{n+1}) + \varphi(\eta_n) + \varphi(\eta_{n+1})) = \gamma = 0,\quad (28)$$

which implies that $\lim_{n \rightarrow +\infty} \mathcal{Q}(\eta_n, \eta_{n+1}) = 0$ and $\lim_{n \rightarrow +\infty} \varphi(\eta_n) = 0$. In view of hypothesis (v), we have $\alpha(\eta_{n-2}, \eta_n) \geq s^p$. Taking $\xi = \xi_{n-1}$ and $\eta = \xi_{n+1}$ in (14), we obtain

$$\begin{aligned}\theta(\mathcal{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) + \varphi(\eta_{n+1})) &\leq \theta(\alpha(\eta_{n-2}, \eta_n)[\mathcal{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) + \varphi(\eta_{n+1})]) \\ &\leq \beta(\theta(h(\xi_{n-1}, \xi_{n+1}, \mathcal{Q}, O, R, \varphi))\theta(h(\xi_{n-1}, \xi_{n+1}, \mathcal{Q}, O, R, \varphi)) \\ &\quad + L\theta(q(\xi_{n-1}, \xi_{n+1}, \mathcal{Q}, O, R, \varphi))),\end{aligned}\quad (29)$$

where

$$\begin{aligned}h(\xi_{n-1}, \xi_{n+1}, \mathcal{Q}, O, R, \varphi) &= \max \left\{ \mathcal{Q}(O\xi_{n-1}, O\xi_{n+1}) + \varphi(O\xi_{n-1}) + \varphi(O\xi_{n+1}), \right. \\ &\quad \frac{\mathcal{Q}(O\xi_{n+1}, R\xi_{n+1}) + \varphi(O\xi_{n+1}) + \varphi(R\xi_{n+1})}{1 + \mathcal{Q}(O\xi_{n-1}, R\xi_{n-1}) + \varphi(O\xi_{n-1}) + \varphi(R\xi_{n-1})} \\ &\quad \cdot \{ \mathcal{Q}(R\xi_{n-1}, R\xi_{n+1}) + \varphi(R\xi_{n-1}) + \varphi(R\xi_{n+1}) \}, \\ &\quad \frac{1}{2} \{ \mathcal{Q}(O\xi_{n-1}, O\xi_{n+1}) + \varphi(O\xi_{n-1}) + \varphi(O\xi_{n+1}) \\ &\quad \left. + \mathcal{Q}(R\xi_{n-1}, R\xi_{n+1}) + \varphi(R\xi_{n-1}) + \varphi(R\xi_{n+1}) \} \right\} \\ &\leq \max \{ \mathcal{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) + \varphi(\eta_{n+1}), \mathcal{Q}(\eta_{n-2}, \eta_n) \\ &\quad + \varphi(\eta_{n-2}) + \varphi(\eta_n) \},\end{aligned}\quad (30)$$

$$\begin{aligned}
q(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi) &= \frac{1}{2} \min \{ \mathbf{Q}(O\xi_{n-1}, O\xi_{n+1}) + \varphi(O\xi_{n-1}) \\
&\quad + \varphi(O\xi_{n+1}), \mathbf{Q}(R\xi_{n-1}, R\xi_{n+1}) + \varphi(R\xi_{n-1}) \\
&\quad + \varphi(R\xi_{n+1}) \} \\
&= \frac{1}{2} \min \{ \mathbf{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) \\
&\quad + \varphi(\eta_{n+1}), \mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n) \}.
\end{aligned} \tag{31}$$

If for some $n \in \mathbb{N}$, $\mathbf{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) + \varphi(\eta_{n+1}) > \mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n)$, according to (29), (30), and (31), we get

$$\begin{aligned}
&\theta(\mathbf{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) + \varphi(\eta_{n+1})) \\
&< \frac{1}{s} \theta(h(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi)) + L\theta(q(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi)) \\
&< \theta(\mathbf{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) + \varphi(\eta_{n+1})),
\end{aligned} \tag{32}$$

which is a contradiction. It follows that

$$\mathbf{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) + \varphi(\eta_{n+1}) \leq \mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n), \tag{33}$$

$$h(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi) \leq \mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n), \tag{34}$$

$$q(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi) < \mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n). \tag{35}$$

Inequality (33) yields that $\{\mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n)\}$ is non-increasing and which yields that there exists $\epsilon \geq 0$ satisfying

$$\lim_{n \rightarrow +\infty} (\mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n)) = \epsilon. \tag{36}$$

In light of (32), (34), and (35), one can deduce

$$\begin{aligned}
&\theta(\mathbf{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) + \varphi(\eta_{n+1})) \\
&\leq \beta(\theta(h(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi))\theta(h(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi)) \\
&\quad + L\theta(q(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi))) \\
&< \theta(\mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n)).
\end{aligned} \tag{37}$$

Assume that $\epsilon > 0$. Letting $n \rightarrow \infty$ in (37), we derive

$$\begin{aligned}
\theta(\epsilon) &= \lim_{n \rightarrow +\infty} \theta(\mathbf{Q}(\eta_{n-1}, \eta_{n+1}) + \varphi(\eta_{n-1}) + \varphi(\eta_{n+1})) \\
&\leq \lim_{n \rightarrow +\infty} \beta(\theta(h(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi))\theta(h(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi)) \\
&\quad + L \lim_{n \rightarrow +\infty} \theta(q(\xi_{n-1}, \xi_{n+1}, \mathbf{Q}, O, R, \varphi))) \\
&< \lim_{n \rightarrow +\infty} \theta(\mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n)) = \theta(\epsilon),
\end{aligned} \tag{38}$$

which gives a contradiction. This yields that

$$\lim_{n \rightarrow +\infty} (\mathbf{Q}(\eta_{n-2}, \eta_n) + \varphi(\eta_{n-2}) + \varphi(\eta_n)) = \epsilon = 0. \tag{39}$$

It follows that $\lim_{n \rightarrow +\infty} \mathbf{Q}(\eta_{n-2}, \eta_n) = 0$.

Now, we aim to show that $\{\eta_n\}$ is a Cauchy sequence. Assume on the contrary that, $\{\eta_n\}$ is not Cauchy. So, there exists $\epsilon > 0$ for which we can choose sequences $\{\eta_{m_k}\}$ and $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$,

$$\epsilon \leq \mathbf{Q}(\eta_{m_k}, \eta_{n_k}), \tag{40}$$

$$\mathbf{Q}(\eta_{m_k}, \eta_{n_k-1}) < \epsilon. \tag{41}$$

In light of the rectangular inequality and (40) and (41), we have

$$\begin{aligned}
\epsilon &\leq \mathbf{Q}(\eta_{m_k}, \eta_{n_k}) \leq s [\mathbf{Q}(\eta_{m_k}, \eta_{n_k-1}) + \mathbf{Q}(\eta_{n_k-1}, \eta_{n_k+1}) + \mathbf{Q}(\eta_{n_k+1}, \eta_{n_k})] \\
&< s\epsilon + s\mathbf{Q}(\eta_{n_k-1}, \eta_{n_k+1}) + s\mathbf{Q}(\eta_{n_k+1}, \eta_{n_k}).
\end{aligned} \tag{42}$$

Taking the superior limit as $k \rightarrow +\infty$, we have

$$\epsilon \leq \limsup_{k \rightarrow +\infty} \mathbf{Q}(\eta_{m_k}, \eta_{n_k}) \leq s\epsilon. \tag{43}$$

Similarly,

$$\mathbf{Q}(\eta_{m_k}, \eta_{n_k}) \leq s [\mathbf{Q}(\eta_{m_k}, \eta_{m_k+1}) + \mathbf{Q}(\eta_{m_k+1}, \eta_{m_k-1}) + \mathbf{Q}(\eta_{m_k-1}, \eta_{n_k})], \tag{44}$$

$$\mathbf{Q}(\eta_{m_k}, \eta_{n_k}) \leq s [\mathbf{Q}(\eta_{m_k}, \eta_{m_k-1}) + \mathbf{Q}(\eta_{m_k-1}, \eta_{n_k-1}) + \mathbf{Q}(\eta_{n_k-1}, \eta_{n_k})], \tag{45}$$

$$\mathbf{Q}(\eta_{m_k-1}, \eta_{n_k}) \leq s [\mathbf{Q}(\eta_{m_k-1}, \eta_{m_k}) + \mathbf{Q}(\eta_{m_k}, \eta_{n_k-1}) + \mathbf{Q}(\eta_{n_k-1}, \eta_{n_k})]. \tag{46}$$

It follows from (40), (41), and (42) that

$$\frac{\epsilon}{s} \leq \limsup_{k \rightarrow +\infty} \mathbf{Q}(\eta_{m_k}, \eta_{n_k-1}) \leq \epsilon. \tag{47}$$

By (40), (41), (44), and (46), we get

$$\frac{\epsilon}{s} \leq \limsup_{k \rightarrow +\infty} \mathbf{Q}(\eta_{m_k-1}, \eta_{n_k}) \leq s\epsilon. \tag{48}$$

By the similar method, we have

$$\mathbf{Q}(\eta_{m_k-1}, \eta_{n_k-1}) \leq s [\mathbf{Q}(\eta_{m_k-1}, \eta_{m_k}) + \mathbf{Q}(\eta_{m_k}, \eta_{n_k}) + \mathbf{Q}(\eta_{n_k}, \eta_{n_k-1})], \tag{49}$$

so is

$$\frac{\varepsilon}{s} \leq \limsup_{k \rightarrow +\infty} \varrho(\eta_{m_k-1}, \eta_{n_k-1}) \leq s^2 \varepsilon. \quad (50)$$

According to the definition of $h(\xi, \eta, \varrho, O, R, \varphi)$, we get

$$\begin{aligned} h(\xi_{m_k}, \xi_{n_k}, \varrho, O, R, \varphi) &= \max \left\{ \varrho(O\xi_{m_k}, O\xi_{n_k}) + \varphi(O\xi_{m_k}) \right. \\ &\quad \left. + \varphi(O\xi_{n_k}), \frac{\varrho(O\xi_{m_k}, R\xi_{n_k}) + \varphi(O\xi_{m_k}) + \varphi(R\xi_{n_k})}{1 + \rho(O\xi_{m_k}, R\xi_{m_k}) + \varphi(O\xi_{m_k}) + \varphi(R\xi_{m_k})} \right. \\ &\quad \cdot \left\{ \varrho(R\xi_{m_k}, R\xi_{n_k}) + \varphi(R\xi_{m_k}) + \varphi(R\xi_{n_k}) \right\}, \frac{1}{2} \left\{ \varrho(O\xi_{m_k}, O\xi_{n_k}) + \varphi(O\xi_{m_k}) \right. \\ &\quad \left. + \varphi(O\xi_{n_k}) + \varrho(R\xi_{m_k}, R\xi_{n_k}) + \varphi(R\xi_{m_k}) + \varphi(R\xi_{n_k}) \right\} \Big\} \\ &= \max \left\{ \varrho(\eta_{m_k}, \eta_{n_k}) + \varphi(\eta_{m_k}) + \varphi(\eta_{n_k}), \frac{\varrho(\eta_{m_k}, \eta_{n_k-1}) + \varphi(\eta_{m_k}) + \varphi(\eta_{n_k-1})}{1 + \varrho(\eta_{m_k}, \eta_{m_k-1}) + \varphi(\eta_{m_k}) + \varphi(\eta_{m_k-1})} \right. \\ &\quad \cdot \left\{ \varrho(\eta_{m_k-1}, \eta_{n_k-1}) + \varphi(\eta_{m_k-1}) + \varphi(\eta_{n_k-1}) \right\}, \frac{1}{2} \left\{ \varrho(\eta_{m_k}, \eta_{n_k}) + \varphi(\eta_{m_k}) + \varphi(\eta_{n_k}) \right. \\ &\quad \left. + \varrho(\eta_{m_k-1}, \eta_{n_k-1}) + \varphi(\eta_{m_k-1}) + \varphi(\eta_{n_k-1}) \right\} \Big\}. \end{aligned} \quad (51)$$

Taking the superior limit as $k \rightarrow +\infty$ in (51), we get

$$\limsup_{k \rightarrow +\infty} h(\xi_{m_k}, \xi_{n_k}, \varrho, O, R, \varphi) \leq \max \left\{ s\varepsilon, 0, \frac{s\varepsilon + s^2\varepsilon}{2} \right\} < s^2\varepsilon. \quad (52)$$

Also, we have

$$\begin{aligned} q(\xi_{m_k}, \xi_{n_k}, \varrho, O, R, \varphi) &= \frac{1}{2} \min \left\{ \varrho(O\xi_{m_k}, O\xi_{n_k}) + \varphi(O\xi_{m_k}) + \varphi(O\xi_{n_k}), \varrho(R\xi_{m_k}, R\xi_{n_k}) \right. \\ &\quad \left. + \varphi(R\xi_{m_k}) + \varphi(R\xi_{n_k}) \right\} \\ &= \frac{1}{2} \min \left\{ \varrho(\eta_{m_k}, \eta_{n_k}) + \varphi(\eta_{m_k}) + \varphi(\eta_{n_k}), \varrho(\eta_{m_k-1}, \eta_{n_k-1}) \right. \\ &\quad \left. + \varphi(\eta_{m_k-1}) + \varphi(\eta_{n_k-1}) \right\}. \end{aligned} \quad (53)$$

It follows that

$$\limsup_{k \rightarrow +\infty} q(\xi_{m_k}, \xi_{n_k}, \varrho, O, R, \varphi) < s^2\varepsilon. \quad (54)$$

The transitivity property of α yields that $\alpha(R\xi_{m_k}, R\xi_{n_k}) \geq s^p$. Taking $\xi = \xi_{m_k}$ and $\eta = \xi_{n_k}$ in (14), one can deduce

$$\begin{aligned} \theta(s^2\varepsilon) &\leq \theta(s^p\varepsilon) \leq \theta \left(\alpha(R\xi_{m_k}, R\xi_{n_k}) \limsup_{n \rightarrow +\infty} \left[\varrho(\eta_{m_k}, \eta_{n_k}) + \varphi(\eta_{m_k}) + \varphi(\eta_{n_k}) \right] \right) \\ &\leq \limsup_{n \rightarrow +\infty} \theta \left(h(\xi_{m_k}, \xi_{n_k}, \varrho, O, R, \varphi) \right) \theta \left(h(\xi_{m_k}, \xi_{n_k}, \varrho, O, R, \varphi) \right) \\ &\quad + \limsup_{n \rightarrow +\infty} L\theta \left(q(\xi_{m_k}, \xi_{n_k}, \varrho, O, R, \varphi) \right) \\ &\leq \frac{1}{s} \theta(s^2\varepsilon) + L\theta(s^2\varepsilon) < \theta(s^2\varepsilon), \end{aligned} \quad (55)$$

a contradiction. Hence, $\{\eta_n\}$ is Cauchy. Since (\mathcal{M}, ρ) is

complete, there is a $\vartheta \in \mathcal{M}$ such that

$$\lim_{n \rightarrow +\infty} \varrho(\eta_n, \vartheta) = \lim_{n \rightarrow +\infty} \varrho(O\xi_n, \vartheta) = \lim_{n \rightarrow +\infty} \varrho(R\xi_{n+1}, \vartheta) = \lim_{n, m \rightarrow +\infty} \varrho(\eta_n, \eta_m) = 0. \quad (56)$$

Since $R(\mathcal{M})$ is closed, we have $\vartheta \in R(\mathcal{M})$. Hence, we choose a $z \in \mathcal{M}$ satisfying $\vartheta = Rz$. We write (56) as

$$\lim_{n \rightarrow +\infty} \varrho(\eta_n, Rz) = \lim_{n \rightarrow +\infty} \varrho(O\xi_n, Rz) = \lim_{n \rightarrow +\infty} \varrho(R\xi_{n+1}, Rz) = 0. \quad (57)$$

It follows from the definition of φ that

$$\varphi(Rz) = \varphi(\vartheta) \leq \liminf_{n \rightarrow +\infty} \varphi(\eta_n) = 0, \quad (58)$$

which implies that $\varphi(Rz) = \varphi(\vartheta) = 0$.

The property (A_{s^p}) ensures that there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ with $\alpha(\eta_{n_k-1}, Rz) \geq s^p$ for $k \in \mathbb{N}$. If $\varrho(Oz, Rz) + \varphi(Oz) \neq 0$, taking $\xi = \xi_{n_k}$ and $\eta = z$ in (14), one deduce that

$$\begin{aligned} &\theta(\varrho(O\xi_{n_k}, Oz) + \varphi(O\xi_{n_k}) + \varphi(Oz)) \\ &\leq \theta(s^p [\varrho(O\xi_{n_k}, Oz) + \varphi(O\xi_{n_k}) + \varphi(Oz)]) \\ &\leq \theta \left(\alpha(\eta_{n_k-1}, Rz) [\varrho(O\xi_{n_k}, Oz) + \varphi(O\xi_{n_k}) + \varphi(Oz)] \right) \\ &\leq \beta(\theta(h(\xi_{n_k}, z, \varrho, O, R, \varphi))) \theta(h(\xi_{n_k}, z, \varrho, O, R, \varphi)) \\ &\quad + L\theta(q(\xi_{n_k}, z, \varrho, O, R, \varphi)), \end{aligned} \quad (59)$$

where

$$\begin{aligned} h(\xi_{n_k}, z, \varrho, O, R, \varphi) &= \max \left\{ \varrho(O\xi_{n_k}, Oz) + \varphi(O\xi_{n_k}) + \varphi(Oz), \right. \\ &\quad \frac{\varrho(Oz, Rz) + \varphi(Oz) + \varphi(Rz)}{1 + \rho(O\xi_{n_k}, R\xi_{n_k}) + \varphi(O\xi_{n_k}) + \varphi(R\xi_{n_k})} \\ &\quad \cdot \left\{ \varrho(R\xi_{n_k}, Rz) + \varphi(R\xi_{n_k}) + \varphi(Rz) \right\}, \\ &\quad \cdot \frac{1}{2} \left\{ \varrho(O\xi_{n_k}, Oz) + \varphi(O\xi_{n_k}) + \varphi(Oz) \right. \\ &\quad \left. + \varrho(Rz, R\xi_{n_k}) + \varphi(Rz) + \varphi(R\xi_{n_k}) \right\} \Big\} \\ &= \max \left\{ \varrho(\eta_{n_k}, Oz) + \varphi(\eta_{n_k}) + \varphi(Oz), \right. \\ &\quad \frac{\varrho(Oz, Rz) + \varphi(Oz) + \varphi(Rz)}{1 + \varrho(\eta_{n_k}, \eta_{n_k-1}) + \varphi(\eta_{n_k}) + \varphi(\eta_{n_k-1})} \\ &\quad \cdot \left\{ \varrho(\eta_{n_k-1}, Rz) + \varphi(\eta_{n_k-1}) + \varphi(Rz) \right\}, \\ &\quad \cdot \frac{1}{2} \left\{ \varrho(\eta_{n_k}, Oz) + \varphi(\eta_{n_k}) + \varphi(Oz) + \varrho(\eta_{n_k-1}, Rz) \right. \\ &\quad \left. + \varphi(\eta_{n_k-1}) + \varphi(Rz) \right\} \Big\}, \end{aligned} \quad (60)$$

$$\begin{aligned}
q(\xi_{n_k}, z, \mathcal{Q}, O, R, \varphi) &= \frac{1}{2} \min \left\{ \mathcal{Q}(O\xi_{n_k}, Oz) + \varphi(O\xi_{n_k}) \right. \\
&\quad \left. + \varphi(Oz), \mathcal{Q}(R\xi_{n_k}, Rz) + \varphi(R\xi_{n_k}) + \varphi(Rz) \right\} \\
&= \frac{1}{2} \min \left\{ \mathcal{Q}(\eta_{n_k}, Oz) + \varphi(\eta_{n_k}) + \varphi(Oz), \mathcal{Q}(\eta_{n_{k-1}}, Rz) \right. \\
&\quad \left. + \varphi(\eta_{n_{k-1}}) + \varphi(Rz) \right\}.
\end{aligned} \quad (61)$$

By simple calculation, we get

$$\limsup_{k \rightarrow +\infty} h(\xi_{n_k}, z, \mathcal{Q}, O, R, \varphi) \leq s(\mathcal{Q}(Oz, Rz) + \varphi(Oz)), \quad (62)$$

$$\limsup_{k \rightarrow +\infty} q(\xi_{n_k}, z, \mathcal{Q}, O, R, \varphi) < s(\rho(Oz, Rz) + \varphi(Oz)). \quad (63)$$

Letting $k \rightarrow +\infty$ in (59), using (62) and (63), we obtain

$$\begin{aligned}
\theta(s(\mathcal{Q}(Oz, Rz) + \varphi(Oz))) &\leq \theta\left(s^2 \limsup_{k \rightarrow \infty} (\mathcal{Q}(O\xi_{n_k}, Oz) + \varphi(O\xi_{n_k}) + \varphi(Oz))\right) \\
&\leq \theta\left(\limsup_{k \rightarrow \infty} (\alpha(\eta_{n_{k-1}}, Rz) [\mathcal{Q}(O\xi_{n_k}, Oz) + \varphi(O\xi_{n_k}) + \varphi(Oz)])\right) \\
&< \frac{1}{s} \theta(s(\mathcal{Q}(Oz, Rz) + \varphi(Oz))) + L\theta(s(\mathcal{Q}(Oz, Rz) + \varphi(Oz))) \\
&< \theta(s(\mathcal{Q}(Oz, Rz) + \varphi(Oz))).
\end{aligned} \quad (64)$$

It follows that $\mathcal{Q}(Oz, Rz) + \varphi(Oz) = 0$, which implies that $Oz = Rz$, $\varphi(Oz) = 0$.

Next, we show that O and R possess the unique point of coincidence ϑ . Assume on the contrary, there exist $z, z' \in C(O, R)$ and $Oz \neq Oz'$. By the property of $(B_{\mathcal{S}^p})$, one can obtain

$$\alpha(Rz', Rz) \geq s^p. \quad (65)$$

Taking $\xi = z'$ and $\eta = z$ in (14), we obtain

$$\begin{aligned}
&\theta(\mathcal{Q}(Oz', Oz) + \varphi(Oz') + \varphi(Oz)) \\
&\leq \theta(s^p [\mathcal{Q}(Oz', Oz) + \varphi(Oz') + \varphi(Oz)]) \\
&\leq \theta(\alpha(Rz', Rz) [\mathcal{Q}(Oz', Oz) + \varphi(Oz') + \varphi(Oz)]) \\
&\leq \beta(\theta(h(z', z, \mathcal{Q}, O, R, \varphi))) \theta(h(z', z, \mathcal{Q}, O, R, \varphi)) \\
&\quad + L\theta(q(z', z, \mathcal{Q}, O, R, \varphi)),
\end{aligned} \quad (66)$$

where

$$\begin{aligned}
h(z', z, \mathcal{Q}, O, R, \varphi) &= \max \left\{ \mathcal{Q}(Oz', Oz) + \varphi(Oz') + \varphi(Oz), \right. \\
&\quad \cdot \frac{\mathcal{Q}(Oz, Rz) + \varphi(Oz) + \varphi(Rz)}{1 + \mathcal{Q}(Oz', Rz') + \varphi(Oz') + \varphi(Rz')} \\
&\quad \cdot \left\{ \mathcal{Q}(Rz', Rz) + \varphi(Rz') + \varphi(Rz) \right\}, \\
&\quad \cdot \frac{1}{2} \left\{ \mathcal{Q}(Oz', Oz) + \varphi(Oz') + \varphi(Oz) \right. \\
&\quad \left. + \mathcal{Q}(Rz', Rz) + \varphi(Rz') + \varphi(Rz) \right\} \Big\} \\
&\leq \mathcal{Q}(Rz', Rz) + \varphi(Rz'), \\
q(z', z, \mathcal{Q}, O, R, \varphi) &= \frac{1}{2} \min \left\{ \mathcal{Q}(Oz', Oz) + \varphi(Oz') \right. \\
&\quad \left. + \varphi(Oz), \mathcal{Q}(Rz', Rz) + \varphi(Rz') + \varphi(Rz) \right\} \\
&< \mathcal{Q}(Rz', Rz) + \varphi(Rz').
\end{aligned} \quad (67)$$

In view of (66), we have

$$\begin{aligned}
\theta(\mathcal{Q}(Rz', Rz) + \varphi(Rz')) &< \frac{1}{s} \theta(\mathcal{Q}(Rz', Rz) + \varphi(Rz')) \\
&\quad + L\theta(\mathcal{Q}(Rz', Rz) + \varphi(Rz')) \\
&< \theta(\mathcal{Q}(Rz', Rz) + \varphi(Rz')).
\end{aligned} \quad (68)$$

Therefore, one can obtain that $\mathcal{Q}(Rz, Rz') + \varphi(Rz') = 0$, that is, $Rz = Rz' = \vartheta$ and $\varphi(Rz') = 0$. Hence, ϑ is a unique point of coincidence for O and R . Furthermore, if O and R are weak compatible mappings, it is easy to prove that O and R have a unique common fixed point z . The proof is complete. \square

Example 3. Let $(\mathcal{M}, \mathcal{Q})$ be the same as it is in Example 1. Define mappings $O, R : \mathcal{M} \rightarrow \mathcal{M}$ by

$$\begin{aligned}
O\xi &= \begin{cases} \frac{1}{5}, & \xi \in A, \\ \frac{1}{3}, & \xi \in B, \end{cases} \\
R\xi &= \begin{cases} \frac{1}{5}, & \xi = \frac{1}{5}, \\ \frac{1}{3}, & \xi = \frac{1}{4}, \\ \frac{1}{2}, & \xi = \frac{1}{3}, \\ 1, & \xi \in \left\{ \frac{1}{2} \right\} \cup B. \end{cases}
\end{aligned} \quad (69)$$

Define mappings $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ by

$$\alpha(\xi, \eta) = \begin{cases} s^p, \xi, \eta \in \left\{ \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \right\} \text{ with } \xi \neq \eta, \text{ or } \xi = \eta = \frac{1}{5}, \text{ or } \xi = \eta = 1, \text{ or } \xi = \eta = \frac{1}{3}, \\ 0, \text{ otherwise.} \end{cases} \quad (70)$$

Define $\theta : [0, +\infty) \longrightarrow [0, +\infty)$, $\varphi : \mathcal{M} \longrightarrow [0, +\infty)$ as follows:

$$\theta(\xi) = \frac{\xi}{2},$$

$$\varphi(\xi) = \begin{cases} 0, & \xi \in \left[0, \frac{1}{5}\right], \\ 0.15\xi - 0.03, & \xi \in \left(\frac{1}{5}, \frac{1}{3}\right], \\ 11.97\xi - 3.97, & \xi \in \left(\frac{1}{3}, +\infty\right). \end{cases} \quad (71)$$

Defined $\beta(\xi) = 1/5$ for all $\xi \geq 0$, then $\beta \in \Omega$. We can show that $O(\mathcal{M}) \subset R(\mathcal{M})$, $R(\mathcal{M})$ is closed. For $\xi, \eta \in \mathcal{M}$ such that $\alpha(R\xi, R\eta) \geq s^p$, we get that $R\xi, R\eta \in \{1/5, 1/4, 1/3, 1/2\}$ with $R\xi \neq R\eta$, or $R\xi = R\eta = 1/5$, or $R\xi = R\eta = 1$. This implies that $\xi, \eta \in \{1/5, 1/4, 1/3\}$ with $\xi \neq \eta$, or $\xi = \eta = 1/5$, or $\xi = \eta = 1/4$, or $\xi, \eta \in \{1/2\} \cup B$. So we obtain $O\xi, O\eta \in \{1/5, 1/3\}$ and $\alpha(O\xi, O\eta) \geq s^p$. Combining with the condition $q(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta) \neq 0$, the following cases are considered:

Case 1. $\xi = 1/2, \eta \in B$ (or $\eta = 1/2, \xi \in B$).

$$\begin{aligned} & \theta(\alpha(R\xi, R\eta)[q(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta)]) \\ &= \frac{1}{2} \cdot 16 \cdot \left[q\left(\frac{1}{5}, \frac{1}{3}\right) + \varphi\left(\frac{1}{5}\right) + \varphi\left(\frac{1}{3}\right) \right] = 0.8, \end{aligned}$$

$$\begin{aligned} & \beta(\theta(h(\xi, \eta, q, O, R, \varphi)))\theta(h(\xi, \eta, q, O, R, \varphi)) + L\theta(q(\xi, \eta, q, O, R, \varphi)) \\ & \geq \beta(\theta(h(\xi, \eta, q, O, R, \varphi)))\theta(h(\xi, \eta, q, O, R, \varphi)) \\ & \geq \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{2} \left\{ q\left(\frac{1}{5}, \frac{1}{3}\right) + \varphi\left(\frac{1}{5}\right) + \varphi\left(\frac{1}{3}\right) + q(1, 1) + \varphi(1) + \varphi(1) \right\} \\ &= \frac{1}{20} \cdot (0.1 + 8 + 8) > 0.8. \end{aligned} \quad (72)$$

In view of above inequalities, one can get that

$$\begin{aligned} & \theta(\alpha(R\xi, R\eta)[q(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta)]) \\ & \leq \beta(\theta(h(\xi, \eta, q, O, R, \varphi)))\theta(h(\xi, \eta, q, O, R, \varphi)) + L\theta(q(\xi, \eta, q, O, R, \varphi)), \end{aligned} \quad (73)$$

with $L \geq 0, 1/s + L < 1$ and $s = 4, p = 2$.

Case 2. $\xi, \eta \in B$.

$$\begin{aligned} & \theta(\alpha(R\xi, R\eta)[q(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta)]) \\ &= \frac{1}{2} \cdot 16 \cdot \left[q\left(\frac{1}{3}, \frac{1}{3}\right) + \varphi\left(\frac{1}{3}\right) + \varphi\left(\frac{1}{3}\right) \right] = 0.32, \end{aligned}$$

$$\begin{aligned} & \beta(\theta(h(\xi, \eta, q, O, R, \varphi)))\theta(h(\xi, \eta, q, O, R, \varphi)) + L\theta(q(\xi, \eta, q, O, R, \varphi)) \\ & \geq \beta(\theta(h(\xi, \eta, q, O, R, \varphi)))\theta(h(\xi, \eta, q, O, R, \varphi)) \\ & \geq \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{2} \left\{ q\left(\frac{1}{3}, \frac{1}{3}\right) + \varphi\left(\frac{1}{3}\right) + \varphi\left(\frac{1}{3}\right) + q(1, 1) + \varphi(1) + \varphi(1) \right\} \\ &= \frac{1}{20} \cdot (0.04 + 8 + 8) > 0.32. \end{aligned} \quad (74)$$

That is, for $\xi, \eta \in B$,

$$\begin{aligned} & \theta(\alpha(R\xi, R\eta)[q(O\xi, O\eta) + \varphi(O\xi) + \varphi(O\eta)]) \\ & \leq \beta(\theta(h(\xi, \eta, q, O, R, \varphi)))\theta(h(\xi, \eta, q, O, R, \varphi)) \\ & \quad + L\theta(q(\xi, \eta, q, O, R, \varphi)), \end{aligned} \quad (75)$$

with $L \geq 0, 1/s + L < 1$ and $s = 4, p = 2$.

In summary, all requirements of Theorem 14 are satisfied. O and R have a unique common fixed point $1/5$.

In Theorem 14, letting $\varphi = 0$, we can obtain the following result.

Corollary 15. Let (\mathcal{M}, ρ) be a complete rectangular b -metric space with coefficient $s \geq 1$. Let $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ and $O, R : \mathcal{M} \longrightarrow \mathcal{M}$ be given mappings with $O(\mathcal{M}) \subset R(\mathcal{M})$ and $R(\mathcal{M})$ is closed. Assume that $p \geq 2$ is a arbitrary constant and $\varphi : \mathcal{M} \longrightarrow [0, +\infty)$ is a lower semi-continuous function. If

(i) O is R - α_{sp} -admissible

(ii) for $\xi, \eta \in \mathcal{M}$ such that $\alpha(R\xi, R\eta) \geq s^p$ and $q(O\xi, O\eta) \neq 0$, there exist $\theta \in \Theta, \beta \in \Omega$ and $L \geq 0, 1/s + L < 1$ satisfying:

$$\begin{aligned} & \theta(\alpha(R\xi, R\eta)q(O\xi, O\eta)) \leq \beta(\theta(m(\xi, \eta, q, O, R)))\theta(m(\xi, \eta, q, O, R)) \\ & \quad + L\theta(n(\xi, \eta, q, O, R)), \end{aligned} \quad (76)$$

where

$$\begin{aligned} m(\xi, \eta, q, O, R) &= \max \left\{ q(O\xi, O\eta), \frac{q(O\eta, R\eta)}{1 + q(O\xi, R\xi)} \right. \\ & \quad \left. \cdot q(R\xi, R\eta), \frac{1}{2}(q(O\xi, R\eta) + q(R\xi, R\eta)) \right\}, \end{aligned} \quad (77)$$

and

$$n(\xi, \eta, \mathcal{Q}, O, R) = \frac{1}{2} \min \{ \mathcal{Q}(O\xi, O\eta), \mathcal{Q}(R\xi, R\eta) \} \quad (78)$$

- (iii) there is $\xi_0 \in \mathcal{M}$ such that $\alpha(R\xi_0, O\xi_0) \geq s^p$
- (iv) properties (A_{s^p}) and (B_{s^p}) are fulfilled
- (v) α satisfies transitive property, i.e., for $\xi, \eta, z \in \mathcal{M}$

$$\alpha(\xi, \eta) \geq s^p \text{ and } \alpha(\eta, z) \geq s^p \Rightarrow \alpha(\xi, z) \geq s^p, \quad (79)$$

then, O and R possess a unique point of coincidence in \mathcal{M} . Further, if O and R are weakly compatible, then O and R possess a unique common fixed point

If $\varphi = 0$, $R = I$, and $L = 0$ in Theorem 14, we have the following.

Corollary 16. Let $(\mathcal{M}, \mathcal{Q})$ be a complete rectangular b -metric space with coefficient $s \geq 2$ and \mathcal{M} be closed. Let $\alpha : \mathcal{M} \times \mathcal{M} \rightarrow [0, +\infty)$ and $O : \mathcal{M} \rightarrow \mathcal{M}$ be given mappings. Suppose $p \geq 2$ is a arbitrary constant. If

- (i) O is α_{s^p} -admissible
- (ii) for $\xi, \eta \in \mathcal{M}$ such that $\alpha(\xi, \eta) \geq s^p$ and $\mathcal{Q}(O\xi, O\eta) \neq 0$, there exists $\theta \in \Theta$ satisfying:

$$\theta(\alpha(\xi, \eta)\mathcal{Q}(O\xi, O\eta)) \leq \beta\theta(m^*(\xi, \eta, \mathcal{Q}, O)), \quad (80)$$

where $\beta \in (0, 1/s)$ is a constant and

$$m^*(\xi, \eta, \mathcal{Q}, O) = \max \left\{ \mathcal{Q}(O\xi, O\eta), \frac{\mathcal{Q}(O\eta, \eta)}{1 + \mathcal{Q}(O\xi, \xi)} \cdot \mathcal{Q}(\xi, \eta), \frac{1}{2}(\mathcal{Q}(O\xi, \eta) + \mathcal{Q}(\xi, \eta)) \right\} \quad (81)$$

- (iii) there is $\xi_0 \in \mathcal{M}$ with $\alpha(\xi_0, O\xi_0) \geq s^p$
- (iv) properties (A_{s^p}) and (B_{s^p}) are fulfilled when $R = I$
- (v) α satisfies transitive property, i.e., for $\xi, \eta, z \in \mathcal{M}$

$$\begin{aligned} \alpha(\xi, \eta) &\geq s^p, \\ \alpha(\eta, z) &\geq s^p \Rightarrow \alpha(\xi, z) \geq s^p, \end{aligned} \quad (82)$$

then O possesses a unique fixed point

Definition 17. The self-mappings $O, R : \mathcal{M} \rightarrow \mathcal{M}$ are called α_s orbital admissible mapping, if the following condition

holds:

$$\alpha_s(\xi, O\xi) \geq s^p, \alpha_s(\xi, R\xi) \geq s^p \text{ imply } \alpha_s(O\xi, RO\xi) \geq s^p, \alpha_s(R\xi, OR\xi) \geq s^p, \quad (83)$$

for a constant $p \geq 3$.

Definition 18. Let $O, R : \mathcal{M} \rightarrow \mathcal{M}$ be two self-maps and $p \geq 3$ be a real number. The pair (O, R) is said to be triangular α_s orbital admissible if

- (i) $O, R : \mathcal{M} \rightarrow \mathcal{M}$ are α_s orbital admissible
- (ii) $\alpha_s(\xi, \eta) \geq s^p, \alpha_s(\eta, O\eta) \geq s^p$ and $\alpha_s(\eta, R\eta) \geq s^p$ imply $\alpha_s(\xi, O\eta) \geq s^p, \alpha_s(\xi, R\eta) \geq s^p$

Lemma 19. Let $O, R : \mathcal{M} \rightarrow \mathcal{M}$ be two self-mappings satisfying the pair (O, R) is triangular α_s orbital admissible. Suppose that there is $\xi_0 \in \mathcal{M}$ satisfying $\alpha_s(\xi_0, O\xi_0) \geq s^p$. Define a sequence $\{\xi_n\}$ in \mathcal{M} by $\xi_{2i+1} = O\xi_{2i}, \xi_{2i+2} = R\xi_{2i+1}$ where $i = 0, 1, 2, \dots$. Then $\alpha_s(\xi_n, \xi_m) \geq s^p$ for $m, n \in \mathbb{N} \cup \{0\}$ with $m > n$.

Proof. Since $\alpha_s(\xi_0, O\xi_0) = \alpha_s(\xi_0, \xi_1) \geq s^p$ and (O, R) is triangular α_s orbital admissible, $\alpha_s(\xi_0, O\xi_0) \geq s^p$ implies $\alpha_s(O\xi_0, RO\xi_0) = \alpha_s(\xi_1, R\xi_1) = \alpha_s(\xi_1, \xi_2) \geq s^p$, $\alpha_s(\xi_1, R\xi_1) \geq s^p$ implies $\alpha_s(R\xi_1, OR\xi_1) = \alpha_s(\xi_2, O\xi_2) = \alpha_s(\xi_2, \xi_3) \geq s^p$, $\alpha_s(\xi_2, O\xi_2) \geq s^p$ implies $\alpha_s(O\xi_2, RO\xi_2) = \alpha_s(\xi_3, R\xi_3) = \alpha_s(\xi_3, \xi_4) \geq s^p$. Applying the above argument repeated, we obtain $\alpha_s(\xi_n, \xi_{n+1}) \geq s^p$ for all $n \in \mathbb{N} \cup \{0\}$. Since (O, R) is triangular α_s orbital admissible, $\alpha_s(\xi_n, \xi_m) \geq s^p$ for $m, n \in \mathbb{N} \cup \{0\}$ with $m > n$. \square

Definition 20. Let $(\mathcal{M}, \mathcal{Q})$ be a rectangular b -metric space with coefficient $s \geq 1$. Let $O, R : \mathcal{M} \rightarrow \mathcal{M}$ be two self-mappings. Suppose that $\alpha_s : \mathcal{M} \times \mathcal{M} \rightarrow [0, +\infty)$ and $\varphi : \mathcal{M} \rightarrow [0, +\infty)$ is a lower semicontinuous function and $p \geq 3$ is an arbitrary constant. The mappings O, R are said to be generalized α_s - θ -Geraghty contractions, if there exist $\theta \in \Theta, \beta, L \geq 0$ and $\beta + L < 1, 0 < \lambda < 1/4$ such that

$$\begin{aligned} \alpha_s(\xi, \eta)[\mathcal{Q}(O\xi, R\eta) + \varphi(O\xi) + \varphi(R\eta)] \\ \leq \beta\theta(s(\xi, \eta, \mathcal{Q}, O, R, \varphi)) + L\theta(t(\xi, \eta, \mathcal{Q}, O, \varphi)), \end{aligned} \quad (84)$$

for all $\xi, \eta \in \mathcal{M}$ with $\alpha_s(\xi, \eta) \geq s^p$ and $\mathcal{Q}(O\xi, R\eta) + \varphi(O\xi) + \varphi(R\eta) \neq 0$, where

$$\begin{aligned} s(\xi, \eta, \mathcal{Q}, O, R, \varphi) &= \lambda \max \{ \mathcal{Q}(\xi, \eta) + \varphi(\xi) + \varphi(\eta), \\ &\quad \frac{1 + \mathcal{Q}(\xi, O\xi) + \varphi(\xi) + \varphi(O\xi)}{1 + \mathcal{Q}(\xi, \eta) + \varphi(\xi) + \varphi(\eta)} \\ &\quad \cdot [\mathcal{Q}(O\xi, R\eta) + \varphi(O\xi) + \varphi(R\eta)], \\ &\quad \frac{\mathcal{Q}(\eta, R\eta) + \varphi(\eta) + \varphi(R\eta)}{1 + \mathcal{Q}(\xi, \eta) + \varphi(\xi) + \varphi(\eta)} \\ &\quad \cdot [\mathcal{Q}(\xi, O\xi) + \varphi(\xi) + \varphi(O\xi)] \}, \end{aligned}$$

$$t(\xi, \eta, \mathbf{Q}, O, \varphi) = \lambda \min \{ \mathbf{Q}(\xi, O\xi) + \varphi(\xi) + \varphi(O\xi), \mathbf{Q}(\eta, O\xi) + \varphi(\eta) + \varphi(O\xi) \}. \quad (85)$$

Let $\alpha_s : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ be a given mapping. Set.

(C_s) For all $\xi^* \in \mathcal{M}$, we have $\alpha_s(\xi^*, \xi^*) \geq s^p$

(D_s) For all $x, y \in C(O, R)$, one can get the condition of $\alpha_s(x, y) \geq s^p$ or $\alpha_s(y, x) \geq s^p$

Theorem 21. Let $(\mathcal{M}, \mathbf{Q})$ be a complete rectangular b -metric space with coefficient $s \geq 1$. Let $O, R : \mathcal{M} \longrightarrow \mathcal{M}$ be generalized α_s - θ -Geraghty contractions and one of O and R is continuous. If

- (i) O, R are triangular α_s orbital admissible
- (ii) there is $\xi_0 \in \mathcal{M}$ with satisfying $\alpha_s(\xi_0, O\xi_0) \geq s^p$
- (iii) properties (C_s) and (D_s) are satisfied

then O and R possess a unique common fixed point

Proof. For $\xi_0 \in \mathcal{M}$, define a sequence $\{\xi_n\}$ by $\xi_{2j+1} = O\xi_{2j}$, $\xi_{2j+2} = R\xi_{2j+1}$ for $j = 0, 1, 2, \dots$. We show that O and R have at most one common fixed point. Assume that $v \neq w$ are two common fixed points, then $R(v) = O(v) = v \neq w = R(w) = O(w)$. Therefore, $\mathbf{Q}(O(v), R(w)) = \mathbf{Q}(v, w) > 0$. It follows from the property of (D_s) that $\alpha_s(v, w) \geq s^p$ or $\alpha_s(w, v) \geq s^p$. Without loss of generality, suppose that $\alpha_s(v, w) \geq s^p$. Letting $\xi = v$ and $\eta = w$ in (84), we have

$$\begin{aligned} \mathbf{Q}(v, w) + \varphi(v) + \varphi(w) &\leq s^p [\mathbf{Q}(Ov, Rw) + \varphi(Ov) + \varphi(Rw)] \\ &\leq \alpha_s(v, w) [\mathbf{Q}(Ov, Rw) + \varphi(Ov) + \varphi(Rw)] \\ &\leq \beta\theta(s(v, w, \mathbf{Q}, O, R, \varphi)) + L\theta(t(v, w, \mathbf{Q}, O, \varphi)), \end{aligned} \quad (86)$$

where

$$\begin{aligned} s(v, w, \mathbf{Q}, O, R, \varphi) &= \lambda \max \left\{ \mathbf{Q}(v, w) + \varphi(v) + \varphi(w), \frac{1 + \mathbf{Q}(v, Ov) + \varphi(v) + \varphi(Ov)}{1 + \mathbf{Q}(v, w) + \varphi(v) + \varphi(w)} \right. \\ &\quad \cdot [\mathbf{Q}(Ov, Rw) + \varphi(Ov) + \varphi(Rw)], \frac{\mathbf{Q}(w, Rw) + \varphi(w) + \varphi(Rw)}{1 + \mathbf{Q}(v, w) + \varphi(v) + \varphi(w)} \\ &\quad \cdot [\mathbf{Q}(v, Ov) + \varphi(v) + \varphi(Ov)] \left. \right\} \\ &< \frac{1}{4} \max \{ \mathbf{Q}(v, w) + \varphi(v) + \varphi(w), 2[\mathbf{Q}(v, w) + \varphi(v) + \varphi(w)], 4[\mathbf{Q}(v, w) + \varphi(v) + \varphi(w)] \} = \mathbf{Q}(v, w) + \varphi(v) + \varphi(w), \end{aligned} \quad (87)$$

and

$$\begin{aligned} t(v, w, \mathbf{Q}, O, \varphi) &= \lambda \min \{ \mathbf{Q}(v, Ov) + \varphi(v) + \varphi(Ov), \mathbf{Q}(w, Ov) + \varphi(w) + \varphi(Ov) \} \\ &< \frac{1}{4} \min \{ \mathbf{Q}(v, v) + \varphi(v) + \varphi(v), \mathbf{Q}(w, v) + \varphi(w) + \varphi(v) \} \\ &< \mathbf{Q}(v, w) + \varphi(v) + \varphi(w). \end{aligned} \quad (88)$$

In view of (86), we have

$$\begin{aligned} \mathbf{Q}(v, w) + \varphi(v) + \varphi(w) &< \theta(\mathbf{Q}(v, w) + \varphi(v) + \varphi(w)) \\ &< \mathbf{Q}(v, w) + \varphi(v) + \varphi(w), \end{aligned} \quad (89)$$

which implies that $\mathbf{Q}(v, w) + \varphi(v) + \varphi(w) = 0$. That is, $v = w$ and $\varphi(v) = 0$. Hence, O, R have at most one common fixed point.

Now, assume $\mathbf{Q}(\xi_n, \xi_{n+1}) > 0$ for $n \in \mathbb{N}$. Otherwise, for some k , $\xi_{2k} = \xi_{2k+1}$, by assumption (ii) and Lemma 19, we have $\alpha_s(\xi_{2k}, \xi_{2k+1}) \geq s^p$. According to (84), if $\xi_{2k+1} \neq \xi_{2k+2}$, we obtain

$$\begin{aligned} &\mathbf{Q}(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+2}) \\ &\leq s^p [\mathbf{Q}(O\xi_{2k}, R\xi_{2k+1}) + \varphi(O\xi_{2k}) + \varphi(R\xi_{2k+1})] \\ &\leq \alpha_s(\xi_{2k}, \xi_{2k+1}) [\mathbf{Q}(O\xi_{2k}, R\xi_{2k+1}) + \varphi(O\xi_{2k}) + \varphi(R\xi_{2k+1})] \\ &\leq \beta\theta(s(\xi_{2k}, \xi_{2k+1}, \mathbf{Q}, O, R, \varphi)) + L\theta(t(\xi_{2k}, \xi_{2k+1}, \mathbf{Q}, O, \varphi)), \end{aligned} \quad (90)$$

where

$$\begin{aligned} s(\xi_{2k}, \xi_{2k+1}, \mathbf{Q}, O, R, \varphi) &= \lambda \max \{ \mathbf{Q}(\xi_{2k}, \xi_{2k+1}) + \varphi(\xi_{2k}) + \varphi(\xi_{2k+1}), \\ &\quad \frac{1 + \mathbf{Q}(\xi_{2k}, O\xi_{2k}) + \varphi(\xi_{2k}) + \varphi(O\xi_{2k})}{1 + \mathbf{Q}(\xi_{2k}, \xi_{2k+1}) + \varphi(\xi_{2k}) + \varphi(\xi_{2k+1})} \\ &\quad \cdot [\mathbf{Q}(O\xi_{2k}, R\xi_{2k+1}) + \varphi(O\xi_{2k}) + \varphi(R\xi_{2k+1})], \\ &\quad \frac{\mathbf{Q}(\xi_{2k+1}, R\xi_{2k+1}) + \varphi(\xi_{2k+1}) + \varphi(R\xi_{2k+1})}{1 + \mathbf{Q}(\xi_{2k}, \xi_{2k+1}) + \varphi(\xi_{2k}) + \varphi(\xi_{2k+1})} \\ &\quad \cdot [\mathbf{Q}(\xi_{2k}, O\xi_{2k}) + \varphi(\xi_{2k}) + \varphi(O\xi_{2k})] \left. \right\} \\ &< \frac{1}{4} \max \{ \mathbf{Q}(\xi_{2k}, \xi_{2k+1}) + \varphi(\xi_{2k}) + \varphi(\xi_{2k+1}), \\ &\quad \frac{1 + \mathbf{Q}(\xi_{2k}, \xi_{2k+1}) + \varphi(\xi_{2k}) + \varphi(\xi_{2k+1})}{1 + \mathbf{Q}(\xi_{2k}, \xi_{2k+1}) + \varphi(\xi_{2k}) + \varphi(\xi_{2k+1})} \\ &\quad \cdot [\mathbf{Q}(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+2})], \\ &\quad \frac{\mathbf{Q}(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+2})}{1 + \mathbf{Q}(\xi_{2k}, \xi_{2k+1}) + \varphi(\xi_{2k}) + \varphi(\xi_{2k+1})} \\ &\quad \cdot [\mathbf{Q}(\xi_{2k}, \xi_{2k+1}) + \varphi(\xi_{2k}) + \varphi(\xi_{2k+1})] \left. \right\} \\ &\leq \mathbf{Q}(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+2}), \end{aligned} \quad (91)$$

$$\begin{aligned} t(\xi_{2k}, \xi_{2k+1}, \mathbf{Q}, O, \varphi) &= \lambda \min \{ \mathbf{Q}(\xi_{2k}, O\xi_{2k}) + \varphi(\xi_{2k}) + \varphi(O\xi_{2k}), \mathbf{Q}(\xi_{2k+1}, O\xi_{2k}) \\ &\quad + \varphi(\xi_{2k+1}) + \varphi(O\xi_{2k}) \left. \right\} \\ &< \frac{1}{4} \min \{ \mathbf{Q}(\xi_{2k}, \xi_{2k+1}) + \varphi(\xi_{2k}) + \varphi(\xi_{2k+1}), \mathbf{Q}(\xi_{2k+1}, \xi_{2k+1}) \\ &\quad + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+1}) \} \leq \mathbf{Q}(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+2}). \end{aligned} \quad (92)$$

In light of (90) and above inequalities, we have

$$\begin{aligned} &\mathbf{Q}(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+2}) \leq \beta\theta(s(\xi_{2k}, \xi_{2k+1}, \mathbf{Q}, O, R, \varphi)) \\ &\quad + L\theta(t(\xi_{2k}, \xi_{2k+1}, \mathbf{Q}, O, \varphi)) + L\theta(\mathbf{Q}(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+2})) \\ &< \theta(\mathbf{Q}(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+2})) < \mathbf{Q}(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) \\ &\quad + \varphi(\xi_{2k+2}), \end{aligned} \quad (93)$$

which yields that $Q(\xi_{2k+1}, \xi_{2k+2}) + \varphi(\xi_{2k+1}) + \varphi(\xi_{2k+2}) = 0$. It follows that $\xi_{2k+1} = \xi_{2k+2}$.

