

Advances in Mathematical Physics

Geometry of Warped Product Manifolds: Theory and Applications

Lead Guest Editor: Meraj Ali Khan

Guest Editors: Ali H. Al-Khaldi and Shyamal Kumar Hui





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























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Research Article

Imperfect Fluid Generalized Robertson Walker Spacetime Admitting Ricci-Yamabe Metric

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In the present paper, we investigate the nature of Ricci-Yamabe soliton on an imperfect fluid generalized Robertson-Walker spacetime with a torse-forming vector field ξ . Furthermore, if the potential vector field ξ of the Ricci-Yamabe soliton is of the gradient type, the Laplace-Poisson equation is derived. Also, we explore the harmonic aspects of η -Ricci-Yamabe soliton on an imperfect fluid GRW spacetime with a harmonic potential function ψ . Finally, we examine necessary and sufficient conditions for a 1-form η , which is the g -dual of the vector field ξ on imperfect fluid GRW spacetime to be a solution of the Schrödinger-Ricci equation.

1. Introduction

Symmetry is a beautiful property of the universe. It is also one of the fundamental concepts that can describe the laws of nature such as from general relativity to other physical theories. In 1915, Albert Einstein introduced the theory, namely, “General Relativity of gravity” (GR). In GR, the field of gravity with its source is the spacetime curvature and energy-momentum tensor, respectively. Einstein equations which explain spacetime curvature evolution lead to current particle physics, equations in nuclear physics [1], astrophysics [2], and plasma physics [3]. To understand the general theory of relativity, we study the model of relativistic fluids from the view of differential geometry. The general theory of relativity is based on the concept that spacetime is a curved manifold.

According to J. A. Wheeler, “Matter tells spacetime how to bend and spacetime returns the complement by telling matter how to move.”

The spacetime of GR and cosmology is modeled as a connected 4-dimension Lorentzian manifold considered as a specific subclass of pseudo-Riemannian manifolds among Lorentzian metric g , where its signatures $(-, +, +, +)$ play an important role in GR. The geometry of Lorentzian manifold is connected to the nature of vectors of manifold. As a result, Lorentzian manifold is the best model to investigate GR.

Alias et al. [4] presented the concept of generalized Robertson-Walker spacetime (GRW, in short) that generalizes the Robertson-Walker (RW) spacetime which is a direct application of *warped product manifolds*.

Definition 1 [4]. A Lorentzian manifold M of dimension $n \geq 3$ is said to be a GRW spacetime if it is the warped product $M = I \times_f M^*$ with an open interval $(I, -dt^2)$ of \mathbb{R} and a Riemannian manifold M^* with warping function $f : I \rightarrow \mathbb{R}^+$.

Definition 2 [5]. An n -dimensional Lorentzian manifold is named GRW spacetime if the metric takes the local form

$$ds^2 = -(dt)^2 + r(t)^2 g_{ij}^* dx^i dx^j, \quad (1)$$

where $g^* = g_{ij}^*(x^k)$ are functions of x^k only ($i, j, k = 2, 3, \dots, n$) and r is the function of t only. If g_{ij}^* has dimension 3 and has constant curvature, the space is a Robertson-Walker spacetime.

Definition 3 [6]. A nonflat semi-Riemannian manifold of dimension $n > 2$ is known as pseudo quasi Einstein manifolds, if its nonzero Ricci tensor satisfies

$$S(U, V) = ag(U, V) + b\eta(U)\eta(V) + cD(U, V), \quad (2)$$

where a, b , and c are nonzero scalars, η is a nonzero 1-form associated with the unit timelike vector field ξ defined by $\eta(U) = g(U, \xi)$, and D is a symmetric tensor with zero trace defined as $D(U, \xi) = 0$.

A GRW spacetime with dimension n is the n -dimensional Lorentzian manifold M . According to Sanchez [7], the GRW spacetime has applications in the homogeneous spacetime with an isotropic radiation. O'Neil [8] in his book listed that an RW spacetime is the imperfect fluid spacetime. If the dimension of GRW spacetime is $n = 4$, then it becomes a perfect fluid spacetime if and only if it is an RW spacetime [5].

In geometry, symmetry is used to describe the distribution of physical objects, particularly in relation to the geometry of spacetime. In most cases, the metric of symmetry simplifies the solution in various studies. More specifically, Ricci curvature has several applications in GR, for instance, in the solution of Einstein field equations. Solitons are one of the most important symmetry patterns, which are related to the geometrical flow of spacetime geometry. For instance, Ricci flow and Yamabe flow in which vital terms have been used to understand energy and entropy in GR. Moreover, various studies in GR show that Ricci soliton and Yamabe soliton are focused because their curvatures preserve self-similarity.

In [9], Ali and Ahsan studied the symmetries of space time manifold via Ricci solitons. However, Blaga [10] discussed geometrical aspects for the perfect fluid spacetime by using Einstein solitons and Ricci solitons. In addition, Venkatesha and Aruna [11] used Ricci solitons in the study of perfect fluid spacetime admitting the potential vector field. Many researchers have performed extensive research on solitons with spacetimes by using different methods (for more details, see [10, 12–14]).

As a result, we concentrate on the geometry of an imperfect fluid spacetime admitting the Ricci-Yamabe soliton and an eta-Ricci-Yamabe soliton to continue the work initiated in the past studies. We develop a new notion of Ricci-Yamabe soliton and its extension η -Ricci-Yamabe soliton with the help of Ricci-Yamabe maps studied by Güler and Crasmareanu [15].

