Abstract and Applied Analysis

Recent Developments in Integral Transforms, Special Functions, and Their Extensions to Distributions Theory

Guest Editors: Adem Kılıçman, Mustafa Bayram, and Hassan Eltayeb

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Editorial Board

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Editorial

Recent Developments in Integral Transforms, Special Functions, and Their Extensions to Distributions Theory

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The distributions are also known as generalized functions which generalize the idea of classical functions and allow us to extend the concept of derivative not only to all continuous functions but also to the discontinuous functions in the classical sense. The theory of distributions has applications in various fields especially in science and engineering where there are many noncontinuous phenomena which might naturally lead to differential equations whose solutions are distributions. For example, such as the delta distribution is not mathematical function in classical sense. However it is very useful in several applications, therefore the distributions can help us to develop an operational calculus in order to investigate linear ordinary differential equations as well as partial differential equations with constant and variable coefficients through their fundamental solutions.

By considering some regular operations which are valid for ordinary functions such as addition, multiplication by scalar scan is extended into distributions. Other operations can only be defined for certain restricted subclasses; these are also known as irregular operations. However, generalized functions are useful tools to extend the concept of derivatives and further to use them to formulate the generalized solutions of partial differential equations. They are also very important in physics and engineering where many noncontinuous problems naturally might lead to differential equations whose solutions are distributions, such as the Dirac delta distribution. This special issue is focused on some integrals transform, special functions, and their applications with fractional orders. In some papers, further relationship between linear and nonlinear partial differential equations and generalized functions was also considered.

We are pleased to announce the completion of this special issue on Recent Developments in Integral Transforms, Special Functions, and Their Extensions to Distributions Theory. This special issue was opened in late August of 2012 and closed in late May of 2013.

In this special issue, a total of 33 articles were published and they cover a wide range of special functions changes from theoretical sides of special functions to their applications in solutions of certain differential equations by mostly focusing on some special functions in several areas. There were also some submissions on the different types of integral transforms and their applications including some works on generalized functions. For example, the modeling of thermal distributions around a barrier at the interface of coating and substrate was studied by A. Sahin; A. Secer studied the numerical solution and simulation of second-order parabolic PDEs with Sinc-Galerkin method which covers to solve second-order PDEs by numerical method. Similarly, some integrals involving q-Laguerre polynomials and applications were studied by J. Cao.

An efficient pseudospectral method was reported to solve a class of nonlinear optimal control problems, which were presented by E. Tohidi et al. By M. Inc et al., numerical solutions of the second-order one-dimensional telegraph equation were studied which was based on the reproducing kernel

Hilbert space method; in an interesting work by H.-L. Wu and J.-C. Lan, it was reported that Lipschitz estimates can be applied to fractional multilinear singular integrals on variable exponent Lebesgue spaces. In a paper by D. Kumar et al., an efficient approach for fractional harry dym equation by using Sumudu transform was presented. The stability of trigonometric functional equations in distributions and hyperfunctions was studied by J. Chung and J. Chang.

Further, by A. Atangana and A. Secer, the fractional order derivatives and table of fractional derivatives of some special functions were reported.

Since it would be very lengthy to list all the contributions, thus, we suggest that the readers read the full special issue to see the further details of each study.

Acknowledgments

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Research Article

A Proposal to the Solution of Multiobjective Linear Fractional Programming Problem

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We have proposed a new solution to the Multiobjective Linear Fractional Programming Problem (MOLFPP). The proposed solution is based on a theorem that deals with nonlinear fractional programming with single objective function and studied in the work by Dinkelbach, 1967. As a new contribution, we have proposed that $\overline{x}$ is an efficient solution of MOLFPP if $\overline{x}$ is an optimal solution of problem $\mathrm{Max}_{x\in X}\sum_{i=1}^{k}(N_i(x) - Z_i^* D_i(x))$, where is $Z_i^* = N_i(x_i^*)/D_i(x_i^*)$ for all i. Hence, MOLFPP is simply reduced to linear programming problem (LPP). Some numerical examples are provided in order to illustrate the applications of the proposed method. The optimization software package, namely, WinQSB (Chang, 2001), has been employed in the computations.

1. Introduction

Fractional programming concerns with the optimization problem of one or several ratios of functions subject to some constraints. These ratios are quantities that measure the efficiency of system, such as cost/profit, cost/time, cost/volume, and output/worker, while several ratios of functions are measured in different scales at the existence of some conflicts. The optimal solution for an objective function may not be an optimal solution for some other objective functions. Therefore, one needs to find the notion of the best compromise solution, also known as nondominant solution [1, 2].

In the literature, for various types of fractional programming, there are many different sorts of studies; some of them deal with theory [3–6], and some of them concern with solution methods [2, 7–18] and applications [19]. Dinkelbach [7] presented the algorithm based on a theorem by Jagannathan [20] concerning the relationship between fractional and parametric programming and restated and proved this theorem in somewhat simpler way. Leber et al. proposed [19] to use a fractional programming algorithm (the Dinkelbach algorithm) to calculate the melting temperature of pairings of two single DNA strands in biology.

If both the numerator and dominator of these ratios of functions in fractional programming are linear functions under some technological linear restrictions, then we have the multiple objective linear fractional programming (MOLFP) problems. There are so many studies including different approaches to solve different models of MOLFP problems in literature. Kornbluth and Steuer [21] proposed some possible linear fractional criteria [1] and have presented a generalized approach for solving a goal programming with linear fractional criteria [22]. Luhandjula [23] proposed a linguistic variable approach to solve a MOLFP problem. This approach simply and adequately describes imprecise aspirations of the decision maker to obtain a solution that is in some sense good in his/her opinion. These linguistic descriptions are considered as fuzzy objectives and are aggregated as in fuzzy linear programming [1].

Dutta et al. [24] modified the linguistic approach of Luhandjula such as to develop a method which yields always an efficient solution for optimising MOLFP problem. Stancu-Minasian and Pop [2] pointed out certain shortcomings in the work of Dutta et al. and have given the correct proof of theorem, which validates the obtaining of the efficient solutions. Lee and Tcha [25] developed iterative solution method to generate a sequence of linear inequality problems by parameterizing objective values to obtain a compromise solution of MOLFP problem. Chakraborty and Gupta [22] have presented a different methodology that always yields an efficient solution for solving MOLFP problem. In this methodology, MOLFPP may be solved easily with the transformation $y = tx, t > 0$ resulting in a multiple objective linear programming (MOLP) problem. t has been considered as the least value of both $1/D_i(x)$ if objective function $Z_i(x) \geq 0$ for some x in the feasible region and $1/-N_i(x)$ if objective function $Z_i(x) < 0$, for each x in the feasible region. After original MOLFP problem reduces to an equivalent MOLP problem, the resulting MOLP problem is solved using fuzzy set theoretical approach by suitably defined membership functions and using min operator introduced by Zimmerman.

In this paper, we have investigated a solution to the MOLFP problem based on a theorem previously studied by Dinkelbach [7]. We have proposed that a feasible solution $\overline{x}$ of MOLFPP is an efficient solution of MOLFPP if $\overline{x} \in R^n$ is an optimal solution of problem $\text{Max}_{x\in X} \sum_{i=1}^{k}(N_i(x) - Z_i^* D_i(x))$, where is $Z_i^* = N_i(x_i^*)/D_i(x_i^*)$ for all i. Thus, MOLFPP is reduced to linear programming problem (LPP), and its solution procedure can be easily applied.

1.1. Linear Fractional Programming Problem (LFPP). The general LFPP is defined as follows:

$$\underset{X\in x}{\text{Maximize}} \frac{N(x)}{D(x)}. \tag{1}$$

$N(x) = c^T x + \alpha$, $D(x) = d^T x + \beta$ are valued and continuous functions on X and $d^T x + \beta \neq 0$ for all X and $X = \{x \mid Ax = b, x \geq 0\}$, $x \in R^n$, $b \in R^m$, $A \in R^{n\times m}$.

$c^T, d^T \in R^n$, $\alpha, \beta \in R$ are assumed to be nonempty convex and compact set in R^n.

Theorem 1. *Consider*

$$\begin{aligned} \text{Max} \quad & \frac{N(x)}{D(x)} \\ \text{s.t.} \quad & Ax \leq b, \\ & x \geq 0, x \in X = \{x \mid Ax = b, x \geq 0\} \\ & \Longrightarrow D(x) > 0. \end{aligned} \tag{2a}$$

$$\begin{aligned} \text{Max} \quad & tN\left(\frac{y}{t}\right) \\ \text{s.t.} \quad & A\left(\frac{y}{t}\right) - b \leq 0, \\ & tD\left(\frac{y}{t}\right) \leq 1, \\ & t > 0, y \geq 0. \end{aligned} \tag{2b}$$

For some $\xi \in X$, $N(\xi) \geq 0$, if (2a) *reaches a (global) maximum at $x = x^*$, then* (2b) *reaches a (global) maximum at point $(t, y) = (t^*, y^*)$, where $y^*/t^* = x^*$ and the objective functions at these points are equal [22, 26].*

Theorem 2 (see [22, 26]). *If* (2a) *is a standard concave-convex fractional programming problem which reaches a (global) maximum at point x^*, then the corresponding transformed problem* (2b) *attains the same maximum value at a point (t^*, y^*), where $y^*/t^* = x^*$. Moreover* (2b) *has a concave objective function and a convex feasible set.*

Theorem 3 (see [7]). $z^* = N(x^*)/D(x^*) = \max\{N(x)/D(x) \mid x \in X\}$ *if and only if* $F(z^*, x^*) = \max\{N(x) - z^* D(x) \mid x \in X\} = 0$.

2. Proposed Approach for Objective Functions of MOLFP Problem

The vector-maximum Multiple Objective Linear Fractional programming (MOLFP) problem is defined as follows:

$$\text{Maximize} \left\{ Z(x) = \left(\frac{N_1(x)}{D_1(x)}, \frac{N_2(x)}{D_2(x)}, \ldots, \frac{N_k(x)}{D_k(x)} \right) \mid x \in X \right\}, \tag{3}$$

where $X = \{x \in R^n / Ax \leq b, x \geq 0\}$ is convex and nonempty bounded set, A is an $m \times n$ constraint matrix, x is an n-dimensional vector of decision variable, and $b \in R^m$, $k \geq 2$, $N_i(x) = c_i^T x + \alpha_i$, $D_i(x) = d_i^T x + \beta_i$, for all $i = 1, \ldots, k$, $c_i^T, d_i^T \in R^n$, $\alpha_i, \beta_i \in R$, for all $i = 1, \ldots, k$, $D_i(x) = d_i x + \beta_i > 0$, for all $i = 1, \ldots, k$, for all $x \in X$.

Definition 4. $\overline{x} \in R^n$ is an efficient solution of MOLFP if there is no $\hat{x} \in R^n$ such that $N_i(\hat{x})/D_i(\hat{x}) \geq N_i(\overline{x})/D_i(\overline{x})$, $i = 1, 2, \ldots, k$, $N_i(\hat{x})/D_i(\hat{x}) > N_i(\overline{x})/D_i(\overline{x})$, for at least one i.

In this study, in order to solve MOLFP problem in (3), we can maximize each objective function $Z_i(x)$ subject to the given set of constraints using one of the methods proposed for single fractional objective function in [27] or others. Let x_i^* and Z_i^* be the global maximum points and values of each objective function $\text{Max}\{Z_i(x) = (c_i x + \alpha_i)/(d_i x + \beta_i) \mid x \in X\}$ for all $i = 1, 2, \ldots, k$. Now, we can prove that the solution $\overline{x}$ is an efficient solution of $\text{Max}\{Z_i(x) = (c_i x + \alpha_i)/(d_i x + \beta_i), i = 1, 2, \ldots, k \mid x \in X\}$.

If $\overline{x}$ is an optimal solution of problem: $\text{Max}\{\sum_{i=1}^{k}(N_i(x) - Z_i^* D_i(x)), \mid x \in X\}$, where is $Z_i^* = N_i(x_i^*)/D_i(x_i^*)$ for all $i = 1, 2, \ldots, k$.

Let $\overline{x}$ maximise problem $\text{Max}\{\sum_{i=1}^{k}(N_i(x) - Z_i^* D_i(x)) \mid x \in X\}$; then we can write inequality $\sum_{i=1}^{k}(N_i(x) - Z_i^* D_i(x)) \leq \sum_{i=1}^{k}(N_i(\overline{x}) - Z_i^* D_i(\overline{x}))$ for any feasible solution $x \in X$. Hence,

$$\begin{aligned} \sum_{i=1}^{k}(N_i(x) - Z_i^* D_i(x)) &\leq \sum_{i=1}^{k}(N_i(\overline{x}) - Z_i^* D_i(\overline{x})) \\ &\leq \sum_{i=1}^{k} \max\{N_i(x) - Z_i^* D_i(x)\} \\ &\leq \sum_{i=1}^{k}(N_i(x_i^*) - Z_i^* D_i(x_i^*)) = 0 \\ & \text{for } x \in X. \end{aligned} \tag{4}$$

From these inequalities, one obtains $N_i(x) - Z_i^* D_i(x) \leq N_i(\overline{x}) - Z_i^* D_i(\overline{x}) \leq 0$, for all i, $x \in X$.

We have $D_i(x)[N_i(x)/D_i(x) - Z_i^*] \leq D_i(\overline{x})[N_i(\overline{x})/D_i(\overline{x}) - Z_i^*]$ for all i:

$$\left[\frac{N_i(x)}{D_i(x)} - Z_i^*\right] \leq \frac{D_i(\overline{x})}{D_i(x)}\left[\frac{N_i(\overline{x})}{D_i(\overline{x})} - Z_i^*\right]. \quad (5)$$

Both via Theorem 3 and the inequality $D_i(\overline{x})/D_i(x) \geq 1$, one can write that $[N_i(x)/D_i(x) - Z_i^*] \leq [N_i(\overline{x})/D_i(\overline{x}) - Z_i^*]$ and $[N_i(x)/D_i(x)] \leq [N_i(\overline{x})/D_i(\overline{x})]$ for all i. If $\overline{x}$ maximise the problem $\text{Max}\{\sum_{i=1}^{k}(N_i(x) - Z_i^* D_i(x)) \mid x \in X\}$, then it is an efficient solution of $\text{Max}\{Z_i(x) = (c_i x + \alpha_i)/(d_i x + \beta_i), i = 1, 2, \ldots, k \mid x \in X\}$. Now, assume that $\overline{x}$ is not an efficient of MOLFPP; then there exists a feasible solution x of MOLFPP and $N_i(\overline{x})/D_i(\overline{x}) \leq N_i(x)/D_i(x)$ for all i and $N_j(\overline{x})/D_{ij}(\overline{x}) < N_j(x)/D_j(x)$ at least one j, where $x, \overline{x} \in X$. It follows that $N_i(\overline{x})/D_i(\overline{x}) \leq N_i(x)/D_i(x) \leq Z^*$, $N_i(\overline{x}) - Z_i^* D_i(\overline{x}) \leq N_i(x) - Z_i^* D_i(x)$ for all i and $N_j(\overline{x}) - Z_j^* D_i(\overline{x}) < N_j(x) - Z_i^* D_i(x)$ at least one j. Summing the k-inequalities, we have $\sum_{i=1}^{k}(N_i(x) - Z_i^* D_i(x)) \leq \sum_{i=1}^{k}(N_i(\overline{x}) - Z_i^* D_i(\overline{x}))$. This inequality leads to a contradiction.

Thus, we have made a proposal for the solution of MOLFP based on the above proof. These examples considered by Chakraborty and Gupta in [22] use Zimmermann's min operator for the fuzzy model.

3. Numerical Examples

Example 1. Let us consider a MOLFPP with two objectives as follows:

$$\begin{aligned} &\text{Max} \quad \left\{Z_1(x) = \frac{-3x_1 + 2x_2}{x_1 + x_2 + 3},\ Z_2(x) = \frac{7x_1 + x_2}{5x_1 + 2x_2 + 1}\right\} \\ &\text{s.t.} \quad x_1 - x_2 \geq 1 \\ &\qquad 2x_1 + 3x_2 \leq 15 \\ &\qquad x_1 \geq 3 \\ &\qquad x_1, x_2 \geq 0. \end{aligned} \quad (6)$$

It is observed that $Z_1 < 0$, $Z_2 \geq 0$, for each x in the feasible region:

$$\frac{-15}{7} \leq Z_1 \leq \frac{-14}{23}, \qquad \frac{139}{121} \leq Z_2 \leq \frac{105}{77}, \quad (7)$$

This MOLFPP is equivalent to the following LPP. The given MOLFP problem can be written as

$$\begin{aligned} \text{Max} \quad & \left\{-3x_1 + 2x_2 + \frac{14}{23}(x_1 + x_2 + 3)\right. \\ & \left. +7x_1 + x_2 - \frac{105}{77}(5x_1 + 2x_2 + 1)\right\}, \\ \text{s.t.} \quad & x_1 - x_2 \geq 1, \\ & 2x_1 + 3x_2 \leq 15, \\ & x_1 \geq 3, \\ & x_1, x_2 \geq 0. \end{aligned} \quad (8)$$

The solution of the ealier linear programming problem is obtained as $x_1 = 3$, $x_2 = 2$.

The solution for original problem is given by

$$x_1 = 3, \quad x_2 = 2, \quad Z_1 = \frac{-5}{8}, \quad Z_2 = \frac{23}{20}. \quad (9)$$

Example 2. Let us consider a MOLFPP with three objectives as follows:

$$\begin{aligned} \text{Max} \quad & \left\{Z_1(x) = \frac{-3x_1 + 2x_2}{x_1 + x_2 + 3},\right. \\ & Z_2(x) = \frac{7x_1 + x_2}{5x_1 + 2x_2 + 1}, \\ & \left. Z_3(x) = \frac{x_1 + 4x_2}{2x_1 + 3x_2 + 2}\right\}, \\ \text{s.t.} \quad & x_1 - x_2 \geq 1, \\ & 2x_1 + 3x_2 \leq 15, \\ & x_1 + 9x_2 \geq 9, \\ & x_1 \geq 3, \\ & x_1, x_2 \geq 0. \end{aligned} \quad (10)$$

It is observed that $Z_1 < 0$, $Z_2 \geq 0$, $Z_3 \geq 0$ for each x in the feasible region. These values are $-53/26 \leq Z_1 \leq -14/23$, $139/121 \leq Z_2 \leq 23/17$, and $8/17 \leq Z_3 \leq 14/17$.

The earlier MOLFP problem is equivalent to the following LP problem:

$$\begin{aligned} \text{Max} \quad & \left\{-3x_1 + 2x_2 - \frac{14}{23}(x_1 + x_2 + 3)\right. \\ & + 7x_1 + x_2 - \frac{23}{17}(5x_1 + 2x_2 + 1) + x_1 \\ & \left. + 4x_2 - \frac{14}{17}(2x_1 + 3x_2 + 2)\right\}, \\ \text{s.t.} \quad & x_1 - x_2 \geq 1 \\ & 2x_1 + 3x_2 \leq 15 \\ & x_1 + 9x_2 \geq 9 \\ & x_1 \geq 3. \end{aligned} \quad (11)$$

The solution of the above linear programming problem is obtained as $x_1 = 3, x_2 = 2$. The solution for the given MOLFP problem is given by

$$x_1 = 3, \quad x_2 = 2, \quad Z_1 = \frac{-5}{8}, \quad Z_2 = \frac{23}{20}, \quad Z_3 = \frac{11}{14}. \quad (12)$$

4. Conclusion

In this paper, we have presented a new solution to the Multiobjective Linear Fractional Programming Problem (MOLFPP). The solution is based on a theorem proposed in [7] dealing with nonlinear fractional programming with single objective function. With the help of this suggested approach, all of linear fractional objective functions of MOLFP problem become a single objective function. Furthermore the MOLFP problem is transformed into LPP. Thus, the complexity and the computations in solving MOLFP problem reduce in a certain amount. We used two numerical examples solved with different methods in [18, 22, 28].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

An Efficient Pseudospectral Method for Solving a Class of Nonlinear Optimal Control Problems

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This paper gives a robust pseudospectral scheme for solving a class of nonlinear optimal control problems (OCPs) governed by differential inclusions. The basic idea includes two major stages. At the first stage, we linearize the nonlinear dynamical system by an interesting technique which is called linear combination property of intervals. After this stage, the linearized dynamical system is transformed into a multi domain dynamical system via computational interval partitioning. Moreover, the integral form of this multidomain dynamical system is considered. Collocating these constraints at the Legendre Gauss Lobatto (LGL) points together with using the Legendre Gauss Lobatto quadrature rule for approximating the involved integrals enables us to transform the basic OCPs into the associated nonlinear programming problems (NLPs). In all parts of this procedure, the associated control and state functions are approximated by piecewise constants and piecewise polynomials, respectively. An illustrative example is provided for confirming the accuracy and applicability of the proposed idea.

1. Introduction

Optimal control problems (OCPs) have received considerable attention during the last four decades because of their applications. Such problems arise in many areas of science and engineering and play an important role in the modeling of real-life phenomena in other fields of science. The principal difficulty in studying OCPs via traditional and classical methods lies in their special nature. Obviously, most of OCPs cannot be solved by the well-known indirect methods [1, 2]. Therefore, it is highly desirable to design accurate direct numerical approaches to approximate the solutions of OCPs [3].

Among all of the numerical techniques for solving smooth OCPs, orthogonal functions and polynomials have been applied in a huge size of research works. High accuracy and ease of applying these polynomials and functions for OCPs are two important advantages which have encouraged many authors to use them for different types of problems. For solving smooth OCPs, there exist a broad class of methods based on orthogonal polynomials which were presented by famous applied mathematics scientists such as [4, 5]. The fundamental idea of these methods is based upon pseudospectral (or spectral collocation) operational matrices of differentiation. However, Legendre spectral operational matrix of differentiation was used in [6] (for other applications of spectral operational matrices of differentiation see [7]). The best property of the spectral operational matrices of differentiation is the sparsity, while the pseudospectral ones are relatively filled matrices. Another computational approach for solving OCPs which is based on high order Gauss quadrature rules was presented in [8]. However, high order of accuracy may be obtained by this method, but suitable preconditionings should be explored because of its ill-conditioning of the associated algebraic system.

In many real mathematical models, the controller should be restricted. In other words, the control functions of OCPs are bounded in many cases. According to the classical theory of optimal control [9], if the control functions are bounded and appear linearly in the cost functionals and dynamical systems, the resulting problem is a Bang-Bang OCP. In this case, the control functions are discontinuous. Therefore, we deal with a nonsmooth OCP. For dealing with such nonsmooth OCPs, some new numerical methods have been proposed in the literature such as [10, 11]. These approaches are based on finite difference methods (FDMs). Simplicity of the discretization by FDMs is usually easy to handle, but lower order of accuracy may make them unsuccessful. Therefore, we should look at high order numerical methods such as spectral or pseudospectral techniques. But, as it is mentioned in the literature, spectral schemes are the best tools just for the problems with smooth solutions and data. In other words, if we apply these methods for approximating nonsmooth functions we usually observe the Gibbs phenomena. The following example illustrates this fact.

Example 1 (see [9]). We consider the following OCP:

$$\begin{aligned} \text{Min} \quad & J = \int_0^2 (3u(\tau) - 2y(\tau))\, d\tau \\ \text{s.t.} \quad & \dot{y}(\tau) = y(\tau) + u(\tau), \quad 0 \le \tau \le 2, \\ & y(0) = 4, \quad y(2) = 39.392, \\ & u(\tau) \in [0, 2], \quad 0 \le \tau \le 2. \end{aligned} \tag{1}$$

Since the computational interval is $[0, 2]$, we should change it into $[-1, 1]$ by a simple transformation as follows:

$$\begin{aligned} \text{Min} \quad & J = \int_{-1}^1 (3u(t) - 2y(t))\, dt \\ \text{s.t.} \quad & \dot{y}(t) = y(t) + u(t), \quad -1 \le t \le 1, \\ & y(-1) = 4, \quad y(1) = 39.392, \\ & u(t) \in [0, 2], \quad -1 \le t \le 1. \end{aligned} \tag{2}$$

The optimal control of the above-mentioned problem is given in the following form:

$$u^*(t) = \begin{cases} 2 & -1 \le t \le 0.096, \\ 0 & 0.096 \le t \le 1. \end{cases} \tag{3}$$

For approximating the control function of this problem, we use the classical spectral method [6]. As it is depicted in Figures 1 and 2, the desired optimal control cannot be obtained in a good manner. From these Figures one can observe that not only the exact value of switching point (i.e., $t = 0.096$) is not detected with a high accuracy, but also the obtained solutions have additional jumps in the boundary of domain. These are the disadvantages of the applying the classical spectral methods for solving nonsmooth problems.

To delete these mentioned disadvantages, a robust spectral method is presented in [12] for solving a class of nonsmooth OCPs that has some fundamental differences with the classical spectral techniques. First, the computational interval is partitioned into subintervals where the size of each subinterval is considered as an unknown parameter, and this enables us to compute the switching times more efficiently. Second, in contrast with the classical spectral schemes,

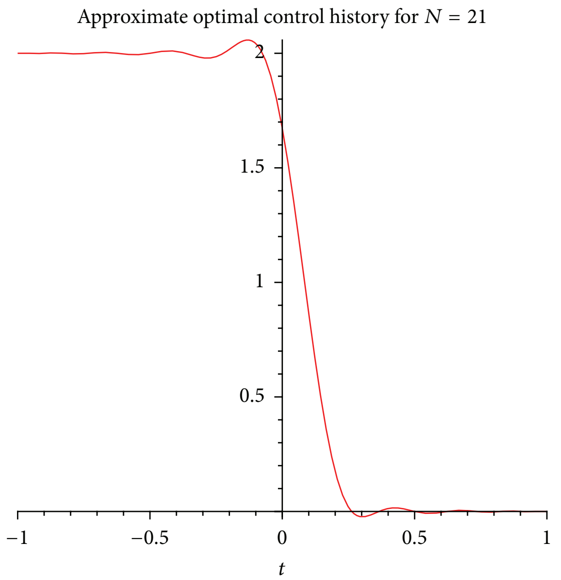

FIGURE 1: Approximate optimal control history of Example 1 for $N = 21$.

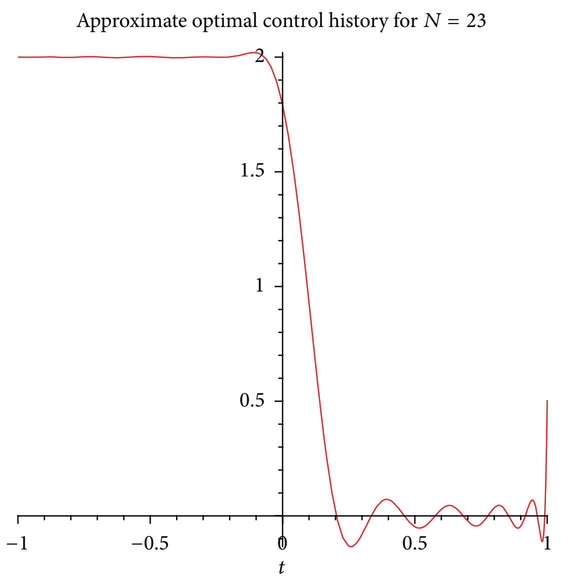

FIGURE 2: Approximate optimal control history of Example 1 for $N = 23$.

the integral form of the dynamical system is considered. This equivalent form is found by integrating the differential dynamics and adding the initial conditions.

Our fundamental goal of this paper is to extend a new idea which was introduced in [12] to approximate the control and state functions of the following nonsmooth OCP:

$$\begin{aligned} \text{Min} \quad & J = \int_0^{t_f} f(t, y(t))\, dt \\ \text{s.t.} \quad & \dot{y}(t) \in d(t), \quad t \in [0, t_f], \\ & (y(0), y(t_f)) \in S, \end{aligned} \tag{4}$$

where $d(t)$ is a set of continuous functions on $[0, t_f]$ and S is a set which contains boundary points of state variable $y(t)$. Also, $y(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^n$, and $f(t, y(t)) \in \mathfrak{R}$. According to discussions in [11], we can assume that

$$d(t) = \{D(t, u(t)) : u(t) \in U\}, \quad t \in [0, t_f], \tag{5}$$

where $U \subset \mathfrak{R}^n$ is a compact set and $D(t, u(t)) = (D_1(t, u(t)), D_2(t, u(t)), \ldots, D_n(t, u(t)))^T$ is a continuous function on $[0, t_f] \times U$. Thus, OCP (4) can be rewritten in the form

$$\begin{aligned} \text{Min} \quad & J = \int_0^{t_f} f(t, y(t))\, dt \\ \text{s.t.} \quad & \dot{y}(t) = D(t, u(t)), \quad u(t) \in U, \\ & y(0) = y_0, \quad y(t_f) = y_f. \end{aligned} \tag{6}$$

It should be noted that the dynamical system of (6) is nonlinear in terms of control $u(t)$. For handling OCP (6) in a proper manner, we first linearize the nonlinear dynamical system by an interesting technique which is called linear combination property of intervals (LCPI). After this stage, the linearized dynamical system is transformed into a multidomain dynamical system via computational interval partitioning. Collocating these constraints at the Legendre Gauss Lobatto (LGL) points together with using the Legendre Gauss Lobatto quadrature rule for approximating the involved integrals enables us to transform the basic OCPs into the associated nonlinear programming problems (NLPs).

The paper is organized as follows. Section 2 is devoted to linearize the nonlinear dynamical system by using LCPI. In Section 3, we design our basic idea which is based on approximation of the associated control and state functions by piecewise constant and piecewise polynomials, respectively. It should be noted that Legendre Gauss Lobatto points are used for collocating the linearized dynamical system. In Section 4, we present a numerical example, demonstrating the efficiency of the suggested numerical algorithm. Concluding remarks are given in Section 5.

2. Dynamical System Linearization

Since $D : [0, t_f] \times U \rightarrow \mathfrak{R}^n$ is continuous and $[0, t_f] \times U$ is a compact and connected subset of $\mathfrak{R}^{n+1}$, then $\{D(t, u(t)) : u \in U\}$ is a closed set in $\mathfrak{R}^n$. Thus, $\{D_i(t, u(t)) : u \in U\}$ for $i = 1, 2, \ldots, n$ is closed in $\mathfrak{R}$. Now, suppose that the lower and upper bounds of the $\{D_i(t, u(t)) : u \in U\}$ are $l_i(t)$ and $v_i(t)$, respectively. Therefore,

$$l_i(t) \le D_i(t, u(t)) \le v_i(t), \quad t \in [0, t_f]. \tag{7}$$

In other words,

$$\begin{aligned} l_i(t) &= \min_u \{D_i(t, u(t)) : u \in U\}, \quad t \in [0, t_f], \\ v_i(t) &= \max_u \{D_i(t, u(t)) : u \in U\}, \quad t \in [0, t_f]. \end{aligned} \tag{8}$$

By using LCPI, which was first introduced in [10], $D_i(t, u(t))$ can be approximated as a convex linear combination of its minimum $l_i(t)$ and maximum $v_i(t)$ in the following form:

$$\begin{aligned} D_i(t, u(t)) &\approx \lambda_i(t) v_i(t) + (1 - \lambda_i(t)) l_i(t) \\ &= \lambda_i(t) \alpha_i(t) + l_i(t), \end{aligned} \tag{9}$$

where $\alpha_i(t) = v_i(t) - l_i(t)$ and $\lambda_i(t) \in [0, 1]$. It should be mentioned that all the $\lambda_i(t)$ are the new associated control variables. Now, the main problem (6) is approximated by the following OCP:

$$\begin{aligned} \text{Min} \quad & \int_0^{t_f} f(t, y(t))\, dt \\ \text{s.t.} \quad & \dot{y}(t) = A(t) \Lambda(t) + l(t), \\ & \Lambda(t) \in \overbrace{[0,1] \times [0,1] \times \cdots \times [0,1]}^{n \text{ times}}, \quad t \in [0, t_f], \\ & y(0) = y_0, \quad y(t_f) = y_f, \end{aligned} \tag{10}$$

where $A(t) = \operatorname{diag}(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))_{n\times n}$, $\Lambda(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))_{n\times 1}$, and $l(t) = (l_1(t), l_2(t), \ldots, l_n(t))_{n\times 1}$. Note that problem (10) is a Bang-Bang OCP, because in this problem the new control $\Lambda(t)$ has lower 0 and upper 1 bounds and appears linearly in the dynamical equations. As soon as the controls are assumed to be bang-bang, the problem of finding the required controls becomes one of finding the switching times.

3. Discretization of the New OCP Containing Linearized Dynamical System

In the sequel and for simplicity in the discretization procedure, we assume that $n = 1$ (in other words, $\Lambda(t) = \lambda(t)$) and suppose that problem (10) has an optimal solution with $m \ge 1$ switching points denoted by $t_1, t_2, \ldots, t_m$. So if we set $t_0 = 0$ and $t_{m+1} = t_f$, then the interval $[0, t_f]$ breaks into $m+1$ subintervals. That is,

$$[0, t_f] = [t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_m, t_{m+1}], \tag{11}$$

where on each subinterval, $\lambda(t)$ is constant. We denote $\lambda(t)$ in kth subinterval with constant b^k. Since $0 \le \lambda(t) \le 1$, we have

$$0 \le b^k \le 1, \quad k = 1, 2, \ldots, m + 1. \tag{12}$$

Moreover, we take the restriction of $y(t)$ to the kth subinterval by $y^k(t)$. By considering (11), the dynamical system of (10) is conveyed as

$$\begin{aligned} \dot{y}^k(t) &= A(t) b^k + l(t), \\ t_{k-1} &\le t \le t_k, \quad k = 1, 2, \ldots, m + 1, \end{aligned} \tag{13}$$

$$y^1(0) = y_0, \tag{14}$$

$$y^k(t_{k-1}) = y^{k-1}(t_{k-1}), \quad k = 2, 3, \ldots, m + 1. \tag{15}$$

It should be noted that (15) is assumed to guarantee the continuity of state functions. Integrating (13) gives rise to the following dynamic equations:

$$y^k(t) = c^{k-1} + \int_{t_{k-1}}^{t} \left(A(s)\, b^k + l(s)\right) ds, \quad k = 1, \ldots, m+1, \tag{16}$$

where $t_{k-1} \le t \le t_k$ and

$$c^k = \begin{cases} y_0 & k = 0, \\ y^k(t_k) & k = 1, 2, \ldots, m. \end{cases} \tag{17}$$

The final condition $y(t_f) = y_f$ is imposed only on $y^{m+1}(t)$ as

$$y^{m+1}(t_{m+1}) = y_f. \tag{18}$$

Therefore, problem (10) is transformed into the following optimization problem:

$$\begin{aligned} \text{Min} \quad & J = \sum_{k=1}^{s+1} \int_{t_{k-1}}^{t_k} f\left(s, y^k(s)\right) ds \\ \text{s.t.} \quad & y^k(t) = c^{k-1} + \int_{t_{k-1}}^{t} \left(A(s)\, b^k + l(s)\right) ds, \\ & t_{k-1} \le t \le t_k \quad k = 1, 2, \ldots, m+1, \\ & y^{s+1}(t_{s+1}) = y_f, \\ & 0 \le b^k \le 1, \quad k = 1, 2, \ldots, m+1. \end{aligned} \tag{19}$$

For discretizing (19), we assume $\hat{s}_i^k$, $i = 0, 1, \ldots, N$, to be the shifted LGL nodes to subinterval $[t_{k-1}, t_k]$; that is, $\hat{s}_i^k = (s_i)((t_k - t_{k-1})/2) + ((t_k + t_{k-1})/2)$. By using Lagrange interpolation, we approximate $y^k(t)$ by

$$y^k(t) \approx \sum_{i=0}^{N} y_i^k \hat{L}_i^k(t), \tag{20}$$

where $y_i^k = y^k(\hat{s}_i^k)$ and

$$\hat{L}_i^k(t) = L_i\left(\frac{2}{t_k - t_{k-1}} t - \frac{t_k + t_{k-1}}{t_k - t_{k-1}}\right). \tag{21}$$

It should be noted that $L_i(t)$ is the ith Lagrange basis function. Since $y^k(t)$ is approximated, therefore f can be approximated in the kth subinterval as follows:

$$\begin{aligned} f\left(s, y^k(s)\right) &\approx \sum_{j=0}^{N} f\left(\hat{s}_j^k, y^k\left(\hat{s}_j^k\right)\right) \hat{L}_j^k(t) \\ &= \sum_{j=0}^{N} f\left(\hat{s}_j^k, y_j^k\right) \hat{L}_j^k(t). \end{aligned} \tag{22}$$

Now by substituting approximations (20) and (22) into (19), we get

$$\sum_{j=0}^{N} y_j^k \hat{L}_j^k(t) = c^{k-1} + \int_{t_{k-1}}^{t} \left(A(s)\, b^k + l(s)\right) ds, \quad t_{k-1} \le t \le t_k. \tag{23}$$

From (17) and (20) for $k = 1, \ldots, m$, we have $c^k = y_N^k$. Now if we set $y_N^0 = y_0$, then we obtain

$$c^k = y_N^k, \quad k = 0, \ldots, m+1. \tag{24}$$

Collocating (23) at $\hat{s}_i^k$, $i = 0, \ldots, N$, $k = 1, \ldots, m+1$, yields

$$y_i^k = y_N^{k-1} + b^k \int_{t_{k-1}}^{\hat{s}_i^k} A(s)\, ds + \int_{t_{k-1}}^{\hat{s}_i^k} l(s)\, ds, \quad i = 0, 1, \ldots, N. \tag{25}$$

Now, by applying a simple linear transformation, we transform the interval $[t_{k-1}, \hat{s}_i^k]$ into $[-1, 1]$ as follows:

$$s = \frac{\hat{s}_i^k - t_{k-1}}{2}\eta + \frac{\hat{s}_i^k + t_{k-1}}{2}, \quad i = 0, 1, \ldots, N. \tag{26}$$

Therefore, the Legendre Gauss Lobatto quadrature rule can be applied in the following form:

$$\begin{aligned} y_i^k &= y_N^{k-1} + \frac{\hat{s}_i^k - t_{k-1}}{2}\left(b^k \int_{-1}^{1} \hat{A}(\eta)\, d\eta + \int_{-1}^{1} \hat{l}(\eta)\, d\eta\right) \\ &\approx y_N^{k-1} + \frac{\hat{s}_i^k - t_{k-1}}{2}\left\{\sum_{q=0}^{N} w_q \left(b^k \hat{A}(s_q) + \hat{l}(s_q)\right)\right\}, \\ & \qquad 0 \le i \le N, \end{aligned} \tag{27}$$

where $\hat{A}(\eta) = A(((\hat{s}_i^k - t_{k-1})/2)\eta + ((\hat{s}_i^k + t_{k-1})/2))$, $\hat{l}(\eta) = l(((\hat{s}_i^k - t_{k-1})/2)\eta + (\hat{s}_i^k + t_{k-1})/2)$, $w_q = (2/N(N+1))(1/P_N^2(s_q))$ for $q = 0, 1, \ldots, N$ are the LGL weights and $P_N(x)$ is the Nth degree Legendre Polynomial.

So by considering $y^{m+1}(t_f) = y_N^{m+1}$, problem (19) is discretized to the following NLP:

$$\begin{aligned} \text{Min} \quad & J_{N,m} = \sum_{k=1}^{m+1} \sum_{j=0}^{N} \sum_{q=0}^{N} \frac{t_k - t_{k-1}}{2} f\left(\hat{s}_j^k, y_j^k\right) w_q L_j(s_q) \\ \text{s.t.} \quad & y_i^k - y_N^{k-1} - \frac{\hat{s}_i^k - t_{k-1}}{2}\left\{\sum_{q=0}^{N} w_q \left(b^k \hat{A}(s_q) + \hat{l}(s_q)\right)\right\} \\ & \quad = 0 \\ & i = 0, 1, \ldots, N, \quad k = 1, 2, \ldots, m+1, \\ & y_N^{s+1} - y_f = 0, \\ & 0 \le b^k \le 1, \quad k = 1, 2, \cdots, m+1. \end{aligned} \tag{28}$$

Here, b^k, y_i^k, $i = 0, \ldots, N$, $k = 1, \ldots, m$, and parameters $t_1, \ldots, t_s, t_f$ are unknown variables in the NLP. Note that y_N^0 is known and $y_N^0 = y_0$.

In the above discretization procedure, the number of switching points, s, is considered as a known parameter. So at first we should guess the number of switching points. This is the disadvantage of the proposed method. It should be noted that if the number of switching points, s, is chosen correctly, then the resulting value of b^k is equal to its lower or upper bounds; furthermore, b^k changes in each switching point.

4. Numerical Example

We now apply the proposed idea for solving a nonlinear OCP governed by differential inclusion. This example was first introduced in [11]. In the mentioned work, the authors used the simplest form of FDMs. One of the advantages of [11] is that we finally solve a Linear Programming (LP) problem. However, this method has other disadvantages such as needing higher values of approximations (i.e., N), and this leads to ill-conditioning of the associated discrete problem. Our presented ideas do not contains the difficulties of the classical spectral methods and FDMs for solving nonsmooth OCPs and also achieve superior results with respect to at least 3 other methods. These advantages confirm the efficiency of this modern spectral approximation. The following example is modeled using the mathematical software package MAPLE, and the corresponding nonlinear programming problem is solved using the command NLPSolve. It should be noted that if the NLP is univariate and unconstrained except for finite bounds, quadratic interpolation method may be used. If the problem is unconstrained and the gradient of the objective function is available, the preconditioned conjugate gradient (PCG) method may be used. Otherwise, the sequential quadratic programming (SQP) method can be used. According to the structure of our NLP, the SQP method is used.

Example 2. We consider the following nonlinear OCP governed by differential inclusion:

$$\begin{aligned} \text{Min} \quad & J = \int_0^1 \sin(3\pi t)\, y(t)\, dt \\ \text{s.t.} \quad & \dot{y}(t) \in \left\{ -\tan\left(\frac{\pi}{8} u^3(t) + t\right) : u(t) \in [0,1] \right\}, \\ & y(0) = 1, \quad y(1) = 0. \end{aligned} \tag{29}$$

According to discussions in [11], the above OCP can be rewritten in the following form:

$$\begin{aligned} \text{Min} \quad & J = \int_0^1 \sin(3\pi t)\, y(t)\, dt \\ \text{s.t.} \quad & \dot{y}(t) = -\tan\left(\frac{\pi}{8} u^3(t) + t\right), \\ & u(t) \in [0,1], \quad y(0) = 1, \quad y(1) = 0. \end{aligned} \tag{30}$$

Table 1: Numerical results of Example 2.

N	t_1	t_2	t_3	J_N
6	0.2218	0.4538	0.8874	0.0970
8	0.2214	0.4653	0.8683	0.0960
10	0.2214	0.4317	0.8341	0.0968
12	0.2214	0.4591	0.8890	0.0963
14	0.2214	0.4473	0.8762	0.0969
16	0.2214	0.4481	0.8971	0.0962

In this problem, the control function appears nonlinearly, and we should linearize the initial dynamical system. According to the idea of LCPI, the above OCP can be reduced to a linear OCP which is Bang-Bang. Here, $D(t, u(t)) = -\tan((\pi/8)u^3(t) + t)$. Thus,

$$\begin{aligned} l(t) &= \underset{u}{\text{Min}} \left\{ -\tan\left(\frac{\pi}{8} u^3(t) + t\right) : u(t) \in [0,1] \right\} \\ &= -\tan\left(\frac{\pi}{8} + t\right), \\ v(t) &= \underset{u}{\text{Max}} \left\{ -\tan\left(\frac{\pi}{8} u^3(t) + t\right) : u(t) \in [0,1] \right\} \\ &= -\tan(t), \end{aligned} \tag{31}$$

and hence, $\alpha(t) = v(t) - l(t) = \tan((\pi/8) + t) - \tan(t)$. Therefore, $D(t, u(t))$ can be approximated as follows:

$$\begin{aligned} D(t, u(t)) &\approx \alpha(t)\lambda(t) + l(t) \\ &\approx \left(\tan\left(\frac{\pi}{8} + t\right) - \tan(t)\right)\lambda(t) - \tan\left(\frac{\pi}{8} + t\right). \end{aligned} \tag{32}$$

It should be noted that $\lambda(t) \in [0, 1]$ is the new control function, which is called associated control. By considering this approximation for $D(t, u(t))$, the basic OCP is approximated by the following Bang-Bang OCP:

$$\begin{aligned} \text{Min} \quad & J = \int_0^1 \sin(3\pi t)\, y(t)\, dt \\ \text{s.t.} \quad & \dot{y}(t) = \left(\tan\left(\frac{\pi}{8} + t\right) - \tan(t)\right)\lambda(t) - \tan\left(\frac{\pi}{8} + t\right), \\ & \lambda(t) \in [0,1], \quad y(0) = 1, \quad y(1) = 0. \end{aligned} \tag{33}$$

According to our experiences in [11], we assume that the number of switching points is $s = 3$. Since by applying this assumption we reach to the exact results in which the new associated control $\lambda(t)$ is switched between its lower and upper bounds, the numerical results related to the values of switching points and objective function for different values of N are provided in Table 1. Moreover, the associated control $\lambda(t)$, control $u(t)$, and optimal state $y(t)$ are depicted in Figures 3, 4, and 5, respectively. Moreover, in Table 2 comparisons of the numerical results of the proposed method with respect to the methods of [6, 10, 13] are given. From

Table 2: Comparisons of the methods in evaluating the objective function J^*.

N	Proposed method	Method of [6]	Method of [13]	Method of [10]
6	0.0970	0.1196	0.1371	—
8	0.0960	0.1058	0.1289	—
10	0.0968	0.0964	0.1152	—
11	0.0960	0.1009	0.1094	—
100	—	—	—	0.0985

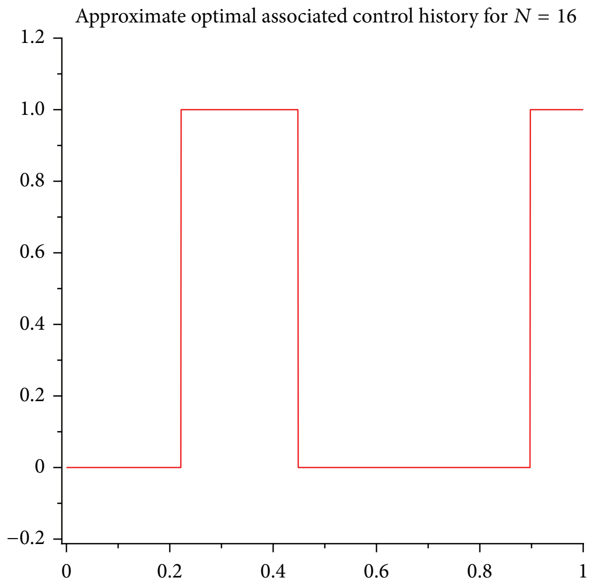

Figure 3: Approximate optimal associated control $\lambda(t)$ history of Example 2 for $N = 16$.

this table one can see the efficiency and applicability of the suggested method for solving nonlinear OCPs governed by differential inclusions.

5. Concluding Remarks

In this study, a robust numerical technique has been used for solving a class of optimal control problems (OCPs) governed by differential inclusions. The proposed idea includes linearizing the dynamical system in which the resulting problem is a Bang-Bang OCP. After obtaining this nonsmooth OCP, we use the general idea of [12] for dealing with such problems in the best manner. As observed in the numerical example, the proposed scheme has superior results with regard to at least 3 methods which confirm the applicability of the method. One of the disadvantages of our method is more sensitivity to initial guess in comparison with the classical spectral schemes. However, our idea is terminated successfully by considering an initial guess from the solution of the traditional spectral techniques, even for small values of N.

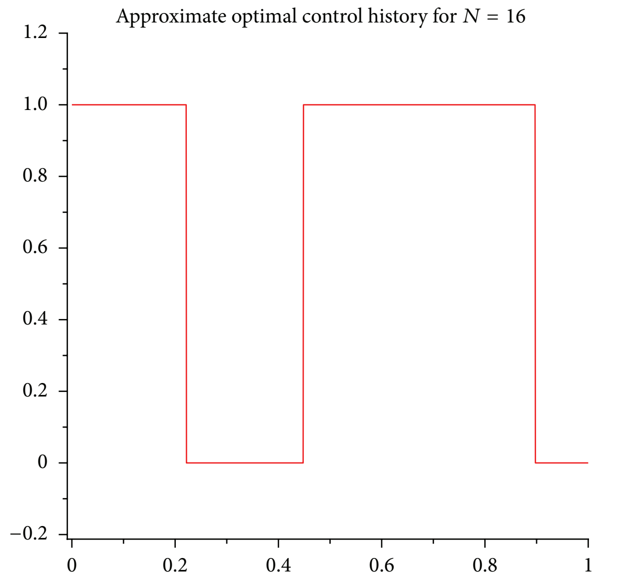

Figure 4: Approximate optimal control $u(t)$ history of Example 2 for $N = 16$.

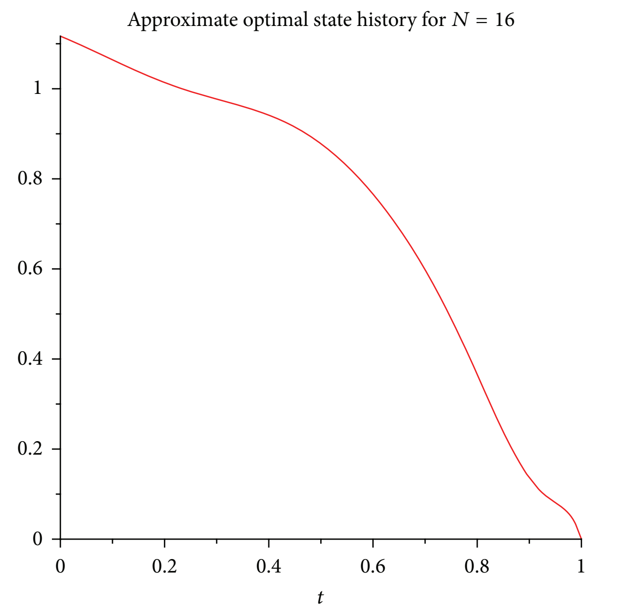

Figure 5: Approximate optimal state $y(t)$ history of Example 2 for $N = 16$.

Conflict of Interests

The authors declare that they do not have any conflict of interests in their submitted paper.

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Research Article

On a Kind of Dirichlet Character Sums

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Let $p \geq 3$ be a prime and let χ denote the Dirichlet character modulo p. For any prime q with $q < p$, define the set $E(q,p) = \{a \mid 1 \leq a, \overline{a} \leq p, a\overline{a} \equiv 1 (\mathrm{mod} p)$ and $a \equiv \overline{a} (\mathrm{mod} q)\}$. In this paper, we study a kind of mean value of Dirichlet character sums $\sum_{a \leq p\ a \in E(q,p)} \chi(a)$, and use the properties of the Dirichlet L-functions and generalized Kloosterman sums to obtain an interesting estimate.

1. Introduction

Let $k \geq 3$ be an integer and let χ denote the Dirichlet character modulo k, for any real number $x \geq 1$, many scholars have studied the following sums:

$$\sum_{n \leq x} \chi(n), \tag{1}$$

where n are positive integers.

Perhaps one of the most famous results is Pólya's inequality [1]. That is, when χ is the primitive character modulo k, we have

$$\sum_{n \leq x} \chi(n) < k^{1/2} \log k. \tag{2}$$

In fact, the result can be extended to the nonprincipal character χ modulo k [2]. Further details about the estimates of character sums can be found in the literature, for example, [3, 4].

For any fixed integer $H > 0$ and any positive integer $k \geq 3$, define the following set:

$$\begin{aligned} L(H,k) = \{ & a \mid 1 \leq a, \overline{a} \leq k-1, \\ & (a,k) = 1, a\overline{a} \equiv 1 (\mathrm{mod} k), |a - \overline{a}| \leq H \}, \end{aligned} \tag{3}$$

let χ denote the Dirichlet character modulo k, define the sums as follows:

$$\sum_{\substack{n \leq k \\ n \in L(H,k)}} \chi(n). \tag{4}$$

Xi and Yi [5] studied the problem for χ the nonprincipal Dirichlet character modulo k, and got

$$\sum_{\substack{n \leq k \\ n \in L(H,k)}} \chi(n) \ll k^{1/2} d(k) \log H, \tag{5}$$

where $0 < H \leq q$ was a constant and $d(k)$ was the divisor function. Before this, Wenpeng [6] got an asymptotic formula for the case that χ was the principal Dirichlet character modulo k.

On the other hand, for each integer a with $1 \leq a \leq k$ and $(a,k) = 1$, we know that there exists one and only one b with $1 \leq b \leq k$ such that $ab \equiv 1 (\mathrm{mod} k)$. Let $r_2(k)$ be the number of solutions of the congruent equation $ab \equiv 1 (\bmod k)$ for $1 \leq a, b \leq k$ in which a and b are of opposite parity, this can be expressed as follows:

$$r_2(k) = \sum_{\substack{a=1 \\ a\overline{a} \equiv 1 (\bmod k) \\ 2 \dagger (a + \overline{a})}}^{k} 1. \tag{6}$$

Richard [7] asks us to find $r_2(k)$ or at least to say something nontrivial about it. About this problem, a lot of scholars have studied it [8–12]. Now we let m be another integer with $m < k$ and let $r_m(k)$ denote the number of all pairs of integers a, b satisfying $ab \equiv 1 (\mathrm{mod} k)$, $1 \leq a, b \leq k$, and $m\dagger(a+b)$. Lu and Yi [13] have obtained the asymptotic formula of generalized D. H. Lehmer problem as follows:

$$r_m(k) = \sum_{\substack{a=1 \\ a\bar{a}\equiv 1(\bmod k) \\ m\dagger(a+\bar{a})}}^{k} 1 = \left(1-\frac{1}{m}\right)\phi(k) + O\left(k^{1/2}\log^2 k\right), \tag{7}$$

where the O constant only depends on m.

In this paper, let p be an odd prime and let q be a fixed prime with $q < p$, define the set $E(q,p)$ for $a(1 \le a \le p)$ such that $a\bar{a} \equiv 1(\bmod p)$ and $a \equiv \bar{a}(\bmod q)$, that is,

$$\begin{aligned} E(q,p) = \{ & a \mid 1 \le a, \bar{a} \le p-1, \\ & a\bar{a} \equiv 1(\bmod p), a \equiv \bar{a}(\bmod q)\}. \end{aligned} \tag{8}$$

As another case of (7), we will consider the mean value of Dirichlet character sums as follows:

$$\sum_{\substack{a\le p-1 \\ a\in E(q,p)}} \chi(a), \tag{9}$$

and get an interesting estimate. That is, we will prove the following theorem.

Theorem 1. *Let p be an odd prime and let q be a fixed prime with $q < p$, and let χ denote the Dirichlet character modulo p. Let $E(q,p)$ denote the following set:*

$$\begin{aligned} E(q,p) = \{ & a \mid 1 \le a \le p-1, \\ & a\bar{a} \equiv 1(\bmod p), a \equiv \bar{a}(\bmod q)\}, \end{aligned} \tag{10}$$

then, for any nonprincipal Dirichlet character $\chi \bmod p$, we have the following estimate:

$$\sum_{\substack{a\le p-1 \\ a\in E(q,p)}} \chi(a) = O\left(p^{1/2+\epsilon}\right), \tag{11}$$

where the O constant only depends on q.

From this Theorem we can get

$$\begin{aligned} \sum_{\substack{a=1 \\ a\bar{a}\equiv 1(\bmod p) \\ q\dagger(a-\bar{a})}}^{p-1} \chi(a) &= \sum_{\substack{a=1 \\ a\bar{a}\equiv 1(\bmod p)}}^{p-1} \chi(a) - \sum_{\substack{a\le p-1 \\ a\in E(q,p)}} \chi(a) \\ &= O\left(p^{1/2+\epsilon}\right). \end{aligned} \tag{12}$$

For any integer k and fixed integer m such that $(m,k)=1$, whether or not there exists an estimate for

$$\sum_{\substack{a\le k \\ a\in E(m,k)}} \chi(a) \tag{13}$$

is still an open problem.

2. Some Lemmas

In this section, we will give several lemmas which are necessary in the proof of the theorem.

Lemma 2. *Let Q be an integer, and let χ be a primitive character modulo Q. Then, for any real number u and v with $u < v$, we have*

$$\begin{aligned} \sum_{uQ<n\le vQ} \chi(n) = \tau(\chi) \sum_{0<|h|\le H} \bar{\chi}(h) \frac{e(-hu)-e(-hv)}{2\pi i h} \\ + O\left(1+\frac{Q\log Q}{H}\right), \end{aligned} \tag{14}$$

where $e(x) = e^{2\pi i x}$ and $\tau(\chi) = \sum_{a=1}^{Q} \chi(a)e(a/Q)$ are Gauss sums.

Especially, let $u = 0$, we have a slight modification

$$\sum_{0<n\le vQ} \chi(n) = \begin{cases} \dfrac{\tau(\chi)}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\sin(2\pi n v)}{n} + O(1), & \text{if } \chi(-1) = 1, \\ \dfrac{\tau(\chi)}{\pi i} \displaystyle\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)(1-\cos(2\pi n v))}{n} + O(1), & \text{if } \chi(-1) = -1. \end{cases} \tag{15}$$

Proof. (See [1]). □

Lemma 3. *Let q be a prime, let Q be an integer with $Q > q$, and let χ be a primitive character modulo Q, then, we have*

$$\sum_{n\le Q/q} \chi(n) = \begin{cases} \dfrac{\tau(\chi)}{(q-1)\pi} \displaystyle\sum_{l=1}^{q-1} \sum_{\chi_2 \bmod q} \left(\chi_2(l)\sin\frac{2\pi l}{q}\right) \times L(1, \overline{\chi\chi_2}) + O(1), & \chi(-1) = 1, \\ \dfrac{\tau(\chi)}{(q-1)\pi i} \displaystyle\sum_{l=1}^{q-1} \sum_{\chi_2 \bmod q} \chi_2(l)\left(1-\cos\frac{2\pi l}{q}\right) \times L(1, \overline{\chi\chi_2}) + O(1), & \chi(-1) = -1, \end{cases} \tag{16}$$

where $L(1,\chi)$ are the Dirichlet L-functions corresponding to χ.

Proof. From Lemma 2, we take $u = 0$, and $v = 1/q$ and get

$$\begin{aligned} \sum_{n\le Q/q} \chi(n) = \tau(\chi) \sum_{0<h\le H} \bar{\chi}(h) \frac{1-e(-h/q)}{2\pi i h} \\ + \tau(\chi) \sum_{-H<h\le 0} \bar{\chi}(h) \frac{1-e(-h/q)}{2\pi i h} \\ + O\left(1+\frac{Q\log Q}{H}\right). \end{aligned} \tag{17}$$

When $\chi(-1)=1$, we have

$$\begin{aligned}\sum_{n\le Q/q}\chi(n) &= \tau(\chi)\sum_{0<h\le H}\overline{\chi}(h)\frac{\sin(2\pi h/q)}{\pi h} \\ &\quad + O\left(1+\frac{Q\log Q}{H}\right) \\ &= \frac{\tau(\chi)}{\pi}\sum_{l=1}^{q-1}\left(\sin\frac{2\pi l}{q}\right)\sum_{\substack{0<h\le H\\ h\equiv l(\bmod q)}}\frac{\overline{\chi}(h)}{h} \\ &\quad + O\left(1+\frac{Q\log Q}{H}\right) \\ &= \frac{\tau(\chi)}{(q-1)\pi}\sum_{l=1}^{q-1}\left(\sin\frac{2\pi l}{q}\right)\sum_{0<h\le H}\frac{\overline{\chi}(h)}{h} \\ &\quad \times\sum_{\chi_2 \bmod q}\overline{\chi}_2(h)\chi_2(l)+O\left(1+\frac{Q\log Q}{H}\right) \\ &= \frac{\tau(\chi)}{(q-1)\pi}\sum_{l=1}^{q-1}\sum_{\chi_2 \bmod q}\left(\chi_2(l)\sin\frac{2\pi l}{q}\right) \\ &\quad \times\sum_{0<h\le H}\frac{\overline{\chi\chi_2}(h)}{h}+O\left(1+\frac{Q\log Q}{H}\right).\end{aligned} \tag{18}$$

Let $H\to\infty$, then, we have

$$\sum_{n\le Q/q}\chi(n) = \frac{\tau(\chi)}{(q-1)\pi}\sum_{l=1}^{q-1}\sum_{\chi_2 \bmod q}\left(\chi_2(l)\sin\frac{2\pi l}{q}\right)L(1,\overline{\chi\chi_2})+O(1). \tag{19}$$

The case of $\chi(-1)=-1$ can be treated in the same way. This proves Lemma 3. □

Lemma 4. *Let p,q be odd primes and let χ_1,χ_2 be the Dirichlet characters modulo p and q, respectively, such that $(p,q)=1$, denote $\chi=\chi_1\chi_2$, $k=pq$, and χ is the Dirichlet character modulo pq, the famous Gauss sums are defined as follows:*

$$G(n,\chi)=\sum_{a=1}^{k}\chi(a)e\left(\frac{na}{k}\right), \tag{20}$$

where $e(y)=e^{2\pi iy}$. Hence, we have

$$G(n,\chi)=\chi_1(q)\chi_2(p)G(n,\chi_1)G(n,\chi_2). \tag{21}$$

When $n=1$, we denote $\tau(\chi)=G(1,\chi)$; therefore, we have

$$\tau(\chi)=\chi_1(q)\chi_2(p)\tau(\chi_1)\tau(\chi_2). \tag{22}$$

Proof. (See [14]). □

Lemma 5. *Let m, and n be integers and let $q\ge 3$ be prime, let χ denote the Dirichlet character modulo q, the generalized Kloosterman sums are defined by*

$$S_\chi(m,n;q)=\sum_{a \bmod q}\chi(a)e\left(\frac{m\bar{a}+na}{q}\right), \tag{23}$$

where $a\bar{a}\equiv 1(\bmod q)$ and $e(y)=e^{2\pi iy}$.

Then, we have the following estimate:

$$S_\chi(m,n;q)\ll q^{1/2+\epsilon}(m,n,q)^{1/2}, \tag{24}$$

where (m,n,q) denotes the gcd of m, n, and q.

Proof. (See [15]). □

Lemma 6. *Let $p\ge 3$ be an odd prime, let χ, χ_1 be a Dirichlet character modulo p, and $\chi\chi_1\ne\chi_0^p$. For any odd prime q with $q<p$, let χ_2,χ_3,χ_4 be any Dirichlet characters with χ_2 mod q, χ_3 mod q and χ_4 mod q, respectively, then, no matter χ is odd character or even character modulo p, we have*

$$\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi_1\chi_2(-1)=1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \times L(1,\overline{\chi\chi_1\chi_2\chi_3})L(1,\overline{\chi}_1\chi_2\overline{\chi}_4)\ll p^{3/2+\epsilon}; \tag{25}$$

$$\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi_1\chi_2(-1)=-1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \times L(1,\overline{\chi\chi_1\chi_2\chi_3})L(1,\overline{\chi}_1\chi_2\overline{\chi}_4)\ll p^{3/2+\epsilon}; \tag{26}$$

$$\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi(-1)=1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \times L(1,\overline{\chi\chi_1\chi_3})L(1,\overline{\chi_1\chi_4})\ll p^{3/2+\epsilon}; \tag{27}$$

$$\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi(-1)=-1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \times L(1,\overline{\chi\chi_1\chi_3})L(1,\overline{\chi_1\chi_4})\ll p^{3/2+\epsilon}, \tag{28}$$

where the $\ll$ constant only depends on q.

Proof. For any integer n with $(n,k)=1$ ($k\ge 3$ is any positive integer), we have

$$\sum_{\substack{\chi \bmod k\\ \chi(-1)=1}}\chi(n)=\begin{cases}\frac{1}{2}\phi(k), & n\equiv\pm 1(\bmod k),\\ 0, & \text{otherwise.}\end{cases}$$

$$\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}}\chi(n)=\begin{cases}\frac{1}{2}\phi(k), & n\equiv 1(\bmod k),\\ -\frac{1}{2}\phi(k), & n\equiv -1(\bmod k),\\ 0, & \text{otherwise.}\end{cases} \tag{29}$$

Now let $y > p$ and let $A(y,\chi) = \sum_{p<n\le y}\chi(n)$. Then, from the Pólya-Vinogradov inequality, we obtain

$$A(y,\chi) \ll \sqrt{p}\log p. \tag{30}$$

Hence, from Abel's identity, for any $\operatorname{Re}(s) \ge 1$, we can easily get

$$\begin{aligned} L(s,\chi) &= \sum_{n\le p}\frac{\chi(n)}{n^s} + \int_p^{\infty}\frac{A(y,\chi)}{y^{s+1}}dy \\ &= \sum_{n\le p}\frac{\chi(n)}{n^s} + O\left(\frac{\log p}{p^{s-1/2}}\right) \\ &= \sum_{n\le p}\frac{\chi(n)}{n^s} + O\left(\frac{\log p}{p^{1/2}}\right). \end{aligned} \tag{31}$$

We will take (25), for example, to prove this lemma. For $(q,p) = 1$, from the definition of $\tau(\chi)$ and (31), since χ_1 is not the principle character modulo p and $\chi\chi_1$ is not the principle character modulo p, we have

$$\begin{aligned} &\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi_1\chi_2(-1)=1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \\ &\quad\times L(1,\overline{\chi\chi_1\chi_2\chi_3})L(1,\overline{\chi}_1\chi_2\overline{\chi}_4) \\ &= \sum_{\substack{\chi_1 \bmod p\\ \chi_1\chi_2(-1)=1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \\ &\quad\times\left(\sum_{k\le p}\frac{\overline{\chi\chi_1\chi_2\chi_3}(k)}{k} + O\left(\frac{\log pq}{(pq)^{1/2}}\right)\right) \\ &\quad\times\left(\sum_{n\le p}\frac{\overline{\chi}_1\chi_2\overline{\chi}_4(n)}{n} + O\left(\frac{\log pq}{(pq)^{1/2}}\right)\right) \\ &\quad-\chi(q)\tau(\chi)\tau(\chi_0^p)L(1,\overline{\chi\chi_2\chi_3})L(1,\chi_2\overline{\chi}_4) \\ &= \sum_{\substack{\chi_1 \bmod p\\ \chi_1\chi_2(-1)=1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \\ &\quad\times\sum_{k\le p}\frac{\overline{\chi\chi_1\chi_2\chi_3}(k)}{k}\sum_{n\le p}\frac{\overline{\chi}_1\chi_2\overline{\chi}_4(n)}{n} + O\left(p^{3/2}\log^2 p\right) \\ &= \sum_{k\le p}\sum_{n\le p}\frac{\overline{\chi\chi_1\chi_3}(k)\chi_2\overline{\chi}_4(n)\chi(qa)}{kn} \\ &\quad\times\sideset{}{'}\sum_{a=1}^{p}\sideset{}{'}\sum_{b=1}^{p}\chi(a)e\left(\frac{a+b}{p}\right) \\ &\quad\times\sum_{\substack{\chi_1 \bmod p\\ \chi_1\chi_2(-1)=1}}\chi_1(q^2ab)\overline{\chi}_1(kn) + O\left(p^{3/2}\log^2 p\right) \\ &= \frac{\phi(p)}{2}\sum_{k\le q}\sum_{n\le p}\frac{\overline{\chi\chi_1\chi_3}(k)\chi_2\overline{\chi}_4(n)\chi(qa)}{kn} \\ &\quad\times\left(\sideset{}{'}\sum_{a=1}^{p}\sideset{}{'}\sum_{\substack{b=1\\ q^2ab\equiv kn(\bmod p)}}^{p}\chi(a)e\left(\frac{a+b}{p}\right)\right. \\ &\quad\left.\pm\sideset{}{'}\sum_{a=1}^{p}\sideset{}{'}\sum_{\substack{b=1\\ q^2ab\equiv -kn(\bmod p)}}^{p}\chi(a)e\left(\frac{a+b}{p}\right)\right) \\ &\quad+O\left(p^{3/2}\log^2 p\right) \\ &= \frac{\phi(p)}{2}\sum_{k\le p}\sum_{n\le p}\frac{\overline{\chi\chi_1\chi_3}(k)\chi_2\overline{\chi}_4(n)\chi(qa)}{kn} \\ &\quad\times\left(S_\chi\left(1,\overline{q}^2kn;p\right) \pm S_\chi\left(1,-\overline{q}^2kn;p\right)\right) \\ &\quad+O\left(p^{3/2}\log^2 p\right), \end{aligned} \tag{32}$$

where $abq^2\overline{kn} \equiv 1(\bmod p)$. From Lemma 5, we can easily obtain

$$\begin{aligned} &\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi_1\chi_2(-1)=1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1)L(1,\overline{\chi\chi_1\chi_2\chi_3}) \\ &\quad\times L(1,\overline{\chi}_1\chi_2\overline{\chi}_4) \ll p^{3/2+\epsilon_1}\log^2 p \ll p^{3/2+\epsilon}, \end{aligned} \tag{33}$$

where the $\ll$ constant only depends on q. Therefore, this completes the proof of Lemma 6. □

Lemma 7. *Let q be a fixed odd prime and let p be a prime with $p > q$, let χ denote the nonprincipal Dirichlet character modulo p, and χ_1 denote the Dirichlet character modulo p, then, we have*

$$\begin{aligned} &\sum_{\substack{\chi_1 \bmod p\\ \chi_1\ne\chi_p^0}}\chi\chi_1(q)\chi_1(q) \\ &\quad\times\sum_{a\le(p-1)/q}\chi\chi_1(a)\sum_{b\le(p-1)/q}\chi_1(b) \ll p^{3/2+\epsilon}, \end{aligned} \tag{34}$$

where the $\ll$ constant only depends on q.

Proof. For primes p and q, χ_1 and $\chi\chi_1$ are nonprincipal and primitive characters modulo p, hence, from Lemmas 3 and 6, when $\chi(-1)=1$, we can get

$$\begin{aligned}
&\sum_{\substack{\chi_1 \bmod p\\ \chi_1\neq\chi_p^0}} \chi\chi_1(q)\chi_1(q)\sum_{a\le (p-1)/q}\chi\chi_1(a)\sum_{b\le (p-1)/q}\chi_1(b)\\
&=\sum_{\substack{\chi_1(-1)=1\\ \chi_1\neq\chi_p^0}}\chi\chi_1(q)\chi_1(q)\\
&\quad\times\left(\frac{\tau(\chi\chi_1)}{(q-1)\pi}\sum_{u=1}^{q-1}\sum_{\chi_3 \bmod q}\left(\chi_3(u)\sin\frac{2\pi u}{q}\right)\right.\\
&\qquad\left.\times L\left(1,\overline{\chi\chi_1\chi_3}\right)+O(1)\right)\\
&\quad\times\left(\frac{\tau(\chi_1)}{(q-1)\pi}\sum_{v=1}^{q-1}\sum_{\chi_4 \bmod q}\left(\chi_4(v)\sin\frac{2\pi v}{q}\right)\right.\\
&\qquad\left.\times L\left(1,\overline{\chi_1\chi_4}\right)+O(1)\right)\\
&\quad+\sum_{\substack{\chi_1(-1)=-1\\ \chi_1\neq\chi_p^0}}\chi\chi_1(q)\chi_1(q)\\
&\quad\times\left(\frac{\tau(\chi\chi_1)}{(q-1)\pi i}\sum_{u=1}^{q-1}\sum_{\chi_3 \bmod q}\left(\chi_3(u)\left(1-\cos\frac{2\pi u}{q}\right)\right)\right.\\
&\qquad\left.\times L\left(1,\overline{\chi\chi_1\chi_3}\right)+O(1)\right)\\
&\quad\times\left(\frac{\tau(\chi_1)}{(q-1)\pi i}\sum_{v=1}^{q-1}\sum_{\chi_4 \bmod q}\left(\chi_4(v)\left(1-\cos\frac{2\pi v}{q}\right)\right)\right.\\
&\qquad\left.\times L\left(1,\overline{\chi_1\chi_4}\right)+O(1)\right)\\
&=\frac{1}{(q-1)^2\pi^2}\\
&\quad\times\sum_{u=1}^{q-1}\sum_{v=1}^{q-1}\sum_{\chi_3 \bmod q}\sum_{\chi_4 \bmod q}\chi_3(u)\chi_4(v)\sin\frac{2\pi u}{q}\sin\frac{2\pi v}{q}\\
&\quad\times\sum_{\substack{\chi_1\neq\chi_p^0\\ \chi_1(-1)=1}}\chi\chi_1(q)\chi_1(m)\tau(\chi\chi_1)\tau(\chi_1)\\
&\qquad\times L\left(1,\overline{\chi\chi_1\chi_3}\right)L\left(1,\overline{\chi_1\chi_4}\right)\\
&\quad+\frac{-1}{(q-1)^2\pi^2}\\
&\quad\times\sum_{u=1}^{q-1}\sum_{v=1}^{q-1}\sum_{\chi_3 \bmod q}\sum_{\chi_4 \bmod m}\chi_3(u)\chi_4(v)\\
&\qquad\times\left(1-\cos\frac{2\pi u}{q}\right)\left(1-\cos\frac{2\pi v}{q}\right)\\
&\quad\times\sum_{\substack{\chi_1\neq\chi_p^0\\ \chi_1(-1)=-1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1)\\
&\qquad\times L\left(1,\overline{\chi\chi_1\chi_3}\right)L\left(1,\overline{\chi_1\chi_4}\right)+O\left(p^{1/2+\epsilon}\right)\\
&\ll p^{3/2+\epsilon},
\end{aligned}\tag{35}$$

where the $\ll$ constant is only concerned with q.

When $\chi(-1)=-1$, by the similar method, we can also obtain

$$\begin{aligned}
&\sum_{\substack{\chi_1 \bmod p\\ \chi_1\neq\chi_p^0}} \chi\chi_1(q)\chi_1(q)\sum_{a\le (p-1)/q}\chi\chi_1(a)\sum_{b\le (p-1)/q}\chi_1(b)\\
&\quad\ll p^{3/2+\epsilon},
\end{aligned}\tag{36}$$

where the $\ll$ constant is only concerned with q. Combining (35) and (36), we can obtain Lemma 7. This completes the proof of Lemma 7. □

Lemma 8. *Let q be a fixed odd prime and let p be a prime with $p>q$, let χ denote the nonprincipal Dirichlet character modulo p, let χ_1,χ_2 denote the Dirichlet character modulo p,q respectively, then, we have*

$$\sum_{\substack{\chi_1 \bmod p\\ \chi_1\neq\chi_p^0}}\sum_{\substack{\chi_2 \bmod q\\ \chi_2\neq\chi_q^0}}\sum_{\substack{a\le p-1\\ q\nmid a}}\sum_{\substack{b\le p-1\\ q\nmid b}}\chi\chi_1\chi_2(a)\chi_1\overline{\chi}_2(b)\ll p^{3/2+\epsilon},\tag{37}$$

where the $\ll$ constant is only concerned with q.

Proof. According to the properties of Dirichlet character, we can get

$$\begin{aligned}
&\sum_{\substack{\chi_1 \bmod p\\ \chi_1\neq\chi_p^0}}\sum_{\substack{\chi_2 \bmod q\\ \chi_2\neq\chi_q^0}}\sum_{\substack{a\le p-1\\ q\nmid a}}\sum_{\substack{b\le p-1\\ q\nmid b}}\chi\chi_1\chi_2(a)\chi_1\overline{\chi}_2(b)\\
&=\sum_{\substack{\chi_1 \bmod p\\ \chi_1\neq\chi_p^0}}\sum_{\substack{\chi_2 \bmod q\\ \chi_2\neq\chi_q^0}}\left(\sum_{a\le p-1}\chi\chi_1\chi_2(a)-\sum_{\substack{a\le p-1\\ q\mid a}}\chi\chi_1\chi_2(a)\right)\\
&\quad\times\left(\sum_{b\le p-1}\chi_1\overline{\chi}_2(b)-\sum_{\substack{b\le p-1\\ q\mid b}}\chi_1\overline{\chi}_2(b)\right)
\end{aligned}$$

$$= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \left(\sum_{a \le p-1} \chi\chi_1\chi_2(a) - \chi\chi_1\chi_2(q) \times \sum_{a \le (p-1)/q} \chi\chi_1\chi_2(a) \right) \times \left(\sum_{b \le p-1} \chi_1\overline{\chi}_2(b) - \chi_1\overline{\chi}_2(q) \sum_{b \le (p-1)/q} \chi_1\overline{\chi}_2(b) \right)$$
$$= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{a \le p-1} \chi\chi_1\chi_2(a) \sum_{b \le p-1} \chi_1\overline{\chi}_2(b). \tag{38}$$

For primes p and q, let $\chi' = \chi\chi_1$ be a nonprincipal and primitive character modulo p, χ_2 is also a primitive character modulo q, so $\chi\chi_1\chi_2$ is a primitive character modulo pq; therefore, from Lemmas 3, 4, and 6 and from (38), it is clear that when $\chi(-1) = 1$, we can obtain

$$\sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1 \\ q \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1\chi_2(a)\chi_1\overline{\chi}_2(b)$$
$$= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0 \\ \chi_1\chi_2(-1)=1}} \times \left(\frac{\tau(\chi\chi_1\chi_2)}{(q-1)\pi} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3(u) \sin\frac{2\pi u}{q} \right) \times L\left(1, \overline{\chi\chi_1\chi_2\chi_3}\right) + O(1) \right)$$
$$\times \left(\frac{\tau(\chi_1\overline{\chi}_2)}{(q-1)\pi} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4(v) \sin\frac{2\pi v}{q} \right) \times L\left(1, \overline{\chi}_1\chi_2\overline{\chi}_4\right) + O(1) \right)$$
$$+ \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0 \\ \chi_1\chi_2(-1)=-1}} \times \left(\frac{\tau(\chi\chi_1\chi_2)}{(q-1)\pi i} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3(u) \left(1 - \cos\frac{2\pi u}{q}\right) \right) \times L\left(1, \overline{\chi\chi_1\chi_2\chi_3}\right) + O(1) \right)$$
$$\times \left(\frac{\tau(\chi_1\overline{\chi}_2)}{(q-1)\pi i} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4(v) \left(1 - \cos\frac{2\pi v}{q}\right) \right) \times L\left(1, \overline{\chi}_1\chi_2\overline{\chi}_4\right) + O(1) \right)$$
$$= \frac{1}{(q-1)^2\pi^2} \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod q} \times \chi_2(p)\tau(\chi_2)\overline{\chi}_2(p)\tau(\overline{\chi}_2)\chi_3(u)\chi_4(v) \sin\frac{2\pi u}{q} \times \sin\frac{2\pi v}{q} \sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1\chi_2(-1)=1}} \chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \times L\left(1, \overline{\chi\chi_1\chi_2\chi_3}\right) L\left(1, \overline{\chi}_1\chi_2\overline{\chi}_4\right)$$
$$+ \frac{-1}{(q-1)^2\pi^2} \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod q} \times \chi_2(p)\tau(\chi_2)\overline{\chi}_2(p)\tau(\overline{\chi}_2)\chi_3(u)\chi_4(v) \times \left(1 - \cos\frac{2\pi u}{q}\right)\left(1 - \cos\frac{2\pi v}{q}\right) \times \sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1\chi_2(-1)=-1}} \chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \times L\left(1, \overline{\chi\chi_1\chi_2\chi_3}\right) L\left(1, \overline{\chi}_1\chi_2\overline{\chi}_4\right)$$
$$\ll p^{3/2+\epsilon}, \tag{39}$$

where the $\ll$ constant is only concerned with q.

When $\chi(-1) = -1$, in the similar way, we can also obtain

$$\sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1 \\ q \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1\chi_2(a)\chi_1\overline{\chi}_2(b) \ll p^{3/2+\epsilon}, \tag{40}$$

where the $\ll$ constant is only concerned with q.

Therefore, from (39) and (40), we can easily get Lemma 8. This completes the proof of Lemma 8. □

3. Proof of Theorem

In this section, we will complete the proof of the theorem. According to the orthogonality relation for character sums, we have

$$\sum_{\substack{a \le p \\ a \in E(q,p)}} \chi(a) = \sum_{\substack{b=1 \\ a \equiv b \pmod q \\ ab \equiv 1 \pmod p}}^{p-1} \sum_{a=1}^{p-1} \chi(a)$$
$$= \frac{1}{p-1} \sum_{\chi_1 \bmod p} \sum_{\substack{b=1 \\ a \equiv b \pmod q}}^{p-1} \sum_{a=1}^{p-1} \chi\chi_1(a)\chi_1(b)$$

$$= \frac{1}{p-1} \sum_{\substack{a=1 \\ a \equiv b (\bmod q)}}^{p-1} \sum_{b=1}^{p-1} \chi(a)$$
$$+ \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a=1 \\ a \equiv b (\bmod q)}}^{p-1} \sum_{b=1}^{p-1} \chi\chi_1(a)\chi_1(b)$$
$$= \frac{1}{p-1} \sum_{a=1}^{p-1} \chi(a) \left(\sum_{\substack{b=1 \\ b \equiv a (\bmod q)}}^{p-1} 1 \right)$$
$$+ \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a=1 \\ q \mid a}}^{p-1} \sum_{\substack{b=1 \\ q \mid b}}^{p-1} \chi\chi_1(a)\chi_1(b)$$
$$+ \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a=1 \\ q \nmid a \\ a \equiv b (\bmod q)}}^{p-1} \sum_{\substack{b=1 \\ q \nmid b}}^{p-1} \chi\chi_1(a)\chi_1(b)$$
$$= \frac{1}{p-1} \sum_{a=1}^{p-1} \chi(a) \left(\frac{p}{q} + O(1) \right)$$
$$+ \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q)$$
$$\times \sum_{a \le (p-1)/q} \chi\chi_1(a) \sum_{b \le (p-1)/q} \chi_1(b)$$
$$+ \frac{1}{(p-1)\phi(q)}$$
$$\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\chi_2 \bmod q} \sum_{\substack{a \le p-1 \\ q \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b)$$
$$\times \chi_2(a)\overline{\chi}_2(b)$$
$$= \frac{p}{(p-1)q} \sum_{a=1}^{p-1} \chi(a) + O(1)$$
$$+ \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q)$$
$$\times \sum_{a \le (p-1)/q} \chi\chi_1(a) \sum_{b \le (p-1)/q} \chi_1(b)$$
$$+ \frac{1}{(p-1)\phi(q)}$$
$$\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a \le p-1 \\ m \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b)$$
$$+ \frac{1}{(p-1)\phi(q)}$$
$$\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1 \\ q \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b)$$
$$\times \chi_2(a)\overline{\chi}_2(b)$$
$$= O(1) + \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q)$$
$$\times \sum_{a \le (p-1)/q} \chi\chi_1(a) \sum_{b \le (p-1)/q} \chi_1(b)$$
$$+ \frac{1}{(p-1)\phi(q)}$$
$$\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a \le p-1 \\ q \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b)$$
$$+ \frac{1}{(p-1)\phi(q)}$$
$$\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1 \\ q \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1\chi_2(a)\chi_1\overline{\chi}_2(b). \tag{41}$$

Note that for any nonprincipal Dirichlet character χ modulo k ($k \ge 3$ is an positive integer), we have $\sum_{n=1}^{k} \chi(n) = 0$, hence, we obtain

$$\sum_{\substack{a \le p-1 \\ q \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b)$$
$$= \left(\sum_{a \le p-1} \chi\chi_1(a) - \sum_{\substack{a \le p-1 \\ q \mid a}} \chi\chi_1(a) \right)$$
$$\times \left(\sum_{b \le p-1} \chi_1(b) - \sum_{\substack{b \le p-1 \\ q \mid b}} \chi_1(b) \right) \tag{42}$$
$$= \sum_{\substack{a \le p-1 \\ q \mid a}} \chi\chi_1(a) \sum_{\substack{b \le p-1 \\ q \mid b}} \chi_1(b)$$
$$= \chi\chi_1(q)\chi_1(q) \sum_{a \le (p-1)/q} \chi\chi_1(a) \sum_{b \le (p-1)/q} \chi_1(b).$$

From (41) and (42) and Lemmas 7 and 8, we get

$$\sum_{\substack{a \le p \\ a \in E(q,p)}} \chi(a)$$
$$= \frac{1}{p-1} \left(1 + \frac{1}{\phi(q)} \right)$$

$$\begin{aligned}
&\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\,\chi_1(q) \sum_{a \le (p-1)/q} \chi\chi_1(a) \sum_{b \le (p-1)/q} \chi_1(b) \\
&+ \frac{1}{(p-1)\,\phi(q)} \\
&\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1 \\ q \dagger a}} \sum_{\substack{b \le p-1 \\ q \dagger b}} \chi\chi_1\chi_2(a)\,\chi_1\overline{\chi}_2(b) + O(1) \\
&\ll p^{1/2+\epsilon},
\end{aligned} \tag{43}$$

where the $\ll$ constant only depends on q. This completes the proof of Theorem.

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Research Article

Lipschitz Estimates for Fractional Multilinear Singular Integral on Variable Exponent Lebesgue Spaces

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We obtain the Lipschitz boundedness for a class of fractional multilinear operators with rough kernels on variable exponent Lebesgue spaces. Our results generalize the related conclusions on Lebesgue spaces with constant exponent.

1. Introduction and Results

Let $0 < \alpha < n$, $\Omega \in L^s(S^{n-1})$ $(s > n/(n-\alpha))$ is homogeneous of degree zero on R^n, S^{n-1} denotes the unit sphere in R^n, the fractional multilinear singular integral operator with rough kernel $T_{\Omega,\alpha,A}$ is defined by

$$T_{\Omega,\alpha,A} f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha+m-1}} R_m(A;x,y) f(y)\,dy, \quad (1)$$

where $R_m(A;x,y)$ denotes the mth remainder of the Taylor series of a function A defined on R^n at x about y. More precisely,

$$R_m(A;x,y) = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma, \quad (2)$$

and the corresponding fractional multilinear maximal operator is defined by

$$M_{\Omega,\alpha,A} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha+m-1}} \times \int_{|x-y|<r} |\Omega(x-y)| |R_m(A;x,y)| |f(y)| dy. \quad (3)$$

Multilinear operator was first introduced by Calderón in [1], and then Meyer [2] studied it in depth and extended such type of operators. Multilinear singular integral operator was later introduced by Professor Lu during 1999 [3]. Especially as $m = 1$, the fractional multilinear singular integral operator $T_{\Omega,\alpha,A}$ is obviously the commutator operator

$$[A, T_{\Omega,\alpha}] f(x) = A(x) T_{\Omega,\alpha} f(x) - T_{\Omega,\alpha}(Af)(x), \quad (4)$$

the commutator is a typical non-convolution singular operator. Since the commutator has a close relation with partial differential equations and pseudo-differential operator, multilinear operator has been receiving more widely attention.

It is well known that the boundedness of $T_{\Omega,\alpha,A}$ and $M_{\Omega,\alpha,A}$ had been obtained on Lebesgue spaces in [4–7]. However, the corresponding results have not been obtained on $L^{p(\cdot)}(R^n)$. Nowadays, there is an evident increase of investigations related to both the theory of the spaces $L^{p(\cdot)}$ themselves and the operator theory in these spaces [8–11]. This is caused by possible applications to models with non-standard local growth in elasticity theory, fluid mechanics, and differential equations [12–14]. The purpose of this paper is to study the behaviour of $T_{\Omega,\alpha,A}$ and $M_{\Omega,\alpha,A}$ on variable Lebesgue spaces.

To state the main results of this paper, we need to recall some notions.

Definition 1. Suppose a measurable function $p(\cdot) : R^n \to [1, \infty)$, for some $\lambda > 0$, then, the variable exponent Lebesgue space $L^{p(\cdot)}(R^n)$ is defined by

$$L^{p(\cdot)}(R^n) = \left\{ f \text{ is measurable} : \int_{R^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \right\}, \tag{5}$$

with norm

$$\|f\|_{L^{p(\cdot)}(R^n)} = \inf \left\{ \lambda > 0 : \int_{R^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}. \tag{6}$$

We denote

$$\begin{aligned} p_- &= \operatorname{essinf} \{ p(x) : x \in R^n \}, \\ p_+ &= \operatorname{esssup} \{ p(x) : x \in R^n \}. \end{aligned} \tag{7}$$

Using this notation we define a class of variable exponent as follows:

$$\Phi(R^n) = \{ p(\cdot) : R^n \longrightarrow [1, \infty), p_- > 1, p_+ < \infty \}. \tag{8}$$

The exponent $p'(\cdot)$ means the conjugate of $p(\cdot)$, namely, $1/p(x) + 1/p'(x) = 1$ holds.

Definition 2. For $\beta > 0$, the homogeneous Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f, such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x,h \in R^n, h \ne 0} \frac{\left| \Delta_h^{[\beta]+1} f(x) \right|}{|h|^\beta} < \infty, \tag{9}$$

where $\Delta_h^1 f(x) = f(x+h) - f(x)$, $\Delta_h^{k+1} f(x) = \Delta_h^k f(x+h) - \Delta_h^k f(x)$, $k \ge 1$.

Definition 3. For $0 < \alpha < n$, the fractional integral operator with rough kernel is defined by

$$\begin{aligned} T_{\Omega,\alpha} f(x) &= \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\, dy, \\ \overline{T}_{\Omega,\alpha} f(x) &= \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\, dy. \end{aligned} \tag{10}$$

The corresponding fractional maximal operator with rough kernel is defined by

$$M_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|<r} |\Omega(x-y)| |f(y)|\, dy. \tag{11}$$

When $\alpha = 0$, $T_{\Omega,\alpha}$ is much more closely related to the elliptic partial equations of second order with variable coefficients. In 1955, Calderón and Zygmund [15] proved the L^p boundedness. In 1971, Muckenhoupt and Wheeden [16] proved the (L^p, L^q) boundedness of $T_{\Omega,\alpha}$ with power weights.

In this paper, we state some properties of variable exponents belonging to class $B(R^n)$.

Proposition 4. *If $p(\cdot) \in \Phi(R^n)$ satisfies*

$$\begin{aligned} |p(x) - p(y)| &\le \frac{-C}{\log(|x-y|)}, \quad |x-y| \le \frac{1}{2}, \\ |p(x) - p(y)| &\le \frac{C}{\log(e + |x|)}, \quad |y| \ge |x|, \end{aligned} \tag{12}$$

Then, one has $p(\cdot) \in B(R^n)$.

Recently, Mitsuo Izuki has proved the condition as below.

Theorem A (see [17]). *Suppose that $p(\cdot) \in \Phi(R^n)$ satisfies conditions* (12) *in Proposition 4. Let $0 < \alpha < n/p_+$, and define the variable exponent $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}. \tag{13}$$

Then, one has that for all $f \in L^{p(\cdot)}(R^n)$,

$$\left\| [b, I^\alpha] f \right\|_{L^{q(\cdot)}(R^n)} \le C \|b\|_{BMO(R^n)} \|f\|_{L^{p(\cdot)}(R^n)} \tag{14}$$

for all $f \in L^{p(\cdot)}(R^n)$ and $b \in BMO(R^n)$.

Next, we will discuss the boundedness of $T_{\Omega,\alpha,A}$ and $M_{\Omega,\alpha,A}$ on variable Lebesgue spaces. We can get $T_{\Omega,\alpha,A}$ and $M_{\Omega,\alpha,A}$ are bounded from $L^{p(\cdot)}(R^n)$ to $L^{q(\cdot)}(R^n)$. In fact, the results generalize Theorem A in [17] from classical Lebesgue spaces to variable exponent Lebesgue spaces. Now, let us formulate our results as follows.

Theorem 5. *Suppose that $p(\cdot) \in \Phi(R^n)$ satisfies conditions* (12) *in Proposition 4. Let $0 < \alpha < n/p_+$, $0 < \beta < 1$, $0 < \alpha + \beta < n/p_+$, and $1 < p_+ < n/(\alpha + \beta)$, and define the variable exponent $q(\cdot)$ by*

$$\frac{1}{q(x)} - \frac{1}{p(x)} = \frac{\alpha + \beta}{n}. \tag{15}$$

If $D^\gamma A \in \dot{\Lambda}_\beta (|\gamma| = m-1)$, then, there is a $C > 0$, independent of f and A, such that

$$\left\| T_{\Omega,\alpha,A} f \right\|_{L^{q(\cdot)}(R^n)} \le C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|f\|_{L^{p(\cdot)}(R^n)}. \tag{16}$$

Theorem 6. *Suppose that $p(\cdot) \in \Phi(R^n)$ satisfies conditions* (12) *in Proposition 4. Let $0 < \alpha < n/p_+$, $0 < \beta < 1$, $0 < \alpha + \beta < n/p_+$, and $1 < p_+ < n/(\alpha + \beta)$, and define the variable exponent $q(\cdot)$ by*

$$\frac{1}{q(x)} - \frac{1}{p(x)} = \frac{\alpha + \beta}{n}. \tag{17}$$

If $D^\gamma A \in \dot{\Lambda}_\beta (|\gamma| = m-1)$, then, there is a $C > 0$, independent of f and A, such that

$$\left\| M_{\Omega,\alpha,A} f \right\|_{L^{q(\cdot)}(R^n)} \le C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|f\|_{L^{p(\cdot)}(R^n)}. \tag{18}$$

Remark 7. We point out that C will denote positive constants whose values may change at different places.

2. Lemmas and Proof of Theorems

Lemma 8 (see [15]). *Let $A(x)$ be a function on R^n with mth order derivatives in $L^l_{loc}(R^n)$ for some $l > n$. Then,*

$$\left|R_m(A;x,y)\right| \leq C|x-y|^m \sum_{|r|=m} \left(\frac{1}{|Q^y_x|}\int_{Q^y_x} \left|D^\gamma A(z)\right|^l dz\right)^{1/l}, \quad (19)$$

where Q^y_x is the cube centered at x and having diameter $5\sqrt{n}|x-y|$.

Lemma 9 (see [18]). *For $0 < \beta < 1$, $1 \leq q < \infty$, we have*

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q \left|f(x) - m_Q(f)\right| dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|}\int_Q \left|f(x)-m_Q(f)\right|^q dx\right)^{1/q}. \quad (20)$$

Lemma 10 (see [18]). *Let $Q^* \subset Q$, $g \in \dot{\Lambda}_\beta\ (0<\beta<1)$, then,*

$$\left|m_{Q^*}(g) - m_Q(g)\right| \leq C|Q|^{\beta/n}\|g\|_{\dot{\Lambda}_\beta}. \quad (21)$$

We state the following important lemma.

Lemma 11. *Suppose $0 < \alpha < n$, $0 < \beta < 1$, with $0 < \alpha + \beta < n$, $\Omega \in L^s(S^{n-1})(s > n/(n-(\alpha+\beta)))$, $D^\gamma A \in \dot{\Lambda}_\beta$. Then, there exists a constant C only depends on m, n, α, and β, such that*

$$\left|T_{\Omega,\alpha,A} f(x)\right| \leq C\left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \overline{T}_{\Omega,\alpha+\beta} f(x). \quad (22)$$

Proof. For any $x \in R^n$, let the cube be centered at x and having the diameter be l, where $l > 0$, we have

$$\begin{aligned} T_{\Omega,\alpha,A} f(x) &= \left(\int_Q + \int_{Q^c}\right) \frac{\Omega(x-y)}{|x-y|^{n-\alpha+m-1}} R_m(A;x,y) f(y)\,dy \\ &:= H_1 + H_2. \end{aligned} \quad (23)$$

Below, we give estimates of H_1. Let

$$\begin{aligned} |H_1| &\leq \sum_{j=0}^{\infty} \int_{2^{-j}Q\setminus 2^{-j-1}Q} \frac{|\Omega(x-y)|\,|R_m(A;x,y)|}{|x-y|^{n-\alpha+m-1}} |f(y)|\,dy \\ &\leq \sum_{j=0}^{\infty} \int_{2^{-j}Q\setminus 2^{-j-1}Q} \frac{|\Omega(x-y)|\,|R_m(A_{2^{-j}Q};x,y)|}{|x-y|^{n-\alpha+m-1}} |f(y)|dy. \end{aligned} \quad (24)$$

Note that $A_{2^{-j}Q}(y) = A(y) - \sum_{|\gamma|=m-1}(1/\gamma!) m_{2^{-j}Q}(D^\gamma A) y^r$. When $y \in 2^{-j}Q \setminus 2^{-j-1}Q$, by Lemmas 8, 9, and 10, we have

$$\left|R_m(A_{2^{-j}Q};x,y)\right| \leq C\left(2^{-j}l\right)^\beta |x-y|^{m-1} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}. \quad (25)$$

Note that $|x-y| \geq 2^{-j-1}$, we have $|x-y|^\beta \geq 2^{-\beta}(2^{-j}l)^\beta$, such that

$$\begin{aligned} |H_1| &\leq C\left(\sum_{|\gamma|=m-1} \|D^\gamma A_k\|_{\dot{\Lambda}_\beta}\right) \times \sum_{j=0}^{\infty} \left(2^{-j}l\right)^\beta \int_{2^{-j}Q\setminus 2^{-j-1}Q} \frac{|\Omega(x-y)|\,|f(y)|}{|x-y|^{n-\alpha}} dy \\ &\leq C\left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \times \sum_{j=0}^{\infty} \int_{2^{-j}Q\setminus 2^{-j-1}Q} \frac{\left(2^{-j}l\right)^\beta |\Omega(x-y)|\,|f(y)|}{|x-y|^{n-\alpha}} dy \\ &\leq C\left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \times \sum_{j=0}^{\infty} \int_{2^{-j}Q\setminus 2^{-j-1}Q} \frac{2^\beta |x-y|^\beta |\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\,dy \\ &\leq C\left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \int_Q \frac{|\Omega(x-y)|\,|f(y)|}{|x-y|^{n-(\alpha+\beta)}} dy \\ &\leq C\left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \int_{R^n} \frac{|\Omega(x-y)|\,|f(y)|}{|x-y|^{n-(\alpha+\beta)}} dy \\ &= C\left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \overline{T}_{\Omega,\alpha+\beta} f(x). \end{aligned} \quad (26)$$

Below, we give the estimates of H_2. For $0 < \alpha + \beta < n$, we get

$$\begin{aligned} |H_2| &\leq \sum_{j=0}^{\infty} \int_{2^{j+1}Q\setminus 2^jQ} \frac{|\Omega(x-y)|\,|R_m(A;x,y)|}{|x-y|^{n-\alpha+m-1}} |f(y)|\,dy \\ &\leq \sum_{j=0}^{\infty} \int_{2^{j+1}Q\setminus 2^jQ} \frac{|\Omega(x-y)|\,|R_m(A_{2^{j+1}Q};x,y)|}{|x-y|^{n-\alpha+m-1}} |f(y)|dy. \end{aligned} \quad (27)$$

For any $y \in 2^{j+1}Q \setminus 2^jQ$,

$$A_{2^{j+1}Q}(y) = A(y) - \sum_{|\gamma|_{m-1}} \frac{1}{\gamma!} m_{2^{j+1}Q}(D^\gamma A). \quad (28)$$

Thus, by Lemmas 8 and 9, we obtain

$$\left|R_m(A_{2^{j+1}Q};x,y)\right| \leq C\left(2^jl\right)^\beta |x-y|^{m-1} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}. \quad (29)$$

And for $|x-y| \geq 2^j l$, we have $|x-y|^\beta \geq (2^j l)^\beta$. Hence,

$$\begin{aligned}
|H_2| &\leq \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \\
&\quad \times \sum_{j=0}^{\infty} (2^j l)^\beta \int_{2^{j+1}Q \backslash 2^j Q} \frac{|\Omega(x-y)| |f(y)|}{|x-y|^{n-\alpha}} dy \\
&\leq C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \\
&\quad \times \sum_{j=0}^{\infty} \int_{2^{j+1}Q \backslash 2^j Q} \frac{|x-y|^\beta |\Omega(x-y)| |f(y)|}{|x-y|^{n-\alpha}} dy \\
&\leq C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \\
&\quad \times \sum_{j=0}^{\infty} \int_{2^{j+1}Q \backslash 2^j Q} \frac{|\Omega(x-y)| |f(y)|}{|x-y|^{n-(\alpha+\beta)}} dy \\
&\leq C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \int_{R^n} \frac{|\Omega(x-y)| |f(y)|}{|x-y|^{n-(\alpha+\beta)}} dy \\
&\leq C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \overline{T}_{\Omega,\alpha+\beta} f(x).
\end{aligned} \tag{30}$$

□

From the proof above, we obtain

$$\begin{aligned}
|T_{\Omega,\alpha,A} f(x)| &\leq |H_1| + |H_2| \\
&\leq C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \overline{T}_{\Omega,\alpha+\beta} f(x).
\end{aligned} \tag{31}$$

Lemma 12 (see [19]). *If $p(\cdot) \in \Phi(R^n)$, for all $f \in L^{p(\cdot)}(R^n)$, then, the norm $\|f\|_{L^{p(\cdot)}(R^n)}$ has the following equivalence:*

$$\begin{aligned}
\|f\|_{L^{p(\cdot)}(R^n)} &\leq \sup \left\{ \int_{R^n} |f(x) g(x)| dx : \|g\|_{L^{p'(\cdot)}(R^n)} \leq 1 \right\} \\
&\leq r_p \|f\|_{L^{p'(\cdot)}(R^n)},
\end{aligned} \tag{32}$$

where $r_p := 1 + 1/p_- - 1/p_+$.

Lemma 13 (see [19], the generalized Hölder inequality). *If $p(\cdot) \in \Phi(R^n)$, then, for all $f \in L^{p(\cdot)}(R^n)$ and for all $g \in L^{p'(\cdot)}(R^n)$, we have*

$$\int_{R^n} |f(x) g(x)| dx \leq C \|f\|_{L^{p(\cdot)}(R^n)} \|g\|_{L^{p'(\cdot)}(R^n)}. \tag{33}$$

By a similar method of Ding and Lu [20], it is easy to verify the following result.

Lemma 14. *For any $\varepsilon > 0$ with $0 < \alpha + \beta - \varepsilon < \alpha + \beta + \varepsilon < n$, we have*

$$\left|\overline{T}_{\Omega,\alpha+\beta} f(x)\right| \leq C \left[M_{\Omega,\alpha+\beta+\varepsilon} f(x)\right]^{1/2} \left[M_{\Omega,\alpha+\beta-\varepsilon} f(x)\right]^{1/2}, \tag{34}$$

where C depends only on $\alpha, \beta, \varepsilon$, and n.

Lemma 15 (see [19]). *Given that $p(\cdot) : R^n \to [1, \infty)$, such that $p_+ < \infty$, then, $\|f\|_{L^{p(\cdot)}(R^n)} < C_1$ if and only if $|f|_{L^{p(\cdot)}(R^n)} < C_2$. In particular, if either constant equals 1, one can make the other equals 1 as well.*

Remark 16. We denote $|f|_{L^{p(\cdot)}(R^n)} = \int_{R^n} |f(y)|^{p(y)} dy$.

Lemma 17 (see [21]). *Suppose that $p(\cdot) \in \Phi(R^n)$ satisfies conditions (12) in Proposition 4. Let $0 < \alpha + \beta < n/p_+$, and define the variable exponent $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha+\beta}{n}. \tag{35}$$

Then, one has that for all $f \in L^{p(\cdot)}(R^n)$,

$$\left\|M_{\Omega,\alpha+\beta} f\right\|_{L^{q(\cdot)}(R^n)} \leq C \|f\|_{L^{p(\cdot)}(R^n)}. \tag{36}$$

Lemma 18. *Let $0 < \alpha < n$, $\Omega \in L^s(S^{n-1})$, then, for $x \in R^n$,*

$$\overline{T}_{\Omega,\alpha,A} f(x) \geq M_{\Omega,\alpha,A} f(x), \tag{37}$$

where

$$\overline{T}_{\Omega,\alpha,A} f(x) = \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}} |R_m(A; x, y)| |f(y)| dy. \tag{38}$$

Proof. Since

$$\begin{aligned}
&\overline{T}_{\Omega,\alpha,A} f(x) \\
&\quad = \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}} |R_m(A; x, y)| |f(y)| dy \\
&\quad \geq \int_{|x-y|<r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}} |R_m(A; x, y)| |f(y)| dy \\
&\quad \geq \frac{1}{r^{n-\alpha+m-1}} \int_{|x-y|<r} |\Omega(x-y)| |R_m(A; x, y)| |f(y)| dy,
\end{aligned} \tag{39}$$

then,

$$\overline{T}_{\Omega,\alpha,A} f(x) \geq M_{\Omega,\alpha,A} f(x). \tag{40}$$

□

Proof of Theorem 5. Since

$$|T_{\Omega,\alpha,A} f(x)| \leq C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \overline{T}_{\Omega,\alpha+\beta} f(x), \tag{41}$$

by Lemma 12, then, we have

$$\begin{aligned}&\|T_{\Omega,\alpha,A}f(x)\|_{L^{q(\cdot)}(R^n)}\\&\leq C\left(\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}}\right)\\&\quad\times\sup\left\{\int_{R^n}\overline{T}_{\Omega,\alpha+\beta}f(x)\,|g(x)|\,dx:\|g\|_{L^{q'(\cdot)}(R^n)}\leq 1\right\}.\end{aligned}\tag{42}$$

Using the generalized Hölder inequality, then,

$$\begin{aligned}&\|T_{\Omega,\alpha,A}f\|_{L^{q(\cdot)}(R^n)}\\&\leq C\left(\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}}\right)\left\|\overline{T}_{\Omega,\alpha+\beta}f\right\|_{L^{q(\cdot)}(R^n)}\|g\|_{L^{q'(\cdot)}(R^n)}\\&\leq C\left(\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}}\right)\left\|\overline{T}_{\Omega,\alpha+\beta}f\right\|_{L^{q(\cdot)}(R^n)}.\end{aligned}\tag{43}$$

Next, we will prove $\|\overline{T}_{\Omega,\alpha+\beta}f\|_{L^{q(\cdot)}(R^n)} \leq \|f\|_{L^{p(\cdot)}(R^n)}$. Fix $f \in L^{p(\cdot)}(R^n)$, without loss of generality we may assume that $\|f\|_{L^{p(\cdot)}(R^n)} = 1$. Since $q_+ < \infty$, by Lemma 15 it will suffice to prove that $|T_{\Omega,\alpha+\beta}f|_{L^{q(\cdot)}(R^n)} \leq C$.

Fix ε, $0 < \varepsilon < \min(\alpha+\beta, n-(\alpha+\beta))$, such that

$$\frac{2}{(\varepsilon q_+/n)+1} > 1,\tag{44}$$

define $r(\cdot) : R^n \to [1,+\infty)$ by

$$r(x) = \frac{2}{(\varepsilon q(x)/n)+1}.\tag{45}$$

Then, by (44), we have $r_- > 1$. Moreover, by elementary algebra, for all $x \in R^n$,

$$\frac{1}{p(x)} - \frac{1}{r(x)q(x)/2} = \frac{\alpha+\beta-\varepsilon}{n},\tag{46}$$

$$\frac{1}{p(x)} - \frac{1}{r'(x)q(x)/2} = \frac{\alpha+\beta+\varepsilon}{n}.\tag{47}$$

So that by Lemma 14, we have

$$\begin{aligned}&\int_{R^n}\left|\overline{T}_{\Omega,\alpha+\beta}f(x)\right|^{q(x)}dx\\&\leq C\int_{R^n}\left[M_{\Omega,\alpha+\beta-\varepsilon}f(x)\right]^{q(x)/2}\left[M_{\Omega,\alpha+\beta+\varepsilon}f(x)\right]^{q(x)/2}dx.\end{aligned}\tag{48}$$

By Lemma 13, then,

$$\begin{aligned}&\int_{R^n}\left|\overline{T}_{\Omega,\alpha+\beta}f(x)\right|^{q(x)}dx\\&\leq C\left\|\left[M_{\Omega,\alpha+\beta-\varepsilon}f(x)\right]^{q(x)/2}\right\|_{L^{r(\cdot)}(R^n)}\\&\quad\times\left\|\left[M_{\Omega,\alpha+\beta+\varepsilon}f(x)\right]^{q(x)/2}\right\|_{L^{r'(\cdot)}(R^n)}.\end{aligned}\tag{49}$$

Without loss of generality, we may assume that each is greater than 1, since, otherwise, there is nothing to prove. In this case, in the definition of each norm we may assume that the infimum is taken over by values of λ which are greater than 1. But then, since for all $x \in R^n$ and $\lambda > 1$, $\lambda^{2/q(x)} \geq \lambda^{2/q_+}$, we have

$$\begin{aligned}&\int_{R^n}\left(\frac{\left[M_{\Omega,\alpha+\beta-\varepsilon}f(x)\right]^{q(x)/2}}{\lambda}\right)^{r(x)}dx\\&=\int_{R^n}\left(\frac{M_{\Omega,\alpha+\beta-\varepsilon}f(x)}{\lambda^{2/q(x)}}\right)^{r(x)q(x)/2}dx\\&\leq\int_{R^n}\left(\frac{M_{\Omega,\alpha+\beta-\varepsilon}f(x)}{\lambda^{2/q_+(x)}}\right)^{r(x)q(x)/2}dx.\end{aligned}\tag{50}$$

Therefore, by (46) and Lemma 17, we have

$$\begin{aligned}\left\|\left[M_{\Omega,\alpha+\beta-\varepsilon}f(x)\right]^{q(x)/2}\right\|_{L^{r(x)}(R^n)}&\leq\left\|\left[M_{\Omega,\alpha+\beta-\varepsilon}f(x)\right]\right\|^{q_+/2}_{L^{r(x)q(x)/2}(R^n)}\\&\leq C\|f\|^{q_+/2}_{L^{p(x)}(R^n)}\leq C.\end{aligned}\tag{51}$$

In the same way, we have

$$\begin{aligned}&\int_{R^n}\left(\frac{\left[M_{\Omega,\alpha+\beta+\varepsilon}f(x)\right]^{q(x)/2}}{\lambda}\right)^{r'(x)}dx\\&=\int_{R^n}\left(\frac{M_{\Omega,\alpha+\beta+\varepsilon}f(x)}{\lambda^{2/q(x)}}\right)^{r'(x)q(x)/2}dx\\&\leq\int_{R^n}\left(\frac{M_{\Omega,\alpha+\beta+\varepsilon}f(x)}{\lambda^{2/(q)_+(x)}}\right)^{r'(x)q(x)/2}dx.\end{aligned}\tag{52}$$

Therefore, by (47) and Lemma 17, then,

$$\begin{aligned}\left\|\left[M_{\Omega,\alpha+\beta+\varepsilon}f(x)\right]^{q(x)/2}\right\|_{L^{r'(x)}(R^n)}&\leq\left\|\left[M_{\Omega,\alpha+\beta+\varepsilon}f(x)\right]\right\|^{q_+/2}_{L^{r'(x)q(x)/2}(R^n)}\\&\leq C\|f\|^{q_+/2}_{L^{p(x)}(R^n)}\leq C.\end{aligned}\tag{53}$$

Hence,

$$\left|\overline{T}_{\Omega,\alpha+\beta}f\right|_{L^{q(\cdot)}(R^n)} = \int_{R^n}\left|\overline{T}_{\Omega,\alpha+\beta}f(x)\right|^{q(x)}dx \leq C.\tag{54}$$

So, we have

$$\begin{aligned}&\left\|\overline{T}_{\Omega,\alpha+\beta}f\right\|_{L^{q(\cdot)}(R^n)}\leq\|f\|_{L^{p(\cdot)}(R^n)},\\&\|T_{\Omega,\alpha,A}f\|_{L^{q(\cdot)}(R^n)}\\&\leq C\left(\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}}\right)\left\|\overline{T}_{\Omega,\alpha+\beta}f\right\|_{L^{q(\cdot)}(R^n)}\\&\leq C\left(\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}}\right)\|f\|_{L^{p(\cdot)}(R^n)}.\end{aligned}\tag{55}$$

This completes the proof of Theorem 5. □

By Lemmas 15 and 18 and Theorem 5, the proof of Theorem 6 is directly deduced.

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Research Article

Modeling of Thermal Distributions around a Barrier at the Interface of Coating and Substrate

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Due to constant heat flux, the thermal distribution around an insulated barrier at the interface of substrate and functionally graded material (FGM) which are essentially two-phase particulate composites is examined in such a way that the volume fractions of the constituents vary continuously in the thickness direction. Using integral transform method, two-dimensional steady-state diffusion equation with variable conductivity is turned into constant coefficient differential equation. Reducing that equation to a singular integral equation with Cauchy type, the temperature distribution around the barrier is obtained by defining an unknown function, which is called density function, as a series expansion of orthogonal polynomials. Results are shown for different thickness and nonhomogeneity parameters of FGM.

1. Introduction

There are many engineering applications under severe thermal loading that require high temperature resistant materials in various forms of composites and bonded materials such as power generation, transportation, aerospace, and thermal barrier coatings. New developments in science and technology rely on the developments of new materials. Composites appear to provide the necessary flexibility in the design of these new materials, which are essentials that every part of the material in use exhibits uniform properties. In the mid-1980s, a new composite material, which was initially designed as a kind of thermal barrier coating used in aerospace structural applications and fusion reactors, was found by a group of scientists in Japan. Because of the material's structure, it is named as functionally graded material (FGM). FGMs were used in modern technologies as advanced structures where the composition or the microstructure is locally varied so that a certain variation of the local material properties is achieved [1]. FGMs are also developed for general use as structural components in extremely high-temperature environments. The concept is to make a composite material by varying the microstructure from one material to another material with a specific gradient. The transition between the two materials can usually be approximated by means of a power series or an exponential function [2–5].

The aircraft and aerospace industry and the computer circuit industry are very interested in the possibility of materials that can withstand very high thermal gradients. This is normally achieved by using a ceramic layer connected with a metallic layer. The composition profile which varies from 0% ceramic at the interface to 100% ceramic near the surface, in turn, is selected in such a way that the resulting nonhomogeneous material exhibits the desired thermomechanical properties. The concept of FGMs could provide great flexibility in material design by controlling both composition profile and microstructure [6].

A number of reviews dealing with various aspects of FGMs have been published in the past few decades. They show that most of early research studies in FGMs had focused more on thermal stress analysis and fracture mechanics. Fracture mechanics of FGMs have been studied analytically by Erdogan and coworkers [7–9]. Erdogan identified a number of typical problem areas relating to the fracture of FGMs by considering mainly the investigation of the nature of stress singularity near the tip of a crack in a different geometry [10]. Erdogan also investigated the nature of the crack-tip stress field in a nonhomogeneous medium having a shear modulus with a discontinuous derivative. The problem was considered for the simplest possible loading and geometry, namely, the antiplane shear loading of two bonded half spaces, in which the crack is perpendicular to the interface. It was shown that the square-root singularity of the crack-tip stress field is unaffected by the discontinuity in the derivative of the shear modulus [11]. Related to the fracture problems in composite materials, the solution of integral equations with strongly singular kernels is examined by Kaya and Erdogan [12, 13]. In an axisymmetric coordinate system, an embedded axisymmetric crack in a nonhomogeneous infinite medium was studied by Ozturk and Erdogan [14]. They showed the effect of the material nonhomogeneity on the stress intensity factors under constant Poisson's ratio.

Due to the material mismatch at the interface of substrate and coating, the thermal distributions and thermal stresses on the crack or insulated barrier at the interface are examined by researchers. A general analysis of one-dimensional steady-state thermal stresses in a hollow thick cylinder made of functionally graded material is developed by Jabbari et al. [15]. The material properties were assumed to be nonlinear with a power law distribution. The mechanical and thermal stresses were obtained through the direct method of solution of the Navier equation. The problem of general solution for the mechanical and thermal stresses in a short length functionally graded hollow cylinder due to the two-dimensional axisymmetric steady-state loads was solved using the Bessel functions by Jabbari et al. [16]. A standard method was used to solve a nonhomogeneous system of partial differential Navier equations with nonconstants coefficients, using Fourier series.

Jin and Noda [17] examined an internal crack problem in nonhomogeneous half-plane under thermal loading using the airy stress function method and Fourier transform. They reduced the problem to a system of singular integral equations and solved it by numerical methods. They used superposition method by defining the problem in two cases. One is the linear one-dimensional heat conduction problem under constant heat flux without crack, and the other one is the two-dimensional heat conduction problem with an insulated crack subject to constant heat flux which is acted in the opposite direction. It was also considered the problem of an axisymmetric penny-shaped crack embedded in an isotropic graded coating bonded to a semi-infinite homogeneous medium by Rekik et al. [18]. The coating's material gradient is parallel to the axisymmetric direction and is orthogonal to the crack plane. They used Hankel transform to convert the equations into coupled singular integral equations along with the density function, and they solved it numerically.

In this study, the Hankel integral transform method will be used to solve the heat equation in axisymmetric coordinate system. Problem will be examined as a one-dimensional and a two-dimensional heat conduction problem that is a mixed boundary value problem over the real line. Using mixed boundary conditions a Fredholm integral equation will be obtained with Cauchy type singularity and then it will be solved by using some known numerical techniques [19, 20].

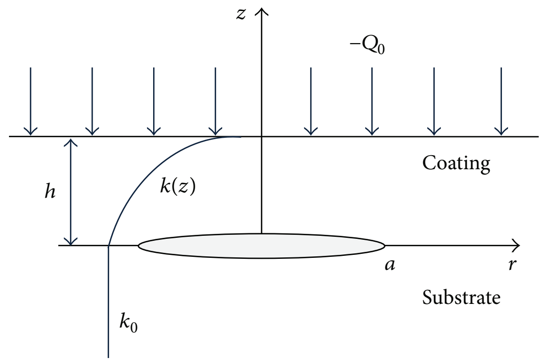

Figure 1: Geometry of the heat conduction problem.

2. Definition of the Problem

The thermal distribution around a penny-shaped barrier at the interface of graded composite coating and a substrate is given by the following steady-state heat equation in axisymmetric coordinate system:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk(z)\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial z}\left(k(z)\frac{\partial T}{\partial z}\right)=0, \quad (1)$$

and the conductivities of the substrate and the graded composite coating are, respectively, given as

$$\begin{aligned} k(z) &= k_0, \quad z \le 0, \\ k(z) &= k_0 e^{\delta z}, \quad z \ge 0, \end{aligned} \quad (2)$$

where k_0 is a constant and δ is the nonhomogeneity parameter related to the graded coating. Note that the conductivity is continuous at the interface of substrate and the graded composite coating. As shown in Figure 1, it is considered a penny-shaped barrier with radius a, centered at the origin of the axisymmetric coordinate system. It is assumed that a uniform heat flux is applied over the stress free boundary, and the barrier faces remain insulated.

The solution can be obtained using superposition method which is an addition of one- and two-dimensional heat conduction problems, $T_1(z)$ and $T_2(r,z)$, respectively, as shown in Figures 2(a) and 2(b). As in Figure 2(a), it will be assumed that there will be no barrier and in flux causes thermal distribution only z direction. On the other hand, in Figure 2(b), flux will be assumed in an opposite direction on the barrier that causes thermal distribution in (r,z) plane.

Rewriting (1) assuming that no changing in r direction:

$$\begin{aligned} \delta\frac{dT_1}{dz}+\frac{d^2T_1}{dz^2} &= 0, \quad 0 \le z \le h; \\ \frac{d^2T_1}{dz^2} &= 0, \quad z \le 0, \end{aligned} \quad (3)$$

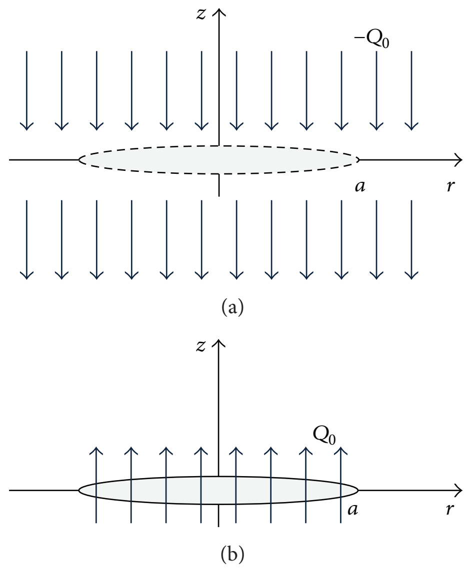

Figure 2: (a) One-dimensional heat conduction without an insulated barrier, (b) Two-dimensional heat conduction with an insulated barrier.

along with suitable boundary conditions:

$$\begin{aligned} &k(z)\frac{d}{dz}T_1(z) \longrightarrow \text{finite}, \quad z \longrightarrow -\infty, \\ &k(z)\frac{d}{dz}T_1(z) = -Q_0, \quad z = h, \\ &T_1(0^+) = T_1(0^-), \end{aligned} \tag{4}$$

the solution of the one-dimensional heat conduction is obtained straightforward as

$$T_1(z) = \begin{cases} \dfrac{Q_0}{k_0\delta}e^{-\delta z}, & 0 \le z \le h, \\ \dfrac{Q_0}{k_0\delta}, & z \le 0, \end{cases} \tag{5}$$

where the continuity can be seen as $z \rightarrow 0$. For two-dimensional heat conduction problem, (1) can be simplified for $0 < z \le h$ as

$$\begin{aligned} &\frac{\partial^2}{\partial r^2}T_2(r,z) + \frac{1}{r}\frac{\partial}{\partial r}T_2(r,z) + \frac{\partial^2}{\partial z^2}T_2(r,z) \\ &\quad + \delta\frac{\partial}{\partial z}T_2(r,z) = 0, \end{aligned} \tag{6}$$

and for $z < 0$,

$$\frac{\partial^2}{\partial r^2}T_2(r,z) + \frac{1}{r}\frac{\partial}{\partial r}T_2(r,z) + \frac{\partial^2}{\partial z^2}T_2(r,z) = 0 \tag{7}$$

with standard boundary conditions:

$$\begin{aligned} &k(z)\frac{\partial}{\partial z}T_2(r,z) \longrightarrow \text{finite}, \quad z \longrightarrow -\infty, \\ &k(z)\frac{\partial}{\partial z}T_2(r,z) = 0, \quad z = h, \ 0 \le r < \infty, \\ &\frac{\partial}{\partial z}T_2(r,0^+) = \frac{\partial}{\partial z}T_2(r,0^-), \quad a < r < \infty, \end{aligned} \tag{8}$$

and mixed boundary conditions:

$$k(z)\frac{\partial}{\partial z}T_2(r,z) = Q_0, \quad z \longrightarrow 0, \ 0 \le r < a, \tag{9}$$

$$T_2(r,0^+) = T_2(r,0^-), \quad a < r < \infty. \tag{10}$$

Equation (6) will be solved using Hankel integral transform such that $T_2(r,z)$ denote the Hankel transform of zero order and $\tau(\rho,z)$ denote inverse Hankel transform of zero order [21] shown as below, respectively,

$$\begin{aligned} T_2(r,z) &= \int_0^\infty \tau(\rho,z)J_0(r\rho)\rho d\rho, \\ \tau(\rho,z) &= \int_0^\infty T_2(r,z)J_0(r\rho)rdr. \end{aligned} \tag{11}$$

Using Hankel transform, the solution of (6) along with boundary conditions (8) can be obtained as

$$\tau(\rho,z) = \begin{cases} A(\rho)e^{-(m_2+m_1)z} + A(\rho)\dfrac{m_2+m_1}{m_2-m_1} \\ \quad \times e^{-2m_2h+(m_2-m_1)z}, & 0 < z \le h, \\ A(\rho)\dfrac{1}{\rho}\left(e^{-2m_2h}-1\right)(m_2-m_1) \\ \quad \times e^{\rho z}, & -\infty < z < 0, \end{cases} \tag{12}$$

where $m_1 = \delta/2$, $m_2 = \rho\sqrt{(\delta/2\rho)^2+1}$ and observing that $m_2 + m_1 > 0$ and $m_2 - m_1 > 0$.

3. Evaluation of Integral Equation

The unknown value $A(\rho)$ can be obtained by defining a new function, which is called density function [11], such as

$$\psi(r) = \frac{\partial}{\partial r}\left(T_2(r,0^+) - T_2(r,0^-)\right), \tag{13}$$

where $\psi(r)$ satisfies the following conditions:

$$\begin{aligned} &\int_0^a \psi(r)\,dr = 0, \quad \psi(r) = 0, \quad \text{at } a < r < \infty, \\ &\psi(r) = -\psi(-r). \end{aligned} \tag{14}$$

Substituting (12) into (13) along with the conditions given in (14), the unknown function $A(\rho)$ can be obtained:

$$A(\rho) = \frac{1}{F(\rho)}\int_0^a \psi(s)J_1(s\rho)\,sds, \tag{15}$$

where

$$F(\rho) = -\rho - \rho\left(\frac{m_2+m_1}{m_2-m_1}\right)e^{-2m_2h} + \left(e^{-2m_2h}-1\right)(m_2-m_1). \tag{16}$$

Using the boundary condition (9) in transformed domain as $z \to 0^-$, it can be obtained as

$$\int_0^\infty \left(e^{-2m_2h}-1\right)(m_2-m_1)A(\rho)J_0(r\rho)\rho d\rho = \frac{Q_0}{k_0}, \tag{17}$$

and substituting the value of $A(\rho)$ into (17), the integral equation to be solved for unknown $\psi(s)$ can be obtained as

$$\int_0^a \left(\int_0^\infty \left(e^{-2m_2h}-1\right)\eta(\rho)J_0(r\rho)J_1(s\rho)\rho\, d\rho\right)\psi(s)sds = \frac{Q_0}{k_0}, \tag{18}$$

where

$$\eta(\rho) = \frac{m_2-m_1}{F(\rho)}. \tag{19}$$

Defining new normalized variables and parameters such as

$$s' = \frac{s}{a}, \quad r' = \frac{r}{a}, \quad \rho' = a\rho, \quad h' = \frac{h}{a}, \quad \delta' = a\delta, \quad \psi(s) = \psi(as') = \psi(s'), \tag{20}$$

the integral equation in (18) can be expressed in the form of

$$\begin{aligned}\int_0^1 \psi(s)sds\int_0^\infty J_0(r\rho)J_1(s\rho)\rho d\rho &+ \int_0^1 \psi(s)sds\int_0^\infty 2e^{-2m_2h}\,\eta(\rho)J_0(r\rho)J_1(s\rho)\rho d\rho \\ &- \int_0^1 \psi(s)sds\int_0^\infty (2\eta(\rho)+1)J_0(r\rho)J_1(s\rho)\rho d\rho \\ &= q_0,\end{aligned} \tag{21}$$

where $q_0 = 2Q_0/k_0$ and the prime sign is removed for the sake of simplicity. The first double integral in (21) can be expressed in terms of the first and second kinds of elliptic integrals, K and E, respectively. As $k \to 1$, the second kind of elliptic integral $E(k)$ has a finite value while the first kind of elliptic integral $K(k)$ has a logarithmic singularity such as

$$K(k) = \log\left(\frac{4}{\sqrt{1-k^2}}\right), \qquad E(k) = 1. \tag{22}$$

Defining a new function $M(s,r)$, given in the appendix, and using some algebraic manipulations, a Cauchy type singularity can obtained by using the properties of $\psi(s)$ in (14). Hence, the first integral of (21) becomes

$$\begin{aligned}\int_0^1 \psi(s)sds\int_0^\infty J_0(r\rho)J_1(s\rho)\rho d\rho &= \frac{1}{\pi}\int_{-1}^1 \frac{\psi(s)}{s-r}ds \\ &+ \frac{1}{\pi}\int_0^1\left(\frac{M(s,r)-1}{s-r}+\frac{M(s,r)-1}{s+r}\right)\psi(s)ds.\end{aligned} \tag{23}$$

The second double integral in (21) has an exponential integrand with negative exponent so that as $\rho \to \infty$, the integrand asymptotically approaches to zero. The asymptotic expansion of the integrand can be expressed as

$$\begin{aligned}e^{-2h\rho}\rho\Bigg(-1 &+ \frac{h\delta^2-\delta}{2^2\rho} - \frac{h^2\delta^4-2h\delta^3}{2^5\rho^2} \\ &+ \frac{(1/3)h^3\delta^6 - h^2\delta^5 - 2h\delta^4 + 2\delta^3}{2^7\rho^3} \\ &- \frac{(1/3)h^4\delta^8 + (4/3)h^3\delta^7 - 8h^2\delta^6 - 40h\delta^5}{2^{11}\rho^4} \\ &+ O\left(\rho^{-5}\right)\Bigg).\end{aligned} \tag{24}$$

Since there is no singularity and any discontinuity over the interval, and due to the asymptotic expansion as $\rho \to \infty$, the infinite integral can be approximated over the interval $[0, B]$. Depending on parameters h and δ, the value of B can be chosen large enough to obtain small enough value of the integral (less than 10^{-25}) over $[B, \infty)$. Hence, the second double integral in (21) may be expressed as

$$\begin{aligned}\int_0^1 \psi(s)sds\int_0^\infty 2e^{-2m_2h}\eta(\rho)J_0(r\rho)J_1(s\rho)\rho d\rho \\ = \int_0^1 \psi(s)sds\int_0^{B(h,\delta)} 2e^{-2m_2h}\eta(\rho)J_0(r\rho)J_1(s\rho)\rho d\rho.\end{aligned} \tag{25}$$

Finally, the last integral in (21) can be evaluated by using a series

$$\begin{aligned}G_A(\rho) = &-\frac{\delta}{2^2} + \frac{\delta^3}{2^6}\frac{1}{\rho^2} - \frac{\delta^5}{2^9}\frac{1}{\rho^4} + \frac{5\delta^7}{2^{14}}\frac{1}{\rho^6} \\ &- \cdots + \frac{429\delta^{15}}{2^{30}}\frac{1}{\rho^{14}} - O\left(\rho^{-16}\right),\end{aligned} \tag{26}$$

an asymptotic expansion of the integrand $G(\rho) = \rho(2\eta(\rho) + 1)$ as $\rho \to \infty$. Let us define a value C that depends on parameters h and δ and superposes the infinite integral as follows:

$$\begin{aligned}\int_0^{\infty} G(\rho) J_0(r\rho) J_1(s\rho)\, d\rho &= \int_0^{C(h,\delta)} G(\rho) J_0(r\rho) J_1(s\rho)\, d\rho \\ &\times \int_{C(h,\delta)}^{\infty} G_A(\rho) J_0(r\rho) J_1(s\rho)\, d\rho \\ &+ \int_{C(h,\delta)}^{\infty} \left[G(\rho) - G_A(\rho)\right] J_0(r\rho) J_1(s\rho)\, d\rho.\end{aligned} \tag{27}$$

For a particular value of C, $G(C)-G_A(C) \approx 0$ can be obtained; then the last infinite integral can be expressed as

$$\begin{aligned}\int_0^{\infty} G(\rho) J_0(r\rho) J_1(s\rho)\, d\rho &= \int_0^{C(h,\delta)} G(\rho) J_0(r\rho) J_1(s\rho)\, d\rho \\ &+ \int_{C(h,\delta)}^{\infty} G_A(\rho) J_0(r\rho) J_1(s\rho)\, d\rho.\end{aligned} \tag{28}$$

Since the function $G(\rho)$ is a smooth function over the interval $[0, C]$, it can be evaluated numerically using Gaussian quadrature along with Chebyshev orthogonal polynomials. On the other hand, the evaluation of the integral over the interval $[C, \infty)$ can be evaluated separately for each term in $G_A(\rho)$ like

$$\begin{aligned}\int_{C(h,\delta)}^{\infty} G_A(\rho) J_0(r\rho) J_1(s\rho)\, d\rho &= -\frac{\delta}{2^2}\int_{C(h,\delta)}^{\infty} J_0(r\rho) J_1(s\rho)\, d\rho \\ &+ \int_{C(h,\delta)}^{\infty} \left(\frac{\delta^3}{64}\frac{1}{\rho^2} - \frac{\delta^5}{512}\frac{1}{\rho^4} + \frac{5\delta^7}{16384}\frac{1}{\rho^6} - \cdots\right) \\ &\times J_0(r\rho) J_1(s\rho)\, d\rho,\end{aligned} \tag{29}$$

where

$$\begin{aligned}H_0(r,s) &= \int_{C(h,\delta)}^{\infty} J_0(r\rho) J_1(s\rho)\, d\rho \\ &= \begin{cases} \dfrac{1}{s} - \displaystyle\int_0^{C(h,\delta)} J_0(r\rho) J_1(s\rho)\, d\rho, & s > r, \\ \dfrac{1}{2s} - \displaystyle\int_0^{C(h,\delta)} J_0(r\rho) J_1(s\rho)\, d\rho, & s = r, \\ -\displaystyle\int_0^{C(h,\delta)} J_0(r\rho) J_1(s\rho)\, d\rho, & s < r, \end{cases}\end{aligned} \tag{30}$$

and the other integrals in (29) can be evaluated iteratively for $k = 1, 2, 3, \ldots$ as follows:

$$\begin{aligned}H_k(r,s) &= \int_{C(h,\delta)}^{\infty} \frac{J_0(r\rho) J_1(s\rho)}{\rho^k}\, d\rho \\ &= \frac{J_0(rC) J_1(sC)}{kC^{k-1}} + \frac{s}{k} L_{k-1}(r,s) + \frac{r}{k} N_{k-1}(r,s),\end{aligned} \tag{31}$$

where

$$\begin{aligned}L_k(r,s) &= \int_{C(h,\delta)}^{\infty} \frac{J_0(r\rho) J_0(s\rho)}{\rho^k}\, d\rho \\ &= \frac{J_0(rC) J_0(sC)}{(k-1) C^{k-1}} - \frac{rH_{k-1}(s,r)}{k-1} - \frac{sH_{k-1}(r,s)}{1-k},\end{aligned} \tag{32}$$

$$\begin{aligned}N_k(r,s) &= \int_{C(h,\delta)}^{\infty} \frac{J_1(r\rho) J_1(s\rho)}{\rho^k}\, d\rho \\ &= \frac{J_1(rC) J_1(sC)}{(k+1) C^{k-1}} + \frac{rH_{k-1}(r,s)}{k+1} + \frac{sH_{k-1}(s,r)}{k+1}.\end{aligned} \tag{33}$$

The initial values of the integral $H_1(r,s)$, $L_2(r,s)$, and $N_3(r,s)$ are shown in the appendix.

4. Numerical Evaluation of Integral Equation

The integral equation in (21) can be solved using Gaussian quadrature method. Using the condition given in (14), the unknown function $\psi(s)$ can be defined in terms of the truncated Nth term series expansion of Chebyshev orthogonal polynomials of the first kind, $T_n(s)$, as

$$\psi(s) = \sum_{n=1}^{N} A_{2n-1} \frac{T_{2n-1}(s)}{\sqrt{1-s^2}}, \quad -1 < s < 1, \tag{34}$$

where A_{2n-1} are coefficients. Substituting the series expansion of the unknown function $\psi(s)$ into the first integral in (23), it can be found that

$$\begin{aligned}\frac{1}{\pi}\int_{-1}^{1} \frac{\psi(s)}{s-r}\, ds &= \sum_{n=1}^{N} A_{2n-1} \frac{1}{\pi}\int_{-1}^{1} \frac{T_{2n-1}(s)}{(s-r)\sqrt{1-s^2}}\, ds \\ &= \sum_{n=1}^{N} A_{2n-1} U_{2n-2}(r),\end{aligned} \tag{35}$$

where $U_{2n-2}(r)$ is the Chebyshev polynomials of the second kind, and eliminating the logarithmic singularities, shown in appendix, the second integral in (23) can be written as

$$\begin{aligned}
&\frac{1}{\pi}\int_0^1\left(\frac{M(s,r)-1}{s-r}+\frac{M(s,r)-1}{s+r}\right)\psi(s)\,ds\\
&\quad=\sum_{n=1}^{N}A_{2n-1}\frac{1}{\pi}\\
&\qquad\times\int_0^1\left(\frac{M(r,s)-1}{s-r}+\frac{M(r,s)-1}{s+r}+\frac{1}{2r}\log\left|\frac{s-r}{s+r}\right|\right)\\
&\qquad\times\frac{T_{2n-1}(s)}{\sqrt{1-s^2}}ds.
\end{aligned}\tag{36}$$

Finally, substituting the truncated series representation of $\psi(s)$ into (25) and (29) system of algebraic equation can be obtained the to be solved for A_{2n-1},

$$\begin{aligned}
&\sum_{n=1}^{N}A_{2n-1}\left(U_{2n-2}(r_i)+\frac{1}{2r_i}\frac{T_{2n-1}(r_i)}{2n-1}+Z_1(r_i)\right.\\
&\qquad\left.+Z_2(r_i)-Z_3(r_i)\right)=q_0,
\end{aligned}\tag{37}$$

where r_i, $i=1,2,3,\ldots$, are collocation points and

$$\begin{aligned}
Z_1(r_i)&=\frac{1}{\pi}\int_0^1\left(\frac{M(r_i,s)-1}{s-r_i}+\frac{M(r_i,s)-1}{s+r_i}\right.\\
&\qquad\left.+\frac{1}{2r_i}\log\left|\frac{s-r_i}{s+r_i}\right|\right)\frac{T_{2n-1}(s)}{\sqrt{1-s^2}}ds,\\
Z_2(r_i)&=\int_0^1\left(\int_0^{B(h,\delta)}2e^{-2m_2h}\,\eta(\rho)J_0(r_i\rho)J_1(s\rho)\rho\,d\rho\right)\\
&\qquad\times\frac{sT_{2n-1}(s)}{\sqrt{1-s^2}}ds,\\
Z_3(r_i)&=\int_0^1\left(\int_0^{C(h,\delta)}G(\rho)J_0(r_i\rho)J_1(s\rho)\,d\rho\right.\\
&\qquad\left.+\int_{C(h,\delta)}^{\infty}G_A(\rho)J_0(r_i\rho)J_1(s\rho)\,d\rho\right)\\
&\qquad\times\frac{sT_{2n-1}(s)}{\sqrt{1-s^2}}ds.
\end{aligned}\tag{38}$$

5. Results

Because of the nature of the problem it is necessary to increase the density of the collocation points near the ends $r=\pm1$. Thus, these points may be selected as follows:

$$T_n(r_i)=0,\quad r_i=\cos\left(\frac{(2i-1)\pi}{2N}\right),\quad i=1,2,\ldots,N.\tag{39}$$

Then, we get a $(N\times N)$ system of equations whose solution gives the coefficients A_{2n+1}. With known coefficient values, the temperature distribution around the insulated barrier may be obtained by integrating (13) such as

$$\begin{aligned}
&T_2(r,0^+)-T_2(r,0^-)\\
&\quad=\sum_{n=1}^{N}A_{2n-1}\int_{-1}^{r/a}\frac{aT_{2n-1}(s)}{\sqrt{1-s^2}}ds.
\end{aligned}\tag{40}$$

Defining new variable like

$$s=\cos\theta,\quad \pi\le\theta\le\arccos\left(\frac{r}{a}\right),\tag{41}$$

the integral in (40) can be evaluated using the relation

$$U_n(t)=\frac{\sin\{(n+1)\arccos t\}}{\sin(\arccos t)},\tag{42}$$

and the difference in temperature distribution on the plane of the insulated barrier can be obtained as

$$\begin{aligned}
T^*(r)&=\frac{T_2(r,0^+)-T_2(r,0^-)}{a}\\
&=-\sqrt{1-\left(\frac{r}{a}\right)^2}\sum_{n=1}^{N}A_{2n-1}\frac{U_{2n-2}(r/a)}{2n-1}.
\end{aligned}\tag{43}$$

Appendix

The function $M(s,r)$ in (23) can be defined [14] as

$$M(s,r)=\begin{cases}\dfrac{r}{s}E\left(\dfrac{s}{r}\right)+\dfrac{s^2-r^2}{rs}K\left(\dfrac{s}{r}\right), & s<r,\\[2mm] E\left(\dfrac{r}{s}\right), & s>r,\end{cases}\tag{A.1}$$

in terms of the complete elliptic integrals of the first and second kinds, respectively,

$$\begin{aligned}
K(k)&=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},\\
E(k)&=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta.
\end{aligned}\tag{A.2}$$

For $k \geq 2$, the values of the integrals in (31), (32), and (33), respectively, can be obtained by solving the initial value of each integral for $k = 1$ such that

$$H_1(r,s) = \int_{C(h,\delta)}^{\infty} \frac{J_0(r\rho) J_1(s\rho)}{\rho} d\rho$$
$$= \frac{1}{\pi} \int_0^{\pi} \frac{s - r\cos\phi}{R} \left(1 + J_1(RC(h,\delta)) - \int_0^{RC(h,\delta)} J_0(v)\,dv\right) d\phi,$$
$$L_1(r,s) = \int_{C(h,\delta)}^{\infty} \frac{J_0(r\rho) J_0(s\rho)}{\rho} d\rho$$
$$= \frac{1}{\pi} \int_0^{\pi} \left(-\gamma - \log\left(\frac{RC(h,\delta)}{2}\right) + \int_0^{RC(h,\delta)} \frac{1 - J_0(u)}{u} du\right) d\phi,$$
$$N_1(r,s) = \int_{C(h,\delta)}^{\infty} \frac{J_1(r\rho) J_1(s\rho)}{\rho} d\rho$$
$$= \frac{rs}{\pi} \int_0^{\pi} \frac{\sin^2\phi}{R} \left(\int_{C(h,\delta)}^{\infty} J_1(R\rho)\,d\rho\right) d\phi, \tag{A.3}$$

where $R^2 = r^2 + s^2 - 2rs\cos\phi$.