Thus, ξ_{2k} is a common fixed point. Similarly, we can prove that ξ_{2k+1} is a common fixed point of O and R when $\xi_{2k+1} = \xi_{2k+2}$.

Now, assume that $Q(\xi_n, \xi_{n+1}) > 0$ for each $n \in \mathbb{N}$. Applying (84) with $\xi = \xi_{2n}, \eta = \xi_{2n+1}$, we get

$$\begin{aligned} & s(Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})) \\ & \leq \beta\theta(s(\xi_{2n}, \xi_{2n+1}, Q, O, R, \varphi)) + L\theta(t(\xi_{2n}, \xi_{2n+1}, Q, O, \varphi)), \end{aligned} \quad (94)$$

where

$$\begin{aligned} s(\xi_{2n}, \xi_{2n+1}, Q, O, R, \varphi) &= \lambda \max \{Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1}), \\ & \quad \frac{1 + Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1})}{1 + Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1})} \\ & \quad \cdot [Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})], \\ & \quad \frac{Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})}{1 + Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1})} \\ & \quad \cdot [Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1})]\} \\ & < \frac{1}{4} \max \{Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) \\ & \quad + \varphi(\xi_{2n+1}), Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})\}, \end{aligned} \quad (95)$$

$$\begin{aligned} t(\xi_{2n}, \xi_{2n+1}, Q, O, \varphi) &= \lambda \min \{Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) \\ & \quad + \varphi(\xi_{2n+1}), Q(\xi_{2n+1}, \xi_{2n+1}) + \varphi(\xi_{2n+1}) \\ & \quad + \varphi(\xi_{2n+1})\} < Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) \\ & \quad + \varphi(\xi_{2n+1}). \end{aligned} \quad (96)$$

If $Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2}) > \rho(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1})$ for some n , then by (94), (95), and (96), we have

$$\begin{aligned} & s(Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})) \\ & \leq \beta\theta(Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})) + L\theta(Q(\xi_{2n+1}, \xi_{2n+2}) \\ & \quad + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})) < \theta(Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})) \\ & < Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2}), \end{aligned} \quad (97)$$

which yields that

$$Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2}) = 0. \quad (98)$$

That is, $Q(\xi_{2n+1}, \xi_{2n+2}) = 0$, a contradiction. Therefore,

$$Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2}) \leq Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1}), \quad (99)$$

for $n \in \mathbb{N}$. It follows from (94), (95), and (96) that

$$s(Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})) < Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1}). \quad (100)$$

Using the same technique, we have

$$s(Q(\xi_{2n+2}, \xi_{2n+3}) + \varphi(\xi_{2n+2}) + \varphi(\xi_{2n+3})) < Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2}), \quad (101)$$

which implies that $\{Q(\xi_n, \xi_{n+1}) + \varphi(\xi_n) + \varphi(\xi_{n+1})\}$ is a non-increasing sequence satisfying

$$s(Q(\xi_{n+1}, \xi_{n+2}) + \varphi(\xi_{n+1}) + \varphi(\xi_{n+2})) < Q(\xi_n, \xi_{n+1}) + \varphi(\xi_n) + \varphi(\xi_{n+1}). \quad (102)$$

So there exists a $\lambda \geq 0$ satisfying

$$\lim_{n \rightarrow +\infty} (Q(\xi_n, \xi_{n+1}) + \varphi(\xi_n) + \varphi(\xi_{n+1})) = \lambda. \quad (103)$$

Now, we suppose $\lambda > 0$. By virtue of (94), (95), (96), and (99), one can get that

$$\begin{aligned} Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2}) &\leq \beta\theta(Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) \\ & \quad + \varphi(\xi_{2n+1})) + L\theta(Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1})). \end{aligned} \quad (104)$$

Letting $n \rightarrow +\infty$ in (104), we have

$$\begin{aligned} \lambda &= \lim_{n \rightarrow +\infty} (Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})) \\ &\leq \beta \lim_{n \rightarrow +\infty} \theta(Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1})) \\ & \quad + L \lim_{n \rightarrow +\infty} \theta(Q(\xi_{2n}, \xi_{2n+1}) + \varphi(\xi_{2n}) + \varphi(\xi_{2n+1})) \\ &< \lim_{n \rightarrow +\infty} \theta(Q(\xi_{2n+1}, \xi_{2n+2}) + \varphi(\xi_{2n+1}) + \varphi(\xi_{2n+2})) = \theta(\lambda) < \lambda, \end{aligned} \quad (105)$$

a contradiction. It follows that

$$\lim_{n \rightarrow +\infty} (Q(\xi_n, \xi_{n+1}) + \varphi(\xi_n) + \varphi(\xi_{n+1})) = 0. \quad (106)$$

Hence,

$$\lim_{n \rightarrow +\infty} Q(\xi_n, \xi_{n+1}) = 0, \quad \lim_{n \rightarrow +\infty} \varphi(\xi_n) = 0. \quad (107)$$

□

Now, we will show that $\{\xi_n\}$ is Cauchy. It is sufficient to show that $\{\xi_{3n}\}$, $\{\xi_{3n+1}\}$, and $\{\xi_{3n+2}\}$ are Cauchy. First of all, we prove $\{\xi_{3n}\}$ is Cauchy. We take into consideration the following:

Case 1. $k = 2m + 1$, where $m \geq 1$ and $3n$ is an odd number. By means of rectangular inequality and (102), we deduce that

$$\begin{aligned}
 \mathbf{Q}(\xi_{3n}, \xi_{3n+3k}) &\leq s[\mathbf{Q}(\xi_{3n}, \xi_{3n+1}) + \mathbf{Q}(\xi_{3n+1}, \xi_{3n+2}) + \mathbf{Q}(\xi_{3n+2}, \xi_{3n+3k})] \\
 &\leq s[\mathbf{Q}(\xi_{3n}, \xi_{3n+1}) + \mathbf{Q}(\xi_{3n+1}, \xi_{3n+2})] + s^2[\mathbf{Q}(\xi_{3n+2}, \xi_{3n+3}) \\
 &\quad + \mathbf{Q}(\xi_{3n+3}, \xi_{3n+4}) + \mathbf{Q}(\xi_{3n+4}, \xi_{3n+5})] + s^3[\mathbf{Q}(\xi_{3n+5}, \xi_{3n+6})] + \dots \\
 &\quad + s^{3m+1}[\mathbf{Q}(\xi_{3n+3(2m+1)-3}, \xi_{3n+3(2m+1)-2}) \\
 &\quad + \mathbf{Q}(\xi_{3n+3(2m+1)-2}, \xi_{3n+3(2m+1)-1}) + \mathbf{Q}(\xi_{3n+3(2m+1)-1}, \xi_{3n+3(2m+1)})] \\
 &\leq \left[s \left(\left(\frac{1}{s} \right)^{3n} + \left(\frac{1}{s} \right)^{3n+1} \right) + s^2 \left(\left(\frac{1}{s} \right)^{3n+2} + \left(\frac{1}{s} \right)^{3n+3} \right) \right. \\
 &\quad \left. + \dots + s \cdot s^{3m+1} \cdot \left(\frac{1}{s} \right)^{3n+6m+2} \right] (\mathbf{Q}(\xi_0, \xi_1) + \varphi(\xi_0) + \varphi(\xi_1)) \\
 &\leq \left(\frac{1}{s} \right)^{3n} \left[1 + \frac{1}{s} + \left(\frac{1}{s} \right)^2 + \dots \right] (\mathbf{Q}(\xi_0, \xi_1) + \varphi(\xi_0) + \varphi(\xi_1)) \\
 &\quad + s \left(\frac{1}{s} \right)^{3n+1} \left[1 + \frac{1}{s} + \left(\frac{1}{s} \right)^2 + \dots \right] (\mathbf{Q}(\xi_0, \xi_1) + \varphi(\xi_0) + \varphi(\xi_1)) \\
 &= \left(\frac{1}{s} \right)^{3n} \cdot \frac{1+s}{1-1/s} (\mathbf{Q}(\xi_0, \xi_1) + \varphi(\xi_0) + \varphi(\xi_1)) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
 \end{aligned} \tag{108}$$

The case that $3n$ is an even number is similar to the case that $3n$ is an odd number.

Case 2. $k = 2m$, where $m \geq 1$ and $3n$ is an odd number. In view of rectangular inequality again, we get that

$$\begin{aligned}
 \mathbf{Q}(\xi_{3n}, \xi_{3n+3k}) &\leq s[\mathbf{Q}(\xi_{3n}, \xi_{3n+1}) + \mathbf{Q}(\xi_{3n+1}, \xi_{3n+2}) + \mathbf{Q}(\xi_{3n+2}, \xi_{3n+3k})] \\
 &\leq s[\mathbf{Q}(\xi_{3n}, \xi_{3n+1}) + \mathbf{Q}(\xi_{3n+1}, \xi_{3n+2})] + s^2[\mathbf{Q}(\xi_{3n+2}, \xi_{3n+3}) \\
 &\quad + \mathbf{Q}(\xi_{3n+3}, \xi_{3n+4}) + \mathbf{Q}(\xi_{3n+4}, \xi_{3n+5})] + s^3[\mathbf{Q}(\xi_{3n+5}, \xi_{3n+6})] \\
 &\quad + \dots + s^{3m}[\mathbf{Q}(\xi_{3n+6m-3}, \xi_{3n+6m-2}) + \mathbf{Q}(\xi_{3n+6m-2}, \xi_{3n+6m-1}) \\
 &\quad + \mathbf{Q}(\xi_{3n+6m-1}, \xi_{3n+6m})] \leq \left[s \left(\left(\frac{1}{s} \right)^{3n} + \left(\frac{1}{s} \right)^{3n+1} \right) \right. \\
 &\quad \left. + s^2 \left(\left(\frac{1}{s} \right)^{3n+2} + \left(\frac{1}{s} \right)^{3n+3} \right) + \dots + s \cdot s^{3m+1} \cdot \left(\frac{1}{s} \right)^{3n+6m+2} \right] (\mathbf{Q}(\xi_0, \xi_1) \\
 &\quad + \varphi(\xi_0) + \varphi(\xi_1)) \leq s \left(\frac{1}{s} \right)^{3n} \left[1 + \frac{1}{s} + \left(\frac{1}{s} \right)^2 + \dots \right] (\mathbf{Q}(\xi_0, \xi_1) + \varphi(\xi_0) \\
 &\quad + \varphi(\xi_1)) + s \left(\frac{1}{s} \right)^{3n+1} \left[1 + \frac{1}{s} + \left(\frac{1}{s} \right)^2 + \dots \right] (\mathbf{Q}(\xi_0, \xi_1) + \varphi(\xi_0) + \varphi(\xi_1)) \\
 &= \left(\frac{1}{s} \right)^{3n} \cdot \frac{1+s}{1-(1/s)} (\mathbf{Q}(\xi_0, \xi_1) + \varphi(\xi_0) + \varphi(\xi_1)) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
 \end{aligned} \tag{109}$$

The case that $3n$ is an even number is similar to the case that $3n$ is an odd number.

Hence, $\{\xi_{3n}\}$ is a Cauchy sequence. Similarly, $\{\xi_{3n+1}\}$, $\{\xi_{3n+2}\}$ are also Cauchy sequences. That is, $\{\xi_n\}$ is Cauchy. According to the completeness of (\mathcal{M}, ρ) , we obtain that there is a ξ^* in \mathcal{M} satisfying

$$\lim_{n \rightarrow +\infty} O\xi_{2n} = \lim_{n \rightarrow +\infty} R\xi_{2n+1} = \xi^*. \tag{110}$$

In view of the definition of φ , we have

$$\varphi(\xi^*) \leq \liminf_{n \rightarrow +\infty} \varphi(\xi_n) = 0. \tag{111}$$

Next, we show that $O\xi^* = R\xi^* = \xi^*$ provided O or R is continuous. Without loss of generality, assume that O is continuous. By (110), we get

$$\xi^* = \lim_{n \rightarrow +\infty} O\xi_{2n} = O \left(\lim_{n \rightarrow +\infty} \xi_{2n} \right) = O(\xi^*). \tag{112}$$

This implies that ξ^* is a fixed point of O .

Using property (C_{s^p}) , we have $\alpha_s(\xi^*, \xi^*) \geq s^p$. If $\xi^* \neq R\xi^*$, from condition (84), one can deduce

$$\begin{aligned}
 \mathbf{Q}(\xi^*, R\xi^*) + \varphi(\xi^*) + \varphi(R\xi^*) &\leq s[\mathbf{Q}(O\xi^*, R\xi^*) + \varphi(O\xi^*) + \varphi(R\xi^*)] \\
 &\leq \alpha_s(\xi^*, \xi^*) [\mathbf{Q}(O\xi^*, R\xi^*) + \varphi(O\xi^*) + \varphi(R\xi^*)] \\
 &\leq \beta\theta(s(\xi^*, \xi^*, O, R, \varphi)) + L\theta(t(\xi^*, \xi^*, O, \varphi)),
 \end{aligned} \tag{113}$$

where

$$\begin{aligned}
 s(\xi^*, \xi^*, O, R, \varphi) &= \lambda \max \{ \mathbf{Q}(\xi^*, \xi^*) + \varphi(\xi^*) + \varphi(\xi^*), \\
 &\quad \cdot \frac{1 + \mathbf{Q}(\xi^*, O\xi^*) + \varphi(\xi^*) + \varphi(O\xi^*)}{1 + \mathbf{Q}(\xi^*, \xi^*) + \varphi(\xi^*) + \varphi(\xi^*)} \\
 &\quad \cdot [\mathbf{Q}(O\xi^*, R\xi^*) + \varphi(O\xi^*) + \varphi(R\xi^*)], \\
 &\quad \cdot \frac{\mathbf{Q}(\xi^*, R\xi^*) + \varphi(\xi^*) + \varphi(R\xi^*)}{1 + \mathbf{Q}(\xi^*, \xi^*) + \varphi(\xi^*) + \varphi(\xi^*)} \\
 &\quad \cdot [\mathbf{Q}(\xi^*, O\xi^*) + \varphi(\xi^*) + \varphi(O\xi^*)] \} \\
 &< \mathbf{Q}(\xi^*, R\xi^*) + \varphi(R\xi^*),
 \end{aligned} \tag{114}$$

$$t(\xi^*, \xi^*, O, \varphi) = 0 \leq \mathbf{Q}(\xi^*, R\xi^*) + \varphi(R\xi^*). \tag{115}$$

It follows from (113) that

$$\begin{aligned}
 \mathbf{Q}(\xi^*, R\xi^*) + \varphi(R\xi^*) &\leq \beta\theta(\rho(\xi^*, R\xi^*) + \varphi(R\xi^*)) + L\theta(\mathbf{Q}(\xi^*, R\xi^*) \\
 &\quad + \varphi(R\xi^*)) < \theta(\mathbf{Q}(\xi^*, R\xi^*) + \varphi(R\xi^*)) \\
 &< \mathbf{Q}(\xi^*, R\xi^*) + \varphi(R\xi^*),
 \end{aligned} \tag{116}$$

which implies that $\mathbf{Q}(\xi^*, R\xi^*) + \varphi(R\xi^*) = 0$, that is, $\xi^* = R\xi^*$ and $\varphi(R\xi^*) = 0$. It follows that O and R possess the unique common fixed point ξ^* . This completes the proof.

Example 4. Let $\mathcal{M} = [0, +\infty)$ and $\mathbf{Q}(\xi, \eta) = (\xi - \eta)^2$. Define mappings $O, R : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$O\xi = \frac{\xi}{72}, R\xi = \frac{\xi}{63}, \xi \in [0, +\infty). \tag{117}$$

Define mappings $\alpha_s : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ by

$$\alpha_s(\xi, \eta) = s^3, \xi, \eta \in [0, +\infty). \tag{118}$$

Define $\theta : [0, +\infty) \longrightarrow [0, +\infty)$ and $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$ with $\theta(\xi) = \xi/2$, $\varphi(\xi) = \xi^2$. Let $\beta = 1/3$ and $\lambda = 1/6$.

For $\xi, \eta \in \mathcal{M}$ with $\alpha_s(\xi, \eta) \geq s^3$, we can know that for $\xi, \eta \in [0, +\infty)$ such that $\xi \neq 0$ or $\eta \neq 0$,

$$\begin{aligned} \alpha_s(\xi, \eta)[\varrho(O\xi, R\eta) + \varphi(O\xi) + \varphi(R\eta)] &= 3^3 \cdot \left[\left(\frac{\xi}{72} - \frac{\eta}{63} \right)^2 + \left(\frac{\xi}{72} \right)^2 + \left(\frac{\eta}{63} \right)^2 \right] \\ &= 27 \cdot \frac{1}{81} \cdot \frac{3}{49} \left(\frac{49\xi^2}{64} + \eta^2 \right) \leq \frac{1}{49} (\xi^2 + \eta^2), \end{aligned}$$

$$\begin{aligned} \beta\theta(s(\xi, \eta, \varrho, O, R, \varphi)) + L\theta(t(\xi, \eta, \varrho, O, \varphi)) &\geq \beta\theta(s(\xi, \eta, \varrho, O, R, \varphi)) \\ &\geq \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{6} [\varrho(\xi, \eta) + \varphi(\xi) + \varphi(\eta)] = \frac{1}{36} [(\xi - \eta)^2 + \xi^2 + \eta^2] \\ &\geq \frac{1}{36} (\xi^2 + \eta^2). \end{aligned} \quad (119)$$

In view of above inequalities, one can obtain that

$$\begin{aligned} \alpha_s(\xi, \eta)[\varrho(O\xi, R\eta) + \varphi(O\xi) + \varphi(R\eta)] &\leq \beta\theta(s(\xi, \eta, \varrho, O, R, \varphi)) \\ &\quad + L\theta(t(\xi, \eta, \varrho, O, \varphi)), \end{aligned} \quad (120)$$

for $\xi, \eta \in \mathcal{M}$ with $\alpha_s(\xi, \eta) \geq s^p$ and $\varrho(O\xi, R\eta) + \varphi(O\xi) + \varphi(R\eta) \neq 0$. Hence, all requirements of Theorem 21 are fulfilled with $p = 3$, $s = 3$, and $L < 2/3$. It is obvious that O and R possess the unique common fixed point 0.

Taking $\varphi = 0$ in Theorem 21, one can obtain the following result.

Corollary 22. Let (\mathcal{M}, ϱ) be a complete rectangular b -metric space with coefficient $s \geq 1$. Suppose $O, R : \mathcal{M} \longrightarrow \mathcal{M}$ are two given mappings and one of O and R is a continuous mapping. Assume that $\alpha_s : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ and $p \geq 1$ is a constant and there exist $\theta \in \Theta$, $\beta, L \geq 0$ and $\beta + L < 1$, $0 < \lambda < 1/4$ such that

$$\alpha_s(\xi, \eta)\varrho(O\xi, R\eta) \leq \beta\theta(s^*(\xi, \eta, \varrho, O, R)) + L\theta(t^*(\xi, \eta, \varrho, O)), \quad (121)$$

for $\xi, \eta \in \mathcal{M}$ with $\alpha_s(\xi, \eta) \geq s^p$ and $\varrho(O\xi, R\eta) \neq 0$, where

$$\begin{aligned} s^*(\xi, \eta, \varrho, O, R) &= \lambda \max \left\{ \varrho(\xi, \eta), \frac{1 + \varrho(\xi, O\xi)}{1 + \varrho(\xi, \eta)} \cdot \varrho(O\xi, R\eta), \frac{\varrho(\eta, R\eta)}{1 + \varrho(\xi, \eta)} \cdot \varrho(\xi, O\xi) \right\}, \\ t^*(\xi, \eta, \varrho, O) &= \lambda \min \{ \varrho(\xi, O\xi), \varrho(\eta, O\xi) \}. \end{aligned} \quad (122)$$

If

- (i) O, R are triangular α_s orbital admissible
- (ii) there is $\xi_0 \in \mathcal{M}$ with satisfying $\alpha_s(\xi_0, O\xi_0) \geq s^p$
- (iii) properties (C_{sp}) and (D_{sp}) are satisfied

then O and R possess a unique common fixed point

4. Application

In this part, we shall prove the existence and uniqueness of solution to the integral equation:

$$\xi(\vartheta) = \int_0^K F(\vartheta, \lambda, \xi(\lambda)) d\lambda. \quad (123)$$

Let $\mathcal{M} = C([0, K])$ denote the collection of all continuous mappings on $[0, K]$. For $p \geq 2$, define

$$\varrho(\xi, \eta) = \sup_{\vartheta \in [0, K]} |\xi(\vartheta) - \eta(\vartheta)|^p \text{ for all } \xi, \eta \in \mathcal{M}. \quad (124)$$

Hence, (\mathcal{M}, ρ) is a complete rectangular b -metric space with $s = 3^{p-1}$.

Theorem 23. Let $O : \mathcal{M} \longrightarrow \mathcal{M}$ by $O\xi(\vartheta) = \int_0^K F(\vartheta, \lambda, \xi(\lambda)) d\lambda$ and $\delta : (-\infty, +\infty) \times (-\infty, +\infty) \longrightarrow (-\infty, +\infty)$ be a given mapping. Suppose that

- (i) $F : [0, K] \times [0, K] \times (-\infty, +\infty) \longrightarrow [0, +\infty)$ is continuous
- (ii) for all $\vartheta \in [0, K]$ and $\xi, \eta \in \mathcal{M}$, $\delta(\xi(\vartheta), \eta(\vartheta)) \geq 0$ implies $\delta(O\xi(\vartheta), O\eta(\vartheta)) \geq 0$
- (iii) there exists $\xi_0 \in \mathcal{M}$ satisfying $\delta(\xi_0(\vartheta), O\xi_0(\vartheta)) \geq 0$ for $\vartheta \in [0, K]$,
- (iv) properties (A_{sp}) and (B_{sp}) are fulfilled when $R = I$
- (v) there is a continuous mapping $\gamma : [0, K] \times [0, K] \longrightarrow [0, +\infty)$ such that

$$\sup_{\vartheta \in [0, K]} \int_0^K \gamma(\vartheta, \lambda) d\lambda \leq 1 \quad (125)$$

- (vi) there exists a real number $\beta \in (0, 1/s)$ satisfying for $(\vartheta, \lambda) \in [0, K] \times [0, K]$,

$$|F(\vartheta, \lambda, \xi(\lambda)) - F(\vartheta, \lambda, \eta(\lambda))| \leq \sqrt[p]{\frac{\beta}{2s^p}} \gamma(\vartheta, \lambda) |\xi(\lambda) - \eta(\lambda)| \quad (126)$$

Then, the integral equation (123) possesses a unique solution $\xi(\vartheta) \in \mathcal{M}$.

Proof. Define $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ by

$$\alpha(\xi, \eta) = \begin{cases} s^p, & \text{if } \delta(\xi(\vartheta), \eta(\vartheta)) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (127)$$

For $\xi, \eta \in \mathcal{M}$, according to assumptions (i)-(vi), we

obtain

$$\begin{aligned}
 s^p \mathcal{Q}(O\xi(\vartheta), O\eta(\vartheta)) &= s^p \sup_{\vartheta \in [0, K]} |O\xi(\vartheta) - O\eta(\vartheta)|^p \\
 &= s^p \sup_{\vartheta \in [0, K]} \left| \int_0^K F(\vartheta, \lambda, \xi(\lambda)) d\lambda \right. \\
 &\quad \left. - \int_0^K F(\vartheta, \lambda, \eta(\lambda)) d\lambda \right|^p \\
 &\leq s^p \sup_{\vartheta \in [0, K]} \left(\int_0^K |F(\vartheta, \lambda, \xi(\lambda)) - F(\vartheta, \lambda, \eta(\lambda))| d\lambda \right)^p \\
 &\leq s^p \sup_{\vartheta \in [0, K]} \left(\int_0^K \sqrt[p]{\frac{\beta}{2s^p}} \gamma(\vartheta, \lambda) |\xi(\lambda) - \eta(\lambda)| d\lambda \right)^p \\
 &\leq s^p \sup_{\vartheta \in [0, K]} \left(\int_0^K \sqrt[p]{\frac{\beta}{2s^p}} \gamma(\vartheta, \lambda) d\lambda \right)^p \\
 &\quad \cdot \sup_{\vartheta \in [0, K]} |\xi(\vartheta) - \eta(\vartheta)|^p \\
 &= \beta \cdot \frac{1}{2} \mathcal{Q}(\xi, \eta) \leq \beta \theta(m^*(\xi, \eta, \mathcal{Q}, O)).
 \end{aligned}
 \tag{128}$$

Letting $\theta(\vartheta) = \vartheta/2$, one can verify that all the conditions of Corollary 16 hold. As a result, O possesses a unique fixed point $\xi \in \mathcal{M}$, that is, $\xi(\vartheta)$ is the unique solution of integral equation (123). This completes the proof. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors equally contributed to this paper and approved the final version.

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Research Article

On Solution of Boundary Value Problems via Weak Contractions

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The aim is to present a new relational variant of fixed point result that generalizes various fixed point results of the existing theme for contractive type mappings. As an application, we solve a periodic boundary value problem and validate all assertions with the help of nontrivial examples. We also highlight the close connections of the fixed point results equipped with a binary relation to that of graph related metrical fixed point results. Radically, these investigations unify the theory of metrical fixed points for contractive type mappings.

1. Introduction

Alber and Guerre-Delabriere [1] presented the notion of weak contraction in Hilbert spaces and established the compatible fixed point results. Afterwards, Rhoades [2] stated that these results are still valid in the settings of metric spaces which are complete. Weak contractions are also connected to the mappings of Boyd and Wong [3], Geraghty [4], and that of Reich [5]. Further, generalizations of these fixed point results for weakly contractive mappings on this theme was obtained by Dutta and Choudhury [6]. In this continuation, an ordered analog of results due to Reich [5] and Geraghty [4] were presented by Amini-Harandi and Emami [7]. However, an analog of Banach contraction principle [8] in the same settings was investigated by Turinici [9] which was later explored by several authors (see Ran and Reurings [10], Nieto and Rodríguez-López [11], Sabetghadam and Masiha [12], Sabetghadam et al. [13], Harjani and Sadarangani [14], Samet and Turinici [15], Alam and Imdad [16, 17], and Prasad [18, 19]) and this process is still on. Meanwhile, Jachymski [20] presented an interesting metrical fixed point result by incorporating the notion of graphical contraction mapping, and there exist detailed generalization of this settings too (see for instance [21–23]).

Among all these generalizations, we must recite Alam and Imdad [17] in which the authors utilized relational var-

iants of metrical definitions of continuity, contractions, and completeness to obtain some interesting generalizations of the fixed point results. Noticeably, Alam and Imdad [16] presented a relational variant of fixed point result due to Boyd and Wong [3] to such settings. The objective of this work is to investigate a new fixed point theorem in relational metric spaces and to solve a boundary value problem in the light of obtained results. Moreover, we highlight the connection of such findings to the fixed point results obtained under graphical contraction mappings. In this way, we utilize the contractive assumption enjoying only on those elements which are associated with either a binary relation or some graph related structure instead of the entire space.

2. Preliminaries

We use notations \mathfrak{R} for a nonempty binary relation, \mathbb{N}_0 for the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and \mathbb{R} for the set of real numbers thoroughly in this paper. Also, the triple $(\mathbb{F}, \varsigma, \mathfrak{R})$ denotes an \mathfrak{R} -metric space where \mathfrak{R} is a binary relation on a nonempty set \mathbb{F} , and ς is a metric on \mathbb{F} .

Definition 1 (see [24]). Let \mathbb{F} be a nonempty set and $\mathfrak{R} \subseteq \mathbb{F} \times \mathbb{F}$. Then,

- (a) \mathfrak{R} is a binary relation on \mathbb{F} and “ \check{u} relates \check{v} under \mathfrak{R} ” iff $(\check{u}, \check{v}) \in \mathfrak{R}$
- (b) \check{u} and \check{v} are \mathfrak{R} -comparative, if either $(\check{u}, \check{v}) \in \mathfrak{R}$ or $(\check{v}, \check{u}) \in \mathfrak{R}$, and denoted by $[\check{u}, \check{v}] \in \mathfrak{R}$
- (c) The inverse of \mathfrak{R} is defined by $\mathfrak{R}^{-1} := \{(\check{u}, \check{v}) \in \mathbb{F}^2 : (\check{v}, \check{u}) \in \mathfrak{R}\}$
- (d) The symmetric closure of \mathfrak{R} is defined by $\mathfrak{R}^s := \mathfrak{R} \cup \mathfrak{R}^{-1}$
- (e) $(\check{u}, \check{v}) \in \mathfrak{R}^s$ iff $[\check{u}, \check{v}] \in \mathfrak{R}$

Definition 2 (see [17]). Consider a binary relation \mathfrak{R} and a self-map θ on a nonempty set \mathbb{F} . Then, for $\check{u}, \check{v} \in \mathbb{F}$,

- (a) \mathfrak{R} is θ -closed if

$$(\check{u}, \check{v}) \in \mathfrak{R} \Rightarrow (\theta\check{u}, \theta\check{v}) \in \mathfrak{R}, \check{u}, \check{v} \in \mathbb{F}. \quad (1)$$

- (b) \mathfrak{R} is θ -closed iff \mathfrak{R}^s is θ -closed

Definition 3 (see [17]). Consider a binary relation \mathfrak{R} and a sequence $\{\check{u}_n\}$ on a nonempty set \mathbb{F} . Then, $\{\check{u}_n\}$ is an \mathfrak{R} -preserving sequence (shortly, \mathfrak{R} -sequence) if $(\check{u}_n, \check{u}_{n+1}) \in \mathfrak{R}, n \in \mathbb{N}_0$.

Definition 4 (see [17]). Consider an \mathfrak{R} -sequence $\{\check{u}_n\}$ on an \mathfrak{R} -metric space $(\mathbb{F}, \varsigma, \mathfrak{R})$. Then, $(\mathbb{F}, \varsigma, \mathfrak{R})$ is \mathfrak{R} -complete if every \mathfrak{R} -Cauchy sequence converges to a point in \mathbb{F} .

Remark 5. Every \mathfrak{R} -complete metric space is a complete metric space, and in respect to the universal relation, these notions are the same.

Proposition 6 ([17]). Consider a binary relation \mathfrak{R} and a self-map θ on a nonempty set \mathbb{F} . If \mathfrak{R} is θ -closed, then \mathfrak{R} is θ^n -closed, where $n \in \mathbb{N}_0$ and θ^n denotes n th iterate of θ .

Definition 7 (see [17]). Consider a self-map θ on an \mathfrak{R} -metric space $(\mathbb{F}, \varsigma, \mathfrak{R})$. Then, θ is \mathfrak{R} -continuous at \check{u} if for any \mathfrak{R} -sequence $\{\check{u}_n\}$ with $\check{u}_n \xrightarrow{\varsigma} \check{u}$, we have $\theta\check{u}_n \xrightarrow{\varsigma} \theta\check{u}$. Moreover, θ is \mathfrak{R} -continuous if it is \mathfrak{R} -continuous at each point of \mathbb{F} .

Remark 8. Noticeably, continuity of θ implies \mathfrak{R} -continuity, and in respect to the universal relation, these notions are the same.

Definition 9 (see [17]). Consider an \mathfrak{R} -metric space $(\mathbb{F}, \varsigma, \mathfrak{R})$. Then, \mathfrak{R} is ς -self-closed if for any \mathfrak{R} -sequence $\{\check{u}_n\} \subset \mathbb{F}$ with $\check{u}_n \xrightarrow{\varsigma} \check{u}$, there exists a subsequence $\{\check{u}_{n_k}\}$ of $\{\check{u}_n\}$ with $[\check{u}_{n_k}, \check{u}] \in \mathfrak{R}, k \in \mathbb{N}_0$.

Definition 10 (see [17]). Consider a binary relation \mathfrak{R} on a nonempty set \mathbb{F} . A subset \mathcal{E} of \mathbb{F} is \mathfrak{R} -connected if for each pair $\check{u}, \check{v} \in \mathcal{E}$, there exists a path in \mathfrak{R} from \check{u} to \check{v} .

Definition 11 (see [25]). Consider a binary relation \mathfrak{R} on a nonempty set \mathbb{F} . Then, a subset \mathcal{E} of \mathbb{F} is \mathfrak{R} -directed if for each pair $\check{u}, \check{v} \in \mathcal{E}$, there exists $\check{w} \in \mathbb{F}$ so that $(\check{u}, \check{w}) \in \mathfrak{R}$ and $(\check{v}, \check{w}) \in \mathfrak{R}$.

Definition 12 ([16]). Consider a self-map θ on an \mathfrak{R} -metric space $(\mathbb{F}, \varsigma, \mathfrak{R})$. Then, \mathfrak{R} is θ -transitive if for any $\check{u}, \check{v}, \check{w} \in \mathbb{F}$, $(\theta\check{u}, \theta\check{v}), (\theta\check{v}, \theta\check{w}) \in \mathfrak{R} \Rightarrow (\theta\check{u}, \theta\check{w}) \in \mathfrak{R}$.

Motivated by Turinici [26], Alam and Imdad [16] notified the subsequent weaker form of transitivity.

Definition 13 ([16]). A binary relation \mathfrak{R} on a nonempty set \mathbb{F} is locally transitive if for each (effectively) \mathfrak{R} -sequence $\{\check{u}_n\} \subset \mathbb{F}$ (with range $\mathcal{U} := \{\check{u}_n : n \in \mathbb{N}_0\}$), the binary relation $\mathfrak{R}|_{\mathcal{U}}$ is transitive, where $\mathfrak{R}|_{\mathcal{U}}$ is the restriction of \mathfrak{R} to \mathcal{U} .

Definition 14 ([16]). Consider a self-map θ on an \mathfrak{R} -metric space $(\mathbb{F}, \varsigma, \mathfrak{R})$. Then, \mathfrak{R} is locally θ -transitive if for each (effectively) \mathfrak{R} -sequence $\{\check{u}_n\} \subset \theta(\mathbb{F})$ (with range $\mathcal{U} := \{\check{u}_n : n \in \mathbb{N}_0\}$), the binary relation $\mathfrak{R}|_{\mathcal{U}}$ is transitive.

Definition 15 (see [25]). Consider a binary relation \mathfrak{R} on a nonempty set \mathbb{F} . For $\check{u}, \check{v} \in \mathbb{F}$, a path of length $k (k \in \mathbb{N})$, in \mathfrak{R} from \check{u} to \check{v} is a finite sequence $\{\check{w}_0, \check{w}_1, \check{w}_2, \dots, \check{w}_k\} \subset \mathbb{F}$ satisfying the following:

- (i) $\check{w}_0 = \check{u}$ and $\check{w}_k = \check{v}$
- (ii) $(\check{w}_i, \check{w}_{i+1}) \in \mathfrak{R}$ for each $i (0 \leq i \leq k-1)$.

Noticeably, a path of length k has $k+1$ elements of \mathbb{F} , though they are not necessarily distinct.

Lemma 16 (see [16, 27]). Consider a sequence $\{\check{u}_n\}$ on a metric space (\mathbb{F}, ς) . If $\{\check{u}_n\}$ is not a Cauchy, then there exist $\varepsilon > 0$ and two subsequences $\{\check{u}_{n_k}\}$ and $\{\check{u}_{m_k}\}$ of $\{\check{u}_n\}$ so that for $k \in \mathbb{N}$,

$$\begin{aligned} k &\leq m_k < n_k, \\ \varsigma(\check{u}_{m_k}, \check{u}_{n_k}) &> \varepsilon, \\ \varsigma(\check{u}_{m_k}, \check{u}_{n_k-1}) &\leq \varepsilon. \end{aligned} \quad (2)$$

- (i) Moreover, if $\lim_{n \rightarrow \infty} \varsigma(\check{u}_n, \check{u}_{n+1}) = 0$, then

$$\begin{aligned} \lim_{k \rightarrow \infty} \varsigma(\check{u}_{m_k}, \check{u}_{n_k}) &= \varepsilon, \\ \lim_{k \rightarrow \infty} \varsigma(\check{u}_{m_k+1}, \check{u}_{n_k+1}) &= \varepsilon. \end{aligned} \quad (3)$$

Consider a binary relation \mathfrak{R} and a self-map θ on a nonempty set \mathbb{F} . We use the following notations in the subsequent sections:

- (i) $\mathcal{F}(\theta) := \{\check{u} \in \mathbb{F} : \check{u} = \theta\check{u}\}$ (the set of all fixed points of θ),
- (ii) $\mathcal{M}(\theta, \mathfrak{R}) := \{\check{u} \in \mathbb{F} : (\check{u}, \theta\check{u}) \in \mathfrak{R}\}$.

Also, \mathcal{C} is the class of functions $\varphi : [0, +\infty) \rightarrow [0, 1]$ satisfying the assumption $\varphi(s_n) \rightarrow 1$ implies $s_n \rightarrow 0$.

3. Main Results

In this section, we first consider the existence and uniqueness of fixed points for contractive mappings in relational metric spaces. Secondly, we present results related to graphical structure in the similar metric settings.

Theorem 17. *Consider a self-map θ on an \mathfrak{R} -metric space $(\mathbb{F}, \varsigma, \mathfrak{R})$. Assume that the subsequent assumptions hold:*

- (a) $(\mathbb{F}, \varsigma, \mathfrak{R})$ is \mathfrak{R} -complete
- (b) \mathfrak{R} is θ -closed and locally θ -transitive
- (c) either θ is \mathfrak{R} -continuous or \mathfrak{R} is ς -self-closed
- (d) $\mathcal{M}(\theta, \mathfrak{R})$ is nonempty
- (e) there exists $\varphi \in \mathcal{C}$ so that

$$\varsigma(\theta\check{u}, \theta\check{v}) \leq \varphi(\varsigma(\check{u}, \check{v}))\varsigma(\check{u}, \check{v}), \quad (4)$$

for each $\check{u}, \check{v} \in \mathbb{F}$ with $(\check{u}, \check{v}) \in \mathfrak{R}$. Then, θ has a fixed point.

Proof. In the light of assumption (d), let $\check{u}_0 \in \mathcal{M}(\theta, \mathfrak{R})$. Define a sequence $\{\check{u}_n\}$ of joint iterates with initial point \check{u}_0 , that is,

$$\check{u}_n = \theta^n \check{u}_0, n \in \mathbb{N}_0. \quad (5)$$

Since $(\check{u}_0, \theta\check{u}_0) \in \mathfrak{R}$ and \mathfrak{R} is θ -closed, we have

$$(\theta\check{u}_0, \theta^2\check{u}_0), (\theta^2\check{u}_0, \theta^3\check{u}_0), \dots, (\theta^n\check{u}_0, \theta^{n+1}\check{u}_0), \dots \in \mathfrak{R}, \quad (6)$$

so that

$$(\check{u}_n, \check{u}_{n+1}) \in \mathfrak{R}, n \in \mathbb{N}_0. \quad (7)$$

So, $\{\check{u}_n\}$ is \mathfrak{R} -sequence.

If there exists $n_0 \in \mathbb{N}$ so that $\varsigma(\check{u}_{n_0}, \check{u}_{n_0-1}) = 0$, then $\check{u}_{n_0} = \theta\check{u}_{n_0-1} = \check{u}_{n_0-1}$ is a fixed point of θ , so the proof is accomplished.

In the other case, assume that $\varsigma(\check{u}_n, \check{u}_{n-1}) \neq 0, n \in \mathbb{N}$. From (e), we have

$$\varsigma(\check{u}_{n+1}, \check{u}_n) = \varsigma(\theta\check{u}_n, \theta\check{u}_{n-1}) \leq \varphi(\varsigma(\check{u}_n, \check{u}_{n-1}))\varsigma(\check{u}_n, \check{u}_{n-1}) < \varsigma(\check{u}_n, \check{u}_{n-1}). \quad (8)$$

Put $s_n := \varsigma(\check{u}_{n+1}, \check{u}_n)$. Then, we have

$$s_n \leq \varphi(s_{n-1})s_{n-1} < s_{n-1}. \quad (9)$$

So, $\{s_n\}$ is a nonnegative nonincreasing and bounded below which possesses the limit s . From the inequality (9), taking $n \rightarrow \infty$, we have

$$s \leq \varphi(s)s < s, \quad (10)$$

implies $\varphi(s) = 1$, and so, $s = 0$.

Now, we shall show that $\{\check{u}_n\}$ is Cauchy. On contrary, assume that $\{\check{u}_n\}$ is not Cauchy. So, by Lemma 16, there exist $\varepsilon > 0$ and two subsequences $\{\check{u}_{n_k}\}$ and $\{\check{u}_{m_k}\}$ of $\{\check{u}_n\}$ so that

$$k \leq m_k \leq n_k, \varsigma(\check{u}_{m_k}, \check{u}_{n_k}) > \varepsilon \geq \varsigma(\check{u}_{m_k}, \check{u}_{n_k-1}), k \in \mathbb{N}. \quad (11)$$

Next, in view of Lemma 16, we have

$$\lim_{k \rightarrow \infty} \varsigma(\check{u}_{m_k}, \check{u}_{n_k}) = \lim_{k \rightarrow \infty} \varsigma(\check{u}_{m_k+1}, \check{u}_{n_k+1}) = \varepsilon. \quad (12)$$

Since $\{\check{u}_n\}$ is \mathfrak{R} -sequence and $\{\check{u}_n\} \subset \theta(\mathbb{F})$, so the local θ -transitivity of \mathfrak{R} gives rise that $(\check{u}_{m_k}, \check{u}_{n_k}) \in \mathfrak{R}$. By triangular inequality and (e), we obtain

$$\begin{aligned} \varsigma(\check{u}_{m_k}, \check{u}_{n_k}) &\leq \varsigma(\check{u}_{m_k}, \check{u}_{m_k+1}) + \varsigma(\check{u}_{m_k+1}, \check{u}_{n_k+1}) + \varsigma(\check{u}_{n_k}, \check{u}_{n_k+1}) \\ &\leq \varsigma(\check{u}_{m_k}, \check{u}_{m_k+1}) + \varphi(\varsigma(\check{u}_{m_k}, \check{u}_{n_k}))\varsigma(\check{u}_{m_k}, \check{u}_{n_k}) + \varsigma(\check{u}_{n_k}, \check{u}_{n_k+1}), \end{aligned} \quad (13)$$

that is,

$$\varsigma(\check{u}_{m_k}, \check{u}_{n_k}) \leq (1 - \varphi(\varsigma(\check{u}_{m_k}, \check{u}_{n_k})))^{-1} [\varsigma(\check{u}_{m_k}, \check{u}_{m_k+1}) + \varsigma(\check{u}_{n_k}, \check{u}_{n_k+1})]. \quad (14)$$

Using the facts that $\limsup_{k \rightarrow +\infty} \varsigma(\check{u}_{m_k}, \check{u}_{n_k}) > 0$ and $\lim_{n \rightarrow +\infty} \varsigma(\check{u}_{n_k}, \check{u}_{n_k+1}) = 0$, we have

$$\limsup_{k \rightarrow +\infty} (1 - \varphi(\varsigma(\check{u}_{m_k}, \check{u}_{n_k})))^{-1} = +\infty, \quad (15)$$

which implies that $\limsup_{k \rightarrow +\infty} \varphi(\varsigma(\check{u}_{m_k}, \check{u}_{n_k})) = 1$. Since $\varphi \in \mathcal{C}$, we obtain

$$\limsup_{k \rightarrow +\infty} \varsigma(\check{u}_{m_k}, \check{u}_{n_k}) = 0, \quad (16)$$

which is a contradiction in the light of (12). So, $\{\check{u}_n\}$ is \mathfrak{R} -Cauchy. As $(\mathbb{F}, \varsigma, \mathfrak{R})$ is \mathfrak{R} -complete, there exists $\check{u} \in \mathbb{F}$ so that $\check{u}_n \xrightarrow{\varsigma} \check{u}$.

Next, we assert that \check{u} is a fixed point of θ . At first, we consider θ is \mathfrak{R} -continuous. As $\{\check{u}_n\}$ is \mathfrak{R} -sequence with $\check{u}_n \xrightarrow{\varsigma} \check{u}$, \mathfrak{R} -continuity of θ implies that $\check{u}_{n+1} = \theta\check{u}_n \xrightarrow{\varsigma} \theta\check{u}$. From the uniqueness of the limit, we obtain $\theta\check{u} = \check{u}$, that is, \check{u} is a fixed point of θ .

Alternately, assume that \mathfrak{R} is ς -self-closed. So, there exists subsequence $\{\check{u}_{n_k}\}$ of $\{\check{u}_n\}$ with $(\check{u}_{n_k}, \check{u}) \in \mathfrak{R}, k \in \mathbb{N}_0$.

By using the fact that $[\check{u}_{n_k}, \check{u}] \in \mathfrak{R}$ in the light of (e), we have

$$\varsigma(\check{u}_{n_k+1}, \theta\check{u}) = \varsigma(\theta\check{u}_{n_k}, \theta\check{u}) \leq \varphi(\varsigma(\check{u}_{n_k}, \check{u}))\varsigma(\check{u}_{n_k}, \check{u}) < \varsigma(\check{u}_{n_k}, \check{u}). \quad (17)$$

Taking limit $k \rightarrow \infty$ and $\check{u}_{n_k} \xrightarrow{\varsigma} \check{u}$, we have $\check{u}_{n_k+1} \xrightarrow{\varsigma} \theta\check{u}$, and hence, $\theta\check{u} = \check{u}$. \square

Remark 18. Theorem 17 remains valid if we consider θ -transitive, locally transitive or simply transitive assumption in place of the locally θ -transitivity of \mathfrak{R} besides retaining all other assumptions.

3.1. Uniqueness Result

Theorem 19. Along with the assumptions of Theorem 17, assume that the subsequent assumption holds:

(u) $\theta(\mathbb{F})$ is \mathfrak{R}^s -connected. Then, θ has a unique fixed point.

Proof. Let \check{u} and \check{v} be two distinct fixed points of θ , that is, $\mathcal{F}(\theta) \neq \emptyset$ and $\check{u}, \check{v} \in \mathcal{F}(\theta)$, then for $n \in \mathbb{N}_0$, we have

$$\theta^n \check{u} = \check{u}, \theta^n \check{v} = \check{v}. \quad (18)$$

Noticeably, $\check{u}, \check{v} \in \theta(\mathbb{F})$. By assumption (u), there exists a path (say $\check{w}_0, \check{w}_1, \check{w}_2, \dots, \check{w}_k$) of finite length k in \mathfrak{R}^s from \check{u} to \check{v} so that

$$\check{w}_0 = \check{u}, \check{w}_k = \check{v} \text{ and } [\check{w}_i, \check{w}_{i+1}] \in \mathfrak{R} \text{ for each } i (0 \leq i \leq k-1). \quad (19)$$

As \mathfrak{R} is θ -closed, then in the light of Proposition 6, we obtain

$$[\theta^n \check{w}_i, \theta^n \check{w}_{i+1}] \in \mathfrak{R} \text{ for each } i (0 \leq i \leq k-1), n \in \mathbb{N}_0. \quad (20)$$

Now, applying the contractive assumption (e) to (20), we obtain

$$\varsigma(\theta^n \check{w}_i, \theta^n \check{w}_{i+1}) \leq \varphi(\varsigma(\theta^{n-1} \check{w}_i, \theta^{n-1} \check{w}_{i+1}))\varsigma(\theta^{n-1} \check{w}_i, \theta^{n-1} \check{w}_{i+1}). \quad (21)$$

For convenience, we put $t_n^i = \varsigma(\theta^n \check{w}_i, \theta^n \check{w}_{i+1})$.

We have two cases: Firstly, assume that $t_{n_0}^i = \varsigma(\theta^{n_0} \check{w}_i, \theta^{n_0} \check{w}_{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, that is, $\theta^{n_0} \check{w}_i = \theta^{n_0} \check{w}_{i+1}$, which implies that $\theta^{n_0+1} \check{w}_i = \theta^{n_0+1} \check{w}_{i+1}$. In this way, $t_{n_0+1}^i = \varsigma(\theta^{n_0+1} \check{w}_i, \theta^{n_0+1} \check{w}_{i+1}) = 0$. Thus, by induction, we get $t_n^i = 0$ for every $n \geq n_0$. Hence, $\lim_{n \rightarrow \infty} t_n^i = 0$.

Secondly, assume that $t_n^i > 0$ for $n \in \mathbb{N}_0$, then using (20), in view of (e) and taking $n \rightarrow \infty$ on the inequality (21), we have $\lim_{n \rightarrow \infty} t_n^i = 0$ for each $i (0 \leq i \leq k-1)$.

Finally, utilizing the triangular inequality of metric ς , in view of above conclusion, we obtain

$$\varsigma(\check{u}, \check{v}) = \varsigma(\theta^n \check{w}_0, \theta^n \check{w}_k) \leq t_n^0 + t_n^1 + \dots + t_n^{k-1} \rightarrow 0, \quad (22)$$

as $n \rightarrow \infty$. Hence, θ has a unique fixed point. \square

Remark 20. Theorem 19 remains valid if we consider $\mathfrak{R}|_{\theta(\mathbb{F})}$ is complete or $\theta(\mathbb{F})$ is \mathfrak{R}^s -directed in place of the assumption (u) besides retaining the all other assumptions.