2. Development of Ricci-Yamabe Solitons

In 1988, Hamilton [16] first time introduced the concept of Ricci flow and Yamabe flow simultaneously. Ricci soliton and Yamabe soliton appear in the limiting case of the Ricci flow and Yamabe flow, respectively. If dimension of Yamabe soliton is $n = 2$, then it turns to Ricci soliton, but when $n > 2$, Yamabe and Ricci solitons are not identical in general, because Yamabe soliton keeps the conformal class of the metric while Ricci soliton does not.

Over the past twenty years, many geometers and physicists have been fascinated by the theory of geometric flow such as Ricci flow and Yamabe flow. A singularity study is a part of the solution where the metric develops through dilation and diffeomorphism, soliton solution is a common term for this type of solution.

In 2019, Güler and Crasmareanu [15] introduced the study of a new geometric flow, namely, Ricci-Yamabe map, which is a scalar combination of Ricci and Yamabe flow; this is also called Ricci-Yamabe flow of the type (α, β) . The Ricci-Yamabe flow is the evolution of metrics at the Riemannian or semi-Riemannian manifold defined as [15].

$$\frac{\partial}{\partial t} g(t) = -2\alpha S(t) + \beta R(t)g(t), \quad g_0 = g(0), \quad t \in (a, b). \quad (3)$$

Due to the involvement of scalars α and β , the Ricci-Yamabe flow may be a Riemannian or semi-Riemannian or singular Riemannian flow; multiple options like this can be advantageous in some geometrical or physical models, for example, relativistic theories. As a result, the Ricci-Yamabe soliton appears as the limit of soliton for the Ricci-Yamabe flow; this is a strong motivation for us to develop the concept of Ricci-Yamabe solitons. In [17], Catino and Mazzieri presented an interpolation solitons between Ricci and Yamabe soliton, where the Ricci-Bourguignon soliton corresponds to Ricci-Bourguignon flow, but it depends on a scalar.

A soliton of Ricci-Yamabe flow which moves just by one parameter group of diffeomorphism and scaling is named Ricci-Yamabe soliton. To be precise, the Ricci-Yamabe soliton at Riemannian manifold (M, g) is the structure $(g, V, \lambda, \alpha, \beta)$ satisfying

$$\mathcal{L}_V g + 2\alpha S + (2\lambda - \beta R)g = 0, \quad (4)$$

where S is Ricci tensor, R is scalar curvature, \mathcal{L}_V is Lie-derivative along the vector field V , and α and β are constant. The (M, g) is called *Ricci-Yamabe shrinker*, *Ricci-Yamabe expander*, or *Ricci-Yamabe steady soliton*, depending on whether $\lambda > 0$, $\lambda < 0$, or $\lambda = 0$, respectively. Therefore, equation (4) is called Ricci-Yamabe soliton of (α, β) -type, which is a generalization of Ricci and Yamabe solitons. We note that Ricci-Yamabe solitons of type $(\alpha, 0)$ and $(0, \beta)$ -type are α -Ricci soliton and β -Yamabe soliton, respectively.

A notion for η -Ricci soliton defined as in [18] is an advance extension for Ricci soliton. Therefore, we can define a new notion in an analogous fashion by perturbing the equation (4) that defines the type of soliton using the multiplication of the certain $(0, 2)$ -tensor field $\eta \otimes \eta$; a slightly

more general notion is obtained, specifically η -Ricci-Yamabe soliton of type (α, β) defined as:

$$\mathcal{L}_V g + 2\alpha S + (2\lambda - \beta R)g + 2\mu\eta \otimes \eta = 0. \quad (5)$$

Again, let us remark that η -Ricci-Yamabe solitons of type $(\alpha, 0)$ or $(1, 0)$ and $(0, \beta)$ or $(0, 1)$ -type are α - η -Ricci soliton (or η -Ricci soliton) and β - η -Yamabe soliton (or η -Yamabe soliton), respectively; for more details about these particular cases, one can follow ([19–24]).

Example 4. Let us consider the case of Einstein soliton, that is generating self-similar solutions of Einstein flow [17], which is given by

$$\frac{\partial}{\partial t} g(t) = -2 \left(S - \frac{R}{2} g \right). \quad (6)$$

Therefore, an Einstein soliton appears as the solution limit of the Einstein flow, which is governed by the following formula

$$\frac{1}{2} \mathcal{L}_V g + S + \left(\lambda - \frac{R}{2} \right) g = 0. \quad (7)$$

In this case, comparing equation (7) with (4), we have $\alpha = 1$ and $\beta = 1$, i.e., it is a type of $(1, 1)$ -Ricci-Yamabe soliton.

Example 5. Let us consider the conformal Ricci flow equation which was studied in [25], which is characterized by the following tensorial equation

$$\frac{\partial g}{\partial t} = -2S - \left(p + \frac{2}{n} \right) g, \quad (8)$$

where $r(g) = -1$, $t \in (a, b)$, a time interval including $g(0)$, p is a nondynamical scalar field (time dependent scalar field), $r(g)$ is the scalar curvature of the manifold, and n is the dimension of the manifold M . The notion of conformal Ricci soliton is governed by the following equation

$$\mathcal{L}_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n} \right) \right] g = 0. \quad (9)$$

On comparing equation (9) with (4), we have $\alpha = 1$ and $\beta = -1$, i.e., it is a type $(1, -1)$ Ricci-Yamabe soliton.