Due to the logarithmic singularity as $s \to r$ in (23), the ratio $((M(s,r) - 1)/(s - r)) \to (0/0)$ has undetermined limiting case. Using (22) along with L'Hospital's rule, we have

$$\lim_{s \to r} \frac{M(s,r) - 1}{s - r} = -\frac{1}{2r} \log|s - r| + \frac{1}{r}\left(\log\sqrt{8r} - 1\right). \tag{A.4}$$

Now, by adding and subtracting the leading part of the logarithmic function,

$$\frac{1}{\pi} \int_0^1 \left(\frac{M(r,s) - 1}{s - r} + \frac{M(r,s) - 1}{s + r} + \frac{1}{2r} \log|s - r|\right) \psi(s)\,ds$$
$$- \frac{1}{2\pi r} \int_0^1 \left(\log|s - r|\right) \psi(s)\,ds, \tag{A.5}$$

and using the symmetry property of $\psi(s)$ like

$$-\int_0^1 \left(\log|s - r|\right) \psi(s)\,ds = -\int_{-1}^1 \left(\log|s - r|\right) \psi(s)\,ds$$
$$- \int_0^1 \left(\log|s + r|\right) \psi(s)\,ds, \tag{A.6}$$

we have

$$\frac{1}{\pi} \int_0^1 \left(\frac{M(r,s) - 1}{s - r} + \frac{M(r,s) - 1}{s + r} + \frac{1}{2r} \log\left|\frac{s - r}{s + r}\right|\right) \psi(s)\,ds$$
$$- \frac{1}{\pi} \int_{-1}^1 \frac{1}{2r} \left(\log|s - r|\right) \psi(s)\,ds, \tag{A.7}$$

where

$$\frac{1}{\pi} \int_{-1}^1 \log|s - r| \frac{T_{2n-1}(s)}{\sqrt{1 - s^2}} ds = -\frac{T_{2n-1}(r)}{2n - 1}, \tag{A.8}$$

using the series expansion of unknown function $\psi(s)$ in (34).

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Research Article

Numerical Solutions of the Second-Order One-Dimensional Telegraph Equation Based on Reproducing Kernel Hilbert Space Method

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We investigate the effectiveness of reproducing kernel method (RKM) in solving partial differential equations. We propose a reproducing kernel method for solving the telegraph equation with initial and boundary conditions based on reproducing kernel theory. Its exact solution is represented in the form of a series in reproducing kernel Hilbert space. Some numerical examples are given in order to demonstrate the accuracy of this method. The results obtained from this method are compared with the exact solutions and other methods. Results of numerical examples show that this method is simple, effective, and easy to use.

1. Introduction

The hyperbolic partial differential equations model the vibrations of structures (e.g., buildings, beams, and machines). These equations are the basis for fundamental equations of atomic physics. In this paper, we consider the telegraph equation of the form

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2}(x,t) + 2\alpha\frac{\partial u}{\partial t}(x,t) + \beta^2 u(x,t)\\&\quad = \frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t), \quad 0 \le x,\ t \le 1,\ \alpha > \beta \ge 0,\end{aligned} \tag{1}$$

with initial conditions

$$u(x,0) = \varphi_1(x), \qquad u_t(x,0) = \varphi_2(x), \tag{2}$$

and appropriate boundary conditions

$$u(0,t) = g_0(t), \qquad u(1,t) = g_1(t), \quad t \ge 0 \tag{3}$$

by using reproducing kernel method (RKM). In recent years, much attention has been given in the literature to the development, analysis, and implementation of stable methods for the numerical solution of (1)–(3) [1–3]. Mohanty carried out a new technique to solve the linear one-space-dimensional hyperbolic equation (1) [4]. High-order accurate method for solving linear hyperbolic equation is presented in [5]. A compact finite difference approximation of fourth order for discretizing spatial derivative of linear hyperbolic equation and a collocation method for the time component are used in [6]. A numerical scheme is developed to solve the one-dimensional hyperbolic telegraph equation using the collocation points and approximating the solution using thin plate splines radial basis function [7]. Several test problems were given, and the results of numerical experiments were compared with analytical solutions to confirm the good accuracy of their scheme. Yao [8] investigated a nonlinear hyperbolic telegraph equation with an integral condition by reproducing kernel space at $\alpha = \beta = 0$. Yousefi presented a numerical method for solving the one-dimensional hyperbolic telegraph equation by using Legendre multiwavelet Galerkin method [9]. Dehghan and Lakestani presented a numerical technique for the solution of the second-order one-dimensional linear hyperbolic equation [10]. Lakestani and Saray used interpolating scaling functions for solving (1)–(3) [11]. Dehghan provided a solution of the second-order one-dimensional hyperbolic telegraph equation by using the dual reciprocity boundary integral equation (DRBIE) method [12]. The problem has explicit solution that can be obtained by the method of separation of variables in [13].

In this paper, the problem is solved easily and elegantly by using RKM. The technique has many advantages over the classical techniques. It also avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculation, and avoidance of physically unrealistic assumptions. In the next section, we will describe the procedure of this method.

The theory of reproducing kernels was used for the first time at the beginning of the 20th century by Zaremba in his work on boundary value problems for harmonic and biharmonic functions [14]. Reproducing kernel theory has important application in numerical analysis, differential equations, probability, and statistics. Recently, using the RKM, some authors discussed telegraph equation [15], Troesch's porblem [16], MHD Jeffery-Hamel flow [17], Bratu's problem [18], KdV equation [19], fractional differential equation [20], nonlinear oscillator with discontinuity [21], nonlinear two-point boundary value problems [22], integral equations [23], and nonlinear partial differential equations [24].

The paper is organized as follows. Section 2 introduces several reproducing kernel spaces. The representation in $W(\Omega)$ and a linear operator are presented in Section 3. Section 4 provides the main results. The exact and approximate solutions of (1)–(3) and an iterative method are developed for the kind of problems in the reproducing kernel space. We have proved that the approximate solution converges to the exact solution uniformly. Numerical experiments are illustrated in Section 5. Some conclusions are given in Section 6.

2. Reproducing Kernel Spaces

In this section, some useful reproducing kernel spaces are defined.

Definition 1 (reproducing kernel function). Let $E \neq \emptyset$. A function $K : E \times E \rightarrow \mathbb{C}$ is called *a reproducing kernel function* of the Hilbert space H if and only if

(a) $K(\cdot, t) \in H$ for all $t \in E$,

(b) $\langle \varphi, K(\cdot, t)\rangle = \varphi(t)$ for all $t \in E$ and all $\varphi \in H$.

The last condition is called "the reproducing property" as the value of the function φ at the point t is reproduced by the inner product of φ with $K(\cdot, t)$.

Definition 2. Hilbert function space H is a reproducing kernel space if and only if for any fixed $x \in X$, the linear functional $I(f) = f(x)$ is bounded [25, page 5].

Definition 3. We define the space $H_2^1[0,1]$ by

$$H_2^1[0,1] = \left\{u \in AC[0,1] : u' \in L^2[0,1]\right\}. \quad (4)$$

The inner product and the norm in $H_2^1[0,1]$ are defined by

$$\langle u, g\rangle_{H_2^1} = u(0)\, g(0) + \int_0^1 u'(t)\, g'(t)\, \mathrm{d}t, \quad u, g \in H_2^1[0,1],$$
$$\|u\|_{H_2^1} = \sqrt{\langle u, u\rangle_{H_2^1}}, \quad u \in H_2^1[0,1]. \quad (5)$$

Lemma 4. *The space $H_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function q_s is given by [25, page 123]*

$$q_s(t) = \begin{cases} 1+t, & t \le s, \\ 1+s, & t > s. \end{cases} \quad (6)$$

Definition 5. We define the space $F_2^3[0,T]$ by

$$F_2^3[0,T] = \left\{u \in AC[0,T] : u', u'' \in AC[0,T], u^{(3)} \in L^2[0,T], u(0) = u'(0) = 0\right\}. \quad (7)$$

The inner product and the norm in $F_2^3[0,T]$ are defined by

$$\langle u, g\rangle_{F_2^3} = \sum_{i=0}^{2} u^{(i)}(0)\, g^{(i)}(0) + \int_0^T u^{(3)}(t)\, g^{(3)}(t)\, \mathrm{d}t, \quad u, g \in F_2^3[0,T], \quad (8)$$
$$\|u\|_{F_2^3} = \sqrt{\langle u, u\rangle_{F_2^3}}, \quad u \in F_2^3[0,T].$$

Lemma 6. *The space $F_2^3[0,T]$ is a reproducing kernel space, and its reproducing kernel function r_s is given by [25, page 148]*

$$r_s(t) = \begin{cases} \frac{1}{4}s^2t^2 + \frac{1}{12}s^2t^3 - \frac{1}{24}st^4 + \frac{1}{120}t^5, & t \le s, \\ \frac{1}{4}s^2t^2 + \frac{1}{12}s^3t^2 - \frac{1}{24}ts^4 + \frac{1}{120}s^5, & t > s. \end{cases} \quad (9)$$

Definition 7. We define the space $W_2^3[0,1]$ by

$$W_2^3[0,1] = \left\{u \in AC[0,1] : u', u'' \in AC[0,1], u^{(3)} \in L^2[0,1], u(0) = u(1) = 0\right\}. \quad (10)$$

The inner product and the norm in $W_2^3[0,1]$ are defined by

$$\langle u, g\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)\, g^{(i)}(0) + \int_0^1 u^{(3)}(x)\, g^{(3)}(x)\, \mathrm{d}x, \quad u, g \in W_2^3[0,1],$$
$$\|u\|_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}, \quad u \in W_2^3[0,1]. \quad (11)$$

The space $W_2^3[0,1]$ is a reproducing kernel space, and its reproducing kernel function R_y is given by the following theorem.

Theorem 8. *The space $W_2^3[0,1]$ is a reproducing kernel space, and its reproducing kernel function R_y is given by*

$$R_y(x) = \begin{cases} \sum_{i=1}^{5} c_i(y)x^i, & x \le y, \\ \sum_{i=0}^{5} d_i(y)x^i, & x > y, \end{cases} \tag{12}$$

where

$$\begin{aligned}
c_1(y) &= \frac{5}{156}y^4 - \frac{1}{156}y^5 - \frac{5}{26}y^2 - \frac{5}{78}y^3 + \frac{3}{13}y, \\
c_2(y) &= \frac{5}{624}y^4 - \frac{1}{624}y^5 + \frac{21}{104}y^2 - \frac{5}{312}y^3 - \frac{5}{26}y, \\
c_3(y) &= \frac{5}{1872}y^4 - \frac{1}{1872}y^5 + \frac{7}{104}y^2 - \frac{5}{936}y^3 - \frac{5}{78}y, \\
c_4(y) &= -\frac{5}{3744}y^4 + \frac{1}{3744}y^5 + \frac{5}{624}y^2 + \frac{5}{1872}y^3 - \frac{1}{104}y, \\
c_5(y) &= \frac{5}{3744}y^4 - \frac{1}{18720}y^5 - \frac{1}{624}y^2 \\
&\quad - \frac{1}{1872}y^3 - \frac{1}{156}y + \frac{1}{120}, \\
d_0(y) &= \frac{1}{120}y^5, \\
d_1(y) &= -\frac{1}{104}y^4 - \frac{1}{156}y^5 - \frac{5}{26}y^2 - \frac{5}{78}y^3 + \frac{3}{13}y, \\
d_2(y) &= \frac{7}{104}y^3 + \frac{5}{624}y^4 - \frac{1}{624}y^5 + \frac{21}{104}y^2 - \frac{5}{26}y, \\
d_3(y) &= \frac{5}{1872}y^4 - \frac{1}{1872}y^5 - \frac{5}{312}y^2 - \frac{5}{936}y^3 - \frac{5}{78}y, \\
d_4(y) &= -\frac{5}{3744}y^4 + \frac{1}{3744}y^5 + \frac{5}{624}y^2 + \frac{5}{1872}y^3 + \frac{5}{156}y, \\
d_5(y) &= \frac{1}{3744}y^4 - \frac{1}{18720}y^5 - \frac{1}{624}y^2 - \frac{1}{1872}y^3 - \frac{1}{156}y.
\end{aligned} \tag{13}$$

Proof. Let $u \in W_2^3[0,1]$ and let $0 \le y \le 1$. By Definition 7 and integrating by parts two times, we obtain that

$$\begin{aligned}
\langle u, R_y \rangle_{W_2^3} &= \sum_{i=0}^{2} u^{(i)}(0) R_y^{(i)}(0) + \int_0^1 u^{(3)}(x) R_y^{(3)}(x)\,dx \\
&= u(0)R_y(0) + u'(0)R_y'(0) \\
&\quad + u''(0)R_y''(0) + u''(1)R_y^{(3)}(1) \\
&\quad - u''(0)R_y^{(3)}(0) \\
&\quad - u'(1)R_y^{(4)}(1) + u'(0)R_y^{(4)}(0) \\
&\quad + \int_0^1 u'(x) R_y^{(5)}(x)\,dx.
\end{aligned} \tag{14}$$

After substituting the values of $R_y(0)$, $R_y'(0)$, $R_y''(0)$, $R_y^{(3)}(0)$, $R_y^{(4)}(0)$, $R_y^{(3)}(1)$, and $R_y^{(4)}(1)$ into the above equation, we get

$$\begin{aligned}
\langle u, R_y \rangle_{W_2^3} &= u(0)0 + u'(0) \\
&\quad \times \left(\frac{3}{13}y - \frac{5}{78}y^3 - \frac{5}{26}y^2 + \frac{5}{156}y^4 - \frac{1}{156}y^5\right) \\
&\quad + u''(0)\left(\frac{-5}{13}y - \frac{5}{156}y^3 + \frac{21}{52}y^2 + \frac{5}{312}y^4 - \frac{1}{312}y^5\right) \\
&\quad + u''(1)0 - u''(0) \\
&\quad \times \left(\frac{-5}{13}y - \frac{5}{156}y^3 + \frac{21}{52}y^2 + \frac{5}{312}y^4 - \frac{1}{312}y^5\right) \\
&\quad - u'(1)0 + u'(0) \\
&\quad \times \left(-\frac{3}{13}y + \frac{5}{78}y^3 + \frac{5}{26}y^2 - \frac{5}{156}y^4 + \frac{1}{156}y^5\right) \\
&\quad + \int_0^1 u'(x) R_y^{(5)}(x)\,dx;
\end{aligned} \tag{15}$$

thus we obtain that

$$\begin{aligned}
\langle u, R_y \rangle_{W_2^3} &= \int_0^1 u'(x) R_y^{(5)}(x)\,dx \\
&= \int_0^y u'(x) R_y^{(5)}(x)\,dx + \int_y^1 u'(x) R_y^{(5)}(x)\,dx \\
&= \int_0^y u'(x)\left(1 - \frac{10}{13}y - \frac{5}{78}y^3 - \frac{5}{26}y^2 \right. \\
&\qquad \left. + \frac{5}{156}y^4 - \frac{1}{156}y^5\right)dx \\
&\quad + \int_y^1 u'(x)\left(-\frac{10}{13}y - \frac{5}{78}y^3 - \frac{5}{26}y^2 \right. \\
&\qquad \left. + \frac{5}{156}y^4 - \frac{1}{156}y^5\right)dx \\
&= (u(y) - u(0)) \\
&\quad \times \left(1 - \frac{10}{13}y - \frac{5}{78}y^3 - \frac{5}{26}y^2 \right. \\
&\qquad \left. + \frac{5}{156}y^4 - \frac{1}{156}y^5\right)
\end{aligned}$$

$$\begin{aligned}
&+ (u(1) - u(y)) \\
&\quad\times\left(-\frac{10}{13}y - \frac{5}{78}y^3 - \frac{5}{26}y^2 + \frac{5}{156}y^4 - \frac{1}{156}y^5\right) \\
&= u(y) - u(0) \\
&\quad\times\left(1 - \frac{10}{13}y - \frac{5}{78}y^3 - \frac{5}{26}y^2 + \frac{5}{156}y^4 - \frac{1}{156}y^5\right) \\
&\quad + u(1)\left(-\frac{10}{13}y - \frac{5}{78}y^3 - \frac{5}{26}y^2 + \frac{5}{156}y^4 - \frac{1}{156}y^5\right).
\end{aligned} \tag{16}$$

By Definition 7, we have $u(0) = u(1) = 0$. So

$$\left\langle u, R_y \right\rangle_{W_2^3} = u(y). \tag{17}$$

This completes the proof. □

Definition 9. We define the binary space $W(\Omega)$ by

$$\begin{aligned}
W(\Omega) = \Bigg\{ &u : \frac{\partial^4 u}{\partial x^2 \partial t^2} \text{ is completely continuous in} \\
&\Omega = [0,1]\times[0,1],\ \frac{\partial^6 u}{\partial x^3 \partial t^3} \in L^2(\Omega), \\
&u(x,0) = 0,\ \frac{\partial u(x,0)}{\partial t} = 0, \\
&u(0,t) = 0,\ u(1,t) = 0 \Bigg\}.
\end{aligned} \tag{18}$$

The inner product and the norm in $W(\Omega)$ are defined by

$$\begin{aligned}
\langle u, g\rangle_W = &\sum_{i=0}^{2}\int_0^1 \left[\frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}u(0,t) \times \frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}g(0,t)\right]dt \\
&+ \sum_{j=0}^{2}\left\langle \frac{\partial^j}{\partial t^j}u(x,0), \frac{\partial^j}{\partial t^j}g(x,0)\right\rangle_{W_2^3} \\
&+ \iint_0^1 \left[\frac{\partial^3}{\partial x^3}\frac{\partial^3}{\partial t^3}u(x,t) \times \frac{\partial^3}{\partial x^3}\frac{\partial^3}{\partial t^3}g(x,t)\right]dt\,dx,
\end{aligned} \tag{19}$$

$$\|u\|_W = \sqrt{\langle u, u\rangle_W}, \quad u \in W(\Omega). \tag{20}$$

Lemma 10. *$W(\Omega)$ is a reproducing kernel space, and its reproducing kernel function $K_{(y,s)}$ is given by [25, page 148]*

$$K_{(y,s)} = R_y r_s. \tag{21}$$

Definition 11. We define the binary space $\widehat{W}(\Omega)$ by

$$\begin{aligned}
\widehat{W}(\Omega) = \Bigg\{ &u : u \text{ is completely continuous in} \\
&\Omega = [0,1]\times[0,1],\ \frac{\partial^2 u}{\partial x \partial t} \in L^2(\Omega) \Bigg\}.
\end{aligned} \tag{22}$$

The inner product and the norm in $\widehat{W}(\Omega)$ are defined by

$$\begin{aligned}
\langle u, g\rangle_{\widehat{W}} = &\int_0^1 \left[\frac{\partial}{\partial t}u(0,t)\frac{\partial}{\partial t}g(0,t)\right]dt \\
&+ \langle u(x,0), g(x,0)\rangle_{H_2^1} \\
&+ \iint_0^1 \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t}u(x,t)\frac{\partial}{\partial x}\frac{\partial}{\partial t}g(x,t)\right]dt\,dx,
\end{aligned} \tag{23}$$

$$\|u\|_{\widehat{W}} = \sqrt{\langle u, u\rangle_{\widehat{W}}}, \quad u \in \widehat{W}(\Omega).$$

Lemma 12. *$\widehat{W}(\Omega)$ is a reproducing kernel space, and its reproducing kernel function $G_{(y,s)}$ is given as [25, page 148]*

$$G_{(y,s)} = Q_y q_s. \tag{24}$$

Remark 13. Hilbert spaces can be completely classified: there is a unique Hilbert space up to isomorphism for every cardinality of the base. Since finite-dimensional Hilbert spaces are fully understood in linear algebra and since morphisms of Hilbert spaces can always be divided into morphisms of spaces with Aleph-null (χ_0) dimensionality, functional analysis of Hilbert spaces mostly deals with the unique Hilbert space of dimensionality Aleph-null and its morphisms. One of the open problems in functional analysis is to prove that every bounded linear operator on a Hilbert space has a proper invariant subspace. Many special cases of this invariant subspace problem have already been proven [26].

3. Solution Representation in $W(\Omega)$

In this section, the solution of (1) is given in the reproducing kernel space $W(\Omega)$. On defining the linear operator $L : W(\Omega) \to \widehat{W}(\Omega)$ by

$$Lv = \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + 2\alpha\frac{\partial v}{\partial t} + \beta^2 v(x,t), \tag{25}$$

after homogenizing the initial and boundary conditions, model problem (1)–(3) changes to the problem

$$\begin{aligned}
&Lv = M(x,t), \quad (x,t) \in [0,1]\times[0,1], \\
&v(x,0) = \frac{\partial v}{\partial t}(x,0) = v(0,t) = v(1,t) = 0,
\end{aligned} \tag{26}$$

where

$$M(x,t) = \frac{\partial^2 Z}{\partial t^2}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) + 2\alpha \frac{\partial Z}{\partial t}(x,t) + \beta^2 Z(x,t) + f(x,t); \quad (27)$$

for convenience, we again write u instead of v in (26)

Lemma 14. *L is a bounded linear operator.*

Proof. Let $u \in W(\Omega)$ and let $(x,t) \in \Omega$. By Lemma 10, we have

$$u(x,t) = \langle u, K_{(x,t)} \rangle_W, \quad (28)$$

and thus

$$\begin{aligned} Lu(x,t) &= \langle u, LK_{(x,t)} \rangle_W, \\ \frac{\partial}{\partial x} Lu(x,t) &= \left\langle u, \frac{\partial}{\partial x} LK_{(x,t)} \right\rangle_W, \\ \frac{\partial}{\partial t} Lu(x,t) &= \left\langle u, \frac{\partial}{\partial t} LK_{(x,t)} \right\rangle_W, \\ \frac{\partial}{\partial t}\frac{\partial}{\partial x} Lu(x,t) &= \left\langle u, \frac{\partial}{\partial t}\frac{\partial}{\partial x} LK_{(x,t)} \right\rangle_W. \end{aligned} \quad (29)$$

Hence there exist a_0, b_0, a_1, $b_1 > 0$ such that

$$\begin{aligned} |Lu(x,t)| &\le a_0 \|u\|_W, \\ \left|\frac{\partial}{\partial t} Lu(x,t)\right| &\le b_0 \|u\|_W, \\ \left|\frac{\partial}{\partial x} Lu(x,t)\right| &\le a_1 \|u\|_W, \\ \left|\frac{\partial}{\partial t}\frac{\partial}{\partial x} Lu(x,t)\right| &\le b_1 \|u\|_W. \end{aligned} \quad (30)$$

Therefore,

$$\begin{aligned} \|Lu\|_{\widehat{W}}^2 &= \int_0^1 \left[\frac{\partial}{\partial t} Lu(0,t)\right]^2 dt + \langle Lu(x,0), Lu(x,0) \rangle_{H_2^1} + \iint_0^1 \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t} Lu(x,t)\right]^2 dt\, dx \\ &= \int_0^1 \left[\frac{\partial}{\partial t} Lu(0,t)\right]^2 dt + [Lu(0,0)]^2 + \int_0^1 \left[\frac{\partial}{\partial x} Lu(x,0)\right]^2 dx + \iint_0^1 \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t} Lu(x,t)\right]^2 dt\, dx \\ &\le \left(a_0^2 + a_1^2 + b_0^2 + b_1^2\right) \|u\|_W^2. \end{aligned} \quad (31)$$

This completes the proof. □

Now, choose a countable dense subset $\{(x_1,t_1),(x_2,t_2),\ldots\}$ in Ω and define

$$\varphi_i = G_{(x_i,t_i)}, \qquad \Psi_i = L^* \varphi_i, \quad (32)$$

where L^* is the adjoint operator of L. The orthonormal system $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$ of $W(\Omega)$ can be derived from the process of Gram-Schmidt orthogonalization of $\{\Psi_i\}_{i=1}^{\infty}$ as

$$\widehat{\Psi}_i = \sum_{k=1}^{i} \beta_{ik} \Psi_k. \quad (33)$$

Theorem 15. *Suppose that $\{(x_i,t_i)\}_{i=1}^{\infty}$ is dense in Ω. Then $\{\Psi_i\}_{i=1}^{\infty}$ is a complete system in $W(\Omega)$, and*

$$\Psi_i = LK_{(x_i,t_i)}(x,t). \quad (34)$$

Proof. We have

$$\begin{aligned} \Psi_i = L^* \varphi_i &= \langle L^* \varphi_i, K_{(x,t)} \rangle_W \\ &= \langle \varphi_i, LK_{(x,t)} \rangle_{\widehat{W}} \\ &= \left\langle LK_{(x,t)}, G_{(x_i,t_i)} \right\rangle_{\widehat{W}} \\ &= LK_{(x,t)}(x_i,t_i) \\ &= LK_{(x_i,t_i)}(x,t). \end{aligned} \quad (35)$$

Clearly, $\Psi_i \in W(\Omega)$. For each fixed $u \in W(\Omega)$, if

$$\langle u, \Psi_i \rangle_W = 0, \quad i = 1,2,\ldots, \quad (36)$$

then

$$\begin{aligned} 0 &= \langle u, \Psi_i \rangle_W \\ &= \langle u, L^* \varphi_i \rangle_W \\ &= \langle Lu, \varphi_i \rangle_{\widehat{W}} \\ &= Lu(x_i,t_i), \quad i = 1,2,\ldots. \end{aligned} \quad (37)$$

Note that $\{(x_i,t_i)\}_{i=1}^{\infty}$ is dense in Ω. Hence, $Lu = 0$. From the existence of L^{-1}, it follows that $u = 0$. The proof is completed. □

Theorem 16. *If $\{(x_i,t_i)\}_{i=1}^{\infty}$ is dense in Ω, then the solution of (26) is given by*

$$u = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M(x_k,t_k) \widehat{\Psi}_i. \quad (38)$$

Proof. By Theorem 15, $\{\Psi_i(x,t)\}_{i=1}^{\infty}$ is a complete system in $W(\Omega)$. Thus,

$$
\begin{aligned}
u &= \sum_{i=1}^{\infty} \left\langle u, \widehat{\Psi}_i \right\rangle_W \widehat{\Psi}_i \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u, \Psi_k \rangle_W \widehat{\Psi}_i \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u, L^* \varphi_k \rangle_W \widehat{\Psi}_i \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Lu, \varphi_k \rangle_{\widehat{W}} \widehat{\Psi}_i \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle Lu, G_{(x_k,t_k)} \right\rangle_{\widehat{W}} \widehat{\Psi}_i \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} Lu\,(x_k, t_k)\, \widehat{\Psi}_i \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M\,(x_k, t_k)\, \widehat{\Psi}_i .
\end{aligned} \tag{39}
$$

This completes the proof. □

Now the approximate solution u_n can be obtained from the n-term intercept of the exact solution u and

$$u_n = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} M\,(x_k, t_k)\, \widehat{\Psi}_i . \tag{40}$$

Obviously

$$\|u_n - u\|_W \longrightarrow 0, \quad n \longrightarrow \infty. \tag{41}$$

Theorem 17. *If $u \in W(\Omega)$, then*

$$\|u_n - u\|_W \longrightarrow 0, \quad n \longrightarrow \infty. \tag{42}$$

Moreover, a sequence $\|u_n - u\|_W$ is monotonically decreasing in n.

Proof. From (38) and (40), it follows that

$$\|u_n - u\|_W = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M\,(x_k, t_k)\, \widehat{\Psi}_i\,(x,t) \right\|_W . \tag{43}$$

Thus,

$$\|u_n\,(x,t) - u\,(x,t)\|_W \longrightarrow 0, \quad n \longrightarrow \infty. \tag{44}$$

In addition

$$
\begin{aligned}
\|u_n - u\|_W^2 &= \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M\,(x_k, t_k)\, \widehat{\Psi}_i\,(x,t) \right\|_W^2 \\
&= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} M\,(x_k, t_k)\, \widehat{\Psi}_i\,(x,t) \right)^2 .
\end{aligned} \tag{45}
$$

Clearly, $\|u_n(x,t) - u(x,t)\|_W$ is monotonically decreasing in n. □

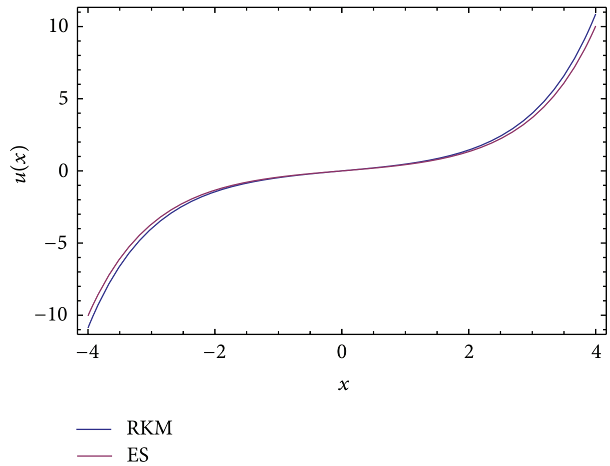


Figure 1: Graph of numerical results for Example 20 ($\alpha = 10$, $\beta = 20$, $t = 0.5$).

4. The Method Implementation

(i) If (26) is linear, then the analytical solution of (26) can be obtained directly by (38).

(ii) If (26) is nonlinear, then the solution of (26) can be obtained by the following iterative method.

We construct an iterative sequence u_n, putting

$$
\begin{aligned}
&\text{any fixed } u_0 \in W_2^3\,[0,1]\,, \\
&u_n = \sum_{i=1}^{n} A_i \widehat{\Psi}_i ,
\end{aligned} \tag{46}
$$

where

$$
\begin{aligned}
A_1 &= \beta_{11} M\,(x_k, t_k, u\,(x_k, t_k)) \\
A_2 &= \sum_{k=1}^{2} \beta_{2k} M\,(x_k, t_k, u_{k-1}\,(x_k, t_k))\,, \\
&\vdots \\
A_n &= \sum_{k=1}^{n} \beta_{nk} M\,(x_k, t_k, u_{k-1}\,(x_k, t_k))\,.
\end{aligned} \tag{47}
$$

Next we will prove that u_n given by the iterative formula (46) converges to the exact solution.

Theorem 18. *Suppose that $\|u_n\|$ defined by (46) is bounded and (26) has a unique solution. If $\{(x_i, t_i)\}_{i=1}^{\infty}$ is dense in Ω, then u_n converges to the analytical solution u of (26), and*

$$u = \sum_{i=1}^{\infty} A_i \widehat{\Psi}_i , \tag{48}$$

where A_i is given by (47).

TABLE 1: Numerical results for Example 20 for $t = 0.5$.

x	CPU time (s)	AE $\alpha = 20$ $\beta = 10$	RE $\alpha = 20$ $\beta = 10$	AE $\alpha = 10$ $\beta = 5$	RE $\alpha = 10$ $\beta = 5$	AE $\alpha = 50$ $\beta = 2$	RE $\alpha = 50$ $\beta = 2$
0.9829	2.168	4.9×10^{-9}	1.1×10^{-7}	8.9×10^{-8}	2.0×10^{-6}	9.6×10^{-9}	2.1×10^{-7}
0.9903	2.324	3.8×10^{-8}	1.5×10^{-7}	4.6×10^{-8}	1.9×10^{-7}	9.8×10^{-7}	4.0×10^{-6}
0.9938	2.215	5.9×10^{-8}	3.6×10^{-6}	5.0×10^{-8}	3.1×10^{-6}	8.5×10^{-9}	5.2×10^{-7}
0.9957	2.262	6.8×10^{-7}	2.5×10^{-7}	1.4×10^{-7}	5.3×10^{-7}	5.2×10^{-7}	1.9×10^{-6}

TABLE 2: Numerical results for Example 20 for $t = 1.0$.

x	CPU time (s)	AE $\alpha = 20$ $\beta = 10$	RE $\alpha = 20$ $\beta = 10$	AE $\alpha = 10$ $\beta = 5$	RE $\alpha = 10$ $\beta = 5$	AE $\alpha = 50$ $\beta = 2$	RE $\alpha = 50$ $\beta = 2$
0.9829	2.340	5.1×10^{-9}	3.1×10^{-7}	5.5×10^{-8}	3.3×10^{-6}	4.5×10^{-7}	2.7×10^{-5}
0.9903	2.278	7.1×10^{-7}	7.8×10^{-6}	1.6×10^{-8}	1.8×10^{-7}	9.5×10^{-7}	1.0×10^{-5}
0.9938	2.293	4.4×10^{-7}	7.3×10^{-5}	1.7×10^{-6}	2.9×10^{-4}	2.2×10^{-7}	3.6×10^{-5}
0.9957	2.277	2.2×10^{-8}	2.2×10^{-7}	1.4×10^{-8}	1.4×10^{-7}	7.2×10^{-7}	7.4×10^{-6}

TABLE 3: A comparison between interpolating scaling function method [11] and RKM for different values of α, β, and t for Example 20.

x	CPU time (s)	AE [11] $\alpha = 20$ $t = 0.5$	AE RKM $\alpha = 20$ $t = 0.5$	AE [11] $\beta = 10$ $t = 1$	AE RKM $\beta = 10$ $t = 1$
0.0	2.512	2×10^{-21}	0.0	2×10^{-6}	0.0
0.1	2.262	5×10^{-4}	3×10^{-7}	1×10^{-3}	3×10^{-8}
0.2	2.309	7×10^{-4}	2×10^{-8}	2×10^{-3}	3×10^{-7}
0.3	2.293	1×10^{-3}	2×10^{-7}	2×10^{-4}	2×10^{-7}
0.4	2.278	2×10^{-3}	5×10^{-5}	4×10^{-4}	1×10^{-7}
0.5	2.883	3×10^{-3}	1×10^{-5}	5×10^{-4}	8×10^{-8}
0.6	2.821	3×10^{-3}	5×10^{-8}	8×10^{-4}	5×10^{-7}
0.7	2.805	4×10^{-3}	2×10^{-7}	8×10^{-4}	1×10^{-7}
0.8	2.231	3×10^{-3}	6×10^{-7}	6×10^{-4}	1×10^{-7}
0.9	2.277	2×10^{-3}	2×10^{-6}	3×10^{-4}	8×10^{-8}
1.0	2.169	2×10^{-4}	2×10^{-8}	9×10^{-5}	4×10^{-10}

TABLE 4: A comparison between interpolating scaling function method [11] and RKM for different values of α, β, and t for Example 20.

x	AE [11] $\alpha = 10$ $t = 0.5$	AE RKM $\alpha = 10$ $t = 0.5$	AE [11] $\beta = 5$ $t = 1$	AE RKM $\beta = 5$ $t = 1$
0.0	0.0	0.0	2×10^{-6}	0.0
0.1	3×10^{-4}	1×10^{-7}	1×10^{-4}	2×10^{-7}
0.2	1×10^{-3}	3×10^{-8}	7×10^{-4}	2×10^{-6}
0.3	1×10^{-3}	9×10^{-8}	6×10^{-4}	6×10^{-7}
0.4	2×10^{-3}	2×10^{-8}	9×10^{-4}	7×10^{-8}
0.5	2×10^{-3}	2×10^{-7}	1×10^{-3}	5×10^{-8}
0.6	2×10^{-3}	5×10^{-7}	1×10^{-3}	9×10^{-8}
0.7	2×10^{-3}	7×10^{-8}	8×10^{-4}	3×10^{-8}
0.8	2×10^{-3}	7×10^{-8}	7×10^{-4}	6×10^{-8}
0.9	1×10^{-3}	4×10^{-7}	3×10^{-4}	2×10^{-8}
1.0	1×10^{-4}	8×10^{-9}	9×10^{-5}	7×10^{-8}

Proof. First, we prove the convergence of u_n. From (46) and the orthonormality of $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$, we infer that

$$\begin{aligned} \|u_{n+1}\|^2 &= \sum_{i=1}^{n+1} A_i^2 = \sum_{i=1}^{n} A_i^2 + A_{n+1}^2 \\ &= \|u_n\|^2 + A_{n+1}^2 \geq \|u_n\|. \end{aligned} \tag{49}$$

By (49), $\|u_n\|$ is nondecreasing, and by the assumption, $\|u_n\|$ is bounded. Thus $\|u_n\|$ is convergent. By (49), there exists a constant c such that

$$\sum_{i=1}^{\infty} A_i^2 = c. \tag{50}$$

This implies that

$$\{A_i\}_{i=1}^{\infty} \in \ell^2. \tag{51}$$

If $m > n$, then

$$\begin{aligned} \|u_m - u_n\|^2 &= \left\| \sum_{k=n}^{m-1} u_{k+1} - u_k \right\|^2 \\ &\leq \sum_{k=n}^{m-1} \|u_{k+1} - u_k\|^2 \\ &= \sum_{k=n}^{m-1} A_{k+1}^2 \longrightarrow 0, \quad m, n \longrightarrow \infty. \end{aligned} \tag{52}$$

Table 5: Numerical results for Example 21 for $t = 0.5$.

x	CPU time (s)	AE $\alpha = 20$ $\beta = 10$	RE $\alpha = 20$ $\beta = 10$	AE $\alpha = 10$ $\beta = 5$	RE $\alpha = 10$ $\beta = 5$	AE $\alpha = 50$ $\beta = 2$	RE $\alpha = 50$ $\beta = 2$
0.9829	2.215	7.23×10^{-8}	6.8×10^{-7}	3.4×10^{-9}	3.2×10^{-8}	8.41×10^{-8}	7.99×10^{-7}
0.9903	2.246	3.51×10^{-9}	6.8×10^{-9}	1.8×10^{-9}	3.5×10^{-9}	3.52×10^{-7}	6.86×10^{-7}
0.9938	2.231	7.56×10^{-7}	1.9×10^{-5}	7.6×10^{-8}	1.9×10^{-6}	2.84×10^{-9}	7.28×10^{-8}
0.9957	2.433	5.45×10^{-8}	9.9×10^{-8}	2.6×10^{-8}	4.8×10^{-8}	4.53×10^{-8}	8.27×10^{-8}

Table 6: Numerical results for Example 21 for $t = 1.0$.

x	CPU time (s)	AE $\alpha = 20$ $\beta = 10$	RE $\alpha = 20$ $\beta = 10$	AE $\alpha = 10$ $\beta = 5$	RE $\alpha = 10$ $\beta = 5$	AE $\alpha = 50$ $\beta = 2$	RE $\alpha = 50$ $\beta = 2$
0.9829	2.184	1.3×10^{-7}	2.08×10^{-6}	6.32×10^{-8}	9.7×10^{-7}	2.78×10^{-8}	4.2×10^{-7}
0.9903	2.293	5.1×10^{-9}	1.61×10^{-8}	7.31×10^{-8}	2.3×10^{-7}	2.07×10^{-7}	6.5×10^{-7}
0.9938	2.512	8.0×10^{-9}	3.34×10^{-7}	8.10×10^{-10}	3.3×10^{-8}	4.37×10^{-7}	1.8×10^{-5}
0.9957	2.215	1.6×10^{-8}	4.83×10^{-8}	3.71×10^{-9}	1.0×10^{-8}	4.04×10^{-7}	1.2×10^{-6}

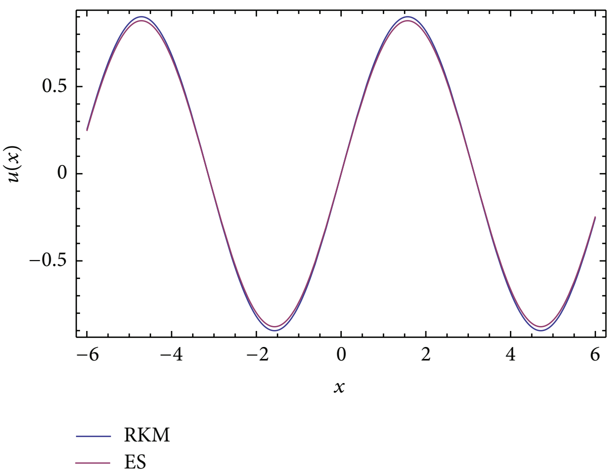


Figure 2: Graph of numerical results for Example 21 ($\alpha = 10$, $\beta = 20$, $t = 0.5$).

The completeness of $W(\Omega)$ shows that there exists $\widehat{u} \in W(\Omega)$ such that $u_n \to \widehat{u}$ as $n \to \infty$. Now, we prove that $\widehat{u}$ solves (26). Taking limits in (40), we get

$$\widehat{u} = \sum_{i=1}^{\infty} A_i \widehat{\Psi}_i. \tag{53}$$

Note that

$$L\widehat{u} = \sum_{i=1}^{\infty} A_i L\widehat{\Psi}_i, \tag{54}$$

$$\begin{aligned}(L\widehat{u})(x_k, t_k) &= \sum_{i=1}^{\infty} A_i L\widehat{\Psi}_i (x_k, t_k) \\ &= \sum_{i=1}^{\infty} A_i \left\langle L\widehat{\Psi}_i, G_{(x_k,t_k)} \right\rangle_{\widehat{W}} \\ &= \sum_{i=1}^{\infty} A_i \left\langle L\widehat{\Psi}_i, \varphi_k \right\rangle_{\widehat{W}} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\Psi}_i, L^* \varphi_k \right\rangle_{W} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\Psi}_i, \Psi_k \right\rangle_{W}.\end{aligned} \tag{55}$$

In view of (47), we have

$$L\widehat{u}(x_l, t_l) = M(x_l, t_l, u_{l-1}(x_l, t_l)). \tag{56}$$

Since $\{(x_i, t_i)\}_{i=1}^{\infty}$ is dense in Ω, for each $(y, s) \in \Omega$, there exists a subsequence $\{(x_{n_j}, t_{n_j})\}_{j=1}^{\infty}$ such that

$$\left(x_{n_j}, t_{n_j}\right) \longrightarrow (y, s), \qquad j \longrightarrow \infty. \tag{57}$$

We know that

$$L\widehat{u}\left(x_{n_j}, t_{n_j}\right) = M\left(x_{n_j}, t_{n_j}, u_{n_j-1}\left(x_{n_j}, t_{n_j}\right)\right). \tag{58}$$

Let $j \to \infty$. By the continuity of f, we have

$$(L\widehat{u})(y, s) = M(y, s, \widehat{u}(y, s)), \tag{59}$$

which indicates that $\widehat{u}$ satisfies (26). □

Remark 19. In the same manner, it can be proved that

$$\left\| \frac{\partial u_n}{\partial x} - \frac{\partial u}{\partial x} \right\| \longrightarrow 0, \quad n \longrightarrow \infty, \tag{60}$$

where

$$\frac{\partial u}{\partial x} = \sum_{i=1}^{\infty} A_i \frac{\partial \widehat{\Psi}_i}{\partial x}, \qquad \frac{\partial u_n}{\partial x} = \sum_{i=1}^{n} A_i \frac{\partial \widehat{\Psi}_i}{\partial x}, \tag{61}$$

and A_i is given by (47).

TABLE 7: RMS errors for Example 21.

N	[10] $\alpha = 20$ $\beta = 10$ $t = 0.5$	RKM $\alpha = 20$ $\beta = 10$ $t = 0.5$	[10] $\alpha = 20$ $\beta = 10$ $t = 1.0$	RKM $\alpha = 20$ $\beta = 10$ $t = 1.0$	[10] $\alpha = 10$ $\beta = 5$ $t = 0.5$	RKM $\alpha = 10$ $\beta = 5$ $t = 0.5$
5	3.2×10^{-7}	2.8×10^{-8}	3.4×10^{-6}	2.3×10^{-8}	3.6×10^{-6}	1.1×10^{-7}
7	2.0×10^{-10}	5.7×10^{-7}	3.7×10^{-9}	5.9×10^{-11}	4.1×10^{-10}	9.6×10^{-11}
9	2.3×10^{-13}	2.7×10^{-16}	2.4×10^{-12}	4.8×10^{-13}	3.5×10^{-13}	1.9×10^{-12}
11	1.1×10^{-16}	3.8×10^{-20}	1.0×10^{-15}	2.3×10^{-16}	3.4×10^{-16}	1.3×10^{-16}

TABLE 8: Numerical results for Example 22 for $t = 0.5$.

x	AE $\alpha = 20$ $\beta = 10$	RE $\alpha = 20$ $\beta = 10$	CPU time (s)	AE $\alpha = 10$ $\beta = 5$	RE $\alpha = 10$ $\beta = 5$	AE $\alpha = 50$ $\beta = 2$	RE $\alpha = 50$ $\beta = 2$
0.9829	6.3×10^{-8}	1.9×10^{-7}	2.231	1.8×10^{-5}	5.8×10^{-5}	6.5×10^{-6}	2.0×10^{-5}
0.9903	7.2×10^{-7}	1.1×10^{-6}	2.540	4.5×10^{-4}	7.2×10^{-4}	7.5×10^{-4}	1.1×10^{-3}
0.9938	4.9×10^{-8}	1.7×10^{-7}	2.230	3.6×10^{-5}	1.2×10^{-4}	8.9×10^{-6}	3.2×10^{-5}
0.9957	6.4×10^{-8}	9.6×10^{-8}	2.680	2.5×10^{-6}	3.8×10^{-6}	1.5×10^{-4}	2.3×10^{-4}

5. Numerical Results

To test the accuracy of the present method, some numerical experiments are presented in this section. Using our method, we chose 36 points in Ω and obtained the approximate solution u_{36}. The comparison between interpolating scaling function method [11] and RKM for different values of α, β, and t is given in Tables 4 and 5. We solve these examples for a set of points

$$\{x_1 = 0, \dots, x_i = (i-1)h, \dots, x_N = 1\}, \quad h = \frac{1}{N-1}. \tag{62}$$

In Tables 7 and 10 we calculate the RMS error by the following formula:

$$\text{RMS error} = \sqrt{\frac{\sum_{i=1}^{N+1}(u - u_{36})^2}{N+1}}. \tag{63}$$

It can be seen from Tables 4 and 5 and 7–10 that the results obtained by the RKM are more accurate than those obtained by the methods in [10, 11]. This indicates that RKM is a reliable method. The CPU time (s) is given in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Numerical solutions are described in the extended domain $[-4, 4] \times [-3, 3]$. The comparison of RMS error is given for our method and Chebyshev method.

Example 20. Consider the following telegraph equation with initial and boundary conditions:

$$\begin{gathered}
\frac{\partial^2 u}{\partial t^2}(x,t) + 2\alpha \frac{\partial u}{\partial t}(x,t) + \beta^2 u(x,t) = \frac{\partial^2 u}{\partial x^2} + f(x,t), \\
u(x,0) = \varphi_1(x) = \sinh(x), \\
\frac{\partial u}{\partial t}(x,0) = \varphi_2(x) = -2\sinh(x), \\
u(0,t) = g_0(t) = 0, \quad t \geq 0, \\
u(1,t) = g_1(t) = \exp(-2t)\sinh(1), \quad t \geq 0,
\end{gathered} \tag{64}$$

where

$$f(x,t) = (3 - 4\alpha + \beta^2)\exp(-2t)\sinh(x). \tag{65}$$

The exact solution of (64) is given by [11]

$$u(x,t) = \exp(-2t)\sinh(x). \tag{66}$$

If we apply

$$\begin{aligned}
v(x,t) = {} & u(x,t) - x\sinh(1)(\exp(-2t) - 1 + 2t) \\
& + \sinh(x)(2t-1)
\end{aligned} \tag{67}$$

to (64), then the following equation (68) is obtained:

$$\begin{gathered}
\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + 2\alpha\frac{\partial v}{\partial t} + \beta^2 v(x,t) = M(x,t), \\
v(x,0) = \frac{\partial v}{\partial t}(x,0) = v(0,t) = v(1,t) = 0,
\end{gathered} \tag{68}$$

where

$$\begin{aligned}
M(x,t) = {} & (3 - 4\alpha + \beta^2)\exp(-2t)\sinh(x) \\
& - \sinh(x)(2t-1) \\
& - 2\alpha\sinh(1)(-2\exp(-2t) + 2)x \\
& + 4\alpha\sinh(x) \\
& - \beta^2\sinh(1)(\exp(-2t) - 1 + 2t)x \\
& + \beta^2\sinh(x)(2t-1) - 4x\exp(-2t)\sinh(1).
\end{aligned} \tag{69}$$

Table 9: Numerical results for Example 22 for $t = 1.0$.

x	CPU time (s)	AE $\alpha = 20$ $\beta = 10$	RE $\alpha = 20$ $\beta = 10$	AE $\alpha = 10$ $\beta = 5$	RE $\alpha = 10$ $\beta = 5$	AE $\alpha = 50$ $\beta = 2$	RE $\alpha = 50$ $\beta = 2$
0.9829	2.587	2.7×10^{-7}	4.3×10^{-7}	3.2×10^{-5}	5.1×10^{-5}	6.9×10^{-5}	1.1×10^{-4}
0.9903	2.058	9.2×10^{-7}	8.7×10^{-7}	1.0×10^{-5}	9.6×10^{-6}	1.0×10^{-4}	9.6×10^{-5}
0.9938	2.262	2.5×10^{-5}	4.4×10^{-5}	6.9×10^{-5}	1.2×10^{-4}	1.1×10^{-4}	1.9×10^{-4}
0.9957	2.246	1.7×10^{-4}	1.6×10^{-4}	6.9×10^{-4}	6.3×10^{-4}	1.4×10^{-4}	1.3×10^{-4}

Table 10: RMS errors for Example 22.

N	[10] $\alpha = 10$ $\beta = 5$ $t = 1.0$	RKM $\alpha = 10$ $\beta = 5$ $t = 1.0$	[10] $\alpha = 50$ $\beta = 2$ $t = 0.5$	RKM $\alpha = 50$ $\beta = 2$ $t = 0.5$	[10] $\alpha = 50$ $\beta = 2$ $t = 1.0$	RKM $\alpha = 50$ $\beta = 2$ $t = 1.0$
5	2.8×10^{-5}	9.8×10^{-6}	3.1×10^{-5}	2.3×10^{-6}	6.9×10^{-4}	1.1×10^{-5}
7	1.3×10^{-5}	6.7×10^{-7}	1.5×10^{-6}	5.9×10^{-7}	2.2×10^{-5}	9.6×10^{-6}
9	2.8×10^{-7}	7.8×10^{-8}	1.8×10^{-8}	4.8×10^{-10}	6.3×10^{-7}	1.9×10^{-9}
11	6.1×10^{-9}	5.3×10^{-11}	7.8×10^{-10}	2.3×10^{-11}	1.6×10^{-8}	1.3×10^{-10}

After homogenizing the initial and boundary conditions and using the above method, we obtain Tables 1–4 and Figure 1.

Example 21. Consider the following telegraph equation with initial and boundary conditions:

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u(x,t) = \frac{\partial^2 u}{\partial x^2} + f(x,t), \\ u(x,0) = \varphi_1(x) = \sin(x), \\ \frac{\partial u}{\partial t}(x,0) = \varphi_2(x) = 0, \\ u(0,t) = g_0(t) = 0, \quad t \geq 0, \\ u(1,t) = g_1(t) = \cos(t)\sin(1), \quad t \geq 0,\end{gathered} \tag{70}$$

where

$$f(x,t) = -2\alpha \sin(t)\sin(x) + \beta^2 \cos(t)\sin(x). \tag{71}$$

The exact solution of (70) is given by [11]

$$u(x,t) = \cos(t)\sin(x). \tag{72}$$

If we apply

$$v(x,t) = u(x,t) - x\sin(1)(\cos(t) - 1) - \sin(x) \tag{73}$$

to (70), then the following (74) is obtained:

$$\begin{gathered}\frac{\partial^2 v}{\partial t^2}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) + 2\alpha \frac{\partial v}{\partial t} \\ + \beta^2 v(x,t) = M(x,t),\end{gathered} \tag{74}$$

$$v(x,0) = \frac{\partial v}{\partial t}(x,0) = v(0,t) = v(1,t) = 0,$$

where

$$\begin{aligned}M(x,t) = & -2\alpha \sin(t)\sin(x) + \beta^2 \cos(t)\sin(x) - \sin(x) \\ & - 2\alpha \sin(t)\sinh(1)x + x\cos(t)\sin(1) \\ & - \beta^2 x\sin(1)(\cos(t) - 1) - \beta^2 \sin(x).\end{aligned} \tag{75}$$

After homogenizing the initial and boundary conditions and using the above method, we obtain Tables 5–7 and Figures 2 and 3.

Example 22. Consider the following telegraph equation with initial and boundary conditions:

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u(x,t) = \frac{\partial^2 u}{\partial x^2} + f(x,t), \\ u(x,0) = \varphi_1(x) = \tan\left(\frac{x}{2}\right), \\ \frac{\partial u}{\partial t}(x,0) = \varphi_2(x) = \frac{\left(1 + \tan^2(x/2)\right)}{2}, \\ u(0,t) = g_0(t) = \tan\left(\frac{t}{2}\right), \quad t \geq 0, \\ u(1,t) = g_1(t) = \tan\left(\frac{1+t}{2}\right), \quad t \geq 0,\end{gathered} \tag{76}$$

where

$$f(x,t) = \alpha\left(1 + \tan^2\left(\frac{x+t}{2}\right)\right) + \beta^2 \tan\left(\frac{x+t}{2}\right). \tag{77}$$

The exact solution of (76) is given by [10]

$$u(x,t) = \tan\left(\frac{x+t}{2}\right). \tag{78}$$

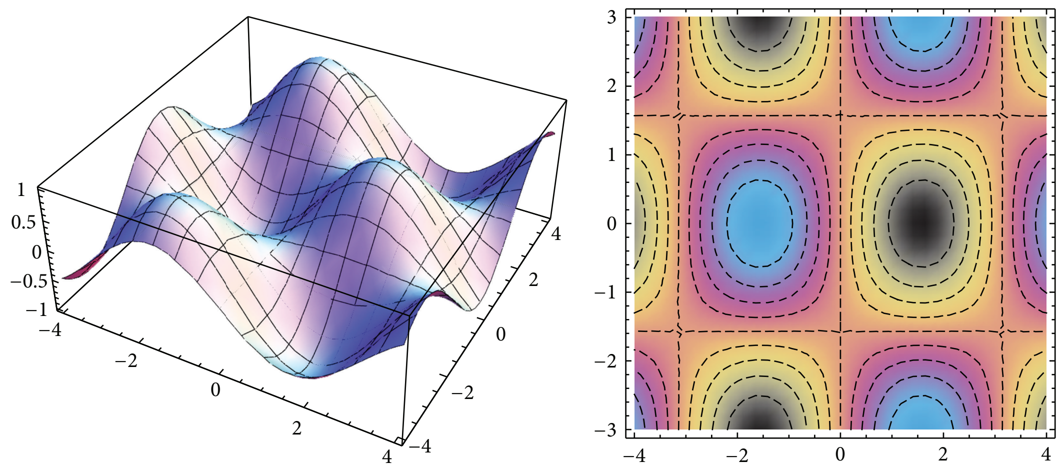


FIGURE 3: Plots of numerical results for Example 21 ($\alpha = 6$, $\beta = 2$).

If we apply

$$\begin{aligned} v(x,t) &= u(x,t) + (x-1)\tan\left(\frac{t}{2}\right) \\ &\quad - x\tan\left(\frac{1+t}{2}\right) - \tan\left(\frac{x}{2}\right) \\ &\quad \times\left(1+\left(\frac{t\tan(x/2)}{2}\right)\right) \\ &\quad + x\tan\left(\frac{1}{2}\right)\left(1+\left(\frac{t\tan(1/2)}{2}\right)\right) \end{aligned} \tag{79}$$

to (76), then the following equation (80) is obtained

$$\begin{gathered} \frac{\partial^2 v}{\partial t^2}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) + 2\alpha\frac{\partial v}{\partial t} \\ + \beta^2 v(x,t) = M(x,t), \\ v(x,0) = \frac{\partial v}{\partial t}(x,0) = v(0,t) = v(1,t) = 0, \end{gathered} \tag{80}$$

where

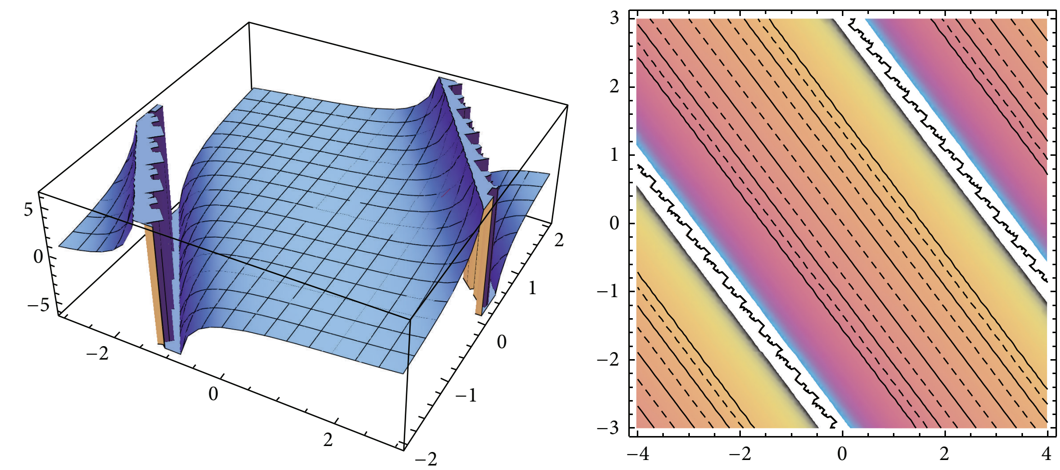
$$\begin{aligned} M(x,t) &= \alpha\left(1+\tan^2\left(\frac{x+t}{2}\right)\right) + \beta^2\tan\left(\frac{x+t}{2}\right) \\ &\quad + \left(\frac{\tan(x/2)}{2} + \frac{t}{4} + \frac{3t}{4}\tan^2\left(\frac{x}{2}\right)\right) \\ &\quad \times\left(1+\tan^2\left(\frac{x}{2}\right)\right) \\ &\quad + (x-1)\alpha\left(1+\tan^2\left(\frac{t}{2}\right)\right) \\ &\quad - \alpha x\left(1+\tan^2\left(\frac{t+1}{2}\right)\right) - \alpha\tan^2\left(\frac{x}{2}\right) \\ &\quad + \alpha x\tan^2\left(\frac{1}{2}\right) + \beta^2(x-1)\tan\left(\frac{t}{2}\right) \\ &\quad - \beta^2 x\tan\left(\frac{1+t}{2}\right) - \beta^2\tan\left(\frac{x}{2}\right) \\ &\quad \times\left(1+\frac{t\tan(x/2)}{2}\right) \\ &\quad + \beta^2 x\tan\left(\frac{1}{2}\right)\left(1+\frac{t\tan(1/2)}{2}\right) \\ &\quad \times\frac{x-1}{2}\tan\left(\frac{t}{2}\right)\left(1+\tan^2\left(\frac{t}{2}\right)\right) \\ &\quad - \frac{x}{2}\tan\left(\frac{1+t}{2}\right)\left(1+\tan^2\left(\frac{1+t}{2}\right)\right). \end{aligned} \tag{81}$$

After homogenizing the initial and boundary conditions and using the above method, we obtain Tables 8–10 and Figure 4.

Remark 23. In Tables 1–9, we abbreviate the exact solution and the approximate solution by AS and ES, respectively. AE stands for the absolute error, that is, the absolute value of the difference of the exact solution and the approximate solution, while RE indicates the relative error, that is, the absolute error divided by the absolute value of the exact solution.

6. Conclusion

In this study, a second-order one-dimensional telegraph equation with initial and boundary conditions was solved by reproducing kernel Hilbert space method. We described the method and used it in some test examples in order to show its applicability and validity in comparison with exact and other numerical solutions. The obtained results show that this approach can solve the problem effectively and need few computations. The satisfactory results that we obtained were compared with the results that were obtained by [10, 11]. Numerical experiments on test examples show that our proposed schemes are of high accuracy and support the theoretical results. As shown in Tables 7 and 10 our results are better than the results that were obtained by [10]. According to these results, it is possible to apply RKM to linear and nonlinear differential equations with initial and



(a) (b)

Figure 4: Plots of numerical results for Example 22 ($\alpha = 10$, $\beta = 5$).

boundary conditions. It has been shown that the obtained results are uniformly convergent and the operator that was used is a bounded linear operator. From the results, RKM can be applied to high dimensional partial differential equations, integral equations, and fractional differential equations without any transformation or discretization, and good results can be obtained.

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Research Article

Data Dependence Results for Multistep and CR Iterative Schemes in the Class of Contractive-Like Operators

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We intend to establish some results on the data dependence of fixed points of certain contractive-like operators for the multistep and CR iterative processes in a Banach space setting. One of our results generalizes the corresponding results of Soltuz and Grosan (2008) and Chugh and Kumar (2011).

1. Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of all nonnegative integers. Let X be a Banach space, $E \subset X$ a nonempty closed, convex subset of X, and T a self-map on E. Suppose that $F_T := \{p \in X : p = Tp\}$ is the set of all fixed points of T. Iterative schemes abound in the literature of fixed point theory for which the fixed points of operators have been approximated over the years by many authors.

It is well known that the Picard iteration procedure [1] is defined by

$$\begin{gathered} x_0 \in E, \\ x_{n+1} = Tx_n, \quad n \in \mathbb{N}. \end{gathered} \tag{1}$$

Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$, $i = \overline{1, k-2}$, $k \geq 2$ be the real sequences in $[0, 1)$ satisfying certain conditions.

The Mann iterative scheme [2] is defined by

$$\begin{gathered} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \quad n \in \mathbb{N}. \end{gathered} \tag{2}$$

If $\alpha_n = \lambda$ (constant) in (2), then the resulting iteration will be called Krasnoselkij iteration procedure [3].

A sequence $\{x_n\}_{n=0}^{\infty}$, defined by

$$\begin{gathered} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{gathered} \tag{3}$$

is commonly known as the Ishikawa iterative method [4].

The Noor iterative procedure [5] is defined by

$$\begin{gathered} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n T z_n, \\ z_n = (1 - \gamma_n) x_n + \gamma_n T x_n, \quad n \in \mathbb{N}. \end{gathered} \tag{4}$$

In 2004, Rhoades and Soltuz [6] introduced a multistep iterative process as follows:

$$\begin{gathered} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n^1, \\ y_n^i = \left(1 - \beta_n^i\right) x_n + \beta_n^i T y_n^{i+1}, \\ y_n^{k-1} = \left(1 - \beta_n^{k-1}\right) x_n + \beta_n^{k-1} T x_n, \quad n \in \mathbb{N}. \end{gathered} \tag{5}$$

The iteration processes (2), (3), and (4) can be viewed as the special cases of the iteration procedure (5).

Recently, Chugh et al. introduced a CR iterative scheme in [7] as follows:

$$\begin{aligned} & x_0 \in E, \\ & x_{n+1} = (1-\alpha_n) y_n + \alpha_n T y_n, \\ & y_n = (1-\beta_n) T x_n + \beta_n T z_n, \\ & z_n = (1-\gamma_n) x_n + \gamma_n T x_n, \quad n \in \mathbb{N}. \end{aligned} \tag{6}$$

Now we mention some important contractive type operators.

Any mapping T is called a Kannan mapping, see [8], if there exists $b \in (0, 1/2)$ such that, for all $x, y \in X$,

$$\|Tx - Ty\| \le b (\|x - Tx\| + \|y - Ty\|). \tag{7}$$

Similar mapping is called a Chatterjea mapping, see [9], if there exists $c \in (0, 1/2)$ such that, for all $x, y \in X$,

$$\|Tx - Ty\| \le c (\|x - Ty\| + \|y - Tx\|). \tag{8}$$

In [10] Zamfirescu collected these classes of operators and proved an important result which states that an operator $T : X \to X$ satisfies condition Z (Zamfirescu condition) if and only if there exist the real numbers $a, b,$ and c satisfying $0 < a < 1$, $0 < b$, and $c < 1/2$ such that, for each pair $x, y \in X$, at least one of the following conditions is true:

(z_1) $\|Tx - Ty\| \le a\|x - y\|$,

(z_2) $\|Tx - Ty\| \le b(\|x - Tx\| + \|y - Ty\|)$,

(z_3) $\|Tx - Ty\| \le c(\|x - Ty\| + \|y - Tx\|)$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1) converges to p, for any $x_0 \in X$.

It is well known, see [11], that the conditions (z_1), (z_2), and (z_3) are independent.

Let $x, y \in X$. Since T satisfies condition Z, at least one of the conditions from (z_1), (z_2), and (z_3) is satisfied. Then T satisfies the inequalities

$$\|Tx - Ty\| \le \delta \|x - y\| + 2\delta \|x - Tx\|, \tag{9}$$

$$\|Tx - Ty\| \le \delta \|x - y\| + 2\delta \|x - Ty\|, \tag{10}$$

for all $x, y \in X$ where $\delta := \max\{a, b/(1-b), c/(1-c)\}$, $\delta \in [0, 1)$, and it was shown that this class of operators is wider than the class of Zamfirescu operators; see [12]. Any mapping satisfying condition (9) or (10) is called a quasi-contractive operator.

Osilike and Udomene [13] extended the previous definition by considering an operator satisfying the condition that there exist $L \ge 0$ and $\delta \in [0, 1)$ such that, for all $x, y \in X$,

$$\|Tx - Ty\| \le \delta \|x - y\| + L \|x - Tx\|. \tag{11}$$

Thereafter, Imoru and Olatinwo [14] further generalized and extended the previous definition as follows: an operator T is called contractive-like operator if there exist a constant $\delta \in [0, 1)$ and a strictly increasing and continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, such that, for each $x, y \in X$,

$$\|Tx - Ty\| \le \delta \|x - y\| + \varphi (\|x - Tx\|). \tag{12}$$

Remark 1 (see [15]). A map satisfying (12) need not have a fixed point. However, using (12) it is obvious that if T has a fixed point, then it is unique.

It is important to know whether an iterative scheme converges to fixed points of its associated map. In this context, there are numerous works dealing with the convergence of various iterative schemes in the literature, such as [6, 10, 12, 16–27].

As shown by Soltuz and Grosan [26, Theorem 3.1], in a real Banach space X, the Ishikawa iteration $\{x_n\}_{n=0}^{\infty}$ given by (3) converges to the fixed point of T, where $T : E \to E$ is a mapping satisfying condition (12).

It is known from [28, Corollary 2] that there is equivalence between convergence of iterative procedures (3), (5) and that of some other well-known iterative procedures for the class of operators satisfying (12).

Hussain et al. [29] introduced a Kirk-CR iterative scheme and proved the convergence of this iteration for the class of operators satisfying (12).

Remark 2. Putting $s = t = i = 1$ in [29, Theorem 2.5], convergence of the CR iteration to a fixed point of contractive-like operators satisfying (12) can be obtained easily.

2. Preliminaries

Definition 3 (see [30]). Let X be a Banach space and $T, \widetilde{T} : X \to X$ two operators. We say that $\widetilde{T}$ is an approximate operator of T if for all $x \in X$ and for a fixed $\varepsilon > 0$ we have

$$\|Tx - \widetilde{T}x\| \le \varepsilon. \tag{13}$$

Suppose that there exists a certain fixed point iteration that converges to some fixed point $p \in F_T$ and $\widetilde{T}$ has a fixed point $q \in F_{\widetilde{T}}$ which can be computed by certain method. If it cannot compute fixed point p of T due to various results, then approximate operator $\widetilde{T}$ can be used. One can find some of works done under this title in the following list [15, 24–26, 31].

In this paper, we prove the data dependence results for the multistep and CR iterative procedures utilizing the contractive-like operators satisfying (12).

The following lemma will be useful to prove the main results of this work.

Lemma 4 (see [26]). *Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0(\epsilon) \in \mathbb{N}$, such that for all $n \ge n_0$ one has satisfied the inequality*

$$a_{n+1} \le (1 - \mu_n) a_n + \mu_n \eta_n, \tag{14}$$

where $\mu_n \in (0,1)$, *for all* $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \mu_n = \infty$ *and* $\eta_n \geq 0$, *for all* $n \in \mathbb{N}$. *Then the following holds:*

$$0 \leq \lim_{n\to\infty} \sup\ a_n \leq \lim_{n\to\infty} \sup \eta_n. \tag{15}$$

3. Main Results

For simplicity we use the following notation through this section.

For any iterative process, $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ denote iterative sequences associated to T and $\widetilde{T}$, respectively.

Theorem 5. *Let* $T : E \to E$ *be a map satisfying* (12) *with* $F_T \neq \emptyset$, *and let* $\widetilde{T}$ *be an approximate operator of* T *as in Definition 3. Let* $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ *be two iterative sequences defined by the multistep iteration* (5) *and with real sequences* $\{\beta_n^i\}_{n=0}^{\infty} \subset [0,1)$, $i = \overline{1,k-1}$, *and* $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1)$ *satisfying* $\sum \alpha_n = \infty$. *If* $p = Tp$ *and* $q = \widetilde{T}q$, *then one has*

$$\|p-q\| \leq \frac{k\varepsilon}{1-\delta}. \tag{16}$$

Proof. For a given $x_0 \in E$ and $u_0 \in E$ we consider the following multistep iteration for T and $\widetilde{T}$:

$$\begin{aligned}
&x_0 \in E,\\
&x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1, \quad n \in \mathbb{N},\\
&y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \quad i = \overline{1,k-2},\\
&y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T x_n, \quad k \geq 2,\\
&u_0 \in E,\\
&u_{n+1} = (1-\alpha_n)u_n + \alpha_n \widetilde{T} v_n^1, \quad n \in \mathbb{N},\\
&v_n^i = (1-\beta_n^i)u_n + \beta_n^i \widetilde{T} v_n^{i+1}, \quad i = \overline{1,k-2},\\
&v_n^{k-1} = (1-\beta_n^{k-1})u_n + \beta_n^{k-1} \widetilde{T} u_n, \quad k \geq 2.
\end{aligned} \tag{17}$$

Then from (17), we get

$$x_{n+1} - u_{n+1} = (1-\alpha_n)(x_n - u_n) + \alpha_n \left(T y_n^1 - \widetilde{T} v_n^1\right). \tag{18}$$

Thus, we have the following estimates by using (18) and (12):

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\|\\
&\quad = \left\|(1-\alpha_n)(x_n-u_n) + \alpha_n\left(Ty_n^1 - \widetilde{T}v_n^1\right)\right\|\\
&\quad \leq (1-\alpha_n)\|x_n-u_n\| + \alpha_n\left\|Ty_n^1 - \widetilde{T}v_n^1\right\|\\
&\quad = (1-\alpha_n)\|x_n-u_n\| + \alpha_n\left\|Ty_n^1 - Tv_n^1 + Tv_n^1 - \widetilde{T}v_n^1\right\|\\
&\quad \leq (1-\alpha_n)\|x_n-u_n\| + \alpha_n\left\|Ty_n^1 - Tv_n^1\right\|\\
&\qquad + \alpha_n\left\|Tv_n^1 - \widetilde{T}v_n^1\right\|\\
&\quad \leq (1-\alpha_n)\|x_n-u_n\| + \alpha_n\delta\left\|y_n^1 - v_n^1\right\|\\
&\qquad + \alpha_n\varphi\left(\left\|y_n^1 - Ty_n^1\right\|\right) + \alpha_n\varepsilon,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\left\|y_n^1 - v_n^1\right\|\\
&\quad = \left\|\left(1-\beta_n^1\right)(x_n-u_n) + \beta_n^1\left(Ty_n^2 - \widetilde{T}v_n^2\right)\right\|\\
&\quad \leq \left(1-\beta_n^1\right)\|x_n-u_n\| + \beta_n^1\left\|Ty_n^2 - \widetilde{T}v_n^2\right\|\\
&\quad \leq \left(1-\beta_n^1\right)\|x_n-u_n\| + \beta_n^1\left\|Ty_n^2 - Tv_n^2\right\|\\
&\qquad + \beta_n^1\left\|Tv_n^2 - \widetilde{T}v_n^2\right\|\\
&\quad \leq \left(1-\beta_n^1\right)\|x_n-u_n\| + \beta_n^1\delta\left\|y_n^2 - v_n^2\right\|\\
&\qquad + \beta_n^1\varphi\left(\left\|y_n^2 - Ty_n^2\right\|\right) + \beta_n^1\varepsilon,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\left\|y_n^2 - v_n^2\right\|\\
&\quad = \left\|\left(1-\beta_n^2\right)(x_n-u_n) + \beta_n^2\left(Ty_n^3 - \widetilde{T}v_n^3\right)\right\|\\
&\quad \leq \left(1-\beta_n^2\right)\|x_n-u_n\| + \beta_n^2\left\|Ty_n^3 - \widetilde{T}v_n^3\right\|\\
&\quad \leq \left(1-\beta_n^2\right)\|x_n-u_n\| + \beta_n^2\left\|Ty_n^3 - Tv_n^3\right\|\\
&\qquad + \beta_n^2\left\|Tv_n^3 - \widetilde{T}v_n^3\right\|\\
&\quad \leq \left(1-\beta_n^2\right)\|x_n-u_n\| + \beta_n^2\delta\left\|y_n^3 - v_n^3\right\|\\
&\qquad + \beta_n^2\varphi\left(\left\|y_n^3 - Ty_n^3\right\|\right) + \beta_n^2\varepsilon,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\left\|y_n^3 - v_n^3\right\|\\
&\quad = \left\|\left(1-\beta_n^3\right)(x_n-u_n) + \beta_n^3\left(Ty_n^4 - \widetilde{T}v_n^4\right)\right\|\\
&\quad \leq \left(1-\beta_n^3\right)\|x_n-u_n\| + \beta_n^3\left\|Ty_n^4 - \widetilde{T}v_n^4\right\|\\
&\quad \leq \left(1-\beta_n^3\right)\|x_n-u_n\| + \beta_n^3\left\|Ty_n^4 - Tv_n^4\right\|\\
&\qquad + \beta_n^3\left\|Tv_n^4 - \widetilde{T}v_n^4\right\|\\
&\quad \leq \left(1-\beta_n^3\right)\|x_n-u_n\| + \beta_n^3\delta\left\|y_n^4 - v_n^4\right\|\\
&\qquad + \beta_n^3\varphi\left(\left\|y_n^4 - Ty_n^4\right\|\right) + \beta_n^3\varepsilon.
\end{aligned} \tag{22}$$

Combining (19), (20), (21), and (22) we obtain

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\|\\
&\quad \leq \left[(1-\alpha_n) + \alpha_n\delta\left(1-\beta_n^1\right) + \alpha_n\delta^2\beta_n^1\left(1-\beta_n^2\right)\right.\\
&\qquad \left. + \alpha_n\delta^3\beta_n^1\beta_n^2\left(1-\beta_n^3\right)\right]\|x_n-u_n\|\\
&\qquad + \alpha_n\delta^4\beta_n^1\beta_n^2\beta_n^3\left\|y_n^4 - v_n^4\right\|
\end{aligned}$$

$$+\alpha_n\delta^3\beta_n^1\beta_n^2\beta_n^3\varphi\left(\left\|y_n^4-Ty_n^4\right\|\right)+\alpha_n\delta^3\beta_n^1\beta_n^2\beta_n^3\varepsilon$$
$$+\alpha_n\delta^2\beta_n^1\beta_n^2\varphi\left(\left\|y_n^3-Ty_n^3\right\|\right)+\alpha_n\delta^2\beta_n^1\beta_n^2\varepsilon$$
$$+\alpha_n\delta\beta_n^1\varphi\left(\left\|y_n^2-Ty_n^2\right\|\right)+\alpha_n\delta\beta_n^1\varepsilon$$
$$+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)+\alpha_n\varepsilon. \tag{23}$$

Thus, inductively, we get

$$\begin{aligned}
&\|x_{n+1}-u_{n+1}\|\\
&\quad\le\Big[(1-\alpha_n)+\delta\alpha_n\left(1-\beta_n^1\right)+\delta^2\alpha_n\beta_n^1\left(1-\beta_n^2\right)\\
&\qquad+\delta^3\alpha_n\beta_n^1\beta_n^2\left(1-\beta_n^3\right)+\delta^4\alpha_n\beta_n^1\beta_n^2\beta_n^3\left(1-\beta_n^4\right)\\
&\qquad+\cdots+\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-3}\left(1-\beta_n^{k-2}\right)\Big]\|x_n-u_n\|\\
&\quad+\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-2}\left\|y_n^{k-1}-v_n^{k-1}\right\|\\
&\quad+\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-2}\varphi\left(\left\|y_n^{k-1}-Ty_n^{k-1}\right\|\right)\\
&\quad+\cdots+\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3\varphi\left(\left\|y_n^4-Ty_n^4\right\|\right)\\
&\quad+\delta^2\alpha_n\beta_n^1\beta_n^2\varphi\left(\left\|y_n^3-Ty_n^3\right\|\right)\\
&\quad+\delta\alpha_n\beta_n^1\varphi\left(\left\|y_n^2-Ty_n^2\right\|\right)+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)\\
&\quad+\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-2}\varepsilon+\cdots+\delta^2\alpha_n\beta_n^1\beta_n^2\varepsilon\\
&\quad+\delta\alpha_n\beta_n^1\varepsilon+\alpha_n\varepsilon.
\end{aligned}\tag{24}$$

Using now (17) and (12), we get

$$\begin{aligned}
&\left\|y_n^{k-1}-v_n^{k-1}\right\|\\
&\quad=\left\|\left(1-\beta_n^{k-1}\right)(x_n-u_n)+\beta_n^{k-1}\left(Tx_n-\widetilde{T}u_n\right)\right\|\\
&\quad\le\left(1-\beta_n^{k-1}\right)\|x_n-u_n\|+\beta_n^{k-1}\left\|Tx_n-\widetilde{T}u_n\right\|\\
&\quad\le\left(1-\beta_n^{k-1}\right)\|x_n-u_n\|+\beta_n^{k-1}\left\|Tx_n-Tu_n\right\|\\
&\qquad+\beta_n^{k-1}\left\|Tu_n-\widetilde{T}u_n\right\|\\
&\quad\le\left(1-\beta_n^{k-1}\right)\|x_n-u_n\|+\beta_n^{k-1}\delta\|x_n-u_n\|\\
&\qquad+\beta_n^{k-1}\varphi\left(\|x_n-Tx_n\|\right)+\beta_n^{k-1}\varepsilon.
\end{aligned}\tag{25}$$

Substituting (25) in (24) we have

$$\begin{aligned}
&\|x_{n+1}-u_{n+1}\|\\
&\quad\le\Big[(1-\alpha_n)+\delta\alpha_n\left(1-\beta_n^1\right)+\delta^2\alpha_n\beta_n^1\left(1-\beta_n^2\right)\\
&\qquad+\delta^3\alpha_n\beta_n^1\beta_n^2\left(1-\beta_n^3\right)+\delta^4\alpha_n\beta_n^1\beta_n^2\beta_n^3\left(1-\beta_n^4\right)\\
&\qquad+\cdots+\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-3}\left(1-\beta_n^{k-2}\right)\\
&\qquad+\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-2}\left(1-\beta_n^{k-1}\right)\\
&\qquad+\delta^{k}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-2}\beta_n^{k-1}\Big]\|x_n-u_n\|\\
&\quad+\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-2}\beta_n^{k-1}\varphi\left(\|x_n-Tx_n\|\right)\\
&\quad+\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-2}\varphi\left(\left\|y_n^{k-1}-Ty_n^{k-1}\right\|\right)\\
&\quad+\cdots+\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3\varphi\left(\left\|y_n^4-Ty_n^4\right\|\right)\\
&\quad+\delta^2\alpha_n\beta_n^1\beta_n^2\varphi\left(\left\|y_n^3-Ty_n^3\right\|\right)\\
&\quad+\delta\alpha_n\beta_n^1\varphi\left(\left\|y_n^2-Ty_n^2\right\|\right)+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)\\
&\quad+\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-2}\beta_n^{k-1}\varepsilon\\
&\quad+\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\cdots\beta_n^{k-2}\varepsilon\\
&\quad+\cdots+\delta^2\alpha_n\beta_n^1\beta_n^2\varepsilon+\delta\alpha_n\beta_n^1\varepsilon+\alpha_n\varepsilon.
\end{aligned}\tag{26}$$

If this inequality is rearranged using $\{\beta_n^i\}_{n=0}^{\infty}\subset[0,1)$, $\delta^{i+1}<\delta^i$ for each $i=\overline{1,k-1}$, then we get the following inequality as follows:

$$\begin{aligned}
&\|x_{n+1}-u_{n+1}\|\\
&\quad\le\left[1-\alpha_n(1-\delta)\right]\|x_n-u_n\|\\
&\qquad+\alpha_n(1-\delta)\frac{\varphi\left(\|x_n-Tx_n\|\right)}{1-\delta}\\
&\qquad+\alpha_n(1-\delta)\\
&\qquad\times\frac{\left\{\varphi\left(\left\|y_n^{k-1}-Ty_n^{k-1}\right\|\right)+\cdots+\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)+k\varepsilon\right\}}{1-\delta}.
\end{aligned}\tag{27}$$

Denote

$$\begin{aligned}
&a_n:=\|x_n-u_n\|,\\
&\mu_n:=\alpha_n(1-\delta)\in(0.1),\\
&\eta_n:=\frac{\varphi\left(\|x_n-Tx_n\|\right)}{1-\delta}\\
&\qquad+\frac{\varphi\left(\left\|y_n^{k-1}-Ty_n^{k-1}\right\|\right)+\cdots+\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)+k\varepsilon}{1-\delta}.
\end{aligned}\tag{28}$$

From [26, Theorem 3.1] and [28, Corollary 2] we have $\lim_{n\to\infty}\|x_n-p\|=0$. Since T satisfies condition (12), and $p\in F_T$, that is, $Tp=p$, it follows from (12) that

$$\begin{aligned}
0&\le\|x_n-Tx_n\|\\
&\le\|x_n-p\|+\|Tp-Tx_n\|\\
&\le\|x_n-p\|+\delta\|p-x_n\|+\varphi\left(\|p-Tp\|\right)\\
&=(1+\delta)\|x_n-p\|\longrightarrow0\quad\text{as }n\longrightarrow\infty.
\end{aligned}\tag{29}$$

Considering $\beta_n^i \in [0,1)$, for all $n \in \mathbb{N}$, $i = \overline{1, k-1}$, and using (12) and (5) we have

$$
\begin{aligned}
0 &\le \left\| y_n^1 - T y_n^1 \right\| = \left\| y_n^1 - p + p - T y_n^1 \right\| \\
&\le \left\| y_n^1 - p \right\| + \left\| Tp - T y_n^1 \right\| \\
&\le \left\| y_n^1 - p \right\| + \delta \left\| p - y_n^1 \right\| + \varphi\left(\left\| p - Tp \right\|\right) \\
&= (1+\delta) \left\| y_n^1 - p \right\| \\
&= (1+\delta) \left\| \left(1-\beta_n^1\right) y_n^2 + \beta_n^1 T y_n^2 - p\left(1 - \beta_n^1 + \beta_n^1\right) \right\| \\
&\le (1+\delta) \left\{ \left(1-\beta_n^1\right) \left\| y_n^2 - p \right\| + \beta_n^1 \left\| T y_n^2 - Tp \right\| \right\} \\
&\le (1+\delta) \left\{ \left(1-\beta_n^1\right) \left\| y_n^2 - p \right\| + \beta_n^1 \delta \left\| y_n^2 - p \right\| \right\} \\
&= (1+\delta) \left[1 - \beta_n^1 (1-\delta) \right] \left\| y_n^2 - p \right\| \\
&= (1+\delta) \left[1 - \beta_n^1 (1-\delta) \right] \\
&\quad \times \left\| \left(1-\beta_n^2\right) y_n^3 + \beta_n^2 T y_n^3 - p\left(1 - \beta_n^2 + \beta_n^2\right) \right\| \\
&\le (1+\delta) \left[1 - \beta_n^1 (1-\delta) \right] \\
&\quad \times \left\{ \left(1-\beta_n^2\right) \left\| y_n^3 - p \right\| + \beta_n^2 \left\| T y_n^3 - Tp \right\| \right\} \\
&\le (1+\delta) \left[1 - \beta_n^1 (1-\delta) \right] \\
&\quad \times \left[1 - \beta_n^2 (1-\delta) \right] \left\| y_n^3 - p \right\| \\
&\ \ \vdots \\
&\le (1+\delta) \left[1 - \beta_n^1 (1-\delta) \right] \\
&\quad \cdots \left[1 - \beta_n^{k-2} (1-\delta) \right] \left\| y_n^{k-1} - p \right\| \\
&\le (1+\delta) \left[1 - \beta_n^1 (1-\delta) \right] \\
&\quad \cdots \left[1 - \beta_n^{k-1} (1-\delta) \right] \left\| x_n - p \right\| \\
&\le (1+\delta) \left\| x_n - p \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{30}
$$

It is easy to see from (30) that this result is also valid for $\|Ty_n^2 - y_n^2\|, \ldots, \|Ty_n^{k-1} - y_n^{k-1}\|$.

Making use of the fact that φ is a continuous map we have

$$
\begin{aligned}
\lim_{n\to\infty} \varphi\left(\left\| x_n - T x_n \right\|\right) &= \lim_{n\to\infty} \varphi\left(\left\| y_n^1 - T y_n^1 \right\|\right) \\
&= \cdots = \lim_{n\to\infty} \varphi\left(\left\| y_n^{k-1} - T y_n^{k-1} \right\|\right) = 0.
\end{aligned} \tag{31}
$$

Hence an application of Lemma 4 to (27) leads to

$$
\| p - q \| \le \frac{k\varepsilon}{1-\delta}. \tag{32}
$$

□

Now we prove result on data dependence for the CR iterative procedure.

Theorem 6. *Let $T : E \to E$ be a map satisfying* (12) *with $F_T \neq \emptyset$, and let $\widetilde{T}$ be an approximate operator of T as in Definition 3. Let $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ be two iterative sequences defined by the CR iteration* (6) *and with real sequences $\{\beta_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$ satisfying* (i)*$1/2 \le \alpha_n$, for all $n \in \mathbb{N}$, and* (ii)*$\sum_{n=0}^{\infty} \alpha_n = \infty$. If $p = Tp$ and $q = \widetilde{T} q$, then one has*

$$
\| p - q \| \le \frac{5\varepsilon}{1-\delta}. \tag{33}
$$

Proof. For a given $x_0 \in E$ and $u_0 \in E$ we consider the following iteration for T and $\widetilde{T}$:

$$
\begin{aligned}
x_{n+1} &= (1-\alpha_n) y_n + \alpha_n T y_n, \\
y_n &= (1-\beta_n) T x_n + \beta_n T z_n, \\
z_n &= (1-\mu_n) x_n + \mu_n T x_n, \\
u_{n+1} &= (1-\alpha_n) v_n + \alpha_n \widetilde{T} v_n, \\
v_n &= (1-\beta_n) \widetilde{T} u_n + \beta_n \widetilde{T} w_n, \\
w_n &= (1-\mu_n) u_n + \mu_n \widetilde{T} u_n.
\end{aligned} \tag{34}
$$

Then from (34) we have

$$
\begin{aligned}
x_{n+1} - u_{n+1} &= (1-\alpha_n)(y_n - v_n) + \alpha_n \left(T y_n - \widetilde{T} v_n\right), \\
y_n - v_n &= (1-\beta_n)\left(T x_n - \widetilde{T} u_n\right) + \beta_n \left(T z_n - \widetilde{T} w_n\right), \\
z_n - w_n &= (1-\mu_n)(x_n - u_n) + \mu_n \left(T x_n - \widetilde{T} u_n\right).
\end{aligned} \tag{35}
$$

Thus, by considering (35), it follows from (6) and (12) that

$$
\begin{aligned}
&\left\| x_{n+1} - u_{n+1} \right\| \\
&\quad = \left\| (1-\alpha_n)(y_n - v_n) + \alpha_n \left(T y_n - \widetilde{T} v_n\right) \right\| \\
&\quad \le (1-\alpha_n) \left\| y_n - v_n \right\| + \alpha_n \left\| T y_n - \widetilde{T} v_n \right\| \\
&\quad = (1-\alpha_n) \left\| y_n - v_n \right\| + \alpha_n \left\| T y_n - T v_n + T v_n - \widetilde{T} v_n \right\| \\
&\quad \le (1-\alpha_n) \left\| y_n - v_n \right\| + \alpha_n \left\| T y_n - T v_n \right\| \\
&\qquad + \alpha_n \left\| T v_n - \widetilde{T} v_n \right\| \\
&\quad \le (1-\alpha_n) \left\| y_n - v_n \right\| + \alpha_n \delta \left\| y_n - v_n \right\| \\
&\qquad + \alpha_n \varphi\left(\left\| y_n - T y_n \right\|\right) + \alpha_n \varepsilon
\end{aligned}
$$

$$= [1 - \alpha_n (1 - \delta)] \|y_n - v_n\| + \alpha_n \varphi (\|y_n - Ty_n\|) + \alpha_n \varepsilon, \tag{36}$$

$$\begin{aligned}
\|y_n - v_n\| &= \left\|(1 - \beta_n)\left(Tx_n - \widetilde{T}u_n\right) + \beta_n \left(Tz_n - \widetilde{T}w_n\right)\right\| \\
&\le (1 - \beta_n) \left\|Tx_n - \widetilde{T}u_n\right\| + \beta_n \left\|Tz_n - \widetilde{T}w_n\right\| \\
&= (1 - \beta_n) \left\|Tx_n - Tu_n + Tu_n - \widetilde{T}u_n\right\| \\
&\quad + \beta_n \left\|Tz_n - Tw_n + Tw_n - \widetilde{T}w_n\right\| \\
&\le (1 - \beta_n) \left\{\|Tx_n - Tu_n\| + \left\|Tu_n - \widetilde{T}u_n\right\|\right\} \\
&\quad + \beta_n \left\{\|Tz_n - Tw_n\| + \left\|Tw_n - \widetilde{T}w_n\right\|\right\} \\
&\le (1 - \beta_n) \delta \|x_n - u_n\| + \beta_n \delta \|z_n - w_n\| \\
&\quad + (1 - \beta_n) \varphi (\|x_n - Tx_n\|) \\
&\quad + (1 - \beta_n) \varepsilon + \beta_n \varphi (\|Tz_n - Tw_n\|) + \beta_n \varepsilon,
\end{aligned} \tag{37}$$

$$\begin{aligned}
\|z_n - w_n\| &= \left\|(1 - \mu_n)(x_n - u_n) + \mu_n \left(Tx_n - \widetilde{T}u_n\right)\right\| \\
&\le (1 - \mu_n) \|x_n - u_n\| + \mu_n \left\|Tx_n - \widetilde{T}u_n\right\| \\
&= (1 - \mu_n) \|x_n - u_n\| + \mu_n \left\|Tx_n - Tu_n + Tu_n - \widetilde{T}u_n\right\| \\
&\le (1 - \mu_n) \|x_n - u_n\| \\
&\quad + \mu_n \left\{\|Tx_n - Tu_n\| + \left\|Tu_n - \widetilde{T}u_n\right\|\right\} \\
&\le (1 - \mu_n) \|x_n - u_n\| \\
&\quad + \mu_n \{\delta \|x_n - u_n\| + \varphi (\|x_n - Tx_n\|) + \varepsilon\} \\
&= [1 - \mu_n (1 - \delta)] \|x_n - u_n\| \\
&\quad + \mu_n \varphi (\|x_n - Tx_n\|) + \mu_n \varepsilon.
\end{aligned} \tag{38}$$

Combining (36), (37), and (38) we obtain

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\le [1 - \alpha_n (1 - \delta)] \\
&\quad \times [(1 - \beta_n) \delta + \beta_n \delta [1 - \mu_n (1 - \delta)]] \|x_n - u_n\| \\
&\quad + [1 - \alpha_n (1 - \delta)] \\
&\quad \times \{[1 - \beta_n (1 - \delta\mu_n)] \varphi (\|x_n - Tx_n\|) + \beta_n \mu_n \delta \varepsilon \\
&\qquad + (1 - \beta_n) \varepsilon + \beta_n \varphi (\|z_n - Tz_n\|) + \beta_n \varepsilon\} \\
&\quad + \alpha_n \varphi (\|y_n - Ty_n\|) + \alpha_n \varepsilon.
\end{aligned} \tag{39}$$

It may be noted that for $\{\beta_n\}_{n=0}^{\infty}, \{\mu_n\}_{n=0}^{\infty} \subset [0, 1)$, and $\delta \in [0, 1)$ the following inequalities are always true:

$$1 - \mu_n (1 - \delta) < 1, \quad \beta_n \mu_n \delta < 1, \quad \delta\mu_n < \delta. \tag{40}$$

Using now the inequality $\delta\mu_n < \delta$ we get

$$1 - \beta_n (1 - \delta\mu_n) < 1 - \beta_n (1 - \delta) < 1. \tag{41}$$

Therefore an application of (40) and (41) to (39) gives us

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\le [1 - \alpha_n (1 - \delta)] \|x_n - u_n\| \\
&\quad + [1 - \alpha_n (1 - \delta)] \\
&\quad \times \{\varphi (\|x_n - Tx_n\|) + \varepsilon + \varepsilon + \varphi (\|z_n - Tz_n\|)\} \\
&\quad + \alpha_n \varphi (\|y_n - Ty_n\|) + \alpha_n \varepsilon.
\end{aligned} \tag{42}$$

Now, by the condition (i) $1/2 \le \alpha_n$, for all $n \in \mathbb{N}$ we have

$$1 - \alpha_n \le \alpha_n. \tag{43}$$

Utilizing (43) in (42) we obtain

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\le [1 - \alpha_n (1 - \delta)] \|x_n - u_n\| \\
&\quad + \alpha_n (1 - \delta) \frac{\{2\varphi (\|x_n - Tx_n\|) + \varphi (\|y_n - Ty_n\|)\}}{1 - \delta} \\
&\quad + \alpha_n (1 - \delta) \frac{\{2\varphi (\|z_n - Tz_n\|) + 5\varepsilon\}}{1 - \delta}.
\end{aligned} \tag{44}$$

Denote

$$\begin{aligned}
a_n &:= \|x_n - u_n\|, \\
\mu_n &:= \alpha_n (1 - \delta) \in (0, 1), \\
\eta_n &:= \frac{\{2\varphi (\|x_n - Tx_n\|) + \varphi (\|y_n - Ty_n\|) + 2\varphi (\|z_n - Tz_n\|) + 5\varepsilon\}}{1 - \delta}.
\end{aligned} \tag{45}$$

From Remark 2, we have $\lim_{n\to\infty} \|x_n - p\| = 0$. Since T satisfies condition (12), and $p \in F_T$, that is, $Tp = p$, using similar arguments as in the proof of Theorem 5, we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|y_n - Ty_n\| = \lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{46}$$

Making use of the fact that φ is a continuous map we have

$$\begin{aligned}
\lim_{n\to\infty} \varphi (\|x_n - Tx_n\|) &= \lim_{n\to\infty} \varphi (\|y_n - Ty_n\|) \\
&= \lim_{n\to\infty} \varphi (\|z_n - Tz_n\|) = 0.
\end{aligned} \tag{47}$$

Hence an application of Lemma 4 to (44) leads to

$$\|p - q\| \le \frac{5\varepsilon}{1 - \delta}. \tag{48}$$

□

Corollary 7. *Since the Mann* (2)*, Ishikawa* (3)*, and Noor* (4) *iterative processes are special cases of the multistep iterative scheme* (5)*, the data dependence results of these iterative processes can be obtained similarly.*

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Research Article

Numerical Solution and Simulation of Second-Order Parabolic PDEs with Sinc-Galerkin Method Using Maple

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An efficient solution algorithm for sinc-Galerkin method has been presented for obtaining numerical solution of PDEs with Dirichlet-type boundary conditions by using Maple Computer Algebra System. The method is based on Whittaker cardinal function and uses approximating basis functions and their appropriate derivatives. In this work, PDEs have been converted to algebraic equation systems with new accurate explicit approximations of inner products without the need to calculate any numeric integrals. The solution of this system of algebraic equations has been reduced to the solution of a matrix equation system via Maple. The accuracy of the solutions has been compared with the exact solutions of the test problem. Computational results indicate that the technique presented in this study is valid for linear partial differential equations with various types of boundary conditions.

1. Introduction

Sinc methods for differential equations were originally introduced by Stenger in [1–3]. The sinc functions were first analyzed in [4, 5] and a detailed research of the method for two-point boundary-value problems can be found in [6, 7]. In [8], parabolic and hyperbolic problems are presented in detail. To solve a problem arising from chemical reactor theory, the properties of the sinc-Galerkin method are used to reduce the computation of nonlinear two-point boundary-value problems to some algebraic equations in [9]. A computer algorithm for sinc method to solve numerically the linear and nonlinear ODEs and their simulations has been presented in [7, 10], respectively. The full sinc-Galerkin method is developed for a family of complex-valued partial differential equations with time-dependent boundary conditions [9]. A study of the performance of the Galerkin method using sinc basis functions for solving Bratu's problem is presented in [11]. In [12] a numerical algorithm has been presented for recovering the unknown function and obtaining a solution to the inverse ill-posed problem. They have presented a Galerkin method with the sinc basis functions in both space and time domains for solving the direct problem. A sinc-collocation method has been developed for solving linear systems of integrodifferential equations of Fredholm and Volterra type with homogeneous boundary conditions in [13].

2. Sinc-Approximation Formula for PDEs

We use the sinc-Galerkin method as mentioned in [1] to derive an approximate solution of the following:

$$\begin{gathered} u_t - u_{xx} = F(x,t), \\ u(0,t) = u(1,t) = 0, \quad 0 < x < 1, \\ u(x,0) = f(x), \quad t > 0. \end{gathered} \tag{1}$$

For the equation given above, the sinc-Galerkin scheme can be developed in both space and time directions as follows.

In general, approximations can be constructed for infinite, semi-infinite, and infinite intervals and both spatial and time spaces will be introduced. Define the function

$$\phi(z) = \ln\left(\frac{z}{1-z}\right) \tag{2}$$

which is a conformal mapping from D_E, the eye-shaped domain in the z-plane, onto the infinite strip, D_S, where

$$D_E = \left\{ z = x + iy : \left| \arg\left(\frac{z}{1-z}\right) \right| < d \le \frac{\pi}{2} \right\}. \tag{3}$$

A more general form of sinc basis according to intervals can be given as follows:

$$S(m,h_x)\circ\phi(x)=\operatorname{Sinc}\left(\frac{\phi(x)-mh_x}{h_x}\right),\quad m=-N_x,\ldots,N_x,$$
$$S(k,h_t)\circ\gamma(t)=\operatorname{Sinc}\left(\frac{\gamma(t)-kh_t}{h_t}\right),\quad k=-N_t,\ldots,N_t, \tag{4}$$

where

$$\operatorname{Sinc}(z)=\begin{cases}\dfrac{\sin(\pi z)}{\pi z}, & z\neq 0\\ 1, & z=0,\end{cases}$$
$$\begin{aligned}\operatorname{Sinc}(k,h)(z) &= \operatorname{Sinc}\left(\frac{z-kh}{h}\right)\\ &= \begin{cases}\dfrac{\sin(\pi((z-kh)/h))}{\pi((z-kh)/h)}, & z\neq kh\\ 1, & z=kh,\end{cases}\quad k=0,\mp1,\mp2,\mp3,\ldots\end{aligned} \tag{5}$$

and the conformal maps for both directions

$$\phi(x)=\ln\left(\frac{x}{l-x}\right),\quad x\in(0,l),$$
$$\gamma(t)=\ln(t),\quad t\in(0,\infty) \tag{6}$$

are used to define the basis functions on the intervals $(0,l)$ and $(0,\infty)$, respectively. $h_x, h_t>0$ represents the mesh sizes in the space direction and the time direction, respectively. The sinc nodes x_i and t_j are chosen so that $x_i=\phi^{-1}(ih_x)$, $t_j=\gamma^{-1}(jh_t)$.

Here the function $x=\phi^{-1}(x)=e^x/(1+e^x)$ is an inverse mapping of $\phi=\phi(x)$. We may define the range of ϕ^{-1} on the real line as

$$\Gamma_1=\left\{\phi^{-1}(u)\in D_E:-\infty<u<\infty\right\}. \tag{7}$$

For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k=\phi^{-1}(kh)=\frac{e^{kh}}{1+e^{kh}}, \tag{8}$$

where $0<x_k<1$, for all k.

The sinc basis functions in (4) do not have a derivative when x tends to 0 or 1. We modify the sinc basis functions as

$$\frac{S(m,h_x)\circ\phi(x)}{\phi'(x)}=\frac{\operatorname{Sinc}((\phi(x)-mh_x)/h_x)}{\phi'(x)}, \tag{9}$$

where

$$\frac{1}{\phi'(x)}=x(1-x). \tag{10}$$

Table 1: Conformal mappings and nodes for several subintervals of R.

(a,b)		$\phi(z)$	z_k
a	b	$\ln\left(\frac{z-a}{b-z}\right)$	$\frac{a+be^{kh}}{1+e^{kh}}$
0	1	$\ln\left(\frac{z}{1-z}\right)$	$\frac{e^{kh}}{1+e^{kh}}$
0	∞	$\ln(z)$	e^{kh}
0	∞	$\ln(\sinh(z))$	$\ln\left(e^{kh}+\sqrt{e^{2kh}+1}\right)$
$-\infty$	∞	z	kh
$-\infty$	∞	$\sinh^{-1}(z)$	kh

For the temporal space, we construct an approximation by defining the function

$$w=\gamma(r)=\ln(r) \tag{11}$$

which is a conformal mapping from D_W, the wedge-shaped domain in the r-plane, onto the infinite strip, D_S, where

$$D_W=\left\{r=t+is:\left|\arg(r)\right|<d<\frac{\pi}{2}\right\}, \tag{12}$$

derived from composite translated functions

$$S(k,h_t)\circ\gamma(t)=\operatorname{Sinc}\left(\frac{\gamma(t)-kh_t}{h_t}\right),\quad k=-N_t,\ldots,N_t, \tag{13}$$

for $r\in D_W$.

Here $w=\gamma(r)$ and $\gamma^{-1}(w)=r=e^w$. We may define γ^{-1} on the real line as

$$\Gamma_2=\left\{\gamma^{-1}(u)\in D_w:-\infty<u<\infty\right\}. \tag{14}$$

For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$t_k=\gamma^{-1}(kh)=e^{kh}, \tag{15}$$

where $0<t_k<\infty$, for all k.

A list of conformal mappings may be found in Table 1 [14].

Definition 1. Let $B(D_E)$ be the class of functions F that are analytic in D_E and satisfy

$$\int_{\psi(L+u)}|F(z)|\,dz\longrightarrow 0,\quad \text{as } u=\mp\infty, \tag{16}$$

where

$$L=\left\{iy:|y|<d\leq\frac{\pi}{2}\right\} \tag{17}$$

and on the boundary of D_E satisfy

$$T(F)=\int_{\partial D_E}|F(z)\,dz|<\infty. \tag{18}$$

The proof of the following theorems can be found in [1].

Theorem 2. *Let Γ be $(0,1)$, $F \in B(D_E)$, and then for $h > 0$ sufficiently small*

$$\int_\Gamma F(z)\,dz - h\sum_{j=-\infty}^{\infty}\frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2}\int_{\partial D}\frac{F(z)\,k(\phi,h)(z)}{\sin(\pi\phi(z)/h)}dz \equiv I_F, \tag{19}$$

where

$$\left|k(\phi,h)\right|_{z\in\partial D} = \left|e^{[(i\pi\phi(z)/h)\,\mathrm{sgn}\,(\mathrm{Im}\,\phi(z))]}\right|_{z\in\partial D} = e^{-\pi d/h}. \tag{20}$$

For the sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum; the following theorem indicates the conditions under which exponential convergence results.

Theorem 3. *If there exist positive constants α, β, and C such that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C\begin{cases} e^{-\alpha|\phi(x)|}, & x\in\psi(-\infty,\infty)\\ e^{-\beta|\phi(x)|}, & x\in\psi(0,\infty),\end{cases} \tag{21}$$

then the error bound for the quadrature rule (19) *is*

$$\left|\int_\Gamma F(x)\,dx - h\sum_{j=-N}^{N}\frac{F(x_j)}{\phi'(x_j)}\right| \le C\left(\frac{e^{-\alpha Nh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta}\right) + \left|I_F\right|. \tag{22}$$

The infinite sum in (19) *is truncated with the use of* (20) *to arrive at* (22).

Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \qquad N \equiv \left\|\frac{\alpha N}{\beta} + 1\right\|, \tag{23}$$

where $\|\cdot\|$ is integer part of statement,

$$\int_\Gamma F(x)\,dx = h\sum_{j=-N}^{N}\frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi\alpha dN)^{1/2}}\right). \tag{24}$$

Theorems 2 and 3 can be used to approximate the integrals that arise in the formulation of the discrete systems corresponding to two-point BVPs.

3. Discrete Solutions Scheme for Two-Point BVPs

In ordinary differential equations

$$Lu = f \tag{25}$$

on Γ_1, sinc solution is assumed as an approximate solution u_m in the form of series with $m = 2N + 1$ terms

$$u_m(z) = \sum_{j=-N}^{N} c_j S(j,h)\circ\phi(z). \tag{26}$$

The coefficients $\{c_j\}_{j=-N}^{N}$ are determined by orthogonalizing the residual $Lu - f$ with respect to the sinc basis functions $\{S_k\}_{k=-N}^{N}$ where $S_k(z) = S(k,h)\circ\phi(z)$. An inner product for two continuous functions such as f_1 and f_2 can be given by the following formula

$$\langle f_1, f_2\rangle = \int_\Gamma f_1 f_2 w\,dz, \tag{27}$$

where w is the weight function and is chosen depending on boundary conditions. If we implement the above inner product rule in orthogonalization, this yields the discrete sinc-Galerkin system:

$$\int_\Gamma (Lu_m - f)(z)\,S(k,h)\circ\phi(z)\cdot w(z)\,dz = 0, \quad -N\le k\le N. \tag{28}$$

Now, we are going to derive discrete sinc-Galerkin system for PDEs. Assume u_{m_z,m_t} is the approximate solution of (1).Then, the discrete system takes the following form:

$$u_{m_z,m_t}(z,t) = \sum_{j=-N}^{N}\sum_{k=-N}^{N} c_{jk} S(j,h)\circ\phi(z)\cdot S(k,h)\circ\gamma(t). \tag{29}$$

The coefficients $\{c_{jk}\}_{j,k=-N}^{N}$ are determined by orthogonalizing the residual $Lu_{m_z,m_t} - f$ with respect to the sinc basis functions $\{S_k S_h\}_{k,h=-N}^{N}$ where $S_j S_h(z,t) = S(j,h)\circ\phi(z)S(k,h)\circ\gamma(t)$ for $-N \le j, k \le N$. In this case the inner product takes the following form:

$$\langle f_1, f_2\rangle = \int_{\Gamma_t}\int_{\Gamma_z} f_1(z,t)\,f_2(z,t)\,w(z,t)\,dz\,dt. \tag{30}$$

The choice of the weight function $w(z,t)$ in the double integrand depends on the boundary conditions, the domain, and the partial differential equation. Therefore, the discrete Galerkin system is

$$\int_{\Gamma_t}\int_{\Gamma_z}\left(Lu_{m_z m_t} - f\right)(z,t)\cdot S(j,h)\circ\phi(z)\cdot S(k,h)\circ\gamma(t)\cdot w(z,t)\,dz\,dt = 0. \tag{31}$$

4. Matrix Representation of the Derivatives of Sinc Basis Functions at Nodal Points

The sinc-Galerkin method actually requires the evaluated derivatives of sinc basis functions at the sinc nodes, $z = z_j$. The rth derivative of $S_k(z) = S(k,h)\circ\phi(z)$ with respect to ϕ, evaluated at the nodal point z_j, is denoted by

$$\frac{1}{h^r}\delta_{pj}^{(r)} = \frac{d^r}{d\phi^r}\left(S(p,h)\circ\phi(z)\right)\Big|_{z=z_j}. \tag{32}$$

```
restart:
with(linalg):
with(LinearAlgebra):
N:=16:
d:=Pi/2:
h:=0.75/sqrt(N);
s:=.5/sqrt(N);
SIZE:=2*N+1;
delta0:=(i,j)->piecewise(j=i,1,j<>i,0):
delta1:=(i,j)->piecewise(i=j,0,i<>j,((-1)^(i-j))/(i-j)):
delta2:=(i,j)->piecewise(i=j,(-Pi^2)/3,i<>j,-2*(-1)^(i-j)/(i-j)^2):
I_0:=Matrix(SIZE,delta0):
I_1:=Matrix(SIZE,delta1):
I_2:=Matrix(SIZE,delta2):
xk:=1/2+1/2*tanh(k*h/2);
xk_func:=unapply(xk,k);
phi:=unapply(log((x)/(1-x)),x);
Dphi:=unapply(simplify(diff(phi(x),x)),x);
g:=unapply(simplify(1/diff(phi(x),x)),x);
Dg:=unapply(diff(g(x),x),x);
gk:=unapply(subs(x=xk,g(x)),k);
Dgk:=unapply(subs(x=xk,Dg(x)),k);
g_Div_Dphi:=unapply(g(x)/Dphi(x),x);
g_Div_Dphi_k:=unapply(subs(x=xk,g_Div_Dphi(x)),k);
#Temporal Spaces;
tl:=exp(l*s);
tl_func:=unapply(tl,l);
gamm:=unapply(log(t),t);
Dgam:=unapply(diff(gamm(t),t),t);
g_gamm:=unapply(1/diff(gamm(t),t),t);
g_gamm_l:=unapply(subs(t=tl,g_gamm(t)),l);
gamm_Div_Dgam:=unapply(g_gamm(t)/Dgam(t),t);
gamm_Div_Dgam:= t-> t^2;
gamm_Div_Dgam_l:=unapply(subs(t=tl,gamm_Div_Dgam(t)),l);
GenerateDiagonalAm := proc( x )
                          local i:=1:
                           local A:=Matrix(SIZE):
                             for i from 1 by 1 to SIZE
                              do
                               A[i,i]:=evalf(x(-N+i-1)):
                              end do:
                             return A;
                      end proc;
B:= -2*h*MatrixMatrixMultiply(I_0,GenerateDiagonalAm(gk))+
   MatrixMatrixMultiply(I_1,GenerateDiagonalAm(Dgk))+1/h*I_2:
DD:=h*GenerateDiagonalAm(g_Div_Dphi_k):
C:=MatrixMatrixMultiply(GenerateDiagonalAm(g_gamm_l),s*(I_0-I_1)):
E:=s*GenerateDiagonalAm(gamm_Div_Dgam_l):
Fkl:=unapply((Pi^2-4)*sin(Pi*xk_func(k-N-1))*exp(-tl_func(l-N-1)),k,l);
F:=evalf(Matrix(SIZE,Fkl)):
V:=Matrix(SIZE,v):
EQN_SYS:=evalf(
  MatrixMatrixMultiply(
            MatrixMatrixMultiply(
                Matrix(inverse(DD)),B
                               ),V
```

Algorithm 1: Continued.

```
                )
            )
        +evalf(
            MatrixMatrixMultiply(
                MatrixMatrixMultiply(
                    V,C),Matrix(inverse(E)
                                )
                            )
            ):
SYS:=[]:
      for i from 1 by 1 to SIZE
          do
             for j from 1 by 1 to SIZE
                 do
                    SYS:=[op(SYS),EQN_SYS(i,j)=F(i,j)];
                 end do:
          end do:
vars:=seq(seq(v(i,j),i=1..2*N+1),j=1..2*N+1):
A,b:LinearAlgebra[GenerateMatrix](evalf(SYS),[vars]):
c:=linsolve(A,b):
CoeffMatrix=Matrix(SIZE):
cnt:=1;
for i to SIZE do
    for j to SIZE do
      CoeffMatrix[j,i]:=c[cnt]:
      cnt:=cnt+1
    end do;
end do;
CoeffMatrix:=Matrix(CoeffMatrix,SIZE):
ApproximateSol:=unapply(
                     evalf(
                            sum(
                              sum("CoeffMatrix"[m+N+1,n+N+1]
                 *sin(Pi*(phi(x)-m*h)/h)/(Pi*(phi(x)-m*h)/h)
               *sin(Pi*(gamm(t)-n*s)/s)/(Pi*(gamm(t)-n*s)/s)
                       ,m=-N..N),
                     n=-N...N)
                          )
                           +exp(-4*t)*sin(Pi*x)
                      ,x,t):
plot3d(ApproximateSol(x,t),x=0..1,t=0..1):
Exact:=unapply(exp(-Pi^2*t)*sin(Pi*x),x,t);
plot3d(Exact(x,t),x=0..1,t=0..1);
plot3d({Exact(x,t),ApproximateSol(x,t)},x=0..1,t=0..1);
XX:=.6;
#Numerical Comparision EXACT
for j from 0.1 to 10 by 1
   do
    evalf(Exact(XX,j)):
   od;
#Numerical Comparision APPROX
for j from 0.1 to 10 by 1
   do
    evalf(ApproximateSol(XX,j)):
   od;
#Numerical Comparision ERROR
for j from 0.1 to 10 by 1
   do
    abs(evalf(evalf(ApproximateSol(XX,j)-Exact(XX,j)))):
   od;
```

Algorithm 1

The expressions in (14) for each p and j can be stored in a matrix $I^{(r)} = [\delta_{pj}^{(r)}]$. For $r = 0, 1, 2, \ldots$

$$I^{(0)} = \delta_{jk}^{(0)} = \left[S(j,h)\circ\phi(x)\right]\big|_{x=x_k} = \begin{cases}1, & k=j\\ 0, & k\neq j,\end{cases}$$

$$I^{(1)} = \delta_{jk}^{(1)} = h\frac{d}{d\phi}\left[S(j,h)\circ\phi(x)\right]\Big|_{x=x_k} = \begin{cases}0, & k=j\\ \dfrac{(-1)^{k-j}}{(k-j)}, & k\neq j,\end{cases}$$

$$\begin{aligned}I^{(2)} = \delta_{jk}^{(2)} &= h\frac{d^2}{d\phi^2}\left[S(j,h)\circ\phi(x)\right]\Big|_{x=x_k}\\ &= \begin{cases}\dfrac{-\pi^2}{3}, & k=j\\ \dfrac{-2(-1)^{k-j}}{(k-j)^2}, & k\neq j,\end{cases}\end{aligned} \tag{33}$$

where

$$I_m^{(0)} = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix} = \left[\delta_{jk}^{(0)}\right],$$

$$I_m^{(1)} = \begin{bmatrix}0 & -1 & \frac{1}{2} & \cdots & \frac{1}{2N}\\ 1 & 0 & -1 & \cdots & -\frac{1}{2N-1}\\ -\frac{1}{2} & 1 & 0 & \cdots & \frac{1}{2N-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -\frac{1}{2N} & \frac{1}{2N-1} & \frac{1}{2N-2} & \cdots & 0\end{bmatrix} = \left[\delta_{jk}^{(1)}\right],$$

$$\begin{aligned}I_m^{(2)} &= \begin{bmatrix}-\frac{\pi^2}{3} & \frac{2}{1^2} & -\frac{2}{2^2} & \cdots & -\frac{2}{(2N)^2}\\ \frac{2}{1^2} & -\frac{\pi^2}{3} & \frac{2}{1^2} & \cdots & \frac{2}{(2N-1)^2}\\ -\frac{2}{2^2} & \frac{2}{1^2} & -\frac{\pi^2}{3} & \cdots & -\frac{2}{(2N-2)^2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -\frac{2}{(2N)^2} & \frac{2}{(2N-1)^2} & -\frac{2}{(2N-2)^2} & \cdots & -\frac{\pi^2}{3}\end{bmatrix}\\ &= \left[\delta_{jk}^{(2)}\right].\end{aligned} \tag{34}$$

The chain rule has been used for the z-derivative of product sinc functions. For example, when $S_k(z) = S(k,h)\circ\phi(z)$,

$$\begin{aligned}\frac{d\left(S_j(z)w(z)\right)}{dz} &= \left(\frac{dS_j(z)}{d\phi(z)}\cdot\frac{d\phi(z)}{dz}\right)w(z)\\ &\quad + S_j(z)\frac{dw(z)}{dz}\\ &= \frac{dS_j(z)}{d\phi}\phi'(z)w(z) + S_j(z)w'(z),\\ \frac{d^2\left(S_j(z)w(z)\right)}{dz^2} &= \frac{d}{dz}\left(\frac{dS_j(z)}{d\phi}\phi'(z)w(z) + S_j(z)w'(z)\right)\\ &= \frac{d^2S_j(z)}{d\phi^2}\left(\phi'(z)\right)^2 w(z)\\ &\quad + \frac{dS_j(z)}{d\phi}\phi''(z)w(z)\\ &\quad + 2\cdot\frac{dS_j(z)}{d\phi}\phi'(z)w'(z) + S_j(z)w''(z).\end{aligned} \tag{35}$$

Now, we are going to develop discrete form for (1). We choose for special case the parameters as follows for the spatial dimension:

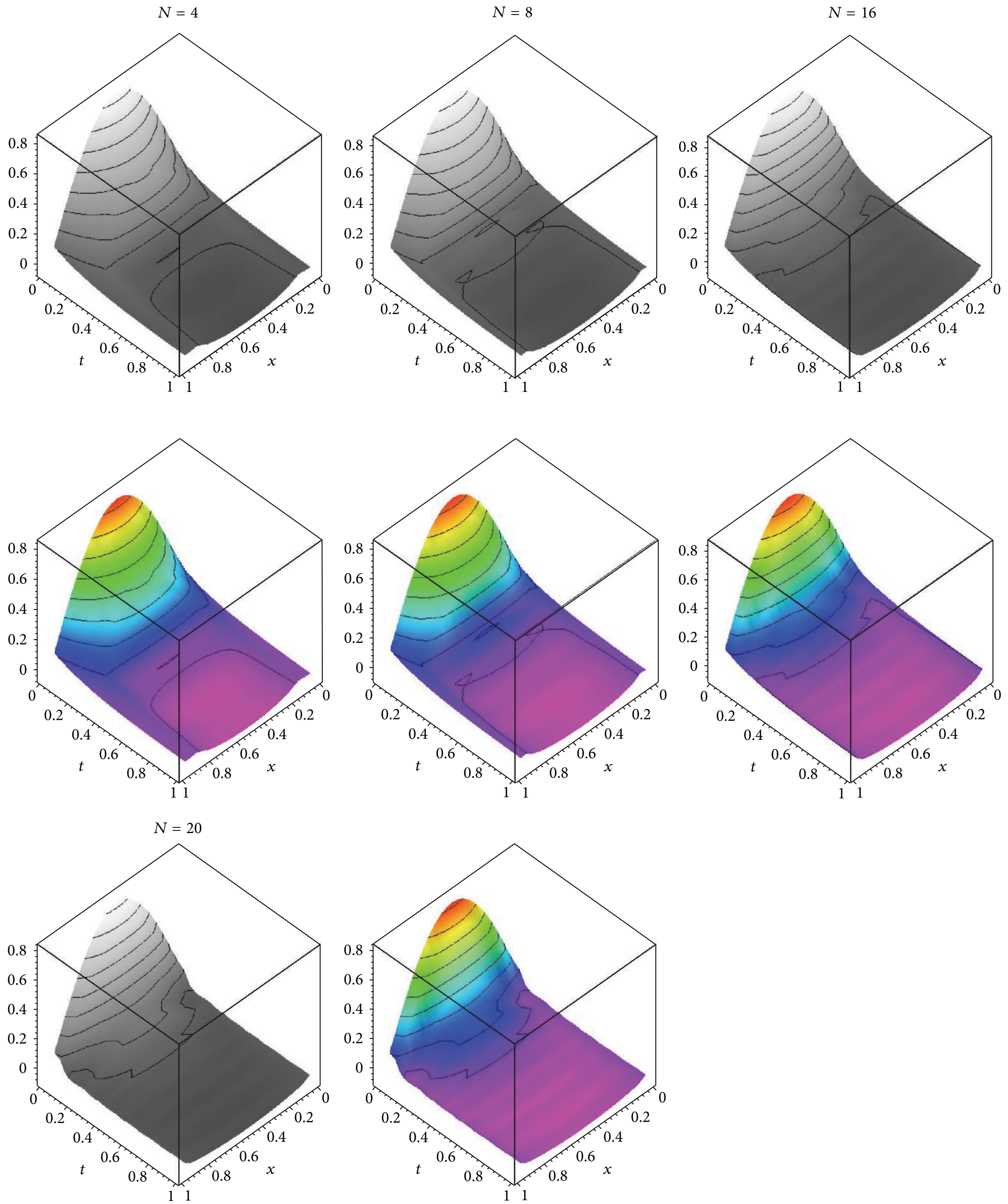
$$\begin{aligned}\phi(z) &= \ln\left(\frac{z}{1-z}\right),\\ w_X(z) &= \frac{1}{\phi'(z)},\\ \frac{1}{\phi'(z)} &= z(1-z),\end{aligned} \tag{36}$$

and for the temporal space as

$$\begin{aligned}\gamma(t) &= \ln(t),\\ w_T(t) &= \frac{1}{\gamma'(t)},\\ \frac{1}{\gamma'(t)} &= t.\end{aligned} \tag{37}$$

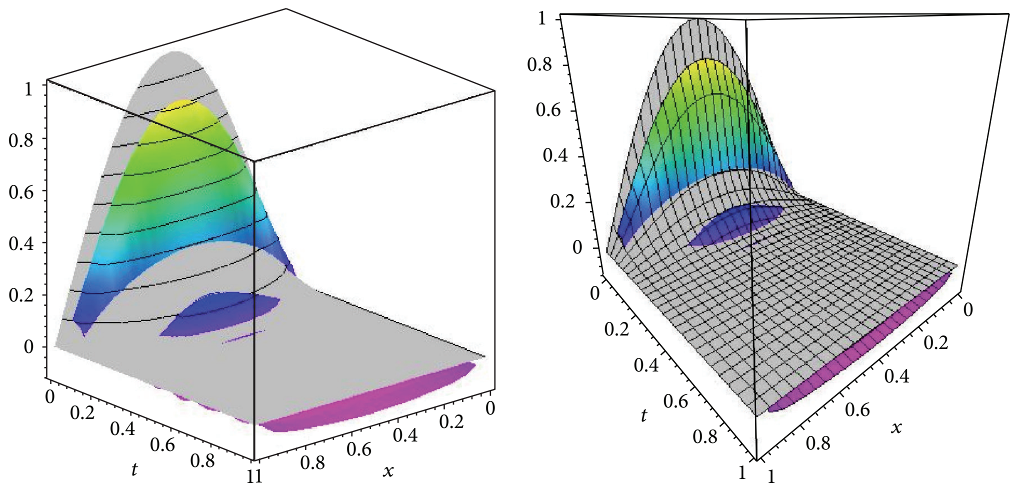
The discrete form of (1) can be given the following form:

$$\begin{aligned}&\langle Lu - F, S_k\cdot S_l\rangle\\ &\quad = \int_{\Gamma_t}\int_{\Gamma_z}(Lu-F)\,S(k,h)\circ\phi(z)\\ &\qquad\cdot w_X(x)\,S(l,s)\circ\gamma(t)\cdot w_T(t)\,dz\,dt\\ &\quad = \int_{\Gamma_t}\int_{\Gamma_z}\left(u_t - u_{xx} - F\right)S(k,h)\circ\phi(z)\\ &\qquad\cdot w_X(x)\,S(l,s)\circ\gamma(t)\cdot w_T(t)\,dz\,dt.\end{aligned} \tag{38}$$

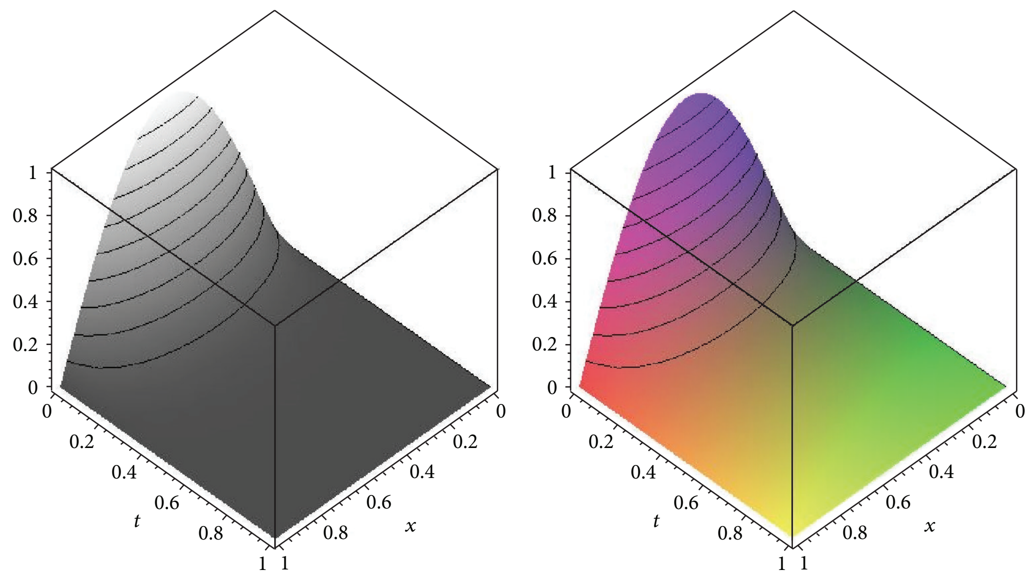


(a) The sinc-Galerkin solutions according to the grid points size N

Figure 1: Continued.



(b) Intersection of surfaces for $N = 20$. The left-side figure is normal perspective and the second one has medium perspective view



(c) The exact solution (or particular solution)

Figure 1: Simulation of approximate solution.

We solve this by taking our approximating basis functions to be

$$S_k(x) = w_X S(k,h) \circ \phi(x), \qquad w_X = \frac{1}{\phi'(x)} = x(1-x),$$
$$\phi(x) = \ln\left(\frac{x}{1-x}\right),$$
$$S_l(t) = w_T S(l,s) \circ \gamma(t), \qquad w_T = \frac{1}{\gamma'(t)} = t,$$
$$\gamma(t) = \ln(t). \tag{39}$$

If we apply sinc-quadrature rules with the help of (32)–(37) on the definite integral given (38) by using (39), we can get the following matrix system.

Let $A_m(u)$ denote a diagonal matrix, whose diagonal elements are $u(x_{-N}), u(x_{-N+1}), \dots, u(x_N)$ and nondiagonal elements are zero. Then (38) reproduces the following matrixes accordingly.

Firstly we set the coefficient matrix as follows:

$$C = \begin{pmatrix} c_{-N,-N} & c_{-N,-N+1} & c_{-N,-N+2} & \cdots & c_{-N,N} \\ c_{-N+1,-N} & c_{-N+1,-N+1} & c_{-N+1,-N+2} & \cdots & c_{-N+1,N} \\ c_{-N+2,-N} & c_{-N+2,-N+1} & c_{-N+2,-N+2} & \cdots & c_{-N+2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{N,-N} & c_{N,-N+1} & c_{N,-N+2} & \cdots & c_{N,N} \end{pmatrix},$$

$$B = -2hI_m^{(0)}\left(A_m(w_X)\right) + I_m^{(1)}\left(A_m\left(w_X'\right)\right) + \frac{I_m^{(2)}}{h},$$

$$G = A_m(w_T)\left[sI_m^{(0)} - I_m^{(1)}\right],$$

Table 2: Numerical results.

	t	Exact solution	Sinc-Galerkin solution	Error
$N = 16, x = 0.6$	0.1	0.35446621870000	0.59102026517764600	0.23655404647764600
	0.11	0.00001833412226	−0.08327827240548060	0.08329660652774060
	0.21	$0.9482992112 \times 10^{-9}$	−0.03068110846399060	0.03068110941228980
	0.31	$0.4904905672 \times 10^{-13}$	−0.01083286679129120	0.01083286679134030
	0.41	$0.2536973471 \times 10^{-17}$	−0.000420168677914081	0.000420168677914083
	0.51	$0.1312203514 \times 10^{-21}$	−0.00108069489771883	0.00108069489771883
	0.61	$0.6787134677 \times 10^{-26}$	−0.00492511086695295	0.00492511086695295
	0.71	$0.3510522275 \times 10^{-30}$	0.00331706758955819	0.00331706758955819
	0.81	$0.1815753976 \times 10^{34}$	−0.00303031084399243	0.00303031084399243
	0.91	$0.9391658013 \times 10^{-39}$	0.00342519240062265	0.00342519240062265

$$D = hA_m\left(\frac{w_X}{\phi'}\right),$$
$$E = sA_m\left(\frac{w_T}{\gamma'}\right). \tag{40}$$

Finally, for the right side function F given (1) can be written in the following matrix form:

$$F = \begin{pmatrix} F_{-N,-N} & F_{-N,-N+1} & F_{-N,-N+2} & \cdots & F_{-N,N} \\ F_{-N+1,-N} & F_{-N+1,-N+1} & F_{-N+1,-N+2} & \cdots & F_{-N+1,N} \\ F_{-N+2,-N} & F_{-N+2,-N+1} & F_{-N+2,-N+2} & \cdots & F_{-N+2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Fc_{N,-N} & F_{N,-N+1} & F_{N,-N+2} & \cdots & F_{N,N} \end{pmatrix}. \tag{41}$$

Using (32)–(37) we arrive at a matrix system given in [1] as follows:

$$D^{-1}BC + CGE^{-1} = F. \tag{42}$$

Finally, by using Maple Computer Algebra Software, the matrix system (42) can be solved by using *LU* or *QR* decomposition method and unknown coefficients can be found. After calculation of C we get approximate solution as follows:

$$u_{x,t} = \sum_{j=-N}^{N}\sum_{k=-N}^{N} c_{jk} S(j,h)\circ\phi(x)\cdot S(k,h)\circ\gamma(t). \tag{43}$$

5. Numerical Simulation

The example in this section will illustrate the sinc method.

Example 4. This problem has been addressed in [1]. The following equation is given in Dirichlet-type boundary condition:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad 0 < x < 1,\ 0 < t,$$
$$u(0,t) = u(1,t) = 0, \tag{44}$$
$$u(x,0) = \sin(\pi x).$$

The particular solution of (44) can be calculated via separation variables rules and can be given as follows:

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x). \tag{45}$$

For (44) we choose sinc components here in the following:

$$h = s = \frac{0.75}{\sqrt{N}}, \qquad x_k = \frac{e^{kh}}{1+e^{kh}}, \qquad t_l = e^{sl},$$
$$\phi(x) = \ln\left(\frac{x}{1-x}\right), \qquad \gamma(t) = \ln(t), \qquad w_X = \frac{1}{\phi'(x)},$$
$$w_T = \frac{1}{\gamma'(t)}. \tag{46}$$

According to the above parameters, the approximate solution simulation of (44) has been given in Figure 1 and numerical results also can be found in Table 2.

6. Conclusions

We have developed a Maple algorithm to solve and simulate second-order parabolic PDEs with Dirichlet-type boundary conditions based on sinc-Galerkin approximation on some closed real intervals and the method has been compared with the exact solutions. When compared with other computational approaches, this method turns out to be more efficient in the sense that selection parameters and changing boundary conditions and also giving different problems to the algorithms. The accuracy of the solutions improves by increasing the number of sinc grid points N. The method presented here is simple and uses sinc-Galerkin method that gives a numerical solution, which is valid for various boundary conditions. Several PDEs have been solved by using our technique in less than 20 seconds. All computations and graphical representations have been prepared automatically by our algorithm.

Appendix

See Algorithm 1.

Conflict of Interests

The author declares that he has no conflict of interests.

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Research Article

Computational Solution of a Fractional Integro-Differential Equation

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Although differential transform method (DTM) is a highly efficient technique in the approximate analytical solutions of fractional differential equations, applicability of this method to the system of fractional integro-differential equations in higher dimensions has not been studied in detail in the literature. The major goal of this paper is to investigate the applicability of this method to the system of two-dimensional fractional integral equations, in particular to the two-dimensional fractional integro-Volterra equations. We deal with two different types of systems of fractional integral equations having some initial conditions. Computational results indicate that the results obtained by DTM are quite close to the exact solutions, which proves the power of DTM in the solutions of these sorts of systems of fractional integral equations.

1. Introduction

The subject of the present paper is to investigate the applicability of the differential transform method to the systems of the two-dimensional Volterra integro-differential equations of the second kind. To the best of our knowledge, the Volterra-integro differential equations considered in this paper was not studied with any method in the literature. Therefore, solving a new equation with differential transform method is our main purpose in this paper. For this propose, we consider the system of two-dimensional fractional Volterra integro-differential equations in the form of

$$\begin{aligned} &F_i\left(D_{11}^{\alpha}u_1(x,t), D_{11}^{\beta}u_1(x,t), \dots, D_{1m}^{\alpha}u_m(x,t),\right.\\ &\qquad \left. D_{1m}^{\beta}u_m(x,t)\right)\\ &\quad - \lambda_i \int_{t_0}^{t}\int_{x_0}^{x}\sum_{j=0}^{n} v_{ij}(x,t)\, w_{ij}(y,z)\, G_i\left(D_{11}u_1(y,z), \dots,\right.\\ &\qquad \left. D_{1m}u_m(y,z)\right) dy\, dz = f_i(x,t),\\ &\qquad i = 1, \dots, m, \end{aligned} \tag{1}$$

where α, β is a parameter describing the fractional derivative. If $0 < \alpha, \beta \leq 1$, then the resulting system of fractional integro-differential equations is known as a system of the two-dimensional fractional Volterra integro-differential equations of the second kind.

Fractional calculus basically deals with a generalization of the concept of the ordinary and partial derivative (or differentiation) and integration to arbitrary order including a fractional order. Although the origin of the subject dates back to almost a hundred years ago, recently, this subject has been broadly employed in the various fields of engineering and science. Fractional calculus has a broad range of application areas including nonlinear control theory, mathematical biology, plasma physics and fusion, computational fluid mechanics, and images processing.

Integral equations are useful and significant equations in many applications. Problems in which integral equations are encountered include electromagnetic waves, radiative energy transfer, and the oscillation problems. The Volterra integral equations are a special sort of integral equations which are significantly important and useful equations having broad application areas in different branches of science. The Volterra integral equations were first introduced by Volterra and then studied by Lalescu in his thesis. The Volterra integral equations find application in many different areas including sorption kinetics, demography, viscoelastic materials, oscillation of a spring, financial mathematics, stochastic dynamical systems, and mathematical biology.

Fractional differential equations (e.g., see [1, 2]) and fractional integral equations (e.g., see [3]) are a significant research area of recent times. The organization of this paper might be briefly summarized as describing the problem, the DTM method, and applying the method to the problem. Having defined the problem previously, next, we describe the differential transform method shortly.

2. Differential Transform Method

The DTM constructs analytical solutions of fractional differential equations in an iterative way in the form of polynomials. DTM is different from the traditional higher order Taylor series techniques which usually demand symbolic computations.

Consider a function of two variables $u(x, y)$, and suppose that it can be represented as a product of two single-variable functions as $u(x, y) = f(x)g(y)$. Now, using the properties of DTM, it is not hard to show that the function $u(x, y)$ can be represented as

$$\begin{aligned} u(x, y) &= \sum_{k=0}^{\infty} F_{\alpha}(k)(x - x_0)^{k\alpha} \sum_{h=0}^{\infty} G_{\beta}(h)(y - y_0)^{h\beta} \\ &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha\beta}(k, h)(x - x_0)^{k\alpha}(y - y_0)^{h\beta}, \end{aligned} \tag{2}$$

where $0 < \alpha, \beta \leq 1$, $U_{\alpha\beta}(k, h) = F_{\alpha}(k)G_{\beta}(h)$ is called the spectrum of $u(x, y)$. The generalized two-dimensional differential transform of the function $u(x, y)$ is given by

$$\begin{aligned} U_{\alpha,\beta}(k, h) &= \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} \\ &\quad \times \left[\left(D^{\alpha}_{*x_0}\right)^k \left(D^{\beta}_{*y_0}\right)^h u(x, y)\right]_{(x_0, y_0)}, \end{aligned} \tag{3}$$

where $(D^{\alpha}_{x_0})^k = D^{\alpha}_{x_0} D^{\alpha}_{x_0} \cdots D^{\alpha}_{x_0}$, k-times. Let us notice that if $u(x, y) = D^{\alpha}_{x_0} v(x, y)$, $0 < \alpha \leq 1$, then $U_{\alpha,\beta}(k, h) = (\Gamma(\alpha(k + 1) + 1)/\Gamma(\alpha k + 1))V_{\alpha,\beta}(k + 1, h)$, and the generalized differential transform (3) might be written as

$$\begin{aligned} U_{\alpha,\beta}(k, h) &= \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} \\ &\quad \times \left[D^{\alpha k}_{*x_0} \left(D^{\beta}_{*y_0}\right)^h u(x, y)\right]_{(x_0, y_0)}. \end{aligned} \tag{4}$$

If $u(x, y) = D^{\gamma}_{x_0} v(x, y)$, $m - 1 < \gamma \leq m$ and $v(x, y) = f(x)g(y)$, then

$$U_{\alpha,\beta}(k, h) = \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}\left(k + \frac{\gamma}{\alpha}, h\right). \tag{5}$$

If $u(x, y, t) = D^{\alpha}_{*x_0} v(x, y, t)$, $0 < \alpha \leq 1$, then

$$U_{\alpha,\beta,\gamma}(k, h, m) = \frac{\Gamma(\alpha(k + 1) + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta,m}(k + 1, h, m). \tag{6}$$

If $u(x, y) = a(x, y)(\partial^2 v(x, y)/\partial v^2(x, y))$, then

$$\begin{aligned} U(k, h) &= \sum_{i=0}^{k} \sum_{j=0}^{h} (k - i + 2)(k - i + 1) A(i, j) \\ &\quad \times U(k - i + 2, h - j). \end{aligned} \tag{7}$$

The proofs of some of these properties can be found in [4]. The application of DTM was successfully extended to obtain analytical approximate solutions to some other differential equations of fractional order. Interested reader can take a look at the related papers at [5–7]. As a related work, we can show the work by Bandrowski et al. who studied the numerical solutions of the fractional perturbed Volterra equations in [8]. Next, we illustrate the application of DTM to the systems of fractional integral equations.

3. Computational Applications

Example 1. Consider the system of integro-differential equations:

$$\begin{aligned} &\frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} + v(x, t) \\ &\qquad - \int_0^t \int_0^x y \sin z \left(u^2(y, z) - v^2(y, z)\right) dy\, dz \\ &\quad = \frac{1}{12}\left(1 + 2\cos^3 t - 3\cos t\right) x^4 \\ &\frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} + \frac{\partial^{\alpha} v(x, t)}{\partial t^{\alpha}} + u(x, t) \\ &\qquad - \int_0^t \int_0^x y \cos z \left(u(y, z) - \frac{\partial v(y, z)}{\partial z}\right) dy\, dz \\ &\quad = x(2\cos t - \sin t), \end{aligned} \tag{8}$$

with $x, t \in [0, 1]$ $u_1(x, 0) = x$, $u_2(x, 0) = 0$, $x \in [0, 1]$ which has the exact solutions $u_1(x, t) = x\cos t$ and

$u_2(x,t) = x\sin t$. Taking the two-dimensional transform of this equation by using the related theorem, we have

$$\frac{\Gamma(\alpha(m+1)+1)}{\Gamma(\beta m+1)}U_{\alpha,1}(m+1,n) - V_{\alpha,1}(m,n)$$
$$-\frac{1}{mn}\sum_{l}^{n-1}\sum_{k}^{m-1}\sum_{s}^{l}\sum_{r}^{k}\sum_{q}^{n-l-1}\sum_{p}^{m-k-1}\frac{\delta_{r,1}\delta_{s,0}\delta_{k-r,0}}{(l-s)!}\sin\left(\frac{(l-s)\pi}{2}\right)$$
$$\times\left[U_{\alpha,1}(p,q)U_{\alpha,1}(m-k-q-1,n-l-q-1)\right.$$
$$\left.-V_{\alpha,1}(m-k-q-1,n-l-q-1)\right]$$
$$=\frac{1}{6}\sum_{l}^{n}\sum_{k}^{m}\sum_{s}^{m-l}\sum_{r}^{m-k}\sum_{q}^{n-l-m}\sum_{p}^{m-k-r}\frac{\delta_{k,4}\delta_{l,0}\delta_{r,0}\delta_{q,0}\delta_{m-k-r-q,0}}{s!q!(n-l-s-q)!}$$
$$\times\cos\left(\frac{s\pi}{2}\right)\cos\left(\frac{q\pi}{2}\right)\cos\left(\frac{(n-l-s-q)\pi}{2}\right)$$
$$+\frac{1}{12}\delta_{m,4}\delta_{n,0} - \frac{1}{4}\sum_{l}^{n}\sum_{k}^{m}\frac{\delta_{k,4}\delta_{l,0}\delta_{m-k,0}}{(n-l)!}\cos\left(\frac{(n-l)\pi}{2}\right),$$
$$m = 1,\dots,N-1, n = 1,\dots,N,$$
$$\frac{\Gamma(\alpha(m+1)+1)}{\Gamma(\alpha m+1)}U_{\alpha,1}(m,n+1)$$
$$+\frac{\Gamma(\alpha(m+1)+1)}{\Gamma(\alpha m+1)}V_{\alpha,1}(m,n+1) - U_{\alpha,1}(m,n)$$
$$-\frac{1}{mn}\sum_{l}^{n-1}\sum_{k}^{m-1}\sum_{s}^{l}\sum_{r}^{k}\frac{\delta_{r,1}\delta_{s,0}\delta_{k-r,0}}{(l-s)!}\cos\left(\frac{(l-s)\pi}{2}\right)$$
$$\times\left[U_{\alpha,1}(m-k-1,n-l-1)\right.$$
$$\left.-(n-1)V_{\alpha,1}(m-k-1,n-l)\right]$$
$$=\sum_{l}^{n}\sum_{k}^{m}\frac{\delta_{k,1}\delta_{l,0}\delta_{m-k,0}}{(n-l)!}\left[2\cos\left(\frac{(n-l)\pi}{2}\right)-\sin\left(\frac{(n-l)\pi}{2}\right)\right],$$
$$m = 1,\dots,N, n = 1,\dots,N-1. \quad (9)$$

Substituting initial conditions in the equations, we obtain

$$u_N(x,t) = x\left(1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots+\frac{t^{2N\alpha}}{\Gamma(2N\alpha+1)}\right),$$
$$v_N(x,t) = x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{t^{5\alpha}}{\Gamma(5\alpha)}+\cdots+\frac{(-1)^{N-1}t^{(2N-1)\alpha}}{\Gamma((2N-1)\alpha+1)}\right). \quad (10)$$

As a special case, if $\alpha = 1$, we get the following exact solutions:

$$u_1(x,t) = x\cos t, \qquad u_2(x,t) = x\sin t. \quad (11)$$

Example 2. In this example, we consider that

$$\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}} - v(x,t) - \int_0^t\int_0^x y^2u^2(y,z)v^2(y,z)\,dy\,dz$$
$$= e^t - te^{-t} - \frac{1}{15}x^5t^3,$$
$$\frac{\partial^{\beta}v(x,t)}{\partial x^{\beta}} + u(x,t) - \int_0^t\int_0^x z^2u(y,z)v(y,z)\,dy\,dz$$
$$= te^t - \frac{1}{8}x^2t^4, \quad (12)$$

for $x, t \in [0,1]$ and initial conditions $u(x,0) = 0$, $v(0,t) = te^{-t}$, $t \in [0,1]$ which have the exact solutions $u(x,t) = xe^t$ and $v(x,t) = te^{-t}$. Using similar methods as in Example 1, we get

$$\frac{\Gamma(\beta(m+1)+1)}{\Gamma(\beta m+1)}U_{1,\beta}(m+1,n) - V_{1,\beta}(m,n)$$
$$-\frac{1}{mn}\sum_{l}^{n-1}\sum_{k}^{m-1}\sum_{s}^{n-l-1}\sum_{r}^{m-k-1}\sum_{j}^{s}\sum_{i}^{s}\sum_{p}^{n-l-s-1}\sum_{q}^{m-k-r-1}$$
$$\times\left[\delta_{k,2}\delta_{l,0}U_{1,\beta}(i,j)U_{1,\beta}(r-i,s-j)V_{1,\beta}(p,q)\right.$$
$$\left.V_{1,\beta}(m-k-r-q-1,n-l-s-q-1)\right]$$
$$-\frac{1}{n!}\delta_{m,0} - \frac{(-1)^{n-1}}{(n-1)!} - \frac{1}{15}\delta_{m,5}\delta_{n,3},$$
$$m = 1,\dots,N-1,\ n = 1,\dots,N \quad (13)$$

and for the second equation,

$$\frac{\Gamma(\beta(m+1)+1)}{\Gamma(\beta m+1)}V_{1,\beta}(m+1,n) + U_{1,\beta}(m,n)$$
$$-\frac{1}{mn}\sum_{l}^{n-1}\sum_{k}^{m-1}\sum_{s}^{n-l-1}\sum_{r}^{m-k-1}$$
$$\times\left[\delta_{k,0}\delta_{l,2}U_{1,\beta}(i,j)U_{1,\beta}(r,s)V_{1,\beta}(p,q)\right.$$
$$\left.V_{1,\beta}(m-k-r-1,n-l-s-1)\right]$$
$$=\frac{1}{n!}\delta_{m,1} - \frac{1}{8}\delta_{m,2}\delta_{n,4}, \quad m = 1,\dots,N-1,\ n = 1,\dots,N \quad (14)$$

From the initial conditions, we get

$$U(0,n) = 0, \quad n = 0,1,2,\dots,N,$$
$$V(0,0) = 0, \qquad V(0,n) = \frac{(-1)^{n-1}}{(n-1)!}, \quad n = 1,2,\dots,N. \quad (15)$$

Substituting $x = 0$ in the second equation and using the second condition, we obtain

$$\frac{\partial u_1}{\partial x}(0,t) = e^t. \quad (16)$$

Substituting initial conditions into the equations, we obtain

$$\begin{aligned} u_N(x,t) &= x\left(1+\frac{t}{\Gamma(1+1)}+\frac{t^2}{\Gamma(2+1)}+\cdots\right.\\ &\qquad\left.+\frac{t^N}{\Gamma(N+1)}\right),\\ v_N(x,t) &= \frac{t}{\Gamma(+1)}-\frac{t^2}{\Gamma(2+1)}\\ &\qquad+\frac{t^{3\beta}}{\Gamma(3+1)}+\cdots+\frac{(-1)^{N-1}t^N}{\Gamma(N)}, \end{aligned} \tag{17}$$

where u_N and v_N are approximate solutions of u and v.

