Abstract and Applied Analysis

Analysis and Models in Interdisciplinary Mathematics 2016

Guest Editors: J. C. Cortés, B. Chen-Charpentier, R. Company, Francisco J. Solis, J. R. Torregrosa, and R. J. Villanueva

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Editorial Board

Contents

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Editorial

Analysis and Models in Interdisciplinary Mathematics 2016

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According to the traditional course of action of this journal, this special issue includes a wide range of topics related to mathematical modelling, paying particular attention to social sciences, economics, finance, and uncertainty quantification. The aim of this issue is to provide a selected number of contributed papers that cover the area of mathematical modelling, giving a rather fair picture of the current research interests in this scientific field. It is expected that all the manuscripts comprised in this piece can contribute to enlarging both the foundations and applications of mathematical modeling.

This special issue has the vocation of continuing the previous ones entitled "Analysis and Models in Interdisciplinary Mathematics" and "Analysis and Models in Interdisciplinary Mathematics 2015" published in this journal in 2014 and 2015, respectively. The accepted papers have been selected after reviewing in order to integrate novel mathematical models and methods not only in the scope of traditional natural sciences but also in opening the scope to the social sciences framework. Also, theory and data-driven models, even in a synergy that gives rise to producing fertile, multidisciplinary, and hybrid models, have been considered.

Traditionally, mathematical modelling has been understood just as part of applied mathematics. This could be justified because, at its beginning, engineers were the main practitioners of this area of mathematics, developing mathematical models for solving engineering problems in natural sciences. However, this special issue also pursues providing mathematical modeling in contexts as social behavior, economics, and finance. Analysis methods and models in social sciences are similar to those of natural sciences including engineering, with the only difference that instead of using principles of nature one uses principles or theories from experts of such sciences. The special issue has also been dedicated to provide a window for the scientific community where mathematical models that account for uncertainty quantification within the framework of random differential equations and modelling can be shown.

It is our professional hope that the papers published in this volume can inspire and help other colleagues to some extent.

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We thank all authors, participants, sponsors, editorial team, anonymous referees, and organizing team for their contribution in this special issue.

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Research Article

Pricing Strategy versus Heterogeneous Shopping Behavior under Market Price Dispersion

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We consider the ubiquitous problem of a seller competing in a market of a product with dispersed prices and having limited information about both his competitors' prices and the shopping behavior of his potential customers. Given the distribution of market prices, the distribution of consumers' shopping behavior, and the seller's cost as inputs, we find the computational solution for the pricing strategy that maximizes his expected profits. We analyze the seller's solution with respect to different exogenous perturbations of parametric and functional inputs. For that purpose, we produce synthetic price data using the family of Generalized Error Distributions that includes normal and quasiuniform distributions as particular cases, and we also generate consumers' shopping data from different behavioral assumptions. Our analysis shows that, beyond price mean and dispersion, the shape of the price distribution plays a significant role in the seller's pricing solution. We focus on the seller's response to an increasing diversity in consumers' shopping behavior. We show that increasing heterogeneity in the shopping distribution typically lowers seller's prices and expected profits.

1. Introduction

More often than not, economic agents must operate in the market under lack of relevant information. Pricing is the primary key strategic variable for any profit maximizing seller with some market power. Each seller sets his own price—typically different across sellers—and each buyer searches for the lowest price while having a limited capacity to visit shops and learn different prices.

In this paper we adopt the point of view of a single seller and analyze his optimal pricing policy under limited knowledge both about its competitors' prices and the shopping behavior of the typical buyer. The seller has overall information about rival prices represented by a continuous probability distribution $\mathcal{F}$, which is nondegenerate. This fact is referred to in the literature as market *price dispersion*. Additionally, the seller has an idiosyncratic lower bound c: he is not willing to sell at any price below c. The parameter c can be interpreted as the seller's unitary production cost, assuming linear production technology. More generally, c represents the seller's valuation of not selling. Also, we will consider a single representative buyer, who eventually visits a (random) number of shops n from a total sample of size N that has been previously drawn from $\mathcal{F}$. Notice that considering a representative consumer is not a restrictive condition for our analysis, since the relevant decision variable for the seller is the price he should offer to a buyer visiting his shop. The rival prices observed by the buyer are private information; the seller only knows that prices are drawn from $\mathcal{F}$.

The basic setting above can be framed within the literature of price dispersion in economics, business, or marketing. A first issue is why price dispersion should exist in a market for a homogeneous good. Classical economic theory postulates the "law of one price" since the seminal competition model by Bertrand [1]. If buyers look at all prices, technologies are linear, and the good on sale is homogeneous, only the shop offering the lowest price operates in the market. However, evidence from decades of empirical studies denies the law of one price, see, for example, the exhaustive survey by Baye et al. [2] or, more recently, the systematic study by Kaplan and Menzio [3]. Stigler [4] introduced search costs on the buyer's side: if each additional price observation is costly, buyers will generally check only a subset of all existing prices. Later research by Burdett and Judd [5] completed the picture by showing that, under the existence of search costs, buyers will only check a subcollection of all existing prices, which in turn induces some price dispersion among sellers.

While theoretical economics may explain *why* price dispersion occurs—justifying the underlying assumptions in this paper—there is still the issue of *how* to behave under price dispersion. The optimal behavior of buyers under search cost has been analyzed in the literature with the focus on characterizing a (nondegenerate) price distribution that conforms an equilibrium. Here we opt for a more practical view: a seller in the market must define a pricing policy *even* off equilibrium. This angle turns out to be more relevant in real markets with large and persistent disequilibria.

A key feature in our pricing model is the fact that shopping behavior is diverse (or heterogeneous), that is, the number of sellers visited to make their purchase varies across consumers. Alternatively, a representative consumer selects the number of sellers to visit according to some (nondegenerate) probability distribution Φ. The seller's problem thus consists of selecting the price that maximizes his expected profit in an environment of price dispersion and, also, of diverse shopping behavior.

In the proposed model, a basic pricing policy maps an input quadruple to the price p^* that yields maximal expected profits, namely, $(c, \mathcal{F}, \Phi, N) \rightarrow p^* \in [c, \infty)$, where c is the seller's unit cost, $\mathcal{F}$ is the market *price distribution*, Φ is the *shopping distribution*, that is, the distribution of the number of shops which are visited by a representative consumer, and N is the size of the sample of prices.

We solve numerically the seller's problem for different costs, price distributions, and shopping distributions. We consider $\mathcal{F}$ to be a Generalized Error Distribution (GED hereafter), which allows us to consider a wide variety of price distributions by changing the first, second, and higher order moments. The GED family includes normal and quasuniform distributions as particular cases. We are particularly interested in the role of the shape of $\mathcal{F}$ in the seller's solution. Apparently, the question whether the shape of the price distribution does or does not condition the optimal pricing behavior has not been considered in the literature. Furthermore, it was shown in Alvarez et al. [6] that changes in the first two moments of $\mathcal{F}$, but not in its shape, affect the consumer's shopping behavior (determined via efficient time allocation). We will show below that this is not the case for the seller's pricing problem.

We assume that Φ is a discrete distribution defined on the set $\{2, \ldots, N\}$, so that a representative consumer selects *a priori* a sample of N shops and then visits a number n of them with probability $\Phi(n)$ (to keep notation simple we will use Φ interchangeably for the shopping distribution and its probability mass function). This amounts to consider that the consumer's behavior is perceived as probabilistic by the seller. In turn, this probabilistic perception might be exclusively due to the seller's lack of knowledge or, alternatively, it might be that the buyer's behavior is intrinsically probabilistic. Our main finding here is that, once a buyer goes shopping, more diversity in his shopping behavior, or, alternatively, more uncertainty in the knowledge of the seller about the consumer's behavior, entails lowering prices and expected profits.

The pricing model above can be implemented in real markets once accurate estimates of market prices and consumer behavior are available. Estimates of the two basic inputs, $\mathcal{F}$ and Φ, of the pricing model above can in principle be obtained for specific markets. Price histograms for a large number of goods and markets can be obtained from available consumer data sets and they can be typically fitted by some $\mathcal{F}$ within the GED family, as shown by Kaplan and Menzio [3]. Shopping data required to estimate Φ are not commonly available in the literature or public databases. Yet, some studies on choice overload can provide partial information about Φ (see, e.g., the references in the survey by Scheibehenne et al. [7]).

The manuscript is organized as follows. In Section 2 we describe the problem of a seller maximizing his expected profits in an environment of dispersed prices, which we solve numerically as explained in Section 3. In Section 4 we discuss the results of the numerical analysis. We conclude in Section 5 with final remarks.

2. The Seller's Problem

We adopt the point of view of a seller (or a firm) with linear production technology that must decide the price of his product in a market with an indeterminate number of competitors. The main elements of the scenario in which the seller makes his pricing decision are the existence of market price dispersion and imperfect knowledge about rivals' prices and about the shopping behavior of his potential customers.

Specifically, the seller sets his price in a market in which a representative consumer with a fixed demand of, say, one unit, goes shopping. The seller has information about rivals' prices represented by a probability distribution $\mathcal{F}$. The consumer selects *a priori* N sellers or shops to visit, obtained as a random draw from $\mathcal{F}$. The products offered by the different sellers are indistinguishable for the consumer; that is, the market product is homogeneous. Consequently, he just searches for the lowest price. The consumer eventually observes n price quotes, obtained as a subcollection of the preliminary sample of size N.

In general, shopping behavior is diverse, so that a population of consumers is expected to be nonhomogeneous in terms of the number of shops to be visited. In turn, the shopping behavior of a representative consumer can be understood as probabilistic and heterogeneous. This heterogeneity is represented by a distribution Φ supported on the set $\{2, \ldots, N\}$, so that a consumer will visit n shops with probability $\Phi(n)$. We assume that $n > 1$ in the shopping behavior: if $n = 1$, the seller acts as a monopolist regardless of its rivals prices (regardless of $\mathscr{F}$) since the consumer only checks one price. From the seller's perspective, this probability distribution also reflects his uncertainty about the type of consumer who will visit his shop: he just knows that a consumer that visits n shops will show up with probability $\Phi(n)$.

The seller's problem thus consists of finding the mapping $(c, \mathscr{F}, \Phi, N) \to p^*$, where p^* is the optimal price; that is, p^* maximizes the seller's expected profits. Expected profits are defined by

$$\pi(p) = (p - c) S(p), \tag{1}$$

where $S(p)$ is the probability of success in selling at price p. Specifically, $S(p)$ is the probability of making the sale at price p, given that a consumer will buy at the lowest price after having visited a number of n shops, $n = 2, 3, \ldots, N$, (with probability $\Phi(n)$), from a sample of size N obtained from $\mathscr{F}$. Notice that $S(p)$ can be written as

$$\begin{aligned} S(p) &= \sum_{n=2}^{N} \Phi(n) \frac{\binom{N-1}{n-1}}{\binom{N}{n}} (1 - F(p))^{n-1} \\ &= \sum_{n=2}^{N} \Phi(n) \frac{n}{N} (1 - F(p))^{n-1}, \end{aligned} \tag{2}$$

where F is the cumulative distribution function of the distribution $\mathscr{F}$. Now, given a price p, the first term in equation (1) is simply the *mark-up*, the difference between price and unitary cost. The chances $S(p)$ that the seller earns his mark-up—making the sale—competing in price with other $N-1$ sellers require that (i) the consumer visits the seller's shop given that he will visit n shops out of N, and (ii) the seller's price is the lowest among those n prices checked by the consumer. The second factor in each term of the sum in (2) corresponds to (i), whereas the third factor corresponds to (ii). Given that the consumer will visit n shops with probability $\Phi(n)$, (2) gives the probability of making the sale at price p. In the case that any of the events (i) or (ii) does not occur, $S(p) = 0$, and consequently the seller makes zero profits, so that (1) can be interpreted as expected profit.

Notice that the seller faces a trade-off when choosing his price: increasing the price (the mark-up) implies increasing effective profits in the case he sells the unit, but also lowering the probability of selling the unit.

Shopping behavior is understood here as the choice of the number n of shops to visit (alternatively, prices to check). Even in the case that the consumer has selected a total number N of shops to visit *a priori*, he may actually end up visiting a smaller number n. The classical literature of search cost (e.g., Stigler [4]) gives a rational explanation for this shopping behavior: assuming that the consumer minimizes her total expected cost—search cost plus expected price to be paid—the optimal n may be typically lower than N. Alternatively, the consideration of time as a fundamental constraint affecting consumer's behavior can also explain the rational choice of a number of visits n below N. This applies in particular when the total number of shops N is large and the consumer optimally allocates his time among several alternative uses. Indeed, it was shown in Sanchis et al. [8] and Alvarez et al. [6] that assuming that n is determined from the time allocation that maximizes well-being, a consumer might choose not to visit the total number N of available shops.

No particular optimal behavior on the consumer's part will be assumed here. In fact, behavioral scientists claim that consumers' behavior may be far from rational in this kind of shopping environment. The so-called choice overload phenomenon is a focal example in which consumers are worse off when they visit all available shops (see, e.g., Schwartz [9] and Iyengar and Lepper [10]). A consumer who visits *all* available shops is called a *maximizer* by social psychologists; otherwise he may be generically called a *satisficer* [11]. A consumers' population is not expected to be composed by maximizers only, but rather by a mixture of maximizers and different types of satisficers. This mixture is represented in this paper by the shopping distribution Φ: given a sample of N shops in the market, $\Phi(N)$ is the probability that the representative consumer is a maximizer, whereas $\Phi(n)$, for $2 < n < N$, are the probabilities that he is some type of satisficer.

3. Numerical Analysis: Input Data and Problems under Study

We follow a numerical approach for the analysis of the seller's problem described above. Given the unitary cost c, a distribution of market prices $\mathscr{F}$, the number N of sampled sellers, and a shopping distribution Φ, a core computational routine must produce a numerical solution to problem (1). Since an explicit solution can only be obtained in very special cases, the numerical analysis becomes essential. We have used R as a programming environment (R Core Team [12]).

We consider the market price distribution $\mathscr{F}$ to be a Generalized Error Distribution GED (μ, s, ν), whose parameters are location (μ), scale (s), and shape (ν) [13]. GED is a family of symmetrical unimodal distributions with domain in $(-\infty, +\infty)$ whose probability density function is given by

$$f(p) = \frac{\nu}{2s\Gamma(1/\nu)} \exp\left(-\left(\frac{|p-\mu|}{s}\right)^{\nu}\right), \tag{3}$$

where Γ denotes the Gamma function. The parameter μ locates the mean, the median, and the mode of the distribution. The variance of the distribution depends on both scale s and shape ν, as follows:

$$\sigma^2 := \operatorname{Var}[p] = \frac{s^2\Gamma(3/\nu)}{\Gamma(1/\nu)}. \tag{4}$$

Our choice is due to the fact that GED is a flexible family of distributions that includes Laplace (double exponential) distributions if $\nu = 1$, normal distributions if $\nu = 2$, or quasiuniform distributions when ν is large enough. In fact, the GED converges pointwise to a uniform distribution on $(\mu - s, \mu + s)$ as $\nu \to +\infty$. Several GED probability densities for different shapes are displayed in Figure 2(a).

The skewness of a GED is zero, while the excess of kurtosis depends exclusively on the shape ν. In fact, GEDs with shape $\nu < 2$ are leptokurtic. This is a significant case, since distribution of prices for individual goods appears to be leptokurtic in many cases (see Kaplan and Menzio [3]). In the model analysis with respect to shape below both leptokurtic (e.g., Laplace) and platykurtic distributions (e.g., quasiuniform) are considered. We set the mesokurtic case $\nu = 2$ (normal distributions) as the benchmark case below for other analyses where shape is constant.

The solution to our seller's problem can be written in parametric form as a mapping

$$(c, (\mu, s, \nu), (\Phi(n))_{n=2,\dots,N}, N) \longrightarrow (p^*, \pi^*), \qquad (5)$$

where p^* is the maximum of (1) and $\pi^* := \pi(p^*)$ is the seller's optimal (expected) profit. We are mainly interested in analyzing the response of the output variables (p^*, π^*) as the diversity of satisficers in the market increases. This amounts to study the seller's solution with respect to suitable variations in Φ while keeping everything else constant. Notice that a homogeneous population of maximizers can be defined by $\Phi(N) = 1$. When the probability mass is spread over the set $\{2, \dots, n\}$, the heterogeneity of the distribution Φ increases with respect to the maximizer case $\Phi(N) = 1$. It will be useful to characterize the degree of heterogeneity using some sensible parameter. We analyze the effects on price and profits of an increase in shopping heterogeneity which is due to two different sources, in turn controlled by two different parameters. First we consider in Section 4.2.1 uniform spreads over an increasing range of n's, $n \in \{2, \dots, K\}$, with $1 < K \le N$, which can be parameterized using the Shannon's entropy of the distribution. Second, in Section 4.2.2, we analyze distributional spreads generated by shopping fatigue that can be parameterized by the probability ρ of not visiting a new shop after having visited a number of them.

As a preliminary study, in Section 4.1, we consider a homogeneous population of maximizers, that is, $\Phi(N) = 1$, and analyze the effect of a variation in some basic parameters of the problem on the seller's solution. We are particularly interested in learning whether the seller's solution depends only on the mean and dispersion—typical deviation—of prices or whether the shape of the price distribution also matters.

4. Results and Discussion

Initially, we assume a homogeneous population of maximizers, so that $\Phi(N) = 1$, and we analyze the seller's response with respect to changes in the cost or in a distributional parameter of prices, while keeping everything else constant except for the market size N—the number of available sellers *a priori*. The price distribution $\mathcal{F}$ is assumed to be GED with shape $\nu = 2$, which corresponds with a normal distribution. The results do not differ if a different GED is considered. We are particularly interested in the effect of the shape of the price distribution on price and profits. Typically, for a set of values of N, we compute and compare the seller's solution within a range of equispaced values of each parameter, while keeping the rest of parameters at their benchmark values. The benchmark case and the range of variation considered for each parameter are shown in Table 1. The benchmark value of c in Table 1 corresponds to the price level which is higher than 15% of all prices.

Table 1: Benchmark and range of analysis for price distribution parameters, cost, and market size.

	GED (μ, s, ν)			Cost	Market size
	μ	σ^2	ν	c	N
Benchmark	600	125	2	470.44	30
Range of analysis	575–625	100–150	2	425–475	2–30

Table 2: Benchmark case for Section 4.1.

	GED (μ, s, ν)			Competitiveness	Market size
	μ	σ^2	ν	α_c	N
Shape analysis (range)	600	125	1–10	0.15	2–30

Assuming a maximizer profile for the representative consumer, it can be checked consistently that an increase in the market size N leads to more competitive prices—that is closer to the unitary cost c—whereas an increase in the unitary cost or in the mean price μ leads to higher prices. In turn, the associated profits shrink as N increases or as c increases, but they grow as μ increases. These facts account for responses of the seller's solution that could be somehow expected. The corresponding graphs are displayed in Figure 1.

4.1. Shape Variation in Price Distribution. As mentioned above, we are particularly interested in the seller's response with respect to variations in the shape of the price distribution. This is a relevant question from the economic perspective. The use of GED (μ, s, ν) as price distributions is particularly useful here, since the shape ν of the distribution can be changed while keeping mean μ and variance σ^2 constant. Notice that a GED with a given σ^2 can be obtained for any shape ν by adjusting the scale parameter s suitably in (4). Again, the representative consumer is a maximizer here.

In order to isolate the shape effect on the seller's solution, the unitary cost c needs to be adjusted so that $F(c)$ remains constant at a certain level α_c. Such α_c represents the fraction of rivals whose costs are lower than the cost of the seller under analysis. It can be interpreted as a competitiveness level, since *de facto* the seller could not compete in price with $100 \times \alpha_c$ percent of other rivals in the market. The competitiveness level will remain invariant for different shapes in the analysis below. Also, the mean and dispersion of the price distribution remain constant. As mentioned above, this can be controlled within the GED family by adjusting the scale parameter s to keep variance fixed as ν varies. Table 2 shows the benchmark and the range of variations in shape ν and market size N.

The typical output of the analysis in this section is displayed in Figure 2(b) for $N = 30$. The main message of the analysis is that shape matters; that is, when it comes to choose the optimal price, information about mean and dispersion of the market prices is not enough. A seller should also have an estimate of the shape of the price distribution. Thus, while the total mass of effective competitors (i.e., $100 \times (1 - \alpha_c)$ percent of the sellers) is important, also the shape according

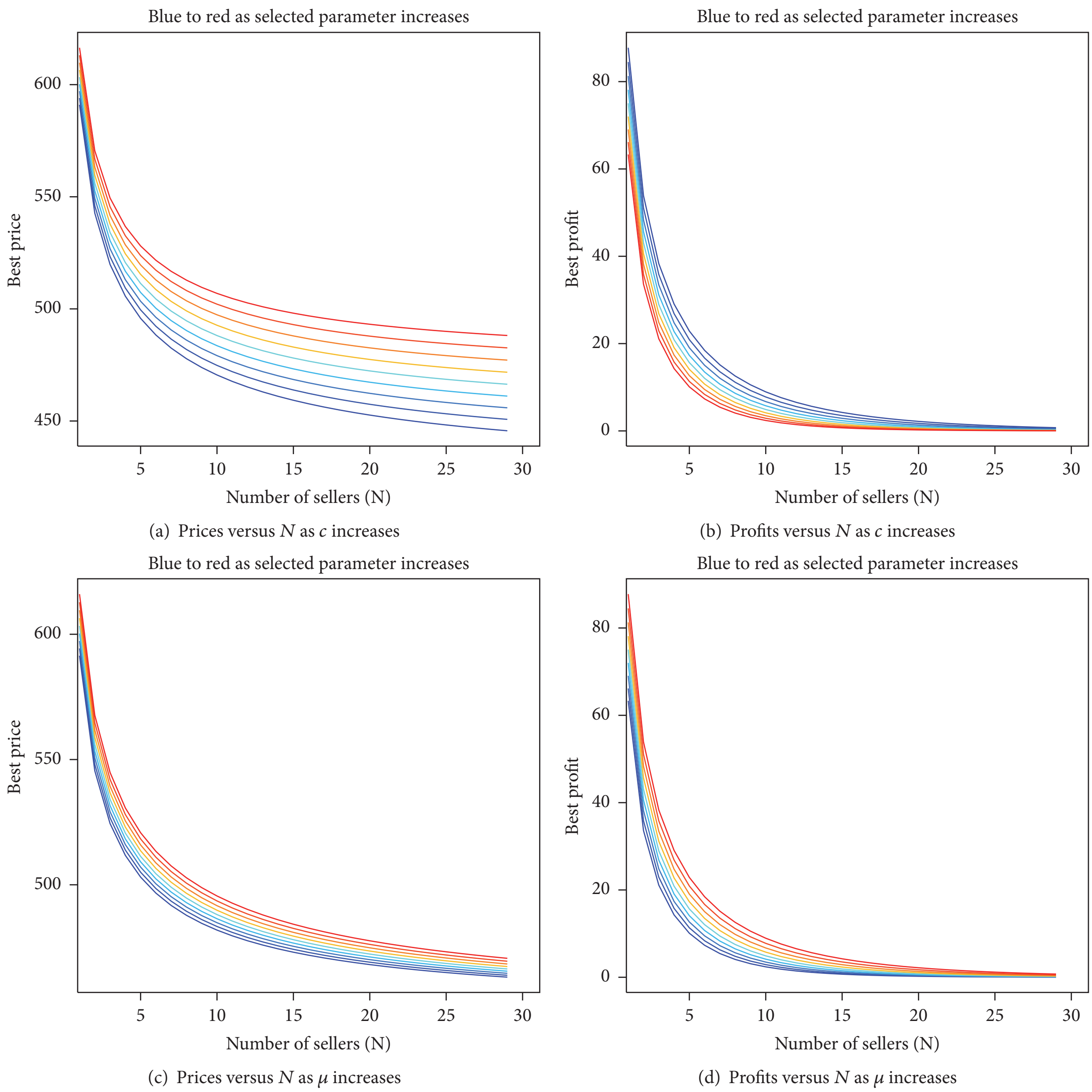
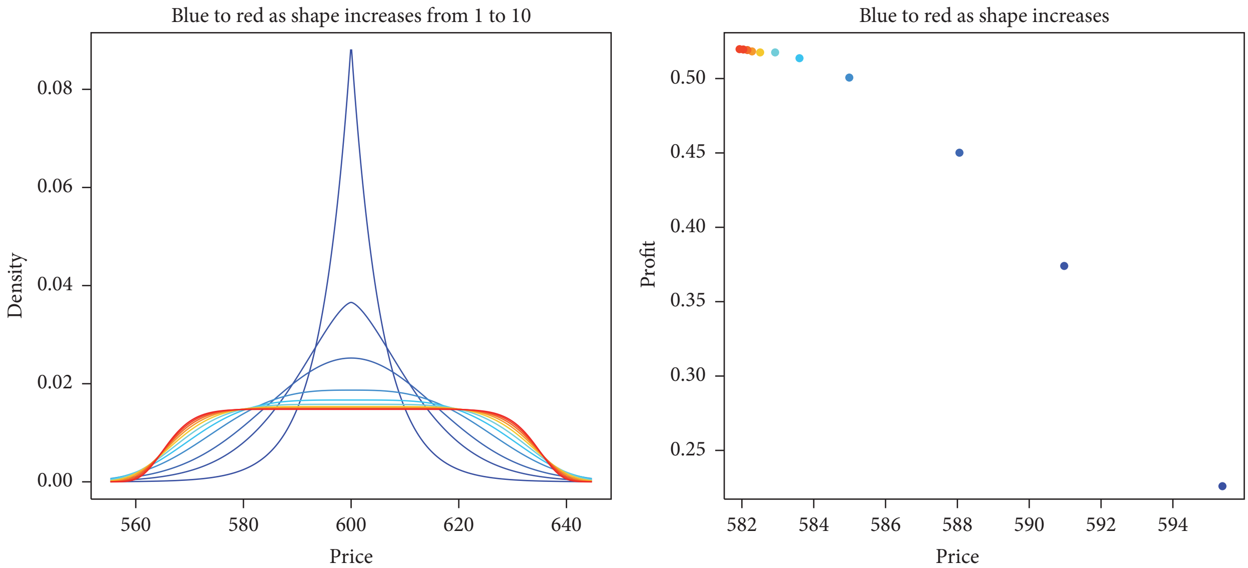


Figure 1: Responses in optimal price and (expected) profits to changes in cost c, price mean μ, and market size N. From blue to red as cost and price mean increase within the range of analysis and with parameter values as in the benchmark case ($\mu = 600$, $\sigma^2 = 125$, $\nu = 2$, $N = 30$); see Table 1.

to which this probability is distributed is relevant for the seller's problem.