Example 21. Let $\mathbb{F} = [0, +\infty)$ equipped with the usual metric $\varsigma(\check{u}, \check{v}) = |\check{u} - \check{v}|$ for $\check{u}, \check{v} \in \mathbb{F}$. Define a binary relation $\mathfrak{R} = \{(\check{u}, \check{v}) : \check{u} \geq \check{v} \text{ and } \check{u}, \check{v} \in [0, 1]\}$ on \mathbb{F} and a mapping $\theta : \mathbb{F} \rightarrow \mathbb{F}$ by

$$\theta\check{u} = \begin{cases} \frac{\check{u}}{1+\check{u}}, & \text{if } \check{u} \in [0, 1], \\ 3\check{u}, & \text{if } \check{u} \in (1, +\infty). \end{cases} \quad (23)$$

Clearly, $(\mathbb{F}, \varsigma, \mathfrak{R})$ is an \mathfrak{R} -complete metric space and θ is \mathfrak{R} -continuous. Let $\check{u}, \check{v} \in \mathbb{F}$. If $(\check{u}, \check{v}) \in \mathfrak{R}$, that is, $\check{u}, \check{v} \in [0, 1]$, then $(\theta\check{u}, \theta\check{v}) \in \mathfrak{R}$. Also, for all $\check{u} \in [0, 1]$, we have $\theta\check{u} \leq 1$. This implies that $(\check{u}, \theta\check{u}) \in \mathfrak{R}$. Thus, the claim holds. In consequence of the above reasonings, $(0, \theta 0) \in \mathfrak{R}$. Also, we can easily verify that \mathfrak{R} is θ -transitive and locally θ -transitive.

Let $\check{u}, \check{v} \in \mathbb{F}$ with $(\check{u}, \check{v}) \in \mathfrak{R}$. Define $\varphi(t) = (1/(1+t))$, $t \in [0, +\infty)$, we have

$$\begin{aligned} \varsigma(\theta\check{u}, \theta\check{v}) &= |\theta\check{u} - \theta\check{v}| = \left| \frac{\check{u}}{1+\check{u}} - \frac{\check{v}}{1+\check{v}} \right| = \frac{\check{u} - \check{v}}{1+\check{u}+\check{v}+\check{u}\check{v}} \\ &\leq \frac{\check{u} - \check{v}}{1+\check{u}-\check{v}} = \varphi(\varsigma(\check{u}, \check{v}))\varsigma(\check{u}, \check{v}). \end{aligned} \quad (24)$$

Thus, all the assumptions of Theorems 17 and 19 are satisfied. Hence, θ has a unique fixed point.

Example 22. Let $\mathbb{F} = [1, 4]$ equipped with the usual metric $\varsigma(\check{u}, \check{v}) = |\check{u} - \check{v}|$. Define a binary relation $\mathfrak{R} = \{(1, 1), (3/2, 2), (2, 1), (2, 2), (5/2, 3), (3, 3), (7/2, 4)\}$ on \mathbb{F} and the mapping $\theta : \mathbb{F} \rightarrow \mathbb{F}$ by

$$\theta\check{u} = \begin{cases} 1, & \text{if } \check{u} \in [1, 2], \\ 2, & \text{if } \check{u} \in (2, 3], \\ 3, & \text{if } \check{u} \in (3, 4]. \end{cases} \quad (25)$$

Let $(\check{u}, \check{v}) \in \mathfrak{R}$. Then, $(\theta\check{u}, \theta\check{v}) \in \{(1, 1), (2, 2), (3, 3)\}$ which implies that \mathfrak{R} is θ -closed. Observe that \mathfrak{R} is not reflexive, antisymmetric, and neither transitive. So, \mathfrak{R} is not partial order.

Now, we shall show \mathfrak{R} is ς -self-closed. Let $\{\check{u}_n\}$ be any \mathfrak{R} -sequence with $\check{u}_n \xrightarrow{\varsigma} \check{u}$, so that $(\check{u}_n, \check{u}_{n+1}) \in \mathfrak{R}$, $n \in \mathbb{N}_0$ which implies that $\{\check{u}_n\} \subset \{1, 2, 3\}$. As $\{1, 2, 3\}$ is closed, we can take a subsequence $\{\check{u}_{n_k}\}$ of $\{\check{u}_n\}$ so that $\check{u}_{n_k} = \check{u}$, k

$\in \mathbb{N}_0$, which implies that $[\check{u}_{n_k}, \check{u}] \in \mathfrak{R}, k \in \mathbb{N}_0$. Hence, \mathfrak{R} is ς -self-closed.

Notice that, for $\varphi(t) = 1/(1+t)$, $t \in [0, +\infty)$, we have

$$\varsigma(\theta 1, \theta 3) = |\theta 1 - \theta 3| = 1 > \varphi(\varsigma(1, 3))\varsigma(1, 3) = \frac{2}{3}, \quad (26)$$

that is, θ does not satisfy the contractive assumption (e) of Theorem 17. However, if $(\check{u}, \check{v}) \in \mathfrak{R}$, then the assumption (e) is satisfied for all $(\check{u}, \check{v}) \in \mathfrak{R}$. Also, by usual calculations, we can easily verify that $\theta(\mathbb{F})$ is \mathfrak{R}^s -connected.

So, θ satisfies all assumptions of Theorems 17 and 19. Thus, θ has a unique fixed point at $\check{u} = 1$.

Remark 23. Noticeably, in Example 22 binary relation \mathfrak{R} is nonreflexive, nonsymmetric, nonantisymmetric, and non-transitive. So, it is not a partial order, quasiorder, and near-order which indicate the utility of such generalizations over the corresponding several prominent recent fixed point results on this theme.

3.2. Fixed Point Result under Graphical Structure. Jachymski [11] introduced the graphical variant of Banach contraction principle in metric spaces by transforming the order structure into a graphical structure on such spaces.

Let \mathbb{F} be a nonempty set and Δ denotes the diagonal points of $\mathbb{F} \times \mathbb{F}$. Then, \mathcal{G} is a directed graph with the vertex set $\mathcal{V}(\mathcal{G})$ which coincides with \mathbb{F} , and the edge set $\mathcal{E}(\mathcal{G})$ containing its edges with all loops, that is, $\mathcal{E}(\mathcal{G}) \supseteq \Delta$. Additionally, assuming that \mathcal{G} has no parallel edges, so we can symbolize \mathcal{G} as a pair $(\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$. Also, we assume \mathcal{G} as a weighted graph by assigning to each edge the distance between its vertices. If \check{u} and \check{v} are any vertices of a graph \mathcal{G} , then a path in \mathcal{G} from \check{u} to \check{v} of length k ($k \in \mathbb{N}$) is a sequence $\{\check{u}_i\}_{i=0}^k$ of $k+1$ vertices so that $\check{u}_0 = \check{u}$, $\check{u}_k = \check{v}$ and $(\check{u}_{i-1}, \check{u}_i) \in \mathcal{E}(\mathcal{G})$ for $i = 1, 2, 3, \dots, k$. Graph \mathcal{G} is connected if there is a path between any two of its vertices, and \mathcal{G} is weakly connected if $\tilde{\mathcal{G}}$ is connected (see for details [21–23]).

The triple $(\mathbb{F}, \varsigma, \mathcal{G})$ denotes a \mathcal{G} -metric space where \mathcal{G} is a graph on a nonempty set \mathbb{F} and ς is a metric on \mathbb{F} .

Definition 24 (see [20]). Consider a self-map θ on a \mathcal{G} -metric space $(\mathbb{F}, \varsigma, \mathcal{G})$. Then, θ is said to be \mathcal{G} -contraction if there exists $k \in (0, 1)$ such that

$$\forall \check{u}, \check{v} \in \mathbb{F}, \quad (\check{u}, \check{v}) \in \mathcal{E}(\mathcal{G}) \Rightarrow \varsigma(\theta \check{u}, \theta \check{v}) \leq k\varsigma(\check{u}, \check{v}), \quad (27)$$

and \mathcal{G} is θ -closed, that is,

$$\forall \check{u}, \check{v} \in \mathbb{F}, \quad (\check{u}, \check{v}) \in \mathcal{E}(\mathcal{G}) \Rightarrow (\theta \check{u}, \theta \check{v}) \in \mathcal{E}(\mathcal{G}). \quad (28)$$

Definition 25 (see [20]). Consider a sequence $\{\check{u}_n\}$ on a \mathcal{G} -metric space $(\mathbb{F}, \varsigma, \mathcal{G})$. Then, $\{\check{u}_n\}$ is said to be edge-preserving sequence (shortly, \mathcal{G} -sequence) if $(\check{u}_n, \check{u}_{n+1}) \in \mathcal{E}(\mathcal{G})$, $n \in \mathbb{N}_0$.

Also, $(\mathbb{F}, \varsigma, \mathcal{G})$ is \mathcal{G} -complete if every \mathcal{G} -Cauchy sequence converges in \mathbb{F} .

Definition 26 (see [20]). Consider a self-map θ on a \mathcal{G} -metric space $(\mathbb{F}, \varsigma, \mathcal{G})$. Then, θ is \mathcal{G} -continuous at \check{u} if for any \mathcal{G} -sequence $\{\check{u}_n\}$ with $\check{u}_n \xrightarrow{\varsigma} \check{u}$, we have $\theta \check{u}_n \xrightarrow{\varsigma} \theta \check{u}$. Moreover, θ is \mathcal{G} -continuous if it is \mathcal{G} -continuous at each point of \mathbb{F} .

Definition 27. Consider a \mathcal{G} -metric space $(\mathbb{F}, \varsigma, \mathcal{G})$. Then, \mathcal{G} is ς -self-closed if for any \mathcal{G} -sequence $\{\check{u}_n\} \subset \mathbb{F}$ with $\check{u}_n \xrightarrow{\varsigma} \check{u}$, there exists a subsequence $\{\check{u}_{n_k}\}$ of $\{\check{u}_n\}$ with $(\check{u}_{n_k}, \check{u}) \in \mathcal{E}(\mathcal{G})$, $k \in \mathbb{N}_0$.

Definition 28 ([23]). Consider a graph \mathcal{G} on a nonempty set \mathbb{F} . Then, \mathcal{G} is transitive if, for any $\check{u}, \check{v}, \check{w} \in \mathcal{V}(\mathcal{G})$ with $(\check{u}, \check{v}), (\check{v}, \check{w}) \in \mathcal{E}(\mathcal{G}) \Rightarrow (\check{u}, \check{w}) \in \mathcal{E}(\mathcal{G})$.

Definition 29. Consider a self-map θ on a \mathcal{G} -metric space $(\mathbb{F}, \varsigma, \mathcal{G})$. Then, \mathcal{G} is θ -transitive if for any $\check{u}, \check{v}, \check{w} \in \mathbb{F}$, $(\theta \check{u}, \theta \check{v}), (\theta \check{v}, \theta \check{w}) \in \mathcal{E}(\mathcal{G}) \Rightarrow (\theta \check{u}, \theta \check{w}) \in \mathcal{E}(\mathcal{G})$.

Definition 30. A graph \mathcal{G} on a nonempty set \mathbb{F} is locally transitive if for each (effectively) \mathcal{G} -sequence $\{\check{u}_n\} \subset \mathbb{F}$ (with range $\mathcal{U} := \{\check{u}_n : n \in \mathbb{N}_0\}$), the graph $\mathcal{G}|_{\mathcal{U}}$ is transitive.

Definition 31. Consider a self-map θ on a \mathcal{G} -metric space $(\mathbb{F}, \varsigma, \mathcal{G})$. Then, \mathcal{G} is locally θ -transitive if for each (effectively) \mathcal{G} -sequence $\{\check{u}_n\} \subset \theta(\mathbb{F})$ (with range $\mathcal{U} := \{\check{u}_n : n \in \mathbb{N}_0\}$), the graph $\mathcal{G}|_{\mathcal{U}}$ is transitive.

Theorem 32. Consider a self-map θ on a \mathcal{G} -metric space $(\mathbb{F}, \varsigma, \mathcal{G})$. Assume that the subsequent assumptions hold:

- (a) $(\mathbb{F}, \varsigma, \mathcal{G})$ is \mathcal{G} -complete
- (b) \mathcal{G} is θ -closed and locally θ -transitive
- (c) either θ is \mathcal{G} -continuous or \mathcal{G} is ς -self-closed
- (d) $\mathcal{M}(\theta, \mathcal{G})$ is nonempty, that is, there exists \check{u}_0 in \mathbb{F} so that $(\check{u}_0, \theta \check{u}_0) \in \mathcal{E}(\mathcal{G})$,
- (e) there exists $\varphi \in \mathcal{C}$ so that

$$\varsigma(\theta \check{u}, \theta \check{v}) \leq \varphi(\varsigma(\check{u}, \check{v}))\varsigma(\check{u}, \check{v}), \quad (29)$$

for all $\check{u}, \check{v} \in \mathbb{F}$ with $(\check{u}, \check{v}) \in \mathcal{E}(\mathcal{G})$,

- (f) \mathcal{G} is weakly connected

Then, θ has a unique fixed point.

Proof. Define $\mathfrak{R} = \{(\check{u}, \check{v}) : (\check{u}, \check{v}) \in \mathcal{E}(\mathcal{G}), \check{u}, \check{v} \in \mathbb{F}\}$. Then, clearly, the contractive assumption (e) is same as in Theorem 17. Similarly, \mathcal{G} -completeness of metric space implies the \mathfrak{R} -completeness. From (d), we have $(\check{u}_0, \theta \check{u}_0) \in \mathcal{E}(\mathcal{G})$, which implies that $\mathcal{M}(\theta, \mathfrak{R})$ is nonempty. For $\check{u}, \check{v} \in \mathbb{F}$ with $(\check{u}, \check{v}) \in \mathcal{E}(\mathcal{G}) \Rightarrow (\check{u}, \check{v}) \in \mathfrak{R}$, then in the light of assumption (b), $(\theta \check{u}, \theta \check{v}) \in \mathcal{E}(\mathcal{G}) \Rightarrow (\theta \check{u}, \theta \check{v}) \in \mathfrak{R}$, that is, if \mathcal{G} is θ -closed and locally θ -transitive, then \mathfrak{R} is θ -closed and locally θ -transitive. Also, one can easily verify that \mathcal{G} -continuity of θ implies \mathfrak{R} -continuity and ς -self-closedness of \mathcal{G} implies ς

-self-closedness of \mathfrak{R} . Moreover, the assumption (f) implies that $\theta(\mathbb{F})$ is \mathfrak{R}^s -connected which validates that θ has only one fixed point. \square

Remark 33. In view of the above discussion, if we define a binary relation \mathfrak{R} so that $\mathfrak{R} = \{(\check{u}, \check{v}) : (\check{u}, \check{v}) \in \mathcal{E}(\mathcal{G}), \check{u}, \check{v} \in \mathbb{F}\}$. Then, under this assumption of \mathfrak{R} , Theorem 32 reduces to Theorems 17 and 19. This implies that edge preserving structure of a graph is considered as a particular case of a binary relation \mathfrak{R} .

Remark 34. Noticeably, if we define $\mathcal{E}(\mathcal{G})$ so that $\mathcal{E}(\mathcal{G}) = \{(\check{u}, \check{v}) : \check{u} \leq \check{v}, \check{u}, \check{v} \in \mathbb{F}\}$. Then, under this assumption of $\mathcal{E}(\mathcal{G})$, Theorem 32 reduces to their corresponding partial ordered analogous. This implies that partial-order relation-related metrical notions can be considered as a particular case of an edge-preserving structure related to a graph.

4. An Application

The theory of boundary value problems is a substantial field of mathematics, having various applications in numerous branches of physics, biology, chemistry, engineering, and other fields related to the real life problems. Based on this fact, we present a unique solution for the first order periodic boundary value problem by utilizing the main result. For this, we consider a periodic boundary value problem of first order as follows:

$$\check{u}'(s) = q(s, \check{u}(s)), s \in J = [0, S], \check{u}(0) = \check{u}(S), \quad (30)$$

where $S > 0$ and $q : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Let $C(J)$ denote the space of all continuous functions defined on J . We recall the subsequent definitions.

Definition 35 (see [14]). A function $\beta \in C^1(J)$ is a lower solution of (30), if

$$\begin{aligned} \beta'(s) &\leq q(s, \beta(s)), s \in J, \\ \beta(0) &\leq \beta(S). \end{aligned} \quad (31)$$

Definition 36 ([14]). A function $\beta \in C^1(J)$ is an upper solution of (30), if

$$\begin{aligned} \beta'(s) &\geq q(s, \beta(s)), s \in J, \\ \beta(0) &\geq \beta(S). \end{aligned} \quad (32)$$

Now, we prove the existence of solution for the problem (30). Let \mathcal{A} be a class of functions $\phi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the subsequent assumptions:

- (i) ϕ is increasing
- (ii) for each $\check{u} > 0$, $\phi(\check{u}) < \check{u}$

Examples of such functions are $\phi(s) = \mu s$, $s \in [0, 1)$, $\phi(s) = s/(1+s)$ and $\phi(s) = \ln(1+s)$.

Theorem 37. In addition to the problem (30), assume that there exists $\delta > 0$ so that $\check{u}, \check{v} \in \mathbb{R}$ with $\check{u} \leq \check{v}$,

$$0 \leq q(s, \check{v}) + \delta \check{v} - [q(s, \check{u}) + \delta \check{u}] \leq \delta \phi(\check{v} - \check{u}), \quad (33)$$

where $\phi \in \mathcal{A}$. Then, the existence of a lower or an upper solution of problem (30) validates the existence and uniqueness of a solution of problem (30).

Proof. Problem (30) can be rewritten as

$$\check{u}'(s) + \delta \check{u}(s) = q(s, \check{u}(s)) + \delta \check{u}(s), s \in J = [0, S], \check{u}(0) = \check{u}(S). \quad (34)$$

This can be transformed to the integral equation

$$\check{u}(s) = \int_0^S Q(s, r) [q(r, \check{u}(r)) + \delta \check{u}(r)] dr, \quad (35)$$

where

$$Q(s, r) = \begin{cases} \frac{e^{\delta(S+r-s)}}{e^{\delta S} - 1}, & 0 \leq r < s \leq S, \\ \frac{e^{\delta(r-s)}}{e^{\delta S} - 1}, & 0 \leq s < r \leq S. \end{cases} \quad (36)$$

Define $\theta : C(J) \rightarrow C(J)$ by

$$(\theta \check{u})(s) = \int_0^S Q(s, r) [q(r, \check{u}(r)) + \delta \check{u}(r)] dr, \quad (37)$$

and a binary relation

$$\mathfrak{R} = \{(\check{u}, \check{v}) \in C(J) \times C(J) : \check{u}(s) \leq \check{v}(s), s \in J\}. \quad (38)$$

- (i) Note that $C(J)$ with supmetric, that is, $\zeta(\check{u}, \check{v}) = \sup |\check{u}(s) - \check{v}(s)|$ for $s \in J$ and $\check{u}, \check{v} \in C(J)$, is an \mathfrak{R} -complete metric space
- (ii) For an \mathfrak{R} -preserving sequence $\{\check{u}_n\}$ so that $\check{u}_n \xrightarrow{\zeta} \check{w}$. Then, for $s \in J$, we have

$$\check{u}_0(s) \leq \check{u}_1(s) \leq \check{u}_2(s) \leq \dots \leq \check{u}_n(s) \leq \check{u}_{n+1}(s) \leq \dots \quad (39)$$

which converges to $\check{w}(s)$. This implies that $\check{u}_n(s) \leq \check{w}(s)$, $n \in \mathbb{N}_0$. So, $[\check{u}_n, \check{w}] \in \mathfrak{R}$, $n \in \mathbb{N}_0$. Hence, \mathfrak{R} is ζ -self-closed.

- (iii) For $(\check{u}, \check{v}) \in \mathfrak{R}$, that is, $\check{u}(s) \leq \check{v}(s)$, then in the light of inequality (33), we have

$$q(s, \check{u}(s)) + \delta \check{u}(s) \leq q(s, \check{v}(s)) + \delta \check{v}(s), s \in J, \quad (40)$$

and $Q(s, r) > 0$ for $(s, r) \in J \times J$, we have

$$\begin{aligned} (\theta\check{u})(s) &= \int_0^S Q(s, r)[q(r, \check{u}(r)) + \delta\check{u}(r)]dr \\ &\leq \int_0^S Q(s, r)[q(r, \check{v}(r)) + \delta\check{v}(r)]dr \\ &= (\theta\check{v})(s), s \in J, \end{aligned} \quad (41)$$

so that $(\theta\check{u}, \theta\check{v}) \in \mathfrak{R}$, that is, \mathfrak{R} is θ -closed.

(iv) Let $\beta \in C^1(J)$ be a lower solution of (30), then we must have

$$\beta'(s) + \delta\beta(s) \leq q(s, \beta(s)) + \delta\beta(s), s \in J. \quad (42)$$

Multiplying both sides by $e^{\delta s}$, we have

$$(\beta(s)e^{\delta s})' \leq [q(s, \beta(s)) + \delta\beta(s)]e^{\delta s}, s \in J, \quad (43)$$

so that

$$\beta(s)e^{\delta s} \leq \beta(0) + \int_0^s [q(r, \beta(r)) + \delta\beta(r)]e^{\delta r} dr, s \in J. \quad (44)$$

As $\beta(0) \leq \beta(S)$, we have

$$\beta(0)e^{\delta S} \leq \beta(S)e^{\delta S} \leq \beta(0) + \int_0^S [q(r, \beta(r)) + \delta\beta(r)]e^{\delta r} dr, \quad (45)$$

so that

$$\beta(0) \leq \int_0^S \frac{e^{\delta r}}{e^{\delta S} - 1} [q(r, \beta(r)) + \delta\beta(r)]dr. \quad (46)$$

Using (43) and (45), we have

$$\begin{aligned} \beta(s)e^{\delta s} &\leq \int_0^s \frac{e^{\delta(S+r)}}{e^{\delta S} - 1} [q(r, \beta(r)) + \delta\beta(r)]dr \\ &\quad + \int_s^S \frac{e^{\delta r}}{e^{\delta S} - 1} [q(r, \beta(r)) + \delta\beta(r)]dr, \end{aligned} \quad (47)$$

that is,

$$\begin{aligned} \beta(s) &\leq \int_0^s \frac{e^{\delta(S+r-s)}}{e^{\delta S} - 1} [q(r, \beta(r)) + \delta\beta(r)]dr \\ &\quad + \int_s^S \frac{e^{\delta(r-s)}}{e^{\delta S} - 1} [q(r, \beta(r)) + \delta\beta(r)]dr \\ &\leq \int_0^S Q(s, r)[q(r, \beta(r)) + \delta\beta(r)]dr = (\theta\beta)(s). \end{aligned} \quad (48)$$

Thus, $(\beta(s), \theta\beta(s)) \in \mathfrak{R}$, $s \in J$ and so $\mathcal{M}(\theta, \mathfrak{R}) \neq \emptyset$.

For $(\check{u}, \check{v}) \in \mathfrak{R}$,

$$\begin{aligned} \varsigma(\theta\check{u}, \theta\check{v}) &= \sup_{s \in J} |(\theta\check{u})(s) - (\theta\check{v})(s)| = \sup_{s \in J} |(\theta\check{v})(s) - (\theta\check{u})(s)| \\ &\leq \sup_{s \in J} \int_0^S Q(s, r)[q(r, \check{v}(r)) + \delta\check{v}(r) - q(r, \check{u}(r)) - \delta\check{u}(r)]dr \\ &\leq \sup_{s \in J} \int_0^S Q(s, r)\delta\phi(\check{v}(r) - \check{u}(r))dr \leq \delta\phi(\varsigma(\check{u}, \check{v})) \sup_{s \in J} \int_0^S Q(s, r)dr \\ &= \delta\phi(\varsigma(\check{u}, \check{v})) \sup_{s \in J} \frac{1}{e^{\delta S} - 1} \left(\frac{1}{\delta} e^{\delta(S+r-s)} \Big|_0^s + \frac{1}{\delta} e^{\delta(r-s)} \Big|_s^S \right) \\ &= \delta\phi(\varsigma(\check{u}, \check{v})) \frac{1}{\delta(e^{\delta S} - 1)} (e^{\delta S} - 1) \\ &= \phi(\varsigma(\check{u}, \check{v})) = \frac{\phi(\varsigma(\check{u}, \check{v}))}{\varsigma(\check{u}, \check{v})} \varsigma(\check{u}, \check{v}). \end{aligned} \quad (49)$$

Define $\varphi(\check{u}) = \phi(\check{u})/\check{u}$, then $\varphi \in \mathcal{C}$. By the last inequality, we derive

$$\varsigma(\theta\check{u}, \theta\check{v}) \leq \varphi(\varsigma(\check{u}, \check{v}))\varsigma(\check{u}, \check{v}), \check{u}, \check{v} \in C(J) \text{ with } (\check{u}, \check{v}) \in \mathfrak{R}. \quad (50)$$

Thus, all assumptions of Theorem 17 are satisfied, so θ has a fixed point. Finally, in view of the proof of Theorem 19, θ has a unique fixed point, which is indeed a unique solution of the problem (30). \square

5. Conclusion

In this work, we have proved new relational and graphical variants of fixed point results and validated all the assertions with the help of nontrivial examples. We have also provided a view to connect the theory of fixed point results equipped with a binary relation with that of graph related metrical fixed point theory. Further, inspired by the fact that boundary value problems appear in various branches of science and engineering, we resolve them to verify the genuineness and utility of the established conclusions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

No competing interests are associated with the article.

Authors' Contributions

All authors done the equal contributions to the article.

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Research Article

The Ulam Stability of Fractional Differential Equation with the Caputo-Fabrizio Derivative

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The aim of this paper is to establish the Ulam stability of the Caputo-Fabrizio fractional differential equation with integral boundary condition. We also present the existence and uniqueness results of the solution for the Caputo-Fabrizio fractional differential equation by Krasnoselskii's fixed point theorem and Banach fixed point theorem. Some examples are provided to illustrate our theorems.

1. Introduction

Ulam [1] proposed to study the approximation degree of the approximate solution and the exact solution of the equation in 1940. Hyers [2] responded to Ulam's proposal and defined the Hyers-Ulam stability of equation in 1941. Later on, Rassias [3] extended Hyers's work and defined the Hyers-Ulam-Rassias stability of equation in 1978. The Hyers-Ulam stability and Hyers-Ulam-Rassias stability are collectively referred to as the Ulam stability. Subsequently, researchers initiated a research on the Ulam stability of integer-order differential equations (see [4–10]). Obloza [4], Cemil and Emel [5] proved the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the first-order differential equation, respectively. Wang et al. [6] studied the Ulam stability of the first-order differential equation with a boundary value condition. Otrocol and Ilea [7] obtained the Ulam stability of the first-order delay differential equation. Huang and Li [8] also obtained the Hyers-Ulam stability of another class of the first-order delay differential equation. Zada et al. [9] studied the Hyers-Ulam-Rassias stability of the higher order delay differential equation. However, the study on the Ulam stability of fractional differential equations is in its infancy.

Fractional differential equations are widely applied in physics [11, 12], control systems [13], chemical technology [14], and biosciences [15]. Fractional integral boundary

value problems have been explored by many researchers. In particular, the integral boundary value problem provides a feasible method for the modeling of population dynamics and chemical engineering problems (see [16–18]). Although fractional integral boundary value problems are widely used, it is not easy to solve the equation, and the exact solution is often not obtained. Therefore, it is necessary to study the Ulam stability of fractional differential equations and use the approximate solution to replace the exact solution. So far, researchers have studied the Ulam stability and the existence and uniqueness of a solution for fractional differential equations with Hilfer-Hadamard, Caputo, and Caputo-Fabrizio fractional derivatives (see [19–22]). Abbas et al. [19] proved the existence and the Ulam stability of a fractional differential equation with the Hilfer-Hadamard derivative.

In [20], Wang et al. established the Ulam stability and data dependence for the Caputo fractional differential equation

$${}^c D^\beta x(t) = k(t, x(t)), t \in [a, +\infty). \quad (1)$$

In [21], Dai et al. studied the Ulam stability of the Caputo fractional differential equation with an integral boundary condition

$$\begin{cases} x'(t) + {}^c D_{0+}^{\beta} x(t) = k(t, x(t)), t \in [0, 1], \\ x(1) = I_{0+}^{\gamma} x(\eta), \end{cases} \quad (2)$$

where $I_{0+}^{\gamma}(\cdot)$ is the Riemann-Liouville fractional integral, $\gamma > 0$.

In [22], Liu et al. obtained the Hyers-Ulam stability and the existence of solutions for the Caputo-Fabrizio fractional differential equation

$${}^{CF}D^{\beta} x(t) = k(t, x(t)), t \in [0, T], \quad (3)$$

where ${}^{CF}D^{\beta}(\cdot)$ is the Caputo-Fabrizio fractional derivative, $\beta \in (0, 1)$.

Motivated by [20–22], in this paper, our purpose is to study the existence and uniqueness of a solution and the Ulam stability of the following Caputo-Fabrizio fractional differential equation with boundary value condition:

$$\begin{cases} x'(t) + {}^{CF}D^{\beta} x(t) = k(t, x(t)), t \in [0, 1], \\ x(1) = I_{0+}^{\gamma} x(\xi), \end{cases} \quad (4)$$

where $x(t)$ is a continuous differentiable function on $[0, 1]$; $k: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous; ${}^{CF}D^{\beta}(\cdot)$ is the Caputo-Fabrizio fractional derivative, $\beta \in (0, 1)$; and $I_{0+}^{\gamma}(\cdot)$ is the Riemann-Liouville fractional integral, $\gamma > 0$, $\xi \in [0, 1]$.

Equation (4) is a new kind of the Korteweg-de Vries-Burgers (KDVB) equation model. In [23], Equation (4) is used to describe unusual irregularities and nonlinearities in wave dynamics and liquids motions.

The main contributions are as follows: Firstly, we give the definitions of the Hyers-Ulam stability and Hyers-Ulam-Rassias stability for Equation (4). Then, we obtain a sufficient condition to derive the uniqueness of the solution for Equation (4) by the Banach contraction principle. Next, we give a sufficient condition to prove the existence of the solution for Equation (4) by Krasnoselskii's fixed point theorem. On this basis, we give the Ulam stability results for Equation (4) by the Laplace transform and inequality results.

The rest of our article is arranged as follows. Some basic definitions and necessary theorems are presented in Section 2. We establish sufficient conditions to show existence and uniqueness of solution for the Caputo-Fabrizio fractional differential equation in Section 3. In Section 4, we prove the Ulam stability of the Caputo-Fabrizio fractional differential equation. Two examples are provided in Section 5 to illustrate our theorems.

2. Preliminaries

We will denote by $C^1[0, 1]$ the space of continuous differentiable functions on $[0, 1]$ with norm

$$\|x\| = \sup \{|x(t)|, t \in [0, 1]\}. \quad (5)$$

Definition 1 [24]. The Caputo-Fabrizio fractional derivative of order β of a continuous differentiable function x is given by

$${}^{CF}D^{\beta} x(t) = \frac{(2-\beta)M(\beta)}{2(1-\beta)} \int_0^t \exp\left(-\frac{\beta}{1-\beta}(t-\tau)\right) x'(\tau) d\tau, t \geq 0, \quad (6)$$

the normalization function $M(\beta)$ depends on β .

Definition 2 [25]. The Riemann-Liouville fractional integral of order γ of a function x is given by

$$I_{0+}^{\gamma} x(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} x(\tau) d\tau, t \geq 0. \quad (7)$$

Based on Definition 2 in [5] and Definition 2.1 in [9], we give the definitions of the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability for Equation (4).

Definition 3. Equation (4) has the Hyers-Ulam stability if and only if for any solution $x(t)$ of

$$|x'(t) + {}^{CF}D^{\beta} x(t) - k(t, x(t))| \leq \varepsilon, t \in [0, 1], \quad (8)$$

where $\varepsilon > 0$, there is a constant $C > 0$ and a solution $y(t)$ of Equation (4) satisfying

$$|x(t) - y(t)| \leq C \times \varepsilon, t \in [0, 1]. \quad (9)$$

Definition 4. Equation (4) has the Hyers-Ulam-Rassias stability if and only if for any solution $x(t)$ of

$$|x'(t) + {}^{CF}D^{\beta} x(t) - k(t, x(t))| \leq \delta(t), t \in [0, 1], \quad (10)$$

where $\delta(t) \in C([0, 1], \mathbb{R}_+)$, there is a constant $K_{k,\delta} > 0$ and a solution $y(t)$ of Equation (4) satisfying

$$|x(t) - y(t)| \leq K_{k,\delta} \times \delta(t), t \in [0, 1]. \quad (11)$$

Theorem 5 [26]. If x is a piecewise continuous function and there exist $K > 0$ and μ such that

$$|x(t)| \leq K e^{\mu t}, t \geq t_0, \quad (12)$$

then the Laplace transform $L[x(t)](s)$ exists.

Theorem 6 [27]. Let $\beta \in (0, 1)$. The Laplace transform of ${}^{CF}D^{\beta} x(t)$ is

$$L[{}^{CF}D^{\beta} x(t)](s) = \frac{(2-\beta)M(\beta)}{2(s+\beta(1-s))} (sL[x(t)](s) - x(0)), s > 0, \quad (13)$$

where $L[x(t)](s)$ is the Laplace transform of $x(t)$.

Theorem 7. *The solution of the following fractional problem*

$$\begin{cases} x'(t) + {}^{CF}D^\beta x(t) = k(t, x(t)), t \in [0, 1], \\ x(1) = I_{0^+}^\gamma x(\xi). \end{cases} \quad (14)$$

is given by

$$x(t) = I_{0^+}^\gamma x(\xi) + \int_0^1 G(t, s)k(s, x(s))ds, \quad (15)$$

where

$$G(t, s) = \begin{cases} \left(1 - \frac{b_\beta}{a_\beta}\right) \exp(-a_\beta(t-s)) - \left(1 - \frac{b_\beta}{a_\beta}\right) \exp(-a_\beta(1-s)), & 0 \leq s \leq t, \\ -\left(1 - \frac{b_\beta}{a_\beta}\right) \exp(-a_\beta(1-s)) - \frac{b_\beta}{a_\beta}, & t \leq s \leq 1, \end{cases} \quad (16)$$

$$a_\beta = \frac{2\beta + (2-\beta)M(\beta)}{2(1-\beta)}, \quad b_\beta = \frac{2\beta}{2(1-\beta)}. \quad (17)$$

Proof. Since $x(t)$ is continuous differentiable function on $[0, 1]$, $x'(t)$ is bounded function on $[0, 1]$. By Definition 1, ${}^{CF}D^\beta x(t)$ is also a bounded function. Then, there exist constants $k_1, k_2 > 0$ and μ_1, μ_2 such that

$$\begin{aligned} |x'(t)| &\leq k_1 e^{\mu_1 t}, t \geq t_0 > 0, \\ |{}^{CF}D^\beta x(t)| &\leq k_2 e^{\mu_2 t}, t \geq t_0 > 0. \end{aligned} \quad (18)$$

From Theorem 5, the Laplace transform of $x'(t)$ and ${}^{CF}D^\beta x(t)$ exists.

Taking the Laplace transform for the first formula of Equation (14), we conclude

$$s\tilde{x}(s) - x(0) + \frac{(2-\beta)M(\beta)}{2(s+\beta(1-s))}(s\tilde{x}(s) - x(0)) = \tilde{k}(s, x(s)), \quad (19)$$

or

$$\begin{aligned} \tilde{x}(s) &= \frac{1}{s}x(0) + \frac{1}{s + (2\beta + (2-\beta)M(\beta)/2(1-\beta))}\tilde{k}(s, x(s)) \\ &\quad + \frac{2\beta/2(1-\beta)}{s(s + (2\beta + (2-\beta)M(\beta)/2(1-\beta)))}\tilde{k}(s, x(s)). \end{aligned} \quad (20)$$

Taking the Laplace inverse transform for the above equation, we conclude

$$\begin{aligned} x(t) &= x(0) + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s))k(s, x(s))ds \\ &\quad + \frac{b_\beta}{a_\beta} \int_0^t k(s, x(s))ds. \end{aligned} \quad (21)$$

Then

$$\begin{aligned} x(1) &= x(0) + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^1 \exp(-a_\beta(1-s))k(s, x(s))ds \\ &\quad + \frac{b_\beta}{a_\beta} \int_0^1 k(s, x(s))ds. \end{aligned} \quad (22)$$

Since $x(1) = I_{0^+}^\gamma x(\xi)$, thus

$$\begin{aligned} x(0) &= I_{0^+}^\gamma x(\xi) - \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^1 \exp(-a_\beta(1-s))k(s, x(s))ds \\ &\quad - \frac{b_\beta}{a_\beta} \int_0^1 k(s, x(s))ds. \end{aligned} \quad (23)$$

Then

$$\begin{aligned} x(t) &= I_{0^+}^\gamma x(\xi) - \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^1 \exp(-a_\beta(1-s))k(s, x(s))ds \\ &\quad - \frac{b_\beta}{a_\beta} \int_0^1 k(s, x(s))ds + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp \\ &\quad \cdot (-a_\beta(t-s))k(s, x(s))ds + \frac{b_\beta}{a_\beta} \int_0^t k(s, x(s))ds. \end{aligned} \quad (24)$$

By the definition of $G(t, s)$, we conclude

$$x(t) = I_{0^+}^\gamma x(\xi) + \int_0^1 G(t, s)k(s, x(s))ds. \quad (25)$$

□

Remark 8.

$$\begin{aligned} \int_0^t |G(t, s)|ds &= \int_0^t \left| \left(1 - \frac{b_\beta}{a_\beta}\right) \exp(-a_\beta(t-s)) \right. \\ &\quad \left. - \left(1 - \frac{b_\beta}{a_\beta}\right) \exp(-a_\beta(1-s)) \right| ds \\ &\leq \int_0^t \left[\left(1 - \frac{b_\beta}{a_\beta}\right) \exp(-a_\beta(t-s)) \right. \\ &\quad \left. + \left(1 - \frac{b_\beta}{a_\beta}\right) \exp(-a_\beta(1-s)) \right] ds \\ &\leq \int_0^t [\exp(-a_\beta(t-s)) + \exp(-a_\beta(1-s))] ds \\ &= [\exp(-a_\beta t) + \exp(-a_\beta)] \cdot \int_0^t \exp(a_\beta s) ds \\ &= \frac{1}{a_\beta} \cdot [\exp(-a_\beta t) + \exp(-a_\beta)] \\ &\quad \cdot [\exp(a_\beta t) - 1] \leq \frac{2}{a_\beta} \cdot [1 - \exp(-a_\beta)] = E. \end{aligned} \quad (26)$$

Thus, there exists a constant $E > 0$ such that

$$\int_0^t |G(t, s)| ds \leq E, \quad t \in [0, 1]. \quad (27)$$

Theorem 9 (Krasnoselskii's fixed point theorem). *Let S be a bounded convex closed subset of a Banach space W , and $P, Q : S \longrightarrow W$ satisfy the following:*

- (i) $Px + Qy \in S$, for all $x, y \in S$
- (ii) P is completely continuous
- (iii) Q is a contraction mapping

Then, $P + Q$ has at least one fixed point.

3. Existence and Uniqueness Theorems for Fractional Differential Equation

The following assumption will be needed throughout the paper:

(S₁): $k : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

(S₂): $k(t, x)$ satisfies the following Lipschitz condition for the second variable:

$$|k(t, x_1) - k(t, x_2)| \leq c_k |x_1 - x_2|, \quad x_1, x_2 \in \mathbb{R}, \quad t \in [0, 1]. \quad (28)$$

(S₃): Let $\delta(t) : [0, 1] \longrightarrow \mathbb{R}_+$ satisfy

$$\int_0^t \delta(s) ds \leq L_\delta \cdot \delta(t), \quad L_\delta > 0, \quad t \in [0, 1]. \quad (29)$$

Theorem 10. *Suppose that (S₁) and (S₂) are satisfied; then Equation (4) has a unique solution provided that $\xi^\gamma / (\Gamma(\gamma + 1)) + Ec_k < 1$.*

Proof. Since $k \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, there exists $T > 0$ such that

$$T = \max_{t \in [0, 1], s \in \mathbb{R}} |k(t, s)|. \quad (30)$$

Similar to the proof of Theorem 3 in [22]. Let operator F be given by

$$(Fx)(t) = I_{0+}^\gamma x(\xi) + \int_0^1 G(t, s) k(s, x(s)) ds. \quad (31)$$

Firstly, we prove that F maps a closed set into a closed set.

Let $U_b = \{x \in C^1([0, 1], \mathbb{R}) \mid \|x\| \leq b, b \geq ET/1 - \xi^\gamma/\Gamma(\gamma + 1) > 0\}$. For $x \in U_b$, it follows that

$$\begin{aligned} |(Fx)(t)| &\leq \frac{1}{\Gamma(\gamma)} \int_0^\xi (\xi - s)^{\gamma-1} |x(s)| ds + \int_0^1 |G(t, s)| |k(s, x(s))| ds \\ &\leq \frac{\xi^\gamma}{\Gamma(\gamma + 1)} b + ET \leq b. \end{aligned} \quad (32)$$

This implies $FU_b \subseteq U_b$.

Then, we prove that F is a strict contraction.

Let $x_1, x_2 \in C^1([0, 1], \mathbb{R})$, for any $t \in [0, 1]$; it follows that

$$\begin{aligned} |(Fx_1)(t) - (Fx_2)(t)| &\leq \left| \frac{1}{\Gamma(\gamma)} \int_0^\xi (\xi - s)^{\gamma-1} (x_1(s) - x_2(s)) ds \right. \\ &\quad \left. + \int_0^1 G(t, s) (k(s, x_1(s)) - k(s, x_2(s))) ds \right| \\ &\leq \left(\frac{\xi^\gamma}{\Gamma(\gamma + 1)} + Ec_k \right) \|x_1 - x_2\|. \end{aligned} \quad (33)$$

As $\xi^\gamma / (\Gamma(\gamma + 1)) + Ec_k < 1$, for $x_1, x_2 \in C^1([0, 1], \mathbb{R})$, F is a strict contraction. From the Banach fixed point theorem, F has a unique fixed point $x^*(t) \in C^1([0, 1], \mathbb{R})$; accordingly, Equation (4) has a unique solution. \square

Theorem 11. *Suppose that (S₁) and (S₂) are satisfied; then Equation (4) has at least one solution provided that $\xi^\gamma / (\Gamma(\gamma + 1)) + Ec_k < 1$.*

Proof. Since $k \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, there exists $T > 0$ such that

$$T = \max_{t \in [0, 1], s \in \mathbb{R}} |k(t, s)|. \quad (34)$$

Let $U_c = \{x \in C^1[0, 1] \mid \|x\| \leq c, c \geq ET/1 - \xi^\gamma/\Gamma(\gamma + 1) > 0\}$.

Let operators P and Q be given by

$$\begin{aligned} (Px)(t) &= - \left(1 - \frac{b_\beta}{a_\beta} \right) \int_t^1 \exp(-a_\beta(1-s)) k(s, x(s)) ds \\ &\quad - \frac{b_\beta}{a_\beta} \int_t^1 k(s, x(s)) ds, \\ (Qx)(t) &= I_{0+}^\gamma x(\xi) + \left(1 - \frac{b_\beta}{a_\beta} \right) \int_0^t \exp(-a_\beta(t-s)) k(s, x(s)) \\ &\quad - \exp(-a_\beta(1-s)) k(s, x(s)) ds. \end{aligned} \quad (35)$$

Firstly, for all $x_1, x_2 \in U_c$, using Remark 8, it follows that

$$\begin{aligned}
\|Px_1 + Qx_2\| &= \sup \left| -\left(1 - \frac{b_\beta}{a_\beta}\right) \int_t^1 \exp(-a_\beta(1-s)) \right. \\
&\quad \cdot k(s, x_1(s)) ds - \frac{b_\beta}{a_\beta} \int_t^1 k(s, x_1(s)) ds + I_{0^+}^\gamma x_2(\xi) \\
&\quad + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) k(s, x_2(s)) \\
&\quad \left. - \exp(-a_\beta(1-s)) k(s, x_2(s)) ds \right| \\
&\leq \sup \left| \int_0^t G(t, s) k(s, x_2(s)) ds \right. \\
&\quad \left. + \int_t^1 G(t, s) k(s, x_1(s)) ds \right| \\
&\quad + \sup \left| \frac{1}{\Gamma(\gamma)} \int_0^\xi (\xi-s)^{\gamma-1} x_2(s) ds \right| \\
&\leq \sup \left\{ \int_0^t |G(t, s)| |k(s, x_2(s))| ds \right. \\
&\quad \left. + \int_t^1 |G(t, s)| |k(s, x_1(s))| ds \right\} \\
&\quad + \sup \left| \frac{1}{\Gamma(\gamma)} \int_0^\xi (\xi-s)^{\gamma-1} ds \right| \cdot c \\
&\leq \sup \left\{ \int_0^1 |G(t, s)| ds \right\} \cdot T + \frac{\xi^\gamma}{\Gamma(\gamma+1)} c \\
&\leq \frac{\xi^\gamma}{\Gamma(\gamma+1)} c + ET \leq c.
\end{aligned} \tag{36}$$

Hence, we have $Px_1 + Qx_2 \in U_c$.

Then, for all $x_1, x_2 \in C^1[0, 1]$,

$$\begin{aligned}
\|Qx_1 - Qx_2\| &= \sup \left| I_{0^+}^\gamma x_1(\xi) - I_{0^+}^\gamma x_2(\xi) + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \right. \\
&\quad \cdot [\exp(-a_\beta(t-s)) - \exp(-a_\beta(1-s))] \\
&\quad \cdot [k(s, x_1(s)) - k(s, x_2(s))] ds \left| \right. \\
&\leq \sup \left| \frac{1}{\Gamma(\gamma)} \int_0^\xi (\xi-s)^{\gamma-1} |x_1(s) - x_2(s)| ds \right| \\
&\quad + \sup \left| \int_0^t |G(t, s)| \cdot |k(s, x_1(s)) - k(s, x_2(s))| ds \right| \\
&\leq \left(\frac{\xi^\gamma}{\Gamma(\gamma+1)} + Ec_k \right) \|x_1 - x_2\|.
\end{aligned} \tag{37}$$

As $\xi^\gamma/(\Gamma(\gamma+1)) + Ec_k < 1$, Q is a contraction mapping. Finally, we prove operator P is completely continuous.

Step 1. Operator P is continuous.

Let x_n be a convergent sequence, $x_n \rightarrow x \in C^1([0, 1], \mathbb{R})$, by Remark 8 and (S_2) ; it follows that

$$\begin{aligned}
|(Px_n)(t) - (Px)(t)| &= \left| \left(1 - \frac{b_\beta}{a_\beta}\right) \int_t^1 \exp(-a_\beta(1-s)) (k(s, x_n(s)) \right. \\
&\quad \left. - k(s, x(s))) ds + \frac{b_\beta}{a_\beta} \int_t^1 (k(s, x_n(s)) \right. \\
&\quad \left. - k(s, x(s))) ds \right| \leq \int_t^1 |G(t, s)| |k(s, x_n(s)) \\
&\quad - k(s, x(s))| ds \leq Ec_k \|x_n - x\|.
\end{aligned} \tag{38}$$

Since $x_n \rightarrow x$, we have $Px_n \rightarrow Px$; then operator P is continuous.

Step 2. Operator P is bounded on U_c .

$$\begin{aligned}
|(Px)(t)| &= \left| -\left(1 - \frac{b_\beta}{a_\beta}\right) \int_t^1 \exp(-a_\beta(1-s)) k(s, x(s)) ds \right. \\
&\quad \left. - \frac{b_\beta}{a_\beta} \int_t^1 k(s, x(s)) ds \right| = \left| \int_t^1 G(t, s) k(s, x(s)) ds \right| \\
&\leq \int_t^1 |G(t, s)| |k(s, x(s))| ds \leq ET.
\end{aligned} \tag{39}$$

Step 3. Operator P is equicontinuous in $C^1([0, 1], \mathbb{R})$.

Let $t_1, t_2 \in [0, 1]$ and $t_2 < t_1$, $x \in U_c$; it follows that

$$\begin{aligned}
|(Px)(t_1) - (Px)(t_2)| &= \left| \left(1 - \frac{b_\beta}{a_\beta}\right) \int_{t_2}^{t_1} \exp(-a_\beta(1-s)) k(s, x(s)) ds \right. \\
&\quad \left. + \frac{b_\beta}{a_\beta} \int_{t_2}^{t_1} k(s, x(s)) ds \right| \\
&\leq \left[\left(1 - \frac{b_\beta}{a_\beta}\right) \int_{t_2}^{t_1} |\exp(-a_\beta(1-s))| ds + \frac{b_\beta}{a_\beta} \int_{t_2}^{t_1} ds \right] \\
&\quad \cdot T \leq T \cdot |t_1 - t_2|.
\end{aligned} \tag{40}$$

Then, operator P is equicontinuous.

From Step 1-Step 3 and the Arzela-Ascoli theorem, P is completely continuous. By Theorem 9, $P + Q$ has at least one fixed point, since

$$(Px + Qx)(t) = I_{0^+}^\gamma x(\xi) + \int_0^1 G(t, s) k(s, x(s)) ds. \tag{41}$$

From Theorem 7, Equation (4) has at least one solution. \square

4. Stability Results

Theorem 12. Suppose that (S_1) and (S_2) are satisfied; then Equation (4) has the Hyers-Ulam stability on $[0, 1]$.