For the special case ($\beta = 1$), we can reproduce the series solution of Example 2 and the solution in a closed form $u(x,t) = xe^t$ and $v(x,t) = te^{-t}$.

4. Conclusions

The application of the differential transform method has been successfully employed to obtain the approximate analytical solutions for two classes of systems of the two-dimensional fractional Volterra integro-differential equations of the second kind. The method was used in a direct way without using linearization, perturbation, or restrictive assumptions. When $\alpha = 1$ and $\beta = 1$, we conclude that our approximate solutions are in good agreement with the exact values.

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Research Article

Fast Spectral Collocation Method for Solving Nonlinear Time-Delayed Burgers-Type Equations with Positive Power Terms

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Since the collocation method approximates ordinary differential equations, partial differential equations, and integral equations in physical space, it is very easy to implement and adapt to various problems, including variable coefficient and nonlinear differential equations. In this paper, we derive a Jacobi-Gauss-Lobatto collocation method (J-GL-C) to solve numerically nonlinear time-delayed Burgers-type equations. The proposed technique is implemented in two successive steps. In the first one, we apply $(N-1)$ nodes of the Jacobi-Gauss-Lobatto quadrature which depend upon the two general parameters $(\theta, \vartheta > -1)$, and the resulting equations together with the two-point boundary conditions constitute a system of $(N-1)$ ordinary differential equations (ODEs) in time. In the second step, the implicit Runge-Kutta method of fourth order is applied to solve a system of $(N-1)$ ODEs of second order in time. We present numerical results which illustrate the accuracy and flexibility of these algorithms.

1. Introduction

Spectral methods have emerged as powerful techniques used in applied mathematics and scientific computing to numerically solve linear and nonlinear differential equations [1–4] and integral equations [5–7]. Also, they have become increasingly popular for solving fractional differential equations [8–10]. The main idea of spectral methods is to put the solution of the problem as a sum of certain basic functions and then to choose the coefficients in the sum in order to minimize the difference between the exact solution and the approximate one as well as possible. The choice of test functions leads to the three well-known types of spectral methods, namely, the Galerkin, tau, and collocation methods [11–14]. Spectral collocation method has an exponential convergence rate, which is very useful in providing highly accurate solutions to nonlinear differential equations even using a small number of grids.

The Jacobi polynomials satisfy the orthogonality condition on the interval $[-1, 1]$ with respect to the weight function $(1-x)^{\theta}(l+x)^{\vartheta}$. There are many special cases of the Jacobi polynomials such as Gegenbauer, Legendre, Zernike, ultraspherical, and Chebyshev polynomials [15]. In recent decades, the use of Jacobi polynomials for solving differential equations has gained increasing popularity due to obtaining the solution in terms of the Jacobi parameters θ and ϑ (see, e.g., [16, 17]).

Time-delay partial differential equations are a type of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. They have a wide range of applications in science and engineering such as physical, chemical, and biological sciences [18, 19]. Also, time-delayed nonlinear partial differential equations described the propagation and transport phenomena or population dynamics.

The solution of delay differential equations has been presented by many authors, but we briefly review some of them. In [20], Ghasemi and Kajani developed a numerical approach of time-varying delay systems using Chebyshev wavelets. Hybrid functions together with Legendre polynomials are investigated in [21–23] to obtain efficient numerical solution of delay systems. Sedaghat et al. [24] presented a numerical scheme based on Chebyshev polynomials to treat the delay differential equations of pantograph type. The authors in [25] proposed a Bernoulli operational matrix method for solving generalized pantograph equation. Recently, Ali et al. [26] implemented spectral Legendre approach for solving pantograph-type differential and integral equations with studying the error analysis of the method. More recently, the work of Trif [27] discussed the application of the Tau method based on operational matrix of Chebyshev polynomials for solving DDEs of pantograph-type.

Recently, the authors of [28] and [29] presented some new travelling wave solutions of Burgers equation with finite transport memory and the Korteweg-de Vries-Burgers equation, respectively. Pandey et al. [30] investigated Du Fort-Frankel finite difference approach for solving Burgers equation in finite domain. Meanwhile, in [31], Sun and Wu presented and applied an efficient numerical solution for the Burgers equation based on a difference scheme in an unbounded domain. More recently, a differential quadrature scheme for Burgers equations was proposed in [32]. The idea of commutative hypercomplex mathematics and the homotopy perturbation method were combined to investigate solutions of time-delayed Burgers equation by Rostamy and Karimi [33]. The Darboux transformation was described in [34] to determine the exact solutions to the Burgers equation. Caglar and Ucar [35] proposed the nonpolynomial cubic spline scheme to develop a numerical solution of Burgers equation. Regarding the two-dimensional Burgers equation, Wang and Zhao [36] presented a novel combination of two-dimensional Haar wavelet functions based on tensorial products for solving two-dimensional Burgers equation.

Time-delayed Burgers equation has a wide range of application in many areas of applied sciences, for example, forest fire, population growth models, and Neolithic transitions [37, 38]. The time-delayed Burgers-Fisher equation is a very important model to forest fire, population growth, Neolithic transitions, the interaction between the reaction mechanism, convection effect and diffusion transport, and so forth [37]. Fahmy et al. [39] used improved tanh function, variational iteration, and the Adomian decomposition methods to present some exact solutions and numerical solutions of the time-delayed Burgers equation. With the aid of a subsidiary high-order ODE, Zhang et al. [37] obtained the exact solutions of the generalized time-delayed Burgers-Fisher equation with positive fractional power terms. Jawad et al. [40] introduced the exact solution of time-delayed Burgers equation using complex tanh method. Moreover, (G'/G)-expansion method is applied by Kim and Sakthivel [41] to find the exact solutions of time-delayed Burgers equation.

The goal of this paper is to propose an orthogonal collocation scheme for solving three nonlinear time-delay partial differential equations based on Jacobi family in which the roots of the Jacobi orthogonal polynomials whose distributions can be tuned by two parameters θ and ϑ. Firstly, we apply the Jacobi-Gauss-Lobatto collocation (J-GL-C) method to the model equation for discretizing spatial derivatives, using $(N-1)$ nodes of the Jacobi-Gauss-Lobatto quadrature which depends upon the two general parameters $(\theta, \vartheta > -1)$; these equations together with the two-point boundary conditions constitute system of $(N-1)$ ordinary differential equations (ODEs) in time. Secondly, the Runge-Kutta method of fourth order is investigated for the time integration of the resulting system of $(N-1)$ nonlinear second-order ODEs.

Indeed, the main advantage of the proposed technique is that the Legendre and Chebyshev collocation methods and other methods can be obtained as special cases from our proposed technique. Comparison of the results obtained by various choices of Jacobi parameters θ and ϑ reveals that the present method is very effective and convenient for all choices of θ and ϑ. Finally, the accuracy of the proposed method is showed by test problems. From the results, these algorithms are extremely efficient and accurate for solving nonlinear time-delayed Burgers'-type equations.

The rest of this paper is structured as follows. In the next section, some basic properties of Jacobi polynomials, which are required in our paper, are given. Section 3 is devoted to the development of Gauss-Lobatto collocation technique for a general form of time-delay partial differential equation based on the Jacobi polynomials, and in Section 4 the proposed method is applied to obtain some numerical results for three problems of time-delay partial differential equations with known exact solutions. Finally, a brief conclusion and some remarks are provided in Section 5.

2. Preliminaries

In this section, we briefly recall some properties of the Jacobi polynomials $(J_k^{(\theta,\vartheta)}(x), k = 0, 1, \dots \theta > -1, \vartheta > -1)$, which are satisfying the following relations:

$$\begin{aligned} J_k^{(\theta,\vartheta)}(-x) &= (-1)^k J_k^{(\theta,\vartheta)}(x), \\ J_k^{(\theta,\vartheta)}(-1) &= \frac{(-1)^k \Gamma(k+\vartheta+1)}{k!\Gamma(\vartheta+1)}, \\ J_k^{(\theta,\vartheta)}(1) &= \frac{\Gamma(k+\theta+1)}{k!\Gamma(\theta+1)}. \end{aligned} \tag{1}$$

The qth derivative of Jacobi polynomials of degree $k(J_k^{(\theta,\vartheta)}(x))$ can be given by

$$D^{(q)} J_k^{(\theta,\vartheta)}(x) = \frac{\Gamma(j+\theta+\vartheta+q+1)}{2^q \Gamma(j+\theta+\vartheta+1)} J_{k-q}^{(\theta+q,\vartheta+q)}(x). \tag{2}$$

These polynomials are the only polynomials arising as eigenfunctions of the following singular Sturm-Liouville equation:

$$\begin{aligned} &\left(1-x^2\right)\phi''(x) + [\vartheta - \theta + (\theta+\vartheta+2)x]\phi'(x) \\ &\quad + n(n+\theta+\vartheta+1)\phi(x) = 0. \end{aligned} \tag{3}$$

Let $w^{(\theta,\vartheta)}(x) = (1-x)^\theta (1+x)^\vartheta$; then we define the weighted space $L^2_{w^{(\theta,\vartheta)}}$ as usual. The inner product and the norm of $L^2_{w^{(\theta,\vartheta)}}$ with respect to the weight function are defined as follows:

$$\begin{aligned} (u,v)_{w^{(\theta,\vartheta)}} &= \int_{-1}^{1} u(x)\, v(x)\, w^{(\theta,\vartheta)}(x)\, dx, \\ \|u\|_{w^{(\theta,\vartheta)}} &= (u,u)^{1/2}_{w^{(\theta,\vartheta)}}. \end{aligned} \tag{4}$$

The set of Jacobi polynomials forms a complete $L^2_{w^{(\theta,\vartheta)}}$-orthogonal system, and

$$\left\|J_k^{(\theta,\vartheta)}\right\|_{w^{(\theta,\vartheta)}} = h_k = \frac{2^{\theta+\vartheta+1}\Gamma(k+\theta+1)\Gamma(k+\vartheta+1)}{(2k+\theta+\vartheta+1)\Gamma(k+1)\Gamma(k+\theta+\vartheta+1)}. \quad (5)$$

3. Jacobi Spectral Collocation Method

The main objective of this section is to develop the J-GL-C method to numerically solve the nonlinear time-delayed Burgers-type equations:

$$\begin{aligned}\tau\partial_{tt}v(y,t) &+ p(1-\tau+qv(y,t))\partial_t v(y,t)\\ &= \zeta\partial_{yy}v(y,t) - \lambda(v(y,t))^s \\ &\quad\times\partial_y v(y,t) + qv(y,t)(1-v(y,t)),\end{aligned} \quad (6)$$

where

$$(y,t)\in D\times[0,T], \quad D=\{y: A\le y\le B\}, \quad (7)$$

with the boundary-initial conditions

$$v(A,t)=g_1(t), \qquad v(B,t)=g_2(t), \quad (8)$$

$$v(y,0)=f_1(y), \qquad \partial_t v(y,0)=f_2(y), \quad y\in D, \quad (9)$$

where p, q, s, and λ are real numbers and $\tau > 0$ is the time delay. Now, suppose that the change of variables $x=(2/(B-A))y+(A+B)/(A-B)$, $u(x,t)=v(y,t)$, which will be used to transform problem (6)–(9) into another one in the classical interval $[-1,1]$ for the space variable, to directly implement collocation method based on Jacobi family defined in $[-1,1]$,

$$\begin{aligned}\tau\partial_{tt}u(x,t) &+ p(1-\tau+qu(x,t))\partial_t u(x,t)\\ &= \zeta\left(\frac{2}{B-A}\right)^2\partial_{xx}u(x,t) - \lambda(u(x,t))^s\\ &\quad\times\left(\frac{2}{B-A}\right)\partial_x u(x,t) + qu(x,t)(1-u(x,t)),\end{aligned} \quad (10)$$

where

$$(x,t)\in D^*\times[0,T], \quad D^*=\{x:-1\le x\le 1\}, \quad (11)$$

with the boundary-initial conditions

$$\begin{aligned}&u(-1,t)=g_1(t), \qquad u(1,t)=g_2(t),\\ &u(x,0)=f_3(x), \quad \partial_t u(x,0)=f_4(x), \quad x\in D^*.\end{aligned} \quad (12)$$

The aim of this work is to consider the advantage of the collocation point distribution in a specified domain $[-1,1]$ using the roots of the Jacobi orthogonal polynomials whose distributions can be tuned by two parameters, θ and ϑ. Now, we outline the main step of the J-GL-C method for solving nonlinear time-delayed Burgers-type equation. Let us expand the dependent variable in a Jacobi series,

$$u(x,t)=\sum_{j=0}^{N}a_j(t)J_j^{(\theta,\vartheta)}(x), \quad (13)$$

and in virtue of (4)-(5), we deduce that

$$a_j(t)=\frac{1}{h_j}\int_{-1}^{1}u(x,t)w^{(\theta,\vartheta)}(x)J_j^{(\theta,\vartheta)}(x)\,dx. \quad (14)$$

To evaluate the previous integral accurately, we present the Jacobi-Gauss-Lobatto quadrature. For any $\phi\in S_{2N+1}(-1,1)$,

$$\int_{-1}^{1}w^{(\theta,\vartheta)}(x)\phi(x)\,dx=\sum_{j=0}^{N}\varpi_{N,j}^{(\theta,\vartheta)}\phi\left(x_{N,j}^{(\theta,\vartheta)}\right), \quad (15)$$

where $S_N(-1,1)$ is the set of polynomials of degree less than or equal to N, $x_{N,j}^{(\theta,\vartheta)}$ $(0\le j\le N)$ and $\varpi_{N,j}^{(\theta,\vartheta)}$ $(0\le j\le N)$ are the nodes and the corresponding Christoffel numbers of the Jacobi-Gauss-Lobatto quadrature formula on the interval $(-1,1)$, respectively.

In accordance with (15), the coefficients $a_j(t)$ in terms of the solution at the collocation points can be approximated by

$$a_j(t)=\frac{1}{h_j}\sum_{i=0}^{N}J_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right)\varpi_{N,i}^{(\theta,\vartheta)}u\left(x_{N,i}^{(\theta,\vartheta)},t\right). \quad (16)$$

Therefore, (13) can be rewritten as

$$\begin{aligned}u(x,t) &= \sum_{i=0}^{N}\left(\sum_{j=0}^{N}\frac{1}{h_j}J_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right)J_j^{(\theta,\vartheta)}(x)\varpi_{N,i}^{(\theta,\vartheta)}\right)\\ &\quad\times u\left(x_{N,i}^{(\theta,\vartheta)},t\right).\end{aligned} \quad (17)$$

Furthermore, if we differentiate (17) once (using (2)) and evaluate it at all Jacobi-Gauss-Lobatto collocation points, it is easy to compute the first spatial partial derivative in terms of the values at these collocation points as

$$\begin{aligned}u_x\left(x_{N,n}^{(\theta,\vartheta)},t\right) &= \sum_{i=0}^{N}\left(\sum_{j=0}^{N}\frac{j+\theta+\vartheta+1}{2h_j}J_j^{(\theta,\vartheta)}\right.\\ &\quad\left.\times\left(x_{N,i}^{(\theta,\vartheta)}\right)J_{j-1}^{(\theta+1,\vartheta+1)}\left(x_{N,n}^{(\theta,\vartheta)}\right)\varpi_{N,i}^{(\theta,\vartheta)}\right)\\ &\quad\times u\left(x_{N,i}^{(\theta,\vartheta)},t\right), \quad n=0,1,\dots,N,\end{aligned} \quad (18)$$

or it can be shortened to

$$u_x\left(x_{N,n}^{(\theta,\vartheta)},t\right)=\sum_{i=0}^{N}A_{ni}u\left(x_{N,i}^{(\theta,\vartheta)},t\right), \quad n=0,1,\dots,N, \quad (19)$$

where

$$A_{ni} = \sum_{j=0}^{N} \frac{j+\theta+\vartheta+1}{2h_j} J_j^{(\theta,\vartheta)} \times \left(x_{N,i}^{(\theta,\vartheta)}\right) J_{j-1}^{(\theta+1,\vartheta+1)} \left(x_{N,n}^{(\theta,\vartheta)}\right) \varpi_{N,i}^{(\theta,\vartheta)}. \tag{20}$$

Similar steps can be applied to the second spatial partial derivative to get

$$\begin{aligned} u_{xx}\left(x_{N,n}^{(\theta,\vartheta)}, t\right) &= \sum_{i=0}^{N} \left(\sum_{j=0}^{N} \frac{(j+\theta+\vartheta+2)(j+\theta+\vartheta+1)}{4h_j} \times J_j^{(\theta,\vartheta)} \left(x_{N,i}^{(\theta,\vartheta)}\right) J_{j-2}^{(\theta+2,\vartheta+2)} \left(x_{N,n}^{(\theta,\vartheta)}\right) \varpi_{N,i}^{(\theta,\vartheta)} \right) \\ &\quad \times u\left(x_{N,i}^{(\theta,\vartheta)}, t\right) = \sum_{i=0}^{N} B_{ni} u\left(x_{N,i}^{(\theta,\vartheta)}, t\right), \\ &\qquad n = 0, 1, \ldots, N, \end{aligned} \tag{21}$$

where

$$B_{ni} = \sum_{j=0}^{N} \frac{(j+\theta+\vartheta+2)(j+\theta+\vartheta+1)}{4h_j} J_j^{(\theta,\vartheta)} \times \left(x_{N,i}^{(\theta,\vartheta)}\right) J_{j-2}^{(\theta+2,\vartheta+2)} \left(x_{N,n}^{(\theta,\vartheta)}\right) \varpi_{N,i}^{(\theta,\vartheta)}. \tag{22}$$

In the proposed Jacobi-Gauss-Lobatto collocation method, the residual of (6) is set to zero at $N-1$ of Jacobi-Gauss-Lobatto points; moreover, the boundary conditions (8) will be enforced at the two collocation points -1 and 1. Therefore, adopting (19)–(22) enables one to write (6)–(8) in the form

$$\begin{aligned} &\tau \ddot{u}_n(t) + p\left(1-\tau+qu_n(t)\right)\dot{u}_n(t) \\ &\quad = \zeta \left(\frac{2}{B-A}\right)^2 \sum_{i=0}^{N} B_{ni} u_i(t) \\ &\qquad - \lambda \left(\frac{2}{B-A}\right) u_n(t)^s \sum_{i=0}^{N} A_{ni} u_i(t) \\ &\qquad + q\, u_n(t)\left(1-u_n(t)\right), \end{aligned} \tag{23}$$

where

$$\begin{aligned} u_k(t) = u\left(x_{N,k}^{(\theta,\vartheta)}, t\right), \quad & k = 1, \ldots, N-1, \\ & n = 1, \ldots, N-1. \end{aligned} \tag{24}$$

This provides an $(N-1)$ system of second-order ordinary differential equations in the expansion coefficients $a_j(t)$, namely,

$$\begin{aligned} &\tau \ddot{u}_n(t) + p\left(1-\tau+qu_n(t)\right)\dot{u}_n(t) \\ &\quad = \zeta \left(\frac{2}{B-A}\right)^2 \left(\sum_{i=1}^{N-1} B_{ni} u_i(t) + \tilde{d}_n(t)\right) \\ &\qquad + qu_n(t)\left(1-u_n(t)\right) - \lambda \left(\frac{2}{B-A}\right) u_n(t)^s \\ &\qquad \times \left(\sum_{i=1}^{N-1} A_{ni} u_i(t) + d_n(t)\right), \end{aligned} \tag{25}$$

where

$$\begin{aligned} d_n(t) &= A_{no} g_1(t) + A_{nN} g_2(t), \\ \tilde{d}_n(t) &= B_{no} g_1(t) + B_{nN} g_2(t). \end{aligned} \tag{26}$$

This means that problem (6)–(9) is transformed to the following system of ordinary differential equations (SODEs):

$$\begin{aligned} &\tau \ddot{u}_n(t) + p\left(1-\tau+qu_n(t)\right)\dot{u}_n(t) \\ &\quad = \zeta \left(\frac{2}{B-A}\right)^2 \left(\sum_{i=1}^{N-1} B_{ni} u_i(t) + \tilde{d}_n(t)\right) \\ &\qquad + qu_n(t)\left(1-u_n(t)\right) \\ &\qquad - \lambda \left(\frac{2}{B-A}\right) u_n(t)^s \left(\sum_{i=1}^{N-1} A_{ni} u_i(t) + d_n(t)\right), \end{aligned} \tag{27}$$

subject to the initial values

$$\begin{aligned} u_n(0) &= f_3\left(x_{N,n}^{(\theta,\vartheta)}\right), \\ \dot{u}_n(0) &= f_4\left(x_{N,n}^{(\theta,\vartheta)}\right), \quad n = 1, \ldots, N-1. \end{aligned} \tag{28}$$

Finally, (27)–(28) can be rewritten into a matrix form of $N-1$ second-order ordinary differential equations with their vectors of initial values:

$$\begin{aligned} \ddot{\mathbf{u}}(t) + \dot{\mathbf{u}}(t) &= \mathbf{F}(t, u(t)), \\ \mathbf{u}(0) &= \mathbf{f}_3, \\ \dot{\mathbf{u}}(0) &= \mathbf{f}_4, \end{aligned} \tag{29}$$

where

$$\mathbf{\ddot{u}}(t) = [\tau \ddot{u}_1(t), \tau \ddot{u}_2(t), \ldots, \tau \ddot{u}_{N-1}(t)]^T,$$
$$\mathbf{\dot{u}}(t) = [p(1-\tau+qu_n(t))\dot{u}_1(t), p(1-\tau+qu_n(t))\dot{u}_2(t), \ldots, p(1-\tau+qu_n(t))\dot{u}_{N-1}(t)]^T,$$
$$\mathbf{f}_3 = \left[f_3\left(x_{N,1}^{(\theta,\vartheta)}\right), f_3\left(x_{N,2}^{(\theta,\vartheta)}\right), \ldots, f_3\left(x_{N,N-1}^{(\theta,\vartheta)}\right)\right]^T,$$
$$\mathbf{f}_4 = \left[f_4\left(x_{N,1}^{(\theta,\vartheta)}\right), f_4\left(x_{N,2}^{(\theta,\vartheta)}\right), \ldots, f_4\left(x_{N,N-1}^{(\theta,\vartheta)}\right)\right]^T,$$
$$\mathbf{F}(t,u(t)) = [F_1(t,u(t)), F_1(t,u(t)), \ldots, F_{N-1}(t,u(t))]^T, \quad (30)$$

where

$$F_n(t,u(t)) = \zeta\left(\frac{2}{B-A}\right)^2\left(\sum_{i=1}^{N-1} B_{ni}u_i(t) + \tilde{d}_n(t)\right) - \lambda\left(\frac{2}{B-A}\right)u_n(t)^s\left(\sum_{i=1}^{N-1} A_{ni}u_i(t) + d_n(t)\right) + qu_n(t)(1-u_n(t)). \quad (31)$$

Remark 1. We can replace the Jacobi polynomials by the Legendre, Chebyshev of the first, second, third, and fourth kinds, or Gegenbauer polynomials and replace the nodes of Jacobi-Gauss-Lobatto quadrature by the Legendre-, Chebyshev-, or Gegenbauer-Gauss-Lobatto quadrature (cf. [42–45]), just by taking the special cases $\theta = \vartheta = 0$, $\theta = \vartheta = -0.5$, $\theta = -0.5$, $\vartheta = 0.5$, $\theta = 0.5$, $\vartheta = -0.5$, or $\theta = \vartheta$, respectively, in the resulting system of ordinary differential equations.

The SODEs (29) can be solved by using implicit Runge-Kutta method of fourth order, which is an important family of implicit and explicit iterative methods for the approximation of solution of system of ordinary differential equations. In the next section, we recall that the difference between the measured or inferred value of approximate solution and its actual value (absolute error) is given by

$$E(x,t) = |u(x,t) - \tilde{u}(x,t)|, \quad (32)$$

where $u(x,t)$ and $\tilde{u}(x,t)$ are the exact solution and the approximate solution at the point (x,t), respectively. Moreover, the maximum absolute error is given by

$$M_E = \operatorname{Max}\{E(x,t) : \forall (x,t) \in D \times [0,T]\}. \quad (33)$$

4. Numerical Results and Discussions

This section considers three numerical examples to demonstrate the accuracy and applicability of the proposed method in the present paper. Comparison of the results obtained by various choices of Jacobi parameters θ and ϑ reveals that the present method is very effective and convenient for all choices of θ and ϑ.

We consider the following three examples.

Example 2. As a first example, we consider the nonlinear time-delayed one-dimensional Burgers equation in the form

$$\tau u_{tt} + u_t + \lambda_1 u u_x - u_{xx} = 0, \quad (x,t) \in D \times [0,T], \quad (34)$$

subject to the initial and boundary values

$$u(A,t) = \frac{1}{2}\left(1 - \tanh\left(\frac{2\lambda_1}{2(4-\tau\lambda^2)}\left(A - \frac{\lambda_1}{2}t\right)\right)\right),$$
$$u(B,t) = \frac{1}{2}\left(1 - \tanh\left(\frac{2\lambda_1}{2(4-\tau\lambda^2)}\left(B - \frac{\lambda_1}{2}t\right)\right)\right), \quad (35)$$

$$u(x,0) = \frac{1}{2}\left(1 - \tanh\left(\frac{2\lambda_1}{2(4-\tau\lambda^2)}(x)\right)\right), \quad x \in D. \quad (36)$$

$$u_t(x,0) = \frac{\lambda_1^2}{4(4-\tau\lambda^2)} \times \operatorname{sech}^2\left(\frac{2\lambda_1}{2(4-\tau\lambda^2)}\left(x - \frac{\lambda_1}{2}t\right)\right), \quad x \in D. \quad (37)$$

The exact solution [39] of (34) is

$$u(x,t) = \frac{1}{2}\left(1 - \tanh\left(\frac{2\lambda_1}{2(4-\tau\lambda^2)}\left(x - \frac{\lambda_1}{2}t\right)\right)\right). \quad (38)$$

Maximum absolute errors of (34) subject to (35) and (36) are introduced in Table 1 using J-GL-C method with various choices of N, θ and ϑ, and $-A = B = 10$; this table indicates that the obtained results are very accurate for a small choice of N, while the absolute errors of problem (34) are presented in Table 2 for $\theta = \vartheta = 1/2$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 12$ with different values of (x,t) in the interval $[0,100]$.

In Figure 1, we see that the approximate solution and the exact solution for $t = 0.5$ of problem (34) coincide with values of parameters listed in their caption. Moreover, the absolute error of problem (34) where $\theta = \vartheta = -1/2$ and $N = 2$ is displayed in Figure 2. In Figure 3, we plotted the approximate solution $\tilde{u}(x,t)$. This assertion shows that the obtained numerical results are accurate and compare favorably with the exact solution.

Example 3. Consider the nonlinear time-delayed one-dimensional generalized Burgers equation in the form

$$\tau u_{tt} + u_t + \lambda_1 u^s u_x - u_{xx} = 0, \quad (x,t) \in D \times [0,T], \quad (39)$$

TABLE 1: Maximum absolute errors with various choices of N, θ, and ϑ with $\tau = 0.5$ and $\lambda_1 = 0.01$ for Example 2.

N	A	B	θ	ϑ	M_E	N	A	B	θ	ϑ	M_E
2	−10	10	0	0	2.59×10^{-7}	2	−10	10	0.5	0.5	2.59×10^{-7}
4					2.70×10^{-11}	4					2.31×10^{-11}
6					2.22×10^{-15}	6					1.89×10^{-15}
8					1.11×10^{-16}	8					1.11×10^{-16}
2	−10	10	−0.5	−0.5	2.59×10^{-7}	2	−10	10	−0.5	0.5	8.95×10^{-7}
4					2.23×10^{-11}	4					5.97×10^{-11}
6					3.22×10^{-15}	6					3.77×10^{-15}
8					1.11×10^{-16}	8					1.11×10^{-16}

TABLE 2: Absolute errors with $\theta = \vartheta = 1/2$, $N = 12$ and various choices of x, t with $\tau = 0.5$ and $\lambda_1 = 0.01$ for Example 2.

x	t	N	A	B	E	x	t	N	A	B	E
0	0.1	12	0	100	3.33×10^{-16}	0	0.2	12	0	100	7.77×10^{-16}
10					5.55×10^{-17}	10					5.55×10^{-17}
02					5.55×10^{-17}	20					0
30					5.55×10^{-17}	30					5.55×10^{-17}
40					5.55×10^{-17}	40					0
50					0	50					5.55×10^{-17}
60					1.67×10^{-16}	60					5.55×10^{-17}
70					0	70					5.55×10^{-17}
80					1.11×10^{-16}	80					0
90					1.67×10^{-16}	90					0
100					5.00×10^{-16}	100					5.55×10^{-17}

subject to the boundary conditions

$$u(A,t) = \sqrt[s]{\frac{1}{2}\left(1-\tanh\left(\frac{s(s+1)\lambda_1}{2\left((s+1)^2-\tau\lambda^2\right)}\left(A-\frac{\lambda_1}{(s+1)}t\right)\right)\right)},$$

$$u(B,t) = \sqrt[s]{\frac{1}{2}\left(1-\tanh\left(\frac{s(s+1)\lambda_1}{2\left((s+1)^2-\tau\lambda^2\right)}\left(B-\frac{\lambda_1}{(s+1)}t\right)\right)\right)}, \tag{40}$$

and the initial conditions

$$u(x,0) = \sqrt[s]{\frac{1}{2}\left(1-\tanh\left(\frac{s(s+1)\lambda_1}{2\left((s+1)^2-\tau\lambda^2\right)}(x)\right)\right)}, \quad x \in D, \tag{41}$$

$$u_t(x,0) = \frac{s\lambda_1^2 \operatorname{sech}^2\left(\left(s(s+1)\lambda_1/2\left((s+1)^2-\tau\lambda^2\right)\right)(x)\right)}{2\left((s+1)^2-\tau\lambda^2\right)} \times \sqrt[s]{\left(\frac{1}{2}\left(1-\tanh\left(\frac{s(s+1)\lambda_1}{2\left((s+1)^2-\tau\lambda^2\right)}(x)\right)\right)\right)^{1-s}}, \quad x \in D. \tag{42}$$

The exact solution [39] of (39) is

$$u(x,t) = \sqrt[s]{\frac{1}{2}\left(1-\tanh\left(\frac{s(s+1)\lambda_1}{2\left((s+1)^2-\tau\lambda^2\right)}\left(x-\frac{\lambda_1}{(s+1)}t\right)\right)\right)}. \tag{43}$$

Table 3 lists the maximum absolute errors of (39) subject to (40) and (41) in $[-10,10]$, using J-GL-C method for different values of N, θ, and ϑ. Moreover, in Tables 4 and 5, we evaluate the absolute errors of (39) for $s = 3$ and 5, respectively, with different values of (x,t) in the interval $[-100,100]$.

Figure 4 plots the approximate and the exact solutions at $t = 0.5$ of (39) with the special values $\theta = \vartheta = -1/2$,

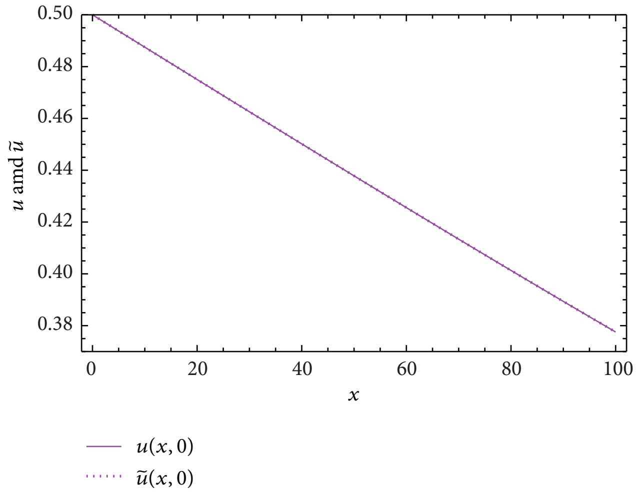

FIGURE 1: The approximate solution and the exact solution for $t = 0.5$ of problem (34) with $\theta = \vartheta = 1/2$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 12$.

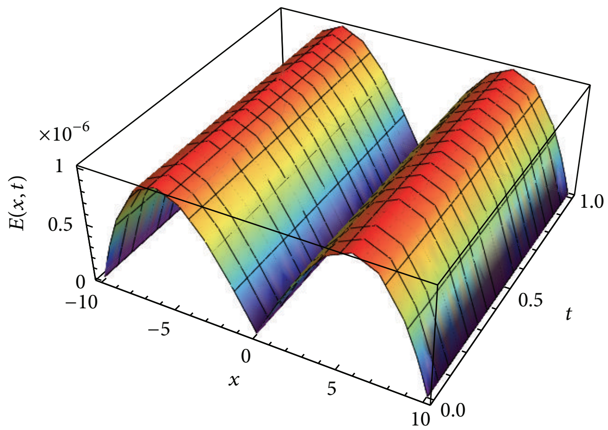

FIGURE 2: The absolute error between the approximate and the exact solutions of problem (34) with $\theta = \vartheta = -1/2$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 2$.

$\lambda_1 = 0.01$, $s = 2$, and $N = 12$. Moreover, the absolute errors between the approximate and the exact solutions of (39) with $\theta = \vartheta = 1/2$, $\lambda_1 = 0.01$, $s = 3$, and $N = 8$ are shown in Figure 5.

Example 4. Consider the nonlinear time-delayed Burgers-Fisher equation in the form

$$\begin{aligned}\tau u_{tt} + (1-\tau) u_t + 2\tau u u_t \\ + u u_x - u_{xx} - u + u^2 = 0, \\ (x,t) \in D \times [0,T],\end{aligned} \tag{44}$$

subject to the boundary conditions

$$\begin{aligned}u(A,t) &= \frac{1}{2}\left(1 + \tanh\left(\frac{\tau+1}{\tau-4}\left(A - \frac{5}{2(\tau+1)}t\right)\right)\right), \\ u(B,t) &= \frac{1}{2}\left(1 + \tanh\left(\frac{\tau+1}{\tau-4}\left(B - \frac{5}{2(\tau+1)}t\right)\right)\right),\end{aligned} \tag{45}$$

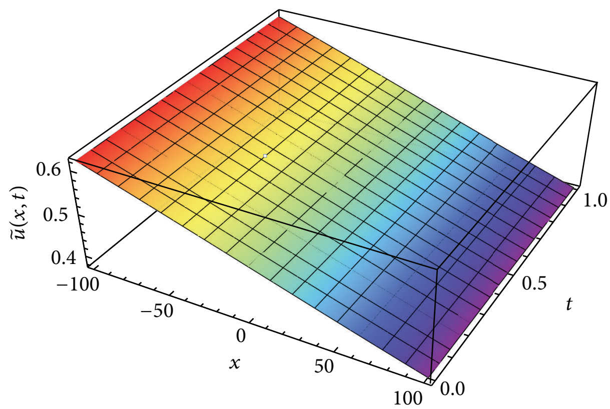

FIGURE 3: The approximate solution of problem (34) with $\theta = \vartheta = 0$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 12$.

and the initial condition

$$u(x,0) = \frac{1}{2}\left(1 + \tanh\left(\frac{\tau+1}{\tau-4}(x)\right)\right), \quad x \in D, \tag{46}$$

Table 3: Maximum absolute errors with various choices of N, θ, and ϑ with $\tau = 0.5$, $\lambda_1 = 0.01$, and $s = 3$ for Example 3.

N	A	B	θ	ϑ	M_E	N	A	B	θ	ϑ	M_E
2	−10	10	0	0	6.20×10^{-5}	2	−10	10	0.5	0.5	6.20×10^{-5}
4					5.53×10^{-9}	4					4.84×10^{-9}
6					5.89×10^{-13}	6					4.86×10^{-13}
8					2.22×10^{-16}	8					2.22×10^{-16}
2	−10	10	−0.5	−0.5	6.20×10^{-5}	2	−10	10	−0.5	0.5	5.202×10^{-5}
4					6.45×10^{-9}	4					4.24×10^{-9}
6					7.29×10^{-13}	6					4.42×10^{-13}
8					3.33×10^{-16}	8					1.11×10^{-16}

Table 4: Absolute errors with $-\theta = \vartheta = 1/2$, $N = 12$ and various choices of x, t for Example 3 for $\tau = 0.5$, $\lambda_1 = 0.01$, and $s = 3$.

x	t	N	A	B	E	x	t	N	A	B	E
−100	0.1	12	−100	100	1.46×10^{-11}	−100	0.2	12	−100	100	1.45×10^{-11}
−80					1.85×10^{-12}	−80					1.85×10^{-12}
−60					1.66×10^{-12}	−60					1.16×10^{-12}
−40					1.06×10^{-12}	−40					1.05×10^{-12}
−20					8.93×10^{-13}	−20					8.92×10^{-13}
0					4.89×10^{-13}	0					4.87×10^{-13}
20					7.77×10^{-15}	20					8.55×10^{-15}
40					3.54×10^{-13}	40					3.54×10^{-13}
60					4.50×10^{-13}	60					4.50×10^{-13}
80					1.89×10^{-13}	80					1.90×10^{-13}
100					5.83×10^{-13}	100					5.81×10^{-13}

Table 5: Absolute errors with $\theta = \vartheta = -1/2$, $N = 12$ and various choices of x, t for Example 3 for $\tau = 0.5$, $\lambda_1 = 0.01$, and $s = 5$.

x	t	N	A	B	E	x	t	N	A	B	E
−100	0.1	12	−100	100	2.13×10^{-12}	−100	0.2	12	−100	100	2.13×10^{-12}
−80					3.24×10^{-13}	−80					3.20×10^{-13}
−60					2.00×10^{-13}	−60					2.01×10^{-13}
−40					5.94×10^{-13}	−40					5.92×10^{-13}
−20					1.70×10^{-12}	−20					1.69×10^{-12}
0					2.13×10^{-12}	0					2.13×10^{-12}
20					1.46×10^{-12}	20					1.45×10^{-12}
40					2.66×10^{-13}	40					2.64×10^{-13}
60					4.87×10^{-13}	60					4.88×10^{-13}
80					1.13×10^{-13}	80					1.08×10^{-13}
100					2.13×10^{-12}	100					2.13×10^{-12}

$$u_t(x,0) = -\frac{5}{4(\tau-4)}\operatorname{sech}^2 \times \left(\frac{\tau+1}{\tau-4}\left(x - \frac{5}{2\tau+1}t\right)\right), \quad x \in D. \tag{47}$$

The exact solution [39] of (44) is

$$u(x,t) = \frac{1}{2}\left(1 + \tanh\left(\frac{\tau+1}{\tau-4}\left(x - \frac{5}{2(\tau+1)}t\right)\right)\right). \tag{48}$$

Maximum absolute errors of (44) subject to (45) and (46) are tabulated in Table 6 using J-GL-C method with various choices of N, θ, and ϑ, while the absolute errors of (44) are presented in Table 7 for $\theta = \vartheta = 0$ and $N = 40$ with different values of (x, t) in the interval $[-5, 5]$.

In Figure 6, we see that the approximate solution and the exact solution of (44) coincide for different values of $t = 0$, 0.5, and 0.9 and $\theta = \vartheta = 0$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 40$. The approximate solution of (44) with $\theta = \vartheta = 0$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 40$ is displayed in Figure 7, while the absolute error of (44) with $\theta = \vartheta = 0$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 40$ is displayed in Figure 8.

TABLE 6: Maximum absolute errors with various choices of N, θ, and ϑ with $\tau = 0.5$ for Example 4.

N	A	B	θ	ϑ	M_E	N	A	B	θ	ϑ	M_E
8	−10	10	0	0	2.25×10^{-3}	8	−10	10	0.5	0.5	1.92×10^{-3}
20					6.10×10^{-7}	20					5.16×10^{-7}
32					2.83×10^{-10}	32					3.01×10^{-10}
8	−10	10	−0.5	−0.5	2.73×10^{-3}	8	−10	10	0.5	0.5	3.09×10^{-3}
20					7.70×10^{-7}	20					8.81×10^{-7}
32					3.36×10^{-10}	32					2.67×10^{-10}

TABLE 7: Absolute errors with $\theta = \vartheta = 0$, $N = 40$ and various choices of x, t with $\tau = 0.5$ for Example 4.

x	t	N	A	B	E	x	t	N	A	B	E
−5	0.1	12	−5	5	4.16×10^{-12}	−5	0.2	12	−5	5	1.66×10^{-12}
−4					1.84×10^{-13}	−4					6.58×10^{-13}
−3					5.07×10^{-13}	−3					3.95×10^{-13}
−2					3.32×10^{-13}	−2					5.74×10^{-13}
−1					2.82×10^{-13}	−1					9.45×10^{-13}
0					3.59×10^{-13}	0					3.07×10^{-13}
1					7.25×10^{-13}	1					2.60×10^{-13}
2					1.01×10^{-12}	2					3.42×10^{-13}
3					9.92×10^{-13}	3					3.12×10^{-13}
4					6.08×10^{-14}	4					7.46×10^{-13}
5					4.16×10^{-12}	5					1.66×10^{-12}

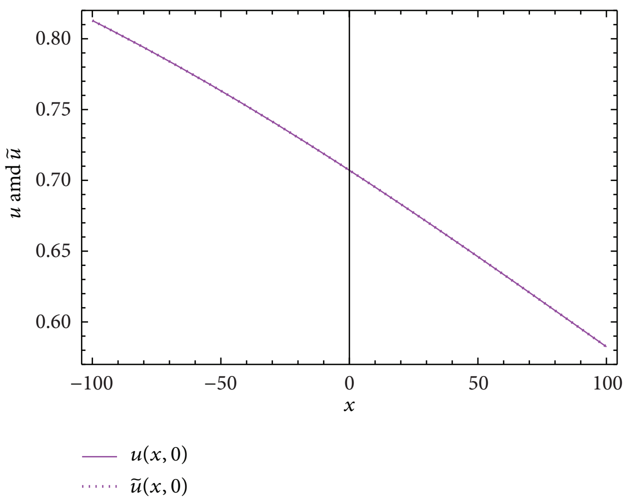

FIGURE 4: The approximate and the exact solutions for different values of $t = 0.5$ of problem (39) with $\theta = \vartheta = -1/2$, $\lambda_1 = 0.01$, $s = 2$, and $N = 12$.

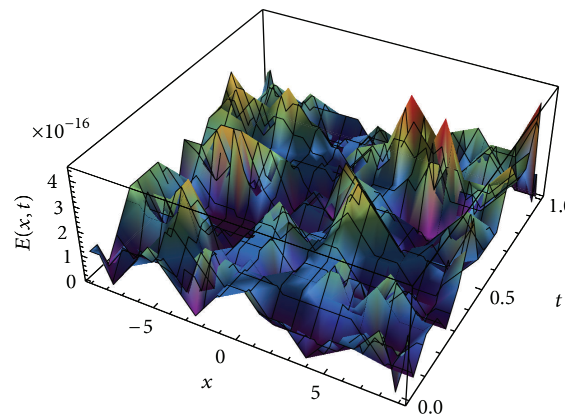

FIGURE 5: The absolute errors between the approximate and the exact solutions of problem (39) with $\theta = \vartheta = 1/2$, $\lambda_1 = 0.01$, $s = 3$, and $N = 8$.

5. Conclusions and Future Works

We have constructed in this paper an efficient spectral-collocation algorithm for solving nonlinear time-delayed Burgers-type equations with positive power terms subject to initial and boundary conditions. The Jacobi-Gauss-Lobatto collocation methods based upon two general parameters, θ and ϑ, are developed and applied to the time-delayed Burgers-type equations for reformulating the problem to a system of second-order ordinary differential equations.

In fact, Jacobi-Gauss-Lobatto collocation method is slightly more complicated to implement than other orthogonal collocation methods but is more efficient and can be applied to a wider class of problems. We presented some advantages of the presented algorithm, as well as numerical results which demonstrate its accuracy and flexibility.

While the new algorithm presented in this paper only applied to a specific class of nonlinear time-delayed equations, it is very efficient and accurate whenever applicable. Moreover, high accuracy in long computational intervals and the stability of the proposed method encourage us to apply a similar scheme for the numerical solution of coupled

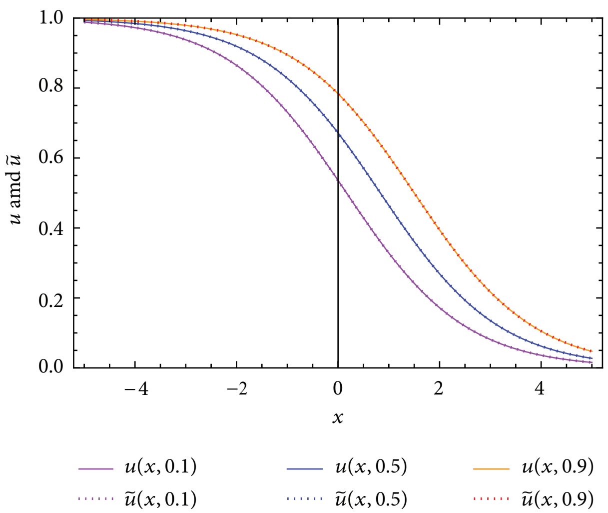

FIGURE 6: The approximate and the exact solutions for different values of t (0, 0.5, and 0.9) of problem (44) with $\theta = \vartheta = 0$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 40$.

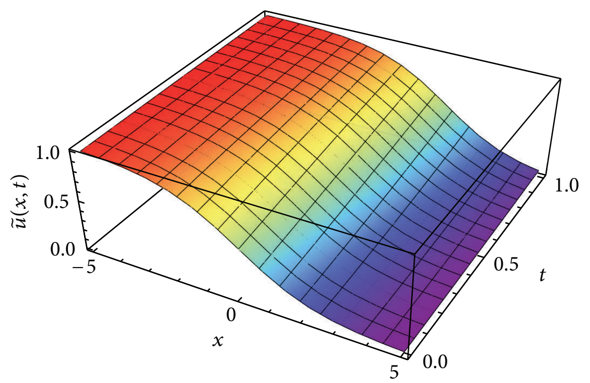

FIGURE 7: The approximate solution of problem (44) with $\theta = \vartheta = 0$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 40$.

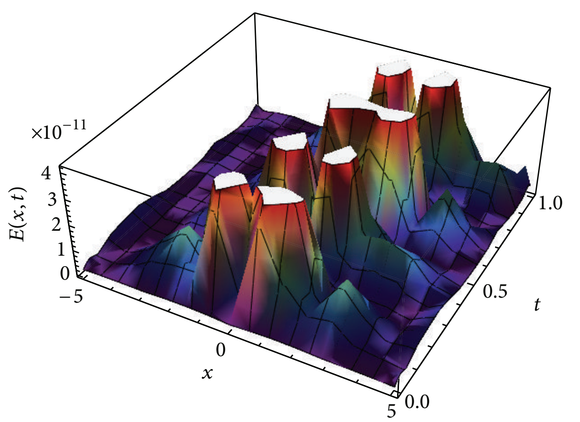

FIGURE 8: The absolute error between the approximate and the exact solutions of problem (44) with $\theta = \vartheta = 0$, $\lambda_1 = 0.01$, $\tau = 0.5$, and $N = 40$.

nonlinear partial differential equations and other applied mathematics problems (see, e.g, [46–49] and references therein) in the future.

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Research Article

Some Integrals Involving q-Laguerre Polynomials and Applications

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The integrals involving multivariate q-Laguerre polynomials and then auxiliary ones are studied. In addition, the representations of q-Hermite polynomials by q-Laguerre polynomials and their related integrals are given. At last, some generalized integrals associated with generalized q-Hermite polynomials are deduced.

Dedicated to Srinivasa Ramanujan on the occasion of his 125th birth anniversary

1. Introduction

The q-Laguerre polynomials are important q-orthogonal polynomials whose applications and generalizations arise in many applications such as quantum group (oscillator algebra, etc.), q-harmonic oscillator, and coding theory. For example, covariant oscillator algebra can be expressed by q-Laguerre polynomials [1]. The q-deformed radial Schrödinger is analyzed by q-Laguerre polynomials [2]. The q-Laguerre polynomials are the eigenvectors of an $su_q(1 \mid q)$-representation by [3]. For more information, please refer to [1–5].

The q-Laguerre polynomials are defined by [6, equation (1.0.1)]

$$\mathscr{L}_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}} (xq^{n+\alpha+1})^k}{(q^{\alpha+1};q)_k (q;q)_k}, \quad (1)$$

which belong to the Askey scheme of basic hypergeometric orthogonal polynomials and according to Koekoek and Swarttouw [7, equation (3.21.1)]. The case of x in (1) replaced by $(1-q)x$ is studied by Moak [8, equation (2.3)].

In this paper, we first define the auxiliary q-Laguerre polynomials as follows:

$$\mathscr{M}_n^{(\alpha)}(x;q) = q^{-n\alpha} \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k x^k}{(q^{\alpha+1};q)_k (q;q)_k}. \quad (2)$$

It is easy to see the validity of the following:

$$\mathscr{L}_n^{(\alpha)}((1-q)x;q) = \mathscr{M}_n^{(\alpha)}\left(\left(1-q^{-1}\right)x;q^{-1}\right), \quad (3)$$

$$\lim_{q\to 1-} \mathscr{L}_n^{(\alpha)}((1-q)x;q) = \lim_{q\to 1+} \mathscr{M}_n^{(\alpha)}((1-q)x;q) = L_n^{(\alpha)}(x), \quad (4)$$

where the classical Laguerre polynomials are defined by [9, page 201]

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k}{k!\,(n-k)!(1+\alpha)_k}. \quad (5)$$

For more information about classical Laguerre polynomials, please refer to [9–15] and the references therein.

The well-known orthogonality of q-Laguerre polynomials reads the following.

Proposition 1 (see [6, equation (2.0.7)] and [8, equation (2.4)]). *For $\alpha > -1$ and for $m, n \in \mathbb{N}$, one has*

$$\int_0^\infty \mathscr{L}_m^{(\alpha)}(x;q)\,\mathscr{L}_n^{(\alpha)}(x;q)\,\frac{x^\alpha}{(-x;q)_\infty}dx = -q^{-n}\frac{(q^{\alpha+1};q)_n(q^{-\alpha};q)_\infty}{(q;q)_n(q;q)_\infty}\frac{\pi\delta_{m,n}}{\sin\pi\alpha}. \tag{6}$$

Hahn [16] discovered the previous q-extensions of the Laguerre polynomials, although he said little about them. Moak [8] found that the q-Laguerre polynomials are orthogonal with respect to the discrete measures (Dirac measure). Koekoek and Meijer [17–19] studied systematically the inner product of q-Laguerre polynomials. Ismail and Rahman [20] studied the indeterminate Hamburger moment problems related to q-Laguerre polynomials. For more information, please refer to [6–8, 16–21] and the references therein.

In this paper, we first generalize Proposition 1 and the auxiliary ones as follows.

Theorem 2. *For $\Re\{\mu\} > -1$ and $m, n \in \mathbb{N}$, one has*

$$\begin{aligned}\int_0^\infty &\mathscr{L}_m^{(\alpha)}(xy;q)\,\mathscr{L}_n^{(\beta)}(xz;q)\,\frac{x^\mu}{(-x;q)_\infty}dx \\ &= (1-q)^{1+\mu}\frac{\pi\csc(-\mu\pi)}{\Gamma_q(-\mu)} \\ &\times \sum_{k=0}^{\min\{m,n\}}\begin{bmatrix} m+\alpha \\ m-k\end{bmatrix}\begin{bmatrix} n+\beta \\ n-k\end{bmatrix}\begin{bmatrix} k+\mu \\ k\end{bmatrix} q^{(\alpha+\beta-2\mu-1)k}y^k z^k \\ &\times {}_2\phi_1\left[\begin{matrix} q^{k-m}, q^{\mu+1+k} \\ q^{\alpha+k+1}\end{matrix};\ yq^{\alpha-\mu+m-k}\right] \\ &\times {}_2\phi_1\left[\begin{matrix} q^{k-n}, q^{\mu+1+k} \\ q^{\beta+k+1}\end{matrix};\ zq^{\beta-\mu+n-k}\right].\end{aligned} \tag{7}$$

Theorem 3. *For $\Re\{\mu\} > -1$ and $m, n \in \mathbb{N}$, one has*

$$\begin{aligned}\int_0^\infty &\mathscr{M}_m^{(\alpha)}(xy;q)\,\mathscr{M}_n^{(\beta)}(xz;q)\,x^\mu(x;q)_\infty dx \\ &= q^{\binom{\mu+2}{2}-\alpha m-\beta n}(1-q)^{1+\mu}\frac{\pi\csc(-\mu\pi)}{\Gamma_q(-\mu)} \\ &\times \sum_{k=0}^{\min\{m,n\}}\begin{bmatrix} m+\alpha \\ m-k\end{bmatrix}\begin{bmatrix} n+\beta \\ n-k\end{bmatrix}\begin{bmatrix} k+\mu \\ k\end{bmatrix} \\ &\times q^{(2k+\mu-m-n+1)k}y^k z^k \\ &\times {}_2\phi_1\left[\begin{matrix} q^{k-m}, q^{\mu+1+k} \\ q^{\alpha+k+1}\end{matrix};\ yq\right]{}_2\phi_1\left[\begin{matrix} q^{k-n}, q^{\mu+1+k} \\ q^{\beta+k+1}\end{matrix};\ zq\right].\end{aligned} \tag{8}$$

Corollary 4 (see [15, equation (14)]). *For $\Re\{\gamma\} > -1$, $\Re\{\sigma\} > 0$, and $m, n \in \mathbb{N}$, one has*

$$\begin{aligned}\int_0^\infty &x^\gamma e^{-\sigma x}L_m^{(\alpha)}(\lambda x)\,L_n^{(\beta)}(\mu x)\,dx \\ &= \frac{\Gamma(\gamma+1)}{\sigma^{\gamma+1}}\sum_{k=0}^{\min\{m,n\}}\binom{m+\alpha}{m-k} \\ &\times \binom{n+\beta}{n-k}\binom{k+\gamma}{k}\left(\frac{\lambda\mu}{\sigma^2}\right)^k \\ &\times {}_2F_1\left[\begin{matrix} -m+k, \gamma+k+1 \\ \alpha+k+1\end{matrix};\ \frac{\lambda}{\sigma}\right] \\ &\times {}_2F_1\left[\begin{matrix} -n+k, \gamma+k+1 \\ \beta+k+1\end{matrix};\ \frac{\mu}{\sigma}\right].\end{aligned} \tag{9}$$

Remark 5. Theorems 2 and 3 reduce to Proposition 1 and formula (41), respectively, if letting $y = z = 1$ and $\alpha = \beta = \mu$, and become Corollary 4 by setting $q \to 1$ and taking $(\mu, x, y, z) = (\gamma, \sigma x, \lambda/\sigma, \mu/\sigma)$.

The discrete q-Hermite polynomials $h_n(x;q)$ and $\widetilde{h}_n(x;q)$ are defined by [7, pages 90-91]

$$\begin{aligned} h_n(x;q) &= x^n{}_2\phi_0\left[\begin{matrix} q^{-n}, q^{-n+1} \\ - \end{matrix};\ q^2, \frac{q^{2n-1}}{x^2}\right] \\ &\triangleq q^{\binom{n}{2}}\mathscr{G}_n\left(x\sqrt{1-q^2};q^2\right), \\ \widetilde{h}_n(x;q) &= x^n{}_2\phi_1\left[\begin{matrix} q^{-n}, q^{-n+1} \\ 0 \end{matrix};\ q^2, \frac{q^2}{x^2}\right] \\ &\triangleq q^{-\binom{n}{2}}\mathscr{H}_n\left(x\sqrt{1-q^2};q^2\right),\end{aligned} \tag{10}$$

which are equivalent to Al-Salam-Carlitz polynomials with $a = -1$ (please refer to [22, page 53] also), and the relation between them is $h_n(ix;q^{-1}) = i^n\widetilde{h}_n(x;q)$. For more information about the Al-Salam-Carlitz polynomials and the discrete q-Hermite polynomials, please refer to [7, 22–30] and the references therein.

In this paper, we also define new q-Hermite polynomials $\mathscr{H}_n(x;q)$ and $\mathscr{G}_n(x;q)$, whose names come from the facts

$$\begin{aligned}\lim_{q\to 1}&(1-q)^{-n/2}\mathscr{H}_n((1-q)x;q) \\ &= \lim_{q\to 1}(1-q)^{-n/2}\mathscr{G}_n((1-q)x;q) = 2^{-n}H_n(x),\end{aligned} \tag{11}$$

$$\begin{aligned}\lim_{\substack{x\to(1-q)x,\\ t\to\sqrt{(1-q)}t,\ q\to 1}} &\sum_{n=0}^\infty \frac{t^n}{(q^{1/2};q^{1/2})_n}\mathscr{H}_n(x;q) \\ &= \exp\left(2xt-t^2\right)\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$
$$= \lim_{\substack{x \to (1-q)x, \\ t \to \sqrt{(1-q)}t,\ q \to 1}} \sum_{n=0}^{\infty} \frac{t^n q^{n(n-1)/4}}{(q^{1/2}; q^{1/2})_n} \mathscr{G}_n(x;q), \tag{12}$$

then we deduce the representations of $\mathscr{H}_n(x;q)$ and $\mathscr{G}_n(x;q)$ by q-Laguerre polynomials; see Theorems 15 and 16.

As an application, using the orthogonality of q-Laguerre polynomials (6), and (41), and combining the expressions of q-Hermite polynomials (52) and (54), we can obtain the following results immediately.

Theorem 6. *For $\alpha > -1$ and $j \le n \in \mathbb{N}$, one has*

$$\int_0^\infty \mathscr{H}_n(x;q) \mathscr{L}_j^{(\alpha)}(x;q) \frac{x^\alpha}{(-x;q)_\infty} dx = -\frac{(q^{\alpha+1}, q; q)_n (q^{\alpha+1}; q)_j (q^{-\alpha}; q)_\infty (1-q)^{-n/2}}{(q;q)_j (q;q)_\infty} \times \frac{\pi C(n,j)}{\sin \pi\alpha}, \tag{13}$$

where $C(n,k)$ is defined by (52).

Theorem 7. *For $\alpha > -1$ and $j \le n \in \mathbb{N}$, one has*

$$\int_0^\infty \mathscr{G}_n(x;q) \mathscr{M}_j^{(\alpha)}(x;q) x^\alpha (x;q)_\infty dx = -q^{-(1/2)\binom{n}{2} + \binom{\alpha+2}{2} + j - \alpha j} \times \frac{(q^{\alpha+1}, q; q)_n (q^{\alpha+1}; q)_j (q^{-\alpha}; q)_\infty (1-q)^{-n/2}}{(q;q)_j (q;q)_\infty} \times \frac{\pi D(n,j)}{\sin \pi\alpha}, \tag{14}$$

where $D(n,k)$ is defined by (54).

The generalized Hermite polynomials were introduced by Szegö [31], (see also [23, equation (1.1)]) as follows:

$$H_{2n}^{(\mu)}(x) = (-1)^n 2^{2n} n! L_n^{(\mu-1/2)}(x^2),$$
$$H_{2n+1}^{(\mu)}(x) = (-1)^n 2^{2n+1} n! x L_n^{(\mu+1/2)}(x^2). \tag{15}$$

The authors [23, equation (2.7)] defined the following generalized q-Hermite polynomials:

$$\mathscr{H}_{2n}^{(\mu)}(x;q) = (-1)^n (q;q)_n \mathscr{L}_n^{(\mu-1/2)}(x^2;q),$$
$$\mathscr{H}_{2n+1}^{(\mu)}(x;q) = (-1)^n (q;q)_n x \mathscr{L}_n^{(\mu+1/2)}(x^2;q) \tag{16}$$

and deduced their orthogonal relations; see Proposition 19 below.

In this paper, we continue to define the auxiliary polynomials according to (16) as follows:

$$\mathscr{G}_{2n}^{(\mu)}(x;q) = q^{-\binom{n+1}{2}} (q;q)_n \mathscr{M}_n^{(\mu-1/2)}(x^2;q),$$
$$\mathscr{G}_{2n+1}^{(\mu)}(x;q) = q^{-\binom{n+1}{2}} (q;q)_n x \mathscr{M}_n^{(\mu+1/2)}(x^2;q). \tag{17}$$

With the aid of (15)–(17) and (4), one readily verifies that

$$\lim_{q \to 1} (1-q)^{-n/2} \mathscr{H}_n^{(\mu)}(\sqrt{1-q}x;q) = \lim_{q \to 1} (1-q)^{-n/2} \mathscr{G}_n^{(\mu)}(\sqrt{1-q}x;q) = 2^{-n} H_n^{(\mu)}(x). \tag{18}$$

As another application of this paper, we gain the general q-Laguerre polynomials of several variables by Theorems 2 and 3, and we also deduce the orthogonal polynomials of $\mathscr{G}_n^{(\mu)}(x;q)$. For more details of the results, see Theorems 20 and 21 and Corollary 23.

The structure of this paper is organized as follows. In Section 2, we show how to prove the integrals involving q-Laguerre polynomials of several variables. In Section 3, we represent discrete q-Hermite polynomials by q-Laguerre polynomials and their related integral results. In Section 4, we study the general integrals of q-Hermite polynomials involving several variables.

2. Notations and Proof of Theorems 2 and 3

Throughout this paper, we follow the notations and terminology in [32] and assume that $0 < q < 1$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathbb{R}$ is rational number. The q-series and its compact factorials are defined [32, page 6], respectively, by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$
$$(a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \tag{19}$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n$, where m is a positive integer and n is a nonnegative integer or ∞.

The basic hypergeometric series ${}_r\phi_s$ is given by

$${}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} z^n \left[(-1)^n q^{\binom{n}{2}} \right]^{s+1-r}. \tag{20}$$

For convergence of the infinite series in (20), $|q| < 1$ and $|z| < \infty$ when $r \le s$, or $|q| < 1$ and $|z| < 1$ when $r = s + 1$, provided that no zeros appear in the denominator. Letting $(a_i, b_i) = (q^{a_i}, q^{b_i})$ and setting $q \to 1$, (20) reduces to the classical Gauss' hypergeometric series

$${}_rF_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{n! (b_1)_n \cdots (b_s)_n} z^n, \tag{21}$$

where Pochhammer symbol $(z)_n$ is defined by $(z)_n = z(z+1)\cdots(z+n-1) = \Gamma(z+n)/\Gamma(z)$.

The q-analogue of the gamma function is defined by (see [32, equation (1.10.1)]) as follows:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0<q<1. \tag{22}$$

The q-Chu-Vandermonde formula [32, equations (II.6) and (II.7)] reads that

$$\begin{aligned} {}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q\right] &= \frac{(c/b;q)_n}{(c;q)_n} b^n, \\ {}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, \frac{cq^n}{b}\right] &= \frac{(c/b;q)_n}{(c;q)_n}. \end{aligned} \tag{23}$$

The ${}_3\phi_2$ transformations [32, equations (III.12) and (III.13)] stats that

$$\begin{aligned} {}_3\phi_2\left[\begin{matrix} q^{-n}, a, c \\ b, d \end{matrix}; q, q\right] &= \frac{a^n(d/a;q)_n}{(d;q)_n} \\ &\quad \times {}_3\phi_2\left[\begin{matrix} q^{-n}, a, \dfrac{b}{c} \\ b, \dfrac{aq^{1-n}}{d} \end{matrix}; q, \frac{cq}{d}\right], \\ {}_3\phi_2\left[\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix}; q, \frac{deq^n}{bc}\right] &= \frac{(e/c;q)_n}{(e;q)_n} \\ &\quad \times {}_3\phi_2\left[\begin{matrix} q^{-n}, c, \dfrac{d}{b} \\ d, \dfrac{cq^{1-n}}{e} \end{matrix}; q, \frac{cq}{d}\right]. \end{aligned} \tag{24}$$

The q-analogue of the Pfaff-Kummer transformation [32, equation (III.4)] is as follows:

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty} {}_2\phi_2\left[\begin{matrix} a, \dfrac{c}{b} \\ c, az \end{matrix}; q, bz\right]. \tag{25}$$

The Ramanujan beta integral is stated as follows [33, equation (2.8)]:

$$\int_0^\infty t^{x-1}\frac{(-at;q)_\infty}{(-t;q)_\infty}dt = \frac{(a, q^{1-x};q)_\infty}{(q, aq^{-x};q)_\infty}\frac{\pi}{\sin \pi x}, \quad (0<a<q^x, x>0). \tag{26}$$

Lemma 8 (see [33, equation (4.2)]). *One has*

$$\begin{aligned} \int_0^\infty \frac{x^\alpha}{(-x;q)_\infty}dx &= \frac{\Gamma(-\alpha)\Gamma(\alpha+1)(1-q)^{1+\alpha}}{\Gamma_q(-\alpha)} \\ &= -\frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin \pi\alpha}, \end{aligned} \tag{27}$$

$$\int_0^\infty x^\alpha (x;q)_\infty dx = -q^{\binom{\alpha+2}{2}}\frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin\pi\alpha}. \tag{28}$$

Proof. Taking $(a,t) = (-1/a, at)$ in (26), then letting $a \to 0$, we obtain (28) immediately. The proof is complete. □

Lemma 9. *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$\mathscr{L}_n^{(\alpha)}(xy;q) = \sum_{k=0}^{n}\frac{(q^{\alpha+1};q)_n (y;q)_{n-k} y^k \mathscr{L}_k^{(\alpha)}(x;q)}{(q;q)_{n-k}(q^{\alpha+1};q)_k}, \tag{29}$$

$$\begin{aligned} \mathscr{M}_n^{(\alpha)}(xy;q) &= \sum_{k=0}^{n}\frac{(-1)^{n-k}(q^{\alpha+1};q)_n (1/y;q)_{n-k} y^n \mathscr{M}_k^{(\alpha)}(x;q)}{(q;q)_{n-k}(q^{\alpha+1};q)_k} \\ &\quad \times q^{\binom{k}{2}+\alpha k - \binom{n}{2} - \alpha n}. \end{aligned} \tag{30}$$

Proof. Letting $\gamma = 0$ in [6, Proposition 4.1],

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{(\gamma t;q)_n q^{\alpha n+n^2}}{(\gamma t, q^{\alpha+1}, q;q)_n}(-xt)^n \\ &\quad = \frac{(t;q)_\infty}{(\gamma t;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_n t^n}{(q^{\alpha+1};q)_n}\mathscr{L}_n^{(\alpha)}(x;q), \end{aligned} \tag{31}$$

then replacing x by xy, we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{\mathscr{L}_n^{(\alpha)}(xy;q)}{(q^{\alpha+1};q)_n}t^n \\ &\quad = \frac{(yt;q)_\infty}{(t;q)_\infty}\frac{1}{(yt;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{\alpha n+n^2}}{(q^{\alpha+1}, q;q)_n}(-xyt)^n \\ &\quad = \sum_{k=0}^{\infty}\frac{(y;q)_k}{(q;q)_k}t^k \sum_{n=k}^{\infty}\frac{L_{n-k}^{(\alpha)}(x;q)(yt)^{n-k}}{(q^{\alpha+1};q)_{n-k}}. \end{aligned} \tag{32}$$

Comparing the coefficients of t^n on both sides of (32) yields (29). Similar to (32), by the definition (1), we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{(\gamma;q)_n (tq^\alpha)^n}{(q^{\alpha+1};q)_n}\mathscr{M}_n^{(\alpha)}(x;q) \\ &\quad = \frac{(\gamma t;q)_\infty}{(t;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_n x^n}{(q/t, q^{\alpha+1}, q;q)_n}. \end{aligned} \tag{33}$$

By taking $(\gamma, t, x) = (1/\gamma, \gamma t, xy)$ and letting $\gamma \to 0$ in (33), we obtain (30). The proof of Lemma 9 is complete. □

Lemma 10. *For* $\min\{\alpha,\beta\} > -1$ *and* $m, n \in \mathbb{N}$, *one has*

$$\int_0^\infty \mathscr{L}_m^{(\alpha)}(x;q)\,\mathscr{L}_n^{(\beta)}(x;q)\,\frac{x^\beta}{(-x;q)_\infty}dx = (-1)^{m+n} q^{(\alpha-\beta)m+\binom{m}{2}+\binom{n}{2}-mn} \times (1-q)^{1+\beta}\begin{bmatrix}\beta-\alpha\\ m-n\end{bmatrix}\begin{bmatrix}n+\beta\\ n\end{bmatrix}\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}, \tag{34}$$

$$\int_0^\infty \mathscr{M}_j^{(\alpha)}(x;q)\,\mathscr{M}_k^{(\beta)}(x;q)\,x^\beta(x;q)_\infty dx = (-1)^{k+j} q^{\binom{j+1}{2}+\binom{k+1}{2}-(\alpha+j)k+\binom{\beta+2}{2}} \times (1-q)^{1+\beta}\begin{bmatrix}\beta-\alpha\\ j-k\end{bmatrix}\begin{bmatrix}k+\beta\\ k\end{bmatrix}\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}. \tag{35}$$

Proof. Interchanging the integral and summation by definition, the left hand side of (34) equals

$$\begin{aligned}
&\frac{(q^{\alpha+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}\sum_{k=0}^{m}\sum_{j=0}^{n}\frac{(q^{-m};q)_k q^{\binom{k}{2}}q^{k(m+\alpha+1)}}{(q^{\alpha+1},q;q)_k}
\times\frac{(q^{-n};q)_j q^{\binom{j}{2}}q^{j(n+\beta+1)}}{(q^{\beta+1},q;q)_j}\int_0^\infty\frac{x^{\beta+k+j}dx}{(-x;q)_\infty}\\
&=\frac{(q^{\alpha+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}\sum_{k=0}^{m}\sum_{j=0}^{n}\frac{(q^{-m};q)_k q^{\binom{k}{2}}q^{k(m+\alpha+1)}}{(q^{\alpha+1},q;q)_k}
\times\frac{(q^{-n};q)_j q^{\binom{j}{2}}q^{j(n+\beta+1)}}{(q^{\beta+1},q;q)_j}\\
&\quad\times\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}(1-q)^{1+\beta}q^{-j\beta-\binom{j+1}{2}-k(\beta+j)-\binom{k+1}{2}}
\times\left(q^{\beta+1};q\right)_{j+k}\\
&=\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}(1-q)^{1+\beta}\frac{(q^{\alpha+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}
\times\sum_{j=0}^{n}\frac{(q^{-n};q)_j q^{jn}}{(q;q)_j}\sum_{k=0}^{m}\frac{(q^{-m},q^{\beta+j+1};q)_k q^{k(m+\alpha-\beta-j)}}{(q^{\alpha+1},q;q)_k}\\
&=(-1)^m q^{(\alpha-\beta)m+\binom{m}{2}}\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}(1-q)^{1+\beta}
\times\frac{(q^{\beta-\alpha-m+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}
\times\sum_{j=0}^{n}\frac{(q^{-n},q^{\beta-\alpha+1};q)_j}{(q,q^{\beta-\alpha-m+1};q)_j}q^{j(n-m)},
\end{aligned}\tag{36}$$

which is the right hand side of (34) by using the second formula of (23) and simplification. Similar to (34), the right hand side of (35) is equal to

$$\begin{aligned}
&q^{-m\alpha-n\beta}\frac{(q^{\alpha+1};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}
\times\sum_{k=0}^{m}\sum_{j=0}^{n}\frac{(q^{-m};q)_k(q^{-n};q)_j}{(q^{\alpha+1},q;q)_k(q^{\beta+1},q;q)_j}
\times\int_0^\infty x^{\beta+k+j}(x;q)_\infty dx\\
&=q^{-m\alpha-n\beta}\frac{(q^{\alpha-\beta};q)_m(q^{\beta+1};q)_n}{(q;q)_m(q;q)_n}(1-q)^{1+\beta}
\times\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}q^{(\beta+1)m}q^{\binom{\beta+2}{2}}{}_2\phi_1\left[\begin{matrix}q^{-n},q^{\beta-\alpha+1}\\ q^{\beta-\alpha-m+1}\end{matrix};q\right],
\end{aligned}\tag{37}$$

which is equivalent to the right hand side of (35) by using the first formula of (23) and simplification. The proof of Lemma 10 is complete. □

Lemma 11. *If* $f(x)$ *is a polynomial of degree* m *about* x *and is defined in the infinite interval* $(0,\infty)$, *which can be expanded in a series of the form*

$$f(x)=\sum_{n=0}^{m}C_{mn}\mathscr{L}_n^{(\alpha)}(x;q) \quad \text{or} \quad f(x)=\sum_{n=0}^{m}D_{mn}\mathscr{M}_n^{(\alpha)}(x;q), \tag{38}$$

where C_{mn} *and* D_{mn} *are the nth Fourier-Laguerre coefficients, and both of them are independent of* x, *then one has*

$$C_{mn}=\frac{\Gamma_q(-\alpha)(q;q)_n q^n}{\Gamma(\alpha+1)\Gamma(-\alpha)(q^{\alpha+1};q)_n(1-q)^{\alpha+1}}\times\int_0^\infty\frac{x^\alpha}{(-x;q)_\infty}f(x)\mathscr{L}_n^{(\alpha)}(x;q)\,dx, \tag{39}$$

$$D_{mn}=\frac{\Gamma_q(-\alpha)(q;q)_n q^{\alpha n-n-\binom{\alpha+2}{2}}}{\Gamma(\alpha+1)\Gamma(-\alpha)(q^{\alpha+1};q)_n(1-q)^{\alpha+1}}\times\int_0^\infty x^\alpha(x;q)_\infty f(x)\mathscr{M}_n^{(\alpha)}(x;q)\,dx. \tag{40}$$

Proof. Multiplying (38) by $x^\alpha\mathscr{L}_n(x;q)/(-x;q)_\infty$ and integrating term by term over the interval $(0,\infty)$, using (6), we obtain the proof of (39). Similarly, taking $\beta=\alpha$ in (35), we deduce

$$\int_0^\infty \mathscr{M}_m^{(\alpha)}(x;q)\,\mathscr{M}_n^{(\alpha)}(x;q)\,x^\alpha(x;q)_\infty dx = -q^{\binom{\alpha+2}{2}+n-\alpha n}\frac{(q^{\alpha+1};q)_n(q^{-\alpha};q)_\infty}{(q;q)_n(q;q)_\infty}\frac{\pi\delta_{m,n}}{\sin\pi\alpha}, \tag{41}$$

so we also gain the proof of (40). The proof of Lemma 11 is complete. □

Lemma 12. *For* $\min\{\alpha,\beta\} > -1$, *one has*

$$\begin{aligned}\mathscr{L}_n^{(\alpha)}(xy;q) &= \sum_{k=0}^{n}\begin{bmatrix} n+\alpha \\ n-k\end{bmatrix} y^k q^{(\alpha-\beta)k}\mathscr{L}_k^{(\beta)}(x;q) \\ &\quad \times {}_2\phi_1\left[\begin{matrix} q^{k-n}, q^{\beta+1+k} \\ q^{\alpha+k+1}\end{matrix}; yq^{\alpha-\beta+n-k}\right],\end{aligned} \tag{42}$$

$$\begin{aligned}\mathscr{M}_n^{(\alpha)}(xy;q) &= \sum_{k=0}^{n}\begin{bmatrix} n+\alpha \\ n-k\end{bmatrix} y^k q^{(k-n+\beta)k-\alpha n}\mathscr{M}_k^{(\beta)}(x;q) \\ &\quad \times {}_2\phi_1\left[\begin{matrix} q^{k-n}, q^{\beta+1+k} \\ q^{\alpha+k+1}\end{matrix}; yq\right].\end{aligned} \tag{43}$$

Remark 13. Replacing x by $(1-q)x$ and letting $q \to 1$, we have [15, equation (11)]

$$\begin{aligned}L_n^{(\alpha)}(xy) &= \sum_{k=0}^{n}\binom{n+\alpha}{n-k} y^k L_k^{(\beta)}(x)\ {}_2F_1 \\ &\quad \times\left[\begin{matrix} -n+k, \beta+k+1 \\ \alpha+k+1\end{matrix}; y\right].\end{aligned} \tag{44}$$

Setting $\alpha = \beta$, (42) and (43) reduce to (29) and (30), respectively.

Proof. Let

$$\begin{aligned}A &= \frac{\Gamma_q(-\beta)(q;q)_k q^k}{\pi\csc(-\beta\pi)(q^{\beta+1};q)_k(1-q)^{1+\beta}}, \\ B &= (1-q)^{1+\beta}\frac{\pi\csc(-\beta\pi)}{\Gamma_q(-\beta)}.\end{aligned} \tag{45}$$

By Lemmas 10 and 11, the coefficient of $\mathscr{L}_n^{(\alpha)}(xy;q)$ expanded by $\mathscr{L}_n^{(\beta)}(x;q)$

$$\begin{aligned}C_{nk} &= A\cdot\sum_{j=0}^{n}\frac{(q^{\alpha+1};q)_n(y;q)_{n-j}y^j}{(q;q)_{n-j}(q^{\alpha+1};q)_j} \\ &\quad\times\int_0^\infty \mathscr{L}_j^{(\alpha)}(x;q)\mathscr{L}_k^{(\beta)}(x;q)\frac{x^\beta}{(-x;q)_\infty}dx \\ &= AB\cdot\sum_{j=0}^{n}\frac{(q^{\alpha+1};q)_n(y;q)_{n-j}y^j}{(q;q)_{n-j}(q^{\alpha+1};q)_j} \\ &\quad\times(-1)^{k+j}q^{(\alpha-\beta)j+\binom{j}{2}+\binom{k}{2}} \\ &\quad\times\begin{bmatrix}\beta-\alpha \\ j-k\end{bmatrix}\begin{bmatrix}k+\beta \\ k\end{bmatrix}q^{-kj} \\ &= AB\cdot\sum_{j=0}^{n-k}\frac{(q^{\alpha+1};q)_n(y;q)_{n-j-k}y^{j+k}}{(q;q)_{n-j-k}(q^{\alpha+1};q)_{j+k}} \\ &\quad\times\begin{bmatrix}\beta-\alpha \\ j\end{bmatrix}\begin{bmatrix}k+\beta \\ k\end{bmatrix}(-1)^j q^{(\alpha-\beta)(j+k)-k+\binom{j}{2}} \\ &= AB\cdot\frac{(q^{\alpha+1};q)_n(y;q)_{n-k}y^k}{(q;q)_{n-k}(q^{\alpha+1};q)_k}q^{(\alpha-\beta-1)k} \\ &\quad\times\begin{bmatrix}k+\beta \\ k\end{bmatrix}{}_3\phi_2\left[\begin{matrix} q^{k-n}, q^{\alpha-\beta}, 0 \\ \dfrac{q^{k-n+1}}{y}, q^{\alpha+k+1}\end{matrix}; q\right] \\ &= AB\cdot\frac{(q^{\alpha+1};q)_n y^k}{(q;q)_{n-k}(q^{\alpha+1};q)_k}q^{(\alpha-\beta-1)k} \\ &\quad\times\begin{bmatrix}k+\beta \\ k\end{bmatrix}{}_2\phi_1\left[\begin{matrix} q^{k-n}, q^{\beta+1+k} \\ q^{\alpha+k+1}\end{matrix}; yq^{\alpha-\beta+n-k}\right]\end{aligned} \tag{46}$$

is equal to the right hand side of (42). Similarly, we have

$$\begin{aligned}D_{nk} &= A\cdot q^{-\binom{\beta+2}{2}-2k+\beta k} \\ &\quad\times\int_0^\infty \mathscr{M}_n^{(\alpha)}(xy;q)\mathscr{M}_k^{(\beta)}(x;q)x^\beta(x;q)_\infty dx \\ &= AB\cdot q^{-\binom{\beta+2}{2}-2k+\beta k} \\ &\quad\times\sum_{j=0}^{n}\frac{(-1)^{n-j}(q^{\alpha+1};q)_n(1/y;q)_{n-j}y^n}{(q;q)_{n-j}(q^{\alpha+1};q)_j}q^{\binom{j}{2}+\alpha j-\binom{n}{2}-\alpha n} \\ &\quad\times(-1)^{j+k}q^{\binom{j+1}{2}+\binom{k+1}{2}-(\alpha+j)k+\binom{\beta+2}{2}} \\ &\quad\times\begin{bmatrix}\beta-\alpha \\ j-k\end{bmatrix}\begin{bmatrix}k+\beta \\ k\end{bmatrix} \\ &= AB\cdot q^{-2k+\beta k-\alpha n-\binom{n}{2}+\binom{k+1}{2}-\alpha k} \\ &\quad\times\sum_{j=0}^{n-k}\frac{(-1)^{n+k}(q^{\alpha+1};q)_n(1/y;q)_{n-k}(q^{n-j-k+1};q)_j y^n}{(q^{n-j-k}/y;q)_j(q;q)_{n-k}(q^{\alpha+1};q)_{j+k}} \\ &\quad\times q^{(j+k)^2+\alpha(j+k)-(j+k)k} \\ &\quad\times\frac{(q^{\beta-\alpha-j+1};q)_j}{(q;q)_j}\begin{bmatrix}k+\beta \\ k\end{bmatrix}\end{aligned}$$

$$= AB \cdot q^{-2k+\beta k-\alpha n-\binom{n}{2}+\binom{k+1}{2}} \times \frac{(-1)^{n+k}(q^{\alpha+1};q)_n(1/y;q)_{n-k}y^n}{(q;q)_{n-k}(q^{\alpha+1};q)_k} \begin{bmatrix} k+\beta \\ k \end{bmatrix} \times {}_2\phi_2 \left[\begin{matrix} q^{k-n}, q^{\alpha-\beta} \\ yq^{k+1-n}, q^{\alpha+k+1} \end{matrix} ; yq^{\beta+k+2} \right], \tag{47}$$

which is equivalent to the right hand side of (43) by (25) and simplification. The proof of Lemma 12 is complete. □

Proof of Theorems 2 and 3. By using formula (42), the left hand side of (7) is equal to

$$\sum_{k=0}^{m} \begin{bmatrix} m+\alpha \\ m-k \end{bmatrix} y^k q^{(\alpha-\mu)k} \times {}_2\phi_1 \left[\begin{matrix} q^{k-m}, q^{\mu+1+k} \\ q^{\alpha+k+1} \end{matrix} ; yq^{\alpha-\mu+m-k} \right] \times \sum_{j=0}^{n} \begin{bmatrix} n+\beta \\ n-j \end{bmatrix} z^j q^{(\beta-\mu)j} \times {}_2\phi_1 \left[\begin{matrix} q^{j-n}, q^{\mu+1+j} \\ q^{\beta+j+1} \end{matrix} ; zq^{\beta-\mu+n-j} \right] \times \int_0^\infty \mathscr{L}_k^{(\mu)}(x;q)\, \mathscr{L}_j^{(\mu)}(x;q) \frac{x^\mu}{(-x;q)_\infty} dx. \tag{48}$$

Similarly, with the help of formula (43), the left hand side of (8) equals

$$\sum_{k=0}^{m} \begin{bmatrix} m+\alpha \\ m-k \end{bmatrix} y^k q^{(k-m+\mu)k-\alpha m} {}_2\phi_1 \times \left[\begin{matrix} q^{k-m}, q^{\mu+1+k} \\ q^{\alpha+k+1} \end{matrix} ; yq \right] \times \sum_{j=0}^{n} \begin{bmatrix} n+\beta \\ n-j \end{bmatrix} z^j q^{(j-n+\mu)j-\beta n} \times {}_2\phi_1 \left[\begin{matrix} q^{j-n}, q^{\mu+1+j} \\ q^{\beta+j+1} \end{matrix} ; zq \right] \times \int_0^\infty \mathscr{M}_k^{(\mu)}(x;q)\, \mathscr{M}_j^{(\mu)}(x;q)\, x^\mu (x;q)_\infty dx. \tag{49}$$

Using formulas (41) and (43) and noticing that the orthogonality of previous two types of q-Laguerre polynomials for the case of $k = j$, we can deduce (7) and (8). The proof of Theorems 2 and 3 is complete. □

3. Representations of q-Hermite Polynomials

Doha [34, page 5460] deduced the following result by third-order recurrence relation of the coefficients.

Proposition 14 (see [34, equation (49)]). *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$H_n(x) = 2^n (1+\alpha)_n \times \sum_{k=0}^{n} {}_2F_2 \left[\begin{matrix} \frac{-(n-k)}{2}, \frac{-(n-k-1)}{2} \\ \frac{-(\alpha+n)}{2}, \frac{-(\alpha+n-1)}{2} \end{matrix} ; -\frac{1}{4} \right] \times \frac{(-n)_k L_k^{(\alpha)}(x)}{(1+\alpha)_k}. \tag{50}$$

In this section, we employ the technique of rearrangement of series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathscr{A}(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathscr{A}(k,n-k) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \mathscr{A}(k,n-2k) \tag{51}$$

to derive the following q-analogue of Proposition 14.

Theorem 15. *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$\mathscr{H}_n(x;q) = (q^{\alpha+1}, q; q)_n (1-q)^{-n/2} \sum_{k=0}^{n} C(n,k)\, \mathscr{L}_k^{(\alpha)}(x;q), \tag{52}$$

where

$$C(n,k) = \frac{(-1)^k q^{-(n^2+n+4\alpha n+4nk-2k^2)/4}}{(q^{\alpha+1};q)_k (q;q)_{n-k}} \times \sum_{s=0}^{[(n-k)/2]} \frac{(-1)^s (q^{k-n};q)_{2s} (1-q)^s}{(q^{-(n+\alpha)};q)_{2s} (-q^{-n/2};q^{1/2})_{2s} (q;q)_s}, \tag{53}$$

and $[x]$ denotes the greatest integer not exceeding x.

Theorem 16. *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$\mathscr{G}_n(x;q) = (q^{\alpha+1}, q; q)_n (1-q)^{-n/2} q^{-n(n-1)/4} \times \sum_{k=0}^{n} D(n,k)\, \mathscr{M}_k^{(\alpha)}(x;q), \tag{54}$$

where

$$D(n,k) = \frac{(-1)^k q^{\binom{k}{2}+\alpha k}}{(q^{\alpha+1};q)_k (q;q)_{n-k}} \times \sum_{s=0}^{[(n-k)/2]} \frac{(-1)^s q^{\binom{s}{2}} (1-q)^s (q^{k-n};q)_{2s} q^{(2s^2-s-4sk-4s\alpha)/2}}{(q;q)_s (q^{-(n+\alpha)};q)_{2s} (-q^{-n/2};q^{1/2})_{2s}}. \tag{55}$$

Before the proof of Theorem 15 the following lemma is necessary.

Lemma 17. *For $\alpha > -1$ and $n \in \mathbb{N}$, one has*

$$x^n = \left(q^{\alpha+1}, q; q\right)_n q^{-(\alpha n+n^2)} \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{n-k}{2}} \mathscr{L}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k (q;q)_{n-k}},$$
$$x^n = \left(q^{\alpha+1}, q; q\right)_n q^n \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}+\alpha k} \mathscr{M}_k^{(\alpha)}(x;q)}{(q;q)_{n-k} (q^{\alpha+1};q)_k}. \quad (56)$$

Proof. Letting $f(x) = x^m$ in (38) and using the following fact [8, page 23]:

$$\int_0^\infty \mathscr{L}_n^{(\alpha)}(x;q) \frac{x^{\alpha+m}}{(-x;q)_\infty} dx = \frac{(q^{-m};q)_n (q^{\alpha+1};q)_m \Gamma(-\alpha)\Gamma(\alpha+1)}{(q;q)_n \Gamma_q(-\alpha) q^{\alpha m+\binom{m+1}{2}} (1-q)^{-1-\alpha}}, \quad (57)$$

similarly, we deduce the explicit representation of (39) and (40), respectively,

$$C_{mn} = \frac{(q^{\alpha+1};q)_m (q^{-m};q)_n q^n}{(q^{\alpha+1};q)_n q^{\alpha m+\binom{\alpha+1}{2}}},$$
$$D_{mn} = \frac{(q^{\alpha+1};q)_m (q^{-m};q)_n q^{m(n+1)}}{(q^{\alpha+1};q)_n q^{-\alpha n}}, \quad (58)$$

so we obtain the formula (56). The proof is complete. □

Lemma 18 (see [7, equations (3.29.5) and (3.28.5)]). *One has*

$$\sum_{n=0}^{\infty} \frac{t^n}{(q^{1/2};q^{1/2})_n} \mathscr{H}_n(x;q) = \frac{\left(-xt(1-q)^{-1/2};q^{1/2}\right)_\infty}{(-t^2;q)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{t^n q^{n(n-1)/4}}{(q^{1/2};q^{1/2})_n} \mathscr{G}_n(x;q) = \frac{(t^2;q)_\infty}{\left(xt(1-q)^{-1/2};q^{1/2}\right)_\infty}. \quad (59)$$

Proof. By using [7, equations (3.29.5) and (3.28.5)]

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} \widetilde{h}_n(x;q)\, t^n = \frac{(-xt;q)_\infty}{(-t^2;q^2)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{h_n(x;q)}{(q;q)_n} t^n = \frac{(t^2;q^2)_\infty}{(xt;q)_\infty}. \quad (60)$$

and replacing, respectively, by

$$\mathscr{H}_n(x;q) = q^{n(n-1)/4} \widetilde{h}_n\left(\frac{x}{\sqrt{1-q}};q^{1/2}\right),$$
$$\mathscr{G}_n(x;q) = q^{-n(n-1)/4} h_n\left(\frac{x}{\sqrt{1-q}};q^{1/2}\right), \quad (61)$$

we deduce the proof of Lemma 18. The proof is complete. □

Proof of Theorem 15. From the generating function of $\mathscr{H}_n(x;q)$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{t^n}{(q^{1/2};q^{1/2})_n} \mathscr{H}_n(x;q) \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}(xt)^n}{(q^{1/2};q^{1/2})_n} (1-q)^{-n/2} \sum_{s=0}^{\infty} \frac{(-t^2)^s}{(q;q)_s} \\
&= \sum_{n,s=0}^{\infty} \frac{q^{n(n-1)/4} t^{n+2s} (-1)^s x^n}{(q^{1/2};q^{1/2})_n (q;q)_s} \\
&= \sum_{n,s=0}^{\infty} \frac{q^{n(n-1)/4} t^{n+2s} (-1)^s (1-q)^{-n/2}}{(q^{1/2};q^{1/2})_n (q;q)_s} \\
&\quad \times \left(q^{\alpha+1}, q; q\right)_n q^{-(\alpha n+n^2)} \\
&\quad \times \sum_{k=0}^{n} \frac{(-1)^j q^{\binom{n-k}{2}} \mathscr{L}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k (q;q)_{n-k}} \\
&= \sum_{n,s,k=0}^{\infty} \Bigg[\Big(q^{(n+k)(n+k-1)/4} (-1)^s t^{n+2s+k} \left(q^{\alpha+1};q\right)_{n+k} \\
&\qquad \times q^{-[\alpha(n+k)+(n+k)^2]} (1-q)^{-(n+k)/2} \Big) \\
&\qquad \times \left(\left(q^{1/2};q^{1/2}\right)_{n+k} (q;q)_s \right)^{-1} \Bigg] \\
&\quad \times \frac{(-1)^k q^{\binom{n}{2}} \mathscr{L}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k (q;q)_n} \\
&= \sum_{n,k=0}^{\infty} \sum_{s=0}^{[n/2]} \Bigg[\Big(q^{(n+k-2s)(n+k-2s-1)/4} (-1)^s t^{n+k} \\
&\qquad \times \left(q^{\alpha+1};q\right)_{n-2s+k} q^{-[\alpha(n-2s+k)+(n-2s+k)^2]} \Big) \\
&\qquad \times \left(\left(q^{1/2};q^{1/2}\right)_{n-2s+k} (q;q)_s \right)^{-1} \Bigg] \\
&\quad \times (1-q)^{-(n+k-2s)/2} \frac{(-1)^k q^{\binom{n-2s}{2}} \mathscr{L}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k (q;q)_{n-2s}} \\
&= \sum_{n,k=0}^{\infty} \frac{(-1)^k t^{n+k} \mathscr{L}_k^{(\alpha)}(x;q) (1-q)^{-(n+k)/2}}{(q^{\alpha+1};q)_k} \\
&\quad \times \sum_{s=0}^{[n/2]} \frac{(-1)^s \left(q^{\alpha+1}, q; q\right)_{n-2s+k} (1-q)^s}{(q^{1/2};q^{1/2})_{n+k-2s} (q;q)_{n-2s} (q;q)_s} \\
&\qquad \times q^{(n+k-2s)(n+k-2s-1)/4} \\
&\qquad \times q^{-[\alpha(n+k-2s)+(n+k-2s)^2]+\binom{n-2s}{2}} \\
&= \sum_{n,k=0}^{\infty} \Bigg[\Big((-1)^k t^{n+k} \mathscr{L}_k^{(\alpha)}(x;q) (1-q)^{-(n+k)/2}
\end{aligned}$$

$$\times\left(q^{\alpha+1};q\right)_{n+k}\left(-q^{1/2};q^{1/2}\right)_{n+k}\Big)$$
$$\times\left(\left(q^{\alpha+1};q\right)_k(q;q)_n\right)^{-1}\Big]$$
$$\times\sum_{s=0}^{[n/2]}\frac{(-1)^s(q^{-n};q)_{2s}(1-q)^s}{(q^{-(n+k+\alpha)};q)_{2s}(-q^{-(n+k)/2};q^{1/2})_{2s}(q;q)_s}$$
$$\times q^{(n+k-2s)(n+k-2s-1)/4-[\alpha(n+k-2s)+(n+k-2s)^2]}$$
$$\times q^{+\binom{n-2s}{2}-2s(\alpha+k)-s(n+k)+s(2s-1)/2}$$
$$=\sum_{n=0}^{\infty}\left(q^{\alpha+1};q\right)_n\left(-q^{1/2};q^{1/2}\right)_n(1-q)^{-n/2}t^n$$
$$\times\sum_{k=0}^{n}\frac{(-1)^k\mathscr{L}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k(q;q)_{n-k}}$$
$$\times\sum_{s=0}^{[(n-k)/2]}\frac{(-1)^s\left(q^{k-n};q\right)_{2s}(1-q)^s}{(q^{-(n+\alpha)};q)_{2s}(-q^{-n/2};q^{1/2})_{2s}(q;q)_s}$$
$$\times q^{(n-2s)(n-2s-1)/4-[\alpha(n-2s)+(n-2s)^2]}$$
$$\times q^{+\binom{n-k-2s}{2}-2s(\alpha+k)-sn+s(2s-1)/2}. \tag{62}$$

Comparing the coefficients of t^n on both sides of (62), we obtain the results. □

Proof of Theorem 16. From the generating function of $\mathscr{G}_n(x;q)$, we have

$$\sum_{n=0}^{\infty}\frac{t^nq^{n(n-1)/4}}{(q^{1/2};q^{1/2})_n}\mathscr{G}_n(x;q)$$
$$=\sum_{n,s=0}^{\infty}\frac{(-1)^sq^{\binom{s}{2}}x^n(1-q)^{-n/2}t^{n+2s}}{(q^{1/2};q^{1/2})_n(q;q)_s}$$
$$=\sum_{n,s,k=0}^{\infty}\Big[\Big((-1)^sq^{\binom{s}{2}+n}(1-q)^{-(n+k)/2}t^{n+k+2s}$$
$$\times\left(q^{\alpha+1};q\right)_{n+k}(-1)^kq^{\binom{k}{2}+\alpha k}\mathscr{M}_k^{(\alpha)}(x;q)\Big)$$
$$\times\left(\left(q^{1/2};q^{1/2}\right)_{n+k}(q;q)_n\left(q^{\alpha+1};q\right)_k(q;q)_s\right)^{-1}\Big]$$
$$=\sum_{n,k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}+\alpha k+n}\mathscr{M}_k^{(\alpha)}(x;q)(1-q)^{-(n+k)/2}t^{n+k}}{(q^{\alpha+1};q)_k}$$
$$\times\sum_{s=0}^{[n/2]}\frac{(-1)^sq^{\binom{s}{2}-2s}(1-q)^s\left(q^{\alpha+1};q\right)_{n-2s+k}}{(q^{1/2};q^{1/2})_{n+k-2s}(q;q)_{n-2s}(q;q)_s}$$
$$=\sum_{n,k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}+\alpha k+n}\mathscr{M}_k^{(\alpha)}(x;q)(1-q)^{-(n+k)/2}t^{n+k}}{(q^{\alpha+1};q)_k}$$
$$\times\frac{\left(q^{\alpha+1};q\right)_{n+k}\left(-q^{1/2};q^{1/2}\right)_{n+k}}{(q;q)_n}$$
$$\times\sum_{s=0}^{[n/2]}\frac{(-1)^sq^{\binom{s}{2}-2s}(1-q)^s(q^{-n};q)_{2s}}{(q;q)_s(q^{-(n+k+\alpha)};q)_{2s}(-q^{-(n+k)/2};q^{1/2})_{2s}}$$
$$\times q^{-2s(k+\alpha)+s(2s-1)/2}$$
$$=\sum_{n=0}^{\infty}\left(q^{\alpha+1};q\right)_n\left(-q^{1/2};q^{1/2}\right)_n(1-q)^{-n/2}t^n$$
$$\times\sum_{k=0}^{n}\frac{(-1)^kq^{\binom{k}{2}+\alpha k+n-k}\mathscr{M}_k^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_k(q;q)_{n-k}}$$
$$\times\sum_{s=0}^{[(n-k)/2]}\frac{(-1)^sq^{\binom{s}{2}-2s}(1-q)^s\left(q^{k-n};q\right)_{2s}}{(q;q)_s(q^{-(n+\alpha)};q)_{2s}(-q^{-n/2};q^{1/2})_{2s}}$$
$$\times q^{-2s(k+\alpha)+s(2s-1)/2}. \tag{63}$$

Equating the coefficients of t^n on both sides of (63), we obtain the results. □

Proof of Corollary 23. In view of the fact that

$$\left(q^{\beta};q\right)_{2n}=\left(q^{\beta/2},-q^{\beta/2},q^{(\beta+1)/2},-q^{(\beta+1)/2};q\right)_n,\quad |\beta|\in\mathbb{N}, \tag{64}$$

we have

$$\lim_{q\to1}\sum_{s=0}^{[(n-k)/2]}\frac{(-1)^s\left(q^{k-n};q\right)_{2s}(1-q)^sq^{\binom{s}{2}-2s}}{(q^{-(n+\alpha)};q)_{2s}(-q^{-n/2};q^{1/2})_{2s}(q;q)_s}$$
$$=\lim_{q\to1}\sum_{s=0}^{[(n-k)/2]}\Big[\Big((-1)^s\Big(q^{-(n-k)/2},-q^{-(n-k)/2},q^{-(n-k-1)/2},$$
$$-q^{-(n-k-1)/2};q\Big)_S\Big)$$
$$\times\Big(\Big(q^{-(n+\alpha)/2},-q^{-(n+\alpha)/2},$$
$$q^{-(n+\alpha-1)/2},-q^{-(n+\alpha-1)/2};q\Big)_S\Big)^{-1}\Big]$$
$$\times\frac{(1-q)^sq^{\binom{s}{2}-2s}}{(-q^{-n/2};q^{1/2})_{2s}(q;q)_s}$$
$$={}_2F_2\left[\begin{matrix}\dfrac{-(n-k)}{2},\dfrac{-(n-k-1)}{2}\\ \dfrac{-(\alpha+n)}{2},\dfrac{-(\alpha+n-1)}{2}\end{matrix};-\frac{1}{4}\right],$$

$$\lim_{\substack{x\to(1-q)x,\\ q\to 1}} \frac{\left(q^{\alpha+1};q\right)_n (q;q)_n}{(1-q)^n} \sum_{k=0}^{n} \frac{(-1)^k \mathscr{L}_k^{(\alpha)}(x;q)}{\left(q^{\alpha+1};q\right)_k (q;q)_{n-k}}$$
$$\times q^{-n^2/4-n/4-\alpha n-nk+k^2/2}$$
$$= \lim_{\substack{x\to(1-q)x,\\ q\to 1}} \frac{\left(q^{\alpha+1},q;q\right)_n}{(1-q)^n} q^{-n(n-1)/4}$$
$$\times \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}+\alpha k} \mathscr{M}_k^{(\alpha)}(x;q)}{\left(q^{\alpha+1};q\right)_k (q;q)_{n-k}}$$
$$= (1+\alpha)_n \sum_{k=0}^{n} \frac{(-n)_k L_k^{(\alpha)}(x)}{(1+\alpha)_k}. \tag{65}$$

Combing (65) and (2), we deduce (50). The proof of Corollary 23 is complete. □

4. Integrals Related to Generalized q-Hermite Polynomials

The authors [23] deduced the following interesting result inspired by the relation of (16) and the orthogonality of q-Laguerre polynomials (6).

Proposition 19 (see [23, Theorem 1]). *The sequence of the q-polynomials $\{\mathscr{H}_n^{(\mu)}(x;q)\}$, which are defined by the relations* (16), *satisfies the orthogonality relation*

$$\int_{-\infty}^{\infty} \mathscr{H}_m^{(\mu)}(x;q)\, \mathscr{H}_n^{(\mu)}(x;q) \frac{|x|^{2\mu}}{\left(-x^2;q\right)_\infty} dx$$
$$= \frac{\pi}{\cos \pi\mu} \frac{\left(q^{1/2-\mu};q\right)_\infty}{(q;q)_\infty} q^{-n/2-\mu\theta_n} \tag{66}$$
$$\times (q;q)_{[n/2]} \left(q^{\mu+1/2};q\right)_{[(n+1)/2]} \delta_{mn}$$

on the whole real line $\mathbb{R}$, where $\theta_n = n - 2[n/2]$.

In this section, we will further consider multivariate q-Hermite polynomials by Theorems 2 and 3.

Theorem 20. *For $\theta_n = n - 2[n/2]$, one has*

$$\int_{-\infty}^{\infty} \mathscr{H}_m^{(\alpha)}(xy;q)\, \mathscr{H}_n^{(\beta)}(xz;q) \frac{|x|^{2\mu}}{\left(-x^2;q\right)_\infty} dx$$
$$= (-1)^{[m/2]+[n/2]} (q;q)_{[m/2]} (q;q)_{[n/2]}$$
$$\times (1-q)^{\mu+1/2+\theta_n} \frac{\pi \csc\left(-\mu+1/2-\theta_n\right)\pi}{\Gamma_q\left(-\mu+1/2-\theta_n\right)}$$
$$\times \sum_{k=0}^{\min\{[m/2],[n/2]\}} q^{(\alpha+\beta-2\mu-1)k} (yz)^{2k+\theta_n}$$
$$\times {}_2\phi_1\left[\begin{matrix} q^{k-[m/2]}, q^{\mu+1/2+k+\theta_n} \\ q^{\alpha+k+1/2+\theta_n}\end{matrix}; y^2 q^{\alpha-\mu+[m/2]-k}\right]$$
$$\times {}_2\phi_1\left[\begin{matrix} q^{k-[n/2]}, q^{\mu+1/2+k+\theta_n} \\ q^{\beta+k+1/2+\theta_n}\end{matrix}; z^2 q^{\beta-\mu+[n/2]-k}\right]$$
$$\times \left[\begin{matrix}\left[\frac{m}{2}\right]+\alpha-\frac{1}{2}+\theta_n \\ \left[\frac{m}{2}\right]-k\end{matrix}\right] \left[\begin{matrix}\left[\frac{n}{2}\right]+\beta-\frac{1}{2}+\theta_n \\ \left[\frac{n}{2}\right]-k\end{matrix}\right]$$
$$\times \left[\begin{matrix} k+\mu-\frac{1}{2}+\theta_n \\ k\end{matrix}\right]. \tag{67}$$

Theorem 21. *For $\theta_n = n - 2[n/2]$, one has*

$$\int_{-\infty}^{\infty} \mathscr{G}_m^{(\alpha)}(xy;q)\, \mathscr{G}_n^{(\beta)}(xz;q) |x|^{2\mu} \left(x^2;q\right)_\infty dx$$
$$= (q;q)_{[m/2]} (q;q)_{[n/2]} (1-q)^{\mu+1/2+\theta_n}$$
$$\times q^{\binom{\mu+3/2+\theta_n}{2}-(\alpha-1/2+\theta_n)[m/2]-(\beta-1/2+\theta_n)[n/2]}$$
$$\times \frac{\pi \csc\left(-\mu+1/2-\theta_n\right)\pi}{\Gamma_q\left(-\mu+1/2-\theta_n\right)}$$
$$\times \sum_{k=0}^{\min\{[m/2],[n/2]\}} q^{(2k+\mu+1/2-[m/2]-[n/2]+\theta_n)k}$$
$$\times (yz)^{2k+\theta_n} {}_2\phi_1\left[\begin{matrix} q^{k-[m/2]}, q^{\mu+1/2+k+\theta_n} \\ q^{\alpha+k+1/2+\theta_n}\end{matrix}; y^2 q\right] \tag{68}$$
$$\times {}_2\phi_1\left[\begin{matrix} q^{k-[n/2]}, q^{\mu+1/2+k+\theta_n} \\ q^{\beta+k+1/2+\theta_n}\end{matrix}; z^2 q\right]$$
$$\times \left[\begin{matrix}\left[\frac{m}{2}\right]+\alpha-\frac{1}{2}+\theta_n \\ \left[\frac{m}{2}\right]-k\end{matrix}\right] \left[\begin{matrix}\left[\frac{n}{2}\right]+\beta-\frac{1}{2}+\theta_n \\ \left[\frac{n}{2}\right]-k\end{matrix}\right]$$
$$\times \left[\begin{matrix} k+\mu-\frac{1}{2}+\theta_n \\ k\end{matrix}\right].$$

Remark 22. For $\alpha = \beta = \mu$ and $y = z = 1$, Theorems 20 and 21 reduce to Proposition 19 and Corollary 23 respectively.

Corollary 23. *The sequence of the q-polynomials $\{\mathscr{G}_n^{(\mu)}(x;q)\}$, which are defined by the relations* (16), *satisfies the orthogonality relation*

$$\int_{-\infty}^{\infty} \mathscr{G}_m^{(\mu)}(x;q)\, \mathscr{G}_n^{(\mu)}(x;q) \frac{|x|^{2\mu}}{\left(-x^2;q\right)_\infty} dx$$
$$= \frac{\pi}{\cos \pi\mu} \frac{\left(q^{1/2-\mu};q\right)_\infty}{(q;q)_\infty}$$

$$\times q^{\binom{\mu+3/2}{2}-[n/2](\mu-3/2+\theta_n)+\theta_n}$$
$$\times (q;q)_{[n/2]}\left(q^{\mu+1/2};q\right)_{[(n+1)/2]}\delta_{mn} \tag{69}$$

on the whole real line $\mathbb{R}$, *where* $\theta_n = n - 2[n/2]$.

Proof of Theorems 20 and 21. Let $\mathcal{I}(\alpha, y, m; \beta, z, n; \mu)$ and $\mathcal{J}(\alpha, y, m; \beta, z, n; \mu)$ represent the right hand side of (7) and (8) respectively. Since the weight function in (67) is an even function of the independent variable x by the definition (16), so the polynomials are evidently orthogonal to each other when the degrees m and n are either simultaneously even or odd. From (16) and Theorem 2 we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathscr{H}_{2m}^{(\alpha)}(xy;q)\,\mathscr{H}_{2n}^{(\beta)}(xz;q)\,\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx\\
&= (-1)^{m+n}(q;q)_m(q;q)_n\\
&\quad\times\int_{-\infty}^{\infty}\mathscr{L}_m^{(\alpha-1/2)}\left(x^2y^2;q\right)\mathscr{L}_n^{(\beta-1/2)}\\
&\qquad\times\left(x^2z^2;q\right)\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx\\
&= 2(-1)^{m+n}(q;q)_m(q;q)_n\\
&\quad\times\int_0^{\infty}\mathscr{L}_m^{(\alpha-1/2)}\left(x^2y^2;q\right)\mathscr{L}_n^{(\beta-1/2)}\\
&\qquad\times\left(x^2z^2;q\right)\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx\\
&= (-1)^{m+n}(q;q)_m(q;q)_n\\
&\quad\times\int_0^{\infty}\mathscr{L}_m^{(\alpha-1/2)}\left(y^2t;q\right)\mathscr{L}_n^{(\beta-1/2)}\left(z^2t;q\right)\frac{t^{\mu-1/2}}{(-t;q)_\infty}dt\\
&= (-1)^{m+n}(q;q)_m(q;q)_n\\
&\quad\times\mathcal{I}\left(\alpha-\frac{1}{2},y^2,m;\beta-\frac{1}{2},z^2,n;\mu-\frac{1}{2}\right),
\end{aligned} \tag{70}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathscr{H}_{2m+1}^{(\alpha)}(xy;q)\,\mathscr{H}_{2n+1}^{(\beta)}(xz;q)\,\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx\\
&= 2(-1)^{m+n}yz(q;q)_m(q;q)_n\\
&\quad\times\int_0^{\infty}\mathscr{L}_m^{(\alpha+1/2)}\left(x^2y^2;q\right)\mathscr{L}_n^{(\beta+1/2)}\left(x^2z^2;q\right)\\
&\qquad\times\frac{|x|^{2\mu+2}}{(-x^2;q)_\infty}dx\\
&= (-1)^{m+n}yz(q;q)_m(q;q)_n\\
&\quad\times\int_0^{\infty}\mathscr{L}_m^{(\alpha+1/2)}\left(y^2t;q\right)\mathscr{L}_n^{(\beta+1/2)}\left(z^2t;q\right)\frac{t^{\mu+1/2}}{(-t;q)_\infty}dt\\
&= (-1)^{m+n}yz(q;q)_m(q;q)_n\\
&\quad\times\mathcal{I}\left(\alpha+\frac{1}{2},y^2,m;\beta+\frac{1}{2},z^2,n;\mu+\frac{1}{2}\right).
\end{aligned} \tag{71}$$

Putting (70) and (71) together and using Theorem 2, we complete the proof of Theorem 20 after some simplification. In the same way we find that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathscr{G}_{2m}^{(\alpha)}(xy;q)\,\mathscr{G}_{2n}^{(\beta)}(xz;q)\,|x|^{2\mu}\left(-x^2;q\right)_\infty dx\\
&= q^{-\binom{m+1}{2}-\binom{n+1}{2}}(q;q)_m(q;q)_n\mathcal{J}\\
&\quad\times\left(\alpha-\frac{1}{2},y^2,m;\beta-\frac{1}{2},z^2,n;\mu-\frac{1}{2}\right),\\
&\int_{-\infty}^{\infty} \mathscr{G}_{2m+1}^{(\alpha)}(xy;q)\,\mathscr{G}_{2n+1}^{(\beta)}(xz;q)\,|x|^{2\mu}\left(-x^2;q\right)_\infty dx\\
&= q^{-\binom{m+1}{2}-\binom{n+1}{2}}yz(q;q)_m(q;q)_n\\
&\quad\times\mathcal{J}\left(\alpha+\frac{1}{2},y^2,m;\beta+\frac{1}{2},z^2,n;\mu+\frac{1}{2}\right),
\end{aligned} \tag{72}$$

which are two cases of Theorem 21; thus we obtain the proof. □

Proof of Proposition 19 and Corollary 23. Let us consider first that the case of both m and n is even, and just take $\alpha = \beta = \mu$ and $y = z = 1$ in Theorem 20. We have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathscr{H}_{2m}^{(\mu)}(x;q)\,\mathscr{H}_{2n}^{(\mu)}(x;q)\,\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx\\
&= (-1)^{m+n}(q;q)_m(q;q)_n(1-q)^{\mu+1/2}\\
&\quad\times\frac{\pi\csc(-\mu+1/2)\pi}{\Gamma_q(-\mu+1/2)}\sum_{k=0}^{\min\{m,n\}}q^{-k}\\
&\quad\times{}_1\phi_0\left[\begin{matrix}q^{k-m}\\ -\end{matrix};q^{m-k}\right]{}_1\phi_0\left[\begin{matrix}q^{k-n}\\ -\end{matrix};q^{n-k}\right]\\
&\quad\times\begin{bmatrix}m+\alpha-\frac{1}{2}\\ m-k\end{bmatrix}\begin{bmatrix}n+\beta-\frac{1}{2}\\ n-k\end{bmatrix}\begin{bmatrix}k+\mu-\frac{1}{2}\\ k\end{bmatrix}\\
&= (q;q)_n(q;q)_n(1-q)^{\mu+1/2}\\
&\quad\times\frac{\pi\csc(-\mu+1/2)\pi}{\Gamma_q(-\mu+1/2)}q^{-n}\begin{bmatrix}n+\mu-\frac{1}{2}\\ n\end{bmatrix}\delta_{mn}\\
&= \frac{\pi}{\cos\mu\pi}\frac{\left(q^{1/2-\mu};q\right)_\infty}{(q;q)_\infty}q^{-n}(q;q)_n\left(q^{\mu+1/2};q\right)_n\delta_{mn},
\end{aligned} \tag{73}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathscr{H}_{2m+1}^{(\mu)}(x;q)\,\mathscr{H}_{2n+1}^{(\mu)}(x;q)\,\frac{|x|^{2\mu}}{(-x^2;q)_\infty}dx \\
&\quad = (q;q)_n (q;q)_n (1-q)^{\mu+3/2} \\
&\qquad \times \frac{\pi \csc(-\mu-1/2)\pi}{\Gamma_q(-\mu-1/2)} q^{-n} \begin{bmatrix} n+\mu+\frac{1}{2} \\ n \end{bmatrix} \delta_{mn} \\
&\quad = -\frac{\pi}{\cos\mu\pi}\frac{(q^{-\mu-1/2};q)_\infty}{(q;q)_\infty} \\
&\qquad \times q^{-n}(q;q)_n \frac{(q;q)_{n+\mu+1/2}}{(q;q)_{\mu+1/2}}\delta_{mn} \\
&\quad = \frac{\pi}{\cos\mu\pi}\frac{(q^{1/2-\mu};q)_\infty}{(q;q)_\infty} \\
&\qquad \times q^{-n-\mu-1/2}(q;q)_n (q^{\mu+1/2};q)_{n+1}\delta_{mn}.
\end{aligned} \tag{74}$$

Putting (73) and (74) together results in the orthogonality relation (66). The proof of Proposition 19 is complete. By the same way, we can deduce Corollary 23. This completes the proof. □

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Research Article

A Comparison between Adomian Decomposition and Tau Methods

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We present a comparison between Adomian decomposition method (ADM) and Tau method (TM) for the integro-differential equations with the initial or the boundary conditions. The problem is solved quickly, easily, and elegantly by ADM. The numerical results on the examples are shown to validate the proposed ADM as an effective numerical method to solve the integro-differential equations. The numerical results show that ADM method is very effective and convenient for solving differential equations than Tao method.

1. Introduction

The decomposition method was introduced by Adomian in [1–3] in the 1980s in order to solve linear and nonlinear functional equations (algebraic, differential, partial differential equations and systems, integral, delay, integro-differential equations, etc.) [1–10]. This method leads to computable, accurate, approximate convergence solutions to linear and nonlinear deterministic and stochastic operator equations. The solution can verify any stage of approximation. The convergence of this method was proved by Cherruault and coauthors in [11–13].

In this paper we will be concerned with approximate solutions of the linear or nonlinear Volterra integro-differential equations. Firstly, this type of equation was introduced by Volterra [14] in the early 1900s. These equations can be found in physics, biology, and engineering applications such as heat transfer, diffusion process in general, and neutron diffusion [4].

Many authors have compared the ADM with some existing methods in solving different linear or nonlinear evolution equations, integral and integro-differential equations. Bellomo and Monaco [15] compared the ADM and the perturbation techniques. Advantages of the ADM over Picard's method have been shown by Rach [16]. Edwards et al. [17] compared the ADM and the Runge-Kutta methods for approximate solutions of some predator-prey models. Additionally, Wazwaz [18] presented a comparison between the ADM and the Taylor series methods. He showed that the ADM minimizes the computational difficulties of the Taylor series in that the components of the solution were determined elegantly by using simple integrals. More recently, El-Sayed and Abdel-Aziz [19] introduced a comparison of the ADM and the Wavelet-Galerkin method for the solution of integro-differential equations. They showed that the ADM was simple and easy to use.

In [20], Hosseini and Shahmorad employed Tau method to obtain a numerical solution to the integro-differential equations given by (1). Batiha et al. [21] presented the variational iteration method (VIM) and the ADM for solving nonlinear integro-differential equations. Fariborzi Araghi and Sadigh Behzadi [22–24] solved nonlinear Volterra-Fredholm integro-differential equations by using the modified ADM, the VIM, and the homotopy analysis method, respectively. Borhanifar and Abazari [25] implemented the differential transform method for solving nonlinear integro-differential equations with the kernel functions including derivative type of unknown solution. Ben Zitoun and Cherruault [26] presented a method for solving nonlinear integro-differential equations with constant or variable coefficients with initial or boundary conditions. El-Kalla [27] introduced a new technique for solving a class of quadratic integral and integro-differential equations.

In this work, we will describe and adapt Adomian's decomposition method to obtain an approximate solution for (1). As we will see, the method converges rapidly. The balance of this paper is as follows: in Section 2, we will give analysis of ADM for the problem; in Section 3 we will give three examples to demonstrate the method. Concluding remarks are given in the last section.

2. Analysis

We consider the nonlinear Volterra integro-differential equations of the form in [4] as follows:

$$\begin{aligned} u^{(n)}(x) &= f(x) + \int_0^x K(x,t)\, Nu(t)\, dt, \\ u^{(m)}(0) &= c_m, \quad 0 \le m \le (n-1), \end{aligned} \tag{1}$$

where $u^{(n)}(x)$ indicates the nth derivative of $u(x)$ with respect to x, c_m constants that define the initial conditions, and Nu is nonlinear operator. In this work we take Nu equal to u^2 or uu_x. Thus, applying the inverse operator L^{-1} to (1) yields

$$\begin{aligned} u(x) = {} & \sum_{k=0}^{n-1} \frac{1}{k!} c_m x^k + L^{-1}\left[f(x)\right] \\ & + L^{-1}\left(\int_0^x K(x,t)\, Nu(t)\, dt\right), \end{aligned} \tag{2}$$

where $\sum_{k=0}^{n-1}(1/k!)c_m x^k$ is obtained by using the initial conditions in [4] and L^{-1} is n-fold integration operator; that is,

$$L^{-1}(\cdot) = \underbrace{\int_0^x \cdots \int_0^x}_{n\text{-times}} (\cdot)\, \underbrace{dx \cdots dx}_{n\text{-times}}. \tag{3}$$

We obtain the zeroth component

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} c_m x^k + L^{-1}\left[f(x)\right], \tag{4}$$

which is defined by all terms that arise from the initial conditions and from integrating the source terms. Then, decomposing the unknown function $u(x)$ gives a sum of the component defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{5}$$

Since the nonlinear terms $Nu = u^2$ or $Nu = uu_x$, then it can be expressed as

$$F(u) = Nu = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where A_n appropriate Adomian is polynomial which is generated form of the following formula [1–3, 6]:

$$\begin{aligned} A_0 &= F(u_0), \\ A_1 &= u_1 F'(u_0), \\ A_2 &= u_2 F'(u_0) + \frac{1}{2} u_1^2 F''(u_0), \\ A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0), \\ &\ \vdots \end{aligned} \tag{7}$$

Substituting (5) and (6) into (2) yields,

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) = {} & \sum_{k=0}^{n-1} \frac{1}{k!} c_m x^k + L^{-1}\left[f(x)\right] \\ & + L^{-1}\left(\int_0^x K(x,t) \sum_{n=0}^{\infty} A_n(t)\, dt\right). \end{aligned} \tag{8}$$

The components $u_1(x), u_2(x), \ldots$ are completely determined by using the recurrent formula

$$\begin{aligned} u_1(x) &= L^{-1}\left(\int_0^x K(x,t)\, A_0(t)\, dt\right), \\ &\ \vdots \\ u_n(x) &= L^{-1}\left(\int_0^x K(x,t)\, A_{n-1}(t)\, dt\right), \end{aligned} \tag{9}$$

for $n \ge 0$. It is useful to note that the recursive formula is constructed on the basis that the zeroth component $u_0(x)$ is defined by all terms that arise from the initial conditions and from integrating the source terms. The remaining components $u_n(x)$, $n \ge 0$, can be completely determined such that each term is computed by using the previous term. As a result, the components $u_0(x), u_1(x), u_2(x), \ldots$ are identified, and the series solutions are thus entirely determined.

The n-term approximation ϕ_n is defined by

$$\phi_n = \sum_{k=0}^{n-1} u_k(x), \tag{10}$$

which can be used for numerical approximation.

3. Test Problems

In this section, we report on numerical results of some examples, selected through integral and integro-differential equations, solved by ADM. These examples can be solved analytically by reducing them to differential equations, and they are also solved numerically by Tau method in [20]. Here the aim is to solve these examples using the ADM given Section 2 and compare these results with the presented results in [20].

Problem 1. We mainly present the method using the algorithm given in Section 2. As a first example, consider the equation

$$y(s) - \int_0^s y(t)\,dt = 1, \quad 0 \le s \le 1. \tag{11}$$

In order to illustrate the proposed method, we get zeroth component

$$y_0 = 1 \tag{12}$$

and obtain $y_1(x), y_2(x), \ldots$ by using (9) to determine the other individual terms of the decomposition series. Thus

$$\begin{aligned} y_1 &= \int_0^s y_0(t)\,dt = s, \\ y_2 &= \int_0^s y_1(t)\,dt = \frac{1}{2!}s^2, \\ y_3 &= \int_0^s y_2(t)\,dt = \frac{1}{3!}s^3, \\ &\vdots \end{aligned} \tag{13}$$

and so on. Consequently, the series solution is obtained as

$$\begin{aligned} y(s) &= \sum_{n=0}^{\infty} y_n(s) = y_0 + y_1 + y_2 + y_3 + \cdots \\ &= 1 + s + \frac{1}{2!}s^2 + \frac{1}{3!}s^3 + \cdots, \end{aligned} \tag{14}$$

so that the closed form of the solution is

$$y(s) = e^s. \tag{15}$$

Problem 2. We consider Fredholm integro-differential equation which is given as follows [20]:

$$\begin{aligned} y''(s) - y(s) &+ \frac{1}{20}\int_0^1 t^{39} y(t)\,dt \\ &= -s^2 - 2s + \frac{2111}{344400}, \quad 0 \le s \le 1, \\ y(0) - y'(0) &= 0, \qquad y(1) + y'(1) = 9. \end{aligned} \tag{16}$$

Proceeding as before, we obtain

$$\begin{aligned} y_0 &= y(0) + sy'(0) + L^{-1}\left(-s^2 - 2s + \frac{2111}{344400}\right) \\ &= 2 + 2s + \frac{2111}{688800}s^2 - \frac{1}{3}s^3 - \frac{1}{12}s^4, \\ y_1 &= \int_0^s\int_0^s (y_0(s))\,dsds \\ &\quad - \frac{1}{20}\int_0^s\int_0^s\left(\int_0^1 t^{39} y_0(t)\,dt\right)dsds \\ &= s^2 - \frac{29772923}{13349952000}s^2 + \frac{1}{3}s^3 + \frac{2111}{8265600}s^4 \\ &\quad - \frac{1}{60}s^5 - \frac{1}{360}s^5, \\ y_2 &= \int_0^s\int_0^s (y_1(s))\,dsds \\ &\quad - \frac{1}{20}\int_0^s\int_0^s\left(\int_0^1 t^{39} y_1(t)\,dt\right)dsds \\ &= \frac{-5478328678327}{7049842652160000}s^2 + \frac{13320179077}{160199424000}s^4 \\ &\quad + \frac{1}{60}s^5 + \frac{2111}{247968000}s^6 - \frac{1}{2520}s^7 - \frac{1}{201160}s^8, \\ &\vdots \end{aligned} \tag{17}$$

Consequently, the series solution is found as

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0 + y_1 + y_2 + \cdots \\ &= 2 + 2s + \frac{2111}{688800}s^2 - \frac{1}{3}s^3 - \frac{1}{12}s^4 \\ &\quad + s^2 - \frac{29772923}{13349952000}s^2 + \frac{1}{3}s^3 + \cdots. \end{aligned} \tag{18}$$

In Table 1, ADM and TM values are presented which correspond to the various values of s. As it is seen in this table, the values obtained by [20] and the results we obtained which are close and but present method better accuracy and easy to use than the TM. It is to be noted that only few iterations were needed to obtain the accuracy for approximate solutions. The overall errors can be made even much smaller by adding new terms of the decomposition. Thus the convergence would be seen more rapidly.

The numerical solutions showed that ADM is a very convenient method for such linear and nonlinear integral and integro-differential equations. By using this method, it is possible to obtain more precise results than the traditional methods, with less calculations and consuming the less time.

Problem 3. In [4, 28], Wazwaz proposed that the construction of the zeroth component of the decomposition series can be defined in a slightly different way. In [4, 28], he assumed that if

Table 1: Numerical results for Problem 2.

s	Exact	Tau (n)	ADM (ϕ)	Tau-Er.	ADM-Er.
			$n = 2, \phi_2$		
0.00	2.00000	1.99878	2.00000	$1.2200e-03$	$0.000e+00$
0.20	2.44000	2.43858	2.43989	$1.4152e-03$	$1.051e-04$
0.40	2.96000	2.95849	2.95782	$1.5128e-03$	$2.175e-03$
0.60	3.56000	3.55849	3.54811	$1.5128e-03$	$1.200e-02$
			$n = 5, \phi_3$		
0.00	2.00000	1.99885	2.00000	$1.1525e-03$	$0.000e+00$
0.20	2.44000	2.43867	2.44000	$1.3342e-03$	$2.228e-06$
0.40	2.96000	2.95858	2.95999	$1.4226e-03$	$1.049e-06$
0.60	3.56000	3.55858	3.55988	$1.4211e-03$	$1.114e-04$
0.80	4.24000	4.23867	4.23924	$1.3308e-03$	$7.522e-04$
1.00	5.00000	4.99885	4.99691	$1.1485e-03$	$3.089e-03$
			$n = 15, \phi_5$		
0.00	2.00000	1.99884	2.00000	$1.1567e-03$	$0.000e+00$
0.20	2.44000	2.43866	2.44000	$1.3394e-03$	$6.597e-08$
0.40	2.96000	2.95857	2.96000	$1.4289e-03$	$2.671e-07$
0.60	3.56000	3.55857	3.56000	$1.4288e-03$	$6.143e-07$
0.80	4.24000	4.23866	4.24000	$1.3394e-03$	$1.940e-06$
1.00	5.00000	4.99884	5.00000	$1.1567e-03$	$1.380e-06$

the zeroth component is $y_0 = f$ and the function f is possible to be divided into two parts such as f_1 and f_2, then one can formulate the recursive algorithm in a form of a modified recursive scheme as follows:

$$\begin{gathered} y_0(s) = f_1, \\ y_1(s) = f_2 + L^{-1}\left[f(x)\right] + L^{-1}\left(\int_0^x K(x,t)\, y_1(t)\, dt\right), \\ y_{n+1}(s) = L^{-1}\left(\int_0^x K(x,t)\, y_n(t)\, dt\right), \quad n \geq 1. \end{gathered} \tag{19}$$

We finally consider the Volterra integral equation in the following form [20]:

$$\begin{aligned} y(s) = {} & 1 + 120s - 100\left(1 - e^{-s}\right) \\ & + \int_0^s \left(100e^{t-s} - 120\right) y(t)\, dt, \quad 0 \leq s \leq 20. \end{aligned} \tag{20}$$

Using the modified decomposition method, we first decompose the function $f(s)$ into two parts as f_1 and f_2, namely,

$$\begin{gathered} f_1(s) = 1, \\ f_2(s) = 120s - 100\left(1 - e^{-s}\right). \end{gathered} \tag{21}$$

Consequently, we obtain

$$\begin{aligned} y_0(s) &= 1, \\ y_1(s) &= 120s - 100\left(1 - e^{-s}\right) \\ &\quad + \int_0^s \left(100e^{t-s} - 120\right) y_0(t)\, dt = 0. \end{aligned} \tag{22}$$

Other components $y_n(s) = 0$ for $n \geq 2$. Therefore, the exact solution

$$y(s) = 1 \tag{23}$$

follows immediately. It is clear that two components are calculated to determine the exact solution.

4. Concluding Remarks

In this paper, we calculated the approximate solutions of the integral and Volterra integro-differential equations by using Adomian decomposition method. We demonstrated that the decomposition procedure is quite efficient in order to determine the solution in closed form by using initial and boundary conditions. Our present method avoids the tedious work needed by traditional techniques. In the studies by Hosseini and Shahmorad in [20], they spent more time, and boring operations were done to get approximate solutions by using TM. In our study, however, we got more accurate approximate solutions by using the initial condition in this method; Hosseini and Shahmorad in [20] obtained the approximate solutions for Problems 1 and 3, such that, the Tau-error is $2.73127e-08$ for $n = 10$ and $s = 1.00$ in Problem 1. Moreover, the exact solutions are obtained by our present method for Problems 1 and 3. Our method avoids the difficulties and massive computational work that usually arise from Wavelet-Galerkin, Tau, and finite difference methods.

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Research Article

On Solution of Fredholm Integrodifferential Equations Using Composite Chebyshev Finite Difference Method

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A new numerical method is introduced for solving linear Fredholm integrodifferential equations which is based on a hybrid of block-pulse functions and Chebyshev polynomials using the well-known Chebyshev-Gauss-Lobatto collocation points. Composite Chebyshev finite difference method is indeed an extension of the Chebyshev finite difference method and can be considered as a nonuniform finite difference scheme. The main advantage of the proposed method is reducing the given problem to a set of algebraic equations. Some examples are given to approve the efficiency and the accuracy of the proposed method.

1. Introduction

Linear and nonlinear Fredholm integrodifferential equations can be used to model many problems of science and theoretical physics such as engineering, biological models, electrostatics, control theory of industrial mathematics, economics, fluid dynamics, heat and mass transfer, oscillation theory, and queuing theory [1].

In recent years, many authors have considered different numerical methods to solve these kinds of problems. In 2012, Dehghan and Salehi employed [2] the meshless moving least square method for solving nonlinear Fredholm integrodifferential equations. A sequential method for the solution of Fredholm integrodifferential equations was presented by Berenguer et al. [3] in 2012. The formulation of the Fredholm integrodifferential equation in terms of an operator and the use of Schauder bases are the main tools of this method.

In [4], the operational Adomian-Tau method with Pade approximation was used for solving nonlinear Fredholm integrodifferential equations. This approach is based on two matrices, and Pade approximation was used to improve the accuracy of the method. Chebyshev finite difference method was proposed in [5] in order to solve Fredholm integrodifferential equations. In this scheme the problem is reduced to a set of algebraic equations. In [6], Legendre collocation matrix method was introduced for solving high-order linear Fredholm integrodifferential equations. In this way, the equation and its conditions are converted to matrix equations using collocation points on the interval $[-1, 1]$. Atabakan et al. [7, 8] proposed a modification of homotopy analysis method (HAM) known as spectral homotopy analysis method (SHAM) to solve linear Volterra and Fredholm integrodifferential equations. In this procedure, the Chebyshev pseudospectral method was used to obtain an approximation of solutions to higher-order equation. The semiorthogonal spline method was discussed in [9]. This approach is used to solve Fredholm integral and integrodifferential equations.

In this paper, we applied a composite Chebyshev finite difference (ChFD) method for solving Fredholm integrodifferential equations. Fredholm integro differential equation is given by

$$\begin{aligned}
&F\left(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)\right) \\
&\quad = f(x) + \lambda \int_0^T k(x,t)\, y(t)\, dt, \\
&G_r\left(y(\tau_0), \dots, y^{(n-1)}(\tau_0), \dots, y(\tau_n), \dots, y^{(n-1)}(\tau_n)\right) = 0, \\
&\qquad\qquad r = 0, \dots, n-1,
\end{aligned} \tag{1}$$

where $k(x,t)$, $f(x)$, and $y(x)$ are analytic functions, λ is a constant value, G_r, $r = 0,\ldots,n-1$, are linear functions and the points $\tau_0, \tau_1, \ldots, \tau_n$ lie in $[0,T]$. It will always be assumed that (1) possesses a unique solution $y \in C^n[0,T]$.

The base of the proposed method is a hybrid of block-pulse functions and Chebyshev polynomials using Chebyshev-Gauss-Lobatto points. This method was introduced and applied for solving the optimal control of delay systems with a quadratic performance index in [10, 11].

Chebyshev polynomials which are the eigenfunctions of a singular Sturm-Liouville problem have many advantages. They can be considered as a good representation of smooth functions by finite Chebyshev expansions provided that the function is infinitely differentiable. The Chebyshev expansion coefficients converge faster than any finite power of $1/m$ as m goes to infinity for problems with smooth solutions. The numerical differentiation and integration can be performed. Moreover, they have been applied to solve different kinds of boundary value problems [12–14].

The paper is organized in the following way. Section 2 includes some necessary preliminaries and notations. Chebyshev finite difference method and composite Chebyshev finite difference method for solving Fredholm integrodifferential equations are described in Sections 3 and 4, respectively. Convergence analysis of the proposed method is presented in Section 5. In Section 6 discretization of the method is introduced, and some numerical examples are presented in Section 7. In Section 8, concluding remarks are given.

2. Preliminaries and Notations

In this section, we present some notations, definitions, and preliminary facts that will be used further in this work.

2.1. Block-Pulse Functions (BPF). In order to introduce block-pulse functions, we first suppose the interval $[0,T)$ to be divided into K equidistant subintervals $[((k-1)/K)T, (k/K)T)$, $k = 1,2,\ldots,K$. A set of block-pulse functions $B_{(K)}(t)$ composed of K orthogonal functions with piecewise constant values is defined on the semiopen interval $[0,T)$ as follows:

$$B_{(K)}(t) = [b_1(t), b_2(t), \ldots, b_k(t), \ldots, b_K(t)], \quad (2)$$

where the kth component is given by

$$b_k(t) = \begin{cases} 1, & \left(\frac{k-1}{K}\right)T \le t < \left(\frac{k}{K}\right)T, \\ 0, & \text{otherwise}. \end{cases} \quad (3)$$

Block-pulse functions have some nice characteristics. They are disjoint and orthogonal; that is,

$$b_k(t)\, b_l(t) = \begin{cases} b_k(t), & k = l, \\ 0, & k \ne l, \end{cases}$$
$$\int_0^T b_k(t)\, b_l(t)\, dt = \|b_k(t)\|^2 \delta_{k,l}, \quad (4)$$

where $\delta_{k,l}$ is the Kronecker delta function. In addition, the set of block-pulse functionies satisfy Parseval's identity when K tends to infinity. In other words, for any function $v \in £^2[0,T)$,

$$\|v\|^2 = \int_0^T v^2(t)\, dt$$
$$= \sum_{k=1}^{\infty} \left(\int_0^T v(t)\, b_k(t)\, dt \right)^2 = \sum_{k=1}^{\infty} c_k^2 \|b_k(t)\|^2, \quad (5)$$

where

$$c_k = \frac{1}{\|b_k(t)\|} \int_0^T v(t)\, b_k(t)\, dt, \quad k = 1,2,3,\ldots, \quad (6)$$

so they are complete. For more information about block-pulse functions, interested reader is referred to [20–30].

2.2. Chebyshev Polynomials. Chebyshev polynomials of the first kind of degree m can be defined as follows [12]:

$$T_m(t) = \cos m\beta, \qquad \beta = \arccos t, \quad (7)$$

which are orthogonal with respect to the weight function $w(t) = 1/\sqrt{1-t^2}$, where

$$\langle T_m, T_n \rangle_{£_w^2[-1,1]} = \begin{cases} 0, & m \ne n, \\ \pi, & m = n = 0, \\ \frac{\pi}{2}, & m = n \ge 1. \end{cases} \quad (8)$$

Chebyshev polynomials also satisfy the following recursive formula:

$$T_0(t) = 1, \qquad T_1(t) = t,$$
$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t). \quad (9)$$

The set of Chebyshev polynomials is a complete orthogonal set in the Hilbert space $£_w^2[-1,1]$. The Chebyshev expansion of a function $f \in £_w^2[-1,1]$ is

$$f(t) = \sum_{m=0}^{\infty} \widehat{f}_m T_m(t),$$
$$\widehat{f}_m = \frac{2}{\pi c_m} \int_{-1}^{1} f(t)\, T_m(t)\, w(t)\, dx, \quad (10)$$

where

$$c_m = \begin{cases} 2, & m = 0, \\ 1, & m \ge 1. \end{cases} \quad (11)$$

They have also another useful characteristic; see [14]. If

$$h(t) = \frac{1}{2} h_0 T_0(t) + \sum_{m=1}^{\infty} h_m T_m(t), \quad (12)$$

then

$$\int_{-1}^{1} h(t)\, dt = h_0 - \sum_{m=2}^{\infty} \frac{1 + (-1)^m}{m^2 - 1} h_m. \quad (13)$$

TABLE 1: A comparison of absolute errors between Wc, WG, Cfd, and present method.

x	Wavelet collocation [15]	Wavelet Galerkin [15]	Chebyshev finite difference [5]	Present method
0.125	9.3×10^{-4}	7.9×10^{-7}	2.1×10^{-9}	1.16×10^{-15}
0.250	1.6×10^{-3}	1.3×10^{-6}	2.0×10^{-8}	2.28×10^{-15}
0.375	2.0×10^{-3}	1.6×10^{-6}	1.8×10^{-7}	1.27×10^{-15}
0.500	1.9×10^{-3}	1.6×10^{-6}	1.9×10^{-8}	3.15×10^{-16}
0.625	1.6×10^{-3}	1.5×10^{-6}	1.9×10^{-7}	2.79×10^{-17}
0.750	1.1×10^{-3}	1.1×10^{-6}	4.9×10^{-8}	1.63×10^{-16}
0.875	5.5×10^{-4}	6.5×10^{-7}	4.2×10^{-8}	1.52×10^{-15}

TABLE 2: The maximum errors of E_{KM} for different values of K and M.