The study above suggests that knowledge of mean and dispersion of prices does not suffice to set prices rationally. This is in contrast with the analysis of shopping behavior of consumer—that is their choice of the subsample size—searching for the lowest price under optimal allocation of their time. The analysis in Alvarez et al. [6] suggests that their optimal choices depend only on the first and second order moments of the (symmetric) price distribution. Therefore, while knowledge of the mean and the dispersion of prices seems sufficient for the consumer to solve his shopping problem—as a by-product of optimal time allocation—it is not enough for the seller to define his pricing policy efficiently.

4.2. Heterogeneity in Shopping Behavior. The seller's response is analyzed next with respect to variations of parameters of the shopping distribution Φ, keeping everything else fixed. We will analyze the effect of increasing the heterogeneity of the shopping distribution, that is, augmenting the variety of consumer's profiles in terms of the number of visited shops. We consider below two ways of increasing heterogeneity in Φ: in Section 4.2.1 we consider uniform spreads over



(a) Probability density functions

(b) Profits versus prices

FIGURE 2: Probability density functions of prices and output from the analysis in Section 4.1 for GED (μ, s, ν) distribution with varying shape (from blue ($\nu = 0.5$) to red ($\nu = 10$, quasiuniform)) and the rest of parameters as in the benchmark case ($\mu = 600$, $\sigma^2 = 125$, $N = 10$); see Table 2.

TABLE 3: Heterogeneous consumer's shopping distributions in Sections 4.2.1 and 4.2.2 and their respective benchmarks.

	Shopping distributions	Heterogeneity analysis (range)	Benchmark
Spread-of-search (Section 4.2.1)	Φ_K	$K = 2, 3, \ldots, 30$	$K = 2$
Search fatigue (Section 4.2.2)	Φ_ρ	$\rho = 0.05i,\ i = 1, 2, \ldots, 20$	$\rho = 1$

an increasing range of n's, and in Section 4.2.2 we analyze distributions generated by a fatigue effect on consumer's shopping behavior.

4.2.1. Increasing Spread of Shopping Distribution. We assume here that $\Phi = \Phi_K$ is the uniform distribution on the set $\{2, 3, \ldots, K\}$, for every integer $2 \leq K \leq N$, that is $\Phi_K(n) = (K-1)^{-1}$ for $n \in \{2, 3, \ldots, K\}$. The benchmark case is $K = 2$; that is, the representative consumer visits 2 shops. The benchmark values for other relevant parameters are as in Table 1. We analyze the seller's response with respect to a uniform spread in the number of possible visits; that is, we compute his pricing policy and associated profits when a representative consumer chooses randomly a number n of shops to visit between 2 and K, with equal probability. The seller is totally uncertain about the precise number of shops (between 2 and K) the consumer will eventually visit: any number is equally likely. As K increases, the Shannon's entropy of Φ_K, given by $H_K = \log(K-1)$, also increases. The entropy H_K serves as a measure of the heterogeneity of the distribution and, alternatively, of the uncertainty of the seller with regard to the consumer side of the market (see, e.g., Gray [14]).

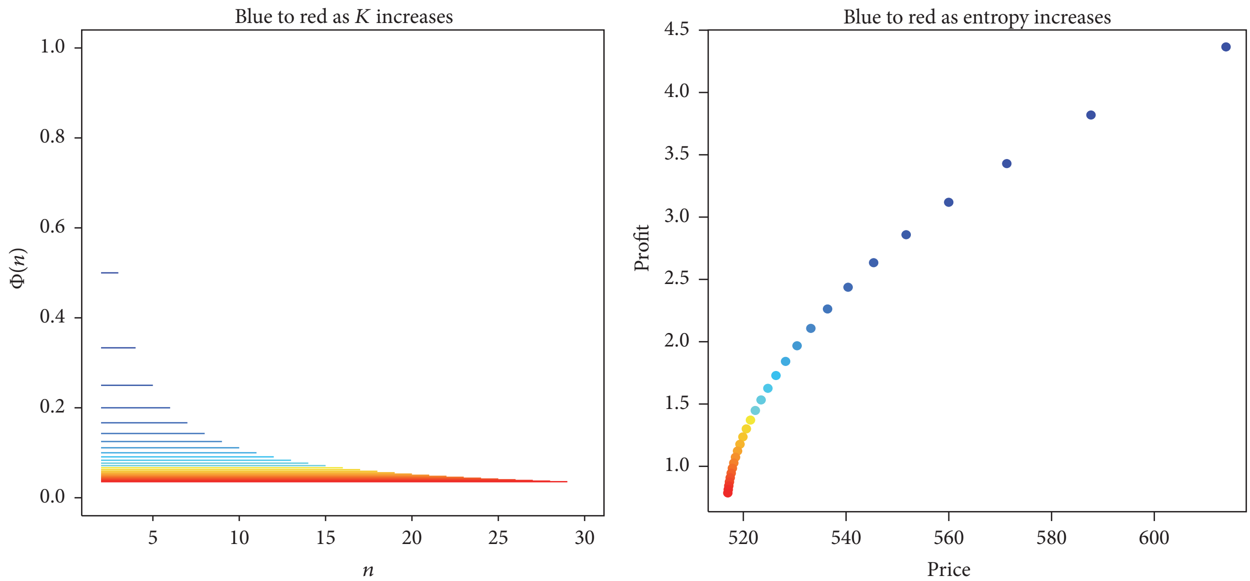
Figure 3(b) summarizes the seller's optimal response—prices and associated (expected) profits—as the entropy of the distribution increases. It can be observed that prices and profits decrease as the heterogeneity of the distribution increases. As the demand side of the market becomes more competitive (i.e., an extra shop is equally likely to be visited), the seller's profits shrink.

4.2.2. Search Fatigue in Shopping Behavior. We consider here a behavioral effect as a possible source of the heterogeneity in the shopping distribution. Assume that a representative consumer has selected a sample of N potential sellers, or shops, to visit (i.e., prices to check) from $\mathcal{F}$ and that he will visit at least 2 shops. We further assume that, after having visited any number of shops, there is a probability $0 < \rho < 1$ of not visiting one more shop. In principle, the fatigue parameter could have memory of the number of shops already visited, that is, $\rho = \rho(n)$. We consider here the simple case in which ρ is constant.

The simple behavioral assumption above produces a shopping probability distribution Φ_ρ with support $\{2, 3, \ldots, N\}$ if $0 < \rho < 1$. Indeed, the probability that a representative consumer will visit n shops, given that he knows that a total of N shops is available is given by

$$\Phi_\rho(n) = \frac{\rho(1-\rho)^{n-1}}{1-(1-\rho)^{N-1}}, \tag{6}$$

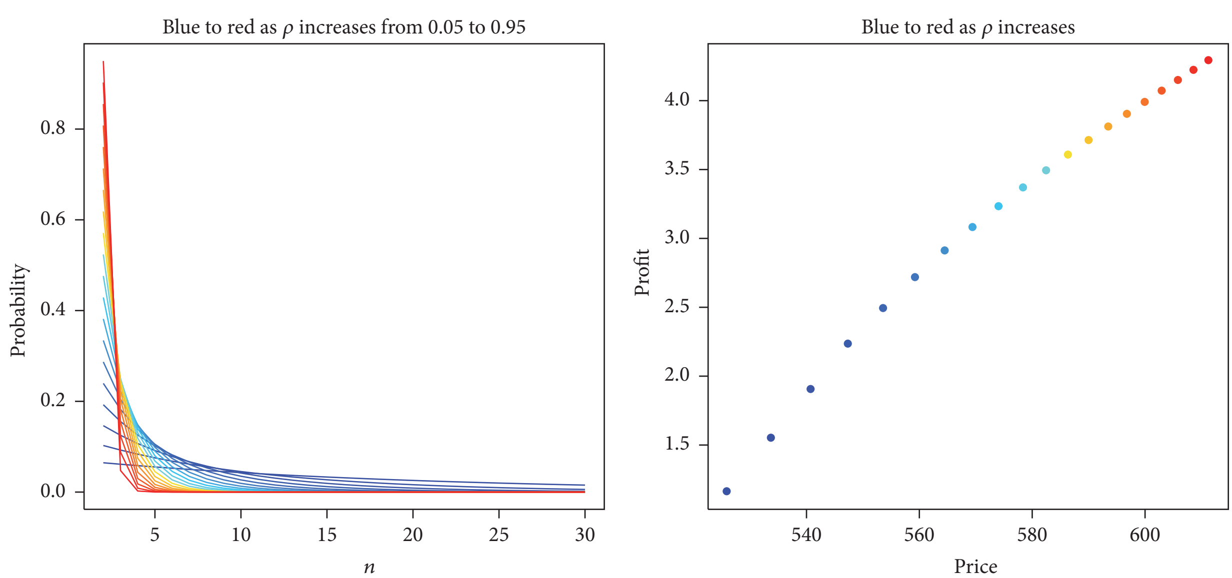
which is a version of the geometric distribution [13]. Notice that if $\rho = 1$ the consumer will visit just 2 shops almost surely, which is the benchmark case considered in Table 3.



(a) Probability mass functions as the number of visited shops K increases from blue ($K = 2$) to red ($K = 30$)

(b) Profits versus prices (optimal) for an increasing spread of the shopping distribution Φ_K as entropy increases from blue ($K = 2$) to red ($K = 30$)

FIGURE 3: The seller's solution under a uniform spread of shopping behavior. Main parameters in Table 3 and those used earlier as benchmark ($\mu = 600$, $\sigma^2 = 125$, $\nu = 2$, $\alpha_c = 0.15$, $N = 30$).



(a) Probability mass functions as ρ increases, from blue ($\rho = 0.05$) to red ($\rho = 1$)

(b) Profits versus prices (optimal) for an increasing shopping fatigue (ρ increasing from blue ($\rho = 0.05$) to red ($\rho = 1$))

FIGURE 4: The seller's problem under shopping fatigue. Main parameters in Table 3 and those used earlier as benchmark ($\mu = 600$, $\sigma^2 = 125$, $\nu = 2$, $\alpha_c = 0.15$, $N = 30$).

The probability mass functions for an increasing sequence of values of ρ, from blue ($\rho = 0.05$) to red ($\rho = 1$), can be seen in Figure 4(a). As ρ decreases, the entropy of the distribution increases; in turn the shopping distribution becomes more heterogeneous.

Figure 4(b) displays the seller's response with respect to an increasing fatigue, that is, a higher propensity of the representative consumer to stop shopping in the market. The red point corresponds to the benchmark case ($\rho = 1$) in which the consumer visits 2 shops almost surely. As the fatigue is reduced (ρ decreases) the probability of visiting more shops increases and the shopping distribution becomes more uniform. The seller reacts to the fatigue reduction by lowering prices which entails lower profits. In particular, as the shopping distribution becomes more heterogeneous, the seller's profits diminish.

Notice that the benchmark case $\rho = 1$ of Section 4.2.2 (red point in Figure 4(b)) is equivalent to the benchmark case $K = 2$ of Section 4.2.1 (blue point in Figure 3(b)), and the pattern of the seller's solution is similar in both cases: as the shopping distribution becomes more heterogeneous, profits are reduced.

5. Conclusions

In this paper we have introduced a general framework to select the optimal pricing strategy of a seller competing in a market of a homogeneous product in which prices are dispersed and he has limited information about both his competitors' prices and the shopping behavior of his potential customers. In order to solve his problem in a real setting, the seller needs to estimate the distribution of prices in the market and the distribution of shopping behavior of consumers. By computing the solution to the problem, he determines the price that maximizes his expected profits. This theoretical setting describes many prevailing situations in real scenarios.

In general, the solution to the seller's problem depends in a delicate manner on many parametric and functional inputs. We have tested the model response under different scenarios in which some of the parameters or functions are modified. In order to analyze the seller's response under different exogenous perturbations, we have produced synthetic price data using a wide family of distributions and have generated consumer's shopping data from some naive behavioral assumptions.

Regarding the supply side of the market, a significant issue for the seller is how much (limited) information is required about his rivals' prices in order to determine his own price. In real markets it is not unusual that sellers set their prices looking only at the mean price in the market (and maybe also at some measure of price dispersion). Our computational study has showed that optimal pricing is sensitive not only to the mean and dispersion of prices but also to the shape of the distribution. Thus, in order to price efficiently, the seller must estimate the whole distribution of prices. This fact is in contrast with the behavior of the demand side of the market when shopping behavior is determined from rational time allocation. Indeed, the analysis in Alvarez et al. [6] suggested that consumers' shopping behavior can be determined from price mean and variance only, so the shape of the distribution does not seem to play a role here.

Regarding the demand side of the market, an important question for the seller is understanding how diversity in shopping behavior affects his pricing policy. In real markets, consumers' shopping behavior is typically expected to be far from rational and in turn nonhomogeneous. We have analyzed two different types of heterogeneity in the shopping distribution. Our computational analysis has showed that more diversity in the shopping behavior—parameterized, for example, by the Shannon's entropy—entails lowering optimal prices and profits.

Price distributions and shopping distributions are the main inputs to run the pricing model. The real utility of the model proposed in this paper should come from its use in real markets. It is apparent from our analysis how the model could be implemented in real markets once accurate estimates of market prices and consumer behavior are available.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

The Impact of Cocreation on the Student Satisfaction: Analysis through Structural Equation Modeling

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The objective of this study is to apply the cocreation initiative as a marketing tool in the context of university undergraduate programs. Considering that cocreation is a practice that involves stakeholders in different phases of product production or service, this research analyzes the interactions between some of the factors during the cocreation process as students collaborate with the university. These factors are participation, communication, cocreation, and satisfaction, and this study focuses on how they fuse together at the moment of cocreation. After a literature review, which supplied the basis for creating a model, we used exploratory and confirmatory factor analysis and structural equation modeling to validate the hypothesized relations between the variables; finally, the proposed cocreation model was verified. The results could empower academic institutions to develop managerial strategies in order to increase students' collaboration and satisfaction.

1. Cocreation and the University

Higher education has been involved in recent trends such as the increasing competition in the university market, budget reductions, the internalization of education, the growth of quality standards, and clients (students) becoming more demanding and competitive in the recruitment market. Facing this situation, universities need to reevaluate their strategies and gain a marketing orientation [1, 2] in order to avoid the intense competitive force [3].

Higher education institutions generate alternatives to increase their loyalty rates through active interaction with the student. Considering that consumer satisfaction positively affects loyalty [4], a strategic goal for universities is to enhance student satisfaction.

At the current research, cocreation is conducted as a marketing alternative to increase the institutions' service satisfaction at the educational level. Cocreation assures interactions and connections among different stakeholders, generating communications and collaborative ties among them [5]. This approach allows the companies to generate value through client participation, with an active role during the service process or product production [6] assuring a competitive advantage in the market [7].

Although the university world differs considerably from the business sector, academic institutions are looking to increase their service quality and stakeholder satisfaction in order to gain a competitive advantage in the current situation [8]. Thus cocreation is analyzed with the objective to research the impact of students' inclusion in activities such as curriculum and program development and the teaching-learning process. The importation of cocreation to higher education institutions allows universities to adopt a marketing orientation to seek excellence and recognize quality levels.

The purpose of this investigation is to fill the existing gap in the academic market and to determine whether it is plausible to apply cocreation at higher education institutions. This viability is explored in terms of the impact of the two principal factors (participation and communication) on the cocreation process and the impact of cocreation on student satisfaction. Researching the links among those elements will permit us to confirm whether cocreation is applicable in this sector. The principal research questions are do communication and student participation have a positive impact under cocreation in the university context? What are the consequences of applying cocreation in higher education institutions? Does student satisfaction increase due to the cocreation experience?

Although studies by [9, 10] have researched cocreation at the university level, they have only focused on postgraduate programs. The current investigation aims to respond to the aforementioned research questions by analyzing the relationships between four principal constructs (participation, communication, cocreation, and satisfaction) in undergraduate programs as the target. Through the study of these relationships, it is possible to validate the proposed model, which has cocreation as its cornerstone. The principal qualitative tools of exploratory and confirmatory factor analysis and structural equation modeling (SEM) were used to confirm or reject the different hypotheses and to validate the proposed cocreation model. The research was developed by examining a case study of undergraduate students from 11 Ecuadorian universities.

2. Relationships between Communication, Participation, Cocreation, and Satisfaction

In this section, the theoretical basis of the proposed cocreation model will be analyzed. Four principal constructs were identified (participation, communication, cocreation, and satisfaction), which have been detailed below by comparing the conceptual relationships existing among them at the university level.

2.1. Communication versus Participation. Communication and participation are two elements that have important impacts on cocreation when applied to the business world. Reference [11] commented about the positive effect of communication on cooperation between stakeholders. Communication with customers allows for positive client participation in open innovation processes [12], and the Internet allows broad communication with a higher user-participation rate [13].

In their research, [14] revealed that communication technologies have a positive influence on the interaction process, allowing the generation of new products. Terblanche [15] reflected on the employer's role as an important generator in the communications process.

We find that the direct link between these two elements is maintained in the educational environment. To strengthen the relationship between university and student, it is actually a necessity to consolidate value through cocreation. Through the communication, dialogue, and participation of the involved stakeholders, it is possible to develop strategies such as knowledge cocreation in this field [16].

Authors in [17] have commented on the application to universities of methodologies such as blended learning, which integrates the traditional face-to-face system with online courses. The online learning approach, supported by the Internet and solid communication with students, guarantees quality and effective education. In this sense, the cocreation concept comes to life because the student plays an important role when he collaborates actively in the teaching-learning process.

On the other hand, student participation in formal and informal education on campus not only contributes to education quality but also positively affects the key competencies that students acquire [18]. Junco [19, p.168] described the effect of participation in social media such as Facebook, where it has been demonstrated that students' active roles are "related to out-of-class engagement."

Regarding the relationship between communication and participation, we hypothesize the following:

> H1: communication has a direct, positive impact on participation.

2.2. Participation versus Cocreation. At the market and university context, participation refers to the client's collaboration with the institution, which is important in order to develop a solid exchange of information to know the consumer's (students) desires and ideas and to avoid misunderstandings and ambiguous situations [20].

The user's involvement in different steps of the processes allows the coproduction development [21], leading customers to become partial employees [22]. Several studies (e.g., [23–29]) had been analyzing the interrelationship between participation and cocreation and found an interesting result that supported the link between these two constructs at several industries.

The ties existing between participation and cocreation in the university context have been addressed in some studies. For example, students' behavior is predominantly active in what is called Education 3.0, in which collaboration allows them to gain a "strong sense of ownership of own education, cocreation of resources and opportunities" [30, p. 2]. In this standard the main objective is the generation of a more open and free learning system. That is why one of the conditions for developing this education level is the promotion of cocreation by creating multidirectional participation involving the affected parts.

Educational services include stakeholders such as students and professors; the students are emotionally and behaviorally involved during the service consumption, playing a dynamic role during the interaction. Some the benefits of such a collaboration are the facilitation of learner control, enhancement of program adaptation, and learning flexibility [8].

Another study [31] remarked on the positive impact of student participation in the curriculum design process. Across this collaboration in the cocreation activities, the teacher becomes a facilitator of learning, giving the students more responsibility at the individual and collective levels. Student collaboration and participation in different processes during the educational exchange allow satisfactory results in "both pedagogical and business outcomes" [8, p. 36].

Yeo [32, p. 72] commented that, in the transformative view, students participate actively, improve their knowledge and skills, and have the "ability to think critically," so collaboration leads to cocreation of knowledge.

Based on the findings obtained by the aforementioned author, we hypothesize the following:

> H2: participation has a direct, positive impact on cocreation.

2.3. Communication versus Cocreation. As [5] commented, communication had evolved from one-way to participatory conversations, principally considering the Internet as an important channel of information flow. The positive influences that communication has under cocreation have been noted.

Communication between firms and clients (students) has an important influence in the cocreation process [33, 34] and constitutes one of the four building blocks (dialogue) identified by [35] in the DART model (dialogue, access, risk, and transparency). This model was established to consider the blocks in an accurate application of cocreation.

Social networks are a tool used to create content, as different participants can communicate and thus cocreate knowledge. Applied to the educative framework, to assure an effective dialogue, "universities/colleges and the customer must become equal and joint problem solvers" to cocreate value [36, p. 50]. With this perspective shift, the student goes from having a passive role to becoming a live participant who can promote his or her opinion and initiatives through communication to foster the cocreation process.

As [22] commented, cocreation has been fomented by different communication media, such as blogs, e-distribution, and home videos; therefore, people in environments with greater access to communications instruments are better able to collaborate in coproduction activities. Considering that universities need to create physical and virtual spaces where students can obtain information, documents, and news as well as give their feedback or news ideas, enhancing the communication channels with the institution. If communication is a mandatory condition to implement cocreation, the institution is responsible for eliminating the existing barriers to dialogue and to create a space for facilitating a proper exchange of information. Based on the research on the relationship between communication and cocreation, we hypothesize the following:

> H3: communication has a direct, positive impact on cocreation.

2.4. Cocreation versus Satisfaction. Satisfaction refers to a positive reaction in front of a state of fulfillment [37] and as [38] reflected, the cocreation benefits are as follows: cost diminution, response time and sales improvement, and the induction of higher satisfaction and enjoyment. Studies by [8, 39, 40] support the aforementioned relationship.

At higher education institutions, satisfaction is linked with "a short term attitude which arises from the students' evaluation of the educational experience, which is subjective in nature" [8, p. 38]. Some valuable impacts of satisfaction are student loyalty, cost reduction, increase in revenue, and continued education.

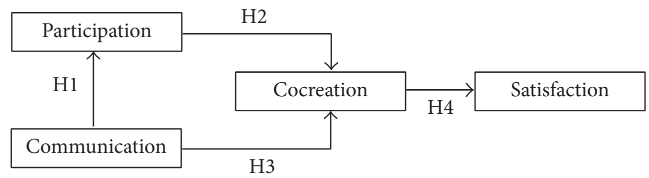


Figure 1: The cocreation research model.

In the academic context, it has been proven that when curriculum is cocreated with student collaboration, the satisfaction level increases for both teachers and students [31]. The cocreation concept empowers the university to understand what the student wants and needs, and in consequence, it is possible to deliver a superior service that directly influences student satisfaction. By tailoring its educational offers to students' needs, an institution can provide a valuable learning experience [8].

In their study, [41] reformulated Porter's value chain by coupling it to the higher education sector. The primary services/attributes they proposed were programs, regulation recognition, moment of truth, learning spirit, and service competition; the supporting services were professional recruitment, modern tools and infrastructure, library, and after-sales service. Under this proposed change, the university and the students will be able to cocreate value that satisfies both parties.

As [42, p. 728] commented, value cocreation is a learning process characterized as "emergent, unstructured, interactive, and uncertain." For that reason, the delivery of activities is important, and faculty-student and student-student interactions are key to the learning experience. The accomplishment of student expectations generates satisfaction growth, but when the institution exceeds what the client/student wants, then loyalty is reached.

On the relationship between cocreation and student satisfaction, we hypothesize the following:

> H4: cocreation has a direct, positive impact on satisfaction.

2.5. The Hypothesized Cocreation Model. The model to be validated is shown in Figure 1, which reflects the different relationships to be analyzed. The different ties existing between the constructs communication, participation, cocreation, and satisfaction have theoretical support from different authors, as mentioned above. The principal objective of this model is to analyze the impact of cocreation on satisfaction in the higher education world.

The hypotheses to be studied are four and presented in Table 1; the different constructs involved in the model and the questionnaire items related to the components are also shown.

Table 1: Hypothetical links in the research model, the constructs analyzed, and the related questionnaire items.

Hypothesis	Construct	Items
H1: communication has a direct, positive impact on participation.	Communication	com1, com2, com3, com4
	Participation	par1, par2, par3, par4
H2: participation has a direct, positive impact on cocreation.	Cocreation	cocre1, cocre2, cocre3, cocre4
	Participation	par1, par2, par3, par4
H3: communication has a direct, positive impact on cocreation.	Cocreation	cocre1, cocre2, cocre3, cocre4
	Communication	com1, com2, com3, com4
H4: cocreation has a direct, positive impact on satisfaction.	Cocreation	cocre1, cocre2, cocre3, cocre4
	Satisfaction	sat1, sat2, sat3

3. Methodology, Data Collection, and Technique

The technique applied during the investigation's development to recollect information was a structured questionnaire comprising of 31 questions; only 12 questions were analyzed in the present research related to the variables studied. A Likert scale with 7-level items, from strongly disagree (1) to strongly agree (7), was applied. The questionnaire composition proceeded from previously accomplished investigations [9, 43–46] and was distributed in two ways, physically and electronically, in 11 public and private Ecuadorian universities. We obtained 395 responses (92 women and 303 men) among the different versions distributed in order to prevent possible bias and to randomize the question order [47]. The questionnaire was applied only in undergraduate programs, including students in their fourth through tenth semesters, considering those scholars have a solid perception about the university's services.

3.1. Measures. Our measurements were adapted from existing scales developed in other studies in order to measure the four constructs (communication, participation, cocreation, and satisfaction). Participation was adapted from a validated questionnaire created by [43] measuring the degree of the information students shared with the university and how much they were involved in the institution's process. Communication was extracted from [44, 48] investigations, analyzing the exchange of information among the parties involved. Cocreation's construct checked how the students were involved in the different academic and administrative processes, and it was measured by four items adapted from [9, 45, 46]. Satisfaction is comprised of items extracted from [45, 46] and studied the contentment that the user has with the institution. Since all questions were originally in English, they were translated to Spanish for this study.

4. Empirical Results: Multivariate Analysis

To analyze the results, we applied a confirmatory factor analysis to explore the associations between items and constructs and, lastly, SEM to investigate the causal relationships existing between constructs.

4.1. Exploratory Factor Analysis (EFA). EFA was applied in order to check whether the principal components detected by this technique were similar to the components identified by the authors, recognizing that items that are pooled jointly measure the same factor [49, 50]. Every variable was included, taking into account the theoretical basis and allowing the EFA to corroborate whether those statements were correct. EFA granted the validity of each construct through the principal components method [51]. It used the SPSS v19 program, and the results showed that there are four principal components, as established in the proposed model (participation, communication, cocreation, and satisfaction).

A Varimax rotation and the maximum likelihood extraction method were used with the four fixed components. Table 2 reflects the results of the first and second iteration. In the first iteration, problems with four items were detected (par3: I have a high level of participation in the service process, com1: the information provided by the university can be trusted, com2: in case of any problem, the university provides me with enough information, and sat3: I think I did the right thing when I enrolled in this university). The items par3, com1, and com2 had loading differences under 0.1 with several constructs. The item sat3 had a loading difference above 0.1, but its highest loading values do not correspond with the construct relative to satisfaction. For that reason, those four items were dropped; the rest of them (11 items) remained in the analysis.