2.1. Geometrical and Physical Effects of Ricci-Yamabe Solitons. Geometry of Ricci-Yamabe solitons can develop a bridge between the curvature inheritance symmetry for the imperfect fluid spacetime (semi-Riemannian manifold) and class of the Ricci-Yamabe solitons. For this, three mathematical forms are constructed for semiconformally flat Ricci-Yamabe soliton manifolds. To investigate the kinematic and dynamic properties of spacetime in order to apply relativity, the physical model is presented for the three classes, namely, expanding,

steady, and shrinking of perfect fluid solution for Ricci-Yamabe soliton spacetime.

To deal with these specific classes of Ricci-Yamabe solitons, specifically shrinking ($\lambda < 0$), where it happens on the maximal time interval $-\infty < t < b$ and $b < \infty$, steady ($\lambda = 0$) where it happens for every time or expanding ($\lambda > 0$) that occurs at maximal time interval $a < t < \infty$, where $a > -\infty$ [26]. These classes yield examples of *ancient*, *eternal*, and *immortal solution*, in the same order. Additionally, shrinking or expanding Ricci-Yamabe solitons are linked to Einstein gravity coupled to a free mass less scalar field with nonzero cosmological constant.

3. Preliminaries

The energy-momentum tensor is used as a basic tool of the spacetime, assuming the fluid which have density, pressure, dynamical, and kinematic quantities such as velocity, acceleration, vorticity, shear, and expansion [27]. If the viscosity terms are non-zero, the fluid is named an imperfect fluid [28]. Imperfect fluid spacetime can give an adequate details for cosmological models beyond the standard model like perfect fluid spacetimes. The complete idea of the nature deals with the behavior of the perfect fluid and imperfect fluid space-time in standard cosmological models. The Brans-Dicke-like field of scalar-tensor gravity is identified as the imperfect fluid that is described by an effective Einstein equation. In Einstein's theory, the effective imperfect fluid explanation is presented for the canonical; GRW spaces applied at Friedmannian cosmology [29].

Now, we should state the following definitions, which will come in handy in the next parts.

Definition 6 [30]. A vector field γ_j on the semi-Riemannian manifold is called torse-forming vector field in case

$$\nabla_k \gamma_j = \omega_k \gamma_j + \varphi g_{kj}, \quad (10)$$

where φ is the scalar function and ω_k is a nonvanishing 1-form.

Clearly, the unit timelike torse-forming vector field u_i on the semi-Riemannian manifold M is given as:

$$\nabla_k u_j = \varphi (g_{kj} + u_k u_j). \quad (11)$$

In addition, we have some significant results based on GRW spacetime.

Theorem 7 [22]. A Lorentzian manifold M with $\dim(M) \geq 3$ is a GRW spacetime if and only if it admits a timelike concircular vector field.

In 2017, Mantica and Molinari [5] have established the necessary and sufficient conditions for a Lorentzian manifold to admit the unit timelike torse-forming vector field to be GRW spacetime, that is also an eigenvector of Ricci tensor.

In a spacetime with Lorentzian metric g_{ij} , the stress-energy-momentum tensor T with heat flow for an imperfect fluid GRW spacetime can be written as [8, 29, 31]

$$T(U, V) = pg(U, V) + (\sigma + p)\eta(U)\eta(V) + P(U, V), \quad (12)$$

where σ and p define the energy density and isotropic pressure, respectively, and P defines the tensor of isotropic pressure for the viscous fluid [28].

The Einstein's gravitational equation which controls the fluid motion is given by the following relation [8]

$$S(U, V) + \left(\lambda - \frac{R}{2}\right)g(U, V) = \kappa T(U, V), \quad (13)$$

for all $U, V \in \chi(M)$, where λ is the cosmological constant, κ is the gravitational constant (that is taken $8\pi G$, G represents the universal gravitational constant), S is Ricci tensor, and R is scalar curvature of g . Basically, the universe is filled with the mysterious components, and these are called dark energy DE and dark matter DM; they are considered to be the main reason for the accelerated expansion of the universe and balance the mass-energy ratio.

Also, using equation (12) as well as (13) for an imperfect fluid GRW spacetime, we get

$$S(U, V) = ag(U, V) + b\eta(U)\eta(V) + cP(U, V), \quad (14)$$

where $a = (-\lambda + R/2 + \kappa p)$, $b = \kappa(\sigma + p)$, and $c = \kappa$. Thus, in the light of (14) and (2), we can state the following result.

Theorem 8. *An imperfect fluid GRW spacetime with stress energy tensor described by (12), obeying Einstein's field equation with cosmological constant, is a pseudo quasi-Einstein GRW spacetime.*

4. Imperfect Fluid Generalized Robertson-Walker Spacetime

Here, we discuss the basic concepts of GRW spacetime.