Proof. Since (S_1) and (S_2) hold, by Theorems 10 and 11, Equation (4) has a unique solution. From Theorem 7, Equation (4) has the unique solution

$$x(t) = x(0) + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) k(s, x(s)) ds + \frac{b_\beta}{a_\beta} \int_0^t k(s, x(s)) ds. \quad (42)$$

Let $y(t)$ satisfy $y(0) = x(0)$ and be a solution of the inequality

$$|y'(t) + {}^{CF}D^\beta y(t) - k(t, y(t))| \leq \varepsilon, t \in [0, 1]. \quad (43)$$

Set

$$G(t) = y'(t) + {}^{CF}D^\beta y(t) - k(t, y(t)), t \in [0, 1]. \quad (44)$$

Then

$$y'(t) + {}^{CF}D^\beta y(t) = G(t) + k(t, y(t)), t \in [0, 1], \quad (45)$$

$$|G(t)| \leq \varepsilon, t \in [0, 1].$$

From the proof of Theorem 7, we conclude

$$y(t) = y(0) + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) [G(s) + k(s, y(s))] ds + \frac{b_\beta}{a_\beta} \int_0^t [G(s) + k(s, y(s))] ds. \quad (46)$$

Then

$$\begin{aligned} & \left| y(t) - y(0) - \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) k(s, y(s)) ds - \frac{b_\beta}{a_\beta} \int_0^t k(s, y(s)) ds \right| \\ &= \left| \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) G(s) ds + \frac{b_\beta}{a_\beta} \int_0^t G(s) ds \right| \leq \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t |G(s)| ds + \frac{b_\beta}{a_\beta} \int_0^t |G(s)| ds \\ &\leq \int_0^t |G(s)| ds \leq \varepsilon. \end{aligned} \quad (47)$$

Thus

$$\begin{aligned} |y(t) - x(t)| &= \left| y(t) - x(0) - \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) k(s, x(s)) ds - \frac{b_\beta}{a_\beta} \int_0^t k(s, x(s)) ds \right| \\ &\leq \left| y(t) - y(0) - \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) k(s, y(s)) ds - \frac{b_\beta}{a_\beta} \int_0^t k(s, y(s)) ds \right| \\ &\quad + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) |k(s, y(s)) - k(s, x(s))| ds + \frac{b_\beta}{a_\beta} \int_0^t |k(s, y(s)) - k(s, x(s))| ds \\ &\leq \varepsilon + c_k \int_0^t |y(s) - x(s)| ds. \end{aligned} \quad (48)$$

From the Gronwall-Bellman inequality, we conclude

$$|y(t) - x(t)| \leq \left[\exp \left(\int_0^t c_k ds \right) \right] \cdot \varepsilon \leq \exp(c_k) \cdot \varepsilon. \quad (49)$$

From Definition 3, Equation (4) has the Hyers-Ulam stability. \square

Theorem 13. Suppose that (S_1) , (S_2) , and (S_3) are satisfied; then Equation (4) has the Hyers-Ulam-Rassias stability on $[0, 1]$.

Proof. Since (S_1) and (S_2) hold, by Theorems 10 and 11, Equation (4) has a unique solution. From Theorem 7, Equation (4) has the unique solution

$$x(t) = x(0) + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) k(s, x(s)) ds + \frac{b_\beta}{a_\beta} \int_0^t k(s, x(s)) ds. \quad (50)$$

Let $y(t)$ satisfy $y(0) = x(0)$ and be a solution of the inequality

$$|y'(t) + {}^{CF}D^\beta y(t) - k(t, y(t))| \leq \delta(t), t \in [0, 1]. \quad (51)$$

Set

$$G(t) = y'(t) + {}^{CF}D^\beta y(t) - k(t, y(t)), t \in [0, 1]. \quad (52)$$

Then

$$\begin{aligned} y'(t) + {}^{CF}D^\beta y(t) &= G(t) + k(t, y(t)), t \in [0, 1], \\ |G(t)| &\leq \delta(t), t \in [0, 1]. \end{aligned} \quad (53)$$

From the proof of Theorem 7, we conclude

$$\begin{aligned} y(t) &= y(0) + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) [G(s) + k(s, y(s))] ds \\ &\quad + \frac{b_\beta}{a_\beta} \int_0^t [G(s) + k(s, y(s))] ds. \end{aligned} \quad (54)$$

Then by (S_3) , it follows that

$$\begin{aligned} \left| y(t) - y(0) - \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) k(s, y(s)) ds \right. \\ \left. - \frac{b_\beta}{a_\beta} \int_0^t k(s, y(s)) ds \right| &= \left| \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) G(s) ds \right. \\ &\quad \left. + \frac{b_\beta}{a_\beta} \int_0^t G(s) ds \right| \leq \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t |G(s)| ds + \frac{b_\beta}{a_\beta} \int_0^t |G(s)| ds \\ &\leq \int_0^t |G(s)| ds \leq \int_0^t \delta(s) ds \leq L_\delta \cdot \delta(t). \end{aligned} \quad (55)$$

Thus

$$\begin{aligned} |y(t) - x(t)| &= \left| y(t) - x(0) - \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) k(s, x(s)) ds \right. \\ &\quad \left. - \frac{b_\beta}{a_\beta} \int_0^t k(s, x(s)) ds \right| \leq \left| y(t) - y(0) - \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) k(s, y(s)) ds \right. \\ &\quad \left. - \frac{b_\beta}{a_\beta} \int_0^t k(s, y(s)) ds \right| \\ &\quad + \left(1 - \frac{b_\beta}{a_\beta}\right) \int_0^t \exp(-a_\beta(t-s)) |k(s, y(s)) - k(s, x(s))| ds \\ &\quad + \frac{b_\beta}{a_\beta} \int_0^t |k(s, y(s)) - k(s, x(s))| ds \leq L_\delta \cdot \delta(t) + c_k \int_0^t |y(s) - x(s)| ds. \end{aligned} \quad (56)$$

From the Gronwall-Bellman inequality, we conclude

$$\begin{aligned} |y(t) - x(t)| &\leq L_\delta \times \delta(t) + \int_0^t \left[L_\delta \times \delta(s) \times c_k \exp\left(\int_s^t c_k dt\right) \right] ds \\ &\leq [L_\delta + L_\delta^2 c_k \exp(c_k)] \times \delta(t). \end{aligned} \quad (57)$$

From Definition 4, Equation (4) has the Hyers-Ulam-Rassias stability on $[0, 1]$. \square

5. Example

In this section, we give two examples to illustrate our main results.

Example 1. Consider the following problem of the Caputo-Fabrizio fractional differential equation of form

$$\begin{cases} x'(t) + {}^{CF}D^{\frac{1}{2}} x(t) = \frac{e^{-t}}{|x|+8}, t \in [0, 1], \\ x(1) = I_{0+}^{\frac{1}{2}} x\left(\frac{1}{4}\right), \end{cases} \quad (58)$$

and the following inequality

$$\left| y'(t) + {}^{CF}D^{\frac{1}{2}} y(t) - \frac{e^{-t}}{|y|+8} \right| \leq \delta(t), t \in [0, 1]. \quad (59)$$

Let

$$\beta = \frac{1}{3}, \gamma = \frac{1}{2}, \xi = \frac{1}{4}. \quad (60)$$

Then

$$M\left(\frac{1}{3}\right) = \frac{6}{5}, a_{\frac{1}{3}} = 2, b_{\frac{1}{3}} = \frac{1}{2}, \quad (61)$$

since

$$k(t, x) = \frac{e^{-t}}{|x|+8}, (t, x) \in [0, 1] \times \mathbb{R}. \quad (62)$$

Then, it follows that

$$\begin{aligned} |k(t, x_1) - k(t, x_2)| &= e^{-t} \left| \frac{1}{|x_1|+8} - \frac{1}{|x_2|+8} \right| \\ &\leq e^{-t} \left| \frac{1}{(|x_1|+8)(|x_2|+8)} \right| |x_1 - x_2| \\ &\leq \frac{e^{-t} |x_1 - x_2|}{64} \leq \frac{1}{64} |x_1 - x_2|. \end{aligned} \quad (63)$$

Hence, $c_k = 1/64$.

Therefore, (S_1) and (S_2) are satisfied, $\xi^\gamma / (\Gamma(\gamma+1)) + Ec_k = (1/4)^{1/2} / \Gamma(1/2+1) + 3 \times 1/64 < 1$. By Theorems 10 and 11, Equation (4) has a unique solution

$$x(t) = x(0) + \frac{3}{4} \int_0^t \exp(-2(t-s)) \frac{e^{-s}}{|x|+8} ds + \frac{1}{4} \int_0^t \frac{e^{-s}}{|x|+8} ds. \quad (64)$$

Set $\delta(t) = e^t \in C([0, 1], (0, +\infty))$, $\int_0^t \delta(s) ds = \int_0^t e^s ds = e^t - 1 \leq e^t$; we conclude $L_\delta = 1 > 0$.

Because $y(t)$ satisfies the following inequality:

$$\left| y'(t) + {}^{CF}D^{\frac{1}{3}}y(t) - \frac{e^{-t}}{|y|+8} \right| \leq \delta(t), t \in [0, 1], \quad (65)$$

it follows that

$$\begin{aligned} & \left| y(t) - x(0) - \left(1 - \frac{b_{1/3}}{a_{1/3}} \right) \int_0^t \exp \left(-a_{\frac{1}{3}}(t-s) \right) k(s, y(s)) ds \right. \\ & \quad \left. - \frac{b_{1/3}}{a_{1/3}} \int_0^t k(s, y(s)) ds \right| \leq e^t. \end{aligned} \quad (66)$$

Because (S_1) , (S_2) , and (S_3) are satisfied, by Theorem 13, it follows that

$$|y(t) - x(t)| \leq [L_\delta + L_\delta^2 c_k \exp(c_k)] \cdot e^t \leq \left(1 + \frac{1}{64} e^{\frac{1}{64}} \right) \cdot e^t. \quad (67)$$

Consequently, the equation has the Hyers-Ulam-Rassias stability.

Example 2. Consider the following problem of the Caputo-Fabrizio fractional differential equation of form

$$\begin{cases} x'(t) + {}^{CF}D^{\frac{1}{3}}x(t) = \frac{t}{|x|+8}, t \in [0, 1], \\ x(1) = I_{0^+}^{\frac{1}{3}}x\left(\frac{1}{2}\right), \end{cases} \quad (68)$$

and the following inequality

$$\left| y'(t) + {}^{CF}D^{\frac{1}{3}}y(t) - \frac{t}{|y|+8} \right| \leq \varepsilon, t \in [0, 1]. \quad (69)$$

Let

$$\beta = \frac{1}{2}, \gamma = \frac{1}{3}, \xi = \frac{1}{2}. \quad (70)$$

Then

$$M\left(\frac{1}{2}\right) = \frac{4}{3}, a_{\frac{1}{2}} = 3, b_{\frac{1}{2}} = 1, \quad (71)$$

since

$$k(t, x) = \frac{t}{|x|+8}, (t, x) \in [0, 1] \times \mathbb{R}. \quad (72)$$

Then, it follows that

$$\begin{aligned} |k(t, x_1) - k(t, x_2)| &= t \left| \frac{1}{|x_1|+8} - \frac{1}{|x_2|+8} \right| \\ &\leq t \left| \frac{1}{(|x_1|+8)(|x_2|+8)} \right| |x_1 - x_2| \\ &\leq \frac{t|x_1 - x_2|}{64} \leq \frac{1}{64} |x_1 - x_2|. \end{aligned} \quad (73)$$

Hence, $c_k = 1/64$.

Therefore, (S_1) and (S_2) are satisfied, $\xi^\gamma/(\Gamma(\gamma+1)) + Ec_k = (1/2)^{1/3}/\Gamma(1/3+1) + 3 \times 1/64 < 1$. By Theorems 10 and 11, Equation (4) has a unique solution

$$x(t) = x(0) + \frac{2}{3} \int_0^t \exp(-3(t-s)) \frac{s}{|x|+8} ds + \frac{1}{3} \int_0^t \frac{s}{|x|+8} ds. \quad (74)$$

Set $y(t) = e^t \in C([0, 1], (0, +\infty))$, and fix $\varepsilon = 9/32$; it follows that

$$\left| y'(t) + {}^{CF}D^{\frac{1}{3}}y(t) - \frac{t}{|y|+8} \right| \leq \frac{9}{32} = \varepsilon, t \in [0, 1]. \quad (75)$$

Because (S_1) and (S_2) are satisfied, by Theorem 12, we conclude

$$|y(t) - x(t)| \leq \exp\left(\frac{1}{64}\right) \cdot \frac{9}{32} = \exp\left(\frac{1}{64}\right) \cdot \varepsilon. \quad (76)$$

Consequently, the equation has the Hyers-Ulam stability.

6. Conclusions

In this article, we established the Ulam stability of the Caputo-Fabrizio fractional differential equation with an integral boundary condition by the Laplace transform method. Krasnoselskii's fixed point theorem and Banach fixed point theorem are employed to prove the existence and uniqueness results of the solution for the Caputo-Fabrizio fractional differential equation. Besides, we constructed a solution for the equation via new Green's function $G(t, s)$. The Ulam stability of the Caputo-Fabrizio fractional differential equation is used to study unusual irregularities and nonlinearities in wave dynamics and liquids motions. Because the Ulam stability is widely used, we will study the Ulam stability of the ABC fractional differential equation in the future study.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that she has no competing interests.

Authors' Contributions

The author read and approved the final manuscript.

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Research Article

A Forward-Backward-Forward Algorithm for Solving Quasimonotone Variational Inequalities

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In this paper, we continue to investigate the convergence analysis of Tseng-type forward-backward-forward algorithms for solving quasimonotone variational inequalities in Hilbert spaces. We use a self-adaptive technique to update the step sizes without prior knowledge of the Lipschitz constant of quasimonotone operators. Furthermore, we weaken the sequential weak continuity of quasimonotone operators to a weaker condition. Under some mild assumptions, we prove that Tseng-type forward-backward-forward algorithm converges weakly to a solution of quasimonotone variational inequalities.

1. Introduction

Let H be a real Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H . Let $f : H \rightarrow H$ be an operator. Our purpose of this paper is to investigate the following Stampacchia-type variational inequality (shortly, $VI(C, f)$).

Find $u \in C$ such that

$$\langle f(u), x - u \rangle \geq 0, \quad \forall x \in C. \quad (1)$$

Denote the solution set of (1) by $\text{Sol}(C, f)$.

Variational inequality problem (1) was introduced by Stampacchia [1] in 1964. Now it is well-known that variational inequality problem (1) provides a natural, convenient, and unified framework for the study of a large number of problems in economics, operation research, and engineering (see [2–5]). Variational inequality (1) contains, as special cases, such well-known problems in mathematical programming as systems of nonlinear equations, optimization problems ([3, 6]), complementarity problems ([7–9]), and fixed-point problems ([10–20]). Many iterative algorithms for solving variational inequalities and related problems have been proposed and investigated (see, for example, [1, 6, 9,

16, 21–40]). Among them, one of the influential algorithms for solving $VI(C, f)$ is the projection-gradient algorithm ([28, 39, 40]) which defines a sequence $\{u^k\}$ by

$$u^{k+1} = P_C \left(u^k - \lambda f(u^k) \right), \quad \forall k \geq 0, \quad (2)$$

where P_C is the orthogonal projection operator onto C and $\lambda > 0$ is the step size.

The projection-gradient algorithm guarantees the convergence of the sequence $\{u^k\}$ defined by (2) if f is strongly (pseudo-)monotone (see [8, 41]) or f is inverse strongly monotone (see [3, 42]). However, if f is plain monotone, then the sequence $\{u^k\}$ generated by (2) does not necessarily converge. Consequently, Korpelevich [43] proposed an extragradient algorithm which generates a sequence $\{u^k\}$ by

$$\begin{cases} u^0 \in H, \\ v^k = P_C \left(u^k - \lambda f(u^k) \right), \\ u^{k+1} = P_C \left(u^k - \lambda f(v^k) \right), \quad \forall k \geq 0. \end{cases} \quad (3)$$

This algorithm guarantees the convergence of the sequence $\{u^k\}$ defined by (3) if f is pseudomonotone. Since then, Korpelevich's algorithm has attracted so much attention by many scholars, who modified it in several different forms (see, e.g., [34, 44–47]). Especially, Vuong [31] proved that Korpelevich's extragradient method has weak convergence provided that f is sequentially weakly continuous and pseudomonotone.

A challenging task when devise efficient algorithms for solving variational inequalities is to avoid to compute the projection operators at each iteration because the computation of the projection operator may be very expensive. In this respect, Tseng [30] modified extragradient algorithm with the following form:

$$\begin{cases} u^0 \in H, \\ v^k = P_C(u^k - \lambda f(u^k)), \\ u^{k+1} = u^k + \lambda(f(u^k) - f(v^k)), \quad \forall k \geq 0. \end{cases} \quad (4)$$

Boţ et al. [48] approach the solution of $VI(C, f)$ from a continuous perspective by means of trajectories generated by the following dynamical system of forward-backward-forward type:

$$\begin{cases} u(0) = u^0, \\ v(t) = P_C(u(t) - \lambda f(u(t))), \\ \dot{u}(t) + u(t) = v(t) + \lambda(f(u(t)) - f(v(t))), \end{cases} \quad (5)$$

where $\lambda > 0$ and $u^0 \in H$.

Note that (5) has its roots and the existence and uniqueness of the trajectory $x \in C^1([0, +\infty), H)$ generated by (5) has been obtained (see [49]). The explicit time discretization of the dynamical system (5) yields the following Tseng-type forward-backward-forward algorithm:

$$\begin{cases} u^0 \in H, \\ v^k = P_C(u^k - \lambda f(u^k)), \\ u^{k+1} = \mu_k(v^k + \lambda(f(u^k) - f(v^k))) + (1 - \mu_k)u^k, \quad \forall k \geq 0. \end{cases} \quad (6)$$

Bot et al. ([48]) proved that the sequence $\{u^k\}$ generated by (6) converges weakly to an element in $\text{Sol}(C, f)$ provided f is pseudomonotone and sequentially weakly continuous. On the other hand, for solving (1) and related problems, some self-techniques have been used to relax the step size without prior knowledge of the Lipschitz constant of the operator f (see [50–53]).

Let $\text{Sol}^d(C, f)$ be the solution set of the dual variational inequality of (1), that is,

$$\text{Sol}^d(C, f) := \{u \in C \mid \langle f(x), x - u \rangle \geq 0, \forall x \in C\}. \quad (7)$$

Note that $\text{Sol}^d(C, f)$ is closed convex. If C is convex and f is continuous, then $\text{Sol}^d(C, f) \subset \text{Sol}(C, f)$.

To prove the convergence of the sequence $\{u^k\}$, a common assumption $\text{Sol}(C, f) \subset \text{Sol}^d(C, f)$ has been used, that is,

$$\langle f(x), x - u \rangle \geq 0, \quad \forall u \in \text{Sol}(C, f), \quad x \in C, \quad (8)$$

which is a direct consequence of the pseudomonotonicity of f . But this conclusion (that is, $\text{Sol}(C, f) \subset \text{Sol}^d(C, f)$) is false, if f is quasimonotone.

In this paper, we introduce a self-adaptive Tseng-type forward-backward-forward algorithm to solve quasimonotone variational inequalities (1). The algorithm is designed such that the step sizes are dynamically chosen and its convergence is guaranteed without prior knowledge of the Lipschitz constant of f . Moreover, we replace the sequential weak continuity imposed on f by a weaker condition. We show that the proposed algorithm converges weakly to a solution of quasimonotone variational inequalities under some additional conditions.

2. Preliminaries

Let C be a nonempty convex and closed subset of a real Hilbert space H . Use “ \rightharpoonup ” and “ \rightarrow ” to denote weak convergence and strong convergence, respectively. Let $f : H \rightarrow H$ be an operator. Recall that f is said to be

- (i) strongly monotone if there exists a positive constant α such that

$$\langle f(u) - f(v), u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H \quad (9)$$

- (ii) α -inverse strongly monotone if there exists a positive constant α such that

$$\langle f(u) - f(v), u - v \rangle \geq \alpha \|f(u) - f(v)\|^2, \quad \forall u, v \in H \quad (10)$$

- (iii) monotone if

$$\langle f(u) - f(v), u - v \rangle \geq 0, \quad \forall u, v \in H \quad (11)$$

- (iv) pseudomonotone if

$$\langle f(v), u - v \rangle \geq 0 \text{ implies } \langle f(u), u - v \rangle \geq 0, \quad \forall u, v \in H \quad (12)$$

- (v) quasimonotone if

$$\langle f(v), u - v \rangle > 0 \text{ implies } \langle f(u), u - v \rangle \geq 0, \quad \forall u, v \in H \quad (13)$$

It is easy to see that strongmonotonicity \Rightarrow monotonicity \Rightarrow pseudomonotonicity \Rightarrow quasimonotonicity.

But the reverse assertions are not true in general.

Example 1 (see [50]). Let $H = \mathbb{R}^4$ and $C = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - x_2 - x_3 \geq 1\}$. Let $f : C \rightarrow \mathbb{R}^4$ be defined by $f(x) = (\|x\|^2 + 2)u$ for all $x \in C$, where $u = (1, -1, -1, 0)^T$. Then, f is pseudomonotone on C . But f is not monotone on C .

Example 2 (see [33]). The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is quasimonotone on \mathbb{R} , but not pseudomonotone on \mathbb{R} .

An operator $f : H \rightarrow H$ is said to be η -Lipschitz continuous if there exists a positive constant η such that

$$\|f(u) - f(v)\| \leq \eta \|u - v\|, \quad \forall u, v \in H. \quad (14)$$

If $\eta = 1$, then f is said to be nonexpansive.

An operator $f : H \rightarrow H$ is said to be sequentially weakly continuous if for given sequence $\{u^k\}$: $u^k \rightharpoonup u$ implies that $f(u^k) \rightharpoonup f(u)$.

For $\forall x \in H$, there exists a unique point in C , denoted by $P_C(x)$ satisfying

$$\|x - P_C(x)\| \leq \|y - x\|, \quad \forall y \in C. \quad (15)$$

Moreover, P_C has the following property:

$$\langle x - P_C(x), y - P_C(x) \rangle \leq 0, \quad \forall x \in H, \forall y \in C. \quad (16)$$

3. Main Results

In this section, we present our main results.

Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Assume that the following conditions are satisfied:

(C1) The operator f is quasimonotone on H .

(C2) The operator f is η -Lipschitz continuous on H .

(C3) $\text{Sol}^d(C, f) \neq \emptyset$ and $\{u \in C : f(u) = 0\} \setminus \text{Sol}^d(C, f)$ is a finite set.

Assume that the operator f possesses the following property: for any given sequence $\{u^k\} \subset H$,

$$\left. \begin{array}{l} u^k \rightharpoonup u^\dagger \in H, \\ \liminf_{k \rightarrow +\infty} \|f(u^k)\| = 0 \end{array} \right\} \text{ imply that } f(u^\dagger) = 0. \quad (17)$$

Remark 1. If the operator f is sequentially weakly continuous, then f satisfies the above property (17).

Next, we propose a self-adaptive Tseng-type forward-backward-forward algorithm for solving the quasimonotone variational inequality (1).

Remark 2. If $v^k = u^k$, that is, $u^k = P_C(u^k - \lambda_k f(u^k))$, then $u^k \in \text{Sol}(C, f)$. In what follows, we assume that $v^k \neq u^k$. In this case, we can obtain an infinite sequence $\{u^k\}$ generated by Algorithm 1.

Remark 3. According to the definition (3.4) of $\{\lambda_k\}$, λ_k is monotonically decreasing and therefore converges. Set $\lim_{k \rightarrow +\infty} \lambda_k = \tilde{\lambda}$. It is obvious that $\min \{\delta/\eta, \lambda_0\} \leq \tilde{\lambda} \leq \lambda_0$.

Next, we prove the convergence of the sequence $\{u^k\}$ generated by Algorithm 1.

Theorem 4. Suppose that the conditions (C1)-(C3) and (17) are satisfied. Then, the sequence $\{u^k\}$ generated by Algorithm 1 converges weakly to a point in $\text{Sol}(C, f)$.

Proof. Let $x^* \in \text{Sol}^d(C, f)$. Set $w^k = v^k + \lambda_k(f(u^k) - f(v^k))$, $\forall k \geq 0$. Then, we have

$$\begin{aligned} \|w^k - x^*\|^2 &= \|v^k + \lambda_k(f(u^k) - f(v^k)) - x^*\|^2 \\ &= \|v^k - x^*\|^2 + 2\lambda_k \langle f(u^k) - f(v^k), v^k - x^* \rangle \\ &\quad + \lambda_k^2 \|f(u^k) - f(v^k)\|^2 = \|u^k - x^*\|^2 \\ &\quad + \|v^k - u^k\|^2 + 2\langle v^k - u^k, u^k - x^* \rangle \\ &\quad + 2\lambda_k \langle f(u^k) - f(v^k), v^k - x^* \rangle \\ &\quad + \lambda_k^2 \|f(u^k) - f(v^k)\|^2 = \|u^k - x^*\|^2 \\ &\quad - \|v^k - u^k\|^2 + 2\langle v^k - u^k, v^k - x^* \rangle \\ &\quad + 2\lambda_k \langle f(u^k) - f(v^k), v^k - x^* \rangle \\ &\quad + \lambda_k^2 \|f(u^k) - f(v^k)\|^2. \end{aligned} \quad (18)$$

Since $x^* \in \text{Sol}^d(C, f) \subset C$, from (16) and (3.2), we achieve $\langle u^k - \lambda_k f(u^k) - v^k, x^* - v^k \rangle \leq 0$. It follows that

$$\langle u^k - v^k, x^* - v^k \rangle \leq \lambda_k \langle f(u^k), x^* - v^k \rangle. \quad (19)$$

Using $v^k \in C$ and $x^* \in \text{Sol}^d(C, f)$, we obtain

$$\langle f(v^k), v^k - x^* \rangle \geq 0. \quad (20)$$

By (18), (19), and (20), we receive

$$\begin{aligned} \|w^k - x^*\|^2 &\leq \|u^k - x^*\|^2 - \|v^k - u^k\|^2 + 2\lambda_k \langle f(u^k), x^* - v^k \rangle \\ &\quad + 2\lambda_k \langle f(u^k) - f(v^k), v^k - x^* \rangle \\ &\quad + \lambda_k^2 \|f(u^k) - f(v^k)\|^2 \end{aligned}$$

Step 1. Let u^k and λ_k be given. Compute
 $v^k = P_c(u^k - \lambda_k f(u^k)).$

Criterion: if $v^k = u^k$, then stop.

Step 2. Compute

$$u^{k+1} = \mu_k(v^k + \lambda_k(f(u^k) - f(v^k))) + (1 - \mu_k)u^k,$$

$$\lambda_{k+1} = \begin{cases} \min\{(\delta\|u^k - v^k\|/\|f(u^k) - f(v^k)\|), \lambda_k\}, & f(u^k) \neq f(v^k), \\ \lambda_k, & \text{else,} \end{cases}$$

update k to $k + 1$ and go to Step 1.

ALGORITHM 1: Let $\lambda_0 > 0$ and $\delta \in (0, 1)$. Select the starting point $u^0 \in H$ and the sequence of relaxation parameters $\{\mu_k\}_{k \geq 0} \subset (0, 1]$ satisfying $\liminf_{k \rightarrow +\infty} \mu_k > 0$. Set $k = 0$.

$$\begin{aligned} &= \|u^k - x^*\|^2 - \|v^k - u^k\|^2 + 2\lambda_k \langle f(v^k), x^* - v^k \rangle \\ &\quad + \lambda_k^2 \|f(u^k) - f(v^k)\|^2 \\ &\leq \|u^k - x^*\|^2 - \|v^k - u^k\|^2 + \lambda_k^2 \|f(u^k) - f(v^k)\|^2. \end{aligned} \quad (21)$$

From (3.4), we have $\|f(u^k) - f(v^k)\| \leq \delta/\lambda_{k+1} \|u^k - v^k\|$. This together with (21) implies that

$$\|w^k - x^*\|^2 \leq \|u^k - x^*\|^2 - \left(1 - \delta^2 \frac{\lambda_k^2}{\lambda_{k+1}^2}\right) \|v^k - u^k\|^2. \quad (22)$$

Note that $\lim_{k \rightarrow +\infty} \lambda_k/\lambda_{k+1} = 1$. So, there exists an integer K such that $(1 - \delta^2(\lambda_k^2/\lambda_{k+1}^2)) > 0$ when $k \geq K$. Hence, from (22), we deduce $\|w^k - x^*\| \leq \|u^k - x^*\|$ when $k \geq K$.

In terms of (3.3), we get

$$\begin{aligned} \|u^{k+1} - x^*\| &= \|\mu_k(w^k - x^*) + (1 - \mu_k)(u^k - x^*)\| \\ &\leq \mu_k \|w^k - x^*\| + (1 - \mu_k) \|u^k - x^*\| \\ &\leq \|u^k - x^*\|. \end{aligned} \quad (23)$$

Thus, the sequence $\{\|u^k - x^*\|\}$ is monotonically decreasing and $\lim_{k \rightarrow +\infty} \|u^k - x^*\|$ exists. So, the sequence $\{u^k\}$ is bounded.

By virtue of (22) and (23), we have

$$\begin{aligned} \|u^{k+1} - x^*\|^2 &\leq \mu_k \|w^k - x^*\|^2 + (1 - \mu_k) \|u^k - x^*\|^2 \\ &\leq \|u^k - x^*\|^2 - \mu_k \left(1 - \delta^2 \frac{\lambda_k^2}{\lambda_{k+1}^2}\right) \|v^k - u^k\|^2. \end{aligned} \quad (24)$$

It follows that

$$\mu_k \left(1 - \delta^2 \frac{\lambda_k^2}{\lambda_{k+1}^2}\right) \|v^k - u^k\|^2 \leq \|u^k - x^*\|^2 - \|u^{k+1} - x^*\|^2. \quad (25)$$

Since $\lim_{k \rightarrow +\infty} (1 - \delta^2(\lambda_k^2/\lambda_{k+1}^2)) = 1 - \delta^2 > 0$, $\liminf_{k \rightarrow +\infty} \mu_k > 0$, and $\lim_{k \rightarrow +\infty} \|u^k - x^*\|$ exist, it follows from (25) that

$$\lim_{k \rightarrow +\infty} \|v^k - u^k\| = 0. \quad (26)$$

Since f is Lipschitz, from (26), we obtain

$$\lim_{k \rightarrow +\infty} \|f(v^k) - f(u^k)\| = 0. \quad (27)$$

Thanks to (3.3), we derive

$$\|u^{k+1} - u^k\| \leq \mu_k \|u^k - v^k\| + \mu_k \lambda_k \|f(u^k) - f(v^k)\|. \quad (28)$$

Based on (26)–(28), we deduce

$$\lim_{k \rightarrow +\infty} \|u^{k+1} - u^k\| = 0. \quad (29)$$

According to (16) and (3.2), we have

$$\langle u^k - \lambda_k f(u^k) - v^k, x - v^k \rangle \leq 0, \quad \forall x \in C. \quad (30)$$

It follows that

$$\begin{aligned} &\frac{1}{\lambda_k} \langle u^k - v^k, x - v^k \rangle + \langle f(u^k), v^k - u^k \rangle \\ &\leq \langle f(u^k), x - u^k \rangle, \quad \forall x \in C. \end{aligned} \quad (31)$$

Since $\{u^k\}$ is bounded, by (26), $\{v^k\}$ is also bounded. At the same time, using the Lipschitz continuity of f , $\{f(u^k)\}$ is bounded. Combining (26), (27), and (31), we attain

$$\liminf_{k \rightarrow +\infty} \langle f(u^k), x - u^k \rangle \geq 0, \quad \forall x \in C. \quad (32)$$

Since $\{u^k\}$ is bounded, there exists a subsequence $\{u^{k_i}\}$ of $\{u^k\}$ such that $u^{k_i} \rightharpoonup \hat{u} \in C$ as $i \rightarrow +\infty$. By virtue of

(32), we have

$$\liminf_{i \rightarrow +\infty} \langle f(u^{k_i}), x - u^{k_i} \rangle \geq 0, \quad \forall x \in C. \quad (33)$$

Next, we consider two possible cases.

Case 1. $\liminf_{i \rightarrow +\infty} \|f(u^{k_i})\| = 0$. Since $u^{k_i} \rightarrow \hat{u}$ and f satisfies (17), we deduce that $f(\hat{u}) = 0$.

Case 2. $\liminf_{i \rightarrow +\infty} \|f(u^{k_i})\| > 0$. In this case, $\exists I_0 > 0$ such that $f(u^{k_i}) \neq 0$ for all $i \geq I_0$. From (33), we obtain

$$\liminf_{i \rightarrow +\infty} \left\langle \frac{f(u^{k_i})}{\|f(u^{k_i})\|}, x - u^{k_i} \right\rangle \geq 0, \quad \forall x \in C. \quad (34)$$

Choose a positive strictly decreasing sequence $\{\varepsilon_j\}$ such that $\varepsilon_j \rightarrow 0$ as $j \rightarrow +\infty$. Thanks to (34), there exists a strictly increasing subsequence $\{k_{i_j}\}$ with the property that $k_{i_j} \geq I_0$ and

$$\left\langle \frac{f(u^{k_{i_j}})}{\|f(u^{k_{i_j}})\|}, x - u^{k_{i_j}} \right\rangle + \varepsilon_j > 0, \quad \forall j \geq 0. \quad (35)$$

It follows that

$$\langle f(u^{k_{i_j}}), x - u^{k_{i_j}} \rangle + \varepsilon_j \|f(u^{k_{i_j}})\| > 0, \quad \forall j \geq 0. \quad (36)$$

Set $y^j = f(u^{k_{i_j}}) / \|f(u^{k_{i_j}})\|^2$ for all $j \geq 0$. Thus, we have $\langle f(u^{k_{i_j}}), y^j \rangle = 1$ for each $j \geq 0$. From (36), we deduce

$$\langle f(u^{k_{i_j}}), x + \varepsilon_j \|f(u^{k_{i_j}})\| y^j - u^{k_{i_j}} \rangle > 0, \quad \forall j \geq 0. \quad (37)$$

Since f is quasimonotone on H , by (37), we get

$$\langle f(x + \varepsilon_j \|f(u^{k_{i_j}})\| y^j), x + \varepsilon_j \|f(u^{k_{i_j}})\| y^j - u^{k_{i_j}} \rangle \geq 0, \quad \forall j \geq 0. \quad (38)$$

Observe that $\lim_{j \rightarrow +\infty} \varepsilon_j \|f(u^{k_{i_j}})\| \|y^j\| = \lim_{j \rightarrow +\infty} \varepsilon_j = 0$. Since f is Lipschitz continuous, $f(x + \varepsilon_j \|f(u^{k_{i_j}})\| y^j) \rightarrow f(x)$ as $j \rightarrow +\infty$. Thus, taking the limit as $j \rightarrow +\infty$ in (38), we obtain that

$$\langle f(x), x - \hat{u} \rangle \geq 0, \quad \forall x \in C. \quad (39)$$

So, $\hat{u} \in \text{Sol}^d(C, f)$.

Next, we prove $\{u^k\}$ has finite weak cluster points in $\text{Sol}(C, f)$. First, we show that $\{u^k\}$ has at most one weak cluster point in $\text{Sol}^d(C, f)$. Let $\hat{u} \in \text{Sol}^d(C, f)$ and $\tilde{u} \in \text{Sol}^d(C, f)$ be two distinct weak cluster points of $\{u^k\}$. There exist two sequences $\{u^{k_i}\}$ and $\{u^{k_j}\}$ of $\{u^k\}$ satisfying $u^{k_i} \rightharpoonup \hat{u}$ as

$i \rightarrow +\infty$ and $u^{k_j} \rightharpoonup \tilde{u}$ as $j \rightarrow +\infty$. Note that for all $k \geq 0$,

$$2 \langle u^k, \hat{u} - \tilde{u} \rangle = \|u^k - \tilde{u}\|^2 - \|u^k - \hat{u}\|^2 + \|\hat{u}\|^2 - \|\tilde{u}\|^2. \quad (40)$$

Since $\lim_{k \rightarrow +\infty} \|u^k - \hat{u}\|$ and $\lim_{k \rightarrow +\infty} \|u^k - \tilde{u}\|$ exist, by (40), we conclude that $\lim_{k \rightarrow +\infty} \langle u^k, \hat{u} - \tilde{u} \rangle$ exists, denoted by l . Thus,

$$l = \lim_{i \rightarrow +\infty} \langle u^{k_i}, \hat{u} - \tilde{u} \rangle = \lim_{j \rightarrow +\infty} \langle u^{k_j}, \hat{u} - \tilde{u} \rangle. \quad (41)$$

Since $u^{k_i} \rightharpoonup \hat{u}$ and $u^{k_j} \rightharpoonup \tilde{u}$, from (41), we have

$$l = \langle \hat{u}, \hat{u} - \tilde{u} \rangle = \langle \tilde{u}, \hat{u} - \tilde{u} \rangle, \quad (42)$$

which implies that $\|\hat{u} - \tilde{u}\|^2 = 0$ and hence, $\hat{u} = \tilde{u}$. Therefore, $\{u^k\}$ has at most one weak cluster point in $\text{Sol}^d(C, f)$. By the condition (C3), $\{u \in C, f(u) = 0\} \setminus \text{Sol}^d(C, f)$ is a finite set. Therefore, $\{u^k\}$ has finite weak cluster points in $\text{Sol}(C, f)$.

Let p_1, p_2, \dots, p_t be the finite weak cluster points of $\{u^k\}$ in $\text{Sol}(C, f)$. Set $I = \{1, 2, \dots, t\}$ and

$$\sigma = \min \left\{ \frac{\|p_n - p_m\|}{3}, n, m \in I, n \neq m \right\}. \quad (43)$$

Taking any weak cluster point $p_n, n \in I$, there exists a subsequence $\{u_n^{k_i}\}$ of $\{u^k\}$ such that $u_n^{k_i} \rightharpoonup p_n$ as $i \rightarrow +\infty$. Then, we have

$$\lim_{i \rightarrow +\infty} \left\langle u_n^{k_i}, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle = \left\langle p_n, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle, \quad \forall m \in I. \quad (44)$$

Observe that $\forall m \neq n$,

$$\begin{aligned} \left\langle p_n, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle &= \frac{\|p_n - p_m\|}{2} + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|} \\ &> \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|}. \end{aligned} \quad (45)$$

According to (44) and (45), there exists a large enough positive integer $n(i)$ such that when $i \geq n(i)$,

$$u_n^{k_i} \in \left\{ x : \left\langle x, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|} \right\}, \quad m \in I, m \neq n. \quad (46)$$

Set

$$R_n = \bigcap_{m=1, m \neq n}^t \left\{ x : \left\langle x, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|} \right\}. \quad (47)$$

Step 1. Let u^k and λ_k be given. Compute $v^k = P_C(u^k - \lambda_k f(u^k))$.
 Criterion: if $v^k = u^k$, then stop.
 Step 2. Compute $u^{k+1} = v^k + \lambda_k(f(u^k) - f(v^k))$,
 $\lambda_{k+1} = \begin{cases} \min \{(\delta \|u^k - v^k\| / \|f(u^k) - f(v^k)\|), \lambda_k\}, & f(u^k) \neq f(v^k), \\ \lambda_k, & \text{else,} \end{cases}$
 update k to $k+1$ and go to Step 1.

ALGORITHM 2: Let $\lambda_0 > 0$ and $\delta \in (0, 1)$. Select the starting point $u^0 \in H$. Set $k = 0$.

In the light of (46) and (47), we have $u_{n_i}^{k_i} \in R_n$ when $i \geq \max \{n(i), n \in I\}$.

Now, we show that $u^k \in \bigcup_{n=1}^t R_n$ for a large enough k . Assume that there exists a subsequence $\{u^{k_j}\}$ of $\{u^k\}$ such that $u^{k_j} \notin \bigcup_{n=1}^t R_n$. By the boundedness of $\{u^{k_j}\}$, there exists a subsequence of $\{u^{k_j}\}$ convergent weakly to p^\dagger . Without loss of generality, we still denote the subsequence as $\{u^{k_j}\}$. According to assumptions, $u^{k_j} \notin \bigcup_{n=1}^t R_n$, so $u^{k_j} \notin R_n$ for any $n \in I$. Therefore, there exists a subsequence $\{u^{k_{j_s}}\}$ of $\{u^{k_j}\}$ such that $\forall s \geq 0$,

$$u^{k_{j_s}} \notin \left\{ x : \left\langle x, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|} \right\}, \quad m \in I, m \neq n. \quad (48)$$

Thus,

$$p^\dagger \notin \left\{ x : \left\langle x, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|} \right\}, \quad m \in I, m \neq n, \quad (49)$$

which implies that $p^\dagger \neq p_n, n \in I$. This is impossible. So, for a large enough positive integer K_0 , $u^k \in \bigcup_{n=1}^t R_n$ when $k \geq K_0$.

Next, we show that $\{u^k\}$ has a unique weak cluster point in $\text{Sol}(C, f)$. First, from (29), there exists a positive integer $K_1 \geq K_0$ such that $\|u^{k+1} - u^k\| < \sigma$ for all $k \geq K_1$. Assume that $\{u^k\}$ has at least two weak cluster points in $\text{Sol}(C, f)$. Then, there exists $\hat{K} \geq K_1$ such that $u^{\hat{K}} \in R_n$ and $u^{\hat{K}+1} \in R_m$, where $n, m \in I$ and $t \geq 2$, that is,

$$\begin{aligned} u^{\hat{K}} \in R_n &= \bigcap_{m=1, m \neq n}^t \left\{ x : \left\langle x, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|} \right\}, \\ u^{\hat{K}+1} \in R_m &= \bigcap_{n=1, n \neq m}^t \left\{ x : \left\langle x, \frac{p_m - p_n}{\|p_m - p_n\|} \right\rangle > \sigma + \frac{\|p_m\|^2 - \|p_n\|^2}{2\|p_m - p_n\|} \right\}. \end{aligned} \quad (50)$$

Therefore,

$$\left\langle u^{\hat{K}}, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|}, \quad (51)$$

$$\left\langle u^{\hat{K}+1}, \frac{p_m - p_n}{\|p_m - p_n\|} \right\rangle > \sigma + \frac{\|p_m\|^2 - \|p_n\|^2}{2\|p_m - p_n\|}. \quad (52)$$

Combining (51) and (52), we achieve

$$\left\langle u^{\hat{K}} - u^{\hat{K}+1}, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle > 2\sigma. \quad (53)$$

At the same time, we have

$$\|u^{\hat{K}+1} - u^{\hat{K}}\| < \sigma. \quad (54)$$

Based on (53) and (54), we deduce

$$2\sigma < \left\langle u^{\hat{K}} - u^{\hat{K}+1}, \frac{p_n - p_m}{\|p_n - p_m\|} \right\rangle \leq \|u^{\hat{K}} - u^{\hat{K}+1}\| < \sigma. \quad (55)$$

This leads to a contradiction. Thus, $\{u^k\}$ has a unique weak cluster point in $\text{Sol}(C, f)$. Therefore, $\{u^k\}$ converges weakly to a point in $\text{Sol}(C, f)$. This completes the proof. \square

Corollary 5. Suppose that the conditions (C1)-(C3) and (17) are satisfied. Then, the sequence $\{u^k\}$ generated by Algorithm 2 converges weakly to a point in $\text{Sol}(C, f)$.

Remark 6. If f is pseudomonotone, then Theorem 4 and Corollary 5 hold.

Remark 7. If the operator f is sequentially weakly continuous and also satisfies conditions (C1)-(C3), then Theorem 4 and Corollary 5 still hold.

Remark 8. Our main purpose is to solve (1); hence, a natural condition is $\text{Sol}(C, f) \neq \emptyset$. In order to prove our main theorem, we assume that $\text{Sol}^d(C, f) \neq \emptyset$. Note that $\text{Sol}^d(C, f) \subset \text{Sol}(C, f)$. This means that even if $\text{Sol}(C, f) \neq \emptyset$, $\text{Sol}^d(C, f) \neq \emptyset$ does not necessarily hold. A question is under what

conditions $\text{Sol}^d(C, f) \neq \emptyset$ holds. In fact, we have the following results:

- (i) If f is pseudomonotone on C and $\text{Sol}(C, f) \neq \emptyset$, then $\text{Sol}^d(C, f) \neq \emptyset$
- (ii) If f is quasimonotone on C , $\text{int } C \neq \emptyset$ and $\{u \mid f(u) = 0\} \neq \emptyset$, then $\text{Sol}^d(C, f) \neq \emptyset$ (see [35])

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Some Compatible and Weakly-Compatible Four Self-Mapping Results Approach to Nonlinear Integral Equations in Fuzzy Cone Metric Spaces

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This paper is aimed at proving some unique common fixed point theorems by using the compatible and weakly-compatible four self-mappings in fuzzy cone metric (FCM) space. We prove the results under the generalized rational contraction conditions in FCM spaces with the help of one self-map are continuous. Furthermore, we prove some rational contraction results with the weaker condition of the self-mapping continuity. Ultimately, our theoretical work has been utilized to prove the existence solution of the two nonlinear integral equations. This is an illustrative application of how FCM spaces can be used in other integral type operators.

1. Introduction

The theory of fixed-point theory was introduced by Banach [1]. He proved a “Banach contraction principle,” which is stated as follows: “A self-mapping on a complete metric space verifying the contraction condition has a unique fixed point (FP).” Later on, many researchers have been generalized this principle in many directions and proved different contractive type FP and common fixed point (CFP) for single-valued and multivalued mappings in the context of metric spaces. Chatterjea [2], Chatterjea [3], and Kannan [4] proved some single-valued contractive type FP theorems. While Ali et al. [5], Covitz and Nadler [6], Daker and Kaneko [7, 8], Khan [9], and Patle et al. [10] proved multivalued contractive type FP and CFP results by using different types of spaces.

Zadeh [11], in 1965, introduced the concept of fuzzy sets. Later on, this concept was used in topology and functional

analysis by many researchers. Kramosil and Michalek [12] introduced the notion of fuzzy metric FM space, and they established some basic properties. After that, George and Veeramani [13] presented the stronger form of the FM. Grabiec [14] proved two FP theorems by using the concept of complete and compact FM spaces. Gregori and Sapena [15] established some FP contraction results in the sense of [13, 15]. Hadzic and Pap [16] proved a FP theorem for multivalued mappings in probabilistic metric spaces and presented applications in FM spaces. Imdad and Ali [17] and Rehman et al. [18] proved some FP theorems in complete FM spaces. Pant and Chauhan [19] established some CFP theorems by using weakly-compatible mappings in menger spaces and FM spaces. Kiyani et al. [20] and Sadeghi et al. [21] proved some results for set-valued contractive type mappings in FM spaces.

The concept of cone metric space (CMS) was proposed by many researchers but it became popular after being redis-

covered by Huang and Zhang [22]. They proved the convergence properties and FP theorems for nonlinear contractive type mappings. By using the concept of Huang and Zhang [22], many authors have contributed their work to the problems on CMSs. Some of such works can be found in ([23–28]).

In 2015, the notion of fuzzy cone metric space (FCM space) was introduced by Oner et al. [29]. They proved the key attributes of FCM space and a “fuzzy cone Banach contraction theorem for FP” in FCM space. In [30], Rehman and Li extended and improved a “fuzzy cone Banach contraction theorem” and established some generalized-contraction results for FP in FCM spaces. Rehman et al. [31, 32] proved different contractive type CFP-theorems in FCM spaces. Recently, the concept of weakly compatible self-mappings in FCM spaces was given by Jabeen et al. [33].

This paper is aimed at proving some unique CFP-theorems under the generalized rational contraction conditions in FCM spaces by using compatibility and weak-compatibility of four self-mappings. We prove our results by using the one self-map are continuous. Furthermore, we prove some results without the continuity of self-mappings with supportive examples. In addition, we present an application of two nonlinear integral equations (NIEs) for the existence of a common solution to support our main work. This paper is managed as follows: in Section 2, we present the basic preliminary concept. While in Section 3, we prove our main results for unique CFP-theorems under the generalized rational contraction conditions in FCM spaces by using compatibility and weak-compatibility of four self-mappings. In Section 4, we present NIEs as an application to support our main work.

2. Preliminaries

In this section, we recall some basic definitions and lemmas.

Definition 1 (see [34]). An operation $*$: $[0, 1]^2 \rightarrow [0, 1]$ is called a continuous t -norm if:

- (i) $*$ is associative, commutative, and continuous
- (ii) $1 * q_1 = q_1$ and $q_1 * q_2 \leq q_3 * q_4$, whenever $q_1 \leq q_3$ and $q_2 \leq q_4$, for all $q_1, q_2, q_3, q_4 \in [0, 1]$

Schweizer and Sklar [34] define the following basic continuous t -norms are

- (i) The minimum; $q_1 * q_2 = \min \{q_1, q_2\}$
- (ii) The product; $q_1 * q_2 = q_1 q_2$
- (iii) The Lukasiewicz; $q_1 * q_2 = \max \{q_1 + q_2 - 1, 0\}$

For detail study (see [34]).

Definition 2 (see [29]). A 3-tuple $(U, M_o, *)$ is called a FCM space if C is a cone of \mathbb{E} , U is an arbitrary set, $*$ is a contin-

uous t -norm and M_o is a fuzzy set on $U^2 \times \text{int}(P)$ satisfying the following conditions:

- (1) $M_o(\lambda_1, \lambda_2, t) > 0$ and $M_o(\lambda_1, \lambda_2, t) = 1 \Leftrightarrow \lambda_1 = \lambda_2$
 - (2) $M_o(\lambda_1, \lambda_2, t) = M_o(\lambda_2, \lambda_1, t)$
 - (3) $M_o(\lambda_1, \lambda_2, t) * M_o(\lambda_2, \lambda_3, s) \leq M_o(\lambda_1, \lambda_3, t + s)$
 - (4) $M_o(\lambda_1, \lambda_2, \cdot) : \text{int}(P) \rightarrow [0, 1]$ is continuous
- $\forall \lambda_1, \lambda_2, \lambda_3 \in U$ and $t, s \in \text{int}(P)$.

Definition 3 (see [29]). Let $(U, M_o, *)$ be a FCM space, $\exists \lambda_1 \in U$ and $\{\lambda_j\}$ be any sequence in U .

- (i) $\{\lambda_j\}$ converges to λ_1 if for any $c \in (0, 1)$, $t \gg \theta$, and $\exists j_1 \in \mathbb{N}$ such that $M_o(\lambda_j, \lambda_1, t) > 1 - c$, for $j \geq j_1$. This can be written as $\lim_{j \rightarrow \infty} \lambda_j = \lambda_1$, or $\lambda_j \rightarrow \lambda_1$ as $j \rightarrow \infty$
- (ii) (λ_j) is Cauchy if for any $c \in (0, 1)$, $t \gg \theta$, and $\exists j_1 \in \mathbb{N}$ such that $M_o(\lambda_j, \lambda_k, t) > 1 - c$, for $j, k \geq j_1$
- (iii) $(U, M_o, *)$ is complete if every Cauchy sequence is convergent in U
- (iv) $\{\lambda_j\}$ is FC contractive if $\exists a \in (0, 1)$ so that

$$\frac{1}{M_o(\lambda_j, \lambda_{j+1}, t)} - 1 \leq a \left(\frac{1}{M_o(\lambda_{j-1}, \lambda_j, t)} - 1 \right), \text{ for } t \gg \theta, j \geq 1. \quad (1)$$

Lemma 4 (see [29]). “Let $(U, M_o, *)$ be a FCM space and a sequence $\lambda_j \rightarrow \lambda_1 \in U$ iff $M_o(\lambda_j, \lambda_1, t) \rightarrow 1$ as $j \rightarrow \infty$ for each $t \gg \theta$ ”.