Cronbach's alpha [52] is an indicator that reflects the homogeneity in the instrument's consistency; the second and last iteration had an excellent value of 0.906 (above 130 0.7). The explained variance of the four principal components is about 64.4%. The KMO value is 0.910, higher than 0.5 [53], and the Bartlett test returned as $p = 0.000$. The differences in these indicators between the first and the second EFA are minimal, and despite the diminished Cronbach's alpha (from 0.926 to 0.906) and decreased KMO (from 0.936 to 0.9910), both indicators had excellent values.

4.2. Confirmatory Factor Analysis. A confirmatory factor analysis (CFA) was carried out with the remaining items. In this step the SPSS Analysis of Moment Structures (AMOS) program was used, allowing us to assess the overall measurement model.

A convergent and discriminant analysis to evaluate the model's validity was used. The convergent validity was studied through the composite reliability (CR), average variance extracted (AVE), and the factor loading of each item. Table 3 shows AVE values for the four constructs, and all of them had values above 0.5, proving that the variance captured by the constructs is larger than the variance due to measurement errors, as stated by [54]. The CR, as [55] mentioned, brings a proportion of variance attributable to only the latent variable

TABLE 2: Exploratory factor analysis results.

	EFA first iteration Components				EFA second iteration Components			
	1	2	3	4	1	2	3	4
par1			0.552				0.521	
par2			0.714				0.615	
par3			0.504	0.485				
par4			0.660				0.766	
com1	0.406	0.236						
com2		0.179		0.489				
com3		0.502				0.582		
com4		0.501				0.563		
cocre1	0.707				0.710			
cocre2	0.766				0.786			
cocre3	0.616				0.608			
cocre4	0.647				0.654			
sat1				0.679				0.667
sat2				0.792				0.746
sat3	0.448			0.457				
Cronbach's alpha	0.890	0.781	0.783	0.822	0.890	0.766	0.743	0.835
Cronbach's alpha (general)	0.926				0.906			
KMO	0.936				0.910			
Bartlett test	$\chi^2 = 3889.291$, gl = 105, $p = 0.000$				$\chi^2 = 2466.155$, gl = 55, $p = 0.000$			
Variance explained	59.34				64.40			

TABLE 3: Confirmatory factor analysis, CR, and AVE.

Constructs	Factor loadings	t-values
Participation (CR = 0.764, AVE = 0.519, Squared Root of AVE = 0.720)		
par1 I put a lot of effort into expressing my personal needs to the staff during the service process[a].	0.753	—
par2 I always provide suggestions to the staff for improving the service outcome.	0.696	10.784
par3 I have a high level of participation in the service process[b].	—	—
par4 I am very much involved in deciding how the services should be provided.	0.711	10.157
Communication (CR = 0.781, AVE = 0.642, Squared Root of AVE = 0.801)		
com1 The information provided by the university can be trusted[b].	—	—
com2 In case of any problem, the university provides me with enough information[b].	—	—
com3 The university allows me to have an interactive communication with it[a].	0.844	—
com4 The university maintains a regular contact with me.	0.756	14.735
Cocreation (CR = 0.892, AVE = 0.676, Squared Root of AVE = 0.822)		
cocre1 Overall, I would describe my relationship with this university as involving a high level of cocreation.	0.866	24.495
cocre2 The final purchase solution was arrived at mainly through the joint effort of the university and myself[a].	0.901	—
cocre3 What I receive from this university is due to work jointly between the university and student.	0.790	19.829
cocre4 I contribute actively to the final solution in the educational service I receive.	0.721	17.235
Satisfaction (CR = 0.839, AVE = 0.723, Squared Root of AVE = 0.850)		
sat1 Overall, I am pleased with the services offered by this university.	0.904	17.844
sat2 The service offered by my university meets my expectations[a].	0.793	—
sat3 I think I did the right thing when I enrolled in this university[b].	—	—

Notes. CR: composite reliability. AVE: average variance extracted.
[a]Initial loading is fixed to 1 to set the scale of the construct.
[b]Deleted after AFE.

Table 4: Means, correlations (above diagonal), and covariance (below diagonal) among construct.

	Mean	Participation	Communication	Cocreation	Satisfaction
Participation	4.63	1	0.680	0.630	0.568
Communication	4.86	0.789	1	0.743	0.770
Cocreation	5.39	0.751	0.945	1	0.806
Satisfaction	5.27	0.633	0.916	0.984	1

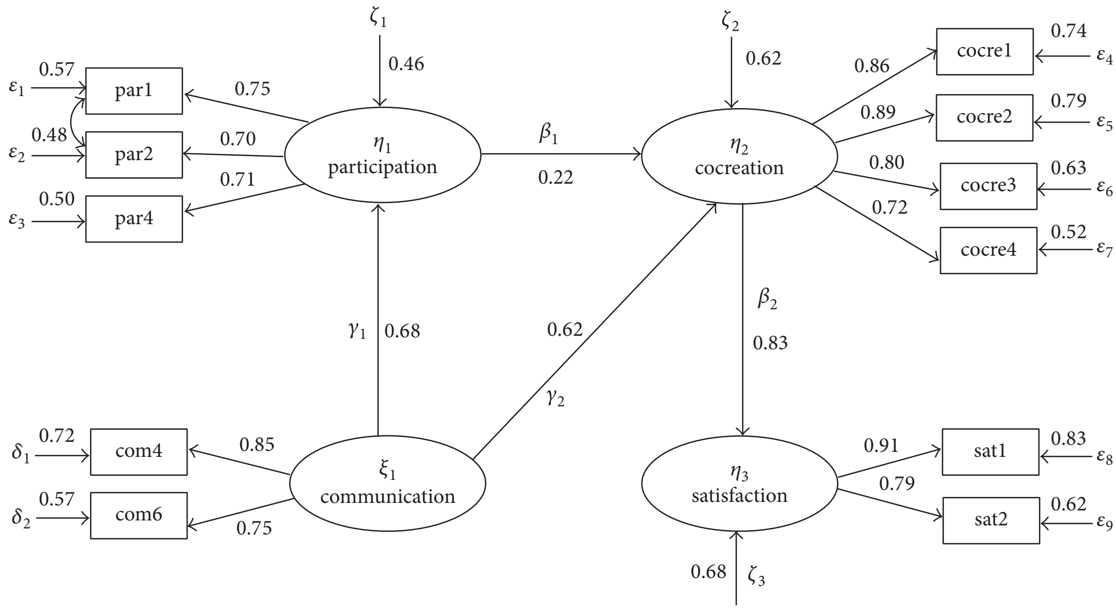


Figure 2: SEM model.

with a recommended value greater than 0.7, putting the four constructs' CR values above the upper bound and confirming the model's reliability. Also, all the factor loadings are higher than 0.5, and the estimated coefficients of each item are all significant (t-value > 2.0; [21, 56]).

In looking for the discriminant validity, we noticed that the square roots of the AVEs had higher values than the correlations among the constructs. For example, cocreation and satisfaction have AVE values of 0.676 and 0.723, the square roots of AVE are 0.822 and 0.850, respectively, and both values are higher than the correlation between cocreation and satisfaction (0.806). The same occurred for the other constructs' relationships, assuring the discriminant validity. These analyses are shown in Tables 3 and 4.

4.3. The Structural Model. The SEM approach was used in order to validate the proposed model and to confirm the relationship between the proposed construct, with the application of the SPSS AMOS software. SEM is widely used to build and validate theories [57, 58]. The SPSS AMOS module was used, since it was primarily created for SEM analysis.

In order to obtain a better model fit, the item errors from par1 and par2 were correlated. Figure 2 shows the results of the SEM model and Table 5 shows the model fit indices and the structural model estimates.

The proposed model fits the data well. The comparative fix index (CFI) had an excellent value (0.962, over 0.95), and the adjusted goodness-of-fit index (AGFI) also had a good value (0.903 > 0.8). The root mean square residual (RMR) was 0.076, under 0.09; the normative fit index (NFI) was 0.948, and the root mean square error of approximation (RMSEA) = 0.077 (less than 0.08; [59]).

The squared multiple correlation (SMC) of cocreation showed that 62% (SMC = 0.623) of this element is explained by the direct effect of participation and the direct and indirect effects of communication, with a high value. Half of the variance of participation (46%; 0.459) was explained by the direct impact of communication; more than half of satisfaction's variance (68%, 0.682) was explained by the direct effect of cocreation.

The four relationship studies have significant and positive impacts such as communication under cocreation with a value of $\gamma = 0.62$ ($p < 0.001$), as many authors had highlighted [21, 26–28]. Communication had a high impact on participation ($\gamma = 0.68$, p value < 0.001; [12, 13, 56]), and participation also had a positive effect on cocreation, though with a lesser impact ($\beta = 0.22$, p value $= 0.003$), supporting the relationship established by authors like in [5, 22, 60]. Cocreation had the highest impact on satisfaction ($\beta = 0.83$, p value < 0.001), as authors like in [24, 61] had remarked.

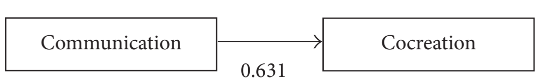
Table 5: Structural model results. Estimates and model fit.

	Hypothesis	Standardized coefficients (β, γ)	SE	p value
Direct effects				
Communication → participation (γ_1)	H1	0.677	0.067	<0.001
Participation → cocreation (β_1)	H2	0.219	0.080	0.003
Communication → cocreation (γ_2)	H3	0.625	0.080	<0.001
Cocreation → satisfaction (β_2)	H4	0.826	0.053	<0.001
Model fit indices				
CMIN/DF	$3.346 \lesssim 3$			
CFI	$0.962 > 0.95$			
GFI	$0.943 \gtrsim 0.95$			
AGFI	$0.903 > 0.8$			
RMR	$0.076 < 0.09$			
RMSEA	$0.077 < 0.08$			

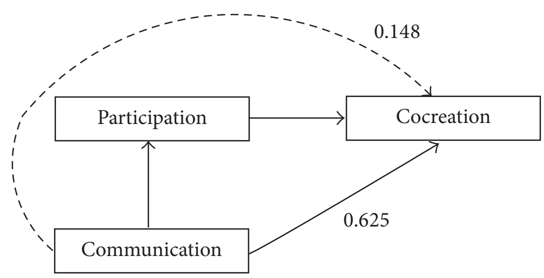
Table 6: Participation mediation between communication and cocreation.

Hypothesis	Direct effect w/o mediator (1st situation)	Direct w mediator (2nd sit.)	Indirect effect (3th sit.)	Mediation type observed
Partial mediation communication, participation, cocreation	0.631^{**}	0.625^{**}	0.148^{*}	Partial mediation

$^{*}p$ value < 0.05; $^{**}p$ value < 0.01.



(a) Without mediator



(b) With participation as a mediator

Figure 3: Direct effect of communication in cocreation.

We researched mediation by participation in Hypothesis H3, studying the relationship between communication and cocreation. Authors like in [62, 63] had pointed to the importance of the mediation analysis. Table 6 and Figure 3 reflect the resulting analysis, where a poor but significant partial mediation existed by participation between communication and cocreation.

The obtained results allowed us to conclude that the four hypotheses raised in the initial phase of the research are accepted. Communication had a positive and significant impact on participation (0.68), and participation had a positive and significant influence on cocreation (0.22). Communication also significantly and positively affected cocreation (0.62), and cocreation in turn affected satisfaction (0.83), with the highest regression coefficient indicating that this relationship was the strongest of all analyzed.

5. Conclusions and Contributions

Taking into account the principal objective of the research, the positive relationships existing between communication and participation, participation and cocreation, communication and cocreation, and lastly cocreation and satisfaction in the undergraduate context were verified. The research also validated a cocreation model, considering that participation and communication were the most important promoters of cocreation; cocreation also had a high impact on student satisfaction. This model assured the importance of a change to a management practice focused on cocreation, as was the original intent.

To face the reality of student satisfaction, higher education institutions are looking for innovative ways to improve their administration. Considering that cocreation had been studied previously by many authors with favorable effects in terms of satisfaction, trust, and loyalty, it is a pragmatic tool to be considered and implemented in the university context. It will be important to notice that the lowest detected interaction was between participation and cocreation. Based on this, undergraduate students mostly valued communication as a cocreation precursor. At this point, universities need to develop open dialogues and bidirectional conversations with students to enhance open talks and forums and to improve the communication channels based on information or virtual systems, Internet, or other portals where the scholar can interact with the school.

Despite satisfaction as a valuable factor in terms of competitive advantage, its existence is essential to obtain high loyalty levels. That is why it would be interesting and innovative to investigate loyalty inclusion as a new construct within the cocreation model aforementioned in further studies. Despite these relationships having been analyzed previously in postgraduate programs, they had never been researched in undergraduate programs.

It will be useful and timely to deepen our understanding of how we must change the institution's process or how to move from the actual vision of rigid value chains to newer ones, with the objective of materializing and concretely practicing the cocreation approach. The benefits of strategic management oriented to this trend have been tested, but the implementation and the actions to be undertaken are a poorly explored field.

It is important to foment and explore methodologies for applying strategies such as cocreation in the university context in order to increase the level of retention, word of mouth, and student loyalty.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

An Efficient Method for Solving Spread Option Pricing Problem: Numerical Analysis and Computing

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This paper deals with numerical analysis and computing of spread option pricing problem described by a two-spatial variables partial differential equation. Both European and American cases are treated. Taking advantage of a cross derivative removing technique, an explicit difference scheme is developed retaining the benefits of the one-dimensional finite difference method, preserving positivity, accuracy, and computational time efficiency. Numerical results illustrate the interest of the approach.

1. Introduction

Multiasset option pricing problem has two main challenges arising from their high dimensions and the correlations between assets. This is a type of option that derives its value from the difference between the prices of two or more assets. Spread options can be written on all types of financial products including equities, bonds, and currencies. Spread options are frequently traded in the energy market [1]. Two examples are as follows.

Crack Spreads. Crack spreads are the options on the spread between refined petroleum products and crude oil. The spread represents the refinement margin made by the oil refinery by "cracking" the crude oil into a refined petroleum product.

Spark Spreads. Spark spreads are the options on the spread between electricity and some type of fuel. The spread represents the margin of the power plant, which takes fuel to run its generator to produce electricity.

Since the closed form solution is not available for such problems, many numerical methods are employed. Probably the most popular methods of solving multiasset option pricing problems are those around Monte Carlo method and its modifications [2, 3]. However, their simulation are usually time consuming and slow convergent [4].

Analytical-numerical methods are also existing in the literature as it occurs in the one asset case. So, Kirk and Aron [5] provide an approximation method to price a bivariate spread option problem, but it is inaccurate when the strike prices are high [1, 6, 7]. Alexander and Venkatramanan improve Kirk's approach in [6] based on the hypothesis that the spread option price is the sum of the prices of two compound exchange options which avoids any strike convention.

Fourier Transform approach and numerical integration techniques are used by Chiarella and Ziveyi in [8, 9] for numerical evaluation of American options written on two underlying assets.

Among the numerical methods we mention the so-called lattice methods [10, 11]. Recently authors in [4] improve the results of [10, 11] and give fast and accurate results although it requires some minor correlation restrictions.

Dealing with pure finite difference methods the existence of the cross derivative term in the partial differential equation (PDE) problem due to asset correlation introduces additional challenges in the numerical solution due to the existence of negative coefficient terms in the numerical scheme as well as involving expensive stencil schemes. Both facts may produce numerical drawbacks coming from the dominance of the convection versus the diffusion as well as more expensive computational cost [12–15].

In order to overcome these difficulties the authors have recently proposed several strategies dealing with finite difference approach: high order compact schemes are proposed in [16] using central difference approximations for stochastic volatility Heston model. The authors in [17] use ADI methods and clever discretizations to obtain stable numerical schemes under minor coefficients restrictions in the PDE. The authors of [14, 18] use special finite difference approximations of the cross derivative term based on a seven-point stencil instead of the nine-point stencil scheme resulting from the central finite difference approach.

In this paper we continue removing cross derivative approach based on the transformation of the PDE problem initiated in [12] for the case of one asset stochastic volatility Heston model and applied to the Bates stochastic volatility jump diffusion model in [19]. Apart from avoiding negative coefficients in the proposed scheme, it has the additional computational advantage that uses only a five-point stencil.

Here we consider spread option pricing problems for both European and American cases. The paper is organized as follows. The first part of the paper deals with European spread call option with payoff $g(S_1, S_2) = (S_1 - S_2 - E)^+$, depending on two assets S_1 and S_2 and strike price $E \geq 0$. In Section 2 the PDE with the boundary conditions is established for the spread option pricing problem. In Section 3 we remove the cross derivative term of the PDE using the canonical form transformation method [20]. In that section we also develop the discretization of the continuous European spread call option problem achieving an explicit finite difference scheme. Section 4 is devoted to the study of the properties of the method like consistence and conditional stability and positivity. In Section 5 we extend the study of previous sections to the European spread put option summarizing the main results including the treatment of the boundary conditions. Section 6 deals with American spread option pricing problem using the LCP technique based on the proposed difference scheme. Finally, in Section 7 we illustrate the numerical results computing, simulating, and comparing accuracy and CPU time with other well acknowledge methods.

Throughout the paper, we use the following notation: $\|A\|_\infty = \max\{|a_{ij}| : \ 1 \leq i \leq N, \ 1 \leq j \leq M\}$ will denote a supremum norm for a given matrix $A = (a_{ij}) \in \mathbb{R}^{N\times M}$.

2. Spread Option Pricing Problem and Boundary Conditions

The price $U(S_1, S_2, \tau)$ of a spread option is the solution of the PDE

$$\begin{aligned} \frac{\partial U}{\partial \tau} &= \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 U}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 U}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 U}{\partial S_1 \partial S_2} \\ &\quad + (r - q_1) S_1 \frac{\partial U}{\partial S_1} + (r - q_2) S_2 \frac{\partial U}{\partial S_2} - rU, \\ &\qquad 0 < S_1 < \infty,\ 0 < S_2 < \infty,\ 0 < \tau \leq T, \end{aligned} \tag{1}$$

where $\tau = T - t$ is the time to maturity T, r is the interest rate, σ_i is the volatility of the asset S_i, q_i is the dividend yield of asset S_i, and ρ is the correlation parameter; see [1]. Since we consider backwards time, it changes the signs of the coefficients of (1) and instead of a terminal condition the following initial condition is considered:

$$\begin{aligned} U(S_1, S_2, 0) = g(S_1, S_2) &= \max(S_1 - S_2 - E, 0) \\ &= (S_1 - S_2 - E)^+. \end{aligned} \tag{2}$$

The case $E = 0$ corresponds to exchange options that can be reduced to 1D problem by the change of variables $x_1 = S_1/S_2$, $c(x_1, \tau) = U(S_1, S_2, \tau)/S_2$.

Since the numerical solution of European multiasset options requires the selection of a bounded numerical domain and the translation of the boundary conditions to the boundary of the domain, it is important to pay attention to such conditions. For the initial problem (1)-(2) suitable boundary conditions areas follows.

(1) For $S_1 = 0$ the payoff $(S_1 - S_2 - E)^+$ suggests Dirichlet's condition

$$U(0, S_2, \tau) = 0, \quad 0 < S_2 < \infty,\ 0 < \tau \leq T, \tag{3}$$

as one-dimensional call option.

(2) Taking the ideas developed by some authors, for instance, Duffy for the case of basket option (see [21], p. 270), for $S_2 = 0$ we take the value of the closed form solution of the basic Black-Scholes equation of a call for S_1 with given strike E,

$$U(S_1, 0, \tau) = e^{-r\tau}(FN(\phi_1) - EN(\phi_2)), \tag{4}$$

where

$$\begin{aligned} F &= S_1 e^{(r-q_1)\tau}, \\ \phi_1 &= \frac{1}{\sigma_1\sqrt{\tau}}\left[\log\frac{F}{E} + \frac{1}{2}\sigma_1^2\tau\right], \\ \phi_2 &= \phi_1 - \sigma_1\sqrt{\tau}, \end{aligned} \tag{5}$$

where $N(x)$ is the standard normal cumulative distribution function.

(3) As S_1 is large versus $S_2 + E$, then the behaviour of the solution looks like the asymptotic value of a one-dimensional vanilla call option with the strike $(S_2 + E)$ for $S_1 \gg S_2 + E$ (see [22], p. 157),

$$U(S_1, S_2, \tau) \approx e^{-q_1\tau} S_1 - e^{-r\tau}(S_2 + E), \quad S_1 \gg S_2 + E. \tag{6}$$

(4) For large values of S_2 we assume that the values are almost constants when S_2 changes; therefore we can use Neuman's boundary condition:

$$\frac{\partial U}{\partial S_2} = 0. \tag{7}$$

These boundary conditions will be validated with the numerical examples.

3. Removing the Correlation Term and Problem Discretization

We begin this section by transforming the PDE problem (1) in order to remove the cross derivative term. Firstly we eliminate the reaction term rU by means of the substitution

$$V = e^{r\tau} U, \tag{8}$$

obtaining

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} \\ &\quad + (r - q_1) S_1 \frac{\partial V}{\partial S_1} + (r - q_2) S_2 \frac{\partial V}{\partial S_2}. \end{aligned} \tag{9}$$

In order to remove the cross derivative term, note that the right-hand side of (9) is a linear differential operator of two variables and classical techniques to obtain the canonical form of second-order linear PDEs can be applied; see, for instance, chapter 3 of [20]. Under the assumption of correlated variables with $-1 < \rho < 1$ the sign of the discriminant is

$$a_{12}^2 - 4a_{11}a_{22} = \sigma_1^2\sigma_2^2 S_1^2 S_2^2 (\rho^2 - 1) < 0, \tag{10}$$

where

$$\begin{aligned} a_{11} &= \frac{1}{2}\sigma_1^2 S_1^2; \\ a_{12} &= \rho\sigma_1\sigma_2 S_1 S_2; \\ a_{22} &= \frac{1}{2}\sigma_2^2 S_2^2. \end{aligned} \tag{11}$$

Therefore, right-hand side of (9) becomes of elliptic type and a convenient substitution to obtain the canonical form is given by solving the ordinary differential equation

$$\frac{dS_2}{dS_1} + \frac{\sigma_2 S_2}{\sigma_1 S_1}(-\rho \pm i\tilde{\rho}) = 0, \tag{12}$$

where

$$\tilde{\rho} = \sqrt{1 - \rho^2}, \tag{13}$$

and $(\sigma_2 S_2/\sigma_1 S_1)(-\rho \pm i\tilde{\rho})$ are the conjugate roots of $a_{11}x^2 + a_{12}x + a_{22} = 0$. Solving (12) one gets

$$\frac{1}{\sigma_2}\log S_2 + \frac{-\rho \pm i\tilde{\rho}}{\sigma_1}\log S_1 = C_0, \tag{14}$$

where one can relate the integration constant C_0 to the new variables by

$$\begin{aligned} x &= \operatorname{Im}(C_0), \\ y &= -\operatorname{Re}(C_0). \end{aligned} \tag{15}$$

From (14) and (15) it follows the expression of the new variables

$$\begin{aligned} x &= \frac{\tilde{\rho}}{\sigma_1}\log S_1; \\ y &= \frac{\rho}{\sigma_1}\log S_1 - \frac{1}{\sigma_2}\log S_2. \end{aligned} \tag{16}$$

Note that

$$y - mx = -\frac{1}{\sigma_2}\log S_2, \quad m = \frac{\rho}{\tilde{\rho}}. \tag{17}$$

By denoting

$$W(x, y, \tau) = V(S_1, S_2, \tau), \tag{18}$$

(9) takes the following form without cross derivative term

$$\begin{aligned} \frac{\partial W}{\partial \tau} &= \frac{\tilde{\rho}^2}{2}\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) + \frac{\tilde{\rho}}{\sigma_1}\left(r - q_1 - \frac{\sigma_1^2}{2}\right)\frac{\partial W}{\partial x} \\ &\quad + \left(\frac{\rho}{\sigma_1}\left(r - q_1 - \frac{\sigma_1^2}{2}\right) - \frac{1}{\sigma_2}\left(r - q_2 - \frac{\sigma_2^2}{2}\right)\right)\frac{\partial W}{\partial y}, \\ &\qquad (x, y) \in \mathbb{R}^2, \ 0 < \tau \le T. \end{aligned} \tag{19}$$

In accordance with [23, 24] and (6) we choose the rectangular domain $(S_1, S_2) \in [a_1, b_1] \times [a_2, b_2] = \Omega$, where $a_i > 0$, $i = 1, 2$, are small positive values; b_2 about $3E$ and b_1 about $3(b_2 + E)$. For the transformed problem (19) due to (16), the rectangular domain Ω becomes a rhomboid with vertices $ABCD$ (see Figure 1). From (16) let us denote

$$\begin{aligned} c_1 &= \frac{\tilde{\rho}}{\sigma_1}\log a_1, \\ c_2 &= \frac{\log a_2}{\sigma_2}, \\ d_1 &= \frac{\tilde{\rho}}{\sigma_1}\log b_1, \\ d_2 &= \frac{\log b_2}{\sigma_2}. \end{aligned} \tag{20}$$

By (17) the rhomboid domain has the sides:

$$\begin{aligned} \overline{AD} &= \{(x, y) : x = c_1, \ mc_1 - d_2 \le y \le mc_1 - c_2\}, \\ \overline{AB} &= \{(x, y) : c_1 \le x \le d_1, y = mx - d_2\}, \\ \overline{BC} &= \{(x, y) : x = d_1, md_1 - d_2 \le y \le md_1 - c_2\}, \\ \overline{DC} &= \{(x, y) : c_1 \le x \le d_1, y = mx - c_2\}. \end{aligned} \tag{21}$$

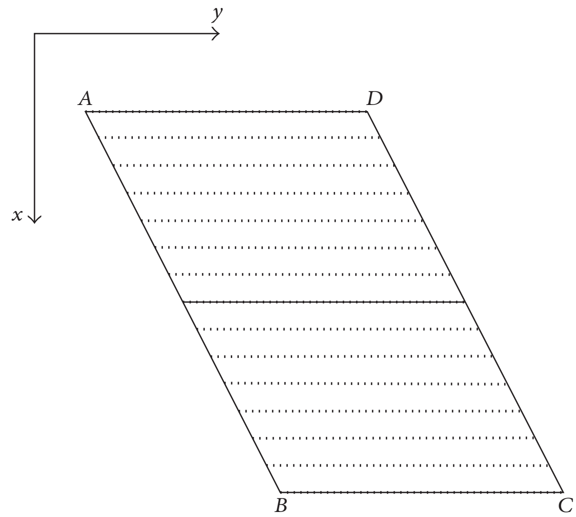

FIGURE 1: Transformed numerical domain.