K	4	10	8
M	8	8	10
E_{KM}	6.66×10^{-10}	3.01×10^{-13}	2.28×10^{-15}

2.3. Hybrid Functions of Block-Pulse and Chebyshev Polynomials. The orthogonal set of hybrid functions $b_{km}(t)$, $k = 1, 2, \ldots, K$, $m = 0, 1, \ldots M$, is defined on the interval $[0, T)$ as

$$b_{km}(t) = \begin{cases} T_m\left(\dfrac{2K}{T}t - 2k + 1\right), & t \in \left[\left(\dfrac{k-1}{K}\right)T, \left(\dfrac{k}{K}\right)T\right), \\ 0, & \text{otherwise}, \end{cases} \tag{14}$$

where k and m are the order of block-pulse functions and Chebyshev polynomials, respectively. The set of hybrid functions of block-pulse and Chebyshev polynomials is a complete orthogonal set in the Hilbert space $\pounds^2_{w_k}[0, T)$ because the set of block-pulse functions and the set of Chebyshev polynomials are completely orthogonal. In view of the following formula:

$$\int_0^T b_{kl}(t)\, b_{pq}(t)\, w_k(t)\, dt = \frac{\pi T}{4K} c_l \delta_{kp} \delta_{lq}, \tag{15}$$

in which δ_{kp} is the Kronecker delta function and $w_k(t)$, $k = 1, 2, \ldots, K$, are the corresponding weight functions on the kth subinterval $[((k-1)/K)T, (k/K)T)$ and defined as

$$w_k(t) = w\left(\frac{2K}{T}t - 2k + 1\right), \tag{16}$$

we can conclude that the hybrid functions are orthogonal with respect to weight functions w_k. The set of hybrid functions is complete, so any function $v \in \pounds^2_{w_k}[0, T)$ can be written as follows:

$$v(t) = \sum_{k=1}^{K} \sum_{m=0}^{\infty} \widehat{v}_{km} b_{km}(t), \tag{17}$$

in which

$$\widehat{v}_{km} = \frac{(v, b_{km})\, w_k}{(b_{km}, b_{km})\, w_k} = \frac{4K}{\pi c_m T} \int_{((k-1)/K)T}^{(k/K)T} v(t)\, b_{km}(t)\, w_k(t)\, dt, \tag{18}$$

where $(\cdot, \cdot)\, w_k$ is the weighted inner product.

3. Chebyshev Finite Difference Method

We can approximate a function $f(t)$ in terms of Chebyshev polynomials as follows [31]:

$$(P_M) f(t) = {\sum_{k=0}^{M}}'' f_k T_k(t), \qquad f_k = \frac{2}{M} {\sum_{k=0}^{M}}'' f(t_k)\, T_m(t_k), \tag{19}$$

with double primes meaning that the first and last terms should be halved. Moreover, t_k are the extrema of the Mth-order Chebyshev polynomial $T_M(t)$ and defined as

$$t_k = \cos\left(\frac{k\pi}{M}\right), \quad k = 0, 1, 2, \ldots, M. \tag{20}$$

In view of (7), we have

$$T_m(t_k) = \cos\left(\frac{mk\pi}{M}\right), \tag{21}$$

so f_m can be rewritten as

$$f_m = \frac{2}{M} {\sum_{k=0}^{M}}'' f(t_k) \cos\left(\frac{mk\pi}{M}\right). \tag{22}$$

The first three derivatives of the function $f(t)$ at the points t_m, $m = 0, 1, \ldots, M$, are given in [32, 33] as

$$f^{(i)}(t_m) = \sum_{j=0}^{M} d^{(i)}_{m,j} f(t_j), \quad i = 1, 2, 3, \tag{23}$$

Table 3: A comparison of absolute errors between Wc, WG, Cfd, and present method.

x	Wavelet collocation [15]	Wavelet Galerkin [15]	Chebyshev finite difference [5]	Present method
0.125	2.6×10^{-2}	2.7×10^{-4}	1.8×10^{-10}	1.45×10^{-15}
0.250	1.5×10^{-2}	3.0×10^{-5}	4.4×10^{-10}	1.78×10^{-15}
0.375	9.3×10^{-3}	2.6×10^{-4}	1.4×10^{-9}	1.71×10^{-15}
0.500	5.1×10^{-3}	4.3×10^{-4}	2.4×10^{-10}	5.00×10^{-15}
0.625	2.5×10^{-3}	5.6×10^{-4}	1.7×10^{-9}	1.22×10^{-15}
0.750	1.0×10^{-3}	6.5×10^{-4}	7.7×10^{-10}	8.66×10^{-16}
0.875	2.3×10^{-4}	7.2×10^{-4}	1.3×10^{-9}	4.52×10^{-16}

Table 4: The maximum errors of E_{KM} for different values of K and M.

K	8	10	8
M	6	6	7
E_{KM}	6.01×10^{-13}	8.72×10^{-14}	1.22×10^{-15}

where

$$\begin{aligned}
d^{(1)}_{m,j} &= \frac{4\theta_j}{M}\sum_{n=1}^{M}\sum_{\substack{l=0\\(n+l)\text{ odd}}}^{n-1}\frac{n\theta_n}{c_l}T_n(t_j)T_l(t_m)\\
&= \frac{4\theta_j}{M}\sum_{n=1}^{M}\sum_{\substack{l=0\\(n+l)\text{ odd}}}^{n-1}\frac{n\theta_n}{c_l}\cos\left(\frac{nj\pi}{M}\right)\cos\left(\frac{lm\pi}{M}\right),\\
d^{(2)}_{m,j} &= \frac{2\theta_j}{M}\sum_{n=2}^{M}\sum_{\substack{l=0\\(n+l)\text{ even}}}^{n-2}\frac{n\left(n^2-l^2\right)\theta_n}{c_l}T_n(t_j)T_l(t_m)\\
&= \frac{2\theta_j}{M}\sum_{n=2}^{M}\sum_{\substack{l=0\\(n+l)\text{ even}}}^{n-2}\frac{n\left(n^2-l^2\right)\theta_n}{c_l}\cos\left(\frac{nj\pi}{M}\right)\\
&\quad\times\cos\left(\frac{lm\pi}{M}\right),\\
d^{(3)}_{m,j} &= \frac{4\theta_j}{M}\sum_{n=2}^{M}\sum_{\substack{l=1\\(n+l)\text{ even}}}^{n-2}\sum_{\substack{r=0\\(l+r)\text{ odd}}}^{l-1}\frac{nl\left(n^2-l^2\right)\theta_n}{c_lc_r}\\
&\quad\times T_n(t_j)T_r(t_m)\\
&= \frac{4\theta_j}{M}\sum_{n=2}^{M}\sum_{\substack{l=1\\(n+l)\text{ even}}}^{n-2}\\
&\quad\times\sum_{\substack{r=0\\(l+r)\text{ odd}}}^{l-1}\frac{nl\left(n^2-l^2\right)\theta_n}{c_lc_r}\cos\left(\frac{nj\pi}{M}\right)\\
&\quad\times\cos\left(\frac{lm\pi}{M}\right),
\end{aligned}\tag{24, 25}$$

with $\theta_0=\theta_M=1/2$, $\theta_j=1$ for $j=1,2,\ldots M-1$.

As can be seen from (23), the first three derivatives of the function $f(t)$ at any point of the Chebyshev-Gauss-Lobatto points is expanded as a linear combination of the values of the function at these points.

In view of (13) and (19), we have

$$\int_{-1}^{1}f(t)\,dt\approx f_0-\sum_{m=2}^{M-1}\frac{1+(-1)^m}{m^2-1}f_m-\frac{1+(-1)^M}{2\left(M^2-1\right)}f_M. \tag{26}$$

4. Composite Chebyshev Finite Difference Method

In this Section, we present the composite Chebyshev finite difference (ChFD) method. Consider t_{km}, $k=1,2,\ldots,K$, $m=0,1,\ldots,M$, as the corresponding Chebyshev-Gauss-Lobatto collocation points at the kth subinterval $[(k-1)/K, k/K]$ such that

$$t_{km}=\frac{T}{2K}\left(t_m+2k-1\right). \tag{27}$$

A function $f(t)$ can be written in terms of hybrid basis functions as follows:

$$\left(P_M\right)f(t)=\sum_{k=1}^{K}{\sum_{m=0}^{M}}'' f_{km}b_{km}(t), \tag{28}$$

where $f_{km}, n=1,2,\ldots,K$, $m=0,1,\ldots,M$, are the expansion coefficients of the function $f(t)$ at the subinterval $[(k-1)/K,k/K]$ and $b_{k,m}(t)$, $k=1,2,\ldots,K$, $m=0,1,\ldots,M$, are defined in (14).

In view of (14) and (19), we can obtain the coefficients f_{km} as

$$\begin{aligned}
f_{km} &= \frac{2}{M}{\sum_{p=0}^{M}}'' f\left(t_{kp}\right)b_{km}\left(t_{kp}\right)\\
&= \frac{2}{M}{\sum_{p=0}^{M}}'' f\left(t_{kp}\right)\cos\left(\frac{mp\pi}{M}\right).
\end{aligned}\tag{29}$$

Table 5: A comparison of absolute errors between Tm, Cfm, and present method.

x	Exact solution	Tau method [16]	Chebyshev finite difference [5]	Present method
−1.0	0.367879441	1.52×10^{-6}	1.19×10^{-8}	1.32×10^{-16}
−0.8	0.449328964	1.74×10^{-6}	1.33×10^{-8}	1.36×10^{-16}
−0.6	0.548811636	1.95×10^{-6}	1.29×10^{-8}	1.38×10^{-16}
−0.4	0.670320046	2.02×10^{-6}	1.43×10^{-8}	1.40×10^{-16}
0.2	0.818730753	1.97×10^{-6}	1.27×10^{-8}	1.20×10^{-16}
0.0	1.000000000	1.83×10^{-6}	1.02×10^{-8}	9.99×10^{-16}
0.2	1.221402758	1.63×10^{-6}	1.04×10^{-8}	7.91×10^{-17}
0.4	1.491824698	1.36×10^{-6}	8.68×10^{-9}	7.31×10^{-17}
0.6	1.822118800	1.04×10^{-6}	2.92×10^{-9}	3.08×10^{-17}
0.8	2.225540928	5.56×10^{-7}	1.65×10^{-9}	3.69×10^{-17}
1.0	2.718281828	1.52×10^{-6}	1.19×10^{-8}	1.32×10^{-16}

Table 6: The maximum errors of E_{KM} for different values of K and M.

K	4	10	8
M	8	8	10
E_{KM}	4.07×10^{-12}	3.93×10^{-15}	2.79×10^{-17}

Using (23)–(25), the first three derivatives of the function $f(t)$ at the points t_{km}, $k = 1, 2, \ldots, K$, $m = 0, 1, \ldots, M$, can be obtained as

$$f^{(i)}\left(t_{km}\right) = \sum_{j=0}^{M} d_{k,m,j}^{(i)} f\left(t_{kj}\right), \quad i = 1, 2, 3, \tag{30}$$

where

$$\begin{aligned} d_{k,m,j}^{(1)} &= \frac{8N\theta_j}{TM} \sum_{n=1}^{M} \sum_{\substack{l=0 \\ (n+l)\ \text{odd}}}^{n-1} \frac{n\theta_n}{c_l} b_{kn}\left(t_{kj}\right) b_{kl}\left(t_{km}\right) \\ &= \frac{8N\theta_j}{TM} \sum_{n=1}^{M} \sum_{\substack{l=0 \\ (n+l)\ \text{odd}}}^{n-1} \frac{n\theta_n}{c_l} \cos\left(\frac{nj\pi}{M}\right) \cos\left(\frac{lm\pi}{M}\right), \\ d_{m,j}^{(2)} &= \frac{8K^2\theta_j}{T^2 M} \sum_{n=2}^{M} \sum_{\substack{l=0 \\ (n+l)\ \text{even}}}^{n-2} \frac{n\left(n^2 - l^2\right)\theta_n}{c_l} T_n\left(t_j\right) T_l\left(t_m\right) \\ &= \frac{8N^2\theta_j}{T^2 M} \sum_{n=2}^{M} \sum_{\substack{l=0 \\ (n+l)\ \text{even}}}^{n-2} \frac{n\left(n^2 - l^2\right)\theta_n}{c_l} \cos\left(\frac{nj\pi}{M}\right) \\ &\quad \times \cos\left(\frac{lm\pi}{M}\right), \\ d_{m,j}^{(3)} &= \frac{32K^3\theta_j}{T^3 M} \sum_{n=2}^{M} \sum_{\substack{l=1 \\ (n+l)\ \text{even}}}^{n-2} \times \sum_{\substack{r=0 \\ (l+r)\ \text{odd}}}^{l-1} \frac{nl\left(n^2 - l^2\right)\theta_n}{c_l c_r} T_n\left(t_j\right) T_r\left(t_m\right) \\ &= \frac{32N^3\theta_j}{T^3 M} \sum_{n=2}^{M} \sum_{\substack{l=1 \\ (n+l)\ \text{even}}}^{n-2} \times \sum_{\substack{r=0 \\ (l+r)\ \text{odd}}}^{l-1} \frac{nl\left(n^2 - l^2\right)\theta_n}{c_l c_r} \\ &\quad \times \cos\left(\frac{nj\pi}{M}\right) \cos\left(\frac{lm\pi}{M}\right). \end{aligned} \tag{31}$$

In view of (26) and (28), we get

$$\int_0^T f(t)\,dt \approx \frac{T}{2N} \sum_{k=1}^{K} f_{k0} - \sum_{m=2}^{M-1} \frac{1 + (-1)^m}{m^2 - 1} f_{km} - \frac{1 + (-1)^M}{2\left(M^2 - 1\right)} f_{kM}. \tag{32}$$

5. Convergence Analysis

A detailed proof of the following results can be found in [11].

Lemma 1. *If the hybrid expansion of a continuous function $h(t)$ converges uniformly, then it converges to the function $h(t)$.*

Theorem 2. *A function $h(t) \in \mathcal{L}^2_{w_k}[0, T)$ with bounded second derivative, say $|h''(t)| \le B$, can be expanded as an infinite sum*

TABLE 7: A comparison of absolute errors between DTM, IHPM, Sa, and present method.

x	CAS wavelet method [17]	DT method [18]	Improved homotopy perturbation [19]	Sequential approach [3]	Present method
0.1	1.34×10^{-3}	1.00×10^{-2}	0.23×10^{-5}	1.01×10^{-7}	1.25×10^{-17}
0.2	1.15×10^{-3}	2.78×10^{-2}	0.92×10^{-5}	4.82×10^{-7}	4.27×10^{-17}
0.3	5.67×10^{-3}	5.08×10^{-2}	0.20×10^{-4}	1.017×10^{-6}	1.46×10^{-16}
0.4	5.93×10^{-2}	7.08×10^{-2}	0.37×10^{-4}	1.61×10^{-6}	1.53×10^{-16}
0.5	1.32×10^{-2}	9.71×10^{-2}	0.57×10^{-4}	2.30×10^{-6}	1.44×10^{-16}
0.6	4.39×10^{-2}	1.09×10^{-1}	0.83×10^{-4}	3.09×10^{-6}	1.68×10^{-16}
0.7	1.41×10^{-2}	1.04×10^{-1}	0.11×10^{-3}	3.97×10^{-6}	1.74×10^{-16}
0.8	1.34×10^{-2}	6.94×10^{-2}	0.14×10^{-3}	4.90×10^{-6}	5.40×10^{-17}
0.9	1.32×10^{-2}	1.00×10^{-2}	0.18×10^{-3}	6.13×10^{-6}	1.72×10^{-17}

TABLE 8: A comparison of absolute errors between Lps and ChFd.

x	Legendre polynomial solutions [6]	Present method
−1.0	1.00×10^{-8}	0
−0.8	1.00×10^{-8}	2.98×10^{-13}
−0.6	0.00	6.56×10^{-13}
−0.4	1.00×10^{-8}	9.80×10^{-13}
−0.2	0.00	1.13×10^{-13}
0.0	0.00	1.18×10^{-12}
0.2	1.00×10^{-8}	9.18×10^{-12}
0.4	0.00	8.34×10^{-13}
0.6	2.00×10^{-8}	7.80×10^{-13}
0.8	4.60×10^{-7}	4.75×10^{-13}
1.0	5.25×10^{-6}	0

of hybrid functions and the series converges uniformly to $h(t)$, that is,

$$h(t) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \widehat{h}_{km} b_{km}(t). \tag{33}$$

Theorem 3. *Suppose that $h(t) \in \pounds^2_{w_k}[0,T)$ with bounded second derivative, say $|h''(t)| \le B$, and then its hybrid expansion converges uniformly to $h(t)$; that is,*

$$\sum_{k=1}^{K}\sum_{m=0}^{\infty}{}' h_{km} b_{km}(t) = h(t), \tag{34}$$

where the summation symbol with prime denotes a sum with the first term halved.

Theorem 4 (accuracy estimation). *Suppose that $h(t) \in L^2_{w_k}[0,T)$ with bounded second derivative, say $|h''(t)| \le B$, and then one has the following accuracy estimation:*

$$\sigma_{K,M} \le \left(S + \sum_{k=K+1}^{\infty}\sum_{m=M+1}^{\infty} \frac{C^2}{(m^2-1)^2} \times \frac{\pi T c_m}{4K} \right)^{1/2}, \tag{35}$$

where

$$\begin{aligned}
\sigma_{K,M} &= \left(\int_0^T \left[h(t) - \sum_{k=1}^{K}\sum_{m=0}^{M}{}'' f_{km} b_{km}(t) \right]^2 w_k(t)\,dt \right)^{1/2}, \\
C &= \frac{R\pi^3}{12} + \frac{BT^2}{k^2 c_m}, \\
R &= \max\left\{ \frac{d^2}{d\beta^2}\left(h\left(\frac{\cos(\beta) + 2k - 1}{2K} T \right) \cos(m\beta) \right), \right. \\
&\qquad \left. 0 \le \beta \le \pi \right\} \\
S &= \frac{1}{4} h_{10}^2 \frac{\pi T}{2K} + \frac{1}{4} h_{KM}^2 \frac{\pi T}{4K}.
\end{aligned} \tag{36}$$

Proof. Consider

$$\begin{aligned}
\sigma_{kM}^2 &= \frac{1}{4} h_{10}^2 \int_0^T b_{10}^2(t)\, w_k(t)\, dt \\
&\quad + \frac{1}{4} h_{KM}^2 \int_0^T b_{KM}^2(t)\, w_k(t)\, dt \\
&\quad + \int_0^T \left[h(t) - \sum_{k=1}^{K}\sum_{m=0}^{M} h_{km} b_{km}(t) \right]^2 w_k(t)\, dt \\
&= \frac{1}{4} h_{10}^2 \frac{\pi T}{2K} + \frac{1}{4} h_{KM}^2 \frac{\pi T}{4K} \\
&\quad + \int_0^T \sum_{k=K+1}^{\infty}\sum_{m=M+1}^{\infty} h_{km}^2 b_{km}^2(t)\, w_k(t)\, dt \\
&= S + \sum_{k=K+1}^{\infty}\sum_{m=M+1}^{\infty} h_{km}^2 \int_0^T b_{km}^2(t)\, w_k(t)\, dt \\
&= S + \sum_{k=K+1}^{\infty}\sum_{m=M+1}^{\infty} h_{km}^2 \times \frac{\pi T c_m}{4K}.
\end{aligned} \tag{37}$$

Table 9: A comparison of absolute errors between Lps and present method.

x	Exact solution	Legendre polynomial solutions [6]	Present method
−1.0	−0.8414709848	4.39×10^{-9}	7.00×10^{-20}
−0.8	−0.7173560909	4.69×10^{-9}	1.50×10^{-15}
−0.6	−0.5646424734	1.19×10^{-9}	4.91×10^{-15}
−0.4	−0.3894183423	2.30×10^{-9}	8.38×10^{-15}
−0.2	−0.1986693308	9.50×10^{-11}	1.05×10^{-14}
0.0	0.0	1.99×10^{-17}	1.07×10^{-14}
0.2	0.1986693308	1.04×10^{-10}	8.79×10^{-15}
0.4	0.3894183423	1.06×10^{-8}	5.34×10^{-15}
0.6	0.5646424734	5.00×10^{-8}	1.48×10^{-15}
0.8	0.7173560909	1.35×10^{-6}	1.11×10^{-15}
1.0	0.8414709848	4.65×10^{-7}	7.00×10^{-20}

Table 10: The maximum errors of E_{KM} for different values of K and M.

K	4	5	9	10
M	5	5	8	9
E_{KM}	4.13×10^{-6}	3.32×10^{-6}	3.77×10^{-14}	1.07×10^{-14}

With the aid of (15) and the proof of Theorem 3, we will have

$$\sigma_{kM}^2 \leq S + \sum_{k=K+1}^{\infty} \sum_{m=M+1}^{\infty} \frac{C^2}{(m^2-1)^2} \times \frac{\pi T c_m}{4K}, \tag{38}$$

where

$$\begin{aligned} C &= \frac{R\pi^3}{12} + \frac{BT^2}{k^2 c_m}, \\ R &= \max\left\{ \frac{d^2}{d\beta^2}\left(h\left(\frac{\cos(\beta) + 2k - 1}{2K} T \right) \cos(m\beta) \right), \right. \\ &\qquad \left. 0 \leq \beta \leq \pi \right\}, \\ S &= \frac{1}{4} h_{10}^2 \frac{\pi T}{2K} + \frac{1}{4} h_{KM}^2 \frac{\pi T}{4K}. \end{aligned} \tag{39}$$

□

6. Discretization of Problem

In this section, we apply the composite ChFD method to solve Fredholm integrodifferential equations of the form (1). For this purpose, we approximate integral part in (1) using (32). We expand $k(x,t)y(t)$ in terms of hybrid functions:

$$k(x,t)\, y(t) \approx \sum_{k=1}^{K} \sum_{m=0}^{M}{}'' f_{km} b_{km}(t), \tag{40}$$

where

$$\begin{aligned} f_{km} &= \frac{2}{M} \sum_{p=0}^{M}{}'' \left(k\left(x, t_{kp}\right) y\left(t_{kp}\right) \right) b_{km}\left(t_{kp}\right) \\ &= \frac{2}{M} \sum_{p=0}^{M}{}'' \left(k\left(x, t_{kp}\right) y\left(t_{kp}\right) \right) \cos\left(\frac{mp\pi}{M} \right); \end{aligned} \tag{41}$$

with aid of (32), we will have

$$\begin{aligned} \int_0^T k(x,t)\, y(t)\, dt \approx \frac{T}{2K} \sum_{k=1}^{K} f_{k0} - \sum_{m=2}^{M-1} \frac{1+(-1)^m}{m^2-1} f_{km} \\ - \frac{1+(-1)^M}{2(M^2-1)} f_{kM}. \end{aligned} \tag{42}$$

In order to obtain the solution $y(x)$ in (1), by applying the composite ChFD method, we first collocate (1) in Chebyshev-Gauss-Lobatto collocation points t_{km}, $k = 1, \ldots, K, m = 0, 1, \ldots, M - n$. In addition, substituting (28) and (30) into boundary conditions (1), we get n equations. Moreover, the approximate solution and its first n derivatives should be continuous at the interface of subintervals; that is,

$$\begin{aligned} y^{(i)}\left(t_{k,0}\right) = y^{(i)}\left(t_{k+1,M}\right), \quad & k = 1, 2, \ldots, K-1, \\ & i = 0, 1, \ldots n-1. \end{aligned} \tag{43}$$

Therefore, we have a system of $K(M+1)$ algebraic equations, which can be solved by using Newton's iterative method for the unknowns $y(t_{km})$, $k = 0, 1, \ldots, K$, $m = 0, 1, \ldots, M$. Consequently, the approximate solution $y(x)$ of (1) can be calculated.

7. Numerical Examples

In this section, we apply the technique described in Section 6 to some illustrative examples of higher-order linear Fredholm integrodifferential equations.

Example 1. Consider the second-order Fredholm integrodifferential equation [5, 15]

$$y''(x) + 4xy'(x) = -\frac{8x^4}{(x^2+1)^3} - 2\int_0^1 \frac{t^2+1}{(x^2+1)^2} y(t)\,dt, \quad 0 \le x, t \le 1 \tag{44}$$

subject to the boundary conditions

$$y(0) = 1, \qquad y(1) = 1, \tag{45}$$

with the exact solution $y(x) = 1/(x^2+1)$.

We solve the problem with $M = 10$, and $K = 8$. A comparison between absolute errors in solutions obtained by composite Chebyshev finite difference method, wavelet collocation method, wavelet Galerkin and Chebyshev finite difference method is tabulated in Table 1. As can been seen in Table 1, our results are much more accurate than those K obtained by other methods specially wavelet collocation method.

The maximum errors for approximate solution $y_{KM}(x)$ can be defined as

$$\begin{aligned} E_{KM} &= \| y_{KM} - y_{\text{exact}}(x) \|_\infty \\ &= \max\{|y_{KM}(x) - y_{\text{exact}}(x)|, 0 \le x \le 1\}, \end{aligned} \tag{46}$$

where the computed result with K is shown by y_{KM} and $y_{\text{exact}}(x)$ is the exact solution. For different values of K, the errors of E_{KM} are presented in Table 2.

Example 2. Consider the second-order Fredholm integrodifferential equation [5, 15]

$$\begin{aligned} x^2 y''(x) + 50xy'(x) - 35y(x) &= \frac{1-e^{x+1}}{x+1} + (x^2 + 50x - 35)e^x \\ &+ \int_0^1 e^{xt} y(t)\,dt, \quad 0 \le x, t \le 1, \end{aligned} \tag{47}$$

subject to the boundary condition

$$y(0) = 1, \qquad y(1) = e. \tag{48}$$

The exact solution of this equation is $y(x) = e^x$.

The problem is solved with $M = 7$, and $K = 8$. A comparison between absolute errors in solutions by composite Chebyshev finite difference method, wavelet collocation method, wavelet Galerkin and Chebyshev finite difference method is tabulated in Table 3. It is clear from Table 3 that our method is reliable and applicable to handle Fredholm integrodifferential equations. For different values of K, the errors of E_{KM} are shown in Table 4.

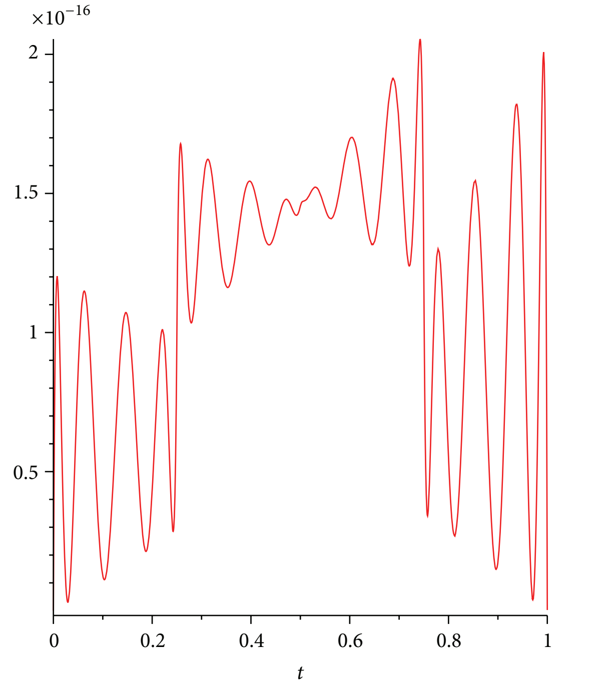

FIGURE 1: The graph of absolute errors for Example 4 for $K = 4$, and $M = 9$.

Example 3. Consider the second-order Fredholm integrodifferential equation [5, 16]

$$\begin{aligned} e^x y''(x) + \cos(x)\, y'(x) + \sin(x)\, y(x) + \int_{-1}^1 e^{(x+1)t} y(t)\,dt &= (\cos(x) + \sin(x) + e^x)e^x \\ &- 2\frac{\sin h(x+2)}{x+2}, \quad -1 \le x, t \le 1, \end{aligned} \tag{49}$$

subject to

$$\begin{aligned} y(-1) + y(1) &= e + \frac{1}{e}, \\ y(-1) + y'(-1) + y(1) &= e, \end{aligned} \tag{50}$$

with the exact solution $y(x) = e^x$.

In order to apply the presented method for solving the given problem, we should transform using an appropriate change of variables as

$$x = 2\zeta - 1, \quad \zeta \in [0, 1]. \tag{51}$$

In this example, we set $M = 9$, and $K = 10$. In Table 5, absolute errors in solutions obtained by composite Chebyshev finite difference method are compared with Tau method and Chebyshev finite difference method. According to Table 5 using the proposed method, we can obtain approximate solution which is almost same as exact solution. For different values of K the errors of E_{KM} are shown in Table 6.

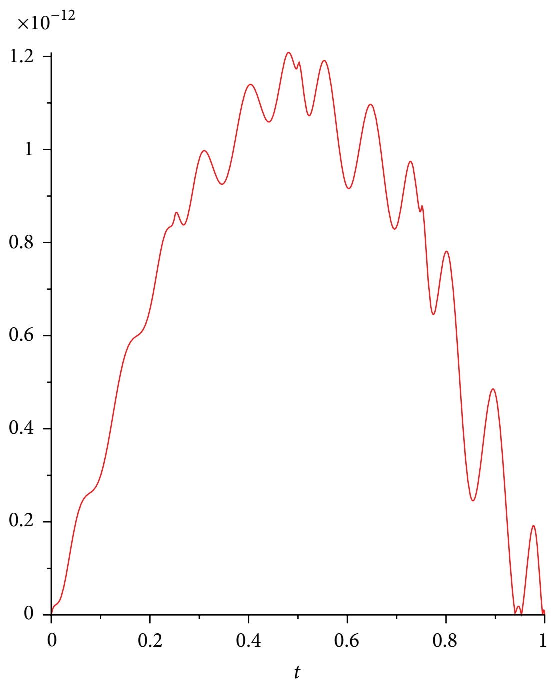

FIGURE 2: The graph of absolute errors for Example 5 for $K = 4$, and $M = 8$.

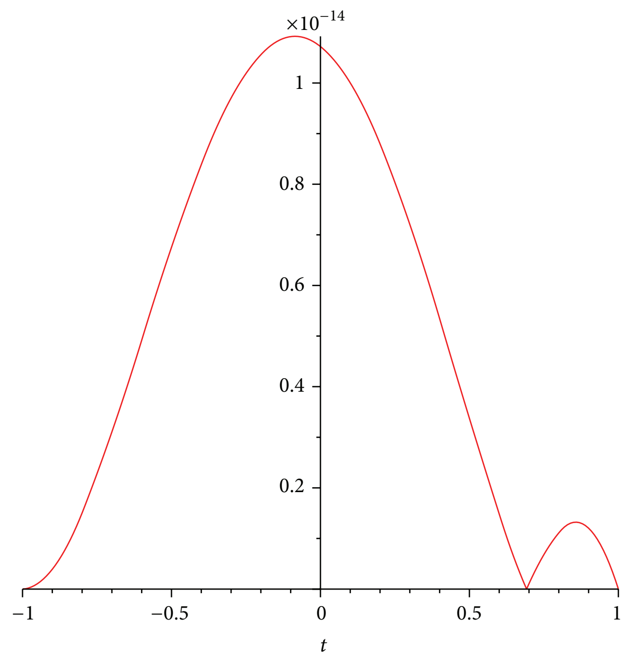

FIGURE 3: The graph of absolute errors for Example 6 for $K = 10$, and $M = 9$.

Example 4. Consider the first-order Fredholm integrodifferential equation [3, 17–19]

$$y'(x) = (x+1)e^{x} - x + \int_0^1 xy(t)\,dt, \quad 0 \le x, t \le 1, \tag{52}$$

subject to

$$y(0) = 0. \tag{53}$$

$M = 9$, and $K = 4$ are considered to solve Example 4. In Table 7, absolute errors in solutions obtained by composite Chebyshev finite difference method are compared with CAS wavelet method, differential transfer method, Improved homotopy perturbation method, and a sequential method. It is illustrated in Table 7 that the results obtained using current method are very closed to exact solution. The graph of absolute errors for $M = 9$, and $K = 4$ is shown in Figure 1.

Example 5. Consider the first-order Fredholm integrodifferential equation [6, 34, 35]

$$\begin{aligned} y''(x) &+ xy'(x) - xy \\ &= e^{x} - 2\sin(x) \\ &\quad + \int_{-1}^{1} \sin(x)e^{-t}y(t)\,dt, \quad -1 \le x, t \le 1, \end{aligned} \tag{54}$$

subject to

$$y(0) = 1, \qquad y'(0) = 1. \tag{55}$$

This example is solved for $M = 8$ and $K = 4$. In order to apply the presented method for solving the given problem, we should transform using an appropriate change of variables as

$$x = 2\zeta - 1, \quad \zeta \in [0, 1]. \tag{56}$$

In Table 8, absolute errors in solutions obtained by composite Chebyshev finite difference method are compared with Legendre polynomial method. As can be shown in Table 8, the introduced method is more efficient than Legendre polynomial method, and the numerical results are in good agreement with exact solutions up to 13 decimal places. The graph of absolute errors for $K = 4$, and $M = 8$ is shown in Figure 2.

Example 6. Consider the third-order linear Fredholm integrodifferential equation [6]

$$\begin{aligned} y'''(x) - y'(x) &= 2x(\cos 1 - \sin 1) - 2\cos x \\ &\quad + \int_{-1}^{1} xty(t)\,dt, \end{aligned} \tag{57}$$

subject to

$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) - 2y'(0) = -2. \tag{58}$$

The exact solution for this problem is $y(x) = \sin x$. We solve the problem with $m = 9$, and $n = 10$. In Table 9, absolute errors in solutions obtained by composite Chebyshev finite difference method are compared with Legendre polynomial solutions. For different values of K the errors of E_{KM} are shown in Table 10. The graph of absolute errors for $K = 10$, and $M = 9$ is shown in Figure 3.

8. Conclusion

In this paper, we presented the composite Chebyshev finite difference method for solving Fredholm integrodifferential equations. The composite ChFD method is indeed an extension of the ChFD scheme with $K = 1$. This method is based on a hybrid of block-pulse functions and Chebyshev polynomials using Chebyshev-Gauss-Lobatto collocation points.

The useful properties of Chebyshev polynomials and block-pulse functions make it a computationally efficient method to approximate the solution of Fredholm integrodifferential equations. We converted the given problem to a system of algebraic equations using collocation points.

The main advantage of the present method is the ability to represent smooth and especially piecewise smooth functions properly. It was also shown that the accuracy can be improved either by increasing the number of subintervals or by increasing the number of collocation points in subintervals. Several examples have been provided to demonstrate the powerfulness of the proposed method. A comparison was made among the present method, some other well-known approaches, and exact solution which confirms that the introduced method is more accurate and efficient.

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Research Article

Fractional Complex Transform and exp-Function Methods for Fractional Differential Equations

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The exp-function method is presented for finding the exact solutions of nonlinear fractional equations. New solutions are constructed in fractional complex transform to convert fractional differential equations into ordinary differential equations. The fractional derivatives are described in Jumarie's modified Riemann-Liouville sense. We apply the exp-function method to both the nonlinear time and space fractional differential equations. As a result, some new exact solutions for them are successfully established.

1. Introduction

Fractional differential equations (FDEs) are viewed as alternative models to nonlinear differential equations. Varieties of them play important roles and serve as tools not only in mathematics but also in physics, biology, fluid flow, signal processing, control theory, systems identification, and fractional dynamics to create the mathematical modeling of many nonlinear phenomena. Besides, they are employed in social sciences such as food supplement, climate, finance, and economics. Oldham and Spanier first considered the fractional differential equations arising in diffusion problems [1]. The fractional differential equations have been investigated by many authors [2–4].

In recent decades, some effective methods for fractional calculus appeared in open literature, such as the exp-function method [5], the fractional subequation method [6–8], the (G'/G)-expansion method [9, 10], and the first integral method [11].

The fractional complex transform [12, 13] is the simplest approach; it is to convert the fractional differential equations into ordinary differential equations, making the solution procedure extremely simple. Recently, the fractional complex transform has been suggested to convert fractional-order differential equations with modified Riemann-Liouville derivatives into integer order differential equations, and the reduced equations can be solved by symbolic computation. The exp-function method [14–20] can be used to construct the exact solutions for fractional differential equations. The present paper investigates the applicability and efficiency of the exp-function method on fractional nonlinear differential equations. The aim of this paper is to extend the application of the exp-function method to obtain exact solutions to some fractional differential equations in mathematical physics and biology.

This paper is organized as follows. In Section 2, some basic properties of Jumarie's modified Riemann-Liouville derivative are given. The main steps of the exp-function method are given in Section 3. In Sections 4–6, we construct the exact solutions of the fractional-order biological population model, fractional Burgers equation, and fractional Cahn-Hilliard equation via this method. Some conclusions are shown in Section 7.

2. Modified Riemann-Liouville Derivative

In the last few decades, in order to improve the local behavior of fractional types, a few local versions of fractional derivatives have been proposed, that is, Caputo's fractional derivative [21], Grünwald-Letnikov's fractional derivative [22], the Riemann-Liouville derivative [22], Jumarie's modified Riemann-Liouville derivative [23, 24]. Jumarie's derivative is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} [f(\xi) - f(0)]\, d\xi, \quad 0 < \alpha < 1, \tag{1}$$

where $f : R \to R, t \to f(t)$ denotes a continuous (but not necessarily first-order-differentiable) function. We list some important properties for the modified Riemann-Liouville derivative as follows.

(1) Assume that $f(t)$ denotes a continuous $R \to R$ function. We use the following equality for the integral with respect to $(dt)^\alpha$:

$$\begin{aligned} I_t^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi)\, d\xi, \\ I_t^\alpha f(t) &= \frac{1}{\Gamma(\alpha+1)} \int_0^t f(\xi)\, (dt)^\alpha, \quad 0 < \alpha \le 1. \end{aligned} \tag{2}$$

(2) Some useful formulas include

$$f^{(\alpha)}[g(t)] = \frac{df}{dt} g^{(\alpha)}(t), \tag{3}$$

$$D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \tag{4}$$

$$\int (dt)^\beta = t^\beta. \tag{5}$$

(3) Let $u(t)$ and $v(t)$ satisfy the definition of the modified Riemann-Liouville derivative, and let $f(t)$ be an α-order-differentiable function:

$$\begin{aligned} &D_t^\alpha (u(t) v(t)) \\ &\quad = v(t) D_t^\alpha u(t) + u(t) D_t^\alpha v(t), \\ &D_t^\alpha f[u(t)] \\ &\quad = f_u'[u(t)] D_t^\alpha u(t) = D_u^\alpha f[u(t)] \left(u'(t)\right)^\alpha. \end{aligned} \tag{6}$$

Function $f(t)$ should be differentiable with respect to $g(t)$, and $g(t)$ is fractional differentiable in (3). The previous results are employed in the following sections.

3. Fractional Complex Transform and exp-Function Method

We consider the following nonlinear FDE of the type

$$F\left(u, D_t^\alpha u, D_x^\beta u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta u, \ldots\right) = 0, \quad 0 < \alpha, \beta < 1, \tag{7}$$

where u is an unknown function and F is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of the exp-function method.

Step 1. Li and He [25, 26] suggested a fractional complex transform to convert fractional differential equations into ordinary differential equations, so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The complex wave variable was as follows:

$$\begin{aligned} u(x,t) &= U(\xi), \\ \xi &= \frac{\tau x^\beta}{\Gamma(1+\beta)} + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \tag{8}$$

where τ and λ are nonzero arbitrary constants; we can rewrite (7) in the following nonlinear ordinary differential equation:

$$Q\left(U, U', U'', U''', \ldots\right) = 0, \tag{9}$$

where the prime denotes the derivation with respect to ξ. If possible, we should integrate (9) term by term one or more times.

Step 2. According to exp-function method, which was developed by He and Wu [14], we assume that the wave solution can be expressed in the following form:

$$U(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp[n\xi]}{\sum_{m=-p}^{q} b_m \exp[m\xi]}, \tag{10}$$

where p, q, c, and d are positive integers which are known to be further determined and a_n and b_m are unknown constants. We can rewrite (10) in the following equivalent form:

$$U(\xi) = \frac{a_{-c} \exp[-c\xi] + \cdots + a_d \exp[d\xi]}{b_{-p} \exp[-p\xi] + \cdots + b_q \exp[q\xi]}. \tag{11}$$

Step 3. This equivalent formulation plays a significant and fundamental part for finding the exact solution of mathematical problems. To determine the values of c and p, we balance the linear term of highest order of (9) with the highest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of lowest order of (9) with lowest order nonlinear term [27–29].

In the following sections, we present three examples to illustrate the applicability of the exp-function method and fractional complex transform to solve nonlinear fractional differential equations.

4. Fractional-Order Biological Population Model

We consider a time fractional biological population model of the form [30, 31]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2}{\partial x^2}\left(u^2\right) + \frac{\partial^2}{\partial y^2}\left(u^2\right) + h\left(u^2 - r\right), \quad t > 0, \quad 0 < \alpha \le 1,\ x, y \in R, \tag{12}$$

where u denotes the population density $h(u^2 - r)$ represents the population supply due to births and deaths, and h, r are constants.

For our goal, we present the following transformation:

$$u(x, y, t) = U(\xi), \quad \xi = \nu x + i\nu y - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{13}$$

where c and ν are constants and $i^2 = -1$.

Then by the use of (13), (12) can be turned into an ODE:

$$cU' + hU^2 - hr = 0, \tag{14}$$

where "U'" $= dU/d\xi$.

Balancing the order of U' and U^2 in (14), we get

$$\begin{gathered} U' = \frac{c_1 \exp[-(c+p)\xi] + \cdots}{c_2 \exp[-2p\xi] + \cdots}, \\ U^2 = \frac{c_3 \exp[-2c\xi] + \cdots}{c_4 \exp[-2p\xi] + \cdots}, \end{gathered} \tag{15}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of exp-function in (15), we obtain

$$-(p+c) = -2c, \tag{16}$$

which leads to the result that

$$p = c. \tag{17}$$

In the same way to determine the values of d and q, we balance the linear term of the lowest order in (14):

$$\begin{gathered} U' = \frac{\cdots + d_1 \exp[(q+d)\xi]}{\cdots + d_2 \exp[2q\xi]}, \\ U^2 = \frac{\cdots + d_3 \exp[2d\xi]}{\cdots + d_4 \exp[2q\xi]}, \end{gathered} \tag{18}$$

where d_i are determined coefficients only for simplicity. From (18), we have

$$q + d = 2d, \tag{19}$$

and this gives

$$q = d. \tag{20}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so (11) reduces to

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{21}$$

Substituting (21) into (14) and by the help of symbolic computation, we have

$$\begin{aligned} \frac{1}{A} [R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0 + R_{-1} \exp(-\xi) \\ + R_{-2} \exp(-2\xi)] = 0, \end{aligned} \tag{22}$$

where

$$\begin{aligned} A &= (b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi))^2, \\ R_2 &= ha_1^2 - hrb_1^2, \\ R_1 &= ca_1 b_0 - ca_0 b_1 + 2ha_1 a_0 - 2hrb_1 b_0, \\ R_0 &= -2ca_{-1} b_1 + 2ca_1 b_{-1} - 2hrb_1 b_{-1} - hrb_0^2 \\ &\quad + 2ha_1 a_{-1} + ha_0^2, \\ R_{-1} &= -ca_{-1} b_0 + ca_0 b_{-1} + 2ha_0 a_{-1} - 2hrb_0 b_{-1}, \\ R_{-2} &= ha_{-1}^2 - hrb_{-1}^2. \end{aligned} \tag{23}$$

Solving this system of algebraic equations by using symbolic computation, we get the following results.

Case 1. Consider

$$\begin{gathered} a_1 = \frac{rb_0^2 - a_0^2}{4\sqrt{r}b_{-1}}, \qquad a_0 = a_0, \qquad a_{-1} = -\sqrt{r}b_{-1}, \\ b_1 = \frac{rb_0^2 - a_0^2}{4rb_{-1}}, \qquad b_0 = b_0, \qquad b_{-1} = b_{-1}, \\ c = 2h\sqrt{r}, \end{gathered} \tag{24}$$

where a_0, b_0, and b_{-1} are free parameters which exist provided that $b_{-1} \neq 0$. Substituting these results into (21), we get the following exact solution:

$$\begin{aligned} u&(x, y, t) \\ &= \frac{rb_0^2 - a_0^2}{4\sqrt{r}b_{-1}} \exp\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^{\alpha}}{\Gamma(1+\alpha)}\right) \\ &\quad + a_0 - \sqrt{r}b_{-1} \exp\left(-\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^{\alpha}}{\Gamma(1+\alpha)}\right)\right) \\ &\quad \times \left(\frac{rb_0^2 - a_0^2}{4rb_{-1}} \exp\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^{\alpha}}{\Gamma(1+\alpha)}\right)\right. \\ &\quad \left. + b_0 + b_{-1} \exp\left(-\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}. \end{aligned} \tag{25}$$

Case 2. Consider

$$\begin{gathered} a_1 = \frac{a_0^2}{4\sqrt{r}b_{-1}}, \qquad a_0 = a_0, \qquad a_{-1} = -\sqrt{r}b_{-1}, \\ b_1 = -\frac{a_0^2}{4rb_{-1}}, \qquad b_0 = 0, \qquad b_{-1} = b_{-1}, \\ c = 2h\sqrt{r}, \end{gathered} \tag{26}$$

where a_0 and b_{-1} are free parameters, which exist provided that $b_{-1} \neq 0$. Substituting these results into (21), we obtain the following exact solution:

$$\begin{aligned}
&u(x,y,t)\\
&= \frac{a_0^2}{4\sqrt{r}b_{-1}} \exp\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^\alpha}{\Gamma(1+\alpha)}\right) + a_0\\
&\quad - \sqrt{r}b_{-1} \exp\left(-\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^\alpha}{\Gamma(1+\alpha)}\right)\right)\\
&\quad \times \left(-\frac{a_0^2}{4rb_{-1}} \exp\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^\alpha}{\Gamma(1+\alpha)}\right)\right.\\
&\quad \left. + b_{-1} \exp\left(-\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}.
\end{aligned} \tag{27}$$

5. Time Fractional Burgers Equation

We consider the one-dimensional time fractional Burgers equation with the value problem [32]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0,\ 0 < \alpha \leq 1, \tag{28}$$

$$u(x,0) = g(x), \tag{29}$$

where α is a parameter describing the order of the fractional time derivative. The function $u(x,t)$ is assumed to be a causal function of time.

For our purpose, we introduce the following transformations:

$$u(x,t) = U(\xi), \quad \xi = \lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{30}$$

where λ and c are nonzero constants.

Substituting (30) into (28), we can show that (28) reduced into the following ODE:

$$-cU' + \lambda\varepsilon UU' - \lambda^2 \nu U'' = 0, \tag{31}$$

where "U'" $= dU/d\xi$.

Integrating (31) with respect to ξ yields

$$-cU + \frac{\lambda\varepsilon}{2}U^2 - \lambda^2 \nu U' + \xi_0 = 0, \tag{32}$$

where ξ_0 is a constant of integration.

By the same procedure as illustrated in Section 3, we can determine values of c and p by balancing terms U^2 and U' in (32). By symbolic computation, we have

$$\begin{aligned}
U' &= \frac{c_1 \exp[-(c+p)\xi] + \cdots}{c_2 \exp[-2p\xi] + \cdots},\\
U^2 &= \frac{c_3 \exp[-2c\xi] + \cdots}{c_4 \exp[-2p\xi] + \cdots},
\end{aligned} \tag{33}$$

where c_i are determined coefficients only for simplicity. Balancing the highest order of exp-function in (33), we have

$$-(p+c) = -2c, \tag{34}$$

which leads to the result that

$$p = c. \tag{35}$$

Similarly, to determine the values of d and q, we balance the linear term of the lowest order in (32):

$$\begin{aligned}
U' &= \frac{\cdots + d_1 \exp[(q+d)\xi]}{\cdots + d_2 \exp[2q\xi]},\\
U^2 &= \frac{\cdots + d_3 \exp[2d\xi]}{\cdots + d_4 \exp[2q\xi]},
\end{aligned} \tag{36}$$

where d_i are determined coefficients only for simplicity. From (36), we obtain

$$q + d = 2d, \tag{37}$$

and this gives

$$q = d. \tag{38}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so (11) reduces to

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{39}$$

Substituting (39) into (32) and by the help of symbolic computation, we have

$$\begin{aligned}
&\frac{1}{A}\left[R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0\right.\\
&\quad \left. + R_{-1}\exp(-\xi) + R_{-2}\exp(-2\xi)\right] = 0,
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
A &= \left(b_{-1}\exp(-\xi) + b_0 + b_1 \exp(\xi)\right)^2,\\
R_2 &= \xi_0 b_1^2 - ca_1 b_1 + \frac{1}{2}\lambda\varepsilon a_1^2,\\
R_1 &= -\lambda^2 \nu a_1 b_0 + \lambda\varepsilon a_1 a_0 + \lambda^2 \nu a_0 b_1 - ca_0 b_1 - ca_1 b_0\\
&\quad + 2\xi_0 b_1 b_0,\\
R_0 &= -2\lambda^2 \nu a_1 b_{-1} + 2\lambda^2 \nu a_{-1} b_1 - ca_0 b_0 + \lambda\varepsilon a_1 a_{-1}\\
&\quad + \xi_0 b_0^2 - ca_1 b_{-1} - ca_{-1} b_1 + \frac{1}{2}\lambda\varepsilon a_0^2 + 2\xi_0 b_1 b_{-1},\\
R_{-1} &= \lambda\varepsilon a_0 a_{-1} + \lambda^2 \nu a_{-1} b_0 - \lambda^2 \nu a_0 b_{-1} + 2\xi_0 b_0 b_{-1}\\
&\quad - ca_0 b_{-1} - ca_{-1} b_0,\\
R_{-2} &= \xi_0 b_{-1}^2 - ca_{-1} b_{-1} + \frac{1}{2}\lambda\varepsilon a_{-1}^2
\end{aligned} \tag{41}$$

Solving this system of algebraic equations by using symbolic computation, we obtain the following results.

Case 1. Consider

$$a_1 = \frac{b_1(\varepsilon a_{-1} - 4\lambda\nu b_{-1})}{\varepsilon b_{-1}}, \qquad a_0 = 0, \qquad a_{-1} = a_{-1},$$
$$b_1 = b_1, \qquad b_0 = 0, \qquad b_{-1} = b_{-1},$$
$$\lambda = \lambda, \qquad \varepsilon = \varepsilon, \qquad \nu = \nu,$$
$$\xi_0 = \frac{\lambda a_{-1}(\varepsilon a_{-1} - 4\lambda\nu b_{-1})}{2b_{-1}^2}, \qquad c = \frac{\lambda(\varepsilon a_{-1} - 2\lambda\nu b_{-1})}{b_{-1}}, \tag{42}$$

where a_{-1} and b_{-1} are free parameters which exist provided that $b_{-1} \neq 0$ and $\varepsilon a_{-1} - 2\lambda\nu b_{-1} \neq 0$. Substituting these results into (39), we obtain the following exact solution:

$$\begin{aligned} u(x,t) &= \frac{b_1(\varepsilon a_{-1} - 4\lambda\nu b_{-1})}{\varepsilon b_{-1}} \exp\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) \\ &\quad + a_{-1}\exp\left(-\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right) \\ &\quad \times \left(b_1 \exp\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right. \\ &\quad \left. + b_{-1}\exp\left(-\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}. \end{aligned} \tag{43}$$

Case 2. Consider

$$\begin{aligned} a_1 &= -\left(\varepsilon a_{-1}^2 b_0^2 - 2a_{-1}b_0^2\lambda\nu b_{-1} - 2\varepsilon a_0 b_0 b_{-1} a_{-1}\right. \\ &\quad \left. + 2\lambda\nu a_0 b_0 b_{-1}^2 + \varepsilon a_0^2 b_{-1}^2\right) \\ &\quad \times (\varepsilon a_{-1} - \lambda\nu b_{-1}) \\ &\quad \times \left(4b_{-1}^2\lambda^2\nu^2\right)^{-1}, \\ b_1 &= -\varepsilon\left(\varepsilon a_{-1}^2 b_0^2 - 2a_{-1}b_0^2\lambda\nu b_{-1} - 2\varepsilon a_0 b_0 b_{-1} a_{-1}\right. \\ &\quad \left. + 2\lambda\nu a_0 b_0 b_{-1}^2 + \varepsilon a_0^2 b_{-1}^2\right) \\ &\quad \times \left(4b_{-1}^3\lambda^2\nu^2\right)^{-1}, \\ a_0 &= a_0, \qquad a_{-1} = a_{-1}, \qquad b_0 = b_0, \\ b_{-1} &= b_{-1}, \qquad \lambda = \lambda, \qquad \varepsilon = \varepsilon, \qquad \nu = \nu, \\ \xi_0 &= \frac{\lambda a_{-1}(\varepsilon a_{-1} - 2\lambda\nu b_{-1})}{2b_{-1}^2}, \qquad c = \frac{\lambda(\varepsilon a_{-1} - \lambda\nu b_{-1})}{b_{-1}}, \end{aligned} \tag{44}$$

where a_{-1}, a_0, b_{-1}, and b_0 are free parameters which exist provided that $b_{-1} \neq 0$ and $\varepsilon a_{-1} - 2\lambda\nu b_{-1} \neq 0$. Substituting these results into (39), we get the following exact solution:

$$\begin{aligned} u(x,t) &= -\left(\varepsilon a_{-1}^2 b_0^2 - 2a_{-1}b_0^2\lambda\nu b_{-1} - 2\varepsilon a_0 b_0 b_{-1} a_{-1}\right. \\ &\quad \left. + 2\lambda\nu a_0 b_0 b_{-1}^2 + \varepsilon a_0^2 b_{-1}^2\right)(\varepsilon a_{-1} - \lambda\nu b_{-1}) \\ &\quad \times \left(4b_{-1}^2\lambda^2\nu^2\right)^{-1} \\ &\quad \times \exp\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) + a_0 \\ &\quad + a_{-1}\exp\left(-\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right) \\ &\quad \times \left(-\varepsilon\left(\varepsilon a_{-1}^2 b_0^2 - 2a_{-1}b_0^2\lambda\nu b_{-1} - 2\varepsilon a_0 b_0 b_{-1} a_{-1}\right.\right. \\ &\quad \left. + 2\lambda\nu a_0 b_0 b_{-1}^2 + \varepsilon a_0^2 b_{-1}^2\right) \\ &\quad \times \left(4b_{-1}^3\lambda^2\nu^2\right)^{-1} \\ &\quad \times \exp\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) + b_0 \\ &\quad \left. + b_{-1}\exp\left(-\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}. \end{aligned} \tag{45}$$

6. Space-Time Fractional Cahn-Hilliard Equation

We consider the space-time fractional Cahn-Hilliard equation [33]

$$D_t^\alpha u - \gamma D_x^\alpha u - 6u(D_x^\alpha u)^2 - \left(3u^2 - 1\right) D_x^{2\alpha} u + D_x^{4\alpha} u = 0, \tag{46}$$

where $0 < \alpha \leq 1$ and u is the function of (x,t). For the case corresponding to $\alpha = 1$, this equation is related to a number of interesting physical phenomena like the spinodal decomposition, phase separation, and phase ordering dynamics. Moreover, it becomes important in material sciences [34]. Nevertheless we notice that this equation is very difficult to be solved and several articles investigated it [35].

Firstly, we consider the following transformations:

$$u(x,t) = U(\xi), \qquad \xi = \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{47}$$

where c is a nonzero constant.

Substituting (47) into (46), we can know this equation reduced into an ODE:

$$-cU' - \gamma U' - 6U\left(U'\right)^2 - 3U^2U'' + U'' + U'''' = 0, \tag{48}$$

where "U'" $= dU/d\xi$.

Integrating (48) with respect to ξ yields

$$-cU - \gamma U - 3U^2U' + U' + U''' + \xi_0 = 0, \tag{49}$$

where ξ_0 is a constant of integration.

Here take notice of nonlinear term in (49), and we can balance U^2U' and U''' by the idea of the exp-function method

[14] to determine the values of p, q, c, and d. By simple calculation, we have

$$\begin{aligned} U''' &= \frac{c_1 \exp[(-7p-c)\xi] + \cdots}{c_2 \exp[-8p\xi] + \cdots} \\ &= \frac{c_1 \exp[-(7p+c)\xi] + \cdots}{c_2 \exp[-8p\xi] + \cdots}, \\ U^2U' &= \frac{c_3 \exp[(-p-3c)\xi] + \cdots}{c_4 \exp[-4p\xi] + \cdots} \\ &= \frac{c_3 \exp[-(5p+3c)\xi] + \cdots}{c_4 \exp[-8p\xi] + \cdots}, \end{aligned} \tag{50}$$

where c_i are determined coefficients only for simplicity. Balancing the highest order of exp-function in (50), we have

$$-(7p+c) = -(5c+3p), \tag{51}$$

which leads to the result that

$$p = c. \tag{52}$$

Similarly, to determine the values of d and q, we balance the linear term of lowest order in (49)

$$\begin{aligned} U''' &= \frac{\cdots + d_1 \exp[(7q+d)\xi]}{\cdots + d_2 \exp[8q\xi]}, \\ U^2U' &= \frac{\cdots + d_3 \exp[(2d+6q)\xi]}{\cdots + d_4 \exp[8q\xi]}, \end{aligned} \tag{53}$$

where d_i are determined coefficients only for simplicity. From (53), we obtain

$$(6q+2d) = (d+7q), \tag{54}$$

and this gives

$$q = d. \tag{55}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so (11) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{56}$$

Substituting (56) into (49) and by the help of symbolic computation, we obtain

$$\begin{aligned} \frac{1}{A} [&R_4 \exp(4\xi) + R_3 \exp(3\xi) + R_2 \exp(2\xi) + R_1 \exp(\xi) \\ &+ R_0 + R_{-1}\exp(-\xi) + R_{-2}\exp(-2\xi) + R_{-3}\exp(-3\xi) \\ &+ R_{-4}\exp(-4\xi)] = 0, \end{aligned} \tag{57}$$

where

$$\begin{aligned}
A &= \left(b_{-1}\exp(-\xi) + b_0 + b_1\exp(\xi)\right)^4, \\
R_4 &= \xi_0 b_1^4 - c a_1 b_1^3 - \gamma a_1 b_1^3, \\
R_3 &= -3a_1^3 b_0 - 2a_0 b_1^3 - c a_0 b_1^3 - \gamma a_0 b_1^3 + 2a_1 b_1^2 b_0 \\
&\quad + 3a_1^2 a_0 b_1 + 4\xi_0 b_1^3 b_0 - 3\gamma a_1 b_1^2 b_0 - 3c a_1 b_1^2 b_0, \\
R_2 &= -10a_{-1} b_1^3 - 6a_1^3 b_{-1} - c a_{-1} b_1^3 - \gamma a_{-1} b_1^3 \\
&\quad + 10a_1 b_1^2 b_{-1} + 6a_1^2 a_{-1} b_1 - 2a_1 b_1 b_0^2 + 4\xi_0 b_1^3 b_{-1} \\
&\quad + 2a_0 b_1^2 b_0 - 6a_1^2 a_0 b_0 + 6a_1 a_0^2 b_1 + 6\xi_0 b_1^2 b_0^2 \\
&\quad - 3c a_1 b_1 b_0^2 - 3c a_0 b_1^2 b_0 - 3\gamma a_1 b_1 b_0^2 - 3\gamma a_0 b_1^2 b_0 \\
&\quad - 3c a_1 b_1^2 b_{-1} - 3\gamma a_1 b_1^2 b_{-1}, \\
R_1 &= -3c a_0 b_1 b_0^2 - 3\gamma a_0 b_1 b_0^2 - 2a_0 b_1 b_0^2 - c a_1 b_0^3 \\
&\quad - \gamma a_1 b_0^3 - 3a_0^2 a_1 b_0 + 4\xi_0 b_1 b_0^3 + 2a_1 b_0^3 + 3a_0^3 b_1 \\
&\quad - 6c a_1 b_1 b_0 b_{-1} - 6\gamma a_1 b_1 b_0 b_{-1} + 22a_0 b_1^2 b_{-1} \\
&\quad - 10a_{-1} b_1^2 b_0 - 3a_1^2 a_{-1} b_0 - 15a_1^2 a_0 b_{-1} \\
&\quad - 12a_1 b_1 b_0 b_{-1} - 3c a_0 b_1^2 b_{-1} - 3c a_{-1} b_1^2 b_0 \\
&\quad - 3\gamma a_0 b_1^2 b_{-1} - 3\gamma a_{-1} b_1^2 b_0 + 18a_1 b_1 a_0 a_{-1} \\
&\quad + 12\xi_0 b_1^2 b_0 b_{-1}, \\
R_0 &= -3c a_{-1} b_1 b_0^2 - 6\gamma a_0 b_1 b_0 b_{-1} + \xi_0 b_0^4 - c a_0 b_0^3 \\
&\quad + 28a_{-1} b_1^2 b_{-1} - 28a_1 b_{-1}^2 b_1 + 8a_1 b_{-1} b_0^2 \\
&\quad - 6c a_0 b_1 b_0 b_{-1} - 8a_{-1} b_1 b_0^2 \\
&\quad - 3c a_1 b_{-1}^2 b_1 - 3c a_1 b_0^2 b_{-1} - 3c a_{-1} b_1^2 b_{-1} \\
&\quad - \gamma a_0 b_0^3 - 3\gamma a_1 b_{-1}^2 b_1 - 3\gamma a_1 b_0^2 b_{-1} - 3\gamma a_{-1} b_1^2 b_{-1} \\
&\quad - 3\gamma a_{-1} b_0^2 b_1 + 12\xi_0 b_1 b_0^2 b_{-1} - 12a_1 a_0^2 b_{-1} \\
&\quad - 12a_1^2 b_{-1} a_{-1} + 12a_1 a_{-1}^2 b_1 + 12b_1 a_0^2 a_{-1}, \\
R_{-1} &= -3c a_0 b_{-1} b_0^2 - 3\gamma a_0 b_{-1} b_0^2 + 2a_0 b_{-1} b_0^2 \\
&\quad - c a_{-1} b_0^3 - \gamma a_{-1} b_0^3 + 3a_0^2 a_{-1} b_0 \\
&\quad + 4\xi_0 b_0^3 b_{-1} - 2a_{-1} b_0^3 - 3a_0^3 b_{-1} - 6c a_{-1} b_1 b_0 b_{-1} \\
&\quad - 6\gamma a_{-1} b_1 b_0 b_{-1} + 10a_1 b_{-1}^2 b_0 - 22a_0 b_{-1}^2 b_1 \\
&\quad + 3a_1 a_{-1}^2 b_0 + 15a_0 a_{-1}^2 b_1 - 12a_{-1} b_1 b_0 b_{-1} \\
&\quad - 3c a_1 b_0 b_{-1}^2 - 3c a_0 b_1 b_{-1}^2 - 3\gamma a_1 b_{-1}^2 b_0 \\
&\quad - 3\gamma a_0 b_1 b_{-1}^2 - 18a_1 a_{-1} a_0 b_{-1} + 12\xi_0 b_{-1}^2 b_0 b_1,
\end{aligned}$$

$$\begin{aligned}
R_{-2} &= 10a_1b_{-1}^3 + 6a_{-1}^3b_1 - ca_1b_{-1}^3 - \gamma a_1b_{-1}^3 \\
&\quad - 10a_{-1}b_1b_{-1}^2 - 6a_1a_{-1}^2b_{-1} + 4\xi_0b_{-1}^3b_1 - 2a_0b_0b_{-1}^2 \\
&\quad + 2a_{-1}b_0^2b_{-1} + 6a_0a_{-1}^2b_0 - 6a_{-1}a_0^2b_{-1} + 6\xi_0b_{-1}^2b_0^2 \\
&\quad - 3ca_0b_0b_{-1}^2 - 3ca_{-1}b_{-1}b_0^2 - 3\gamma a_0b_0b_{-1}^2 \\
&\quad - 3\gamma a_{-1}b_0^2b_{-1} - 3ca_{-1}b_{-1}^2b_1 - 3\gamma a_{-1}b_{-1}^2b_1, \\
R_{-3} &= 2a_0b_{-1}^3 + 3a_{-1}^3b_0 - ca_0b_{-1}^3 - \gamma a_0b_{-1}^3 - 2a_{-1}b_0b_{-1}^2 \\
&\quad - 3a_{-1}^2a_0b_{-1} + 4\xi_0b_0b_{-1}^3 - 3ca_{-1}b_0b_{-1}^2 \\
&\quad - 3\gamma a_{-1}b_0b_{-1}^2, \\
R_{-4} &= \xi_0b_{-1}^4 - ca_{-1}b_{-1}^3 - \gamma a_{-1}b_{-1}^3.
\end{aligned} \tag{58}$$

Solving this system of algebraic equations by using symbolic computation, we obtain the following results:

$$\begin{gathered}
a_1 = \frac{a_0^2\sqrt{6}}{8b_{-1}}, \qquad a_0 = a_0, \qquad a_{-1} = b_{-1}\sqrt{\frac{2}{3}}, \\
b_1 = \frac{3a_0^2}{8b_{-1}}, \qquad b_0 = -a_0\sqrt{6}, \qquad b_{-1} = b_{-1}, \\
c = c, \qquad \gamma = -c, \qquad \xi_0 = 0,
\end{gathered} \tag{59}$$

where a_0 and $b_{-1} \neq 0$ are free parameters.

From (59), substituting these results into (56), we obtain the following exact solution:

$$\begin{aligned}
&u(x,t) \\
&= \frac{a_0^2\sqrt{6}}{8b_{-1}} \exp\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) + a_0 \\
&\quad + b_{-1}\sqrt{\frac{2}{3}} \exp\left(-\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right) \\
&\quad \times \left(\frac{3a_0^2}{8b_{-1}} \exp\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) - a_0\sqrt{6}\right. \\
&\qquad \left. + b_{-1} \exp\left(-\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}.
\end{aligned} \tag{60}$$

7. Conclusion

In this paper, we have successfully developed fractional complex transform with the help of exp-function method to obtain exact solution of some fractional differential equations. The fractional complex transform and exp-function methods are extremely simple but effective and powerful for solving fractional differential equations. These methods are accessible to solve other similar nonlinear equations in fractional calculus. To our knowledge, these new solutions have not been reported in former literature; they may be of significant importance for the explanation of some special physical phenomena.

Conflict of Interests

The authors declare that there is no conflict of interests in this paper.

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Research Article

Coefficient Estimates and Other Properties for a Class of Spirallike Functions Associated with a Differential Operator

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For $0 \le \eta < 1$, $0 \le \lambda < 1$, $-\pi/2 < \gamma < \pi/2$, $0 \le \beta \le \alpha$, and $m \in \mathbb{N} \cup \{0\}$, a new class $S^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ of analytic functions defined by means of the differential operator $D^m_{\alpha,\beta}$ is introduced. Our main object is to provide sharp upper bounds for Fekete-Szegö problem in $S^m_{\alpha,\beta}(\eta, \gamma, \lambda)$. We also find sufficient conditions for a function to be in this class. Some interesting consequences of our results are pointed out.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is said to be in the class of γ-spirallike functions of order λ in $\mathcal{U}$, denoted by $\mathcal{S}^*(\gamma, \lambda)$, if

$$\Re\left(e^{i\gamma} \frac{zf'(z)}{f(z)}\right) > \lambda \cos\gamma, \quad z \in \mathcal{U}, \quad (2)$$

for $0 \le \lambda < 1$ and some real γ with $|\gamma| < \pi/2$.

The class $\mathcal{S}^*(\gamma, \lambda)$ was studied by Libera [1] and Keogh and Merkes [2].

Note that $\mathcal{S}^*(\gamma, 0)$ is the class of spirallike functions introduced by Špaček [3], $\mathcal{S}^*(0, \lambda) = \mathcal{S}^*(\lambda)$ is the class of starlike functions of order λ, and $\mathcal{S}^*(0, 0) = \mathcal{S}^*$ is the familiar class of starlike functions.

For the constants λ, γ with $0 \le \lambda < 1$ and $|\gamma| < \pi/2$, denote

$$p_{\lambda,\gamma}(z) = \frac{1 + e^{-i\gamma}\left(e^{-i\gamma} - 2\lambda\cos\gamma\right)z}{1 - z}, \quad z \in \mathcal{U}. \quad (3)$$

The function $p_{\lambda,\gamma}(z)$ maps the open unit disk onto the half-plane $H_{\lambda,\gamma} = \{z \in \mathbb{C} : \Re(e^{i\gamma}z) > \lambda\cos\gamma\}$. If

$$p_{\lambda,\gamma}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad (4)$$

then it is easy to check that

$$p_n = 2e^{-i\gamma}(1 - \lambda)\cos\gamma, \quad \forall n \ge 1. \quad (5)$$

For $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (6)$$

the Hadamard product (or convolution), denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}. \quad (7)$$

Denote by $\mathscr{B}$ the family of all analytic functions $w(z)$ that satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathscr{U}$.

A function $f \in \mathscr{A}$ is said to be subordinate to a function $g \in \mathscr{A}$, written $f \prec g$, if there exists a function $w \in \mathscr{B}$ such that $f(z) = g(w(z))$, $z \in \mathscr{U}$.

A classical theorem of Fekete and Szegö (see [4]) states that if $f \in \mathscr{S}$ is given by (1), then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 3 - 4\mu, & \text{if } \mu \le 0, \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right), & \text{if } 0 \le \mu \le 1, \\ 4\mu - 3, & \text{if } \mu \ge 1. \end{cases} \quad (8)$$

This inequality is sharp in the sense that for each μ there exists a function in $\mathscr{S}$ such that the equality holds. Later Pfluger (see [5]) has considered the same problem but for complex values of μ. The problem of finding sharp upper bounds for the functional $|a_3 - \mu a_2^2|$ for different subclasses of $\mathscr{A}$ is known as the Fekete-Szegö problem. Over the years, this problem has been investigated by many authors including [6–12].

For a function $f \in \mathscr{A}$, we consider the following differential operator introduced by Răducanu and Orhan [13]:

(i) $D^0_{\alpha,\beta} f(z) = f(z)$,

(ii) $D^1_{\alpha,\beta} f(z) = D_{\alpha,\beta} f(z) = \alpha\beta z^2 f''(z) + (\alpha - \beta) z f'(z) + (1 - \alpha + \beta) f(z)$,

(iii) $D^m_{\alpha,\beta} f(z) = D_{\alpha,\beta}(D^{m-1}_{\alpha,\beta} f(z))$, $z \in \mathscr{U}$,

where $0 \le \beta \le \alpha$ and $m \in \mathbb{N}_0 = \{0, 1, \ldots\}$.

If the function f is given by (1), then, from the definition of the operator $D^m_{\alpha,\beta} f$, it is easy to observe that

$$D^m_{\alpha,\beta} f(z) = z + \sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, m)\, a_n z^n, \quad (9)$$

where

$$\Phi_n(\alpha, \beta, m) = \left[1 + (\alpha\beta n + \alpha - \beta)(n-1)\right]^m, \quad n \ge 2. \quad (10)$$

It should be remarked that the operator $D^m_{\alpha,\beta} f$ generalizes other differential operators considered earlier. For $f \in \mathscr{A}$, we have

(i) $D^m_{1,0} f(z) = D^m f(z)$, the operator introduced by Sălăgean [14];

(ii) $D^m_{\alpha,0} f(z) = D^m_\alpha f(z)$, the operator studied by Al-Oboudi [15].

In view of (9), $D^m_{\alpha,\beta} f(z)$ can be written in terms of convolution as

$$D^m_{\alpha,\beta} f(z) = \left(g_{\alpha,\beta} * f\right)(z), \quad z \in \mathscr{U}, \quad (11)$$

where

$$g_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, m)\, z^n, \quad z \in \mathscr{U}. \quad (12)$$

Define the function $g^{(-1)}_{\alpha,\beta}$ such that

$$g^{(-1)}_{\alpha,\beta}(z) * g_{\alpha,\beta}(z) = \frac{z}{1-z}, \quad z \in \mathscr{U}. \quad (13)$$

It is easy to observe that

$$f(z) = g^{(-1)}_{\alpha,\beta}(z) * D^m_{\alpha,\beta} f(z). \quad (14)$$

Making use of the differential operator $D^m_{\alpha,\beta} f$, we define the following class of functions.

Definition 1. For $0 \le \eta < 1$, $0 \le \lambda < 1$, and $|\gamma| < \pi/2$, denote by $\mathscr{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ the class of functions $f \in \mathscr{A}$ which satisfy the condition

$$\Re\left(e^{i\gamma} \frac{z\left(D^m_{\alpha,\beta} f(z)\right)'}{(1-\eta) D^m_{\alpha,\beta} f(z) + \eta z\left(D^m_{\alpha,\beta} f(z)\right)'}\right) > \lambda \cos\gamma, \quad z \in \mathscr{U}. \quad (15)$$

The class $\mathscr{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ contains as particular cases the following classes of functions:

$$\mathscr{S}^0_{\alpha,\beta}(0, \gamma, \lambda) = \mathscr{S}^*(\gamma, \lambda), \qquad \mathscr{S}^0_{\alpha,\beta}(0, \gamma, 0) = \mathscr{S}^*(\gamma), \qquad \mathscr{S}^0_{\alpha,\beta}(0, 0, 0) = \mathscr{S}^*. \quad (16)$$

Also, the class $\mathscr{S}^0_{\alpha,\beta}(\eta, \gamma, \lambda)$ consists of functions $f \in \mathscr{A}$ satisfying the inequality

$$\Re\left(e^{i\gamma} \frac{z f'(z)}{(1-\eta) f(z) + \eta z f'(z)}\right) > \lambda \cos\gamma, \quad z \in \mathscr{U}. \quad (17)$$

An analogous of the class $\mathscr{S}^0_{\alpha,\beta}(\eta, \gamma, \lambda)$ has been recently studied by Murugusundaramoorthy [16].

The main object of this paper is to obtain sharp upper bounds for the Fekete-Szegö problem for the class $\mathscr{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$. We also find sufficient conditions for a function to be in this class.

2. Membership Characterizations

In this section, we obtain several sufficient conditions for a function $f \in \mathscr{A}$ to be in the class $\mathscr{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$.

Theorem 2. *Let $f \in \mathscr{A}$, and let δ be a real number with $0 \le \delta < 1$. If*

$$\left|\frac{z\left(D^m_{\alpha,\beta} f(z)\right)'}{(1-\eta) D^m_{\alpha,\beta} f(z) + \eta z\left(D^m_{\alpha,\beta} f(z)\right)'} - 1\right| \le 1 - \delta, \quad z \in \mathscr{U}, \quad (18)$$

then $f \in \mathscr{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ provided that

$$|\gamma| \le \cos^{-1}\left(\frac{1-\delta}{1-\lambda}\right). \quad (19)$$

Proof. From (18), it follows that

$$\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)\,D^m_{\alpha,\beta}f(z)+\eta z\left(D^m_{\alpha,\beta}f(z)\right)'}=1+(1-\delta)\,w(z),\tag{20}$$

where $w(z)\in\mathcal{B}$. We have

$$\begin{aligned}&\Re\left(e^{i\gamma}\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)\,D^m_{\alpha,\beta}f(z)+\eta z\left(D^m_{\alpha,\beta}f(z)\right)'}\right)\\&\quad=\Re\left[e^{i\gamma}\left(1+(1-\delta)\,w(z)\right)\right]\\&\quad=\cos\gamma+(1-\delta)\,\Re\left(e^{i\gamma}w(z)\right)\\&\quad\geq\cos\gamma-(1-\delta)\left|e^{i\gamma}w(z)\right|\\&\quad>\cos\gamma-(1-\delta)\geq\lambda\cos\gamma,\end{aligned}\tag{21}$$

provided that $|\gamma|\leq\cos^{-1}((1-\delta)/(1-\lambda))$. Thus, the proof is completed. □

If in Theorem 2 we take $\delta=1-(1-\lambda)\cos\gamma$, we will obtain the following result.

Corollary 3. *Let $f\in\mathcal{A}$. If*

$$\left|\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)\,D^m_{\alpha,\beta}f(z)+\eta z\left(D^m_{\alpha,\beta}f(z)\right)'}-1\right|\leq(1-\lambda)\cos\gamma,\qquad z\in\mathcal{U}\tag{22}$$

then $f\in\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$.

A sufficient condition for a function $f\in\mathcal{A}$ to be in the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$, in terms of coefficients inequality, is obtained in the next theorem.

Theorem 4. *If a function $f\in\mathcal{A}$ given by* (1) *satisfies the inequality*

$$\begin{aligned}&\sum_{n=2}^{\infty}\left[(1-\eta)(n-1)\sec\gamma+(1-\lambda)\left(1+\eta(n-1)\right)\right]\\&\quad\times\Phi_n(\alpha,\beta,m)\left|a_n\right|\leq1-\lambda,\end{aligned}\tag{23}$$

where $0\leq\eta<1$, $0\leq\lambda<1$, $|\gamma|<\pi/2$, and $\Phi_n(\alpha,\beta,m)$ is defined by (10)*, then it belongs to the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$.*

Proof. In virtue of Corollary 3, it suffices to show that the condition (22) is satisfied. We have

$$\begin{aligned}&\left|\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)\,D^m_{\alpha,\beta}f(z)+\eta z\left(D^m_{\alpha,\beta}f(z)\right)'}-1\right|\\&\quad=(1-\eta)\left|\frac{\sum_{n=2}^{\infty}(n-1)\,\Phi_n(\alpha,\beta,m)\,a_nz^{n-1}}{1+\sum_{n=2}^{\infty}(1-\eta+\eta n)\,\Phi_n(\alpha,\beta,m)\,a_nz^{n-1}}\right|\\&\quad<(1-\eta)\frac{\sum_{n=2}^{\infty}(n-1)\,\Phi_n(\alpha,\beta,m)\left|a_n\right|}{1-\sum_{n=2}^{\infty}(1-\eta+\eta n)\,\Phi_n(\alpha,\beta,m)\left|a_n\right|}.\end{aligned}\tag{24}$$

The last expression is bounded previously by $(1-\lambda)\cos\gamma$, if

$$\begin{aligned}&\sum_{n=2}^{\infty}(1-\eta)(n-1)\,\Phi_n(\alpha,\beta,m)\left|a_n\right|\\&\quad\leq(1-\lambda)\cos\gamma\left(1-\sum_{n=2}^{\infty}(1-\eta+\eta n)\,\Phi_n(\alpha,\beta,m)\left|a_n\right|\right),\end{aligned}\tag{25}$$

which is equivalent to

$$\begin{aligned}&\sum_{n=2}^{\infty}\left[(1-\eta)(n-1)\sec\gamma+(1-\lambda)\left(1+\eta(n-1)\right)\right]\\&\quad\times\Phi_n(\alpha,\beta,m)\left|a_n\right|\leq1-\lambda.\end{aligned}\tag{26}$$

□

For special values of m, η, γ, and λ, from Theorem 4, we can derive the following sufficient conditions for a function $f\in\mathcal{A}$ to be in the classes $\mathcal{S}^0_{\alpha,\beta}(\eta,\gamma,\lambda)$, $\mathcal{S}^0_{\alpha,\beta}(0,\gamma,\lambda)=\mathcal{S}^*(\gamma,\lambda)$, and $\mathcal{S}^0_{\alpha,\beta}(0,\gamma,0)=\mathcal{S}^*(\gamma)$, respectively.

Corollary 5. *Let $f\in\mathcal{A}$. If*

$$\begin{aligned}&\sum_{n=2}^{\infty}\left[(1-\eta)(n-1)\sec\gamma+(1-\lambda)\left(1+\eta(n-1)\right)\right]\left|a_n\right|\\&\quad\leq1-\lambda,\end{aligned}\tag{27}$$

where $0\leq\eta<1$, $0\leq\lambda<1$, and $|\gamma|<\pi/2$, then $f\in\mathcal{S}^0_{\alpha,\beta}(\eta,\gamma,\lambda)$.

Corollary 6 (see [17])**.** *Let $f\in\mathcal{A}$. If*

$$\sum_{n=2}^{\infty}\left[(n-1)\sec\gamma+1-\lambda\right]\left|a_n\right|\leq1-\lambda,\tag{28}$$

where $0\leq\lambda<1$, $|\gamma|<\pi/2$, then $f\in\mathcal{S}^*(\gamma,\lambda)$.

Corollary 7 (see [18])**.** *Let $f\in\mathcal{A}$. If*

$$\sum_{n=2}^{\infty}\left[1+(n-1)\sec\gamma\right]\left|a_n\right|\leq1,\tag{29}$$

where $|\gamma|<\pi/2$, then $f\in\mathcal{S}^*(\gamma)$.

A necessary and sufficient condition for a function to be in the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ can be given in terms of integral representation.

Theorem 8. *A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ if and only if there exists $w \in \mathcal{B}$ such that*

$$f(z) = g^{(-1)}_{\alpha,\beta}(z) * z\exp\left(\int_0^z \left[\frac{p_{\lambda,\gamma}(w(\zeta)) - 1}{1 - \eta p_{\lambda,\gamma}(w(\zeta))}\right]\frac{d\zeta}{\zeta}\right), \quad z \in \mathcal{U}, \tag{30}$$

where $p_{\lambda,\gamma}(z)$ and $g^{(-1)}_{\alpha,\beta}(z)$ are defined by (3) *and* (13)*, respectively.*

Proof. In virtue of (15), $f \in \mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ if and only if there exists $w \in \mathcal{B}$ such that

$$\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)D^m_{\alpha,\beta}f(z) + \eta z\left(D^m_{\alpha,\beta}f(z)\right)'} = p_{\lambda,\gamma}(w(z)). \tag{31}$$

From the last equality, we obtain

$$D^m_{\alpha,\beta}f(z) = z\exp\left(\int_0^z \left[\frac{p_{\lambda,\gamma}(w(\zeta)) - 1}{1 - \eta p_{\lambda,\gamma}(w(\zeta))}\right]\frac{d\zeta}{\zeta}\right). \tag{32}$$

Making use of (14) and (32), we have

$$f(z) = g^{(-1)}_{\alpha,\beta}(z) * z\exp\left(\int_0^z \left[\frac{p_{\lambda,\gamma}(w(\zeta)) - 1}{1 - \eta p_{\lambda,\gamma}(w(\zeta))}\right]\frac{d\zeta}{\zeta}\right), \quad z \in \mathcal{U}, \tag{33}$$

and thus, the proof is completed. □

For $0 \le \theta \le 2\pi$, $0 \le \tau \le 1$, define the function

$$\begin{aligned}\Psi(z,\theta,\tau) &= g^{(-1)}_{\alpha,\beta}(z) \\ &* z\exp\left(\int_0^z \left[\frac{p_{\lambda,\gamma}\left(e^{i\theta}\zeta(\zeta+\tau)/(1+\tau\zeta)\right) - 1}{1 - \eta p_{\lambda,\gamma}\left(e^{i\theta}\zeta(\zeta+\tau)/(1+\tau\zeta)\right)}\right]\frac{d\zeta}{\zeta}\right),\end{aligned} \tag{34}$$

where $p_{\lambda,\gamma}(z)$ and $g^{(-1)}_{\alpha,\beta}(z)$ are defined by (3) and (13), respectively.

In virtue of Theorem 8, the function $\Psi(z,\theta,\tau)$ belongs to the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$. Note that $\Psi(z,0,0)$ is an odd function.

3. The Fekete-Szegö Problem

In order to obtain sharp upper bounds for the Fekete-Szegö functional for the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$, the following lemma is required (see, e.g., [19, page 108]).

Lemma 9. *Let the function $w \in \mathcal{B}$ be given by*

$$w(z) = \sum_{n=1}^{\infty} w_n z^n, \quad z \in \mathcal{U}. \tag{35}$$

Then

$$|w_1| \le 1, \qquad |w_2| \le 1 - |w_1|^2, \tag{36}$$

$$|w_2 - s w_1^2| \le \max\{1, |s|\}, \quad \textit{for any complex number } s. \tag{37}$$

The functions $w(z) = z$ and $w(z) = z^2$, or one of their rotations, show that both inequalities (36) *and* (37) *are sharp.*

First we obtain sharp upper bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ with μ real parameter.

Theorem 10. *Let $f \in \mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ be given by* (1)*, and let μ be a real number. Then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{(1-\lambda)\cos\gamma}{(1-\eta)^2\Phi_3(\alpha,\beta,m)} \times \left[\eta + 3 - 2\lambda(1+\eta) - \mu\dfrac{4(1-\lambda)\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right], & \text{if } \mu \le \sigma_1, \\ \dfrac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)}, & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ \dfrac{(1-\lambda)\cos\gamma}{(1-\eta)^2\Phi_3(\alpha,\beta,m)} \times \left[\mu\dfrac{4(1-\lambda)\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} + 2\lambda(1+\eta) - \eta - 3\right], & \text{if } \mu \ge \sigma_2, \end{cases} \tag{38}$$

where

$$\sigma_1 = (1+\eta)\frac{\Phi_2^2(\alpha,\beta,m)}{2\Phi_3(\alpha,\beta,m)}, \tag{39}$$

$$\sigma_2 = \frac{2-\lambda(1+\eta)}{1-\lambda}\,\frac{\Phi_2^2(\alpha,\beta,m)}{2\Phi_3(\alpha,\beta,m)}, \tag{40}$$

and $\Phi_2(\alpha,\beta,m)$, $\Phi_3(\alpha,\beta,m)$ are defined by (10) *with $n = 2$ and $n = 3$, respectively.*

All estimates are sharp.

Proof. Suppose that $f \in \mathcal{S}_{\alpha,\beta}^{m}(\eta,\gamma,\lambda)$ is given by (1). Then, from the definition of the class $\mathcal{S}_{\alpha,\beta}^{m}(\eta,\gamma,\lambda)$, there exist $w \in \mathcal{B}$, $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots$ such that

$$\frac{z\left(D_{\alpha,\beta}^{m} f(z)\right)'}{(1-\eta) D_{\alpha,\beta}^{m} f(z) + \eta z\left(D_{\alpha,\beta}^{m} f(z)\right)'} = p_{\lambda,\gamma}(w(z)), \quad z \in \mathcal{U}. \tag{41}$$

Set $p_{\lambda,\gamma}(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$. Equating the coefficients of z and z^2 on both sides of (41), we obtain

$$\begin{aligned} a_2 &= \frac{p_1 w_1}{(1-\eta)\Phi_2(\alpha,\beta,m)}, \\ a_3 &= \frac{1}{2(1-\eta)\Phi_3(\alpha,\beta,m)}\left[\left(\frac{1+\eta}{1-\eta}p_1^2 + p_2\right)w_1^2 + p_1 w_2\right]. \end{aligned} \tag{42}$$

From (5), we have $p_1 = p_2 = 2e^{-i\gamma}(1-\lambda)\cos\gamma$, and thus we obtain

$$\begin{aligned} a_2 &= \frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{(1-\eta)\Phi_2(\alpha,\beta,m)} w_1, \\ a_3 &= \frac{e^{-i\gamma}(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)} \\ &\quad \times \left[\left(2e^{-i\gamma}(1-\lambda)\cos\gamma\frac{1+\eta}{1-\eta} + 1\right)w_1^2 + w_2\right]. \end{aligned} \tag{43}$$

It follows that

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &\quad \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)} \\ &\quad \times \left\{\left|\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1-\eta}\left(1+\eta-\mu\frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right)+1\right| \right. \\ &\qquad \left. \times \left|w_1\right|^2 + \left|w_2\right|\right\}. \end{aligned} \tag{44}$$

Making use of Lemma 9 (36), we have

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &\quad \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)} \\ &\quad \times \left\{1 + \left[\left|\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1-\eta}\right.\right.\right. \\ &\qquad \left.\left. \times\left(1+\eta-\mu\frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right)+1\right| - 1\right] \\ &\qquad \left. \times \left|w_1\right|^2\right\} \end{aligned} \tag{45}$$

or

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)} \\ &\quad \times \left[1 + \left(\sqrt{1+M(2+M)\cos^2\gamma} - 1\right)\left|w_1\right|^2\right], \end{aligned} \tag{46}$$

where

$$M = \frac{2(1-\lambda)}{1-\eta}\left(1+\eta-\mu\frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right). \tag{47}$$

Denote

$$F(x,y) = 1 + \left(\sqrt{1+M(2+M)x^2} - 1\right)y^2, \tag{48}$$

where $x = \cos\gamma$, $y = |w_1|$, and $(x,y) \in [0,1]\times[0,1]$.

Simple calculation shows that the function $F(x,y)$ does not have a local maximum at any interior point of the open rectangle $(0,1)\times(0,1)$. Thus, the maximum must be attained at a boundary point. Since $F(x,0) = 1$, $F(0,y) = 1$, and $F(1,1) = |1+M|$, it follows that the maximal value of $F(x,y)$ may be $F(0,0) = 1$ or $F(1,1) = |1+M|$.

Therefore, from (46), we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)}\max\{1, |1+M|\}, \tag{49}$$

where M is given by (47).

Consider first the case $|1+M| \ge 1$. If $\mu \le \sigma_1$, where σ_1 is given by (39), then $M \ge 0$, and from (49), we obtain

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &\quad \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)^2\Phi_3(\alpha,\beta,m)} \\ &\quad \times \left[\eta + 3 - 2\lambda(1+\eta) - \mu\frac{4(1-\lambda)\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right], \end{aligned} \tag{50}$$

which is the first part of the inequality (38). If $\mu \ge \sigma_2$, where σ_2 is given by (40), then $M \le -2$, and it follows from (49) that

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &\quad \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)^2\Phi_3(\alpha,\beta,m)} \\ &\quad \times \left[\mu\frac{4(1-\lambda)\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} + 2\lambda(1+\eta) - \eta - 3\right], \end{aligned} \tag{51}$$

and this is the third part of (38).

Next, suppose that $\sigma_1 \le \mu \le \sigma_2$. Then, $|1+M| \le 1$, and thus, from (49), we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)}, \tag{52}$$

which is the second part of the inequality (38).

In view of Lemma 9, the results are sharp for $w(z) = z$ and $w(z) = z^2$ or one of their rotations. From (41), we obtain that the extremal functions are $\Psi(z,\theta,1)$ and $\Psi(z,\theta,0)$ defined by (34) with $\tau = 1$ and $\tau = 0$. □

Next, we consider the Fekete-Szegö problem for the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ with μ complex parameter.

Theorem 11. *Let $f \in \mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ be given by (1), and let μ be a complex number. Then,*

$$\begin{aligned}&\left|a_3 - \mu a_2^2\right| \\ &\quad\leq \frac{(1-\lambda)\cos\gamma}{(1-\eta)\,\Phi_3(\alpha,\beta,m)} \\ &\quad\quad\times \max\left\{1, \left|\frac{2(1-\lambda)\cos\gamma}{1-\eta}\right.\right. \\ &\quad\quad\quad\left.\left.\times\left(\mu\frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} - 1 - \eta\right) - e^{i\gamma}\right|\right\}.\end{aligned} \tag{53}$$

The result is sharp.

Proof. Assume that $f \in \mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$. Making use of (43), we obtain

$$\begin{aligned}&\left|a_3 - \mu a_2^2\right| \\ &\quad\leq \frac{(1-\lambda)\cos\gamma}{(1-\eta)\,\Phi_3(\alpha,\beta,m)} \\ &\quad\quad\times\left|w_2 - \left[\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1-\eta}\right.\right. \\ &\quad\quad\quad\left.\left.\times\left(\mu\frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} - 1 - \eta\right) - 1\right] w_1^2\right|.\end{aligned} \tag{54}$$

The inequality (53) follows as an application of Lemma 9 (37) with

$$s = \frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1-\eta}\left(\mu\frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} - 1 - \eta\right) - 1. \tag{55}$$

The functions $\Psi(z,\theta,1)$ and $\Psi(z,\theta,0)$ defined by (34) with $\tau = 1$ and $\tau = 0$ show that the inequality (53) is sharp. □

Our Theorems 10 and 11 include several various results for special values of m, η, γ, and λ. For example, taking $m = \eta = \gamma = \lambda = 0$, in Theorem 10, we obtain the Fekete-Szegö inequalities for the class $\mathcal{S}^*$ (see [2, 11]). The special case $m = \eta = \lambda = 0$ leads to the Fekete-Szegö inequalities for the class $\mathcal{S}^*(\gamma)$ (see [2]). The Fekete-Szegö inequalities for the class $\mathcal{S}^*(\gamma,\lambda)$ (see [2]) are also included in Theorems 10 and 11.

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Research Article

Classification of Ordered Type Soliton Metric Lie Algebras by a Computational Approach

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We classify some soliton nilpotent Lie algebras and possible candidates in dimensions 8 and 9 up toisomorphy. We focus on $1 < 2 < \cdots < n$ type of derivations, where n is the dimension of the Lie algebras. We present algorithms to generate possible algebraic structures.

1. Introduction

In this paper, we compute and classify n-dimensional ($n = 8, 9$) nilsoliton metric Lie algebras with eigenvalue type $1 < 2 < \cdots < n$, which will be called "ordered type of Lie algebra" throughout this paper. We use MATLAB to achieve this goal. In the literature, *six*-dimensional nilpotent Lie algebras have been classified by algorithmic approaches [1]. In dimension *seven* and lower, nilsoliton metric Lie algebras have been classified [2–9]. Summary and details of some other classifications can be found in [10]. In our paper, we focus on dimensions eight and nine. We note that we have found that our algorithm gives consistent results with the literature in lower dimensions. We use a computational procedure that is similar to the one that we have used in our previous paper [4].

In our previous paper, we classified all the soliton and nonsoliton metric Lie algebras where the corresponding Gram matrix is invertible and of dimensions 7 and 8 up to isomorphism. If corresponding Gram matrix is invertible, then the soliton metric condition $Uv = [1]$ has a unique solution. So in this case, it is easy to check if the algebra is soliton or not. But in noninvertible case, there is more than one solution. Therefore it is hard to guess if one of the solutions provides the soliton condition without solving Jacobi identity which is nonlinear. On the other hand, it may be easy if we can eliminate some algebras which admit a derivation D that does not have ordered eigenvalues without solving the following soliton metric condition $Uv = [1]$. For this, we prove that if the nilpotent Lie algebra admits a soliton metric with corresponding Gram matrix of η being noninvertible, all the solutions of $Uv = [1]$ have a unique derivation. This theorem allows us to omit several cases that come from nonordered eigenvalues without considering Jacobi identity.

This paper is organized as follows. In Section 2, we provide some preliminaries that we use for our classifications. In Section 3, we give specific Jacobi identity conditions for Lie algebras up to dimension *nine*. This allows us to decide whether the Lie algebra has a soliton metric or not. In Section 4, we give details of our classifications with specific examples and provide algorithmic procedures. Section 5 contains our concluding remarks.

2. Preliminaries

Let (η_μ, Q) be a metric algebra, where $\mu \in \Lambda^2\eta \otimes \eta^*$. Let $B = \{X_i\}_{i=1}^n$ be a Q-orthonormal basis of η_μ (we always assume that bases are ordered). The nil-Ricci endomorphism Ric_μ is defined as $\langle \mathrm{Ric}_\mu X, Y\rangle = \mathrm{ric}_\mu(X, Y)$, where

$$\begin{aligned}\mathrm{ric}_\mu(X,Y) = &-\frac{1}{2}\sum_{i=1}^{n}\left\langle [X, X_i], [Y, X_i]\right\rangle \\ &+\frac{1}{4}\sum_{i=1}^{n}\left\langle \left[X_i, X_j\right], X\right\rangle \left\langle \left[X_i, X_j\right], Y\right\rangle\end{aligned} \tag{1}$$

for $X, Y \in \eta$ (we often write an inner product $Q(\cdot,\cdot)$ as $\langle\cdot,\cdot\rangle$). When η is a nilpotent Lie algebra, the nil-Ricci endomorphism is the Ricci endomorphism. If all elements of the basis are eigenvectors for the nil-Ricci endomorphism Ric_μ, we call the orthonormal basis a Ricci eigenvector basis.

Now we define some combinatorial objects associated to a set of integer triples $\Lambda \subset \{(i,j,k) \mid 1 \le i,j,k \le n\}$. For $1 \le i,j,k \le n$, define $1 \times n$ row vector y_{ij}^k to be $\epsilon_i^T + \epsilon_j^T - \epsilon_k^t$, where $\{\epsilon\}_{i=1}^n$ is the standard orthonormal basis for $\mathbb{R}^n$. We call the vectors in $\{y_{ij}^k \mid (i,j,k) \in \Lambda\}$ *root vectors* for Λ. Let $y_1, y_2, \ldots, y_m$ (where $m = |\Lambda|$) be an enumeration of the root vectors in dictionary order. We define *root matrix* Y_Λ for Λ to be the $m \times n$ matrix whose rows are the root vectors $y_1, y_2, \ldots, y_m$. The *Gram matrix* U_Λ for Λ is the $m \times m$ matrix defined by $U_\Lambda = Y_\Lambda Y_\Lambda^T$; the (i,j) entry of U_Λ is the inner product of the ith and jth root vectors. It is easy to see that U is a symmetric matrix. It has the same rank as the root matrix; that is, $\mathrm{Rank}(U_\Lambda) = \mathrm{Rank}(Y_\Lambda)$. Diagonal elements of U are all three, and the off-diagonal entries of U are in the set $\{-2,-1,0,1,2\}$. For more information, see [11]. Let D have distinct real positive eigenvalues, and let Λ index the structure constants for η with respect to eigenvector basis B. If $(i_1, j_1, k_1) \in \Lambda$ and $(i_2, j_2, k_2) \in \Lambda$, then $\langle y_{i_1,j_1}^{k_1}, y_{i_2,j_2}^{k_2}\rangle \neq 2$. Thus U does not contain two as an entry [4].

Lemma 1. *Let (η, Q) be an n-dimensional inner product space, and let μ be an element of $\Lambda^2\eta^* \otimes \eta$. Suppose that η_μ admits a symmetric derivation D having n distinct eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ with corresponding orthonormal eigenvectors $X_1, X_2, \ldots, X_n$. Let α_{ij}^k denote the structure constants for η with respect to the ordered basis $B = \{X_i\}_{i=1}^n$. Let $1 \le i < j \le n$. Then*

(1) *if there is some $k \in \{1,2,\ldots,n\}$ such that $\lambda_k = \lambda_i + \lambda_j$, then $[X_i, X_j]$ is a scalar multiple of X_k; otherwise X_i and X_j commute;*

(2) *$\alpha_{ij}^k \neq 0$ if and only if $X_k \in [\eta_\mu, \eta_\mu]$.*

Theorem 2 (see [11]). *Let η be a vector space, and let $B = X_{ii=1}^n$ be a basis for η. Suppose that a set of nonzero structure constants $\alpha_{i,j}^k$ relative to B, indexed by Λ, defines a skew symmetric product on η. Assume that if $(i,j,k) \in \Lambda$, then $i < j < k$. Then the algebra is a Lie algebra if and only if whenever there exists m so that the inner product of root vectors $\langle y_{ij}^l, y_{lk}^m\rangle = -1$ for triples (i,j,l) and (l,k,m) or (k,l,m) in Λ, the equation*

$$\sum_{s<m} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0 \tag{2}$$

holds. Furthermore, a term of form $\alpha_{i,j}^l \alpha_{l,k}^m$ is nonzero if and only if $\langle y_{i,j}^l, y_{l,k}^m\rangle = -1$

Theorem 3 (see [11]). *Let (η_μ, Q) be a metric algebra and $B = \{X_i\}_{i=1}^n$ a Ricci eigenvector basis for η_μ. Let Y be the root matrix for η_μ. Then the eigenvalues of the nil-Ricci endomorphism are given by*

$$\mathrm{Ric}_\mu^B = -\frac{1}{2} Y^T v, \tag{3}$$

where $v = [\alpha^2]$.

Theorem 4 (see [4, 11]). *Let (η, Q) be a nonabelian metric algebra with Ricci eigenvector basis B. The following are equivalent.*

(1) *(η_μ, Q) satisfies the nilsoliton condition with nilsoliton constant β.*

(2) *The eigenvalue vector V_D for $D = \mathrm{Ric} - \beta\,\mathrm{Id}$ with respect to B lies in the kernel of the root matrix for (η_μ, Q) with respect to B.*

(3) *For noncommuting eigenvectors X and Y for the nil-Ricci endomorphism with eigenvalues κ_X and κ_Y, the bracket $[X, Y]$ is an eigenvector for the nil-Ricci endomorphism with eigenvalue $\kappa_X + \kappa_Y - \beta$.*

(4) *$\beta = y_{ij}^k \,\mathrm{Ric}$ for all (i,j,k) in $\Lambda(\eta_\mu, B)$.*

Theorem 5 (see [4]). *Let η be an n-dimensional nonabelian nilpotent Lie algebra which admits a derivation D having distinct real positive eigenvalues. Let B be a basis consisting of eigenvectors for the derivation D, and let Λ index the nonzero structure constants with respect to B. Let U be the $m \times m$ Gram matrix. If U is invertible, then the following hold:*

(i) *$|\Lambda| \le n - 1$;*

(ii) *if $(i_1, j_1, k_1) \in \Lambda$ and $(i_2, j_2, k_2) \in \Lambda$, then $\langle y_{i_1,j_1}^{k_2}, y_{i_2,j_2}^{k_2}\rangle \neq -1$.*

3. Theory

This section provides some theorems and their proofs that allow us to consider fewer cases for our algoritm. The following theorem gives a pruning method while Gram matrix is noninvertible.

Theorem 6. *Let η be an n-dimensional nilsoliton metric Lie algebra, and U the corresponding Gram matrix which is noninvertible. Then $\mathrm{Ker}(Y^T) = \mathrm{Ker}(U)$. Furthermore all of the solutions of $Uv = [1]$ correspond to a unique derivation.*

Proof. Since rank of a matrix is equal to the rank of its Gram matrix, then $p = \mathrm{Rank}(Y) = \mathrm{Rank}(U)$. Let $U : \mathbb{R}^p \to \mathbb{R}^p$ and $Y^T : \mathbb{R}^p \to \mathbb{R}^n$ denote the linear functions (with respect to the standard basis) that correspond to the Gram matrix U and the transpose of the root matrix Y, respectively. Since $\mathrm{Rank}(U) = \mathrm{Rank}(Y) = \mathrm{Rank}(Y^T)$ and by rank-nullity theorem, we have

$$\mathrm{Ker}(U) = \mathrm{Ker}\left(Y^T\right). \tag{4}$$

Let v be a particular solution and v_0 the last column of reduced row echelon matrix $[U, [1]]$. Then v_0 is also a solution of $Uv_0 = [1]$. Therefore $U(v-v_0) = 0$; that is, $(v-v_0) \in \mathrm{Ker}(U)$. Using (4), then $(v - v_0) \in \mathrm{Ker}(Y^T)$. For the solution v_0, suppose that we denote D_0 for the Nikolayevsky derivation, Ric_0 for the Ricci tensor, and β_0 for the soliton constant. Then using (3), we have

$$(\mathrm{Ric} - \mathrm{Ric}_0) = \frac{1}{2} Y^t (v - v_0) = 0. \tag{5}$$

Then $\mathrm{Ric} = \mathrm{Ric}_0$. Using Theorem 4, we have $\beta = \beta_0$, which implies that $D = D_0$. □

Lemma 7. *If nilsoliton metric Lie algebra η has ordered type of derivations $1 < 2 < \cdots < n$, then its index set Λ consists of triples $(i, j, i + j)$.*

Proof. If $V_D = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the eigenvalue vector of D with eigenvector basis $B = \{X_i\}_{i=1}^n$ for η, then by Theorem 4, V_D lies in the kernel of Y. Thus for each element $(i, j, k) \in \Lambda$, $\lambda_i + \lambda_j - \lambda_k = 0$; that is, $\lambda_i + \lambda_j = \lambda_k$. By Lemma 1, $[X_i, X_j] = \lambda X_k$ for some $\lambda \in \mathbb{R}$. Since $\lambda_i = i$ for all $i \in \{1, 2, \ldots, n\}$, $k = i + j$ and $[X_i, X_j] = \lambda X_{i+j}$. Hence, the index set for ordered type of derivations is of form $(i, j, i + j)$. □

The next corollary describes the index triples (i, j, k) and the Jacobi identity for algebras with ordered type of derivations.

Corollary 8. *The algebra η is a Lie algebra if and only if for all pairs of form (i, j, l) and (l, k, m) or (i, j, l) and (k, l, m) in Λ_B with $k \notin \{i, j\}$ and for all $m \geq \max\{i+3, j+2, 5\}$, the following equation holds:*

$$\sum_{3 \leq s < m, s \notin \{i,j,k\}} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0. \tag{6}$$

If in addition $\lambda_i = i$ for $i = 1, \ldots, n$, then the algebra η is a Lie algebra if and only if for all pairs of form $(i, j, i + j)$ and $(i+j, k, i+j+k)$ or $(i, j, i+j)$ and $(k, i+j, i+j+k)$ in Λ_B with $k \notin \{i, j, i+j\}$ and for all $m = i+j+k \geq \max\{2i+2, j+2, 6\}$, the equation

$$\sum_{4 \leq s < m, s \notin \{i,j\}} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0 \tag{7}$$

holds.

Proof. By Theorem 7 of [11], the algebra η_μ defined by μ is a Lie algebra if and only if whenever there exists m so that $\langle y_{ij}^l, y_{lk}^m \rangle = -1$ for triples (i, j, l) and (l, k, m) or (k, l, m) in Λ_B, (2)

$$\sum_{s<m} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0 \tag{8}$$

holds. Furthermore, if i, j, and k are distinct, the product $\alpha_{ij}^l \alpha_{lk}^m$ is nonzero if and only if $\langle y_{ij}^l, y_{lk}^m \rangle = -1$.

Suppose that $\langle y_{ij}^l, y_{lk}^m \rangle = -1$ for $(i, j, l) \in \Lambda_B$ and (l, k, m) or (k, l, m) in Λ_B. By definition of Λ_B, we have $i < j$. By Lemma 1, $j < l$, $l < m$, and $k < m$. Since $i < j < l < m$, we know that $m \geq i + 3$. Similarly, $j < l < m$ implies that $j + 2 \leq m$. If $i = k$ or $j = k$, then $\langle y_{ij}^l, y_{lk}^m \rangle = 0$, and so i, j, and k must be distinct. Since i, j, k, and l are all distinct and less than m, we know that $m \geq 5$. Thus an expression of form

$$\alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m \tag{9}$$

is nonzero only if $m \geq \max\{i + 3, j + 2, 5\}$, and $k \notin \{i, j\}$.

Suppose that $\lambda_i = i$, and (i, j, l) and (l, k, m) are in the index set. Then from Lemma 7, $l = i + j$ which implies that $m = i + j + k$. We know that $1 \leq i < j < l < m$. Then, since $j \geq i + 1$ and $k \geq 1$, we have $2i + 2 \leq i + j + k = m$. Since $2i + 2 \leq m$, $m = 5$ implies that $i = 1$. So there is no possible (i, j, k), where all i, j, k are distinct and $i < j < m$ with $i+j+k$. Thus if $m = 5$, then $\langle y_{ij}^l, y_{lk}^m \rangle \neq -1$. Therefore, if $\lambda_i = i$, an expression of form

$$\alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m \tag{10}$$

is nonzero only if $m \geq \max\{2i + 2, j + 2, 6\}$, and $k \notin \{i, j\}$.

By Lemma 1, X_1 and X_2 are in $[\eta_\mu, \eta_\mu]^\perp$, and so $\alpha_{rt}^s \neq 0$ which implies that $s \geq 3$ and $r \neq s$, $t \neq s$. Therefore all expressions in (10) with $s < 3$ or $s \in \{i, j, k\}$ are identically zero and may be omitted from the summation for any m. □

The next corollary describes some equations in the structure constants of a nilpotent metric Lie algebra that are equivalent to the Jacobi identity. Each of the terms $\alpha_{ij}^s \alpha_{sk}^m$ in the following equations corresponds to each of -1 entry in the Gram matrix U. Therefore, the following equations are useful for noninvertible case since there is no -1 entry in the Gram matrix for the invertible case.

Corollary 9. *Let $(\eta_\mu, \langle \cdots, \cdots \rangle)$ be an n-dimensional inner product space where $n \leq 9$, and, μ be an element of $\Lambda^2 \eta^* \otimes \eta$. Suppose that the algebra η_μ defined by μ admits a symmetric derivation D having n eigenvalues $1 < 2 < \cdots < n$ with corresponding orthonormal eigenvectors $X_1, X_2, \ldots, X_n$. $\alpha_{i,j}^k$ denote the structure constants for η with respect to the ordered basis $B = \{X_i\}_{i=1}^n$, and let λ_B index the nonzero structure constants as defined in (2). The algebra η_μ is a Lie algebra if and only if*

$$\alpha_{13}^4 \alpha_{42}^6 + \alpha_{23}^5 \alpha_{51}^6 = 0, \tag{11}$$

$$\alpha_{12}^3 \alpha_{3,4}^7 - \alpha_{14}^5 \alpha_{52}^7 - \alpha_{24}^6 \alpha_{61}^7 = 0, \tag{12}$$

$$\alpha_{12}^3 \alpha_{35}^8 + \alpha_{14}^5 \alpha_{53}^8 + \alpha_{15}^6 \alpha_{62}^8 + \alpha_{25}^7 \alpha_{71}^8 + \alpha_{34}^7 \alpha_{71}^8 = 0, \tag{13}$$

$$\alpha_{12}^3 \alpha_{36}^9 + \alpha_{13}^4 \alpha_{45}^9 - \alpha_{23}^5 \alpha_{45}^9 - \alpha_{24}^6 \alpha_{36}^9 - \alpha_{34}^7 \alpha_{27}^9 - \alpha_{35}^8 \alpha_{18}^9 = 0 \tag{14}$$

holds.

TABLE 1: Possible (i, j, k) for $m = 6$.

Case	i	j	k	PT
(a)	1	2	3	—
(b)	1	3	2	√
(c)	1	4	1	—
(d)	2	3	1	√
(e)	2	4	0	—

Proof. Following Lemma 7, the index set consists of elements of form $(i, j, i + j)$. Therefore the number s equals to $i + j$ in the expression $\alpha_{i,j}^{s}\alpha_{s,k}^{m}$. From the previous corollary,

$$\alpha_{i,j}^{s}\alpha_{s,k}^{m} + \alpha_{j,k}^{s}\alpha_{s,i}^{m} + \alpha_{k,i}^{s}\alpha_{s,j}^{m} \tag{15}$$

is nonzero only if $m \geq 6$, and $k \notin \{i, j\}$. Also $k \neq i+j$; otherwise $\alpha_{s,k}^{m} = 0$.

If $m = 6$, $2i + 2 \leq m$ implies that $i \leq 2$; that is, possible numbers for "i" are 1 and 2. Possible and not possible (i, j, k) triples, which are being used in

$$\sum_{3\leq s<m, s\notin\{i,j,k\}} \alpha_{i,j}^{s}\alpha_{s,k}^{m} + \alpha_{j,k}^{s}\alpha_{s,i}^{m} + \alpha_{k,i}^{s}\alpha_{s,j}^{m} = 0 \tag{16}$$

and where $i+j+k = 6$, are illustrated in Table 1. The notations in the table are as follows √ := yes; − = no; PT = possible triple.

In the case (a), $i + j = k$, and then it is not a possible triple. In the case (b), $i < j$, i, j, k are distinct, and $k \neq i + j$. So it is a possible triple. In the case (c), $i = k$, and so it is not a possible triple. In the case (d), $i < j$, and all i, j, k are distinct as $i + j \neq k$. Thus it is a possible triple. In the case (e), k is not a natural number, and so it is not a possible triple. Therefore only possible (i, j, k) triples are $(1, 3, 2)$ and $(2, 3, 1)$. Triples $(1, 3, 2)$ and $(2, 3, 1)$ correspond to nonzero products $\alpha_{13}^{4}\alpha_{42}^{6}$ and $\alpha_{23}^{5}\alpha_{5,1}^{6}$, respectively. Using the skew-symmetry, (2) turns into the following equation:

$$\alpha_{13}^{4}\alpha_{24}^{6} + \alpha_{23}^{5}\alpha_{15}^{6} = 0, \tag{17}$$

which gives (11). Using the same procedure for $m = 7$, possible (i, j, k) triples are $(1, 2, 4)$, $(1, 4, 2)$ and $(2, 4, 1)$, which correspond to nonzero $\alpha_{12}^{3}\alpha_{34}^{7}$, $\alpha_{14}^{5}\alpha_{52}^{7}$, and $\alpha_{24}^{6}\alpha_{61}^{7}$ respectively. Therefore (12) is obtained. Equations (13) and (14) can be obtained by the same way. □

As an illustration, we show how to use the results of this section in the following example.

Example 10. Let η be an 8-dimensional algebra with nonzero structure constants relative to eigenvector basis B indexed by

$$\begin{aligned}\Lambda = \{&(1,2,3),(1,3,4),(1,4,5),(1,6,7),\\ &(2,3,5),(2,6,8),(3,4,7),(3,5,8)\}.\end{aligned} \tag{18}$$

Computation shows that the structure vector $[\alpha^2]$ is a solution to $Uv = [1]_{8\times 1}$ if and only if it is of form

$$[\alpha^2] = \begin{pmatrix} (\alpha_{12}^{3})^2 \\ (\alpha_{13}^{4})^2 \\ (\alpha_{14}^{5})^2 \\ (\alpha_{16}^{7})^2 \\ (\alpha_{23}^{5})^2 \\ (\alpha_{26}^{8})^2 \\ (\alpha_{34}^{7})^2 \\ (\alpha_{35}^{8})^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{17} \\ \frac{10}{17} \\ \frac{15}{17} \\ -\frac{8}{17} \\ -\frac{5}{17} \\ \frac{12}{17} \\ \frac{7}{17} \\ 0 \\ 0 \end{pmatrix} + X\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + Y\cdot\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \tag{19}$$

Equation (13) from the previous corollary leads

$$\begin{aligned}\left(-\frac{3}{17} + X + Y\right)\cdot Y &= \left(\frac{15}{17} - X - Y\right)\cdot Y\\ \Longrightarrow X + Y &= \frac{9}{17}.\end{aligned} \tag{20}$$

Moreover, using (12), we find that $X = 3/17$ and $Y = 6/17$, which means that $(\alpha_{16}^{7})^2 = -2/17$. Thus η is not a Lie algebra.

4. Algorithm and Classifications

In this section, we describe our computational procedure and give the results in dimensions 8 and 9.

4.1. Algorithm. Now we describe the algorithm. The following algorithm can be used for both invertible and noninvertible cases.

Input. The input is the integer n which represents the dimension.

Output. The output is two $0-1$ matrices Wsoliton and Uninv listing characteristic vectors for index sets Λ of Θ_n. The matrix Wsoliton has as its rows all possible characteristic vectors for canonical index sets Λ for nilpotent Lie algebras of dimension n with ordered type nonsingular nilsoliton

Table 2: 8-dimensional nilsoliton metric Lie algebras.

	Lie bracket	Index	Nullity
1	$(0,0,\sqrt{274/2223}\cdot 12, \sqrt{99/764}\cdot 13, \sqrt{527/8179}\cdot 14+\sqrt{1532/9311}\cdot 23, \sqrt{101/2154}\cdot 15+\sqrt{250/4199}\cdot 24, \sqrt{150/3151}\cdot 16+\sqrt{110/367}\cdot 25+\sqrt{110/367}\cdot 34, \sqrt{7/17}\cdot 17)$	6	3
2	$(0,0,\sqrt{82/7253}\cdot 12, \sqrt{4/34}\cdot 13, \sqrt{97/299}\cdot 14+\sqrt{6/34}\cdot 23, \sqrt{1/34}\cdot 15+\sqrt{9/34}\cdot 24, \sqrt{10/34}\cdot 16+\sqrt{1998/12097}\cdot 25+\sqrt{397/13917}\cdot 34, \sqrt{728/2883}\cdot 17+\sqrt{637/4000}\cdot 35)$	6	4
3	$(0,0,\sqrt{163/702}\cdot 12, \sqrt{263/2572}\cdot 13, \sqrt{343/1334}\cdot 14+\sqrt{547/2871}\cdot 23, \sqrt{175/2358}\cdot 15+\sqrt{160/1149}\cdot 24, \sqrt{388/2509}\cdot 16+\sqrt{518/6185}\cdot 25, \sqrt{43/13870}\cdot 17+\sqrt{915/2239}\cdot 35)$	6	3

Table 3: The 9-dimensional nilsoliton metric Lie algebras of nullities 1 and 2.

	Lie bracket	Index	Nullity
1	$(0,0,0,0,\sqrt{45}\cdot 23, \sqrt{14}\cdot 24, \sqrt{91}\cdot 25+\sqrt{91}\cdot 34, \sqrt{136}\cdot 17+\sqrt{29}\cdot 26+\sqrt{14}\cdot 35, \sqrt{104}\cdot 18+4.27)$	4	1
2	$(0,0,1.12,0,1.14+\sqrt{3}\cdot 23,0,\sqrt{6}\cdot 25+\sqrt{6}\cdot 34, \sqrt{8}\cdot 17+1.26+\sqrt{2}\cdot 35, \sqrt{6}\cdot 18+\sqrt{2}\cdot 27)$	5	2
3	$(0,0,\sqrt{21}\cdot 12,0,\sqrt{21}\cdot 14+\sqrt{39}\cdot 23,0,\sqrt{13}\cdot 16+\sqrt{70}\cdot 25-\sqrt{70}\cdot 34, \sqrt{88}\cdot 17+\sqrt{42}\cdot 35, \sqrt{65}\cdot 18+\sqrt{39}\cdot 27)$	5	2

derivation whose canonical Gram matrix U is invertible. The matrix Uninv has as its rows all possible characteristic vectors for canonical index sets Λ for nilpotent Lie algebras of dimension n with ordered type nonsingular nilsoliton derivation whose canonical Gram matrix U is noninvertible. In the dimensions 8 and 9, there is no example for invertible case. Thus Wsoliton is an empty matrix. Therefore we give the algorithm for the noninvertible case.

Algorithm for the Noninvertible Case. Consider the following.

(i) Enter the dimension n.

(ii) Compute the matrix Z_n.

(iii) Compute the matrix W.

(iv) Delete all rows of W containing abelian factor which is the row that represents direct sums of Lie algebras.

(v) Remove all rows of W such that the canonical Gram matrix U associated to the index set Λ is invertible.

(vi) Define eigenvalue vector $v_D = (1,2,3,\dots,n)^T$ in dimension n.

(vii) Remove all rows of W if $v(i) = v_0(i) \le 0$ where v is the general solution of $U_\Lambda v = [1]_{m\times 1}$ and v_0 is the vector that we have defined in the proof of Theorem 6.

(viii) Remove all the rows of W such that the corresponding algebra does not have a derivation of eigenvalue type $1 < 2 < \cdots < n$.

(ix) Remove all the rows of W such that the corresponding algebra does not satisfy Jacobi identity condition, which is obtained in Corollary 9.

After this process, we solve nonlinear systems which follow from Jacobi identity. In order to see how the algorithm works, we give the following example for $n = 6$.

Example 11. Let $n = 6$. Then

$$\Theta_6 = \{(1,2,3),(1,3,4),(1,4,5),(1,5,6),(2,3,5),(2,4,6)\}. \tag{21}$$

Table 4: The 8-dimensional nilsoliton metric Lie algebra candidates.

	Lie bracket	Index	Nullity
1	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35)	6	4
2	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26 + 1.35)	6	4
3	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26)	6	4
4	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35)	6	5

Table 5: Number of 9-dimensional nilsoliton metric Lie algebra candidates.

Nullity	Number of Lie algebras
3	98
4	81
5	45
6	22
7	7
8	1

So, matrix Z_6 is 6×3 of form

$$Z_6 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 5 & 6 \\ 2 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}. \tag{22}$$

Since $|\Theta_6| = 6$, the matrix W is of size $2^6 \times 6$ as follows

$$W_\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{23}$$

Table 6: The 9-dimensional candidates of nullities 6 and 8.

	Lie bracket	Index	Nullity
1	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	6	6
2	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	7	6
3	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	7	6
4	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.36 + 1.45)	7	6
5	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36 + 1.45)	7	6
6	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.45)	7	6
7	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36)	7	6
8	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.36 + 1.45)	7	6
9	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	7	6
10	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	7	6
11	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)	7	6
12	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	7	6
13	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26 + 1.35, 1.18 + 1.36 + 1.45)	7	6
14	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	7	6
15	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	7	6
16	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.36)	7	6
17	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26, 1.18 + 1.27 + 1.36 + 1.45)	7	6
18	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.45)	7	6
19	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36)	7	6
20	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	7	6
21	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	7	6
22	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	7	6
23	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	7	8

The first row of W_θ represents empty matrix, row two represents the subset $\{(2,4,6)\}$ of Θ_6, and so forth. Eliminating rows that represent direct sums, we have 33 rows in W matrix. Therefore none of the rows of W corresponds to Lie algebras that can be written as direct sums. These algebras correspond both to invertible and noninvertible Gram matrices. There is no example for the invertible case. For the noninvertible case, there is one ordered type nilsoliton metric Lie algebra η. Let B be the eigenvector basis for η, whose nonzero structure constants are indexed by

$$\Lambda_\eta = \{(1,2,3),(1,3,4),(1,4,5),(1,5,6),(2,3,5),(2,4,6)\}. \tag{24}$$

Computation shows that the structure vector $[\alpha^2]$ is a solution to $Uv = [1]_{6\times 1}$ if and only if it is of form

$$[\alpha^2] = \begin{pmatrix} (\alpha_{12}^3)^2 \\ (\alpha_{13}^4)^2 \\ (\alpha_{14}^5)^2 \\ (\alpha_{15}^6)^2 \\ (\alpha_{23}^5)^2 \\ (\alpha_{24}^6)^2 \end{pmatrix} = \frac{1}{143}\begin{pmatrix} 2 \\ 1 \\ 2 \\ 5 \\ 5 \\ 0 \end{pmatrix} + \frac{t}{143}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}. \tag{25}$$

By Corollary 9, η satisfies (11). Solving the equation for t, we find that

$$\begin{gathered}(\alpha_{13}^4\alpha_{24}^6)^2 = (\alpha_{23}^5\alpha_{15}^6)^2,\\ (11+t)\,t = (55-t)^2,\\ t = 25.\end{gathered} \tag{26}$$

After rescaling and solving for structure constants from $[\alpha^2]$, we see that letting

$$\begin{aligned} [X_1,X_2] &= \sqrt{22}X_3, & [X_1,X_3] &= 6X_4,\\ [X_1,X_4] &= \sqrt{22}X_5, & [X_1,X_5] &= \sqrt{30}X_6,\\ [X_2,X_3] &= \sqrt{30}X_5, & [X_2,X_4] &= 5X_6 \end{aligned} \tag{27}$$

defines a nilsoliton metric Lie algebra, previously found in [3].

4.2. Classifications. Classification results for dimensions 8 and 9 appear in Tables 2 and 3, respectively. We use vector notations to represent Lie algebra structures. For example, the list

$$\begin{aligned}\big(&0,0,0,0,\sqrt{45}\cdot 23,\sqrt{14}\cdot 24,\sqrt{91}\cdot 25+\sqrt{91}\cdot 34,\sqrt{136}\cdot 17\\ &+\sqrt{29}\cdot 26+\sqrt{14}\cdot 35,\sqrt{104}\cdot 18+4\cdot 27\big)\end{aligned} \tag{28}$$

Table 7: The 9-dimensional nilsoliton metric Lie algebra candidates of nullity 3.

	Lie bracket	Index	Nullity
1	(0, 0, 0, 0, 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18, 1.27 + 1.36 + 1.45)	4	3
2	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	5	3
3	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)	5	3
4	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	5	3
5	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	5	3
6	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	5	3
7	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.36)	5	3
8	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	5	3
9	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	5	3
10	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	5	3
11	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36 + 1.45)	5	3
12	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.45)	5	3
13	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36)	5	3
14	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	5	3
15	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	5	3
16	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	5	3
17	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	5	3
18	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	5	3
19	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	5	3
20	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	5	3
21	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	5	3
22	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	5	3
23	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)	5	3
24	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	5	3
25	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	5	3
26	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	5	3
27	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	5	3
28	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	5	3
29	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	5	3
30	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)	4	3
31	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	4	3
32	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	4	3
33	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.36)	4	3
34	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	4	3
35	(0, 0, 1.12, 1.13, 1.23, 0, 1.16 + 1.25, 1.17 + 1.26 + 1.35, 1.27 + 1.36 + 1.45)	4	3
36	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27 + 1.45)	5	3
37	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27 + 1.36)	5	3
38	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.27 + 1.36 + 1.45)	5	3
39	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.27 + 1.45)	5	3
40	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.27 + 1.36)	5	3
41	(0, 0, 1.12, 1.13, 1.14, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36 + 1.45)	5	3
42	(0, 0, 1.12, 1.13, 1.14, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.45)	6	3
43	(0, 0, 1.12, 1.13, 1.14, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36)	6	3
44	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	6	3
45	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	6	3
46	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	6	3
47	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	6	3
48	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	6	3
49	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.27 + 1.45)	5	3
50	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.27 + 1.36)	5	3
51	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.35, 1.18 + 1.27 + 1.45)	5	3
52	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.35, 1.18 + 1.27 + 1.36)	5	3

Table 7: Continued.

	Lie bracket	Index	Nullity
53	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.45)	5	3
54	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.36)	5	3
55	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27)	5	3
56	(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.27 + 1.36 + 1.45)	5	3
57	(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.27 + 1.45)	5	3
58	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.27 + 1.36 + 1.45)	5	3
59	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.27 + 1.45)	5	3
60	(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	6	3
61	(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)	6	3
62	(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	6	3
63	(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	6	3
64	(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	6	3
65	(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	6	3
66	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.36 + 1.45)	6	3
67	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.45)	6	3
68	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.45)	6	3

Table 8: The 9-dimensional nilsoliton metric Lie algebra candidates of nullity 3.

	Lie bracket	Index	Nullity
69	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.45)	6	3
70	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36)	6	3
71	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27)	6	3
72	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.45)	6	3
73	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18)	6	3
74	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.26, 1.18 + 1.27 + 1.36 + 1.45)	6	3
75	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27 + 1.45)	6	3
76	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27 + 1.36)	6	3
77	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	6	3
78	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	6	3
79	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26, 1.27 + 1.36 + 1.45)	7	3
80	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.27 + 1.45)	7	3
81	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.27 + 1.36)	7	3
82	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.35, 1.18 + 1.27 + 1.45)	7	3
83	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.35, 1.18 + 1.27 + 1.36)	7	3
84	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26, 1.18 + 1.27 + 1.45)	7	3
85	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26, 1.18 + 1.27 + 1.36)	7	3
86	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27)	7	3
87	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17, 1.27 + 1.36 + 1.45)	6	3
88	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.27 + 1.45)	6	3
89	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.27 + 1.36)	6	3
90	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.45)	7	3
91	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36)	7	3
92	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27)	7	3
93	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.35, 1.18)	7	3
94	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26, 1.18)	7	3
95	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17, 1.18 + 1.45)	7	3
96	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17, 1.18 + 1.36)	7	3
97	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.35, 1.18)	7	3
98	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17, 1.18)	7	3

in the first row of Table 3 is meant to encode the metric Lie algebra $(n, \langle\cdot,\cdot\rangle)$ with orthonormal basis $B = \{X_i\}_{i=1}^{8}$ and bracket relations

$$\begin{gathered}
[X_2, X_3] = \sqrt{45}X_5, \qquad [X_2, X_4] = \sqrt{14}X_6, \\
[X_2, X_5] = \sqrt{91}X_7, \qquad [X_3, X_4] = \sqrt{91}X_7, \\
[X_1, X_7] = \sqrt{136}X_8, \qquad [X_2, X_6] = \sqrt{29}X_8, \\
[X_3, X_5] = \sqrt{14}X_8, \qquad [X_1, X_8] = \sqrt{104}X_9, \\
[X_2, X_7] = 4X_9.
\end{gathered} \tag{29}$$

4.2.1. Candidates of Nilsoliton Metrics. Table 5 illustrates how many possible candidates of Lie algebras appear in dimension 9 up to the nullity of its Gram matrix. The algebras illustrated in Table 4 are possible candidates of nilsoliton metric Lie algebras with ordered type of derivations in dimension 8. Here, as an example we give potential Lie algebra structures when the nullity of their corresponding Gram matrices are 3, 6 and 8 in Tables 6, 7, and 8 respectively for dimension *nine*.

5. Conclusion

In this work, we have focused on nilpotent metric Lie algebras of dimensions eight and nine with ordered type of derivations. We have given specific Jacobi identity conditions for Lie algebras which allowed us to simplify the Jacobi identity condition. We have classified nilsoliton metric Lie algebras for the corresponding Gram matrix U being invertible and noninvertible. For dimension 8, we have focused on nilsoliton metric Lie algebras with noninvertable Gram matrix which leads to more than one solution for $Uv = [1]$. We have proved that if the nilpotent Lie algebra admits a soliton metric with corresponding Gram matrix being noninvertible, all the solutions of $Uv = [1]$ correspond to a unique derivation. This theorem has allowed us to omit several cases that come from nonordered eigenvalues without considering Jacobi condition. Moreover, we have classified some nilsoliton metric Lie algebras with derivation types $1 < 2 < \cdots < n$ and provided some candidates that may be classified. We are currently working on an algorithm that provides a full list of classifications for dimensions *eight* and *nine*.

Appendix

See Tables 5, 6, 7, and 8.

Conflict of Interest

Author declares that she has no competing interest.

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Research Article

An Efficient Approach for Fractional Harry Dym Equation by Using Sumudu Transform

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An efficient approach based on homotopy perturbation method by using sumudu transform is proposed to solve nonlinear fractional Harry Dym equation. This method is called homotopy perturbation sumudu transform (HPSTM). Furthermore, the same problem is solved by Adomian decomposition method (ADM). The results obtained by the two methods are in agreement, and, hence, this technique may be considered an alternative and efficient method for finding approximate solutions of both linear and nonlinear fractional differential equations. The HPSTM is a combined form of sumudu transform, homotopy perturbation method, and He's polynomials. The nonlinear terms can be easily handled by the use of He's polynomials. The numerical solutions obtained by the HPSTM show that the approach is easy to implement and computationally very attractive.

1. Introduction

Fractional differential equations have gained importance and popularity, mainly due to its demonstrated applications in science and engineering. For example, these equations are increasingly used to model problems in fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing, and many other physical processes. The most important advantage of using fractional differential equations in these and other applications is their nonlocal property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is nonlocal. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular [1–9].

In this paper, we consider the following nonlinear time-fractional Harry Dym equation of the form

$$D_t^{\alpha} U(x,t) = U^3(x,t)\, D_x^3 U(x,t), \quad 0 < \alpha \le 1, \tag{1}$$

with the initial condition

$$U(x,0) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3}, \tag{2}$$

where α is parameter describing the order of the fractional derivative and $U(x,t)$ is a function of x and t. The fractional derivative is understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$, the fractional Harry Dym equation reduces to the classical nonlinear Harry Dym equation. The exact solution of the Harry Dym equation is given by [10]

$$U(x,t) = \left(a - \frac{3\sqrt{b}}{2}(x+bt)\right)^{2/3}, \tag{3}$$

where a and b are suitable constants. The Harry Dym is an important dynamical equation which finds applications in several physical systems. The Harry Dym equation first appeared in Kruskal and Moser [11] and is attributed in an unpublished paper by Harry Dym in 1973-1974. It represents a system in which dispersion and nonlinearity are coupled together. Harry Dym is a completely integrable nonlinear evolution equation. The Harry Dym equation is very interesting because it obeys an infinite number of conversion laws; it does not posses, the Painleve property. The Harry Dym equation has strong links to the Korteweg-de Vries equation, and applications of this equation were found to the problems of hydrodynamics [12]. The Lax pair of the Harry Dym equation is associated with the Sturm-Liouville operator. The Liouville transformation transforms this operator spectrally into the Schrödinger operator [13]. Recently, a fractional model of Harry Dym equation was studied by Kumar et al. [14], and approximate analytical solution was obtained by using homotopy perturbation method (HPM).

In the present paper, the homotopy perturbation sumudu transform method (HPSTM) basically illustrates how the sumudu transform can be used to approximate the solutions of the linear and nonlinear fractional differential equations by manipulating the homotopy perturbation method. The homotopy perturbation method (HPM) was first introduced and developed by He [15–17]. The HPM was also studied by many authors to handle linear and nonlinear equations arising in various scientific and technological fields [18–24]. The homotopy perturbation sumudu transform method (HPSTM) is a combination of sumudu transform method, HPM, and He's polynomials and is mainly due to Ghorbani [25, 26]. In recent years, many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform [27–30] and sumudu transform [31, 32].

In this paper, we apply the homotopy perturbation sumudu transform method (HPSTM) and Adomian decomposition method (ADM) to solve the nonlinear time-fractional Harry Dym equation. The objective of the present paper is to extend the application of the HPSTM to obtain analytic and approximate solutions to the nonlinear time-fractional Harry Dym equation. The advantage of the HPSTM is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation, or restrictive assumptions. It is worth mentioning that the HPSTM is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

2. Sumudu Transform

In early 1990s, Watugala [33] introduced a new integral transform, named the sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The sumudu transform is defined over the set of functions

$$\begin{aligned} A = &\left\{ f(t) \mid \exists\, M, \tau_1, \tau_2 > 0, \right. \\ &\left. |f(t)| < Me^{|t|/\tau_j},\ \text{if } t \in (-1)^j \times [0,\infty) \right\} \end{aligned} \tag{4}$$

by the following formula:

$$\overline{f}(u) = S[f(t)] = \int_0^\infty f(ut)\, e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \tag{5}$$

Some of the properties were established by Weerakoon in [34, 35]. Furthermore, fundamental properties of this transform were also established by Asiru [36]. This transform was applied to the one-dimensional neutron transport equation in [37] by Kadem. In fact it was shown that there is strong relationship between sumudu and other integral transform methods; see Kılıçman et al. [38]. In particular the relation between sumudu transform and Laplace transforms was proved in Kılıçman and Gadain [39]. Next, in Eltayeb et al. [40], the sumudu transform was extended to the distributions and some of their properties were also studied in Kılıçman and Eltayeb [41]. Recently, this transform is applied to solve the system of differential equations; see Kılıçman et al. [42]. Note that a very interesting fact about sumudu transform is that the original function and its sumudu transform have the same Taylor coefficients except for the factor n; see Zhang [43]. Thus, if $f(t) = \sum_{n=0}^{\infty} a_n t^n$, then $\overline{f}(u) = \sum_{n=0}^{\infty} n! a_n u^n$; see Kılıçman et al. [38]. Similarly, the sumudu transform sends combinations, $C(m,n)$, into permutations, $P(m,n)$, and, hence, it will be useful in the discrete systems.

3. Basic Definitions of Fractional Calculus

In this section, we mention the following basic definitions of fractional calculus.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu$, $\mu \geq -1$, is defined as [3]

$$\begin{gathered} J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \quad (\alpha > 0), \\ J^0 f(t) = f(t). \end{gathered} \tag{6}$$

For the Riemann-Liouville fractional integral, we have

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}. \tag{7}$$

Definition 2. The fractional derivative of $f(t)$ in the Caputo sense is defined as [6]

$$\begin{aligned} D_t^\alpha f(t) &= J^{m-\alpha} D^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau, \end{aligned} \tag{8}$$

for $m - 1 < \alpha \leq m$, $m \in N$, $t > 0$.

For the Riemann-Liouville fractional integral and the Caputo fractional derivative, we have the following relation:

$$J_t^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0+) \frac{t^k}{k!}. \tag{9}$$

Definition 3. The sumudu transform of the Caputo fractional derivative is defined as follows [44]:

$$S\left[D_t^{\alpha} f(t)\right] = u^{-\alpha} S\left[f(t)\right] - \sum_{k=0}^{m} u^{-\alpha+k} f^{(k)}(0+), \quad (m-1 < \alpha \le m). \tag{10}$$

4. Solution by Homotopy Perturbation Sumudu Transform Method (HPSTM)

4.1. Basic Idea of HPSTM. To illustrate the basic idea of this method, we consider a general fractional nonlinear nonhomogeneous partial differential equation with the initial condition of the form

$$D_t^{\alpha} U(x,t) + RU(x,t) + NU(x,t) = g(x,t), \tag{11}$$

$$U(x,0) = f(x), \tag{12}$$

where $D_t^{\alpha} U(x,t)$ is the Caputo fractional derivative of the function $U(x,t)$, R is the linear differential operator, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Applying the sumudu transform (denoted in this paper by S) on both sides of (11), we get

$$S\left[D_t^{\alpha} U(x,t)\right] + S\left[RU(x,t)\right] + S\left[NU(x,t)\right] = S\left[g(x,t)\right]. \tag{13}$$

Using the property of the sumudu transform, we have

$$\begin{aligned} S\left[U(x,t)\right] = {} & f(x) + u^{\alpha} S\left[g(x,t)\right] \\ & - u^{\alpha} S\left[RU(x,t) + NU(x,t)\right]. \end{aligned} \tag{14}$$

Operating with the sumudu inverse on both sides of (14) gives

$$U(x,t) = G(x,t) - S^{-1}\left[u^{\alpha} S\left[RU(x,t) + NU(x,t)\right]\right], \tag{15}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the HPM

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t), \tag{16}$$

and the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{n=0}^{\infty} p^n H_n(U), \tag{17}$$

for some He's polynomials $H_n(U)$ [26, 45] that are given by

$$H_n(U_0, U_1, \dots, U_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{\infty} p^i U_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots. \tag{18}$$

Substituting (16) and (17) in (15), we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,t) & \\ = G(x,t) & \\ - p\left(S^{-1}\left[u^{\alpha} S\left[R \sum_{n=0}^{\infty} p^n U_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(U) \right] \right] \right), & \end{aligned} \tag{19}$$

which is the coupling of the sumudu transform and the HPM using He's polynomials. Comparing the coefficients of like powers of p, the following approximations are obtained:

$$\begin{aligned} p^0 &: U_0(x,t) = G(x,t), \\ p^1 &: U_1(x,t) = -S^{-1}\left[u^{\alpha} S\left[RU_0(x,t) + H_0(U)\right]\right], \\ p^2 &: U_2(x,t) = -S^{-1}\left[u^{\alpha} S\left[RU_1(x,t) + H_1(U)\right]\right], \\ p^3 &: U_3(x,t) = -S^{-1}\left[u^{\alpha} S\left[RU_2(x,t) + H_2(U)\right]\right], \\ &\vdots \end{aligned} \tag{20}$$

Proceeding in this same manner, the rest of the components $U_n(x,t)$ can be completely obtained and the series solution is thus entirely determined. Finally, we approximate the analytical solution $U(x,t)$ by truncated series

$$U(x,t) = \lim_{N \to \infty} \sum_{n=0}^{N} U_n(x,t). \tag{21}$$

The previous series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [46].

4.2. Solution of the Problem. Consider the following nonlinear time-fractional Harry Dym equation:

$$D_t^{\alpha} U(x,t) = U^3(x,t) D_x^3 U(x,t), \quad 0 < \alpha \le 1, \tag{22}$$

with the initial condition

$$U(x,0) = \left(a - \frac{3\sqrt{b}}{2} x \right)^{2/3}. \tag{23}$$

Applying the sumudu transform on both sides of (22), subject to initial condition (23), we have

$$S\left[U(x,t)\right] = \left(a - \frac{3\sqrt{b}}{2} x \right)^{2/3} + u^{\alpha} S\left[U^3(x,t) D_x^3 U(x,t)\right]. \tag{24}$$

The inverse Sumudu transform implies that

$$U(x,t)=\left(a-\frac{3\sqrt{b}}{2}x\right)^{2/3}+S^{-1}\left[u^{\alpha}S\left[U^{3}(x,t)D_{x}^{3}U(x,t)\right]\right]. \tag{25}$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty}p^{n}U_{n}(x,t)=\left(a-\frac{3\sqrt{b}}{2}x\right)^{2/3}+p\left(S^{-1}\left[u^{\alpha}S\left[\sum_{n=0}^{\infty}p^{n}\ H_{n}(U)\right]\right]\right), \tag{26}$$

where $H_n(U)$ are He's polynomials that represent the nonlinear terms. So, the He's polynomials are given by

$$\sum_{n=0}^{\infty}p^{n}\ H_{n}(U)=U^{3}D_{x}^{3}U. \tag{27}$$

The first few components of He's polynomials are given by

$$\begin{aligned}
&H_{0}(U)=U_{0}^{3}D_{x}^{3}U_{0},\\
&H_{1}(U)=U_{0}^{3}D_{x}^{3}U_{1}+3U_{0}^{2}U_{1}D_{x}^{3}U_{0},\\
&H_{2}(U)=U_{0}^{3}D_{x}^{3}U_{2}+3U_{0}^{2}U_{1}D_{x}^{3}U_{1}+\left(3U_{0}U_{1}^{2}+3U_{0}^{2}U_{2}\right)D_{x}^{3}U_{0},\\
&\vdots
\end{aligned} \tag{28}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned}
p^{0}&:U_{0}(x,t)=\left(a-\frac{3\sqrt{b}}{2}x\right)^{2/3},\\
p^{1}&:U_{1}(x,t)=S^{-1}\left[u^{\alpha}S\left[H_{0}(U)\right]\right]\\
&\qquad=-b^{3/2}\left(a-\frac{3\sqrt{b}}{2}x\right)^{-1/3}\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
p^{2}&:U_{2}(x,t)=S^{-1}\left[u^{\alpha}S\left[H_{1}(U)\right]\right]\\
&\qquad=-\frac{b^{3}}{2}\left(a-\frac{3\sqrt{b}}{2}x\right)^{-4/3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
p^{3}&:U_{3}(x,t)=S^{-1}\left[u^{\alpha}S\left[H_{2}(U)\right]\right]\\
&\qquad=b^{9/2}\left(a-\frac{3\sqrt{b}}{2}x\right)^{-7/3}\\
&\qquad\quad\times\left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^{2}}-16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\vdots
\end{aligned} \tag{29}$$

In this manner the rest of components of the HPSTM solution can be obtained. Thus, the solution $U(x,t)$ of the (22) is given as

$$\begin{aligned}
U(x,t)={}&\left(a-\frac{3\sqrt{b}}{2}x\right)^{2/3}-b^{3/2}\left(a-\frac{3\sqrt{b}}{2}x\right)^{-1/3}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&-\frac{b^{3}}{2}\left(a-\frac{3\sqrt{b}}{2}x\right)^{-4/3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&+b^{9/2}\left(a-\frac{3\sqrt{b}}{2}x\right)^{-7/3}\\
&\times\left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^{2}}-16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots.
\end{aligned} \tag{30}$$

The series solution converges very rapidly. The rapid convergence means that only few terms are required to get analytic function. Now, we calculate numerical results of the approximate solution $U(x,t)$ for different values of $\alpha = 1/3, 1/2, 1$ and for various values of t and x. The numerical results for the approximate solution obtained by using HPSTM and the exact solution given by Mokhtari [10] for constant values of $a = 4$ and $b = 1$ for various values of t, x, and α are shown in Figures 1(a)–1(d), and those for different values of x and α at $t = 1$ are depicted in Figure 2. It is observed from Figures 1(a)–1(c) that $U(x,t)$ decreases with the increase in both x and t for $\alpha = 1/3, 1/2$, and $\alpha = 1$. Figures 1(c)-1(d) clearly shows that, when $\alpha = 1$, the approximate solution obtained by the HPSTM is very near to the exact solution. It is also seen from Figure 2 that as the value of α increases, the displacement $U(x,t)$ increases. It is to be noted that only the third order term of the HPSTM was used in evaluating the approximate solutions for Figure 1. It is evident that the efficiency of the present method can be dramatically enhanced by computing further terms of $U(x,t)$ when the HPSTM is used.