Definition 5 (see [30]). Let $(U, M_o, *)$ be a FCM space. The FCM M_o is triangular if

$$\frac{1}{M_o(\lambda_1, \lambda_3, t)} - 1 \leq \left(\frac{1}{M_o(\lambda_1, \lambda_2, t)} - 1 \right) + \left(\frac{1}{M_o(\lambda_2, \lambda_3, t)} - 1 \right), \forall \lambda_1, \lambda_2, \lambda_3 \in U, t \gg \theta. \quad (2)$$

Definition 6 (see [29]). Let $(U, M_o, *)$ be a FCM space and $A : U \rightarrow U$. Then, A is said to be FC contractive if there is $a \in (0, 1)$ so that

$$\frac{1}{M_o(A\lambda_1, A\lambda_2, t)} - 1 \leq a \left(\frac{1}{M_o(\lambda_1, \lambda_2, t)} - 1 \right), \quad (3)$$

$\forall \lambda_1, \lambda_2 \in U$, and $t \gg \theta$.

Definition 7 (see [23]). Let $U \neq \emptyset$ set and let $B, h : U \rightarrow U$ be the self-mappings on U . If there exists $\xi \in U$ such that $B\rho = h\rho = \xi$ for some $\rho \in U$. Then, ρ is called a coincidence

point of B and h , and ξ is known as a point of coincidence of the mappings B, h . A pair of self-mappings (B, h) is known to be weakly-compatible if the self-mappings commute at their coincidence point, i.e., $hB(\rho) = Bh(\rho)$ for $\rho \in U$.

Proposition 8 (see [23]). *Let B, h be weakly-compatible self-mappings on U . If B and h have a unique point of coincidence, that is, $B\rho = h\rho = \xi$, then, ξ is a unique CFP of the mappings B and h .*

Definition 9 (see [32]). A self-mapping pair (h, B) is said to be compatible on a FCM space $(U, M_o, *)$, if, $\lim_{j \rightarrow \infty} M_o(Bh\lambda_j, hB\lambda_j, t) = 1$ for $t \gg \theta$, whenever $\{\lambda_j\}$ is a sequence in U so that $\lim_{j \rightarrow \infty} B\lambda_j = \lim_{j \rightarrow \infty} h\lambda_j = \xi$ for some $\xi \in U$.

3. Main Results

Now, we are in the position to present our main results.

Theorem 10. *Let $A, B, g, h : U \rightarrow U$ be the four self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies*

$$\begin{aligned} \frac{1}{M_o(A\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (4)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, c, d < 1$ with $(a + b + 2c + 2d) < 1$. If $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$ and consider that [(1)]

- (1) h is a continuous self-mapping
- (2) A pair (A, h) is compatible, and
- (3) A pair (B, g) is weakly-compatible

Then, the mappings A, B, g , and h have a unique CFP in U .

Proof. Fix $\lambda_0 \in U$ and by the hypothesis $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$, we define the iterative sequences in U so that

$$\xi_{2j+1} = g\lambda_{2j+1} = A\lambda_{2j} \text{ and } \xi_{2j+2} = h\lambda_{2j+2} = B\lambda_{2j+1}, j \geq 0. \quad (5)$$

Then, by (4),

$$\begin{aligned} \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 &= \frac{1}{M_o(A\lambda_{2j}, B\lambda_{2j+1}, t)} - 1 \\ &\leq a \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)}{M_o(h\lambda_{2j}, B\lambda_{2j+1}, 2t) * M_o(g\lambda_{2j+1}, A\lambda_{2j}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_o(h\lambda_{2j}, A\lambda_{2j}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, B\lambda_{2j+1}, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_o(g\lambda_{2j+1}, A\lambda_{2j}, t)} - 1 + \frac{1}{M_o(h\lambda_{2j}, B\lambda_{2j+1}, t)} - 1 \right) \\ &= a \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 \right) + b \left(\frac{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)}{M_o(h\lambda_{2j}, h\lambda_{2j+2}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_o(h\lambda_{2j}, h\lambda_{2j+2}, t)} - 1 \right). \end{aligned} \quad (6)$$

By Definition 2 (3), $M_o(h\lambda_{2j}, h\lambda_{2j+2}, 2t) \geq M_o(h\lambda_{2j}, g\lambda_{2j+1}, t) * M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)$, for $t \gg \theta$. One writes

$$\begin{aligned} \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t) * M_r(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right). \end{aligned} \quad (7)$$

This implies that

$$\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \leq \beta \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 \right), \text{ for } t \gg \theta, \quad (8)$$

where $\gamma = a + c + d/1 - b - c - d < 1$ since $(a + b + 2c + 2d) < 1$

$d) < 1$. Similarly,

$$\begin{aligned}
& \frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 = \frac{1}{M_o(A\lambda_{2j+2}, B\lambda_{2j+1}, t)} - 1 \\
& \leq a \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\
& \quad + b \left(\frac{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)}{M_o(h\lambda_{2j+2}, B\lambda_{2j+1}, 2t) * M(g\lambda_{2j+1}, A\lambda_{2j+2}, 2t)} - 1 \right) \\
& \quad + c \left(\frac{1}{M_o(h\lambda_{2j+2}, A\lambda_{2j+2}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, B\lambda_{2j+1}, t)} - 1 \right) \\
& \quad + d \left(\frac{1}{M_o(g\lambda_{2j+1}, A\lambda_{2j+2}, t)} - 1 + \frac{1}{M_o(h\lambda_{2j+2}, B\lambda_{2j+1}, t)} - 1 \right) \\
& = a \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) + b \left(\frac{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)}{M(g\lambda_{2j+1}, g\lambda_{2j+3}, 2t)} - 1 \right) \\
& \quad + c \left(\frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\
& \quad + d \left(\frac{1}{M_o(g\lambda_{2j+1}, g\lambda_{2j+3}, t)} - 1 \right). \tag{9}
\end{aligned}$$

Again, by Definition 2 (3), $M_o(g\lambda_{2j+1}, g\lambda_{2j+3}, 2t) \geq M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t) * M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)$, for $t \gg \theta$. We have

$$\begin{aligned}
& \frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \leq a \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\
& \quad + b \left(\frac{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t) * M_r(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \right) \\
& \quad + c \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \right) \\
& \quad + d \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \right). \tag{10}
\end{aligned}$$

This implies that

$$\frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \leq \beta \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right), \text{ for } t \gg \theta, \tag{11}$$

where the value of β is the same as in (8). Now, from (3), (8), (11), and by induction, we have

$$\begin{aligned}
& \frac{1}{M_o(\xi_{2j+2}, \xi_{2j+3}, t)} - 1 \leq \beta \left(\frac{1}{M_o(\xi_{2j+1}, \xi_{2j+2}, t)} - 1 \right) \\
& \leq \beta^2 \left(\frac{1}{M_o(\xi_{2j+1}, \xi_{2j+2}, t)} - 1 \right) \\
& \leq \dots \leq \beta^{2j+2} \left(\frac{1}{M_o(\xi_0, \xi_1, t)} - 1 \right) \longrightarrow 0, \text{ as } j \longrightarrow \infty. \tag{12}
\end{aligned}$$

Its prove that a sequence $\{\xi_j\}_{j \geq 0}$ is a FC contractive, and we get that

$$\lim_{j \longrightarrow \infty} M_o(\xi_j, \xi_{j+1}, t) = 1, \text{ for } t \gg \theta. \tag{13}$$

Since M_o is triangular, then $\forall k > j \geq j_0$,

$$\begin{aligned}
& \frac{1}{M_o(\xi_j, \xi_k, t)} - 1 \leq \left(\frac{1}{M_o(\xi_j, \xi_{j+1}, t)} - 1 \right) + \left(\frac{1}{M_o(\xi_{j+1}, \xi_{j+2}, t)} - 1 \right) \\
& \quad + \dots + \left(\frac{1}{M_o(\xi_{k-1}, \xi_k, t)} - 1 \right) \\
& \leq (\beta^j + \beta^{j+1} + \dots + \beta^{k-1}) \left(\frac{1}{M_o(\xi_0, \xi_1, t)} - 1 \right) \\
& \leq \frac{\beta^j}{1 - \beta} \left(\frac{1}{M_o(\xi_0, \xi_1, t)} - 1 \right) \longrightarrow 0, \text{ as } j \longrightarrow \infty. \tag{14}
\end{aligned}$$

Hence, proved that $\{\xi_j\}$ is a Cauchy sequence. Now by the completeness of FCM space $(U, M_o, *)$, $\exists \xi \in U$ so that $\xi_j \longrightarrow \xi$ as $j \longrightarrow \infty$. Now for its subsequences, we have that

$$g\lambda_{2j+1} \longrightarrow \xi, h\lambda_{2j+2} \longrightarrow \xi, A\lambda_{2j} \longrightarrow \xi, \text{ and } B\lambda_{2j+1} \longrightarrow \xi \text{ as } j \longrightarrow \infty. \tag{15}$$

Since, a self-mapping $h : U \longrightarrow U$ is continuous, therefore

$$h(g\lambda_{2j+1}) \longrightarrow h\xi, h(h\lambda_{2j+2}) \longrightarrow h\xi, h(A\lambda_{2j}) \longrightarrow h\xi, \text{ and } h(B\lambda_{2j+1}) \longrightarrow h\xi \text{ as } j \longrightarrow \infty. \tag{16}$$

By hypothesis (2), a (A, h) is compatible, therefore,

$$\begin{aligned}
& \lim_{j \longrightarrow \infty} M_o(A(h\lambda_{2j}), h(A\lambda_{2j}), t) = \lim_{j \longrightarrow \infty} M_r(A(h\lambda_{2j}), h\xi, t) \\
& = 1, \Rightarrow \lim_{j \longrightarrow \infty} M_r(h(A\lambda_{2j}), h\xi, t) = 1, \text{ for } t \gg \theta. \tag{17}
\end{aligned}$$

Next, we have to prove that $h\xi = \xi$, then, by Definition 2 (3),

$$M_o(h\xi, \xi, 2t) \geq M_r(h\xi, A(h\lambda_{2j}, t) * M_r(A(h\lambda_{2j}, \xi, t) \text{ for } t \gg \theta. \tag{18}$$

Since, a pair (A, h) is compatible, by using limit $j \longrightarrow \infty$, and by the view of (15), (17), and (18), we have

$$\begin{aligned}
& M_o(h\xi, \xi, 2t) \geq \lim_{j \longrightarrow \infty} M_r(h\xi, A(h\lambda_{2j}, t) * \lim_{j \longrightarrow \infty} M_r(A(h\lambda_{2j}, \xi, t) \\
& = 1 * 1 = 1 \text{ for } t \gg \theta. \tag{19}
\end{aligned}$$

Hence, $M_o(h\xi, \xi, 2t) = 1 \Rightarrow h\xi = \xi$, for $t \gg \theta$. Now, we

prove that $A\xi = \xi$, then again by Definition 2 (3),

$$M_o(A\xi, \xi, 2t) \geq M_r(A\xi, h(A\lambda_{2j}, t) * M_r(h(A\lambda_{2j}, \xi, t) \text{ for } t \gg \theta. \quad (20)$$

Again by using limit $j \rightarrow \infty$, and by the view of (15), (17), and (20), we have

$$\begin{aligned} M_o(A\xi, \xi, 2t) &\geq \lim_{j \rightarrow \infty} M_r(A\xi, h(A\lambda_{2j}), t) * \lim_{j \rightarrow \infty} M_r(h(A\lambda_{2j}), \xi, t) \\ &= 1 * 1 = 1 \text{ for } t \gg \theta. \end{aligned} \quad (21)$$

Hence, $M_o(A\xi, \xi, 2t) = 1 \Rightarrow A\xi = \xi$, for $t \gg \theta$. Thus, we get that $A\xi = h\xi = \xi$. Next, we have to prove that $B\xi = g\xi$. Now by hypothesis (1), i.e., $A(U) \subseteq g(U)$, and there exists $\rho \in U$ such that $\xi = A\xi = g\rho$. Then, by view of (4), for $t \gg \theta$,

$$\begin{aligned} \frac{1}{M_o(B\rho, g\rho, t)} - 1 &= \frac{1}{M_o(A\xi, B\rho, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, g\rho, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(h\xi, g\rho, t)}{M_o(h\xi, B\rho, 2t) * M_o(g\rho, A\xi, 2t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_o(h\xi, A\xi, t)} - 1 + \frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_o(g\rho, A\xi, t)} - 1 + \frac{1}{M_o(h\xi, B\rho, t)} - 1 \right) \\ &= a \left(\frac{1}{M_o(h\xi, \xi, t)} - 1 \right) + b \left(\frac{M_o(h\xi, g\rho, t)}{M_o(h\xi, B\rho, 2t) * M_o(\xi, A\xi, 2t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_o(h\xi, \xi, t)} - 1 + \frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_o(\xi, A\xi, t)} - 1 + \frac{1}{M_o(h\xi, B\rho, t)} - 1 \right) \\ &= b \left(\frac{M_o(h\xi, g\rho, t)}{M_o(h\xi, B\rho, 2t)} - 1 \right) + c \left(\frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_o(g\rho, B\rho, t)} - 1 \right). \end{aligned} \quad (22)$$

Again, by Definition 2 (3), $M_o(h\xi, B\rho, 2t) \geq M_o(h\xi, g\rho, t) * M_o(g\rho, B\rho, t)$, for $t \gg \theta$. It follows that

$$\begin{aligned} \frac{1}{M_o(B\rho, g\rho, t)} - 1 &\leq b \left(\frac{M_o(h\xi, g\rho, t)}{M_o(h\xi, g\rho, t) * M_o(g\rho, B\rho, t)} - 1 \right) \\ &\quad + (c + d) \left(\frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \\ &= (b + c + d) \left(\frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \text{ for } t \gg \theta. \end{aligned} \quad (23)$$

Noticing that $(b + c + d) < 1$, therefore, $M_o(B\rho, g\rho, t) = 1 \Rightarrow B\rho = g\rho$ for $t \gg \theta$, hence, $B\rho = g\rho = \xi$. Now by hypothesis (3), a pair (B, g) is weakly compatible, therefore,

$$g\xi = g(B\rho) = B(g\rho) = B\xi. \quad (24)$$

Next, we have to prove that $B\xi = \xi$, then again by view of

(4) and by using Definition 2 (3), for $t \gg \theta$,

$$\begin{aligned} \frac{1}{M_o(B\xi, \xi, t)} - 1 &= \frac{1}{M_o(B\xi, A\xi, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, g\xi, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(h\xi, g\xi, t)}{M_o(h\xi, B\xi, 2t) * M_o(g\xi, A\xi, 2t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_r(h\xi, A\xi, t)} - 1 + \frac{1}{M_r(g\xi, B\xi, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_r(g\xi, A\xi, t)} - 1 + \frac{1}{M_r(h\xi, B\xi, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(\xi, B\xi, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(h\xi, g\xi, t)}{M_o(h\xi, g\xi, t) * M_o(g\xi, B\xi, t) * M_o(g\xi, \xi, t) * M_o(\xi, A\xi, t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_r(h\xi, \xi, t)} - 1 + \frac{1}{M_r(g\xi, B\xi, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 + \frac{1}{M_r(\xi, B\xi, t)} - 1 \right). \end{aligned} \quad (25)$$

After simplification, we obtain

$$\begin{aligned} \frac{1}{M_r(B\xi, \xi, t)} - 1 &\leq (a + b + 2d) \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 \right), \\ &\Rightarrow (1 - a - b - 2d) \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 \right) \leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (26)$$

Since $(1 - a - b - 2d) \neq 0$, therefore, $M_r(B\xi, \xi, t) = 1 \Rightarrow B\xi = \xi$, for $t \gg \theta$, which further implies that $g\xi = \xi$. Hence, proved that $h\xi = g\xi = A\xi = B\xi = \xi$, that is, ξ is the CFP of the mappings A, B, g , and h . \square

Uniqueness: let $\eta \in U$ be the other CFP of the mappings A, B, g , and h in U such that $h\eta = g\eta = A\eta = B\eta = \eta$. Then by view of (4) and by using Definition 2 (3), for $t \gg \theta$,

$$\begin{aligned} \frac{1}{M_r(\xi, \eta, t)} - 1 &= \frac{1}{M_r(A\xi, B\eta, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, g\eta, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(h\xi, g\eta, t)}{M_o(h\xi, B\eta, 2t) * M_o(g\eta, A\xi, 2t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_r(h\xi, A\xi, t)} - 1 + \frac{1}{M_r(g\eta, B\eta, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_r(g\eta, A\xi, t)} - 1 + \frac{1}{M_r(h\xi, B\eta, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(\xi, \eta, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(\xi, \eta, t)}{M_o(\xi, \xi, t) * M_o(\xi, \eta, t) * M_o(\eta, \eta, t) * M_o(\eta, \xi, t)} - 1 \right) \\ &\quad + 2d \left(\frac{1}{M_r(\eta, \xi, t)} - 1 \right). \end{aligned} \quad (27)$$

After simplification, we obtain

$$\begin{aligned} \frac{1}{M_r(\xi, \eta, t)} - 1 &\leq (a + b + 2d) \left(\frac{1}{M_r(\xi, \eta, t)} - 1 \right), \\ \Rightarrow (1 - a - b - 2d) \left(\frac{1}{M_r(\xi, \eta, t)} - 1 \right) &\leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (28)$$

Since $(1 - a - b - 2d) \neq 0$, therefore, $M_r(\xi, \eta, t) = 1 \Rightarrow \xi = \eta$, for $t \gg \theta$. This completes the proof.

Corollary 11. Let $A, B, g, h : U \longrightarrow U$ be the four self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(A\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right), \end{aligned} \quad (29)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, c < 1$ with $(a + b + 2c) < 1$. If $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$ and consider that

- (1) h is a continuous self-mapping
- (2) A pair (A, h) is compatible, and
- (3) A pair (B, g) is weakly-compatible

Then, the mappings A, B, g , and h have a unique CFP in U .

Corollary 12. Let $A, B, g, h : U \longrightarrow U$ be the four self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(A\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (30)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, d < 1$ with $(a + b + 2d) < 1$. If $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$ and consider that

- (1) h is a continuous self-mapping
- (2) A pair (A, h) is compatible, and
- (3) A pair (B, g) is weakly-compatible

Then, the mappings A, B, g , and h have a unique CFP in U .

Corollary 13. Let $A, B, g, h : U \longrightarrow U$ be the four self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(A\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (31)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, c, d < 1$ with $(a + 2c + 2d) < 1$. If $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$ and consider that

- (1) h is a continuous self-mapping
- (2) A pair (A, h) is compatible, and
- (3) A pair (B, g) is weakly-compatible

Then, the mappings A, B, g , and h have a CFP in U .

Example 14. Assume that $U = [0, \infty)$, $*$ is a continuous t -norm and $M_o : U \times U \times (0, \infty) \longrightarrow [0, 1]$ be written as

$$M_o(\lambda, \mu, t) = \frac{t}{t + |\lambda - \mu|}, \forall \lambda, \mu \in U, t \gg \theta. \quad (32)$$

Then, it is easy to verify that FCM M_o is triangular and $(U, M_o, *)$ is a complete FCM space. Now, the mappings, $A, g, h, B : U \longrightarrow U$, be defined by, (for all $\lambda \in U$);

$$A(\lambda) = B(\lambda) = \begin{cases} \frac{1}{3} \left(\frac{3\lambda}{4} + \frac{1}{8} \right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0, \end{cases} \quad (33)$$

and

$$h(\lambda) = g(\lambda) = \begin{cases} \left(\frac{3\lambda}{4} + \frac{1}{8} \right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases} \quad (34)$$

Since, from the above equation, $A(U) = B(U)$ and $g(U) = h(U)$, so that we conclude that $A(U) \subseteq g(U)$ or $B(U) \subseteq h(U)$. Then,

$$\frac{1}{M_o(h\lambda, g\mu, t)} - 1 = \frac{|h\lambda - g\mu|}{t} = \frac{3|\lambda - \mu|}{4t}, \text{ for } t \gg \theta, \quad (35)$$

and

$$\frac{1}{M_o(A\lambda, B\mu, t)} - 1 = \frac{|A(\lambda) - B(\mu)|}{t} = \frac{1}{3} \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \text{ for } t \gg \theta. \quad (36)$$

Hence, the mappings A, B, g , and h on U satisfying the

fuzzy cone-contraction condition in FCM space. Now, from Definition 2 (3), $M_r(h\lambda, B\mu, 2t) \geq M_r(h\lambda, g\mu, t) * M_r(g\mu, B\mu, t)$ and $M_r(g\mu, A\lambda, 2t) \geq M_r(g\mu, h\lambda, t) * M_r(h\lambda, A\lambda, t)$, for $t \gg \theta$. Now, we calculate the following terms of (4), for $t \gg \theta$,

$$\begin{aligned} & \frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_r(g\mu, A\lambda, 2t)} - 1 \\ & \leq \frac{1}{M_o(h\lambda, g\mu, t) * M_r(h\lambda, A\lambda, t) * M_r(g\mu, B\mu, t)} - 1 \\ & = \frac{1}{(t/t + |h\lambda - g\mu|) \cdot (t/t + |h\lambda - A\lambda|) \cdot (t/t + |g\mu - B\mu|)} - 1 \\ & = \frac{(t + |h\lambda - g\mu|) \cdot (t + |h\lambda - A\lambda|) \cdot (t + |g\mu - B\mu|)}{t^3} - 1 \\ & = \frac{(t + 3/4|\lambda - \mu|) \cdot [(t + 1/12(6\lambda + 1)) \cdot (t + 1/12(6\mu + 1))] }{t^3} - 1 \\ & = \frac{(t + 3/4|\lambda - \mu|) \cdot [1/144(6\lambda + 1) \cdot (6\mu + 1) + (t/6)(3\lambda + 3\mu + 1) + (3t^2/4)|\lambda - \mu|] }{t^3} \\ & = \frac{(t + 3/4|\lambda - \mu|) \cdot [1/144(6\lambda + 1) \cdot (6\mu + 1) + (t/6)(3\lambda + 3\mu + 1)] }{t^3} + \frac{3}{4t}|\lambda - \mu|. \end{aligned} \quad (37)$$

Next, we calculate

$$\begin{aligned} & \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right) \\ & = \frac{1}{t/t + |h\lambda - A\lambda|} - 1 + \frac{1}{t/t + |g\mu - B\mu|} - 1 \\ & = \frac{|h\lambda - A\lambda|}{t} + \frac{|g\mu - B\mu|}{t} = \frac{1}{6t}(3\lambda + 3\mu + 1) \text{ for } t \gg \theta. \end{aligned} \quad (38)$$

Similarly,

$$\begin{aligned} & \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right) \\ & = \frac{1}{t/t + |g\mu - A\lambda|} - 1 + \frac{1}{t/t + |h\lambda - B\mu|} - 1 \\ & = \frac{|g\mu - A\lambda|}{t} + \frac{|h\lambda - B\mu|}{t} = \frac{1}{6t}(3\lambda + 3\mu + 1) \text{ for } t \gg \theta. \end{aligned} \quad (39)$$

Thus, after routine calculation, all the conditions of Theorem 10 are satisfied with $a = 1/3$, $b = 2/7$, and $c = d = 1/14$, and the mappings A, B, g , and h on U have a unique CFP, i.e., $0 \in U$.

If we choose the self-mappings $B = A$ and $g = h$ in Theorem 10, we obtain the following corollary.

Corollary 15. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 & \leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ & + b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right) \\ & + c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right) \\ & + d \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (40)$$

$\forall \lambda, \mu \in U$, $t \gg \theta$, and $0 \leq a, b, c, d < 1$ with $(a + b + 2c + 2d) < 1$. If $B(U) \subseteq h(U)$, h is a continuous self-mapping, and a pair (B, h) is weakly-compatible. Then, the mappings B and h have a unique CFP in U .

Next result, we shall prove without the continuity of self-mapping, i.e., h and we replaced the completeness of U by the completeness of $B(U)$ or $h(U)$.

Theorem 16. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 & \leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ & + b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right) \\ & + c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right) \\ & + d \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (41)$$

$\forall \lambda, \mu \in U$, $t \gg \theta$, and $0 \leq a, b, c, d < 1$ with $(a + b + 2c + 2d) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is weakly-compatible. Then, the mappings B and h have a unique CFP in U .

Proof. From the proof of Theorem 10, we assume that $\{\xi_j\}_{j \geq 0}$ is a Cauchy sequence in $h(U)$, and the iterative sequences are earlier defined in the proof of Theorem 10, that are,

$$\xi_{2j+1} = h\lambda_{2j+1} = B\lambda_{2j} \text{ and } \xi_{2j+2} = h\lambda_{2j+2} = B\lambda_{2j+1}, j \geq 0. \quad (42)$$

We know that $h(U)$ is complete, and $\exists \xi, \rho \in U$, so that $\xi_{2j+1} \longrightarrow \xi = h\rho$, as $j \longrightarrow \infty$. Therefore,

$$\lim_{j \longrightarrow \infty} M_o(\xi_{2j+1}, \xi, t) = \lim_{j \longrightarrow \infty} M_o(h\lambda_{2j+1}, \xi, t) = 1 \text{ for } t \gg \theta. \quad (43)$$

Since by M_o triangular property,

$$\frac{1}{M_o(h\rho, B\rho, t)} - 1 \leq \left(\frac{1}{M_o(h\rho, \xi_{2j+2}, t)} - 1 \right) + \left(\frac{1}{M_o(\xi_{2j+2}, B\rho, t)} - 1 \right) \text{ for } t \gg \theta. \quad (44)$$

Now by the view of (41), (43), and by using Definition 2 (3), for $t \gg \theta$, we have that

$$\begin{aligned} \frac{1}{M_o(\xi_{2j+2}, B\rho, t)} - 1 &= \frac{1}{M_o(B\lambda_{2j+1}, B\rho, t)} - 1 \leq a \left(\frac{1}{M_o(h\lambda_{2j+1}, h\rho, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda_{2j+1}, h\rho, t)}{M_o(h\lambda_{2j+1}, B\rho, 2t) * M_o(h\rho, B\lambda_{2j+1}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda_{2j+1}, B\lambda_{2j+1}, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\rho, B\lambda_{2j+1}, t)} - 1 + \frac{1}{M_r(h\lambda_{2j+1}, B\rho, t)} - 1 \right) \\ &= a \left(\frac{1}{M_o(h\lambda_{2j+1}, h\rho, t)} - 1 \right) + b \left(\frac{M_o(h\lambda_{2j+1}, h\rho, t)}{M_o(h\lambda_{2j+1}, B\rho, 2t) * M_o(h\rho, h\lambda_{2j+2}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\rho, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_r(h\lambda_{2j+1}, B\rho, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(h\lambda_{2j+1}, h\rho, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda_{2j+1}, h\rho, t)}{M_o(h\lambda_{2j+1}, h\rho, t) * M_o(h\rho, B\rho, t) * M_o(h\rho, h\lambda_{2j+2}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\rho, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_r(h\lambda_{2j+1}, B\rho, t)} - 1 \right) \\ &\longrightarrow (b + c + d) \left(\frac{1}{M_r(h\rho, B\rho, t)} - 1 \right), \text{ as } j \longrightarrow \infty. \end{aligned} \quad (45)$$

Hence,

$$\limsup_{j \longrightarrow \infty} \left(\frac{1}{M_o(\xi_{2j+2}, B\rho, t)} - 1 \right) \leq (b + c + d) \left(\frac{1}{M_r(h\rho, B\rho, t)} - 1 \right), \text{ for } t \gg \theta. \quad (46)$$

Now from (44) and (46),

$$\begin{aligned} \frac{1}{M_o(h\rho, B\rho, t)} - 1 &\leq (b + c + d) \left(\frac{1}{M_r(h\rho, B\rho, t)} - 1 \right), \\ &\Rightarrow (1 - b - c - d) \left(\frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (47)$$

Since, $(1 - b - c - d) \neq 0$, therefore, $M_r(h\rho, B\rho, t) = 1 \Rightarrow h\rho = B\rho = \xi$, for $t \gg \theta$. Now by the weak-compatibility of (B, h) ,

$$B\xi = B(h\rho) = h(B\rho) = h\xi. \quad (48)$$

Next, we have to prove that $B\xi = \xi$. Then, again by view

of (41) and by using Definition 2 (3), for $t \gg \theta$, we have

$$\begin{aligned} \frac{1}{M_o(B\xi, \xi, t)} - 1 &= \frac{1}{M_o(B\xi, B\rho, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, h\rho, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\xi, h\rho, t)}{M_o(h\xi, B\rho, 2t) * M_o(h\rho, B\xi, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\xi, B\xi, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\rho, B\xi, t)} - 1 + \frac{1}{M_r(h\xi, B\rho, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(B\xi, \xi, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\xi, h\rho, t)}{M_o(h\xi, h\rho, t) * M_o(h\rho, B\rho, t) * M_o(h\rho, \xi, t) * M_o(\xi, B\xi, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\xi, B\xi, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(\xi, B\xi, t)} - 1 + \frac{1}{M_r(B\xi, \xi, t)} - 1 \right) \\ &= (a + b + 2d) \left(\frac{1}{M_r(\xi, B\xi, t)} - 1 \right). \end{aligned} \quad (49)$$

Hence,

$$\begin{aligned} \frac{1}{M_o(B\xi, \xi, t)} - 1 &\leq (a + b + 2d) \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 \right), \\ &\Rightarrow (1 - a - b - 2d) \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 \right) \\ &\leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (50)$$

As $(1 - a - b - 2d) \neq 0$, therefore, $M_r(B\xi, \xi, t) = 1 \Rightarrow B\xi = \xi$, for $t \gg \theta$. Hence, proved that $B\xi = h\xi = \xi$. \square

Uniqueness: let $\exists \eta \in U$ be the other CFP of the mapping B and h in U such that $B\eta = h\eta = \eta$. Then by the view of (41) and by using Definition 2 (3),

$$\begin{aligned} \frac{1}{M_o(\xi, \eta, t)} - 1 &= \frac{1}{M_o(B\xi, B\eta, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, h\eta, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\xi, h\eta, t)}{M_o(h\xi, B\eta, 2t) * M_o(h\eta, B\xi, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\xi, B\xi, t)} - 1 + \frac{1}{M_r(h\eta, B\eta, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\eta, B\xi, t)} - 1 + \frac{1}{M_r(h\xi, B\eta, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(\xi, \eta, t)} - 1 \right) \\ &+ b \left(\frac{M_o(\xi, \eta, t)}{M_o(\xi, \eta, t) * M_o(\eta, \eta, t) * M_o(\eta, \xi, t) * M_o(\xi, \xi, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(\xi, \xi, t)} - 1 + \frac{1}{M_r(\eta, \eta, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(\eta, \xi, t)} - 1 + \frac{1}{M_r(\xi, \eta, t)} - 1 \right) = (a + b + 2d) \left(\frac{1}{M_r(\eta, \xi, t)} - 1 \right). \end{aligned} \quad (51)$$

Hence,

$$\begin{aligned} \frac{1}{M_r(\eta, \xi, t)} - 1 &\leq (a + b + 2d) \left(\frac{1}{M_r(\eta, \xi, t)} - 1 \right), \\ \Rightarrow (1 - a - b - 2d) \left(\frac{1}{M_r(\eta, \xi, t)} - 1 \right) &\leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (52)$$

As $(1 - a - b - 2d) \neq 0$, therefore, $M_r(\xi, \eta, t) = 1 \Rightarrow \xi = \eta$, for $t \gg \theta$. Hence, proved that the mappings B, h have a unique CFP in U .

Corollary 17. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right), \end{aligned} \quad (53)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, c < 1$ with $(a + b + 2c) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is weakly-compatible. Then, the mappings B, h have a unique CFP in U .

Corollary 18. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (54)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, d < 1$ with $(a + b + 2d) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is weakly-compatible. Then, the mappings B, h have a unique CFP in U .

Corollary 19. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right), \end{aligned} \quad (55)$$

for all $\lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b < 1$ with $(a + b) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is

weakly-compatible. Then, the mappings B, h have a unique CFP in U .

Corollary 20. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (56)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, c, d < 1$ with $(a + 2c + 2d) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is weakly-compatible. Then, the mappings B, h have a unique CFP in U .

Example 21. From Example 14, let us define $B, h : U \longrightarrow U$, for all $\lambda \in U$ such that

$$B(\lambda) = \begin{cases} \frac{1}{6} \left(\frac{3\lambda}{4} + \frac{1}{4} \right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0, \end{cases} \quad (57)$$

and

$$h(\lambda) = \begin{cases} \left(\frac{3\lambda}{4} + \frac{1}{4} \right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases} \quad (58)$$

Hence, from the above equations, we have that $B(U) \subseteq h(U)$. Then, for $t \gg \theta$, we have that

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &= \frac{|B\lambda - B\mu|}{t} = \frac{|\lambda - \mu|}{8t} \leq \frac{12\lambda + 3\mu + 5}{36t} \\ &\leq \frac{1}{8t} |\lambda - \mu| + \frac{5}{72t} (3\lambda + 3\mu + 2) \\ &= \frac{1}{6t} \left| \frac{3\lambda}{4} - \frac{3\mu}{4} \right| + \frac{5}{144t} (3\lambda + 3\mu + 2) + \frac{5}{144t} (3\lambda + 3\mu + 2) \\ &= \frac{1}{6t} \left| \frac{3\lambda}{4} - \frac{3\mu}{4} \right| + \frac{1}{6t} \left(\frac{5}{24} (3\lambda + 3\mu + 2) \right) + \frac{1}{6t} \left(\frac{5}{24} (3\lambda + 3\mu + 2) \right) \\ &\leq \frac{1}{6} \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) + \frac{1}{6} \left(\frac{5}{24t} (3\lambda + 1) + \frac{5}{24t} (3\mu + 1) \right) \\ &\quad + \frac{1}{6} \left(\frac{1}{24t} |18\mu - 3\lambda + 5| + \frac{1}{24t} |18\lambda - 3\mu + 5| \right) \\ &= \frac{1}{6} \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) + \frac{1}{6} \left(\left| \frac{h\lambda - B\lambda}{t} \right| + \left| \frac{h\mu - B\mu}{t} \right| \right) \\ &\quad + \frac{1}{6} \left(\left| \frac{h\mu - B\lambda}{t} \right| + \left| \frac{h\lambda - B\mu}{t} \right| \right) = \frac{1}{6} \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right) \\ &\quad + \frac{1}{6} \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right). \end{aligned} \quad (59)$$

Thus, after simplification, we conclude that all the

conditions of Corollary 20 are satisfied with $a = c = d = 1/6$ and the mappings B, h have a unique CFP, i.e., $0 \in U$.

4. Supportive Application

This section will deal with the nonlinear integral equations (NIEs) to support our main work. Let $U = C([0, q], \mathbb{R})$ be the space of all \mathbb{R} -valued continuous functions on $[0, q]$, where $0 < q \in \mathbb{R}$. The two NIEs are

$$\lambda(\tau) = \int_0^p Q_1(\tau, v, \lambda(v)) dv, \text{ and } \mu(\tau) = \int_0^p Q_2(\tau, v, \mu(v)) dv, \forall \lambda, \mu \in U, \quad (60)$$

where $\tau \in [0, q]$ and $Q_1, Q_2 : [0, q] \times [0, q] \times \mathbb{R} \rightarrow \mathbb{R}$, and a metric $d : U \times U \rightarrow \mathbb{R}$ is defined by

$$d(\lambda, \mu) = \sup_{\tau \in [0, q]} |\lambda(\tau) - \mu(\tau)| = \|\lambda - \mu\|, \text{ where } \lambda, \mu \in C([0, q], \mathbb{R}) = U. \quad (61)$$

The operation $*$ is defined by $\rho_1 * \rho_2 = \rho_1 \rho_2$ for all $\rho_1, \rho_2 \in [0, q]$. A FM $M_o : U \times U \times (0, \infty) \rightarrow [0, 1]$ is defined by

$$M_r(\lambda, \mu, t) = \frac{t}{t + d(\lambda, \mu)}, \text{ for } t > 0, \forall \lambda, \mu \in U. \quad (62)$$

Then, FM M_o is triangular and $(U, M_o, *)$ is a complete FCM space.

Theorem 22. *The two NIEs are*

$$\lambda(\tau) = \int_0^q Q_1(\tau, v, x(v)) dv, \text{ and } \mu(\tau) = \int_0^q Q_2(\tau, v, \mu(v)) dv, \quad (63)$$

where $\tau, v \in [0, q]$ and $\lambda, \mu \in U$. Assume that $Q_1, Q_2 : [0, q] \times [0, q] \times \mathbb{R} \rightarrow \mathbb{R}$ are so that $A_\lambda, B_\mu \in U$ for all $\lambda, \mu \in U$, where

$$A_\lambda(\tau) = \int_0^q Q_1(\tau, v, \lambda(v)) dv, \text{ and } B_\mu(\tau) = \int_0^q Q_2(\tau, v, \mu(v)) dv. \quad (64)$$

If $\exists 0 < k^* < 1$ such that $\forall \lambda, \mu \in U$,

$$\|A_\lambda - B_\mu\| \leq k^* N_{(A, B, g, h, \lambda, \mu)}, \quad (65)$$

where

$$N_{(A, B, g, h, \lambda, \mu)} = \max \left\{ \begin{aligned} &\|\lambda - \mu\|, \|\lambda - A_\lambda\| + \|\mu - B_\mu\|, \|\mu - A_\lambda\| + \|\lambda - B_\mu\|, \\ &\frac{1}{t^2} (t + \|\lambda - \mu\|) (t \|\lambda - A_\lambda\| + t \|\mu - B_\mu\| + \|\lambda - A_\lambda\| \cdot \|\mu - B_\mu\|) \end{aligned} \right\}, \quad (66)$$

then, the two nonlinear integral equations defined in (60) have a unique common solution in U .

Proof. Define the integral operators $A, B, g, h : U \rightarrow U$ by

$$A(\lambda) = A_\lambda, B(\mu) = B_\mu, g(\mu) = \mu \text{ and } h(\lambda) = \lambda. \quad (67)$$

The NIEs in (60) have a unique common solution iff A, B, g , and h have a unique CFP in U . Now, we prove that Theorem 10 applies to the integral operators A, B, g , and h . Then, $\forall \lambda, \mu \in U$, we may have the following four cases arises:

(a) If $N_{(A, B, g, h, \lambda, \mu)} = \|\lambda - \mu\|$ in (66). Then, by using (62), (65), and (67), we have

$$\begin{aligned} \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &= \frac{d(A\lambda, B\mu)}{t} \leq k^* \frac{N_{(A, B, g, h, \lambda, \mu)}}{t} = k^* \frac{\|\lambda - \mu\|}{t} \\ &= k^* \left(\frac{1}{M_r(h\lambda, g\mu, t)} - 1 \right). \end{aligned} \quad (68)$$

This implies that

$$\frac{1}{M_r(A\lambda, B\mu, t)} - 1 \leq k^* \left(\frac{1}{M_r(h\lambda, g\mu, t)} - 1 \right), \text{ for } t \gg \theta, \quad (69)$$

$\forall \lambda, \mu \in U$. Hence, the integral operators A, B, g , and h satisfy the conditions of Theorem 10 with $k^* = a$ and $b = c = d = 0$ in (4). The operators A, B, g , and h have a unique CFP, i.e., the unique common solution of the two NIEs (60) in U .

(b) If $N_{(A, B, g, h, \lambda, \mu)} = 1/t^2 (t + \|\lambda - \mu\|) (t \|\lambda - A_\lambda\| + t \|\mu - B_\mu\| + \|\lambda - A_\lambda\| \cdot \|\mu - B_\mu\|)$ in (66). Then, by using (62) and (65), we have

$$\begin{aligned} \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &= \frac{d(A\lambda, B\mu)}{t} \leq k^* \frac{N_{(A, B, g, h, \lambda, \mu)}}{t} \\ &= k^* \frac{1}{t^3} (t + \|\lambda - \mu\|) (t \|\lambda - A_\lambda\| + t \|\mu - B_\mu\| \\ &\quad + \|\lambda - A_\lambda\| \cdot \|\mu - B_\mu\|). \end{aligned} \quad (70)$$

It yields that

$$\begin{aligned} \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &\leq k^* \frac{1}{t^3} (t + \|\lambda - \mu\|) (t \|\lambda - A_\lambda\| + t \|\mu - B_\mu\| \\ &\quad + \|\lambda - A_\lambda\| \cdot \|\mu - B_\mu\|), \end{aligned} \quad (71)$$

$\forall \lambda, \mu \in U$, and $t \gg \theta$. Now first, we have to simplify the term $M_o(h\lambda, g\mu, t)/M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t) - 1$ by

using Definition 2 (3) and (62), for $t \gg \theta$,

$$\begin{aligned}
 & \frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \\
 & \leq \frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, g\mu, t) * M_o(g\mu, B\mu, t) * M_r(g\mu, h\lambda, t) * M_o(h\lambda, A\lambda, t)} - 1 \\
 & = \frac{1}{M_r(h\lambda, g\mu, t) * M_r(g\mu, B\mu, t) * M_r(h\lambda, A\lambda, t)} - 1 \\
 & = \frac{1}{(t + \|h\lambda - g\mu\|)(t + \|g\mu - B\mu\|)(t + \|h\lambda - A\lambda\|)} - 1 \\
 & = \frac{1}{t^3} \left(t^2 \|h\lambda - g\mu\| + t^2 (\|g\mu - B\mu\| + \|h\lambda - A\lambda\|) + t \|h\lambda - g\mu\| (\|g\mu - B\mu\| + \|h\lambda - A\lambda\|) \right) \\
 & = \frac{1}{t^3} \left(t^2 \|h\lambda - g\mu\| + t^2 (\|g\mu - B\mu\| + \|h\lambda - A\lambda\|) + t \|h\lambda - g\mu\| \cdot \|g\mu - B\mu\| + t \|h\lambda - g\mu\| \cdot \|h\lambda - A\lambda\| + t \|g\mu - B\mu\| \cdot \|h\lambda - A\lambda\| \right) \\
 & = \frac{1}{t^3} (t^2 \|h\lambda - g\mu\| + (t + \|h\lambda - g\mu\|)(t \|g\mu - B\mu\| + t \|h\lambda - A\lambda\| + \|h\lambda - A\lambda\| \cdot \|g\mu - B\mu\|)).
 \end{aligned} \tag{72}$$

This implies that

$$\begin{aligned}
 & \frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \\
 & \leq \frac{1}{t^3} (t^2 \|h\lambda - g\mu\| + (t + \|h\lambda - g\mu\|)(t \|g\mu - B\mu\| \\
 & \quad + t \|h\lambda - A\lambda\| + \|h\lambda - A\lambda\| \cdot \|g\mu - B\mu\|)),
 \end{aligned} \tag{73}$$

$\forall \lambda, \mu \in U$, and $t \gg \theta$. Now, from (71) and (73), we have

$$\frac{1}{M_r(A\lambda, B\mu, t)} - 1 \leq k^* \left(\frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \right), \text{ for } t \gg \theta, \tag{74}$$

$\forall \lambda, \mu \in U$. Hence, the integral operators A, B, g , and h satisfy the conditions of Theorem 10 with $k^* = b$ and $a = c = d = 0$ in (4). The operators A, B, g , and h have a unique CFP, i.e., the unique common solution of the two NIEs (60) in U .

(c) If $\mathbf{N}_{(A,B,g,h,\lambda,\mu)} = \|\lambda - A\lambda\| + \|\mu - B\mu\|$ in (66). Then by using (62), (65), and (67), we have

$$\begin{aligned}
 \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &= \frac{d(A\lambda, B\mu)}{t} \leq k^* \frac{\mathbf{N}_{(A,B,g,h,\lambda,\mu)}}{t} \\
 &= k^* \frac{\|\lambda - A\lambda\| + \|\mu - B\mu\|}{t} \\
 &= k^* \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right).
 \end{aligned} \tag{75}$$

This implies that

$$\frac{1}{M_r(A\lambda, B\mu, t)} - 1 \leq k^* \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right), \text{ for } t \gg \theta, \tag{76}$$

for all $\lambda, \mu \in U$. Hence, the integral operators A, B, g , and h satisfy the conditions of Theorem 10 with $k^* = c$ and $a = b = d = 0$ in (4). The operators A, B, g , and h have a unique CFP, i.e., the unique common solution of the two NIEs (60) in U .

(d) If $\mathbf{N}_{(A,B,g,h,\lambda,\mu)} = \|\mu - A\lambda\| + \|\lambda - B\mu\|$ in (66). Then, by using (62), (65), and (67), we have

$$\begin{aligned}
 \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &= \frac{d(A\lambda, B\mu)}{t} \leq k^* \frac{\mathbf{N}_{(A,B,g,h,\lambda,\mu)}}{t} \\
 &= k^* \frac{\|\mu - A\lambda\| + \|\lambda - B\mu\|}{t} \\
 &= k^* \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right).
 \end{aligned} \tag{77}$$

This implies that

$$\frac{1}{M_r(A\lambda, B\mu, t)} - 1 \leq k^* \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \text{ for } t \gg \theta, \tag{78}$$

for all $\lambda, \mu \in U$. Hence, the integral operators A, B, g , and h satisfy the conditions of Theorem 10 with $k^* = d$ and $a = b = c = 0$ in (4). The operators A, B, g , and h have a unique CFP, i.e., the unique common solution of the two NIEs (60) in U . \square

5. Conclusion

In this paper, we proved some unique CFP theorems by using the compatible and weakly-compatible four self-mappings in FCM space. We proved the results under the generalized rational contraction conditions in FCM spaces with help of one self-map are continuous. Moreover, we proved some rational contraction results with the weaker condition of self-map continuity. Ultimately, we provide an application of the two NIEs for our theoretical results that have been utilized to prove the existence common solution of the two NIEs to support our main work. This is an illustrative application of how FCM spaces can be used in other integral type operators.

Data Availability

Data sharing does not apply to this article as no data set were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The authors have equally contributed to the final manuscript.

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Research Article

Infra Soft Semiopen Sets and Infra Soft Semicontinuity

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To contribute to the area of infra soft topology, we introduce one of the generalizations of infra soft open sets called infra soft semiopen sets. We establish some characterizations of them and study their main properties. We determine under what condition this class is closed under finite intersection and show that this class is preserved under infra soft continuous mappings and finite product of soft spaces. Then, we present the concepts of infra semi-interior, infra semiclosure, infra semilimit, and infra semiboundary soft points of a soft set and elucidate the relationships between them. Finally, we exploit infra soft semiopen and infra soft semiclosed sets to define new types of soft mappings. We characterize each one of these soft mappings and explore main features.

1. Introduction

In 1999, Molodtsov [1] presented a novel mathematical tool to address vagueness, namely, soft sets. He discussed its relationship with fuzzy sets and showed some applications in different fields. Then, many scholars and researchers have studied some applications of soft set in different scopes such as decision-making problems [2], computer science [3], and medical science [4].

In 2003, Maji et al. [5] began studying the main concepts and notions of soft set theory. They explored the intersection and union operators, difference of two soft sets, and a complement of a soft set. However, some shortcomings appeared in their definitions, which led to reformulate most of these definitions and present new kinds of them. Ali et al. [6] originated new operators and operations between to preserve some properties and results of the (crisp) set theory in the soft set theory. Attempts were still in this path to produce new operators and relations like those introduced in [7].

In 2011, Çağman et al. [8] and Shabir and Naz [9] made use of soft sets to define soft topological spaces. Whereas, Çağman et al.'s definition given over an absolute soft set

and different sets of parameters, Shabir and Naz' definition given over a fixed set of universe and a fixed set of parameters. This paper follows the definition of Shabir and Naz. Later on, many studies which investigated the topological concepts in soft topologies have been done such as soft compactness [10], soft connectedness [11], soft separation axioms, soft basis [12], Caliber and chain conditions [13], soft bioperators, and generalized soft open sets [14]. Also, uniformity and Menger structures were introduced in the context of soft sets in [15, 16], respectively.

Soft topology was generalized to some structures; one of them is an infra soft topology [17]. The motivations of continuously investigating infra soft topological structure are that many topological properties are kept in the frame of infra soft topologies as well as the easy construction of examples that illustrate the relationships among the topological concepts. This matter was investigated for the concepts of infra soft compactness and infra soft connectedness in [18, 19].

Generalizations of (soft) open sets are a major topic in (soft) topology. One of the important generalizations is a soft semiopen set [20] which was studied in classical

topology by Levine [21]. In this article, we aim to explore the properties of this type of generalizations in the frame of infra soft topology. We elucidate the soundness of several properties of semiopen sets via infra soft topological spaces. This means that the infra soft topological spaces are flexible area to discuss the topological ideas and explore the relationships between them.

The arrangement of this article is as follows: Section 2 is allocated to mention some definitions and results relating to soft set theory and infra soft topology. In Section 3, we define a class of infra soft semiopen sets and establish some of characterizes. The concepts of infra semi-interior, infra semiclosure, infra semilimit, and infra semiboundary soft points of a soft set are introduced and probed in Section 4. In Section 5, we study the concepts of infra soft semicontinuous, infra soft semiopen, infra soft semiclosed, and infra soft semihomomorphism mappings. Also, we formulate and study the concept of semifixed soft points in the frame of infra soft topologies. Finally, some conclusions and the possible upcoming works are given in Section 6.

2. Preliminaries

This section mentions the concepts and findings that we need to understand this manuscript.