Suppose (M^4, g) is the relativistic imperfect fluid GRW spacetime satisfying (14), then by (14) and the assumption that $g(\xi, \xi) = -1$, we have

$$R = 4\lambda - \kappa[3p - \sigma + J], \quad (15)$$

where $J = \text{trace}(P)$. Now, we can deduce that

$$S(U, V) = \left(\lambda - \frac{\kappa}{2}[p - \sigma + 2J]\right)g(U, V) + \kappa(\sigma + p)\eta(U)\eta(V) + \kappa P(U, V), \quad (16)$$

$$QU = aU + b\eta(U)\xi, \quad (17)$$

where $a = (\lambda - \kappa/2(p - \sigma + 2J))$ and $b = \kappa(1 + \sigma + p)$. Also

$$S(\xi, \xi) = \frac{\kappa}{2}[3p + \sigma + 2(J + I)] - \lambda, \quad (18)$$

where $I = P(\xi, \xi)$.

It is noted that, in a GRW spacetime, the velocity field of a perfect fluid or an imperfect fluid described by the stress-energy tensor (12) is a torse-forming and proportional to Chen's vector which is defined in [5, 29].

Motivated by the results of (see [5, 22, 29]) together with the above facts and Definition 6, regarding to global expressions, the next theorem for an imperfect GRW spacetime is stated. [32]:

Theorem 9. *On an imperfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ , the following relations hold*

$$\eta(\nabla_U \xi) = 0, \nabla_\xi \xi = 0, \quad (19)$$

$$(\nabla_U \eta)(V) = \varphi[g(U, V) + \eta(U)\eta(V)], \quad (20)$$

$$R(U, V)\xi = \varphi[\eta(V)U - \eta(U)V], \quad (21)$$

$$R(\xi, U)V = \varphi[\eta(V)U - g(U, V)\xi], \quad (22)$$

$$\eta(R(U, V)W) = \varphi[\eta(U)g(V, W) - \eta(V)g(U, W)], \quad (23)$$

$$(\mathcal{L}_\xi g)(U, V) = 2\varphi[g(U, V) + \eta(U)\eta(V)], \quad (24)$$

$$S(U, \xi) = -3\varphi\eta(U). \quad (25)$$

Proof. To compute $(\nabla_U \eta)(V) = U(\eta(V) - \eta(\nabla_U V)) = U(g(V, \xi) - g(\nabla_U V, \xi)) = g(V, \nabla_U \xi) = \varphi[g(U, V) + \eta(U)\eta(V)]$. In particular $(\nabla_\xi \eta)(V) = 0$, equation (19) is given as (11).

Substituting the term of $\nabla_U \xi$ from (11) into $R(U, V)\xi = \nabla_U \nabla_V \xi - \nabla_V \nabla_U \xi - \nabla_{[U, V]}\xi$ and by direct calculations, we find the relation (21), (22), and (25).

Here, the Lie derivative of g with respect to ξ is followed by straight forward computation, which is (24). \square

5. Geometrical Characteristics of Imperfect Fluid GRW Spacetime

In this section, we discuss the properties of a new curvature tensor called *semiconformal curvature tensor* and its relationship with imperfect fluid GRW spacetime.

In 2017, Kim [33] presented curvature like-tensor field that remains invariant with respect to conharmonic transformation. The new tensor is named as *semiconformal curvature tensor* denoted by H . However, for the semi-Riemannian manifold M with metric g , the tensor H is given by the following formula [34]

$$H(U, V)W = -(n-2)\delta C(U, V)W + [\varepsilon + (n-2)\delta]L(U, V)W, \quad (26)$$

provided the constants ε and δ are not simultaneously zero,

where C and L are conformal curvature tensor as well as conharmonic curvature tensor in the same order.

Now, an imperfect fluid GRW spacetime with a unit timelike torse-forming vector field of dimension 4 named semiconformally flat imperfect fluid GRW spacetime; if the semiconformal curvature tensor H vanishes then, we get the following information by equation (26).

Suppose (M^4, g) is the semiconformally flat imperfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ .

As $H(U, V)W = 0$, it gives $\text{div } H = 0$, where div denotes the divergence of a vector. Now, (26) leads to

$$(\nabla_U S)(V, W) - (\nabla_V S)(U, W) = \frac{\delta}{3\epsilon} [U(R)g(V, W) - V(R)g(U, W)], \quad (27)$$

or

$$g((\nabla_U Q)V - (\nabla_V Q)U, W) = \frac{\delta}{3\epsilon} [U(R)g(V, W) - V(R)g(U, W)]. \quad (28)$$

Since, in light of (15), scalar curvature R is constant, and from (17), equation (28) leads to

$$0 = (\nabla_U Q) - (\nabla_V Q)U = b[(\nabla_U \eta)(V)\xi + \eta(V)\nabla_U \xi - (\nabla_V \eta)(U)\xi - \eta(V)\nabla_U \xi]. \quad (29)$$

Then, between (11) and (20), we find

$$b[\eta(V)U - \eta(U)V] = 0, \quad (30)$$

that gives $b=0$ and leads to $p = -(\sigma + 1)$; the energy-momentum tensor is Lorentz-invariant, and in such a case, we discuss about vacuum.