5. Solution by Adomian Decomposition Method (ADM)

5.1. Basic Idea of ADM. To illustrate the basic idea of Adomian decomposition method [47, 48], we consider a general fractional nonlinear nonhomogeneous partial differential equation with the initial condition of the form

$$D_{t}^{\alpha}U(x,t)+RU(x,t)+NU(x,t)=g(x,t), \tag{31}$$

where $D_t^{\alpha}U(x,t)$ is the Caputo fractional derivative of the function $U(x,t)$, R is the linear differential operator, N represents the general nonlinear differential operator, and $g(x,t)$ is the source term.

Applying the operator J_t^{α} on both sides of (31) and using result (9), we have

$$U(x,t)=\sum_{k=0}^{m-1}\left(\frac{\partial^{k}U}{\partial t^{k}}\right)_{t=0}\frac{t^{k}}{k!}+J_{t}^{\alpha}g(x,t)-J_{t}^{\alpha}\left[RU(x,t)+NU(x,t)\right]. \tag{32}$$

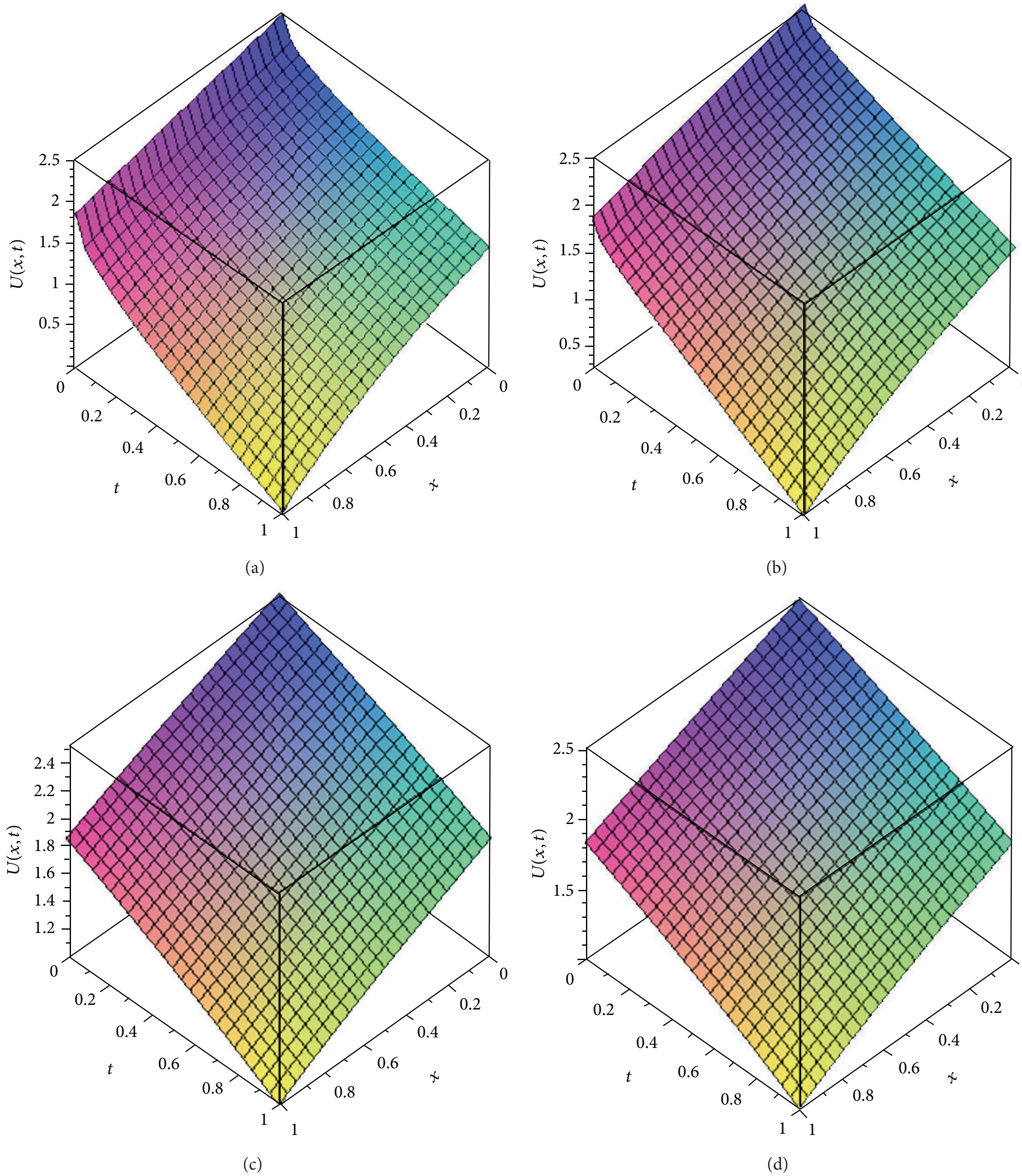

Figure 1: The behaviour of the $U(x,t)$ with respect to x and t being obtained, with (a) $\alpha = 1/3$; (b) $\alpha = 1/2$; (c) $\alpha = 1$; (d) exact solution.

Next, we decompose the unknown function $U(x,t)$ into sum of an infinite number of components given by the decomposition series

$$U = \sum_{n=0}^{\infty} U_n, \tag{33}$$

and the nonlinear term can be decomposed as

$$NU = \sum_{n=0}^{\infty} A_n, \tag{34}$$

where A_n are Adomian polynomials that are given by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i U_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots. \tag{35}$$

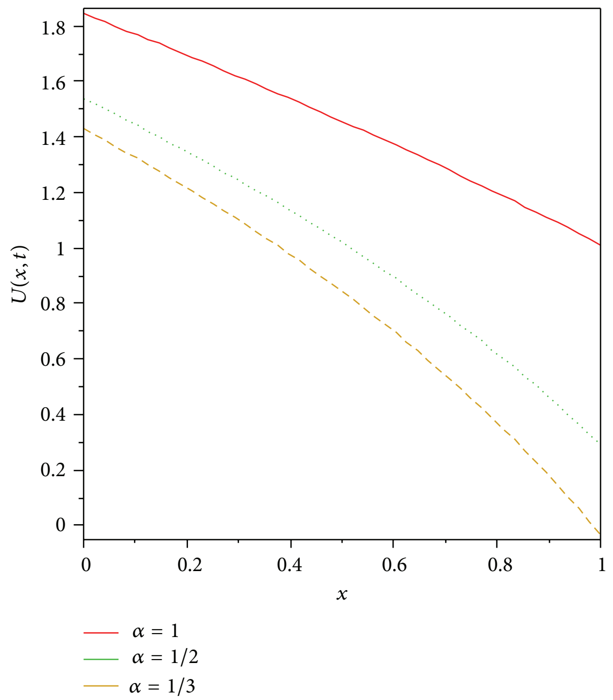

FIGURE 2: Plots of $U(x,t)$ versus x at $t = 1$ for different values of α.

The components $U_0, U_1, U_2, \ldots$ are determined recursively by substituting (33) and (34) into (32) leading to

$$\begin{aligned}\sum_{n=0}^{\infty} U_n = &\sum_{k=0}^{m-1} \left(\frac{\partial^k U}{\partial t^k}\right)_{t=0} \frac{t^k}{k!} + J_t^{\alpha} g(x,t) \\ &- J_t^{\alpha}\left[R\left(\sum_{n=0}^{\infty} U_n\right) + \sum_{n=0}^{\infty} A_n\right].\end{aligned} \tag{36}$$

This can be written as

$$\begin{aligned}&U_0 + U_1 + U_2 + \cdots \\ &= \sum_{k=0}^{m-1} \left(\frac{\partial^k U}{\partial t^k}\right)_{t=0} \frac{t^k}{k!} + J_t^{\alpha} g(x,t) \\ &\quad - J_t^{\alpha}\left[R\left(U_0 + U_1 + U_2 + \cdots\right) + \left(A_0 + A_1 + A_2 + \cdots\right)\right].\end{aligned} \tag{37}$$

Adomian method uses the formal recursive relations as

$$\begin{aligned}U_0 &= \sum_{k=0}^{m-1} \left(\frac{\partial^k U}{\partial t^k}\right)_{t=0} \frac{t^k}{k!} + J_t^{\alpha} g(x,t), \\ U_{n+1} &= -J_t^{\alpha}\left[R\left(U_n\right) + A_n\right], \quad n \geq 0.\end{aligned} \tag{38}$$

5.2. Solution of the Problem. To solve the nonlinear time-fractional Harry Dym equation (22)-(23), we apply the operator J_t^{α} on both sides of (22) and use result (9) to obtain

$$U = \sum_{k=0}^{1-1} \frac{t^k}{k!}\left[D_t^k U\right]_{t=0} + J_t^{\alpha}\left[U^3 D_x^3 U\right]. \tag{39}$$

This gives the following recursive relations using (38):

$$\begin{aligned}U_0 &= \sum_{k=0}^{0} \frac{t^k}{k!}\left[D_t^k U\right]_{t=0}, \\ U_{n+1} &= J_t^{\alpha}\left[A_n\right], \quad n = 0, 1, 2, \ldots,\end{aligned} \tag{40}$$

where

$$\sum_{n=0}^{\infty} A_n(U) = U^3 D_x^3 U. \tag{41}$$

The first few components of Adomian polynomials are given by

$$\begin{aligned}A_0(U) &= U_0^3 D_x^3 U_0, \\ A_1(U) &= U_0^3 D_x^3 U_1 + 3U_0^2 U_1 D_x^3 U_0, \\ A_2(U) &= U_0^3 D_x^3 U_2 + 3U_0^2 U_1 D_x^3 U_1 \\ &\quad + \left(3U_0 U_1^2 + 3U_0^2 U_2\right) D_x^3 U_0, \\ &\vdots\end{aligned} \tag{42}$$

The components of the solution can be easily found by using the previous recursive relations as

$$\begin{aligned}U_0(x,t) &= \left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3}, \\ U_1(x,t) &= -b^{3/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-1/3} \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ U_2(x,t) &= -\frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-4/3} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ U_3(x,t) &= b^{9/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-7/3} \\ &\quad \times \left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^2} - 16\right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ &\vdots\end{aligned} \tag{43}$$

and so on. In this manner the rest of components of the decomposition solution can be obtained. Thus, the ADM

Table 1: Comparison study between HPSTM, ADM, and the exact solution, when $\alpha = 1$ and for constant values of $a = 4$ and $b = 1$.

x	t	HPSTM	ADM	Exact solution
0	1	1.843946953	1.843946953	1.843946953
0.2	1	1.694117377	1.694117377	1.691538112
0.4	1	1.537581542	1.537581542	1.534036644
0.6	1	1.373028020	1.373028020	1.367980757
0.8	1	1.198654865	1.198654865	1.191138425
1.0	1	1.011880649	1.011880649	1.000000000

solution $U(x,t)$ of (22) is given as

$$\begin{aligned} U(x,t) &= \left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3} \\ &\quad - b^{3/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-1/3}\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad - \frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-4/3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad + b^{9/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-7/3} \\ &\quad \times \left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^2} - 16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots, \end{aligned} \tag{44}$$

which is the same solution as obtained by using HPSTM.

From Table 1, it is observed that the values of the approximate solution at different grid points obtained by the HPSTM and ADM are close to the values of the exact solution with high accuracy at the third term approximation. It can also be noted that the accuracy increases as the order of approximation increases.

6. Conclusions

In this paper, the homotopy perturbation sumudu transform method (HPSTM) and the Adomian decomposition method (ADM) are successfully applied for solving nonlinear time-fractional Harry Dym equation. The comparison between the third order terms solution of the HPSTM, ADM, and exact solution is given in Table 1. It is observed that for $t = 1$ and $\alpha = 1$, there is a good agreement between the HPSTM, ADM, and exact solution. Therefore, these two methods are very powerful and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, HPSTM has an advantage over the Adomian decomposition method (ADM) such that it solves the nonlinear problems without using Adomian polynomials. In conclusion, the HPSTM may be considered as a nice refinement in existing numerical techniques and might find wide applications.

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Research Article

The Use of Sumudu Transform for Solving Certain Nonlinear Fractional Heat-Like Equations

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We make use of the properties of the Sumudu transform to solve nonlinear fractional partial differential equations describing heat-like equation with variable coefficients. The method, namely, homotopy perturbation Sumudu transform method, is the combination of the Sumudu transform and the HPM using He's polynomials. This method is very powerful, and professional techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering.

1. Introduction

In the literature one can find a wide class of methods dealing with the problem of approximate solutions to problems described by nonlinear fractional differential equations, for instance, asymptotic methods and perturbation methods [1]. The perturbation methods have some limitations; for instance, the approximate solution engages series of small parameters which causes difficulty since most nonlinear problems have no small parameters at all [1]. Even though a suitable choice of small parameters occasionally lead to ideal solution, in most cases unsuitable choices leads to serious effects in the solutions [1]. Therefore, an analytical method which does not require a small parameter in the equation modeling of the phenomenon is welcome [2–4]. To deal with the pitfall presented by these perturbation methods for solving nonlinear equations, a literature review in some new asymptotic methods for the search for the solitary solutions of nonlinear differential equations, nonlinear differential-difference equations, and nonlinear fractional differential equations is presented in [5]. The homotopy perturbation method (HPM) was first initiated by He [6]. The HPM was also studied by many authors to present approximate and exact solution of linear and nonlinear equations arising in various scientific and technological fields [7–13]. The Adomian decomposition method (ADM) [14–19] and variational iteration method (VIM) [2–4] have also been applied to study the various physical problems. The homotopy decomposition method (HDM) was recently proposed by [20, 21] to solve the groundwater flow equation and the modified fractional KDV equation [20, 21]. The homotopy decomposition method is actually the combination of the perturbation method and Adomian decomposition method. Singh et al. [22] have made used of studying the solutions of linear and nonlinear partial differential equations by using the homotopy perturbation Sumudu transform method (HPSTM). The HPSTM is a combination of Sumudu transform, HPM, and He's polynomials.

2. Sumudu Transform

The Sumudu transform is an integral transform similar to the Laplace transform, introduced in the early 1990s by Watugala [23] to solve differential equations and control engineering problems.

First we will summon up the following useful definitions and theorems for this integral transform operator. Note that these theorems and definitions will be used in the rest of the paper.

2.1. Definitions and Theorems

Definition 1. The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F_s(u)$, defined by

$$S(f(t)) = F_s(u) = \int_0^{\infty} \frac{1}{u} \exp\left[-\frac{t}{u}\right] f(t)\, dt. \quad (1)$$

Definition 2. The double Sumudu transform of a function $f(x,t)$, defined for all real numbers $(x \geq 0, t \geq 0)$, is defined by

$$\begin{aligned} F(u,v) &= S_2[f(x,t),(u,v)] \\ &= \frac{1}{vu}\iint_0^{\infty} \exp\left[-\left(\frac{t}{v}+\frac{x}{u}\right)\right] f(x,t)\, dx\, dt. \end{aligned} \quad (2)$$

In the same line of ideas, the double Sumudu transform of second partial derivative with respect to x is of form [24]

$$\begin{aligned} S_2&\left[\frac{\partial^2 f(x,t)}{\partial x^2};(u,v)\right] \\ &= \frac{1}{u^2}F(u,v) - \frac{1}{u^2}F(0,v) - \frac{1}{u}\frac{\partial F(0,v)}{\partial x}. \end{aligned} \quad (3)$$

Similarly, the double Sumudu transform of second partial derivative with respect to t is of form [24]

$$\begin{aligned} S_2&\left[\frac{\partial^2 f(x,t)}{\partial t^2};(u,v)\right] \\ &= \frac{1}{v^2}F(u,v) - \frac{1}{v^2}F(u,0) - \frac{1}{u}\frac{\partial F(u,0)}{\partial t}. \end{aligned} \quad (4)$$

Theorem 3. *Let $G(u)$ be the Sumudu transform of $f(t)$ such that*

(i) *$G(1/s)/s$ is a meromorphic function, with singularities having $\mathrm{Re}[s] \leq \gamma$ and*

(ii) *there exist a circular region Γ with radius R and positive constants M and $K$with $|G(1/s)/s| < MR^{-K}$; then the function $f(t)$ is given by*

$$\begin{aligned} S^{-1}(G(s)) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp[st]\, G\left(\frac{1}{s}\right)\frac{ds}{s} \\ &= \sum \mathit{residual}\left[\exp[st]\frac{G(1/s)}{s}\right]. \end{aligned} \quad (5)$$

For the proof see [23].

2.2. Properties of Sumudu Transform [25–28]

(i) The transform of a Heaviside unit step function is a Heaviside unit step function in the transformed domain [26, 27].

(ii) The transform of a Heaviside unit ramp function is a Heaviside unit ramp function in the transformed domain [26, 27].

(iii) The transform of a monomial t^n is the scaled monomial $S(t^n) = n!u^n$ [26, 27].

(iv) If $f(t)$ is a monotonically increasing function, so is $F(u)$, and the converse is true for decreasing functions [26, 27].

(v) The Sumudu transform can be defined for functions which are discontinuous at the origin. In that case the two branches of the function should be transformed separately. If $f(t)$ is C^n continuous at the origin, so is the transformation $F(u)$ [26, 27].

(vi) The limit of $f(t)$ as t tends to zero is equal to the limit of $F(u)$ as u tends to zero provided both limits exist [26, 27].

(vii) The limit of $f(t)$ as t tends to infinity is equal to the limit of $F(u)$ as u tends to infinity provided both limits exist [26, 27].

(viii) Scaling of the function by a factor $c > 0$ to form the function $f(ct)$ gives a transform $F(cu)$ which is the result of scaling by the same factor [26, 27].

2.3. Basic Definition of Fractional Calculus

Definition 4. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0,\infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 5. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$\begin{gathered} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, dt, \quad \alpha > 0,\ x > 0, \\ J^0 f(x) = f(x). \end{gathered} \quad (6)$$

Properties of the operator can be found in [30–33] one mentions only the following.

For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma > -1$

$$\begin{gathered} J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \\ J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \\ J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \end{gathered} \quad (7)$$

Lemma 6. *If $m-1<\alpha\leq m$, $m\in\mathbb{N}$ and $f\in C_\mu^m$, and $\mu\geq -1$, then*

$$D^\alpha J^\alpha f(x) = f(x),$$
$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x>0. \tag{8}$$

Definition 7 (partial derivatives of fractional order). Assume now that $f(\mathbf{x})$ is a function of n variables x_i $i=1,\ldots,n$ also of class C on $D\in\mathbb{R}_n$

$$a\partial_{\underline{\mathbf{x}}}^\alpha f = \frac{1}{\Gamma(m-\alpha)}\int_a^{x_i}(x_i-t)^{m-\alpha-1}\partial_{x_i}^m f(x_j)\Big|_{x_j=t} dt, \tag{9}$$

where $\partial_{x_i}^m$ is the usual partial derivative of integer order m.

Definition 8. The Sumudu transform of the Caputo fractional derivative is defined as follows [28]:

$$S[D_t^\alpha f(t)] = u^{-\alpha}S[f(t)] - \sum_{k=0}^{m-1}u^{-\alpha+k}f^{(k)}(0^+) \quad (m-1<\alpha\leq m). \tag{10}$$

3. Solution by HPSTM

3.1. Basic Idea of HPSTM. We illustrate the basic idea of this method, by considering a general fractional nonlinear nonhomogeneous partial differential equation with the initial condition of the form of general form

$$D_t^\alpha U(x,t) = L(U(x,t)) + N(U(x,t)) + f(x,t), \quad \alpha>0 \tag{11}$$

subject to the initial condition

$$D_0^k U(x,0) = g_k, \quad (k=0,\ldots,n-1),$$
$$D_0^n U(x,0) = 0, \quad n=[\alpha], \tag{12}$$

where, D_t^α denotes without loss of generality the Caputo fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator, and L represents a linear fractional differential operator.

Applying the Sumudu Transform on both sides of (11), we obtain

$$S[D_t^\alpha U(x,t]) = S[L(U(x,t))] + S[N(U(x,t))] + S[f(x,t)]. \tag{13}$$

Using the property of the Sumudu transform, we have

$$S[U(x,t)] = u^\alpha S[L(U(x,t))] + u^\alpha S[N(U(x,t))] + u^\alpha S[f(x,t)] + g(x,t). \tag{14}$$

Now applying the Sumudu inverse on both sides of (24) we obtain

$$U(x,t) = S^{-1}[u^\alpha S[L(U(x,t))] + u^\alpha S[N(U(x,t))]] + G(x,t), \tag{15}$$

where $G(x,t)$ represents the term arising from the known function $f(x,t)$ and the initial conditions [1].

Now we apply the HPM

$$U(x,t) = \sum_{n=0}^{\infty}p^n U_n(x,t). \tag{16}$$

The nonlinear term can be decomposed into

$$NU(x,t) = \sum_{n=0}^{\infty}p^n \mathscr{H}_n(U) \tag{17}$$

using the He's polynomial $\mathscr{H}_n(U)$ [17, 18] given as

$$\mathscr{H}_n(U_0,\ldots,U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{j=0}^{\infty}p^jU_j(x,t)\right)\right], \quad n=0,1,2,\ldots. \tag{18}$$

Substituting (16) and (17)

$$\sum_{n=0}^{\infty}p^nU_n(x,t) = G(x,t) + p\left[S^{-1}\left[u^\alpha S\left[L\left(\sum_{n=0}^{\infty}p^nU_n(x,t)\right)\right] + u^\alpha S\left[N\left(\sum_{n=0}^{\infty}p^nU_n(x,t)\right)\right]\right]\right] \tag{19}$$

which is the coupling of the Sumudu transform and the HPM using He's polynomials [1]. Comparing the coefficients of like powers of p, the following approximations are obtained:

$$\begin{aligned}
p^0 &: U_0(x,t) = G(x,t),\\
p^1 &: U_1(x,t) = S^{-1}[u^\alpha S[L(U_0(x,t)) + H_0(U)]],\\
p^2 &: U_2(x,t) = S^{-1}[u^\alpha S[L(U_1(x,t)) + H_1(U)]],\\
p^3 &: U_3(x,t) = S^{-1}[u^\alpha S[L(U_2(x,t)) + H_2(U)]],\\
p^n &: U_n(x,t) = S^{-1}[u^\alpha S[L(U_{n-1}(x,t)) + H_{n-1}(U)]].
\end{aligned} \tag{20}$$

Finally, we approximate the analytical solution $U(x,t)$ by truncated series [1]

$$U(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} U_n(x,t). \tag{21}$$

The above series solutions generally converge very rapidly [1, 34–37].

4. Application

In this section we apply this method for solving fractional differential equation in form of (11) together with (12).

Example 9. Consider the following three-dimensional fractional heat-like equation:

$$\partial_t^\alpha u(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right), \quad 0 < x, y, z < 1, \quad 0 < \alpha \le 1 \tag{22}$$

subject to the initial condition

$$u(x,y,z,0) = 0. \tag{23}$$

Following carefully the steps involved in the HDM, we arrive at the following equation:

$$\begin{aligned}
\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) &= \frac{p}{\Gamma(\alpha)} S^{-1} \times \Bigg(u^{-\alpha} \times \Bigg[\int_0^t (t-\tau)^{\alpha-1} \times \Bigg(x^4 y^4 z^4 + \frac{1}{36} \\
&\quad \times \Bigg(x^2 \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \right)_{xx} + y^2 \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \right)_{yy} \\
&\quad + z^2 \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \right)_{zz} \Bigg) \Bigg) d\tau \Bigg] \Bigg).
\end{aligned} \tag{24}$$

Now comparing the terms of the same power of p yields

$$\begin{aligned}
p^0 &: u_0(x,y,z,t), \\
p^1 &: u_1(x,y,z,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x^4 y^4 z^4 \, d\tau, \\
&\vdots \\
p^n &: u_n(x,y,z,t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \times \left(\frac{1}{36} \left(x^2 (u_{n-1})_{xx} + y^2 (u_{n-1})_{yy} + z^2 (u_{n-1})_{zz} \right) \right) d\tau, \\
&u_n(x,y,z,0) = 0, \quad n \ge 2.
\end{aligned} \tag{25}$$

Thus the following components are obtained as results of the above integrals:

$$\begin{aligned}
u_0(x,y,z,t) &= 0, \\
u_1(x,y,z,t) &= \frac{t^\alpha x^4 y^4 z^4}{\Gamma(\alpha+1)}, \\
u_2(x,y,z,t) &= \frac{t^{2\alpha} x^4 y^4 z^4}{\Gamma(2\alpha+1)}, \\
u_3(x,y,z,t) &= \frac{t^{3\alpha} x^4 y^4 z^4}{\Gamma(3\alpha+1)}, \\
&\vdots \\
u_n(x,y,z,t) &= \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)}.
\end{aligned} \tag{26}$$

Therefore the approximate solution of equation for the first n is given as

$$u_N(x,y,z,t) = \sum_{n=1}^{N} \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)}. \tag{27}$$

Now when $N \to \infty$, we obtained the following solution:

$$\begin{aligned}
u(x,y,z,t) &= \sum_{n=0}^{\infty} \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)} - x^4 y^4 z^4 \\
&= x^4 y^4 z^4 \left(E_\alpha(t^\alpha) - 1 \right),
\end{aligned} \tag{28}$$

where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function. Note that in the case $\alpha = 1$

$$u(x,y,z,t) = x^4y^4z^4(\exp(t) - 1). \tag{29}$$

This is the exact solution for this case.

Example 10. We consider the three-dimensional fractional wave-like equation

$$\begin{gathered} \partial_t^\alpha u(x,y,z,t) \\ = x^2+y^2+z^2+\frac{1}{2}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right), \\ 0<x,y,z<1, \quad 1<\alpha\le 2 \end{gathered} \tag{30}$$

subject to the initial condition

$$u(x,y,z,0) = 0, \qquad u_t(x,y,z,0) = x^2+y^2-z^2. \tag{31}$$

Following carefully the steps involved in the HPSTM, we arrive at the following series solutions:

$$\begin{gathered} u_0(x,y,z,t) = \left(x^2+y^2-z^2\right)t, \\ u_1(x,y,z,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}\left(x^2+y^2-z^2\right), \\ u_2(x,y,z,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left(x^2+y^2+z^2\right), \\ u_3(x,y,z,t) = \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\left(x^2+y^2-z^2\right), \\ \vdots \\ u_n(x,y,z,t) = \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\left(x^2+y^2+(-1)^nz^2\right). \end{gathered} \tag{32}$$

Therefore the approximate solution of equation for the first n is given as

$$u_N(x,y,z,t) = \sum_{n=1}^{N}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\left(x^2+y^2+(-1)^nz^2\right). \tag{33}$$

Now when $N \to \infty$, we obtained the following solution:

$$u(x,y,z,t) = \sum_{n=1}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\left(x^2+y^2+(-1)^nz^2\right). \tag{34}$$

In the case of $\alpha = 2$ we obtain

$$\begin{aligned} u(x,y,z,t) = {} & \left(x^2+y^2\right)\exp(t) \\ & + z^2\exp(-t) - \left(x^2+y^2+z^2\right). \end{aligned} \tag{35}$$

This is the exact solution for this case.

Example 11. We consider the one-dimensional fractional wave-like equation

$$\partial_t^\alpha u(x,t) = \frac{1}{2}x^2u_{xx}, \quad 0<x<1,\ 1<\alpha\le 2,\ t>0 \tag{36}$$

with the initial conditions as

$$u(x,0) = x^2. \tag{37}$$

Following carefully the steps involved in the HPSTM, we arrive at the following series solutions:

$$\begin{gathered} u_0(x,t) = x^2, \\ u_1(x,t) = \frac{t^\alpha x^2}{\Gamma(\alpha+1)}, \\ u_2(x,t) = \frac{t^{2\alpha}x^2}{\Gamma(2\alpha+1)}, \\ u_3(x,t) = \frac{t^{3\alpha}x^2}{\Gamma(3\alpha+1)}, \\ \vdots \\ u_n(x,t) = \frac{t^{n\alpha}x^2}{\Gamma(n\alpha+1)}. \end{gathered} \tag{38}$$

Therefore the approximate solution of equation for the first n is given as

$$u_N(x,t) = \sum_{n=1}^{N}\frac{t^{n\alpha}x^2}{\Gamma(n\alpha+1)}. \tag{39}$$

Now when $N \to \infty$, we obtained the following solution:

$$u(x,t) = \sum_{n=0}^{\infty}\frac{t^{n\alpha}x^2}{\Gamma(n\alpha+1)} = x^2E_\alpha(t^\alpha), \tag{40}$$

where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function. Note that in the case $\alpha = 1$

$$u(x,t) = x^2\exp(t). \tag{41}$$

This is the exact solution for this case.

Example 12. In this example we consider the two-dimensional fractional heat-like equation (Figures 1 and 2)

$$\begin{gathered} \partial_t^\alpha u(x,t) = u_{xx}+u_{yy}, \\ 0<x, \quad y<2\pi, \quad t>0, \quad 0<\alpha\le 1. \end{gathered} \tag{42}$$

subject to the initial condition

$$u(x,y,0) = \sin(x)\sin(y). \tag{43}$$

Following carefully the steps involved in the HPSTM, we arrive at the following series solutions:

$$\begin{aligned}
u_0(x,y,t) &= \sin(x)\sin(y),\\
u_1(x,y,t) &= -2\frac{t^{\alpha}\sin(x)\sin(y)}{\Gamma(\alpha+1)},\\
u_2(x,y,t) &= 4\frac{t^{2\alpha}\sin(x)\sin(y)}{\Gamma(2\alpha+1)},\\
u_3(x,y,t) &= -8\frac{t^{3\alpha}\sin(x)\sin(y)}{\Gamma(3\alpha+1)},\\
&\vdots\\
u_n(x,y,z,t) &= (-2)^n\frac{t^{n\alpha}\sin(x)\sin(y)}{\Gamma(n\alpha+1)}.
\end{aligned} \tag{44}$$

Therefore the approximate solution of equation for the first n is given as

$$u_N(x,y,t) = \sum_{n=1}^{N}(-2)^n\frac{t^{n\alpha}\sin(x)\sin(y)}{\Gamma(n\alpha+1)}. \tag{45}$$

Now when $N \to \infty$, we obtained the following solution:

$$u(x,y,t) = \sum_{n=0}^{\infty}\frac{(-2)^n t^{n\alpha}\sin(x)\sin(y)}{\Gamma(n\alpha+1)}. \tag{46}$$

Note that in the case $\alpha = 1$

$$u(x,y,z,t) = \sin(x)\sin(y)\exp(-2t). \tag{47}$$

This is the exact solution for this case.

Example 13. Consider the following time-fractional derivative in x, y-plane as

$$D_t^{\alpha}u(x,y,t) = \frac{1}{2}\nabla^2 u(x,y,t), \quad 1<\alpha\le 2,\ x,y\in\mathbb{R},\ t>0 \tag{48}$$

subject to the initial conditions

$$u(x,y,0) = \sin(x+y), \qquad u_t(x,y,0) = -\cos(x+y). \tag{49}$$

Applying the steps involved in HPSTM as presented in Section 3.1 to (49) we obtain

$$\begin{aligned}
p^0:\ & u_0(x,y,t) = \sin(x+y) - \cos(x+y)t,\\
p^1:\ & u_1(x,t) = S^{-1}\left[u^{\alpha}S\left(\frac{1}{2}x^2\left[(u_0)_{xx}+(u_0)_{yy}\right]\right)\right]\\
&= -\sin(x+y)\frac{t^2}{2} + \cos(x+y)\frac{t^3}{3!},\\
p^2:\ & u_2(x,t) = S^{-1}\left[u^{\alpha}S\left(\frac{1}{2}x^2\left[(u_1)_{xx}+(u_1)_{yy}\right]\right)\right]\\
&= \sin(x+y)\left[-\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^{4-\alpha}}{\Gamma(5-\alpha)}\right]\\
&\quad + \cos(x+y)\left[-\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^{5-\alpha}}{\Gamma(6-\alpha)}\right],\\
p^3:\ & u_3(x,t)\\
&= S^{-1}\left[u^{\alpha}S\left(\frac{1}{2}x^2\left[(u_2)_{xx}+(u_2)_{yy}\right]\right)\right]\\
&= \sin(x+y)\left[-\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\frac{2t^{4-\alpha}}{\Gamma(5-\alpha)}-\frac{2t^{6-\alpha}}{\Gamma(7-\alpha)}\right.\\
&\qquad\left. -\frac{4^{\alpha-2}\sqrt{\pi}t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)}\right]\\
&\quad + \cos(x+y)\\
&\quad \times\left[\frac{t^3}{3!}-\frac{t^5}{5!}+\frac{t^7}{7!}+\frac{2t^{7-\alpha}}{\Gamma(7-\alpha)}+\frac{2t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right].
\end{aligned} \tag{50}$$

Therefore the series solution is given as

$$\begin{aligned}
u&(x,y,t)\\
&= \sin(x+y)\\
&\quad\times\left[1-\frac{3t^2}{2!}+\frac{t^4}{8}+\frac{t^6}{6!}+\frac{3t^{4-\alpha}}{\Gamma(5-\alpha)}-\frac{2t^{6-\alpha}}{\Gamma(7-\alpha)}\right.\\
&\qquad\left.-\frac{4^{\alpha-2}\sqrt{\pi}t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)}\right]\\
&\quad+\cos(x+y)\\
&\quad\times\left[-t+\frac{t^3}{3!}-\frac{t^5}{5!}+\frac{t^7}{7!}+\frac{3t^{7-\alpha}}{\Gamma(7-\alpha)}+\frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right]+\cdots.
\end{aligned} \tag{51}$$

It is important to point out that, if $\alpha = 2$, the above solution takes the form

$$u_{N=4}(x,y,t) = \sin(x+y)\left[1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}\right] - \cos(x+y)\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}\right] \tag{52}$$

which is the first four terms of the series expansion of the exact solution $u(x,y,t) = \sin(x+y-t)$.

Example 14. Consider the following two-dimensional heat-like equation:

$$D_{tt}^{\alpha}u = \frac{1}{12}\left(x^2u_{xx}+y^2u_{yy}\right), \quad 0<x, \quad y<1, \quad 1<\alpha\le 2, \quad t>0 \tag{53}$$

subject to the initial conditions

$$u(x,y,0) = x^2, \qquad u_t(x,y,0) = y^2. \tag{54}$$

The exact solution is given as

$$u(x,y,t) = x^2\cosh(t) + y^2\sinh(t). \tag{55}$$

Applying the Sumudu transform on both sides of (53), we obtain the following:

$$S[u(x,y,t)] = x^2t^2 + y^2 + u^{\alpha}\left[\frac{1}{12}S\left(x^2u_{xx}+y^2u_{yy}\right)\right] \tag{56}$$

Applying the inverse Sumudu transform on both sides of (56), we obtain the following:

$$u(x,y,t) = x^2t^2 + y^2 + S^{-1} \times \left[u^{\alpha}\left[\frac{1}{12}S\left(x^2u_{xx}+y^2u_{yy}\right)\right]\right]. \tag{57}$$

Now applying the homotopy perturbation technique on the above equation we obtain the following:

$$\sum_{n=0}^{\infty}p^nu_n(x,y,t) = x^2t^2 + y^2 + S^{-1} \times \left[u^{\alpha}\left[\frac{1}{12}S\left(x^2\left[\sum_{n=0}^{\infty}p^nu_n(x,y,t)\right]_{xx} + y^2\left[\sum_{n=0}^{\infty}p^nu_n(x,y,t)\right]_{yy}\right)\right]\right]. \tag{58}$$

By comparing the coefficients of like powers of p, we have

$$\begin{aligned}
p^0 &: u_0(x,y,t) = x^2t^2+y^2,\\
p^1 &: u_1(x,y,t)\\
&= S^{-1}\left[u^{\alpha}\left[\frac{1}{12}S\left(x^2(u_0)_{xx}+y^2(u_0)_{yy}\right)\right]\right]\\
&= x^2t^2+y^2+\frac{1}{6}y^4t^3+\frac{1}{2}x^4t^2,\\
p^2 &: u_2(x,y,t)\\
&= S^{-1}\left[u^{\alpha}\left[\frac{1}{12}S\left(x^2(u_1)_{xx}+y^2(u_1)_{yy}\right)\right]\right]\\
&= \frac{1}{3}y^4t^3+\frac{1}{2}x^4t^2+\frac{(yt)^4}{24}+\frac{(yt)^5}{120}\\
&\quad+\frac{x^4t^{5-\alpha}}{\Gamma(3-\alpha)(4-\alpha)}+\frac{y^4t^{5-\alpha}}{\Gamma(5-\alpha)(5-\alpha)}\\
&\quad-\frac{x^4t^{4-\alpha}}{\Gamma(3-\alpha)(3-\alpha)}-\frac{y^4t^{5-\alpha}}{\Gamma(4-\alpha)(4-\alpha)}\\
&\ \ \vdots
\end{aligned} \tag{59}$$

Example 15. Consider the following one-dimensional fractional heat-like equation:

$$D_{tt}^{\alpha}u = x^2\frac{\partial\left[u_xu_{xx}\right]}{\partial x} - x^2(u_{xx})^2 - u, \quad 0\le x\le 1, \quad 0\le t, \quad 1<\alpha\le 2 \tag{60}$$

Subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = x^2. \tag{61}$$

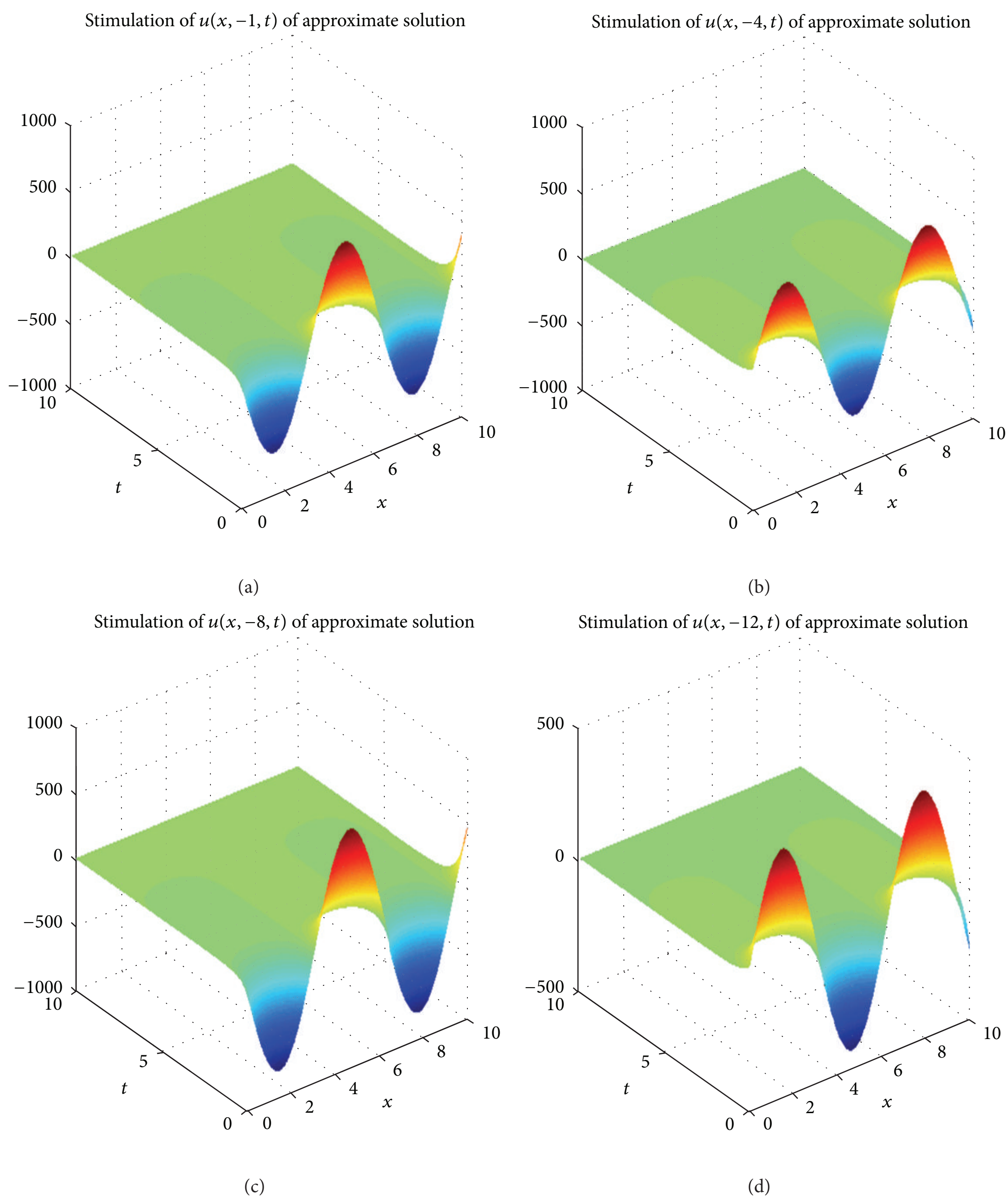

Figure 1: Numerical simulations of the approximate solution of (42) for a fixed y.

The exact solution is given as

$$u(x,t) = x^2 \sin[t]. \tag{62}$$

Applying the Sumudu transform on both sides of (60), we obtain the following:

$$S[u(x,t)] = x^2 + u^\alpha \left[x^2 \frac{\partial [u_x u_{xx}]}{\partial x} - x^2 (u_{xx})^2 - u \right]. \tag{63}$$

Applying the inverse Sumudu transform on both sides of (63), we obtain the following:

$$u(x,y,t) = x^2 + S^{-1} \times \left[u^\alpha \left[S\left(x^2 \frac{\partial [u_x u_{xx}]}{\partial x} - x^2 (u_{xx})^2 - u \right) \right] \right]. \tag{64}$$

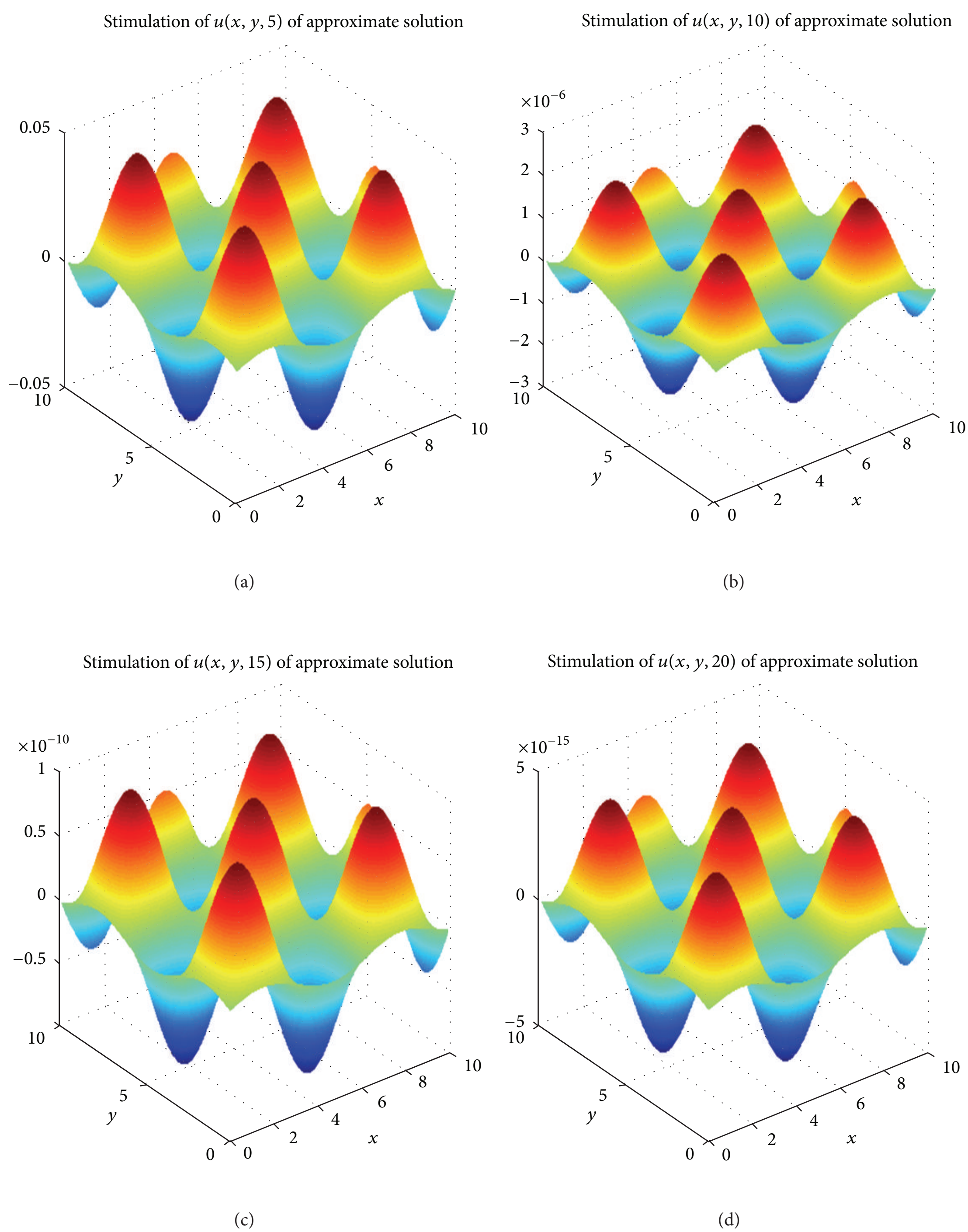

FIGURE 2: Numerical simulations of the approximate solution of (42) for a fixed t.

Now applying the homotopy perturbation technique on the above equation we obtain the following:

$$\sum_{n=0}^{\infty} p^n u_n(x, y, t) = x^2 + S^{-1} \times \left[u^{\alpha} \left[S \left(x^2 \sum_{n=0}^{\infty} p^n H_n - x^2 \sum_{n=0}^{\infty} p^n H_n^1 - \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right] \right]. \quad (65)$$

TABLE 1: Numerical values for approximate solutions of (49) via ADM, VIM, and HPSTM.

t	x	y	ADM [29]	VIM [29]	HPSTM	Exact
0.25	0.5	0.5	0.68163209	0.68163219	0.681639	0.681639
	0.5	1.0	0.94898215	0.94898245	0.948985	0.948985
	1.0	0.5	0.94898215	0.94898245	0.948985	0.948985
	1.0	1.0	0.98398623	0.98398643	0.983986	0.983986
0.5	0.5	0.5	0.47942925	0.47942985	0.479425	0.479426
	0.5	1.0	0.84147331	0.84147361	0.841471	0.841471
	1.0	0.5	0.84147331	0.84147361	0.841471	0.841471
	1.0	1.0	0.99749205	0.99749235	0.997495	0.997495
0.75	0.5	0.5	0.2474231	0.2474232	0.247402	0.247404
	1.0	1.0	0.68163452	0.68163456	0.681636	0.681639
	0.5	0.5	0.68163453	0.68163456	0.681636	0.681639
	1.0	1.0	0.94898533	0.94898532	0.948982	0.948985
1.0	0.5	0.5	−0.000001905	−0.000001925	−0.0000019	−0.00000018
	1.0	1.0	0.4794205	0.4794215	0.479401	0.479426
	0.5	0.5	0.4794205	0.4794215	0.479401	0.479426
	1.0	1.0	0.8414352	0.8414582	0.841448	0.841448

By comparing the coefficients of like powers of p, we have

$$\begin{aligned}
p^0 &: u_0(x,y,t) = x^2,\\
p^1 &: u_1(x,y,t)\\
&= S^{-1}\left[u^{\alpha}\left[S\left[x^2H_0 - x^2H_0^1 - u_0\right]\right]\right] = -x^2 t,\\
p^2 &: u_2(x,y,t)\\
&= S^{-1}\left[u^{\alpha}\left[u^{\alpha}\left[S\left[x^2H_1 - x^2H_1^1 - u_1\right]\right]\right]\right]\\
&= x^2\left[-\frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)}\right],\\
p^3 &: u_3(x,y,t)\\
&= S^{-1}\left[u^{\alpha}\left[\left[S\left[x^2H_2 - x^2H_2^1 - u_2\right]\right]\right]\right]\\
&= x^2\left[\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{2t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{2t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right],\\
u&(x,t)\\
&= x^2\left[-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!}\right.\\
&\qquad\left. + \frac{3t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)} + \cdots + \cdots\right].
\end{aligned} \tag{66}$$

Now if we replace $\alpha = 2$, we recover the following series approximation:

$$u(x,t) = x^2\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + \cdots\right] = x^2 \sin[t] \tag{67}$$

which is the exact solution of this case.

5. Conclusion

The aim of this work was to make use of the properties of the so-called Sumudu transform to solve nonlinear fractional heat-like equations. The basic idea of the method combines Sumudu transform and the HPM using He's polynomials. In addition the method is friendly user, and it does not require anything like Adomian polynomial. From the numerical comparison in Table 1, we can see that, these three methods are very powerful, and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, the HPSTM has an advantage over the ADM and VIM which is that it solves the nonlinear problems without anything like the Lagrangian multiplier as in the case of VIM. We do not need to calculate anything like Adomian polynomial as in the case of ADM. In addition the calculations involved in HPSTM are very simple and straightforward.

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Research Article

A Generalization of Lacunary Equistatistical Convergence of Positive Linear Operators

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In this paper we consider some analogs of the Korovkin approximation theorem via lacunary equistatistical convergence. In particular we study lacunary equi-statistical convergence of approximating operators on H_{w_2} spaces, the spaces of all real valued continuous functions f dened on $K = [0, \infty)^m$ and satisfying some special conditions.

1. Introduction

Approximation theory has important applications in the theory of polynomial approximation, in various areas of functional analysis, numerical solutions of integral and differential equations [1–6]. In recent years, with the help of the concept of statistical convergence, various statistical approximation results have been proved [7]. In the usual sense, every convergent sequence is statistically convergent, but its converse is not always true. And, statistical convergent sequences do not need to be bounded.

Recently, Aktuğlu and Gezer [8] generalized the idea of statistical convergence to lacunary equi-statistical convergences. In this paper, we first study some Korovkin type approximation theorems via lacunary equi- statistical convergence in H_{w_2} spaces. Then using the modulus of continuity, we study rates of lacunary equi-statistically convergence in H_{w_2}.

We recall here the concepts of equi-statistical convergence and lacunary equi-statistical convergence.

Let f and f_r belong to $C(X)$, which is the space of all continuous real valued functions on a compact subset X of the real numbers. $\{f_r\}$ is said to be equi-statistically convergent to f on X and denoted by $f_r \twoheadrightarrow f$ (equistat) if for every $\varepsilon > 0$, the sequence of real valued functions

$$p_{r,\varepsilon}(x) := \frac{1}{r}\left|\{m \le r : f_m(x) - f(x) \ge \varepsilon\}\right| \tag{1}$$

converges uniformly to the zero function on X, which means that

$$\lim_{r\to\infty}\left\|p_{r,\varepsilon}(\cdot)\right\|_{C(X)} = 0. \tag{2}$$

A lacunary sequence $\theta = \{k_r\}$ is an integer sequence such that

$$k_0 = 0, \qquad h_r = k_r - k_{r-1} \longrightarrow \infty \quad \text{as } r \longrightarrow \infty. \tag{3}$$

In this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r.

Let θ be a lacunary sequence then $\{f_r\}_{r\in\mathbb{N}}$ is said to be lacunary equi-statistically convergent to f on X and denoted by $f_r \twoheadrightarrow f$ (θ-equistat) if for every $\varepsilon > 0$, the sequence of real valued functions $\{s_{r,\varepsilon}\}_{n\in\mathbb{N}}$ defined by

$$s_{r,\varepsilon}(x) := \frac{1}{h_r}\left|\{m \in I_r : f_m(x) - f(x) \ge \varepsilon\}\right| \tag{4}$$

uniformly converges to zero function on X, which means that

$$\lim_{n\to\infty}\left\|s_{r,\varepsilon}(\cdot)\right\|_{C(X)} = 0. \tag{5}$$

A Korovkin type approximation theorem by means of lacunary equi-statistical convergence was given in [8]. We can state this theorem now. An operator L defined on a linear space of functions Y is called linear if $L(\alpha f + \beta g, x) = \alpha L(f, x) + \beta L(g, x)$, for all $f, g \in Y$, $\alpha, \beta \in \mathbb{R}$ and is called positive, if $L(f, x) \ge 0$, for all $f \in Y$, $f \ge 0$. Let X be a compact subset of $\mathbb{R}$, and let $C(X)$ be the space of all continuous real valued functions on X.

Lemma 1 (see [8]). *Let θ be a lacunary sequence, and let $L_r : C(X) \to C(X)$ be a sequence of positive linear operators satisfying*

$$L_r(t^\nu, x) \twoheadrightarrow x^\nu, \ (\theta\text{-equistat}), \quad \nu = 0, 1, 2, \tag{6}$$

then for all $f \in C(X)$,

$$L_r(f, x) \twoheadrightarrow f, \ (\theta\text{-equistat}). \tag{7}$$

We turn to introducing some notation and the basic definitions used in this paper. Throughout this paper $I = [0, \infty)$. Let

$$C(I) := \{f : f \text{ is a real-valued continuous function on } I\}, \tag{8}$$

and

$$C_B(I) := \{f \in C(I) : f \text{ is bounded function on } I\}. \tag{9}$$

Consider the space H_w of all real-valued functions f defined on I and satisfying

$$|f(x) - f(y)| \le w\left(f; \left|\frac{x}{1+x} + \frac{y}{1+y}\right|\right), \tag{10}$$

where w is the modulus of continuity defined by

$$w(f;\delta) := \sup_{\substack{x,y\in I\\ |x-y|\le\delta}} |f(x) - f(y)|, \quad \text{for any } \delta > 0 \tag{11}$$

(see [9]). Let $K := I^2 = [0,\infty) \times [0,\infty)$, then the norm on $C_B(K)$ is given by

$$\|f\| := \sup_{(x,y)\in K} |f(x,y)|, \quad f \in C_B(K), \tag{12}$$

and also denote the valued of Lf at a point $(x, y) \in K$ is denoted by $L(f; x, y)$ [10, 11].

$w_2(f; \delta_1, \delta_2)$ is the type of modulus of continuity for the functions of two variables satisfying the following properties: for any real numbers δ_1, δ_2, δ_1', δ_2', δ_1'', and $\delta_2'' > 0$,

(i) $w_2(f; \delta_1, \cdot)$ and $w_2(f; \cdot, \delta_2)$ are nonnegative increasing functions on $[0, \infty)$,

(ii) $w_2(f; \delta_1' + \delta_1'', \delta_2) \le w_2(f; \delta_1', \delta_2) + w_2(f; \delta_1'', \delta_2)$,

(iii) $w_2(f; \delta_1, \delta_2' + \delta_2'') \le w_2(f; \delta_1, \delta_2') + w_2(f; \delta_1, \delta_2'')$,

(iv) $\lim_{\delta_1,\delta_2 \to 0} w_2(f; \delta_1, \delta_2) = 0$.

The space H_{w_2} is of all real-valued functions f defined on K and satisfying

$$\begin{aligned} &|f(u,v) - f(x,y)| \\ &\quad \le w_2\left(f; \left|\frac{u}{1+u} - \frac{x}{1+x}\right|, \left|\frac{v}{1+v} - \frac{y}{1+y}\right|\right). \end{aligned} \tag{13}$$

It is clear that any function in H_{w_2} is continuous and bounded on K.

2. Lacunary Equistatistical Approximation

In this section, using the concept of Lacunary equistatistical convergence, we give a Korovkin type result for a sequence of positive linear operators defined on $C(I^m)$, the space of all continuous real valued functions on the subset I^m of R^m, and the real m-dimensional space. We first consider the case of $m = 2$. Following [7] we can state the following theorem.

Theorem 2. *Let $\theta = \{k_r\}$ be a lacunary sequence, and let $\{L_r\}$ be a sequence of positive linear operators from H_{w_2} into $C_B(K)$. L_r is satisfying $L_r(f_\nu; x, y) \twoheadrightarrow f_\nu(x, y)$ (θ-equistat), $\nu = 0, 1, 2$, where $f_k(u, v) \in H_{w_2}$, $k = 0, 1, 2, 3$,*

$$\begin{aligned} f_0(u,v) &= 1, \\ f_1(u,v) &= \frac{u}{u+1}, \\ f_2(u,v) &= \frac{v}{v+1}, \\ f_3(u,v) &= \left(\frac{u}{u+1}\right)^2 + \left(\frac{v}{v+1}\right)^2, \end{aligned} \tag{14}$$

then for all $f \in H_{w_2}$,

$$L_r(f; x, y) \twoheadrightarrow f(x,y), \ (\theta\text{-equistat}). \tag{15}$$

Proof. Let $(x, y) \in K$ be a fixed point, $f \in H_{w_2}$, and assume that (14) holds. For every $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|f(u, v) - f(x, y)| < \varepsilon$ holds for all $(u, v) \in K$ satisfying

$$\left|\frac{u}{u+1} - \frac{x}{x+1}\right| < \delta_1, \qquad \left|\frac{v}{v+1} - \frac{y}{y+1}\right| < \delta_2. \tag{16}$$

Let

$$\begin{aligned} K_{\delta_1,\delta_2} := \Bigg\{(u,v) \in K : &\left|\frac{u}{1+u} - \frac{x}{1+x}\right| < \delta_1, \\ &\left|\frac{v}{v+1} - \frac{y}{y+1}\right| < \delta_2\Bigg\}. \end{aligned} \tag{17}$$

Hence,

$$\begin{aligned} |f(u,v) - f(x,y)| &= |f(u,v) - f(x,y)|_{\chi_{K_{\delta_1,\delta_2}}(u,v)} \\ &\quad + |f(u,v) - f(x,y)|_{\chi_{K\setminus K_{\delta_1,\delta_2}}(u,v)} \\ &< \varepsilon + 2M_{\chi_{K\setminus K_{\delta_1,\delta_2}}(u,v)}, \end{aligned} \tag{18}$$

where χ_P denotes the characteristic function of the set P. Observe that

$$\chi_{K\setminus K_{\delta_1,\delta_2}}(u,v) \le \frac{1}{\delta_1^2}\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \frac{1}{\delta_2^2}\left(\frac{v}{v+1} - \frac{y}{y+1}\right)^2. \tag{19}$$

Using (18), (19), and $M := \|f\|$ we have

$$|f(u,v)-f(x,y)| \leq \varepsilon + \frac{2M}{\delta^2}\left\{\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2+\left(\frac{v}{v+1}-\frac{y}{y+1}\right)^2\right\}, \tag{20}$$

where $\delta := \min\{\delta_1, \delta_2\}$.

By the linearity and positivity of the operators $\{L_r\}$ and by (18), we have

$$\begin{aligned}
L_r&\left(\left(f_1-\frac{u}{1+u}f_0\right)^2+\left(f_2-\frac{v}{1+v}f_0\right)^2;x,y\right)\\
&\leq L_r(f_3;x,y)\\
&\quad -2\left[\frac{x}{1+x}L_r(f_1;x,y)+\frac{y}{1+y}L_r(f_2;x,y)\right]L_r(f_3;x,y)\\
&\quad -2\left[\frac{x}{1+x}L_r(f_1;x,y)+\frac{y}{1+y}L_r(f_2;x,y)\right]\\
&\quad +\left[\left(\frac{x}{1+x}\right)^2+\left(\frac{y}{1+y}\right)^2\right]L_r(f_0;x,y).
\end{aligned} \tag{21}$$

Hence, we get

$$\begin{aligned}
|L_r&(f;x,y)-f(x,y)|\\
&\leq L_r(|f(u,v)-f(x,y)|;x,y)\\
&\quad +|f(x,y)|\,|L_r(f_0;x,y)-f_0(x,y)|\\
&\leq L_r\left(\varepsilon+\frac{2M}{\delta^2}\left\{\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2\right.\right.\\
&\qquad\qquad \left.\left.+\left(\frac{v}{v+1}-\frac{y}{1+y}\right)^2\right\};x,y\right)\\
&\quad +M\,|L_r(f_0;x,y)-f_0(x,y)|\\
&\leq L_r\left(\varepsilon+\frac{2M}{\delta^2}\left\{\left(f_1-\frac{x}{1+x}\cdot f_0\right)^2\right.\right.\\
&\qquad\qquad \left.\left.+\left(f_2-\frac{y}{1+y}\cdot f_0\right)^2\right\};x,y\right)\\
&\quad +M\,|L_r(f_0;x,y)-f_0(x,y)|\\
&= L_r(\varepsilon;x,y)+L_r\left(\frac{2M}{\delta^2}\left(f_1-\frac{x}{1+x}f_0\right)^2\right.\\
&\qquad\qquad \left.+\left(f_2-\frac{y}{1+y}f_0\right)^2;x,y\right)\\
&\quad +M\,|L_r(f_0;x,y)-f_0(x,y)|\\
&\leq \varepsilon+\frac{2M}{\delta^2}|L_r(f_3;x,y)-f_3(x,y)|\\
&\quad +\frac{4M}{\delta^2}|L_r(f_2;x,y)-f_2(x,y)|\\
&\quad +\frac{4M}{\delta^2}|L_r(f_1;x,y)-f_1(x,y)|\\
&\quad +\left(\varepsilon+M+\frac{4M}{\delta^2}\right)|L_r(f_0;x,y)-f_0(x,y)|\\
&= \frac{2M}{\delta^2}|L_r(f_3;x,y)-f_3(x,y)|\\
&\quad +\frac{4M}{\delta^2}|L_r(f_2;x,y)-f_2(x,y)|\\
&\quad +\frac{4M}{\delta^2}|L_r(f_1;x,y)-f_1(x,y)|\\
&\quad +\varepsilon+N\,|L_r(f_0;x,y)-f_0(x,y)|,
\end{aligned} \tag{22}$$

where $N := \varepsilon + M + 4M/\delta^2$. For a given $\mu > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \mu$. Define the following sets:

$$\begin{aligned}
D_\mu(x,y) &:= \{m\in\mathbb{N} : |L_m(f;x,y)-f(x,y)|\geq\mu\},\\
D_\mu^\nu(x,y) &:= \left\{m\in\mathbb{N} : |L_m(f;x,y)-f(x,y)|\geq\frac{\mu-\varepsilon}{4N}\right\},
\end{aligned} \tag{23}$$

where $\nu = 0, 1, 2, 3$. Then from (22) we clearly have

$$D_\mu(x,y)\subseteq\bigcup_{\nu=0}^{3}D_\mu^\nu(x,y). \tag{24}$$

Therefore define the following real valued functions:

$$\begin{aligned}
s_{r,\mu}(x,y) &:= \frac{1}{h_r}\left|\{m\in I_r : |L_m(f;x,y)-f(x,y)|\geq\mu\}\right|,\\
s_{r,\mu}^\nu(x,y) &\\
&:= \frac{1}{h_r}\left|\left\{m\in I_r : |L_m(f_\nu;x,y)-f(x,y)|\geq\frac{\mu-\varepsilon}{4N}\right\}\right|,
\end{aligned} \tag{25}$$

where $\nu = 0, 1, 2, 3$. Then by the monotonicity and (24) we get

$$s_{r,\mu}(x,y)\leq\sum_{\upsilon=0}^{3}s_{r,\mu}^\nu(x,y) \tag{26}$$

for all $x \in X$, and this implies the inequality

$$\left\|s_{r,\mu}(\cdot)\right\|_K\leq\sum_{\nu=0}^{3}\left\|s_{r,\mu}^\nu(\cdot)\right\|_K. \tag{27}$$

Taking limit in (27) as $r \to \infty$ and using (14) we have

$$\lim_{r\to\infty}\left\|s_{r,\mu}(\cdot)\right\|_K=0. \tag{28}$$

Then for all $f \in H_{w_2}$, we conclude that

$$L_r(f;x,y) \twoheadrightarrow f(x,y), \ (\theta\text{-equistat}). \tag{29}$$

□

Now replace I^2 by $I^m = [0,\infty) \times \cdots \times [0,\infty)$ and by an induction, we consider the modulus of continuity type function w_m as in the function w_2. Then let H_{w_m} be the space of all real-valued functions f satisfying

$$\begin{aligned}&|f(u_1,u_2,\ldots,u_m) - f(x_1,x_2,\ldots,x_m)| \\ &\quad \le w_2\left(f;\left|\frac{u_1}{u_1+1} - \frac{x_1}{x_1+1}\right|,\ldots,\left|\frac{u_m}{u_m+1} - \frac{x_m}{x_m+1}\right|\right).\end{aligned} \tag{30}$$

Therefore, using a similar technique in the proof of Lemma 1 one can obtain the following result immediately.

Theorem 3. *Let $\theta = \{k_r\}$ be a lacunary sequence, and let $\{L_r\}$ be a sequence of positive linear operators from H_{w_m} into $C_B(I^m)$. L_r is satisfying*

$$\begin{aligned}&L_r(f_v;x,y) \twoheadrightarrow f_v(x,y), \ (\theta\text{-equistat}), \\ &\qquad v = 0,1,2,\ldots,m+1,\end{aligned} \tag{31}$$

where $f_k(u_1,u_2,\ldots,u_m) \in H_{w_m}$, $k = 0,1,2,\ldots m+1$,

$$\begin{aligned}
f_0(u_1,u_2,\ldots,u_m) &= 1, \\
f_1(u_1,u_2,\ldots,u_m) &= \frac{u_1}{u_1+1}, \\
&\vdots \\
f_m(u_1,u_2,\ldots,u_m) &= \frac{u_m}{u_m+1}, \\
f_{m+1}(u_1,u_2,\ldots,u_m) &= \left(\frac{u_1}{u_1+1}\right)^2 \\
&\quad + \left(\frac{u_2}{u_2+1}\right)^2 + \cdots + \left(\frac{u_m}{u_m+1}\right)^2.
\end{aligned} \tag{32}$$

Then for all $f \in H_{w_m}$,

$$L_r(f;u_1,u_2,\ldots,u_m) \twoheadrightarrow f(u_1,u_2,\ldots,u_m), \ (\theta\text{-equistat}). \tag{33}$$

Assume that $I = [0,\infty)$, $K := I \times I$. One considers the following positive linear operators defined on $H_{w_2}(K)$:

$$\begin{aligned}B_n(f;x,y) &= \frac{1}{(1+x)^n(1+y)^n} \\ &\quad \times \sum_{k=0}^{n}\sum_{l=0}^{n} f\left(\frac{k}{n-k+1},\frac{l}{n-l+1}\right)\binom{n}{k}\binom{n}{l}x^k y^l,\end{aligned} \tag{34}$$

where $f \in H_{w_2}$, $(x,y) \in K$ and $n \in \mathbb{N}$.

Lemma 4. *Let $\theta = \{k_r\}$ be a lacunary sequence, and let*

$$\begin{aligned}B_n(f;x,y) &= \frac{1}{(1+x)^n(1+y)^n} \\ &\quad \times \sum_{k=0}^{n}\sum_{l=0}^{n} f\left(\frac{k}{n-k+1},\frac{l}{n-l+1}\right)\binom{n}{k}\binom{n}{l}x^k y^l\end{aligned} \tag{35}$$

be a sequence of positive linear operators from H_{w_2} into $C_B(K)$. If B_n is satisfying

$$\begin{aligned}
&B_n(f_v;x,y) \twoheadrightarrow f_v(x,y), \ (\theta\text{-equistat}), \quad v = 0,1,2,3, \\
&f_0(u,v) = 1, \\
&f_1(u,v) = \frac{u}{u+1}, \\
&f_2(u,v) = \frac{v}{v+1}, \\
&f_3(u,v) = \left(\frac{u}{u+1}\right)^2 + \left(\frac{v}{v+1}\right)^2,
\end{aligned} \tag{36}$$

then for all $f \in H_{w_2}$,

$$B_n(f;x,y) \twoheadrightarrow f(x,y), \ (\theta\text{-equistat}). \tag{37}$$

Proof. Assume that (36) holds, and let $f \in H_{w_2}$. Since

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k, \qquad (1+y)^n = \sum_{l=0}^{n}\binom{n}{l}y^l, \tag{38}$$

it is clear that, for all $n \in \mathbb{N}$,

$$\begin{aligned}B_n(f_0;x,y) &= \frac{1}{(1+x)^n(1+y)^n}\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}x^k\binom{n}{l}y^k \\ &= \frac{1}{(1+x)^n(1+y)^n}\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)\sum_{l=0}^{n}\binom{n}{l}y^k = 1.\end{aligned} \tag{39}$$

Now, by assumption we have

$$B_n(f_0;x,y) \twoheadrightarrow f_0(x,y), \ (\theta\text{-equistat}). \tag{40}$$

Using the definition of B_n, we get

$$\begin{aligned}
&B_n(f_1;x,y)\\
&=\frac{1}{(1+x)^n(1+y)^n}\\
&\quad\times\sum_{k=1}^{n}\frac{k/(n-k+1)}{(k/(n-k+1))+1}\binom{n}{k}x^k\sum_{l=0}^{n}\binom{n}{l}y^l\\
&=\frac{1}{(1+x)^n(1+y)^n}\sum_{k=1}^{n}\frac{k}{n+1}\binom{n}{k}x^k\sum_{l=0}^{n}\binom{n}{l}y^l\\
&=\frac{1}{(1+x)^n}\sum_{k=0}^{n-1}\frac{k+1}{n+1}\binom{n}{k+1}x^{k+1}\\
&=\frac{x}{(1+x)^n}\sum_{k=0}^{n-1}\frac{k+1}{n+1}\binom{n}{k+1}x^k.
\end{aligned} \tag{41}$$

Since

$$\binom{n}{k+1}=\binom{n-1}{k}\frac{n}{k+1} \tag{42}$$

we get

$$\begin{aligned}
B_n(f_1;x,y)&=\frac{x}{(1+x)^n}\sum_{k=0}^{n-1}\frac{k+1}{n+1}\frac{n}{k+1}\binom{n-1}{k}x^k\\
&=\frac{x}{(1+x)^n}\sum_{k=0}^{n-1}\frac{n}{n+1}\binom{n-1}{k}x^k\\
&=\frac{x}{(1+x)(1+x)^{n-1}}\cdot\frac{n}{n+1}\cdot\sum_{k=0}^{n-1}\binom{n-1}{k}x^k\\
&=\frac{n}{n+1}\left(\frac{x}{x+1}\right).
\end{aligned} \tag{43}$$

So, we have

$$\left|B_n(f_1;x,y)-f_1(x,y)\right|=\frac{x}{x+1}\left|\frac{n}{n+1}-1\right|. \tag{44}$$

The fact that $\lim_{n\to\infty}(n/(n+1)) = 1$ and using a similar technique as in the proof of Lemma 1, we get

$$\lim_{r\to\infty}\left\|\frac{1}{h_r}\left|\{m\in I_r:\left|B_n(f_1;x,y)-f_1(x,y)\right|\geq\varepsilon\}\right|\right\|=0. \tag{45}$$

Hence we have

$$B_n(f_2;x,y)\twoheadrightarrow f_2(x,y),\ (\theta\text{-equistat}). \tag{46}$$

Also we have

$$B_n(f_3;x,y)\twoheadrightarrow f_3(x,y),\ (\theta\text{-equistat}). \tag{47}$$

To see this, by the definition of B_n, we first write

$$\begin{aligned}
B_n(f_3;x,y)&=\frac{1}{(1+x)^n(1+y)^n}\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}\binom{n}{l}x^ky^l\\
&\quad\times\left[\frac{k^2}{(n+1)^2}+\frac{l^2}{(n+1)^2}\right]\\
&=\frac{1}{(1+x)^n(1+y)^n}\sum_{k=1}^{n}\frac{k^2}{(n+1)^2}\binom{n}{k}x^k\sum_{l=0}^{n}\binom{n}{l}y^l\\
&\quad+\frac{1}{(1+x)^n(1+y)^n}\sum_{k=0}^{n}\binom{n}{k}x^k\sum_{l=1}^{\infty}\frac{l^2}{(n+1)^2}\binom{n}{l}y^l\\
&=\frac{1}{(1+x)^n}\sum_{k=1}^{n}\frac{k}{(n+1)^2}x^k\\
&\quad+\frac{1}{(1+x)^n}\sum_{k=2}^{n}\frac{k(k-1)}{(n+1)^2}\binom{n}{k}x^k\\
&\quad+\frac{1}{(1+y)^n}\sum_{l=2}^{n}\frac{l(l-1)}{(n+1)^2}\binom{n}{l}y^l\\
&\quad+\frac{1}{(1+y)^n}\sum_{l=1}^{n}\frac{l}{(n+1)^2}y^l.
\end{aligned} \tag{48}$$

Then,

$$\begin{aligned}
B_n(f_3;x,y)&=\frac{n(n-1)}{(n+1)^2}\frac{x^2}{(x+1)^2}\\
&\quad+\frac{n}{(n+1)^2}\frac{x}{(x+1)}+\frac{n}{(n+1)^2}\frac{y}{y+1}\\
&\quad+\frac{y^2}{(y+1)^2}\frac{n(n-1)}{(n+1)^2}
\end{aligned} \tag{49}$$

which implies that

$$\begin{aligned}
&\left|B_n(f_3;x,y)-f_3(x,y)\right|\\
&\quad=\left(\frac{x^2}{(x+1)^2}+\frac{y^2}{(y+1)^2}\right)\left|\frac{n(n-1)}{(n+1)^2}-1\right|\\
&\quad+\frac{n}{(n+1)^2}\left|\frac{x}{x+1}+\frac{y}{y+1}\right|.
\end{aligned} \tag{50}$$

Since

$$\lim_{n\to\infty}\frac{n(n-1)}{(n+1)^2}=1,\qquad \lim_{n\to\infty}\frac{n}{(n+1)^2}=0, \tag{51}$$

we get

$$\lim_{r\to\infty}\left\|\frac{1}{h_r}\left|\{m\in I_r:\left|B_n(f_3;x,y)-f_3(x,y)\right|\geq\varepsilon\}\right|\right\|=0. \tag{52}$$

Thus $B_n(f_3;x,y)\twoheadrightarrow f_3(x,y)$, ($\theta$-equistat). Therefore we obtain that for all $f\in H_{w_2}$, $B_n(f;x,y)\twoheadrightarrow f(x,y)$, ($\theta$-equistat). □

3. Rates of Lacunary Equistatistical Convergence

In this section we study the order of lacunary equi-statistical convergence of a sequence of positive linear operators acting on $H_{w_2}(K)$, where $K = I^m$. To achieve this we first consider the case of $m = 2$.

Definition 5. A sequence $\{f_r\}$ is called lacunary equi-statistically convergent to a function f with rate $0 < \beta < 1$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{s_{r,\varepsilon}(x,y)}{r^{-\beta}} = 0, \tag{53}$$

where $s_{r,\varepsilon}(x,y)$ is given in Lemma 1. In this case it is denoted by

$$f_r - f = o\left(r^{-\beta}\right), \ (\theta\text{-equistat}) \quad \text{on } \ K = I \times I. \tag{54}$$

Lemma 6. *Let $\{f_r\}$ and $\{g_r\}$ be two sequences of functions in $H_{w_2}(K)$, with*

$$\begin{aligned} f_r - f &= o\left(r^{-\beta_1}\right), \ (\theta\text{-equistat}), \\ g_r - g &= o\left(r^{-\beta_2}\right), \ (\theta\text{-equistat}). \end{aligned} \tag{55}$$

Then one has

$$\left(f_r + g_r\right) - \left(f + g\right) = o\left(r^{-\beta}\right), \ (\theta\text{-equistat}), \tag{56}$$

where $\beta = \min\{\beta_1, \beta_2\}$.

Proof. Assume that $f_r - f = o(r^{-\beta_1})$, ($\theta$-equistat) and $g_r - g = o(r^{-\beta_2})$, ($\theta$-equistat) on K. For all $\varepsilon > 0$, consider the following functions:

$$\begin{aligned} & s_{r,\varepsilon}(x,y) \\ &\quad := \frac{1}{h_r}\left|\{n \in I_r : \left|\left(f_n + g_n\right)(x,y) - \left(f+g\right)(x,y)\right| \geq \varepsilon\}\right|, \\ &\quad s^1_{r,\varepsilon}(x,y) := \frac{1}{h_r}\left|\left\{n \in I_r : \left|f_n(x) - f(x)\right| \geq \frac{\varepsilon}{2}\right\}\right|, \\ &\quad s^2_{r,\varepsilon}(x,y) := \frac{1}{h_r}\left|\left\{n \in I_r : \left|g_n(x) - g(x)\right| \geq \frac{\varepsilon}{2}\right\}\right|. \end{aligned} \tag{57}$$

Then we have

$$\begin{aligned} \lim_{r\to\infty} \frac{\left\|s_{r,\varepsilon}(x,y)\right\|_{H_{w_2}(K)}}{r^{-\beta}} &= \lim_{r\to\infty} \frac{\left\|s^1_{r,\varepsilon}(x,y)\right\|}{r^{-\beta}} \\ &\quad + \lim_{r\to\infty} \frac{\left\|s^2_{r,\varepsilon}(x,y)\right\|}{r^{-\beta}}, \\ \frac{s_{r,\varepsilon}(x,y)}{r^{-\beta}} &\leq \frac{s^1_{r,\varepsilon}(x,y)}{r^{-\beta}} + \frac{s^2_{r,\varepsilon}(x,y)}{r^{-\beta}} \\ &\leq \frac{s^1_{r,\varepsilon}(x,y)}{r^{-\beta_1}} + \frac{s^2_{r,\varepsilon}(x,y)}{r^{-\beta_2}}, \end{aligned} \tag{58}$$

and hence

$$\frac{\left\|s_{r,\varepsilon}(x,y)\right\|_{H_{w_2}(K)}}{r^{-\beta}} \leq \frac{\left\|s^1_{r,\varepsilon}(x,y)\right\|_{H_{w_2}(K)}}{r^{-\beta_1}} + \frac{\left\|s^2_{r,\varepsilon}(x,y)\right\|_{H_{w_2}(K)}}{r^{-\beta_2}}. \tag{59}$$

Taking limit as $r \to \infty$ and using the assumption complete the proof. □

Now we give the rate of lacunary equi-statistical convergence of a positive linear operators $L_r(f; x, y)$ to $f(x, y)$ with the help of modulus of continuity.

Theorem 7. *Let $K = I \times I$, and let $L_r : H_{w_2}(K) \to H_{w_2}(K)$ be a sequence of positive linear operators. Assume that*

(i) *$L_r(f_0; x, y) - f_0 = o(r^{-\beta_1})$, ($\theta$-equistat) on K,*

(ii) *$w(f; \delta_{r,x}, \delta_{r,y}) = o(r^{-\beta_2})$, ($\theta$-equistat) on K with*

$$\begin{aligned} \delta_{r,x} &= \sqrt{L_r\left(\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2, x\right)}, \\ \delta_{r,y} &= \sqrt{L_r\left(\left(\frac{v}{1+v} - \frac{y}{1+y}\right)^2, y\right)}. \end{aligned} \tag{60}$$

Then

$$L_r(f; x, y) - f(x, y) = o\left(r^{-\beta}\right), \ (\theta\text{-equistat}) \ \text{ on } K, \tag{61}$$

where $\beta = \min\{\beta_1, \beta_2\}$.

Proof. Let $f \in H_{w_2}(K)$ and $x \in K$. Use

$$\begin{aligned} &\left|L_r(f; x, y) - f(x, y)\right| \\ &\quad \leq L_r\left(\left|f(u, v) - f(x, y)\right|; x, y\right) \\ &\qquad + \left|f(x, y)\right| \left|L_r(f_0; x, y) - f(x, y)\right| \\ &\quad \leq L_r\left(w_2\left(f; \left|\frac{u}{1+u} - \frac{x}{1+x}\right|, \left|\frac{v}{1+v} - \frac{y}{1+y}\right|\right); x, y\right) \\ &\qquad + \left|f(x, y)\right| \left|L_r(f_0; x, y) - f_0(x, y)\right| \\ &\quad \leq \left(1 + L_r(f_0; x, y)\right) w_2\left(f; \delta_{r,x}, \delta_{r,y}\right) \\ &\qquad + M\left|L_r(f_0; x, y) - f_0(x, y)\right| \\ &\quad = 2w_2\left(f; \delta_{r,x}, \delta_{r,y}\right) + M\left|L_r(f_0; x, y) - f_0(x, y)\right| \\ &\qquad + w_2\left(f; \delta_{r,x}, \delta_{r,y}\right)\left|L_r(f_0; x, y) - f_0(x, y)\right|, \end{aligned} \tag{62}$$

where $M = \|f\|_{H_{w_2}(K)}$. Using inequality (62), conditions (i) and (ii) we get

$$\lim_{r\to\infty} \frac{1}{r^{-\beta}} \left\| \frac{1}{h_r} \left|\{r \in I_r : \left|L_r(f; x, y) - f(x, y)\right| \geq \varepsilon\}\right| \right\| = 0, \tag{63}$$

so we have

$$L_r(f;x,y)-f(x,y)=o\left(r^{-\beta}\right),\ (\theta\text{-equistat})\ \text{ on }\ K. \tag{64}$$

□

Finally we give the rate of lacunary equi-statistical convergence for the operators $L_r(f,x)$ by using the Peetre's K-functional in the space $H_{w_2}(K)$. The Peetre K-functional of function $f\in H_{w_2}(K)$ is defined by

$$K\left(f;\delta_{r,x},\delta_{r,y}\right)=\inf_{g\in H_{w_2}(K)}\left\{\|f-g\|_{C_B(K)}+\delta\|g\|_{C_B(K)}\right\}, \tag{65}$$

where

$$\|f\|_{C_B(K)}=\sup_{(x,y)\in K}|f(x,y)|. \tag{66}$$

Theorem 8. *Let $f\in H_{w_2}(K)$ and $\{K(f;\delta_{r,x},\delta_{r,y})\}$ be the sequence of Peetre's K-functional. If*

$$\begin{aligned}\delta_{r,x}&=\left\|L_r\left(\left(f_1-\frac{x}{1+x}\right);x,y\right)\right\|_{C_B(K)}\\&\quad+\left\|L_r\left(\left(f_1-\frac{x}{1+x}\right)^2;x,y\right)\right\|_{C_B(K)}\|g\|_{C_B(K)},\\\delta_{r,y}&=\left\|L_r\left(\left(f_2-\frac{y}{1+y}\right);x,y\right)\right\|_{C_B(K)}\\&\quad+\left\|L_r\left(\left(f_2-\frac{y}{1+y}\right)^2;x,y\right)\right\|_{C_B(K)}\|g\|_{C_B(K)},\\&\lim_{r\to\infty}\left\|\delta_{r,x}\right\|=0,\ (\theta\text{-equistat})\\&\lim_{r\to\infty}\left\|\delta_{r,y}\right\|=0,\ (\theta\text{-equistat})\end{aligned} \tag{67}$$

on $x,y\in K$, then

$$\left\|L_r(f;x,y)-f(x,y)\right\|_{C_B(K)}\le K\left(f;\delta_{r,x},\delta_{r,y}\right). \tag{68}$$

Proof. For each $g\in H_{w_2}(K)$, we get

$$\begin{aligned}&\left\|L_r(g;x,y)-g(x,y)\right\|_{C_B(K)}\\&\le\|g\|_{C_B(K)}\left(\left\|L_r\left(\left(f_1-\frac{x}{1+x}\right);x,y\right)\right\|_{C_B(K)}\right.\\&\qquad\left.+\left\|L_r\left(\left(f_1-\frac{x}{1+x}\right)^2;x,y\right)\right\|\right)\\&\quad+\|g\|_{C_B(K)}\left(\left\|L_r\left(f_2-\frac{y}{1+y}\right);x,y\right\|_{C_B(K)}\right)\\&\quad+\left\|L_r\left(\left(f_2-\frac{y}{1+y}\right)^2\right);x,y\right\|\\&=\left(\delta_{r,x},\delta_{r,y}\right)\|g\|_{C_B(K).}\end{aligned} \tag{69}$$

□

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Research Article

A Numerical Solution to Fractional Diffusion Equation for Force-Free Case

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A collocation finite element method for solving fractional diffusion equation for force-free case is considered. In this paper, we develop an approximation method based on collocation finite elements by cubic B-spline functions to solve fractional diffusion equation for force-free case formulated with Riemann-Liouville operator. Some numerical examples of interest are provided to show the accuracy of the method. A comparison between exact analytical solution and a numerical one has been made.

1. Introduction

Scientific and engineering problems including fractional derivatives have become more important in recent years. Since the description of physical and chemical processes by means of equations including fractional derivatives is more accurate and precise, their numerical solutions have been the primary interest of many recently published articles. The applications are so wide that they include such diverse areas as control theory [1], transport problems [2], tumor development [3], subdiffusive anomalous transport in the presence of an external field [4–7], and viscoelastic and viscoplastic flow [8]. These diverse areas of applications have led to an increase in the number of studies on fractional differential equations and have caused it to be an important topic in mathematics and science. Yuste [9] has used weighted average finite difference methods for fractional diffusion equations and provided some examples in which the new methods' numerical solutions are obtained and compared against exact solutions. Langlands and Henry [10] have investigated the accuracy and stability of an implicit numerical scheme for solving the fractional diffusion equation. Murio [11] has developed an implicit unconditionally stable numerical method to solve the one-dimensional linear time fractional diffusion equation, formulated with Caputo's fractional derivative, on a finite slab. Yuste and Acedo [12] have combined the forward time centered space (FTCS) method, well known for the numerical integration of ordinary diffusion equations, with the Grünwald-Letnikov discretization of the Riemann-Liouville derivative to obtain an FTCS scheme for solving the fractional diffusion equation.

The general form of the fractional diffusion equation for force-free case is given by [4, 13, 14]

$$\frac{\partial}{\partial t}u(x,t) = K_0 D_t^{1-\gamma}\frac{\partial^2}{\partial x^2}u(x,t), \tag{1}$$

where ${}_0D_t^{1-\gamma}f(t)$ is the fractional derivative defined by Riemann-Liouville operator as

$${}_0D_t^{1-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}}d\tau, \tag{2}$$

where K is the diffusion coefficient and $\gamma \in (0,1)$ is anomalous diffusion exponent. In all numerical computations, diffusion coefficient K is going to be taken as 1. In this paper, we will take the boundary conditions of (1) given in the interval $0 \le x \le 1$ as

$$u(0,t) = 0, \qquad u(1,t) = 0 \tag{3}$$

and the initial condition as

$$u(x,0) = x(1-x). \tag{4}$$

The exact analytical solution of (1) is found by the method of separation of variables [9] as

$$u(x,t) = \frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin[(2n+1)\pi x] \times E_\gamma\left(-K(2n+1)^2\pi^2 t^\gamma\right), \tag{5}$$

where E_γ is the Mittag-Leffler function [15].

In our numerical solutions, to obtain a finite element scheme for solving the fractional diffusion equation for force-free case ($0 < \gamma \le 1$), we will also discretize the Riemann-Liouville operator [9, 16] as

$$\begin{aligned} {}_0D_t^{1-\gamma}u\left(x_j,t_n\right) &= {}_0D_t^{1-\gamma}u_j^n \\ &= \frac{1}{(\Delta t)^{1-\gamma}}\sum_{k=0}^{n}\omega_k^{1-\gamma}u_j^{n-k} + O\left(\Delta t^p\right), \end{aligned} \tag{6}$$

where

$$\omega_0^{1-\gamma} = 1, \qquad \omega_k^{1-\gamma} = \left(1 - \frac{2-\gamma}{k}\right)\omega_{k-1}^{1-\gamma}. \tag{7}$$

2. Cubic B-Spline Finite Element Collocation Solutions

To solve (1) with the boundary conditions (3) and the initial condition (4) using collocation finite element method, first of all, we define cubic B-spline base functions. Let us consider that the interval $[a,b]$ is partitioned into N finite elements of uniformly equal length by the knots x_m, $m = 0,1,2,\ldots,N$ such that $a = x_0 < x_1 < \cdots < x_N = b$ and $h = x_{m+1} - x_m$. The cubic B-splines $\phi_m(x)$, $(m = -1(1)N+1)$, over the interval $[a,b]$ and at the knots x_m are defined by [17]

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} \left(x-x_{m-2}\right)^3, & x\in\left[x_{m-2},x_{m-1}\right], \\ h^3 + 3h^2\left(x-x_{m-1}\right) + 3h\left(x-x_{m-1}\right)^2 \\ \quad -3\left(x-x_{m-1}\right)^3, & x\in\left[x_{m-1},x_m\right], \\ h^3 + 3h^2\left(x_{m+1}-x\right) + 3h\left(x_{m+1}-x\right)^2 \\ \quad -3\left(x_{m+1}-x\right)^3, & x\in\left[x_m,x_{m+1}\right], \\ \left(x_{m+2}-x\right)^3, & x\in\left[x_{m+1},x_{m+2}\right], \\ 0 & \text{otherwise.}\end{cases} \tag{8}$$

The set of cubic splines $\{\phi_{-1}(x),\phi_0(x),\ldots,\phi_{N+1}(x)\}$ constitutes a basis for the functions defined over the interval $[a,b]$. Thus, an approximate solution $U_N(x,t)$ over the interval can be written in terms of the cubic B-splines trial functions as

$$U_N(x,t) = \sum_{m=-1}^{N+1}\delta_m(t)\,\phi_m(x), \tag{9}$$

where $\delta_m(t)$'s are unknown, time dependent quantities to be determined from the boundary and cubic B-spline collocation conditions. Due to the fact that each cubic B-spline covers four consecutive elements, each element $[x_m, x_{m+1}]$ is covered by four cubic B-splines. For this problem, the finite elements are identified with the interval $[x_m, x_{m+1}]$ and the elements knots x_m, x_{m+1}. Using the nodal values U_m, U'_m, and U''_m given in terms of the parameter $\delta_m(t)$

$$\begin{aligned} U_m &= U\left(x_m\right) = \delta_{m-1}(t) + 4\delta_m(t) + \delta_{m+1}(t), \\ U'_m &= U'\left(x_m\right) = \frac{3}{h}\left(-\delta_{m-1}(t) + \delta_{m+1}(t)\right), \\ U''_m &= U''\left(x_m\right) = \frac{6}{h^2}\left(\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)\right), \end{aligned} \tag{10}$$

the variation of $U_N(x,t)$ over the typical element $[x_m, x_{m+1}]$ can now be given by

$$U_N(x,t) = \sum_{j=m-1}^{m+2}\delta_j(t)\,\phi_j(x). \tag{11}$$

If we substitute the global approximation (11) and its necessary derivatives (10) into (1), we directly obtain the following set of the first-order ordinary differential equations:

$$\begin{aligned} &\dot\delta_{m-1}(t) + 4\dot\delta_m(t) + \dot\delta_{m+1}(t) \\ &\quad - \frac{6}{h^2}\,{}_0D_t^{1-\gamma}\left[\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)\right] = 0, \end{aligned} \tag{12}$$

where dot stands for derivative with respect to time. In the first place, time parameters $\delta_m(t)$ and their time derivatives $\dot\delta_m(t)$ in (12) are discretized by the following Crank-Nicolson formula and first order difference formula, respectively:

$$\begin{aligned} \delta &= \frac{1}{2}\left(\delta^n + \delta^{n+1}\right), \\ \dot\delta &= \frac{\delta^{n+1} - \delta^n}{\Delta t}. \end{aligned} \tag{13}$$

Then, if we discretize the fractional operator ${}_0D_t^{1-\gamma}\delta^n$ by the following formula:

$${}_0D_t^{1-\gamma}\delta^n = \frac{1}{\Delta t^{1-\gamma}}\sum_{k=0}^{n}\omega_k^{1-\gamma}\delta^{n-k}, \tag{14}$$

we easily obtain a recurrence relationship between successive time levels relating unknown parameters $\delta_m^{n+1}(t)$ as

$$\begin{aligned} &(1-\alpha)\,\delta_{m-1}^{n+1} + (4+2\alpha)\,\delta_m^{n+1} + (1-\alpha)\,\delta_{m+1}^{n+1} \\ &\quad = (1+\alpha)\,\delta_{m-1}^{n} + (4-2\alpha)\,\delta_m^{n} + (1+\alpha)\,\delta_{m+1}^{n} \\ &\qquad + \alpha\sum_{k=1}^{n}\omega_k^{1-\gamma}\left[\left(\delta_{m-1}^{n+1-k} + \delta_{m-1}^{n-k}\right) - 2\left(\delta_m^{n+1-k} + \delta_m^{n-k}\right)\right. \\ &\qquad \left. + \left(\delta_{m+1}^{n+1-k} + \delta_{m+1}^{n-k}\right)\right], \end{aligned} \tag{15}$$

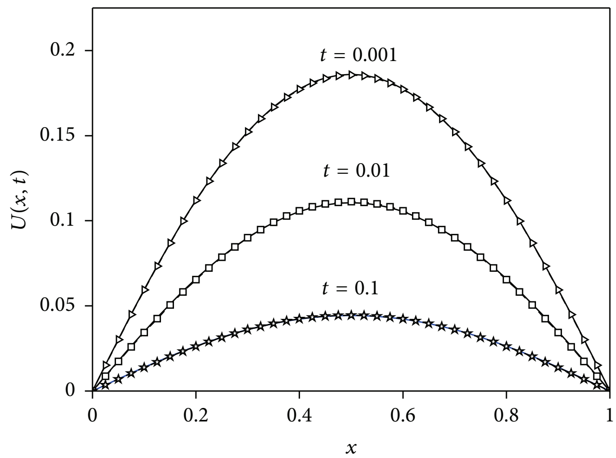

Figure 1: The comparison of the exact (lines) and numerical solutions for $\gamma = 0.50$, $\Delta t = 0.0001$, and $N = 40$ at $t = 0.001$ (triangles), $t = 0.01$ (squares), and $t = 0.1$ (stars).

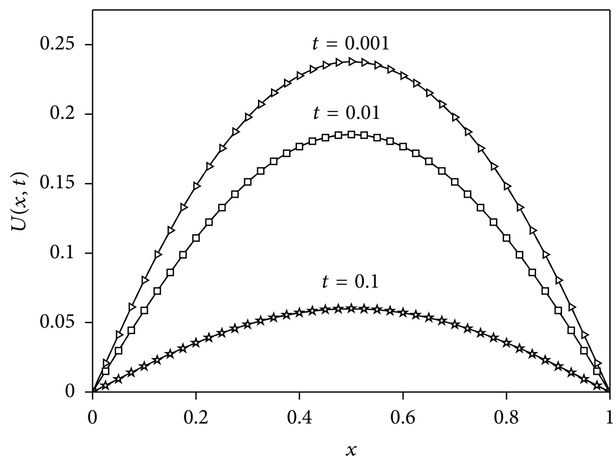

Figure 2: The comparison of the exact and numerical solutions for $\gamma = 0.75$, $\Delta t = 0.0001$, and $N = 40$ at at $t = 0.001$ (triangles), $t = 0.01$ (squares), and $t = 0.1$ (stars).

where

$$\alpha = \frac{3(\Delta t)^{\gamma}}{h^2}. \tag{16}$$

The system (15) is consisted of $N+1$ linear equations including $N+3$ unknown parameters $(\delta_{-1}, \ldots, \delta_{N+1})^T$. To obtain a unique solution to this system, we need two additional constraints. These are obtained from the boundary conditions and can be used to eliminate δ_{-1} and δ_{N+1} from the system.

2.1. Initial State. To start iteration, we do need to evaluate the initial vector at starting time level. The initial vector $\mathbf{d}^0 = (\delta_0, \delta_1, \delta_2, \ldots, \delta_{N-2}, \delta_{N-1}, \delta_N)^T$ is determined from the initial and boundary conditions. Thus, the approximation (11) can be rewritten for the initial condition as

$$U_N(x, 0) = \sum_{m=-1}^{N+1} \delta_m(0)\, \phi_m(x), \tag{17}$$

where the $\delta_m(0)$ are unknown element parameters. We force the initial numerical approximation $U_N(x, 0)$ to meet the following conditions:

$$\begin{aligned} &U_N(x, 0) = U(x_m, 0), \qquad m = 0, 1, \ldots, N, \\ &(U_N)_{xx}(0, 0) = -2, \qquad (U_N)_{xx}(1, 0) = -2. \end{aligned} \tag{18}$$

Using these conditions results in a three-diagonal system of matrix of the form

$$\mathbf{W}\mathbf{d}^0 = \mathbf{b}, \tag{19}$$

where

$$\mathbf{W} = \begin{bmatrix} 6 & 0 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & & \ddots & & & \\ & & & & 1 & 4 & 1 \\ & & & & & 0 & 6 \end{bmatrix},$$

$$\begin{aligned} \mathbf{b} = \Big(&U(x_0, 0) + \frac{h^2}{3}, U(x_1, 0), U(x_2, 0), \ldots, U(x_{N-2}, 0), \\ &U(x_{N-1}, 0), U(x_N, 0) + \frac{h^2}{3} \Big)^T. \end{aligned} \tag{20}$$

Solving this system yields the values of element parameters at $t = 0$. Now, it is time to find out the values of element parameters at different time levels using the iterative system (15).

2.2. Stability Analysis. The study of the stability of the approximation obtained by the present scheme will be based on the von Neumann stability analysis. In this analysis, the growth factor of a typical Fourier mode is defined as

$$\delta_m^n = \xi^n e^{im\varphi}, \tag{21}$$

where $i = \sqrt{-1}$. First of all, substituting the Fourier mode (21) into the recurrence relationship (15) results in the following equation:

$$\begin{aligned} &\xi^{n+1}\left((1-\alpha)e^{-i\varphi} + (4+2\alpha) + (1-\alpha)e^{i\varphi}\right) \\ &\quad = \xi^n\left((1+\alpha)e^{-i\varphi} + (4-2\alpha) + (1+\alpha)e^{i\varphi}\right) \\ &\qquad + \alpha \sum_{k=1}^{n} \omega_k^{1-\gamma}\left[\left(\xi^{n-k+1} + \xi^{n-k}\right)\left(e^{-i\varphi} - 2 + e^{i\varphi}\right)\right]. \end{aligned} \tag{22}$$

Secondly, if we write

$$\xi^{n+1} = \zeta \xi^n \tag{23}$$

and assume that

$$\zeta \equiv \zeta(\varphi) \tag{24}$$

TABLE 1: The comparison of the exact solutions with the numerical solutions with $\gamma = 0.5$, $\Delta t = 0.001$, and $t_f = 0.1$ for different values of N and the error norms L_2 and L_∞.

x	$N = 10$	$N = 20$	$N = 40$	$N = 80$	$N = 100$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.013806	0.013886	0.013920	0.013925	0.013926	0.013765
0.2	0.026128	0.026284	0.026325	0.026335	0.026336	0.026183
0.3	0.035777	0.035982	0.036034	0.036046	0.036048	0.036037
0.4	0.041903	0.042137	0.042195	0.042210	0.042212	0.042364
0.5	0.044001	0.044245	0.044306	0.044321	0.044323	0.044544
0.6	0.041903	0.042137	0.042195	0.042210	0.042212	0.042364
0.7	0.035777	0.035982	0.036034	0.036046	0.036048	0.036037
0.8	0.026128	0.026284	0.026325	0.026335	0.026336	0.026183
0.9	0.013806	0.013886	0.013920	0.013925	0.013926	0.013765
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.294172	0.159576	0.142246	0.139756	0.139501	
$L_\infty \times 10^3$	0.543026	0.299690	0.238942	0.223758	0.221936	

TABLE 2: The comparison of the exact solutions with the numerical solutions with $\gamma = 0.5$, $N = 40$, and $t_f = 0.1$ for different values of Δt and the error norms L_2 and L_∞.

x	$\Delta t = 0.01$	$\Delta t = 0.001$	$\Delta t = 0.0001$	$\Delta t = 0.00001$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.015039	0.013920	0.013929	0.013930	0.013765
0.2	0.026111	0.026325	0.026347	0.026349	0.026183
0.3	0.035392	0.036034	0.036063	0.036065	0.036037
0.4	0.041638	0.042195	0.042229	0.042233	0.042364
0.5	0.043830	0.044306	0.044341	0.044345	0.044544
0.6	0.041638	0.042195	0.042229	0.042233	0.042364
0.7	0.035392	0.036034	0.036063	0.036065	0.036037
0.8	0.026111	0.026325	0.026347	0.026349	0.026183
0.9	0.015039	0.013920	0.013929	0.013930	0.013765
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.836889	0.142246	0.136548	0.136264	
$L_\infty \times 10^3$	1.589369	0.238942	0.203235	0.199723	

TABLE 3: The comparison of the exact solutions with the numerical solutions with $\gamma = 0.75$, $\Delta t = 0.001$, and $t_f = 0.1$ for different values of N and the error norms L_2 and L_∞.

x	$N = 10$	$N = 20$	$N = 40$	$N = 80$	$N = 100$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.018544	0.018677	0.018710	0.018718	0.018719	0.018574
0.2	0.035151	0.035393	0.035453	0.035468	0.035470	0.035331
0.3	0.048210	0.048531	0.048611	0.048631	0.048633	0.048629
0.4	0.056530	0.056900	0.056992	0.057015	0.057018	0.057166
0.5	0.048210	0.059771	0.059868	0.059892	0.059895	0.060108
0.6	0.056530	0.056900	0.056992	0.057015	0.057018	0.057166
0.7	0.048210	0.048531	0.048611	0.048631	0.048633	0.048629
0.8	0.035151	0.035393	0.035453	0.035468	0.035470	0.035331
0.9	0.018544	0.018677	0.018710	0.018718	0.018719	0.018574
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.418167	0.174101	0.136253	0.130959	0.130466	
$L_\infty \times 10^3$	0.722901	0.336912	0.240523	0.216433	0.213542	

Table 4: The comparison of the exact solutions with the numerical solutions with $\gamma = 0.75$, $N = 40$, and $t_f = 0.1$ for different values of Δt and the error norms L_2 and L_∞.

x	$\Delta t = 0.01$	$\Delta t = 0.001$	$\Delta t = 0.0001$	$\Delta t = 0.00001$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.018556	0.018710	0.018722	0.018723	0.018574
0.2	0.035089	0.035453	0.035475	0.035477	0.035331
0.3	0.048219	0.048611	0.048642	0.048644	0.048629
0.4	0.056522	0.056992	0.057028	0.057031	0.057166
0.5	0.059360	0.059868	0.059905	0.059909	0.060108
0.6	0.056522	0.056992	0.057028	0.057031	0.057166
0.7	0.048219	0.048611	0.048642	0.048644	0.048629
0.8	0.035089	0.035453	0.035475	0.035477	0.035331
0.9	0.018556	0.018710	0.018722	0.018723	0.018574
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.449135	0.136253	0.128451	0.127966	
$L_\infty \times 10^3$	0.747758	0.240523	0.203042	0.199418	

is independent of time, then we get the following expression for the amplification factor ζ of the subdiffusion mode:

$$\begin{aligned}\zeta &\left((1-\alpha)e^{-i\varphi} + (4+2\alpha) + (1-\alpha)e^{i\varphi}\right) \\ &= \left((1+\alpha)e^{-i\varphi} + (4-2\alpha) + (1+\alpha)e^{i\varphi}\right) \\ &\quad + \alpha\sum_{k=1}^{n}\omega_k^{1-\gamma}\left[\left(\zeta^{1-k}+\zeta^{-k}\right)\left(e^{-i\varphi}-2+e^{i\varphi}\right)\right].\end{aligned} \tag{25}$$

If we want the given scheme to be stable in terms of Fourier stability analysis, then the condition $|\zeta| \le 1$ must be satisfied. Considering the extreme value $\zeta = 1$, from (22) and (25), we obtain the following inequality:

$$8\alpha\sin^2\left(\frac{\varphi}{2}\right) \ge 0. \tag{26}$$

Since $\alpha > 0$, we can say that the scheme is unconditionally stable.

3. Numerical Examples and Results

Numerical results for the diffusion and diffusion-wave problems are obtained by collocation method using cubic B-spline base functions. The accuracy of the present method is measured by the error norm L_2 as

$$L_2 = \left\|U^{\text{exact}} - U_N\right\|_2 = \sqrt{h\sum_{j=0}^{N}\left|U_j^{\text{exact}} - (U_N)_j\right|^2} \tag{27}$$

and the error norm L_∞ as

$$L_\infty = \left\|U^{\text{exact}} - U_N\right\|_\infty = \max_j\left|U_j^{\text{exact}} - (U_N)_j\right|. \tag{28}$$

Figures 1 and 2 show the graphs of the exact (denoted by lines) solutions and the numerical ones for $\Delta t = 0.0001$ and $N = 40$ at $t = 0.001$ (denoted by triangles), $t = 0.01$ (denoted by squares), and $t = 0.1$ (denoted by stars) for two different values of $\gamma = 0.50$ and $\gamma = 0.75$, respectively. Table 1 shows the comparison of the exact solutions with the numerical ones with $\gamma = 0.5$, $\Delta t = 0.001$, and $t_f = 0.1$ for different values of N. The calculated error norms L_2 and L_∞ at those time levels are also presented in the table. In Table 2, the comparison of the exact solutions with the numerical ones with $\gamma = 0.5$, $N = 40$ and $t_f = 0.1$ for different values of Δt is illustrated and then the error norms L_2 and L_∞ are computed and presented in the table. In Table 3, we have listed the numerical and exact solutions of the problem for $\gamma = 0.75$, $\Delta t = 0.001$, and $t_f = 0.1$ for different values of N and the error norms L_2 and L_∞. Table 4 illustrates the comparison of the exact solutions with the numerical solutions for $\gamma = 0.75$, $N = 40$, and $t_f = 0.1$ for different values of Δt and the error norms L_2 and L_∞.

4. Conclusion

In the present study, first of all, a collocation finite element method has been constructed. Then, the method has been applied using cubic B-spline base functions. During the implementation of the method, Crank-Nicolson formula and first-order difference formula have been applied for discretization process. The stability of the method presented in the paper has been tested using the von Neumann stability analysis in which the growth factor of a typical Fourier mode is used. The accuracy of the method is also measured by the error norms L_2 and L_∞. The successful application of the present method prompts the probability of extending it to other finite element methods and other kinds of fractional differential equations. The available results suggest that this is highly probable.

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Research Article

Unified Treatment of the Krätzel Transformation for Generalized Functions

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We discuss a generalization of the Krätzel transforms on certain spaces of ultradistributions. We have proved that the Krätzel transform of an ultradifferentiable function is an ultradifferentiable function and satisfies its Parseval's inequality. We also provide a complete reading of the transform constructing two desired spaces of Boehmians. Some other properties of convergence and continuity conditions and its inverse are also discussed in some detail.

1. Introduction

Krätzel, in [1, 2], introduced a generalization of the Meijer transform by the integral:

$$\left(K_{\nu}^{p} f\right)(x) = \int_{\mathbf{R}_+} \mathrm{Z}_{p}^{\nu}(xy) f(y)\, dy, \quad x > 0, \tag{1}$$

where

$$\mathrm{Z}_{p}^{\nu}(xy) = \int_{\mathbf{R}_+} t^{\nu-1} e^{-t^{p}-xy/t} dt, \tag{2}$$

$p > 0 (\in \mathbf{N})$, $\nu \in \mathbb{C}$. Then this generalization is known as Krätzel transform.

Let x be in $\mathbf{R}_+$. Denote by $\mathscr{S}_+$, or $\mathscr{S}(\mathbf{R}_+)$, the space of all complex-valued smooth functions $\phi(t)$ on $\mathbf{R}_+$ such that

$$\sup_{x \in \mathbf{K}} \left| \mathscr{D}^{k} \phi(x) \right| < \infty, \tag{3}$$

$x \in \mathbf{R}_+$, where $\mathbf{K}$ runs through compact subsets of $\mathbf{R}_+$; see [3]. The strong dual $\mathscr{S}'_+$ of $\mathscr{S}_+$ consists of distributions of compact supports.

Later, the authors in [4] have studied the K_{ν}^{p} transformation in a space of distributions of compact support inspired by known kernel method. They, also, have obtained its properties of analyticity and boundedness and have established its inversion theorem. In the sense of classical theory, the Meijer transformation and the Laplace transformation in [5] are presented as special forms of the cited transform for $p = 1$ and $p = 1$, $\nu = \pm 1/2$, respectively.

It is worth mentioning in this note that a suitable motivation of the cited transform has thoroughly been discussed in [6] by the aid of a Fréchet space of constituted functions of infinitely differentiable functions over $(0, \infty)$.

This paper is a continuation of the work obtained in [4]. We are concerned with a general study of the transform in the space of ultradistributions and further discuss its extension to Boehmian spaces in some detail. We are employing the adjoint method and method of kernels for our purpose to extend the classical integral transform to generalized functions and hence ultradistributions.

2. Ultradistributions

The theory of ultradistributions is one of generalizations of the theory of Schwartz distributions; see [3, 7]. Since then, in the recent past and even earlier, it was extensively studied by many authors such as Roumieu [8, 9], Komatsu [10], Beurling [11], Carmichael et al. [12], Pathak [13, 14], and Al-Omari [15, 16].

By an ultradifferentiable function we mean an infinitely smooth function whose derivatives satisfy certain growth conditions as the order of the derivatives increases. Unlike sequences presented in [15, 16], a_i, $i = 0, 1, \ldots$, wherever it appears, denotes a sequence of positive real numbers. Such omission of constraints may ease the analysis.

Let α be a real number but fixed and $\mathscr{L}^r$ be the space of Lebesgue integrable functions on $\mathbf{R}_+$. Denote by $\mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ (resp., $\mathcal{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$), $1 \leq r \leq \infty$, the subsets of $\mathcal{S}_+$ of all complex valued infinitely smooth functions on $\mathbf{R}_+$ such that, for some constant $m_1 (> 0)$,

$$\sup_{\alpha, x \in \mathbf{K}} \left\| \mathscr{D}^k \varphi(x) \right\|_{\mathscr{L}^r} \leq m_1 a^\alpha a_\alpha \tag{4}$$

for all $a > 0$ (for some $a > 0$), where $\mathbf{K}$ is a compact set traverses $\mathbf{R}_+$.

The elements of the dual spaces, $\mathcal{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathcal{S}'_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$), are the Beurling-type (Roumieu-type) ultradistributions. It may be noted that $\mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a) \subset \mathcal{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a) \subset \mathcal{S}_+$. Thus, every distribution of compact support is an ultradistribution of Roumieu type and further, and an ultradistribution of Roumieu type is of Beurling-type. Natural topologies on $\mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ (resp., $\mathcal{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) can be generated by the collection of seminorms:

$$\|\varphi\|_{r,a} = \sup_{\alpha, x \in \mathbf{K}} \frac{\left\| \mathscr{D}^k \varphi(x) \right\|_{\mathscr{L}^r}}{a^\alpha a_\alpha}, \quad a > 0. \tag{5}$$

A sequence $(\varphi_n) \to \varphi \in \mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ (resp., $\mathcal{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) if

$$\lim_{n \to \infty} \sup_{\alpha, x \in \mathbf{K}} \left\| \mathscr{D}^k (\varphi_n(x) - \varphi(x)) \right\|_{\mathscr{L}^r} = 0, \tag{6}$$

and there is a constant $m_1 > 0$ independent of n such that

$$\lim_{n \to \infty} \sup_{\alpha, x \in \mathbf{K}} \left\| \mathscr{D}^k (\varphi_n(x) - \varphi(x)) \right\|_{\mathscr{L}^r} \leq m_1 a^\alpha a_\alpha \tag{7}$$

for all $a > 0$. $\mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathcal{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) is dense in $\mathcal{S}_+$, convergence in $\mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathcal{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) implies convergence in $\mathcal{S}_+$, and consequently a restriction of any $f \in \mathcal{S}'_+$ to $\mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathcal{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) is in $\mathcal{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathcal{S}'_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$).

3. The Krätzel Transform of Tempered Ultradistributions

In this section of this paper we define the Krätzel transform of tempered ultradistributions by using both of kernel and adjoint methods. We restrict our investigation to the case of Beurling type since the other investigation for the Roumieu-type tempered ultradistributions is almost similar.

Lemma 1. *Let $\phi \in \mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ and then $K_\nu^p \phi \in \mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$.*

Proof. Let $\phi \in \mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ and x be fixed, and then $K_\nu^p \phi$ certainly exists. By differentiation with respect to x we get

$$\begin{aligned} \mathscr{D}_x (K_\nu^p \phi)(x) &= \iint_{\mathbf{R}_+} \mathscr{D}_x t^{\nu-1} e^{-t^p - xy/t} dt \phi(y)\, dy \\ &= \iint_{\mathbf{R}_+} \frac{-y}{t} t^{\nu-1} e^{-t^p - xy/t} dt \phi(y)\, dy \\ &= \int_{\mathbf{R}_+} -y \int_{\mathbf{R}_+} t^{(\nu-1)-1} e^{-t^p - xy/t} dt \phi(y)\, dy \\ &= \int_{\mathbf{R}_+} -y Z_p^{\nu-1}(xy) \phi(y)\, dy. \end{aligned} \tag{8}$$

Hence the principle of mathematical induction on the kth derivative gives

$$\mathscr{D}_x^k (K_\nu^p \phi)(x) = (-1)^k \int_{\mathbf{R}_+} Z_p^{\nu-k}(xy) y^k \phi(y)\, dy. \tag{9}$$

From [4], we deduce that

$$\begin{aligned} &\left| \mathscr{D}_x^k (K_\nu^p \phi)(x) \right| \\ &\quad \leq \alpha_1 \int_{\mathbf{R}_+} \left| (y)^{(2\nu-p)/(2p+2)} e^{-(xy)^p/(p+1)} \right| y^k \phi(y)\, dy \end{aligned} \tag{10}$$

for some constant α_1. The assumption that $\phi \in \mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ implies that the products under the integral sign, $\Psi = y^k \phi(y)$ and $(xy)^{(2\nu-p)/(2p+2)} e^{-(xy)^p/(p+1)} \Psi$, are also in $\mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$. Moreover, $\phi \in \mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ ensures that the integral:

$$F(x) = \alpha_1 \int_{\mathbf{R}_+} \left| (xy)^{(2\nu-p)/(2p+2)} e^{-(xy)^p/(p+1)} \right| y^k \phi(y)\, dy \tag{11}$$

belongs to $\mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$. Hence

$$\left\| \mathscr{D}_x^k F(x) \right\|_{\mathscr{L}^r} \leq m_1 a^\alpha a_\alpha \tag{12}$$

for some constant m_1. Therefore, from the above inequality we get

$$\left\| \mathscr{D}_x^k (K_\nu^p \phi)(x) \right\|_{\mathscr{L}^r} \leq \left\| \mathscr{D}_x^k F(x) \right\|_{\mathscr{L}^r} \leq m_1 a^\alpha a_\alpha, \tag{13}$$

for certain positive constant m_1. This proves the lemma. □

From Lemma 1 we deduce that the Krätzel transform is bounded and closed from $\mathcal{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ into itself. Next, we establish the Parseval's relation for the Krätzel transform.

Theorem 2. *Let f and g be absolutely integrable functions over $\mathbf{R}_+$ and then*

$$\int_{\mathbf{R}_+} f(x) (K_\nu^p g)(x)\, dx = \int_{\mathbf{R}_+} (K_\nu^p f)(x) g(x)\, dx, \tag{14}$$

where $K_\nu^p f$ and $K_\nu^p g$ are the Krätzel transforms of f and g, respectively.

Proof. It is clear that $K_\nu^p f$ and $K_\nu^p g$ are continuous and bounded on $\mathbf{R}_+$. Moreover, the Fubini's theorem allows us to interchange the order of integration:

$$\int_{\mathbf{R}_+} f(x)\left(K_\nu^p g\right)(x)\,dx = \int_{\mathbf{R}_+}\left(\int_{\mathbf{R}_+} f(x)\,\mathrm{Z}_p^\nu(xy)\,dx\right) g(y)\,dy. \tag{15}$$

Equation (15) follows since the Krätzel kernel $\mathrm{Z}_p^\nu(xy)$ applies for the functions f and g, when the order of integration is interchanged. This completes the proof of the theorem. □

Now, in consideration of Theorem 2, the adjoint method of extending the Krätzel transform can be read as

$$\left\langle K_\nu^p f, \phi\right\rangle = \left\langle f, K_\nu^p \phi\right\rangle, \tag{16}$$

where $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and $\phi \in \mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$.

Theorem 3. *Given that $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ then $K_\nu^p f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$.*

Proof. Consider a zero convergent sequence (ϕ_n) in $\mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$ then certainly $(K_\nu^p \phi_n)$ is a zero-convergent sequence in the same space. It follows from (16) that

$$\left\langle K_\nu^p f, \phi_n\right\rangle = \left\langle f, K_\nu^p \phi_n\right\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{17}$$

Linearity is obvious. This completes the proof. □

From the above theorem we deduce that the Krätzel transform of a tempered ultradistribution is a tempered ultradistribution. Moreover, the boundedness property of $K_\nu^p f$, $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ follows from the following theorem.

Theorem 4. *Let $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and then $K_\nu^p f$ is bounded.*

Proof. See [4, Proposition 2.3]. □

It is interesting to know that the Krätzel transform can be defined in an alternative way, namely, by the kernel method. Let $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$, and then

$$\left(K_\nu^p f\right)(x) = \left\langle f(y), \mathrm{Z}_p^\nu(xy)\right\rangle. \tag{18}$$

In fact, (18) is a straightforward consequence of Lemma 1.

Theorem 5. *Let $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ then $K_\nu^p f$ is infinitely differentiable and*

$$\mathcal{D}_x^k\left(K_\nu^p f\right)(x) = \left\langle f(t), \mathcal{D}_x^k \mathrm{Z}_p^\nu(xt)\right\rangle \tag{19}$$

for every $k \in \mathbf{N}$ and $x \in \mathbf{R}_+$.

Proof. See [4, Proposition 2.2]. □

4. Boehmian Spaces

Boehmians were first constructed as a generalization of regular Mikusinski operators [17]. The minimal structure necessary for the construction of Boehmians consists of the following elements:

(i) a nonempty set A,

(ii) a commutative semigroup $(\mathtt{B}, *)$,

(iii) an aperation $\odot : \mathtt{A} \times \mathtt{B} \to \mathtt{A}$ such that for each $x \in \mathtt{A}$ and $s_1, s_2 \in \mathtt{B}$, $x \odot (s_1 * s_2) = (x \odot s_1) \odot s_2$,

(iv) a collection $\Delta \subset \mathtt{B}^N$ such that

  (a) if $x, y \in \mathtt{A}$, $(s_n) \in \Delta$, $x \odot s_n = y \odot s_n$ for all n, then $x = y$,

  (b) if $(s_n), (t_n) \in \Delta$, then $(s_n * t_n) \in \Delta$.

Elements of Δ are called delta sequences. Consider

$$\mathtt{g} = \left\{(x_n, s_n) : x_n \in \mathtt{A}, (s_n) \in \Delta, x_n \odot s_m = x_m \odot s_n, \forall m, n \in \mathbf{N}\right\}. \tag{20}$$

If $(x_n, s_n), (y_n, t_n) \in \mathtt{g}$, $x_n \odot t_m = y_m \odot s_n$, for all $m, n \in \mathbf{N}$, then we say $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in $\mathtt{g}$. The space of equivalence classes in $\mathtt{g}$ is denoted by β. Elements of β are called Boehmians. Between $\mathtt{A}$ and β there is a canonical embedding expressed as

$$x \longrightarrow \frac{x \odot s_n}{s_n}. \tag{21}$$

The operation $\odot$ can be extended to $\beta \times \mathtt{A}$ by

$$\frac{x_n}{s_n} \odot t = \frac{x_n \odot t}{s_n}. \tag{22}$$

In β, there are two types of convergence:

(δ convergence) a sequence (h_n) in β is said to be δ convergent to h in β, denoted by $h_n \xrightarrow{\delta} h$, if there exists a delta sequence (s_n) such that $(h_n \odot s_n), (h \odot s_n) \in \mathtt{A}$, for all $k, n \in \mathbf{N}$, and $(h_n \odot s_k) \to (h \odot s_k)$ as $n \to \infty$, in $\mathtt{A}$, for every $k \in \mathbf{N}$,

(Δ convergence) a sequence (h_n) in β is said to be Δ convergent to h in β, denoted by $h_n \xrightarrow{\Delta} h$, if there exists a $(s_n) \in \Delta$ such that $(h_n - h) \odot s_n \in \mathtt{A}$, for all $n \in \mathbf{N}$, and $(h_n - h) \odot s_n \to 0$ as $n \to \infty$ in $\mathtt{A}$. For further discussion see [17–21].

5. The Ultra-Boehmian Space β_{s_+}

Denote by $\mathtt{D}_+$, or $\mathtt{D}(\mathbf{R}_+)$, the Schwartz space of $\mathbb{C}^\infty$ functions of bounded support. Let Δ_+ be the family of sequences $(s_n) \in \mathtt{D}(\mathbf{R}_+)$ such that the following holds:

(Δ_1) $\int_{\mathbf{R}_+} s_n(x)dx = 1$, for all $n \in \mathbf{N}$,

(Δ_2) $s_n(x) \geq 0$, for all $n \in \mathbf{N}$,

(Δ_3) supp $s_n \subset (0, \varepsilon_n)$, $\varepsilon_n \to 0$ as $n \to \infty$.

It is easy to see that each (s_n) in Δ_+ forms a delta sequence.

Let $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and $\sigma \in \mathrm{D}(\mathbf{R}_+)$ be related by the expression:

$$(f \cdot \sigma) v = f(\sigma \circledast v), \tag{23}$$

where

$$(\sigma \circledast v)(x) = \int_{\mathbf{R}_+} \sigma(t) v(xt)\, dt \tag{24}$$

for every $v \in \mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$.

Lemma 6. *Let $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and $\sigma \in \mathrm{D}_+$ and then $f \cdot \sigma \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$.*

Proof. Using the weak topology of $\mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$, we write

$$|(f \cdot \sigma) v| = |f(\sigma \circledast v)| \le C\|\sigma \circledast v\|_{r,a}, \tag{25}$$

where $v \in \mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$. Hence, to complete the proof, we are merely required to show that $\sigma \circledast v \in \mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$.

First, if $\sigma \in \mathrm{D}_+$ and $v \in \mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$, then choosing a compact set $\mathbf{K}$ containing the support of σ yields

$$\begin{aligned} &\frac{(\sigma \circledast v)(x + \Delta x) - (\sigma \circledast v)(x)}{\Delta x} \\ &\quad = \int_{\mathbf{R}_+} \sigma(t) \frac{v((x + \Delta x) t) - v(xt)}{\Delta x} dt \end{aligned} \tag{26}$$

which is dominated by $\sigma(t)|\mathcal{D}_x v(x)|$. The dominated convergence theorem and the principle of mathematical induction implies

$$\mathcal{D}_x^k (\sigma \circledast v) = \sigma \circledast \mathcal{D}_x^k v. \tag{27}$$

Finally

$$\begin{aligned} \int_{\mathbf{R}_+} \left|\mathcal{D}^k (\sigma \circledast v)(x)\right|^r dx &= \int_{\mathbf{R}_+} \left|\left(\sigma \circledast \mathcal{D}^k v\right)(x)\right|^r dx \\ &\le \int_{\mathbf{R}_+} \int_{\mathbf{K}} \left|\sigma(t)\, dt \mathcal{D}^k v(xt)\right|^r dx \\ &\le M \int_{\mathbf{R}_+} \left|\mathcal{D}^k v(xt)\right|^r dx. \end{aligned} \tag{28}$$

Therefore

$$\|\sigma \circledast v\|_{r,a} \le d\|v\|_{r,a} < d a^\alpha a_\alpha \text{ for some constant } d. \tag{29}$$

Thus $\sigma \circledast v \in \mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$. This completes the proof of the lemma. □

Lemma 7. *Let $f_1, f_2 \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and $\sigma \in \mathrm{D}_+$ and then*

(i) $\alpha f \cdot \sigma = \alpha(f \cdot \sigma)$, $\alpha \in \mathbb{C}$,

(ii) $(f_1 + f_2) \cdot \sigma = f_1 \cdot \sigma_1 + f_2 \cdot \sigma_2$.

Proof of the above Lemma is obvious.

Lemma 8. *Let $f_n, f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and $\sigma \in \mathrm{D}_+$ and then*

$$f_n \cdot \sigma \longrightarrow f \cdot \sigma. \tag{30}$$

Proof. Let $v \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and then $\sigma \circledast v \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$. Therefore

$$(f_n \cdot \sigma - f \cdot \sigma) v = ((f_n - f) \cdot \sigma) v = (f_n - f)(\sigma \circledast v). \tag{31}$$

Hence $f_n \cdot \sigma \to f \cdot \sigma$ as $n \to \infty$. □

Lemma 9. *Let $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and $(s_n) \in \Delta_+$ then*

$$f \cdot s_n \longrightarrow f \quad \text{in } \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a) \quad \text{as } n \longrightarrow \infty. \tag{32}$$

Proof. Let $v \in \mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$ and supp $s_n \subset (0, \varepsilon_n)$, $n \in \mathbf{N}$ and then

$$(f \cdot s_n) v = f(s_n \circledast \sigma). \tag{33}$$

It is sufficient to establish that $s_n \circledast v \to v$ as $n \to \infty$. By using (27) and Δ_1 imply that

$$\begin{aligned} &\int_{\mathbf{R}_+} \left|\mathcal{D}_x^k (s_n \circledast v - v)(x)\right|^r dx \\ &\quad = \int_{\mathbf{R}_+} \left|\left(s_n \circledast \mathcal{D}_x^k v - \mathcal{D}_x^k v\right)(x)\right|^r dx \\ &\quad = \int_{\mathbf{R}_+} \left|\int_0^{\varepsilon_n} s_n(t) \left(\mathcal{D}_x^k v(xt) - \mathcal{D}_x^k v(x)\right) dt\right|^r dx. \end{aligned} \tag{34}$$

Hence, the mean value theorem implies

$$\begin{aligned} &\int_{\mathbf{R}_+} \left|\mathcal{D}_x^k (s_n \circledast v - v)(x)\right|^r dx \\ &\quad \le \int_{\mathbf{R}_+} \left|\int_0^{\varepsilon_n} \xi s_n(t) \mathcal{D}_x^{k+1} v(x\xi)\, dt\right|^r dx, \end{aligned} \tag{35}$$

$\xi \in (0, t)$. Let $A_1 = \sup_{s \in \mathbf{K}} |\mathcal{D}_x^{k+1} \psi(s)|$, where $\mathbf{K}$ is certain compact set. The calculations show that

$$\|s_n \circledast v - v\|_{r,a} \le F\varepsilon_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{36}$$

where F is certain constant. Hence Lemma 9. The Boehmian space β_{s_+} is therefore constructed. □

6. β_{Z_+} and the Krätzel Transform of Ultra-Boehmians

Denote by $\mathrm{Z}(\mathbf{R}_+)$, or Z_+, the space of functions which are Krätzel transforms of ultradistributions in $\mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$, and then convergence on Z_+ can be defined in such away that $E_n \to E_*$ in Z_+ if $f_n \to f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ as $n \to \infty$, where $E_n = K_\nu^p f_n$ and $E_* = K_\nu^p f$. Let $E \in \mathrm{Z}_+$ and $\sigma \in \mathrm{D}_+$ and then it is proper to define

$$(E \circledcirc \sigma)(u) = \int_{\mathbf{R}_+} E(ut) \sigma(t)\, dt \tag{37}$$

for each $u \in \mathbf{R}_+$.

Lemma 10. *Let $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and $\sigma \in \mathrm{D}_+$ and then $K_\nu^p(f \cdot \sigma) = K_\nu^p f \circledcirc \sigma$.*

Proof. Let $\mathbf{K}$ be a compact set containing the support of σ, and then from (18) it follows that

$$\begin{aligned} K_\nu^p (f \cdot \sigma)(u) &= \left\langle (f \cdot \sigma)(y), \mathrm{Z}_p^\nu (uy) \right\rangle \\ &= \left\langle f(y), \left\langle \sigma(t), \mathrm{Z}_p^\nu ((ut)\, y) \right\rangle \right\rangle \\ &= \int_{\mathbf{R}_+} \left\langle f(y), \mathrm{Z}_p^\nu ((ut)\, y) \right\rangle \sigma(t)\, dt = K_\nu^p f \circledcirc \sigma. \end{aligned} \tag{38}$$

Hence the lemma follows. □

Lemma 11. *Let $E \in \mathrm{Z}_+$ and $\sigma \in \mathrm{D}_+$ and then $E \circledcirc \sigma \in \mathrm{Z}_+$.*

Proof. $E \in \mathrm{Z}_+$ implies $K_\nu^p f = E$, for some $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$. Hence $E \circledcirc \sigma = K_\nu^p f \circledcirc \sigma = K_\nu^p(f \cdot \sigma) \in \mathrm{Z}_+$. □

The following are lemmas which can be easily proved by the aid of the corresponding lemmas from the previous section. Detailed proof is avoided. First, if $\overset{-1}{K_\nu^p}$ is the inverse Krätzel transform of K_ν^p, and then

Lemma 12. *Let $E \in \mathrm{Z}_+$ and $\sigma \in \mathrm{D}_+$ and then*

$$\overset{-1}{K_\nu^p} (E \circledcirc \sigma) = \overset{-1}{K_\nu^p} E \cdot \sigma. \tag{39}$$

Lemma 13. *Let $E_1, E_2 \in \mathrm{Z}_+$ and then for all $\sigma_1, \sigma_2 \in \mathrm{D}_+$ we have*

(1) $(E_1 + E_2) \circledcirc \sigma_1 = E_1 \circledcirc \sigma_1 + E_2 \circledcirc \sigma_1$,

(2) $(aE) \circledcirc \sigma_1 = a(E \circledcirc \sigma_1)$.

Lemma 14. *Let $E_n \to E$ and $(s_n) \in \Delta_+$ and then $E_n \circledcirc s_n \to E$.*

Lemma 15. *Let $E_n \to E$ and $\sigma \in \mathrm{D}_+$ and then $E_n \circledcirc \sigma \to E \circledcirc \sigma$.*

With the previous analysis, the Boehmian space β_{Z_+} is constructed. The sum of two Boehmians and multiplication by a scalar in β_{Z_+} is defined in a natural way $[f_n/\phi_n] + [g_n/\psi_n] = [((f_n \circledcirc \psi_n) + (g_n \circledcirc \phi_n))/(\phi_n \circledcirc \psi_n)]$ and $\alpha[f_n/\phi_n] = [\alpha(f_n/\phi_n)]$, $\alpha \in \mathbb{C}$.

The operation $\circledcirc$ and the differentiation are defined by

$$\left[\frac{f_n}{\phi_n}\right] \circledcirc \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n \circledcirc g_n}{\phi_n \circledcirc \psi_n}\right], \qquad \mathcal{D}^k \left[\frac{f_n}{\phi_n}\right] = \left[\frac{\mathcal{D}^k f_n}{\phi_n}\right]. \tag{40}$$

With the aid of Lemma 10 we define the extended Krätzel transform of a Boehmian $[f_n/s_n] \in \beta_{s_+}$ to be a Boehmian in β_{Z_+} expressed by the relation:

$$\vec{\mathbf{K}}_\nu^p \left[\frac{f_n}{s_n}\right] = \left[\frac{K_\nu^p f_n}{s_n}\right]. \tag{41}$$

Lemma 16. *$\vec{\mathbf{K}}_\nu^p : \beta_{s_+} \to \beta_{\mathrm{Z}_+}$ is well defined and linear mapping.*

Proof is a straightforward conclusion of definitions.

Definition 17. Let $[E_n/s_n] \in \beta_{\mathrm{Z}_+}$ and then the inverse of $\vec{\mathbf{K}}$ is defined as follows:

$$\overset{-1}{\vec{\mathbf{K}}_\nu^p} \left[\frac{E_n}{s_n}\right] = \left[\frac{\overset{-1}{K_\nu^p} E_n}{s_n}\right], \tag{42}$$

for each $(s_n) \in \Delta_+$.

Lemma 18. *$\vec{\mathbf{K}}_\nu^p$ is an isomorphism from β_{s_+} into β_{Z_+}.*

Proof. Assume $\vec{\mathbf{K}}_\nu^p[f_n/s_n] = \vec{\mathbf{K}}_\nu^p[g_n/t_n]$, then it follows from (41) and the concept of quotients of two sequences $K_\nu^p f_n \circledcirc t_m = K_\nu^p g_m \circledcirc s_n$. Therefore, Lemma 10 implies $K_\nu^p(f_n \cdot t_m) = K_\nu^p(g_m \cdot \phi_n)$. Employing properties of K_ν^p implies $f_n \cdot t_m = g_m \cdot s_n$. Thus, $[f_n/s_n] = [g_n/t_n]$. Next we establish that $\vec{\mathbf{K}}_\nu^p$ is onto. Let $[K_\nu^p f_n/s_n] \in \beta_{\mathrm{Z}_+}$ be arbitrary and then $K_\nu^p f_n \circledcirc s_m = K_\nu^p f_m \circledcirc s_n$ for every $m, n \in \mathbf{N}$. Hence $f_n, f_m \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ are such that $K_\nu^p(f_n \cdot s_m) = K_\nu^p(f_m \cdot s_n)$. Hence the Boehmian $[f_n/s_n] \in \beta_{s_+}$ satisfies the equation $\vec{\mathbf{K}}_\nu^p[f_n/s_n] = [K_\nu^p f_n/s_n]$.

This completes the proof of the lemma. □

Lemma 19. *Let $[E_n/s_n] \in \beta_{\mathrm{Z}_+}$, $E_n = K_\nu^p f_n$, and $\phi \in \mathrm{D}_+$ and then*

$$\begin{aligned} \overset{-1}{\vec{\mathbf{K}}_\nu^p} \left(\left[\frac{E_n}{s_n}\right] \circledcirc \phi\right) &= \left[\frac{E_n^{-1}}{s_n}\right] \cdot \phi, \\ \vec{\mathbf{K}}_\nu^p \left(\left[\frac{f_n}{s_n}\right] \bullet \phi\right) &= \left[\frac{E_n}{s_n}\right] \circledcirc \phi. \end{aligned} \tag{43}$$

Proof. It follows from (42) that

$$\overset{-1}{\vec{\mathbf{K}}_\nu^p} \left(\left[\frac{E_n}{s_n}\right] \circledcirc \phi\right) = \overset{-1}{\vec{\mathbf{K}}_\nu^p} \left(\left[\frac{E_n \circledcirc \phi}{s_n}\right]\right) = \left[\frac{\overset{-1}{K_\nu^p} (E_n \circledcirc \phi)}{s_n}\right]. \tag{44}$$

Applying Lemma 12 leads to

$$\overset{-1}{\vec{\mathbf{K}}_\nu^p} \left(\left[\frac{E_n}{s_n}\right] \circledcirc \phi\right) = \left[\frac{\overset{-1}{K_\nu^p} E_n}{s_n}\right] \cdot \phi = \left[\frac{E_n^{-1}}{s_n}\right] \cdot \phi. \tag{45}$$

Proof of the second part is similar. This completes the proof of the lemma. □

Theorem 20. *$\vec{\mathbf{K}}_\nu^p : \beta_{s_+} \to \beta_{\mathrm{Z}_+}$ and $\overset{-1}{\vec{\mathbf{K}}_\nu^p} : \beta_{\mathrm{Z}_+} \to \beta_{s_+}$ are continuous with respect to δ and Δ convergences.*

Proof. First of all, we show that $\vec{\mathbf{K}}_\nu^p : \beta_{s_+} \to \beta_{\mathrm{Z}_+}$ and $\overset{-1}{\vec{\mathbf{K}}_\nu^p} : \beta_{\mathrm{Z}_+} \to \beta_{s_+}$ are continuous with respect to δ convergence. Let $\beta_n \xrightarrow{\delta} \beta$ in β_{s_+} as $n \to \infty$ and then we establish that

$\vec{\mathbf{K}}_\nu^p \beta_n \to \vec{\mathbf{K}}_\nu^p \beta$ as $n \to \infty$. In view of [10], there are $f_{n,k}$ and f_k in $\mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ such that

$$\beta_n = \left[\frac{f_{n,k}}{s_k}\right], \qquad \beta = \left[\frac{f_k}{s_k}\right] \tag{46}$$

such that $f_{n,k} \to f_k$ as $n \to \infty$ for every $k \in \mathbf{N}$. The continuity condition of the Krätzel transform implies $E_{n,k} \to E_k$ as $n \to \infty$, $E_{n,k} = K_\nu^p f_{n,k}$, and $E_k = K_\nu^p f_k$ in the space Z_+. Thus, $[E_{n,k}/s_k] \to [E_k/s_k]$ as $n \to \infty$ in β_{Z_+}.

To prove the second part of the lemma, let $g_n \xrightarrow{\delta} g \in \beta_{\mathrm{Z}_+}$ as $n \to \infty$. From [10], we have $g_n = [E_{n,k}/s_k]$ and $g = [E_k/s_k]$ for some $E_{n,k}, E_k \in \mathrm{Z}_+$ where $E_{n,k} \to E_k$ as $n \to \infty$. Hence $K_\nu^{\overset{-1}{p}} E_{n,k} \to K_\nu^{\overset{-1}{p}} E_k$ in β_{s_+} as $n \to \infty$. That is, $[E_{n,k}^{-1}/s_k] \to [E_k^{-1}/s_k]$ as $n \to \infty$. Using (42) we get $\vec{\mathbf{K}}_\nu^{\overset{-1}{p}} [E_{n,k}/s_k] \to \vec{\mathbf{K}}_\nu^{\overset{-1}{p}} [E_k/s_k]$ as $n \to \infty$.

Now, we establish continuity of $\vec{\mathbf{K}}_\nu^p$ and $\vec{\mathbf{K}}_\nu^{\overset{-1}{p}}$ with respect to Δ_+ convergence. Let $\beta_n \xrightarrow{\Delta} \beta$ in β_{s_+} as $n \to \infty$. Then, we find $f_n \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$, and $(s_n) \in \Delta_+$ such that $(\beta_n - \beta) \cdot s_n = [(f_n \cdot s_k)/s_k]$ and $f_n \to 0$ as $n \to \infty$. Employing (41) we get

$$\vec{\mathbf{K}}_\nu^p \left((\beta_n - \beta) \cdot s_n\right) = \left[\frac{K_\nu^p \left(f_n \cdot s_k\right)}{s_k}\right]. \tag{47}$$

Hence, from (41) and Lemma 19 we have $\vec{\mathbf{K}}_\nu^p((\beta_n - \beta) \cdot s_n) = [(E_n \circledcirc s_k)/s_k] = E_k \to 0$ as $n \to \infty$ in Z_+. Therefore

$$\begin{aligned} &\vec{\mathbf{K}}_\nu^p \left((\beta_n - \beta) \cdot s_n\right) \\ &\quad = \left(\vec{\mathbf{K}}_\nu^p \beta_n - \vec{\mathbf{K}}_\nu^p \beta\right) \circledcirc s_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{48}$$

Hence, $\vec{\mathbf{K}}_\nu^p \beta_n \xrightarrow{\Delta} \vec{\mathbf{K}}_\nu^p \beta$ as $n \to \infty$. Finally, let $g_n \xrightarrow{\Delta} g$ in β_{Z_+} as $n \to \infty$ and then we find $E_k \in \mathrm{Z}_+$ such that $(g_n - g) \circledcirc s_n = [(E_k \circledcirc s_k)/s_k]$ and $E_k \to 0$ as $n \to \infty$ for some $(s_n) \in \Delta_+$ and $E_k = K_\nu^p f_n$.

Next, using (42), we obtain $\vec{\mathbf{K}}_\nu^{\overset{-1}{p}}((g_n - g) \circledcirc s_n) = [K_\nu^{\overset{-1}{p}}(E_k \circledcirc s_k)/s_k]$. Lemma 19 implies that

$$\begin{aligned} &\vec{\mathbf{K}}_\nu^{\overset{-1}{p}} \left((g_n - g) \circledcirc s_n\right) \\ &\quad = \left[\frac{f_n \cdot s_k}{s_k}\right] = f_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \text{ in } \mathcal{S}'_+\left(\mathcal{L}^r, \alpha, \left(a_i\right), a\right). \end{aligned} \tag{49}$$

Thus $\vec{\mathbf{K}}_\nu^{\overset{-1}{p}}((g_n - g) \circledcirc s_n) = (\vec{\mathbf{K}}_\nu^{\overset{-1}{p}} g_n - \vec{\mathbf{K}}_\nu^{\overset{-1}{p}} g) \cdot s_n \to 0$ as $n \to \infty$. Hence, we have $\vec{\mathbf{K}}_\nu^{\overset{-1}{p}} g_n \xrightarrow{\Delta} \vec{\mathbf{K}}_\nu^{\overset{-1}{p}} g$ as $n \to \infty$ in β_{s_+}.

This completes the proof of the theorem. □

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Research Article

On the Stability of Trigonometric Functional Equations in Distributions and Hyperfunctions

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We consider the Hyers-Ulam stability for a class of trigonometric functional equations in the spaces of generalized functions such as Schwartz distributions and Gelfand hyperfunctions.