The mesh points in the rhomboid domain are (x_i, y_j) such that

$$
\begin{aligned}
x_i &= c_1 + ih, \quad x_i \in [c_1, d_1], \quad 0 \le i \le N_x, \\
\Delta_x &= h = \frac{d_1 - c_1}{N_x}, \\
\Delta_y &= |m|\,h, \\
N_y &= \frac{d_2 - c_2}{|m|\,h}, \\
y_j &= mx_i - d_2 + (j - i)\,|m|\,h, \quad i \le j \le N_y + i.
\end{aligned} \tag{22}
$$

The approximations of the derivatives appearing in (19) at the point (x_i, y_j, τ^n) are as follows:

$$
\begin{aligned}
\frac{\partial W}{\partial x} &\sim \frac{w^n_{i+1,j} - w^n_{i-1,j}}{2h}, \\
\frac{\partial W}{\partial y} &\sim \frac{w^n_{i,j+1} - w^n_{i,j-1}}{2\,|m|\,h}, \\
\frac{\partial^2 W}{\partial x^2} &\sim \frac{w^n_{i+1,j} - 2w^n_{i,j} + w^n_{i-1,j}}{h^2}, \\
\frac{\partial^2 W}{\partial y^2} &\sim \frac{w^n_{i,j+1} - 2w^n_{i,j} + w^n_{i,j-1}}{m^2h^2}, \\
\frac{\partial W}{\partial \tau} &\sim \frac{w^{n+1}_{i,j} - w^n_{i,j}}{k},
\end{aligned} \tag{23}
$$

where $w^n_{i,j} \sim W(x_i, y_j, \tau^n)$, $\tau^n = nk$.

Substituting these finite difference approximations of the derivatives into (19) is approximated by the explicit difference scheme with five-point stencil

$$
\begin{aligned}
w^{n+1}_{i,j} &= \alpha_1 w^n_{i-1,j} + \alpha_2 w^n_{i+1,j} + \alpha_3 w^n_{i,j} + \alpha_4 w^n_{i,j-1} \\
&\quad + \alpha_5 w^n_{i,j+1},
\end{aligned} \tag{24}
$$

where

$$
\alpha_{1,2} = \frac{\tilde{\rho}^2}{2}\frac{k}{h^2} \mp \frac{\tilde{\rho}}{\sigma_1}\left(r - q_1 - \frac{\sigma_1^2}{2}\right)\frac{k}{2h}, \tag{25}
$$

$$
\alpha_3 = 1 - \tilde{\rho}^2\frac{k}{h^2}\left(1 + \frac{1}{m^2}\right), \tag{26}
$$

$$
\begin{aligned}
\alpha_{4,5} &= \frac{\tilde{\rho}^2 k}{2m^2h^2} \\
&\quad \mp \left[\frac{\tilde{\rho}\left(r - q_1 - \sigma_1^2/2\right)}{\sigma_1} - \frac{r - q_2 - \sigma_2^2/2}{\sigma_2}\right]\frac{k}{2\,|m|\,h}.
\end{aligned} \tag{27}
$$

The discretization of the spatial boundary is given by

$$
\begin{aligned}
P\left(\overline{AD}\right) &= \left\{\left(x_0, y_j\right) : 0 \le j \le N_y\right\}, \\
P\left(\overline{AB}\right) &= \left\{\left(x_i, y_i\right) : 0 \le i \le N_x\right\}, \\
P\left(\overline{BC}\right) &= \left\{\left(x_{N_x}, y_j\right) : N_x \le j \le N_x + N_y\right\}, \\
P\left(\overline{DC}\right) &= \left\{\left(x_i, y_{N_y+i}\right) : 0 \le i \le N_x\right\}.
\end{aligned} \tag{28}
$$

Both initial and boundary conditions (2), (3), (4), (6), and (7) are transformed throughout (8), (16), (18). For the initial condition we have

$$
\begin{aligned}
w^0_{i,j} &= \left(e^{\sigma_1 x_i/\tilde{\rho}} - e^{-\sigma_2(y_j - mx_i)} - E\right)^+, \\
&\qquad 0 \le i \le N_x, \; i \le j \le i + N_y.
\end{aligned} \tag{29}
$$

Boundary condition for side $\overline{AD}$ is as follows:

$$
w^n_{0,j} = 0, \quad 0 \le j \le N_y, \; 1 \le n \le N_\tau. \tag{30}
$$

For $\overline{DC}$ with small value of S_2, we have transformed Black-Scholes solution, so

$$
\begin{aligned}
w^n_{i,N_y+i} &= F^n_i N\left(\phi^n_{1,i}\right) - EN\left(\phi^n_{2,i}\right), \\
&\qquad 0 < i < N_x, \; 1 \le n \le N_\tau,
\end{aligned} \tag{31}
$$

where

$$
\begin{aligned}
F^n_i &= \exp\left(\frac{\sigma_1}{\tilde{\rho}}x_i + (r - q_1)\,\tau^n\right), \\
\phi^n_{1,i} &= \frac{1}{\sigma_1\sqrt{\tau^n}}\left[\log\frac{F_i}{E} + \frac{1}{2}\sigma_1^2\tau^n\right], \\
\phi^n_{2,i} &= \phi^n_{1,i} - \sigma_1\sqrt{\tau^n}.
\end{aligned} \tag{32}
$$

Side $\overline{BC}$ corresponds to large values of S_1, so behaviour of the option there leads to

$$
\begin{aligned}
w^n_{N_x,j} &= \exp\left(\frac{\sigma_1}{\tilde{\rho}}x_{N_x} + (r - q_1)\,\tau^n\right) \\
&\quad - \left(\exp\left(\sigma_2\left(mx_{N_x} - y_j\right)\right) + E\right), \\
&\qquad N_x + 1 \le j \le N_x + N_y, \; 1 \le n \le N_\tau.
\end{aligned} \tag{33}
$$

Boundary $\overline{AB}$ corresponds with the boundary for large values of S_2. Condition (7) means that component S_2 is constant and the null first derivative is approximated by the forward difference $(w_{i,i+1}^n - w_{i,i}^n)/|m|h = 0$,

$$w_{i,i}^n = w_{i,i+1}^n, \quad 1 \le i \le N_x, \ 1 \le n \le N_\tau. \tag{34}$$

4. Positivity, Stability, and Consistency

In this section we show that the proposed numerical scheme (24)–(27) presents suitable qualitative and computational properties, such as consistency and conditional positivity and stability.

In order to guarantee positivity of the numerical solution let us check the positivity of the coefficients α_i.

From (25) and (27) it is easy to check that coefficients α_1, α_2, α_4, and α_5 of the scheme (24) are positive under the condition

$$h < \min\left\{\frac{\widetilde{\rho}\sigma_1}{\left|r - q_1 - \sigma_1^2/2\right|}; \frac{\widetilde{\rho}^2}{\left|m\left[\widetilde{\rho}\left(r - q_1 - \sigma_1^2/2\right)/\sigma_1 - \left(r - q_2 - \sigma_2^2/2\right)/\sigma_2\right]\right|}\right\}. \tag{35}$$

Coefficient α_3 is positive, if

$$k < \frac{m^2h^2}{\widetilde{\rho}^2\left(m^2+1\right)}. \tag{36}$$

Under conditions (35) and (36), values in interior points of the numerical domain $w_{i,j}^{n+1}$, $1 \le i \le N_x - 1$; $i+1 \le j \le N_y + i - 1$, calculated by the scheme (24), preserve nonnegativity from the previous time level n at all interior and boundary points $w_{i,j}^n \ge 0$, $0 \le i \le N_x$; $i \le j \le N_y + i$.

Let us pay attention now to the positivity of the numerical solution at the boundary of the numerical domain. On the boundary $\overline{AD}$ from (30) we have zero values. On the boundary $\overline{DC}$ formula (31) is used preserving the positivity since it is the transformed Black-Scholes formula throughout expression (16). Along the line $\overline{AB}$ we use Neumann boundary condition (34). Therefore, the positivity at the neighbour interior point guarantees the positivity at the boundary. Finally, on the line $\overline{BC}$ boundary conditions are determined by formula (33). This is transformed expression for the asymptotic behaviour of the call option (6) that is positive under the condition

$$S_{1_{\max}} > \left(S_{2_{\max}} + E\right)\exp\left(\left(q_1 - r\right)\tau^n\right), \quad 1 \le n \le N_\tau. \tag{37}$$

Following the ideas of Kangro and Nicolaides [24], the computational domain has to be large enough to translate the boundary conditions; therefore

$$S_{1_{\max}} \ge \max\left\{\left(S_{2_{\max}} + E\right)\exp\left(\left(q_1 - r\right)T\right), 3\left(S_{2_{\max}} + E\right)\right\}. \tag{38}$$

Therefore, choosing $S_{1_{\max}}$ according to (38) for the transformed problem $w_{N_x,j}^n \ge 0$ for any $N_x + 1 \le j \le N_x + N_y$, $1 \le n \le N_\tau$.

In summary, the nonnegativity of the numerical solution follows from the nonnegativity of the initial values and positivity preservation under the scheme and boundary conditions.

As authors manage several stability criteria in literature, we recall the concept of stability used in this paper.

Definition 1. Let $w^n \in \mathbb{R}^{(N_x+1)\times(N_y+1)}$ be the matrix involving the numerical solution $\{w_{i,j}^n\}$ of PDE problem (19) computed by scheme (24), (29), (30), (31), (33), and (34),

$$w^n = \left(w_{i,j}^n\right)_{0\le i\le N_x, i\le j\le N_y+i}, \quad 0 \le n \le N_\tau. \tag{39}$$

The numerical scheme (24), (29), (30), (31), (33), and (34) is said to be uniformly $\|\cdot\|_\infty$-stable in the domain $[c_1, d_1] \times [c_2, d_2] \times [0, T]$, if for every partition with k, h as defined above, condition

$$\left\|w^n\right\|_\infty \le A, \quad 0 \le n \le N_\tau, \tag{40}$$

holds true, where A is independent of h, k. When the scheme is uniformly stable under a condition imposed to the step size discretization, then the scheme is said to be *conditionally* uniformly stable.

Let us find the maximum value of the $w_{i,j}^n$ with respect to i, j for each fixed n. For $n = 0$ the values $w_{i,j}^n$ are defined by the initial condition (29). Let us consider the following function:

$$f(x, y) = \exp\left(\frac{\sigma_1}{\widetilde{\rho}}x\right) - \exp\left(-\sigma_2\left(y - mx\right)\right) - E \tag{41}$$

and find the maximum value on the domain $[c_1, d_1] \times [c_2, d_2]$. The first derivatives are

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\sigma_1}{\widetilde{\rho}}\exp\left(\frac{\sigma_1}{\widetilde{\rho}}x\right) + \sigma_2 m \exp\left(-\sigma_2\left(y - mx\right)\right) > 0,\\ \frac{\partial f}{\partial y} &= \sigma_2 \exp\left(-\sigma_2\left(y - mx\right)\right) > 0.\end{aligned} \tag{42}$$

Analysing (42) one concludes that the function $f(x, y)$ is increasing with respect to x and y. Therefore, the maximum value is reached at the point (d_1, d_2):

$$\begin{aligned}\max_{(x,y)\in[c_1,d_1]\times[c_2,d_2]} f(x, y) &= f(d_1, d_2)\\ &= \exp\left(\frac{\sigma_1}{\widetilde{\rho}}d_1\right) - \exp\left(-\sigma_2\left(d_2 - md_1\right)\right) - E = C\\ &= \text{const.}\end{aligned} \tag{43}$$

For the next time levels $n \ge 1$ let us consider the numerical scheme (24). Under conditions (35) and (36) the coefficients $\alpha_i > 0$, $i = 1, \ldots, 5$. Moreover, it is easy to check that

$$\sum_{i=1}^{5}\alpha_i = 1. \tag{44}$$

Therefore the values in interior points at the $(n+1)$th level, $n \geq 0$, can be bounded by the following:

$$w_{i,j}^{n+1} \leq \max\left\{w_{i-1,j}^{n}, w_{i+1,j}^{n}, w_{i,j}^{n}, w_{i,j-1}^{n}, w_{i,j+1}^{n}\right\} \leq \max_{i,j} w_{i,j}^{n}. \quad (45)$$

This maximum can be reached on the boundary. Therefore, let us study the boundary conditions. From (30) one gets that

$$\max_{\overline{AD}} w_{i,j}^{n+1} = 0. \quad (46)$$

The values on the boundary $\overline{AB}$ are equal to the interior points; therefore the inequality (45) can be applied.

The values on the boundary $\overline{BC}$ are increasing with respect to the index j and reach the maximum at the point $(N_x, N_x + N_y - 1)$. From the other side, this value can be bounded by the value at the point $(N_x, N_x + N_y)$ that is calculated by formula (31). From the theory of option pricing, the price of European call is increasing with respect to the asset price S and, moreover, it is always greater than its asymptotic function (see [25], p. 268, formula (13.1)). The proposed transformation preserves monotonicity; therefore,

$$\begin{aligned} w_{N_x,N_x+N_y}^{n+1} &= F_{N_x}^{n+1} N\left(\alpha_{1,N_x}^{n+1}\right) - EN\left(\alpha_{2,N_x}^{n+1}\right) \\ &\geq e^{((\sigma_1/\widetilde{\rho})x_{N_x}+(r-q_1)\tau^{n+1})} \\ &\quad - \left(e^{(\sigma_2(m x_{N_x} - y_{N_x+N_y-1}))} + E\right) \\ &= w_{N_x,N_x+N_y-1}^{n+1}. \end{aligned} \quad (47)$$

In summary we can conclude that

$$\max_{i,j} w_{i,j}^{n+1} = w_{N_x,N_x+N_y}^{n+1}. \quad (48)$$

This value is transformed call option price for the asset price $S_{1_{\max}}$, that is, the solution of the 1D Black-Scholes equation at the fixed point. European call option gives a right to the option holder to purchase one share stock for a certain price. The option itself cannot be worth more than the stock. In other words,

$$\begin{aligned} U(S_1, 0, \tau) &= e^{-q_1\tau} S_1 N(d_1) - e^{-r\tau} EN(d_2) \\ &\leq e^{-q\tau} S_1 N(d_1) \leq e^{-q\tau} S_1, \\ w_{N_x,N_x+N_y}^{n+1} &\leq e^{r\tau^{n+1}} e^{-q_1\tau^{n+1}} S_{1_{\max}} \leq e^{|r-q_1|T} S_{1_{\max}}, \\ & \qquad 0 \leq n \leq N_\tau - 1. \end{aligned} \quad (49)$$

Then the following result can be stated.

Theorem 2. *With previous notations, under conditions (35) and (36), numerical scheme (24), (29), (30), (31), (33), and (34) is conditionally uniformly* $\|\cdot\|_\infty$*-stable with upper bound* $A = e^{|r-q_1|T} S_{1_{\max}}$.

Now consistency of the numerical scheme (24) with PDE (19) will be studied [26].

Let $F(w_{i,j}^n) = 0$ be approximating difference equation (24). The scheme (24) is consistent with (19) if the local truncation error $T_{i,j}^n$ satisfies

$$T_{i,j}^n = F\left(W_{i,j}^n\right) - L\left(W_{i,j}^n\right) \longrightarrow 0 \quad \text{as } h \longrightarrow 0,\ k \longrightarrow 0, \quad (50)$$

where $W_{i,j}^n$ denotes the theoretical solution of (19) evaluated at the point (x_i, y_j, τ^n) and L is the operator

$$\begin{aligned} L(W) &= \frac{\partial W}{\partial \tau} - \frac{\widetilde{\rho}^2}{2}\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) \\ &\quad - \frac{\widetilde{\rho}}{\sigma_1}\left(r - q_1 - \frac{\sigma_1^2}{2}\right)\frac{\partial W}{\partial x} \\ &\quad - \left(\frac{\rho}{\sigma_1}\left(r - q_1 - \frac{\sigma_1^2}{2}\right) - \frac{1}{\sigma_2}\left(r - q_2 - \frac{\sigma_2^2}{2}\right)\right)\frac{\partial W}{\partial y}. \end{aligned} \quad (51)$$

Assuming that theoretical solution W of the PDE problem is four times continuously differentiable with respect to x and y and twice with respect to τ and using Taylor expansion about (x_i, y_j, τ^n) it is easy to check that

$$\frac{W_{i,j}^{n+1} - W_{i-1,j}^{n}}{k} = \frac{\partial W}{\partial \tau}\left(x_i, y_j, \tau^n\right) + kE_{i,j}^n(1), \quad (52)$$

where

$$E_{i,j}^n(1) = \frac{1}{2}\frac{\partial^2 W}{\partial \tau^2}\left(x_i, y_j, \delta\right), \quad nk < \delta < (n+1)k, \quad (53)$$

$$\begin{aligned} \left|E_{i,j}^n(1)\right| &\leq \frac{1}{2}\left|D_{i,j}^n(1)\right|_{\max} \\ &= \frac{1}{2}\max\left\{\left|\frac{\partial^2 W}{\partial \tau^2}\left(x_i, y_j, \delta\right)\right|;\ 0 < \delta < T\right\}. \end{aligned} \quad (54)$$

For the spatial discretization one gets

$$\frac{W_{i+1,j}^n - W_{i-1,j}^n}{2h} = \frac{\partial W}{\partial x}\left(x_i, y_j, \tau^n\right) + h^2 E_{i,j}^n(2), \quad (55)$$

where

$$E_{i,j}^n(2) = \frac{1}{6}\frac{\partial^3 W}{\partial x^3}\left(\xi, y_j, \tau^n\right), \quad x_{i-1} < \xi < x_{i+1}, \quad (56)$$

$$\begin{aligned} \left|E_{i,j}^n(2)\right| &\leq \frac{1}{6}\left|D_{i,j}^n(2)\right|_{\max} \\ &= \frac{1}{6}\max\left\{\left|\frac{\partial^3 W}{\partial x^3}\left(\xi, y_j, \tau^n\right)\right|;\ c_1 < \xi < d_1\right\}. \end{aligned} \quad (57)$$

Analogously,

$$\frac{W_{i,j+1}^n - W_{i,j-1}^n}{2|m|h} = \frac{\partial W}{\partial y}\left(x_i, y_j, \tau^n\right) + h^2 E_{i,j}^n(3), \quad (58)$$

where

$$E_{i,j}^n(2) = \frac{m^2}{6}\frac{\partial^3 W}{\partial y^3}(x_i, \eta, \tau^n), \quad y_{j-1} < \eta < y_{j+1}; \tag{59}$$

thus

$$\left|E_{i,j}^n(3)\right| \le \frac{m^2}{6}\max\left\{\left|\frac{\partial^3 W}{\partial y^3}(x_i, \eta, \tau^n)\right|;\ c_2 < \eta < d_2\right\}. \tag{60}$$

The discretization of the second spatial derivatives is as follows:

$$\frac{W_{i+1,j}^n - 2W_{i,j}^n + W_{i-1,j}^n}{h^2} = \frac{\partial^2 W}{\partial x^2}(x_i, y_j, \tau^n) + h^4 E_{i,j}^n(4), \tag{61}$$

where

$$E_{i,j}^n(4) = \frac{1}{12}\frac{\partial^4 W}{\partial x^4}(\xi, y_j, \tau^n), \quad x_{i-1} < \xi < x_{i+1}, \tag{62}$$

$$\left|E_{i,j}^n(4)\right| \le \frac{1}{12}\max\left\{\left|\frac{\partial^4 W}{\partial x^4}(\xi, y_j, \tau^n)\right|;\ c_1 < \xi < d_1\right\}. \tag{63}$$

Analogously,

$$\frac{w_{i,j+1}^n - 2w_{i,j}^n + w_{i,j-1}^n}{m^2h^2} = \frac{\partial^2 W}{\partial y^2}(x_i, y_j, \tau^n) + h^4 E_{i,j}^n(5), \tag{64}$$

where

$$E_{i,j}^n(5) = \frac{m^4}{12}\frac{\partial^4 W}{\partial y^4}(x_i, \eta, \tau^n), \quad y_{j-1} < \eta < y_{j+1}, \tag{65}$$

$$\left|E_{i,j}^n(5)\right| \le \frac{m^4}{12}\max\left\{\left|\frac{\partial^3 W}{\partial y^3}(x_i, \eta, \tau^n)\right|;\ c_2 < \eta < d_2\right\}. \tag{66}$$

Then the local truncation error takes the following form:

$$\begin{aligned} T_{i,j}^n &= F(W_{i,j}^n) - L(W_{i,j}^n) = kE_{i,j}^n(1) - \frac{\widetilde{\rho}^2}{2} \\ &\cdot h^4\left(E_{i,j}^n(4) + E_{i,j}^n(5)\right) - \frac{\widetilde{\rho}}{\sigma_1}\left(r - q_1 - \frac{\sigma_1^2}{2}\right) \\ &\cdot h^2 E_{i,j}^n(2) \\ &- \left(\frac{\rho}{\sigma_1}\left(r - q_1 - \frac{\sigma_1^2}{2}\right) - \frac{1}{\sigma_2}\left(r - q_2 - \frac{\sigma_2^2}{2}\right)\right) \\ &\cdot h^2 E_{i,j}^n(3). \end{aligned} \tag{67}$$

From (54), (57), (60), (63), and (66) one gets

$$\left|T_{i,j}^n\right| = O(k) + O(h^2). \tag{68}$$

Theorem 3. *Assuming that the solution of the PDE (19) admits two times continuous partial derivative with respect to time and up to order four with respect to each of space directions solution, the numerical solution computed by scheme (24) is consistent with (19).*

5. European Spread Put Option Case

For the sake of brevity and in order to avoid repetition of proofs we summarize in this section the results proved in previous sections for the call option case.

Let us consider a European spread put option with payoff

$$U(S_1, S_2, 0) = ((S_2 + E) - S_1)^+, \quad E > 0. \tag{69}$$

Equation (1) and all the transformed versions of it do not change for the put option. Like in the study of the previous call option case, due to different nature of the put option, we need to establish the boundary conditions and translate them to the boundary of the numerical domain.

(1) If $S_2 = 0$ the problem is reduced to the European put option on asset S_1 and strike E. Therefore, the option value on this boundary is

$$U(S_1, 0, \tau) = Ee^{-r\tau}N(-\phi_2) - S_1 e^{q_1\tau}N(-\phi_1), \tag{70}$$

where ϕ_1 and ϕ_2 are defined by (5).

(2) If $S_1 = 0$, we consider $U(0, S_2, \tau)$ as a put option price on asset $S_1 = 0$ with the strike $S_2 + E$:

$$U(0, S_2, \tau) = (S_2 + E)e^{-r\tau}. \tag{71}$$

(3) For large values of S_2 we suppose that the slope of the curve tends to 1; that is,

$$\frac{\partial U}{\partial S_2}(S_1, S_2 \longrightarrow \infty, \tau) = e^{-q_2\tau}. \tag{72}$$

(4) For $S_1 \gg S_2 + E$ we consider $U(S_1, S_2, \tau)$ as a put option price with large values of S_1, and therefore

$$U(S_1, S_2, \tau) = 0. \tag{73}$$

Under transformations (8) and (16) the boundary conditions (70)–(73) are translated to the rhomboid numerical domain $ABCD$ as follows:

$$\begin{aligned} \overline{DC} : w_{i,N_y+i}^n &= EN(-\alpha_{2,i}) \\ &+ e^{(\sigma_1/\widetilde{\rho})x_i - q_1\tau^n}N(-\alpha_{1,i}); \end{aligned} \tag{74}$$

$$\overline{AD} : w_{0,j}^n = E + e^{-\sigma_2(y_j - mx_0)}; \tag{75}$$

$$\overline{AB} : w_{i,i}^n = w_{i,i+1}^n + |m|\, h\sigma_2 e^{-\sigma_2(y_{i+1} - mx_i) - r\tau}; \tag{76}$$

$$\overline{BC} : w_{N_x+1,N_x+j}^n = 0. \tag{77}$$

Positivity. As the numerical scheme (24) for the interior points is the same and the new boundary conditions (74)–(77) are nonnegative, then under conditions (35) and (36) the numerical solution for the European spread put option is nonnegative.

Stability. Stability of the scheme is preserved with new boundary conditions (74)–(77), because analogously to the call option case the maximum value of the transformed function $W(x, y, \tau)$ is reached at a corner (in this case (c_1, d_2), where we use the boundary condition (75)). At the corresponding point of the original domain we use boundary condition (71). In the original coordinates it can be bounded as

$$U(0, S_2, \tau) = (S_2 + E)e^{-r\tau} \le S_{2_{\max}} + E = 4E = \text{const.} \quad (78)$$

Therefore, the transformed function also can be bounded as

$$\max_{i,j} w_{i,j}^n = 4Ee^{rT}. \quad (79)$$

6. American Spread Options

In the case of American type option the holder has the right to exercise option at any moment before expiring. It leads to a free boundary problem [27]. In order to simplify the computational procedure this problem can be considered as a linear complementarity problem (LCP):

$$\begin{gathered} \left(\frac{\partial U}{\partial \tau} - LU\right)(U - g) = 0, \\ \frac{\partial U}{\partial \tau} - LU \ge 0, \\ U - g(S_1, S_2) \ge 0, \end{gathered} \quad (80)$$

where L represents the spatial differential operator of (1).

If we rewrite (19) in the form

$$\frac{\partial W}{\partial \tau} - \mathscr{L}W = 0, \quad (81)$$

then under transformation (16) LCP (80) has the following form:

$$\begin{gathered} \left(\frac{\partial W}{\partial \tau} - \mathscr{L}W\right)(W - f(x, y)) = 0, \\ \frac{\partial W}{\partial \tau} - \mathscr{L}W \ge 0; \\ W - f(x, y) \ge 0, \end{gathered} \quad (82)$$

where $\mathscr{L}$ is the right-hand side of (19) and $f(x, y)$ is given by (41).

Numerical solution for problem (82) can be easily obtained by a small modification of the calculating procedure described above in Sections 3 and 4. On each time level

$$\widetilde{w}_{i,j}^n = \max\left\{w_{i,j}^n, f(x_i, y_j)\right\} \ge 0, \quad (83)$$

where $w_{i,j}^n$ is defined by the finite difference equation (24).

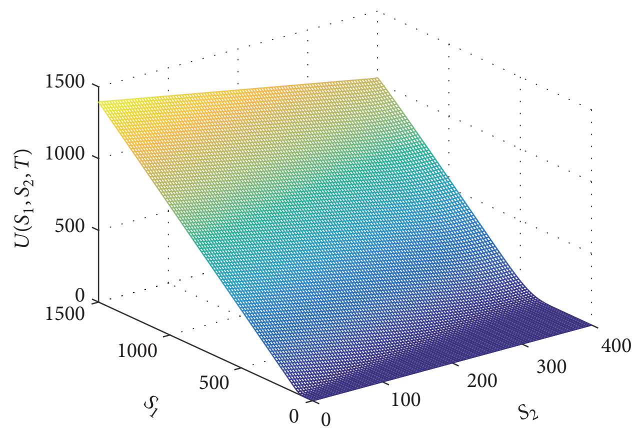

Figure 2: Parameters h and k satisfy (85).