By (17), it is easy to conclude $QU = aU$. Therefore, $H = 0$, and it leads to

$$R(U, V)W = \frac{2\delta(\lambda + \kappa[1/2 + \sigma + J])}{3\epsilon} [g(V, W)U - g(U, W)V], \quad (31)$$

which means (M^4, g) is of constant curvature $2\delta(\lambda + \kappa[1/2 + \sigma + J])/3\epsilon$; by the applications of (31), we have the following result:

Theorem 10. *If imperfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ and constant scalar curvature R is semiconformally flat, then the stress-energy tensor is Lorentz-invariant and is of constant curvature $2\delta(\lambda + \kappa[1/2 + \sigma + J])/3\epsilon$.*

The pseudo-Riemannian manifold is called a quasi-constant curvature, if the curvature tensor of the type $(0, 4)$ satisfies

$$\begin{aligned} R(U, V, W, W') &= m[g(V, W)g(U, W') - g(U, W)g(V, W')] \\ &\quad + n[g(U, W')\eta(V)\eta(W) - g(U, W)\eta(V)\eta(W') \\ &\quad + g(V, W)\eta(U)\eta(W') - g(V, W')\eta(U)\eta(W)], \end{aligned} \quad (32)$$

where m and n are scalars and η is nonzero 1-form that is $g(U, Z) = \eta(U)$, for any unit vector fields U, Z . The concept of the manifold with quasi-constant curvature was presented by Yano [35].

On using equation (31) in (32), we get

Corollary 11. *A semiconformally flat imperfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ is of quasi-constant curvature with $m = 2\delta(\lambda + \kappa[1/2 + \sigma + J])/3\epsilon$ and $n = 0$.*

It is well known that the manifold of constant curvature is Einstein manifold; now by the application of Theorem 10, we state the following theorem:

Theorem 12. *The semiconformally flat imperfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ is an Einstein.*

A pseudo-Riemannian manifold (M, g) is called *semi-symmetric* and *Ricci semisymmetric* if (M, g) hold the conditions $R(U, V) \cdot R = 0$ and $R(U, V) \cdot S = 0$, in the same order. The restriction $R(U, V) \cdot R = 0$ gives $R(U, V) \cdot S = 0$, but the other does not true in general.

Now, we prove the next theorem;

Theorem 13. *A semiconformally flat imperfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ is semisymmetric and Ricci semisymmetric.*

Proof. From equation (31), it is clear that $R(U, V) \cdot R = 0$, and this gives $R(U, V) \cdot S = 0$. \square

6. Ricci-Yamabe Soliton Structure in an Imperfect Fluid GRW Spacetime

In this section, we deal with Ricci-Yamabe soliton of type (α, β) in an imperfect fluid GRW spacetime whose unit timelike velocity vector field ξ is torse-forming.

Now, taking $V = \xi$, equation (4) becomes

$$\mathcal{L}_\xi g(U, V) + 2\alpha S(U, V) + (2\Omega - \beta R)g(U, V) = 0, \quad (33)$$

where R is scalar curvature. Now using (24), we find

$$\alpha S(U, V) + \left(\Omega - \frac{\beta R}{2} + \varphi \right) g(U, V) + \varphi \eta(U)\eta(V) = 0. \quad (34)$$

Putting $U = V = \xi$ in (34) and using (18), we get

$$\Omega = \left\{ \frac{\beta R}{2} \right\} + \alpha \left[\frac{\kappa}{2} \{ \sigma + 3p + 2(J + I) \} - \lambda \right]. \quad (35)$$

Hence, we give the following:

Theorem 14. *Let an imperfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ admitting a Ricci-Yamabe soliton (g, ξ, Ω) of type (α, β) , then the Ricci-Yamabe soliton is expanding.*

Corollary 15. *If an imperfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ admitting the Ricci-Yamabe soliton (g, ξ, Ω) of type $(0, \beta)$, then the β -Yamabe solitons is expanding.*

Corollary 16. *If an imperfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ admitting a Ricci-Yamabe soliton (g, ξ, Ω) of type $(\alpha, 0)$, then the Ricci-Yamabe soliton expands, steady, and shrinks according as*

- (i) $(\kappa/2)\{\sigma + 3p + 2(J + I)\} > \lambda$
- (ii) $(\kappa/2)\{\sigma + 3p + 2(J + I)\} = \lambda$ and
- (iii) $(\kappa/2)\{\sigma + 3p + 2(J + I)\} < \lambda$, respectively

Moreover, if $J = I = 0$ for the perfect fluid GRW spacetime, then we turn up the following:

Theorem 17. *If a perfect fluid GRW spacetime with a unit timelike torse-forming vector field ξ admitting a Ricci-Yamabe soliton (g, ξ, Ω) of type (α, β) , then the Ricci-Yamabe soliton is expanding.*

Remark 18. According to the above corollaries (15) and (16), we can easily obtain the similar results for perfect fluid GRW spacetime.

7. η -Ricci-Yamabe Soliton in an Imperfect Fluid GRW Spacetime

Consider the equation

$$\mathcal{L}_\xi g + 2\alpha S + (2\Omega - \beta R)g + 2\mu\eta \otimes \eta = 0, \quad (36)$$

where g is a Lorentzian metric, S is a Ricci curvature, ξ is the vector field, η is the 1-form, and Ω and μ are real constant. The structure $(g, \xi, \Omega, \mu, \alpha, \beta)$ that satisfies equation (36) called η -Ricci-Yamabe soliton at M [14]. In particular if $\mu = 0$, (g, ξ, Ω) becomes Ricci-Yamabe soliton and it is *shrunked, steady, or expanded* with respect to Ω that is negative, zero, or positive, accordingly.