1. Introduction

Hyers-Ulam stability problems of functional equations go back to 1940 when Ulam proposed the following question [1].

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d\left(f\left(xy\right),f\left(x\right)f\left(y\right)\right)\le\epsilon. \tag{1}$$

Then does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that

$$d\left(f\left(x\right),h\left(x\right)\right)\le\delta_\epsilon \tag{2}$$

for all $x\in G_1$?

This problem was solved affirmatively by Hyers [2] under the assumption that G_2 is a Banach space. After the result of Hyers, Aoki [3] and Bourgin [4, 5] treated with this problem; however, there were no other results on this problem until 1978 when Rassias [6] treated again with the inequality of Aoki [3]. Generalizing Hyers' result, he proved that if a mapping $f : X \to Y$ between two Banach spaces satisfies

$$\left\|f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\|\le\Phi\left(x,y\right),\quad \text{for } x,y\in X \tag{3}$$

with $\Phi(x,y)=\epsilon(\|x\|^p+\|y\|^p)$ ($\epsilon\ge 0$, $0\le p<1$), then there exists a unique additive function $A : X\to Y$ such that $\|f(x)-A(x)\|\le 2\epsilon|x|^p/(2-2^p)$ for all $x\in X$. In 1951 Bourgin [4, 5] stated that if Φ is symmetric in $\|x\|$ and $\|y\|$ with $\sum_{j=1}^{\infty}\Phi(2^jx,2^jx)/2^j<\infty$ for each $x\in X$, then there exists a unique additive function $A : X\to Y$ such that $\|f(x)-A(x)\|\le\sum_{j=1}^{\infty}\Phi(2^jx,2^jx)/2^j$ for all $x\in X$. Unfortunately, there was no use of these results until 1978 when Rassias [7] treated with the inequality of Aoki [3]. Following Rassias' result, a great number of papers on the subject have been published concerning numerous functional equations in various directions [6–10, 10–25]. In 1990 Székelyhidi [24] has developed his idea of using invariant subspaces of functions defined on a group or semigroup in connection with stability questions for the sine and cosine functional equations. We refer the reader to [9, 10, 18, 19, 25] for Hyers-Ulam stability of functional equations of trigonometric type. In this paper, following the method of Székelyhidi [24] we consider a distributional analogue of the Hyers-Ulam stability problem of the trigonometric functional inequalities

$$\begin{aligned}\left|f\left(x-y\right)-f\left(x\right)g\left(y\right)+g\left(x\right)f\left(y\right)\right|&\le\psi\left(y\right),\\ \left|g\left(x-y\right)-g\left(x\right)g\left(y\right)-f\left(x\right)f\left(y\right)\right|&\le\psi\left(y\right),\end{aligned} \tag{4}$$

where $f,g:\mathbb{R}^n\to\mathbb{C}$ and $\psi:\mathbb{R}^n\to[0,\infty)$ is a continuous function. As a distributional version of the inequalities (4), we consider the inequalities for the generalized functions $u, v \in \mathcal{G}'(\mathbb{R}^n)$ (resp., $\mathcal{S}'(\mathbb{R}^n)$),

$$\begin{aligned} &\left\| u \circ (x-y) - u_x \otimes v_y + v_x \otimes u_y \right\| \le \psi(y), \\ &\left\| v \circ (x-y) - v_x \otimes v_y - u_x \otimes u_y \right\| \le \psi(y), \end{aligned} \tag{5}$$

where $\circ$ and $\otimes$ denote the pullback and the tensor product of generalized functions, respectively, and $\psi : \mathbb{R}^n \to [0,\infty)$ denotes a continuous infraexponential function of order 2 (resp., a function of polynomial growth). For the proof we employ the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right), \quad x \in \mathbb{R}^n,\ t > 0. \tag{6}$$

For the first step, convolving $E_t(x)E_s(y)$ in both sides of (5) we convert (5) to the Hyers-Ulam stability problems of *trigonometric-hyperbolic type* functional inequalities, respectively,

$$\begin{aligned} &\left| U(x-y, t+s) - U(x,t)V(y,s) + V(x,t)U(y,s) \right| \\ &\quad \le \Psi(y,s), \\ &\left| V(x-y, t+s) - V(x,t)V(y,s) - U(x,t)U(y,s) \right| \\ &\quad \le \Psi(y,s), \end{aligned} \tag{7}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where U, V are the Gauss transforms of u, v, respectively, given by

$$U(x,t) = u * E_t(x) = \langle u_y, E_t(x-y) \rangle, \tag{8}$$

$$V(x,t) = v * E_t(x), \tag{9}$$

which are solutions of the heat equation, and

$$\Psi(y,s) = \int \psi(\eta) E_s(\eta - y)\, d\eta = (\psi * E_s)(y). \tag{10}$$

For the second step, using similar idea of Székelyhidi [24] we prove the Hyers-Ulam stabilities of inequalities (7). For the final step, taking initial values as $t \to 0^+$ for the results we arrive at our results.

2. Generalized Functions

We first introduce the spaces $\mathcal{S}'$ of Schwartz tempered distributions and $\mathcal{G}'$ of Gelfand hyperfunctions (see [26–29] for more details of these spaces). We use the notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

Definition 1 (see [29]). One denotes by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{\alpha,\beta} = \sup_x \left| x^\alpha \partial^\beta \varphi(x) \right| < \infty \tag{11}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions, and the elements of the dual space $\mathcal{S}'$ are called tempered distributions.

Definition 2 (see [26]). One denotes by $\mathcal{G}$ or $\mathcal{G}(\mathbb{R}^n)$ the Gelfand space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n, \alpha,\beta \in \mathbb{N}_0^n} \frac{\left| x^\alpha \partial^\beta \varphi(x) \right|}{h^{|\alpha|} k^{|\beta|} \alpha!^{1/2} \beta!^{1/2}} < \infty \tag{12}$$

for some $h, k > 0$. One says that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{h,k} \to 0$ as $j \to \infty$ for some h, k, and one denotes by $\mathcal{G}'$ the dual space of $\mathcal{G}$ and calls its elements Gelfand hyperfunctions.

It is well known that the following topological inclusions hold:

$$\mathcal{G} \hookrightarrow \mathcal{S}, \qquad \mathcal{S}' \hookrightarrow \mathcal{G}'. \tag{13}$$

It is known that the space $\mathcal{G}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ which can be extended to an entire function on $\mathbb{C}^n$ satisfying

$$\left| \varphi(x+iy) \right| \le C \exp\left(-a|x|^2 + b|y|^2 \right), \quad x, y \in \mathbb{R}^n \tag{14}$$

for some a, b, and $C > 0$ (see [26]).

By virtue of Theorem 6.12 of [27, p. 134] we have the following.

Definition 3. Let $u_j \in \mathcal{G}'(\mathbb{R}^{n_j})$ for $j = 1, 2$, with $n_1 \ge n_2$, and let $\lambda : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ be a smooth function such that, for each $x \in \mathbb{R}^{n_1}$, the Jacobian matrix $\nabla\lambda(x)$ of λ at x has rank n_2. Then there exists a unique continuous linear map $\lambda^* : \mathcal{G}'(\mathbb{R}^{n_2}) \to \mathcal{G}'(\mathbb{R}^{n_1})$ such that $\lambda^* u = u \circ \lambda$ when u is a continuous function. One calls $\lambda^* u$ the pullback of u by λ which is often denoted by $u \circ \lambda$.

In particular, let $\lambda : \mathbb{R}^{2n} \to \mathbb{R}^n$ be defined by $\lambda(x,y) = x - y$, $x, y \in \mathbb{R}^n$. Then in view of the proof of Theorem 6.12 of [27, p. 134] we have

$$\langle u \circ \lambda, \varphi(x,y) \rangle = \left\langle u, \int \varphi(x-y, y)\, dy \right\rangle. \tag{15}$$

Definition 4. Let $u_x \in \mathcal{G}'(\mathbb{R}^{n_1})$, $u_y \in \mathcal{G}'(\mathbb{R}^{n_2})$. Then the tensor product $u_x \otimes u_y$ of u_x and u_y, defined by

$$\langle u_x \otimes u_y, \varphi(x,y) \rangle = \langle u_x, \langle u_y, \varphi(x,y) \rangle \rangle \tag{16}$$

for $\varphi(x,y) \in \mathcal{G}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, belongs to $\mathcal{G}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

For more details of pullback and tensor product of distributions we refer the reader to Chapter V-VI of [27].

3. Main Theorems

Let f be a Lebesgue measurable function on $\mathbb{R}^n$. Then f is said to be an *infraexponential function of order 2 (resp.,* *a function of polynomial growth)* if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ (resp., there exist positive constants C, N, and d) such that

$$|f(x)| \le C_\epsilon e^{\epsilon |x|^2} \quad [\text{resp. } \le C|x|^N + d] \tag{17}$$

for all $x \in \mathbb{R}^n$. It is easy to see that every infraexponential function f of order 2 (resp., every function of polynomial growth) defines an element of $\mathcal{G}'(\mathbb{R}^n)$ (resp., $\mathcal{S}'(\mathbb{R}^n)$) via the correspondence

$$\langle f, \varphi \rangle = \int f(x) \varphi(x)\, dx \tag{18}$$

for $\varphi \in \mathcal{G}(\mathbb{R}^n)$ (resp. $\mathcal{S}(\mathbb{R}^n)$).

Let $u, v \in \mathcal{G}'(\mathbb{R}^n)$ (resp., $\mathcal{S}'(\mathbb{R}^n)$). We prove the stability of the following functional inequalities:

$$\| u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y \| \le \psi(y), \tag{19}$$

$$\| v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y \| \le \psi(y), \tag{20}$$

where $\circ$ and $\otimes$ denote the pullback and the tensor product of generalized functions, respectively, $\psi : \mathbb{R}^n \to [0, \infty)$ denotes a continuous infraexponential functional of order 2 (resp. a continuous function of polynomial growth) with $\psi(0) = 0$, and $\| \cdot \| \le \psi$ means that $|\langle \cdot, \varphi \rangle| \le \| \psi \varphi \|_{L^1}$ for all $\varphi \in \mathcal{G}(\mathbb{R}^n)$ (resp., $\mathcal{S}(\mathbb{R}^n)$).

In view of (14) it is easy to see that the n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t} \right), \quad t > 0, \tag{21}$$

belongs to the Gelfand space $\mathcal{G}(\mathbb{R}^n)$ for each $t > 0$. Thus the convolution $(u * E_t)(x) := \langle u_y, E_t(x - y) \rangle$ is well defined for all $u \in \mathcal{G}'(\mathbb{R}^n)$. It is well known that $U(x,t) = (u * E_t)(x)$ is a smooth solution of the heat equation $(\partial/\partial_t - \Delta)U = 0$ in $\{(x,t) : x \in \mathbb{R}^n, t > 0\}$ and $(u * E_t)(x) \to u$ as $t \to 0^+$ in the sense of generalized functions that is, for every $\varphi \in \mathcal{G}(\mathbb{R}^n)$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int (u * E_t)(x) \varphi(x)\, dx. \tag{22}$$

We call $(u * E_t)(x)$ the Gauss transform of u.

A function A from a semigroup $\langle S, + \rangle$ to the field $\mathbb{C}$ of complex numbers is said to be *an additive function* provided that $A(x + y) = A(x) + A(y)$, and $m : S \to \mathbb{C}$ is said to be *an exponential function* provided that $m(x + y) = m(x)m(y)$.

For the proof of stabilities of (19) and (20) we need the following.

Lemma 5 (see [15]). *Let S be a semigroup and $\mathbb{C}$ the field of complex numbers. Assume that $f, g : S \to \mathbb{C}$ satisfy the inequality; for each $y \in S$ there exists a positive constant M_y such that*

$$|f(x + y) - f(x) g(y)| \le M_y \tag{23}$$

for all $x \in S$. Then either f is a bounded function or g is an exponential function.

Proof. Suppose that g is not exponential. Then there are $y, z \in S$ such that $g(y + z) \ne g(y)g(z)$. Now we have

$$\begin{aligned} f(x + y + z) &- f(x + y) g(z) \\ &= (f(x + y + z) - f(x) g(y + z)) \\ &\quad - g(z)(f(x + y) - f(x) g(y)) \\ &\quad + f(x)(g(y + z) - g(y) g(z)), \end{aligned} \tag{24}$$

and hence

$$\begin{aligned} f(x) = {} & (g(y + z) - g(y) g(z))^{-1} \\ & \times ((f(x + y + z) - f(x + y) g(z)) \\ & \quad - (f(x + y + z) - f(x) g(y + z)) \\ & \quad + g(z)(f(x + y) - f(x) g(y))). \end{aligned} \tag{25}$$

In view of (23) the right hand side of (25) is bounded as a function of x. Consequently, f is bounded. □

Lemma 6 (see [30, p. 122]). *Let $f(x,t)$ be a solution of the heat equation. Then $f(x,t)$ satisfies*

$$|f(x,t)| \le M, \quad x \in \mathbb{R}^n, \ t \in (0,1) \tag{26}$$

for some $M > 0$, if and only if

$$f(x,t) = (f_0 * E_t)(x) = \int f_0(y) E_t(x - y)\, dy \tag{27}$$

for some bounded measurable function f_0 defined in $\mathbb{R}^n$. In particular, $f(x,t) \to f_0(x)$ in $\mathcal{G}'(\mathbb{R}^n)$ as $t \to 0^+$.

We discuss the solutions of the corresponding trigonometric functional equations

$$u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y = 0, \tag{28}$$

$$v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y = 0, \tag{29}$$

in the space $\mathcal{G}'$ of Gelfand hyperfunctions. As a consequence of the results [8, 31, 32] we have the following.

Lemma 7. *The solutions $u, v \in \mathcal{G}'(\mathbb{R}^n)$ of (28) and (29) are equal, respectively, to the continuous solutions $f, g : \mathbb{R}^n \to \mathbb{C}$ of corresponding classical functional equations*

$$f(x - y) - f(x) g(y) + g(x) f(y) = 0, \tag{30}$$

$$g(x - y) - g(x) g(y) - f(x) f(y) = 0. \tag{31}$$

The continuous solutions (f, g) of the functional equation (30) are given by one of the following:

(i) *$f = 0$ and g is arbitrary,*

(ii) *$f(x) = c_1 \cdot x$, $g(x) = 1 + c_2 \cdot x$ for some $c_1, c_2 \in \mathbb{C}^n$,*

(iii) *$f(x) = \lambda_1 \sin(c \cdot x)$ and $g(x) = \cos(c \cdot x) + \lambda_2 \sin(c \cdot x)$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$, $c \in \mathbb{C}^n$.*

Also, the continuous solutions (f, g) *of the functional equation* (31) *are given by one of the following:*

(i) $g(x) = \lambda$ *and* $f(x) = \pm\sqrt{\lambda - \lambda^2}$ *for some* $\lambda \in \mathbb{C}$,

(ii) $g(x) = \cos(c \cdot x)$ *and* $f(x) = \sin(c \cdot x)$ *for some* $c \in \mathbb{C}^n$.

For the proof of the stability of (19) we need the followings.

Lemma 8. *Let* G *be an Abelian group and let* $U, V : G \times (0, \infty) \to \mathbb{C}$ *satisfy the inequality; there exists a nonnegative function* $\Psi : G \times (0, \infty) \to \mathbb{R}$ *such that*

$$\left|U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s)\right| \leq \Psi(y, s) \tag{32}$$

for all $x, y \in G$, $t, s > 0$. *Then either there exist* $\lambda_1, \lambda_2 \in \mathbb{C}$, *not both are zero, and* $M > 0$ *such that*

$$\left|\lambda_1 U(x, t) - \lambda_2 V(x, t)\right| \leq M, \tag{33}$$

or else

$$U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s) = 0 \tag{34}$$

for all $x, y \in G$, $t, s > 0$.

Proof. Suppose that inequality (33) holds only when $\lambda_1 = \lambda_2 = 0$. Let

$$\begin{aligned} K(x, y, t, s) = {} & U(x + y, t + s) - U(x, t) V(-y, s) \\ & + V(x, t) U(-y, s), \end{aligned} \tag{35}$$

and choose y_1 and s_1 satisfying $U(-y_1, s_1) \neq 0$. Now it can be easily calculated that

$$\begin{aligned} V(x, t) = {} & \lambda_0 U(x, t) + \lambda_1 U(x + y_1, t + s_1) \\ & - \lambda_1 K(x, y_1, t, s_1), \end{aligned} \tag{36}$$

where $\lambda_0 = V(-y_1, s_1)/U(-y_1, s_1)$ and $\lambda_1 = -1/U(-y_1, s_1)$. By (35) we have

$$\begin{aligned} U(x + (y + z), t + (s + r)) = {} & U(x, t) V(-y - z, s + r) \\ & - V(x, t) U(-y - z, s + r) \\ & + K(x, y + z, t, s + r). \end{aligned} \tag{37}$$

Also by (35) and (36) we have

$$\begin{aligned} & U((x + y) + z, (t + s) + r) \\ & = U(x + y, t + s) V(-z, r) - V(x + y, t + s) U(-z, r) \\ & \quad + K(x + y, z, t + s, r) \\ & = (U(x, t) V(-y, s) - V(x, t) U(-y, s) \\ & \quad + K(x, y, t, s)) V(-z, r) \\ & \quad - (\lambda_0 U(x + y, t + s) + \lambda_1 U(x + y + y_1, t + s + s_1) \\ & \quad - \lambda_1 K(x + y, y_1, t + s, s_1)) U(-z, r) \\ & \quad + K(x + y, z, t + s, r) \\ & = (U(x, t) V(-y, s) - V(x, t) U(-y, s) \\ & \quad + K(x, y, t, s)) V(-z, r) \\ & \quad - \lambda_0 (U(x, t) V(-y, s) - V(x, t) U(-y, s) \\ & \quad + K(x, y, t, s)) U(-z, r) \\ & \quad - \lambda_1 (U(x, t) V(-y - y_1, s + s_1) \\ & \quad - V(x, t) U(-y - y_1, s + s_1) \\ & \quad + K(x, y + y_1, t, s + s_1)) U(-z, r) \\ & \quad + \lambda_1 K(x + y, y_1, t + s, s_1) U(-z, r) \\ & \quad + K(x + y, z, t + s, r). \end{aligned} \tag{38}$$

From (37) and (38) we have

$$\begin{aligned} & (V(-y, s) V(-z, r) - \lambda_0 V(-y, s) U(-z, r) \\ & \quad - \lambda_1 V(-y - y_1, s + s_1) U(-z, r) \\ & \quad - V(-y - z, s + r)) U(x, t) \\ & + (-U(-y, s) V(-z, r) + \lambda_0 U(-y, s) U(-z, r) \\ & \quad + \lambda_1 U(-y - y_1, s + s_1) U(-z, r) \\ & \quad + U(-y - z, s + r)) V(x, t) \\ & = -K(x, y, t, s) V(-z, r) + \lambda_0 K(x, y, t, s) U(-z, r) \\ & \quad + \lambda_1 K(x, y + y_1, t, s + s_1) U(-z, r) \\ & \quad - \lambda_1 K(x + y, y_1, t + s, s_1) U(-z, r) \\ & \quad - K(x + y, z, t + s, r) + K(x, y + z, t, s + r). \end{aligned} \tag{39}$$

Since $K(x, y, t, s)$ is bounded by $\Psi(-y, s)$, if we fix y, z, r, and s, the right hand side of (39) is bounded by a constant M, where

$$\begin{aligned} M = {} & \Psi(-y,s)\left|V(-z,r)\right| + \Psi(-y,s)\left|\lambda_0 U(-z,r)\right| \\ & + \Psi(-y-y_1, s+s_1)\left|\lambda_1 U(-z,r)\right| \\ & + \Psi(-y_1,s_1)\left|\lambda_1 U(-z,r)\right| + \Psi(-z,r) \\ & + \Psi(-y-z, r+s). \end{aligned} \tag{40}$$

So by our assumption, the left hand side of (39) vanishes, so is the right hand side. Thus we have

$$\begin{aligned} & \left(-\lambda_0 K(x,y,t,s) - \lambda_1 K(x, y+y_1, t, s+s_1)\right. \\ & \quad \left. +\lambda_1 K(x+y, y_1, t+s, s_1)\right) U(-z,r) \\ & + K(x,y,t,s) V(-z,r) = K(x, y+z, t, s+r) \\ & - K(x+y, z, t+s, r). \end{aligned} \tag{41}$$

Now by the definition of K we have

$$\begin{aligned} & K(x+y, z, t+s, r) - K(x, y+z, t, s+r) \\ & = U(x+y+z, t+s+r) - U(x+y, t+s) V(-z,r) \\ & \quad + V(x+y,t+s) U(-z,r) - U(x+y+z, t+s+r) \\ & \quad + U(x,t) V(-y-z, s+r) - V(x,t) U(-y-z, s+r) \\ & = U(-y-z-x, s+r+t) - U(-y-z, s+r) V(x,t) \\ & \quad + V(-y-z, s+r) U(x,t) - U(-z-x-y, r+t+s) \\ & \quad + U(-z,r) V(x+y, t+s) - V(-z,r) U(x+y, t+s) \\ & = K(-y-z, -x, s+r, t) - K(-z, -x-y, r, t+s). \end{aligned} \tag{42}$$

Hence the left hand side of (41) is bounded by $\Psi(x,t)+\Psi(x+y,t+s)$. So if we fix x, y, t, and s in (41), the left hand side of (41) is a bounded function of z and r. Thus $K(x, y, t, s) \equiv 0$ by our assumption. This completes the proof. □

In the following lemma we assume that $\Psi : \mathbb{R}^n \times (0, \infty) \to [0, \infty)$ is a continuous function such that

$$\psi(x) := \lim_{t \to 0^+} \Psi(x,t) \tag{43}$$

exists and satisfies the conditions $\psi(0) = 0$ and

$$\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k} \psi\left(-2^k x\right) < \infty \tag{44}$$

or

$$\Phi_2(x) := \sum_{k=1}^{\infty} 2^{k} \psi\left(-2^{-k} x\right) < \infty. \tag{45}$$

Lemma 9. *Let $U, V : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be continuous functions satisfying*

$$\begin{aligned} & \left|U(x-y, t+s) - U(x,t) V(y,s) + V(x,t) U(y,s)\right| \\ & \quad \le \Psi(y,s) \end{aligned} \tag{46}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, and there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$\left|\lambda_1 U(x,t) - \lambda_2 V(x,t)\right| \le M. \tag{47}$$

Then (U, V) satisfies one of the followings:

(i) *$U = 0$, V is arbitrary,*

(ii) *U and V are bounded functions,*

(iii) *$V(x,t) = \lambda U(x,t) + e^{ic\cdot x - bt}$ for some $\lambda \in \mathbb{C}^n$, $c(\neq 0) \in \mathbb{R}^n$, and $b \in \mathbb{C}$, and $f(x) := \lim_{t\to 0^+} U(x,t)$ is a continuous function; in particular, there exists $\delta : (0,\infty) \to [0,\infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that*

$$\left|U(x,t) - f(x) e^{-bt}\right| \le \delta(t) \tag{48}$$

for all $x \in \mathbb{R}^n$, $t > 0$, and satisfies the condition; there exists $d \ge 0$ satisfying

$$\left|f(x)\right| \le \psi(-x) + d \tag{49}$$

for all $x \in \mathbb{R}^n$,

(iv) *$V(x,t) = \lambda U(x,t) + e^{-bt}$ for some $\lambda \in \mathbb{C}^n$, $b \in \mathbb{C}$, and $f(x) := \lim_{t\to 0^+} U(x,t)$ is a continuous function; in particular, there exists $\delta : (0,\infty) \to [0,\infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that*

$$\left|U(x,t) - f(x) e^{-bt}\right| \le \delta(t) \tag{50}$$

for all $x \in \mathbb{R}^n$, $t > 0$, and satisfies one of the following conditions; there exists $a_1 \in \mathbb{C}^n$ such that

$$\left|f(x) - a_1 \cdot x\right| \le \Phi_1(x) \tag{51}$$

for all $x \in \mathbb{R}^n$, or there exists $a_2 \in \mathbb{C}^n$ such that

$$\left|f(x) - a_2 \cdot x\right| \le \Phi_2(x) \tag{52}$$

for all $x \in \mathbb{R}^n$.

Proof. If $U = 0$, V is arbitrary which is case (i). If U is a nontrivial bounded function, in view of (46) V is also bounded which gives case (ii). If U is unbounded, it follows from (47) that $\lambda_2 \neq 0$ and

$$V(x,t) = \lambda U(x,t) + R(x,t) \tag{53}$$

for some $\lambda \in \mathbb{C}$ and a bounded function R. Putting (53) in (46) we have

$$\begin{aligned} & \left|U(x-y, t+s) - U(x,t) R(y,s) + R(x,t) U(y,s)\right| \\ & \quad \le \Psi(y,s) \end{aligned} \tag{54}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Replacing y by $-y$ and using the triangle inequality, we have, for some $C > 0$,

$$\begin{aligned} &\left|U(x+y,t+s) - U(x,t)R(-y,s)\right| \\ &\quad \le C\left|U(-y,s)\right| + \Psi(-y,s) \end{aligned} \tag{55}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. By Lemma 5, $R(-y,s)$ is an exponential function. If $R = 0$, putting $y = 0, s \to 0^+$ in (54) we have

$$|U(x,t)| \le \psi(0) = 0. \tag{56}$$

Thus we have $R \ne 0$ since U is unbounded. Given the continuity of U and V we have

$$R(x,t) = e^{ic\cdot x - bt} \tag{57}$$

for some $c \in \mathbb{R}^n$, $b \in \mathbb{C}$ with $\Re b \ge 0$. Putting $y = 0$ and $s = 1$ in (54), dividing $R(0,1)$, and using the triangle inequality we have

$$|U(x,t)| \le |R(0,1)|^{-1}\left(|U(x,t+1)| + C|U(0,1)| + \Psi(0,1)\right) \tag{58}$$

for all $x \in \mathbb{R}^n$, $t > 0$.

From (58) and the continuity of U it is easy to see that

$$\limsup_{t\to 0^+} U(x,t) := f(x) \tag{59}$$

exists. Putting $x = y = 0$ and replacing s and t by $t/2$ in (54) we have

$$|U(0,t)| \le \Psi\left(0, \frac{t}{2}\right) \tag{60}$$

for all $t > 0$.

Fixing x, putting $y = 0$ letting $t \to 0^+$ so that $U(x,t) \to f(x)$ in (54), and using the triangle inequality and (60) we have

$$\left|U(x,s) - f(x)e^{-bs}\right| \le \Psi\left(0, \frac{s}{2}\right) + \Psi(0,s) := \delta(s) \tag{61}$$

for all $x \in \mathbb{R}^n$, $s > 0$. Letting $s \to 0^+$ in (61) we have

$$\lim_{s\to 0^+} U(x,s) = f(x) \tag{62}$$

for all $x \in \mathbb{R}^n$. From (61) the continuity of f can be checked by a usual calculus. Letting $t \to 0^+$ in (60) we see that $f(0) = 0$. Letting $t, s \to 0^+$ in (54) we have

$$\left|f(x-y) - f(x)e^{ic\cdot y} + e^{ic\cdot x} f(y)\right| \le \psi(y) \tag{63}$$

for all $x, y \in \mathbb{R}^n$. Putting $x = 0$ in (63) and replacing y by $-y$ we have

$$\left|f(-y) + f(y)\right| \le \psi(-y) \tag{64}$$

for all $y \in \mathbb{R}^n$.

Replacing y by $-y$ and using (64) and the triangle inequality we have

$$\left|f(x+y) - f(x)e^{-ic\cdot y} - e^{ic\cdot x} f(y)\right| \le 2\psi(-y) \tag{65}$$

for all $x, y \in \mathbb{R}^n$. Now we divide (65) into two cases: $c = 0$ and $c \ne 0$. First we consider the case $c \ne 0$. Replacing x by y and y by x in (65) we have

$$\left|f(x+y) - f(y)e^{-ic\cdot x} - e^{ic\cdot y} f(x)\right| \le 2\psi(-x) \tag{66}$$

for all $x, y \in \mathbb{R}^n$. From (65) and (66), using the triangle inequality and dividing $|e^{ic\cdot y} - e^{-ic\cdot y}|$ we have

$$\left|f(x)\right| \le \frac{2\left(\psi(-x) + \psi(-y) + \left|f(y)\right|\right)}{\left|e^{ic\cdot y} - e^{-ic\cdot y}\right|} \tag{67}$$

for all $x, y \in \mathbb{R}^n$ such that $c \cdot y \ne 0$. Choosing $y_0 \in \mathbb{R}^n$ so that $c \cdot y_0 = \pi/2$ and putting $y = y_0$ in (67) we have

$$\left|f(x)\right| \le \psi(-x) + d, \tag{68}$$

where $d = \psi(\pi/2) + |f(\pi/2)|$, which gives (iii). Now we consider the case $c = 0$. It follows from (65) that

$$\left|f(x+y) - f(x) - f(y)\right| \le 2\psi(-y) \tag{69}$$

for all $x, y \in \mathbb{R}^n$. By the well-known results in [3], there exists a unique additive function $A_1(x)$ given by

$$A_1(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{70}$$

such that

$$\left|f(x) - A_1(x)\right| \le \Phi_1(x) \tag{71}$$

if $\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k}\psi(-2^k x) < \infty$, and there exists a unique additive function $A_2(x)$ given by

$$A_2(x) = \lim_{n\to\infty} 2^{n} f(2^{-n} x) \tag{72}$$

such that

$$\left|f(x) - A_2(x)\right| \le \Phi_2(x) \tag{73}$$

if $\Phi_2(x) := \sum_{k=0}^{\infty} 2^{k}\psi(-2^{-k} x) < \infty$. Now by the continuity of U and inequality (61), it is easy to see that f is continuous. In view of (70) and (72), $A_j(x)$, $j = 1, 2$, are Lebesgue measurable functions. Thus there exist $a_1, a_2 \in \mathbb{C}^n$ such that $A_1(x) = a_1 \cdot x$ and $A_2(x) = a_2 \cdot x$ for all $x \in \mathbb{R}^n$, which gives (iv). This completes the proof. □

In the following we assume that ψ satisfies (44) or (45).

Theorem 10. *Let $u, v \in \mathcal{G}'$ satisfy* (19). *Then (u, v) satisfies one of the followings:*

(i) $u = 0$, *and v is arbitrary,*

(ii) *u and v are bounded measurable functions,*

(iii) $v(x) = \lambda u(x)+e^{ic\cdot x}$ *for some* $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, *where* u *is a continuous function satisfying the condition; there exists* $d \geq 0$

$$|u(x)| \leq \psi(-x) + d \tag{74}$$

for all $x \in \mathbb{R}^n$,

(iv) $v(x) = \lambda u(x) + 1$ *for some* $\lambda \in \mathbb{C}$, *where* u *is a continuous function satisfying one of the following conditions; there exists* $a_1 \in \mathbb{C}^n$ *such that*

$$|u(x) - a_1 \cdot x| \leq \Phi_1(x) \tag{75}$$

for all $x \in \mathbb{R}^n$, *or there exists* $a_2 \in \mathbb{C}^n$ *such that*

$$|u(x) - a_2 \cdot x| \leq \Phi_2(x) \tag{76}$$

for all $x \in \mathbb{R}^n$,

(v) $u = \lambda \sin(c\cdot x)$, $v = \cos(c\cdot x) + \lambda \sin(c \cdot x)$, *for some* $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.

Proof. Convolving in (19) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels in both sides of inequality (19) we have

$$\begin{aligned}
&[u\circ(\xi-\eta) * (E_t(\xi)E_s(\eta))](x,y) \\
&\quad = \left\langle u_\xi, \int E_t(x-\xi-\eta)E_s(y-\eta)\,d\eta \right\rangle \\
&\quad = \left\langle u_\xi, (E_t * E_s)(x-y-\xi) \right\rangle \\
&\quad = \left\langle u_\xi, E_{t+s}(x-y-\xi) \right\rangle \\
&\quad = U(x-y,t+s).
\end{aligned} \tag{77}$$

Similarly we have

$$\begin{aligned}
&[(u\otimes v) * (E_t(\xi)E_s(\eta))](x,y) = U(x,t)V(y,s), \\
&[(v\otimes u) * (E_t(\xi)E_s(\eta))](x,y) = V(x,t)U(y,s),
\end{aligned} \tag{78}$$

where U, V are the Gauss transforms of u, v, respectively. Thus we have the following inequality:

$$\begin{aligned}
&|U(x-y,t+s) - U(x,t)V(y,s) + V(x,t)U(y,s)| \\
&\quad \leq \Psi(y,s)
\end{aligned} \tag{79}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where

$$\begin{aligned}
\Psi(y,s) &= \int \psi(\eta)E_t(x-\xi)E_s(y-\eta)\,d\xi\,d\eta \\
&= \int \psi(\eta)E_s(\eta-y)\,d\eta = (\psi * E_s)(y).
\end{aligned} \tag{80}$$

By Lemma 8 there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$|\lambda_1 U(x,t) - \lambda_2 V(x,t)| \leq M, \tag{81}$$

or else U, V satisfy

$$U(x-y,t+s) - U(x,t)V(y,s) + V(x,t)U(y,s) = 0 \tag{82}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Assume that (81) holds. Applying Lemma 9, case (i) follows from (i) of Lemma 9. Using (ii) of Lemma 9, it follows from Lemma 7 the initial values u, v of $U(x,t), V(x,t)$ as $t \to 0^+$ are bounded measurable functions, respectively, which gives (ii). For case (iii), it follows from (50) that, for all $\varphi \in \mathcal{G}(\mathbb{R}^n)$,

$$\begin{aligned}
&|\langle u,\varphi\rangle - \langle f,\varphi\rangle| \\
&\quad = \left|\lim_{t\to 0^+}\int U(x,t)\varphi(x)\,dx - \int f(x)\varphi(x)\,dx\right| \\
&\quad = \left|\lim_{t\to 0^+}\int \left(U(x,t) - f(x)e^{-bt}\right)\varphi(x)\,dx\right| \\
&\quad \leq \lim_{t\to 0^+}\int \left|U(x,t) - f(x)e^{-bt}\right||\varphi(x)|\,dx \\
&\quad \leq \lim_{t\to 0^+}\delta(t)\int |\varphi(x)|\,dx = 0.
\end{aligned} \tag{83}$$

Thus we have $u = f$ in $\mathcal{G}'(\mathbb{R}^n)$. Letting $t \to 0^+$ in (iii) of Lemma 9 we get case (iii). Finally we assume that (82) holds. Letting $t, s \to 0^+$ in (82) we have

$$u\circ(x-y) - u_x \otimes v_y + v_x \otimes u_y = 0. \tag{84}$$

By Lemma 6 the solutions of (84) satisfy (i), (iv), or (v). This completes the proof. □

Let $\psi(x) = \epsilon|x|^p$, $p > 0$. Then ψ satisfies the conditions assumed in Theorem 10. In view of (44) and (45) we have

$$\Phi_1(x) = \frac{2\epsilon|x|^p}{2-2^p} \tag{85}$$

if $0 < p < 1$, and

$$\Phi_2(x) = \frac{2\epsilon|x|^p}{2^p-2} \tag{86}$$

if $p > 1$. Thus as a direct consequence of Theorem 10 we have the following.

Corollary 11. *Let* $0 < p < 1$ *or* $p > 1$. *Suppose that* $u, v \in \mathcal{G}'$ *satisfy*

$$\left\|u\circ(x-y) - u_x\otimes v_y + v_x \otimes u_y\right\| \leq \epsilon|y|^p. \tag{87}$$

Then (u, v) *satisfies one of the followings:*

(i) $u = 0$, *and* v *is arbitrary,*

(ii) u *and* v *are bounded measurable functions,*

(iii) $v(x) = \lambda u(x) + e^{ic\cdot x}$ *for some* $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, *where* u *is a continuous function satisfying the condition; there exists* $d \geq 0$

$$|u(x)| \leq \epsilon |x|^p + d \tag{88}$$

for all $x \in \mathbb{R}^n$,

(iv) $v(x) = \lambda u(x) + 1$ *for some* $\lambda \in \mathbb{C}$, *where* u *is a continuous function satisfying the conditions; there exists* $a \in \mathbb{C}^n$ *such that*

$$|u(x) - a \cdot x| \leq \frac{2\epsilon |x|^p}{|2^p - 2|} \tag{89}$$

for all $x \in \mathbb{R}^n$,

(v) $u = \lambda \sin(c \cdot x)$, $v = \cos(c \cdot x) + \lambda \sin(c \cdot x)$, *for some* $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.

Now we prove the stability of (20). For the proof we need the following.

Lemma 12. *Let* $U, V : G \times (0, \infty) \to \mathbb{C}$ *satisfy the inequality; there exists a* $\Psi : G \times (0, \infty) \to [0, \infty)$ *such that*

$$\begin{aligned} &\left| V(x-y, t+s) - V(x,t)V(y,s) - U(x,t)U(y,s) \right| \\ &\quad \leq \Psi(y,s) \end{aligned} \tag{90}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. *Then either there exist* $\lambda_1, \lambda_2 \in \mathbb{C}$, *not both are zero, and* $M > 0$ *such that*

$$\left| \lambda_1 U(x,t) - \lambda_2 V(x,t) \right| \leq M, \tag{91}$$

or else

$$V(x-y, t+s) - V(x,t)V(y,s) - U(x,t)U(y,s) = 0 \tag{92}$$

for all $x, y \in G$, $t, s > 0$.

Proof. As in Lemma 9, suppose that $\lambda_1 U(x,t) - \lambda_2 V(x,t)$ is bounded only when $\lambda_1 = \lambda_2 = 0$, and let

$$\begin{aligned} L(x, y, t, s) = {} & V(x+y, t+s) - V(x,t)V(-y,s) \\ & - U(x,t)U(-y,s). \end{aligned} \tag{93}$$

Since we may assume that U is nonconstant, we can choose y_1 and s_1 satisfying $U(-y_1, s_1) \neq 0$. Now it can be easily got that

$$\begin{aligned} U(x,t) = {} & \lambda_0 V(x,t) + \lambda_1 V(x+y_1, t+s_1) \\ & - \lambda_1 L(x, y_1, t, s_1), \end{aligned} \tag{94}$$

where $\lambda_0 = -V(-y_1, s_1)/U(-y_1, s_1)$ and $\lambda_1 = 1/U(-y_1, s_1)$. From the definition of L and the use of (94), we have the following two equations:

$$\begin{aligned} &V((x+y)+z, (t+s)+r) \\ &\quad = V(x+y, t+s)V(-z,r) + U(x+y, t+s)U(-z,r) \\ &\qquad + L(x+y, z, t+s, r) \\ &\quad = (V(x,t)V(-y,s) + U(x,t)U(-y,s) \\ &\qquad + L(x,y,t,s))V(-z,r) \\ &\qquad + (\lambda_0 V(x+y, t+s) + \lambda_1 V(x+y+y_1, t+s+s_1) \\ &\qquad - \lambda_1 L(x+y, y_1, t+s, s_1))U(-z,r) \\ &\qquad + L(x+y, z, t+s, r) \\ &\quad = (V(x,t)V(-y,s) + U(x,t)U(-y,s) \\ &\qquad + L(x,y,t,s))V(-z,r) \\ &\qquad + \lambda_0 (V(x,t)V(-y,s) + U(x,t)U(-y,s) \\ &\qquad + L(x,y,t,s))U(-z,r) \\ &\qquad + \lambda_1 (V(x,t)V(-y-y_1, s+s_1) \\ &\qquad + U(x,t)U(-y-y_1, s+s_1) \\ &\qquad + L(x, y+y_1, t, s+s_1))U(-z,r) \\ &\qquad - \lambda_1 L(x+y, y_1, t+s, s_1)U(-z,r) \\ &\qquad + L(x+y, z, t+s, r), \end{aligned} \tag{95}$$

$$\begin{aligned} &V(x+(y+z), t+(s+r)) \\ &\quad = V(x,t)V(-y-z, s+r) + U(x,t)U(-y-z, s+r) \\ &\qquad + L(x, y+z, t, s+r). \end{aligned} \tag{96}$$

By equating (95) and (96), we have

$$\begin{aligned} &V(x,t)(V(-y,s)V(-z,r) + \lambda_0 V(-y,s)U(-z,r) \\ &\qquad + \lambda_1 V(-y-y_1, s+s_1)U(-z,r) \\ &\qquad - V(-y-z, s+r)) \\ &\quad + U(x,t)(U(-y,s)V(-z,r) + \lambda_0 U(-y,s)U(-z,r) \\ &\qquad + \lambda_1 U(-y-y_1, s+s_1)U(-z,r) \\ &\qquad - U(-y-z, s+r)) \end{aligned}$$

$$\begin{aligned}
&= -L(x,y,t,s)V(-z,r) - \lambda_0 L(x,y,t,s)U(-z,r)\\
&\quad - \lambda_1 L(x,y+y_1,t,s+s_1)U(-z,r)\\
&\quad + \lambda_1 L(x+y,y_1,t+s,s_1)U(-z,r)\\
&\quad - L(x+y,z,t+s,r) + L(x,y+z,t,s+r).
\end{aligned} \tag{97}$$

In (97), when y, s, z, and r are fixed, the right hand side is bounded; so by our assumption we have

$$\begin{aligned}
&L(x,y,t,s)V(-z,r)\\
&\quad + (\lambda_0 L(x,y,t,s) + \lambda_1 L(x,y+y_1,t,s+s_1)\\
&\quad\quad -\lambda_1 L(x+y,y_1,t+s,s_1))U(-z,r)\\
&= L(x,y+z,t,s+r) - L(x+y,z,t+s,r).
\end{aligned} \tag{98}$$

Here, we have

$$\begin{aligned}
&L(x,y+z,t,s+r) - L(x+y,z,t+s,r)\\
&= V(x+y+z,t+s+r) - V(x,t)V(-y-z,s+r)\\
&\quad - U(x,t)U(-y-z,s+r) - V(x+y+z,t+s+r)\\
&\quad + V(x+y,t+s)V(-z,r) + U(x+y,t+s)U(-z,r)\\
&= L(-y-z,-x,s+r,t) - L(-z,-x-y,r,t+s)\\
&\le \Psi(x,t) + \Psi(x+y,t+s).
\end{aligned} \tag{99}$$

Considering (98) as a function of z and r for all fixed x, y, t, and s again, we have $L(x,y,t,s) \equiv 0$. This completes the proof. □

In the following lemma we assume that $\Psi : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ is a continuous function such that

$$\psi(x) := \lim_{t\to 0^+} \Psi(x,t) \tag{100}$$

exists and satisfies the condition $\psi(0) = 0$.

Lemma 13. *Let $U, V : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be continuous functions satisfying*

$$\begin{aligned}
&|V(x-y,t+s) - V(x,t)V(y,s) - U(x,t)U(y,s)|\\
&\quad \le \Psi(y,s)
\end{aligned} \tag{101}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, and there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $M > 0$ such that

$$|\lambda_1 U(x,t) - \lambda_2 V(x,t)| \le M. \tag{102}$$

Then (U, V) satisfies one of the followings:

(i) *U and V are bounded functions in $\mathbb{R}^n \times (0,1)$,*

(ii) *$\pm iU(x,t) = V(x,t) - e^{ia\cdot x - bt}$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{C}$, and $g(x) := \lim_{t\to 0^+} V(x,t)$ is a continuous function; in particular, there exists $\delta : (0,\infty) \to [0,\infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that*

$$\left|V(x,t) - g(x)e^{-bt}\right| \le \delta(t) \tag{103}$$

for all $x \in \mathbb{R}^n$, $t > 0$, and g satisfies

$$|g(x) - \cos(a\cdot x)| \le \frac{1}{2}\psi(x) \tag{104}$$

for all $x \in \mathbb{R}^n$.

Proof. If U is bounded, then in view of inequality (100), for each y, s, $V(x+y,t+s) - V(x,t)V(-y,s)$ is also bounded. It follows from Lemma 5 that V is (101). If V is bounded, case (i) follows. If V is a nonzero exponential function, then by the continuity of V we have

$$V(x,t) = e^{c\cdot x + bt} \tag{105}$$

for some $c \in \mathbb{C}^n$, $b \in \mathbb{C}$. Putting (105) in (101) and using the triangle inequality we have for some $d \ge 0$

$$\left|e^{c\cdot x}e^{b(t+s)}\left(e^{-c\cdot y} - e^{c\cdot y}\right)\right| \le \Psi(y,s) + d \tag{106}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. In view of (106) it is easy to see that $c = ia$, $a \in \mathbb{R}^n$. Thus $V(x,t)$ is bounded on $\mathbb{R}^n \times (0,1)$. If U is unbounded; then in view of (101) V is also unbounded, hence $\lambda_1\lambda_2 \ne 0$ and

$$U(x,t) = \lambda V(x,t) + R(x,t) \tag{107}$$

for some $\lambda \ne 0$ and a bounded function R. Putting (107) in (101), replacing y by $-y$, and using the triangle inequality we have

$$\begin{aligned}
&\left|V(x+y,t+s) - V(x,t)\left(\left(\lambda^2+1\right)V(-y,s) + \lambda R(-y,s)\right)\right|\\
&\quad \le |(\lambda V(-y,s) + R(-y,s))R(x,t)| + \Psi(-y,s).
\end{aligned} \tag{108}$$

From Lemma 5 we have

$$\left(\lambda^2+1\right)V(y,s) + \lambda R(y,s) = m(y,s) \tag{109}$$

for some exponential function m. From (107) and (109), m is continuous, and we have

$$m(x,t) = e^{c\cdot x + bt} \tag{110}$$

for some $c \in \mathbb{C}^n$, $b \in \mathbb{C}$. If $\lambda^2 \ne -1$, we have

$$U = \frac{\lambda m + R}{\lambda^2 + 1}, \qquad V = \frac{m - \lambda R}{\lambda^2 + 1}. \tag{111}$$

Putting (111) in (101), multiplying $|\lambda^2 + 1|$ in the result, and using the triangle inequality we have, for some $d \ge 0$,

$$|m(x,t)(m(-y,s) - m(y,s))| \le \left|\lambda^2 + 1\right|\Psi(y,s) + d \tag{112}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Since m is unbounded, we have

$$m(y, s) = m(-y, s) \tag{113}$$

for all $y \in \mathbb{R}$ and $s > 0$. Thus it follows that $m(x,t) = e^{bt}$ and that U, V are bounded in $\mathbb{R}^n \times (0, 1)$. If $\lambda^2 = -1$, we have

$$U = \pm i (V - m), \tag{114}$$

where m is a bounded exponential function. Putting (114) in (101) we have

$$\begin{aligned} &\left|V(x-y, t+s) - V(x,t) m(y,s) - V(y,s) m(x,t)\right. \\ &\quad \left.+ m(x,t) m(y,s)\right| \leq \Psi(y,s) \end{aligned} \tag{115}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Since m is a bounded continuous function, we have

$$m(x,t) = e^{ia\cdot x - bt} \tag{116}$$

for some $a \in \mathbb{R}^n$, $b \in \mathbb{C}$ with $\Re b \geq 0$.

Similarly as in the proof of Lemma 9, by (101) and the continuity of V, it is easy to see that

$$\limsup_{t\to 0^+} V(x,t) := g(x) \tag{117}$$

exists. Putting $x = y = 0$ in (115), multiplying $|e^{bt}|$ in both sides of the result, and using the triangle inequality we have

$$\left|V(0,s) - e^{-bs}\right| \leq \left|e^{bt}\right| \left(\left|V(0, t+s) - V(0,t) e^{-bs}\right| + \Psi(0,s)\right) \tag{118}$$

for all $t, s > 0$. Letting $s \to 0^+$ in (118) we have

$$\lim_{t\to 0^+} V(0,t) = 1. \tag{119}$$

Putting $y = 0$, fixing x, letting $t \to 0^+$ in (115) so that $V(x,t) \to g(x)$, and using the triangle inequality we have

$$\left|V(x,s) - g(x) e^{-bs}\right| \leq \left|V(0,s) - e^{-bs}\right| + \Psi(0,s) \tag{120}$$

for all $x \in \mathbb{R}^n$, $s > 0$. Letting $s \to 0^+$ in (120) we have

$$\lim_{s\to 0^+} V(x,s) = g(x) \tag{121}$$

for all $x \in \mathbb{R}^n$. The continuity of g follows from (120). Letting $t, s \to 0^+$ in (115) we have

$$\left|g(x-y) - g(x) e^{ia\cdot y} - g(y) e^{ia\cdot x} + e^{ia\cdot(x+y)}\right| \leq \psi(y) \tag{122}$$

for all $x, y \in \mathbb{R}^n$. Replacing y by x in (122) and dividing the result by $2e^{ia\cdot x}$ we have

$$\left|g(x) - \cos(a\cdot x)\right| \leq \frac{1}{2}\psi(x). \tag{123}$$

From (114), (116), (120) and (123) we get (ii). This completes the proof. □

Theorem 14. *Let $u, v \in \mathcal{G}'$ satisfy (20). Then (u, v) satisfies one of the followings:*

(i) *u and v are bounded measurable functions,*

(ii) $v(x) = \cos(a\cdot x) + r(x)$, $\pm u(x) = \sin(a \cdot x) + ir(x)$ *for some $a \in \mathbb{R}^n$, where $r(x)$ is a continuous function satisfying*

$$|r(x)| \leq \frac{1}{2}\psi(x) \tag{124}$$

for all $x \in \mathbb{R}^n$,

(iii) $v(x) = \cos(c\cdot x)$ *and* $u(x) = \sin(c\cdot x)$ *for some $c \in \mathbb{C}^n$.*

Proof. Similarly as in the proof of Theorem 10 convolving in (20) the tensor product $E_t(x)E_s(y)$ we obtain the inequality

$$\begin{aligned} &\left|V(x-y, t+s) - V(x,t) V(y,s) - U(x,t) U(y,s)\right| \\ &\quad \leq \Psi(y,s) \end{aligned} \tag{125}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where U, V are the Gauss transforms of u, v, respectively, and

$$\begin{aligned} \Psi(y,s) &= \int \psi(\eta) E_t(x-\xi) E_s(y-\eta)\, d\xi\, d\eta \\ &= \int \psi(\eta) E_s(\eta - y)\, d\eta = (\psi * E_s)(y). \end{aligned} \tag{126}$$

By Lemma 12 there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $M > 0$ such that

$$\left|\lambda_1 U(x,t) - \lambda_2 V(x,t)\right| \leq M, \tag{127}$$

or else U, V satisfy

$$V(x-y, t+s) - V(x,t) V(y,s) - U(x,t) U(y,s) = 0 \tag{128}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$.

Firstly we assume that (127) holds. Letting $t \to 0^+$ in (i) of Lemma 13, by Lemma 6, the initial values u, v of $U(x,t), V(x,t)$ as $t \to 0^+$ are bounded measurable functions, respectively, which gives case (i). Using the same approach of the proof of case (iii) of Theorem 10, we have $v = g$ in $\mathcal{G}'$. It follows from (104) that

$$v(x) = \cos(a\cdot x) + r(x), \tag{129}$$

where $r(x)$ is a continuous function satisfying

$$|r(x)| \leq \frac{1}{2}\psi(x) \tag{130}$$

for all $x \in \mathbb{R}^n$. Letting $t \to 0^+$ in (ii) of Lemma 13 we have

$$\pm iu(x) = v(x) - e^{ia\cdot x}. \tag{131}$$

Putting (129) in (131) we have

$$\pm u(x) = \sin(a\cdot x) + ir(x). \tag{132}$$

Secondly we assume that (128) holds. Letting $t, s \to 0^+$ in (127) we have

$$v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y = 0. \tag{133}$$

By Lemma 7 the solution of (133) satisfies (i) or (iii). This completes the proof. □

Every infraexponential function f of order 2 defines an element of $\mathscr{G}'(\mathbb{R}^n)$ via the correspondence

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)\,dx \tag{134}$$

for $\varphi \in \mathscr{G}$. Thus as a direct consequence of Corollary 11 and Theorem 14 we have the followings.

Corollary 15. *Let $0 < p < 1$ or $p > 1$. Suppose that f, g are infraexponential functions of order 2 satisfying the inequality*

$$\left| f(x - y) - f(x) g(y) + g(x) f(y) \right| \leq \epsilon |x|^p \tag{135}$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. Then (f, g) satisfies one of the following:

(i) *$f(x) = 0$, almost everywhere $x \in \mathbb{R}^n$, and g is arbitrary,*

(ii) *f and g are bounded in almost everywhere,*

(iii) *$f(x) = f_0(x)$, $g(x) = \lambda f_0(x) + e^{ic\cdot x}$ for almost everywhere $x \in \mathbb{R}^n$, where $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, and f_0 is a continuous function satisfying the condition; there exists $d \geq 0$*

$$\left| f_0(x) \right| \leq \epsilon |x|^p + d \tag{136}$$

for all $x \in \mathbb{R}^n$,

(iv) *$f(x) = f_0(x)$, $g(x) = \lambda f_0(x) + 1$ for a.e. $x \in \mathbb{R}^n$, where $\lambda \in \mathbb{C}$ and f_0 is a continuous function satisfying the condition; there exists $a \in \mathbb{C}^n$ such that*

$$\left| f_0(x) - a \cdot x \right| \leq \frac{2\epsilon |x|^p}{|2^p - 2|} \tag{137}$$

for all $x \in \mathbb{R}^n$,

(v) *$f(x) = \lambda \sin(c \cdot x)$, $g(x) = \cos(c \cdot x) + \lambda \sin(c \cdot x)$ for a.e. $x \in \mathbb{R}^n$, where $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.*

Corollary 16. *Suppose that f, g are infraexponential functions of order 2 satisfying the inequality*

$$\left| g(x - y) - g(x) g(y) - f(x) f(y) \right| \leq \epsilon |y|^p \tag{138}$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. Then (f, g) satisfies one of the followings:

(i) *f and g are bounded in almost everywhere,*

(ii) *there exists $a \in \mathbb{R}^n$ such that*

$$\left| g(x) - \cos(a \cdot x) \right| \leq \frac{1}{2} \epsilon |x|^p, \tag{139}$$

$$\left| f(x) \pm \sin(a \cdot x) \right| \leq \frac{1}{2} \epsilon |x|^p \tag{140}$$

for almost every $x \in \mathbb{R}^n$,

(iii) *$g(x) = \cos(c \cdot x)$ and $f(x) = \sin(c \cdot x)$ for a.e. $x \in \mathbb{R}^n$, where $c \in \mathbb{C}^n$.*

Remark 17. Taking the growth of $u = e^{c\cdot x}$ as $|x| \to \infty$ into account, $u \in \mathscr{S}'(\mathbb{R}^n)$ only when $c = ia$ for some $a \in \mathbb{R}^n$. Thus Theorems 10 and 14 are reduced to the following:

Corollary 18. *Let $u, v \in \mathscr{S}'$ satisfy* (19). *Then (u, v) satisfies one of the followings*:

(i) *$u = 0$, and v is arbitrary,*

(ii) *u and v are bounded measurable functions,*

(iii) *$v(x) = \lambda u(x) + e^{ic\cdot x}$ for some $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, where u is a continuous function satisfying the condition; there exists $d \geq 0$*

$$|u(x)| \leq \psi(-x) + d \tag{141}$$

for all $x \in \mathbb{R}^n$,

(iv) *$v(x) = \lambda u(x) + 1$ for some $\lambda \in \mathbb{C}$, where u is a continuous function satisfying one of the following conditions; there exists $a_1 \in \mathbb{C}^n$ such that*

$$\left| u(x) - a_1 \cdot x \right| \leq \Phi_1(x) \tag{142}$$

for all $x \in \mathbb{R}^n$, or there exists $a_2 \in \mathbb{C}^n$ such that

$$\left| u(x) - a_2 \cdot x \right| \leq \Phi_2(x) \tag{143}$$

for all $x \in \mathbb{R}^n$.

Corollary 19. *Let $u, v \in \mathscr{S}'$ satisfy* (20). *Then (u, v) satisfies one of the followings:*

(i) *u and v are bounded measurable functions,*

(ii) *$v(x) = \cos(a \cdot x) + r(x)$, $\pm u(x) = \sin(a \cdot x) + ir(x)$ for some $a \in \mathbb{R}^n$, where $r(x)$ is a continuous function satisfying*

$$|r(x)| \leq \frac{1}{2} \psi(x) \tag{144}$$

for all $x \in \mathbb{R}^n$.

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Research Article

Some Remarks on the Extended Hartley-Hilbert and Fourier-Hilbert Transforms of Boehmians

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We obtain generalizations of Hartley-Hilbert and Fourier-Hilbert transforms on classes of distributions having compact support. Furthermore, we also study extension to certain space of Lebesgue integrable Boehmians. New characterizing theorems are also established in an adequate performance.

1. Introduction

The classical theory of integral transforms and their applications have been studied for a long time, and they are applied in many fields of mathematics. Later, after [1], the extension of classical integral transformations to generalized functions has comprised an active area of research. Several integral transforms are extended to various spaces of generalized functions, distributions [2], ultradistributions, Boehmians [3, 4], and many more.

In recent years, many papers are devoted to those integral transforms which permit a factorization identity (of Fourier convolution type) such as Fourier transform, Mellin transform, Laplace transform, and few others that have a lot of attraction, the reason that the theory of integral transforms, generally speaking, became an object of study of integral transforms of Boehmian spaces.

The Hartley transform is an integral transformation that maps a real-valued temporal or spacial function into a real-valued frequency function via the kernel

$$k(v; x) = \operatorname{cas}(vx). \tag{1}$$

This novel symmetrical formulation of the traditional Fourier transform, attributed to Hartley 1942, leads to a parallelism that exists between a function of the original variable and that of its transform. In any case, signal and systems analysis and design in the frequency domain using the Hartley transform may be deserving an increased awareness due to the necessity of the existence of a fast algorithm that can substantially lessen the computational burden when compared to the classical complex-valued fast Fourier transform.

The Hartley transform of a function $f(x)$ can be expressed as either [5]

$$\mathscr{A}(v) =: \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \operatorname{cas}(vx)\, dx \tag{2}$$

or

$$\mathscr{A}(f) =: \int_{-\infty}^{\infty} f(x) \operatorname{cas}(2\pi f x)\, dx, \tag{3}$$

where the angular or radian frequency variable v is related to the frequency variable f by $v = 2\pi f$ and

$$\mathscr{A}(f) = \sqrt{2\pi}\mathscr{A}(2\pi f) = \sqrt{2\pi}\mathscr{A}(v). \tag{4}$$

The integral kernel, known as cosine-sine function, is defined as

$$\operatorname{cas}(vx) = \cos vx + \sin(vx). \tag{5}$$

Inverse Hartley transform may be defined as either

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{A}(v) \operatorname{cas}(vx)\, dv \tag{6}$$

or

$$f(x) = \int_{-\infty}^{\infty} \mathscr{A}(f)\operatorname{cas}(2\pi f x)\, df. \tag{7}$$

The theory of convolutions of integral transforms has been developed for a long time and is applied in many fields of mathematics. Historically, the convolution product [2]

$$(f * g)(y) = \int_{-\infty}^{\infty} f(x)\, g(x-y)\, dy \tag{8}$$

has a relationship with the Fourier transform with the factorization property

$$\mathscr{F}(f * g)(y) = \mathscr{F}(f)(y)\,\mathscr{F}(g)(y). \tag{9}$$

The more complicated convolution theorem of Hartley transforms, compared to that of Fourier transforms, is that

$$\mathscr{A}(f * g)(y) = \frac{1}{2}\mathscr{G}(\mathscr{A}f \times \mathscr{A}g)(y), \tag{10}$$

where

$$\begin{aligned}\mathscr{G}(f * g)(y) = {} & f(y)\,g(y) + f(y)\,g(-y) \\ & + f(y)\,g(y) - f(-y)\,g(-y).\end{aligned} \tag{11}$$

Some properties of Hartley transforms can be listed as follows.

(i) Linearity: if f and g are real functions then

$$\mathscr{A}(af + bg)(y) = a\mathscr{A}(f)(y) + b\mathscr{A}(g)(y), \quad a, b \in \mathscr{R}. \tag{12}$$

(ii) Scaling: if f is a real function then

$$\int_{-\infty}^{\infty} f(\alpha\varsigma)\operatorname{cas}(2\pi y\varsigma)\, d\varsigma = \frac{1}{a}(\mathscr{A}f)\left(\frac{y}{a}\right). \tag{13}$$

2. Distributional Hartley-Hilbert Transform of Compact Support

The Hilbert transform via the Hartley transform is defined by [6, 7]

$$\mathscr{B}^{\mathscr{A}}(y) = -\frac{1}{\pi}\int_{0}^{\infty} \left(\mathscr{A}^{o}(x)\cos(xy) + \mathscr{A}^{e}(x)\sin(xy)\right) dx, \tag{14}$$

where

$$\begin{aligned}\mathscr{A}^{o}(x) &= \frac{\mathscr{A}(x) - \mathscr{A}(-x)}{2}, \\ \mathscr{A}^{e}(x) &= \frac{\mathscr{A}(x) + \mathscr{A}(-x)}{2}\end{aligned} \tag{15}$$

are the respective odd and even components of (2).

We denote, $\mathscr{C}(\mathscr{R})$, $\mathscr{C}(\mathscr{R}) = \mathscr{C}$, the space of smooth functions and $\mathscr{C}'(\mathscr{R})$, $\mathscr{C}'(\mathscr{R}) = \mathscr{C}'$, the strong dual of $\mathscr{C}$ of distributions of compact support over $\mathscr{R}$.

Following is the convolution theorem of $\mathscr{B}^{\mathscr{A}}$.

Theorem 1 (Convolution Theorem). *Let f and $g \in \mathscr{C}$ then*

$$\mathscr{B}^{\mathscr{A}}(f * g)(y) = \int_{0}^{\infty} \left(k_1(x)\cos(yx) + k_2(x)\sin(yx)\right) dx, \tag{16}$$

where

$$\begin{aligned}k_1(x) &= \mathscr{A}^{e} f(x)\,\mathscr{A}^{o} g(x) + \mathscr{A}^{o} f(x)\,\mathscr{A}^{e} g(x), \\ k_2(x) &= \mathscr{A}^{e} f(x)\,\mathscr{A}^{e} g(x) - \mathscr{A}^{o} f(x)\,\mathscr{A}^{o} g(x).\end{aligned} \tag{17}$$

Proof. To prove this theorem it is sufficient to establish that

$$k_1(x) = \mathscr{A}^{o}(f * g)(x), \tag{18}$$

$$k_2(x) = \mathscr{A}^{e}(f * g)(x). \tag{19}$$

Therefore, we have

$$\begin{aligned}&\mathscr{A}^{o}(f * g)(x) \\ &\quad = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\gamma)\, g(y-\gamma)\, dr\right)\operatorname{cas}(xy)\, dy \\ &\quad = \int_{-\infty}^{\infty} f(\gamma)\int_{-\infty}^{\infty} g(y-\gamma)\operatorname{cas}(xy)\, dy dr.\end{aligned} \tag{20}$$

The substitution $y - \gamma = z$ and using of (1) together with Fubini theorem imply

$$\begin{aligned}&\mathscr{A}^{o}(f * g)(x) \\ &\quad = \int_{-\infty}^{\infty} f(\gamma)\int_{-\infty}^{\infty} g(z)\left(\cos(x(z+\gamma)) + \sin(x(z+\gamma))\right) \\ &\qquad \times dz dr.\end{aligned} \tag{21}$$

By invoking the formulae

$$\begin{aligned}\cos(x(z+\gamma)) &= \cos(xz)\cos(x\gamma) - \sin(xz)\sin(x\gamma), \\ \sin(x(z+\gamma)) &= \sin(xz)\cos(x\gamma) + \cos(xz)\sin(x\gamma),\end{aligned} \tag{22}$$

then (18) follows from simple computation. Proof of (19) has a similar technique. Hence, the theorem is completely proved. □

It is of interest to know that $\cos(xy)$ and $\sin(xy)$ are members of $\mathscr{C}$ and, therefore, $\mathscr{A}^{o} f, \mathscr{A}^{e} f \in \mathscr{C}'$. This leads to the following statement.

Definition 2. Let $f \in \mathscr{C}'$ then we define the distributional Hartley-Hilbert transform of f as

$$\widehat{\mathscr{B}^{\mathscr{A}}} f(y) = \langle \mathscr{A}^{o} f(x), \cos(xy)\rangle + \langle \mathscr{A}^{e} f(x), \sin(xy)\rangle. \tag{23}$$

The extended transform $\widehat{\mathscr{B}^{\mathscr{A}}} f$ is clearly well defined for each $f \in \mathscr{C}'$.

Theorem 3. *The distributional Hartley-Hilbert transform $\widehat{\mathscr{B}^{\mathscr{A}}} f$ is linear.*

Proof. Let $f, g \in \mathscr{C}'$ then their components $\mathscr{A}^e f, \mathscr{A}^o f, \mathscr{A}^e g$, $\mathscr{A}^o g \in \mathscr{C}'$. Hence,

$$\widehat{\mathscr{B}^{\mathscr{A}}}(f+g)(y) = \langle \mathscr{A}^o (f+g)(x), \cos(xy) \rangle + \langle \mathscr{A}^e (f+g)(x), \sin(xy) \rangle . \tag{24}$$

By factoring and rearranging components we get that

$$\widehat{\mathscr{B}^{\mathscr{A}}}(f+g)(y) = \widehat{\mathscr{B}^{\mathscr{A}}} f(y) + \widehat{\mathscr{B}^{\mathscr{A}}} g(y). \tag{25}$$

Furthermore,

$$\widehat{\mathscr{B}^{\mathscr{A}}}(kf)(y) = \langle k\mathscr{A}^o f(x), \cos(xy) \rangle + \langle k\mathscr{A}^e f(x), \sin(xy) \rangle . \tag{26}$$

Hence,

$$\widehat{\mathscr{B}^{\mathscr{A}}}(kf)(y) = k\widehat{\mathscr{B}^{\mathscr{A}}} f(y). \tag{27}$$

This completes the proof of the theorem. □

Theorem 4. *Let $f \in \mathscr{C}'$ then $\widehat{\mathscr{B}^{\mathscr{A}}} f$ is a continuous mapping on $\mathscr{C}'$.*

Proof. Let $f_n, f \in \mathscr{C}'$, $n \in \mathscr{N}$ and $f_n \to f$ as $n \to \infty$. Then,

$$\begin{aligned}\widehat{\mathscr{B}^{\mathscr{A}}} f_n(y) &= \langle \mathscr{A}^o f_n(x), \cos(xy) \rangle + \langle \mathscr{A}^e f_n(x), \sin(xy) \rangle \\ &\longrightarrow \langle \mathscr{A}^o f(x), \cos(xy) \rangle + \langle \mathscr{A}^e f(x), \sin(xy) \rangle \\ &= \widehat{\mathscr{B}^{\mathscr{A}}} f(y) \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{28}$$

Hence we have the following theorem. □

Theorem 5. *The mapping $\widehat{\mathscr{B}^{\mathscr{A}}} f$ is one-to-one.*

Proof. Let $f, g \in \mathscr{C}'$ and that $\widehat{\mathscr{B}^{\mathscr{A}}} f = \widehat{\mathscr{B}^{\mathscr{A}}} g$ then, using (23) we get

$$\langle \mathscr{A}^o f(x), \cos xy \rangle + \langle \mathscr{A}^e f(x), \sin xy \rangle = \langle \mathscr{A}^o g(x), \cos xy \rangle + \langle \mathscr{A}^e g(x), \sin xy \rangle . \tag{29}$$

Basic properties of inner product implies

$$\langle \mathscr{A}^o f(x) - \mathscr{A}^o g(x), \cos(xy) \rangle + \langle \mathscr{A}^e f(x) - \mathscr{A}^e g(x), \sin(xy) \rangle = 0. \tag{30}$$

Hence,

$$\mathscr{A}^o f(x) = \mathscr{A}^o g(x), \qquad \mathscr{A}^e f(x) = \mathscr{A}^e g(x). \tag{31}$$

Therefore,

$$\begin{aligned}\mathscr{A} f(x) &= \mathscr{A}^o f(x) + \mathscr{A}^e f(x) \\ &= \mathscr{A}^o g(x) + \mathscr{A}^e g(x) = \mathscr{A} g(x)\end{aligned} \tag{32}$$

for all x. This completes the proof of the theorem. □

Theorem 6. *Let $f \in \mathscr{C}'$ then f is analytic and*

$$\mathscr{D}_y^k \widehat{\mathscr{B}^{\mathscr{A}}} f(y) = \left\langle \mathscr{A}^o f(x), \mathscr{D}_y^k \cos(xy) \right\rangle + \left\langle \mathscr{A}^e f(x), \mathscr{D}_y^k \sin(xy) \right\rangle . \tag{33}$$

Proof of this theorem is analogous to that of the previous theorem and is thus avoided.

Denote by δ the dirac delta function then it is easy to see that

$$\mathscr{A}^e \delta(y) = 1, \qquad \mathscr{A}^e \delta(y) = 0. \tag{34}$$

3. Lebesgue Space of Boehmians for Hartley-Hilbert Transforms

The original construction of Boehmians produce a concrete space of generalized functions. Since the space of Boehmians was introduced, many spaces of Boehmians were defined. In references, we list selected papers introducing different spaces of Boehmians. One of the main motivations for introducing different spaces of Boehmians was the generalization of integral transforms. The idea requires a proper choice of a space of functions for which a given integral transform is well defined, a choice of a class of delta sequences that is transformed by that integral transform to a well-behaved class of approximate identities, and finally a convolution product that behaves well under the transform. If these conditions are met, the transform has usually an extension to the constructed space of Boehmians and the extension has desirable properties. For general construction of Boehmians, see [8–12].

Let $\mathscr{D}$ be the space of test functions of bounded support. By delta sequence, we mean a subset of $\mathscr{D}$ of sequences (δ_n) such that

$$\begin{gathered}\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1, \\ \|\delta_n\| = \int_{-\infty}^{\infty} |\delta_n(x)|\, dx < \mathscr{M}, \quad 0 < \mathscr{M} \in \mathscr{R}, \\ \lim_{n\to\infty} \int_{|x|>\varepsilon} |\delta_n(x)|\, dx = 0 \quad \text{for each } \varepsilon > 0,\end{gathered} \tag{35}$$

where $\varepsilon(\delta_n)(x) = \{x \in \mathscr{R} : \delta_n(x) \neq 0\}$.

The set of all such delta sequences is usually denoted as Δ. Each element in Δ corresponds to the dirac delta function δ, for large values of n.

Proposition 7. *Let $(\delta_n) \in \Delta$ then*

$$\begin{aligned}\mathscr{A}^e \delta_n(y) &= \int_{-\infty}^{\infty} \delta_n(x) \cos(xy)\, dx \longrightarrow 1 \quad \textit{as } n \longrightarrow \infty, \\ \mathscr{A}^o \delta_n(y) &= \int_{-\infty}^{\infty} \delta_n(x) \sin(xy)\, dx \longrightarrow 0 \quad \textit{as } n \longrightarrow \infty.\end{aligned} \tag{36}$$

Let $\mathscr{L}^1(\mathscr{R})$, $\mathscr{L}^1(\mathscr{R}) = \mathscr{L}^1$, be the space of complex valued Lebesgue integrable functions. From Proposition 7 we establish the following theorem.

Theorem 8. *Let $f \in \mathscr{L}^1$ then $\mathscr{B}^{\mathscr{A}}(f * \delta_n)(y) \to \mathscr{B}^{\mathscr{A}} f(y)$ as $n \to \infty$.*

Proof. For $f \in \mathscr{L}^1$, $(\delta_n) \in \Delta$, then using of (14) implies

$$\mathscr{B}^{\mathscr{A}}(f * \delta_n)(y) = \int_{-\infty}^{\infty} (\mathscr{A}^o (f * \delta_n)(x) \cos xy + \mathscr{A}^e (f * \delta_n)(x) \sin xy)\, dx. \tag{37}$$

Since

$$(f * \delta_n)(\zeta) = \int_{-\infty}^{\infty} f(t)\, \delta_n(\zeta - t)\, dt \longrightarrow f(\zeta) \tag{38}$$

as $n \to \infty$, we see that

$$\begin{aligned} \mathscr{A}^o (f * \delta_n)(x) &= \int_{-\infty}^{\infty} (f * \delta_n)(\zeta) \sin x\zeta d\zeta \\ &= \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} \delta_n(\zeta - t) \sin(x\zeta)\, d\zeta dt \\ &\longrightarrow \int_{-\infty}^{\infty} f(t) \sin(xt)\, dt \\ &= \mathscr{A}^o(f)(x). \end{aligned} \tag{39}$$

Similarly,

$$\mathscr{A}^e (f * \delta_n)(x) \longrightarrow \mathscr{A}^e(f)(x) \quad \text{as } n \longrightarrow \infty. \tag{40}$$

Therefore, invoking the above equations in (37), we get $\mathscr{B}^{\mathscr{A}}(f * \delta_n)(y) \to \mathscr{B}^{\mathscr{A}} f(y)$ as $n \to \infty$. Hence the theorem. □

Denote by $\rho_{\mathscr{L}^1}$ the space of integrable Boehmians, then $\rho_{\mathscr{L}^1}$ is a convolution algebra when multiplication by scalar, addition, and convolution is defined as [9]

$$\begin{gathered} k\left[\frac{f_n}{\delta_n}\right] = \left[\frac{kf_n}{\delta_n}\right], \\ \left[\frac{f_n}{\delta_n}\right] + \left[\frac{g_n}{\gamma_n}\right] = \left[\frac{f_n * \gamma_n + g_n * \delta_n}{\delta_n * \gamma_n}\right], \\ \left[\frac{f_n}{\delta_n}\right] * \left[\frac{g_n}{\gamma_n}\right] = \left[\frac{f_n * g_n}{\delta_n * \gamma_n}\right]. \end{gathered} \tag{41}$$

Each function $f \in \mathscr{L}^1$ is identified with the Boehmian $[f * \delta_n/\delta_n]$. Also, $[f_n/\delta_n] * \delta_n = f_n \in \mathscr{L}^1$, for every $n \in \mathscr{N}$. Since $[\delta_n/\delta_n]$ corresponds to Dirac delta distribution δ, the kth-derivative of each $\rho \in \rho_{\mathscr{L}^1}$ is defined as

$$\mathscr{D}^k \rho = \rho * \mathscr{D}^k \delta. \tag{42}$$

The integral of a Boehmian $\rho = [f_n/\delta_n] \in \rho_{\mathscr{L}^1}$ is defined as [11]

$$\int_{-\infty}^{\infty} \rho(x)\, dx = \int_{-\infty}^{\infty} f_1(x)\, dx. \tag{43}$$

It is of great interest to observe the following example.

Example 9. Every infinitely smooth function $f(x) \in \mathscr{L}^1$ such that $\mathscr{D}^k f(x) \notin \mathscr{L}^1$ is integrable as Boehmian but not integrable as function.

The following has importance in the sense of analysis.

Theorem 10. *Let $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then the sequence*

$$\mathscr{B}^{\mathscr{A}}(f_n)(y) = \int_{-\infty}^{\infty} (\mathscr{A}^o f_n(x) \cos(xy) + \mathscr{A}^e f_n(x) \sin(xy))\, dx \tag{44}$$

converges uniformly on each compact subset $\mathscr{K}$ of $\mathscr{R}$.

Proof. By aid of the Theorem 8 and the concept of quotient of sequences, we have,

$$\begin{aligned} \mathscr{B}^{\mathscr{A}}(f_n)(y) &= \mathscr{B}^{\mathscr{A}}\left(f_n * \frac{\delta_k}{\delta_k}\right)(y) \\ &= \mathscr{B}^{\mathscr{A}}\left(\frac{f_n * \delta_k}{\delta_k}\right)(y) \\ &= \mathscr{B}^{\mathscr{A}}\left(\frac{f_k}{\delta_k} * \delta_n\right)(y) \\ &\longrightarrow \mathscr{B}^{\mathscr{A}} \frac{f_k}{\delta_k}(y) \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{45}$$

where convergence ranges over compact subsets of $\mathscr{R}$. The theorem is completely proved. □

Let $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then by virtue of Theorem 10 we define the Hartley-Hilbert transform of the Lebesgue Boehmian $[f_n/\delta_n]$ as

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n}{\delta_n}\right] = \lim_{n\to\infty} \mathscr{B}^{\mathscr{A}} f_n. \tag{46}$$

on compact subsets of $\mathscr{R}$.

The next objective is to establish that our definition is well defined. Let $[f_n/\delta_n] = [g_n/\gamma_n]$ in $\rho_{\mathscr{L}^1}$, then

$$f_n * \gamma_m = g_m * \delta_n, \quad \text{for every } m, n \in \mathscr{N}. \tag{47}$$

Hence, applying the Hartley-Hilbert transform to both sides of the above equation and using the concept of quotients of sequences imply

$$\mathscr{B}^{\mathscr{A}}(f_n * \gamma_m) = \mathscr{B}^{\mathscr{A}}(g_m * \delta_n) = \mathscr{B}^{\mathscr{A}}(g_n * \delta_m). \tag{48}$$

Thus, in particular, for $n = m$, and considering Theorem 3, we get

$$\lim_{n\to\infty} \mathscr{B}^{\mathscr{A}} f_n = \lim_{n\to\infty} \mathscr{B}^{\mathscr{A}} g_n. \tag{49}$$

Hence,

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n}{\delta_n}\right] = \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{g_n}{\gamma_n}\right]. \tag{50}$$

Definition (46) is therefore well defined.

Theorem 11. *The generalized transform $\widetilde{\mathscr{B}^{\mathscr{A}}}$ is linear.*

Proof. Let $\rho_1 = [f_n/\delta_n]$ and $\rho_2 = [g_n/\gamma_n]$ be arbitrary in $\rho_{\mathscr{L}^1}$ and $\alpha \in \mathbb{C}$ then $\rho_1 + \rho_2 = [(f_n * \gamma_n + g_n * \delta_n)/\delta_n * \gamma_n]$. Hence, employing (46) yields

$$\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho_1 + \rho_2) = \lim_{n\to\infty}\left(\mathscr{B}^{\mathscr{A}}(f_n * \gamma_n) + \mathscr{B}^{\mathscr{A}}(g_n * \delta_n)\right). \tag{51}$$

By Theorem 8, we get

$$\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho_1 + \rho_2) = \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n + \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} g_n. \tag{52}$$

Hence,

$$\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho_1 + \rho_2) = \widetilde{\mathscr{B}^{\mathscr{A}}}\rho_1 + \widetilde{\mathscr{B}^{\mathscr{A}}}\rho_2. \tag{53}$$

Also, for each complex number α, we have

$$\begin{aligned}\widetilde{\mathscr{B}^{\mathscr{A}}}(\alpha\rho_1) &= \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{\alpha f_n}{\delta_n}\right] \\ &= \alpha\lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n \\ &= \alpha\widetilde{\mathscr{B}^{\mathscr{A}}}\rho_1.\end{aligned} \tag{54}$$

Hence we have the following theorem. □

Theorem 12. *Let $\rho \in \rho_{\mathscr{L}^1}$ and $(\epsilon_n) \in \Delta$, then*

$$\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho * \epsilon_n) = \widetilde{\mathscr{B}^{\mathscr{A}}}\rho = \widetilde{\mathscr{B}^{\mathscr{A}}}(\epsilon_n * \rho). \tag{55}$$

Proof. Let $\rho = [f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then $\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho * \epsilon_n) = \widetilde{\mathscr{B}^{\mathscr{A}}}[f_n * \epsilon_n/\delta_n] = \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}}(f_n * \epsilon_n)$.

Hence, $\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho * \epsilon_n) = \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n = \widetilde{\mathscr{B}^{\mathscr{A}}}\rho$.

Similarly, we proceed for $\widetilde{\mathscr{B}^{\mathscr{A}}}\rho = \widetilde{\mathscr{B}^{\mathscr{A}}}(\epsilon_n * \rho)$.

This completes the theorem. The following theorem is obvious. □

Theorem 13. *If $\widetilde{\mathscr{B}^{\mathscr{A}}}\rho_1 = 0$, then $\rho_1 = 0$.*

Theorem 14. *The Hartley-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{A}}}$ is continuous with respect to the δ-convergence.*

Proof. Let $\rho_n \xrightarrow{\delta} \rho$ in $\rho_{\mathscr{L}^1}$ as $n \to \infty$, then we show that $\widetilde{\mathscr{B}^{\mathscr{A}}}\rho_n \xrightarrow{\delta} \widetilde{\mathscr{B}^{\mathscr{A}}}\rho$ as $n \to \infty$. Using ([11, Theorem 2.6]), we find $f_{n,k}, f_k \in \mathscr{L}^1$, $(\delta_k) \in \Delta$ such that $[f_{n,k}/\delta_k] = \rho_n$, $[f_k/\delta_k] = \rho$ and $f_{n,k} \to f_k$ as $n \to \infty$, $k \in \mathscr{N}$.

Applying the Hartley-Hilbert transform for both sides implies $\mathscr{B}^{\mathscr{A}} f_{n,k} \to \mathscr{B}^{\mathscr{A}} f_k$ in the space of continuous functions. Therefore, considering limits we get

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_{n,k}}{\delta_k}\right] \longrightarrow \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_k}{\delta_k}\right]. \tag{56}$$

This completes the proof of the theorem. □

Theorem 15. *The Hartley-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{A}}}$ is continuous with respect to the Δ-convergence.*

Proof. Let $\rho_n \xrightarrow{\Delta} \rho$ as $n \to \infty$ in $\rho_{\mathscr{L}^1}$, then there is $f_n \in \mathscr{L}^1$ and $\delta_n \in \Delta$ such that

$$(\rho_n - \rho) * \delta_n = \left[\frac{f_n * \delta_n}{\delta_k}\right], \quad f_n \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{57}$$

Thus by the aid of Theorem 3 and the hypothesis of the theorem we have

$$\begin{aligned}\widetilde{\mathscr{B}^{\mathscr{A}}}((\rho_n - \rho) * \delta_n) &= \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n * \delta_n}{\delta_k}\right] \\ &\longrightarrow \mathscr{B}^{\mathscr{A}}(f_n * \delta_n) \quad \text{as } n \longrightarrow \infty \\ &\longrightarrow \mathscr{B}^{\mathscr{A}} f_n \quad \text{as } n \longrightarrow \infty \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{58}$$

Therefore, $\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho_n - \rho) \to 0$ as $n \to \infty$. Thus, $\widetilde{\mathscr{B}^{\mathscr{A}}}\rho_n \xrightarrow{\Delta} \widetilde{\mathscr{B}^{\mathscr{A}}}\rho$ as $n \to \infty$.

This completes the proof. □

Lemma 16. *Let $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, and δ has the usual meaning of (34) then*

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left(\left[\frac{f_n}{\delta_n}\right] * \delta\right) = \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n}{\delta_n}\right]. \tag{59}$$

Proof. Let $\rho = [f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then

$$\begin{aligned}\widetilde{\mathscr{B}^{\mathscr{A}}}\left(\left[\frac{f_n}{\delta_n}\right] * \delta\right) &= \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n * \delta}{\delta_n}\right] \\ &= \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}}(f_n * \delta) \\ &= \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n.\end{aligned} \tag{60}$$

Hence,

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left(\left[\frac{f_n}{\delta_n}\right] * \delta\right) = \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n}{\delta_n}\right]. \tag{61}$$

□

Theorem 17. *The Hartley-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{A}}}$ is one-to-one.*

Proof. Let $\widetilde{\mathscr{B}^{\mathscr{A}}}[f_n/\delta_n] = \widetilde{\mathscr{B}^{\mathscr{A}}}[g_n/\gamma_n]$, then by the aid of (46), we get

$$\lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n = \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} g_n. \tag{62}$$

Hence,

$$\mathscr{B}^{\mathscr{A}}\left(\lim_{n\to\infty} f_n\right) = \mathscr{B}^{\mathscr{A}}\left(\lim_{n\to\infty} g_n\right), \tag{63}$$

that is, $\mathscr{B}^{\mathscr{A}} f = \mathscr{B}^{\mathscr{A}} g$. The fact that $\mathscr{B}^{\mathscr{A}}$ is one-to-one implies $f = g$. Hence we have the following theorem. □

4. A Comparative Study: Fourier-Hilbert Transform

In [6, 7], the Hilbert transform via the Fourier transform of $f(x)$ is defined as

$$\mathscr{B}^{\mathscr{F}}(f)(y) = \frac{1}{\pi}\int_0^\infty \left(\mathscr{F}_i(f)(x)\cos(xy) - \mathscr{F}_r(f)(x)\sin(xy)\right)dx, \tag{64}$$

where

$$\mathscr{F}_r(f)(x) = \int_0^\infty f(t)\cos(xt)\,dt, \qquad \mathscr{F}_i(f)(x) = \int_0^\infty f(t)\sin(xt)\,dt \tag{65}$$

are, respectively, the real and imaginary components of the Fourier transform of f, which are related by

$$\mathscr{F}(f)(x) = \mathscr{F}_r(f)(x) - i\mathscr{F}_i(f)(x). \tag{66}$$

It is interesting to know that a concrete relationship between $\mathscr{F}_r, \mathscr{A}^e$ and $\mathscr{F}_i, \mathscr{A}^o$ is described as $\mathscr{F}_r(x) = \mathscr{A}^e(x)$ and $\mathscr{F}_i(x) = \mathscr{A}^o(x)$ [2]. Those equations, above, justify the following statements of the next theorems.

Theorem 18. *Let $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then the sequence of Fourier-Hilbert transforms of (f_n) satisfies*

$$\mathscr{B}^{\mathscr{F}}(f_n)(y) = \frac{1}{\pi}\int_0^\infty \left(\mathscr{F}_i(f_n)(x)\cos(xy) - \mathscr{F}_r(f_n)(x)\sin(xy)\right)dx \tag{67}$$

and converges uniformly on each compact subset $\mathscr{K}$ of $\mathscr{R}$.

Thus, for $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, the Fourier-Hilbert transform of $[f_n/\delta_n]$ is similarly defined by

$$\widetilde{\mathscr{B}^{\mathscr{F}}}\left[\frac{f_n}{\delta_n}\right] = \lim_{n\to\infty}\mathscr{B}^{\mathscr{F}} f_n \tag{68}$$

on compact subsets of $\mathscr{R}$.

The following theorems are stated and their proofs are justified for similar reasons. We prefer to we omit the details.

Theorem 19. *The generalized Fourier-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{F}}}$ is linear.*

Theorem 20. $\widetilde{\mathscr{B}^{\mathscr{F}}}(\rho * \epsilon_n) = \widetilde{\mathscr{B}^{\mathscr{F}}}\rho = \widetilde{\mathscr{B}^{\mathscr{F}}}(\epsilon * \rho)$, $(\epsilon_n) \in \Delta$.

Theorem 21. *If $\widetilde{\mathscr{B}^{\mathscr{F}}}\rho_1 = 0$, then $\rho_1 = 0$.*

Theorem 22. *If $\rho_n \xrightarrow{\Delta} \rho$ as $n \to \infty$ in $\rho_{\mathscr{L}^1}$, then $\widetilde{\mathscr{B}^{\mathscr{F}}}\rho_n \xrightarrow{\Delta} \widetilde{\mathscr{B}^{\mathscr{F}}}\rho$ as $n \to \infty$ in $\rho_{\mathscr{L}^1}$ on compact subsets.*

Theorem 23. *The Fourier-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{F}}}$ is continuous with respect to the δ-convergence.*

Theorem 24. *The Fourier-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{F}}}$ is continuous with respect to the Δ-convergence.*

Proofs of the above theorems are similar to that given for the corresponding ones of Hartley-Hilbert transform.

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Research Article

A New Application of the Reproducing Kernel Hilbert Space Method to Solve MHD Jeffery-Hamel Flows Problem in Nonparallel Walls

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The present paper emphasizes Jeffery-Hamel flow: fluid flow between two rigid plane walls, where the angle between them is 2α. A new method called the reproducing kernel Hilbert space method (*RKHSM*) is briefly introduced. The validity of the reproducing kernel method is set by comparing our results with HAM, DTM, and HPM and numerical results for different values of H, α, and Re. The results show up that the proposed reproducing kernel method can achieve good results in predicting the solutions of such problems. Comparison between obtained results showed that *RKHSM* is more acceptable and accurate than other methods. This method is very useful and applicable for solving nonlinear problems.

1. Introduction

1.1. Problem Formulation. Consider a system of cylindrical polar coordinates (r, h, z), where the steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at channel walls lies in planes and intersects in z-axis. It is assumed that there are no changes with respect to z, that the motion is purely in radial direction and merely depends on r and θ, and that there is no magnetic field along z-axis. Then the governing equations are given as [1].

$$\frac{\rho\partial}{r\partial r}\left(ru(r,\theta)\right)=0, \tag{1}$$

$$\begin{aligned} u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} &= \nu\left[\frac{\partial^2 u(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,\theta)}{\partial r}\right. \\ &\quad \left.+\frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial\theta^2}-\frac{u(r,\theta)}{r^2}\right] \\ &\quad -\frac{\sigma B_0^2}{\rho r^2}u(r,\theta)-\frac{1}{\rho}\frac{\partial P}{\partial r}, \end{aligned} \tag{2}$$

$$\frac{1}{\rho r}\frac{\partial P}{\partial\theta}-\frac{2\nu}{r^2}\frac{\partial u(r,\theta)}{\partial\theta}=0, \tag{3}$$

where B_0 is the electromagnetic induction, σ is the conductivity of the fluid, $u(r,\theta)$ is the velocity along radial direction, P is the fluid pressure, ν is the coefficient of kinematic viscosity, and ρ is the fluid density. From (1)

$$f(\theta)=ru(r,\theta), \tag{4}$$

using dimensionless parameters

$$f(x)=\frac{f(\theta)}{f_{\max}},\quad x=\frac{\theta}{\alpha}, \tag{5}$$

where α is the semiangle between the two inclined walls as shown in Figure 1. Substituting (5) into (2) and (3) and eliminating P, we obtain an ordinary differential equation for the normalized function profile $F(x)$ [2]:

$$F'''(x)+2\alpha\,\mathrm{Re}\,F(x)F'(x)+(4-H)\alpha^2F'(x)=0, \tag{6}$$

with boundary conditions

$$F(0) = 1, \qquad F'(0) = 0, \qquad F(1) = 0. \tag{7}$$

The Reynolds number is

$$\begin{aligned} \mathrm{Re} &= \frac{f_{\max}\alpha}{\nu} = \frac{U_{\max} r\alpha}{\nu} \\ &= \begin{pmatrix} \text{divergent channel}: & \alpha > 0, \ f_{\max} > 0 \\ \text{convergent channel}: & \alpha < 0, \ f_{\max} < 0 \end{pmatrix}. \end{aligned} \tag{8}$$

The Hartmann number is

$$H = \sqrt{\frac{\alpha B_0^2}{\rho\nu}}. \tag{9}$$

Internal flow between two plates is one of the most applicable cases in mechanics, civil and environmental engineering. In simple cases, the one-dimensional flow through tube and parallel plates, which is known as Couette-Poisseuille flow, has exact solution, but in general, like most of fluid mechanics equations, a set of nonlinear equations must be solved which make some problems for analytical solution. Many authors have shown interest in studying two-dimensional incompressible flow between two inclined plates. Jeffery [1] and Hamel et al. [2] were the first persons who discussed this problem, and so, it is known as Jeffery-Hamel problem. The incompressible viscous fluid flow through convergent and divergent channels is one of the most applicable cases in fluid mechanics, electrical, and biomechanical engineering. The MHD Jeffery-Hamel flows in nonparallel walls were investigated analytically for strongly nonlinear ordinary differential equations using homotopy analysis method (HAM). Results for velocity profiles in divergent and convergent channels were proffered for various values of Hartmann and Reynolds numbers in [3]. The mathematical investigations of this problem were underresearched by [3, 4]. Jeffery-Hamel flows are of the Navier-Stokes equations in the particular case of two dimensional flow through a channel with inclined walls [3–13]. One of the most important examples of Jeffery-Hamel problems is this subjected to an applied magnetic field. The equations of magnetohydrodynamics have been solved exactly for the case of two-dimensional steady flow between nonparallel walls of a viscous, incompressible, electrically conducting fluid; this is a straightforward extension of the famous Jeffrey-Hamel problem in ordinary hydrodynamics [9]. It has been indicated that for the Jeffrey-Hamel problem, the equations of magnetohydrodynamics can be curtailed to a set of three ordinary differential equations, two of which are linear and of first order [10]. In addition, these kinds of problems have been well studied in literature [3–13]. Most recent problems such as Jeffery-Hamel flow and other fluid mechanic problems are inherently nonlinear. Except a limited number of these problems, most of them do not have analytical solutions. So, these nonlinear equations should be solved utilizing other methods.

In this paper, the *RKHSM* [14–31] will be used to investigate MHD Jeffery-Hamel flows Problem. In recent years, a lot of attention has been devoted to the study of *RKHSM* to investigate various scientific models. The *RKHSM* which accurately computes the series solution is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that can be elegantly computed.

Recently, a lot of research work has been devoted to the application of *RKHSM* to a wide class of stochastic and deterministic problems involving fractional differential equation, nonlinear oscillator with discontinuity, singular nonlinear two-point periodic boundary value problems, integral equations and nonlinear partial differential equations and so on [14–31]. The method is well suited to physical problems since it makes unnecessary restrictive methods.

The efficiency of the method was used by many authors to investigate several scientific applications. Cui and Lin [15] applied the *RKHSM* to handle the second-order boundary value problems. Wang et al. [24] investigated a class of singular boundary value problems by this method, and the obtained results were good. In [27], the method was used to solve nonlocal boundary value problems. Geng and Cui [18] investigated the approximate solution of the forced Duffing equation with integral boundary conditions by combining the homotopy perturbation method and the *RKHSM*. Recently, the method was appllied the fractional partial differential equations and multipoint boundary value problems [18–22]. For more details about *RKHSM* and the modified forms and its effectiveness, see [14–31] and the references therein. The paper is organized as follows. Section 2 is devoted to several reproducing kernel spaces. Solution representation in $W_2^4[0,1]$ and a linear operator are introduced in Section 3. Section 4 provides the main results; the exact and approximate solution of system (34) and an iterative method are developed for the kind of problems in the reproducing kernel space. We have proved that the approximate solution converges to the exact solution uniformly. Numerical results are given in Section 5. The last Section is the conclusions.

2. Preliminaries

2.1. Reproducing Kernel Spaces. In this section, we define some useful reproducing kernel spaces.

Definition 1 (reproducing kernel). Let E be a nonempty abstract set. A function $K : E \times E \to C$ is a reproducing kernel of the Hilbert space H if and only if

$$\begin{gathered} \forall t \in E, \quad K(\cdot, t) \in H, \\ \forall t \in E, \ \forall \varphi \in H, \quad (\varphi(\cdot), K(\cdot, t)) = \varphi(t). \end{gathered} \tag{10}$$

The last condition is called "the reproducing property"; the value of the function φ at the point t is reproduced by the inner product of φ with $K(\cdot, t)$.

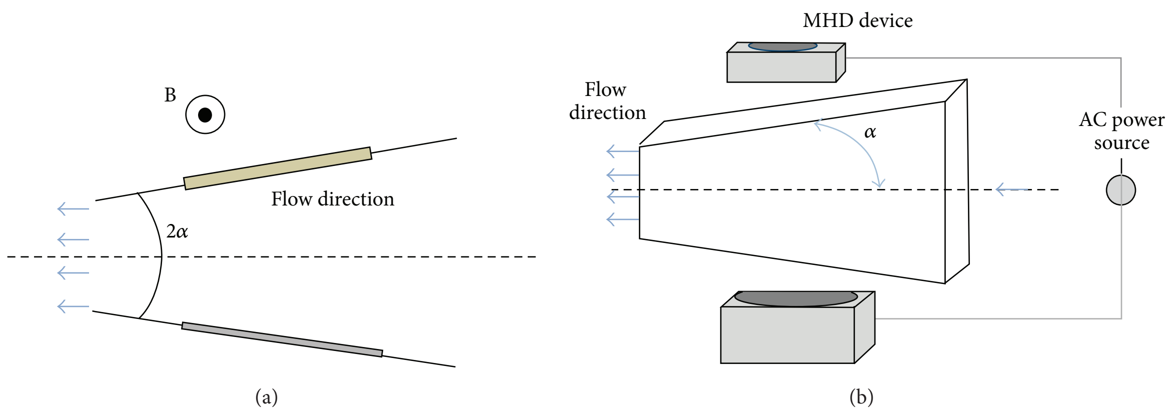

FIGURE 1: Geometry of the MHD Jeffery-Hamel flow in convergent cannel. (a) 2D view and (b) schematic setup of problem.

Definition 2. We define the space W_2^4 by

$$W_2^4[0,1] = \left\{ \begin{array}{l} u \mid u, u', u'', u''', \\ \quad \text{are absolutely continuous in } [0,1] \\ u^{(4)} \in L^2[0,1],\ x \in [0,1], \\ u(0) = 0,\ u(1) = 0,\ u'(0) = 0. \end{array} \right\}. \tag{11}$$

The inner product and the norm in $W_2^4[0,1]$ are defined, respectively, by

$$\begin{aligned} \langle u, g \rangle_{W_2^4} &= \sum_{i=0}^{3} u^{(i)}(0)\, g^{(i)}(0) \\ &\quad + \int_0^1 u^{(4)}(x)\, g^{(4)}(x)\, dx, \quad u, g \in W_2^4[0,1], \\ \|u\|_{W_2^4} &= \sqrt{\langle u, u \rangle_{W_2^4}}, \quad u \in W_2^4[0,1]. \end{aligned} \tag{12}$$

The space $W_2^4[0,1]$ is a reproducing kernel space; that is, for each fixed $y \in [0,1]$ and any $u(x) \in W_2^4[0,1]$, there exists a function $R_y(x)$ such that

$$u(y) = \langle u, R_y \rangle_{W_2^4}. \tag{13}$$

Definition 3. We define the space W_2^2 by

$$W_2^2[0,1] = \left\{ \begin{array}{c} u \mid u, u' \text{are absolutely continuous in } [0,1] \\ u'' \in L^2[0,1],\ x \in [0,1]. \end{array} \right\}. \tag{14}$$

The inner product and the norm in $W_2^2[0,1]$ are defined, respectively, by

$$\langle u, g \rangle_{W_2^2} = u(0)\, g(0) + u'(0)\, g'(0) + \int_0^1 u''(x)\, g''(x)\, dx, \quad \left(u, g \in W_2^2[0,1]\right), \tag{15}$$

$$\|u\|_{W_2^2} = \sqrt{\langle u, u \rangle_{W_2^2}}, \quad u \in W_2^2[0,1]. \tag{16}$$

The space $W_2^2[0,1]$ is a reproducing kernel space and its reproducing kernel function T_x is given by

$$T_x(y) = \begin{cases} 1 + xy + \dfrac{y}{2}x^2 - \dfrac{1}{6}x^3, & x \le y, \\ 1 + xy + \dfrac{x}{2}y^2 - \dfrac{1}{6}y^3, & x > y. \end{cases} \tag{17}$$

Theorem 4. *The space $W_2^4[0,1]$ is a complete reproducing kernel space; that is, for each fixed $y \in [0,1]$, there exists $u(x) \in W_2^4[0,1]$, such that*

$$u(y) = \langle u, R_y \rangle_{W_2^4} \tag{18}$$

for any $u(x) \in W_2^4[0,1]$. The reproducing kernel R_y can be denoted by

$$R_y(x) = \begin{cases} \displaystyle\sum_{i=1}^{8} c_i(y)\, x^{i-1}, & x \le y, \\ \displaystyle\sum_{i=1}^{8} d_i(y)\, x^{i-1}, & x > y, \end{cases} \tag{19}$$

where

$$c_1(y) = 0,$$

$$c_2(y) = 0,$$

$$c_3(y) = \frac{21}{5680}y^5 + \frac{1}{5680}y^7 - \frac{7}{1136}y^4 - \frac{7}{284}y^3 + \frac{2}{71}y^2 - \frac{7}{5680}y^6,$$

$$c_4(y) = \frac{7}{17040}y^5 + \frac{1}{51120}y^7 - \frac{7}{10224}y^4 + \frac{16}{639}y^3 - \frac{7}{284}y^2 - \frac{7}{51120}y^6,$$

$$c_5(y) = \frac{7}{68160}y^5 + \frac{1}{204480}y^7 - \frac{7}{40896}y^4 + \frac{4}{639}y^3 - \frac{7}{1136}y^2 - \frac{7}{204480}y^6,$$

$$c_6(y) = \frac{-7}{113600}y^5 - \frac{1}{340800}y^7 + \frac{7}{68160}y^4 + \frac{7}{17040}y^3 - \frac{1}{2130}y^2 + \frac{7}{340800}y^6,$$

$$c_7(y) = \frac{7}{340800}y^5 + \frac{1}{1022400}y^7 - \frac{7}{204480}y^4 - \frac{7}{51120}y^3 - \frac{7}{5680}y^2 - \frac{7}{1022400}y^6 + \frac{1}{720}y,$$

$$c_8(y) = \frac{-1}{340800}y^5 - \frac{1}{7156800}y^7 + \frac{7}{204480}y^4 + \frac{1}{51120}y^3 + \frac{1}{5680}y^2 + \frac{1}{10224000}y^6 - \frac{1}{5040},$$

$$d_1(y) = \frac{-1}{5040}y^7,$$

$$d_2(y) = \frac{1}{720}y^6,$$

$$d_3(y) = \frac{-1}{2130}y^5 + \frac{1}{5680}y^7 - \frac{7}{1136}y^4 - \frac{7}{284}y^3 + \frac{7}{21}y^2 - \frac{7}{5680}y^6,$$

$$d_4(y) = \frac{4}{639}y^4 + \frac{1}{17040}y^5 + \frac{7}{40896}y^7 + \frac{16}{639}y^3 - \frac{7}{284}y^2 - \frac{7}{51120}y^6,$$

$$d_5(y) = \frac{7}{68160}y^5 + \frac{1}{24480}y^7 - \frac{7}{40896}y^4 - \frac{7}{10224}y^3 - \frac{7}{1136}y^2 - \frac{7}{204480}y^6,$$

$$d_6(y) = \frac{-7}{113600}y^5 - \frac{1}{340800}y^7 + \frac{7}{68160}y^4 + \frac{7}{17040}y^3 + \frac{21}{5680}y^2 + \frac{7}{340800}y^6,$$

$$d_7(y) = \frac{7}{34080}y^5 + \frac{1}{1022400}y^7 - \frac{7}{204480}y^4 - \frac{7}{51120}y^3 - \frac{7}{5680}y^2 - \frac{7}{1022400}y^6,$$

$$d_8(y) = \frac{-1}{340800}y^5 - \frac{1}{7156800}y^7 + \frac{1}{204480}y^4 + \frac{1}{51120}y^3 + \frac{1}{5680}y^2 + \frac{7}{1022400}y^6. \tag{20}$$

Proof. By Definition 3, we have

$$\left\langle u, R_y \right\rangle_{W_2^4} = \sum_{i=0}^{3} u^{(i)}(0)\, R_y^{(i)}(0) + \int_0^1 u^{(4)}(x)\, R_y^{(4)}(x)\, dx, \quad \left(u, R_y \in W_2^4[0,1]\right). \tag{21}$$

Through several integrations by parts for (21) we have

$$\begin{aligned}\left\langle u(x), R_y(x) \right\rangle_{W_2^4} &= \sum_{i=0}^{3} u^{(i)}(0)\left[R_y^{(i)}(0) - (-1)^{(3-i)} R_y^{(7-i)}(0)\right] \\ &\quad + \sum_{i=0}^{3} (-1)^{(3-i)} u^{(i)}(1)\, R_y^{(7-i)}(1) \\ &\quad + \int_0^1 u(x)\, R_y^{(8)}(x)\, dx.\end{aligned} \tag{22}$$

Note that property of the reproducing kernel

$$\left\langle u, R_y \right\rangle_{W_2^4} = u(y), \tag{23}$$

R_y, is the solution of the following differential equation:

$$R_y^{(8)}(x) = \delta(x - y), \tag{24}$$

with the boundary conditions

$$\begin{aligned} R_y^{(3)}(0) - R_y^{(4)}(0) &= 0, \\ R_y''(0) + R_y^{(5)}(0) &= 0, \\ R_y^{(4)}(1) &= 0, \\ R_y^{(5)}(1) &= 0, \\ R_y^{(6)}(1) &= 0, \end{aligned} \tag{25}$$

when $x \neq y$,

$$R_y^{(8)}(x) = 0, \tag{26}$$

therefore

$$R_y(x) = \begin{cases} \sum_{i=1}^{8} c_i(y)\, x^{i-1}, & x \le y, \\ \sum_{i=1}^{8} d_i(y)\, x^{i-1}, & x > y. \end{cases} \tag{27}$$

Since

$$R_y^{(8)}(x) = \delta(x-y), \tag{28}$$

we have

$$\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0,1,2,3,4,5,6, \tag{29}$$

$$\partial^7 R_{y^+}(y) - \partial^7 R_{y^-}(y) = 1. \tag{30}$$

Since $R_y(x) \in W_2^4[0,1]$, it follows that

$$R_y(0) = 0, \qquad R_y'(0) = 0, \qquad R_y(1) = 0. \tag{31}$$

From (25)–(31), the unknown coefficients $c_i(y)$ ve $d_i(y)(i = 1,2,\ldots,8)$ can be obtained.

$$R_y(x) = \begin{cases} \frac{21}{5680}x^2y^5 + \frac{1}{5680}x^2y^7 - \frac{7}{1136}x^2y^4 - \frac{7}{284}x^2y^3 + \frac{2}{71}x^2y^2 \\ -\frac{7}{5680}x^2y^6 + \frac{7}{17040}x^3y^5 + \frac{1}{51120}x^3y^7 - \frac{7}{10224}x^3y^4 + \frac{16}{639}x^3y^3 \\ -\frac{7}{284}x^3y^2 - \frac{7}{51120}x^3y^6 + \frac{7}{68160}x^4y^5 + \frac{1}{204480}x^4y^7 - \frac{7}{40896}x^4y^4 \\ +\frac{4}{639}x^4y^3 - \frac{7}{1136}x^4y^2 - \frac{7}{204480}x^4y^6 \frac{-7}{113600}x^5y^5 - \frac{1}{340800}x^5y^7 \\ +\frac{7}{68160}x^5y^4 + \frac{7}{17040}x^5y^3 - \frac{1}{2130}x^5y^2 + \frac{7}{340800}x^5y^6 + \frac{7}{340800}x^6y^5 \\ +\frac{1}{1022400}x^6y^7 - \frac{7}{204480}x^6y^4 - \frac{7}{51120}x^6y^3 - \frac{7}{5680}x^6y^2 - \frac{7}{1022400}x^6y^6 \\ +\frac{1}{720}x^6y - \frac{1}{340800}x^7y^5 - \frac{1}{7156800}x^7y^7 + \frac{7}{204480}x^7y^4 + \frac{1}{51120}x^7y^3 \\ +\frac{1}{5680}x^7y^2 + \frac{1}{10224000}x^7y^6 - \frac{x^7}{5040}, \quad x \le y \\ \frac{21}{5680}y^2x^5 + \frac{1}{5680}y^2x^7 - \frac{7}{1136}y^2x^4 - \frac{7}{284}y^2x^3 + \frac{2}{71}y^2x^2 - \frac{7}{5680}y^2x^6 \\ \frac{7}{17040}y^3x^5 + \frac{1}{51120}y^3x^7 - \frac{7}{10224}y^3x^4 + \frac{16}{639}y^3x^3 - \frac{7}{284}y^3x^2 - \frac{7}{51120}y^3x^6 \\ \frac{7}{68160}y^4x^5 + \frac{1}{204480}y^4x^7 - \frac{7}{40896}y^4x^4 + \frac{4}{639}y^4x^3 - \frac{7}{1136}y^4x^2 \\ -\frac{7}{204480}y^4x^6 - \frac{7}{113600}y^5x^5 - \frac{1}{340800}y^5x^7 + \frac{7}{68160}y^5x^4 + \frac{7}{17040}y^5x^3 \\ -\frac{1}{2130}y^5x^2 + \frac{7}{340800}y^5x^6 + \frac{7}{340800}y^6x^5 + \frac{1}{1022400}y^6x^7 - \frac{7}{204480}y^6x^4 \\ -\frac{7}{51120}y^6x^3 - \frac{7}{5680}y^6x^2 - \frac{7}{1022400}y^6x^6 + \frac{1}{720}y^6x - \frac{1}{340800}y^7x^5 - \frac{1}{7156800}x^7y^7 \\ +\frac{7}{204480}y^7x^4 + \frac{1}{51120}y^7x^3 + \frac{1}{5680}y^7x^2 + \frac{1}{10224000}y^7x^6 - \frac{y^7}{5040}, \quad x > y. \end{cases} \tag{32}$$

□

3. Solution Representation in $W_2^4[0,1]$

In this section, the solution of (34) is given in the reproducing kernel space $W_2^4[0,1]$.

On defining the linear operator $L : W_2^4[0,1] \to W_2^2[0,1]$ as

$$\begin{aligned}(Lu)(x) = u'''(x) &+ \left[-2\alpha \operatorname{Re}\left(x^2-1\right) + (4-H)\alpha^2\right] u'(x) \\ &- 4\alpha x \operatorname{Re} u(x).\end{aligned} \tag{33}$$

Model problem (6) changes the following problem:

$$Lu = f\left(x,u,u'\right), \quad x \in [0,1],$$

$$u(0) = 0, \qquad u'(0) = 0, \qquad u(1) = 0, \tag{34}$$

where

$$f(x, u, u') = -2\alpha \operatorname{Re} u(x) u'(x) - 4\alpha \operatorname{Re}(x^3 - x) + 2(4 - H)\alpha^2 x, \tag{35}$$
$$u(x) = F(x) + x^2 - 1.$$

Theorem 5. *The operator L defined by* (33) *is a bounded linear operator.*

Proof. We only need to prove $\|Lu\|^2_{W_2^2} \le M\|u\|^2_{W_2^4}$, where $M > 0$ is a positive constant. By (15) and (16), we have

$$\begin{aligned}\|Lu\|^2_{W_2^2} &= \langle Lu, Lu\rangle_{W_2^2} \\ &= [(Lu)(0)]^2 + [(Lu)'(0)]^2 \\ &\quad + \int_0^1 [(Lu)''(x)]^2 dx.\end{aligned} \tag{36}$$

By (18), we have

$$\begin{aligned}u(x) &= \langle u, R_x\rangle_{W_2^4}, \\ (Lu)(x) &= \langle u, (LR_x)\rangle_{W_2^4}, \\ (Lu)'(x) &= \langle u, (LR_x)'\rangle_{W_2^4},\end{aligned} \tag{37}$$

so

$$\begin{aligned}&|(Lu)(x)| \le \|u\|_{W_2^4} \|LR_x\|_{W_2^4} = M_1\|u\|_{W_2^4}, \\ &(\text{where } M_1 > 0 \text{ is a positive constant}), \\ &|(Lu)'(x)| \le \|u\|_{W_2^4} \|(LR_x)'\|_{W_2^4} = M_2\|u\|_{W_2^4}, \\ &(\text{where } M_2 > 0 \text{ is a positive constant}),\end{aligned} \tag{38}$$

thus

$$(Lu)^2(0) + [(Lu)'(0)]^2 \le (M_1^2 + M_2^2)\|u\|^2_{W_2^4}. \tag{39}$$

Since

$$(Lu)'' = \langle u, (LR_x)''\rangle_{W_2^4}, \tag{40}$$

then

$$\begin{aligned}&|(Lu)''| \le \|u\|_{W_2^4} \|(LR_x)''\|_{W_2^4} = M_3\|u\|_{W_2^4}, \\ &(\text{where } M_3 > 0 \text{ is a positive constant}),\end{aligned} \tag{41}$$

so, we have

$$\begin{aligned}&[(Lu)'']^2 \le M_3^2\|u\|^2_{W_2^4}, \\ &\int_0^1 [(Lu)''(x)]^2 dx \le M_3^2\|u\|^2_{W_2^4},\end{aligned} \tag{42}$$

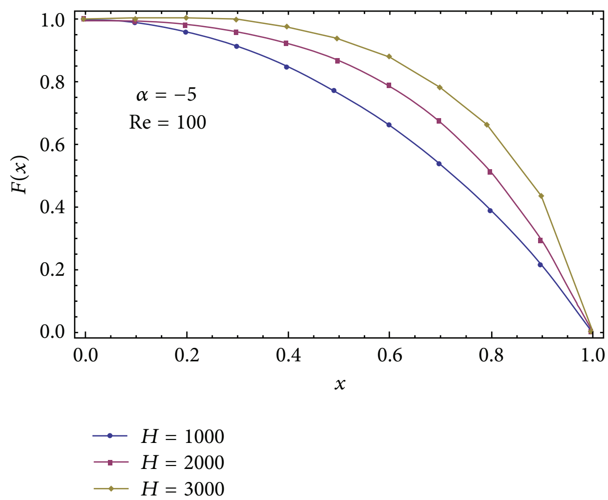

Figure 2: A comparison between increasing Hartmann numbers for the velocity profile Re = 100.

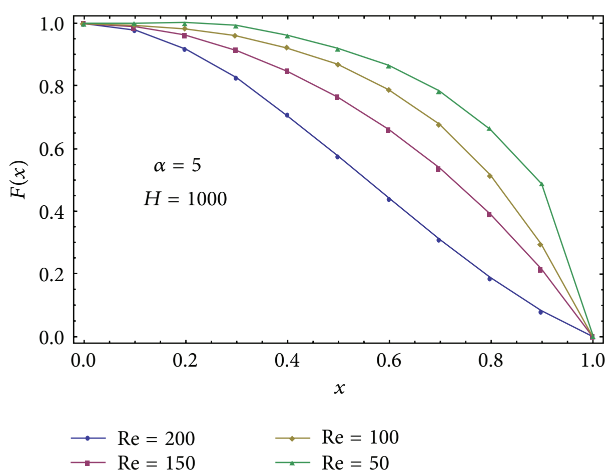

Figure 3: A comparison between the increasing values of Re for the velocity profile $H = 1000$.

that is

$$\begin{aligned}\|Lu\|^2_{W_2^2} &= [(Lu)(0)]^2 + [(Lu)'(0)]^2 + \int_0^1 [(Lu)''(x)]^2 dx \\ &\le (M_1^2 + M_2^2 + M_3^2)\|u\|^2_{W_2^4} = M\|u\|^2_{W_2^4},\end{aligned} \tag{43}$$

where $M = (M_1^2 + M_2^2 + M_3^2) > 0$ is a positive constant. □

4. The Structure of the Solution and the Main Results

In (33) it is clear that $L : W_2^4[0,1] \to W_2^2[0,1]$ is a bounded linear operator. Put $\varphi_i = T_{x_i}$ and $\psi_i = L^*\varphi_i$, where L^* is conjugate operator of L. The orthonormal system

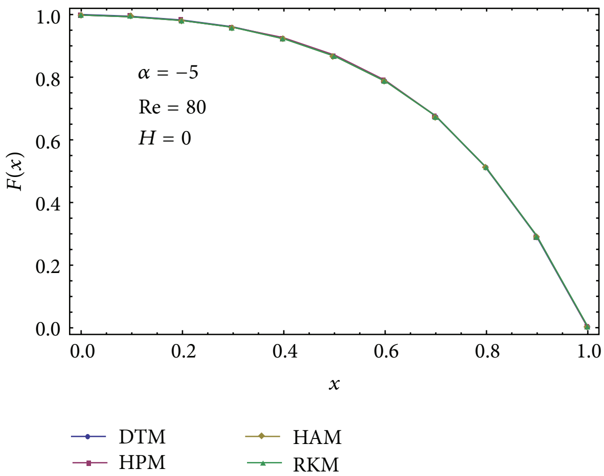

FIGURE 4: A comparison between the DTM, HPM, *RKHSM*, and HAM solutions for the velocity profile Re = 80 and $H = 0$.

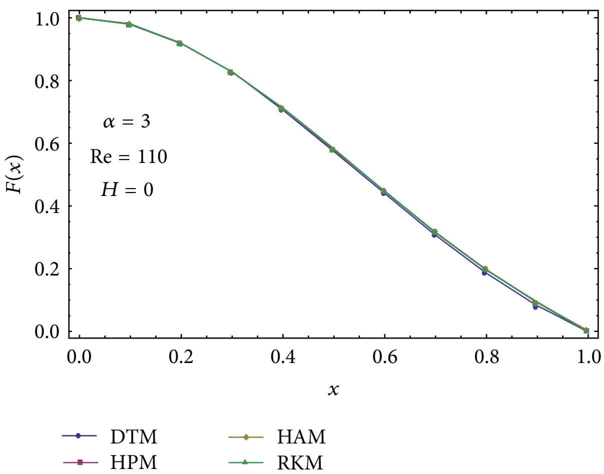

FIGURE 5: A comparison between the DTM, HPM, *RKHSM*, and HAM solutions for the velocity profile Re = 110 and $H = 0$.

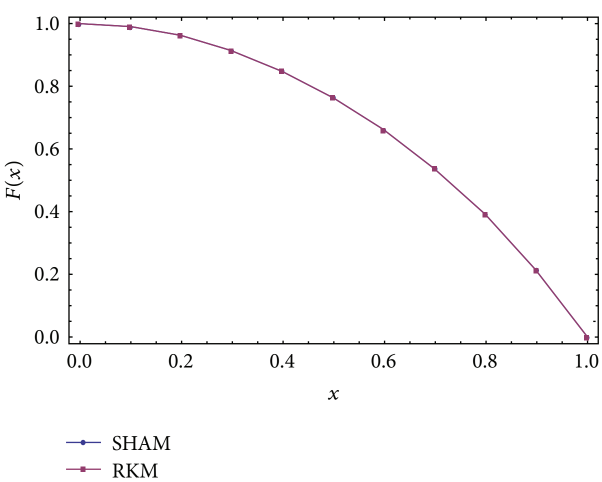

FIGURE 6: A comparison between the *RKHSM* and SHAM solutions for the velocity profile Re = 50, $\alpha = 5$, and $H = 0$.

$\{\overline{\Psi}_i\}_{i=1}^{\infty}$ of $W_2^4[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i\}_{i=1}^{\infty}$ as

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0,\ i = 1, 2, \ldots). \tag{44}$$

Theorem 6. *For* (33), *if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$ *then* $\{\psi_i\}_{i=1}^{\infty}$ *is the complete system of* $W_2^4[0,1]$ *and* $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.

Proof. We have

$$\begin{aligned}\psi_i(x) &= (L^*\varphi_i)(x) = \langle (L^*\varphi_i)(y), R_x(y)\rangle \\ &= \langle (\varphi_i)(y), LyR_x(y)\rangle = L_y R_x(y)\big|_{y=x_i}.\end{aligned} \tag{45}$$

The subscript y by the operator L indicates that the operator L applies to the function of y. Clearly, $\psi_i(x) \in W_2^4[0,1]$. For each fixed $u(x) \in W_2^4[0,1]$, let $\langle u(x), \psi_i(x)\rangle = 0$, $(i = 1, 2, \ldots)$, which means that

$$\langle u, (L^*\varphi_i)\rangle = \langle Lu, \varphi_i\rangle = \langle Lu, T_{x_i}\rangle = (Lu)(x_i) = 0. \tag{46}$$

Note that, $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of Theorem 6 is complete. □

Theorem 7. *If* $u(x)$ *is the exact solution of* (34), *then*

$$u = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, u_k, u'_k)\,\overline{\Psi}_i, \tag{47}$$

where $\{x_i\}_{i=1}^{\infty}$ *is a dense set in* $[0,1]$.

Proof. From (44) and uniqueness of solution of (34) we have

$$\begin{aligned} u &= \sum_{i=1}^{\infty} \langle u, \overline{\psi}_i\rangle_{W_2^4}\overline{\psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle u, L^*T_{x_k}\rangle_{W_2^4}\overline{\psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle Lu, T_{x_k}\rangle_{W_2^2}\overline{\psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle f(x,u,u'), T_{x_k}\rangle_{W_2^2}\overline{\psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, u_k, u'_k)\,\overline{\psi}_i. \end{aligned} \tag{48}$$

Now the approximate solution $u_n(x)$ can be obtained by truncating the n-term of the exact solution $u(x)$:

$$u_n = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k, u_k, u'_k)\,\overline{\psi}_i. \tag{49}$$

□

Table 1: The comparison between the numerical results and DTM, HPM, HAM, and *RKHSM* solutions for Re = 110, $\alpha = 3$, and $H = 0$.

x	DTM [5]	HPM [5]	HAM [5]	*RKHSM*	Numerical [5]
0.0	1.000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.9789771156	0.9791761778	0.9792357062	0.9792357171	0.9792357085
0.2	0.9182598446	0.9190424983	0.9192658842	0.91926585	0.9192658898
0.3	0.8243664466	0.8260939720	0.8265336102	0.82653635	0.8265336182
0.4	0.7065763476	0.7096036928	0.7102211838	0.7102315393	0.7102211890
0.5	0.5751498602	0.5798357741	0.5804994700	0.5804817201	0.5804994634
0.6	0.4397114086	0.4463900333	0.4469350941	0.4468796913	0.4469350697
0.7	0.3081560927	0.3170877938	0.3174084545	0.3174013727	0.3174084270
0.8	0.1862239095	0.1975366451	0.1976410661	0.1976321	0.1976410889
0.9	0.0784362201	0.09124214542	0.09123022879	0.0912030082	0.0912304211
1.0	0.0000000015	0.0000000007	−0.00000047	$8.052549207 \times 10^{-8}$	0.0

Table 2: The numerical results for Re = 50, $H = 1000$.

x	HAM [3]	*RKHSM* ($\alpha = 5$)	Error	HAM [3]	*RKHSM* ($\alpha = -5$)	Error
0	1.000000000	1.0000000000	0.0	1.000000000	1.00000000	0.0
0.05	0.997605126	0.997605447	3.203×10^{-7}	0.999197467	0.99919702	4.432×10^{-7}
0.10	0.990427215	0.990432890	0.56744×10^{-6}	0.99675704	0.9967562	8.409×10^{-7}
0.15	0.978485626	0.9784839628	0.16638×10^{-6}	0.992578975	0.992578	9.754×10^{-7}
0.20	0.961810074	0.96179	0.20074×10^{-5}	0.98649281	0.98649340	5.900×10^{-7}
0.25	0.940436864	0.9403939	0.42964×10^{-5}	0.978250927	0.9782510	7.24×10^{-8}
0.30	0.91440365	0.9145	0.96349×10^{-5}	0.967519314	0.9675443	0.24985×10^{-5}
0.35	0.883742856	0.8833	0.44285×10^{-4}	0.953865319	0.95382	0.45319×10^{-5}
0.40	0.848473706	0.8484738539	1.473×10^{-7}	0.936742176	0.936821	0.78823×10^{-5}
0.45	0.808592961	0.808592834	1.279×10^{-7}	0.915470063	0.915531	0.60936×10^{-5}
0.50	0.764064241	0.7640637445	4.967×10^{-7}	0.889213540	0.889241	0.27459×10^{-5}
0.55	0.714805913	0.7148062	2.867×10^{-7}	0.856955292	0.8565	0.45529×10^{-4}
0.60	0.660677266	0.660670	0.72666×10^{-6}	0.817466464	0.817199	0.26746×10^{-4}
0.65	0.601462467	0.6014683135	0.58461×10^{-6}	0.769274094	0.770	0.7259×10^{-4}
0.70	0.536852087	0.53685274	6.525×10^{-7}	0.710627559	0.710014	0.61355×10^{-4}
0.75	0.466421078	0.4664202	8.783×10^{-7}	0.639465773	0.63946970	0.39331×10^{-6}
0.80	0.389601905	0.389602099	1.934×10^{-7}	0.553390063	0.55336107	0.28992×10^{-5}
0.85	0.305651801	0.305645	0.68011×10^{-6}	0.449648596	0.44963621	0.12386×10^{-5}
0.90	0.213611172	0.2136120	8.277×10^{-7}	0.325142373	0.32516167	0.19298×10^{-5}
0.95	0.112250324	0.112249347	9.775×10^{-7}	0.176465831	0.17656197	0.9614×10^{-5}
1.00	0.000000000	8.3437×10^{-8}	8.3437×10^{-8}	0.000000000	3.614×10^{-7}	3.614×10^{-7}

Table 3: The comparison between the numerical results and DTM, HPM, HAM, and *RKHSM* solutions for Re = 80, $\alpha = -5$, and $H = 0$.

x	DTM [5]	HPM [5]	HAM [5]	*RKHSM*	Numerical
0	1.000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.10	0.9959603887	0.9960671874	0.9959606242	0.99595999	0.9959606278
0.20	0.9832745481	0.9836959424	0.9832755258	0.983275	0.9832755381
0.30	0.9601775551	0.9610758773	0.9601798911	0.96017	0.96017991139
0.40	0.9235170706	0.9249245156	0.9235215737	0.923519	0.9235215894
0.50	0.8684511349	0.8701997697	0.8684588997	0.86845826	0.86845887772
0.60	0.7880785402	0.7898325937	0.7880910186	0.78809	0.78809092032
0.70	0.6731248448	0.6745334968	0.6731437690	0.67314	0.6731436346
0.80	0.5119644061	0.5128373095	0.5119909939	0.5119873503	0.5119910891
0.90	0.2915280122	0.2918936991	0.2915580178	0.2915582665	0.29155874261
1.00	0.0000000000	0.0000000001	−0.000001149	2.851385×10^{-9}	0.0

Lemma 8. *If $u \in W_2^4[0,1]$, then there exists $M_1 > 0$, such that*

$$\|u\|_{C^2[0,1]} \le M_1 \|u\|_{W_2^4}, \tag{50}$$

where $\|u\|_{C^2[0,1]} = \max_{x\in[0,1]}|u(x)| + \max_{x\in[0,1]}|u'(x)| + \max_{x\in[0,1]}|u''(x)|$.

Lemma 9. *If $\|u_n - u\|_{W_2^4} \to 0$, $x_n \to x$, $(n \to \infty)$ and $f(x,u,u')$ is continuous for $x \in 0,1]$, then*

$$f\left(x_n, u_{n-1}(x_n), u'_{n-1}(x_n)\right) \longrightarrow f\left(x, u(x), u'(x)\right) \quad \text{as } n \longrightarrow \infty. \tag{51}$$

Proof. Since $\|u_n - u\|_{W_2^4} \to 0$ $(n \to \infty)$, by Lemma 8, we know that u_n is convergent uniformly to $u(x)$, therefore, the proof is complete. □

Remark 10. (i) If (34) is linear, that is, $f(x,u) = f(x)$, then the analytical solution of (34) can be obtained directly by (47).

(ii) If (34) is nonlinear; that is, f depends on u and u' then the solution of (34) can be obtained by the following iterative method.

We construct an iterative sequence $u_n(x)$, putting

$$\begin{gathered}\text{any fixed } u_0(x) \in W_2^4[0,1], \\ u_n(x) = \sum_{i=1}^{n} A_i \overline{\psi}_i(x),\end{gathered} \tag{52}$$

where

$$\begin{aligned} A_1 &= \beta_{11} f\left(x_1, u_0(x_1), u'_0(x_1)\right), \\ A_2 &= \sum_{k=1}^{2} \beta_{2k} f\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k)\right), \\ &\vdots \\ A_n &= \sum_{k=1}^{n} \beta_{nk} f\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k)\right). \end{aligned} \tag{53}$$

Next we will prove that u_n given by the iterative formula (52) converges to the exact solution (47).

Theorem 11. *Suppose that the following conditions are satisfied: (i) $\|u_n\|_{W_2^4}$ is bounded; (ii) $\{x_i\}_{i=1}^{\infty}$ is a dense in $[0,1]$; (iii) $f(x,u,u') \in W_2^2[0,1]$ for any $u \in W_2^4[0,1]$. Then u_n in iterative formula* (52) *converges to the exact solution of* (47) *in $W_2^4[0,1]$ and*

$$u = \sum_{i=1}^{\infty} A_i \overline{\psi}_i, \tag{54}$$

where A_i is given by (53).

Proof. (i) First, we will prove the convergence of $u_n(x)$. By (52), we have

$$u_{n+1}(x) = u_n(x) + A_{n+1}\overline{\psi}_{n+1}(x). \tag{55}$$

From the orthogonality of $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\begin{aligned} \|u_{n+1}\|_{W_2^4}^2 &= \|u_n\|_{W_2^4}^2 + (A_{n+1})^2 \\ &= \|u_{n-1}\|_{W_2^4}^2 + (A_n)^2 + (A_{n+1})^2 \\ &= \cdots = \sum_{i=1}^{n+1} (A_i)^2. \end{aligned} \tag{56}$$

From boundedness of $\|u_n\|_{W_2^4}$, we have

$$\sum_{i=1}^{\infty} (A_i)^2 < \infty, \tag{57}$$

that is,

$$\{A_i\} \in l^2 \quad (i = 1, 2, \ldots). \tag{58}$$

Let $m > n$, in view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned} &\|u_m - u_n\|_{W_2^4}^2 \\ &\quad = \|u_m - u_{m-1} + u_{m-1} - \mathrm{u}_{m-2} + \cdots + u_{n+1} - u_n\|_{W_2^4}^2 \\ &\quad \le \|u_m - u_{m-1}\|_{W_2^4}^2 + \cdots + \|u_{n+1} - u_n\|_{W_2^4}^2 \\ &\quad = \sum_{i=n+1}^{m} (A_i)^2 \longrightarrow 0 \quad (m, n \longrightarrow \infty). \end{aligned} \tag{59}$$

Considering the completeness of $W_2^4[0,1]$, there exists $u(x) \in W_2^4[0,1]$, such that

$$u_n(x) \xrightarrow{\|\cdot\|_{W_2^4}} u(x), \quad \text{as } n \longrightarrow \infty. \tag{60}$$

(ii) Second, we will prove that $u(x)$ is the solution of (34).

By Lemma 8 and Theorem 11 (i), we know that u_n converges uniformly to u. It follows that, on taking limits in (52),

$$u = \sum_{i=1}^{\infty} A_i \overline{\psi}_i. \tag{61}$$

Since

$$\begin{aligned} (Lu)(x_j) &= \sum_{i=1}^{\infty} A_i \left\langle L\overline{\Psi}_i(x), \varphi_j(x)\right\rangle_{W_2^2} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \overline{\Psi}_i(x), L^*\varphi_j(x)\right\rangle_{W_2^4} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \overline{\Psi}_i(x), \overline{\Psi}_j(x)\right\rangle_{W_2^4}, \end{aligned} \tag{62}$$

Table 4: The errors of DTM, HPM, HAM, and *RKHSM* for $F(x)$ results when $\mathrm{Re} = 110, \alpha = 3$, and $H = 0$.

x	DTM [5]	HPM [5]	HAM [5]	*RKHSM*
0.0	0.0	0.0	0.0	0.0
0.1	0.0002	0.000059	0.0000000023	8.6×10^{-9}
0.2	0.0010	0.00022	0.0000000056	3.98×10^{-8}
0.3	0.0021	0.00043	0.000000008	0.0000027318
0.4	0.0036	0.00061	0.0000000052	0.0000103503
0.5	0.0053	0.00066	0.0000000066	0.0000177433
0.6	0.0072	0.00054	0.000000024	0.0000553784
0.7	0.0092	0.00032	0.000000027	0.0000070543
0.8	0.0114	0.000104	0.000000022	0.0000089889
0.9	0.0127	0.000011	0.00000019	0.0000274129
1.0	0.0000	0.000000	0.0000004	$8.052549207 \times 10^{-8}$

Table 5: The errors of DTM, HPM, HAM, and *RKHSM* for $F(x)$ results when for $\mathrm{Re} = 80, \alpha = -5$, and $H = 0$.

x	DTM [5]	HPM [5]	HAM [5]	*RKHSM*
0.0	0.0	0.0	0.0	0.0
0.1	0.00000023	0.000106	0.000000003	6.378×10^{-7}
0.2	0.00000099	0.00042	0.000000012	5.381×10^{-7}
0.3	0.0000023	0.00089	0.00000002	0.9114×10^{-6}
0.4	0.0000045	0.0014	0.000000015	2.5894×10^{-6}
0.5	0.0000077	0.0017	0.000000021	6.177×10^{-7}
0.6	0.000012	0.0017	0.000000098	9.203×10^{-7}
0.7	0.000018	0.0013	0.00000013	3.6346×10^{-6}
0.8	0.000026	0.0008	0.000000095	7.388×10^{-6}
0.9	0.000030	0.00033	0.00000072	4.761×10^{-7}
1.0	0.0000	0.0000000001	0.0000011	2.8513856×10^{-9}

it follows that

$$\begin{aligned}\sum_{j=1}^{n}\beta_{nj}(Lu)(x_j) &= \sum_{i=1}^{\infty}A_i\left\langle \overline{\Psi}_i(x), \sum_{j=1}^{n}\beta_{nj}\overline{\Psi}_j(x)\right\rangle_{W_2^4} \\ &= \sum_{i=1}^{\infty}A_i\left\langle \overline{\Psi}_i(x), \overline{\Psi}_n(x)\right\rangle_{W_2^4} = A_n.\end{aligned} \tag{63}$$

If $n = 1$, then

$$(Lu)(x_1) = f\left(x_1, u_0(x_1), u_0'(x_1)\right). \tag{64}$$

If $n = 2$, then

$$\begin{aligned}&\beta_{21}(Lu)(x_1) + \beta_{22}(Lu)(x_2) \\ &\quad = \beta_{21}f\left(x_1, u_0(x_1), u_0'(x_1)\right) \\ &\qquad + \beta_{22}f\left(x_2, u_1(x_2), u_1'(x_2)\right).\end{aligned} \tag{65}$$

From (64) and (65), it is clear that

$$(Lu)(x_2) = f\left(x_2, u_1(x_2), u_1'(x_2)\right). \tag{66}$$

Furthermore, it is easy to see by induction that

$$(Lu)(x_j) = f\left(x_j, u_{j-1}(x_j), u_{j-1}'(x_j)\right). \tag{67}$$

Notice that $\{x_i\}_{i=1}^{\infty}$ is dense on interval $[0,1]$, for any $y \in [0,1]$, there exists subsequence $\{x_{n_j}\}$, such that $x_{n_j} \to y$, as $j \to \infty$. Hence, by the convergence of $u_n(x)$ and Lemma 9, we have

$$(Lu)(y) = f\left(y, u(y), u'(y)\right), \tag{68}$$

that is, $u(x)$ is the solution of (34) and

$$u = \sum_{i=1}^{\infty}A_i\overline{\Psi}_i, \tag{69}$$

where A_i is given by (53). □

Corollary 12. *Assume that the conditions of Theorem 11 hold; then u_n in* (52) *satisfies $\|u_n - u\|_{C^2[0,1]} \to 0$, $n \to \infty$, where u is the solution of* (34).

Theorem 13. *Assume that u is the solution of* (34) *and r_n is the error between the approximate solution u_n and the exact solution u. Then the error sequence r_n is monotone decreasing in the sense of $\|\cdot\|_{W_2^4}$ and $\|r_n(x)\|_{W_2^4} \to 0$.*

Proof. From (47) and (49), it follows that

$$\begin{aligned}\|r_n\|_{W_2^4} &= \left\|\sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f\left(x_k, u_k, u_k'\right)\overline{\Psi}_i(x)\right\|_{W_2^4} \\ &= \sum_{i=n+1}^{\infty}\left(\sum_{k=1}^{i}\beta_{ik}f\left(x_k, u_k, u_k'\right)\right)^2.\end{aligned} \tag{70}$$

Equation (70) shows that the error r_n is decreasing in the sense of $\|\cdot\|_{W_2^4}$. □

5. Numerical Results

All computations are performed by Maple 15. Results obtained by the method are compared with the homotopy analysis method [3], three analytical methods [5], homotopy perturbation method [6], and a new spectral-homotopy analysis method [8]. The *RKHSM* does not require discretization of the variables, that is, time and space; it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The accuracy of the *RKHSM* for the MHD Jeffery-Hamel flows problem is controllable and absolute errors are small with present choice of x (see Tables 1–5). The numerical results that we obtained justify the advantage of this methodology.

5.1. Result and Discussion. In this study the purpose is to apply the *RKHSM* to obtain an approximate solution of the Jeffery-Hamel problem. The obtained results of *RKHSM* solution and numerical ones are shown in the tables and figures. In Table 2 a comparison of the HAM and *RKHSM* is shown. Tables 1 and 3 show the comparison between the numerical results and DTM, HPM, HAM, and *RKHSM* solutions. Tables 4 and 5 indicate the errors of DTM, HPM, HAM, and *RKHSM* for $F(x)$ results. Our results further show

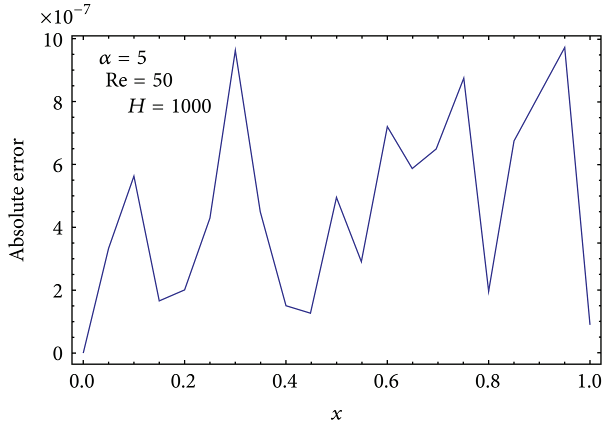

FIGURE 7: Absolute error for Re = 50 and $H = 1000$.

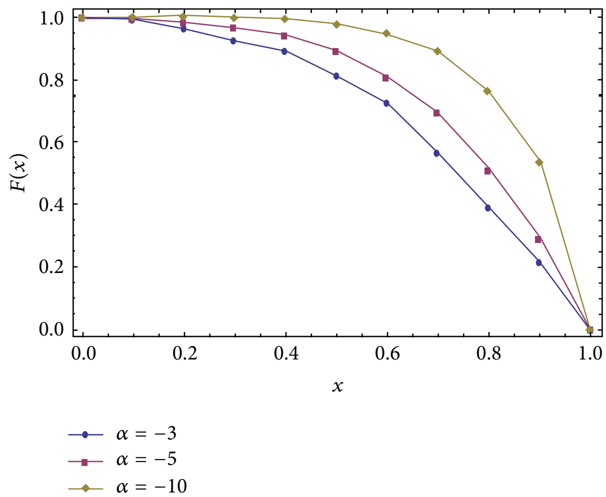

FIGURE 8: A comparison between different values of α for velocity in convergent channel for Re = 50 and $H = 1000$.

that the fluid velocity increases with increasing Hartman numbers. Numerical simulations show that for fixed Hartmann numbers, the fluid velocity increases with Reynolds numbers in the case of convergent channels but decreases with Re in the case of divergent channels. Figure 2 indicates that increasing the Hartmann number leads to higher velocity which has a great effect on the performance of the system. In Figure 3 we give a comparison between the *RKHSM* and the HAM solutions for several Re numbers at $H = 1000$. In Figure 4 we can see a comparison between the DTM, HPM, *RKHSM* and HAM solutions for the velocity profile Re = 80 and $H = 0$. There is a comparison between the DTM, HPM, *RKHSM*, and HAM solutions for the velocity profile Re = 110 and $H = 0$ in Figure 5. In Figure 6 we compare *RKHSM* and SHAM solutions. We can see absolute error for Re = 50 and $H = 1000$ in Figure 7. The comparison of numerical results and *RKHSM* solution for velocity in convergent channel for Re = 50 and $H = 1000$ is given with Figure 8. The solutions show that the results of the present method are in excellent agreement with those of the numerical ones. Moreover, *RKHSM* has been used to investigate the effects of the parameters of the problem.

6. Conclusion

In this paper, we introduce an algorithm for solving the MHD Jeffery-Hamel flows problem with boundary conditions by using the *RKHSM*. The approximate solution obtained by the present method is uniformly convergent. Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems. However, if the problem becomes nonlinear, then the *RKHSM* does not require discretization or perturbation and it does not make closure approximation. Results show that the present method is an accurate and reliable analytical method for MHD Jeffery-Hamel flows problem with boundary conditions.

Conflict of Interests

The authors declare that they do not have any competing or conflict of interests.

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Research Article

On an Extension of Kummer's Second Theorem

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The aim of this paper is to establish an extension of Kummer's second theorem in the form $e^{-x/2}\,{}_2F_2\left[\begin{smallmatrix} a, & 2+d; \\ 2a+2, & d; \end{smallmatrix}\; x\right] = {}_0F_1\left[\begin{smallmatrix} -; \\ a+3/2; \end{smallmatrix}\; x^2/16\right] + ((a/d - 1/2)/(a+1))x\,{}_0F_1\left[\begin{smallmatrix} -; \\ a+3/2; \end{smallmatrix}\; x^2/16\right] + (cx^2/2(2a+3))\,{}_0F_1\left[\begin{smallmatrix} -; \\ a+5/2; \end{smallmatrix}\; x^2/16\right]$, where $c = (1/(a+1))(1/2 - a/d) + a/d(d+1)$, $d \neq 0, -1, -2, \ldots$. For $d = 2a$, we recover Kummer's second theorem. The result is derived with the help of Kummer's second theorem and its contiguous results available in the literature. As an application, we obtain two general results for the terminating ${}_3F_2(2)$ series. The results derived in this paper are simple, interesting, and easily established and may be useful in physics, engineering, and applied mathematics.

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by [1]

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix}\; z\right] &= {}_pF_q\left[a_1, \ldots, a_p; b_1, \ldots, b_q; z\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \end{aligned} \tag{1}$$

where $(a)_n$ denotes Pochhammer's symbol (or the shifted or raised factorial, since $(1)_n = n!$) defined by

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1), & n \in \mathbb{N}, \\ 1, & n = 0. \end{cases} \tag{2}$$

Using the fundamental properties of Gamma function $\Gamma(a+1) = a\Gamma(a)$, $(a)_n$ can be written in the form

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{3}$$

where Γ is the familiar Gamma function.