7. Numerical Examples

In this section we illustrate numerical results for both European and American spread options.

In Example 1 we show that the stability conditions (35) and (36) for the European spread option can not be disregarded.

Example 1. We tested the algorithm for the European spread call option pricing problem with the parameters

$$\begin{gathered} T = 1; \\ E = 100; \\ \sigma_1 = 0.2; \\ \sigma_2 = 0.1; \\ r = 0.1; \\ q_1 = 0.01; \\ q_2 = 0.05; \\ \rho = 0.1; \end{gathered} \quad (84)$$

and taking S_1 and S_2 large enough, in particular, $S_{2_{\max}} = 400$; $S_{1_{\max}} = 3(S_{2_{\max}} + E) = 1500$.

For parameters (84) the stability conditions (35) and (36) on space and time steps are as follows:

$$h < \min\{2.8428; 23.7358\} = 2.8428, \quad k < 0.0112. \quad (85)$$

These conditions are crucial for the qualitative properties such as positivity and stability of the scheme. In Figure 2 the numerical solution with h and k satisfying (85) is presented. On the other hand, in Figure 3 there is a numerical simulation with time step $k = 0.012 > 0.0112$. The solution is unstable and it has negative values.

Example 2 illustrates the computational time as well as the comparison with analytical approximation, provided by [6].

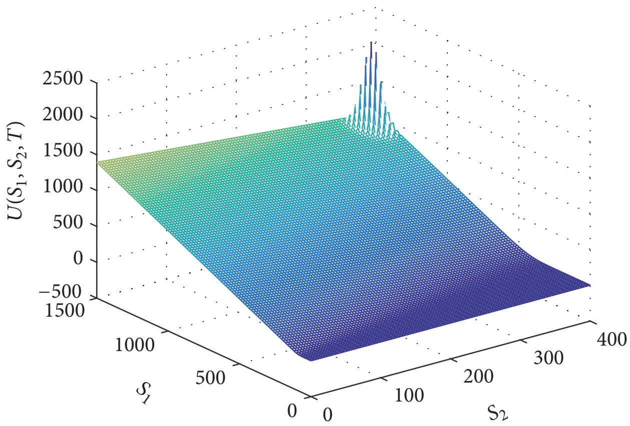

Figure 3: Stability condition is broken.

Table 1: Comparison proposed method (FDM) with analytical approximation (86).

S_2	S_1	FDM	Approximation
100	400	210.5978	209.7261
	200	24.2895	23.4543
	100	0.1150	0.0123
	50	$3.1179e-05$	$-4.7741e-10$
90	400	220.0678	220.1714
	200	29.8745	29.8724
	100	0.1691	0.0372
	50	$5.2150e-05$	$-1.9574e-09$
	CPU time (sec)	106.188	929.467

Example 2. Consider the European spread option with the data of Example 1 (84) and choosing step sizes satisfying the stability condition.

The price of European spread option can be expressed as the price of a compound exchange option (see [6]). The known derived analytical approximation for the spread option reads

$$U(S_1, S_2, T) = S_1 e^{-q_1 T} N(d_1) - \left(Ee^{-rT} + S_2\right) e^{-(r-\tilde{r}-\tilde{q}_2)T} N(d_2), \quad (86)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}} \left(\ln \frac{S_1}{S_2 + Ee^{-rT}} + \left(r - \tilde{r} + \tilde{q}_2 - q_1 + \frac{\sigma^2}{2} \right) T \right), \quad d_2 = d_1 - \sigma\sqrt{T},$$
$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 \frac{S_2}{S_2 + Ee^{-rT}} + \left(\frac{S_2}{S_2 + Ee^{-rT}} \right)^2 \sigma_2^2}, \quad (87)$$
$$\tilde{r} = \frac{S_2}{S_2 + Ee^{-rT}} r,$$
$$\tilde{q}_2 = \frac{S_2}{S_2 + Ee^{-rT}} q_2.$$

The comparison of our proposed numerical method and the analytical approximation (86) is presented in Table 1. Analytical approximation is calculated for the same arrays of S_1 and S_2 using MATLAB-function cdf for calculating cumulative distribution function that requires a lot of computational resources, for each point. In the proposed finite difference method (FDM) cdf is only used for the boundary conditions. Last row in the Table 1 presents the CPU time for each method for the same sizes of array. For approximation method cdf-time is 925.018 sec. It is the main part of computational time. In the case of FDM cdf takes less than half of computational time - 48.867 sec.

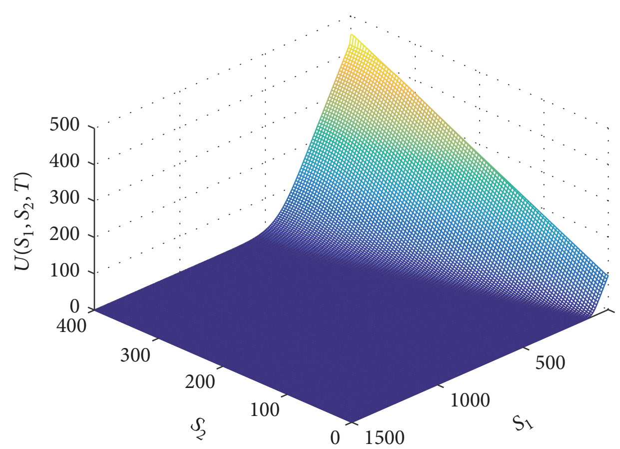

Figure 4: European spread put option with parameters (84).

Furthermore, unsuitable negative values appear in analytical approximation for small S_1 and S_2 while the proposed method preserves nonnegativity of the solution.

In the next example we confirm that for the spread put option case the removing technique is fruitful and the selection of the boundary conditions at the boundary of the numerical domain is appropriated because the behaviour of the solution at the boundaries fits the solution at the interior points as it is shown in Figure 4.

Example 3. With the parameters in (84) of Example 1, but with payoff $(S_2+E-S_1)^+$ in Figure 4, we show the simulation of the numerical solution obtained by the proposed numerical scheme.

Finally, in Example 4 we compare the numerical solution obtained with our scheme (83) using LCP approach with other tested efficient methods presented in [28].

Table 2: Comparing several methods for American option pricing.

E	Analytic	FFT	MC	FDM
0	8.513201	8.513079	8.516613	8.508198
2	7.542296	7.542242	7.545496	7.534064
4	6.653060	6.652976	6.656364	6.648792

Example 4. Let us consider the American spread call option problem with parameters:

$$\begin{aligned} T &= 1; \\ \sigma_1 &= 0.1; \\ \sigma_2 &= 0.2; \\ r &= 0.1; \\ q_1 &= 0.05; \\ q_2 &= 0.05; \\ \rho &= 0.5. \end{aligned} \tag{88}$$

Table 2 shows the option price at fixed asset prices $S_1 = 96$ and $S_2 = 100$ for different values of strike price E evaluated by one-dimensional integration analytic method (Analytic), Fast Fourier Transform (FFT), Monte Carlo (MC) method, and our proposed method. The accuracy is competitive with other methods, such as Fourier Transform with high number of discretization (4096) and Monte Carlo with big number of time steps (2000).

Another set of parameters can be considered in order to compare it with data in [9]:

$$\begin{aligned} T &= 0.5; \\ E &= 100; \\ \sigma_1 &= 0.25; \\ \sigma_2 &= 0.3; \\ r &= 0.03; \\ q_1 &= 0.06; \\ q_2 &= 0.02; \\ \rho &= 0.5. \end{aligned} \tag{89}$$

In [9], three methods are considered to evaluate the spread option price for data 7.4 and fixed values $S_1 = 100$ and $S_2 = 200$: a numerical integration method (NIM), a method of lines (MOL), and a Monte Carlo approach. NIM used in [9] is based in the recursive integration techniques developed by Huang et al. [29]. In NIM the option is treated as a Bermudan that is an option that can be exercised at discrete points of time and Simpsons rule is used. For the experiment, the authors used 1024 spatial nodes; see [8, 9] for more details. The MOL is implemented by a spatial discretization of 1438 × 200 mesh points and a three-time level scheme is used for the integration in time [9]. 50 000 simulations have been used in the Monte Carlo method described in [30]. In our method FDM, the step size discretization considered is $h = 0.1$. To compare FDM with the three previous methods, absolute errors are computed in Table 3 with respect to a reference value obtained by the Fourier space time-stepping approach of Surkov [31] with large number of nodes: 4096 spatial nodes and 300 time steps. From Table 3 we can state that the proposed method has the same order of accuracy as MOL, more efficient, and requires less computational time than Monte Carlo method. It is slightly less efficient than the NIM, but we prove that it possesses a very highly desirable qualitative behaviour in finance.

Table 3: CPU time in sec (first row) and absolute difference (second row) for different methods depending on number of time steps for fixed number of space steps for parameters (89).

N_τ	NIM	MOL	Monte Carlo	FDM
50	296.598	59.20	47 970	26.547
	0.003	0.001	0.1735	0.0036
32	127.069	55.12	19 811	17.911
	0.0035	0.006	0.1148	0.0122
16	33.493	51.718	5 149	9.432
	0.0012	0.0236	0.132	0.0371
8	10.360	49.01	1 350	4.787
	0.0105	0.074	0.1316	0.0907
4	4.117	45.47	355.75	2.815
	0.0364	0.2512	0.1351	0.2254

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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Research Article

Capturing the Data Uncertainty Change in the Cocaine Consumption in Spain Using an Epidemiologically Based Model

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A probabilistic model is proposed to study the transmission dynamics of the cocaine consumption in Spain during the period of 1995–2011. Using the so-called probabilistic fitting technique, we study if the model is able to capture the data uncertainty coming from surveys. The proposed model is formulated through a nonlinear system of difference equations whose coefficients are treated as stochastic processes. A discussion regarding the usefulness and limitations of probabilistic fitting technique in order to capture the data uncertainty of the proposed model is presented.

1. Introduction

In [1], the authors presented an epidemiologically based model to study the transmission dynamics of the cocaine consumption in Spain. This model worked well in the presented scenario, 1995–2005, where the population of cocaine consumers had an increasing trend. However, in 2008, the economic crisis began and the Spanish Health Ministry started to apply the National Drug Plan. As a consequence, the nonconsumer population increased and the consumer populations started to decrease. This sudden change could not be captured by the model as it was shown in [2].

In this work, we propose to use a variation of the model stated in [1] and assume that the model parameters are variable over the time in order to see if the variable model parameters are able to adapt and help the model to capture the observed changes in the cocaine consumption.

With this aim, we use the probabilistic fitting technique recently developed in [3, 4] by some of the authors. With this technique, we expect to capture the data uncertainty (data coming from surveys), despite the change of trend, and provide an estimation to the probability distribution of the model parameters. The latter will help us study if the observed data variation also produces variations in some or all model parameters as well as their quantification.

These types of mathematical models also have been used in the study of epidemics and other drug addictions, such as alcohol, tobacco, ecstasy, or heroin addiction [5–9], and in the approach to other sociological topics that are spread by social contact as obesity or extreme ideological behaviors [10, 11].

The work is organized as follows. In Section 2, we present the data we are going to use about cocaine consumers in Spain. In Section 3, we build a type-epidemiological model describing the cocaine consumption dynamics in Spain. Also, the model is scaled in order to match data and model magnitudes. In Section 4, we recall and adapt the *probabilistic fitting* technique introduced in [3]. This technique is applied to the model in order to obtain both data estimation with uncertainty and an estimation of the probability distribution of the model parameters. In Section 5, we present and discuss the results. In Section 6, conclusions are drawn.

2. Data

In this study, we use data from the Survey on Alcohol and Drugs in Spain (EDADES), part of the Spanish National Drug Plan [12]. This survey is published every two years. Specifically, we have focused on the following key question quoted in this survey: *how often have you consumed cocaine?* The available responses were as follows: "never consumed

Table 1: Survey results. % of people who belong to nonconsumers, occasional consumers, regular consumers, or habitual consumers of cocaine in Spain from 1995 to 2011.

Survey dates	Nonconsumer	Occasional consumer	Regular consumer	Habitual consumer
$t_1 = 1995$	0.944	0.034	0.018	0.004
$t_2 = 1997$	0.948	0.032	0.015	0.005
$t_3 = 1999$	0.948	0.031	0.015	0.006
$t_4 = 2001$	0.911	0.049	0.026	0.014
$t_5 = 2003$	0.903	0.059	0.027	0.011
$t_6 = 2005$	0.884	0.070	0.030	0.016
$t_7 = 2007$	0.874	0.080	0.030	0.016
$t_8 = 2009$	0.860	0.102	0.026	0.012
$t_9 = 2011$	0.879	0.088	0.022	0.011

(nonconsumers)," "at least once in your life (occasional consumers)," "at least once in the past year (regular consumers)," and "at least once in the past 30 days (habitual consumers)." This survey was launched in 1995 and we have data until 2011. The data are collected in Table 1.

In order to simplify the notation, we are going to define the set of time instants where survey data are available:

$$\Omega = \{t_1 = 1995,\ t_2 = 1997,\ t_3 = 1999,\ t_4 = 2001,\ t_5 = 2003,\ t_6 = 2005,\ t_7 = 2007,\ t_8 = 2009,\ t_9 = 2011\}. \quad (1)$$

3. Model Building

From the available survey data, we then introduce an epidemiological model to describe the dynamics of the four subpopulations over the time. While a traditional epidemiological study considers the transmission of a biological contagion, for our study, we will apply a *social contagion* approach. The underlying assumption is the following: we can consider cocaine consumption to be a contagion insofar as the probability that one individual becomes a cocaine consumer depends on the interaction with people who already exhibit this behavior, that is, fellow cocaine consumers [1, 13]. Let us assume *homogeneous population mixing*; that is, each individual can contact with any other individual [14, 15]. With this assumption, we can now build a system of difference equations based on the model introduced in [1] to further describe the transmission dynamics.

As with the survey results, we divide the Spanish 15–65-year-old population (the age group considered in the surveys) into four subpopulations:

(1) N_t, the amount of people who have never consumed cocaine at year t

(2) O_t, the amount of people who occasionally consume cocaine (at least once in their lives) at year t

(3) R_t, the amount of people who regularly consume cocaine (at least once in the past year) at year t

(4) H_t, the amount of people who habitually consume cocaine (at least once in the past month) at year t

Furthermore, we have the total population

$$T_t = N_t + O_t + R_t + H_t, \quad (2)$$

which is not going to be a constant value over the time.

The transitions between the different subpopulations are determined as follows:

(i) Let us consider that the newly recruited 15-year-old individuals become members of N_t subpopulation at rate μ; that is, we consider that they have never consumed cocaine before.

(ii) An individual in the nonconsumer population can become an occasional consumer through contact with occasional, regular, or habitual consumers. Then, this social contagion term can be modeled with $\beta_t(N_t/T_t)(O_t + R_t + H_t)$.

(iii) An occasional consumer can increase their consumption and become a regular consumer. This transition term can be modeled by $\gamma_t O_t$.

(iv) A regular consumer can increase their consumption and become a habitual consumer. We assume that this can be modeled by the term $\sigma_t R_t$.

(v) A habitual consumer can decide to go to drug-treatment therapy. If he/she stays in therapy for 6 months, according to experts, he/she can be considered a nonconsumer [16]. This term is modeled by $\epsilon_t H_t$.

Using the above assumptions, a dynamic cocaine consumption model for the 15–65-year-old Spanish population is given by the following nonlinear system of difference equations:

$$N_{t+1} = N_t + \mu T_t - d_N N_t - \beta_t \frac{N_t}{T_t}\left(O_t + R_t + H_t\right) + \epsilon_t H_t, \quad (3)$$

$$O_{t+1} = O_t - d_O O_t + \beta_t \frac{N_t}{T_t}\left(O_t + R_t + H_t\right) - \gamma_t O_t, \quad (4)$$

$$R_{t+1} = R_t - d_R R_t + \gamma_t O_t - \sigma_t R_t, \quad (5)$$

$$H_{t+1} = H_t - d_H H_t + \sigma_t R_t - \epsilon_t H_t, \quad (6)$$

where

(i) μ is the birth rate (the mortality between 0 years old and 14 years old is so small that it can be discarded),

(ii) d_N is the death rate for nonconsumers,

(iii) d_O is the death rate for occasional consumers,

(iv) d_R is the death rate for regular consumers,

(v) d_H is the death rate for habitual consumers,

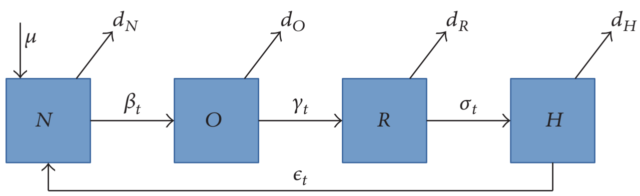

FIGURE 1: Flow diagram of the mathematical model (3)–(6) for the dynamics of cocaine consumption in Spain. The boxes represent the subpopulations and the arrows represent the transitions between the subpopulations. Arrows are labeled by the parameters of the model.

(vi) β_t is the time-dependent transmission rate for cocaine consumption,

(vii) γ_t is the time-dependent transition rate between occasional and regular cocaine consumers,

(viii) σ_t is the time-dependent transition rate between regular and habitual cocaine consumers,

(ix) ϵ_t is the time-dependent rate at which habitual consumers enter and complete drug-treatment therapy.

Figure 1 shows the flow diagram of the dynamic cocaine consumption model. The boxes represent the subpopulations and the arrows represent the transitions between the subpopulations. Arrows are labeled by their corresponding parameters of the model.

Following the idea proposed in [1], we are going to determine bounds for the model parameters. The birth rate $\mu \in [0.009168052, 0.011275930]$; both ends of this interval represent the minimum and maximum values for the birth rate during the period 1995–2011 [17]. Analogously, the death rate $d_N \in [0.008166102, 0.009228115]$. Taking into account the fact that the death rate of the consumers may be up to 6.8% higher than d_N [12] and the fact that the more an individual consumes drugs the higher probability to die because of drug abuse is, we have that $d_N < d_O < d_R < d_H \leq 0.009228115 \times 1.068$.

Also, we have the time-dependent parameters β_t, γ_t, σ_t, and ϵ_t for $t \in \Omega$ for which we consider that all of them lie between 0 and the double of the values given in [1] (maximum likelihood). Then, $\beta_t \in [0, 2 \times 0.09614]$, $\gamma_t \in [0, 2 \times 0.0596]$, $\sigma_t \in [0, 2 \times 0.0579]$, and $\epsilon_t \in [0, 2 \times 0.0000456]$, for $t \in \Omega$.

The time step for simulations will be of two years according to the time instants in Ω. Then, the total number of model parameters is 1 birth rate, 4 death rates, $9\beta_t$, $9\gamma_t$, $9\sigma_t$, and $9\epsilon_t$, that is, 41 model parameters.

3.1. Model Scaling. Despite the fact that (3)–(6) describe the cocaine consumption dynamics in Spain, there is a mismatch between the input for these difference equations, which is formulated for total population counts, and the survey data available, which is stated in percentages. We therefore need to scale the equations. To do this, we add together (3)–(6). Then, taking into account (2), the left-hand side yields

$$N_{t+1} + O_{t+1} + R_{t+1} + H_{t+1} = T_{t+1}, \tag{7}$$

while the right-hand side yields

$$T_t + \mu T_t - d_N N_t - d_O O_t - d_R R_t - d_H H_t. \tag{8}$$

Now, we shall define

$$\begin{aligned} n_t &= \frac{N_t}{T_t}, \\ o_t &= \frac{O_t}{T_t}, \\ r_t &= \frac{R_t}{T_t}, \\ h_t &= \frac{H_t}{T_t}, \end{aligned} \tag{9}$$

from which we get $n_t + o_t + r_t + h_t = 1$. From here, we will scale the equations, starting with (3). Here, using (8)-(9), one obtains

$$\begin{aligned} n_{t+1} &= \frac{N_{t+1}}{T_{t+1}} \\ &= \frac{N_t + \mu T_t - d_N N_t - \beta_t (N_t/T_t)(O_t + R_t + H_t) + \epsilon_t H_t}{T_t + \mu T_t - d_N N_t - d_O O_t - d_R R_t - d_H H_t}, \end{aligned} \tag{10}$$

and by dividing the numerator and denominator by T_t, one gets

$$n_{t+1} = \frac{n_t + \mu - d_N n_t - \beta_t n_t (o_t + r_t + h_t) + \epsilon_t h_t}{1 + \mu - d_N n_t - d_O o_t - d_R r_t - d_H h_t}. \tag{11}$$

Applying the same method to the other equations, the following expressions are obtained:

$$\begin{aligned} o_{t+1} &= \frac{O_{t+1}}{T_{t+1}} \\ &= \frac{O_t - d_O O_t + \beta_t (N_t/T_t)(O_t + R_t + H_t) - \gamma_t O_t}{T_t + \mu T_t - d_N N_t - d_O O_t - d_R R_t - d_H H_t} \\ &= \frac{o_t - d_O o_t + \beta_t n_t (o_t + r_t + h_t) - \gamma_t o_t}{1 + \mu - d_N n_t - d_O o_t - d_R r_t - d_H h_t}, \\ r_{t+1} &= \frac{R_{t+1}}{T_{t+1}} \\ &= \frac{R_t - d_R R_t + \gamma_t O_t - \sigma_t R_t}{T_t + \mu T_t - d_N N_t - d_O O_t - d_R R_t - d_H H_t} \\ &= \frac{r_t - d_R r_t + \gamma_t o_t - \sigma_t r_t}{1 + \mu - d_N n_t - d_O o_t - d_R r_t - d_H h_t}, \end{aligned}$$

$$h_{t+1} = \frac{H_{t+1}}{T_{t+1}} = \frac{H_t - d_H H_t + \sigma_t R_t - \epsilon_t H_t}{T_t + \mu T_t - d_N N_t - d_O O_t - d_R R_t - d_H H_t} = \frac{h_t - d_H h_t + \sigma_t r_t - \epsilon_t h_t}{1 + \mu - d_N n_t - d_O o_t - d_R r_t - d_H h_t}. \tag{12}$$

And the scaled equations can be written in the following form:

$$\begin{aligned} n_{t+1} &= \frac{n_t + \mu - d_N n_t - \beta_t n_t (o_t + r_t + h_t) + \epsilon_t h_t}{1 + \mu - d_N n_t - d_O o_t - d_R r_t - d_H h_t}, \\ o_{t+1} &= \frac{o_t - d_O o_t + \beta_t n_t (o_t + r_t + h_t) - \gamma_t o_t}{1 + \mu - d_N n_t - d_O o_t - d_R r_t - d_H h_t}, \\ r_{t+1} &= \frac{r_t - d_R r_t + \gamma_t o_t - \sigma_t r_t}{1 + \mu - d_N n_t - d_O o_t - d_R r_t - d_H h_t}, \\ h_{t+1} &= \frac{h_t - d_H h_t + \sigma_t r_t - \epsilon_t h_t}{1 + \mu - d_N n_t - d_O o_t - d_R r_t - d_H h_t}. \end{aligned} \tag{13}$$

4. Probabilistic Fitting

This technique consists of using information of the data survey to assign probability distributions to the data. Then, we sample data values from these probability distributions and fit the model to the sampled data. Thus, we find model parameters that fit not only the data but also the uncertainty contained into the intrinsic survey error. Thus, these model parameters will allow the model to capture the data uncertainty (with 95% confidence intervals).

4.1. Data 95% Confidence Intervals (95% CI). Data in Table 1 correspond to the mean percentage obtained from the EDADES survey conducted between 1995 and 2011 every two years. In the technical specifications of each survey, we can see sample sizes of 15000, 15000, 15000, 15000, 15000, 27934, 23715, 20109, and 22128 interviews, respectively.

Taking into account the fact that the sample is not the same for each survey, let us assume that the survey outputs are independent. For each one of the 9 available surveys, let us denote by $X^j = (X_1^j, X_2^j, X_3^j, X_4^j)$, $0 \le X_i^j \le n_j$, $i = 1, 2, 3, 4$, and $j = 1, \dots, 9$, a random vector whose entries are X_1^j = # nonconsumers, X_2^j = # occasional consumers, X_3^j = # regular consumers, and X_4^j = # habitual consumers and $n_1 = 15000$, $n_2 = 15000$, $n_3 = 15000$, $n_4 = 15000$, $n_5 = 15000$, $n_6 = 27934$, $n_7 = 23715$, $n_8 = 20109$, and $n_9 = 22128$ are the sample sizes of surveys. These components represent exclusive selections (events) with probabilities

$$\begin{aligned} P^j\left(X_1^j = x_1\right) &= \theta_1^j, \\ P^j\left(X_2^j = x_2\right) &= \theta_2^j, \\ P^j\left(X_3^j = x_3\right) &= \theta_3^j, \\ P^j\left(X_4^j = x_4\right) &= \theta_4^j, \\ & j = 1, \dots, 9, \end{aligned} \tag{14}$$

where $\theta_1^j, \theta_2^j, \theta_3^j$, and θ_4^j are the percentages collected in Table 1 for each survey j: $j = 1, \dots, 9$. Thus, each random vector X^j follows a multinomial probability distribution. Therefore, the probability that X_1^j occurs x_1 times, X_2^j occurs x_2 times, X_3^j occurs x_3 times, and X_4^j occurs x_4 times is given by

$$P_{n_j}^j (x_1, x_2, x_3, x_4) = \frac{n_j!}{x_1! x_2! x_3! x_4!} \left(\theta_1^j\right)^{x_1} \left(\theta_2^j\right)^{x_2} \left(\theta_3^j\right)^{x_3} \left(\theta_4^j\right)^{x_4}, \quad j = 1, \dots, 9, \tag{15}$$

where $x_1 + x_2 + x_3 + x_4 = n_j$. The resulting multinomials for each EDADES survey can be seen in Table 2.

Now, we compute the quantiles 2.5 and 97.5 (95% CI) of each one of the joint multinomial distributions in Table 2: $j = 1, 2, \dots, 9$. The obtained 95% CI are collected in Table 3.

4.2. Probabilistic Estimation. Let $M(t; \alpha)$ be a short representation of the scaled model (13), where α denotes the list of 41 model parameters; that is,

$$\alpha = \left(\mu, d_N, d_O, d_R, d_H, \beta_{t_1}, \dots, \beta_{t_9}, \gamma_{t_1}, \dots, \gamma_{t_9}, \sigma_{t_1}, \dots, \sigma_{t_9}, \epsilon_{t_1}, \dots, \epsilon_{t_9}\right). \tag{16}$$

Also, we have the data 95% confidence intervals (95% CI) of Table 3 obtained by sampling the joint probability distributions of Table 2 and calculating the percentiles 2.5 and 97.5.