2.1. Soft Set Theory

Definition 1 (see [1]). Consider Θ as a set of parameters, \mathcal{T} a universal set, and $2^{\mathcal{T}}$ the power set of \mathcal{T} . An ordered pair (Ω, Θ) is called a soft set over \mathcal{T} provided that $\Omega : \Theta \rightarrow 2^{\mathcal{T}}$ is a crisp mapping. A soft set is expressed as $(\Omega, \Theta) = \{(\theta, \Omega(\theta)) : \theta \in \Theta \text{ and } \Omega(\theta) \in 2^{\mathcal{T}}\}$. We call $\Omega(\theta)$ a θ -approximate of (Ω, Θ) .

The class of all soft sets over \mathcal{T} under a set of parameters Θ is symbolized by $C(\mathcal{T}_{\Theta})$.

Definition 2 (see [6]). The complement of a soft set (Ω, Θ) , denoted by (Ω^c, Θ) , provided that a mapping $\Omega^c : \Theta \rightarrow 2^{\mathcal{T}}$ is given by $\Omega^c(\theta) = \mathcal{T} \setminus \Omega(\theta)$ for each $\theta \in \Theta$.

Definition 3 (see [5]). We call (Ω, Θ) an absolute (resp., a null) soft set over \mathcal{T} if the image of each parameter of Θ under a mapping $\Omega : \Theta \rightarrow 2^{\mathcal{T}}$ is the universal set \mathcal{T} (resp., empty set).

The absolute and null soft sets are symbolized by $\tilde{\mathcal{T}}$ and Φ , respectively.

Definition 4 (see [22]). We call (Ω, Θ) a stable (resp., finite, countable) soft set if every θ -approximate of (Ω, Θ) is equal (resp., finite, countable). Otherwise, it is called unstable (resp., infinite, uncountable).

Definition 5 (see [23]). Consider (Ω, Θ) as a soft set over \mathcal{T} such that $\Omega(\theta) = t \in \mathcal{T}$ and $\Omega(\theta') = \emptyset$ for each $\theta' \neq \theta$. Then, we call (Ω, Θ) a soft point over \mathcal{T} . It will be symbolized by δ_{θ}^t .

Definition 6 (see [6]). The intersection of soft sets (Ω, Θ) and (Ψ, Δ) which are defined over \mathcal{T} , symbolized by $(\Omega, \Theta) \cap \sim (\Psi, \Delta)$, is a soft set (Y, Σ) , where $\Sigma = \Theta \cap \Delta \neq \emptyset$, and a mapping $Y : \Sigma \rightarrow 2^{\mathcal{T}}$ is given by $Y(\theta) = \Omega(\theta) \cap \Psi(\theta)$ for each $\theta \in \Sigma$.

Definition 7 (see [5]). The union of soft sets (Ω, Θ) and (Ψ, Δ) which are defined over \mathcal{T} , symbolized by $(\Omega, \Theta) \cup \sim (\Psi, \Delta)$, is a soft set (Y, Σ) , where $\Sigma = \Theta \cup \Delta$ and a mapping $Y : \Sigma \rightarrow 2^{\mathcal{T}}$ is given as follows:

$$Y(\theta) = \begin{cases} \Omega(\theta) & \theta \in \Theta \setminus \Delta, \\ \Psi(\theta) & \theta \in \Delta \setminus \Theta, \\ \Omega(\theta) \cup \Psi(\theta) & \theta \in \Theta \cap \Delta. \end{cases} \quad (1)$$

Definition 8 (see [24]). A soft set (Ω, Θ) is a subset of a soft set (Ψ, Δ) , symbolized by $(\Omega, \Theta) \subseteq \sim (\Psi, \Delta)$, if $\Theta \subseteq \Delta$ and $\Omega(\theta) \subseteq \Psi(\theta)$ for all $\theta \in \Theta$. The soft sets (Ω, Θ) and (Ψ, Δ) are called soft equal if $(\Omega, \Theta) \subseteq \sim (\Psi, \Delta)$ and $(\Psi, \Delta) \subseteq \sim (\Omega, \Theta)$.

Definition 9 (see [10]). The Cartesian product of (Ω, Θ) and (Ψ, Δ) , symbolized by $(\Omega \times \Psi, \Theta \times \Delta)$, is defined as $(\Omega \times \Psi)(\theta, \theta') = \Omega(\theta) \times \Psi(\theta')$ for each $(\theta, \theta') \in \Theta \times \Delta$.

The definition of soft mappings given in [25] was reformulated in a way that reduces calculation burden and gives a justification (logical explanation) for some soft concepts such as why we determine that E_{τ} is injective, or surjective according to its two crisp maps E and τ .

Definition 10 (see [26]). Let $E : \mathcal{T} \rightarrow \mathcal{S}$ and $\tau : \Theta \rightarrow \Sigma$ be two crisp mappings. A soft mapping E_{τ} of $C(\mathcal{T}_{\Theta})$ into $C(\mathcal{S}_{\Sigma})$ is a relation such that each soft point in $C(\mathcal{T}_{\Theta})$ is related to one and only one soft point in $C(\mathcal{S}_{\Sigma})$ such that

$$E_{\tau}(\delta_{\theta}^t) = \delta_{\tau(\theta)}^{E(t)} \text{ for each } \delta_{\theta}^t \in C(\mathcal{T}_{\Theta}). \quad (2)$$

In addition, $E_{\tau}^{-1}(\delta_{\sigma}^s) = \cup_{\omega \in \tau^{-1}(\sigma)} \cup_{v \in E^{-1}(s)} \delta_{\omega}^v$ for each $\delta_{\sigma}^s \in C(\mathcal{S}_{\Sigma})$.

Definition 11 (see [25]). A soft mapping $f_{\tau} : C(\mathcal{T}_{\Theta}) \rightarrow C(\mathcal{S}_{\Delta})$ is said to be injective (resp., surjective, bijective) if both f and τ are injective (resp., surjective, bijective).

2.2. Infra Soft Topological Spaces

Definition 12 (see [17]). A family ξ of soft sets over \mathcal{T} with Θ as a parameter set is said to be an infra soft topology on \mathcal{T} if it is closed under finite intersection and Φ is a member of ξ .

The triple $(\mathcal{T}, \xi, \Theta)$ is called an infra soft topological space (briefly, ISTS). We call a member of ξ an infra soft open set and call its complement an infra soft closed set. We call $(\mathcal{T}, \xi, \Theta)$ stable if all its infra soft open sets are stable.

Definition 13 (see [17]). Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$.

- (i) The intersection of all infra soft closed subsets of $(\mathcal{T}, \xi, \Theta)$ which contains a soft set (Ω, Θ) is called the infra soft closure of (Ω, Θ) . It is denoted by $Cl(\Omega, \Theta)$
- (ii) The union of all infra soft open subsets of $(\mathcal{T}, \xi, \Theta)$ which are contained in a soft set (Ω, Θ) is called the infra soft interior of (Ω, Θ) . It is denoted by $Int(\Omega, \Theta)$

It was shown in [17] that $Cl(\Omega, \Theta)$ and $Int(\Omega, \Theta)$ need not be infra soft closed and infra soft open, respectively. Through this paper, (Ω, Θ) is called ξ -infra soft open (resp., ξ -infra soft closed) if $Int(\Omega, \Theta) = (\Omega, \Theta)$ (resp., $Cl(\Omega, \Theta) = (\Omega, \Theta)$).

Proposition 14 (see [17]). Let (Ω, Θ) and (Ψ, Θ) subsets of an ISTS $(\mathcal{T}, \xi, \Theta)$. Then

- (i) $Cl[(\Omega, \Theta) \cup (\Psi, \Theta)] = Cl(\Omega, \Theta) \cup Cl(\Psi, \Theta)$
- (ii) $Int[(\Omega, \Theta) \cap (\Psi, \Theta)] = Int(\Omega, \Theta) \cap Int(\Psi, \Theta)$

Proposition 15 (see [17]). Let (Ω, Θ) be an infra soft open set. Then

$$(\Omega, \Theta) \cap \sim Cl(\Psi, \Theta) \subseteq \sim Cl[(\Omega, \Theta) \cup (\Psi, \Theta)] \quad \text{for any subset } (\Psi, \Theta) \text{ of } (\mathcal{T}, \xi, \Theta). \quad (3)$$

Proposition 16 (see [17]). Let (Ω, Θ) be an infra soft closed set. Then

$$Int[(\Omega, \Theta) \cup (\Psi, \Theta)] \subseteq \sim(\Omega, \Theta) \cap \sim Int(\Psi, \Theta) \quad \text{for any subset } (\Psi, \Theta) \text{ of } (\mathcal{T}, \xi, \Theta). \quad (4)$$

Definition 17. A soft mapping $f_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is said to be an infra soft homeomorphism if it is bijective, infra soft continuous (i.e., the image of every infra soft open set is infra soft open), and infra soft open (i.e., the image of every infra soft open set is infra soft open).

A property is called an infra soft topological property (briefly, IST property) if it is preserved by any infra soft homeomorphism.

Definition 18 (see [17]). Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ be a soft mapping and $\mathcal{M} \neq \emptyset$ be a subset of \mathcal{T} . A soft mapping $E_{\tau, \mathcal{M}} : (\mathcal{M}, \xi_{\mathcal{M}}, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ which given by $E_{\tau, \mathcal{M}}(\delta_\theta^m) = E_\tau(\delta_\theta^m)$ for every $\delta_\theta^m \in \mathcal{M}$ is called a restriction soft mapping of E_τ on \mathcal{M} .

Proposition 19. Let $\{(\mathcal{T}_k, \xi_k, \Theta_k) : k \in K\}$ be a family of ISTSs. Then, $\xi = \{\prod_{k \in K} (\theta_k, \Theta_k) : (\theta_k, \Theta_k) \in \tau_k\}$ is an infra soft topology on $\mathcal{T} = \prod_{k \in K} \mathcal{T}_k$ under a set of parameters $\mathcal{B} = \prod_{k \in K} \Theta_k$.

We call ξ given in proposition above, a product of infra soft topologies, and $(\mathcal{T}, \xi, \mathcal{B})$ a product of infra soft spaces.

3. Infra Soft Semiopen Sets and Basic Properties

In this section, we introduce the concept of infra soft semiopen sets which represents a class of generalizations of infra soft open sets. We give some characterizations of infra soft semiopen and infra soft semiclosed sets and establish main properties. Also, we prove that this class is closed under arbitrary unions and determine under what condition this class is closed under finite intersection. Finally, we show that an infra soft semiopen set and its complement are preserved under infra soft continuous mappings and finite product of soft spaces.

Definition 20. A subset (Ω, Θ) of an ISTS $(\mathcal{T}, \xi, \Theta)$ is said to be infra soft semiopen if $(\Omega, \Theta) \subseteq \sim Cl(Int(\Omega, \Theta))$. Its complement is said to be an infra soft semiclosed set.

The following two propositions give some descriptions for infra soft semiopen and infra soft semiclosed sets.

Proposition 21. Let (Ω, Θ) be a subset of an ISTS $(\mathcal{T}, \xi, \Theta)$. Then, we have the following equivalent properties:

- (i) (Ω, Θ) is an infra soft semiopen set
- (ii) $Cl(Int(\Omega, \Theta)) = Cl(\Omega, \Theta)$
- (iii) There exists an ξ -infra soft open set (Ψ, Θ) such that $(\Psi, \Theta) \subseteq \sim(\Omega, \Theta) \subseteq \sim Cl(\Psi, \Theta)$

Proof. (i) \Leftrightarrow (ii): Let (Ω, Θ) be an infra soft semiopen set. Then, $(\Omega, \Theta) \subseteq \sim Cl(Int(\Omega, \Theta))$. Therefore, $Cl(\Omega, \Theta) \subseteq \sim Cl(Int(\Omega, \Theta))$. It is well known that $Cl(Int(\Omega, \Theta)) \subseteq \sim Cl(\Omega, \Theta)$. Thus, $Cl(Int(\Omega, \Theta)) = Cl(\Omega, \Theta)$. The direction (ii) \Rightarrow (i) is obvious.

(ii) \Leftrightarrow (iii): Since (Ω, Θ) is an infra soft semiopen set, $(\Omega, \Theta) \subseteq \sim Cl(Int(\Omega, \Theta))$. Taking $(\Psi, \Theta) = Int(\Omega, \Theta)$, we obtain $(\Psi, \Theta) \subseteq \sim(\Omega, \Theta) \subseteq \sim Cl(\Psi, \Theta)$. Since $Int(\Psi, \Theta) = Int(Int(\Omega, \Theta)) = Int(\Omega, \Theta) = (\Psi, \Theta)$, (Ψ, Θ) is an ξ -infra soft open set. Conversely, let (Ψ, Θ) is an ξ -infra soft open set such that $(\Psi, \Theta) \subseteq \sim(\Omega, \Theta) \subseteq \sim Cl(\Psi, \Theta)$. Then, $Cl(\Psi, \Theta) \subseteq \sim Cl(Int(\Omega, \Theta))$. By assumption, $(\Omega, \Theta) \subseteq \sim Cl(Int(\Omega, \Theta))$ which means that (Ω, Θ) is an infra soft semiopen set. \square

Proposition 22. Let (Ω, Θ) be a subset of an ISTS $(\mathcal{T}, \xi, \Theta)$. Then, we have the following equivalent properties:

- (i) (Ω, Θ) is an infra soft semiclosed set
- (ii) $Int(Cl(\Omega, \Theta)) \subseteq \sim(\Omega, \Theta)$
- (iii) $Int(Cl(\Omega, \Theta)) = Int(\Omega, \Theta)$
- (iv) There exists an ξ -infra soft closed set (Ψ, Θ) such that $Int(\Psi, \Theta) \subseteq \sim(\Omega, \Theta) \subseteq \sim(\Psi, \Theta)$

Proof. One can prove it following similar arguments given in the proof of Proposition 21. \square

Proposition 23. *The class of infra soft semiopen sets is closed under arbitrary unions.*

Proof. Consider $\{(\Omega_j, \Theta) : j \in J\}$ as a family of infra soft semiopen sets. If the index J is an empty set, then $\bigcup_{j \in \emptyset} (\Omega_j, \Theta) = \Phi$ which is an infra soft semiopen set. Suppose that $J \neq \emptyset$. Then $(\Omega_j, \Theta) \subseteq \sim Cl(Int(\Omega_j, \Theta))$ for each $j \in J$. Consequentially, $\bigcup_{j \in J} (\Omega_j, \Theta) \subseteq \sim \bigcup_{j \in J} Cl(Int(\Omega_j, \Theta)) \subseteq \sim Cl(Int(\bigcup_{j \in J} (\Omega_j, \Theta)))$. Hence, $\bigcup_{j \in J} (\Omega_j, \Theta)$ is infra soft semiopen. \square

Corollary 24. *The class of infra soft semiclosed sets is closed under arbitrary intersections.*

Corollary 25. *The class of infra soft semiopen subsets of an ISTS $(\mathcal{T}, \xi, \Theta)$ forms a supra soft topology over \mathcal{T} .*

To illustrate that the class of infra soft semiopen sets does not form an infra soft topology, we present the following example.

Example 1. Let $\mathcal{T} = \{t_1, t_2, t_3, t_4\}$ and $\Theta = \{\theta_1, \theta_2\}$. Then, $\xi = \{\Phi, \tilde{\mathcal{T}}, (\Omega_1, \Theta), (\Omega_2, \Theta)\}$ is an infra soft topology on \mathcal{T} with Θ as a set of parameters, where

$$(\Omega_1, \Theta) = \{(\theta_1, \{t_1\}), (\theta_2, \{t_1\})\}, \quad (5)$$

and

$$(\Omega_2, \Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \{t_2\})\}. \quad (6)$$

Let $(\Omega_3, \Theta) = \{(\theta_1, \{t_1, t_3\}), (\theta_2, \{t_1, t_3\})\}$ and $(\Omega_4, \Theta) = \{(\theta_1, \{t_2, t_3\}), (\theta_2, \{t_2, t_3\})\}$. Then, (Ω_3, Θ) and (Ω_4, Θ) are infra soft semiopen sets because $Cl(Int(\Omega_3, \Theta)) = \{(\theta_1, \{t_1, t_3, t_4\}), (\theta_2, \{t_1, t_3, t_4\})\}$ and $Cl(Int(\Omega_4, \Theta)) = \{(\theta_1, \{t_2, t_3, t_4\}), (\theta_2, \{t_2, t_3, t_4\})\}$. But $(\Omega_3, \Theta) \cap \sim (\Omega_4, \Theta)$ is not infra soft semiopen because $Cl(Int[(\Omega_3, \Theta) \cap \sim (\Omega_4, \Theta)]) = \Phi$.

Proposition 26. *The intersection of infra soft open and infra soft semiopen sets is an infra soft semiopen set.*

Proof. Let (Ω_1, Θ) be an infra soft open set and (Ω_2, Θ) be an infra soft semiopen set. Then, $(\Omega_1, \Theta) \cap \sim (\Omega_2, \Theta) \subseteq \sim (\Omega_1, \Theta) \cap \sim Cl(Int(\Omega_2, \Theta))$; by Proposition 15, we obtain $(\Omega_1, \Theta) \cap \sim (\Omega_2, \Theta) \subseteq \sim Cl[(\Omega_1, \Theta) \cap \sim Int(\Omega_2, \Theta)] = Cl[Int(\Omega_1, \Theta) \cap \sim Int(\Omega_2, \Theta)] = Cl[Int[(\Omega_1, \Theta) \cap \sim (\Omega_2, \Theta)]]$. Hence, we obtain the desired result. \square

Corollary 27. *The union of infra soft closed and infra soft semiclosed sets is an infra soft semiclosed set.*

Definition 28. An ISTS $(\mathcal{T}, \xi, \Theta)$ is said to be infra soft hyperconnected if the intersection of any two nonnull ξ

-infra soft open sets is nonnull. Otherwise, $(\mathcal{T}, \xi, \Theta)$ is said to be infra soft dishyperconnected.

Proposition 29. *The intersection of two infra soft semiopen subsets of an infra soft hyperconnected space is an infra soft semiopen set.*

Proof. Let (Ω_1, Θ) and (Ω_2, Θ) be infra soft semiopen sets. If one of them is the null soft set, then, we obtain the desired result. Suppose that (Ω_1, Θ) and (Ω_2, Θ) are nonnull soft sets. According to Proposition 21, there are two ξ -infra soft open sets $(\Psi_1, \Theta) \neq \Phi$ and $(\Psi_2, \Theta) \neq \Phi$ such that $(\Psi_1, \Theta) \subseteq \sim (\Omega_1, \Theta) \subseteq \sim Cl(\Psi_1, \Theta)$ and $(\Psi_2, \Theta) \subseteq \sim (\Omega_2, \Theta) \subseteq \sim Cl(\Psi_2, \Theta)$. By hypothesis of infra soft hyperconnectedness, $(\Psi_1, \Theta) \cap \sim (\Psi_2, \Theta)$ is a nonnull ξ -infra soft open set. Now, $(\Psi_1, \Theta) \cap \sim (\Psi_2, \Theta) \subseteq \sim (\Omega_1, \Theta) \cap \sim (\Omega_2, \Theta) \subseteq \sim Cl[(\Psi_1, \Theta) \cap \sim (\Psi_2, \Theta)]$. Hence, $(\Omega_1, \Theta) \cap \sim (\Omega_2, \Theta)$ is an infra soft semiopen set. \square

Lemma 30. Let $E_\tau : (\mathcal{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathcal{T}_2, \xi_2, \Theta_2)$ be an infra soft homeomorphism map. Then, for any subset (Ω, Θ_1) , we have the next two results.

$$(i) E_\tau(Int(\Omega, \Theta_1)) = Int(E_\tau(\Omega, \Theta_1))$$

$$(ii) E_\tau(Cl(\Omega, \Theta_1)) = Cl(E_\tau(\Omega, \Theta_1))$$

Proof. To prove (i), let $\delta_{\theta'}^s \in E_\tau(Int(\Omega, \Theta_1))$. Then, there is $\delta_{\theta'}^t \in Int(\Omega, \Theta_1)$ such that $E_\tau(\delta_{\theta'}^t) = \delta_{\theta'}^s$. This means there exists an infra soft open set (Ψ, Θ_1) such that $\delta_{\theta'}^t \in (\Psi, \Theta_1) \subseteq \sim (\Omega, \Theta_1)$. Therefore, $\delta_{\theta'}^s = E_\tau(\delta_{\theta'}^t) \in E_\tau(\Psi, \Theta_1) \subseteq \sim E_\tau(\Omega, \Theta_1)$. This implies that $\delta_{\theta'}^s \in Int(E_\tau(\Omega, \Theta_1))$. Thus, $E_\tau(Int(\Omega, \Theta_1)) \subseteq \sim Int(E_\tau(\Omega, \Theta_1))$. Conversely, let $\delta_{\theta'}^s \in Int(E_\tau(\Omega, \Theta_1))$. Then, there exists an infra soft open set (Ψ, Θ_2) such that $\delta_{\theta'}^s \in (\Psi, \Theta_2) \subseteq \sim E_\tau(\Omega, \Theta_1)$. Therefore, $E_\tau^{-1}(\delta_{\theta'}^s) \in E_\tau^{-1}(\Psi, \Theta_2) \subseteq \sim (\Omega, \Theta_1)$. Automatically, we obtain $E_\tau^{-1}(\delta_{\theta'}^s) \in Int(\Omega, \Theta_1)$. So that, $\delta_{\theta'}^s \in E_\tau(Int(\Omega, \Theta_1))$. Thus, $Int(E_\tau(\Omega, \Theta_1)) \subseteq \sim E_\tau(Int(\Omega, \Theta_1))$. Hence, the proof is complete.

Following similar arguments, one can prove (ii). \square

Proposition 31. *The infra soft homeomorphism image of an infra soft semiopen set is an infra soft semiopen set.*

Proof. Consider $E_\tau : (\mathcal{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathcal{T}_2, \xi_2, \Theta_2)$ as an infra soft continuous mapping and let (Ω, Θ_1) be an infra soft semiopen subset of $(\mathcal{T}_1, \xi_1, \Theta_1)$. Then, $E_\tau(\Omega, \Theta_1) \subseteq \sim E_\tau(Cl(Int(\Omega, \Theta_1)))$. It follows from the above lemma that $E_\tau(\Omega, \Theta_1) \subseteq \sim Cl(Int(E_\tau(\Omega, \Theta_1)))$. Hence, $E_\tau(\Omega, \Theta_1)$ is an infra soft semiopen subset of $(\mathcal{T}_2, \xi_2, \Theta_2)$, as required. \square

Lemma 32. Consider (Ω_1, Θ_1) and (Ω_2, Θ_2) as subsets of $(\mathcal{T}_1, \xi_1, \Theta_1)$ and $(\mathcal{T}_2, \xi_2, \Theta_2)$, respectively. Then

$$(i) Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] = Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2)$$

$$(ii) \text{Int}[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] = \text{Int}(\Omega_1, \Theta_1) \times \text{Int}(\Omega_2, \Theta_2)$$

Proof. (i): Let $\delta_{(\theta, \theta)}^{(t,s)} \notin Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)]$. Then, there is an infra soft open subset $(\Psi_1, \Theta_1) \times (\Psi_2, \Theta_2)$ of $\widetilde{\mathcal{T}}_1 \times \widetilde{\mathcal{T}}_2$ containing $\delta_{(\theta, \theta)}^{(t,s)}$ such that $[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] \cap \sim[(\Psi_1, \Theta_1) \times (\Psi_2, \Theta_2)] = \Phi_{\Theta_1 \times \Theta_2}$. This implies that $(\Omega_1, \Theta_1) \cap \sim(\Psi_1, \Theta_1) = \Phi_{\Theta_1}$ or $(\Omega_2, \Theta_2) \cap \sim(\Psi_2, \Theta_2) = \Phi_{\Theta_2}$. Therefore, $\delta_{\theta}^t \notin Cl(\Omega_1, \Theta_1)$ or $\delta_{\theta}^s \notin Cl(\Omega_2, \Theta_2)$. Thus, $\delta_{(\theta, \theta)}^{(t,s)} \notin [Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2)]$. Hence, $Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2) \subseteq \sim Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)]$. Conversely, let $\delta_{(\theta, \theta)}^{(t,s)} \notin Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2)$. Then, $\delta_{\theta}^t \notin Cl(\Omega_1, \Theta_1)$ or $\delta_{\theta}^s \notin Cl(\Omega_2, \Theta_2)$. Suppose, without loss of generality, that $\delta_{\theta}^t \notin Cl(\Omega_1, \Theta_1)$. Then, there is an infra soft open subset (Ψ_1, Θ_1) of $\mathcal{T}_1, \xi_1, \Theta_1$ containing δ_{θ}^t such that $(\Omega_1, \Theta_1) \cap \sim(\Psi_1, \Theta_1) = \Phi_{\Theta_1}$. Obviously, $(\Psi_1, \Theta_1) \times \widetilde{\mathcal{T}}_2$ is an infra soft open subset of $\widetilde{\mathcal{T}}_1 \times \widetilde{\mathcal{T}}_2$ containing $\delta_{(\theta, \theta)}^{(t,s)}$ such that $[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] \cap \sim[(\Psi_1, \Theta_1) \times (\Omega_2, \Theta_2)] = \Phi_{\Theta_1 \times \Theta_2}$. Therefore, $\delta_{(\theta, \theta)}^{(t,s)} \notin Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)]$. Thus, $Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] \subseteq \sim Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2)$. Hence, the proof is complete.

Following similar arguments, one can prove (ii). \square

Proposition 33. *The product of infra soft semiopen sets is an infra soft semiopen set.*

Proof. Let (Ω_1, Θ_1) and (Ω_2, Θ_2) be infra soft semiopen subsets of $(\mathcal{T}_1, \xi_1, \Theta_1)$ and $(\mathcal{T}_2, \xi_2, \Theta_2)$, respectively. Then, $(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2) \subseteq \sim Cl(\text{Int}(\Omega_1, \Theta_1) \times \text{Int}(\Omega_2, \Theta_2))$. According to the above lemma, we obtain $(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2) \subseteq \sim Cl(\text{Int}[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)])$ which means that $(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)$ is an infra soft semiopen subset of $\widetilde{\mathcal{T}}_1 \times \widetilde{\mathcal{T}}_2$. \square

4. Infra Semi-Interior, Infra Semiclosure, Infra Semilimit, and Infra Semiboundary Soft Points of a Soft Set

In this section, we first present the infra soft semi-interior and infra soft semiclosure operators and scrutinize their essential properties. Then, we define infra soft semilimit and infra soft semiboundary soft points of a soft set. We discuss their main features and reveal the relationships between them with the help of examples.

Definition 34. Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then

- (i) The infra soft semi-interior of (Ω, Θ) , denoted by $s\text{Int}(\Omega, \Theta)$, is the union of all infra soft semiopen sets that are contained in (Ω, Θ)
- (ii) The infra soft semiclosure of (Ω, Θ) , denoted by $sCl(\Omega, \Theta)$, is the intersection of all infra soft semiclosed sets containing (Ω, Θ)

Proposition 35. *We have the following properties.*

- (i) (Ω, Θ) is an infra soft semiopen subset of $(\mathcal{T}, \xi, \Theta)$ iff $s\text{Int}(\Omega, \Theta) = (\Omega, \Theta)$
- (ii) (Ω, Θ) is an infra soft semiclosed subset of $(\mathcal{T}, \xi, \Theta)$ iff $sCl(\Omega, \Theta) = (\Omega, \Theta)$

Proof. It follows from Proposition 23 and Corollary 24. \square

It should be noted that the infra soft open and infra soft closed sets do not satisfy the above two properties.

Proposition 36. *Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$.*

- (i) $\delta_{\theta}^t \in s\text{Int}(\Omega, \Theta)$ iff there is an infra soft semiopen set (Ψ, Θ) such that $\delta_{\theta}^t \in (\Psi, \Theta) \subseteq \sim(\Omega, \Theta)$
- (ii) $\delta_{\theta}^t \in sCl(\Omega, \Theta)$ iff the intersection of any infra soft semiopen set (Ψ, Θ) containing δ_{θ}^t and (Ω, Θ) is nonnull

Proof. The proof of (i) is obvious, so we prove (ii).

Let $\delta_{\theta}^t \in sCl(\Omega, \Theta)$. Then, every infra soft semiclosed set contains (Ω, Θ) contains δ_{θ}^t as well. Suppose that there exists an infra soft semiopen set (Ψ, Θ) containing δ_{θ}^t such that $(\Omega, \Theta) \cap \sim(\Psi, \Theta) = \Phi$. Therefore, $(\Omega, \Theta) \subseteq \sim(\Psi^c, \Theta)$ which means that $\delta_{\theta}^t \notin sCl(\Omega, \Theta)$. This is a contradiction. Conversely, suppose that there exists an infra soft semiopen set (Ψ, Θ) containing δ_{θ}^t such that $(\Omega, \Theta) \cap \sim(\Psi, \Theta) = \Phi$. Therefore, $sCl(\Omega, \Theta) \subseteq \sim(\Psi^c, \Theta)$ which means that $\delta_{\theta}^t \notin sCl(\Omega, \Theta)$. Hence, we obtain the desired result. \square

Proposition 37. *Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then,*

- (i) $(s\text{Int}(\Omega, \Theta))^c = sCl(\Omega^c, \Theta)$
- (ii) $(sCl(\Omega, \Theta))^c = s\text{Int}(\Omega^c, \Theta)$

Proof. (i): $(s\text{Int}(\Omega, \Theta))^c = \{\cup_{j \in J} (\Psi_j, \Theta) : (\Psi_j, \Theta) \text{ is an infra soft semiopen set contained in } (\Omega, \Theta)\}^c = \cap_{j \in J} \{(\Psi_j^c, \Theta) : (\Psi_j^c, \Theta) \text{ is an infra soft semi-closed set containing } (\Omega^c, \Theta)\} = sCl(\Omega^c, \Theta)$.

The proof of (ii) is similar to (i). \square

Proposition 38. *Let (Ψ, Θ) be an infra soft open set and (Λ, Θ) be an infra soft closed set in $(\mathcal{T}, \xi, \Theta)$. Then,*

- (i) $(\Psi, \Theta) \cap \sim sCl(\Omega, \Theta) \subseteq \sim sCl((\Psi, \Theta) \cap \sim(\Omega, \Theta))$
- (ii) $s\text{Int}((\Lambda, \Theta) \cup \sim(\Omega, \Theta)) \subseteq \sim(\Lambda, \Theta) \cup s\text{Int}(\Omega, \Theta)$

Proof. (i): Let $\delta_{\theta}^t \in (\Psi, \Theta) \cap \sim sCl(\Omega, \Theta)$. Then, $\delta_{\theta}^t \in (\Psi, \Theta)$ and $\delta_{\theta}^t \notin sCl(\Omega, \Theta)$. This implies that $(\Gamma, \Theta) \cap \sim(\Omega, \Theta) \neq \Phi$ for every infra soft semiopen set (Γ, Θ) containing δ_{θ}^t . It

follows from Proposition 26 that $(\Psi, \Theta) \cap \sim(\Gamma, \Theta)$ is an infra soft semiopen set containing δ_θ^t . Therefore, $[(\Psi, \Theta) \cap \sim(\Gamma, \Theta)] \cap \sim(\Omega, \Theta) \neq \Phi$. Now, $(\Gamma, \Theta) \cap [(\Psi, \Theta) \cap \sim(\Omega, \Theta)] \neq \Phi$ which means that $\delta_\theta^t \in sCl((\Psi, \Theta) \cap \sim(\Omega, \Theta))$. Hence, $(\Psi, \Theta) \cap \sim sCl(\Omega, \Theta) \subseteq \sim sCl((\Psi, \Theta) \cap \sim(\Omega, \Theta))$.

One can prove (ii) following similar arguments. \square

Theorem 39. Let (Ω, Θ) and (Ψ, Θ) be subsets of $(\mathcal{T}, \xi, \Theta)$. Then, we have the following properties.

- (i) $sInt(\tilde{\mathcal{T}}) = \tilde{\mathcal{T}}$
- (ii) $sInt(\Omega, \Theta) \subseteq \sim(\Omega, \Theta)$
- (iii) If $(\Psi, \Theta) \subseteq \sim(\Omega, \Theta)$, then $sInt(\Psi, \Theta) \subseteq \sim sInt(\Omega, \Theta)$
- (iv) $sInt(sInt(\Omega, \Theta)) = sInt(\Omega, \Theta)$
- (v) $sInt(\Psi, \Theta) \cap \sim sInt(\Omega, \Theta) \subseteq \sim sInt((\Psi, \Theta) \cap \sim(\Omega, \Theta))$

Proof. (i): Since $\tilde{\mathcal{T}}$ is infra soft semiopen, $sInt(\tilde{\mathcal{T}}) = \tilde{\mathcal{T}}$.

(ii) and (iii) are obvious.

(iv): It is clear that $sInt(sInt(\Omega, \Theta))$ is the largest infra soft semiopen set contained in $sInt(\Omega, \Theta)$; however, $sInt(\Omega, \Theta)$ is an infra soft semiopen set; hence, $sInt(sInt(\Omega, \Theta)) = sInt(\Omega, \Theta)$.

(v): It comes from (iii). \square

Theorem 40. Let (Ω, Θ) and (Ψ, Θ) be subsets of $(\mathcal{T}, \xi, \Theta)$. Then, we have the following properties.

- (i) $sCl(\Phi) = \Phi$
- (ii) $(\Omega, \Theta) \subseteq \sim sCl(\Omega, \Theta)$
- (iii) If $(\Psi, \Theta) \subseteq \sim(\Omega, \Theta)$, then $sCl(\Psi, \Theta) \subseteq \sim sCl(\Omega, \Theta)$
- (iv) $sCl(sCl(\Omega, \Theta)) \subseteq \sim sCl(\Omega, \Theta)$
- (v) $sCl((\Psi, \Theta) \cup \sim(\Omega, \Theta)) = sCl(\Psi, \Theta) \cup \sim sCl(\Omega, \Theta)$

Proof. It can be proved following similar arguments given in the proof of Theorem 39. \square

The next example shows that the inclusion relations given in the above two theorems are proper.

Example 2. Let $\mathcal{T} = \{t_1, t_2\}$ and $\Theta = \{\theta_1, \theta_2\}$. Then, $\xi = \{\Phi, \tilde{\mathcal{T}}, (\Omega_j, \Theta): j = 1, 2, 3\}$ is an infra soft topology on \mathcal{T} over \mathcal{T} with Θ as a set of parameters, where

$$(\Omega_1, \Theta) = \{(\theta_1, \{t_1\}), (\theta_2, \emptyset)\}; \quad (7)$$

$$(\Omega_2, \Theta) = \{(\theta_1, \emptyset), (\theta_2, \{t_1\})\}, \quad (8)$$

and

$$(\Omega_3, \Theta) = \{(\theta_1, \mathcal{T}), (\theta_2, \{t_2\})\}. \quad (9)$$

Let $(\Psi_1, \Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \{t_1\})\}$. Then, $sInt(\Psi_1, \Theta)$

$= \{(\theta_1, \emptyset), (\theta_2, \{t_1\})\} \subset \sim(\Psi_1, \Theta)$ and $sCl(\Psi_1, \Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \mathcal{T})\} \supset \sim(\Psi_1, \Theta)$. Also, consider $(\Psi_2, \Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \emptyset)\}$. Then, $sCl((\Psi_1, \Theta) \cup \sim(\Psi_2, \Theta)) = \{(\theta_1, \{t_2\}), (\theta_2, \mathcal{T})\} \supset \sim sCl(\Psi_1, \Theta) \cap \sim sCl(\Psi_2, \Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \{t_2\})\}$.

Definition 41. A soft point δ_θ^t is said to be an infra soft semilimit point of a subset (Ω, Θ) of $(\mathcal{T}, \xi, \Theta)$ provided that $[(\Psi, \Theta) \setminus \delta_\theta^t] \cap \sim(\Omega, \Theta) \neq \Phi$ for every infra soft semiopen set (Ψ, Θ) containing δ_θ^t .

The soft set of all infra soft semilimit points of (Ω, Θ) is said to be an infra semiderived soft set. It is denoted by $(\Omega, \Theta)^{is'}$.

Proposition 42. Consider (Ψ, Θ) and (Ω, Θ) as subsets of $(\mathcal{T}, \xi, \Theta)$. Then,

- (i) $\Phi^{is'} = \Phi$ and $\mathcal{T}^{is'} \subseteq \sim \tilde{\mathcal{T}}$
- (ii) If $(\Psi, \Theta) \subseteq \sim(\Omega, \Theta)$, then $(\Psi, \Theta)^{is'} \subseteq \sim(\Omega, \Theta)^{is'}$
- (iii) If $\delta_\theta^t \in (\Omega, \Theta)^{is'}$, then $\delta_\theta^t \in ((\Omega, \Theta) \setminus \delta_\theta^t)^{is'}$
- (iv) $(\Psi, \Theta)^{is'} \cup \sim(\Omega, \Theta)^{is'} \subseteq \sim((\Psi, \Theta) \cup \sim(\Omega, \Theta))^{is'}$.

Proof. Straightforward. \square

Theorem 43. Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then,

- (i) If (Ω, Θ) is an infra soft semiclosed set, then, $(\Omega, \Theta)^{is'} \subseteq (\Omega, \Theta)$
- (ii) $((\Omega, \Theta) \cup \sim(\Omega, \Theta))^{is'} \subseteq \sim(\Omega, \Theta) \cup \sim(\Omega, \Theta)^{is'}$
- (iii) $sCl(\Omega, \Theta) = (\Omega, \Theta) \cup \sim(\Omega, \Theta)^{is'}$

Proof.

- (i) Consider (Ω, Θ) as an infra soft semiclosed set such that $\delta_\theta^t \notin (\Omega, \Theta)$. Then, $\delta_\theta^t \in (\Omega^c, \Theta)$. Now, (Ω^c, Θ) is an infra soft semiopen set such that $(\Omega^c, \Theta) \cup \sim(\Omega, \Theta) = \Phi$ which means that $\delta_\theta^t \notin (\Omega, \Theta)^{is'}$. Thus, $(\Omega, \Theta)^{is'} \subseteq \sim(\Omega, \Theta)$
- (ii) Consider $\delta_\theta^t \notin (\Omega, \Theta) \cup \sim(\Omega, \Theta)^{is'}$. Then, $\delta_\theta^t \notin (\Omega, \Theta)$ and $\delta_\theta^t \notin (\Omega, \Theta)^{is'}$. Therefore, there exists an infra soft semiopen set (Ψ, Θ) such that

$$(\Psi, \Theta) \cap \widetilde{(\Omega, \Theta)} = \Phi, \quad (10)$$

This implies that

$$(\Psi, \Theta) \cap \widetilde{(\Omega, \Theta)^{is'}} = \Phi. \quad (11)$$

It follows from (10) and (11) that $(\Psi, \Theta) \cap \sim((\Omega, \Theta) \cup \sim(\Omega, \Theta)^{is'}) = \Phi$. Thus, $\delta_\theta^t \notin ((\Omega, \Theta) \cup \sim(\Omega, \Theta)^{is'})^{is'}$. Hence, $((\Omega, \Theta) \cup \sim(\Omega, \Theta)^{is'})^{is'} \subseteq \sim((\Omega, \Theta) \cup \sim(\Omega, \Theta)^{is'})$, as required.

(iii) It is clear that $(\Omega, \Theta) \cup \sim(\Omega, \Theta)^{is'} \subseteq \sim sCl(\Omega, \Theta)$. Conversely, let $\delta_\theta^t \in sCl(\Omega, \Theta)$. Then, for every infra soft semiopen set containing δ_θ^t , we have $(\Omega, \Theta) \cap \sim(\Psi, \Theta) \neq \Phi$. Without loss of generality, let $\delta_\theta^t \notin (\Omega, \Theta)$. Then, $[(\Omega, \Theta) \setminus \delta_\theta^t] \cap \sim(\Psi, \Theta) \neq \Phi$. Consequently, $\delta_\theta^t \in (\Omega, \Theta)^{is'}$. Hence, the proof is complete.

□

Definition 44. The infra soft semiboundary points of a subset (Ω, Θ) of $(\mathcal{T}, \xi, \Theta)$, denoted by $sB(\Omega, \Theta)$, are all the soft points which belong to the complement of $sInt(\Omega, \Theta) \cup \sim sInt(\Omega^c, \Theta)$.

Proposition 45. Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then,

- (i) $sB(\Omega, \Theta) = sCl(\Omega, \Theta) \cap \sim sCl((\Omega^c, \Theta))$
- (ii) $sB(\Omega, \Theta) = sCl(\Omega, \Theta) \setminus sInt(\Omega, \Theta)$

Proof.

- (i) $sB(\Omega, \Theta) = \{\delta_\theta^t \in \tilde{\mathcal{T}} : \delta_\theta^t \notin sInt(\Omega, \Theta) \text{ and } \delta_\theta^t \notin sInt((\Omega^c, \Theta))\} = \{\delta_\theta^t \in \tilde{\mathcal{T}} : \delta_\theta^t \notin (sCl(\Omega^c, \Theta))^c \text{ and } \delta_\theta^t \notin (sCl(\Omega, \Theta))^c\} = \{\delta_\theta^t \in \tilde{\mathcal{T}} : \delta_\theta^t \in sCl(\Omega^c, \Theta) \text{ and } \delta_\theta^t \in sCl(\Omega, \Theta)\} \cap \sim sCl(\Omega^c, \Theta) = sCl(\Omega, \Theta) \cap \sim sCl(\Omega^c, \Theta)$
- (ii) $sB(\Omega, \Theta) = sCl(\Omega, \Theta) \cap \sim sCl(\Omega^c, \Theta) = sCl(\Omega, \Theta) \cap \sim(sInt(\Omega, \Theta))^c = sCl(\Omega, \Theta) \setminus sInt(\Omega, \Theta)$

□

□

Corollary 46. Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then

- (i) $sB(\Omega, \Theta) = sB(\Omega^c, \Theta)$
- (ii) $sCl(\Omega, \Theta) = sInt(\Omega, \Theta) \cup \sim sB(\Omega, \Theta)$

Proposition 47. Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then,

- (i) (Ω, Θ) is infra soft semiopen iff $sB(\Omega, \Theta) \cap \sim(\Omega, \Theta) = \Phi$
- (ii) (Ω, Θ) is infra soft semiclosed iff $sB(\Omega, \Theta) \subseteq \sim(\Omega, \Theta)$

Proof.

- (i) $sB(\Omega, \Theta) \cap (\Omega, \Theta) = sB(\Omega, \Theta) \cap sInt(\Omega, \Theta) = \Phi$. Conversely, let $\delta_\theta^t \in (\Omega, \Theta)$. Then $\delta_\theta^t \in sInt(\Omega, \Theta)$ or $\delta_\theta^t \in sB(\Omega, \Theta)$. Since $sB(\Omega, \Theta) \cap (\Omega, \Theta) = \Phi$, $\delta_\theta^t \in sInt(\Omega, \Theta)$.

$sInt(\Omega, \Theta)$. Thus, $(\Omega, \Theta) \subseteq sInt(\Omega, \Theta)$ which means that $(\Omega, \Theta) = sInt(\Omega, \Theta)$. Hence, (Ω, Θ) is infra soft semiopen.

- (ii) (Ω, Θ) is infra soft semiclosed $\Leftrightarrow (\Omega^c, \Theta)$ is infra soft semiopen $\Leftrightarrow sB(\Omega^c, \Theta) \cap (\Omega^c, \Theta) = \Phi \Leftrightarrow sB(\Omega, \Theta) \cap (\Omega, \Theta) = \Phi \Leftrightarrow sB(\Omega, \Theta) \subseteq \sim(\Omega, \Theta)$

□

Corollary 48. A subset (Ω, Θ) of $(\mathcal{T}, \xi, \Theta)$ is infra soft semiopen and infra soft semiclosed iff $sB(\Omega, \Theta) = \Phi$.

5. Infra Soft Semihomomorphism Maps

This section introduces the concepts of infra soft semicontinuous, infra soft semiopen, infra soft semiclosed, and infra soft semihomomorphism maps. We give some characterizations of each one of these concepts and demonstrate some interrelations between them. Finally, we study the concept of fixed soft points with respect to infra soft semiopen sets.

Definition 49. A soft mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is said to be infra soft semicontinuous at $\delta_\theta^t \in \tilde{\mathcal{T}}$ if for any infra soft semiopen set (Ψ, Δ) containing $E_\tau(\delta_\theta^t)$, there is an infra soft semiopen set (Ω, Θ) containing δ_θ^t such that $E_\tau(\Omega, \Theta) \subseteq \sim(\Psi, \Delta)$.

If E_τ is infra soft semicontinuous at all soft points of the domain, then, it is called infra soft semicontinuous.

Theorem 50. $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is an infra soft semicontinuous mapping iff the preimage of each infra soft semiopen set is infra soft semiopen.

Proof. Necessity: let (Ω, Δ) be an infra soft semiopen subset of $(\mathcal{S}, \pi, \Delta)$. If $E_\tau^{-1}(\Omega, \Delta) = \Phi$, then, the proof is trivial. So, consider $E_\tau^{-1}(\Omega, \Delta) \neq \Phi$. Now, for any $\delta_\theta^t \in E_\tau^{-1}(\Omega, \Delta)$, there is an infra soft semiopen subset (Ψ, Θ) of $(\mathcal{T}, \xi, \Theta)$ containing δ_θ^t such that $E_\tau(\Psi, \Theta) \subseteq \sim(\Omega, \Delta)$. Thus, $\delta_\theta^t \in (\Psi, \Theta) \subseteq \sim E_\tau^{-1}(\Omega, \Delta)$ and $\bigcup \{(\Psi, \Theta)\} = E_\tau^{-1}(\Omega, \Delta)$. This means that $E_\tau^{-1}(\Omega, \Delta)$ is infra soft semiopen.

Sufficiency: let $\delta_\theta^t \in \tilde{\mathcal{T}}$ and (Ψ, Θ) be an infra soft semiopen set containing $E_\tau(\delta_\theta^t)$. Then, $E_\tau^{-1}(\Psi, \Theta)$ is an infra soft semiopen set containing δ_θ^t such that $E_\tau(E_\tau^{-1}(\Psi, \Theta)) \subseteq \sim(\Psi, \Theta)$. This means that E_τ is infra soft semicontinuous at δ_θ^t . This ends the proof that E_τ is infra soft semicontinuous. □

Example 3. Consider \mathcal{T} is the set of real numbers, \mathcal{S} is the set of natural numbers and $\Theta = \{\theta_1, \theta_2\}$. Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Theta)$ and $F_\nu : (\mathcal{S}, \mu, \Theta) \longrightarrow (\mathcal{S}, \pi, \Theta)$ be two soft mappings such that E, F, τ , and ν are identity mappings, and π is the discrete soft topology (it is also infra soft topology). Let $\xi = \{(\mathcal{T}, (\Omega, \Theta) \subseteq \sim \mathcal{T} : (\Omega, \Theta) \text{ is finite})\}$ and $\mu = \{\Phi, \mathcal{S}, \{(\theta_1, \{2\}), (\theta_2, \{3\})\}, \{(\theta_1, \{3\}), (\theta_2, \{2\})\}\}$ are two infra soft topologies on \mathcal{T} and \mathcal{S} , respectively. It is clear that every subset of $(\mathcal{T}, \xi, \Theta)$ is infra soft semiopen. So that, E_τ is

infra soft semicontinuous. On the other hand, F_v is not infra soft semicontinuous because $(\Psi, \Theta) = \{(\theta_1, \{1\}), (\theta_2, \{1\})\}$ is an infra soft semiopen subset of $(\mathcal{S}, \pi, \Theta)$, whereas its preimage under a soft mapping F_v is not an infra soft semiopen subset of $(\mathcal{S}, \mu, \Theta)$.

Theorem 51. Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ be an infra soft semicontinuous mapping. Then, we have the following five equivalent statements:

- (i) E_τ is an infra soft semicontinuous mapping
- (ii) The preimage of each infra soft semiclosed set is infra soft semiclosed
- (iii) $sCl(E_\tau^{-1}(\Omega, \Delta)) \subseteq \sim E_\tau^{-1}(sCl(\Omega, \Delta))$ for each $(\Omega, \Delta) \subseteq \sim \tilde{\mathcal{S}}$
- (iv) $E_\tau(sCl(\Psi, \Theta)) \subseteq \sim sCl(E_\tau(\Psi, \Theta))$ for each $(\Psi, \Theta) \subseteq \sim \tilde{\mathcal{S}}$
- (v) $E_\tau^{-1}(sInt(\Omega, \Delta)) \subseteq \sim sInt(E_\tau^{-1}(\Omega, \Delta))$ for each $(\Omega, \Delta) \subseteq \sim \tilde{\mathcal{S}}$

Proof. (i) \Rightarrow (ii): Let (Ω, Δ) be an infra soft semiclosed set in $(\mathcal{S}, \pi, \Delta)$.

Then, $E_\tau^{-1}(\Omega^c, \Delta)$ is an infrasoftware semiopen subset of $\tilde{\mathcal{S}}$. Obviously, $E_\tau^{-1}(\Omega^c, \Delta) = \tilde{\mathcal{S}} - E_\tau^{-1}(\Omega, \Delta)$; hence, $E_\tau^{-1}(\Omega, \Delta)$ is an infra soft semiclosed subset of $\tilde{\mathcal{S}}$.

(ii) \Rightarrow (iii): According to (ii), $E_\tau^{-1}(sCl(\Omega, \Delta))$ is an infra soft semiclosed subset of $\tilde{\mathcal{S}}$. Then, $sCl(E_\tau^{-1}(\Omega, \Delta)) \subseteq \sim sCl(E_\tau^{-1}(sCl(\Omega, \Delta))) = E_\tau^{-1}(sCl(\Omega, \Delta))$.

(iii) \Rightarrow (vi): According to (iii), $sCl(E_\tau^{-1}(E_\tau(\Psi, \Theta))) \subseteq \sim E_\tau^{-1}(sCl(E_\tau(\Psi, \Theta)))$. Then, $E_\tau(sCl(\Psi, \Theta)) \subseteq \sim E_\tau(E_\tau^{-1}(sCl(E_\tau(\Psi, \Theta)))) \subseteq \sim sCl(E_\tau(\Psi, \Theta))$.