More specific, the Lie derivative $\mathcal{L}_\xi g$ gives

$$(\mathcal{L}_\xi g)(U, V) = g(\nabla_U \xi, V) + g(U, \nabla_V \xi), \quad (37)$$

and form (36) we obtain

$$\alpha S(U, V) = - \left(\Omega - \frac{\beta R}{2} \right) g(U, V) - \mu \eta(U) \eta(V) - \frac{1}{2} [g(\nabla_U \xi, V) + g(U, \nabla_V \xi)], \quad (38)$$

for any $U, V \in \chi(M)$.

On contracting (38), we get

$$-(\alpha - \beta)R = -4\Omega + \mu - \operatorname{div}(\xi). \quad (39)$$

Let (M^4, g) be a general relativistic imperfect fluid GRW-spacetime and $(g, \xi, \Omega, \mu, \alpha, \beta)$ is a η -Ricci-Yamabe soliton at M , (16) and (38) lead to

$$\begin{aligned} \alpha \left[\lambda - \frac{\kappa(p - \sigma + 2J)}{2} + \left(\Omega - \frac{\beta R}{2} \right) \right] g(U, V) \\ + [\alpha\kappa(\sigma + p) + \mu] \eta(U) \eta(V) + \alpha\kappa P(U, V) \\ + \frac{1}{2} g(\nabla_U \xi, V) + g(U, \nabla_V \xi) = 0, \end{aligned} \quad (40)$$

for any $U, V \in \chi(M)$.

Considering $\{e_i\}_{1 \leq i \leq 4}$ is an orthonormal frame and $\xi = \sum_{i=1}^4 \xi^i e_i$, we have $\sum_{i=1}^4 \varepsilon_{ii}(\xi^i)^2 = -1$ and $\eta(e_i) = \varepsilon_{ii} \xi^i$.

By the multiplication of (40) with ε_{ii} , putting $U = V = e_i$ and summing over i , we have

$$4\alpha\Omega - \mu = -(\alpha + 2\alpha\beta)4\lambda + \kappa[(\alpha + 6\alpha\beta)p - (3\alpha + 2\alpha\beta)(\sigma + J)] - \operatorname{div}(\xi). \quad (41)$$

Writing (40) with $X = Y = \xi$, which leads to

$$\alpha\Omega - \mu = -(\alpha + 2\alpha + \beta)\lambda - \frac{\kappa}{2}[(\alpha - 3\alpha\beta)p + (\alpha + \alpha\beta)\sigma - (2\alpha + \alpha\beta)J + \alpha I]. \quad (42)$$

Therefore

$$\begin{cases} \Omega = -(1 + 2\beta)\lambda + \frac{\kappa}{\alpha} \left(c_1 p - c_2 \sigma + c_3 J + \frac{I}{3} \right) - \frac{\operatorname{div}(\xi)}{3\alpha}, \\ \mu = \frac{\kappa}{3} (3\alpha p - c_4 \sigma + c_5 J - 4\alpha I) - \frac{\operatorname{div}(\xi)}{3}, \end{cases} \quad (43)$$

where $c_1 = (\alpha/2 + 3\alpha\beta/2)$, $c_2 = (7\alpha - \alpha\beta)$, $c_3 = (4\alpha + \alpha\beta)$, $c_4 = (5\alpha + 4\alpha\beta)$, and $c_5 = (7\alpha + 4\alpha\beta)$. Using (43), the coming results are state.

Theorem 19. *If (M^4, g) is the general relativistic imperfect fluid GRW spacetime and let η be the g -dual 1-form of the gradient vector field $\xi = \operatorname{grad}(\psi)$. If (36) defines an η -Ricci-Yamabe soliton with nonvanishing α and β in M^4 , therefore,*

Laplace-Poisson equation insured by ψ turns to

$$\Delta(\psi) = -3 \left[\mu - \frac{\kappa}{3} (3\alpha p - c_4 \sigma + c_5 J - 4\alpha I) \right]. \quad (44)$$

For perfect fluid GRW spacetime J, I and c_5 vanishes, therefore we can turn up the following result.

Corollary 20. Let (M^4, g) is the general relativistic perfect fluid GRW spacetime and let η be the g -dual 1-form of the gradient vector field $\xi = \text{grad}(\psi)$. If (36) defines an η -Ricci-Yamabe soliton with nonvanishing α and β in M^4 , therefore, Laplace-Poisson equation insured by ψ turns to

$$\Delta(\psi) = -3 \left[\mu - \frac{\kappa}{3} (3\alpha p - c_4 \sigma) \right]. \quad (45)$$

Example 21. An η -Ricci-Yamabe soliton $(g, \xi, \Omega, \mu, \alpha, \beta)$ at the radiation fluid is given as

$$\begin{cases} \Omega = -(1 + 2\beta)\lambda + \frac{\kappa}{\alpha} \left(c_1 p - c_2 \sigma + c_3 J + \frac{I}{3} \right) - \frac{\text{div}(\xi)}{3\alpha}, \\ \mu = \frac{\kappa}{3} (3\alpha p - c_4 \sigma + c_5 J - 4\alpha I) - \frac{\text{div}(\xi)}{3}. \end{cases} \quad (46)$$