Given the probability distributions P^i, $i = 1, \dots, 9$, in Table 2, we take a sample $d_{t_i j}^*$, $i = 1, \dots, 9$ and $j = 1, \dots, 4$, and we look for the values to the parameters α^* so that

$$\left\| \begin{bmatrix} M(t_1; \alpha^*) \\ M(t_2; \alpha^*) \\ \vdots \\ M(t_9; \alpha^*) \end{bmatrix} - \begin{bmatrix} d_{t_1 1}^*, d_{t_1 2}^*, d_{t_1 3}^*, d_{t_1 4}^* \\ d_{t_2 1}^*, d_{t_2 2}^*, d_{t_2 3}^*, d_{t_2 4}^* \\ \vdots \\ d_{t_9 1}^*, d_{t_9 2}^*, d_{t_9 3}^*, d_{t_9 4}^* \end{bmatrix} \right\|_F \tag{17}$$

is as minimum as possible, with $\|\cdot\|_F$ being the Frobenius norm [18]:

$$\left\| [a_{ij}] \right\|_F = \sqrt{\sum_i \sum_j a_{ij}^2}. \tag{18}$$

This procedure is a classic optimization problem that can be carried out through genetic algorithms, PSO, Nelder-Mead, and so forth [19, 20] but having the following key

Table 2: Date and data joint multinomial probability function of each survey.

Survey dates	Joint multinomial probability functions
$t_1 = 1995$	$P^1 = P^1_{15000}(x_1,x_2,x_3,x_4) = \frac{15000!}{x_1!x_2!x_3!x_4!}0.944^{x_1}0.034^{x_2}0.018^{x_3}0.004^{x_4}$
$t_2 = 1997$	$P^2 = P^2_{15000}(x_1,x_2,x_3,x_4) = \frac{15000!}{x_1!x_2!x_3!x_4!}0.948^{x_1}0.032^{x_2}0.015^{x_3}0.005^{x_4}$
$t_3 = 1999$	$P^3 = P^3_{15000}(x_1,x_2,x_3,x_4) = \frac{15000!}{x_1!x_2!x_3!x_4!}0.948^{x_1}0.031^{x_2}0.015^{x_3}0.006^{x_4}$
$t_4 = 2001$	$P^4 = P^4_{15000}(x_1,x_2,x_3,x_4) = \frac{15000!}{x_1!x_2!x_3!x_4!}0.911^{x_1}0.049^{x_2}0.026^{x_3}0.014^{x_4}$
$t_5 = 2003$	$P^5 = P^5_{15000}(x_1,x_2,x_3,x_4) = \frac{15000!}{x_1!x_2!x_3!x_4!}0.903^{x_1}0.059^{x_2}0.027^{x_3}0.011^{x_4}$
$t_6 = 2005$	$P^6 = P^6_{27934}(x_1,x_2,x_3,x_4) = \frac{27934!}{x_1!x_2!x_3!x_4!}0.884^{x_1}0.07^{x_2}0.03^{x_3}0.016^{x_4}$
$t_7 = 2007$	$P^7 = P^7_{23715}(x_1,x_2,x_3,x_4) = \frac{23715!}{x_1!x_2!x_3!x_4!}0.874^{x_1}0.08^{x_2}0.03^{x_3}0.016^{x_4}$
$t_8 = 2009$	$P^8 = P^8_{20109}(x_1,x_2,x_3,x_4) = \frac{20109!}{x_1!x_2!x_3!x_4!}0.86^{x_1}0.102^{x_2}0.026^{x_3}0.012^{x_4}$
$t_9 = 2011$	$P^9 = P^9_{22128}(x_1,x_2,x_3,x_4) = \frac{22128!}{x_1!x_2!x_3!x_4!}0.879^{x_1}0.088^{x_2}0.022^{x_3}0.011^{x_4}$

Table 3: 95% CI of the EDADES surveys data using the joint multinomial probability function of each survey.

Survey dates	Nonconsumer	Occasional consumer	Regular consumer	Habitual consumer
$t_1 = 1995$	[0.940, 0.947]	[0.031, 0.037]	[0.016, 0.020]	[0.003, 0.005]
$t_2 = 1997$	[0.944, 0.952]	[0.029, 0.035]	[0.013, 0.017]	[0.004, 0.006]
$t_3 = 1999$	[0.944, 0.952]	[0.028, 0.034]	[0.013, 0.017]	[0.005, 0.007]
$t_4 = 2001$	[0.906, 0.916]	[0.046, 0.053]	[0.023, 0.029]	[0.012, 0.016]
$t_5 = 2003$	[0.898, 0.908]	[0.055, 0.063]	[0.024, 0.030]	[0.009, 0.013]
$t_6 = 2005$	[0.880, 0.888]	[0.067, 0.073]	[0.028, 0.032]	[0.015, 0.018]
$t_7 = 2007$	[0.870, 0.878]	[0.076, 0.083]	[0.028, 0.032]	[0.014, 0.018]
$t_8 = 2009$	[0.855, 0.865]	[0.098, 0.106]	[0.024, 0.028]	[0.011, 0.014]
$t_9 = 2011$	[0.875, 0.883]	[0.084, 0.092]	[0.020, 0.024]	[0.010, 0.012]

Table 4: Model fitting to N samples of the data's probability distributions.

Error	Parameters	Model
e_1^*	α^1	$M(t;\alpha^1)$
e_2^*	α^2	$M(t;\alpha^2)$
$\vdots$	$\vdots$	$\vdots$
e_N^*	α^N	$M(t;\alpha^N)$

difference: now we will fit the data sampled from probability distributions instead of the raw data. We shall perform the fitting N times (N being a large number), storing both the parameter values α^* and the calculated errors e^*, ordered from smallest to largest errors. The result of this procedure is a list of model parameters fitted to a sample of the data with their corresponding errors represented in Table 4.

The value N should be a large number in order to capture as much data uncertainty as possible during the sampling process and this uncertainty could be fitted by the model. In our case, we take $N = 25000$.

Now, we take $M(t;\alpha^1)$ and $M(t;\alpha^2)$ in Table 4 and calculate the outputs for times $t_1, \ldots, t_9$ in Ω, the time instants where data are available. For each time instant, we shall calculate percentiles 2.5 and 97.5, one for each one of the 4 subpopulations. Hence, we will name m_2 the sum of the following:

(i) The Frobenius norm of the difference between the percentiles 2.5 from the model output and from the data percentiles 2.5 in Table 3

(ii) The Frobenius norm of the difference between the percentiles 97.5 from the model output and from the data percentiles 97.5 in Table 3

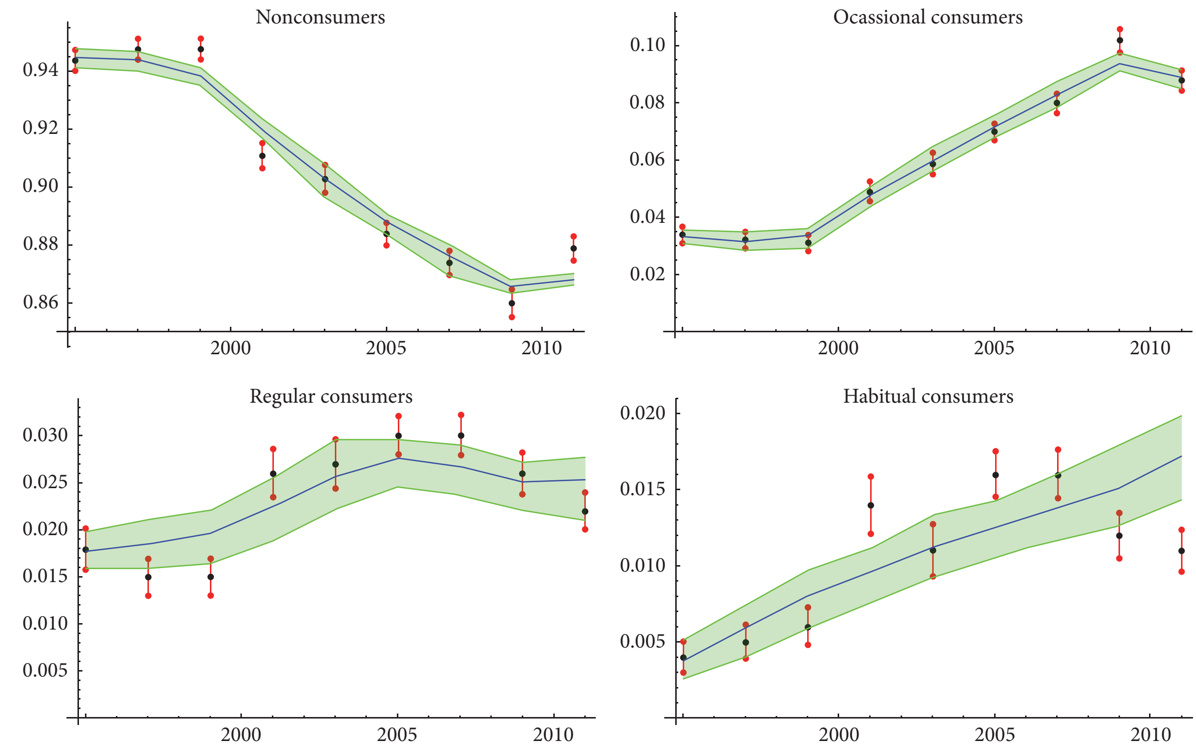

Figure 2: Result of the probabilistic fitting. The green band corresponds to 95% CI model output and the blue line its mean. The red points are the 95% CI data given in Table 3 and the black points their means. We can see how the model tries to capture the data uncertainty, but the goal is not achieved completely.

We will repeat the above process with the outputs from $M(t;\alpha^1)$, $M(t;\alpha^2)$, and $M(t;\alpha^3)$, obtaining m_3, the measure between the confidence bands from the outputs and the data. The same shall be done for m_4 with $M(t;\alpha^1)$, $M(t;\alpha^2)$, $M(t;\alpha^3)$, and $M(t;\alpha^4)$ and so on, until $M(t;\alpha^1), M(t;\alpha^2), \ldots, M(t;\alpha^N)$, obtaining m_N as the measure between the confidence bands from the outputs and the data.

Taking $m_k = \min\{m_1, \ldots, m_N\}$, we ensure that the 95% confidence band of the outputs from $M(t;\alpha^1), M(t;\alpha^2), \ldots, M(t;\alpha^k)$ is the closest to the 95% confidence band from the data, with which our model will capture the maximum uncertainty of the data from the output of the models $M(t;\alpha^1)$, $M(t;\alpha^2), \ldots, M(t;\alpha^k)$.

5. Results

The probabilistic fitting procedure returns, as the best fitting, the one giving for $k = 40$ with an error of $m_k = 0.0502666$. A graphical representation of this result can be seen In Figure 2.

The model 95% CI band captures almost all data uncertainty except 1999 and 2011 in nonconsumers, 2009 in occasional consumers, and 2001 and 2011 in habitual consumers (there is no intersection between the model output band and these data 95% CI).

The decrease in the nonconsumer subpopulation during the period of 1997–2001 is not properly captured by the model. With regard to the economic crisis and the National Drug Plan proposed in 2008 by the Spanish government, their effect does not appear until 2011, where an increase in nonconsumers and a decrease in occasional consumers arise. As can be seen in Figure 2, for these two subpopulations, the model is unable to capture the data uncertainty. Also, in the habitual consumers' subpopulation, the model still does not realize the trend change.

All these comments lead us to say that the proposed model was fairly good to explain what happened when the number of consumers was increasing, mainly in occasional and regular consumers; nevertheless when the trend changed, it could not adapt and explain entirely what was happening.

With respect to the model parameters, in Figure 3, we can see the estimation of the probability density function (PDF) of β_t, γ_t, σ_t, and ϵ_t for every $t \in \Omega$.

Figure 3 has been built taking $k = 40$ sets of model parameters selected by the probabilistic fitting procedure, in particular corresponding to β_t, γ_t, σ_t, and ϵ_t for $t \in \Omega$, and generate a kernel PDF in each time instant in Ω using the *Mathematica* [21] command `KernelMixtureDistribution` [].

In Figures 3 and 4, we can see the variability of the parameters over the time, trying to adapt to the cocaine consumption behavior changes. Note that ϵ_t has a mean and 95% confidence interval fairly stable, which means that the flow of people going to therapy is almost invariable over the time with independence of the changes in the consumption behavior. Therefore, we could make $\epsilon_t = \epsilon$ for all $t \in \Omega$ and transform this random process into a random variable.

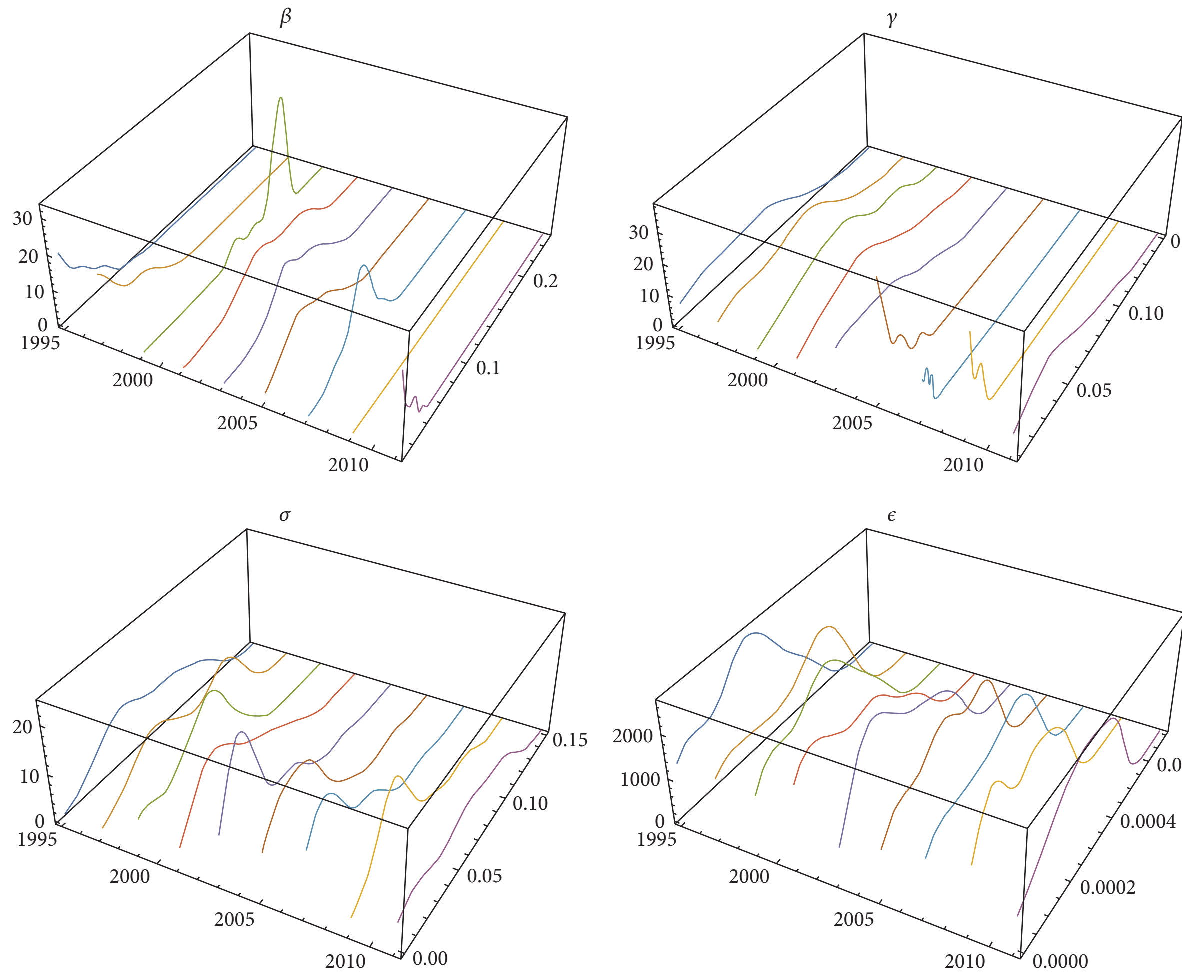

Figure 3: Estimation of the random processes β_t, γ_t, σ_t, and ϵ_t using the estimated PDF of the parameters in the time instants in Ω.

The use of variable parameters may allow us to identify changes in the studied phenomenon which have not been considered in the hypotheses or which could arise unexpectedly, as it seems to happen in this regard to the economic crisis and the National Drug Plan, in this case. Accordingly, we can see in Figure 4 how parameters β_t and γ_t, and with lower changes parameter σ_t, try to adapt in the rightmost part of the graphs to data changes, but, unfortunately, they do not change enough to capture the data uncertainty when the consumption trend varies. This leads us to think that the model may not be appropriate to explain the decreasing trends in cocaine consumption.

6. Conclusion

In this work, we present a model formulated through a system of nonlinear difference equations, where the model parameters are variable over the time. This model intends to describe the cocaine consumption dynamics in Spain in the period of 1995–2011. The model follows fairly well the dynamics until 2009, where the effect of two nonconsidered facts, the effect of the economic crisis and the implantation of the National Drug Plan, introduced a change in the behavior of the individuals, especially in the young people who are the group that is increasing the nonconsumer subpopulation.

We expected that the introduction of variable parameters over the time would allow us to capture the changes in the cocaine consumption dynamics; however, it could not. This indicates that the proposed model may not be appropriate when a decreasing trend in cocaine consumption emerges.

With regard to the parameters, we must say that it was not necessary to consider the parameter ϵ_t as variable, because it is fairly stable over the time. The remaining parameters, β_t, γ_t, and σ_t, are trying to adapt to the data trend changes, but, in nonconsumer and habitual consumer subpopulations, they fail to do it accurately.

Finally, the three sides that this study, a model with variable parameters using the probabilistic fitting technique, provides should be pointed out:

(1) Consider the model parameters by default as random processes (a family of random variables indexed by time), which is more versatile than considering them as random variables, and then we can detect if they really vary over the time.

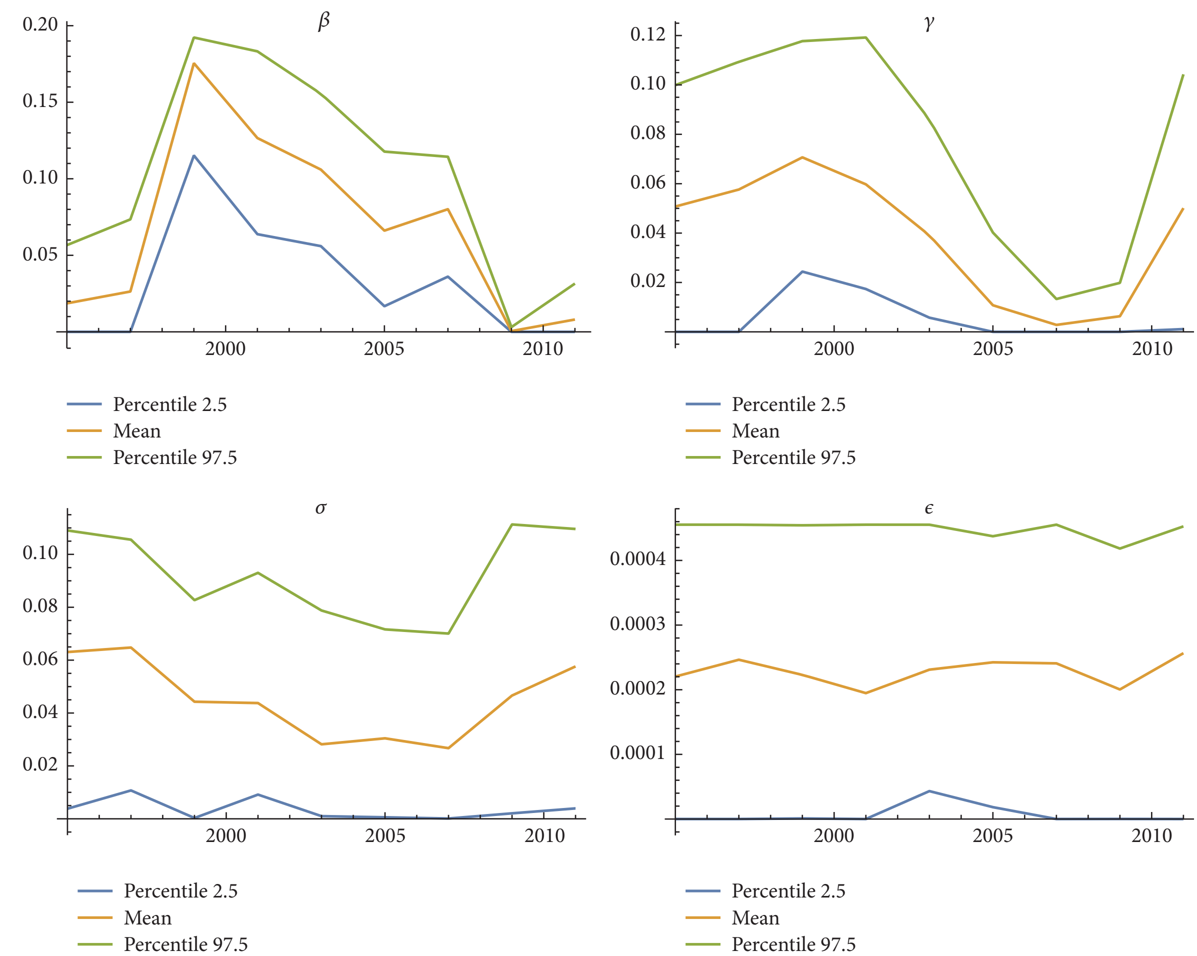

Figure 4: Estimated 95% confidence intervals (band) and means of the random processes β_t, γ_t, σ_t, and ϵ_t plotted in Figure 3.

(2) Study the estimation of the parameter random processes and obtain estimations for the means, the confidence intervals, the PDFs, and hence other higher statistical moments.

(3) Determine the usefulness of the model in several realistic situations or when data changes because unexpected scenarios emerge.

Although it was not mentioned before, a great computational effort has been done to obtain the presented results and this is one of the major drawbacks we want to improve in future works.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this article.

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Research Article

Mathematical Modeling of Hidden Intimate Partner Violence in Spain: A Quantitative and Qualitative Approach

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The fact that women are abused by their male partner is something that happens worldwide in the 21st century. In numerous cases, abuse only becomes publicly known when a fatal event occurs and is beyond any possible remedy, that is, when men murder their female partner. Since 2003, 793 (September 4, 2015) women have been assassinated by their significant other or excouple in Spain. Only 7.2% of murdered women had reported their fear and previous intimate partner violence (IPV) to the police. Even when the number of female victims is comparable to the number of victims by terrorism, the Government has not assigned an equal amount of resources to diminish the magnitude of this hidden social problem. In this paper, a mathematical epidemiological model to forecast intimate partner violence in Spain is constructed. Both psychological and physical aggressor subpopulations are predicted and simulated. The model's robustness versus uncertain parameters is studied by a sensitivity analysis.

1. Introduction

Violence against women occurs in all geographical areas and in all types of society. Violence affects girls and women of all ages and in all stages of life. In western societies, it has not been until quite recently (1979) that the intimate violence partner was institutionally identified and condemned. The origin is found in feminists in the 1950s [1].

Women abuse embraces different types of manifestations, such as physical violence, including sexual abuse, emotional-psychological violence, verbal abuse, religious discrimination, economic deprivation, isolation, gender privilege, and beating [2].

Women abuse takes place in different environments: workplace, public areas, family, and friends, and even at continuous and sustained connectability due to technological appliances. While the former type (workplaces and public areas) is more visible and socially condemned [3, 4], violence that occurs in families, particularly intimate violence partner, is hidden [5, 6] and linked to drunkenness and drug consumption causes [7].

Moreover according to [8] women, the risk of suffering abuse or being raped in an age interval [15, 44] is higher than the risk of suffering cancer, malaria, or a car accident [9]. According to [8], as many as 38% of all women murders are committed by their intimate partners.

The incidence of violence against women in intimate relationships is difficult to diagnose. In fact, five different types can be identified: coercive controlling, violent resistance, situational couple violence, separation-instigated violence, and mutual violent control [10]. Intimate partner violence occurs in all settings and in all socioeconomic, religious, and cultural groups. Indeed the commonest perpetrators of violence against women are male intimate partners or expartners [11, 12].

Even when progress is made from the justice viewpoint, the intimate violence problem remains mainly hidden for several reasons, such as low victimization acceptance, as it takes place at the heart of the couple. It produces social shame and a feeling of emotional failure experienced by women, which are the strength of a male chauvinist culture. There is also common victim tendency to forgive and forget lower levels of violence based on the myth of the romantic love that causes intimate partner violence (IPV) to persist [13]. Furthermore in those cases in which women recognize themselves as victims, lack of immediate public protocols reduces the number of legal complaints about victims' insecurity [14].

It is difficult not only to identify the IPV problem, but to also quantify it in order to strength the claim for public answers to mitigate this social problem. In fact, the only available information is the casualties and police complaints made by victims [15], mostly when couples have broken up in practice, but not officially/socially. In fact, abused women who leave their couples are at more risk of lethal violence than those who do not [16].

As far as we know, previous studies have been qualitative in nature, have been based on sampling women populations by questionnaires [17], or have conducted research by focusing on abused women once they have recovered [18].

For the particular case of Spain, more than 1,000 women have been murdered in the last 15 years by their significant others, a number that exceeds the number of victims murdered by the terrorist group ETA. For the time being, about 4,500 aggressors are in Spanish prisons for committing sexual abuse [19].

There exist recent studies modeling mathematically the domestic violence throughout continuous time differential equation models [20, 21]. However, domestic violence embraces all types of aggressions occurring in the family unit, including violence against children, while intimate partner violence only embraces aggressions among couples. Furthermore, the models developed at [20, 21] are not realistic in the sense that those assume the propagation depends on the proportion of victims. Also, those studies do not take into account demographics or socioeconomic factors.

The aim of this paper was to construct a dynamic mathematical model by forecasting the amount of intimate partner aggressors over the 2012–2017 time horizon in Spain. By means of a population model that split the population into several categories according to different levels of violence and by quantifying by semester the main causes of IPV transits among subpopulations, a difference system provided dynamic quantification in a future time.

This paper is structured as follows: Section 2 shows the hypotheses and model construction; Section 3 illustrates the results and simulation; Section 4 summarizes the conclusions and provides recommendations for short- and long-term reductions of the intimate partner violence (IPV) problem.