It is not out of place to mention here that whenever a generalized hypergeometric or hypergeometric function ${}_2F_1$ reduces to Gamma function, the results are very important from the applicative point of view. Thus, the classical summation theorem for the series ${}_2F_1$ such as those of Gauss, Gauss second, Kummer, and Bailey plays an important role in the theory of hypergeometric series. For generalization and extensions of these classical summation theorems, we refer to [2, 3].

By employing the above mentioned classical summation theorems, Bailey [4] had obtained a large number of very interesting results (including results due to Ramanujan, Gauss, Kummer, and Whipple) involving products of generalized hypergeometric series.

On the other hand, from the theory of differential equations, Kummer [5] established the following very interesting and useful result known in the literature as Kummer's second theorem:

$$e^{-x/2}\,{}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; x\right] = {}_0F_1\left[\begin{matrix} -; \\ a+\frac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right]. \tag{4}$$

Bailey [4] established the result (4) by employing the Gauss second summation theorem, and Choi and Rathie [6] established the result (4) (of course, by changing x to $2x$) by employing the classical Gauss summation theorem. From (4), Rainville [7] deduced the following two useful and classical results:

$$ {}_2F_1\left[\begin{matrix} -2n, & a; \\ 2a; & \end{matrix}\; 2\right] = \frac{(1/2)_n}{(a+1/2)_n}, \quad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right), \tag{5}$$

$$ {}_2F_1\left[\begin{matrix} -2n-1, & a; \\ 2a; & \end{matrix}\; 2\right] = 0, \quad \left(n \in \mathbb{N}_0\right). \tag{6}$$

Using (5) and (6), it is not difficult to establish the following transformation due to Kummer:

$$\begin{aligned} (1-x)^{-r}\, {}_2F_1\left[\begin{matrix} r, & m; \\ 2m; & \end{matrix}\; -\frac{2x}{1-x}\right] \\ = {}_2F_1\left[\begin{matrix} \frac{1}{2}r, & \frac{1}{2}r+\frac{1}{2}; \\ m+\frac{1}{2}; & \end{matrix}\; x^2\right]. \end{aligned} \tag{7}$$

In 1995, Rathie and Nagar [8] obtained two results closely related to Kummer's second theorem (4); one of those results is given below:

$$\begin{aligned} e^{-x/2}\, {}_1F_1\left[\begin{matrix} a; \\ 2a+1; \end{matrix}\; x\right] = {}_0F_1\left[\begin{matrix} -; \\ a+\frac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right] \\ - \frac{x}{2(2a+1)}\, {}_0F_1\left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right]. \end{aligned} \tag{8}$$

In 2010, Kim et al. [1] have generalized the Kummer's second theorem and obtained explicit expressions of

$$ e^{-x/2}\, {}_1F_1\left[\begin{matrix} a; \\ 2a+j; \end{matrix}\; x\right], \tag{9}$$

for $j = 0, \pm 1, \ldots, \pm 5$ by employing the generalized Gauss second summation theorem obtained earlier by Lavoie et al. [9]. We, however, would like to mention one of their results which we will require in our present investigation:

$$\begin{aligned} e^{-x/2}\, {}_1F_1\left[\begin{matrix} a; \\ 2a+2; \end{matrix}\; x\right] \\ = {}_0F_1\left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right] - \frac{x}{2(a+1)}\, {}_0F_1\left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right] \\ + \frac{x^2}{4(a+1)(2a+3)}\, {}_0F_1\left[\begin{matrix} -; \\ a+\frac{5}{2}; \end{matrix}\; \frac{x^2}{16}\right]. \end{aligned} \tag{10}$$

In 2008, Rathie and Pogàny [10] established a new summation formula for ${}_3F_2(1/2)$ and, as an application, obtained the following result which is known as an extension of Kummer's second Theorem (4):

$$\begin{aligned} e^{-x/2}\, {}_2F_2\left[\begin{matrix} a, & 1+d; \\ 2a+1, & d; \end{matrix}\; x\right] \\ = {}_0F_1\left[\begin{matrix} -; \\ a+\frac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right] \\ - \frac{x(1-2a/d)}{2(2a+1)}\, {}_0F_1\left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right], \end{aligned} \tag{11}$$

for $d \neq 0, -1, -2, \ldots$.

It is noted that if in (11) we set $d = 2\alpha$, we immediately recover Kummer's second Theorem (4).

Very recently Rakha [11] rederived the result (11) in its equivalent form by employing the classical Gauss summation theorem, and Kim et al. [12] derived (11) in a very elementary way and, as an application, obtained the following two elegant results:

$$ {}_3F_2\left[\begin{matrix} -2n, & a, & 1+d; \\ 2a+1, & d; & \end{matrix}\; 2\right] = \frac{(1/2)_n}{(a+1/2)_n}, \quad \left(n \in \mathbb{N}_0\right), \tag{12}$$

$$ {}_3F_2\left[\begin{matrix} -2n-1, & a, & 1+d; \\ 2a+1, & d; & \end{matrix}\; 2\right] = \frac{(1-2a/d)}{(2a+1)} \frac{(3/2)_n}{(a+3/2)_n}, \quad \left(n \in \mathbb{N}_0\right). \tag{13}$$

It is interesting to mention here that the right-hand side of (12) is independent of d, where $d \neq 0, -1, -2, \ldots$.

Remark 1. (a) In (12) and (13), if we set $d = 2a$, we recover (5) and (6), respectively.

(b) Using (12) and (13), Kim et al. [12] have obtained the following extension of transformation (7) due to Kummer:

$$(1-x)^{-r}\,{}_3F_2\left[\begin{matrix} r, & m, & d+1; & \\ & & & -\dfrac{2x}{1-x} \\ 2m+1, & d; & & \end{matrix}\right]$$
$$= {}_2F_1\left[\begin{matrix} \dfrac{1}{2}r, & \dfrac{1}{2}r+\dfrac{1}{2}; & \\ & & x^2 \\ m+\dfrac{1}{2}; & & \end{matrix}\right]$$
$$+\frac{xr(1-2m/d)}{(2m+1)}\,{}_2F_1\left[\begin{matrix} \dfrac{1}{2}r+\dfrac{1}{2}, & \dfrac{1}{2}r+1; & \\ & & x^2 \\ m+\dfrac{3}{2}; & & \end{matrix}\right], \quad (14)$$

for $d \neq 0, -1, -2, \ldots$.

The aim of this paper is to establish another extension of Kummer's second Theorem (4) by employing the known results (4), (8), and (10). As an application, we mention two interesting results for the terminating ${}_3F_2(2)$ series. The results established in this paper are simple, interesting, and easily established and may be useful in physics, engineering, and applied mathematics.

2. Main Result

The result to be established in this paper is as follows:

$$e^{-x/2}\,{}_2F_2\left[\begin{matrix} a, & 2+d; & \\ & & x \\ 2a+2, & d; & \end{matrix}\right]$$
$$= {}_0F_1\left[\begin{matrix} -; & \\ & \dfrac{x^2}{16} \\ a+\dfrac{3}{2}; & \end{matrix}\right]$$
$$+\frac{(a/d-1/2)}{(a+1)}x\,{}_0F_1\left[\begin{matrix} -; & \\ & \dfrac{x^2}{16} \\ a+\dfrac{3}{2}; & \end{matrix}\right] \quad (15)$$
$$+\frac{cx^2}{2(2a+3)}\,{}_0F_1\left[\begin{matrix} -; & \\ & \dfrac{x^2}{16} \\ a+\dfrac{5}{2}; & \end{matrix}\right],$$

where $d \neq 0, -1, -2, \ldots$ and c is given by $c = (1/(a+1))(1/2 - a/d) + a/d(d+1)$.

2.1. Derivation. In order to derive (15), we proceed as follows. Denoting the left-hand side of (15) by S and expressing ${}_2F_2$ as a series with the help of (1), we have

$$S = e^{-x/2}\sum_{n=0}^{\infty}\frac{(a)_n}{(2a+2)_n}\frac{x^n}{n!}\left\{\frac{(2+d)_n}{(d)_n}\right\}. \quad (16)$$

Now, it is not difficult to see that

$$\frac{(2+d)_n}{(d)_n} = 1 + \frac{2}{d}n + \frac{n(n-1)}{d(d+1)}; \quad (17)$$

we have

$$S = e^{-x/2}\sum_{n=0}^{\infty}\frac{(a)_n}{(2a+2)_n}\frac{x^n}{n!}\left\{1+\frac{2}{d}n+\frac{n(n-1)}{d(d+1)}\right\}. \quad (18)$$

Separating (18) into three terms, we have

$$S = e^{-x/2}\left[\sum_{n=0}^{\infty}\frac{(a)_n}{(2a+2)_n}\frac{x^n}{n!} + \frac{2}{d}\sum_{n=1}^{\infty}\frac{(a)_n}{(2a+2)_n}\frac{x^n}{(n-1)!} + \frac{1}{d(d+1)}\sum_{n=2}^{\infty}\frac{(a)_n}{(2a+2)_n}\frac{x^n}{(n-2)!}\right]. \quad (19)$$

For the second and third terms on the right-hand side of (19), changing n to $n+1$ and n to $n+2$, respectively, and making use of the following results:

$$\begin{gathered} (a)_{n+1} = a(a+1)_n, \\ (2a+2)_{n+1} = (2a+2)(2a+3)_n, \\ (a)_{n+2} = a(a+1)(a+2)_n, \\ (2a+2)_{n+2} = (2a+2)(2a+3)(2a+4)_n, \end{gathered} \quad (20)$$

we have, after some simplification,

$$S = e^{-x/2}\left[\sum_{n=0}^{\infty}\frac{(a)_n}{(2a+2)_n}\frac{x^n}{n!} + \frac{ax}{d(a+1)}\sum_{n=0}^{\infty}\frac{(a+1)_n}{(2a+3)_n}\frac{x^n}{n!} + \frac{ax^2}{2d(d+1)(2a+3)}\sum_{n=0}^{\infty}\frac{(a+2)_n}{(2a+4)_n}\frac{x^n}{n!}\right]. \quad (21)$$

Now, summing up the series with the help of (1), we have

$$S = e^{-x/2}\,{}_1F_1\left[\begin{matrix} a; & \\ & x \\ 2a+2; & \end{matrix}\right]$$
$$+\frac{ax}{d(a+1)}e^{-x/2}\,{}_1F_1\left[\begin{matrix} a+1; & \\ & x \\ 2a+3; & \end{matrix}\right] \quad (22)$$
$$+\frac{ax^2}{2d(d+1)(2a+3)}e^{-x/2}\,{}_1F_1\left[\begin{matrix} a+2; & \\ & x \\ 2a+4; & \end{matrix}\right].$$

Finally, observing the right-hand side of (22), we see that the first, second, and third expressions can now be evaluated with the help of the results (10), (8), and (4), respectively, and, after some simplification, we arrive at the desired result (15). This completes the proof of (15).

3. New Results for Terminating ${}_3F_2(2)$

In this section, from our newly obtained result (15), we will establish two new results for the terminating ${}_3F_2$ series. These are

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}-2n, & a, & 2+d; \\ 2a+2, & d; & \end{matrix}\; 2\right] = \frac{(1/4c+1)_n(1/2)_n}{(1/4c)_n(a+3/2)_n}, \quad (n\in\mathbb{N}_0),\\
&{}_3F_2\left[\begin{matrix}-2n-1, & a, & 2+d; \\ 2a+2, & d; & \end{matrix}\; 2\right] = \frac{(d-2a)}{d(a+1)}\frac{(3/2)_n}{(a+3/2)_n}, \quad (n\in\mathbb{N}_0),
\end{aligned}
\tag{23}
$$

where $c = (1/(a+1))(1/2 - a/d) + a/d(d+1)$, and $d \neq 0, -1, -2, \ldots$.

3.1. Derivations. In order to derive the results (23), we proceed as follows. Denoting the left-hand side of (15) by S_1, then expressing both of the functions involved in the series, we have

$$
\begin{aligned}
S_1 &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{x^n}{n!}\sum_{m=0}^{\infty}\frac{(a)_m(2+d)_m}{(2a+2)_m(d)_m m!}x^m\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n(a)_m(2+d)_m}{2^n(2a+2)_m(d)_m m!n!}x^{n+m}.
\end{aligned}
\tag{24}
$$

Replacing n by $n-m$ in (24) and using the known result [7, page 56, Lemma 10]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k), \tag{25}$$

we have

$$S_1 = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^{n-m}(a)_m(2+d)_m}{2^{n-m}(2a+2)_m(d)_m m!\,(n-m)!}x^n. \tag{26}$$

Using the identity

$$(n-m)! = \frac{(-1)^m n!}{(-n)_m}, \tag{27}$$

we have, after some simplification,

$$S_1 = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n n!}x^n\sum_{m=0}^{n}\frac{(-n)_m(a)_m(2+d)_m}{(2a+2)_m(d)_m m!}2^m. \tag{28}$$

Expressing the inner series in the last result, we get

$$S_1 = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n n!}x^n\,{}_3F_2\left[\begin{matrix}-n, & a, & 2+d; \\ 2a+2, & d; & \end{matrix}\; 2\right]. \tag{29}$$

Now, separating the ${}_3F_2$ into even and odd powers of x and making use of the results:

$$
\begin{aligned}
(2n)! &= 2^{2n}n!\left(\frac{1}{2}\right)_n,\\
(2n+1)! &= 2^{2n}n!\left(\frac{3}{2}\right)_n,
\end{aligned}
\tag{30}
$$

we finally have

$$
\begin{aligned}
S_1 &= \sum_{n=0}^{\infty}\frac{x^{2n}}{2^{4n}n!(1/2)_n}\,{}_3F_2\left[\begin{matrix}-2n, & a, & 2+d; \\ 2a+2, & d; & \end{matrix}\; 2\right]\\
&\quad - \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^{4n+1}n!(3/2)_n}\,{}_3F_2\left[\begin{matrix}-2n-1, & a, & 2+d; \\ 2a+2, & d; & \end{matrix}\; 2\right].
\end{aligned}
\tag{31}
$$

Also, it is not difficult to see that

$$
\begin{aligned}
&{}_1F_2\left[\begin{matrix}\frac{1}{4c}+1; & \\ \frac{1}{4c}, & a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right]\\
&= {}_0F_1\left[\begin{matrix}-; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right] + \frac{cx^2}{2(2a+3)}\,{}_0F_1\left[\begin{matrix}-; \\ a+\frac{5}{2}; \end{matrix}\; \frac{x^2}{16}\right].
\end{aligned}
\tag{32}
$$

Now, if we denote the right-hand side of (15) by S_2, then with the help of (32), it can be expressed as

$$
\begin{aligned}
S_2 &= {}_1F_2\left[\begin{matrix}\frac{1}{4c}+1; & \\ \frac{1}{4c}, & a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right]\\
&\quad + \frac{(a/d-1/2)}{(a+1)}x\,{}_0F_1\left[\begin{matrix}-; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right].
\end{aligned}
\tag{33}
$$

Thus, from (31) and (33), if we equate the coefficients of x^{2n} and x^{2n+1} on both sides, we at once arrive at the results (23). This completes the proof.

Remark 2. (a) Setting $d = 2a$ in (15), we immediately recover Kummer's second Theorem (4). Thus, (15) can be regarded as the extension of (4).

(b) Also, if we take $d = 2a$ in (12) and (13), we again at once get the result (5) and (6), respectively. Thus, our results (12) and (13) can be regarded as extensions of (5) and (6).

4. Extension of a Transformation due to Kummer

In this section, we will establish a natural extension of Kummer's transformation:

$$(1-x)^{-r}\,{}_3F_2\left[\begin{matrix} r, & m, & d+2; \\ & & & -\dfrac{2x}{1-x} \\ 2m+2, & d; & \end{matrix}\right]$$
$$= {}_3F_2\left[\begin{matrix} \frac{1}{2}r, & \frac{1}{2}r+\frac{1}{2}, & \frac{1}{4c}+1; \\ & & & x^2 \\ m+\frac{3}{2}, & \frac{1}{4c}; & \end{matrix}\right]$$
$$+\frac{xr(d-2m)}{d(m+1)}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}r+\frac{1}{2}, & \frac{1}{2}r+1; \\ & & x^2 \\ m+\frac{3}{2}; & \end{matrix}\right], \quad (34)$$

for $d \neq 0, -1, -2, \ldots$ and c is given by

$$c = \frac{1}{m+1}\left(\frac{1}{2}-\frac{m}{d}\right)+\frac{m}{d(d+1)}. \quad (35)$$

4.1. Derivation. In order to establish the result (34), we proceed as follows. Denote the left-hand side of (34) by S_3; we have

$$S_3 = (1-x)^{-r}\,{}_3F_2\left[\begin{matrix} r, & m, & d+2; \\ & & & -\dfrac{2x}{1-x} \\ 2m+2, & d; & \end{matrix}\right]; \quad (36)$$

expressing ${}_3F_2$ as a series, we have

$$S_3 = \sum_{k=0}^{\infty}\frac{(r)_k(m)_k(d+2)_k(-1)^k 2^k x^k}{(2m+2)_k(d)_k k!}(1-x)^{-(r+k)}. \quad (37)$$

Applying the generalized Binomial theorem

$$(1-z)^a = \sum_{n=0}^{\infty}\frac{(-a)_n}{n!}z^n \quad (|z|<1), \quad (38)$$

we have

$$S_3 = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(r)_k(m)_k(d+2)_k(-2)^k}{(2m+2)_k(d)_k k! n!}(r+k)_n x^{k+n}. \quad (39)$$

Using $(r)_k(r+k)_n = (r)_{k+n}$, we have

$$S_3 = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m)_k(d+2)_k(-2)^k(r)_{k+n}}{(2m+2)_k(d)_k k! n!}x^{k+n}; \quad (40)$$

changing n to $n-k$ and using (25), we have

$$S_3 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(m)_k(d+2)_k(-2)^k(r)_n}{(2m+2)_k(d)_k k!\,(n-k)!}x^n. \quad (41)$$

Using

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k} \quad (0 \le k \le n), \quad (42)$$

we have

$$S_3 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(m)_k(d+2)_k(-2)^k(r)_n(-n)_k}{(2m+2)_k(d)_k k!(-1)^k n!}x^n$$
$$= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}x^n\sum_{k=0}^{n}\frac{(-n)_k(m)_k(d+2)_k}{(2m+2)_k(d)_k k!}2^k. \quad (43)$$

Expressing the inner series, as ${}_3F_2$, we find

$$S_3 = \sum_{n=0}^{\infty}\frac{(r)_n}{n!}x^n\,{}_3F_2\left[\begin{matrix} -n, & m, & d+2; \\ & & & 2 \\ 2m+2, & d; & \end{matrix}\right], \quad (44)$$

from which, we have

$$S_3 = \sum_{n=0}^{\infty}\frac{(r)_{2n}}{(2n)!}x^{2n}\,{}_3F_2\left[\begin{matrix} -2n, & m, & d+2; \\ & & & 2 \\ 2m+2, & d; & \end{matrix}\right]$$
$$+\sum_{n=0}^{\infty}\frac{(r)_{2n+1}}{(2n+1)!}x^{2n+1}\,{}_3F_2\left[\begin{matrix} -2n-1, & m, & d+2; \\ & & & 2 \\ 2m+2, & d; & \end{matrix}\right]. \quad (45)$$

Using the the following identities:

$$(r)_{2n} = 2^{2n}\left(\frac{1}{2}r\right)_n\left(\frac{1}{2}r+\frac{1}{2}\right)_n,$$
$$(r)_{2n+1} = r2^{2n}\left(\frac{1}{2}r+\frac{1}{2}\right)_n\left(\frac{1}{2}r+1\right)_n,$$
$$(2n)! = 2^{2n}\left(\frac{1}{2}\right)_n n!,$$
$$(2n+1)! = 2^{2n}\left(\frac{3}{2}\right)_n n! \quad (46)$$

together with the ${}_3F_2(2)$ result, we have

$$S_3 = \sum_{n=0}^{\infty}\frac{((1/2)\,r)_n((1/2)\,r+1/2)_n}{n!}x^{2n}\frac{(1/4c+1)_n}{(1/4c)_n(m+3/2)_n}$$
$$+x\sum_{n=0}^{\infty}\frac{r((1/2)\,r+1/2)_n\,((1/2)\,r+1)}{n!}x^{2n+1}$$
$$\times\frac{(d-2m)}{d(m+1)(m+3/2)_n}$$

$$= {}_3F_2\left[\begin{matrix}\frac{1}{2}r, & \frac{1}{2}r+\frac{1}{2}, & \frac{1}{4c}+1; \\ & & & x^2 \\ \frac{1}{4c}, & m+\frac{3}{2}; & \end{matrix}\right] + \frac{x(d-2m)r}{d(m+1)}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}r+\frac{1}{2}, & \frac{1}{2}r+1; \\ & & x^2 \\ m+\frac{3}{2}; & \end{matrix}\right], \quad (47)$$

with

$$c = \frac{1}{m+1}\left(\frac{1}{2}-\frac{m}{d}\right)+\frac{m}{d(d+1)}, \quad d \neq 0,-1,-2,\ldots. \quad (48)$$

This completes the proof of (34).

Remark 3. In (34), if we take $d = 2m$, we get (7). Thus, (34) may be regarded as an extension of (7).

Authors' Contribution

All authors contributed equally to this paper. They read and approved the final paper.

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Research Article

Robustness of Operational Matrices of Differentiation for Solving State-Space Analysis and Optimal Control Problems

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The idea of approximation by monomials together with the collocation technique over a uniform mesh for solving *state-space analysis* and *optimal control* problems (OCPs) has been proposed in this paper. After imposing the Pontryagins maximum principle to the main OCPs, the problems reduce to a linear or nonlinear boundary value problem. In the linear case we propose a monomial collocation matrix approach, while in the nonlinear case, the general collocation method has been applied. We also show the efficiency of the operational matrices of differentiation with respect to the operational matrices of integration in our numerical examples. These matrices of integration are related to the Bessel, Walsh, Triangular, Laguerre, and Hermite functions.

1. Introduction

In the last four decades, numerical methods which are based on the operational matrices of integration (especially for orthogonal polynomials and functions) have received considerable attention for dealing with a huge size of applied mathematics problems such as *state-space analysis* and *optimal control*. The key idea of these methods is based on the integral expression

$$\int_0^t \Phi(\tau)\, d\tau \approx \Phi(t)\, P, \tag{1}$$

where $\Phi(t) = [\Phi_1(t), \Phi_2(t), \ldots, \Phi_N(t)]$ is an arbitrary basis vector and P is a $(N+1) \times (N+1)$ constant matrix, called the operational matrix of integration. The matrix P has already been determined for many types of orthogonal (or nonorthogonal) bases such as Walsh functions [1–3], block-pulse functions [4], Laguerre polynomials [5], Chebyshev polynomials [6], Legendre polynomials [7], Hermite polynomials [8], Fourier series [9], Bernstein polynomials [10], and Bessel functions [11]. As a primary research work which was based on the operational matrices of integration, one can refer to the work of Corrington [1]. In [1], the author proposed a method of solving nonlinear differential and integral equations using a set of Walsh functions as the basis. His method is aimed at obtaining piecewise constant solutions of dynamic equations and requires previously prepared tables of coefficients for integrating Walsh functions. To alleviate the need for such tables, Chen and Hsiao [2, 3] introduced an operational matrix to perform integration of Walsh functions. This operational matrix approach has been applied to various problems such as time-domain analysis and synthesis of linear systems, and piecewise constant-feedback-gain determination for optimal control of linear systems and for inverting irrational Laplace transforms.

On the other hand, since the beginning of 1994, the Bernoulli, Chebyshev, Laguerre, Bernstein, Legendre, Taylor, Hermite, and Bessel matrix methods have been used in the works [12–24] to solve high-order linear and nonlinear differential (including hyperbolic partial differential equations) Fredholm Volterra integrodifferential difference delay equations and their systems. The main characteristic of these approaches is based on the operational matrices of differentiation instead of integration. The best advantage of these techniques with respect to the integation methods is that, in the fundamental matrix relations, there is not any approximation symbol, meanwhile in the integration forms such as (1) the approximation symbol could be seen obviously. In other words

$$\Phi'(\tau) = \Phi(\tau) B, \tag{2}$$

where B is the operational matrix of differentiation for any selected basis such as the previously mentioned polynomials, functions, and truncated series. The readers can see that there is no approximation symbol in (2), meanwhile this can be seen in (1) by using operational matrices of integration. For justifying this expression, one can refer to this subject that after differentiating an Nth degree polynomial we usually reach to a polynomial which has less than Nth degree. However, in the integration processes the degree of polynomials would be increased.

In this paper, we generalize a new collocation matrix method that was applied for solving a huge size of applied mathematics models (see for instance [16] and the references therein), to several special classes of systems of ordinary differential equations (ODEs). Two important classes of such systems of ODEs are

(i) *State space analysis,*

(ii) *Hamiltonian system,*

which are the necessary (and also are sufficient in several special cases) conditions for optimality of the solutions of OCPs, originate from the PMP, and have considerable importance in optimal control and calculus of variation.

We again emphasized that the methods that are based on the operational matrices of differentiation are more accurate and effective with regard to the integration ones. We illustrate this fact through several examples for dealing with the previously mentioned systems in the section of numerical examples. It should be noted that one of the best tools for the integration approaches is using high accurate Gauss quadrature rules such as the method of [25, 26]. However, more CPU times are required for using such quadrature rules, and also the matrix coefficient associated to these methods is ill-conditioned usually and should be preconditioned.

The remainder of this paper is organized as follows. In Section 2, the considered problems such as *state-space analysis* and *Hamiltonian system* are introduced. In Section 3, the fundamental matrix relations together with the method of obtaining approximate solutions are described. In Section 4, several numerical examples are provided for confirming high accuracy of the proposed method. The last Section is devoted to the conclusions.

2. Problems Statement

In this section two types of problems are considered. In the first subsection, we show that how the *Hamiltonian systems* can be obtained in both linear and nonlinear forms. In the second subsection, we introduce a general form of *state-space analysis* problems.

2.1. Hamiltonian Systems

2.1.1. Linear Quadratic Optimal Control Problems. In this part, we consider the following linear optimal control problem (OCP):

$$\begin{aligned} \min \quad & J = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T P x + 2x^T Q u + u^T R u\right) dt \\ \text{s.t.} \quad & \dot{x} = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0, \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A(\cdot) \in \mathbb{R}^{n\times n}$ and $B(\cdot) \in R^{m\times n}$. The control $u(t)$ is an admissible control if it is piecewise continuous in t for $t \in [t_0, t_f]$. Its values belong to a given closed subset U of $\mathbb{R}^m$. The input $u(t)$ is derived by minimizing the quadratic performance index J, where $P \in \mathbb{R}^{n\times n}$ is positive semidefinite matrix and $R \in \mathbb{R}^{m\times m}$ is a positive definite matrix. We consider Hamiltonian for system (3) as

$$\begin{aligned} H(x,u,\lambda,t) = {} & \frac{1}{2}\left(x^T P x + 2x^T Q u + u^T R u\right) \\ & + \lambda^T (A(t)x + B(t)u), \end{aligned} \tag{4}$$

where $\lambda \in \mathbb{R}^n$ is the costate vector.

According to the Pontryagin's maximum principle, we have [27]

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H}{\partial x} = -Px - Qu - A(t)^T \lambda, \\ \frac{\partial H}{\partial u} &= Q^T x + Ru + B(t)^T \lambda = 0. \end{aligned} \tag{5}$$

The optimal control is computed by [27]

$$u^* = -R^{-1}Q^T x - R^{-1}B(t)^T \lambda, \tag{6}$$

where λ and x are the solution of the Hamiltonian system:

$$\begin{aligned} \dot{x} &= \left[A(t) - B(t)R^{-1}Q^T\right] x - B(t)R^{-1}B(t)^T \lambda, \\ \dot{\lambda} &= \left[-P + QR^{-1}Q^T\right] x + \left[QR^{-1}B(t)^T - A(t)^T\right] \lambda, \\ x(t_0) &= x_0, \qquad \lambda\left(t_f\right) = 0. \end{aligned} \tag{7}$$

2.1.2. Nonlinear Quadratic Optimal Control Problems. Consider the nonlinear dynamical system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) + g(t, x(t))u(t), \quad t \in \left[t_0, t_f\right], \\ x(t_0) &= x_0, \end{aligned} \tag{8}$$

with $x(t) \in \mathbb{R}^n$ denoting the state variable, $u(t) \in \mathbb{R}^m$ the control variable, and x_0 is the given initial state at t_0. Moreover, $f(t, x(t)) \in \mathbb{R}^n$ and $g(t, x(t)) \in \mathbb{R}^{n\times m}$ are two continuously differentiable functions in all arguments. Our aim is to minimize the quadratic objective functional

$$J[x,u] = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t) Q x(t) + u^T(t) R u(t)\right) dt, \tag{9}$$

subject to the nonlinear system (8), for $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ positive semidefinite and positive definite matrices, respectively. Since the performance index (9) is convex, the following extreme necessary conditions are also sufficient for optimality [28]:

$$\begin{gathered}\dot{x} = f(t,x) + g(t,x)u^*, \qquad \dot{\lambda} = -H_x(x,u^*,\lambda), \\ u^* = \arg\min_u H(x,u,\lambda), \qquad x(t_0) = x_0, \qquad \lambda(t_f) = 0,\end{gathered} \tag{10}$$

where $H(x,u,\lambda) = (1/2)[x^TQx + u^TRu] + \lambda^T[f(t,x) + g(t,x)u]$ is referred to the Hamiltonian. Equivalently, (10) can be written in the form of

$$\begin{aligned}\dot{x} &= f(t,x) + g(t,x)\left[-R^{-1}g^T(t,x)\lambda\right] \\ \dot{\lambda} &= -\left(Qx + \left(\frac{\partial f(t,x)}{\partial x}\right)^T\lambda \right. \\ &\qquad \left. + \sum_{i=1}^{n}\lambda_i\left[-\mathrm{R}^{-1}g^T(t,x)\lambda\right]^T\frac{\partial g_i(t,x)}{\partial x}\right) \\ x(t_0) &= x_0, \qquad \lambda(t_f) = 0,\end{aligned} \tag{11}$$

where $\lambda(t) \in \mathbb{R}^n$ is the costate vector with the ith component $\lambda_i(t)$, $i = 1,\ldots,n$ and $g(t,x) = [g_1(t,x),\ldots,g_n(t,x)]^T$ with $g_i(t,x) \in \mathbb{R}^m$, $i = 1,\ldots,n$.

Also the optimal control law is obtained by

$$u^* = -R^{-1}g^T(t,x)\lambda. \tag{12}$$

For solving such a two-point boundary value problem (TPBVP) in (11), we apply a similar collocation method that was proposed in [29].

2.2. State Space Analysis Problems. In this part, we consider the following *state space analysis* problem:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(0) = x_0, \tag{13}$$

where $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}$ are known, meanwhile $x(t) \in \mathbb{R}^n$ is unknown. The goal is to obtain the approximation of $x(t)$ in (13). The previously mentioned system (13) is similar to *Hamiltonian system* (7) and the scheme of their solutions is the same.

Remark 1. We recall that the main goal of this paper is to approximate the solution of the systems (7), (11), and (13) by applying a new matrix method which is based on the operational matrix of differentiation and also the uniform collocation scheme in the parts of *Hamiltonian systems* and *state space analysis* problems.

3. Fundamental Matrix Relations and Method of the Solution

In this section, by using the collocation points and the matrix relations between the monomials $\{1, t, t^2, \ldots, t^N\}$ and their derivatives, we will find the approximate solution of the system (7) expressed in the truncated monomial series form (assuming that $x(t)$ and $\lambda(t) \in \mathbb{R}$ and also $A(t)$ together with $B(t)$ is independent of time t, that is, $A = A(t)$ and $B = B(t)$)

$$\begin{aligned}x(t) &\approx x_N(t) = \sum_{n=0}^{N}a_{1,n}t^n, \\ \lambda(t) &\approx \lambda_N(t) = \sum_{n=0}^{N}a_{2,n}t^n,\end{aligned} \tag{14}$$

so that $a_{1,n}$ and $a_{2,n}$; $n = 0,1,2,\ldots,N$ are the unknown coefficients.

Let us consider the desired solutions $x(t)$ and $\lambda(t)$, of (7) defined by the truncated monomial series (14). We can write the approximate solutions, which are given in relation (14) in the matrix form

$$x_N(t) = \mathbf{X}(t)\mathbf{A}_1, \qquad \lambda_N(t) = \mathbf{X}(t)\mathbf{A}_2, \tag{15}$$

where $\mathbf{X}(t) = [1\ t\ \cdots\ t^N]$ and $\mathbf{A}_i = [a_{i,0}\ a_{i,1}\ \cdots\ a_{i,N}]^T$, $i = 1,2$.

The matrix form of the relation between the matrix $\mathbf{X}(t)$ and its kth derivative $\mathbf{X}^{(k)}(t)$ is

$$\mathbf{X}^{(k)}(t) = \mathbf{X}(t)\left(\mathbf{B}^T\right)^k \tag{16}$$

so that

$$\mathbf{B}^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{17}$$

$\left(\mathbf{B}^T\right)^0 = [\mathbf{I}]_{(N+1)\times(N+1)}$ is the identity matrix.

By using the relations (15) and (16), we have the following relations:

$$\begin{aligned}x_N^{(i)}(t) &= \mathbf{X}(t)\left(\mathbf{B}^T\right)^i\mathbf{A}_1, \\ \lambda_N^{(i)}(t) &= \mathbf{X}(t)\left(\mathbf{B}^T\right)^i\mathbf{A}_2, \quad i = 0,1.\end{aligned} \tag{18}$$

Thus, we can express the matrices $\mathbf{y}(t)$ and $\mathbf{y}^{(1)}(t)$ as follows:

$$\mathbf{y}_N^{(i)}(t) = \overline{\mathbf{X}}(t)\left(\overline{\mathbf{B}}\right)^i\mathbf{A}, \quad i = 0,1, \tag{19}$$

where

$$\begin{gathered}\mathbf{y}_N^{(i)}(t) = \begin{bmatrix} x_N^{(i)}(t) \\ \lambda_N^{(i)}(t) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}, \\ \overline{\mathbf{X}}(t) = \begin{bmatrix} \mathbf{X}(t) & 0 \\ 0 & \mathbf{X}(t) \end{bmatrix}, \qquad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^T & 0 \\ 0 & \mathbf{B}^T \end{bmatrix}.\end{gathered} \tag{20}$$

Now, we can restate the system (7) in the matrix form

$$\mathbf{y}_N^{(1)}(t) - M\mathbf{y}_N(t) = O, \tag{21}$$

where

$$\mathbf{y}_N^{(1)}(t) = \begin{bmatrix} x_N^{(1)}(t) \\ \lambda_N^{(1)}(t) \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} A - BR^{-1}Q^T & -BR^{-1}B^T \\ -P + QR^{-1}Q^T & QR^{-1}B^T - A^T \end{bmatrix}, \quad (22)$$

$$\mathbf{y}_N(t) = \begin{bmatrix} x_N(t) \\ \lambda_N(t) \end{bmatrix}, \qquad \mathbf{O} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Applying the following collocation points in (21):

$$t_s = t_0 + \frac{t_f - t_0}{N} s, \quad s = 0, 1, \ldots, N \quad (23)$$

yields to $N + 1$ equations as follows:

$$\mathbf{y}_N^{(1)}(t_s) - M\mathbf{y}_N(t_s) = O, \quad s = 0, 1, \ldots, N. \quad (24)$$

All of the these equations can be written in the following matrix form:

$$\mathbf{Y}^{(1)} - \overline{\mathbf{M}}\mathbf{Y} = O, \quad (25)$$

where

$$\mathbf{Y}^{(1)} = \begin{bmatrix} \mathbf{y}_N^{(1)}(t_0) \\ \mathbf{y}_N^{(1)}(t_1) \\ \vdots \\ \mathbf{y}_N^{(1)}(t_N) \end{bmatrix}, \qquad \overline{\mathbf{M}} = I \otimes M = \operatorname{kron}(I, M),$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}_N(t_0) \\ \mathbf{y}_N(t_1) \\ \vdots \\ \mathbf{y}_N(t_N) \end{bmatrix}, \quad (26)$$

where $\otimes$ denotes the Kronecker product and I is the identity matrix of dimension $N + 1$.

With the aid of relation (19) and the collocation points (23), we gain

$$\mathbf{y}_N(t_s) = \overline{\mathbf{X}}(t_s)\mathbf{A}, \quad \mathbf{y}_N^{(1)}(t_s) = \overline{\mathbf{X}}(t_s)\overline{\mathbf{B}}\mathbf{A}, \quad s = 0, 1, \ldots, N, \quad (27)$$

which can be written as

$$\mathbf{Y} = \mathbf{XA}, \qquad \mathbf{Y}^{(1)} = \mathbf{X}\overline{\mathbf{B}}\mathbf{A}, \quad (28)$$

where

$$\mathbf{X} = \begin{bmatrix} \overline{\mathbf{X}}(t_0) \\ \overline{\mathbf{X}}(t_1) \\ \vdots \\ \overline{\mathbf{X}}(t_N) \end{bmatrix}, \qquad \overline{\mathbf{X}}(t_s) = \begin{bmatrix} \mathbf{X}(t_s) & 0 \\ 0 & \mathbf{X}(t_s) \end{bmatrix}, \quad (29)$$

$$s = 0, 1, \ldots, N.$$

If the relation (28) is substituted into (25), the fundamental matrix equation is obtained as

$$\left\{\mathbf{X}\overline{\mathbf{B}} - \overline{\mathbf{M}}\mathbf{X}\right\}\mathbf{A} = \mathbf{O}. \quad (30)$$

Thus, the fundamental matrix equation (30) corresponding to (7) can be written in the form

$$\mathbf{WA} = \mathbf{O} \quad \text{or} \quad [\mathbf{W};\mathbf{O}], \quad (31)$$

which corresponds to a linear system of $2(N + 1)$ algebraic equations in $2(N + 1)$ the unknown monomial coefficients so that

$$\mathbf{W} = \left\{\mathbf{X}\overline{\mathbf{B}} - \overline{\mathbf{M}}\mathbf{X}\right\} = \left[w_{p,q}\right], \quad p, q = 1, 2, \ldots, 2(N + 1). \quad (32)$$

By the aid of the relation (19), the matrix form for the boundary conditions which are given in (7) can be written as

$$\mathbf{UA} = \left[\overline{\mathbf{R}}\right], \quad \text{or} \quad \left[\mathbf{U};\overline{\mathbf{R}}\right]. \quad (33)$$

Finally, by replacing the rows of the matrices $[\mathbf{U};\overline{\mathbf{R}}]$ by the last rows of the matrices $[\mathbf{W};\mathbf{O}]$, we obtain the new augmented matrix

$$\widetilde{\mathbf{W}}\mathbf{A} = \widetilde{\mathbf{O}}. \quad (34)$$

The unknown monomials coefficients which exist in $\mathbf{A}$ are determined by solving this linear system, and hence $a_{i,0}, a_{i,1}, \ldots, a_{i,N}$, $(i = 1, 2)$ are substituted in (14). Therefore, we find the approximated solutions

$$x_N(t) = \sum_{n=0}^{N} a_{1,n} t^n, \qquad \lambda_N(t) = \sum_{n=0}^{N} a_{2,n} t^n. \quad (35)$$

We can easily check the accuracy of the method. Since the truncated monomial series (14) are the approximate solutions of (7), when the functions $x_N(t)$, $\lambda_N(t)$ and their derivatives are substituted in (7), the resulting equation must be satisfied approximately; that is, for $t = t_q \in [t_0, t_f]$, $q = 0, 1, 2, \ldots$

$$\begin{aligned} E_1(t_q) = {} & \left| x^{(1)}(t_q) - \left[A(t) - B(t)R^{-1}Q^T\right] x(t_q) \right. \\ & \left. + B(t)R^{-1}B(t)^T \lambda(t_q) \right| \cong 0, \\ E_2(t_q) = {} & \left| \lambda^{(1)}(t_q) - \left[-P + QR^{-1}Q^T\right] x(t_q) \right. \\ & \left. - \left[QR^{-1}B(t)^T - A(t)^T\right] \lambda(t_q) \right| \cong 0, \end{aligned} \quad (36)$$

and $E_{i,N}(t_q) \le 10^{-k_q}$, $i = 1, 2$ (k_q positive integer).

If $\max 10^{-k_q} = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{i,N}(t_q)$ at each of the points becomes smaller than the prescribed 10^{-k}, see [24].

Remark 2. We must recall that a similar approach can be applied for the *state space analysis* problem (13). Moreover, as we say before, for solving a general nonlinear system of ODEs such as (11), we apply a generalization of the collocation method that was proposed in [29].

4. Numerical Examples

In this section, several numerical examples are given to illustrate the accuracy and effectiveness of the proposed method. All calculations are designed in MAPLE 13 and run on a Pentium 4 PC Laptop with 2 GHz of CPU and 2 GB of RAM. In this regard, in tables and figures, we report the absolute error functions associated to the trajectory and control variables and also the approximated values of performance index. In the first example, we provide an OCP that was recently considered by a new method (which is based on the operational matrix of integration of Triangular functions) [30] and reach to more accurate results. Also, in the second example, we consider another OCP (or *Hamiltonian system*) with time variant dynamical system, in which our results have more accuracy and credit with regard to methods [30, 31]. Moreover, we consider a nonlinear OCP as our third numerical illustration. In the fourth example, we provide a *state space analysis* problem together with a full comparison with the methods that are based on the operational matrices of integration such as Bessel [11] and Laguerre [32].

Example 3 (see [30] linear Hamiltonian system). Consider the problem of minimizing

$$J = \frac{1}{2}\int_0^1 \left(2x^2(t) + u^2(t)\right) dt, \tag{37}$$

subject to

$$\dot{x}(t) = -\frac{1}{2}x(t) + u(t), \qquad x(0) = 1. \tag{38}$$

The purpose is to find the optimal control $u(t)$ which minimizes (37) subject to (38). The Optimal value of performance index for this problem is $J^* = 0.463566653481105$ and also exact solutions have been given in [30] as

$$\begin{aligned} x^*(t) &= \frac{1}{2+e^{-3}}\left(2e^{-(3t/2)} + e^{-3+(3t/2)}\right), \\ u^*(t) &= \frac{1}{2+e^{-3}}\left(-e^{-(3t/2)} + e^{-3+(3t/2)}\right). \end{aligned} \tag{39}$$

Since the objective function of this OCP is convex, therefore the following necessary conditions (i.e., *linear Hamiltonian system*) for optimality are also sufficient:

$$\begin{aligned} \dot{x}(t) &= -\frac{1}{2}x(t) - \lambda(t), \\ \dot{\lambda}(t) &= -2x(t) + \frac{1}{2}\lambda(t), \\ x(0) &= 1, \qquad \lambda(1) = 0. \end{aligned} \tag{40}$$

Hence, we need to solve the previous system of differential equations such that the obtained numerical solution is the optimal solution of problem (37)-(38). It should be noted that according to (6) the optimal control is computed by $u^*(t) = -\lambda(t)$, where $\lambda(t)$ is the solution of the previous system.

We solve this problem by using our proposed method in the cases of $N = 4, 5, 6, 7$, and 8. The approximated solutions corresponding to these values of N are provided below

$$\begin{aligned} x_4(t) &= 0.130043t^4 - 0.4921311t^3 + 1.1171671t^2 \\ &\quad - 1.4266579t + 1.0, \\ u_4(t) &= -0.1020785t^4 + 0.5247564t^3 - 1.0326910t^2 \\ &\quad + 1.5366710t - 0.9266579, \\ x_5(t) &= -0.0317935t^5 + 0.1886526t^4 - 0.5272247t^3 \\ &\quad + 1.123919t^2 - 1.427164t + 1.0, \\ u_5(t) &= 0.04038275t^5 - 0.1772002t^4 + 0.5697892t^3 \\ &\quad - 1.042226t^2 + 1.536418t - 0.9271638, \\ x_6(t) &= 0.009145352t^6 - 0.05359579t^5 + 0.2075571t^4 \\ &\quad - 0.5343086t^3 + 1.124911t^2 - 1.4271300t + 1.0, \\ u_6(t) &= -0.006951979t^6 + 0.05688585t^5 - 0.1914114t^4 \\ &\quad + 0.5750789t^3 - 1.042907t^2 + 1.536435t \\ &\quad - 0.9271298, \\ x_7(t) &= -0.001606958t^7 + 0.01380419t^6 - 0.05883564t^5 \\ &\quad + 0.210416t^4 - 0.5350703t^3 + 1.124991t^2 \\ &\quad - 1.427133t + 1.0, \\ u_7(t) &= 0.002084015t^7 - 0.01300909t^6 + 0.06371979t^5 \\ &\quad - 0.1951577t^4 + 0.5760817t^3 - 1.043019t^2 \\ &\quad + 1.536433t - 0.9271335, \\ x_8(t) &= 0.0003561777t^8 - 0.002824592t^7 + 0.01548942t^6 \\ &\quad - 0.06004358t^5 + 0.2108896t^4 - 0.5351673t^3 \\ &\quad + 1.124999t^2 - 1.427133t + 1.0, \\ u_8(t) &= -0.0002657152t^8 + 0.00299044t^7 - 0.01426032t^6 \\ &\quad + 0.06461366t^5 - 0.1955065t^4 + 0.5761526t^3 \\ &\quad - 1.043024t^2 + 1.536433t - 0.9271333. \end{aligned} \tag{41}$$

The associated performance indexes for the selected values of N are $J_4 = 0.463550469$, $J_5 = 0.4635575131$, $J_6 = 0.4635665731$, $J_7 = 0.463566618$, and $J_8 = 0.4635666532$. We provide the $e_{x_N} = \max_{0\le t\le 1}|x_N(t) - x^*(t)|$, $e_{J_N} = |J_N - J^*|$ associated to our proposed method (PM) and a new method that is based on the operational matrix of integration of Triangular functions [30] for different values of N in Table 1. It can be seen from this table that our obtained results for such considered values of N (i.e., 4, 5, 6, 7, and 8) are the same and equal to the obtained results of [30] for higher values of N such as 4, 8, 16, 32, and 64 in computation of e_{x_N}. Moreover, our results corresponding to the e_{J_N} are more accurate with regard to the method of [30] even by choosing lower values of N.

Example 4 (see [30, 31] linear Hamiltonian system). Consider the linear time-varying system

$$\dot{x}(t) = tx(t) + u(t), \tag{42}$$

with the cost functional

$$J = \frac{1}{2}\int_0^1 \left(x^2(t) + u^2(t)\right) dt. \tag{43}$$

The problem is to obtain the optimal control $u^*(t)$ which minimizes (43) subject to (42). The optimal control is

$$u^*(t) = -K(t)\,x^*(t), \tag{44}$$

where $K(t)$ is the feedback controller gain matrix and the solution of the Riccati equation [30]

$$K(t) = K^2(t) - 2tK(t) - 1, \qquad K(1) = 0. \tag{45}$$

According to the optimality conditions (5) and (6) we have

$$\begin{gathered} \dot{x} = tx(t) - \lambda(t), \qquad \dot{\lambda} = -x(t) - t\lambda(t), \\ x(0) = 1, \qquad \lambda(1) = 0, \qquad u(t) = -\lambda(t). \end{gathered} \tag{46}$$

We first solve the previous system and obtain the numerical solutions $x_N(t)$ and $\lambda_N(t)$ for $N = 4, 5, 6$, and then solve (45) by ODE solver commands which exist in MAPLE 13. Since, $K(t) = -(u(t)/x(t))$ then our numerical results of $K(t)$ are equal to $-(u_N(t)/x_N(t))$, that is, $K_N(t) = -(u_N(t)/x_N(t))$. The numerical results of system (46), which are obtained by the proposed method could be deduced as

$$\begin{aligned} x_4(t) &= 0.20203304t^4 - 0.31263421t^3 + 1.0000762t^2 \\ &\quad - 0.96700804t + 1.0, \\ u_4(t) &= 0.011211016t^4 - 0.055485904t^3 + 0.011282926t^2 \\ &\quad + 1.0t - 0.96700804, \\ x_5(t) &= 0.021615359t^5 + 0.16221315t^4 - 0.28939672t^3 \\ &\quad + 0.99513749t^2 - 0.96857084t + 1.0, \\ u_5(t) &= 0.047811786t^5 - 0.078776302t^4 - 0.00037310025t^3 \\ &\quad - 0.000091547293t^2 + 1.0t - 0.96857084, \\ x_6(t) &= 0.043743856t^6 - 0.083312898t^5 + 0.25396668t^4 \\ &\quad - 0.32429532t^3 + 1.0001826t^2 - 0.96853474t + 1.0, \\ u_6(t) &= -0.002379348t^6 + 0.052612074t^5 - 0.081954336t^4 \\ &\quad + 0.00027888358t^3 - 0.000022529694t^2 + 1.0t \\ &\quad - 0.96853474. \end{aligned} \tag{47}$$

Also, the exact solution $K(t)$ of the Riccati equation (45) at the uniform mesh in the interval $(0, 1)$ are $K(0) = 9.6854e-001$, $K(0.1) = 9.5147e-001$, $K(0.2) = 9.1063e-001$, $K(0.3) = 8.4416e-001$, $K(0.4) = 7.5241e-001$, $K(0.5) = 6.3856e-001$, $K(0.6) = 5.0873e-001$, $K(0.7) = 3.7127e-001$, $K(0.8) = 2.3540e-001$, $K(0.9) = 1.0955e-001$, and $K(1) = 0$. In Table 2, we provide the absolute values of errors at the selected points for the previously considered values of N together with the same errors associated with other methods [30, 31]. Again, we can see the accuracy of method with regard to the methods that are based on operational matrices of integration.

Example 5 (nonlinear Hamiltonian system). As our third illustration, consider the following nonlinear optimal control problem:

$$\begin{aligned} &\min \quad J = \int_0^1 u^2(t)\,dt \\ &\text{s.t.} \quad \dot{x} = \frac{1}{2}x^2(t)\sin x(t) + u(t), \quad t \in [0,1] \\ &\qquad\quad x(0) = 0, \qquad x(1) = 0.5. \end{aligned} \tag{48}$$

Trivially $f(t, x(t)) = (1/2)x^2(t)\sin x(t)$, $g(t, x(t)) = 1$, $Q = 0$, $R = 1$, $t_0 = 0$, and $t_f = 1$. As mentioned in Section 2.2, we solve the following system of ordinary differential equations:

$$\begin{gathered} \dot{x} = \frac{1}{2}x^2(t)\sin x(t) - \frac{1}{2}\lambda(t), \\ \dot{\lambda} = -\lambda(t)\,x(t)\sin x(t) - \frac{1}{2}\lambda(t)\,x^2(t)\cos x(t), \\ t \in [0,1], \\ x(0) = 0, \qquad x(1) = 0.5, \qquad \lambda(1) = 0. \end{gathered} \tag{49}$$

Also the optimal control law is given by

$$u^*(t) = -\frac{1}{2}\lambda(t). \tag{50}$$

Similar to the linear cases, we suppose that the state and costate variables could be written in terms of linear combination of monic polynomials which are defined in Section 3, with the unknown monomial coefficients. These coefficients will be determined after imposing the previous system of differential equations at the uniform mesh in the interval $(0, 1)$. In other words, applying these collocation points to the main system together with the considered

Table 1: Comparison results of Example 3.

N	e_{x_N} of PM	e_{J_N} of PM	N	e_{x_N} of TFM [30]	e_{J_N} of TFM [30]
4	$1.8441e-003$	$1.6184e-005$	4	$7.68816e-03$	$4.27347e-03$
5	$2.2886e-004$	$9.1404e-006$	8	$3.19260e-03$	$1.06808e-03$
6	$3.3538e-005$	$8.0381e-008$	16	$1.74696e-04$	$2.67002e-04$
7	$4.5138e-005$	$3.5481e-008$	32	$4.36429e-05$	$6.67494e-05$
8	$3.0718e-005$	$2.8111e-010$	64	$1.09092e-05$	$1.66873e-05$

Table 2: The error histories at the selected points for Example 4.

t	PM for $N=4$	PM for $N=5$	PM for $N=6$	TFM [30] $N=6$	MGL [31] $N=6$	TFM [30] $N=64$
0.0	$1.5274e-003$	$3.5407e-005$	$6.8533e-007$	$7.4890e-002$	$9.3400e-002$	$7.6980e-003$
0.1	$1.9116e-003$	$6.7725e-005$	$1.4752e-006$	$2.2140e-002$	$1.0020e-001$	$1.6770e-003$
0.2	$2.3352e-003$	$9.5549e-005$	$1.7583e-006$	$5.2150e-002$	$1.0660e-001$	$5.6050e-003$
0.3	$2.6500e-003$	$9.9318e-005$	$1.8482e-006$	$7.1120e-002$	$1.5530e-001$	$5.8040e-003$
0.4	$2.8209e-003$	$1.0115e-004$	$2.2289e-006$	$1.5220e-002$	$2.3530e-001$	$2.0900e-003$
0.5	$2.9118e-003$	$1.1223e-004$	$2.5894e-006$	$9.9270e-002$	$2.9390e-001$	$1.0200e-002$
0.6	$2.9909e-003$	$1.1855e-004$	$2.3877e-006$	$2.8060e-002$	$2.7180e-001$	$1.9860e-003$
0.7	$3.0063e-003$	$1.0774e-004$	$2.1119e-006$	$5.7040e-002$	$1.7540e-001$	$5.9630e-003$
0.8	$2.7289e-003$	$9.2586e-005$	$2.4315e-006$	$6.5130e-002$	$7.5600e-002$	$5.5260e-003$
0.9	$1.8221e-003$	$8.1824e-005$	$1.6928e-006$	$1.5240e-002$	$1.8200e-002$	$1.7450e-003$

Table 3: Numerical results of Example 5.

N	J_N of PM	$x_N(1) - x(1)$ of PM
5	0.2426752	0
7	0.2426750	$1.0000e-007$
9	0.2426740	$2.0000e-015$

boundary conditions on $x(t)$ and $\lambda(t)$ transforms the basic problem to the corresponding system of nonlinear algebraic equations. By assuming different values of N such as 5, 7, and 9, we solve the previously mentioned system. In Table 3, we provide the approximated performance index J_N, which is obtained by our proposed method and also the difference between the approximated $x_N(1)-x(1)=x_N(1)-(1/2)$ for the considered values of N.

Example 6 (see [11, 32] state-space analysis). We consider a linear-time invariant state equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + bu(t), \quad t \in (0,1), \tag{51}$$

where

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{52}$$

We are given that the input $u(t)$ is the unit step function in the interval $(0,1)$ and the initial state is

$$x_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{53}$$

The exact solution for (51) is

$$x_1(t) = x_2(t) = \frac{1+e^{-2t}}{2}. \tag{54}$$

We solve this problem by using our proposed method in the cases of $N = 4, 5, 6, 7, 8, 9, 10, 12$, and 20. The approximated solutions corresponding to the $N = 4, 5, 6, 7, 8, 9$, and 10 are provided later

$$\begin{aligned}
x_{1,4}(t) = x_{2,4}(t) = {} & 0.1658t^4 - 0.58031t^3 + 0.98446t^2 \\
& - 1.0t + 1.0, \\
x_{1,5}(t) = x_{2,5}(t) = {} & -0.06235t^5 + 0.28058t^4 - 0.64844t^3 \\
& + 0.99761t^2 - 1.0t + 1.0, \\
x_{1,6}(t) = x_{2,6}(t) = {} & 0.019951t^6 - 0.10973t^5 + 0.32143t^4 \\
& - 0.66365t^3 + 0.99969t^2 - 1.0t + 1.0, \\
x_{1,7}(t) = x_{2,7}(t) = {} & -0.0055374t^7 + 0.035993t^6 \\
& - 0.12776t^5 + 0.33126t^4 - 0.66626t^3 \\
& + 0.99997t^2 - 1.0t + 1.0, \\
x_{1,8}(t) = x_{2,8}(t) = {} & 0.0013547t^8 - 0.010161t^7 \\
& + 0.042378t^6 - 0.13232t^5 + 0.33304t^4 \\
& - 0.66662t^3 + 1.0t^2 - 1.0t + 1.0,
\end{aligned}$$

Table 4: The error histories at the selected points for Example 6.

t	Bessel Int $N = 8$	Bessel Int $N = 12$	Bessel Int $N = 20$	PM for $N = 8$	PM for $N = 12$	PM for $N = 20$
0.0	$9.6890e-003$	$6.6498e-003$	$4.1214e-003$	0	0	0
0.1	$2.3974e-002$	$1.0755e-002$	$1.6289e-003$	$6.5649e-009$	$6.8945e-014$	$6.1062e-014$
0.2	$7.2706e-003$	$1.6306e-003$	$4.6111e-003$	$4.5248e-009$	$5.0071e-014$	$4.8073e-014$
0.3	$2.5780e-003$	$4.6009e-003$	$8.9332e-003$	$3.6167e-009$	$4.1522e-014$	$3.7859e-014$
0.4	$2.2427e-003$	$7.2766e-003$	$1.1642e-002$	$3.2895e-009$	$3.3973e-014$	$3.0309e-014$
0.5	$6.3782e-003$	$1.0152e-002$	$1.3160e-002$	$2.2422e-009$	$2.7311e-014$	$2.4425e-014$
0.6	$6.9801e-003$	$1.1190e-002$	$1.3796e-002$	$2.4272e-009$	$2.2649e-014$	$1.9873e-014$
0.7	$9.5374e-003$	$1.1427e-002$	$1.3807e-002$	$1.1916e-009$	$1.8097e-014$	$1.6431e-014$
0.8	$8.6910e-003$	$1.1990e-002$	$1.3398e-002$	$1.8602e-009$	$1.0991e-014$	$1.3878e-014$
0.9	$1.0169e-002$	$1.1068e-002$	$1.2680e-002$	$4.0343e-009$	$6.6391e-014$	$1.8874e-014$
1.0	$5.8131e-003$	$8.4105e-003$	$1.0482e-002$	$2.2065e-007$	$3.9901e-012$	$8.3072e-012$

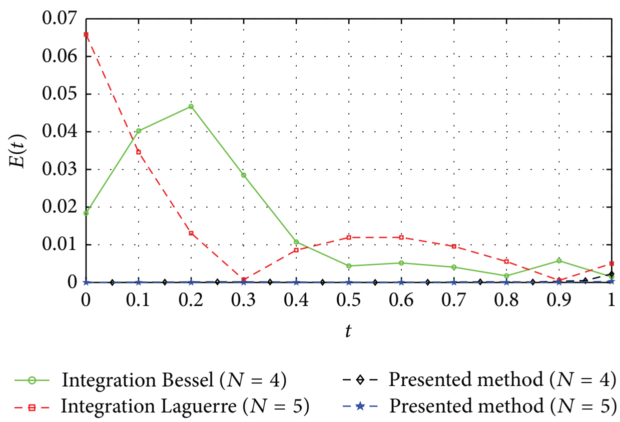

Figure 1: The error histories of our method together with the methods [11, 32] in Example 6.

$$
\begin{aligned}
x_{1,9}(t) = x_{2,9}(t) = & -0.00029605t^9 + 0.0025165t^8 \\
& -0.012061t^7 + 0.044057t^6 - 0.13318t^5 \\
& +0.3333t^4 - 0.66666t^3 + 1.0t^2 \\
& -1.0t + 1.0, \\
x_{1,10}(t) = x_{2,10}(t) = & \ 0.000058425t^{10} - 0.00055504t^9 \\
& +0.003006t^8 - 0.012576t^7 \\
& +0.044386t^6 - 0.13331t^5 + 0.33333t^4 \\
& -0.66667t^3 + 1.0t^2 - 1.0t + 1.0.
\end{aligned}
\tag{55}
$$

An interesting comparison between our presented method (PM) and the method of [11] in the absolute value of the errors at the uniform mesh for $N = 8, 12$, and 20 is considered in Table 4. Moreover, the error histories in the computational interval $(0, 1)$ associated with our method for $N = 4$ and 5 together with the error history of the method [11] (Bessel Integration) for $N = 4$ and also the error history of the method [32] (Laguerre Integration) for $N = 5$ are depicted in Figure 1. From this figure one can see the applicability and high accuracy of the presented method with respect to the methods which are based on operational matrices of integration such as [11, 32].

5. Conclusions

The aim of this paper is to present an indirect approach for solving optimal control problems using truncated monomial series together with the collocation method on a uniform mesh. Our method applies an operational matrix of differentiation which has more efficiency with respect to the integration ones. Operational matrices of differentiation have several specific properties that other integration ones do not have them. One of them is that through using differentiation ones, we solve our problem directly and do not need to integrate the dynamical system. Another property is that the fundamental relations, which are based on differentiation matrices, are the exact relations, meanwhile the methods which are based on integration matrices impose the approximation to the main problem. These properties are shown through several numerical examples such as state-space analysis and specially optimal control problems.

Conflict of Interests

The authors declare that they do not have any conflict of interests in their submitted paper.

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Research Article

A Note on Fractional Order Derivatives and Table of Fractional Derivatives of Some Special Functions

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The purpose of this note is to present the different fractional order derivatives definition that are commonly used in the literature on one hand and to present a table of fractional order derivatives of some functions in Riemann-Liouville sense On other the hand. We present some advantages and disadvantages of these fractional derivatives. And finally we propose alternative fractional derivative definition.

1. Introduction

Fractional calculus has been used to model physical and engineering processes, which are found to be best described by fractional differential equations. It is worth nothing that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. Major topics include anomalous diffusion, vibration and control, continuous time random walk, Levy statistics, fractional Brownian motion, fractional neutron point kinetic model, power law, Riesz potential, fractional derivative and fractals, computational fractional derivative equations, nonlocal phenomena, history-dependent process, porous media, fractional filters, biomedical engineering, fractional phase-locked loops, fractional variational principles, fractional transforms, fractional wavelet, fractional predator-prey system, soft matter mechanics, fractional signal and image processing; singularities analysis and integral representations for fractional differential systems; special functions related to fractional calculus, non-Fourier heat conduction, acoustic dissipation, geophysics, relaxation, creep, viscoelasticity, rheology, fluid dynamics, chaos and groundwater problems. An excellent literature of this can be found in [1–9]. These entire models are making use of the fractional order derivatives that exist in the literature. However, there are many of these definitions in the literature nowadays, but few of them are commonly used, including Riemann-Liouville [10, 11], Caputo [5, 12], Weyl [10, 11, 13], Jumarie [14, 15], Hadamard [10, 11], Davison and Essex [16], Riesz [10, 11], Erdelyi-Kober [10, 11], and Coimbra [17]. All these fractional derivatives definitions have their advantages and disadvantages. The purpose of this note is to present the result of fractional order derivative for some function and from the results establish the disadvantages and advantages of these fractional order derivative definitions. We shall start with the definitions.

2. Definitions

There exists a vast literature on different definitions of fractional derivatives. The most popular ones are the Riemann-Liouville and the Caputo derivatives. For Caputo we have

$$ {}_{0}^{c}D_{x}^{\alpha}\left(f\left(x\right)\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{x}\left(x-t\right)^{n-\alpha-1}\frac{d^{n}f\left(t\right)}{dt^{n}}dt, \quad n-1<\alpha\leq n. \tag{1}$$

For the case of Riemann-Liouville we have the following definition:

$$D_x^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\,dt. \quad (2)$$

Guy Jumarie proposed a simple alternative definition to the Riemann-Liouville derivative:

$$D_x^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} \{f(t) - f(0)\}\,dt. \quad (3)$$

For the case of Weyl we have the following definition:

$$D_x^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^{\infty} (x-t)^{n-\alpha-1} f(t)\,dt. \quad (4)$$

With the Erdelyi-Kober type we have the following definition:

$$D_{0,\sigma,\eta}^{\alpha}(f(x)) = x^{-n\sigma}\left(\frac{1}{\sigma x^{\sigma-1}}\frac{d}{dx}\right)^n x^{\sigma(n+\eta)} I_{0,\sigma,\eta+\sigma}^{n-\alpha}(f(x)), \quad \sigma > 0. \quad (5)$$

Here

$$I_{0,\sigma,\eta+\sigma}^{\alpha}(f(x)) = \frac{\sigma x^{-\sigma(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^x \frac{t^{\sigma\eta+\sigma-1} f(t)}{(t^{\sigma} - x^{\sigma})^{1-\alpha}}\,dt. \quad (6)$$

With Hadamard type, we have the following definition:

$$D_0^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)}\left(x\frac{d}{dx}\right)^n \int_0^x \left(\log\frac{x}{t}\right)^{n-\alpha-1} f(t)\,\frac{dt}{t}. \quad (7)$$

With Riesz type, we have the following definition:

$$D_x^{\alpha}(f(x)) = -\frac{1}{2\cos(\alpha\pi/2)} \times \left\{\frac{1}{\Gamma(\alpha)}\left(\frac{d}{dx}\right)^m \times \left(\int_{-\infty}^x (x-t)^{m-\alpha-1} f(t)\,dt + \int_x^{\infty} (t-x)^{m-\alpha-1} f(t)\,dt\right)\right\}. \quad (8)$$

We will not mention the Grunward-Letnikov type here because it is in series form. This is not more suitable for analytical purpose. In 1998, Davison and Essex [16] published a paper which provides a variation to the Riemann-Liouville definition suitable for conventional initial value problems within the realm of fractional calculus. The definition is as follows:

$$D_0^{\alpha} f(x) = \frac{d^{n+1-k}}{dx^{n+1-k}} \int_0^x \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d^k f(t)}{dt^k}\,dt. \quad (9)$$

In an article published by Coimbra [17] in 2003, a variable order differential operator is defined as follows:

$$D_0^{\alpha(t)}(f(x)) = \frac{1}{\Gamma(1-\alpha(x))} \int_0^x (x-t)^{-\alpha(t)} \frac{df(t)}{dt}\,dt + \frac{(f(0^+) - f(0^-))\,x^{-\alpha(x)}}{\Gamma(1-\alpha(x))}. \quad (10)$$

3. Table of Fractional Order Derivative for Some Functions

In this section we present the fractional of some special functions. The fractional derivatives in Table 1 are in Riemann-Liouville sense.

In Table 1, HypergeometricPFQ [{}, {}, {}] is the generalized hypergeometric function which is defined as follows in the Euler integral representation:

$$F_1(a,b,c,z) = \frac{\Gamma(a)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt, \quad c \in \mathbb{C}\setminus\mathbb{Z}_0^-,$$
$$a,b \in \mathbb{C}\ (0 < \mathrm{Re}[b] < \mathrm{Re}[c]\,;\,|\arg(1-z)| < \pi). \quad (11)$$

The PolyGamma$[n,z]$ and PolyGamma$[z]$ are the logarithmic derivative of gamma function given by

$$\mathrm{PolyGamma}[n,z] = \frac{d^n}{dz^n}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right),$$
$$\mathrm{PolyGamma}[z] = \mathrm{PolyGamma}[0,z]. \quad (12)$$

These functions are meromorphic of z with no branch cut discontinuities. $E_{\alpha}(-t^{\alpha})$ is the generalized Mittag-Leffler function and is defined as

$$E_{\alpha}(-t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-t^{\alpha})^k}{\Gamma(k\alpha+1)}. \quad (13)$$

Γ is denotes the gamma function, which is the Mellin transform of exponential function and is defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\,dt, \quad \mathrm{Re}[z] > 0. \quad (14)$$

$J_n(x)$, $K_n(x)$, and $Y_n(x)$ are Bessel functions first and second kind. Zeta$[s]$ is the zeta function, has no branch cut discontinuities, and is defined as

$$\mathrm{Zeta}[x] = \sum_{n=1}^{\infty} n^{-x}. \quad (15)$$

Table 1: Fractional order derivatives for some functions.

Functions	L-fractional derivatives
$x^{\beta}, \beta > -1$	$\dfrac{x^{-\alpha+\beta}\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)}$
$\mathrm{Cos}(ax), a \in \mathbb{R}$	$\dfrac{x^{-\alpha}\,\mathrm{HypergeometricPFQ}\left[\{1\},\{1-\alpha/2, 3/2-\alpha/2\} - (1/4)a^2x^2\right]}{\Gamma(1-\alpha)} - \dfrac{a^2x^{2-\alpha}\,\mathrm{HypergeometricPFQ}\left[\{2\},\{2-\alpha/2, 5/2-\alpha/2\} - (1/4)a^2x^2\right]}{\Gamma(4-\alpha)}$
$\mathrm{Sin}(ax), a \in \mathbb{R}$	$\dfrac{ax^{1-\alpha}(2-\alpha)\,\mathrm{HypergeometricPFQ}\left[\{1\},\{2-\alpha/2, 3/2-\alpha/2\} - (1/4)a^2x^2\right]}{\Gamma(2-\alpha)} - \dfrac{2a^3x^{3-\alpha}\,\mathrm{HypergeometricPFQ}\left[\{2\},\{5/2-\alpha/2, 3-\alpha/2\} - (1/4)a^2x^2\right]}{\Gamma(5-\alpha)}$
$\ln(x)$	$-\dfrac{x^{-\alpha}\left(\mathrm{EulerGamma} + \pi\mathrm{Cot}[\pi\alpha] - \ln(x) + \mathrm{PolyGamma}[0,\alpha]\right)}{\Gamma(1-\alpha)}$
$e^{ax}, a \in \mathbb{R}$	$\left(\dfrac{a^{\alpha}\left((ax)^{-\alpha} + e^{ax}\left(\Gamma(1-\alpha) - \Gamma(1-\alpha, ax)\right)\right)}{\Gamma(1-\alpha)}\right)$
$\mathrm{Sinh}(ax)$	$\left(\dfrac{e^{-ax}\left(-a^2x^2\right)^{-\alpha}}{2\Gamma(1-\alpha)}\left(\left(-a^2x^2\right)^{\alpha}\right)\left((-a)^{\alpha}\left(\alpha\Gamma(-\alpha) + \Gamma(1-\alpha,-ax)\right) + a^{\alpha}e^{2ax}\left(\Gamma(1-\alpha) - \Gamma(1-\alpha, ax)\right)\right)\right)$
$\mathrm{Cosh}(ax)$	$\left(\dfrac{1}{2\Gamma(1-\alpha)}\left(e^{-ax}\left(-a^2x^2\right)^{-\alpha}\left((-a)^{\alpha}(ax)^{\alpha}\left(e^{ax} + ax\,\mathrm{ExpIntegral}E[\alpha,-ax]\right) + a^{\alpha}e^{ax}(-ax)^{\alpha}\left(1 - ae^{ax}xax\,\mathrm{ExpIntegral}E[\alpha,ax]\right) + \left((-a)^{\alpha} + a^{\alpha}e^{2ax}\right)\left(-a^2x^2\right)^{\alpha}\Gamma(1-\alpha)\right)\right)\right)$
$\mathrm{Arcsin}(x)$, $0 < x < 1$	$\left(\dfrac{x^{1-\alpha}\,\mathrm{HypergeometricPFQ}\left[\{1/2,1/2,1\},\{3/2-\alpha/2, 2-\alpha/2\}, x^2\right]}{\Gamma(2-\alpha)} + \dfrac{2x^{3-\alpha}\mathrm{HypergeometricPFQ}\left[\{3/2,3/2,2\},\{5/2-\alpha/2, 3-\alpha/2\}, x^2\right]}{\Gamma(5-\alpha)}\right)$
$\mathrm{Arccos}(x)$, $0 < x < 1$	$\left(-\dfrac{x^{-\alpha}\left(\pi(-2+\alpha) + 2x\,\mathrm{HypergeometricPFQ}\left[\{1/2,1/2,1\},\{3/2-\alpha/2, 2-\alpha/2\}, x^2\right]\right)}{2(2-\alpha)\Gamma(1-\alpha)} - \dfrac{x^{1-\alpha}\,\mathrm{HypergeometricPFQ}\left[\{1/2,1/2,1\},\{3/2-\alpha/2, 2-\alpha/2\}, x^2\right]}{2\left(2-3\alpha+\alpha^2\right)\Gamma(1-\alpha)}\right.$ $\left. - \dfrac{x^{3-\alpha}\,\mathrm{HypergeometricPFQ}\left[\{3/2,3/2,2\},\{5/2-\alpha/2, 3-\alpha/2\}, x^2\right]}{2\left(2-3\alpha+\alpha^2\right)\Gamma(1-\alpha)\left((3/2-\alpha/2)(2-\alpha/2)\right)}\right)$
$\mathrm{Arctan}(x)$	$-\dfrac{x^{1-\alpha}}{\Gamma(5-\alpha)}\left((-4+\alpha)(-3+\alpha)(-2+\alpha)\,\mathrm{HypergeometricPFQ}\left[\left\{\dfrac{1}{2},1,1\right\},\left\{\dfrac{3}{2}-\dfrac{\alpha}{2}, 2-\dfrac{\alpha}{2}\right\} -x^2\right] + 4x^2\,\mathrm{HypergeometricPFQ}\left[\left\{\dfrac{3}{2},2,2\right\},\left\{\dfrac{5}{2}-\dfrac{\alpha}{2}, 3-\dfrac{\alpha}{2}\right\}, -x^2\right]\right)$
$\int_x^{\infty}\dfrac{e^{-y}}{y}dy$	$\dfrac{1}{\Gamma(3-\alpha)}\left(e^{-x}x^{-1-\alpha}\left((-x)^{\alpha}(-2+\alpha)\left(\Gamma(2-\alpha) - \Gamma(2-\alpha,-x)\right)\right.\right.$ $\left.\left. + e^{x}x(-1+\alpha)\left(x\,\mathrm{HypergeometricPFQ}\left[\{1,1\},\{2,3-\alpha\},-x\right](-2+\alpha)\left(\pi\mathrm{Cot}(\pi\alpha) - \ln(x) + \mathrm{PolyGamma}(0,\alpha)\right)\right)\right)\right)$
$E_{\alpha}\left(-t^{\alpha}\right)$	$\dfrac{x^{-\alpha}}{\Gamma(1-\alpha)} - E_{\alpha}\left(-t^{\alpha}\right)$
$J_n(x)$, $\mathrm{Re}[n] > -1$	$\dfrac{2^{-n}x^{n-\alpha}(1+n-\alpha)}{\Gamma(2+n-\alpha)}\,\mathrm{HypergeometricPFQ}\left[\left\{\dfrac{1}{2}+\dfrac{n}{2}, 1+\dfrac{n}{2}\right\},\left\{1+n, 1+\dfrac{n}{2}-\dfrac{\alpha}{2}, \dfrac{3}{2}+\dfrac{n}{2}-\dfrac{\alpha}{2}\right\}, \dfrac{-x^2}{4}\right]$ $+\dfrac{2^{-1-n}(1/2+n/2)(1+n/2)x^{2+n-\alpha}}{(1+n)(1+n/2-\alpha/2)(3/2+n/2-\alpha/2)\Gamma(2+n-\alpha)}\,\mathrm{HypergeometricPFQ}\left[\left\{\dfrac{3}{2}+\dfrac{n}{2}, 2+\dfrac{n}{2}\right\},\left\{2+n, 2+\dfrac{n}{2}-\dfrac{\alpha}{2}, \dfrac{5}{2}+\dfrac{n}{2}-\dfrac{\alpha}{2}\right\}, \dfrac{-x^2}{4}\right]$
$K_n(x)$, $1 > \mathrm{Re}[n] > -1$	$2^{-1-n}\pi x^{1-n-\alpha}\mathrm{Csc}(n\pi)\left(\dfrac{4^n}{\Gamma(2+n-\alpha)}\,\mathrm{HypergeometricPFQ}\left[\left\{\dfrac{1}{2}-\dfrac{n}{2}, 1-\dfrac{n}{2}\right\},\left\{1-n, 1-\dfrac{n}{2}-\dfrac{\alpha}{2}, \dfrac{3}{2}-\dfrac{n}{2}-\dfrac{\alpha}{2}\right\}, \dfrac{x^2}{4}\right]\right.$ $\left. + \dfrac{x^{2n}}{\Gamma(2+n-\alpha)}\,\mathrm{HypergeometricPFQ}\left[\left\{\dfrac{1}{2}+\dfrac{n}{2}, 1+\dfrac{n}{2}\right\},\left\{1+n, 1+\dfrac{n}{2}-\dfrac{\alpha}{2}, \dfrac{3}{2}+\dfrac{n}{2}-\dfrac{\alpha}{2}\right\}, -\dfrac{x^2}{4}\right]\right)$

Table 1: Continued.

Functions	L-fractional derivatives
$Y_n(x)$, $-1 < \mathrm{Re}[n] < 1$	$2^{-1-n}x^{1-n-\alpha}\left(-\frac{\mathrm{Csc}(n\pi)\,4^n}{\Gamma(2+n-\alpha)}\,\mathrm{HypergeometricPFQ}\left[\left\{\frac{1}{2}-\frac{n}{2},1-\frac{n}{2}\right\},\left\{1-n,1-\frac{n}{2}-\frac{\alpha}{2},\frac{3}{2}-\frac{n}{2}-\frac{\alpha}{2}\right\},-\frac{x^2}{4}\right] + \frac{x^{2n}}{\Gamma(2+n-\alpha)}\,\mathrm{HypergeometricPFQ}\left[\left\{\frac{1}{2}+\frac{n}{2},1+\frac{n}{2}\right\},\left\{1+n,1+\frac{n}{2}-\frac{\alpha}{2},\frac{3}{2}+\frac{n}{2}-\frac{\alpha}{2}\right\},-\frac{x^2}{4}\right]\right)$
$\mathrm{Zeta}[x]$	$\sum_{n=1}^{\infty}\left[\frac{(-1)^{\alpha}(-x)^{-\alpha}}{\Gamma(1-\alpha)}-\frac{n^{-x}\alpha\Gamma(-\alpha)(-\ln(n))^{\alpha}}{\Gamma(1-\alpha)}-\frac{n^{-x}\alpha\Gamma(1-\alpha,-x\ln(n))(-\ln(n))^{\alpha}}{\Gamma(1-\alpha)}\right]$
$\mathrm{Erf}(t)$	$2^{-1+x}x^{1-\alpha}(2-\alpha)\,\mathrm{HypergeometricPFQRegularized}\left[\left\{\frac{1}{2},1\right\},\left\{\frac{3}{2}-\frac{\alpha}{2},2-\frac{\alpha}{2}\right\}-x^2\right]+2^{-1+x}x^{3-\alpha}\,\mathrm{HypergeometricPFQRegularized}\left[\left\{\frac{3}{2},2\right\},\left\{\frac{5}{2}-\frac{\alpha}{2},3-\frac{\alpha}{2}\right\}-x^2\right]$

The above obtained special functions as derivation of Riemann-Liouville fractional derivative are solution of some fractional ordinary differential equation, for instance, Cauchy type.

4. Advantages and Disadvantages

4.1. Advantages. It is very important to point out that all these fractional derivative order definitions have their advantages and disadvantages; here we will include Caputo, variational order, Riemann-Liouville Jumarie and Weyl. We will examine first the Variational order differential operator. Anomalous diffusion phenomena are extensively observed in physics, chemistry, and biology fields [18–21]. To characterize anomalous diffusion phenomena, constant-order fractional diffusion equations are introduced and have received tremendous success. However, it has been found that the constant order fractional diffusion equations are not capable of characterizing some complex diffusion processes, for instance, diffusion process in inhomogeneous or heterogeneous medium [22]. In addition, when we consider diffusion process in porous medium, if the medium structure or external field changes with time, in this situation, the constant-order fractional diffusion equation model cannot be used to well characterize such phenomenon [23, 24]. Still in some biology diffusion processes, the concentration of particles will determine the diffusion pattern [25, 26]. To solve the above problems, the variable-order (VO) fractional diffusion equation models have been suggested for use [27]. The ground-breaking work of VO operator can be traced to Samko et al. by introducing the variable order integration and Riemann-Liouville derivative in [27]. It has been recognized as a powerful modelling approach in the fields of viscoelasticity [17–32] viscoelastic deformation [28], viscous fluid [29] and anomalous diffusion [30]. With the Jumarie definition which is actually the modified Riemann-Liouville fractional derivative, an arbitrary continuous function needs not to be differentiable; the fractional derivative of a constant is equal to zero and more importantly it removes singularity at the origin for all functions for which $f(0) =$ constant for instance, the exponentials functions and Mittag-Leffler functions. With the Riemann-Liouville fractional derivative, an arbitrary function needs not to be continuous at the origin and it needs not to be differentiable. One of the great advantages of the Caputo fractional derivative is that it allows traditional initial and boundary conditions to be included in the formulation of the problem [5, 12]. In addition its derivative for a constant is zero. It is customary in groundwater investigations to choose a point on the centerline of the pumped borehole as a reference for the observations and therefore neither the drawdown nor its derivatives will vanish at the origin, as required [33]. In such situations where the distribution of the piezometric head in the aquifer is a decreasing function of the distance from the borehole, the problem may be circumvented by rather using the complementary, or Weyl, fractional order derivative [33].

4.2. Disadvantages. Although these fractional derivative display great advantages, they are not applicable in all the situations. We shall begin with the Liouville-Riemann type. The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. The Riemann-Liouville derivative of a constant is not zero. In addition, if an arbitrary function is a constant at the origin, its fractional derivation has a singularity at the origin for instant exponential and Mittag-Leffler functions. Theses disadvantages reduce the field of application of the Riemann-Liouville fractional derivative. Caputo's derivative demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first-order derivative might have fractional derivatives of all orders less than one in the Riemann-Liouville sense. With the Jumarie fractional derivative, if the function is not continuous at the origin, the fractional derivative will not exist, for instance what will be the fractional derivative of $\ln(x)$ and many other ones. Variational order differential operator cannot easily be handled analytically. Numerical approach is sometimes needs to deal with the problem under investigation. Although Weyl fractional derivative found its place in groundwater investigation, it still displays a significant disadvantage; because the integral defining these Weyl derivatives is improper, greater restrictions must be placed on a function. For instance, the Weyl derivative of a constant is not defined. On the other hand, general theorem about Weyl derivatives are often more difficult to formulate and be proved than are corresponding theorems for Riemann-Liouville derivatives.

5. Derivatives Revisited

5.1. Variational Order Differential Operator Revisited. Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$ denotes a continuous but necessary differentiable, let $\alpha(x)$ be a continuous function in $(0, 1]$. Then its variational order differential in $[a, \infty)$ is defined as

$$D_a^{\alpha(t)}(f(x)) = \mathrm{FP}\left(\frac{1}{\Gamma(1-\alpha(x))}\frac{d}{dx} \times \int_a^x (x-t)^{-\alpha(t)}(f(t)-f(a))\,dt\right), \tag{16}$$

where FP means finite part of the variational order operator. Notice that the above derivative meets all the requirements of the variational order differential operator; in additional, the derivative of the constant is zero, which was not possible with the standard version.

5.2. Variational Order Fractional Derivatives via Fractional Difference. Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$ denotes a continuous but necessary differentiable, let $\alpha(x)$ be a continuous function in (0, 1], and $h > 0$ denote a constant discretization span. Define the forward operator FWh by the expression

$$\mathrm{FW}(h)(f(x)) := f(x+h) \tag{17}$$

Note that, the symbol means that the left side is defined by the right side. Then the variational order fractional difference of order $\alpha(x)$ of $f(x)$ is defined by the expression

$$\begin{aligned}\Delta^{\alpha(x)} f(x) &= (\mathrm{FW}-1)^{\alpha(x)}\\ &= \sum_{k=0}^{\infty}(-1)^k \frac{1}{\Gamma(k-\alpha(x))} f(x-(\alpha(x)-k)h).\end{aligned} \tag{18}$$

And its variational order fractional derivative of order $\alpha(x)$ is defined by the limit

$$f^{\alpha(x)}(x) = \lim_{h\to 0} \frac{\Delta^{\alpha(x)}(f(x)-f(0))}{h^{\alpha(x)}}. \tag{19}$$

5.3. Jumarie Fractional Derivative Revisited. Recently, Guy Jumarie proposed a simple alternative definition to the Riemann-Liouville derivative. His modified Riemann-Liouville derivative has the advantage of both standard Riemann-Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (nondifferentiable) functions and the fractional derivative of a constant is equal to zero. However if the function is not defined at the origin, the fractional derivative will not exist, therefore in order to circumvent this defeat we propose the following definition. Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$ denotes a continuous but necessary at the origin and not necessary differentiable, then its fractional derivative is defined as:

$$\begin{aligned}&D_0^{\alpha}(f(x))\\ &\quad= \mathrm{FP}\left(\frac{1}{\Gamma(1-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}(f(t)-f(0))\,dt\right),\end{aligned} \tag{20}$$

where FP means finite part of the fractional derivative order operator. Notice that, the above derivative meets all the requirement of the fractional derivative operator; the derivative of the constant is zero, in addition the function needs not to be continuous at the origin. With this definition, the fractional derivative of $\ln(x)$ is given as

$$\begin{aligned}&D_0^{\alpha}(\ln(x))\\ &\quad= -\frac{x^{-\alpha}(\mathrm{EulerGamma} + \pi\mathrm{Cot}[\pi\,\alpha] - \ln(x) + \mathrm{PolyGamma}[0,\alpha])}{\Gamma(1-\alpha)}.\end{aligned} \tag{21}$$

The above fractional order derivative definition can be used in many field for instance in the field of groundwater. Because this definition does not produce a fractional derivative with any kind of singularity as in the case of Jumarie and the traditional Riemann-Liouville fractional order derivative. This concept was introduced by Hadamard [34–36]. The Hadamard regularization [34–36], based on the concept of finite part ("partie finie") of a singular function or a divergent integral, plays an important role in several branches of Mathematical Physics see [29–37]. Typically one deals with functions admitting some non-integrable singularities on a discrete set of isolated points located at finite distances from the origin. The regularization consists of assigning by definition a value for the function at the location of one of the singular points, and for the generally divergent integral of that function. The definition may not be fully deterministic, as the Hadamard "partie finie" depends in general on some arbitrary constants [38].

6. Discussions and Conclusions

We presented the definitions of the commonly used fractional derivatives operators which are ranging from Riemann-Liouville to Guy Jumarie. We presented the disadvantages and advantages of each definition. No definition has fulfilled the entire requirement needed; for example, the Jumarie definition fulfills some interesting requirements including the derivative of a constant is zero, and a nondifferentiable function may have a fractional derivative. However, if the function is not defined at the origin, it may not have a fractional derivative in Jumarie sense. With the Riemann-Liouville fractional derivative, the function needs not to be continuous at the origin and needs not to be differentiable; however, the derivative of a constant is not zero; in addition, his has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Also if an arbitrary function is a zero at the origin, its fractional derivation has a singularity at the origin, for instance exponential and Mittag-Leffler functions. Theses disadvantages reduce the field of application of the Riemann-Liouville fractional derivative. The Caputo derivative is very useful when dealing with real-world problem because, it allows traditional initial and boundary conditions to be included in the formulation of the problem and in addition the derivative of a constant is zero; however, functions that are not differentiable do not have fractional derivative, which reduces the field of application of Caputo derivative. It is in addition important to notice that, to characterize anomalous diffusion phenomena, constant-order fractional diffusion equations have been introduced and have received tremendous success. However, it has been found that the constant order fractional diffusion equations are not capable of characterizing some complex diffusion processes. To solve the above problems, the variable-order (VO) fractional diffusion equation models have been suggested for use; however, the calculations involved in these definitions are very difficult to handle analytically; therefore, numerical attentions are needed for these cases. To solve the problem found in Jumarie definition, we proposed an alternative fractional derivative and we extended the definition to the case of variational differential operator. We provided a table of Liouville fractional derivative of some special functions. Now we can conclude here by observing, that all fractional derivatives examined here are all useful, and they have to be used according to the support of the function.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

Abdon Atangana wrote the first draft and Aydin Secer corrected final version. All authors read and approved the final draft

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Research Article

A Generalized Version of a Low Velocity Impact between a Rigid Sphere and a Transversely Isotropic Strain-Hardening Plate Supported by a Rigid Substrate Using the Concept of Noninteger Derivatives

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A low velocity impact between a rigid sphere and transversely isotropic strain-hardening plate supported by a rigid substrate is generalized to the concept of noninteger derivatives order. A brief history of fractional derivatives order is presented. The fractional derivatives order adopted is in Caputo sense. The new equation is solved via the analytical technique, the Homotopy decomposition method (HDM). The technique is described and the numerical simulations are presented. Since it is very important to accurately predict the contact force and its time history, the three stages of the indentation process, including (1) the elastic indentation, (2) the plastic indentation, and (3) the elastic unloading stages, are investigated.

1. Introduction

The concept of noninteger order derivative has been intensively applied in many fields. It is worth nothing that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, signal image processing, and groundwater problems; an excellent literature of this can be found in [1–9].

However, there exist a quite number of these fractional derivative definitions in the literature which range from Riemann-Liouville to Jumarie [10–17]. The real problem that mathematicians face is that analytical solutions of these equations with noninteger order derivatives are usually not available. Since only limited classes of equations are solved by analytical means, numerical solution of these nonlinear partial differential equations is of practical importance. Though computer science is growing very fast, and numerical simulation is applied everywhere, nonnumerical issues will still play a large role [18–20]. In this paper a possibility of generalization of a low velocity impact between a rigid sphere and transversely isotropic strain-hardening plate supported by a rigid substrate that is generalized to the concept of noninteger derivatives order will be investigated.

There are many physical situations in which a thin plate made of strain-hardening materials resting on a rigid substrate is impacted by a rigid indenter. For example, such a phenomenon may be caused by the impact of hailstones, run way debris, or small stones on the panels of a vehicle or aircraft [21]. Although low velocity impact of a plate by a rigid indenter has been investigated by numerous researchers, the strain-hardening behaviour of the plate material has not been included in the analytical studies yet. Ollson [22] presented a one parameter nondimensional model for small mass impacts. Yigit and Christoforou [23, 24] have investigated the elastoplastic indentation phenomenon. They assumed the plate material to exhibit perfectly plastic behaviour and considered three stages for the indentation process: Hertzian elastic contact, elastic-perfectly plastic indentation, and Hertzian elastic unloading. Christoforou and Yigit [25, 26] used scaling rules for establishing a dynamic similarity between behaviours of the models and prototypes to present a model based on a linearized contact law with two nondimensional parameters that can be used for small as well as large mass impacts. In follow-up work [27], they obtained the nondimensional governing parameters of the low velocity impact response of composite plates through dimensional analysis and simple lumped-parameters models based on asymptotic solutions.

In this paper, approximated solutions for the generalized version of a low velocity impact between a rigid sphere and transversely isotropic strain-hardening plate supported by a rigid substrate will be obtained via the relatively new analytical method HDM.

The remaining of this paper is structured as follows: in Section 2, we present a brief history of the fractional derivative order and their properties. We present the basic ideal of the homotopy decomposition method for solving high order nonlinear fractional partial differential equations, its convergence and stability. We present the application of the HDM for system fractional nonlinear differential equations under investigation and numerical results in Section 4. The conclusions are then given in Section 5.

2. Brief History of Definitions and Properties

There exists a vast literature on different definitions of fractional derivatives. The most popular ones are the Riemann-Liouville and the Caputo derivatives. For Caputo, we have

$$ {}_0^C D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt, \quad n-1 < \alpha \le n. \tag{1}$$

For the case of Riemann-Liouville we have the following definition:

$$ D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\,dt. \tag{2}$$

Guy Jumarie proposed a simple alternative definition to the Riemann-Liouville derivative:

$$ D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} \{f(t)-f(0)\}\,dt. \tag{3}$$

For the case of Weyl we have the following definition:

$$ D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^\infty (x-t)^{n-\alpha-1} f(t)\,dt. \tag{4}$$

With the Erdelyi-Kober type we have the following definition:

$$ D_{0,\sigma,\eta}^\alpha (f(x)) = x^{-n\sigma} \left(\frac{1}{\sigma x^{\sigma-1}} \frac{d}{dx} \right)^n x^{\sigma(n+\eta)} I_{0,\sigma,\eta+\sigma}^{n-\alpha} (f(x)), \quad \sigma > 0. \tag{5}$$

Here

$$ I_{0,\sigma,\eta+\sigma}^\alpha (f(x)) = \frac{\sigma x^{-\sigma(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^x \frac{t^{\sigma\eta+\sigma-1} f(t)}{(t^\sigma - x^\sigma)^{1-\alpha}} dt. \tag{6}$$

With Hadamard type, we have the following definition:

$$ D_0^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \left(x \frac{d}{dx} \right)^n \int_0^x \left(\log \frac{x}{t} \right)^{n-\alpha-1} f(t) \frac{dt}{t}. \tag{7}$$

With Riesz type, we have the following definition:

$$ \begin{aligned} D_x^\alpha (f(x)) = & -\frac{1}{2\cos(\alpha\pi/2)} \\ & \times \left\{ \frac{1}{\Gamma(\alpha)} \left(\frac{d}{dx} \right)^m \right. \\ & \times \left(\int_{-\infty}^x (x-t)^{m-\alpha-1} f(t)\,dt \right. \\ & \left. \left. + \int_x^\infty (t-x)^{m-\alpha-1} f(t)\,dt \right) \right\}. \end{aligned} \tag{8}$$

We will not mention the Grunward-Letnikov type here because it is in series form [28]. This is not more suitable for analytical purpose.

In 1998, Davison and Essex [16] published a paper which provides a variation to the Riemann-Liouville definition suitable for conventional initial value problems within the realm of fractional calculus [28]. The definition is as follows:

$$ D_0^\alpha f(x) = \frac{d^{n+1-k}}{dx^{n+1-k}} \int_0^x \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d^k f(t)}{dt^k} dt. \tag{9}$$

In an article published by Coimbra [17] in 2003, a variable-order differential operator is defined as follows:

$$ \begin{aligned} D_0^{\alpha(t)} (f(x)) = & \frac{1}{\Gamma(1-\alpha(x))} \int_0^x (x-t)^{-\alpha(t)} \frac{df(t)}{dt} dt \\ & + \frac{(f(0^+) - f(0^-))\, x^{-\alpha(x)}}{\Gamma(1-\alpha(x))}. \end{aligned} \tag{10}$$

2.1. Advantages and Disadvantages

2.1.1. Advantages [28]. It is very important to point out that all these fractional derivative order definitions have their advantages and disadvantages; here we will include Caputo, variational order, Riemann-Liouville Jumarie, and Weyl [28]. We will examine first the variational order differential operator. Anomalous diffusion phenomena are extensively observed in physics, chemistry, and biology fields [19, 29]. To characterize anomalous diffusion phenomena, constant-order fractional diffusion equations are introduced and have received tremendous success. However, it has been found that the constant-order fractional diffusion equations are not capable of characterizing some complex diffusion processes, for instance, diffusion process in inhomogeneous or heterogeneous medium [30]. In addition, when we consider diffusion process in porous medium, if the medium structure or external field changes with time, in this situation, the constant-order fractional diffusion equation model cannot be used to well characterize such phenomenon [31, 32]. Still in some biology diffusion processes, the concentration of particles will determine the diffusion pattern [33, 34]. To solve the above problems, the variable-order (VO) fractional diffusion equation models have been suggested for use [34].

With the Jumarie definition which is actually the modified Riemann-Liouville fractional derivative, an arbitrary continuous function needs not to be differentiable; the fractional derivative of a constant is equal to zero and more importantly it removes singularity at the origin for all functions for which $f(0) = \text{constant}$, for instant, the exponentials functions and Mittag-Leffler functions [28].

With the Riemann-Liouville fractional derivative, an arbitrary function needs not to be continuous at the origin and it needs not to be differentiable.

One of the great advantages of the Caputo fractional derivative is that it allows traditional initial and boundary conditions to be included in the formulation of the problem [5, 12]. In addition its derivative for a constant is zero.

It is customary in groundwater investigations to choose a point on the centreline of the pumped borehole as a reference for the observations and therefore neither the drawdown nor its derivatives will vanish at the origin, as required [13]. In such situations where the distribution of the piezometric head in the aquifer is a decreasing function of the distance from the borehole, the problem may be circumvented by rather using the complementary, or Weyl, fractional order derivative [13].

2.1.2. Disadvantages [28]. Although these fractional order derivatives display great advantages, however, they are not applicable in all the situations. We will begin with the Liouville-Riemann type.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations [28]. The Riemann-Liouville derivative of a constant is not zero. In addition, if an arbitrary function is a constant at the origin, its fractional derivation has a singularity at the origin for instant exponential and Mittag-Leffler functions. Theses disadvantages reduce the field of application of the Riemann-Liouville fractional derivative.

Caputo's derivative demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first-order derivative might have fractional derivatives of all orders less than one in the Riemann-Liouville sense.

With the Jumarie fractional derivative, if the function is not continuous at the origin, the fractional derivative will not exist, for instance, what will be the fractional derivative of $\ln(x)$ and many other ones [28].

Variational order differential operator cannot easily be handled analytically. Numerical approach is some time needs to deal with the problem under investigation.

Although Weyl fractional derivative found its place in groundwater investigation, it is still displaying a significant disadvantage; because the integral defining these Weyl derivatives is improper, greater restrictions must be placed on a function [28]. For instance, the Weyl derivative of a constant is not defined. On the other hand general theorems about Weyl derivatives are often more difficult to formulate and prove than are corresponding theorems for Riemann-Liouville derivatives.

3. Method Description [35, 36]

To illustrate the basic idea of this method, we consider a general nonlinear nonhomogeneous fractional partial differential equation with initial conditions of the following form:

$$\frac{\partial^{\alpha} U(x,t)}{\partial t^{\alpha}} = L(U(x,t)) + N(U(x,t)) + f(x,t), \quad \alpha > 0. \tag{11}$$

Subject to the initial condition

$$\begin{gathered} D_0^k U(x,0) = g_k(x), \quad (k = 0, \dots, n-1), \\ D_0^n U(x,0) = 0, \quad n = [\alpha], \end{gathered} \tag{12}$$

where $\partial^{\alpha}/\partial t^{\alpha}$ denotes the Caputo fractional order derivative operator, f is a known function, N is the general nonlinear fractional differential operator, and L represents a linear fractional differential operator. The method first step here is to transform the fractional partial differential equation to the fractional partial integral equation by applying the inverse operator $\partial^{\alpha}/\partial t^{\alpha}$ on both sides of (11) to obtain

$$\begin{aligned} U(x,t) = {} & \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha - j + 1)} t^j \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[L(U(x,\tau)) + N(U(x,\tau)) \right. \\ & \left. + f(x,\tau) \right] d\tau, \end{aligned} \tag{13}$$

or in general by putting

$$f(x,t) = \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha - j + 1)} t^j. \tag{14}$$

We obtain

$$U(x,t) = T(x,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[L(U(x,\tau)) + N(U(x,\tau)) + f(x,\tau)\right]d\tau. \quad (15)$$

In the homotopy decomposition method, the basic assumption is that the solutions can be written as a power series in p

$$U(x,t,p) = \sum_{n=0}^{\infty} p^n U_n(x,t), \quad (16a)$$

$$U(x,t) = \lim_{p\to 1} U(x,t,p), \quad (16b)$$

and the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{n=0}^{\infty} p^n \mathscr{H}_n(U), \quad (17)$$

where $p \in (0,1]$ is an embedding parameter. $\mathscr{H}_n(U)$ is a polynomials that can be generated by

$$\mathscr{H}_n(U_0,\ldots,U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{j=0}^{\infty} p^j U_j(x,t)\right)\right], \quad n = 0,1,2\ldots. \quad (18)$$

The homotopy decomposition method is obtained by the graceful coupling of homotopy technique with Abel integral and is given by

$$\sum_{n=0}^{\infty} p^n U_n(x,t) - T(x,t) = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[f(x,\tau) + L\left(\sum_{n=0}^{\infty} p^n U_n(x,\tau)\right) + N\left(\sum_{n=0}^{\infty} p^n U_n(x,\tau)\right)\right]d\tau. \quad (19)$$

Comparing the terms of same powers of p gives solutions of various orders with the first term

$$U_0(x,t) = T(x,t). \quad (20)$$

4. Application of the Method to Solve the Governing Differential Equations

In this section, the analytical technique described in Section 3 is employed to obtain the solutions of the governing differential equations in each of the mentioned three contact stages. The derivation of this equation can be found in [37].

4.1. Solution of the Governing Differential Equation in the Elastic Indentation Phase. The governing equation under investigation here is given as follows:

$$\partial_t^{\beta}\alpha(t) + \frac{\pi E_z R}{(1 - v_{zr}v_{rz})hm}\alpha^2(t) = 0, \quad 1 < \beta \le 2. \quad (21)$$

Subject to the initial conditions

$$\partial_t \alpha(0) = V_0, \quad \alpha(0) = 0. \quad (22)$$

Here, E and v are Young's modulus and Poisson's ratio of the plate, respectively. $\alpha(t)$ is elastic indentation phase; m and V_0 are the mass of the indenter and the initial velocity, respectively; h is the thickness of the plate and R is the radius of the spherical indenter [38, 39].

Now following the description of the HDM, we arrive at the following equation:

$$\sum_{n=0}^{\infty} p^n\alpha_n(t) - \alpha(0)t - \partial_t\alpha(0) = -\frac{p\gamma}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\left(\sum_{n=0}^{\infty} p^n\alpha_n(\tau)\right)^2 d\tau, \quad \gamma = \frac{\pi E_z R}{(1 - v_{zr}v_{rz})hm}. \quad (23)$$

Comparing the terms of the same power of p we arrive at the following integral equations, which are very easier to compute:

$$p^0 : \alpha_0(t) = V_0 t$$

$$p^1 : \alpha_1(t) = -\frac{\gamma}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\alpha_0^2 d\tau$$

$$\vdots$$

$$p^n : \alpha_n(t) = -\frac{\gamma}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\sum_{j=0}^{n-1}\alpha_j\alpha_{n-j-1}d\tau, \quad n \ge 2. \quad (24)$$

Integrating the above we obtain the following solutions:

$$\alpha_0(t) = V_0 t,$$

$$\alpha_1(t) = -\frac{\gamma t^{\beta+2}V_0^2}{\Gamma(1+\beta)},$$

$$\alpha_2(t) = \frac{2\gamma^2 t^{3+2\beta}V_0^3(1+\beta)(2+\beta)(3+\beta)}{\Gamma(4+2\beta)},$$

$$\alpha_3(t) = \frac{4\gamma^3 t^{4+3\beta}V_0^4}{3}(1+\beta)(3+\beta) \times\left(-\frac{6\Gamma(2+\beta)^2}{\Gamma(5+3\beta)} + \frac{t^{1+\beta}V_0\gamma\Gamma(7+3\beta)}{\Gamma(1+\beta)\Gamma(4+2\beta)\Gamma(6+4\beta)}\right),$$

$$\begin{aligned}
\alpha_4(t) = &-\frac{4\gamma^4 t^{5+4\beta} V_0^5 \Gamma(4+\beta)\Gamma(6+3\beta)}{\Gamma^2(1+\beta)\Gamma(4+2\beta)\Gamma(5+3\beta)\Gamma(6+4\beta)\Gamma(7+5\beta)} \\
&\times \Big(2t^{1+\beta} V_0 \gamma \Gamma(5+3\beta)\Gamma(7+5\beta) \\
&\quad - (2\Gamma(1+\beta)\Gamma(5+2\beta) + \Gamma(5+3\beta)\Gamma(7+5\beta))\Big), \\
\alpha_5(t) = &-1 \times \Big(3\Gamma^3(1+\beta)\Gamma(2+2\beta)\Gamma^2(4+2\beta)\Gamma(5+3\beta) \\
&\quad \times \Gamma(6+4\beta)\Gamma(7+5\beta)\Gamma(8+6\beta)\Big)^{-1} \\
&\times \Big(4\sqrt{\pi}\gamma^5 t^{6+5\beta} V_0^6 \\
&\quad \times \Big(6t^{1+\beta} V_0 \gamma \Gamma(4+\beta)\Gamma(2+2\beta)\Gamma(2+2\beta) \\
&\quad\quad \times \Gamma(4+2\beta)\Gamma(5+3\beta)\Gamma(6+3\beta) \\
&\quad\quad \times (2\Gamma(1+\beta)\Gamma(7+4\beta) + \Gamma(7+5\beta)) \\
&\quad\quad \times \Gamma(8+5\beta) - \Gamma(\beta+1) \\
&\quad\quad \times \Big(2\Gamma(\beta+1)(3+\beta)\Gamma^2(4+2\beta) \\
&\quad\quad\quad \times \Gamma(5+3\beta)\Gamma(7+3\beta) \\
&\quad\quad\quad \times \Gamma(6+4\beta) + 3\Gamma(4+\beta)\Gamma(2+2\beta) \\
&\quad\quad\quad \times (4\Gamma(\beta+1)\Gamma(4+2\beta) \\
&\quad\quad\quad\quad \times \Gamma(5+2\beta)\Gamma(6+3\beta) \\
&\quad\quad\quad\quad + (2\Gamma(4+2\beta)\Gamma(5+2\beta) \\
&\quad\quad\quad\quad\quad + \Gamma(4+\beta)\Gamma(5+3\beta)) \\
&\quad\quad\quad\quad \times \Gamma(6+4\beta))\Gamma(7+4\beta)) \\
&\quad \times \Gamma(8+6\beta))\Big).
\end{aligned} \tag{25}$$

In the same manner one can obtain the rest of the components. But in this case, few terms were computed and the asymptotic solution is given by

$$\alpha(t) = \alpha_0(t) + \alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t) + \alpha_5(t) + \cdots. \tag{26}$$

Remark 1. Equation (21) was solved in [37] via the homotopy perturbation method for = 2. In the HPM, the initial guess or first component of the series solution may not be unique, whereas with the HDM the first component is uniquely defined as the Taylor series expansion of order $n-1$ (n is the order of the partial differential equation). This is one of the advantages that the HDM has over HPM.

The contact force in the elastic indentation phase may be interpreted in terms of the indentation value [37]

$$F(\alpha(t)) = \gamma\alpha^2(t). \tag{27}$$

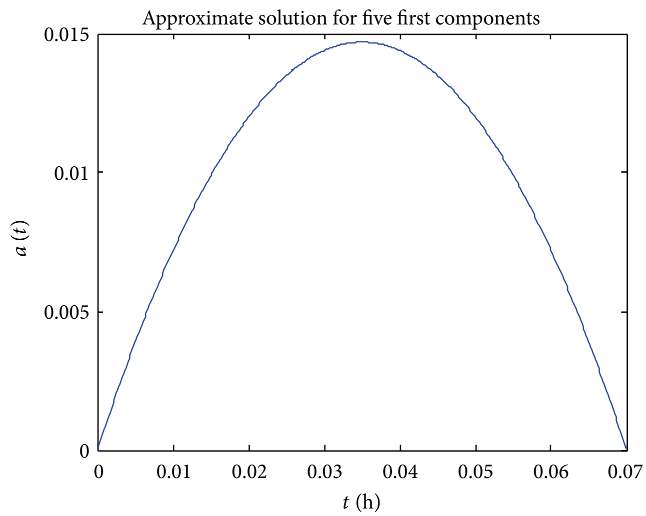

Figure 1: Approximate solution (26) of the governing differential equation in the elastic indentation phase for $\beta = 1.9$.

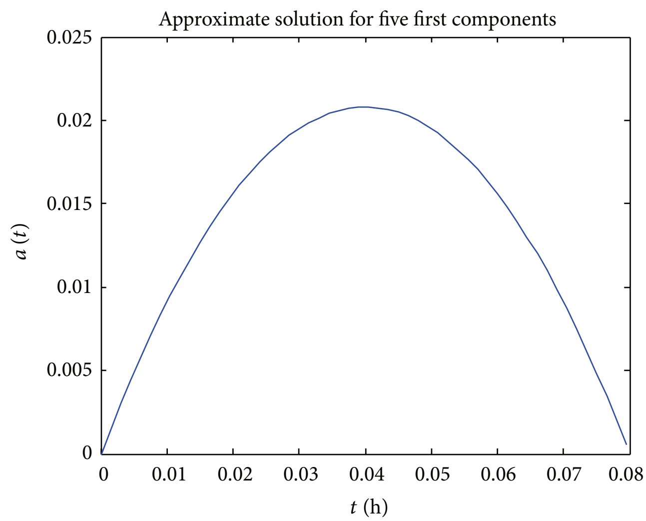

Figure 2: Approximate solution (26) of the governing differential equation in the elastic indentation phase for $\beta = 2$.

Figures 1–6 present the approximate solution for $R = 0.008$ m, $m = 10^{-2}$, $V_0 = 5$ mm/s, $h = 0.0003$, $\nu_{rz} = \nu_{zr} = 0.3$, and $E = 75$ GPa. The approximate solutions of main problem have been depicted in Figures 1, 2, 3, 4, 5, and 6 which plotted according to different β values as function of time for a fixed x and as function of space and time.

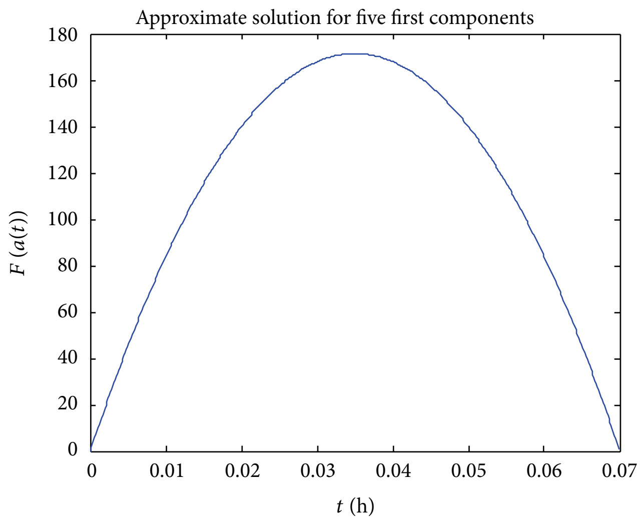

Figure 3: Approximate solution of the contact force in the elastic indentation phase (27) with $\beta = 1.9$.

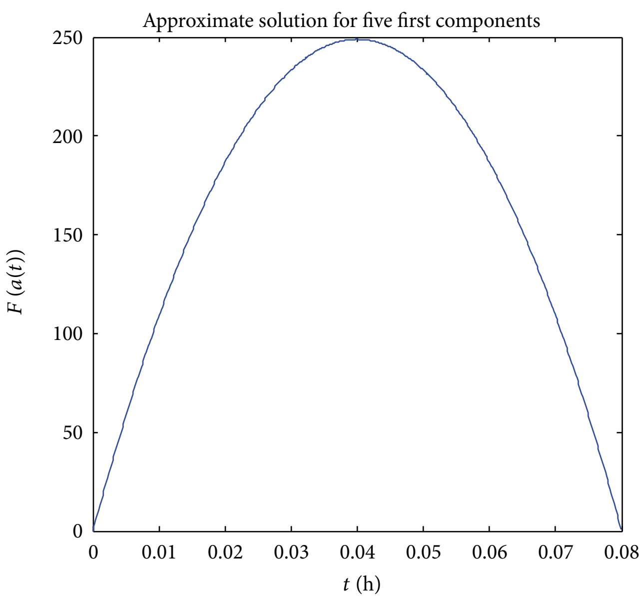

Figure 4: Approximate solution of the contact force in the elastic indentation phase (27) with $\beta = 2$.

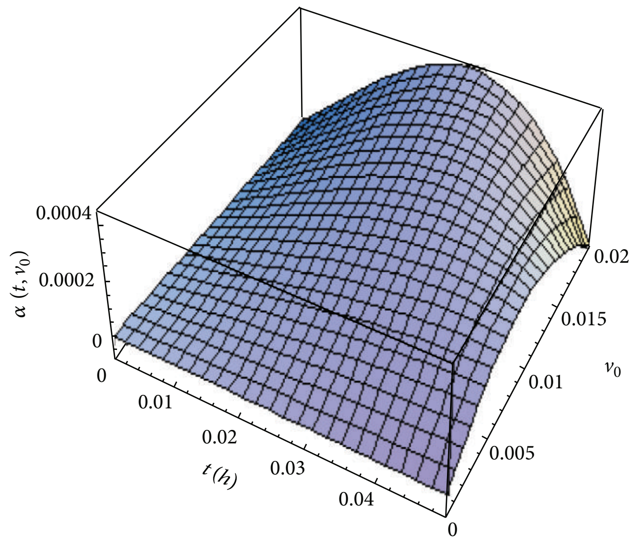

Figure 5: Surface showing the approximate solution of the governing differential equation in the elastic indentation phase equation (21) for $\beta = 1.9$.

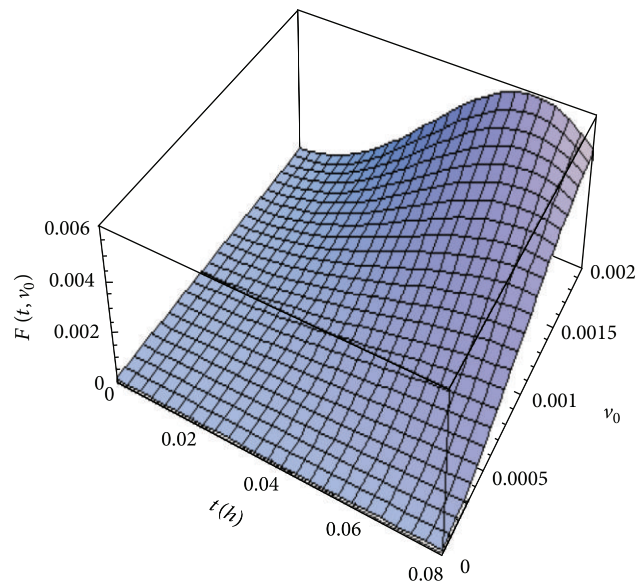

Figure 6: Approximate solution of the contact force in the elastic indentation phase equation (27) for $\beta = 2$.

4.2. Solution of the Governing Differential Equation in the Plastic Indentation Phase. The governing equation under investigation here is given as follows.

$$m_i \partial_t^{\beta} \alpha(t) + 2\pi R S_y \left[2\alpha(t) - \alpha(t_{cr})\right] + \frac{P_z \pi R}{(1 - v_{rz} v_{zr}) h} \left(\alpha(t) - \alpha(t_{cr})\right)^2, \quad 1 < \beta \le 2. \tag{28}$$

Subject to the initial conditions

$$\alpha(t_{cr}) = \alpha_{cr}; \qquad \partial_t \alpha(t_{cr}) = V_{cr}. \tag{29}$$

Here, S_y is the yield stress, P_z is the slope of the stress-strain curve in the plastic region and it may be defined as $P_z = nE_z$, with $0 \le n \le 1$. Therefore, n may be considered as a strain-hardening index. $N = 0$ denotes a perfectly plastic behavior, whereas $n = 1$ represents an elastic material behaviour. By increasing n from 0 to 1, behaviour of the material approaches elastic behaviour. In addition, initial conditions of this phase or the initial velocity correspond to the values attained at the critical indentation at the end of the elastic indentation stage based on (26). For simplicity let

$$a = \frac{4\pi R S_y}{m_i} - \frac{2\pi P_z R \alpha_{cr}}{(1 - v_{rz}v_{zr}) h m_i}, \qquad b = \frac{P_z \pi R}{m_i (1 - v_{rz}v_{zr}) h},$$
$$c = \frac{P_z \pi R}{m_i (1 - v_{rz}v_{zr}) h} \alpha (t_{cr})^2. \tag{30}$$

Such that (28) can be reduced to

$$\partial_t^\beta \alpha (t) + a\alpha (t) + b\alpha (t)^2 + c = 0, \quad 1 < \beta \leq 2. \tag{31}$$

Employing the HDM, we obtain the following integral equations:

$$\alpha_0 (t) = tV_{cr},$$
$$\alpha_1 (t) = -\frac{1}{\Gamma (\beta)} \int_{t_{cr}}^{t} (t - \tau)^{\beta - 1} \left[a\alpha_0 (\tau) + b\alpha_0^2 (\tau) + c \right] d\tau,$$
$$\alpha_1 (t_{cr}) = \partial_t \alpha_1 (t_{cr}) = 0,$$
$$\alpha_n (t) = -\frac{1}{\Gamma (\beta)} \int_{t_{cr}}^{t} (t - \tau)^{\beta - 1} \times \left[a\alpha_{n-1} (\tau) + b \sum_{j}^{n-1} \alpha_j (\tau) \alpha_{n-j-1} (\tau) \right] d\tau,$$
$$\alpha_n (t_{cr}) = \partial_t \alpha_n (t_{cr}) = 0, \quad n \geq 0. \tag{32}$$

Integrating the above we arrived at the following:

$$\alpha_0 (t) = tV_{cr},$$
$$\alpha_1 (t) = -\left((t - t_{cr})^\beta \left(c (1 + \beta) (2 + \beta) + (t - t_{cr}) \times V_0 (2b \ (t - t_{cr}) V_0 + a (2 + \beta)) \right) \right) \times (\Gamma (3 + \beta))^{-1},$$
$$a_2 (t) = \frac{(t - t_{cr})^{2\beta}}{\Gamma (1 + 2\beta) \Gamma (2 + 2\beta) \Gamma (3 + 2\beta) \Gamma (4 + 2\beta)} \times (ac\Gamma (2 + 2\beta) \Gamma (3 + 2\beta) \Gamma (4 + 2\beta) + tV_0 \Gamma (1+2\beta) \times \left(\left(a^2 + 2bc (1 + \beta) \right) \Gamma (3 + 2\beta) \Gamma (4 + 2\beta) + 2b (t - t_{cr}) V_0 (3 + \beta) \Gamma (2 + 2\beta) \times \left(2b (t - t_{cr}) \ V_0 \Gamma (3 + 2\beta) + a\Gamma (4 + 2\beta) \right) \right)). \tag{33}$$

Using the package Mathematica, in the same manner one can obtain the rest of the components. But in this case, few terms were computed and the asymptotic solution is given by

$$\alpha (t) = \alpha_0 (t) + \alpha_1 (t) + \alpha_2 (t) + \alpha_3 (t) + \alpha_4 (t) + \alpha_5 (t) + \cdots. \tag{34}$$

4.3. Solution of the Governing Differential Equation of the Unloading Phase. The governing equation of motion of the indenter mass in the unloading phase under investigation here is given as follows:

$$\partial_t^\beta \alpha (t) + \frac{\pi R E_z}{(1 - v_{rz}v_{zr}) h} \left(\alpha^2 (t) - (1 - n) (\alpha_m - \alpha_{cr})^2 \right) = 0, \quad 1 < \beta \leq 2. \tag{35}$$

Subject to the initial conditions

$$\alpha (t_m) = \alpha_m, \qquad \partial_t \alpha (t_m), \tag{36}$$

where α_m and t_m are the maximum indentation value and its relevant occurrence time, respectively. At the maximum indentation time, the velocity of the indenter becomes zero. Therefore, the values corresponding to this time may be used as initial conditions for the unloading stage [37].

Initial conditions of this phase may be obtained from solutions of the previous stage at the time of the maximum indentation. The velocity of the indenter at the time instant that it attains its maximum indentation is zero. Therefore, time of the maximum indentation may be determined by differentiating (34), with respect to time and setting the resulting equation equal to zero. Solving this equation, the time of the maximum indentation is obtained. Substituting this time into (34) yields the value of the maximum indentation as

$$\alpha (t_m) = \alpha_0 (t_m) + \alpha_1 (t_m) + \alpha_2 (t_m) + \alpha_3 (t_m) + \alpha_4 (t_m) + \alpha_5 (t_m) + \cdots. \tag{37}$$

For simplicity let:

$$a = \frac{\pi R E_z}{m (1 - v_{rz}v_{zr}) h},$$
$$b = \frac{\pi R E_z}{m (1 - v_{rz}v_{zr}) h} (n - 1) (\alpha_m - \alpha_{cr})^2. \tag{38}$$

Thus (35) is reduced to

$$\partial_t^\beta \alpha (t) + a\alpha^2 (t) + b = 0, \quad 1 < \beta \leq 2. \tag{39}$$

Following carefully the steps involved in the HDM we obtain the following integral equations:

$$\begin{aligned}
\alpha_0(t) &= \alpha_m \\
\alpha_1(t) &= -\frac{1}{\Gamma(\beta)}\int_{t_m}^{t}(t-\tau)^{\beta-1}\left[a\alpha_0^2(\tau)+b\right]d\tau \\
&\vdots \\
\alpha_n(t) &= -\frac{1}{\Gamma(\beta)}\int_{t_m}^{t}(t-\tau)^{\beta-1}\left[a\sum_{j=0}^{n-1}\alpha_j\alpha_{n-j-1}\right]d\tau, \\
\alpha_n(t_m) &= \partial_t\alpha(t_m) = 0, \quad n \geq 1.
\end{aligned} \tag{40}$$

Integrating the above we arrive at the following series solutions:

$$\begin{aligned}
\alpha_0(t) &= \alpha_m, \\
\alpha_1(t) &= -\frac{\left(aa_m^2+b\right)(t-t_m)^{\beta}}{\Gamma(1+\beta)}, \\
\alpha_2(t) &= \frac{2aa_m\left(aa_m^2+b\right)(t-t_m)^{2\beta}}{\Gamma(1+2\beta)}, \\
a_3(t) &= -\Big(a\left(aa_m^2+b\right)(t-t_m)^{3\beta} \\
&\quad\times\left(8aa_m^2\Gamma^2(1+\beta)+\left(aa_m^2+b\right)\Gamma(1+2\beta)\right)\Big) \\
&\quad\times\left(\Gamma^2(1+\beta)\Gamma(1+3\beta)\right)^{-1}, \\
\alpha_4(t) &= \frac{aa_m^2(t-t_m)^{4\beta}}{\beta\Gamma(2\beta)\Gamma(4\beta)\Gamma^2(1+\beta)\Gamma(1+2\beta)\Gamma(1+4\beta)} \\
&\quad\times\Big(\Gamma(4\beta)\Gamma(1+2\beta) \\
&\quad\times\Big(\left(aa_m^2+b\right)\Gamma(1+2\beta) \\
&\quad\times\Big(8aa_m^2\Gamma^2(1+\beta)+\left(aa_m^2+b\right)\Gamma(1+2\beta) \\
&\quad+2\left(a^2a_m^4+b^2\right)\Gamma(1+\beta)\Gamma(1+3\beta)\Big) \\
&\quad+2aa_m^2b\Gamma(2\beta)\Gamma(1+\beta)\Gamma(1+3\beta) \\
&\quad\times\Gamma(1+4\beta)\Big)\Big).
\end{aligned} \tag{41}$$

Using the package Mathematica, in the same manner one can obtain the rest of the components. But in this case, few terms were computed and the asymptotic solution is given by

$$\alpha(t) = \alpha_0(t)+\alpha_1(t)+\alpha_2(t)+\alpha_3(t)+\alpha_4(t)+\alpha_5(t)+\cdots. \tag{42}$$

5. Conclusion and Discussion

Low velocity impact between a rigid sphere and a transversely isotropic strain-hardening plate supported by a rigid substrate was extended to the concept of noninteger derivatives. The governing equations of the elastic indentation were obtained by Yigit and Christoforou [23, 24]. The contact was assumed to be elastic, and the stresses through the thickness were assumed to be constant. The stress expressions are only valid when no permanent deformation results due to the impact. The experimental evidence reported by Poe Jr. and Illg [39] and Poe Jr. [40] confirms the maximum value of the transverse. Normal stress has the dominant influence on the failure of a plate subjected to impact loads. The third phase is assumed to be an elastic one again.

A brief history of the fractional derivative orders was presented. Advantages and disadvantages of each definition were presented. The new equations were solved approximately using the relatively new analytical technique, the homotopy decomposition methods. The numerical simulations showed that the approximate solutions are continuous and increase functions of the fractional derivative orders. The method used to derive approximate solution is very efficient, easier to implement, and less time consuming. The HDM is a promising method for solving nonlinear fractional partial differential equations.

Conflict of Interests

The authors declare that they have no conflict interests.

Authors' Contribution

A. Atangana and A. Ahmed made the first draft and N. Bıldık corrected and improved the final version. All the authors read and approved the final draft.

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Research Article

A Numerical Method for Partial Differential Algebraic Equations Based on Differential Transform Method

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We have considered linear partial differential algebraic equations (LPDAEs) of the form $Au_t(t,x)+Bu_{xx}(t,x)+Cu(t,x)=f(t,x)$, which has at least one singular matrix of $A,B\in\mathbb{R}^{n\times n}$. We have first introduced a uniform differential time index and a differential space index. The initial conditions and boundary conditions of the given system cannot be prescribed for all components of the solution vector u here. To overcome this, we introduced these indexes. Furthermore, differential transform method has been given to solve LPDAEs. We have applied this method to a test problem, and numerical solution of the problem has been compared with analytical solution.

1. Introduction

The partial differential algebraic equation was first studied by Marszalek. He also studied the analysis of the partial differential algebraic equations [1]. Lucht et al. [2–4] studied the numerical solution and indexes of the linear partial differential equations with constant coefficients. A study about characteristics analysis and differential index of the partial differential algebraic equations was given by Martinson and Barton [5, 6]. Debrabant and Strehmel investigated the convergence of Runge-Kutta method for linear partial differential algebraic equations [7].

There are numerous LPDAEs applications in scientific areas given, for instance, in the field of Navier-Stokes equations, in chemical engineering, in magnetohydrodynamics, and in the theory of elastic multibody systems [4, 8–12].

On the other hand, the differential transform method was used by Zhou [13] to solve linear and nonlinear initial value problems in electric circuit analysis. Analysis of nonlinear circuits by using differential Taylor transform was given by Köksal and Herdem [14]. Using one-dimensional differential transform, Abdel-Halim Hassan [15] proposed a method to solve eigenvalue problems. The two-dimensional differential transform methods have been applied to the partial differential equations [16–19]. The differential transform method extended to solve differential-difference equations by Arikoglu and Ozkol [20]. Jang et al. have used differential transform method to solve initial value problems [21]. The numerical solution of the differential-algebraic equation systems has been studied by using differential transform method [22, 23].

In this paper, we have considered linear partial differential equations with constant coefficients of the form

$$Au_t(t,x)+Bu_{xx}(t,x)+Cu(t,x)=f(t,x),\qquad (t,x)\in J\times\Omega, \tag{1}$$

where $J=[0,\infty)$, $\Omega=[-l,l]$, $l>0$, and $A,B,C\in\mathbb{R}^{n\times n}$. In (1) at least one of the matrices $A,B\in\mathbb{R}^{n\times n}$ should be singular. If $A=0$ or $B=0$, then (1) becomes ordinary differential equation or differential algebraic equation, so we assume that none of the matrices A or B is the zero matrix.

2. Indexes of Partial Differential Algebraic Equation

Let us consider (1), with initial values and boundary conditions given as follows:

$$u_j(t,\pm l) = 0 \quad \text{for } t \in J, \qquad u_i(0,x) = g(x) \quad \text{for } x \in \Omega, \tag{2}$$

where $j \in \mathfrak{M}_{BC} \subseteq \{1,2,\ldots,n\}$, $\mathfrak{M}_{BC}$ is the set of indices of components of u for which boundary conditions can be prescribed arbitrarily, and $i \in \mathfrak{M}_{IC} \subseteq \{1,2,\ldots,n\}$, $\mathfrak{M}_{IC}$ is the set of indices of components of u for which initial conditions can be prescribed arbitrarily. The initial boundary value problem (IBVP) (1) has only one solution where a function u is a solution of the problem, if it is sufficiently smooth, uniquely determined by its initial values (IVs) and boundary values (BVs), and if it solves the LPDAE point wise.

Definition of the indexes can be given using the following assumptions.

(i) Each component of the vectors u, u_t, and f satisfy the following condition:

$$|y(t,x)| \le Me^{\alpha t}, \quad \alpha \ge 0,\ t \ge 0, \tag{3}$$

where M and α are independent of t and x.

(ii) $(B, \xi A + C)$, $\mathrm{Re}(\xi) > \alpha$, called as the matrix pencil, is regular.

(iii) $(A, \mu_k B + C)$ is regular for all k, where μ_k is an eigenvalue of the operator $\partial^2/\partial x^2$ together with prescribed BCs.

(iv) The vector $f(t,x)$ and the initial vector $g(x)$ are sufficiently smooth.

If we use Laplace transform, from assumption (ii), (1) can be transformed into

$$Bu_\xi''(x) + (\xi A + C)u_\xi(x) = f_\xi(x) + Ag(x), \quad \mathrm{Re}(\xi) > \alpha, \tag{4}$$

if B is a singular matrix, then (4) is a DAE depending on the parameter ξ. To characterize $\mathfrak{M}_{BC}$, we introduce $j \in \mathfrak{M}_{BC}^{(\xi)} \subseteq \{1,2,\ldots,n\}$ as the set of indices of components of u_ξ for which boundary conditions can be prescribed arbitrarily.

In order to define a spatial index, we need the Kronecker normal form of the DAE (4). Assumption (iii) guarantees that there are nonsingular matrices $P_{L,\xi}, Q_{L,\xi} \in \mathbb{C}^{n\times n}$ such that

$$P_{L,\xi} B Q_{L,\xi} = \begin{pmatrix} I_{m_1} & 0 \\ 0 & N_{L,\xi} \end{pmatrix}, \qquad P_{L,\xi}(\xi A + C)Q_{L,\xi} = \begin{pmatrix} R_{L,\xi} & 0 \\ 0 & I_{m_2} \end{pmatrix}, \tag{5}$$

where $R_{L,\xi} \in \mathbb{C}^{m_1\times m_2}$ and $N_{L,\xi} \in \mathbb{R}^{m_2\times m_2}$ is a nilpotent Jordan chain matrix with $m_1 + m_2 = n$. I_k is the unit matrix of order k. The Riesz index (or nilpotency) of $N_{L,\xi}$ is denoted by $\nu_{L,\xi}$ (i.e. $N_{L,\xi}^{\nu_{L,\xi}} = 0, N_{L,\xi}^{\nu_{L,\xi}-1} \ne 0$).

Here, we will assume that there is a real number $\alpha^* \ge \alpha$ such that the index set $\mathfrak{M}_{BC}^{(\xi)}$ is independent of the Laplace parameter ξ, provided $\mathrm{Re}(\xi) \ge \alpha^*$.

Definition 1. Let $\alpha^* \in \mathbb{R}^+$ be a number with $\alpha^* \ge \alpha$, such that for all $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \ge \alpha^*$

(1) the matrix pencil $(B, \xi A + C)$ is regular,

(2) $\mathfrak{M}_{BC}^{(\xi)}$ is independent of ξ, i.e., $\mathfrak{M}_{BC}^{(\xi)} = \mathfrak{M}_{BC}$,

(3) the nilpotency of $N_{L,\xi}$ is $\nu_L \ge 1$.

Then $\nu_{d,x} = 2\nu_L - 1$ is called the "*differential spatial index*" of the LPDAE. If $\nu_L = 0$, then the differential spatial index of LPDAE is defined to be zero.

If we use Fourier transform, (1) can be transformed into

$$A\hat{u}_k'(t) + (\mu_k B + C)\hat{u}_k(t) = \hat{f}_k(t) + B\rho_k(t) \tag{6}$$

with $\rho_k(t) = (\rho_{k1}(t), \ldots, \rho_{kn}(t))^T$ and

$$\rho_{ki}(t) = 0 \quad \text{for } i \in \mathfrak{M}_{BC}, \qquad \rho_{kj}(t) = \frac{1}{l}\left[\phi_k'(x)u_j(t,x) - \phi_k(x)u_{x,j}(t,x)\right]_{x=-l}^{x=l} \tag{7}$$

for $j \notin \mathfrak{M}_{BC}$, which results from partial integration of the term $\int_{-l}^{l} u_{xx}(t,x)\phi_k(x)dx$.

If A is a singular matrix, then (6) is a DAE depending on the parameter μ_k which can be solved uniquely with suitable ICs under the assumptions (iv) and (v). Analogous to the case of the Laplace transform, the above assumption (iv) implies that there exist regular matrices $P_{F,k}, Q_{F,k}$ such that

$$P_{F,k} A Q_{F,k} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & N_{F,k} \end{pmatrix}, \qquad P_{F,k}(\mu_k B + C)Q_{F,k} = \begin{pmatrix} R_{F,k} & 0 \\ 0 & I_{n_2} \end{pmatrix}. \tag{8}$$

With $R_{F,k} \in \mathbb{R}^{n_1\times n_1}$. $N_{F,k} \in \mathbb{R}^{n_2\times n_2}$ is again a nilpotent Jordan chain matrix with Riesz index $\nu_{F,k}$, where $n_1 + n_2 = n$.

To characterize $\mathfrak{M}_{IC}$, we introduce $\mathfrak{M}_{IC}^{(k)} \subseteq \{1,2,\ldots,n\}$ as the set of indices of components of $\hat{u}_k$ for which initial conditions can be prescribed arbitrarily. Therefore, we always assume in the context of a Fourier analysis of u that $\mathfrak{M}_{IC}^{(k)}$ is independent of $k \in \mathbb{N}_+$, i.e., $\mathfrak{M}_{IC}^{(k)} = \mathfrak{M}_{IC}$.

Definition 2. Assume for $k = 1, 2, \ldots$ that

(1) the matrix pencil $(A, \mu_k B + C)$ is regular,

(2) $\mathfrak{M}_{IC}^{(k)}$ is independent of k, i.e., $\mathfrak{M}_{IC}^{(k)} = \mathfrak{M}_{IC}$,

(3) the nilpotency of $N_{F,k}$ is $\nu_{F,k} = \nu_F$.

Then the PDAE (1) is said to have uniform differential *time index* $\nu_{d,t} = \nu_F$.

The differential spatial and time indexes are used to decide which initial and boundary values can be taken to solve the problem.

3. Two-Dimensional Differential Transform Method

The two-dimensional differential transform of function $w(x, y)$ is defined as

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{\substack{x=0\\y=0}}, \quad (9)$$

where it is noted that upper case symbol $W(k,h)$ is used to denote the two-dimensional differential transform of a function represented by a corresponding lower case symbol $w(x,y)$. The differential inverse transform of $W(k,h)$ is defined as

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)\, x^k y^h. \quad (10)$$

From (9) and (10), we obtain

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{x^k y^h}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{\substack{x=0\\y=0}}. \quad (11)$$

The concept of two-dimensional differential transform is derived from two-dimensional Taylor series expansion, but the method doesn't evaluate the derivatives symbolically.

Theorem 3. *Differential transform of the function $w(x,y) = u(x,y) \pm v(x,y)$ is*

$$W(k,h) = U(k,h) \pm V(k,h), \quad (12)$$

see [17].

Theorem 4. *Differential transform of the function $w(x,y) = \lambda u(x,y)$ is*

$$W(k,h) = \lambda U(k,h), \quad (13)$$

see [17].

Theorem 5. *Differential transform of the function $w(x,y) = \partial u(x,y)/\partial x$ is*

$$W(k,h) = (k+1)\, U(k+1,h), \quad (14)$$

see [17].

Theorem 6. *Differential transform of the function $w(x,y) = \partial u(x,y)/\partial y$ is*

$$W(k,h) = (h+1)\, U(k,h+1), \quad (15)$$

see [17].

Theorem 7. *Differential transform of the function $w(x,y) = \partial^{r+s} u(x,y)/\partial x^r \partial y^s$ is*

$$\begin{aligned} W(k,h) = {} & (k+1)(k+2)\cdots(k+r)(h+1) \\ & \times (h+2)\cdots(h+s)\, U(k+r,h+s), \end{aligned} \quad (16)$$

see [17].

Theorem 8. *Differential transform of the function $w(x,y) = u(x,y) \cdot v(x,y)$ is*

$$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)\, V(k-r,s), \quad (17)$$

see [17].

Theorem 9. *Differential transform of the function $w(x,y) = x^m y^n$ is*

$$W(k,h) = \delta(k-m,h-n) = \delta(k-m)\,\delta(h-n), \quad (18)$$

see [17], where

$$\begin{aligned} \delta(k-m) &= \begin{cases} 1, & k = m \\ 0, & k \neq m, \end{cases} \\ \delta(h-n) &= \begin{cases} 1, & h = n \\ 0, & h \neq n. \end{cases} \end{aligned} \quad (19)$$

Theorem 10. *Differential transform of the function $w(x,y) = g(x+a,y)$ is*

$$W(k,h) = \sum_{p=k}^{N} \binom{p}{k} a^{p-k} G(p,h). \quad (20)$$

Proof. From Definition 1, we can write

$$\begin{aligned} w(x,y) = {} & \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} G(k,h)\,(x+a)^k y^h \\ = {} & G(0,0) + G(0,1)\, y + aG(1,0) + G(1,0)\, x \\ & + G(0,2)\, y^2 + G(1,1)\, xy + aG(1,1)\, y \\ & + a^2 G(2,0) + 2aG(2,0)\, x + G(2,0)\, x^2 \\ & + G(0,3)\, y^3 + G(1,2)\, xy^2 + aG(1,2)\, y^2 \\ & + G(2,1)\, x^2 y + 2aG(2,1)\, xy + a^2 G(2,1)\, y \\ & + a^3 G(3,0) + 3a^2 G(3,0)\, x + 3aG(3,0)\, x^2 \\ & + G(3,0)\, x^3 + \cdots, \\ w(x,y) = {} & \left[G(0,0) + aG(1,0) \right. \\ & \left. + a^2 G(2,0) + a^3 G(3,0) + \cdots\right] \\ & + x\left[G(1,0) + 2aG(2,0) + 3a^2 G(3,0) + \cdots\right] \\ & + y\left[G(0,1) + aG(1,1) + a^2 G(2,1) + \cdots\right] \\ & + x^2 \left[G(2,0) + 3aG(3,0) + \cdots\right] \\ & + xy\left[G(1,1) + 2aG(2,1) + \cdots\right] \\ & + y^2 \left[G(0,2) + aG(1,2) + \cdots\right] + \cdots \end{aligned}$$

$$= \sum_{p=0}^{\infty} a^p G(p,0) + x \sum_{p=1}^{\infty} p a^{p-1} G(p,0) + y \sum_{p=0}^{\infty} a^p G(p,1) + x^2 \sum_{p=2}^{\infty} \frac{p!}{(p-2)!2!} \times a^{p-2} G(p,0) + xy \sum_{p=1}^{\infty} p a^{p-1} G(p,1) + \cdots, \quad (21)$$

where

$$w(x,y) = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=k}^{\infty} \binom{p}{k} a^{p-k} G(p,h) x^k y^h \quad (22)$$

hence,

$$W(k,h) = \sum_{p=k}^{N} \binom{p}{k} a^{p-k} G(p,h). \quad (23)$$

□

Theorem 11. *Differential transform of the function $w(x,y) = g(x+a, y+b)$ is*

$$W(k,h) = \sum_{p=k}^{N} \sum_{q=h}^{N} \binom{q}{h} \binom{p}{k} a^{p-k} b^{q-h} G(p,q). \quad (24)$$

Proof. From Definition 2, we can write

$$\begin{aligned} w(x,y) &= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} G(k,h)(x+a)^k (y+b)^h \\ &= G(0,0) + G(1,0)x + aG(1,0) + G(0,1)y + bG(0,1) + G(2,0)x^2 \\ &\quad + 2aG(2,0)x + a^2 G(2,0) + G(1,1)xy + bG(1,1)x + aG(1,1)y + abG(1,1) \\ &\quad + G(0,2)y^2 + 2bG(0,2)y + b^2 G(0,2) + G(3,0)x^3 + 3aG(3,0)x^2 + 3a^2 G(3,0)x \\ &\quad + a^3 G(3,0) + G(2,1)x^2 y + \cdots \\ w(x,y) &= \left[G(0,0) + aG(1,0) + bG(0,1) + a^2 G(2,0) + abG(1,1) + \cdots\right] \\ &\quad + \left[G(1,0) + 2aG(2,0) + bG(1,1) + 3a^2 G(3,0) + 2abG(2,1) + \cdots\right] x \\ &\quad + \left[G(0,1) + aG(1,1) + 2bG(0,2) + a^2 G(2,1) + \cdots\right] y + \cdots \\ &= \sum_{p=0}^{N} \sum_{q=0}^{N} a^p b^q G(p,q) + x \sum_{p=1}^{N} \sum_{q=0}^{N} p a^{p-1} b^q G(p,q) \\ &\quad + y \sum_{p=0}^{N} \sum_{q=1}^{N} q a^p b^{q-1} G(p,q) + x^2 \sum_{p=2}^{N} \sum_{q=0}^{N} \frac{p!}{(p-2)!2!} a^{p-2} b^q G(p,q) \\ &\quad + xy \sum_{p=1}^{N} \sum_{q=1}^{N} pq a^{p-1} b^{q-1} G(p,q) + \cdots. \end{aligned} \quad (25)$$

Hence, we can write

$$w(x,y) = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=k}^{N} \sum_{q=h}^{N} \binom{q}{h} \binom{p}{k} \times a^{p-k} b^{q-h} G(p,q) x^k y^h. \quad (26)$$

Using Definition 2, we obtain

$$W(k,h) = \sum_{p=k}^{N} \sum_{q=h}^{N} \binom{q}{h} \binom{p}{k} a^{p-k} b^{q-h} G(p,q). \quad (27)$$

□

Theorem 12. *Differential transform of the function $w(x,y) = \partial^{r+s} g(x+a, y+b)/\partial x^r \partial y^s$ is*

$$W(k,h) = \frac{(k+r)!}{k!} \frac{(h+s)!}{h!} \times \sum_{p=k+r}^{N} \sum_{q=h+s}^{N} \binom{q}{h+s} \binom{p}{k+r} \times a^{p-k-r} b^{q-h-s} G(p,q). \quad (28)$$

Proof. Let $C(k,h)$ be differential transform of the function $g(x+a, y+b)$. From Theorem 7, we can write that differential transform of the function $w(x,y)$ is

$$W(k,h) = \frac{(k+r)!}{k!} \frac{(h+s)!}{h!} C(k+r, h+s), \quad (29)$$

from Theorem 4, we can write

$$C(k+r, h+s) = \sum_{p=k+r}^{N} \sum_{q=h+s}^{N} \binom{q}{h+s} \binom{p}{k+r} \times a^{p-k-r} b^{q-h-s} G(p,q). \quad (30)$$

If we substitute (30) into (29), we find

$$W(k,h) = \frac{(k+r)!}{k!}\frac{(h+s)!}{h!} \times \sum_{p=k+r}^{N}\sum_{q=h+s}^{N}\binom{q}{h+s}\binom{p}{k+r} \times a^{p-k-r}b^{q-h-s}G(p,q). \quad (31)$$

□

4. Application

We have considered the following PDAE as a test problem:

$$\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}u_t + \begin{pmatrix}-1 & 0\\ -1 & 0\end{pmatrix}u_{xx} + \begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}u = f, \quad t \in [0,\infty), \quad x \in [-1,1], \quad (32)$$

with initial values and boundary values

$$u_1(0,x) = x^3 - x, \qquad u_2(0,x) = x^4 - 1,$$
$$u_1(t,1) = u_1(t,-1) = 0, \qquad u_2(t,1) = u_2(t,-1) = 0. \quad (33)$$

The right hand side function f is

$$f = \left(\left(x^4-1\right)(\cos t - \sin t) - 6xe^{-t}, -e^{-t}\left(x^3+5x\right)\right)^T, \quad (34)$$

and the exact solutions are

$$u_1(t,x) = \left(x^3 - x\right)e^{-t}, \qquad u_2(t,x) = \left(x^4-1\right)\cos t. \quad (35)$$

If nonsingular matrices $P_{F,k}$, $Q_{F,k}$, $P_{L,\xi}$, and $Q_{L,\xi}$ are chosen such as

$$P_{F,k} = \begin{pmatrix}1 & \dfrac{-\mu_k}{\mu_k-1}\\ 0 & \dfrac{1}{\mu_k-1}\end{pmatrix}, \qquad Q_{F,k} = \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix},$$
$$P_{L,\xi} = \begin{pmatrix}0 & \dfrac{1}{\xi+1}\\ \dfrac{1}{\xi+1} & -\dfrac{1}{\xi+1}\end{pmatrix}, \qquad Q_{L,\xi} = \begin{pmatrix}-\xi-1 & 0\\ \xi & 1\end{pmatrix}, \quad (36)$$

matrices $P_{F,k}AQ_{F,k}$ and $P_{L,\xi}BQ_{L,\xi}$ are found as

$$P_{F,k}AQ_{F,k} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad P_{L,\xi}BQ_{L,\xi} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \quad (37)$$

From (38), we have $N_{L,\xi} = 0$ and $N_{F,k} = 0$. Then the PDAE (32) has differential spatial index 1 and differential time index 1. So, it is enough to take $\mathfrak{M}_{BC}^{(\xi)} = \{1\}$ and $\mathfrak{M}_{IC}^{(k)} = \{2\}$ to solve the problem.