8. Harmonic Aspects of the η -Ricci-Yamabe Soliton on an Imperfect Fluid GRW Spacetime

Let η is a g -dual 1-form of the given vector field ξ , considering $g(X, \xi) = \eta(X)$ and $g(\xi, \xi) = -1$. Then, ξ is called a solution of the Schrödinger-Ricci equation if it satisfies

$$\text{div}(\mathcal{L}_\xi g) = 0, \quad (47)$$

where $\mathcal{L}_\xi g$ is Lie derivative for the vector field ξ . In [36], Chow et al. studied the divergence of the Lie derivative such that

$$\text{div}(\mathcal{L}_\xi g) = (\Delta + S)(\xi) + d(\text{div}(\xi)), \quad (48)$$

where Δ represents the Laplace-Hodge operator with respect to the metric g and S is the Ricci curvature tensor field. Now, consider the equation

$$\mathcal{L}_\xi g + 2\alpha S + (2\Omega - \beta R)g + 2\mu\eta \otimes \eta = 0. \quad (49)$$

Taking trace of equation (49), we have

$$\text{div}(\xi) + (\alpha - 4\beta)R + 4\Omega + \mu|\xi|^2 = 0, \quad (50)$$

where R is scalar curvature. By direct calculation, we obtain

$$\text{div}(\eta \otimes \eta) = \text{div}(\xi)\eta + \nabla_\xi \eta. \quad (51)$$

By taking the divergence of (49) and using (48), we obtain

$$\text{div}(\mathcal{L}_\xi g) + (\alpha - 4\beta)d(R) + 2\mu[\text{div}(\xi)\eta + \nabla_\xi \eta] = 0. \quad (52)$$

For Schrödinger-Ricci solution, we say that a 1-form χ is a solution of the Schrödinger-Ricci equation if

$$(\Delta + S)(\chi) + d(\text{div}(\chi)) = 0. \quad (53)$$

Hence, we have next results.

Theorem 22. Let $(g, \xi, \Omega, \mu, \alpha, \beta)$ is an η -Ricci-Yamabe soliton on an imperfect fluid GRW spacetime (M^4, g) with η the g -dual of the vector field ξ . Then, η is the solution of the Schrödinger-Ricci equation if and only if

$$d(\sigma - 3p) = \frac{2\mu}{(\alpha - 4\beta)} \{ [4(\Omega + \lambda) - \mu - k(3p - \sigma + J)]\eta - \nabla_\xi \eta \}. \quad (54)$$

Proof. Using equations (49), (50), (51), and (31) and the fact that $2 \text{div}(S) = (\alpha - 4\beta)d(R)$, it follows that η is a solution of the Schrödinger-Ricci equation if and only if (52) holds. \square

Theorem 23. Let $(g, \xi, \Omega, \mu, \alpha, 0)$ is the η -Ricci soliton on an imperfect fluid GRW spacetime (M^4, g) with η the g -dual of the vector field ξ . Then, η is a solution of the Schrödinger-Ricci equation if and only if

$$d(\sigma - 3p) = \frac{2\mu}{\alpha} \{ [4(\Omega + \lambda) - \mu - k(3p - \sigma + J)]\eta - \nabla_\xi \eta \}. \quad (55)$$

Theorem 24. Let $(g, \xi, \Omega, \mu, 0, \beta)$ is the η -Yamabe soliton on an imperfect fluid GRW spacetime (M^4, g) with η the g -dual of the vector field ξ . Then, η is a solution of the Schrödinger-Ricci equation if and only if

$$d(\sigma - 3p) = -\frac{\mu}{2\beta} \{ [4(\Omega + \lambda) - \mu - k(3p - \sigma + J)]\eta - \nabla_\xi \eta \}. \quad (56)$$

Furthermore, if $J = 0$ on the perfect fluid GRW spacetime, then the coming Corollary is stated as follows:

Corollary 25. Suppose $(g, \xi, \Omega, \mu, \alpha, \beta)$ is the η -Ricci-Yamabe soliton on a perfect fluid GRW spacetime (M^4, g) with η the g -dual of the vector field ξ . Then, η is a solution of the Schrödinger-Ricci equation if and only if

$$d(\sigma - 3p) = \frac{2\mu}{(\alpha - 4\beta)} \{ [4(\Omega + \lambda) - \mu - k(3p - \sigma + J)]\eta - \nabla_\xi \eta \}. \quad (57)$$

By using Theorems (23) and (24) for particular value of α and β , similarly we can obtain the results for a perfect fluid GRW spacetime.

For Schrödinger-Ricci harmonic forms, we say that a 1-form χ is a *Schrödinger-Ricci harmonic form* if

$$(\Delta + S)(\chi) = 0. \quad (58)$$

Furthermore, if $\sigma = 3p$, then the fluid is a radiation fluid if and only if $\mu = 0$, which yields the Ricci-Yamabe soliton

$$\frac{1}{(\alpha - 4\beta)} [4(\Omega + \lambda) - \mu - k(3p - \sigma + J)]\eta = \nabla_{\xi}\eta, \quad (59)$$

which implies that $\mu = 4(\Omega + \lambda) - k(3p - \sigma + J)$. Hence, we introduce the next results.