2. Hypotheses and Model Construction

Violence against women is transversal, with no distinction of social class or academic and professional qualification. We split the population of potential men abusers in Spain whose ages fell in the interval $[16, 74]$ into four compartments according to the level of violence committed against their heterosexual couples. Depending on an increasing degree of violence, the categories were defined as follows.

Gender Egalitarian Man. A category defined as men free of sexist microaggressions. Relationships with gender egalitarian man are characterized by flexibility and negotiability of gender roles, with the primary goal of satisfying individual needs [22].

Regular Men. A subpopulation defined by men influenced by a generalized male chauvinist environment since they could practice sexist microaggression, such as verbal joking without committing either psychological or physical aggression against their couple.

Emotional-Psychological Abuser. A category that embraces those men who attempt to control and reduce their couples' freedom by attacking her self-esteem [16].

Physical Aggressor. A category that includes those men who use physical force against their partner, including undesired sexual relationships [16].

Our study started in January 2012 and split the male population in Spain by semester into the four above-defined categories using the following notation:

$S(n)$: gender egalitarian men at semester n ($n = 0$ corresponds to January 2012)

$N(n)$: regular men at semester n

$\mathrm{AS}(n)$: emotional-psychological abuser men at semester n

$\mathrm{AF}(n)$: physical aggressor men at semester n

The male population whose ages fell in the interval $[16, 74]$ at the beginning of the study (January 2012) came to 16,794,613. We began quantifying subpopulation $\mathrm{AF}(0)$, obtained by the number of legal complaints for gender violence registered by [23], in addition to the number of murdered women who had not previously complained, which came to 134,048 women, which represents 0.8% of the total male population in January 2012 [24]. This amount underestimated the unknown real number of AF aggressors because those women had not legally complained [25].

Secondly, we estimated the AS subpopulation following [26] and sizing the proportion as 36% of the total male population.

The S subpopulation was estimated following [27] by quantifying the proportion of this category as 10.20% of the total male population.

Finally by subtracting the amounts of the three previous categories from the general population, we estimated the number of regular men ($N(0)$) as being 53% of the population, which comes close to 9 million men (8,901,145 men; see Table 1).

Among the factors that influence human behavior, we consider demographic, economic, and psychosocial factors. Some other possible factors are of the genetic type, which was not taken into account [28, 29]. The legal factor was not considered because the legal conditions did not change during the study period.

Our dynamic compartmental model was built by quantifying the semester transits between subpopulations throughout the study period (January 2012–December 2017).

We defined and quantified these transit coefficients. Changes in total population were due to demographic factors, in particular, birth (men who became 16 years old), death (naturally deceased or functional exit from the system

Table 1: Initial subpopulation in January 2012.

$S(0) = r_1$	$N(0) = r_2$	$\mathrm{AS}(0) = r_3$	$\mathrm{AF}(0) = r_4$
10.20%	53%	36%	0.8%

Table 2: Subpopulation proportions of model incomers.

Egalitarian	5.00%
Regular	58.57%
Emotional-psychological abuser	33.00%
Physical aggressor	3.43%

when men became older than 75 years old), and emigration/immigration.

By obtaining the number of male births at each semester [30], we distributed these data among the subpopulations and according to the proportions that resulted from [31] (Table 2). We assumed that these proportions remained constant for the short study period. This coefficient was denoted by

$$\alpha_i (n-32), \quad 1 \leq i \leq 4, \tag{1}$$

where $\alpha_i(n-32)$ represents the number of males born 16 years ago (32 semesters).

This proportion in absolute values sized the quantities shown in Table 3.

Leaving the system occurred when the population became older than 74 years but also when males died in the age interval $[16, 74]$. The total number of outcomers was distributed into the subpopulations according to their proportional size in 2012, $1 \leq i \leq 4$, by assuming that probability of death was exactly the same among all the subpopulations. In both cases, we assumed that these proportions remained constant for the short study period considered. This coefficient is denoted as D_i, $1 \leq i \leq 4$ [30].

Finally, the last demographic transit results from the net emigration impact due to economic reasons [32]. In line with the deterioration of the Spanish economy (high unemployment rate, deterioration of the welfare system) from January 2012 to December 2014, a significant proportion of the Spanish population emigrated abroad to seek work. This process stopped in January 2015 due to an initial recovery of the Spanish economy. Thus we assumed that the net emigration factor due to economic reasons had disappeared since January 2015. We also assumed that the probability of emigration was the same in the four categories and that the number of emigrants was sized according to the 2012 proportion of each subpopulation. According to these data and hypotheses, the emigration coefficient was quantified as follows:

$$\gamma_{1r_i(n)} = \begin{cases} 113203 * r_i; & 1 \leq i \leq 4, \ n = 0, 1 \ (2012) \\ 131204 * r_i; & 1 \leq i \leq 4, \ n = 2, 3 \ (2013) \\ 49736 * r_i; & 1 \leq i \leq 4, \ n = 4, 5 \ (2014) \\ 0; & 11 \geq n \geq 6 \ (2015, 2016, 2017), \end{cases} \tag{2}$$

Table 3: Number of model incomers per semester.

	S	N	AS	AF
01-Jul-12	4,667	54,675	30,805	3,202
01-Jan-13	4,753	55,674	31,368	3,260
01-Jul-13	4,753	55,674	31,368	3,260
01-Jan-14	4,725	55,348	31,185	3,241
01-Jul-14	4,725	55,348	31,185	3,241
01-Jan-15	4,894	57,323	32,297	3,357
01-Jul-15	4,894	57,323	32,297	3,357
01-Jan-16	5,140	60,209	33,923	3,526
01-Jul-16	5,140	60,209	33,923	3,526
01-Jan-17	5,220	61,143	34,450	3,581
01-Jul-17	5,220	61,143	34,450	3,581

where r_i is the proportion of each subpopulation estimated in January 2012 (see Table 1).

Apart from the demographic factors, the subpopulations' dynamical behavior was influenced by the following.

(I) Women's level of permissiveness against abusive behavior by their partners [33, 34].

(II) Men's alcohol and drug consumption [7, 35].

(III) Jealousy as a factor promoted by the chauvinist culture [34].

(IV) Economic stress (long-term unemployment) [30].

(V) The contagion effect on men caused by their close environment where examples of gender violence are experienced or suffered throughout [16, 36].

(VI) Technology: the impact of new technologies on intimate partner psychological abuse (stalking) [37].

After we identified the main factors that influenced the intimate violence partners phenomenon, we proceeded to quantify the transit coefficients among categories from one period (n) to the next ($n+1$).

We began by measuring the transit from the subpopulation of regular $N(n)$ to $S(n+1)$.

We assumed that this effect occurred only after a stable relationship among young partners. In our hypotheses, this possibility only occurred in a stable relationship between a young regular (N) nonjealous man and an egalitarian young woman who did not accept any kind of sexual microaggression. We assumed that the successful transit occurred with a probability of 1/4. This coefficient remained constant throughout the short study period and was computed as follows:

β_1 = (proportion of men whose ages fell in the interval $[16, 35]$) $*$ (proportion of nonjealous men) $*$ (proportion of egalitarian women) $= 0.35 * 0.30 * 0.30 * 1/4 = 0.007875$.

2.1. Transit from N to the AS Subpopulation. β_2 = (proportion of jealous men who consumed drugs and/or alcohol in a stable relationship with a nonegalitarian woman per semester) $= 1/2 * 0.7 * (0.05 + 0.0E) * (0.7) = 0.01225 + 0.00245E$, where E is a perturbation coefficient due to the difficulty of

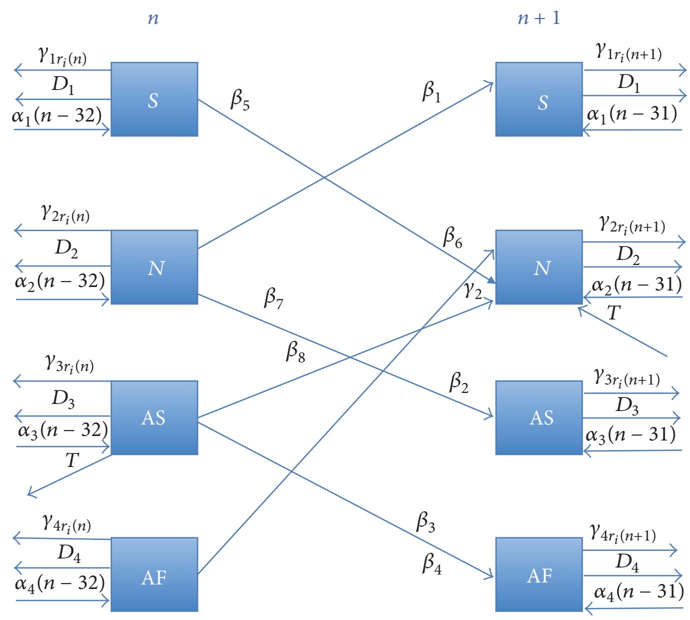

Figure 1: Block diagram model.

measuring the real proportion of consumers of drugs/alcohol. We assumed the variation range of E in the interval $[0, 2]$, which means that we considered a perturbation of the coefficient of 2% at the most [33, 35].

2.2. Transits from the AS to the AF Subpopulation. There were two independent and different types of transit among the same subpopulations.

β_3 = (proportion of men consumers of drugs/alcohol in a stable relationship per semester) = 0.105/2 = 0.0525 [35].

β_4 = (proportion of men whose relationship broke up, combined with long-term unemployment or consumed drugs/alcohol per semester) = $1/2 * [(0.5 * 0.27 * 0.02) + (0.5 * (0.05 + 0.0E))] = 0.026 + 0.025E$ [30, 34].

2.3. Transit from S to the N Subpopulation. β_5 = 6-monthly proportion of men in a relationship with a nonegalitarian woman transiting after a broken relationship due to the infidelity effect = $(1/2) * (0.7 * 0.174 * 0.5) = 0.03045$ [37].

2.4. Transit from the AF to the N Subpopulation. We considered the recovery effect due to therapy and the positive influence of a new egalitarian partner woman.

β_6 = 6-monthly proportion of separated or divorced men who engaged in a new relationship with an egalitarian woman, in addition to the separated men under 50 years of age who engaged in a new relationship with nonegalitarian women who attended therapy [16] per semester = $(1/2) * [(0.5 * 0.3) + (0.5 * 0.7 * 0.05 * 0.55)] = 0.08375$.

2.5. Transit from the AS to the N Subpopulation. γ_2 = 6-monthly proportion of men whose ages fell in the interval [16, 35] and who started a stable relationship with an egalitarian woman for at least 1 year:

$$\gamma_2 = \frac{1}{2}(0.35 * 0.3 * r) = 0.0525 * r, \tag{3}$$

where r is the probability of the couple lasting. We simulated different values for r in the interval $= i/8$, where i varied from 1 to 8.

T = the number of males that attend therapy (cognitive behavioral therapy, psychodynamic therapy) transiting from AS to N. $T = 600$. This amount is considered constant for the period of study [38].

2.6. Transit from N to the AS Subpopulation. β_7 = proportion of psychopathic men (1%) = 0.01. This transit was constant and occurred for genetic reasons [39, 40].

β_8 = proportion of jealous men whose ages fell in the interval [16, 40] in a relationship with a nonegalitarian woman who used technology [40] as the channel to exercise control aggression with their couples. For instance, frequent communication throughout calls and messages in attempt to forbid or persuade a particular behavior and to remain in control [8, 37, 41, 42], quantified approximately as follows: β_8 = technology factor (TF) $*$ proportion of jealous men $*$ proportion of men younger than 40 years $*$ proportion of nonegalitarian women = TF $* 0.7 * 0.5 * 0.7$ = 0.24% year bases = 0.12% per semester.

The above processed coefficient transit allowed the construction of the block diagram model shown in Figure 1.

After defining the transit coefficients, the population dynamics were given by the set of difference equations, expressed as follows.

This system can be written in a matrix form as follows:

$$\begin{aligned}
S(n+1) &= S(n) + \beta_1 N(n) + \alpha_1(n-32) - D_1 \\
&\quad - \gamma_{1r_1}(n) - \beta_5 S(n), \\
N(n+1) &= N(n) - \beta_1 N(n) + \alpha_2(n-32) - D_2 \\
&\quad - \gamma_{1r_2}(n) + \beta_5 S(n) + \beta_6 \mathrm{AF}(n) \\
&\quad + \gamma_2 \mathrm{AS}(n) - \beta_2 N(n) + T, \\
\mathrm{AS}(n+1) &= \mathrm{AS}(n) + \alpha_3(n-32) - D_3 - \gamma_{1r_3}(n) \\
&\quad - (\beta_3+\beta_4)\mathrm{AS}(n) - \gamma_2 \mathrm{AS}(n) \\
&\quad + \beta_2 N(n) - T, \\
\mathrm{AF}(n+1) &= \mathrm{AF}(n) + \alpha_4(n-32) - D_4 - \gamma_{1r_4}(n) \\
&\quad + (\beta_3+\beta_4)\mathrm{AS}(n) - \beta_6 \mathrm{AF}(n).
\end{aligned} \tag{4}$$

With the vector notation,

$$Z(n+1) = [S(n), N(n), \mathrm{AS}(n), \mathrm{AF}(n)]^T. \tag{5}$$

The previous system of difference equations can be written in vector form as follows:

$$Z(n+1) = V(E,r) * Z(n) + b(n), \tag{6}$$

where

$$\begin{gathered}
V = \begin{bmatrix} 1-\beta_5 & \beta_1 & 0 & 0 \\ \beta_5 & 1-\beta_2 & \gamma_2 & \beta_6 \\ 0 & \beta_2 & 1-\gamma_2-\beta_3-\beta_4 & 0 \\ 0 & 0 & \beta_3+\beta_4 & 1 \end{bmatrix}, \\
b(n) = \begin{Bmatrix} \alpha_1(n-32) - \gamma_{1r_1}(n) - D_1 \\ \alpha_2(n-32) - \gamma_{1r_2}(n) - D_2 \\ \alpha_3(n-32) - \gamma_{1r_3}(n) - D_3 \\ \alpha_4(n-32) - \gamma_{1r_4}(n) - D_4 \end{Bmatrix}
\end{gathered} \tag{7}$$

and where the value of the subpopulations at the start of the period was (see Table 1)

$$Z(0) = [1{,}713{,}051; 8{,}901{,}145; 6{,}046{,}061; 134{,}359]. \tag{8}$$

It is easy to check that an explicit closed form solution of the problem takes the form

$$Z(n) = V^n Z(0) + \sum_{j=0}^{n-1} V^{n-j-1} b(j), \quad 1 \le n \le 11, \tag{9}$$

although expression (9) is not necessary to perform the computations of $Z(n)$, for the 11 semesters of the period of study.

Table 4: Results of the model that corresponded to $E = 0$ and $r = 1/8$.

	S	N	AS	AF
01-Jan-12	1,713,051	8,901,145	6,046,061	134,359
01-Jul-12	1,716,898	8,920,041	5,495,299	522,821
01-Jan-13	1,719,020	8,889,527	5,047,893	842,092
01-Jul-13	1,720,848	8,828,147	4,688,071	1,104,939
01-Jan-14	1,730,442	8,790,242	4,429,615	1,322,814
01-Jul-14	1,739,460	8,735,669	4,223,798	1,505,293
01-Jan-15	1,753,035	8,699,773	4,080,490	1,659,531
01-Jul-15	1,765,928	8,655,552	3,968,615	1,791,344
01-Jan-16	1,778,343	8,610,879	3,884,905	1,905,039
01-Jul-16	1,790,044	8,562,543	3,820,648	2,003,658
01-Jan-17	1,801,107	8,515,530	3,774,027	2,089,996
01-Jul-17	1,811,480	8,467,595	3,739,342	2,166,010

Table 5: Subpopulation results.

	S	N	AS	AF
01-Jan-12	10.20%	53.00%	36.00%	0.80%
01-Jul-17	11.19%	52.32%	23.10%	13.38%

3. Results and Simulations

After we modeled the transit coefficients, the subpopulations were computed by solving the model. Thus by taking $E = 0$, which corresponded to the hypotheses of an alcohol rate consumption of 5% and $r = 1/8$, we obtained the subpopulation results shown in Table 4.

As Table 4 shows, the AF subpopulation grew by an average of 200,000 new aggressors each semester until July 2014. Then from July 2014 to July 2017, this growth slowed down to 100,000 new aggressors per semester. The system's initial lack of accuracy due to a poor realistic number of aggressors when only legal complaints were quantified as physical aggressions led to this subpopulation's fast initial growth until the midperiod. Then growth slowed down to approximately half until the end of the period. The subpopulation of regular men was almost stable for the whole study period.

During the considered study period, the proportion of regular and egalitarian men remained fairly stable and changed less than 1%, while AS dropped by 12.90%. In contrast, the AF proportion grew considerably by about 12% (Table 5).

The trend of the studied subpopulations is shown in Figure 2.

Table 6 shows the variations in populations when the alcohol rates changed from 3% to 7%. As we can see, the subpopulations of aggressors grew with an increased alcohol rate. For instance, it was relevant that the physical aggressors population grew by about 0.4% for each 1% increase in the alcohol rate.

One important issue in the proposed model was the probability of the couple's stability, explained when quantifying γ_2. Thus we modeled the subpopulations trend by varying the r-value in July 2017. As Table 7 shows, both physical and

Table 6: Robustness alcohol rate.

Alcohol rate	S	N	AS	AF
3	11.28%	54.34%	21.59%	12.79%
4	11.23%	53.32%	22.36%	13.09%
5	11.19%	52.32%	23.10%	13.38%
6	11.15%	51.34%	23.83%	13.68%
7	11.11%	50.39%	24.53%	13.97%

Table 7: Sensitivity analysis of the model versus the couple's stability.

r	S	N	AS	AF
1	11.11%	50.39%	24.53%	13.97%
2	11.52%	58.83%	18.22%	11.43%
4	12.08%	68.15%	11.60%	8.17%
6	12.41%	72.82%	8.47%	6.29%
8	12.64%	75.56%	6.69%	5.12%

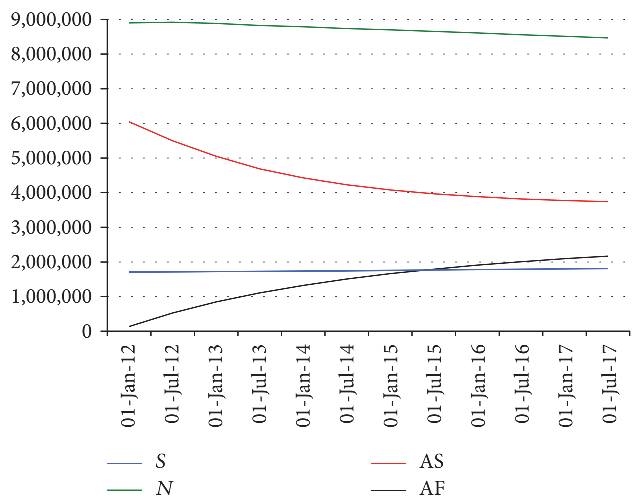

Figure 2: Subpopulation trends.

psychological aggressors became most sensitive to variations in this parameter. When the probability of the couple lasting increased, the AF and AS subpopulations reduced. The maximum reduction for the AF and AS proportions occurred when $r = 1$ (100% couple stability probability). The reason behind those results was that the transit to AS and AF mainly occurred when the couple broke up, and the level of aggression of the jealous and possessive men against women increased. Thus the smaller the number of couples breaking up, the fewer the aggressive behavior transits.

4. Conclusions and Recommendations

Our model quantified the future population of psychological and physical aggressors in Spain by taking into account dynamic factors, such as demographic, economic, and sociocultural (alcohol, drugs, jealousy, marital separations, and poverty index) factors. However, it did not take into account important factors such as genetics and/or legal environments, which most likely had an impact on our results. Data about genetic factors is confidential, while legal environment data can be simulated, but it is a fixed unchangeable factor in location and time period terms.

An important underlying consequence of this study is the prevalent potential and preventive role played by women in each domestic violence event, to the extent that the main recommendation to overcome this dramatic problem lies in women's active decision of breaking up their relationship in early stages of psychological abuse. This long-term recommendation requires strong willpower and Government action by investing in egalitarian education from early stage on childhood.

In the short-term the main recommendation to slow down this dramatic problem is sound and urgent public investment in education and media campaign (TV, social media, radio, public venues, and advertisements).

From our model, the hidden population of aggressors appears which is not quantified by official statistics and which reveals the problem when it is irreparable. Among the main reasons that explain this situation, we find poor recognition of abusive partners' behaviors in early stages of the relationship, combined with the social shame and low levels of self-esteem experienced by abused women. Lack of security resources (economic, legal protection, housing, and psychological resources) means that women barely report their abusive partners.

One of the advantages of having such a model like that presented herein is that the results can be simulated according to certain parameters, such as level of alcoholism consumption and drug use.

The study can be applied to any other geographical area where data are available, and the study period can be changed. However, it is important to take into account that the longer the study period, the less reliable the obtained results.

Competing Interests

The authors declare that they have no conflict of interests as to the publication of this paper.

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Research Article

Approximating the Solution Stochastic Process of the Random Cauchy One-Dimensional Heat Model

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This paper deals with the numerical solution of the random Cauchy one-dimensional heat model. We propose a random finite difference numerical scheme to construct numerical approximations to the solution stochastic process. We establish sufficient conditions in order to guarantee the consistency and stability of the proposed random numerical scheme. The theoretical results are illustrated by means of an example where reliable approximations of the mean and standard deviation to the solution stochastic process are given.

1. Introduction

The heat is the energy which flows from the higher to the lower temperature and the transport coefficient depends on the specific mode transfer. The transfer modes are the diffusive transport of thermal energy (the conduction mode), the exchange of heat between a moving fluid and an adjoining wall (the convection mode), and the radiation mode where all bodies can emit thermal radiation [1, 2]. In a metal rod with nonuniform temperature, heat is transferred from regions of higher temperature to regions of lower temperature. Usually, the physical principles for heat transfer are heat energy of a body with uniform properties, Fourier's law of heat transfer, and conservation of energy [2–4].

When dealing with a partial differential equation together with the initial and boundary conditions, it is crucial to obtain a well-posed problem. The extent of the spatial domain is another division for the partial differential equation that makes one method of solution preferable over another. Spatial domain may be a finite interval or an infinite interval, such as the whole real line. If the spatial domain is unbounded, the boundary conditions are not an important issue and in that case the problem is called initial value problem (IVP). In mathematics, a pure IVP is usually referred to as a Cauchy problem [3, 5]. This paper is concerned with the study of random finite difference schemes (one of the most widely used methods for engineering models) to the Cauchy problem for the one-dimensional random heat equation with unbounded spatial domain

$$u_t(x,t) = \beta u_{xx}(x,t), \quad t > 0, \quad -\infty < x < \infty, \tag{1}$$

with initial condition

$$u(x,0) = u_0(x). \tag{2}$$

In this IVP (1)-(2), t is the time variable, x is the space coordinate, u_t and u_{xx} denote the first and the second derivatives with respect to t and x, respectively, and β is a random variable defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, $u_0(x)$ is an initial deterministic data function. Expression (1) is a random parabolic partial differential equation for temperature $u(x,t)$ in a heat conducting insulated impurity rod along the x-axis since the conductivity coefficient, β, is assumed to be a random variable. The physical significance of thermal diffusion coefficient is associated with the speed of the flux of heat into the material when changes of temperature take place over the time. The heating propagation rate is proportional to the thermal diffusivity [6]. As it is stated in the thermodynamics' laws, β should be a function of two independent and intensive dynamic properties (usually, temperature and pressure) [7]. From the second thermodynamics law, it is required that β be positive. In this paper, we take β as a random variable since the randomness of heat transfer depends on the randomness of the conductivity coefficient. The randomness of β may be from the impurity material properties used to make the rod.

Mathematical models described by means of partial differential equations (PDEs) appear often in many areas of science and engineering and also in medicine and finance, for example, [8–10]. The heat equation has a great deal of application in many branches of sciences, naturally in a variety of models from chemistry, theoretical physics, and others [1]. The Cauchy problem model (1)-(2) is set on an unbounded space domain, so we do not need boundary conditions explicitly. There are analytical and numerical methods for dealing with problem (1)-(2) in case the conductivity diffusive coefficient is a constant or a deterministic function.

In the deterministic scenario, the heat equation on unbounded domains has been studied by different authors [11–13]. This important Cauchy problem appears in areas such as acoustic, electrodynamics, and fluid mechanics [12, 14, 15]. In the random context, the heat model has been studied using analytical techniques based on random Fourier series [16] or random Fourier integral transforms [17, 18]. In all of these contributions uncertainty is considered throughout a very general pattern and they are solved using the so-called L_p-random calculus [19, 20]. The heat model has also been treated considering that randomness is a white noise (the derivative of the Brownian motion or Wiener process). This approach requires the so-called Itô calculus [21, 22]. Under this approach the uncertainty is assumed to be Gaussian. This is a nice statistical property that has enabled the development of both analytic and numerical methods to study the stochastic heat model [23].

In this paper, we propose a random finite numerical scheme to approximate the solution s.p. of the Cauchy problem (1)-(2) and we prove its consistency and stability in a random sense that will be specified later. An important issue regarding our study is that we permit that, apart from the Gaussian distribution, r.v. β that appears in the PDE (1) can also have another quite general probability distributions.

This paper is organized as follows. In Section 2 firstly a random numerical finite difference scheme for the Cauchy problem (1)-(2) is proposed. Secondly, sufficient conditions for the consistency and stability of the random numerical scheme are given. Section 3 addresses the illustration of the theoretical results established by means of an illustrative example. Conclusions are drawn in Section 4.

2. Random Finite Difference Technique

This section is devoted to introducing the numerical technique that will be considered later in order to approximate the solution s.p. to the random IVP (1)-(2). Firstly, it is convenient to introduce some notation that will be used throughout our analysis. With this goal, let us consider a uniform space grid Δx and a uniform time grid Δt which defines a two-dimensional space-time mesh grid where the exact solution s.p. to the random IVP (1)-(2), $u(x,t)$, will be approximated. This approximation at the point $(x_k, t_n) = (k\Delta x, n\Delta t)$ or the mesh grid point (k, n), $k \in \mathbb{Z}$, $n \in \mathbb{N}$, will be denoted by u_k^n; that is, $u_k^n \approx u(x_k, t_n)$.