Taking differential transformation of (32), we obtain

$$(k+1)U_1(k+1,h) + (k+1)U_2(k+1,h) - (h+1)(h+2)U_1(k,h+2) + U_1(k,h) + U_2(k,h) = F_1(k,h), \quad (38)$$

$$-(h+1)(h+2)U_1(k,h+2) - U_1(k,h) = F_2(k,h). \quad (39)$$

Table 1: The numerical and exact solution of the test problem(32), where $u_1(t,x)$ is the exact solution and $u_1^*(t,x)$ is the numerical solution, for $x = 0.1$.

t	$u_1(t,x)$	$u_1^*(t,x)$	$\lvert u_1(t,x) - u_1^*(t,x)\rvert$
0.1	−0.0895789043	−0.0895789043	0
0.2	−0.0810543445	−0.0810543422	0.0000000023
0.3	−0.0733410038	−0.0733409887	0.0000000151
0.4	−0.0663616845	−0.0663616355	0.0000000490
0.5	−0.0600465353	−0.0600464409	0.0000000944
0.6	−0.0543323519	−0.0543322800	0.0000000719
0.7	−0.0491619450	−0.0491621943	0.0000002493
0.8	−0.0444835674	−0.0444849422	0.0000013748
0.9	−0.0402503963	−0.0402546487	0.0000042524
1.0	−0.0364200646	−0.0364305555	0.0000104909

Table 2: The numerical and exact solution of the test problem(32), where $u_2(t,x)$ is exact solution and $u_2^*(t,x)$ is numerical solution, for $x = 0.1$.

t	$u_2(t,x)$	$u_2^*(t,x)$	$\lvert u_2(t,x) - u_2^*(t,x)\rvert$
0.1	−0.9949046649	−0.9949046653	0.0000000004
0.2	−0.9799685711	−0.9799685778	0.0000000067
0.3	−0.9552409555	−0.9552409875	0.0000000320
0.4	−0.9209688879	−0.9209689778	0.0000000899
0.5	−0.8774948036	−0.8774949653	0.0000001637
0.6	−0.8252530813	−0.8252532000	0.0000001187
0.7	−0.7647657031	−0.7647652653	0.0000004378
0.8	−0.6966370386	−0.6966345778	0.0000024608
0.9	−0.6215478073	−0.6215398875	0.0000079198
1.0	−0.5402482757	−0.5402277778	0.0000204979

The Taylor series of functions f_1 and f_2 about $x = 0$, $t = 0$ are

$$\begin{aligned} f_1(t,x) = {} & -1 + t - 6x + \frac{1}{2}t^2 + 6xt \\ & - \frac{1}{6}t^3 - 3xt^2 - \frac{1}{24}t^4 + x^4 + xt^3 \\ & - x^4t + \frac{1}{120}t^5 - \frac{1}{4}xt^4 + \frac{1}{720}t^6 \\ & - \frac{1}{2}x^4t^2 + \frac{1}{20}xt^5 + \frac{1}{6}x^4t^3 \\ & - \frac{1}{5040}t^7 - \frac{1}{120}xt^6 + \cdots, \end{aligned} \quad (40)$$

$$\begin{aligned} f_2(t,x) = {} & -5x + 5xt - \frac{5}{2}xt^2 - x^3 \\ & + \frac{5}{6}xt^3 + x^3t - \frac{1}{2}x^3t^2 - \frac{5}{24}xt^4 \\ & + \frac{1}{6}x^3t^3 + \frac{1}{24}xt^5 - \frac{1}{144}xt^6 \\ & - \frac{1}{24}x^3t^4 + \cdots. \end{aligned} \quad (41)$$

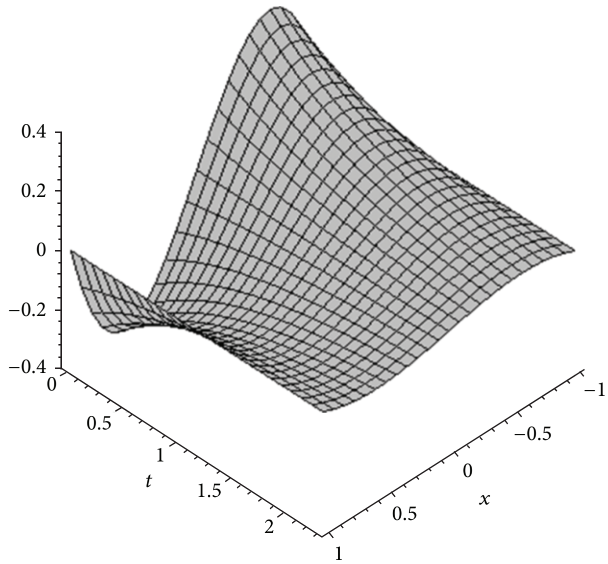

Figure 1: The graphic of the function $u_1(t,x)$ in the test problem (32).

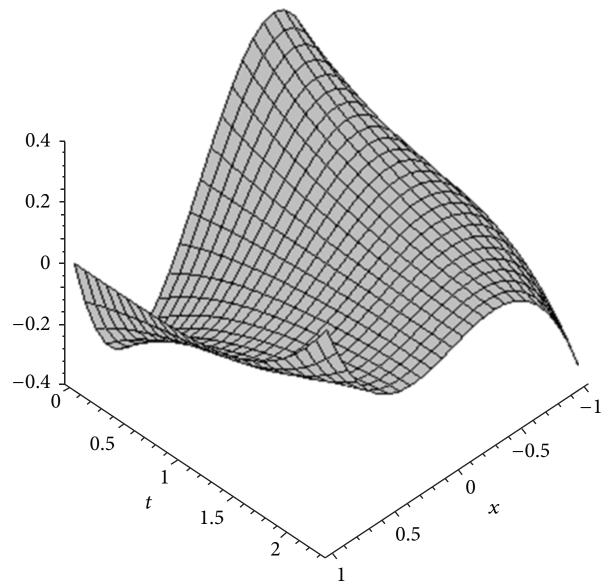

Figure 2: The graphic of the function $u_1^*(t,x)$ in the test problem (32).

The values $F_1(k,h)$ and $F_2(k,h)$ in (39) and (40) are coefficients of polynomials (41) and (42). If we use Theorem 3 for boundary values, we obtain

$$\sum_{i=0}^{7} U_1(j,i) = 0, \quad j = 0,1,\ldots,7, \tag{42}$$

$$\sum_{i=0}^{7} (-1)^i U_1(j,i) = 0, \quad j = 0,1,\ldots,7. \tag{43}$$

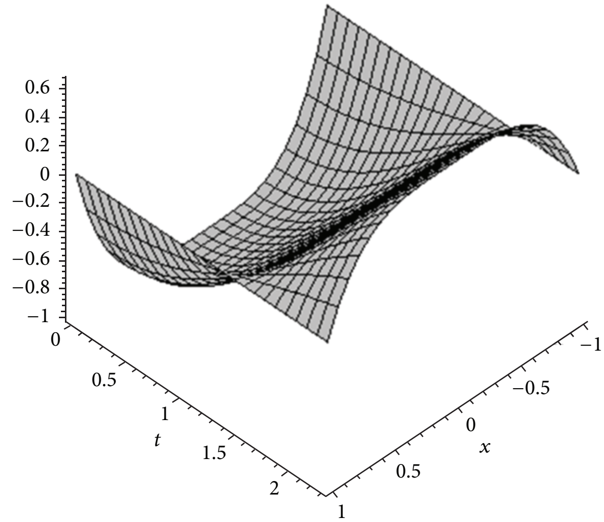

Figure 3: The graphic of the function $u_2(t,x)$ in the test problem (32).

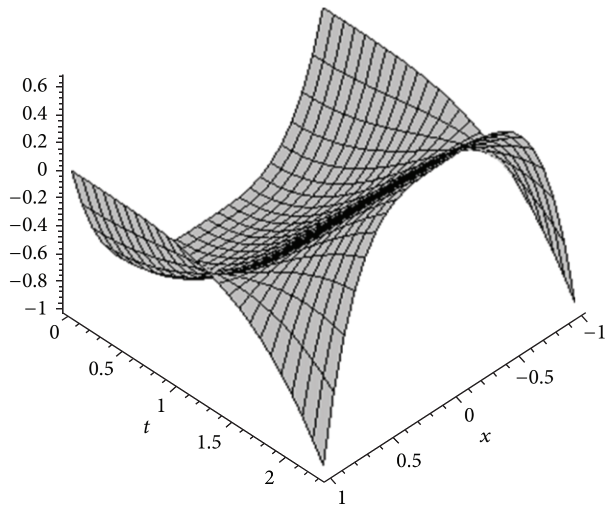

Figure 4: The graphic of the function $u_2^*(t,x)$ in the test problem (32).

In order to write $k = 0$ and $h = 0,1,2,3,4,5$ in (40), we have

$$\begin{aligned} &2U_1(0,2) + U_1(0,0) = 0, && 20U_1(0,5) + U_1(0,3) = 1, \\ &6U_1(0,3) + U_1(0,1) = 5, && 30U_1(0,6) + U_1(0,4) = 0, \\ &12U_1(0,4) + U_1(0,2) = 0, && 42U_1(0,7) + U_1(0,5) = 0. \end{aligned} \tag{44}$$

If we take $j = 0$ in (43) and (44), we obtain

$$\begin{aligned} &U_1(0,0) + U_1(0,1) + U_1(0,2) + U_1(0,3) \\ &\quad + U_1(0,4) + U_1(0,5) + U_1(0,6) + U_1(0,7) = 0, \\ &U_1(0,0) - U_1(0,1) + U_1(0,2) - U_1(0,3) \\ &\quad + U_1(0,4) - U_1(0,5) + U_1(0,6) - U_1(0,7) = 0. \end{aligned} \tag{45}$$

From (45) and (46), we find

$$U_1(0,0)=0,\quad U_1(0,1)=-1,\quad U_1(0,2)=0,$$
$$U_1(0,3)=1,\quad U_1(0,4)=0,\quad U_1(0,5)=0,\quad (46)$$
$$U_1(0,6)=0,\quad U_1(0,7)=0.$$

In this manner, from (40), (44), and (45), the coefficients of the u_1 are obtained as follows:

$$\begin{gathered}U_1(1,0)=0,\quad U_1(1,1)=1,\quad U_1(1,2)=0,\\ U_1(1,3)=-1,\quad U_1(1,4)=0,\quad U_1(1,5)=0,\\ U_1(1,6)=0,\quad U_1(2,0)=0,\quad U_1(2,1)=-\frac{1}{2},\\ U_1(2,2)=0,\quad U_1(2,3)=\frac{1}{2},\quad U_1(2,4)=0,\\ U_1(2,5)=0,\quad U_1(3,0)=0,\quad U_1(3,1)=\frac{1}{6},\\ U_1(3,2)=0,\quad U_1(3,3)=-\frac{1}{6},\quad U_1(3,4)=0,\\ U_1(4,0)=0,\quad U_1(4,1)=-\frac{1}{24},\quad U_1(4,2)=0,\\ U_1(4,3)=\frac{1}{24},\quad U_1(5,0)=0,\quad U_1(5,1)=\frac{1}{120},\\ U_1(5,2)=0,\quad U_1(6,0)=0,\quad U_1(6,1)=-\frac{1}{720},\\ U_1(7,0)=0.\end{gathered}\quad (47)$$

Using the initial values for the second component, we obtain the following coefficients:

$$U_2(0,0)=-1,\quad U_2(0,1)=0,\quad U_2(0,2)=0,$$
$$U_2(0,3)=0,\quad U_2(0,4)=1,\quad U_2(0,5)=0,\quad (48)$$
$$U_2(0,6)=0,\quad U_2(0,7)=0.$$

The coefficients of the u_2 can be found using (47), (48), (49), and taking $k=0,1,2,\ldots$ and $h=0,1,2,\ldots$ in (39) as follows:

$$\begin{gathered}U_2(1,2)=0,\quad U_2(1,3)=0,\quad U_2(1,4)=0,\\ U_2(1,5)=0,\quad U_2(1,6)=0,\quad U_2(2,1)=0,\\ U_2(2,2)=0,\quad U_2(2,3)=0,\quad U_2(2,4)=-\frac{1}{2},\\ U_2(2,5)=0,\quad U_2(3,0)=0,\quad U_2(3,1)=0,\\ U_2(3,2)=0,\quad U_2(3,3)=0,\quad U_2(3,4)=0,\\ U_2(4,1)=0,\quad U_2(4,2)=0,\quad U_2(4,3)=0,\\ U_2(4,0)=-\frac{1}{24},\quad U_2(5,0)=0,\quad U_2(5,1)=0,\\ U_2(5,2)=0,\quad U_2(6,0)=\frac{1}{720},\quad U_2(6,1)=0,\\ U_2(7,0)=0.\end{gathered}\quad (49)$$

If we write the above values in (39) and (40), then we have

$$\begin{aligned}u_1^*(t,x)=&-x+xt-\frac{1}{2}xt^2+x^3+\frac{1}{6}xt^3\\ &-x^3t+\frac{1}{2}x^3t^2-\frac{1}{24}xt^4-\frac{1}{6}x^3t^3\\ &+\frac{1}{120}xt^5-\frac{1}{720}xt^6+\frac{1}{24}x^3t^4+\cdots,\end{aligned}\quad (50)$$

$$\begin{aligned}u_2^*(t,x)=&-1+\frac{1}{2}t^2-\frac{1}{24}t^4\\ &+x^4+\frac{1}{720}t^6-\frac{1}{2}x^4t^2+\cdots.\end{aligned}\quad (51)$$

Numerical and exact solution of the given problem has been compared in Tables 1 and 2, and simulations of solutions have been depicted in Figures 1, 2, 3, and 4, respectively.

5. Conclusion

The computations associated with the example discussed above were performed by using Computer Algebra Techniques [24]. We show the results in Tables 1 and 2 for the solution of (32) by numerical method. The numerical values on Tables 1 and 2 obtained above are in full agreement with the exact solutions of (32). This study has shown that the differential transform method often shows superior performance over series approximants, providing a promising tool for using in applied fields.

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Research Article

The Time-Fractional Coupled-Korteweg-de-Vries Equations

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We put into practice a relatively new analytical technique, the homotopy decomposition method, for solving the nonlinear fractional coupled-Korteweg-de-Vries equations. Numerical solutions are given, and some properties exhibit reasonable dependence on the fractional-order derivatives' values. The fractional derivatives are described in the Caputo sense. The reliability of HDM and the reduction in computations give HDM a wider applicability. In addition, the calculations involved in HDM are very simple and straightforward. It is demonstrated that HDM is a powerful and efficient tool for FPDEs. It was also demonstrated that HDM is more efficient than the adomian decomposition method (ADM), variational iteration method (VIM), homotopy analysis method (HAM), and homotopy perturbation method (HPM).

1. Introduction

Fractional calculus has been used to model physical and engineering processes, which are found to be best described by fractional differential equations. It is worth nothing that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, signal image processing, and groundwater problems. In the past several decades, the investigation of travelling-wave solutions for nonlinear equations has played an important role in the study of nonlinear physical phenomena. In [1], homotopy analysis method is applied to obtain approximate analytical solution of the modified Kuramoto-Sivashinsky equation. In addition to that an excellent literature of this can be found in [2–11]. Analytical solutions of these equations are usually not available. Since only limited classes of equations are solved by analytical means, numerical solution of these nonlinear partial differential equations is of practical importance.

In this paper, we extend the application of the homotopy decomposition method (HDM) in order to derive analytical approximate solutions to nonlinear time-fractional coupled-KDV equations. This coupled system is used to describe iterations of water waves proposed by Hirota and Satsuma [12]. The HDM was recently applied to solve the fractional modified Kawahara equation, fractional model of HIV infection of CD4+T cells, the attractor fractional one-dimensional Keller-Segel equations, the fractional Jaulent-Miodek and Whitham-Broer-Kaup equations, the fractional Riccati differential equation, fractional nonlinear predator-prey population, and the fractional nonlinear system predator-prey population. The relatively new technique that approached the HDM is a promising analytical technique to solve nonlinear fractional partial and ordinary differential equations. The fractional systems of partial differential equations under investigation here are given as

$$\begin{aligned}
&\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + 6au(x,t)\, u_x(x,t) - 2bv(x,t)\, v_x(x,t) \\
&\qquad + au_{x,x,x}(x,t) = 0, \qquad 0 < \alpha \le 1, \\
&\frac{\partial^{\beta} v(x,t)}{\partial t^{\beta}} + 3bu(x,t)\, v_x(x,t) \\
&\qquad + bv_{x,x,x}(x,t) = 0, \qquad 0 < \beta \le 1.
\end{aligned} \tag{1}$$

Subject to the initial conditions

$$u(x,0) = \frac{\lambda}{a}\left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}x\right)\right)^2,$$
$$v(x,0) = \frac{\lambda}{\sqrt{2a}}\left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}x\right)\right)^2. \quad (2)$$

The remaining of this paper is structured as follows: in Section 2 we present a brief history of the fractional derivative order and their properties. We present the basic ideal of the homotopy decomposition method for solving high-order nonlinear fractional partial differential equations. We present the application of the HDM for system fractional nonlinear differential equations (1) and numerical results in Section 4. The conclusions are then given in Section 5.

2. Fractional Derivative Order

2.1. Brief History. In the literature, one can find several definitions of fractional derivatives. The most common used are the Riemann-Liouville and the Caputo derivatives. For Caputo we have

$$ {}_0^C D_x^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\frac{d^n f(t)}{dt^n}dt. \quad (3)$$

For the case of Riemann-Liouville we have the following definition:

$$D_x^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1} f(t)\,dt. \quad (4)$$

Each fractional derivative presents some advantages and disadvantages [13, 14]. The Riemann-Liouville derivative of a constant is not zero while Caputo's derivative of a constant is zero but demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first-order derivative might have fractional derivatives of all orders less than one in the Riemann-Liouville sense [15, 16]. Recently, Jumarie (see [17, 18]) proposed a simple alternative definition to the Riemann-Liouville derivative:

$$D_x^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}\{f(t)-f(0)\}\,dt. \quad (5)$$

His modified Riemann-Liouville derivative seems to have advantages of both the standard Riemann-Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (nondifferentiable) functions and the fractional derivative of a constant is equal to zero. However, the Jumarie fractional derivative gives the fractional derivative of $f(x) - f(0)$ not for $f(x)$, this implies that, there is no fractional derivative for some functions that are not defined at the origin, for instance $\ln(x)$ [19].

We can point out that Caputo and Riemann-Liouville may have their disadvantages but they still remain the best definitions of the fractional derivative. Every definition must be used accordingly [19].

2.2. Properties and Definitions

Definition 1. A real function $f(x)$, $x > 0$ is said to be in the space C_{μ}, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0,\infty)$, and it is said to be in space C_{μ}^m if $f^{(m)} \in C_{\mu}$, $m \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{\mu}$, $\mu \geq -1$, is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad \alpha > 0,\ x > 0,$$
$$J^0 f(x) = f(x). \quad (6)$$

Properties of the operator can be found in [15, 16], and one mentions only the following:

for $f \in C_{\mu}$, $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma > -1$:

$$J^{\alpha}J^{\beta} f(x) = J^{\alpha+\beta} f(x),$$
$$J^{\alpha}J^{\beta} f(x) = J^{\beta}J^{\alpha} f(x)\, J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}. \quad (7)$$

Lemma 3. *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and $f \in C_{\mu}^m$, $\mu \geq -1$, then*

$$D^{\alpha}J^{\alpha} f(x) = f(x),$$
$$J^{\alpha}D_0^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0. \quad (8)$$

Definition 4 (partial derivatives of fractional order). Assume now that $f(\mathbf{x})$ is a function of n variables x_i, $i = 1, \ldots, n$ also of class C on $D \in \mathbb{R}_n$. As an extension of Definition 4, one defines partial derivative of order α for f with respect to x_i the function

$$a\partial_{\underline{\mathbf{x}}}^{\alpha} f = \frac{1}{\Gamma(m-\alpha)}\int_a^{x_i}(x_i-t)^{m-\alpha-1}\partial_{x_i}^m f(x_j)\Big|_{x_j=t}dt, \quad (9)$$

if it exists, where $\partial_{x_i}^m$ is the usual partial derivative of integer-order m.

3. Basic Idea of the HDM

To illustrate the basic idea of this method, we consider a general nonlinear nonhomogeneous fractional partial differential equation with initial conditions of the following form:

$$\frac{\partial^{\alpha}U(x,t)}{\partial t^{\alpha}} = L(U(x,t)) + N(U(x,t)) + f(x,t), \quad \alpha > 0. \quad (10)$$

Subject to the initial condition

$$\begin{gathered} D_0^{\alpha-k}U(x,0) = f_k(x), \quad (k = 0,\dots,n-1), \\ D_0^{\alpha-n}U(x,0) = 0, \quad n = [\alpha], \\ D_0^{k}U(x,0) = g_k(x), \quad (k = 0,\dots,n-1), \\ D_0^{n}U(x,0) = 0, \quad n = [\alpha], \end{gathered} \tag{11}$$

where $\partial^\alpha/\partial t^\alpha$ denotes the Caputo or Riemann-Liouville fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator, and L represents a linear fractional differential operator. The method first step here is to transform the fractional partial differential equation to the fractional partial integral equation by applying the inverse operator $\partial^\alpha/\partial t^\alpha$ of both sides of (10) to obtain the following. In the case of Riemann-Liouville fractional derivative

$$\begin{aligned} U(x,t) = {} & \sum_{j=1}^{n-1} \frac{f_j(x)}{\Gamma(\alpha-j+1)} t^{\alpha-j} \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \Big[L(U(x,\tau)) + N(U(x,\tau)) \\ & + f(x,\tau) \Big] d\tau. \end{aligned} \tag{12}$$

In the case of Caputo fractional derivative

$$\begin{aligned} U(x,t) = {} & \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha-j+1)} t^{j} \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \Big[L(U(x,\tau)) + N(U(x,\tau)) \\ & + f(x,\tau) \Big] d\tau, \end{aligned} \tag{13}$$

or in general by putting

$$\sum_{j=1}^{n-1} \frac{f_j(x)}{\Gamma(\alpha-j+1)} t^{\alpha-j} = f(x,t) \quad \text{or} \quad f(x,t) = \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha-j+1)} t^{j}, \tag{14}$$

we obtain the following:

$$\begin{aligned} U(x,t) = {} & T(x,t) \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \Big[L(U(x,\tau)) + N(U(x,\tau)) \\ & + f(x,\tau) \Big] d\tau. \end{aligned} \tag{15}$$

In the homotopy decomposition method, the basic assumption is that the solutions can be written as a power series in p

$$U(x,t,p) = \sum_{n=0}^{\infty} p^n U_n(x,t), \tag{16a}$$

$$U(x,t) = \lim_{p\to 1} U(x,t,p), \tag{16b}$$

and the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{n=0}^{\infty} p^n \mathscr{H}_n(U), \tag{17}$$

where $p \in (0,1]$ is an embedding parameter. $\mathscr{H}_n(U)$ is the He's polynomials that can be generated by

$$\begin{aligned} & \mathscr{H}_n(U_0,\dots,U_n) \\ & \quad = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{j=0}^{\infty} p^j U_j(x,t) \right) \right], \quad n = 0,1,2,\dots. \end{aligned} \tag{18}$$

The homotopy decomposition method is obtained by the graceful coupling of homotopy technique with the Abel integral and is given by

$$\begin{aligned} & \sum_{n=0}^{\infty} p^n U_n(x,t) - T(x,t) \\ & \quad = \frac{p}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[f(x,\tau) + L\left(\sum_{n=0}^{\infty} p^n U_n(x,\tau) \right) \right. \\ & \qquad \left. + N\left(\sum_{n=0}^{\infty} p^n U_n(x,\tau) \right) \right] d\tau. \end{aligned} \tag{19}$$

Comparison of the terms of same powers of p gives solutions of various orders with the first term:

$$U_0(x,t) = T(x,t). \tag{20}$$

3.1. Convergence of the Method and Unicity of the Solution

Theorem 5 (see [19]). *Assuming that $X \times T \subset \mathbb{R} \times \mathbb{R}^+$ is a Banach space with a well-defined norm $\|\cdot\|$, over which the series sequence of the approximate solution of* (1) *is defined, and the operator $G(U_n(x,t)) = U_{n+1}(x,t)$ defining the series solution of* (16b) *satisfies the Lipschitzian conditions that is $\|G(U_k^*) - G(U_k)\| \le \varepsilon \|U_k^*(x,t) - U_k(x,t)\|$ for all $(x,t,k) \in X \times T \times \mathbb{N}$, then series solution obtained* (16b) *is unique.*

Proof. Assume that $U(x,t)$ and $U^*(x,t)$ are the series solution satisfying (1), then $U^*(x,t,p) = \sum_{n=0}^{\infty} p^n U_n^*(x,t)$ with initial guess $T(x,t)$; $U(x,t,p) = \sum_{n=0}^{\infty} p^n U_n(x,t)$ also with initial guess $T(x,t)$; therefore,

$$\left\| U_n^*(x,t) - U_n(x,t) \right\| = 0, \quad n = 0,1,2,\dots. \tag{21}$$

By the recurrence for $n = 0$, $U_n^*(x,t) = U_n(x,t) = T(x,t)$, assume that for $n > k \geq 0$, $\|U_k^*(x,t) - U_k(x,t)\| = 0$. Then

$$\begin{aligned}\left\|U_{k+1}^*(x,t) - U_{k+1}(x,t)\right\| &= \left\|G(U_k^*) - G(U_k)\right\| \\ &\leq \varepsilon \left\|U_k^*(x,t) - U_k(x,t)\right\| = 0,\end{aligned} \tag{22}$$

which completes the proof. □

3.2. Complexity of the Homotopy Decomposition Method. It is very important to test the computational complexity of a method or algorithm. Complexity of an algorithm is the study of how long a program will take to run, depending on the size of its input and long of loops made inside the code. We compute a numerical example which is solved by the homotopy decomposition method. The code has been presented with Mathematica 8 according to the following code [19].

Step 1. Set $m \leftarrow 0$.

Step 2. Calculating the recursive relation after the comparison of the terms of the same power is done.

Step 3. If $\|U_{n+1}(x,t) - U_n(x,t)\| < r$ with r the ratio of the neighbourhood of the exact solution [5] then go to Step 4, else $m \leftarrow m+1$ and go to Step 2

Step 4. Print out

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t), \tag{23}$$

as the approximate of the exact solution.

Lemma 6. *If the exact solution of the fractional partial differential equation* (10) *exists, then*

$$\left\|U_{n+1}(x,t) - U_n(x,t)\right\| < r \quad \forall (x,t) \in X \times T. \tag{24}$$

Proof. Let $(x,t) \in X \times T$, then since the exact solution exists, then we have that following:

$$\begin{aligned}&\left\|U_{n+1}(x,t) - U_n(x,t)\right\| \\ &\quad = \left\|U_{n+1}(x,t) - U(x,t) + U(x,t) - U_n(x,t)\right\| \\ &\quad \leq \left\|U_{n+1}(x,t) - U(x,t)\right\| + \left\|-U_n(x,t) + U(x,t)\right\| \\ &\quad \leq \frac{r}{2} + \frac{r}{2} = r.\end{aligned} \tag{25}$$

The last inequality follows from [19]. □

Lemma 7. *The complexity of the homotopy decomposition method is of order $O(n)$.*

Proof. The number of computations including product, addition, subtraction, and division are in Step 2

U_o: is 0 because, it is obtained directly form the initial guess $T(x,t)$ [19].

U_1: 3

⋮

U_n: 3.

Now in Step 4, the total number of computations is equal to $\sum_{j=0}^{n} U_j(x,t) = 3n = O(n)$. □

4. Application

In learning science, examples are useful than rules (Isaac Newton). In this section, we apply this method for solving system of fractional differential equation. Following carefully the steps involved in the HDM, we arrive at the following equations:

$$\begin{aligned}&\sum_{n=0}^{\infty} p^n u_n(x,t) \\ &\quad = u(x,0) \\ &\qquad - \frac{p}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(6a \sum_{n=0}^{\infty} p^n u_n \left(\sum_{n=0}^{\infty} p^n u_n\right)_x \right. \\ &\qquad\qquad \left. - 2b \sum_{n=0}^{\infty} p^n v_n \left(\sum_{n=0}^{\infty} p^n v_n\right)_x \right) \\ &\qquad + \left(\sum_{n=0}^{\infty} p^n u_n\right)_{x,x,x}, \\ &\sum_{n=0}^{\infty} p^n v_n(x,t) \\ &\quad = v(x,0) \\ &\qquad - \frac{p}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \left(\left(6a \sum_{n=0}^{\infty} p^n u_n \left(\sum_{n=0}^{\infty} p^n u_n\right)_x \right.\right. \\ &\qquad\qquad \left. + 3b \sum_{n=0}^{\infty} p^n u_n \left(\sum_{n=0}^{\infty} p^n v_n\right)_x \right) \\ &\qquad\qquad \left. \times b \left(\sum_{n=0}^{\infty} p^n u_n\right)_{x,x,x} \right).\end{aligned} \tag{26}$$

If we compare the terms of the same power of p we obtain the following integral equations. Note that when comparing this approach with the methodology of the homotopy perturbation method, one will obtain in this step a set of ordinary differential equations something which needs to be also solved with care, because one will need to choose an appropriate initial guest. But with the current approach, the initial guess is straightforwardly obtained as the Taylor series of the exact solution of the problem under investigation; this is one of the advantages that the approach has over the HPM [22]. On the other hand, when comparing this approach with the variational iteration method [23], one will find out that we do need the Lagrange multiplier here or the correctional function. Also this approach provides us with a convenient way to control the convergence of approximation series without adapting h, as in the case of [24] which is a fundamental qualitative difference in analysis between HDM and other methods. Therefore, comparing the terms of the same power we obtain

$$
\begin{aligned}
&p^0: u_0(x,t) = u(x,0), \quad u_0(x,0) = u(x,0),\\
&p^0: u_0(x,t) = u(x,0), \quad u_0(x,0) = u(x,0),\\
&p^1: u_1(x,t)\\
&\quad = -\frac{1}{\Gamma(\alpha)}\\
&\quad\times \int_0^t (t-\tau)^{\alpha-1}\left(6au_0(u_0)_x - 2bv_0(v_0)_x\right.\\
&\qquad\left.+a(u_0)_{xxx}\right)d\tau,\\
&\qquad u_1(x,0) = 0,\\
&p^1: v_1(x,t)\\
&\quad = -\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\\
&\qquad\times\left(3bu_0(v_0)_x + b(u_0)_{xxx}\right)d\tau,\\
&\qquad v_1(x,0) = 0,\\
&\qquad\vdots\\
&p^n: u_n(x,t)\\
&\quad = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\\
&\qquad\times\left(6a\sum_{i=0}^{n-1}u_i(u_{n-i-1})_x - 2b\sum_{i=0}^{n-1}v_i(v_{n-i-1})_x\right.\\
&\qquad\left.+a(u_{n-1})_{xxx}\right)d\tau,\\
&u_n(x,t) = 0,\\
&p^n: v_n(x,t)\\
&\quad = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\\
&\qquad\times\left(3b\sum_{i=0}^{n-1}u_i(v_{n-i-1})_x + b(v_{n-1})_{xxx}\right)d\tau,\\
&\qquad v_n(x,t) = 0.
\end{aligned}
\tag{27}
$$

Integrating the above, we obtain the following series solutions:

$$
\begin{aligned}
u_0(x,t) &= \frac{\lambda}{a}\left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}x\right)\right)^2,\\
v_0(x,t) &= \frac{\lambda}{\sqrt{2a}}\left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}x\right)\right)^2.
\end{aligned}
\tag{28}
$$

For the sake of simplicity we put the following:

$$
\begin{aligned}
&d = \frac{\lambda}{a}, \qquad d_1 = \frac{\lambda}{\sqrt{2a}}, \qquad m = \frac{1}{2}\sqrt{\frac{\lambda}{a}},\\
&u_1(x,t)\\
&\quad = \frac{4mt^{\alpha}}{\Gamma(1+\alpha)}\\
&\qquad\times\left(-bd_1{}^2 + ad\left(3d-5m^2\right) + adm^2\cosh(2mx)\right)\\
&\qquad\times(\operatorname{sech}(mx))^4\tanh(mx),\\
&v_1(x,t) = \frac{2bd_1mt^{\beta}}{\Gamma(1+\beta)}\left(3d - 10m^2 + 2m^2\cosh(2mx)\right)\\
&\qquad\times(\operatorname{sech}(mx))^4\tanh(mx),\\
&u_2(x,t)\\
&\quad = \frac{1}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(0.5+\alpha)\Gamma(1+\alpha+\beta)}\\
&\qquad\times\left(2^{1-2\alpha}m^2\sqrt{\pi}t^{\alpha}\Gamma(1+\beta)\right.\\
&\qquad\quad\times\left(-2b^2d_1^2t^{\beta}\right.\\
&\qquad\qquad\times\left(-12d + 44m^2 + \left(9d - 38m^2\right)\right.\\
&\qquad\qquad\quad\left.\times\cosh(2mx) + 2m^2\cosh(4mx)\right)\\
&\qquad\qquad\quad\times\Gamma(1+2\alpha) + at^{\alpha}\\
&\qquad\qquad\quad\times\left(-8\left(2bd_1^2\left(-3d+13m^2\right)\right.\right.\\
&\qquad\qquad\qquad\quad\left.+ad\left(18d^2 - 111dm^2 + 151m^4\right)\right)\\
&\qquad\qquad\qquad+\left(4bd_1^2\left(-9d+49m^2\right)\right.\\
&\qquad\qquad\qquad\quad\left.+3ad\left(36d^2 - 272dm^2 + 397m^4\right)\right)\\
&\qquad\qquad\qquad\times\cosh(2mx)\\
&\qquad\qquad\qquad-4m^2\left(4bd_1^2 - 15ad\left(d-2m^2\right)\right)\\
&\qquad\qquad\qquad\left.\times\cosh(4mx) + adm^4\cosh(6xm)\right)\\
&\qquad\qquad\left.\left.\times\Gamma(1+\alpha+\beta)\right)(\operatorname{sech}(mx))^8\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
& v_2(x,t) \\
&= \frac{1}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(0.5+\alpha)\Gamma(1+\alpha+\beta)} \\
&\quad \times \left(2^{1-2\beta}m^2\sqrt{\pi}t^{\beta}\Gamma(1+\alpha)(\operatorname{sech}(mx))^8\right. \\
&\qquad \times \left(bt^{\beta}\left(-27d^2+411dm^2-1208m^4\right.\right. \\
&\qquad\qquad + 3\left(6d^2-124dm^2+397m^4\right) \\
&\qquad\qquad \times \cosh(2mx) + 3m^2\left(9d-40m^2\right) \\
&\qquad\qquad \left.\times \cosh(4mx) + m^4\cosh(6mx)\right) \\
&\qquad \times \Gamma(1+\alpha+\beta) \\
&\qquad + 12t^{\alpha}\left(-bd_1^2+ad\left(3d-5m^2\right)\right. \\
&\qquad\qquad \left.+adm^2\cosh(2mx)\right)\Gamma(1+2\beta) \\
&\qquad \left.\left.\times(\sinh(mx))^2\right)\right).
\end{aligned}
\tag{29}
$$

And so on, using the package Mathematica, in the same manner, one can obtain the rest of the components. But, here, few terms were computed and the asymptotic solution is given by the following:

$$
\begin{aligned}
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots, \\
v(x,t) &= v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+\cdots.
\end{aligned}
\tag{30}
$$

4.1. Numerical Solutions. The following figures show the graphical representation of the approximated solution of the system of the time-fractional coupled-Korteweg-de-Vries equations for $\lambda = 1$, $a = b = 1$.

Note that the below figure show that the coupled solution of KDV equation is not only the function of time and space but also an increasing function of the fractional order derivative, which are α and β. The approximate solution of main problem has been depicted in Figures 1, 2, 3, and 4 which is plotted in Mathematica according to different α and β values.

It is important to note that if $\alpha = \beta$, $a = 1$, and $b = 3$, the exact solution of the coupled-KDV equations is given as

$$
\begin{aligned}
u(x,0) &= \frac{\lambda}{a}\left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}x-\lambda t\right)\right)^2, \\
v(x,0) &= \frac{\lambda}{\sqrt{2a}}\left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}x-\lambda t\right)\right)^2.
\end{aligned}
\tag{31}
$$

Thus, to test the accuracy of the relatively new analytical technique, we represent in Table 1 the numerical values of the approximate and the exact solutions and the results obtained in [20].

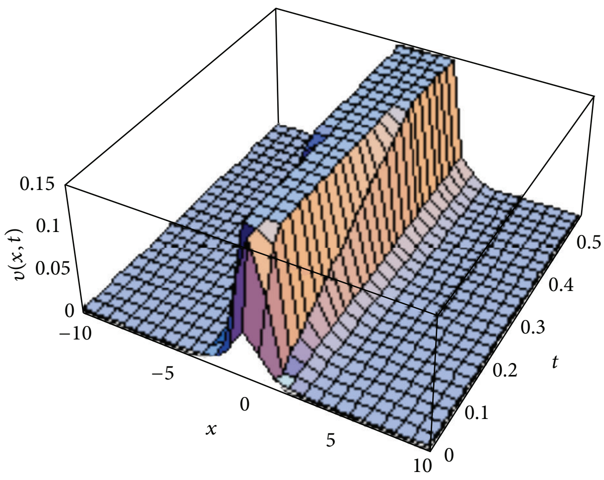

Figure 1: Approximate solution for $\alpha = 0.75$ and $\beta = 0.45$.

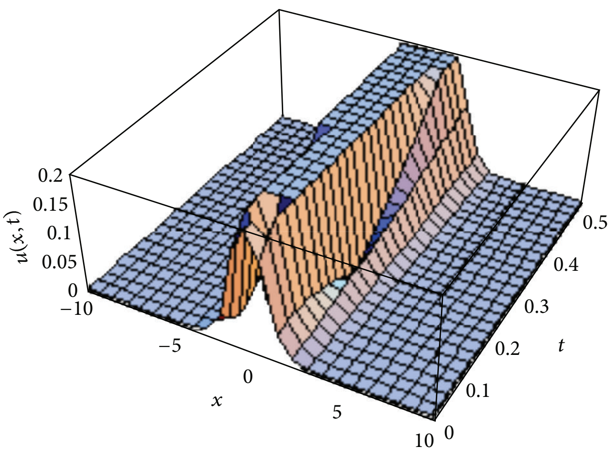

Figure 2: Approximate solution for $\alpha = 0.75$ and $\beta = 0.45$.

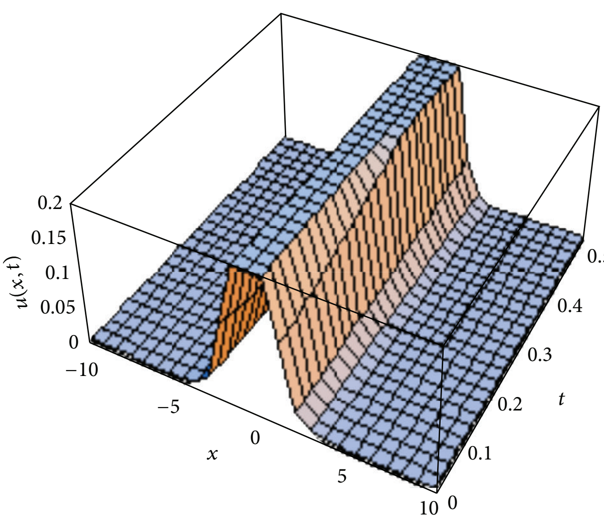

Figure 3: Approximate solution for $\alpha = 1$ and $\beta = 0.9$.

Table 1: Numerical values of the approximate, exact solutions and the results obtained in [20, 21].

x	t	$u(x,t)$ exact	$u(x,t)$ approximate	[20]	Error for [20]	Error approx
−10	0.1	0.000164305	0.000164334	0.000164384	2.99039×10^{-8}	2.95039×10^{-8}
	0.2	0.00014867	0.000148901	0.000148991	2.33335×10^{-7}	2.30335×10^{-7}
−5	0.1	0.0240923	0.0240963	0.02409673	3.96592×10^{-6}	3.93592×10^{-6}
	0.2	0.0218248	0.0218556	0.02185586	0.0000338049	0.0000308049
5	0.1	0.0240923	0.0240963	0.02409653	3.97592×10^{-6}	3.93592×10^{-6}
	0.2	0.0218248	0.0218556	0.02185576	0.0000378049	0.0000308049
10	0.1	0.000164305	0.000164334	0.000164344	2.96039×10^{-8}	2.95039×10^{-8}
	0.2	0.00014867	0.000148901	0.000148931	2.37335×10	2.30335×10^{-7}
x	t	$v(x,t)$ exact	$v(x,t)$ approximate	[20]	Error for [20]	Error
−10	0.1	0.000116181	0.000116202	0.000116232	2.18624×10^{-8}	2.08624×10^{-8}
	0.2	0.000105126	0.000105289	0.000105259	1.64872×10^{-7}	1.62872×10^{-7}
−5	0.1	0.170358	0.0170386	0.0170387	2.88312×10^{-6}	2.78312×10^{-6}
	0.2	0.0154325	0.0154542	0.0154552	0.0000287824	0.0000217824
5	0.1	0.170358	0.0170386	0.0170389	2.98312×10^{-6}	2.78312×10^{-6}
	0.2	0.0154325	0.0154542	0.0154562	0.0000247824	0.0000217824
10	0.1	0.000116181	0.000116202	0.000116252	2.09624×10^{-8}	2.08624×10^{-8}
	0.2	0.000105126	0.000105289	0.000105299	1.72872×10^{-7}	1.62872×10^{-7}

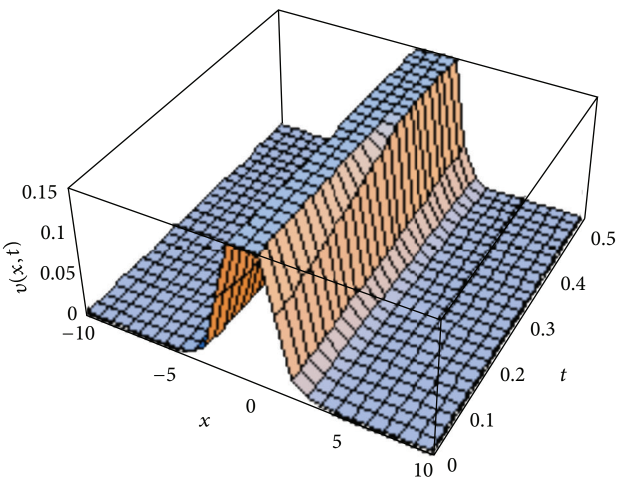

Figure 4: Approximate solution for $\alpha = 1$ and $\beta = 1$.

Table 1 comparison shows that the solutions obtained in this paper are more accurate than those obtained in [20].

5. Conclusions

We derived approximated solutions of nonlinear fractional-coupled KDV equations using the relatively new analytical technique, the HDM. We presented the brief history and some properties of fractional derivative concept. It is demonstrated that HDM is a powerful and efficient tool of FPDEs. In addition, the calculations involved in HDM are very simple and straightforward. Comparing the methodology HDM to HPM, ADM [25], VIM, and HAM have the advantages. Disparate the ADM, the HDM is free from the need to use the Adomian polynomials. In this method, we do not need the Lagrange multiplier, correction functional, stationary conditions, or calculating heavy integrals, as the solutions obtained are noise free [26], which eliminate the complications that exist in the VIM. In contrast to the HAM, this method is not required to solve the functional equations in iteration since the efficiency of HAM is very much dependant on choosing auxiliary parameter. In contract to HPM, we do not need to continuously deform a difficult problem to another that is easier to solve. We can easily conclude that the homotopy decomposition method is a well-organized analytical method for solving exact and approximate solutions of nonlinear fractional partial differential equations.

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Research Article

Multivariate Padé Approximation for Solving Nonlinear Partial Differential Equations of Fractional Order

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Two tecHniques were implemented, the Adomian decomposition method (ADM) and multivariate Padé approximation (MPA), for solving nonlinear partial differential equations of fractional order. The fractional derivatives are described in Caputo sense. First, the fractional differential equation has been solved and converted to power series by Adomian decomposition method (ADM), then power series solution of fractional differential equation was put into multivariate Padé series. Finally, numerical results were compared and presented in tables and figures.

1. Introduction

Recently, differential equations of fractional order have gained much interest in engineering, physics, chemistry, and other sciences. It can be said that the fractional derivative has drawn much attention due to its wide application in engineering physics [1–9]. Some approximations and numerical techniques have been used to provide an analytical approximation to linear and nonlinear differential equations and fractional differential equations. Among them, the variational iteration method, homotopy perturbation method [10, 11], and the Adomian decomposition method are relatively new approaches [5–9, 12, 13].

The decomposition method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations [12, 13]. Recently, the application of the method is extended for fractional differential equations [6–9, 14].

Many definitions and theorems have been developed for multivariate Padé approximations MPA (see [15] for a survey on multivariate Padé approximation). The multivariate Padé Approximation has been used to obtain approximate solutions of linear or nonlinear differential equations [16–19]. Recently, the application of the unvariate Padé approximation is extended for fractional differential equations [20, 21].

The objective of the present paper is to provide approximate solutions for initial value problems of nonlinear partial differential equations of fractional order by using multivariate Padé approximation.

2. Definitions

For the concept of fractional derivative, we will adopt Caputo's definition, which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order, which is the case in most physical processes. The definitions can be seen in [3, 4, 22, 23].

3. Decomposition Method [24]

Consider

$$D^{\alpha}_{*t}u(x,t) = f(u, u_x, u_{xx}) + g(x,t), \quad m-1 < \alpha \le m. \tag{1}$$

The decomposition method requires that a nonlinear fractional differential equation (1) is expressed in terms of operator form as

$$D_{*t}^{\alpha}u(x,t)+Lu(x,t)+Nu(x,t)=g(x,t),\quad x>0,\quad (2)$$

where L is a linear operator which might include other fractional derivatives of order less than α, N is a nonlinear operator which also might include other fractional derivatives of order less than α, $D_{*t}^{\alpha}=\partial^{\alpha}/\partial t^{\alpha}$ is the Caputo fractional derivative of order α, and $g(x,t)$ is the source function [24].

Applying the operator J^{α} [3, 4, 22, 23], the inverse of the operator D_{*t}^{α}, to both sides of (5) Odibat and Momani [24] obtained

$$\begin{aligned} u(x,t)&=\sum_{k=0}^{m-1}\frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!}+J^{\alpha}g(x,t)\\ &\quad -J^{\alpha}\left[Lu(x,t)+Nu(x,t)\right],\\ \sum_{n=0}^{\infty}u_n(x,t)&=\sum_{k=0}^{m-1}\frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!}+J^{\alpha}g(x,t)\\ &\quad -J^{\alpha}\left[L\left(\sum_{n=0}^{\infty}u_n(x,t)\right)+\sum_{n=0}^{\infty}A_n\right]. \end{aligned}\quad (3)$$

From this, the iterates are determined in [24] by the following recursive way:

$$\begin{aligned} u_0(x,t)&=\sum_{k=0}^{m-1}\frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!}+J^{\alpha}g(x,t),\\ u_1(x,t)&=-J^{\alpha}\left(Lu_0+A_0\right),\\ u_2(x,t)&=-J^{\alpha}\left(Lu_1+A_1\right),\\ &\vdots\\ u_{n+1}(x,t)&=-J^{\alpha}\left(Lu_n+A_n\right). \end{aligned}\quad (4)$$

4. Multivariate Padé Aproximation [25]

Consider the bivariate function $f(x,y)$ with Taylor series development

$$f(x,y)=\sum_{i,j=0}^{\infty}c_{ij}x^i y^j \quad (5)$$

around the origin. We know that a solution of unvariate Padé approximation problem for

$$f(x)=\sum_{i=0}^{\infty}c_i x^i \quad (6)$$

is given by

$$p(x)=\begin{vmatrix} \sum_{i=0}^{m}c_i x^i & x\sum_{i=0}^{m-1}c_i x^i & \cdots & x^n\sum_{i=0}^{m-n}c_i x^i\\ c_{m+1} & c_m & \cdots & c_{m+1-n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{vmatrix}, \quad (7)$$

$$q(x)=\begin{vmatrix} 1 & x & \cdots & x^n\\ c_{m+1} & c_m & \cdots & c_{m+1-n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{vmatrix}. \quad (8)$$

Let us now multiply jth row in $p(x)$ and $q(x)$ by x^{j+m-1} $(j=2,\ldots,n+1)$ and afterwards divide jth column in $p(x)$ and $q(x)$ by x^{j-1} $(j=2,\ldots,n+1)$. This results in a multiplication of numerator and denominator by x^{mn}. Having done so, we get

$$\frac{p(x)}{q(x)}=\frac{\begin{vmatrix} \sum_{i=0}^{m}c_i x^i & \sum_{i=0}^{m-1}c_i x^i & \cdots & \sum_{i=0}^{m-n}c_i x^i\\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1\\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}} \quad (9)$$

if $(D=\det D_{m,n}\neq 0)$.

This quotent of determinants can also immediately be written down for a bivariate function $f(x,y)$. The sum $\sum_{i=0}^{k}c_i x^i$ shall be replaced by kth partial sum of the Taylor series development of $f(x,y)$ and the expression $c_k x^k$ by an expression that contains all the terms of degree k in $f(x,y)$. Here a bivariate term $c_{ij}x^i y^j$ is said to be of degree $i+j$. If we define

$$p(x,y) = \begin{vmatrix} \sum_{i+j=0}^{m}c_{ij}x^i y^j & \sum_{i+j=0}^{m-1}c_{ij}x^i y^j & \cdots & \sum_{i+j=0}^{m-n}c_{ij}x^i y^j\\ \sum_{i+j=m+1}c_{ij}x^i y^j & \sum_{i+j=m}c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n}c_{ij}x^i y^j\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{i+j=m+n}c_{ij}x^i y^j & \sum_{i+j=m+n-1}c_{ij}x^i y^j & \cdots & \sum_{i+j=m}c_{ij}x^i y^j \end{vmatrix},$$

$$q(x,y) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=m} c_{ij}x^i y^j \end{vmatrix}. \tag{10}$$

Then it is easy to see that $p(x, y)$ and $q(x, y)$ are of the form

$$p(x,y) = \sum_{i+j=mn}^{mn+m} a_{ij}x^i y^j,$$
$$q(x,y) = \sum_{i+j=mn}^{mn+n} b_{ij}x^i y^j. \tag{11}$$

We know that $p(x, y)$ and $q(x, y)$ are called Padé equations [25]. So the multivariate Padé approximant of order (m, n) for $f(x, y)$ is defined as

$$r_{m,n}(x,y) = \frac{p(x,y)}{q(x,y)}. \tag{12}$$

5. Numerical Experiments

In this section, two methods, ADM and MPA, shall be illustrated by two examples. All the results are calculated by using the software Maple12. The full ADM solutions of examples can be seen from Odibat and Momani [24].

Example 1. Consider the nonlinear time-fractional advection partial differential equation [24]

$$D_{*t}^{\alpha}u(x,t) + u(x,t)\,u_x(x,t) = x + xt^2,$$
$$t > 0,\ x \in R,\ 0 < \alpha \le 1, \tag{13}$$

subject to the initial condition

$$u(x,0) = 0. \tag{14}$$

Odibat and Momani [24] solved the problem using the decomposition method, and they obtained the following recurrence relation [24]:

$$u_0(x,t) = u(x,0) + J^{\alpha}\left(x + xt^2\right) = x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \tag{15}$$
$$u_{j+1}(x,t) = -J^{\alpha}\left(A_j\right), \quad j \ge 0,$$

where A_j are the Adomian polynomials for the nonlinear function $N = uu_x$. In view of (15), the first few components of the decomposition series are derived in [24] as follows:

$$u_0(x,t) = x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right),$$
$$u_1(x,t) = -x\left(\frac{\Gamma(2\alpha+1)\,t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \frac{4\Gamma(2\alpha+3)\,t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+5)\,t^{3\alpha+4}}{\Gamma(\alpha+3)^2\Gamma(3\alpha+5)}\right),$$
$$u_2(x,t) = 2x\left(\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)\,t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+7)\,t^{5\alpha+6}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)} + \cdots\right), \tag{16}$$

and so on; in this manner, the rest of components of the decomposition series can be obtained [24].

The first three terms of the decomposition series are given by [24]

$$u(x,t) = x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{\Gamma(2\alpha+1)\,t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{4\Gamma(2\alpha+3)\,t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \cdots\right). \tag{17}$$

For $\alpha = 1$ (16) is

$$u(x,t) = xt + 0.1\times10^{-9}t^3 - 0.1333333333xt^5. \tag{18}$$

Now, let us calculate the approximate solution of (18) for $m = 4$ and $n = 2$ by using Multivariate Padé approximation. To obtain multivariate Padé equations of (18) for $m = 4$ and $n = 2$, we use (10). By using (10), we obtain

$$p(x,t) = \begin{vmatrix} xt + 0.1\times10^{-9}t^3 & xt & xt \\ 0 & 0.1\times10^{-9}t^3 & 0 \\ -0.1333333333xt^5 & 0 & 0.1\times10^{-9}t^3 \end{vmatrix}$$
$$= 0.1333333333\times10^{-10} \times\left(t^2 + 0.7500000002\times10^{-9}\right)x^3t^7,$$

$$q(x,t) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0.1\times 10^{-9}t^3 & 0 \\ -0.1333333333xt^5 & 0 & 0.1\times 10^{-9}t^3 \end{vmatrix} = 0.1333333333\times 10^{-10} \times \left(t^2 + 0.7500000002\times 10^{-9}\right)x^2t^6. \quad (19)$$

So, the multivariate Padé approximation of order $(4,2)$ for (17), that is,

$$[4,2]_{(x,t)} = \frac{\left(t^2+0.7500000002\times 10^{-9}\right)xt}{t^2+0.7500000002\times 10^{-9}}. \quad (20)$$

For $\alpha = 0.5$ (17) is

$$u(x,t) = 1.128379167xt^{0.5} - 0.9577979850xt^{1.5} + 0.6018022226xt^{2.5} - 0.7005608116xt^{3.5}. \quad (21)$$

For simplicity, let $t^{1/2} = a$; then

$$u(x,a) = 1.128379167xa - 0.9577979850xa^3 + 0.6018022226xa^5 - 0.7005608116xa^7. \quad (22)$$

Using (10) to calculate the multivariate Padé equations for (22) we get

$$p(x,a) = \begin{vmatrix} 1.128379167xa - 0.9577979850xa^3 + 0.6018022226xa^5 & 1.128379167xa - 0.9577979850xa^3 & 1.128379167xa - 0.9577979850xa^3 \\ 0 & 0.6018022226xa^5 & 0 \\ -0.7005608116xa^7 & 0 & 0.6018022226xa^5 \end{vmatrix}$$

$$= -0.4907854507\left(1.201464294a^4 - 1.231832347a^2 - 0.8326662354\right)x^3a^{11},$$

$$q(x,a) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0.6018022226xa^5 & 0 \\ -0.7005608116xa^7 & 0 & 0.6018022226xa^5 \end{vmatrix} = 0.4907854507\left(0.7379312378 + 1.718058483a^2\right)x^3a^{10} \quad (23)$$

recalling that $t^{1/2} = a$, we get multivariate Padé approximation of order $(6,2)$ for (21), that is,

$$[6,2]_{(x,t)} = -\left(1.201464294t^2 - 1.231832347t - 0.8326662354\right)x\sqrt{t} \times (0.7379312378 + 1.718058483t)^{-1}. \quad (24)$$

For $\alpha = 0.75$ (17) is

$$u(x,t) = 0.00007125345441xt^{7.5} + 0.1764791440\times 10^{-5}xt^{9.5} - 0.1238343301\times 10^{-17}xt^{22.5} - 0.2897967272\times 10^{-19}xt^{24.5}. \quad (25)$$

For simplicity, let $t^{1/2} = a$; then

$$u(x,a) = 0.00007125345441xa^{15} + 0.1764791440\times 10^{-5}xa^{19} - 0.1238343301\times 10^{-17}xa^{45} - 0.2897967272\times 10^{-19}xa^{49}. \quad (26)$$

Using (10) to calculate the multivariate Padé equations and then recalling that $t^{1/2} = a$, we get multivariate Padé approximation of order $(49,2)$ for (25), that is,

$$[49,2]_{(x,t)} = -0.8398214310\times 10^{-39}x^3t^{113/2} \times \left(-0.00007125345441 - 0.1764791440 \times 10^{-5}t^2 + 0.1238343301\times 10^{-17}t^{15}\right) \times \left(0.8398214310\times 10^{-39}x^2t^{49}\right)^{-1}. \quad (27)$$

Table 1, Figures 1(a), 1(b), 1(c), 2(a), 2(b), 2(c), and 2(d) shows the approximate solutions for (13) obtained for different values of α using the decomposition method (ADM) and the multivariate Padé approximation (MPA). The value of $\alpha = 1$ is for the exact solution $u(x,t) = xt$ [24].

Example 2. Consider the nonlinear time-fractional hyperbolic equation [24]

$$D^{\alpha}_{*t}u(x,t) = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right), \quad t > 0,\ x\in R,\ 1 < \alpha \le 2, \quad (28)$$

subject to the initial condition

$$u(x,0) = x^2, \qquad u_t(x,0) = -2x^2. \quad (29)$$

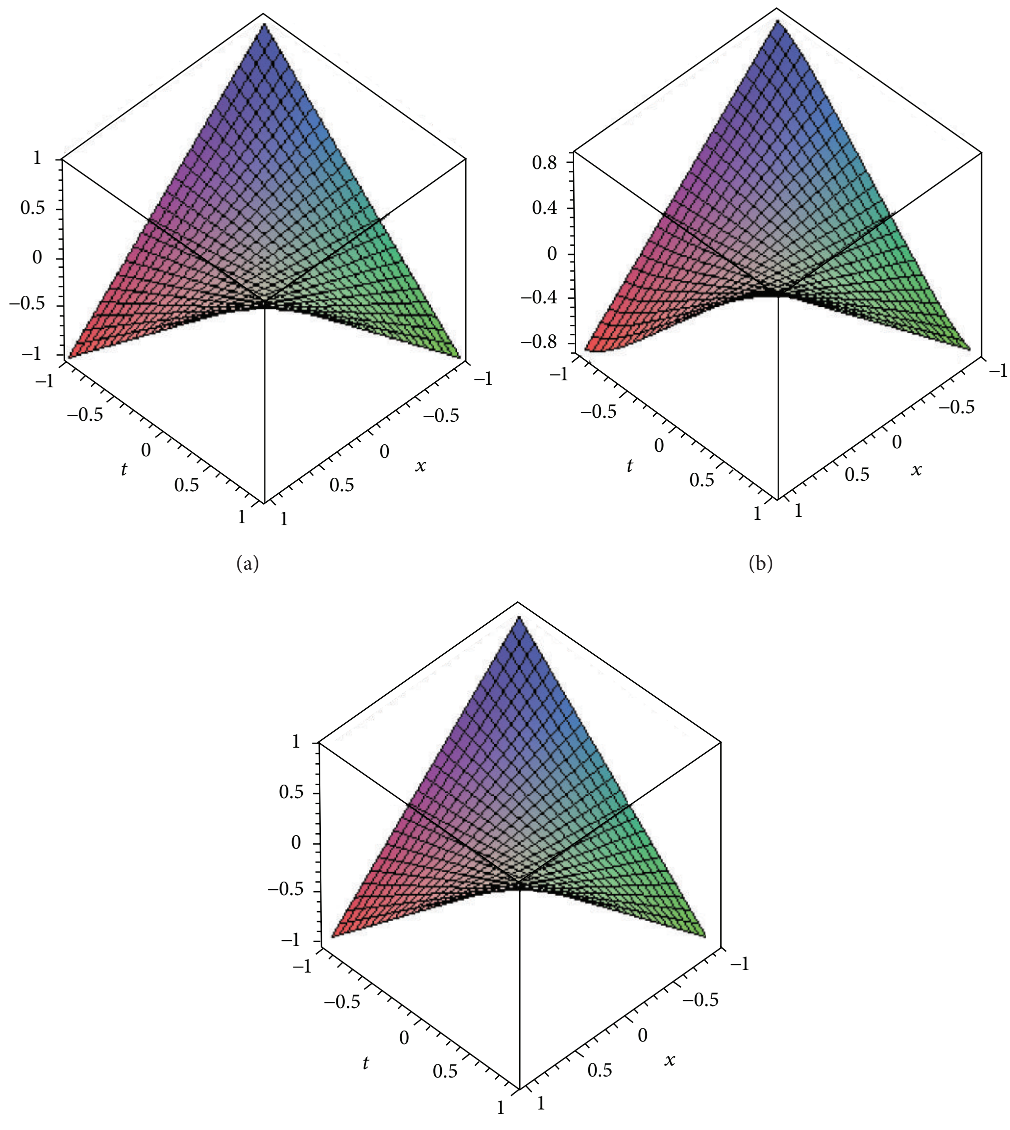

Figure 1: (a) Exact solution of Example 1 for $\alpha = 1$ (b) ADM solution of Example 1 for $\alpha = 1$ (c) Multivariate Padé approximation of ADM solution for $\alpha = 1$ in Example 1.

Odibat and Momani [24] solved the problem using the decomposition method, and they obtained the following recurrence relation in [24]:

$$\begin{gathered} u_0(x,t) = u(x,0) + t u_x(x,0) = x^2(1-2t), \\ u_{j+1}(x,t) = J^{\alpha}\left(A_j\right)_x, \quad j \geq 0, \end{gathered} \tag{30}$$

where A_j are the Adomian polynomials for the nonlinear function $N = uu_x$. In view of (30), the first few components of the decomposition series are derived in [24] as follows:

$$\begin{aligned} u_0(x,t) &= x^2(1-2t), \\ u_1(x,t) &= 6x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \\ u_2(x,t) &= 72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{2\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}\right) \\ &\quad + 72x^2\left(\frac{8\Gamma(\alpha+3)t^{2\alpha+2}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)} - \frac{16\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}\right), \\ &\vdots \end{aligned} \tag{31}$$

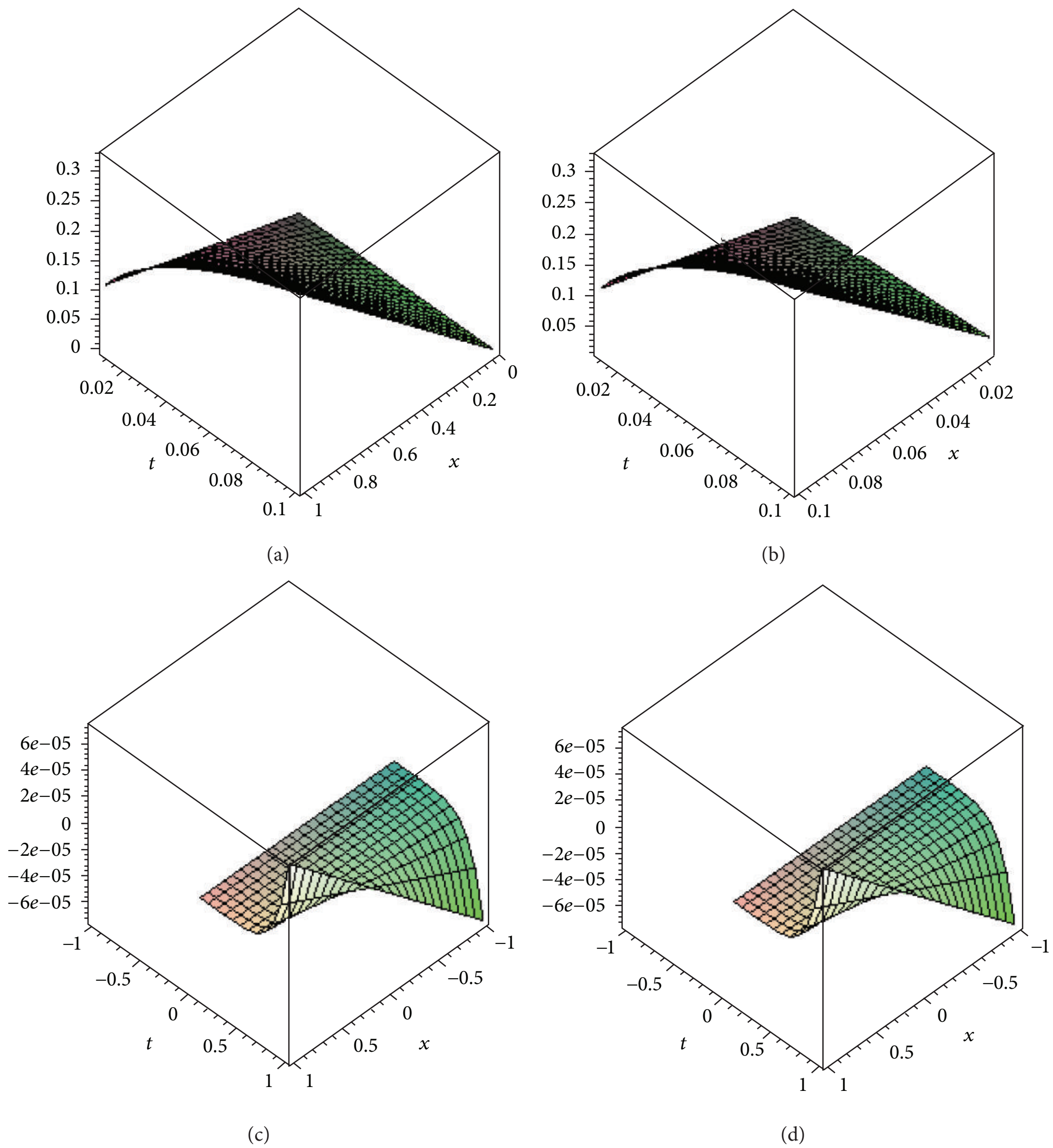

Figure 2: (a) ADM solution of Example 1 for $\alpha = 0.5$ (b) Multivariate Padé approximation of ADM solution for $\alpha = 0.5$ in Example 1 (c) ADM solution of Example 1 for $\alpha = 0.75$ (d) Multivariate Padé approximation of ADM solution for $\alpha = 0.75$ in Example 1.

and so on; in this manner the rest of components of the decomposition series can be obtained.

The first three terms of the decomposition series (7) are given [24] by

$$u(x,t) = x^2(1-2t) + 6x^2 \times \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right) + 72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right). \tag{32}$$

For $\alpha = 2$ (43) is

$$\begin{aligned} u(x,t) = x^2(1-2t) &+ 6x^2\left(0.5000000000t^2 - 0.6666666668t^3 + 0.3333333334t^4\right) \\ &+ 3.000000000x^2t^4. \end{aligned} \tag{33}$$

Now, let us calculate the approximate solution of (33) for $m = 4$ and $n = 2$ by using multivariate Padé approximation. To

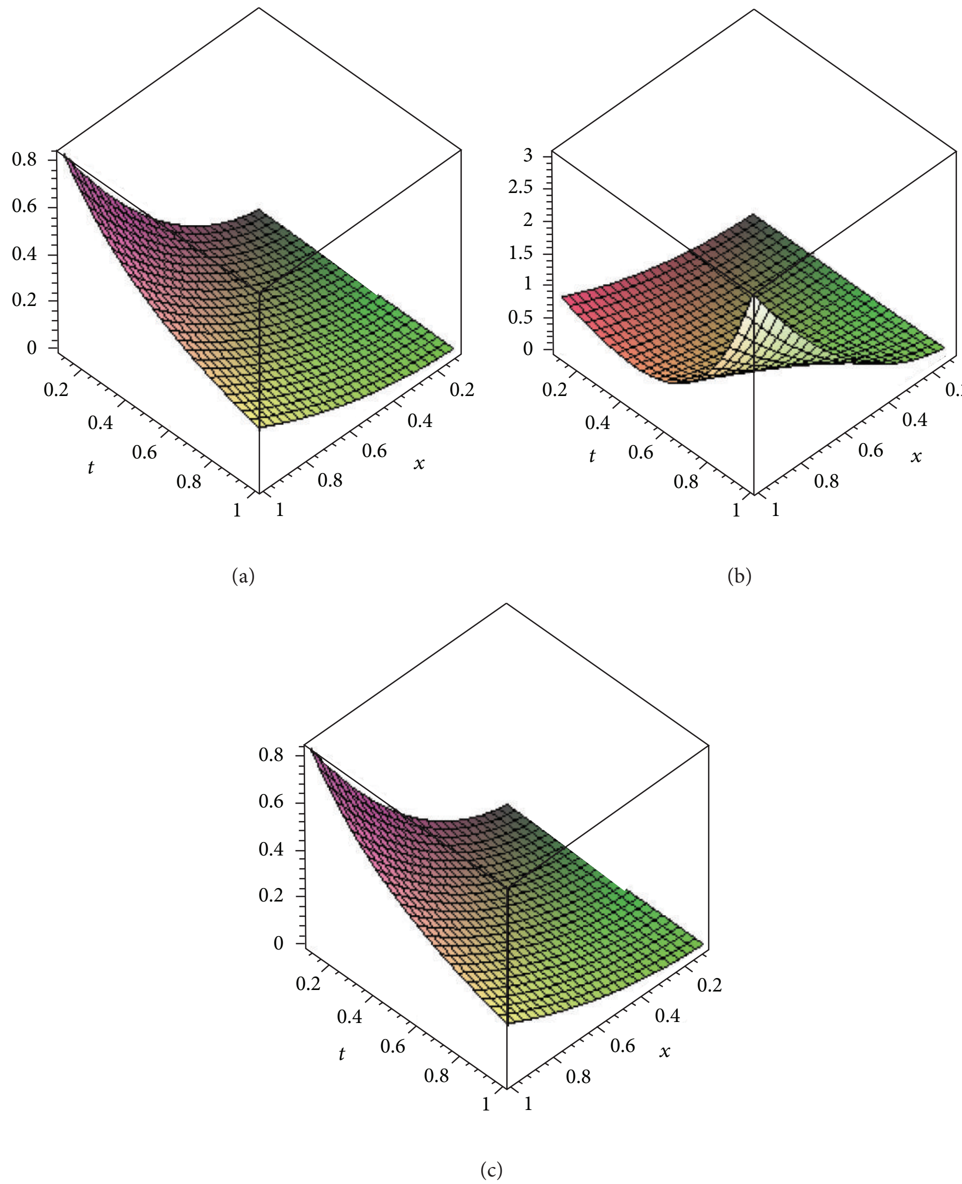

Figure 3: (a) Exact solution of Example 2 for $\alpha = 2.0$ (b) ADM solution of Example 2 for $\alpha = 2.0$ (c) Multivariate Padé approximation of ADM solution for $\alpha = 2.0$ in Example 2.

obtain multivariate Padé equations of (33) for $m = 4$ and $n = 2$, we use (10). By using (10), we obtain

$$
\begin{aligned}
&p(x,t)\\
&= \begin{vmatrix} x^2(1-2t) + 3.000000000x^2t^2 & x^2(1-2t) & x^2 \\ -4.000000001x^2t^3 & 3.000000000x^2t^2 & -2x^2t \\ 5.000000000x^2t^4 & -4.000000001x^2t^3 & 3.000000000x^2t^2 \end{vmatrix}\\
&= -20.00000000t^4\left(0.28\times10^{-9}t^2 - 0.34\times10^{-9}t\ (-0.04999999986)\,x^6,\right.\\
&q(x,t)\\
&= \begin{vmatrix} 1 & 1 & 1 \\ -4.000000001x^2t^3 & 3.000000000x^2t^2 & -2x^2t \\ 5.000000000x^2t^4 & -4.000000001x^2t^3 & 3.000000000x^2t^2 \end{vmatrix}\\
&= 20.00000000t^4\left(0.0499999999 + 0.1000000001t + 0.0500000004t^2\right)x^4.
\end{aligned}
\tag{34}
$$

So, the multivariate Padé approximation of order $(4,2)$ for (33), that is,

$$
\begin{aligned}
[4,2]_{(x,t)} = &-1.000000000\left(0.28\times10^{-9}t^2 - 0.34\times10^{-9}t\right.\\
&\left.-0.04999999986\right)x^2\\
&\times\left(0.0499999999 + 0.1000000001t\right.\\
&\left.+0.0500000004t^2\right)^{-1}.
\end{aligned}
\tag{35}
$$

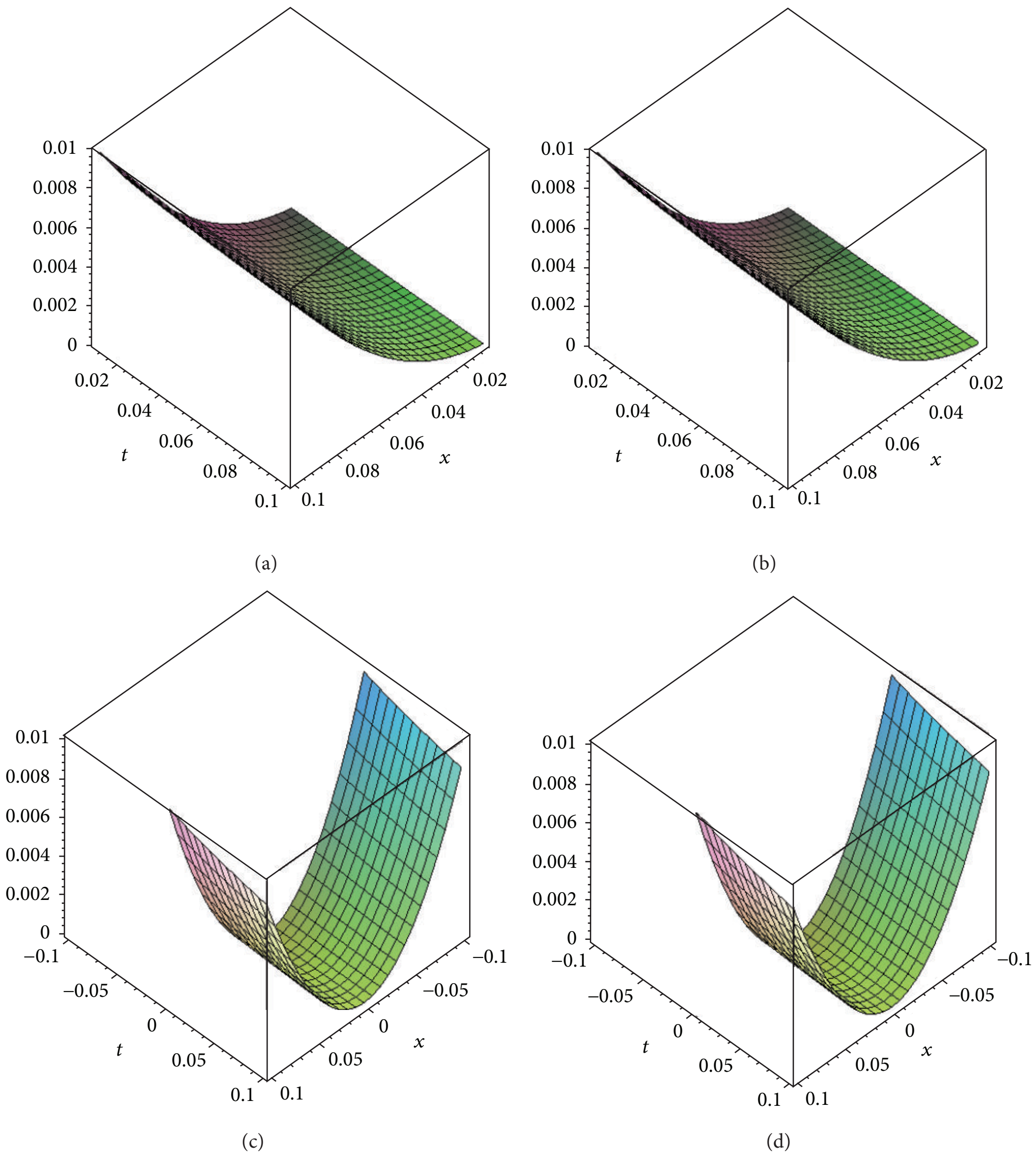

Figure 4: (a) ADM solution of Example 2 for $\alpha = 1.5$ (b) Multivariate Padé approximation of ADM solution for $\alpha = 1.5$ in Example 2 (c) ADM solution of Example 2 for $\alpha = 1.75$ (d) Multivariate Padé approximation of ADM solution for $\alpha = 1.75$ in Example 2.

For $\alpha = 1.5$ (32) is

$$\begin{aligned} u(x,t) = {} & x^2(1-2t) + 6x^2 \\ & \times \left(0.7522527782t^{1.5} - 1.203604445t^{2.5} \right. \\ & \left. +0.6877739683t^{3.5}\right) + 12.00000000x^2t^{3.0}. \end{aligned} \tag{36}$$

For simplicity, let $t^{1/2} = a$; then

$$\begin{aligned} u(x,a) = {} & x^2\left(1-2a^2\right) + 6x^2 \\ & \times \left(0.7522527782a^3 \right. \\ & \left. -1.203604445a^5 + 0.6877739683t^7\right) \\ & + 12.00000000x^2a^6. \\ = {} & x^2 - x^2 2a^2 + 4.513516669x^2a^3 \\ & - 7.221626670x^2a^5 + 4.126643810x^2a^7 \\ & + 12.00000000x^2a^6. \end{aligned} \tag{37}$$

Using (10) to calculate multivariate Padé equations of (37) for $m = 7$ and $n = 2$, we use (10). By using (10), we obtain

TABLE 1: Numerical values when $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1.0$ for (13).

x	t	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		
		u_{ADM}	u_{MPA}	u_{ADM}	u_{MPA}	u_{ADM}	u_{MPA}	u_{Exact}
0.01	0.01	0.001118860666	0.001118859998	$0.7125363089 \times 10^{-21}$	$0.7125363089 \times 10^{-21}$	0.00009999999987	0.0001000000000	0.0001
0.02	0.02	0.003138022015	0.003138007574	$0.2579676158 \times 10^{-18}$	$0.2579676148 \times 10^{-18}$	0.0003999999915	0.0004000000000	0.0004
0.03	0.03	0.005716640278	0.005716554826	$0.8097412617 \times 10^{-17}$	$0.8097412644 \times 10^{-17}$	0.0008999999028	0.0009000000000	0.0009
0.04	0.04	0.008727882362	0.008727584792	$0.9339702880 \times 10^{-16}$	$0.9339702881 \times 10^{-16}$	0.001599999454	0.001600000000	0.0016
0.05	0.05	0.01209607907	0.01209530419	$0.6224118481 \times 10^{-15}$	$0.6224118464 \times 10^{-15}$	0.002499997917	0.002500000000	0.0025
0.06	0.06	0.01576873408	0.01576705530	$0.2931772549 \times 10^{-14}$	$0.2931772552 \times 10^{-14}$	0.003599993779	0.003600000000	0.0036
0.07	0.07	0.01970633078	0.01970312778	$0.1086905954 \times 10^{-13}$	$0.1086905954 \times 10^{-13}$	0.004899984313	0.004900000000	0.0049
0.08	0.08	0.02387754051	0.02387197301	$0.3381735603 \times 10^{-13}$	$0.3381735608 \times 10^{-13}$	0.006399965047	0.006400000000	0.0064
0.09	0.09	0.02825661342	0.02824760190	$0.9203528751 \times 10^{-13}$	$0.9203528751 \times 10^{-13}$	0.008099929141	0.008100000000	0.0081
0.1	0.1	0.03282181204	0.03280802591	$0.2253790147 \times 10^{-12}$	$0.2253790145 \times 10^{-12}$	0.009999866667	0.010000000000	0.01

Table 2: Numerical values when $\alpha = 1.5$, $\alpha = 1.75$, and $\alpha = 2.0$ for (28).

x	t	$\alpha = 1.5$ u_{ADM}	$\alpha = 1.5$ u_{MPA}	$\alpha = 1.75$ u_{ADM}	$\alpha = 1.75$ u_{MPA}	$\alpha = 2.0$ u_{ADM}	$\alpha = 2.0$ u_{MPA}	$\alpha = 2.0$ u_{Exact}
0.01	0.01	0.00009844537131	0.00009844533110	0.00009811632420	0.00009811625316	0.00009802960500	0.00009802960486	0.00009802960494
0.02	0.02	0.0003889833219	0.0003889809228	0.0003855443171	0.0003855410232	0.0003844675200	0.0003844675123	0.0003844675125
0.03	0.03	0.0008664034309	0.0008663775704	0.0008529749012	0.0008529437792	0.0008483364450	0.0008483363176	0.0008483363182
0.04	0.04	0.001527388854	0.001527250241	0.001492265504	0.001492112228	0.001479290880	0.001479289940	0.001479289941
0.05	0.05	0.002370102454	0.002369595331	0.002296248891	0.002295720729	0.002267578126	0.002267573695	0.002267573696
0.06	0.06	0.003393997434	0.003392539471	0.003258613123	0.003257161294	0.003204002880	0.003203987182	0.003203987184
0.07	0.07	0.004599726730	0.004596176215	0.004373819216	0.004370404791	0.004279895445	0.004279849765	0.004279849769
0.08	0.08	0.005989109024	0.005981449991	0.005637043845	0.005629880400	0.005487083520	0.005486968446	0.005486968450
0.09	0.09	0.007565131844	0.007550068937	0.007044140677	0.007030367394	0.006817867605	0.006817607941	0.006817607945
0.1	0.1	0.009331981000	0.009304436794	0.008591616573	0.008566895894	0.008265000000	0.008264462804	0.008264462810

$$
\begin{aligned}
p(x,a) &= \begin{vmatrix} x^2 - x^2 2a^2 + 4.513516669x^2a^3 - 7.221626670x^2a^5 & x^2 - x^2 2a^2 + 4.513516669x^2a^3 & x^2 - x^2 2a^2 + 4.513516669x^2a^3 \\ 12.00000000x^2t^6 & -7.221626670x^2a^5 & 0 \\ 4.126643810x^2a^7 & 12.00000000x^2a^6 & -7.221626670x^2a \end{vmatrix} \\
&= 49.51972572\left(8.235760151a^5 + 0.879185531a^4 + 1.253427636a^3 + 1.403426542a^2 + 1.750000001a + 1.053153890\right)x^6a^{10}, \\
q(x,a) &= \begin{vmatrix} 1 & 1 & 1 \\ 12.00000000x^2a^6 & -7.221626670x^2a^5 & 0 \\ 4.126643810x^2a^7 & 12.00000000x^2a^6 & -7.221626670x^2a \end{vmatrix} \\
&= 49.51972572\left(1.053153890 + 1.750000001a + 3.509734322a^2\right)x^4a^{10},
\end{aligned} \tag{38}
$$

recalling that $t^{1/2} = a$, we get multivariate Padé approximation of order $(7,2)$ for (36), that is,

$$
\begin{aligned}
[7,2]_{(x,t)} = {} & \left(8.235760151t^{5/2} + 0.879185531t^2 \right. \\
& + 1.253427636t^{3/2} + 1.403426542t \\
& \left. +1.750000001\sqrt{t} + 1.053153890\right)x^2 \\
& \times \left(1.053153890 + 1.750000001\sqrt{t} \right. \\
& \left. +3.509734322t\right)^{-1}.
\end{aligned} \tag{39}
$$

For $\alpha = 1.75$ (32) is

$$
\begin{aligned}
u(x,t) = {} & x^2(1-2t) \\
& + 6x^2\left(0.6217515726t^{1.75} - 0.9043659240t^{2.75} \right. \\
& \left. +0.4823284927t^{3.75}\right) \\
& + 6.189965715x^2t^{3.5}.
\end{aligned} \tag{40}
$$

For simplicity, let $t^{1/4} = a$; then

$$
\begin{aligned}
u(x,a) = {} & x^2\left(1-2a^4\right) \\
& + 6x^2\left(0.6217515726a^7 - 0.9043659240a^{11} \right. \\
& \left. +0.4823284927a^{15}\right) + 6.189965715x^2a^{14} \\
= {} & x^2 - 2x^2a^4 + 3.730509436x^2a^7 \\
& - 5.426195544x^2a^{11} + 2.893970956x^2a^{15} \\
& + 6.189965715x^2a^{14}.
\end{aligned} \tag{41}
$$

Using (10) to calculate multivariate Padé equations of (41) for $m = 15$ and $n = 2$, we use (10). By using (10), we obtain

$$
\begin{aligned}
& p(x,a) \\
& = \begin{vmatrix} x^2 - 2x^2a^4 + 3.730509436x^2a^7 - 5.426195544x^2a^{11} & x^2 - 2x^2a^4 + 3.730509436x^2a^7 - 5.426195544x^2a^{11} & x^2 - 2x^2a^4 + 3.730509436x^2a^7 - 5.426195544x^2a^{11} \\ 6.189965715x^2a^{14} & 0 & 0 \\ 2.893970956x^2a^{15} & 6.189965715x^2a^{14} & 0 \end{vmatrix} \\
& = -38.31567556x^6a^{28}\left(-1 + 2a^4 - 3.730509436a^7 + 5.426195544a^{11}\right), \\
& q(x,a) = \begin{vmatrix} 1 & 1 & 1 \\ 6.189965715x^2a^{14} & 0 & 0 \\ 2.893970956x^2a^{15} & 6.189965715x^2a^{14} & 0 \end{vmatrix} = 38.31567556x^4a^{28},
\end{aligned} \tag{42}
$$

recalling that $t^{1/4} = a$, we get multivariate Padé approximation of order $(15,2)$ for (40), that is,

$$
\begin{aligned}
[15,2]_{(x,t)} = {} & -38.31567556x^6t^7 \\
& \times \left(-1 + 2t - 3.730509436t^{7/4} \right. \\
& \left. +5.426195544t^{11/4}\right) \\
& \times \left(38.31567556x^4t^7\right)^{-1}.
\end{aligned} \tag{43}
$$

Table 2, Figures 3(a), 3(b), 3(c), 4(a), 4(b), 4(c), and 4(d) show the approximate solutions for (28) obtained for different values of α using the decomposition method (ADM) and the multivariate Padé approximation (MPA). The value of $\alpha = 2$ is for the exact solution $u(x,t) = (x/t+1)^2$ [24].

6. Concluding Remarks

The fundamental goal of this paper has been to construct an approximate solution of nonlinear partial differential equations of fractional order by using multivariate Padé approximation. The goal has been achieved by using the multivariate Padé approximation and comparing with the Adomian decomposition method. The present work shows the validity and great potential of the multivariate Padé approximation for solving nonlinear partial differential equations of fractional order from the numerical results. Numerical results obtained using the multivariate Padé approximation and the Adomian decomposition method are in agreement with exact solutions.

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Research Article

Starlikeness and Convexity of Generalized Struve Functions

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We give sufficient conditions for the parameters of the normalized form of the generalized Struve functions to be convex and starlike in the open unit disk.

1. Introduction and Preliminary Results

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges of the famous Bieberbach conjecture. The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian, and Kummer hypergeometric functions and the Bessel functions. Many authors have determined sufficient conditions on the parameters of these functions for belonging to a certain class of univalent functions, such as convex, starlike, and close-to-convex functions. More information about geometric properties of special functions can be found in [1–9]. In the present investigation our goal is to determine conditions of starlikeness and convexity of the generalized Struve functions. In order to achieve our goal in this section, we recall some basic facts and preliminary results.

Let $\mathscr{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathscr{U} = \{z : |z| < 1\}$. Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ which are univalent in $\mathscr{U}$. Also let $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)$ denote the subclasses of $\mathscr{A}$ consisting of functions which are, respectively, starlike and convex of order α in $\mathscr{U}$ $(0 \leq \alpha < 1)$. Thus, we have (see, for details, [10]),

$$\begin{aligned} \mathscr{S}^*(\alpha) &= \left\{ f : f \in \mathscr{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \right. \\ &\qquad \left. (z \in \mathscr{U};\ 0 \leq \alpha < 1) \right\}, \\ \mathscr{C}(\alpha) &= \left\{ f : f \in \mathscr{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \right. \\ &\qquad \left. (z \in \mathscr{U};\ 0 \leq \alpha < 1) \right\}, \end{aligned} \tag{2}$$

where, for convenience,

$$\mathscr{S}^*(0) = \mathscr{S}^*, \qquad \mathscr{C}(0) = \mathscr{C}. \tag{3}$$

We remark that, according to the Alexander duality theorem [11], the function $f : \mathscr{U} \to \mathbb{C}$ is convex of order α, where $0 \leq \alpha < 1$ if and only if $z \to zf'(z)$ is starlike of order α. We note that every starlike (and hence convex) function of the form (1) is univalent. For more details we refer to the papers in [10, 12, 13] and the references therein.

Denote by $\mathscr{S}_1^*(\alpha)$, where $\alpha \in [0, 1)$, the subclass of $\mathscr{S}^*(\alpha)$ consisting of functions f for which

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha, \tag{4}$$

for all $z \in \mathcal{U}$. A function f is said to be in $\mathcal{C}_1(\alpha)$ if $zf' \in \mathcal{S}_1^*(\alpha)$.

Lemma 1 (see [4]). *If $f \in \mathcal{A}$ and*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\beta}\left|\frac{zf''(z)}{f'(z)}\right|^{\beta} < (1-\alpha)^{1-2\beta}\left(1-\frac{3\alpha}{2}+\alpha^2\right)^{\beta}, \quad (5)$$

for some fixed $\alpha \in [0,1/2]$ and $\beta \geq 0$, and for all $z \in \mathcal{U}$, then f is in the class $\mathcal{S}^*(\alpha)$.

Lemma 2 (see [14]). *Let $\alpha \in [0,1)$. A sufficient condition for $f(z) = z + \sum_{n\geq 2} a_n z^n$ to be in $\mathcal{S}_1^*(\alpha)$ and $\mathcal{C}_1(\alpha)$, respectively, is that*

$$\sum_{n\geq 2}(n-\alpha)|a_n| \leq 1-\alpha,$$
$$\sum_{n\geq 2} n(n-\alpha)|a_n| \leq 1-\alpha, \quad (6)$$

respectively.

Lemma 3 (see [14]). *Let $\alpha \in [0,1)$. Suppose that $f(z) = z - \sum_{n\geq 2} a_n z^n$, $a_n \geq 0$. Then a necessary and sufficient condition for f to be in $\mathcal{S}_1^*(\alpha)$ and $\mathcal{C}_1(\alpha)$, respectively, is that*

$$\sum_{n\geq 2}(n-\alpha)|a_n| \leq 1-\alpha,$$
$$\sum_{n\geq 2} n(n-\alpha)|a_n| \leq 1-\alpha, \quad (7)$$

respectively. In addition $f \in \mathcal{S}_1^*(\alpha) \Leftrightarrow f \in \mathcal{S}^*(\alpha)$, $f \in \mathcal{C}_1(\alpha) \Leftrightarrow f \in \mathcal{C}(\alpha)$, and $f \in \mathcal{S}^* \Leftrightarrow f \in \mathcal{S}$.

2. Starlikeness and Convexity of Generalized Struve Functions

Let us consider the second-order inhomogeneous differential equation [15, page 341]

$$z^2 w''(z) + zw'(z) + \left(z^2 - p^2\right) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+1/2)} \quad (8)$$

whose homogeneous part is Bessel's equation, where p is an unrestricted real (or complex) number. The function H_p, which is called the Struve function of order p, is defined as a particular solution of (8). This function has the form

$$H_p(z) = \sum_{n\geq 0}\frac{(-1)^n}{\Gamma(n+3/2)\Gamma(p+n+3/2)}\left(\frac{z}{2}\right)^{2n+p+1}, \quad \forall z \in \mathbb{C}. \quad (9)$$

The differential equation

$$z^2 w''(z) + zw'(z) - \left(z^2 + p^2\right) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+1/2)}, \quad (10)$$

which differs from (8) only in the coefficient of w. The particular solution of (10) is called the modified Struve function of order p and is defined by the formula [15, page 353]

$$L_p(z) = -ie^{-ip\pi/2}H_p(iz)$$
$$= \sum_{n\geq 0}\frac{1}{\Gamma(n+3/2)\Gamma(p+n+3/2)}\left(\frac{z}{2}\right)^{2n+p+1}, \quad \forall z \in \mathbb{C}. \quad (11)$$

Now, let us consider the second-order inhomogeneous linear differential equation [16],

$$z^2 w''(z) + bzw'(z) + \left[cz^2 - p^2 + (1-b)p\right] w(z)$$
$$= \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+b/2)}, \quad (12)$$

where $b, c, p \in \mathbb{C}$. If we choose $b = 1$ and $c = 1$, then we get (8), and if we choose $b = 1$ and $c = -1$, then we get (10). So this generalizes (8) and (10). Moreover, this permits to study the Struve and modified Struve functions together. A particular solution of the differential equation (12), which is denoted by $w_{p,b,c}(z)$, is called the generalized Struve function [16] of order p. In fact we have the following series representation for the function $w_{p,b,c}(z)$:

$$w_{p,b,c}(z)$$
$$= \sum_{n\geq 0}\frac{(-1)^n c^n}{\Gamma(n+3/2)\Gamma(p+n+(b+2)/2)}\left(\frac{z}{2}\right)^{2n+p+1}, \quad \forall z \in \mathbb{C}. \quad (13)$$

Although the series defined in (13) is convergent everywhere, the function $w_{p,b,c}(z)$ is generally not univalent in $\mathcal{U}$. Now, consider the function $u_{p,b,c}(z)$ defined by the transformation

$$u_{p,b,c}(z) = 2^p\sqrt{\pi}\Gamma\left(p+\frac{b+2}{2}\right)z^{(-p-1)/2}w_{p,b,c}(\sqrt{z}). \quad (14)$$

By using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions, by $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda) = \lambda(\lambda+1)\cdots(\lambda+n-1)$, we obtain for the function $u_{p,b,c}(z)$ the following form:

$$u_{p,b,c}(z) = \sum_{n\geq 0}\frac{(-c/4)^n}{(3/2)_n(\kappa)_n}z^n$$
$$= b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots, \quad (15)$$

where $\kappa = p + (b+2)/2 \neq 0, -1, -2, \ldots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order inhomogeneous differential equation

$$4z^2 u''(z) + 2(2p+b+3)zu'(z)$$
$$+ (cz + 2p + b)u(z) = 2p + b. \quad (16)$$

Orhan and Yağmur [16] have determined various sufficient conditions for the parameters p, b, and c such that the functions $u_{p,b,c}(z)$ or $z \to zu_{p,b,c}(z)$ to be univalent, starlike, convex, and close to convex in the open unit disk. In this section, our aim is to complete the above-mentioned results.

For convenience, we use the notations: $w_{p,b,c}(z) = w_p(z)$ and $u_{p,b,c}(z) = u_p(z)$.

Proposition 4 (see [16]). *If $b, c, p \in \mathbb{C}$, $\kappa = p + (b+2)/2 \neq 1, 0, -1, -2, \ldots$, and $z \in \mathbb{C}$, then for the generalized Struve function of order p the following recursive relations hold:*

(i) $zw_{p-1}(z) + czw_{p+1}(z) = (2\kappa - 3)w_p(z) + 2(z/2)^{p+1}/\sqrt{\pi}\Gamma(\kappa)$;

(ii) $zw_p'(z) + (p + b - 1)w_p(z) = zw_{p-1}(z)$;

(iii) $zw_p'(z) + czw_{p+1}(z) = pw_p(z) + 2(z/2)^{p+1}/\sqrt{\pi}\Gamma(\kappa)$;

(iv) $[z^{-p}w_p(z)]' = -cz^{-p}w_{p+1}(z) + 1/2^p\sqrt{\pi}\Gamma(\kappa)$;

(v) $u_p(z) + 2zu_p'(z) + (cz/2\kappa)u_{p+1}(z) = 1$.

Theorem 5. *If the function u_p, defined by* (15), *satisfies the condition*

$$\left|\frac{zu_p'(z)}{u_p(z)}\right| < 1 - \alpha, \tag{17}$$

where $\alpha \in [0, 1/2]$ and $z \in \mathcal{U}$, then $zu_p \in \mathcal{S}^*(\alpha)$.

Proof. If we define the function $g : \mathcal{U} \to \mathbb{C}$ by $g(z) = zu_p(z)$ for $z \in \mathcal{U}$. The given condition becomes

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < 1 - \alpha, \tag{18}$$

where $z \in \mathcal{U}$. By taking $\beta = 0$ in Lemma 1, we thus conclude from the previous inequality that $g \in \mathcal{S}^*(\alpha)$, which proves Theorem 5. □

Theorem 6. *If the function u_p, defined by* (15), *satisfies the condition*

$$\left|\frac{zu_p''(z)}{u_p'(z)}\right| < \frac{1 - 3\alpha/2 + \alpha^2}{1 - \alpha}, \tag{19}$$

where $\alpha \in [0, 1/2]$ and $z \in \mathcal{U}$, then it is starlike of order α with respect to 1.

Proof. Define the function $h : \mathcal{U} \to \mathbb{C}$ by $h(z) = [u_p(z) - b_0]/b_1$. Then $h \in \mathcal{A}$ and

$$\left|\frac{zh''(z)}{h'(z)}\right| = \left|\frac{zu_p''(z)}{u_p'(z)}\right| < \frac{1 - 3\alpha/2 + \alpha^2}{1 - \alpha}, \tag{20}$$

where $\alpha \in [0, 1/2]$ and $z \in \mathcal{U}$. By taking $\beta = 1$ in Lemma 1, we deduce that $h \in \mathcal{S}^*(\alpha)$; that is, h is starlike of order α with respect to the origin for $\alpha \in [0, 1/2]$. So, Theorem 6 follows from the definition of the function h, because $b_0 = 1$. □

Theorem 7. *If for $\alpha \in [0, 1/2]$ and $c \neq 0$ one has*

$$\left|\frac{zu_{p+1}'(z)}{u_{p+1}(z)}\right| < 1 - \alpha, \tag{21}$$

for all $z \in \mathcal{U}$, then $u_p + 2zu_p'$ is starlike of order α with respect to 1.

Proof. Theorem 5 implies that $zu_{p+1} \in \mathcal{S}^*(\alpha)$. On the other hand, the part (v) of Proposition 4 yields

$$u_p(z) + 2zu_p'(z) = \frac{-c}{2\kappa}zu_{p+1}(z) + 1. \tag{22}$$

Since the addition of any constant and the multiplication by a nonzero quantity do not disturb the starlikeness. This completes the proof. □

Lemma 8. *If $b, p \in \mathbb{R}$, $c \in \mathbb{C}$, and $\kappa = p + (b+2)/2$ such that $\kappa > |c|/2$, then the function $u_p : \mathcal{U} \to \mathbb{C}$ satisfies the following inequalities:*

$$\frac{6\kappa - 2|c|}{6\kappa - |c|} \le |u_p(z)| \le \frac{6\kappa}{6\kappa - |c|}, \tag{23}$$

$$\frac{|c|(2\kappa - |c|)}{3\kappa(4\kappa - |c|)} \le |u_p'(z)| \le \frac{2|c|}{3(4\kappa - |c|)}, \tag{24}$$

$$|zu_p''(z)| \le \frac{|c|^2}{4\kappa(4\kappa - |c|)}. \tag{25}$$

Proof. We first prove the assertion (23) of Lemma 8. Indeed, by using the well-known triangle inequality:

$$|z_1 + z_2| \le |z_1| + |z_2|, \tag{26}$$

and the inequalities $(3/2)_n \ge (3/2)^n$, $(\kappa)_n \ge \kappa^n$ $(n \in \mathbb{N})$, we have

$$\begin{aligned} |u_p(z)| &= \left|1 + \sum_{n\ge1}\frac{(-c/4)^n}{(3/2)_n(\kappa)_n}z^n\right| \\ &\le 1 + \sum_{n\ge1}\left(\frac{|-c/4|}{(3/2)\kappa}\right)^n \\ &= 1 + \frac{|c|}{6\kappa}\sum_{n\ge1}\left(\frac{|c|}{6\kappa}\right)^{n-1} \\ &= \frac{6\kappa}{6\kappa - |c|}, \quad \left(\kappa > \frac{|c|}{6}\right). \end{aligned} \tag{27}$$

Similarly, by using reverse triangle inequality:

$$|z_1 - z_2| \ge ||z_1| - |z_2||, \tag{28}$$

and the inequalities $(3/2)_n \geq (3/2)^n$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, then we get

$$\begin{aligned} \left|u_p(z)\right| &= \left|1+\sum_{n\geq1}\frac{(-c/4)^n}{(3/2)_n(\kappa)_n}z^n\right| \\ &\geq 1-\sum_{n\geq1}\left(\frac{|-c/4|}{(3/2)\,\kappa}\right)^n \\ &= 1-\frac{|c|}{6\kappa}\sum_{n\geq1}\left(\frac{|c|}{6\kappa}\right)^{n-1} \\ &= \frac{6\kappa-2|c|}{6\kappa-|c|}, \quad \left(\kappa>\frac{|c|}{6}\right), \end{aligned} \tag{29}$$

which is positive if $\kappa > |c|/3$.

In order to prove assertion (24) of Lemma 8, we make use of the well-known triangle inequality and the inequalities $(3/2)_n \geq (3/2)n$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, and we obtain

$$\begin{aligned} \left|u_p'(z)\right| &= \left|\sum_{n\geq1}\frac{n(-c/4)^n}{(3/2)_n(\kappa)_n}z^{n-1}\right| \\ &\leq \frac{2}{3}\sum_{n\geq1}\left(\frac{|c|}{4\kappa}\right)^n \\ &= \frac{2}{3}\frac{|c|}{4\kappa}\sum_{n\geq1}\left(\frac{|c|}{4\kappa}\right)^{n-1} \\ &= \frac{2|c|}{3(4\kappa-|c|)}, \quad \left(\kappa>\frac{|c|}{4}\right). \end{aligned} \tag{30}$$

Similarly, by using the reverse triangle inequality and the inequalities $(3/2)_n \geq (3/2)n$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, we have

$$\begin{aligned} \left|u_p'(z)\right| &= \left|\sum_{n\geq1}\frac{n(-c/4)^n}{(3/2)_n(\kappa)_n}z^{n-1}\right| \\ &\geq \frac{|c|}{6\kappa}-\frac{2}{3}\left(\frac{|c|}{4\kappa}\right)^2\sum_{n\geq2}\left(\frac{|c|}{4\kappa}\right)^{n-1} \\ &= \frac{|c|(2\kappa-|c|)}{3\kappa(4\kappa-|c|)}, \quad \left(\kappa>\frac{|c|}{4}\right), \end{aligned} \tag{31}$$

which is positive if $\kappa > |c|/2$.

We now prove assertion (25) of Lemma 8 by using again the triangle inequality and the inequalities $(3/2)_n \geq n(n-1)$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, and we arrive at the following:

$$\begin{aligned} \left|zu_p''(z)\right| &= \left|\sum_{n\geq2}\frac{n(n-1)(-c/4)^n}{(3/2)_n(\kappa)_n}z^{n-1}\right| \\ &\leq \frac{|c|}{4\kappa}\sum_{n\geq2}\left(\frac{|c|}{4\kappa}\right)^{n-1} \\ &= \frac{|c|^2}{4\kappa(4\kappa-|c|)}, \quad \left(\kappa>\frac{|c|}{4}\right). \end{aligned} \tag{32}$$

Thus, the proof of Lemma 8 is completed. □

Theorem 9. *If $b, p \in \mathbb{R}$, $c \in \mathbb{C}$ and $\kappa = p + (b+2)/2$, then the following assertions are true.*

(i) *If $\kappa > (7/8)|c|$, then $u_p(z)$ is convex in $\mathcal{U}$.*

(ii) *If $\kappa > ((11+\sqrt{41})/24)|c|$, then $zu_p(z)$ is starlike of order $1/2$ in $\mathcal{U}$, and consequently the function $z \to z^{-p}w_p(z)$ is starlike in $\mathcal{U}$.*

(iii) *If $\kappa > ((11+\sqrt{41})/24)|c| - 1$, then the function $z \to u_p(z) + 2zu_p'(z)$ is starlike of order $1/2$ with respect to 1 for all $z \in \mathcal{U}$.*

Proof. (i) By combining the inequalities (24) with (25), we immediately see that

$$\left|\frac{zu_p''(z)}{u_p'(z)}\right| \leq \frac{3|c|}{4(2\kappa-|c|)}. \tag{33}$$

So, for $\kappa > \left(\frac{7}{8}\right)|c|$, we have

$$\left|\frac{zu_p''(z)}{u_p'(z)}\right| < 1. \tag{34}$$

This shows $u_p(z)$ is convex in $\mathcal{U}$.

(ii) If we let $g(z) = zu_p(z)$ and $h(z) = zu_p(z^2)$, then

$$\begin{gathered} h(z) = \frac{g(z^2)}{z} = 2^p\sqrt{\pi}\Gamma(\kappa)z^{-p}w_{p,b,c}(z), \\ \frac{zh'(z)}{h(z)} - 1 = 2\left[\frac{z^2g'(z^2)}{g(z^2)}-1\right] = 2\frac{z^2u_p'(z^2)}{u_p(z^2)}, \end{gathered} \tag{35}$$

so that

$$\left|\frac{zh'(z)}{h(z)}-1\right| < 1, \quad \forall z \in \mathcal{U}, \tag{36}$$

if and only if

$$\left|\frac{z^2u_p'(z^2)}{u_p(z^2)}\right| < \frac{1}{2}, \quad \forall z \in \mathcal{U}. \tag{37}$$

It follows that $zu_p(z)$ is starlike of order $1/2$ if (37) holds. From (24) and (23), we have

$$\left|z^2u_p'(z^2)\right| \leq \frac{2|c|}{3(4\kappa-|c|)}, \quad \left(\kappa>\frac{|c|}{4}\right), \tag{38}$$

$$\frac{6\kappa-2|c|}{6\kappa-|c|} \leq \left|u_p(z^2)\right|, \quad \left(\kappa>\frac{|c|}{3}\right), \tag{39}$$

respectively.

By combining the inequalities (38) with (39), we see that

$$\left|\frac{z^2u_p'(z^2)}{u_p(z^2)}\right| \leq \frac{|c|(6\kappa-|c|)}{3(3\kappa-|c|)(4\kappa-|c|)}, \tag{40}$$

where $\kappa > |c|/3$, and the above bound is less than or equal to $1/2$ if and only if $\kappa > ((11+\sqrt{41})/24)|c|$. It follows that zu_p is starlike of order $1/2$ in $\mathscr{U}$ and $z^{-p}w_{p,b,c}$ is starlike in $\mathscr{U}$.

(iii) The part (ii) of Theorem 9 implies that for $\kappa > ((11+\sqrt{41})/24)|c|-1$, the function $z \to zu_{p+1}(z)$ is starlike of order $1/2$ in $\mathscr{U}$. On the other hand, the part (v) of Proposition 4 yields

$$u_p(z) + 2zu_p'(z) = \frac{-c}{2\kappa} zu_{p+1}(z) + 1. \quad (41)$$

So the function $z \to u_p(z) + 2zu_p'(z)$ is starlike of order $1/2$ with respect to 1 for all $z \in \mathscr{U}$.

This completes the proof. □

Struve Functions. Choosing $b = c = 1$, we obtain the differential equation (8) and the Struve function of order p, defined by (9), satisfies this equation. In particular, the results of Theorem 9 are as follows.

Corollary 10. *Let $\mathscr{H}_p : \mathscr{U} \to \mathbb{C}$ be defined by $\mathscr{H}_p(z) = 2^p\sqrt{\pi}\Gamma(p+3/2)z^{-p-1}H_p(z) = u_{p,1,1}(z^2)$, where H_p stands for the Struve function of order p. Then the following assertions are true.*

(i) *If $p > -5/8$, then $\mathscr{H}_p(z^{1/2})$ is convex in $\mathscr{U}$.*

(ii) *If $p > (-25+\sqrt{41})/24$, then $z\mathscr{H}_p(z^{1/2})$ is starlike of order $1/2$ in $\mathscr{U}$, and consequently the function $z \to z^{-p}H_p(z)$ is starlike in $\mathscr{U}$.*

(iii) *If $p > (-49+\sqrt{41})/24$, then the function $z \to \mathscr{H}_p(z^{1/2}) + 2z\mathscr{H}_p'(z^{1/2})$ is starlike of order $1/2$ with respect to 1 for all $z \in \mathscr{U}$.*

Modified Struve Functions. Choosing $b = 1$ and $c = -1$, we obtain the differential equation (10) and the modified Struve function of order p, defined by (11). For the function $\mathscr{L}_p : \mathscr{U} \to \mathbb{C}$ defined by $\mathscr{L}_p(z) = 2^p\sqrt{\pi}\Gamma(p+3/2)z^{-p-1}L_p(z) = u_{p,1,-1}(z^2)$, where L_p stands for the modified Struve function of order p. The properties are same like for function $\mathscr{H}_p$, because we have $|c| = 1$. More precisely, we have the following results.

Corollary 11. *The following assertions are true.*

(i) *If $p > -5/8$, then $\mathscr{L}_p(z^{1/2})$ is convex in $\mathscr{U}$.*

(ii) *If $p > (-25+\sqrt{41})/24$, then $z\mathscr{L}_p(z^{1/2})$ is starlike of order $1/2$ in $\mathscr{U}$, and consequently the function $z \to z^{-p}L_p(z)$ is starlike in $\mathscr{U}$.*

(iii) *If $p > (-49+\sqrt{41})/24$, then the function $z \to \mathscr{L}_p(z^{1/2}) + 2z\mathscr{L}_p'(z^{1/2})$ is starlike of order $1/2$ with respect to 1 for all $z \in \mathscr{U}$.*

Example 12. If we take $p = -1/2$, then from part (ii) of Corollary 10, the function $z \to z^{1/2}H_{-1/2}(z) = \sqrt{2/\pi}\sin z$ is starlike in $\mathscr{U}$. So the function $f(z) = \sin z$ is also starlike in $\mathscr{U}$. We have the image domain of $f(z) = \sin z$ illustrated by Figure 1.

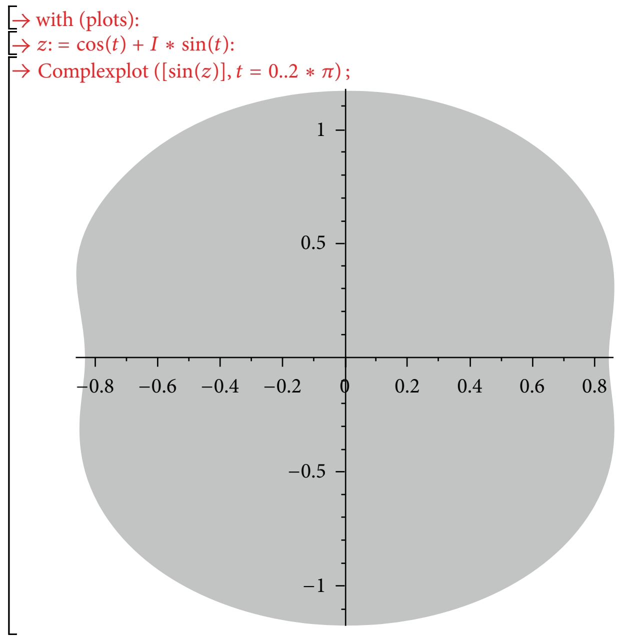

Figure 1: $f(z) = \sin z$.

Theorem 13. *If $\alpha \in [0,1)$, $c < 0$, and $\kappa > 0$, then a sufficient condition for zu_p to be in $\mathscr{S}_1^*(\alpha)$ is*

$$u_p(1) + \frac{u_p'(1)}{1-\alpha} \le 2. \quad (42)$$

Moreover, (42) is necessary and sufficient for $\psi(z) = z[2-u_p(z)]$ to be in $\mathscr{S}_1^*(\alpha)$.

Proof. Since $zu_p(z) = z + \sum_{n\ge2} b_{n-1}z^n$, according to Lemma 2, we need only show that

$$\sum_{n\ge2}(n-\alpha)b_{n-1} \le 1-\alpha. \quad (43)$$

We notice that

$$\begin{aligned}\sum_{n\ge2}(n-\alpha)b_{n-1} &= \sum_{n\ge2}(n-1)b_{n-1} + \sum_{n\ge2}(1-\alpha)b_{n-1}\\ &= \sum_{n\ge2}\frac{(n-1)(-c/4)^{n-1}}{(3/2)_{n-1}(\kappa)_{n-1}} + (1-\alpha)\left[u_p(1)-1\right]\\ &= u_p'(1) + (1-\alpha)\left[u_p(1)-1\right].\end{aligned} \quad (44)$$

This sum is bounded above by $1-\alpha$ if and only if (42) holds. Since

$$z\left[2-u_p(z)\right] = z - \sum_{n\ge2} b_{n-1}z^n, \quad (45)$$

the necessity of (42) for ψ to be in $\mathscr{S}_1^*(\alpha)$ follows from Lemma 3. □

Corollary 14. *If $c < 0$ and $\kappa > 0$, then a sufficient condition for zu_p to be in $\mathcal{S}_1^*(1/2)$ is*

$$u_{p+1}(1) \leq -\frac{2\kappa}{c}. \tag{46}$$

Moreover, (46) *is necessary and sufficient for $\psi(z) = z[2 - u_p(z)]$ to be in $\mathcal{S}_1^*(1/2)$.*

Proof. For $\alpha = 1/2$, the condition (42) becomes $u_p(1) + 2u_p'(1) \leq 2$. From the part (v) of Proposition 4 we get

$$u_p(1) + 2zu_p'(1) = 1 - \frac{c}{2\kappa}u_{p+1}(1). \tag{47}$$

So, $u_p(1) + 2u_p'(1) \leq 2$ if and only if $1 - (c/2\kappa)u_{p+1}(1) \leq 2$. Thus, we obtain the condition (46).

Furthermore, from the proof of Theorem 13, we have necessary and sufficient condition for $\psi(z) = z[2 - u_p(z)]$ to be in $\mathcal{S}_1^*(1/2)$. □

Theorem 15. *If $\alpha \in [0, 1)$, $c < 0$ and $\kappa > 0$, then a sufficient condition for zu_p to be in $\mathcal{C}_1(\alpha)$ is*

$$u_p''(1) + (3-\alpha)u_p'(1) + (1-\alpha)u_p(1) - 2\alpha \leq 2. \tag{48}$$

Moreover, (48) *is necessary and sufficient for $\psi(z) = z[2 - u_p(z)]$ to be in $\mathcal{C}_1(\alpha)$.*

Proof. In view of Lemma 2, we need only to show that

$$\sum_{n\geq 2} n(n-\alpha) b_{n-1} \leq 1 - \alpha. \tag{49}$$

If we let $g(z) = zu_p(z)$, we notice that

$$\begin{aligned}\sum_{n\geq 2} n(n-\alpha) b_{n-1} &= \sum_{n\geq 2} n(n-1) b_{n-1} + (1-\alpha)\sum_{n\geq 2} nb_{n-1} \\ &= g''(1) + (1-\alpha)\left[g'(1) - 1\right] \\ &= u_p''(1) + (3-\alpha)u_p'(1) + (1-\alpha)u_p(1) - 1 + \alpha.\end{aligned} \tag{50}$$

This sum is bounded above by $1 - \alpha$ if and only if (48) holds. Lemma 3 implies that (48) is also necessary for ψ to be in $\mathcal{C}_1(\alpha)$. □

Theorem 16. *If $c < 0$, $\kappa > 0$, and $u_p(1) \leq 2$, then $\int_0^z u_p(t)dt \in \mathcal{S}^*$.*

Proof. Since

$$\int_0^z u_p(t)\,dt = \sum_{n\geq 0}\frac{b_n}{n+1}z^{n+1} = z + \sum_{n\geq 2}\frac{b_{n-1}}{n}z^n, \tag{51}$$

we note that

$$\sum_{n\geq 2} n\frac{b_{n-1}}{n} = \sum_{n\geq 2} b_{n-1} = u_p(1) - 1 \leq 1, \tag{52}$$

if and only if $u_p(1) \leq 2$. □

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Research Article

A Note on Double Laplace Transform and Telegraphic Equations

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Double Laplace transform is applied to solve general linear telegraph and partial integrodifferential equations. The scheme is tested through some examples, and the results demonstrate reliability and efficiency of the proposed method.

1. Introduction

The wave equation is known as one of three fundamental equations in mathematical physics and occurs in many branches of physics, applied mathematics, and engineering. It is also known that there are two types of these equation: the homogenous equation that has constant coefficient with many classical solutions such as separation of variables [1], the methods of characteristics [2, 3], single Laplace transform, and Fourier transform [4] and nonhomogenous equations with constant coefficient solved by double Laplace transform [5] and operation calculus [6].

In this study, we use double Laplace transform to solve telegraph equation and partial integrodifferential equation. We follow the method that was proposed by Kılıçman and Eltayeb [7] where they extended one-dimensional convolution theorem to two-dimensional case [8].

First of all, we recall the following definitions given by Kılıçman and Gadain [9]. The double Laplace transform is defined by

$$L_xL_t\left[f(x,s)\right] = F(p,s) = \int_0^\infty e^{-px}\int_0^\infty e^{-st}f(x,t)\,dt\,dx,$$

$$\text{where } x,t>0 \text{ and } p,\ s \text{ are complex numbers,} \tag{1}$$

and the first-order partial derivative is defined as follows:

$$L_xL_t\left[\frac{\partial f(x,t)}{\partial x}\right] = pF(p,s) - F(0,s). \tag{2}$$

Double Laplace transform for second partial derivative with respect to x is given by

$$L_{xx}\left[\frac{\partial^2 f(x,t)}{\partial^2 x}\right] = p^2F(p,s) - pF(0,s) - \frac{\partial F(0,s)}{\partial x}, \tag{3}$$

and double Laplace transform for second partial derivative with respect to t similarly as the previous is given by

$$L_{tt}\left[\frac{\partial^2 f(x,t)}{\partial^2 t}\right] = s^2F(p,s) - sF(p,0) - \frac{\partial F(p,0)}{\partial t}. \tag{4}$$

In a similar manner, the double Laplace transform of a mixed partial derivative can be deduced from a single Laplace transform as

$$\begin{aligned} L_xL_t&\left[\frac{\partial^2 f(x,t)}{\partial x\partial t}\right] \\ &= psF(p,s) - pF(p,0) - sF(0,s) - F(0,0). \end{aligned} \tag{5}$$

Theorem 1. *If at the point (p,q) the integral*

$$F_1(p,q) = \int_0^\infty\int_0^\infty e^{-px-qy}f_1(x,y)\,dx\,dy \tag{6}$$

is convergent and the integral

$$F_2(p,q) = \int_0^\infty\int_0^\infty e^{-px-qy}f_2(x,y)\,dx\,dy \tag{7}$$

is absolutely convergent, then

$$F(p,q) = F_1(p,q)\,F_2(p,q) \tag{8}$$

is the Laplace transform of the function

$$f(x,y) = \int_0^x \int_0^y f_1(x-\zeta, y-\eta)\, f_2(\zeta,\eta)\, d\zeta\, d\eta \tag{9}$$

and the integral

$$F(p,q) = \int_0^\infty \int_0^\infty e^{-px-qy} f(x,y)\, dx\, dy \tag{10}$$

is convergent at the point (p,q).

Proof. See [4]. □

Next, we study the uniqueness and existences of double Laplace transform. First of all, let $f(x,t)$ be a continuous function on the interval $[0,\infty)$ which is of exponential order, that is, for some $a, b \in \mathbb{R}$. Consider

$$\sup_{\substack{t>0\\x>0}} \frac{|f(x,t)|}{e^{ax+bt}} < \infty. \tag{11}$$

In this case, the double Laplace transform of

$$F(p,s) = \int_0^\infty \int_0^\infty e^{-st-px} f(x,t)\, dx\, dt \tag{12}$$

exists for all $p > a$ and $s > b$ and is in fact infinitely differentiable with respect to $p > a$ and $s > b$. All functions in this study are assumed to be of exponential order. The following theorem shows that $f(x,t)$ can be uniquely recovered from $F(p,s)$.

Theorem 2. *Let $f(x,t)$ and $g(x,t)$ be continuous functions defined for $x,t \ge 0$ and having Laplace transforms, $F(p,s)$, and $G(p,s)$, respectively. If $F(p,s) = G(p,s)$, then $f(x,t) = g(x,t)$.*

Proof. If α and β are sufficiently large, then the integral representation, by

$$f(x,t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st} F(p,s)\, ds\right) dp \tag{13}$$

for the double inverse Laplace transform, can be used to obtain

$$\begin{aligned} f(x,t) &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st} F(p,s)\, ds\right) dp \\ &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st} G(p,s)\, ds\right) dp \\ &= g(x,t), \end{aligned} \tag{14}$$

and the theorem is proven. □

Theorem 3. *A function $f(x,t)$ which is continuous on $[0,\infty)$ and satisfies the growth condition (11) can be recovered from $F(p,s)$ as*

$$f(x,t) = \lim_{\substack{m\to\infty\\n\to\infty}} \frac{(-1)^{m+n}}{m!n!}\left(\frac{m}{x}\right)^{m+1}\left(\frac{n}{t}\right)^{n+1}\Psi^{m+n}\left(\frac{m}{x},\frac{n}{t}\right), \tag{15}$$

where Ψ^{m+n} is denoted by $(m+n)$th mixed partial derivatives of $F(p,s)$, defined by $\Psi^{m+n} = \partial^{m+n}F(p,s)/\partial p^m \partial s^n$ for $x,t \ge 0$ since the previous theorem obtains $f(x,t)$ in terms of $F(p,s)$.

Of course, the main difficulty in using Theorem 3 for computing the inverse Laplace transform is the repeated symbolic differentiation of $F(p,s)$. However, one can apply Theorem 3 in the next type of examples.

Example 4. Let $f(x,t) = e^{-ax-bt}$. The Laplace transform is easily found to be as follows:

$$F(p,s) = \frac{1}{(p+a)(s+b)}. \tag{16}$$

It is also simple to verify that

$$\frac{\partial^{m+n}F(p,s)}{\partial p^m \partial s^n} = m!n!(-1)^{m+n}(p+a)^{-m-1}(s+b)^{-n-1}. \tag{17}$$

Putting this expression for $\partial^{m+n}F(p,s)/\partial p^m\partial s^n$ into Theorem 3 gives the following:

$$\begin{aligned} f(x,t) &= \lim_{\substack{m\to\infty\\n\to\infty}} \frac{m^{m+1}n^{n+1}}{x^{m+1}t^{n+1}}\left(a+\frac{m}{x}\right)^{-m-1}\left(b+\frac{n}{t}\right)^{-n-1} \\ &= \lim_{\substack{m\to\infty\\n\to\infty}} \left(1+\frac{ax}{m}\right)^{-m-1}\left(1+\frac{bt}{n}\right)^{-n-1}. \end{aligned} \tag{18}$$

The last limit is easy to evaluate. Take the natural log of both sides, and write the result in the form of $-(\ln(1+ax/m)/(1/(m+1))) - (\ln(1+bt/n)/(1/(n+1)))$. L'Hopital's rule reveals that the indeterminate from approaches $-ax-bt$. The continuity of the natural logarithm shows that $\ln(f(x,t)) = -ax-bt$; then, $f(x,t) = e^{-ax-bt}$.

2. Properties of Double Laplace Transform

In this part, we consider some of the properties of the double Laplace Transform that will enable us to find further transform pairs $\{f(x,t), F(p,s)\}$ without having to compute consider the following.

$$\begin{aligned} &\text{(I)}\quad F(p+d, s+c) = L_x L_t\left[e^{-dx-ct} f(x,t)\right](p,s), \\ &\text{(II)}\quad \frac{1}{k}F\left(\frac{p}{\alpha},\frac{s}{\beta}\right) = L_x L_t\left[f(\alpha x, \beta t)\right](p,s), \\ &\qquad\qquad \text{where } k = \alpha\beta, \\ &\text{(III)}\quad \frac{\partial^{m+n}F(p,s)}{\partial p^m \partial^n} = L_x L_t\left[(-1)^{m+n}x^m t^n f(x,t)\right](p,s). \end{aligned} \tag{19}$$

Was first verify (I) as

$$\begin{aligned} L_xL_t\left[e^{-dx-ct}f(x,t)\right](p,s) &= \int_0^\infty\int_0^\infty e^{-st-px}e^{-dx-ct}f(x,t)\,dt\,dx \\ &= \int_0^\infty e^{-dx-px}\left(\int_0^\infty e^{-st-ct}f(x,t)\,dt\right)dx. \end{aligned} \tag{20}$$

We calculate the integral inside bracket as

$$\int_0^\infty e^{-st-ct}f(x,t)\,dt = F(x,s+c). \tag{21}$$

By substituting, we obtain

$$\begin{aligned} L_xL_t\left[e^{-dx-ct}f(x,t)\right](p,s) &= \int_0^\infty e^{-dx-px}F(x,s+c)\,dx \\ &= F(p+d,s+c). \end{aligned} \tag{22}$$

Second, the right hand side of (II) can be written in the form of

$$\begin{aligned} L_xL_t\left[f(\alpha x,\beta t)\right](p,s) &= \int_0^\infty\int_0^\infty e^{-st-px}f(\alpha x,\beta t)\,dx\,dt \\ &= \int_0^\infty e^{-st}\left(\int_0^\infty e^{-px}f(\alpha x,\beta t)\,dx\right)dt \\ &= \int_0^\infty e^{-st}\frac{1}{\alpha}F\left(\frac{p}{\alpha},\beta t\right)dt = \frac{1}{\alpha\beta}F\left(\frac{p}{\alpha},\frac{s}{\beta}\right). \end{aligned} \tag{23}$$

The last property, from definition of double Laplace transform

$$F(p,s) = \int_0^\infty\int_0^\infty e^{-st-px}f(x,t)\,dx\,dt, \tag{24}$$

so that

$$\frac{\partial^{m+n}F(p,s)}{\partial p^m\partial s^n} = \frac{\partial^{m+n}}{\partial p^m\partial s^n}\int_0^\infty\int_0^\infty e^{-st-px}f(x,t)\,dx\,dt. \tag{25}$$

Owing to the convergence properties of the improper integral involved, we can interchange the operation of differentiation and integration and differentiate with respect to p, s under the integral sign. Thus,

$$\frac{\partial^{m+n}F(p,s)}{\partial p^m\partial s^n} = \int_0^\infty \frac{\partial^n}{\partial s^n}e^{-st}\left(\int_0^\infty \frac{\partial^m}{\partial p^m}e^{-px}f(x,t)\,dx\right)dt, \tag{26}$$

which, on carrying out the repeated differentiation with respect to p, s, gives the following:

$$\begin{aligned} \frac{\partial^{m+n}F(p,s)}{\partial p^m\partial s^n} &= (-1)^{m+n}\int_0^\infty\int_0^\infty x^mt^ne^{-st-px}f(x,t)\,dx\,dt \\ &= (-1)^{m+n}L_xL_t\left[x^mt^nf(x,t)\right](p,s). \end{aligned} \tag{27}$$

The previous three properties are very useful at the proof of Theorem 3.

Proof of Theorem 3. Let us define the set of functions depending on parameters m, n as

$$\begin{gathered} g_{m,n}(x,t) = \frac{m^{m+1}n^{n+1}}{m!n!}x^mt^ne^{-mx-nt} \\ \text{so} \int_0^\infty\int_0^\infty g_{m,n}(x,t)\,dx\,dt = 1, \\ \lim_{m\to\infty}\lim_{n\to\infty}\int_0^\infty\int_0^\infty g_{m,n}(x,t)\varphi(x,t)\,dx\,dt = \varphi(1,1), \end{gathered} \tag{28}$$

where $\varphi(x,t)$ is any continuous function. Let us denote its Laplace transform as a function of p, s by $L_xL_t[\varphi(x,t)](p,s)$. Now, we define the function $\Psi(x,t) = f(xx_0, tt_0)$, and using property (II), we have

$$\begin{aligned} L_xL_t&\left[\Psi(x,t)\right](p,s) \\ &= L_xL_t\left[f(xx_0,tt_0)\right](p,s) = \frac{1}{x_0t_0}F\left(\frac{p}{x_0},\frac{s}{t_0}\right). \end{aligned} \tag{29}$$

We apply property (III) (we must evaluate the $m+n$ mixed partial derivatives of $F(p,s)$ at the points $p = m/x$ and $s = n/t$) as follows:

$$\begin{aligned} \frac{\partial^{m+n}}{\partial p^m\partial s^n}&\left(L_xL_t\left[\Psi(x,t)\right]\right)(p,s) \\ &= \frac{1}{x_0^{m+1}t_0^{n+1}}\frac{\partial^{m+n}}{\partial p^m\partial s^n}F\left(\frac{p}{x_0},\frac{s}{t_0}\right). \end{aligned} \tag{30}$$

Let $\varphi(x,t) = e^{-px-st}\Psi(x,t)$. By using (28), we have

$$\begin{aligned} \varphi(1,1) &= e^{-p-s}\Psi(1,1) = e^{-p-s}f(x_0,t_0) \\ &= \lim_{m\to\infty}\lim_{n\to\infty}\frac{m^{m+1}n^{n+1}}{m!n!} \\ &\quad\times\int_0^\infty\int_0^\infty x^mt^ne^{-px-st}e^{-mx-nt}\Psi(x,t)\,dx\,dt \\ &= \lim_{m\to\infty}\lim_{n\to\infty}\frac{m^{m+1}n^{n+1}}{m!n!} \\ &\quad\times L_xL_t\left[x^mt^ne^{-mx-nt}\Psi(x,t)\right](p,s). \end{aligned} \tag{31}$$

By using the previous properties (I) and (II) of double Laplace transform, (30), and the definition of $\Psi(x,t)$, we have

$$\begin{aligned} L_xL_t&\left[x^mt^ne^{-mx-nt}\Psi(x,t)\right](p,s) \\ &= (-1)^{m+n}\frac{\partial^{m+n}}{\partial p^m\partial s^n}\left(L_xL_t\left(e^{-mx-nt}\Psi(x,t)\right)\right)(p,s) \\ &= (-1)^{m+n}\frac{\partial^{m+n}}{\partial p^m\partial s^n}\left(L_xL_t\left(\Psi(x,t)\right)\right)(p+m,s+n) \\ &= (-1)^{m+n}\frac{1}{z}\frac{\partial^{m+n}}{\partial p^m\partial s^n}\left(L_xL_t\left(f(xx_0,tt_0)\right)\right)\left(\frac{p+m}{x_0},\frac{s+n}{t_0}\right) \\ &= (-1)^{m+n}\frac{1}{z}\frac{\partial^{m+n}}{\partial p^m\partial s^n}\left(F\left(\frac{p+m}{x_0},\frac{s+n}{t_0}\right)\right), \end{aligned} \tag{32}$$

where $1/z = 1/x_0^{m+1} t_0^{n+1}$. From (31) and (32), with $f(x_0, t_0) = e^{p+s}\varphi(1,1)$, we have

$$f(x_0, t_0) = e^{p+s} \lim_{m\to\infty} \lim_{n\to\infty} \frac{(-1)^{m+n}}{m!n!} \left(\frac{m}{x_0}\right)^{m+1} \left(\frac{n}{t_0}\right)^{n+1} \times \frac{\partial^{m+n}}{\partial p^m \partial s^n} \left(F\left(\frac{p+m}{x_0}, \frac{s+n}{t_0}\right)\right). \quad (33)$$

For any p, s the statement in Theorem 3 is actually just the special case for $p = 0$ and $s = 0$. □

Example 5. Find double Laplace transform for a regular generalized function

$$f(x,t) = H(t) \otimes H(x) \ln(t) \ln(x), \quad (34)$$

$$\frac{\partial^2}{\partial x \partial t} f(x,t) = \frac{\partial^2}{\partial x\, \partial t} [H(t) \otimes H(x) \ln(t) \ln(x)] = pf\left[\frac{H(t) \otimes H(x)}{xt}\right], \quad (35)$$

where $H(x,t) = H(t) \otimes H(x)$ is a Heaviside function, and $\otimes$ is tensor product. The double Laplace transform with respect to x, t of (1) becomes

$$L_x L_t [f(x,t)] = \int_0^\infty e^{-px} \ln(x) \int_0^\infty e^{-st} \ln(t)\, dt\, dx = -\frac{1}{s} \int_0^\infty e^{-px} \ln(x) [\gamma + \ln s]\, dx, \quad (36)$$

$$L_x L_t [f(x,t)] = \frac{1}{sp} \left[\gamma^2 + \ln(p) \ln(s)\right],$$

where γ is Euler's constant [10]. Thus,

$$L_x L_t [f(x,t)] = \frac{1}{sp} \left[\gamma^2 + \ln(p) \ln(s)\right], \quad \text{where } Re > 0. \quad (37)$$

Double Laplace transform of (35) with respect to x and t is obtained as follows:

$$L_x L_t \left[\frac{\partial^2}{\partial x\, \partial t} f(x,t)\right] = L_x L_t [H(t) H(x) \ln(t) \ln(x)] = ps\left[\frac{1}{sp}\left[\gamma^2 + \ln(p)\ln(s)\right]\right] = \gamma^2 + \ln(p) \ln(s). \quad (38)$$

Definition 6. A linear continuous function over the space L of test functions is called a distribution of exponential growth. This dual space of L is denoted by L' [10].

Example 7. Let us find double laplace transform of the function $(x^\alpha + t^\beta) = H(x) \otimes H(t) x^\alpha t^\beta$, where $\alpha, \beta \neq -1, -2, \ldots$ Since $(x^\alpha + t^\beta) \in L'$, then double laplace transform of the function $(x^\alpha + t^\beta) = H(x) \otimes H(t) x^\alpha t^\beta$ is given by

$$L_x L_t \left[\left(x^\alpha + t^\beta\right)\right] = \int_0^\infty x^\alpha e^{-px} \int_0^\infty t^\beta e^{-st} dt\, dx. \quad (39)$$

Letting $u = px$ and $v = st$ for $p, t > 0$, it follows that

$$L_x L_t \left[\left(x^\alpha + t^\beta\right)\right] = \frac{1}{p^{\alpha+1} s^{\beta+1}} \int_0^\infty u^\alpha e^{-u} \int_0^\infty v^\beta e^{-v} dv\, du, = \frac{1}{p^{\alpha+1} s^{\beta+1}} \Gamma(\alpha+1) \Gamma(\beta+1). \quad (40)$$

In particular, if $\alpha, \beta = 0$, (40) becomes

$$L_x L_t [H(x) \otimes H(t)] = \int_0^\infty e^{-px} \int_0^\infty e^{-st} dt\, dx = \frac{1}{ps}. \quad (41)$$

Consider the general telegraph equation in the following form:

$$u_{xx} = u_{tt} + u_t + u + f(x,t), \quad (42)$$

with boundary conditions

$$u(0,t) = f_1(t), \qquad u_x(0,t) = f_2(t), \quad (43)$$

and initial conditions

$$u(x,0) = g_1(x), \qquad u_t(x,0) = g_2(x). \quad (44)$$

We apply double Laplace transform for (42) and single Laplace transform for (43) and (43). after taking double inverse Laplace transform, we obtain the solution of (42) in the form

$$u(x,t) = L_p^{-1} L_s^{-1} \left[\frac{F(p,s) + pF_1(s) + F_2(s)}{(p^2 - s^2 - s - 1)} - \frac{sG_1(p) - G_2(p) - G_1(p)}{(p^2 - s^2 - s - 1)}\right]. \quad (45)$$

Here, we assume that the double inverse Laplace transform exists for each term in the right side of (45).

Example 8. Consider the homogeneous telegraph equation given by

$$u_{xx} - u_{tt} - u_t - u = 0 \quad (46)$$

with boundary conditions

$$u(0,t) = e^{-t}, \qquad u_x(0,t) = e^{-t}, \quad (47)$$

and initial conditions

$$u(x,0) = e^x, \qquad u_x(x,0) = -e^x. \quad (48)$$

Solution 1. By taking double Laplace transform for (46) and single Laplace transform for (47) and (48), we have

$$U(p,s) = \frac{(p^2 - s^2 - s - 1)}{(s+1)(p-1)(p^2 - s^2 - s - 1)} = \frac{1}{(s+1)(p-1)}. \tag{49}$$

By using double inverse Laplace transform for (49), we get the solution as follows:

$$u(x,t) = e^{x-t}. \tag{50}$$

In the next example we apply double Laplace transform for nonhomogenous telegraphic equation as follows.

Example 9. Consider the nonhomogenous telegraphic equation denoted by

$$u_{xx} - u_{tt} - u_t - u = -2e^{x+t} \tag{51}$$

with boundary conditions

$$u(0,t) = e^t, \qquad u_x(0,t) = e^t, \tag{52}$$

and initial conditions

$$u(x,0) = e^x, \qquad u_x(x,0) = e^x. \tag{53}$$

By taking double Laplace transform for (51) and single Laplace transform for (52) and (53), we have

$$U(p,s) = \frac{(p^2 - s^2 - s - 1)}{(s-1)(p-1)(p^2 - s^2 - s - 1)} = \frac{1}{(s-1)(p-1)}. \tag{54}$$

By applying double inverse Laplace transform for (54), we get the solution of (51) in the following form:

$$u(x,t) = e^{x+t}. \tag{55}$$

3. An Application to Partial Integrodifferential Equations

Consider the following partial integrodifferential equation:

$$u_{tt} - u_{xx} + u + \int_0^x \int_0^t g(x-\alpha, t-\beta)\, u(\alpha,\beta)\, d\alpha\, d\beta = f(x,t), \tag{56}$$

with boundary conditions

$$u(0,t) = f_1(t), \qquad u_x(0,t) = f_2(t), \tag{57}$$

and initial conditions

$$u(x,0) = g_1(x), \qquad u_t(x,0) = g_2(x). \tag{58}$$

By taking double Laplace transform for (56) and single Laplace transform for (57) and (58), we get

$$U(p,s) = \frac{p/G_1(p) + 1/G_2(p)}{(p^2 - s^2 + 1 + G(p,s))} - \frac{p/F_1(s) - 1/F_2(s) + F(p,s)}{(p^2 - s^2 + 1 + G(p,s))}. \tag{59}$$

By applying double inverse Laplace transform for (59), we obtain the solution of (56) in the following form:

$$u(x,t) = L_p^{-1} L_s^{-1} \left[\frac{p/G_1(p) + 1/G_2(p)}{(p^2 - s^2 + 1 + G(p,s))} - \frac{p/F_1(s) - 1/F_2(s) + F(p,s)}{(p^2 - s^2 + 1 + G(p,s))} \right]. \tag{60}$$

We provide the double inverse Laplace transform existing for each terms in the right side of (60). In particular, consider the following example.

Example 10. Consider the partial integro-differential equation

$$u_{tt} - u_{xx} + u + \int_0^x \int_0^t e^{x-\alpha+t-\beta} u(\alpha,\beta)\, d\alpha\, d\beta = e^{x+t} + xte^{x+t} \tag{61}$$

with conditions

$$u(x,0) = e^x, \qquad u_t(x,0) = e^x, \qquad u(0,t) = e^t, \qquad u_x(0,t) = e^t. \tag{62}$$

By taking double Laplace transform for (61) and single Laplace transform for (62), we have

$$\left(s^2 - p^2 + 1 + \frac{1}{(p-1)(s-1)}\right) U(p,s) = \frac{s}{p-1} + \frac{1}{p-1} - \frac{p}{s-1} - \frac{1}{s-1} + \frac{1}{(p-1)(s-1)} + \frac{1}{(p-1)^2(s-1)^2}. \tag{63}$$

By simplifying (63), we obtain

$$U(p,s) = \frac{1}{(p-1)(s-1)}. \tag{64}$$

By using double inverse Laplace transform for (64), we obtain the solution of (61) as follows:

$$u(x,t) = e^{x+t}. \tag{65}$$

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