(iv) \Rightarrow (v): According to (iv), $E_\tau(sCl(\tilde{\mathcal{S}} - E_\tau^{-1}(\Omega, \Delta))) \subseteq \sim sCl(E_\tau(\tilde{\mathcal{S}} - E_\tau^{-1}(\Omega, \Delta)))$. Therefore, $E_\tau(\tilde{\mathcal{S}} - sInt(E_\tau^{-1}(\Omega, \Delta))) = E_\tau(sCl(\tilde{\mathcal{S}} - E_\tau^{-1}(\Omega, \Delta))) \subseteq \sim sCl(\tilde{\mathcal{S}} - (\Omega, \Delta)) = \tilde{\mathcal{S}} - sInt(\Omega, \Delta)$. Thus, $\tilde{\mathcal{S}} - sInt(E_\tau^{-1}(\Omega, \Delta)) \subseteq \sim E_\tau^{-1}(\tilde{\mathcal{S}} - sInt(\Omega, \Delta)) = E_\tau^{-1}(\tilde{\mathcal{S}}) - E_\tau^{-1}(sInt(\Omega, \Delta))$. Hence, $E_\tau^{-1}(sInt(\Omega, \Delta)) \subseteq \sim sInt(E_\tau^{-1}(\Omega, \Delta))$.

(v) \Rightarrow (i): Let (Ω, Δ) be an infra soft open subset of $\tilde{\mathcal{S}}$. According to (v), $E_\tau^{-1}(\Omega, \Delta) \subseteq \sim sInt(E_\tau^{-1}(\Omega, \Delta))$. This implies that $E_\tau^{-1}(\Omega, \Delta) = sInt(E_\tau^{-1}(\Omega, \Delta))$. Hence, E_τ is infra soft semicontinuous. \square

Theorem 52. If $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is infra soft semicontinuous, then, the restriction soft mapping $E_{\tau|_{\mathcal{M}}} : (\mathcal{M}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is infra soft semicontinuous provided that $\tilde{\mathcal{M}}$ is an infra soft open set.

Proof. Consider (Ω, Δ) is an infra soft semiopen set in $(\mathcal{S}, \pi, \Delta)$. By hypothesis, $E_\tau^{-1}(\Omega, \Delta)$ is infra soft semiopen. Now, $E_{\tau|_{\mathcal{M}}}^{-1}(\Omega, \Delta) = E_\tau^{-1}(\Omega, \Delta) \cap \sim \tilde{\mathcal{M}}$. Since $\tilde{\mathcal{M}}$ is an infra soft open set, it follows from Proposition 26 that $E_{\tau|_{\mathcal{M}}}^{-1}(\Omega, \Delta)$ is

infra soft semiopen. Hence, $E_{\tau|_{\mathcal{M}}}$ is an infra soft semicontinuous map. \square

Proposition 53. Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ and $F_v : (\mathcal{S}, \pi, \Delta) \longrightarrow (\mathcal{V}, \sigma, \Gamma)$ be infra soft semicontinuous. Then, $F_v \circ E_\tau$ is infra soft semicontinuous.

Proof. Straightforward. \square

Definition 54. A soft mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is said to be infra soft semiopen (resp., infra soft semiclosed) if the image of each infra soft semiopen (resp., infra soft semiclosed) set is infra soft semiopen (resp., infra soft semiclosed).

Proposition 55. $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is an infra soft semiopen mapping iff $E_\tau(sInt(\Omega, \Theta)) \subseteq \sim sInt(E_\tau(\Omega, \Theta))$ for each subset of (Ω, Θ) of $\tilde{\mathcal{S}}$.

Proof. \Rightarrow : Let (Ω, Θ) be a subset of $\tilde{\mathcal{S}}$. Now, $E_\tau(sInt(\Omega, \Theta)) \subseteq \sim E_\tau(\Omega, \Theta)$ and $sInt(\Omega, \Theta)$ are an infra soft semiopen set. By hypothesis, $E_\tau(sInt(\Omega, \Theta))$ is infra soft semiopen. Therefore, $E_\tau(sInt(\Omega, \Theta)) \subseteq \sim sInt(E_\tau(\Omega, \Theta))$.

\Leftarrow : Let (Λ, Θ) be an infra soft open subset of $\tilde{\mathcal{S}}$. Then, $E_\tau(\Omega, \Theta) \subseteq \sim sInt(E_\tau(\Omega, \Theta))$. Therefore, $E_\tau(\Omega, \Theta) = sInt(E_\tau(\Omega, \Theta))$ which means that E_τ is an infra soft semiopen map. \square

Proposition 56. $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is an infra soft semiclosed mapping iff $sCl(E_\tau(\Omega, \Theta)) \subseteq \sim E_\tau(sCl(\Omega, \Theta))$ for each subset (Ω, Θ) of $\tilde{\mathcal{S}}$.

Proof. \Rightarrow : Let E_τ be an infra soft semiclosed mapping and (Ω, Θ) be a subset of $\tilde{\mathcal{S}}$. By hypothesis, $E_\tau(sCl(\Omega, \Theta))$ is infra soft semiclosed. Since $E_\tau(\Omega, \Theta) \subseteq \sim E_\tau(sCl(\Omega, \Theta))$, $sCl(E_\tau(\Omega, \Theta)) \subseteq \sim E_\tau(sCl(\Omega, \Theta))$.

\Leftarrow : Suppose that (Ω, Θ) is an infra soft semiclosed subset of $\tilde{\mathcal{S}}$. By hypothesis, $E_\tau(\Omega, \Theta) \subseteq \sim sCl(E_\tau(\Omega, \Theta)) \subseteq \sim E_\tau(sCl(\Omega, \Theta)) = E_\tau(\Omega, \Theta)$. Therefore, $E_\tau(\Omega, \Theta)$ is infra soft semiclosed. Hence, E_τ is an infra soft semiclosed map. \square

Proposition 57. The concepts of infra soft semiopen and infra soft semiclosed mappings are equivalent under bijectiveness.

Proof. It comes from the fact that a bijective soft mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ implies that $E_\tau(\Omega^c, \Theta) = (E_\tau(\Omega, \Theta))^c$. \square

Proposition 58. Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ and $F_v : (\mathcal{S}, \pi, \Delta) \longrightarrow (\mathcal{V}, \sigma, \Gamma)$ be two soft maps. Then,

- (i) If E_τ and F_v are infra soft semiopen maps, then, $F_v \circ E_\tau$ is an infra soft semiopen map

- (ii) If $F_\nu \circ E_\tau$ is an infra soft semiopen mapping and E_τ is a surjective infra soft semicontinuous map, then, F_ν is an infra soft semiopen map
- (iii) If $F_\nu \circ E_\tau$ is an infra soft semiopen mapping and F_ν is an injective infra soft semicontinuous map, then, E_τ is an infra soft semiopen map

Proof.

- (i) Straightforward
- (ii) Consider (Ω, Δ) as an infra soft semiopen subset of $(\mathcal{S}, \pi, \Delta)$. By hypothesis, $E_\tau^{-1}(\Omega, \Delta)$ is an infra soft semiopen subset of $(\mathcal{T}, \xi, \Theta)$. Again, by hypothesis, $(F_\nu \circ E_\tau)(E_\tau^{-1}(\Omega, \Delta))$ is an infra soft semiopen subset of $(\mathcal{V}, \sigma, \Gamma)$. Since E_τ is surjective, then, $(F_\nu \circ E_\tau)(E_\tau^{-1}(\Omega, \Delta)) = F_\nu(E_\tau(E_\tau^{-1}(\Omega, \Delta))) = F_\nu(\Omega, \Delta)$. Hence, F_ν is an infra soft semiopen map
- (iii) Consider (Ω, Θ) as an infra soft semiopen subset of $(\mathcal{T}, \xi, \Theta)$. By hypothesis, $(F_\nu \circ E_\tau)(\Omega, \Theta)$ is an infra soft semiopen subset of $(\mathcal{V}, \sigma, \Gamma)$. Again, by hypothesis, $F_\nu^{-1}(F_\nu \circ E_\tau(\Omega, \Theta))$ is an infra soft semiopen subset of $(\mathcal{S}, \pi, \Delta)$. Since F_ν is injective, then, $F_\nu^{-1}(F_\nu \circ E_\tau(\Omega, \Theta)) = (F_\nu^{-1} F_\nu)(E_\tau(\Omega, \Theta)) = E_\tau(\Omega, \Theta)$. Hence, E_τ is an infra soft semiopen map.

□

□

The following result can be proved following similar arguments given in proposition' proof above.

Proposition 59. Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \longrightarrow (\mathcal{V}, \sigma, \Gamma)$ be two infra soft maps. Then, the following statements hold.

- (i) If E_τ and F_ν are infra soft semiclosed maps, then, $F_\nu \circ E_\tau$ is an infra soft semiclosed map
- (ii) If $F_\nu \circ E_\tau$ is an infra soft semiclosed mapping and E_τ is a surjective infra soft semicontinuous map, then, F_ν is an infra soft semiclosed map
- (iii) If $F_\nu \circ E_\tau$ is an infra soft semiclosed mapping and F_ν is an injective infra soft semicontinuous map, then, E_τ is an infra soft semiclosed map

Definition 60. A bijective soft mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is said to be an infra soft semihomomorphism if it is infra soft semicontinuous and infra soft semiopen.

We cancel the proofs of the next two results because they are easy.

Proposition 61. Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \longrightarrow (\mathcal{V}, \sigma, \Gamma)$ be infra soft semihomomorphism maps. Then, $F_\nu \circ E_\tau$ is an infra soft semihomomorphism map.

Proposition 62. If $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is a bijective soft map, then, the following statements are equivalent.

- (i) E_τ is an infra soft semihomomorphism
- (ii) E_τ and E_τ^{-1} is infra soft semicontinuous
- (iii) E_τ is infra soft semiclosed and infra soft semicontinuous

Proposition 63. If $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is an infra soft semihomomorphism map, then, the following statements hold for each $(\Omega, \Theta) \in S(X)_A$.

- (i) $E_\tau(sInt(\Omega, \Theta)) = sInt(E_\tau(\Omega, \Theta))$
- (ii) $E_\tau(sCl(\Omega, \Theta)) = sCl(E_\tau(\Omega, \Theta))$

Proof. (i): According to Proposition 55 (i), we obtain $E_\tau(sInt(\Omega, \Theta)) \subseteq \sim sInt(E_\tau(\Omega, \Theta))$.

Conversely, let $\delta_\kappa^s \in sInt(E_\tau(\Omega, \Theta))$. Then, there is an infra soft semiopen set (Ψ, Δ) such that $\delta_\kappa^s \in (\Psi, \Delta) \subseteq \sim E_\tau(\Omega, \Theta)$. By hypothesis, $\delta_\theta^t = E_\tau^{-1}(\delta_\kappa^s) \in E_\tau^{-1}(\Psi, \Delta) \subseteq \sim (\Omega, \Theta)$ such that $E_\tau^{-1}(\Psi, \Delta)$ is an infra soft semiopen set. So that, $\delta_\theta^t \in sInt(\Omega, \Theta)$ which means that $\delta_\kappa^s \in E_\tau(sInt(\Omega, \Theta))$.

One can achieve item (ii) following similar arguments. □

□

Theorem 64. The property of an infra soft semidense set is an infra soft topological invariant.

Proof. Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ be an infra soft semihomomorphism mapping and consider (Ω, Θ) as an infra soft semidense subset of $(\mathcal{T}, \xi, \Theta)$, i.e., $sCl(\Omega, \Theta) = \tilde{\mathcal{T}}$. It comes from Proposition 63 (ii) that $sCl(E_\tau(\Omega, \Theta)) = E_\tau(sCl(\Omega, \Theta)) = E_\tau(\tilde{\mathcal{T}}) = sCl(\tilde{\mathcal{S}}) = \tilde{\mathcal{S}}$. Thus, $E_\tau(\Omega, \Theta)$ is an infra soft semidense set in $(\mathcal{S}, \pi, \Delta)$, as required. □

We complete this section by studying the concept of fixed soft points with respect to infra soft semiopen sets. For more details in fixed soft points in the crisp setting, see [27–29].

Definition 65. We say that $(\mathcal{T}, \xi, \Theta)$ has a semifixed soft point property provided that for every infra soft semicontinuous mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{T}, \xi, \Theta)$, there exists $\delta_\theta^s \in \mathcal{T}$ such that $E_\tau(\delta_\theta^s) = \delta_\theta^s$.

Proposition 66. The property of being a semifixed soft point is preserved under an infra soft semihomomorphism.

Proof. Consider $(\mathcal{T}_1, \xi_1, \Theta_1)$ and $(\mathcal{T}_2, \xi_2, \Theta_2)$ as two infra soft semihomomorphism. This means that there exists a bijective soft mapping $E_\tau : (\mathcal{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathcal{T}_2, \xi_2, \Theta_2)$ such that E_τ and E_τ^{-1} are infra soft semicontinuous. Suppose that $(\mathcal{T}_1, \xi_1, \Theta_1)$ has the property of semifixed soft point.

That is any infra soft semicontinuous mapping $E_\tau : (\mathcal{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathcal{T}_1, \xi_1, \Theta_1)$ has a semifixed soft point. Now, consider $C_\tau : (\mathcal{T}_2, \xi_2, \Theta_2) \longrightarrow (\mathcal{T}_2, \xi_2, \Theta_2)$ is infra soft semicontinuous. It is clear that $C_\tau \circ E_\tau : (\mathcal{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathcal{T}_2, \xi_2, \Theta_2)$ is infra soft semicontinuous. Therefore, $E_\tau^{-1} \circ C_\tau \circ E_\tau : (\mathcal{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathcal{T}_1, \xi_1, \Theta_1)$ is infra soft semicontinuous. Since $(\mathcal{T}_1, \xi_1, \Theta_1)$ has a semifixed soft point property, $E_\tau^{-1}(h_\tau(E_\tau(\delta_\theta^s))) = \delta_\theta^s$ for some $\delta_\theta^s \in \tilde{\mathcal{T}}$. Thus, $E_\tau(E_\tau^{-1}(h_\tau(E_\tau(\delta_\theta^s)))) = E_\tau(\delta_\theta^s)$. This implies that $h_\tau(E_\tau(\delta_\theta^s)) = E_\tau(\delta_\theta^s)$. Hence, $E_\tau(\delta_\theta^s)$ is a semifixed soft point of C_τ which means that $(\mathcal{T}_2, \xi_2, \Theta_2)$ has a semifixed soft point property. \square

6. Concluding Remark and Further Work

This article contributes to the expanding literature on soft topological spaces. The obtained results demonstrate that most soft topological properties of the presented concepts are preserved in structure of infra soft topologies which means we can dispense of some topological stipulations. This gives an advantage of discussing soft topological ideas via infra soft topologies because it relaxes the restrictions imposed in the study. The obtained results in this manuscript and those given in [17–19] validate this viewpoint.

On the other hand, there are a few properties of some topological concepts that are partially losing via infra soft topology such as the equivalence between an infra soft semiopen set (Ω, Θ) and the existence of an infra soft open set (Ψ, Θ) such that $(\Psi, \Theta) \subseteq \sim(\Omega, \Theta) \subseteq \sim Cl(\Psi, \Theta)$. However, we have addressed this matter by defining an ξ -infra soft open set and proving the counterpart equivalence as given in Proposition 21. As we have shown in Corollary 25 that the class of infra soft semiopen subsets on ISTSs forms a new generalization of soft topology called a supra soft topology.

This work considers a promising line for future work; for example, we will complete introducing the main topological concepts using infra soft semiopen sets such as soft separation axioms, soft compact, and soft connected spaces. Our roadmap for research also comprises the examination of the concepts and results initiated herein using another generalization of infra soft open sets such as infra soft α -open and infra soft b -open sets. Moreover, we will introduce new types of rough approximations using these generalizations of infra soft open sets and apply them to improve the accuracy measures of sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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Research Article

Knaster-Kuratowski-Mazurkiewicz Theorem in Generalized Metric Spaces with Applications

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We study Knaster-Kuratowski-Mazurkiewicz theorem in the setting of generalized metric spaces. We establish some results on fixed points of Knaster-Kuratowski-Mazurkiewicz (KKM) mappings. Fan's matching and Schauder's type fixed point theorem in generalized metric spaces are also proved as interesting consequences of our main results. Examples are given to validate our results. We use these results to prove existence result for a given Atangana-Baleanu-Caputo fractional boundary value problem.

1. Introduction

In the large spectrum of mathematical problems, a fixed point of a specific map provides the solution of a mathematical problem. Therefore, fixed point theory is of pronounced significance in numerous fields of mathematics and other disciplines of science and engineering. Fixed point results describe the conditions under which a mathematical problem has a solution. Some fixed point results can be seen in [1–3].

The notion of metric space is fundamental in mathematics and has a major role in understanding and applying the topological concepts in different domains of analysis. This idea has pulled a substantial consideration from mathematicians owing to the notion of fixed point theory in metric spaces. Numerous extension and generalizations of the notion of metric space have been done in the literature. Czerwik [4] in 1993 presented the idea of b -metric spaces. In 1998, Czerwik [5] reintroduced this idea in which the constant 2 associated with the triangular inequality was replaced by a constant $k \geq 1$. In 2010, Khamsi and Hussain [6] generalized the idea of b -metric and named it metric type spaces. After that Hitzler and Seda [7] in 2000 gave the idea about dislocated metric spaces in which distance of a point from itself may or may not be zero. The idea of generalized

metric spaces was given by Jleli and Samet [8], which covers distinctive notable structures including metric type spaces, metric spaces, and dislocated metric spaces, among others.

KKM map was presented by Knaster et al. [9] in 1929 and it established the framework for some notable existing results like Ky Fan Browder's fixed point theorem, Nash's equilibrium theorem, and Ky Fan's minimax inequality theorem [10–14]. The Fan's theorem [15] is a significant result for KKM mappings and is being implemented as a useful technique in the modern nonlinear analysis. The first endeavor to stretch out these types of theorems in metric spaces was done in [16], where the author studied the case of hyperconvex metric spaces. Using the work of Chang and Yen [17] and the idea of Khamsi [16], Amini et al. introduced KKM mappings in metric spaces [18]. After that, Khamsi and Hussain [6] extended those results (of [18]) in the settings of metric type spaces.

In Section 2 of this article, we give some basic definitions and notions for the sake of completeness. We also define and study open set in generalized metric spaces and show that an open ball needs not to be an open set in general. In Section 3, we extend the theorems and the related results of [6] to generalized metric spaces. We also prove some results related to the fixed points of KKM mappings in generalized metric

spaces in this section. Fan's matching and Schauder's type fixed point theorem as interesting consequences of our main results are furnished in Section 4.

2. Basic Definitions and Results

In this section, we recall some basic definitions, which we use to prove our main results.

Definition 1 (see [8]). Let $W \neq \emptyset$ and $d : W \times W \longrightarrow [0, \infty]$ be the given mapping. For every $\lambda \in W$, define the following set:

$$C(d, W, \lambda) = \left\{ \{\lambda_n\} \subset W : \lim_{n \rightarrow \infty} d(\lambda_n, \lambda) = 0 \right\}. \quad (1)$$

Definition 2 (see [8]). Let $W \neq \emptyset$ and $d : W \times W \longrightarrow [0, \infty]$ be the given mapping. Then, d is known as a generalized metric on W , if for all $\beta, \lambda \in W$, the following conditions are satisfied:

- (1) $d(\lambda, \beta) = d(\beta, \lambda)$
- (2) $d(\lambda, \beta) = 0 \Rightarrow \lambda = \beta$
- (3) For $\{\lambda_n\} \in C(d, W, \lambda)$, there exists $S > 0$ such that

$$d(\lambda, \beta) \leq S \lim_{n \rightarrow \infty} \sup d(\lambda_n, \beta). \quad (2)$$

The pair (W, d) is known as a generalized metric space.

Clearly, if the set $C(d, W, \lambda) = \emptyset$ for every $\lambda \in W$, then (W, d) is a generalized metric space if and only if Equations (35) and (44) are satisfied.

Definition 3 (see [8]). Let $\{\lambda_n\}$ be the sequence in the generalized metric space (W, d) . Then, $\{\lambda_n\}$ is d -Cauchy sequence if

$$\lim_{p, q \rightarrow \infty} d(\lambda_p, \lambda_{p+q}) = 0. \quad (3)$$

Definition 4 (see [8]). Consider the generalized metric space (W, d) and $\lambda \in W$. The sequence $\{\lambda_n\}$ in W is d which converges to λ if

$$\{\lambda_n\} \in C(d, W, \lambda). \quad (4)$$

Recall that open and closed balls in the generalized metric space (W, d) are, respectively, defined as $B(\alpha, r) = \{\beta \in W : d(\alpha, \beta) < r\}$ and $B[\alpha, r] = \{\beta \in W : d(\alpha, \beta) \leq r\}$ for any $\alpha \in W$ and $r > 0$. Now, we define open set in a generalized metric space similar as defined in [6].

Definition 5. A nonempty subset Y of the generalized metric space (W, d) is said to be open if for any $\kappa \in Y$, there exists $\varepsilon > 0$ such that

$$B(\kappa, \varepsilon) \subseteq Y. \quad (5)$$

In this case, we denote $\kappa \in Y^\circ$ and read it as κ which is

the interior point of Y . The collection of all such subsets of W will be denoted by τ , which defines topology on (W, d) . The complement of an open set is called closed set and if $\kappa \in \bar{Y}$ is called closure of Y ; then, $\kappa \in Y$ or there exists $\{\kappa_n\} \in Y$ such that $\{\kappa_n\} \in C(d, W, \kappa)$.

Remark 6. It is not necessary for an open ball in a generalized metric space to be an open set.

Example 1. Let $P = [0, 1/2]$, $Q = (1/2, 1]$, and $W = P \cup Q$. Define $d : W \times W \longrightarrow [0, \infty]$ by

$$d(\alpha, \beta) = d(\beta, \alpha) = \begin{cases} 0, & \alpha = \beta, \\ 1, & \alpha \neq \beta, \alpha, \beta \in P, \text{ or } \alpha, \beta \in Q, \\ \alpha, & \alpha \in P, \beta \in Q. \end{cases} \quad (6)$$

Then, (W, d) is the generalized metric space. Now, the open ball having center $\alpha_0 = 1/2$ and radius $\delta = 1$, denoted by the set A , is given by the following:

$$A = B\left(\frac{1}{2}, 1\right) = \left[\frac{1}{2}, 1\right]. \quad (7)$$

Choose $1 = \gamma \in A$ and $\delta' > 0$, then

$$B(1, \delta') = \left\{ \alpha \in W : d(1, \alpha) < \delta' \right\}. \quad (8)$$

For any $\delta' > 0$, we have infinite many $\alpha \in P$ such that $\alpha \in B(1, \delta')$. So, there does not exist $\delta' > 0$ satisfying

$$B(1, \delta') \subseteq A. \quad (9)$$

Thus, the open ball $A = B(1/2, 1)$ is not an open set.

Definition 7. Consider the nonempty subset Y of the generalized metric space (W, d) . Then, Y is referred as sequentially compact if there exists convergent subsequence $\{\alpha_{n_k}\}$ for every sequence $\{\alpha_n\}$ in Y . Y is called compact if Y is sequentially compact. Y is totally bounded if for any $\varepsilon > 0$, we have $\gamma_i \in Y$, $1 \leq i \leq p$ such that

$$Y \subseteq \bigcup_{i=1}^p B(\gamma_i, \varepsilon). \quad (10)$$

3. Main Results

We start with some useful notions which are essential to establish our main results.

Let Y and Z be two topological spaces and $G : Y \longrightarrow 2^Z$ be the nonempty set-valued mapping, where 2^Z represents the collection of all nonempty subsets of Z .

The set-valued mapping $G : Y \longrightarrow 2^Z$ is referred to be as follows:

- (i) Closed if the graph $Gr(G) = \{(y, z) \in Y \times Z; z \in G(y)\}$ is closed
- (ii) Compact if $\overline{G(Y)}$ is compact in Z

The set of all nonempty finite subsets of a set W is denoted by $\langle W \rangle$. For a nonempty bounded subset Y of the generalized metric space (W, d) , we define the following:

$$co(Y) = \bigcap \{D \subset W, \text{ where } D \text{ is closed ball in } W \text{ containing } Y\} \quad (11)$$

and Y is admissible in W , if

$$co(Y) = Y, \quad (12)$$

i.e., Y is admissible if the intersection of all closed balls D containing Y is Y . If for any $D \in \langle Y \rangle$,

$$co(D) \subset Y, \quad (13)$$

then Y is a subadmissible subset of W . Clearly, if Y is admissible in W , then Y is also subadmissible.

Consider the subadmissible subset Y of the generalized metric space (W, d) . A set-valued mapping $G : Y \longrightarrow 2^W$ is known as KKM mapping if

$$co(A) \subset G(A), \quad (14)$$

for any $A \in \langle Y \rangle$. More generally, for the topological space Z , consider the two set-valued mappings $G : Y \longrightarrow 2^Z$ and $H : Y \longrightarrow 2^Z$ such that

$$H(co(A)) \subseteq G(A), \quad (15)$$

for any $A \in \langle Y \rangle$; in this case, G is referred as a generalized KKM mapping with reference to H .

If the set-valued mapping $H : Y \longrightarrow 2^Z$ satisfies the condition that for any generalized KKM mapping $G : Y \longrightarrow 2^Z$ with reference to H , the class $\{\overline{G(y)}, y \in Y\}$ has finite intersection property, then H has KKM property, and we write it as follows:

$$H \in KKM(Y, Z) = \{H : Y \longrightarrow 2^Z, \quad H \text{ has KKM property}\}. \quad (16)$$

Consider the generalized metric space (W, d) and $\phi \neq Y \subset W$. Then, $H : Y \longrightarrow 2^W$ is called to have approximate fixed point property if for any $\varepsilon > 0$, there exists $y \in Y$ such that

$$H(y) \cap B(y, \varepsilon) \neq \phi. \quad (17)$$

We now present approximate fixed point property of KKM type mapping on subadmissible subset of a generalized metric space and generalize the main results of [6, 18].

Theorem 8. Consider the nonempty subadmissible subset Y of the generalized metric space (W, d) . Let $H \in KKM(Y, Y)$ be such that $\overline{H(Y)}$ is totally bounded. Then, H has an approximate fixed point property.

Proof. Consider

$$Z = \overline{H(Y)} \subset \bar{Y}, \quad (18)$$

where Z is totally bounded. Thus, for any $\varepsilon > 0$, Y has a finite subset C such that

$$Z \subseteq \bigcup_{c \in C} B(c, \varepsilon), \quad (19)$$

where $B(c, \varepsilon)$ is an open ball having radius ε and center c .

Now, we define a map $G : Y \longrightarrow 2^Y$ by the following:

$$G(y) = Z \cap \overline{B^c(y, S\varepsilon)}, \quad (20)$$

where S represents the constant associated with inequality and $B^c(y, \varepsilon)$ denotes the complement of $B(y, \varepsilon)$ in W for any $\varepsilon > 0$ and $y \in Y$. Obviously, $G(y)$ is closed.

Now we prove $\bigcap_{c \in C} G(c) = \phi$. On contrary assume that $\bigcap_{c \in C} G(c) \neq \phi$, then we have the following:

$$\kappa \in Z \cap \overline{B^c(c, S\varepsilon)} \Rightarrow \kappa \in B(c, \varepsilon), \quad \kappa \in \overline{B^c(c, S\varepsilon)}. \quad (21)$$

So, there exists $\{\kappa_n\} \in B^c(c, S\varepsilon)$ such that $d(\kappa_n, \kappa) < \varepsilon$ for all $\varepsilon > 0$ and $n \geq N$.

So,

$$\kappa \in B(\kappa_n, \varepsilon) \text{ for all } n \geq N. \quad (22)$$

Thus,

$$B(\kappa_n, \varepsilon) \cap B(c, \varepsilon) \neq \phi \text{ for all } n \geq N. \quad (23)$$

Now, choose $\{z_{m_n}\} \in B(\kappa_n, \varepsilon) \cap B(c, \varepsilon)$ for all $\varepsilon > 0$ and $n \geq N$. Then, $\{z_{m_n}\} \in C(d, W, c)$ and

$$\lim_{n \rightarrow \infty} d(c, \kappa_n) \leq S \lim_{n, m_n \rightarrow \infty} \sup d(z_{m_n}, \kappa_n) < S\varepsilon, \quad (24)$$

which contradicts to $\{\kappa_n\} \in B^c(c, S\varepsilon)$. Thus, we have $\bigcap_{c \in C} G(c) = \phi$.

Hence, G is not a generalized KKM mapping with reference to H . As $H \in KKM(Y, Y)$, so there is a finite nonempty subset $D \subseteq Y$ such that

$$H(co(D)) \not\subseteq \bigcup_{\rho \in D} G(\rho), \quad (25)$$

i.e., we have $y_* \in H(co(D))$ such that $y_* \notin G(\rho)$ for any $\rho \in D$. As $y_* \notin G(\rho) = Z \cap \overline{B^c(\rho, S\varepsilon)}$, so

$$y_* \in \left(\overline{B^c(\rho, S\varepsilon)} \right)^c, \quad (26)$$

for any $\rho \in D$. Now,

$$y_\circ \in \left(\overline{B^c(\rho, S\varepsilon)} \right)^c \subseteq B(\rho, S\varepsilon), \quad (27)$$

for any $\rho \in D$. We may write it as $D \subseteq B(y_\circ, S\varepsilon)$.

As

$$co(D) \subseteq B(y_\circ, S\varepsilon). \quad (28)$$

For $y_\circ \in H(co(D))$, we have $y_\varepsilon \in co(D)$ such that $y_\circ \in H(y_\varepsilon)$. And

$$y_\varepsilon \in co(D) \subseteq B(y_\circ, S\varepsilon). \quad (29)$$

Thus, we have the following:

$$H(y_\varepsilon) \cap B(y_\varepsilon, S\varepsilon) \neq \emptyset. \quad (30)$$

As ε is arbitrary, so, H has an approximate fixed point property. \square

4. Applications of KKM Maps

As the consequence of Theorem 8, we deduce the following fixed point theorem.

Theorem 9. Consider the nonempty subset Y which is subadmissible in the generalized metric space (W, d) and $H \in KKM(Y, Y)$ be such that H is compact and closed. Then, H has a fixed point.

Proof. As H is compact, hence, $\overline{H(Y)}$ is compact. So, $\overline{H(Y)}$ is totally bounded. Hence by Theorem 8, H has an approximate fixed point property; i.e., for any $\varepsilon > 0$, there exists $y_\varepsilon \in Y$ such that

$$H(y_\varepsilon) \cap B(y_\varepsilon, \varepsilon) \neq \emptyset. \quad (31)$$

In particular for $n \geq 1$ and $\varepsilon = 1/n$, we have $y_n \in Y$ such that

$$H(y_n) \cap B\left(y_n, \frac{1}{n}\right) \neq \emptyset \Rightarrow z_n \in H(y_n) \cap B\left(y_n, \frac{1}{n}\right). \quad (32)$$

Since (z_n) is a sequence in $\overline{H(Y)}$ for $n \geq 1$ and $\overline{H(Y)}$ is compact, so, there exists convergent subsequence (z_{n_k}) in $\overline{H(Y)}$ and suppose it converges to z .

Also for $n \geq 1$, we have the following:

$$d(y_n, z_n) < \frac{1}{n}. \quad (33)$$

As the sequence (z_{n_k}) in (W, d) is convergent to z , so,

there exists $S > 0$ such that

$$\begin{aligned} d(y_n, z) &\leq S \lim_{n_k \rightarrow \infty} \sup d(y_n, z_{n_k}) \\ &\leq S \lim_{n \rightarrow \infty} \sup d(y_n, z_n) \Rightarrow d(y_n, z) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (34)$$

So, (y_n) also converges to z .

Since $\{(y_n, z_n)\} \in Gr(H)$ and H is closed, so, $z \in H(z)$, which is our required result. \square

Example 2. Let $([0, \infty), d)$ be the generalized metric space with a generalized metric:

$$d(x, y) = (x - y)^2, \quad (35)$$

and $Y = [0, 1] \subset [0, \infty)$ be subadmissible subset of $([0, \infty), d)$. As d is continuous in first variable from Remark 5.1 in [19], Proposition 3.11 in [20] states that closed ball in $([0, \infty), d)$ is a closed set. Define $H : Y \rightarrow 2^Y$ by the following:

$$H(y) = [0, y]. \quad (36)$$

Then, H is a KKM map and $H \in KKM(Y, Y)$. Also, $H(Y)$ is closed and bounded and clearly $\overline{H(Y)} = [0, 1]$ is totally bounded. So, by Theorem 8, H has an approximate fixed point property. Further, H is closed and compact, so by Theorem 9, H has a fixed point.

The next result will be helpful to present Schauder's type fixed point theorem for generalized metric spaces.

Lemma 10. Let Y be the nonempty subadmissible subset of the generalized metric space (W, d) and Z be the topological space. Suppose that $g : Z \rightarrow Y$ is continuous and $H \in KKM(Y, Z)$. Then, $g \circ H \in KKM(Y, Y)$.

Proof. Consider the generalized KKM mapping $G : Y \rightarrow 2^Y$ with reference to $g \circ H$ such that $G(y)$ is closed for every $y \in Y$. Since G is a generalized KKM mapping with reference to $g \circ H$, so, for any nonempty finite subset A of Y , we have the following:

$$g \circ H(co(A)) \subset \bigcup_{y \in A} G(y) \Rightarrow H(co(A)) \subset \bigcup_{y \in A} g^{-1}(G(y)). \quad (37)$$

Thus, $g^{-1}(G)$ is a generalized KKM mapping with reference to H .

As $H \in KKM(Y, Z)$, then $\{g^{-1}(G(y)), y \in Y\}$ has a finite intersection property.

Also, g is continuous and

$$\bigcap_{j=1}^p g^{-1}(G(y_j)) \neq \emptyset \Rightarrow g^{-1}\left(\bigcap_{j=1}^p G(y_j)\right) \neq \emptyset \Rightarrow \bigcap_{j=1}^p G(y_j) \neq \emptyset. \quad (38)$$

Thus, the collection $\{G(y), y \in Y\}$ has finite intersection property, which gives $g \circ H \in KKM(Y, Y)$.

As the consequence of Theorem 9 and Lemma 10, we obtain Schauder's type fixed point theorem in generalized metric spaces. \square

Theorem 11. *Let Y be the nonempty subadmissible subset of the generalized metric space (W, d) . Suppose that $I \in \text{KKM}(Y, Y)$. Then any continuous map $S : Y \rightarrow Y$ such that $S(Y)$ is compact has a fixed point.*

Proof. From Lemma 10, we have the following:

$$S = S \circ I \in \text{KKM}(Y, Y). \quad (39)$$

As S is continuous and $S(Y)$ is compact, so S is closed and compact. Hence, from Theorem 9, S has a fixed point.

Now, we present the generalized Fan's matching theorem in generalized metric spaces by using KKM property. \square

Theorem 12. *Let Y be the nonempty admissible subset of the generalized metric space (W, d) and Z be the topological space. Suppose that $H \in \text{KKM}(Y, Z)$ is compact and consider the open valued map $K : Y \rightarrow 2^Z$ satisfying $\overline{H(Y)} \subseteq K(Y)$. Then, there exists $A \in \langle Y \rangle$ such that*

$$H(\text{co}(A)) \cap \left(\bigcap_{y \in A} K(y) \right) \neq \emptyset. \quad (40)$$

Proof. Assume that $H \in \text{KKM}(Y, Z)$ and define the multivalued map $G : Y \rightarrow 2^Z$ by $G(y) = \overline{H(Y)} \cap K^c(y)$ for $y \in Y$. Then, $G(y)$ is closed for every $y \in Y$. On the contrary, assume that $H(\text{co}(A)) \cap \left(\bigcap_{y \in A} K(y) \right) = \emptyset$ for any $A \in \langle Y \rangle$. Since $A \in \langle Y \rangle$ and Y is admissible, so,

$$\begin{aligned} \text{co}(A) &\subseteq Y \\ &\Rightarrow \frac{H(\text{co}(A)) \subseteq \overline{H(Y)}}{\bigcap_{y \in A} K(y)} \\ &\Rightarrow H(\text{co}(A)) \subseteq \overline{H(Y)} \cap \left(\bigcap_{y \in A} K(y) \right)^c \\ &\Rightarrow H(\text{co}(A)) \subseteq \bigcup_{y \in A} \left(\overline{H(Y)} \cap K^c(y) \right) = G(A). \end{aligned} \quad (41)$$

Hence, G is a generalized KKM mapping with reference to H . As $H \in \text{KKM}(Y, Z)$, thus, the class $\{G(y) : y \in Y\}$ has a finite intersection property. So,

$$\bigcap_{y \in Y} G(y) \neq \emptyset \Rightarrow y \in \overline{H(Y)} \text{ but } y \notin K(Y), \quad (42)$$

which is contrary to the fact $\overline{H(Y)} \subseteq K(Y)$. So, there exists $A \in \langle Y \rangle$ such that

$$H(\text{co}(A)) \cap \left(\bigcap_{y \in A} K(y) \right) \neq \emptyset. \quad (43)$$

Now, we present an application of Theorem 11 to find

the existence of solutions to the following AB-Caputo fractional BVP:

$$\begin{pmatrix} \text{ABC} \\ 0 \end{pmatrix} D^\alpha u(t) = g(t, u(t)), \quad 1 < \alpha \leq 2, \quad t \in [0, 1], \quad (44)$$

with boundary conditions

$$\begin{aligned} u(0) &= 0, \\ \lambda u'(\eta) &= \gamma u'(1), \end{aligned} \quad (45)$$

where $\begin{pmatrix} \text{ABC} \\ 0 \end{pmatrix} D^\alpha$ represents the AB-Caputo fractional derivative and $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$. Also, $\lambda, \gamma > 0, 0 \leq t \leq \eta \leq 1$. \square

Lemma 13 [21]. *For $0 < \alpha < 1$, we obtain the following:*

$$\begin{aligned} \left(\text{ABI}_a^\alpha \text{ABCD}_a^\alpha \right)(g(t)) &= g(t) - g(a), \\ \left(\text{ABI}_b^\alpha \text{ABCD}_b^\alpha \right)(g(t)) &= g(t) - g(b). \end{aligned} \quad (46)$$

Proposition 14 (see [22]). *For $g(t)$ defined on $[a, b]$ and $\alpha \in (\kappa, \kappa + 1]$ for some $\kappa \in \mathbb{N}$, we have the following:*

$$\begin{aligned} \left(\text{ABRD}_a^\alpha \text{ABI}_a^\alpha \right)(g(t)) &= g(t), \\ \left(\text{ABI}_a^\alpha \text{ABRD}_a^\alpha \right)(g(t)) &= g(t) - \sum_{\gamma=0}^{\kappa-1} \frac{g^\gamma(a)}{\gamma!} (t-a)^\gamma, \\ \left(\text{ABI}_a^\alpha \text{ABCD}_a^\alpha \right)(g(t)) &= g(t) - \sum_{\gamma=0}^{\kappa} \frac{g^\gamma(a)}{\gamma!} (t-a)^\gamma. \end{aligned} \quad (47)$$

Lemma 15 (see [23]). *A subset H in $C(I, \mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on I .*

The following lemma will be crucial for the proof of our next result.

Lemma 16. *Assume that $K : [0, 1] \rightarrow \mathbb{R}$ is continuous. Then, the solution of linear AB-Caputo fractional BVP*

$$\begin{pmatrix} \text{ABC} \\ 0 \end{pmatrix} D^\alpha u(t) = K(t) \quad (48)$$

with boundary Equation (45) is given by the following:

$$u(t) = t \frac{(2-\alpha)}{(\lambda-\gamma)B(\alpha-1)} (\gamma K(1) - \lambda K(\eta)) + \int_0^1 G(t, x) K(x) dx, \quad (49)$$

where

$$G(t, x) = \begin{cases} \frac{\gamma(\alpha-1)^2 t}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)}(1-x)^{\alpha-2} - \frac{\lambda(\alpha-1)^2 t}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)}(\eta-x)^{\alpha-2} & 0 \leq s \leq t, \\ + \frac{(2-\alpha)}{B(\alpha-1)} + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)}(t-x)^{\alpha-1} \\ + \frac{\gamma(\alpha-1)^2 t}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)}(1-x)^{\alpha-2} - \frac{\lambda(\alpha-1)^2 t}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)}(\eta-x)^{\alpha-2} & t \leq s \leq \eta, \\ + \frac{\gamma(\alpha-1)^2 t}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)}(1-x)^{\alpha-2} & \eta \leq s \leq 1. \end{cases} \quad (50)$$

Proof. We have given the following:

$$\left(\begin{smallmatrix} \text{ABC} \\ 0 \end{smallmatrix} D^\alpha u \right)(t) = K(t), \quad 1 < \alpha \leq 2, \quad t \in [0, 1]. \quad (51)$$

From Proposition 14, we get the following:

$$u(t) = c_1 + c_2 t + \frac{(2-\alpha)}{B(\alpha-1)} \int_0^t K(x) dx + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} K(x) dx. \quad (52)$$

Now, using $u(0) = 0$ in Equation (52), which implies $c_1 = 0$, replace value of c_1 in Equation (52):

$$u(t) = c_2 t + \frac{(2-\alpha)}{B(\alpha-1)} \int_0^t K(x) dx + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} K(x) dx. \quad (53)$$

Take first ordinary derivative on both sides.

$$u'(t) = c_2 + \frac{(2-\alpha)}{B(\alpha-1)} K(t) + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-2} K(x) dx. \quad (54)$$

Using boundary condition $\lambda u'(\eta) = \gamma u'$ (Equation (35)), in Equation (54), we get the following:

$$c_2 = \frac{(2-\alpha)}{(\lambda-\gamma)B(\alpha-1)} (\gamma K(1) - \lambda K(\eta)) + \frac{\gamma(\alpha-1)^2}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-2} K(x) dx - \frac{\lambda(\alpha-1)^2}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)} \int_0^\eta (\eta-x)^{\alpha-2} K(x) dx. \quad (55)$$

Putting the value of c_2 in Equation (53),

$$u(t) = t \frac{(2-\alpha)}{(\lambda-\gamma)B(\alpha-1)} (\gamma K(1) - \lambda K(\eta)) + \frac{\gamma(\alpha-1)^2 t}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-2} K(x) dx - \frac{\lambda(\alpha-1)^2 t}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)} \int_0^\eta (\eta-x)^{\alpha-2} K(x) dx + \frac{(2-\alpha)}{B(\alpha-1)} \int_0^t K(x) dx + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} K(x) dx. \quad (56)$$

After simplification, we get the required result which is given in Equation (49); i.e.,

$$u(t) = t + \frac{(2-\alpha)}{(\lambda-\gamma)B(\alpha-1)} (\gamma K(1) - \lambda K(\eta)) + \int_0^1 G(t, x) K(x) dx. \quad (57)$$

In the view of Lemma 16, we transform AB-Caputo fractional BVP (Equations (44) and (45)) into a fixed point problem as follows:

$$u = Tu, \quad (58)$$

where the operator $T : C([0, 1], \mathbb{R}) \longrightarrow C([0, 1], \mathbb{R})$ is defined as follows:

$$T(u(t)) = \delta_1^* + \int_0^1 G(t, x) g(x, u(x)) dx, \quad (59)$$

where

$$\delta_1^* = t \frac{(2-\alpha)}{(\lambda-\gamma)B(\alpha-1)} (\gamma g(1, u(1)) - \lambda g(\eta, u(\eta))), \quad (60)$$

and $G(t, x)$ is defined in Equation (50).

For now and onwards, take $X = \{u \in C([0, 1], \mathbb{R}) : |u(t)| < \infty\}$ be the Banach space with norm defined by the following:

$$\|u\| = \sup_{t \in [0, 1]} |u(t)|. \quad (61)$$

We also use following assumptions:

$$|g(t, u(t))| \leq \mu, \quad \mu > 0, \quad (62)$$

$$\Omega = \frac{2}{(\lambda-\gamma)B(\alpha-1)} [\gamma + \lambda], \quad (63)$$

$$\begin{aligned}
\sup_{t \in [0,1]} \int_0^1 |G|(t, x) dx &\leq \sup_{t \in [0,1]} \int_0^1 \left\{ \left| \frac{\gamma(\alpha-1)^2 t}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)} (1-x)^{\alpha-2} \right| \right. \\
&+ \left| \frac{\lambda(\alpha-1)^2 t}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)} (\eta-x)^{\alpha-2} \right| + \left| \frac{(2-\alpha)}{B(\alpha-1)} \right| \\
&+ \left| \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)} (1-x)^{\alpha-1} \right| \left. \right\} dx \leq \left| \frac{\gamma(\alpha-1)}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)} \right| \\
&+ \left| \frac{\lambda(\alpha-1)}{(\lambda-\gamma)B(\alpha-1)\Gamma(\alpha)} (\eta^{\alpha-1}) \right| + \left| \frac{(2-\alpha)}{B(\alpha-1)} \right| \\
&+ \left| \frac{(\alpha-1)}{B(\alpha-1)\alpha\Gamma(\alpha)} \right| = \mathcal{F}.
\end{aligned} \tag{64}$$

□

Remark 17. Since the generalized metric space is not much explored for finding the results like Arzela Ascoli, therefore, in the next application, we will use metric space (as every metric space is a generalized metric space). This application will constitute a base for application of KKM mappings in the existence theory of differential equations.

Theorem 18. Let $X = C([0, 1], \mathbb{R})$, $Y = B[0, r]$, and $I \in KKM(Y, Y)$. Let $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfying Equations (62)–(64). Then, the AB-Caputo fractional BVP (Equation (44) and Equation (45)) has a solution in X .

Proof. $T : X \rightarrow X$ is defined as follows:

$$T(u(t)) = \delta_1^* + \int_0^1 G(t, x)g(x, u(x))dx, \tag{65}$$

where δ_1^* is defined in Equation (60). Suppose a closed ball $Y = B_r = \{u \in X : \|u\| \leq r\}$ is a convex subset of X . Now consider $u \in B_r$:

$$\begin{aligned}
\|T(u(t))\| &\leq \sup_{t \in [0,1]} \left\{ \left| t \frac{(2-\alpha)}{(\lambda-\gamma)B(\alpha-1)} |\gamma g(1, u(1)) - \lambda g(\eta, u(\eta))| \right| \right. \\
&+ \int_0^1 |G(t, x)g(x, u(x))| dx \leq \frac{2}{(\lambda-\gamma)B(\alpha-1)} \sup_{t \in [0,1]} \{ \gamma |g(1, u(1))| \\
&+ \lambda |g(\eta, u(\eta))| \} + \sup_{t \in [0,1]} \int_0^1 |G(t, x)g(x, u(x))| dx \leq \frac{2}{(\lambda-\gamma)B(\alpha-1)} \\
&\cdot [\gamma + \lambda] \|\mu\| + \mathcal{F} \|\mu\| = [\Omega + \mathcal{F}] \|\mu\|,
\end{aligned} \tag{66}$$

which implies that

$$\|T(u(t))\| \leq \|\mu\| [\Omega + \mathcal{F}] \leq r. \tag{67}$$

Hence, $T(B_r) \subseteq B_r$.

Continuity of g implies continuity of T and

$$\|T(u(t))\| \leq r. \tag{68}$$

Therefore, T is uniformly bounded on B_r .

Now, we show T is equicontinuous. For this, take $0 \leq t_1 \leq t_2 \leq \eta \leq 1$.

$$\begin{aligned}
\|(T u)(t_2) - (T u)(t_1)\| &= \sup_{t \in [0,1]} \left\{ t_2 \frac{(2-\alpha)}{(\lambda-\gamma)B(\alpha-1)} \{ (\gamma g(1, u(1)) \right. \\
&- \lambda g(\eta, u(\eta))) \} + \int_0^1 G(t_2, x)g(x, u(x))dx \\
&- \left. \left\{ t_1 \frac{(2-\alpha)}{(\lambda-\gamma)B(\alpha-1)} \{ (\gamma g(1, u(1)) - \lambda g(\eta, u(\eta))) \} \right. \right. \\
&\quad \left. \left. + \int_0^1 G(t_1, x)g(x, u(x))dx \right\} \right\} \\
&= \sup_{t \in [0,1]} \left\{ (t_2 - t_1) \frac{(2-\alpha)}{(\lambda-\gamma)B(\alpha-1)} \{ (\gamma g(1, u(1)) - \lambda g(\eta, u(\eta))) \} dx \right\} \\
&+ \left\{ \int_0^1 \{ G(t_2, x) - G(t_1, x) \} g(x, u(x)) dx \right\} \\
&\leq \frac{2}{(\lambda-\gamma)B(\alpha-1)} \sup_{t \in [0,1]} \{ \{ (t_2 - t_1) \{ (\gamma g(1, u(1)) - \lambda g(\eta, u(\eta))) \} \} \} \\
&+ \int_0^1 \{ |G(t_2, x) - G(t_1, x)| \} \|g(x, u(x))\| dx \rightarrow 0 \text{ as } t_1 \rightarrow t_2.
\end{aligned} \tag{69}$$

Therefore, T is equicontinuous. Further by virtue of Lemma 15, $\overline{T(B)}$ is compact. Therefore, by Theorem 11, T has a fixed point in Y , which means given AB-Caputo fractional BVP (Equation (44) and Equation (45)) has a solution. □

Data Availability

No data is used in this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Edge Theoretic Extended Contractions and Their Applications

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Edge theoretic extended contractions are introduced and coincidence point theorems and common fixed-point theorems are proved for such contraction mappings in a metric space endowed with a graph. As further applications, we have proved the existence of a solution of a nonlinear integral equation of Volterra type and given a suitable example in support of our result.

1. Introduction and Preliminaries

The celebrated Banach contraction principle is a motivation for many fixed-point theorems. It guarantees the existence and uniqueness of solution of various equations arising in mathematics. The initial generalizations of Banach's result came up in the form of Kannan's contraction, Chatterjea's contraction, Reich's contraction, Ćirić's contraction, Hardy-Roger's contraction, and Ćirić's quasicontraction. Among these, Ćirić's quasicontraction is the most general form in the sense that any mapping which does not satisfy Ćirić's quasicontraction does not satisfy any of the previously mentioned contractions. Further, these results have been widely investigated and many interesting applications have been found by many authors (see [1–7]). F -contraction and fixed-point theorem for F -contraction mappings were introduced by Wardowski [8] as a generalisation of the Banach contraction principle.