The next step is to approximate the solution s.p. to the IVP (1)-(2) on the mesh grid using some kind of approximation of the partial derivatives that appear in the formulation of that problem. In this paper, the following time-forward and space-centered discretization will be considered:

$$\begin{aligned} u_t(k\Delta x, n\Delta x) &\approx \frac{u_k^{n+1} - u_k^n}{\Delta t}, \\ u_{xx}(k\Delta x, n\Delta x) &\approx \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{(\Delta x)^2}. \end{aligned} \tag{3}$$

Substituting the approximations (3) in (1)-(2), one obtains the following random finite difference scheme (RFDS):

$$\begin{aligned} u_k^{n+1} &= (1-2r)\, u_k^n + r u_{k+1}^n + r u_{k-1}^n, \\ u_k^0 &= u_0(k\Delta x), \end{aligned} \tag{4}$$

where

$$r = \beta \frac{\Delta t}{(\Delta x)^2}. \tag{5}$$

As it is well known from the deterministic case, the study of the consistency and stability is a main issue when dealing with numerical schemes. This motivates the analysis of consistency and stability of the random numerical scheme (4)-(5) in a stochastic sense that will be specified later. Since the approximations of the solution s.p. to the IVP (1)-(2) will be constructed in the sense of *fixed station* for the time, hereinafter we will work in the following Banach space $(\ell_2(\Omega), \|\cdot\|_{\mathrm{RV}})$ [24] defined by

$$\ell_2(\Omega) = \{\mathbf{v} = (\ldots, v_{-1}, v_0, v_1, \ldots) : \|\mathbf{v}\|_{\mathrm{RV}} < +\infty\}, \tag{6}$$

$$\|\mathbf{v}\|_{\mathrm{RV}} = \left(\mathbb{E}\left[\left(\sup_k |v_k|\right)^2\right]\right)^{1/2}, \tag{7}$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator. Notice that the supremum in (7) is taken for every $k \in \mathbb{Z}$; however in order to simplify the notation, henceforth we will omit the symbol $\mathbb{Z}$.

2.1. Study of the Consistency of the Random Finite Difference Numerical Scheme. According to the definition of the consistency of a finite difference numerical scheme in the deterministic case, below we extend this definition to the random scenario taking into account the norm (7). For the sake of completeness, we also introduce the definition of the order of a RFDS as a natural generalization of the classical definition.

Definition 1. The random finite difference scheme

$$\mathbf{u}^{n+1} = Q(\mathbf{u}^n) + \Delta t \mathbf{G}^n, \tag{8}$$

being

$$\begin{aligned} \mathbf{u}^n &= (\ldots, u_{-1}^n, u_0^n, u_1^n, \ldots)^\top, \\ \mathbf{G}^n &= (\ldots, G_{-1}^n, G_0^n, G_1^n, \ldots)^\top, \end{aligned} \tag{9}$$

is said to be mean square $\|\cdot\|_{\mathrm{RV}}$ consistent with the random partial differential equation (RPDE) $\mathscr{L}u = F$, if the solution stochastic process (s.p.) of the RPDE, u, satisfies

$$\mathbf{u}^{n+1} = Q(\mathbf{u}^n) + \Delta t \mathbf{G}^n + \Delta t \boldsymbol{\tau}^n, \tag{10}$$

$$\|\boldsymbol{\tau}^n\|_{\mathrm{RV}} \xrightarrow[\Delta t \to 0]{\Delta x \to 0} 0, \tag{11}$$

where the kth component of $\mathbf{u}^n$ in (10) is

$$u_k^n = u(x_k, t_n). \tag{12}$$

Definition 2. In the context of Definition 1, the RFDS is said to be of order (p, q) if

$$\|\boldsymbol{\tau}^n\|_{\mathrm{RV}} = \mathscr{O}((\Delta t)^p) + \mathscr{O}((\Delta x)^q). \tag{13}$$

Next, we shall prove that the RFDS (4)-(5) is mean square $\|\cdot\|_{\mathrm{RV}}$ consistent with the random IVP (1)-(2).

Proposition 3. *Let us consider the random IVP (1)-(2) and assume that its solution s.p.* $u(x,t)$ *satisfies*

$$\begin{gathered} u_{tt}(x,t), \; u_{xxxx}(x,t) \\ \text{are uniformly bounded for every } (x,t), \\ x \in \mathbb{R}, \; t > 0. \end{gathered} \tag{14}$$

Then, the RFDS (4)-(5) is mean square $\|\cdot\|_{RV}$ *consistent. Moreover, this scheme has order* $(p,q) = (1,2)$.

Proof. Let us denote u_k^n as the exact value of the solution s.p. $u(x,t)$ at the mesh grid point (x_k, t_n). Based on expression (10) of Definition 1 with $\mathbf{G} = \mathbf{0}$ and expressions (4) and (5), let us consider the Taylor expansion of the kth component of $\mathbf{u}^{n+1} - Q(\mathbf{u}^n)$, bearing in mind hypotheses (14) and

$$\begin{aligned} &\left(\mathbf{u}^{n+1} - Q(\mathbf{u}^n)\right)_k = u_k^{n+1} - (1-2r) u_k^n - r u_{k+1}^n - r u_{k-1}^n \\ &\quad = u_k^{n+1} - u_k^n - r\{u_{k+1}^n - 2u_k^n + u_{k-1}^n\} = \left[u_k^n + (u_t)_k^n \right. \\ &\quad \left. \cdot \Delta t + \mathscr{O}\left((\Delta t)^2\right)\right] - u_k^n - r\left\{\left[u_k^n + (u_x)_k^n \Delta x \right.\right. \\ &\quad \left. + (u_{xx})_k^n \frac{(\Delta x)^2}{2} + (u_{xxx})_k^n \frac{(\Delta x)^3}{6} + \mathscr{O}\left((\Delta x)^4\right)\right] \\ &\quad - 2u_k^n + \left[u_k^n - (u_x)_k^n \Delta x + (u_{xx})_k^n \frac{(\Delta x)^2}{2} \right. \\ &\quad \left.\left. - (u_{xxx})_k^n \frac{(\Delta x)^3}{6} + \mathscr{O}\left((\Delta x)^4\right)\right]\right\} = \left\{(u_t)_k^n \right. \\ &\quad \left. - \beta (u_{xx})_k^n\right\} \Delta t + \mathscr{O}\left((\Delta t)^2\right) + \mathscr{O}\left(\Delta t (\Delta x)^2\right). \end{aligned} \tag{15}$$

Since u is a solution of random IVP (1)-(2), one gets $(u_t)_k^n - \beta(u_{xx})_k^n = 0$ and hence the first term of the right-hand side of (15) vanishes. Then

$$\begin{aligned} \Delta t \tau_k^n &= \left(\mathbf{u}^{n+1} - Q(\mathbf{u}^n)\right)_k \\ &= \mathscr{O}\left((\Delta t)^2\right) + \mathscr{O}\left(\Delta t (\Delta x)^2\right), \\ \tau_k^n &= \mathscr{O}(\Delta t) + \mathscr{O}\left((\Delta x)^2\right). \end{aligned} \tag{16}$$

Now, taking into account (7) and (16) one gets

$$\|\boldsymbol{\tau}^n\|_{\mathrm{RV}} = \left(\mathbb{E}\left[\left(\sup_k |\tau_k^n|\right)^2\right]\right)^{1/2} \xrightarrow[\Delta t \to 0]{\Delta x \to 0} 0, \tag{17}$$

and the order of the scheme is $(p,q) = (1,2)$. □

2.2. Study of the Stability of the Random Finite Difference Numerical Scheme. Following the same idea we have used for introducing the concept of random consistency, below we extend the deterministic definition of stability of a finite numerical scheme to the random scenario using the norm (7).

Definition 4. The random finite difference scheme (8) is said to be mean square $\|\cdot\|_{\mathrm{RV}}$ stable if there exist positive constants $\epsilon, \delta > 0$, and nonnegative constants η, ξ such that

$$\|\mathbf{u}^n\|_{\mathrm{RV}} \le \eta e^{\xi t} \|\mathbf{u}^0\|_{\mathrm{RV}}, \tag{18}$$

for $0 \le t \le (n+1)\Delta t$, $0 < \Delta x \le \epsilon$, and $0 < \Delta t \le \delta$.

Below, we establish conditions under which the RFDS (8) is mean square $\|\cdot\|_{\mathrm{RV}}$ stable.

Proposition 5. *Let us consider the random IVP (1)-(2) where* β *is a positive and bounded r.v.,*

$$0 < \beta(\omega) \le \beta_1, \quad \omega \in \Omega, \; \beta_1 \in \mathbb{R}. \tag{19}$$

Then, under the condition

$$\Delta t \le \frac{(\Delta x)^2}{2\beta_1}, \tag{20}$$

the RFDS (4)-(5) is mean square $\|\cdot\|_{RV}$ *stable.*

Proof. Taking into account the definition of the norm (7), the definition of mean square $\|\cdot\|_{\mathrm{RV}}$ stability (see (18)), and the RFDS (4)-(5), let us consider

$$\begin{aligned} \left(\|\mathbf{u}^{n+1}\|_{\mathrm{RV}}\right)^2 &= \mathbb{E}\left[\left(\sup_k |u_k^{n+1}|\right)^2\right] = \mathbb{E}\left[\sup_k |u_k^{n+1}|^2\right] \\ &= \mathbb{E}\left[\sup_k \left|(1-2r) u_k^n + r u_{k+1}^n + r u_{k-1}^n\right|^2\right] \\ &\le \mathbb{E}\left[\sup_k \left[\left|(1-2r) u_k^n\right| + \left|r u_{k+1}^n\right| + \left|r u_{k-1}^n\right|\right]^2\right] \end{aligned}$$

$$= \mathbb{E}\left[\sup_k \left[(1-2r)^2\left|u_k^n\right|^2 + r^2\left|u_{k+1}^n\right|^2 + r^2\left|u_{k-1}^n\right|^2\right.\right.$$
$$+ 2r\left|1-2r\right|\left|u_k^n u_{k+1}^n\right| + 2r\left|1-2r\right|\left|u_k^n u_{k-1}^n\right|$$
$$\left.\left.+ 2r^2\left|u_{k+1}^n u_{k-1}^n\right|\right]\right] = \mathbb{E}\left[(1-2r)^2\sup_k\left|u_k^n\right|^2 + r^2\right.$$
$$\cdot\sup_k\left|u_k^n\right|^2 + r^2\sup_k\left|u_k^n\right|^2 + 2r\left|1-2r\right|\sup_k\left|u_k^n\right|^2$$
$$\left.+ 2r\left|1-2r\right|\sup_k\left|u_k^n\right|^2 + 2r^2\sup_k\left|u_k^n\right|^2\right] = \mathbb{E}\left[\left[(1\right.\right.$$
$$\left.-2r)^2 + r^2 + r^2 + 2r\left|1-2r\right| + 2r\left|1-2r\right| + 2r^2\right]$$
$$\left.\cdot\sup_k\left|u_k^n\right|^2\right] = \mathbb{E}\left[\left[(1-2r)^2 + 4r^2 + 4r\left|1-2r\right|\right]\right.$$
$$\left.\cdot\sup_k\left|u_k^n\right|^2\right], \tag{21}$$

where we have used that r.v. r is positive because β is a positive r.v. (see (5) and (19)).

Under hypotheses (19)-(20), we can assure that r.v. $r = r(\omega)$ satisfies

$$0 \le r \le \frac{1}{2}, \tag{22}$$

for all $\omega \in \Omega$; hence $|1-2r| = 1-2r$. Therefore

$$(1-2r)^2 + 4r^2 + 4r\left|1-2r\right| = 1, \tag{23}$$

for all $\omega \in \Omega$. Then applying recursively (21), one obtains

$$\begin{aligned}\left(\left\|\mathbf{u}^{n+1}\right\|_{\mathrm{RV}}\right)^2 &\le \mathbb{E}\left[\sup_k\left|u_k^n\right|^2\right] = \left(\left\|\mathbf{u}^n\right\|_{\mathrm{RV}}\right)^2 \le \cdots \\ &\le \left(\left\|\mathbf{u}^0\right\|_{\mathrm{RV}}\right)^2,\end{aligned} \tag{24}$$

or equivalently

$$\left\|\mathbf{u}^{n+1}\right\|_{\mathrm{RV}} \le \left\|\mathbf{u}^0\right\|_{\mathrm{RV}}. \tag{25}$$

To summarize, condition (18) holds for $\eta = 1$ and $\xi = 0$. □

Remark 6. It is important to point out that the hypothesis of boundedness on r.v. β assumed in (19) in order to guarantee the mean square $\|\cdot\|_{\mathrm{RV}}$ stability of the RFDS (4)-(5) is not restrictive from a practical standpoint. Indeed, the classical Chebyshev inequality assures that any second-order random variable, with mean μ_β and standard deviation σ_β, can be approximated by truncating adequately its domain. Using this result it is easy to prove that the truncated interval $[\mu_\beta - 10\sigma_\beta, \mu_\beta + 10\sigma_\beta]$ contains 99% of the probability of β regardless of the distribution of β. The larger the truncated interval, the better the probabilistic approximation. Naturally, the diameter of the truncation interval can be shortened if the probability distribution of β is known. For example, if β is an unbounded r.v. having a Gaussian distribution, $\beta \sim N(\mu_\beta;\sigma_\beta)$, then the truncation over the domain $[\mu_\beta - 3\sigma_\beta, \mu_\beta + 3\sigma_\beta]$ contains 99.7% of the probability of β.

3. Numerical Example

This section is devoted to illustrating the theoretical results previously established by means of a test example where reliable approximations for the mean and the standard deviation (or equivalently the variance) of the solution s.p. of IVP (1)-(2) are given. These approximations are constructed using the RFDS (4)-(5). These approximations are compared with the corresponding exact values since the example has been chosen in such a way that both the mean and the standard deviation of the solution s.p. are available.

Let us consider the random Cauchy problem (1)-(2) where β is a r.v. of parameters $(a;b) = (2;3)$, $\beta \sim \mathrm{Be}(2;3)$, and the initial condition is $u_0(x) = \exp(-x^2)$. The exact solution s.p. of (1)-(2) is given by

$$u(x,t) = \frac{\exp\left(-x^2/(1+4\beta t)\right)}{\sqrt{1+4\beta t}}. \tag{26}$$

We will approximate the mean and standard deviation of the solution s.p., $u(x,t)$, of the random Cauchy problem (1)-(2) on the spatial domain $-2 \le x \le 2$ using the RFDS (4)-(5). In order to guarantee the mean square $\|\cdot\|_{\mathrm{RV}}$ stability of this scheme, first we fix the space step Δx and we take $\beta_1 = 1$, since $\beta \sim \mathrm{Be}(2;3)$; then according to Proposition 5 (see condition (20)), the time step Δt must be taken satisfying the following condition:

$$\Delta t \le \frac{(\Delta x)^2}{2}. \tag{27}$$

In order to compute approximations of the mean and the standard deviation of the solution s.p. $u(x,t)$ at the mesh grid point (x_k, t_n), we will apply recursively the numerical scheme (4)-(5) and then we will take the expectation operator. The numerical results will be compared with the ones obtained from expression (26) using the following expression:

$$\mathbb{E}[u(x,t)] = \int_0^1 \frac{\exp\left(-x^2/(1+4\beta t)\right)}{\sqrt{1+4\beta t}} f_\beta(\beta)\,\mathrm{d}\beta, \tag{28}$$

for the mean, and

$$\sigma[u(x,t)] = \sqrt{\int_0^1 \frac{\exp\left(-2x^2/(1+4\beta t)\right)}{1+4\beta t} f_\beta(\beta)\,\mathrm{d}\beta - \left(\mathbb{E}[u(x,t)]\right)^2}, \tag{29}$$

for the standard deviation, being

$$f_\beta(\beta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\beta^{a-1}(1-\beta)^{b-1}, \quad a = 2,\ b = 3. \tag{30}$$

In Figure 1 we show a comparison at the time instant $t = 2/5$ (*time fixed station*) of the expectation of the exact solution s.p. and the approximations of the expectations using the random numerical scheme (4)-(5) with different spatial steps

$$\Delta x = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}. \tag{31}$$

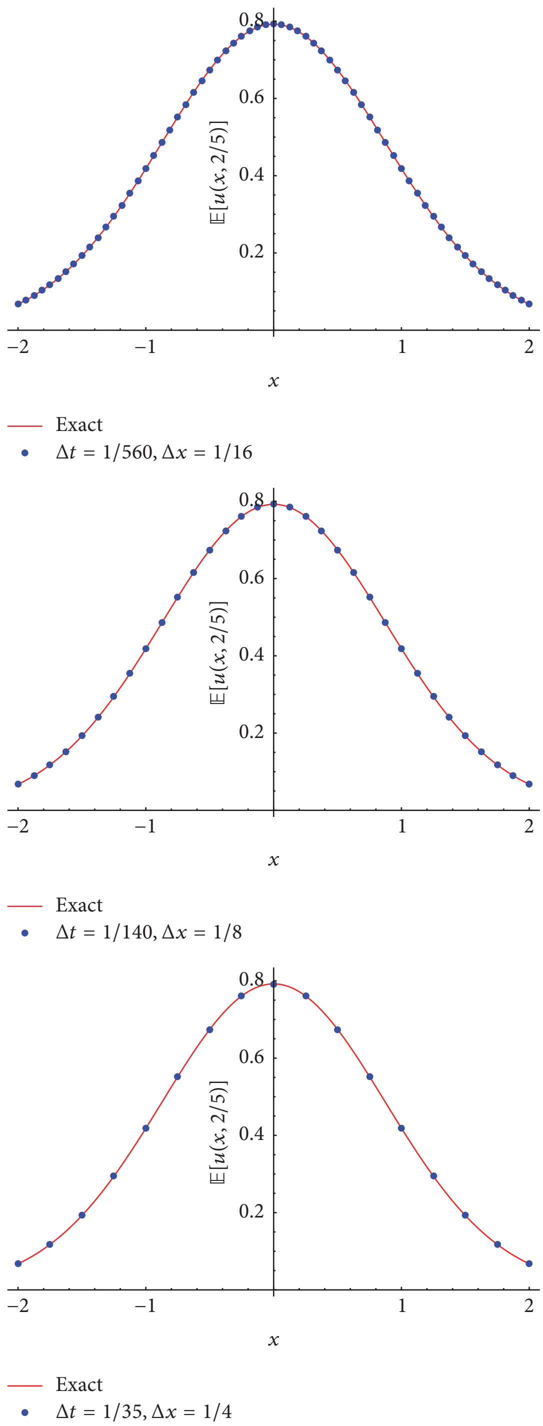

FIGURE 1: Expectation of the exact solution s.p. and the approximations at the time instant $t = 2/5$ for different values of Δx and Δt over the spatial domain $-2 \leq x \leq 2$.

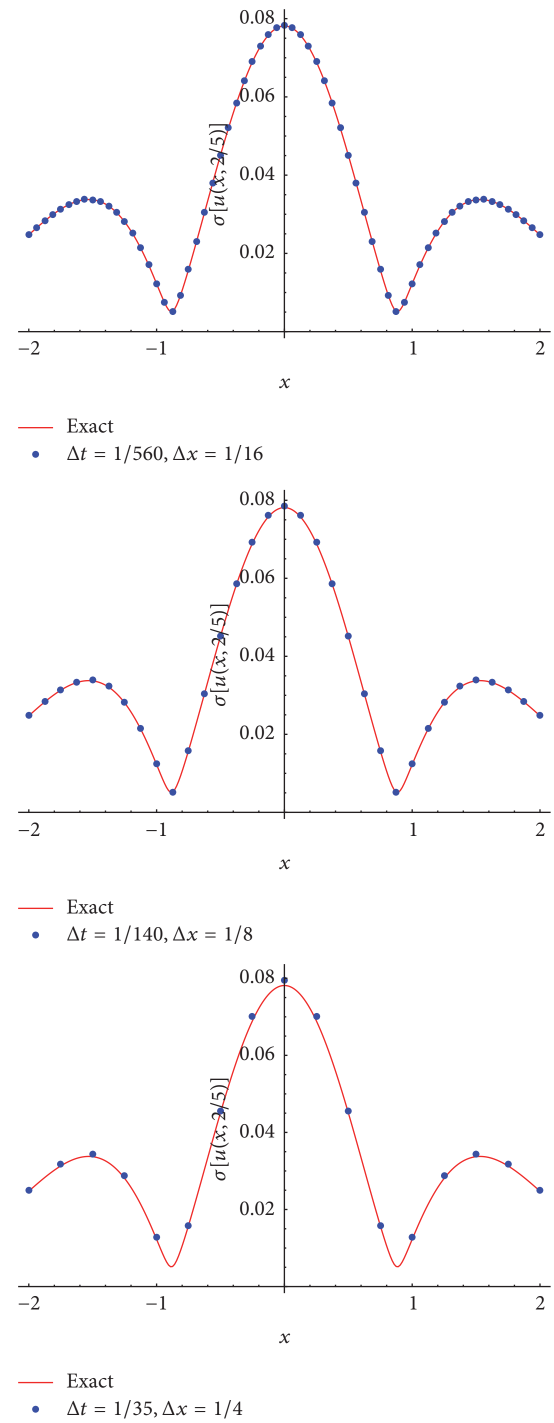

FIGURE 2: Standard deviation of the exact solution s.p. and the approximations at the time instant $t = 2/5$ for different values of Δx and Δt over the spatial domain $-2 \leq x \leq 2$.

The time steps have been chosen as

$$\Delta t = \frac{1}{560}, \frac{1}{140}, \frac{1}{35}, \tag{32}$$

respectively, so that stability condition (27) is guaranteed.

An analogous comparison for the standard deviation at the time instant $t = 2/5$ is shown in Figure 2.

To complete the numerical analysis, in Figures 3 and 4 we have plotted the relative errors for the approximations of the expectation and standard deviation for the spatial and time steps previously chosen, respectively. From these plots

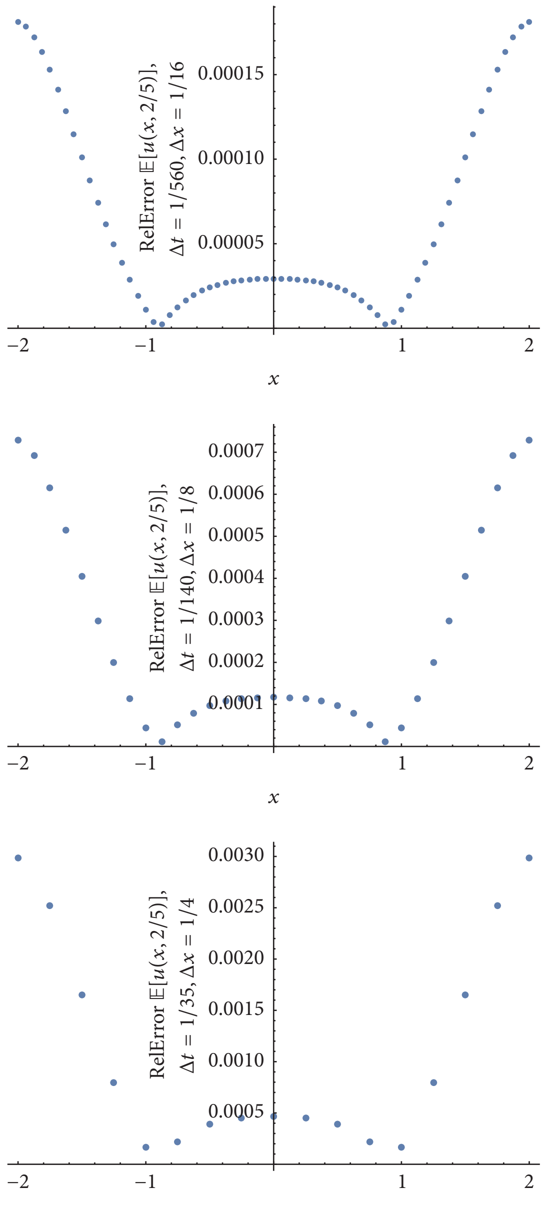

FIGURE 3: Relative errors at the time instant $t = 2/5$ for the approximations of the expectation for different values of Δx and Δt over the spatial domain $-2 \le x \le 2$.

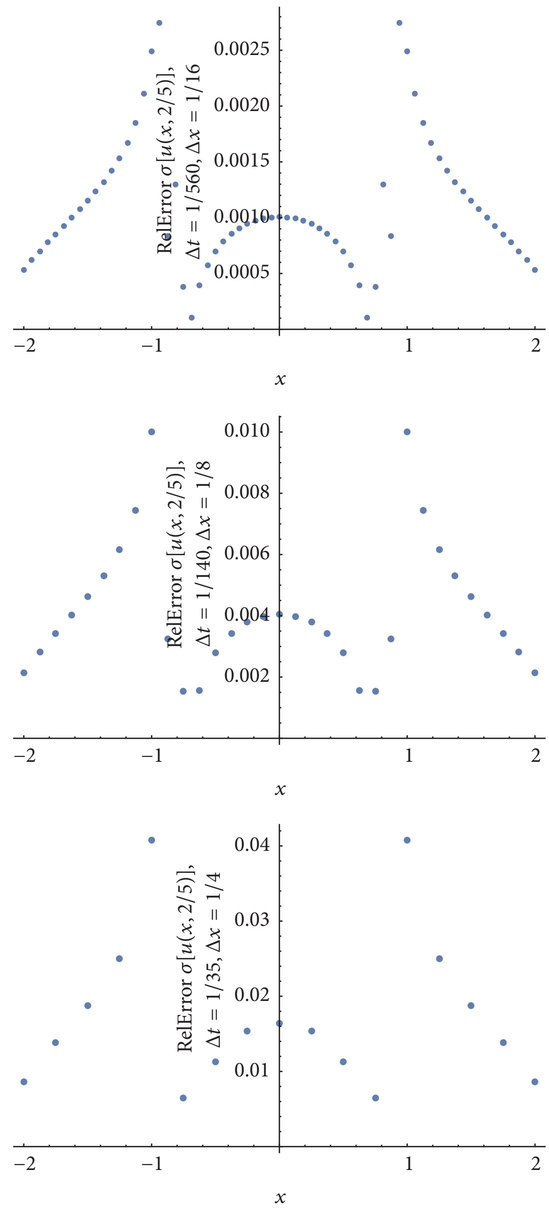

FIGURE 4: Relative errors at the time instant $t = 2/5$ for the approximations of the standard deviation for different values of Δx and Δt over the spatial domain $-2 \le x \le 2$.

we observe that as Δx is divided by 2, the relative error is approximately divided by 4. This confirms the convergence of the random numerical scheme.

4. Conclusions

In this paper we have studied the randomized Cauchy heat model by assuming that the diffusion coefficient is a random variable and considering a deterministic initial condition over an unbounded domain. Thus, boundary conditions have not been required. We have proposed a random finite difference scheme for solving this model. The mean square consistency of the random finite difference scheme has been studied. Sufficient conditions for the mean square stability of the random finite difference scheme have been provided. The numerical experiments show that the proposed random finite difference scheme gives reliable approximations for the mean and the standard deviation of the solution stochastic process.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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