Definition 1 (see [8]). Consider the collection of functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following:

- (F₁) F is strictly increasing
- (F₂) If $\{\alpha_n\} \subset (0, \infty)$ is a sequence, then $\lim_{n \rightarrow \infty} \alpha_n = 0$ iff $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$
- (F₃) There exists $k \in (0, 1)$ such that $\lim_{\gamma \rightarrow 0^+} \gamma^k F(\gamma) = 0$

An operator $T : X^i, d_i \rightarrow X^i$ is an \mathcal{F} -contraction if we can find $\tau > 0$ such that

$$\forall x^i, y^i \in X^i, d_i(Tx^i, Ty^i) > 0 \implies \tau + F(d_i(Tx^i, Ty^i)) \leq F(d_i(x^i, y^i)). \quad (1)$$

Later, the concept of F -weak contraction and ordered F -contractions was introduced by Wardowski and Van Dung [9] and Durmaz et al. [10], respectively. In 2016, Sawangsup et al. [11] extended the F -contraction using a relation theoretic approach which was later generalised by Imdad et al. [12] and Alfaqih et al. [13]. Espinola and Kirk [14] introduced graph theory in fixed-point theory, and Jachymski [15] continued this idea by using different views thereby introducing the G -contraction and proved fixed-point theorem for a G -contraction mapping. These ideas were further extended and generalised by [16–24].

It is interesting to note that all these contraction conditions ensure the existence of a unique fixed point or common fixed point of the mappings under consideration. However, it is observed that a mapping which possesses nonunique fixed points does not satisfy the above contractions, for if x^i and y^i are any two fixed points of a self-map T^i of a metric space (X^i, d^i) , then

$$\begin{aligned}
d^i(T^i x^j, T^i y^j) &= d^i(x^j, y^j) \\
&= \max \left\{ d^i(x^j, y^j), d^i(x^j, T^i x^j), d^i(y^j, T^i y^j), \frac{d^i(x^j, T^i y^j) + d^i(y^j, T^i x^j)}{2} \right\}, \\
d^i(T^i x^j, T^i y^j) &= d^i(x^j, y^j) \\
&= \max \left\{ d^i(x^j, y^j), d^i(x^j, T^i x^j), d^i(y^j, T^i y^j), d^i(x^j, T^i y^j), d^i(y^j, T^i x^j) \right\},
\end{aligned} \tag{2}$$

and thus, we see that T^i does not satisfy Ćirić's quasicontraction, Wardowski's F -contraction, and Wardowski and Van Dung's F -weak contraction. Thus, these contraction conditions cannot be used to prove the existence of nonunique fixed points of a function defined in a metric space. On the other hand, many equations obtained by modeling various problems of engineering and science need not necessarily have a unique solution. Thus, it becomes meaningful to obtain extended forms of above contractions which will ensure the existence of nonunique fixed points of self-maps defined in a metric space.

Motivated by this fact, in this paper, we have introduced extended \mathcal{FW} -contraction (Jungck-Wardowski contraction), extended \mathcal{CW} -contraction (Ćirić-Wardowski contraction), and extended \mathcal{CWLQ} -contraction (Ćirić-Wardowski quasicontraction) and established fixed-point theorems which will ensure the existence of nonunique fixed points of a self-map and coincidence points of a pair of self-maps, respectively, in a metric space endowed with a graph. As an application of our result, we have also proven the existence of solution of a nonlinear integral equation of Volterra type.

Throughout this paper, we consider the metric space (X^j, d_j) to be endowed with the graph $G = (V(G), E(G))$, $V(G) = X^j$, and $\Delta \subseteq E(G)$; $\Delta = \{(x^j, x^j) : x^j \in X^j\}$.

Definition 2 (see [15]). A sequence $\{x_n^j\} \subseteq X^j$ is edge-preserving if $(x_n^j, x_{n+1}^j) \in E(G)$ for all $n \in \mathbb{N}_0$.

Definition 3. Let $g : X^j \rightarrow X^j$. A sequence $\{x_n^j\} \subseteq X^j$ is g -edge-preserving if $(gx_n^j, gx_{n+1}^j) \in E(G)$ for all $n \in \mathbb{N}_0$.

Definition 4. $T : X^j \rightarrow X^j$ is edge-preserving if $(x^j, y^j) \in E(G)$ implies $(Tx^j, Ty^j) \in E(G)$.

Definition 5. $T, g : X^j \rightarrow X^j$ is g -edge-preserving if for all $x^j, y^j \in X$, $(gx^j, gy^j) \in E(G)$ implies $(Tx^j, Ty^j) \in E(G)$.

Definition 6 (see [15]). (X^j, d_j) is edge-complete if every edge-preserving Cauchy sequence in X^j converges to some point in X^j .

Definition 7 (see [15]). $T : X^j \rightarrow X^j$ is edge-continuous at x^j if $\{x_n^j\} \rightarrow x^j$ implies $\{Tx_n^j\} \rightarrow Tx^j$ for any edge-preserving sequence $\{x_n^j\} \subseteq X^j$. If T is edge-continuous at all $x^j \in X^j$, then T is an edge-continuous mapping.

Definition 8. Let $T, g : X^j \rightarrow X^j$ and $x^j \in X^j$. We say that T is g -edge continuous at x^j if $\{gx_n^j\} \rightarrow gx^j$ implies $\{Tx_n^j\}$

$\rightarrow Tx^j$ for any edge-preserving sequence $\{x_n^j\} \subseteq X^j$. If T is g -edge continuous at all $x^j \in X^j$, then T is an g -edge continuous mapping.

Definition 9. (T, g) is edge-compatible if and only if for any sequence T and g edge-preserving sequence $\{x_n^j\} \subseteq X$, $\lim_{n \rightarrow \infty} gx_n^j = \lim_{n \rightarrow \infty} Tx_n^j = x \in X^j$ implies $\lim_{n \rightarrow \infty} d_j(gTx_n^j, Tgx_n^j) = 0$.

We will use the following lemmas taken from [25, 26]:

Lemma 10. (see [25]). Let M be a nonempty set and $g : M \rightarrow M$. Then, there exists a subset $S \subseteq M$ such that $g(s) = g(M)$ and $g : S \rightarrow S$ is one-one.

Lemma 11 (see [26]). Let $\{x_n^j\}$ be a sequence in metric space (X^j, d_j) such that $\lim_{n \rightarrow \infty} d_j(x_n^j, x_{n+1}^j) = 0$. If $\{x_n^j\}$ is not Cauchy in (X^j, d_j) , then there exist $\xi > 0$ and sequences $\{n_k\}$ and $\{p_k\}$ in \mathbb{N} such that $n_k > p_k > k$, and the sequences

$$\begin{aligned}
&\{d_j(x_{n_k}^j, x_{p_k}^j)\}, \{d_j(x_{n_k+1}^j, x_{p_k}^j)\}, \{d_j(x_{n_k}^j, x_{p_k-1}^j)\}, \\
&\{d_j(x_{n_k+1}^j, x_{p_k-1}^j)\}, \{d_j(x_{n_k+1}^j, x_{p_k+1}^j)\},
\end{aligned} \tag{3}$$

tend to be ξ^+ , as $k \rightarrow +\infty$.

2. Edge Theoretic Extended Contractions

Let \mathbb{F} be the collection of all nondecreasing continuous functions $\mathcal{F} : (0, \infty) \rightarrow \mathbb{R}$.

Example 1. Some examples of function belonging to the class \mathbb{F} are

$$\begin{aligned}
\mathcal{F}(y) &= y^2, \\
\mathcal{F}(y) &= \ln y, \\
\mathcal{F}(y) &= y - \frac{1}{y}, \\
\mathcal{F}(y) &= \ln \left(\frac{y}{3} + \sin y \right).
\end{aligned} \tag{4}$$

Let $A \subset [0, \infty)$ and Ξ be the collection of all continuous functions $\xi : A \times A \rightarrow [0, \infty)$ satisfying the following:

- (i) $\alpha = 0$ or $\beta = 0$ implies $\xi(\alpha, \beta) = 0$
- (ii) $\alpha > 0$ and $\beta > 0$ implies $\xi(\alpha, \beta) > 0$

$$\sup_{\alpha, \beta \in A} \xi(\alpha, \beta) = \zeta > 0. \tag{5}$$

Some examples of function ξ are as follows:

Example 2.

- (i) $\xi(\alpha, \beta) = k.\alpha\beta$, for some $k > 0$
- (ii) $\xi(\alpha, \beta) = \min \{\alpha, \beta\}$
- (iii) $\xi(\alpha, \beta) = \alpha/(1 + \ln \beta)$
- (iv) $\xi(\alpha, \beta) = (\alpha + \beta)/(1 + \ln(\alpha\beta))$
- (v) $\xi(\alpha, \beta) = \alpha\beta(\alpha + \beta)$
- (vi) $\xi(\alpha, \beta) = \alpha\beta/(1 + \alpha\beta)$
- (vii) $\xi(\alpha, \beta) = \ln(1 + K.\min \{\alpha, \beta\})$

Let Θ be the family of all functions $\theta : [0, \infty) \rightarrow \mathbb{R}$ which satisfy the following conditions:

- $(\theta_1)\theta$ is strictly increasing
- $(\theta_2)\theta(t) = 0$ iff $t = 0$
- $(\theta_3)\sup_{t>0}\theta(t) = \lambda$ for some $\lambda > 0$

Example 3. Some examples of elements of Θ are

$$\begin{aligned}\theta(t) &= \frac{t}{1+t}, \\ \theta(t) &= \ln\left(1 + \frac{t}{1+t}\right), \\ \theta(t) &= \frac{t}{1 + \ln(1+t)}.\end{aligned}\tag{6}$$

Definition 12. A pair of mappings $T, g : X^j \rightarrow X^j$ is an ξ -extended \mathcal{FW} -contraction pair if we can find $\tau > 0$, $F \in \mathcal{F}$, $\xi \in \Xi$, and $L \geq 0$ such that for all $x^j, y^j \in X^j$,

$$\begin{aligned}d_j(Tx^j, Ty^j) > 0 &\implies \tau + F(d_j(Tx^j, Ty^j)) \\ &\leq \mathcal{F}(d_j(gx^j, gy^j)) + L\xi(d_j(gy^j, Tx^j), d_j(gx^j, Ty^j)),\end{aligned}\tag{7}$$

Definition 13. A pair of mappings $T, g : X^j \rightarrow X^j$ is an ξ -extended \mathcal{CW} -contraction pair if we can find $\tau > 0$, $F \in \mathcal{F}$, $\xi \in \Xi$, and $L \geq 0$ such that for all $x^j, y^j \in X^j$,

$$\begin{aligned}d_j(Tx^j, Ty^j) > 0 &\implies \tau + F(d_j(Tx^j, Ty^j)) \\ &\leq \mathcal{F}(M^j(x^j, y^j)) + L\xi(d_j(gy^j, Tx^j), d_j(gx^j, Ty^j)),\end{aligned}\tag{8}$$

where

$$M^j(x^j, y^j) = \max \left\{ d_j(gx^j, gy^j), d_j(gx^j, Tx^j), d_j(gy^j, Ty^j), \frac{d_j(gx^j, Ty^j) + d_j(gy^j, Tx^j)}{2} \right\}.\tag{9}$$

Definition 14. A pair of mappings $T, g : X^j \rightarrow X^j$ is an ξ -extended \mathcal{CWQ} -contraction pair provided that there is a $\tau > 0$, $F \in \mathcal{F}$, $\xi \in \Xi$, and $L \geq 0$ such that for all $x^j, y^j \in X^j$,

$$\begin{aligned}d_j(Tx^j, Ty^j) > 0 &\implies \tau + F(d_j(Tx^j, Ty^j)) \\ &\leq \mathcal{F}(M^*(x^j, y^j)) + L\xi(d_j(gy^j, Tx^j), d_j(gx^j, Ty^j)),\end{aligned}\tag{10}$$

where

$$\begin{aligned}M^*(x^j, y^j) &= \max \{ d_j(gx^j, gy^j), d_j(gx^j, Tx^j), d_j \\ &\quad \cdot (gy^j, Ty^j), d_j(gx^j, Ty^j), d_j(gy^j, Tx^j) \}.\end{aligned}\tag{11}$$

Definition 15. In Definitions 12, 13, and 14, if conditions (7), (8), and (10) are satisfied only for all $x^j, y^j \in X^j$ with $(x^j, y^j) \in E(G)$, then the pair (T, g) is an ξ -extended \mathcal{FW} -edge contraction, ξ -extended \mathcal{CW} -edge contraction, and ξ -extended \mathcal{CWQ} -edge contraction, respectively.

Definition 16. $T, g : X^j \rightarrow X^j$ is a θ -extended \mathcal{FW} -edge contraction if we can find $\tau > 0$, $F \in \mathcal{F}$, and $\theta \in \Theta$ such that for all $x^j, y^j \in X^j$ with $(gx^j, gy^j) \in E(G)$,

$$\begin{aligned}d_j(Tx^j, Ty^j) > 0 &\implies \tau + F(d_j(Tx^j, Ty^j)) \\ &\leq \mathcal{F}(d_j(gx^j, gy^j)) + L\theta(d_j(gy^j, Tx^j)).\end{aligned}\tag{12}$$

Definition 17. A pair of mappings $T, g : X^j \rightarrow X^j$ is a θ -extended \mathcal{CW} -edge contraction if we can find $\tau > 0$, $F \in \mathcal{F}$, and $\theta \in \Theta$ such that

$$\begin{aligned}d_j(Tx^j, Ty^j) > 0 &\implies \tau + F(d_j(Tx^j, Ty^j)) \\ &\leq \mathcal{F}(M^j(x^j, y^j)) + L\theta(d_j(gy^j, Tx^j)),\end{aligned}\tag{13}$$

for all $x^j, y^j \in X^j$ with $(gx^j, gy^j) \in E(G)$ and $M^j(x^j, y^j)$, is as in (9).

If $g = I$ in the above definitions, then T is an ξ -extended F -contraction mapping, ξ -extended \mathcal{CW} -contraction mapping, θ -extended \mathcal{FW} -edge contraction mapping, and θ -extended \mathcal{CW} -edge contraction mapping, respectively.

Property (*). The space (X^j, d_j) is said to have property(*) if for any edge-preserving sequence $\{x_n^j\} \in X$ such that $\{x_n^j\} \longrightarrow x$; there exists a subsequence $\{x_{n_k}^j\}$ of $\{x_n^j\}$ such that $(x_{n_k}^j, x) \in E(G)|_X$ for all $k \in \mathbb{N}_0$

Example 4. Let $X = [0, 1] \cup \{2\}$, $d_j(x^i, y^j) = |x^i - y^j|$, and $Tx^i = x^{i4}/8$ for all $x^i \in X$. Then, at $x^i = 0$ and $y^j = 2$, T does not satisfy the conditions of Ćirić's quasicontraction, Wardowski's F -contraction, and Wardowski and Van Dung's F -weak contraction. However, T is an ξ -extended F -contraction with $\tau = \ln(2)$, as shown below:

Let $F : (0, \infty) \longrightarrow \mathbb{R}$ be defined by

$$F(t) = \ln(t), \quad (14)$$

and $\xi(\alpha, \beta) = \ln(1 + K \cdot \min\{\alpha, \beta\})$.

Case 1. $x^i, y^j \in [0, 1]$. Clearly,

$$\begin{aligned} d_j(Tx^i, Ty^j) &= \frac{1}{8} |x^{i4} - y^{j4}| \leq \frac{1}{8} |x^i - y^j| |x^i + y^j| |x^{i2} + y^{j2}| \\ &\leq \frac{1}{4} |x^i - y^j| |x^i + y^j| < \frac{1}{2} |x^i - y^j| \leq \frac{1}{2} d_j(x^i, y^j). \end{aligned} \quad (15)$$

Then, we have $\ln(d_j(Tx^i, Ty^j)) < \ln(1/2 d_j(x^i, y^j))$ or

$$\ln 2 + \ln(d_j(Tx^i, Ty^j)) < \ln(d_j(x^i, y^j)) + L\xi(d_j(gy^j, Tx^i), d_j(gx^i, Ty^j)). \quad (16)$$

Case 2. $x^i \in [0, 1]$ and $y^j = 2$. Note that in this case, $d_j(x^i, y^j) \geq 1$.

$$\begin{aligned} d_j(Tx^i, Ty^j) &= \left| \frac{x^{i4}}{8} - 2 \right| \leq \frac{1}{2} + 2 \min \left\{ |x^i - 2|, \left| 2 - \frac{x^{i4}}{8} \right| \right\} \\ &\implies d_j(Tx^i, Ty^j) \\ &\leq \frac{1}{2} d_j(x^i, y^j) \left(1 + 8 \min \left\{ |x^i - 2|, \left| 2 - \frac{x^{i4}}{8} \right| \right\} \right) \\ &\implies \ln(d_j(Tx^i, Ty^j)) \\ &\leq -\ln 2 + \ln(d_j(x^i, y^j)) + \ln \left(1 + 8 \min \left\{ |x^i - 2|, \left| 2 - \frac{x^{i4}}{8} \right| \right\} \right) \\ &\implies \ln 2 + \ln(d_j(Tx^i, Ty^j)) \\ &\leq \ln(d_j(x^i, y^j)) + \ln \left(1 + 8 \min \left\{ |x^i - 2|, \left| 2 - \frac{x^{i4}}{8} \right| \right\} \right) \\ &\implies \ln 2 + F(d_j(Tx^i, Ty^j)) \\ &\leq F(d_j(x^i, y^j)) + \xi(1 + 8 \min\{d_j(x^i, y^j), d_j(y^j, Tx^i)\}). \end{aligned} \quad (17)$$

Example 5. Let $X^j = [0, \infty)$, $d_j(x^j, y^j) = |x^j - y^j|$, $E(G) = \{(n, n), (n, n+1) : n = 0, 1, 2, 3, \dots\}$, and $T, g : X^j \longrightarrow X^j$ be given by

$$Tx^j = \begin{cases} 0, & \text{if } 0 \leq x^j \leq 1, \\ x^j - 1, & \text{if } x^j \geq 1, \end{cases}$$

$$gx^j = x^j + (n+1-x^j)(x^j-n), \quad \text{whenever } n \leq x^j \leq n+1. \quad (18)$$

Let $F : (0, \infty) \longrightarrow \mathbb{R}$ be defined by

$$F(t) = t - \frac{1}{t}, \quad (19)$$

and $\theta \in \Theta$ be defined by $\theta(t) = t/(t+1)$. Then,

$$\begin{aligned} \tau + F(d_j(T(n), T(n+1))) &\leq F(d_j(g(n), g(n+1))) \\ &\quad + L\theta(d_j(g(n+1), T(n))) \\ &\implies \tau + F(d_j(n-1, n)) \leq F(d_j(n, n+1)) \\ &\quad + L\theta(d_j(n+1, n-1)) \\ &\implies \tau \leq F(1) - F(1) + L\theta(2) \implies \tau \leq L\theta(2). \end{aligned} \quad (20)$$

Hence, for any $0 < \tau < 2/3$ and $L = 1$, (13) is satisfied and thus (T, g) is a θ -extended $\mathcal{W}\mathcal{J}$ -edge contraction and θ -extended $\mathcal{W}\mathcal{C}$ -edge contraction. However, the pair (T, g) is neither an ξ -extended $\mathcal{F}\mathcal{W}$ -edge contraction pair nor an ξ -extended $\mathcal{C}\mathcal{W}$ -contraction pair. If we take g to be the identity mapping, then T is a θ -extended $\mathcal{F}\mathcal{W}$ -edge contraction mapping and θ -extended $\mathcal{C}\mathcal{W}$ -edge contraction mapping. However, again T is none of Wardowski's F -contraction, Wardowski and Van Dung's F -weak contraction, and Ćirić's quasicontraction.

3. Main Results

We start by proving the following main theorems:

Theorem 18. Suppose (X^j, d_j) be endowed with a graph G satisfying transitivity property, and the following conditions hold for $T, g : X^j \longrightarrow X^j$.

- (a) $(gx_0^j, Tx_0^j) \in E(G)$ for some $x_0^j \in X^j$
- (b) T is g -edge preserving
- (c) (T, g) is an θ -extended $\mathcal{C}\mathcal{W}$ -edge contraction pair of mappings
- (d) (d_1) There exists an edge-complete subset M^j of X^j for which $T(X^j) \subseteq M^j \subseteq g(X^j)$
- (d₂) One of the following conditions holds:

- (i) T is g -edge continuous
- (ii) T and g are continuous
- (iii) $E(G)|_{X^j}$ satisfies property(*)

Then, the pair (T, g) has a coincidence point.

Proof. In view of the assumption (a), we have $(gx_0^j, Tx_0^j) \in E(G)$. If $Tx_0^j = gx_0^j$, then x_0 is a coincidence point of (T, g) , i.e., $\text{Coin}(T, g) \neq \emptyset$, and there is nothing to prove. Assume

that $Tx_0^j \neq gx_0^j$; then, since $T(X^j) \subseteq g(X^j)$, there exists $x_1^j \in X^j$ such that $gx_1^j = Tx_0^j$. \square

Similarly, there is $x_2^j \in X^j$ such that $gx_2^j = Tx_1^j$ with $(gx_1^j, gx_2^j) \in E(G)$ and consequently $(Tx_0^j, Tx_1^j) \in E(G)$. Inductively, one can construct a sequence $\{x_n^j\} \subseteq X^j$ such that

$$gx_{n+1}^j = Tx_n^j, \text{ for all } n \in \mathbb{N}_0, \quad (21)$$

with

$$(gx_n^j, gx_{n+1}^j) \in E(G) \text{ for all } n \in \mathbb{N}_0, \quad (22)$$

and consequently, as T is g -edge preserving,

$$(Tx_n^j, Tx_{n+1}^j) \in E(G). \quad (23)$$

Now, if $Tx_{n_0}^j = Tx_{n_0}^j$ for some $n_0 \in \mathbb{N}_0$, then x_{n_0} is a coincidence point (T, g) and we are done. Assume that $Tx_n^j \neq Tx_{n+1}^j$, for all $n \in \mathbb{N}_0$. On using (21), (22), (23), and condition (c), we have

$$\begin{aligned} \tau + F(d(gx_n^j, gx_{n+1}^j)) &= \tau + F(d(Tx_{n-1}^j, Tx_n^j)) \\ &\leq F(M(x_{n-1}^j, x_n^j)) + L\theta(d(gx_n^j, Tx_{n-1}^j)). \end{aligned} \quad (24)$$

Now,

$$\begin{aligned} M(x_{n-1}^j, x_n^j) &= \max \left\{ d_j(gx_{n-1}^j, gx_n^j), d_j(gx_{n-1}^j, Tx_{n-1}^j), d_j(gx_n^j, Tx_n^j), \frac{d_j(gx_{n-1}^j, Tx_n^j) + d_j(gx_n^j, Tx_{n-1}^j)}{2} \right\} \\ &= \max \left\{ d_j(gx_{n-1}^j, gx_n^j), d_j(gx_n^j, gx_{n+1}^j) \right\}, \end{aligned}$$

$$\theta(d(gx_n^j, Tx_{n-1}^j)) = \theta(d(gx_n^j, gx_n^j)) = 0. \quad (25)$$

Thus, we get

$$\tau + F(d(gx_n^j, gx_{n+1}^j)) \leq F(\max \{d_j(gx_{n-1}^j, gx_n^j), d_j(gx_n^j, gx_{n+1}^j)\}), \quad (26)$$

i.e.,

$$F(d(gx_n^j, gx_{n+1}^j)) < \tau + F(d(gx_n^j, gx_{n+1}^j)) \leq F(d(gx_{n-1}^j, gx_n^j)). \quad (27)$$

Since F is nondecreasing, we get $d(gx_n^j, gx_{n+1}^j) < d(gx_{n-1}^j, gx_n^j)$. This further means that $d_j(x_n^j, x_{n+1}^j) \rightarrow \delta \geq 0$ as $n \rightarrow +\infty$. If $\delta > 0$, we obtain from (27) that

$$F(\delta +) \leq \tau + F(\delta +) \leq F(\delta +), \quad (28)$$

which is a contradiction. Hence, $\lim_{n \rightarrow +\infty} d_j(x_n^j, x_{n+1}^j) = 0$. Suppose the sequence $\{gx_n^j\}$ is not a Cauchy sequence. By Lemma 11, there exist $\xi > 0$ and sequences $\{n_k\}$ and $\{p_k\}$ in \mathbb{N} such that $n_k > p_k > k$, such that the sequences $d_j(x_{n_k}^j, x_{p_k}^j)$ and $d_j(x_{n_k+1}^j, x_{p_k+1}^j)$ tend to be ξ^+ , as $k \rightarrow +\infty$. By (27) we get

$$\tau + F(\xi^+ +) \leq F(\xi^+ +), \quad (29)$$

which is a contradiction. So sequence $\{gx_n^j\}$ is a Cauchy sequence.

By (21) and (22), $\{gx_n^j\}$ is an edge-preserving Cauchy sequence in $T(X^j) \subset M^j$, and since M^j is edge-complete, there exists $y^j \in M^j$ such that $\{gx_n^j\} \rightarrow y^j$. As $M^j \subseteq g(X^j)$, there exists $u^j \in X^j$ such that $y^j = gu^j$. Hence, on using (21), we obtain

$$\lim_{n \rightarrow \infty} gx_n^j = \lim_{n \rightarrow \infty} Tx_n^j = gu^j. \quad (30)$$

Now, suppose condition $(d_2(i))$ is true. Using (22) and (30), we obtain

$$\lim_{n \rightarrow \infty} Tx_n^j = Tu^j. \quad (31)$$

By (30) and (31), we have

$$Tu^j = gu^j. \quad (32)$$

Suppose condition $(d_2(ii))$ is true. By Lemma 10, there is $S \subseteq X^j$ for which $g(S) = g(X^j)$ and $g : S \rightarrow S$ is one-one. Consider the function $f : g(S) \rightarrow g(X^j)$ given by

$$f(gs) = Ts \text{ (} gs \in g(S), s \in S \text{)}. \quad (33)$$

As $g : S \rightarrow X^j$ is one-one and $T(X^j) \subseteq g(X^j)$, f is well-

defined. Since T and g are continuous, f is also continuous by condition (d_1) of the hypothesis $T(X^j) \subseteq M^j \subseteq g(S)$. Thus, we have $\{x_n^j\} \subseteq S$ and $u^j \in S$. Therefore,

$$Tu^j = f(gu^j) = f\left(\lim_{n \rightarrow \infty} gx_n^j\right) = \lim_{n \rightarrow \infty} f(gx_n^j) = \lim_{n \rightarrow \infty} Tx_n^j = gu^j. \quad (34)$$

Suppose condition $(d_2(\text{iii}))$ is true; that is, $E(G)|_{X^j}$ satisfied Property $(*)$. Since $\{gx_n^j\} \subseteq X$, it follows that $\{gx_n^j\}$ is $E(G)|_{X^j}$ -preserving (due to (22)) and $\{gx_n^j\} \rightarrow gu^j$ (by (30)) and so we have a subsequence $\{gx_{n_k}^j\} \subseteq \{gx_n^j\}$ such that

$$(gx_{n_k}^j, gu^j) \in E(G)|_X, \quad \text{for all } k \in \mathbb{N}_0. \quad (35)$$

Using (35) and condition (b) of the hypothesis, we have

$$(Tx_{n_k}^j, Tu^j) \in E(G)|_{X^j} \subseteq S, \quad \text{for all } k \in \mathbb{N}_0. \quad (36)$$

Now, let $P^j = \{k \in \mathbb{N} : Tx_{n_k}^j = Tu^j\}$.

If P^j is finite, then $\{Tx_{n_k}^j\}$ has a subsequence $\{Tx_{n_{k_i}}^j\}$ such that $Tx_{n_{k_i}}^j \neq Tu^j$ for all $i \in \mathbb{N}$. Also, $(gx_{n_{k_i}}^j, gu^j) \in E(G)|_X \subseteq E(G)$. Thus, we have

$$\begin{aligned} \tau + F(d(Tx_{n_{k_i}}^j, Tu^j)) &\leq F(M(x_{n_{k_i}}^j, u^j)) + L\theta(d(gu^j, Tx_{n_{k_i}}^j)), \\ M(x_{n_{k_i}}^j, u^j) &= \max \left\{ d_j(gx_{n_{k_i}}^j, gu^j), d_j(gx_{n_{k_i}}^j, Tx_{n_{k_i}}^j), d_j(gu^j, Tu^j), \frac{d_j(gx_{n_{k_i}}^j, Tu^j) + d_j(gu^j, Tgx_{n_{k_i}}^j)}{2} \right\}. \end{aligned} \quad (37)$$

Letting $i \rightarrow \infty$, we obtain $M(x_{n_{k_i}}^j, u^j) = d_j(gu^j, Tu^j)$ and $\theta(d(gu^j, Tx_{n_{k_i}}^j)) = 0$. Thus, we get

$$\tau + F(d_j(gu^j, Tu^j)) \leq F(d(gu^j, Tu^j)), \quad (38)$$

which is a contradiction. Hence, P^j is not finite. Thus, P^j is infinite and so $\{Tx_{n_k}^j\}$ has a subsequence $\{Tx_{n_{k_i}}^j\}$ such that $Tx_{n_{k_i}}^j = Tu^j$ for all $i \in \mathbb{N}$. Thus, $\lim_{i \rightarrow \infty} Tx_{n_{k_i}}^j = Tu^j$. As $\lim_{n \rightarrow \infty} Tx_n^j = gu^j$ (by (30)), we get $Tu^j = gu^j$.

Theorem 19. *If, in addition to hypothesis (a) - (d) of Theorem 18, we assume the following:*

(i) *For all $u^j, v^j \in \text{Coin}(T, g)$,*

$$\begin{aligned} d_j(Tu^j, Tv^j) > 0 &\implies \tau + F(d_j(Tu^j, Tv^j)) \\ &\leq \mathcal{F}(M^j(u^j, v^j)) + L\theta(d_j(gu^j, Tu^j)), \end{aligned} \quad (39)$$

(ii) *One of T or g is one-one*

(iii) *T and g are weakly compatible*

then (T, g) has a unique common fixed point.

Proof. In view of Theorem 18, the set $\text{Coin}(T, g)$ is non-empty. Let $u^j, v^j \in \text{Coin}(T, g)$. If $d_j(Tu^j, Tv^j) = 0$, then we

have $Tu^j = gu^j = gv^j = Tv^j$, and hence, $u^j = v^j$ as one of T and g is one-one. Otherwise, using condition (39), we obtain

$$\begin{aligned} \tau + F(d(Tu^j, Tv^j)) &\leq F(d(gu^j, gv^j)) + L\theta(d(gu^j, Tu^j)), \\ &= F(d(Tu^j, Tv^j)), \end{aligned} \quad (40)$$

which is a contradiction. So the coincidence point of T and g is unique.

Let u^j be the unique coincidence point of T and g , and let $z^j \in X$ such that $z^j = Tu^j = gu^j$. As T and g are weakly compatible, we have $Tz^j = Tgu^j = gTu^j = gz^j$. Thus, z^j is a coincidence point of T and g . By the uniqueness of the coincidence point, we conclude $u^j = z^j$; that is, u is a common fixed point of the pair (T, g) which is indeed unique. as the coincidence point of T and g is unique. \square

Remark 20. If we replace condition (d) of Theorem 18 with the following alternate condition:

- $(d^*)(d_1^*)$ There exists a subset Y^j of X^j such that $T(X^j) \subseteq g(X^j) \subseteq Y^j$ and Y^j is edge-complete
 - $(d_2^*)(T, g)$ is an edge-compatible pair
 - (d_3^*) T and g are edge-continuous
- the conclusions of Theorems 18 and 19 still hold.

Proof. Clearly, $\{gx_n^j\}$ is an edge-preserving Cauchy sequence in Y^j , and by edge-completeness of Y , we get $v^j \in Y^j$ such that

$$\lim_{n \rightarrow \infty} gx_n^j = v^j, \quad (41)$$

and then, by (21), we have

$$\lim_{n \rightarrow \infty} Tx_n^j = v^j. \quad (42)$$

Using the edge continuity of g and T , we also have

$$\lim_{n \rightarrow \infty} T(gx_n^j) = T\left(\lim_{n \rightarrow \infty} gx_n^j\right) = Tv^j, \quad (43)$$

$$\lim_{n \rightarrow \infty} g(Tx_n^j) = g\left(\lim_{n \rightarrow \infty} Tx_n^j\right) = gv^j. \quad (44)$$

Then, by edge-compatibility of g and T , we get

$$\lim_{n \rightarrow \infty} d(gTx_n^j, Tgx_n^j) = 0. \quad (45)$$

Finally from (44), (45), and (43), we get

$$d(gv^j, Tv^j) = d\left(\lim_{n \rightarrow \infty} gTx_n^j, \lim_{n \rightarrow \infty} Tgx_n^j\right) = \lim_{n \rightarrow \infty} d(gTx_n^j, Tgx_n^j) = 0. \quad (46)$$

Hence, v^j is a coincidence point of the pair (T, g) . \square

Remark 21. Since every ξ -extended contraction mapping is a θ -extended contraction, the conclusions of Theorems 18 and 19 remain true for an edge theoretic ξ -extended \mathcal{EW} -contraction pair of mappings also.

On setting $g = I$ in Theorem 18, we deduce the following corresponding fixed-point result.

Theorem 22. Let (M, d) be a metric space endowed with a directed graph G and $T : M \rightarrow M$. Assume that the following conditions are fulfilled:

- (a) There exists $x_0 \in M$ such that $(x_0, Tx_0) \in E(G)$
- (b) T is edge-preserving
- (c) T is a θ -extended \mathcal{EW} -edge contraction mapping
- (d) (d_1) There exists a subset X of M such that $T(M) \subseteq X$ and X is edge-complete
- (d₂) One of the following conditions is satisfied:
 - (i) T is edge-continuous
 - (ii) $E(G)|_X$ satisfies Property(*)

Then, T has a fixed point.

Example 6. Let $\{X^j, d_j\}$, $E(G)$, T , and g be as in Example 5. Then, we have the following:

- (1) $(g0, T0) \in E(G)$
- (2) T is g -edge-preserving. In fact, we see that $(gx^j, gy^j) \in E(G)$ implies either $x^j = n, y^j = n$ or $x^j = n, y^j = n$

+ 1. If $n = 0$, then $(T0, T0) \in E(G)$ and $(T0, T1) \in E(G)$. If $n = 1$, then $(T1, T1) \in E(G)$ and $(T1, T2) \in E(G)$. If $n = k > 1$, then $(Tk, Tk) \in E(G)$ and $(Tk, T(k+1)) = (k-1, k) \in E(G)$

(3) (T, g) is a θ -extended \mathcal{EW} -edge contraction mapping

(4) $T(X^j) \subset g(X^j)$

(5) T is g -edge-continuous

Thus, all conditions of Theorem 18 are satisfied and 0 is a coincidence point of T and g . Moreover, we see that T and g satisfy conditions (i), (ii) (g is one-one), and (iii) of Theorem 19, and 0 is the unique common fixed point of T and g .

Remark 23 (an open problem). Prove Theorems 18, 19, and 22 for ξ -extended \mathcal{EQW} -contraction mappings.

4. Application to Nonlinear Integral Equations

Consider the Banach space $M = C([0, 1], R)$ of all continuous functions $x : [0, 1] \rightarrow R$ equipped with norm

$$\|x\| = \max_{s \in [0, 1]} |x(s)|. \quad (47)$$

Define a metric d_j on M by $d_j(x^j, y^j) = \|x^j - y^j\|$ for all $x^j, y^j \in M$. Then, (M, d_j) is a complete metric space.

In this section, we show the applicability of Theorem 19 by investigating the existence and uniqueness of a solution for the following nonlinear integral equation of Volterra type:

$$x^j(s) = \int_0^{\mu(s)} K(s, v, (x^j)(\eta(v))) dv + \int_0^{\sigma(s)} J(s, v, (x^j)(\zeta(v))) dv + f(s), s \in [0, 1], \quad (48)$$

where $K, J : [0, 1] \times [0, 1] \times R \rightarrow R$, $f : [0, 1] \rightarrow R$, and $\mu, \sigma, \eta, \zeta : [0, 1] \rightarrow [0, 1]$.

Definition 24. A lower solution for (48) is a function $x \in M$ such that

$$x^j(s) \leq \int_0^{\mu(s)} K(s, v, (x^j)(\eta(v))) dv + \int_0^{\sigma(s)} J(s, v, (x^j)(\zeta(v))) dv + f(s), s \in [0, 1]. \quad (49)$$

Definition 25. An upper solution for (48) is a function $x \in M$ such that

$$x^j(s) \geq \int_0^{\mu(s)} K(s, v, (x^j)(\eta(v))) dv + \int_0^{\sigma(s)} J(s, v, (x^j)(\zeta(v))) dv + f(s), s \in [0, 1]. \quad (50)$$

Consider the operator $T : M \longrightarrow M$ defined by

$$\begin{aligned} T(x^j(s)) &= \int_0^{\mu(s)} K(s, v, (x^j)(\eta(v))) dv \\ &\quad + \int_0^{\sigma(s)} J(s, v, (x^j)(\zeta(v))) dv + f(s), \text{ for all } x \in M. \end{aligned} \quad (51)$$

Then, x^j is a fixed point of the operator T if and only if it is a solution of the integral equation (48).

Let

$$\begin{aligned} M^\circ(x^j, y^j) &= \max \left\{ |x^j - y^j|, |x^j - T(x^j(s))|, |y^j - T(y^j(s))|, \frac{|x^j - T(y^j(s))| + |y^j - T(x^j(s))|}{2} \right\}, \\ \|M^\circ(x^j, y^j)\| &= \max \left\{ \|x^j - y^j\|, \|x^j - T(x^j(s))\|, \|y^j - T(y^j(s))\|, \frac{\|x^j - T(y^j(s))\| + \|y^j - T(x^j(s))\|}{2} \right\}. \end{aligned} \quad (52)$$

Theorem 26. Assume that K and J are nondecreasing in the third variable, $\mu(t) + \sigma(t) \leq 1$ for all $t \in [0, 1]$, and the following conditions hold:

There exists $\tau > 0$ such that

$$\begin{aligned} |K(s, v, gx^j) - K(s, v, gy^j)| &\leq \frac{M^\circ(x^j, y^j)}{\|M^\circ(x^j, y^j)\| \{ \tau - ((L\|y^j - T(x^j(s))\|)/(1 + \|y^j - T(x^j(s))\|)) \} + 1}, \\ |J(s, v, gx^j) - J(s, v, gy^j)| &\leq \frac{M^\circ(x^j, y^j)}{\|M^\circ(x^j, y^j)\| \{ \tau - L\|y^j - T(x^j(s))\|/1 + \|y^j - T(x^j(s))\| \} + 1}, \end{aligned} \quad (53)$$

for all $s, v \in [0, 1]$, $x^j, y^j \in M$ with $x^j(s) \leq y^j(s)$ and $L \geq 0$. If (48) has a lower solution, e.g., $x^j_0(s)$, then a solution exists for the integral equation (48).

which shows that $(Tx^j, Ty^j) \in E(G)$. Thus, T is edge-preserving. Now, for all $(x^j, y^j) \in E(G)$ and $s \in [0, 1]$, we have

Proof. Consider the graph G in M , with edges $E(G)$ given by

$$E(G) = \{ (x^j, y^j) \in M \times M : x^j(s) \leq y^j(s) \}. \quad (54)$$

For any $(x^j, y^j) \in E(G)$, we have (for all $s \in [0, 1]$)

$$\begin{aligned} T(x^j(s)) &= \int_0^{\mu(s)} K(s, v, (x^j)(\eta(v))) dv \\ &\quad + \int_0^{\sigma(s)} J(s, v, (x^j)(\zeta(v))) dv + f(s) \\ &\leq \int_0^{\mu(s)} K(s, v, (y^j)(\eta(v))) dv \\ &\quad + \int_0^{\sigma(s)} J(s, v, (y^j)(\zeta(v))) dv + f(s) \\ &= T(y^j(s)), \end{aligned} \quad (55)$$

$$\begin{aligned} |T(x^j(s)) - T(y^j(s))| &\leq \int_0^s |(K(s, v, (x^j)(\eta(v))) \\ &\quad - K(s, v, (y^j)(\eta(v))))| dv + \int_0^s |(J(s, v, (x^j)(\zeta(v))) - J(s, v, (y^j)(\zeta(v))))| dv \\ &\leq \int_0^{\mu(s)} \frac{M^\circ(x^j, y^j)}{M^\circ(x^j, y^j) \{ \tau - (L\|y^j - T(x^j(s))\|/(1 + \|y^j - T(x^j(s))\|)) \} + 1} dv \\ &\quad + \int_0^{\sigma(s)} \frac{M^\circ(x^j, y^j)}{M^\circ(x^j, y^j) \{ \tau - (L\|y^j - T(x^j(s))\|/(1 + \|y^j - T(x^j(s))\|)) \} + 1} dv \\ &\leq \int_0^{\mu(s)} \frac{\max_{s \in [0, 1]} M^\circ(x^j, y^j)}{\|M^\circ(x^j, y^j)\| \{ \tau - (L\|y^j - T(x^j(s))\|/(1 + \|y^j - T(x^j(s))\|)) \} + 1} dv \\ &\quad + \int_0^{\sigma(s)} \frac{\max_{s \in [0, 1]} M^\circ(x^j, y^j)}{M^\circ(x^j, y^j) \{ \tau - (L\|y^j - T(x^j(s))\|/(1 + \|y^j - T(x^j(s))\|)) \} + 1} dv \\ &\leq \frac{\|M^\circ(x^j, y^j)\|}{\|M^\circ(x^j, y^j)\| \{ \tau - (L\|y^j - T(x^j(s))\|/(1 + \|y^j - T(x^j(s))\|)) \} + 1} \int_0^{\mu(s)} dv \\ &\quad + \frac{\|M^\circ(x^j, y^j)\|}{M^\circ(x^j, y^j) \{ \tau - (L\|y^j - T(x^j(s))\|/(1 + \|y^j - T(x^j(s))\|)) \} + 1} \int_0^{\sigma(s)} dv \\ &= \frac{\|M^\circ(x^j, y^j)\|}{M^\circ(x^j, y^j) \{ \tau - (L\|y^j - T(x^j(s))\|/(1 + \|y^j - T(x^j(s))\|)) \} + 1} (\mu(s) + \sigma(s)) \\ &\leq \frac{\|M^\circ(x^j, y^j)\|}{M^\circ(x^j, y^j) \{ \tau - (L\|y^j - T(x^j(s))\|/(1 + \|y^j - T(x^j(s))\|)) \} + 1}. \end{aligned} \quad (56)$$

Taking the supremum, we get

$$\|T(x) - T(y)\| \leq \frac{\|M^\circ(x^j, y^j)\|}{M^\circ(x^j, y^j) \{ \tau - (L\|y^j - T(x^j(s))\| / (1 + \|y^j - T(x^j(s))\|)) \} + 1}, \quad (57)$$

or

$$\tau + \frac{1}{\|M^\circ(x^j, y^j)\|} \leq \frac{1}{\|T(x) - T(y)\|} + \frac{L\|y^j - T(x^j(s))\|}{1 + \|y^j - T(x^j(s))\|}, \quad (58)$$

or

$$\tau - \frac{1}{\|T(x) - T(y)\|} \leq \frac{-1}{\|M^\circ(x^j, y^j)\|} + \frac{L\|y^j - T(x^j(s))\|}{1 + \|y^j - T(x^j(s))\|}. \quad (59)$$

That is,

$$\tau - \frac{1}{d_j(Tx^j, Ty^j)} \leq \frac{-1}{\|M^j(x^j, y^j)\|} + \frac{L d_j(y^j, Tx^j(s))}{1 + d_j(y^j, Tx^j(s))}. \quad (60)$$

Thus, inequality (13) is satisfied with $F(\alpha) = -1/\alpha$ and $\theta(\beta) = \beta/(1 + \beta)$, so that $\lambda = \sup_{t>0} \theta(t) = 1$. Also, by Definition 24, we have $(x^j_0, Tx^j_0) \in E(G)$. Therefore, all the assumptions of Theorem 22 are satisfied, and thus, problem (48) has a solution. \square

Theorem 27. Assume that K is nonincreasing in the third variable and there exists $\tau > 0$ such that

$$|K(s, v, gx^j) - K(s, v, gy^j)| \leq \frac{|gx^j - gy^j|}{\tau \|gx^j - gy^j\| + 1}, \quad (61)$$

for all $s, v \in [0, 1]$ and $x, y \in M$. Then, the existence of an upper solution of the integral equation (48) ensures the existence of a solution of (48).

Proof. Define set $E(G)$ of edges on M by

$$E(G) = \{(x, y) \in M \times M : x(s) \geq y(s)\}. \quad (62)$$

Now, following the steps of the proof of Theorem 26 with an analogous procedure, one can check that all the hypotheses of Theorem 22 are validated, and thus, Theorem 22 ensures the existence of a unique solution of the integral equation (48). \square

We now furnish a numerical example to validate the hypothesis of Theorem 27.

Example 7. Consider the function $x \in M$ defined by $x(s) = s^2, s \in [0, 1]$. We show that this function is an upper solution in M for the following integral equation:

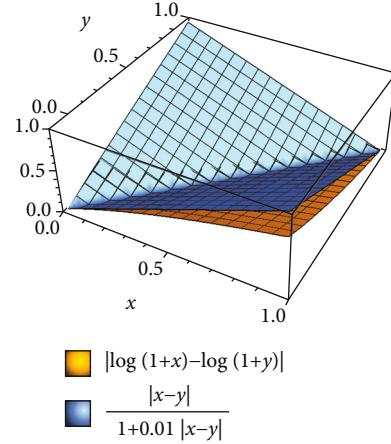


FIGURE 1: Inequality in (66).

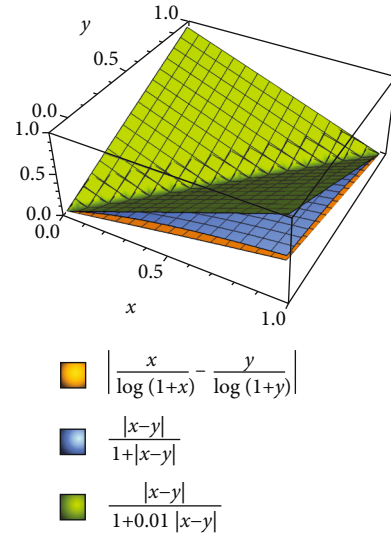


FIGURE 2: Inequality in (67).

$$x(s) = -\frac{1}{2}s + 2s^2 + \arctan\left(\frac{1}{2}s\right) - 3 \arctan\left(\frac{1}{2}s^2\right) - \frac{1}{2}s^2 \ln\left(1 + \frac{1}{4}s^4\right) + \int_0^{s^2/2} \ln(1 + x(v))dv + \int_0^{s^2/2} \frac{x(v)}{1 + x(v)}dv, \quad s \in [0, 1]. \quad (63)$$

Finally, we see that $x_u(s) = s^2 - \arctan(s^2/2)$ is the unique solution of (63).

Proof. Define the operator $T : M \longrightarrow M$ as

$$Tx(s) = -\frac{1}{2}s + 2s^2 + \arctan\left(\frac{1}{2}s\right) - 3 \arctan\left(\frac{1}{2}s^2\right) - \frac{1}{2}s^2 \ln\left(1 + \frac{1}{4}s^4\right) + \int_0^{s^2/2} \ln(1 + x(v))dv + \int_0^{s^2/2} \frac{x(v)}{1 + x(v)}dv, \quad s \in [0, 1]. \quad (64)$$

Now, set $K(s, v, x(v)) = \ln(1 + x(v))$, $J(s, v, x(v)) = x(v)/(1 + x(v))$, $\mu(s) = (1/2)s^2$, $\sigma(s) = (1/2)s$, $f(s) = -(1/2)s + 2s^2 + \arctan((1/2)s) - 3\arctan((1/2)s^2) - (1/2)s^2 \ln(1 + (1/4)s^4)$, and $\tau \leq 0.01$. We observe the following:

- (i) Both the functions $K(s, v, x(v)) = \ln(1 + x(v))$ and $J(s, v, x(v)) = x(v)/(1 + x(v))$ are nondecreasing in the third variable
- (ii) By actual computation, we have

$$\begin{aligned} \int_0^{s^2/2} \ln(1 + x(v)) dv &= -s^2 + 2\arctan\left(\frac{1}{2}s^2\right) + \frac{1}{2}s^2 \ln\left(1 + \frac{1}{4}s^4\right), \quad s \in [0, 1], \\ \int_0^{s^2/2} \frac{x(v)}{1 + x(v)} dv &= \frac{1}{2}s - \arctan\left(\frac{1}{2}s\right), \quad s \in [0, 1]. \end{aligned} \quad (65)$$

- (iii) $s^2 \geq -(1/2)s + 2s^2 + \arctan((1/2)s) - 3\arctan((1/2)s^2) - (1/2)s^2 \ln(1 + (1/4)s^4) + \int_0^{s^2/2} \ln(1 + x(v)) dv + \int_0^{s^2/2} x(v)/(1 + x(v)) dv$, $s \in [0, 1]$ so that $x(s) = s^2$ is an upper solution for (63)

- (iv) The following inequalities hold true for all $x, y \in [0, 1]$ (see Figures 1 and 2):

$$|\ln(1 + x) - \ln(1 + y)| \leq \frac{|x - y|}{1 + 0.01|x - y|}, \quad (66)$$

$$\left| \frac{x}{1 + x} - \frac{y}{1 + y} \right| \leq \frac{|x - y|}{1 + |x - y|} \leq \frac{|x - y|}{1 + 0.01|x - y|}. \quad (67)$$

□

Furthermore, using the nondecreasing function $s \mapsto s/(1 + 0.01s)$, we have

$$\begin{aligned} |\ln(1 + x) - \ln(1 + y)| &\leq \frac{|x - y|}{1 + 0.01|x - y|} \\ &\leq \frac{\max_{s \in [0, 1]} |x - y|}{1 + 0.01 \max_{s \in [0, 1]} |x - y|} \\ &= \frac{\|x - y\|}{1 + 0.01\|x - y\|}. \end{aligned} \quad (68)$$

Similarly, for all $x, y \in [0, 1]$, we have

$$\left| \frac{x}{1 + x} - \frac{y}{1 + y} \right| \leq \frac{\|x - y\|}{1 + 0.01\|x - y\|}. \quad (69)$$

Hence, all the conditions of Theorem 27 are satisfied. It is evident that the integral equation (63) has a unique solution $x_u \in M$ defined by $x_u(s) = s^2 - \arctan(s^2/2)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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