Theorem 26. Suppose $(g, \xi, \Omega, \mu, \alpha, \beta)$ is an η -Ricci-Yamabe soliton on an imperfect fluid GRW spacetime (M^4, g) with η the g -dual of the vector field ξ . Then, η is a solution of the Schrödinger-Ricci harmonic form if and only if $\mu = 0$, which yields Ricci-Yamabe soliton or

$$\frac{1}{(\alpha - 4\beta)} [4(\Omega + \lambda) - \mu - k(3p - \sigma + J)]\eta = \nabla_{\xi}\eta, \quad (60)$$

which implies that $\mu = 4(\Omega + \lambda) - k(3p - \sigma + J)$.

Theorem 27. Let $(g, \xi, \Omega, \mu, 0, \beta)$ is an η -Yamabe soliton on an imperfect fluid GRW spacetime (M^4, g) with η the g -dual of the vector field ξ . Then, η is a solution of the Schrödinger-Ricci harmonic form if and only if $\mu = 0$, which yields Yamabe soliton or

$$\frac{1}{-4\beta} [4(\Omega + \lambda) - \mu - k(3p - \sigma + J)]\eta = \nabla_{\xi}\eta, \quad (61)$$

which implies that $\mu = 4(\Omega + \lambda) - k(3p - \sigma + J)$.

Remark 28. For $J = 0$ and particular value of α and β in the Theorems 26 and 27, we can obtain the Schrödinger-Ricci harmonic form for a perfect fluid GRW spacetime.

9. Some Applications

Equations (3)–(5) describe the deformations of a Riemannian metric g_{ij} with time t . The deformation is driven by Ricci curvature, so that the part of the manifold with greater Ricci curvature will undergo greater deformation. However, the fix points can be computed if the flows are the Ricci flat manifolds $\text{Ric} = 0$. As a result of Perelman's significant work on Ricci-Yamabe (Ricci-Yamabe solitons) [37], the geometrization of these flows has experienced tremendous growth. It is better to see some following physical applications. In fact, the Ricci-Yamabe soliton is used to understand the idea of kinematics and thermodynamics in general relativity [28, 38]. Ricci-Yamabe solitons are focused because their curvatures keep the self-similarity (self-similar solution).

- (1) The spaces are stable or not under geometric flow; this would have application to *tachyon* condensation in string theory [2]
- (2) We can explore Ricci-Yamabe soliton on the manifolds with boundary; in this context, there are applications to *black hole* and thermodynamics as well as it has some relevance to certain formulations of quasilocal mass [38, 39]
- (3) Laplace-Poisson equation follows the principal of relativity; it describes gravitational field. The azimuthally symmetric theory of gravitation (ASTG-model) and Magneto-Hydro-Dynamic (MHD) [39] modeling of molecular clouds are also based on the Laplace-Poisson equation

10. Conclusions

In GR, the matter content of the universe is considered to work like an imperfect fluid in the standard cosmological models such as a time oriented 4-dimensional Lorentzian manifold. In this framework, Einstein's equation plays the fundamental role to construct the cosmological models. Relativistic imperfect GRW fluid spacetime [40] models are of considerable interest in several areas of astrophysics [2, 41, 42], plasma physics [3], and nuclear physics [1, 43, 44].

The propagation of fundamental field on black hole spacetime [45], which is a relevant case of black hole spacetime in the presence of plasma. An ideal MHD is the realization that plasma elements connected by magnetic field line at a given time will remain connected by a magnetic field line at any subsequent time, provided that the plasma velocity field remains smooth. This property arises because a plasma that satisfies the Ohm's law moves with a transport velocity that preserves the magnetic connection between plasma elements [46]. Moreover, plasma elements can be cast in a covariant form in a specific foliation spacetime [47]. Thus, Ohm's law

$$F^c T_{ab} = 0, \quad (62)$$

where F^c is the four fluid velocity vector field and T_{ab} is the electromagnetic field tensor. The above equation provides a simple and effective constitute relation for large scale and low frequency plasma dynamics [3].

On the other side, geometric flows are most effective tools to describe the geometric structures in relativistic imperfect fluid GRW spacetime (semi-Riemannian geometry) [29]. A special class of solutions on which the metric evolves by dilation and diffeomorphisms plays a crucial role in the study of singularities of the flows, as they appear as possible singularity models [16, 37]. They are often called soliton solutions (Ricci-Yamabe solitons).

A 4-dimensional imperfect fluid GRW spacetime manifold model is conceding Ricci-Yamabe soliton. Ricci-Yamabe solitons are the natural extension of the Einstein metric in semi-Riemannian geometry. Therefore, Einstein manifolds arose during the study of exact solution of the Einstein's field equation. We turn up the condition such as

semiconformally flat imperfect fluid GRW spacetime, semi-symmetric, and Ricci semisymmetric imperfect fluid GRW spacetime with Ricci-Yamabe soliton. Finally, we discuss some harmonic significance of imperfect fluid GRW spacetime in terms of η -Ricci-Yamabe soliton.

Data Availability

No data were used to support this study.

Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Research Article

Geometric Mechanics on Warped Product Semi-Slant Submanifold of Generalized Complex Space Forms

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In this study, we develop a general inequality for warped product semi-slant submanifolds of type $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ in a nearly Kaehler manifold and generalized complex space forms using the Gauss equation instead of the Codazzi equation. There are several applications that can be developed from this. It is also described how to classify warped product semi-slant submanifolds that satisfy the equality cases of inequalities (determined using boundary conditions). Several results for connected, compact warped product semi-slant submanifolds of nearly Kaehler manifolds are obtained, and they are derived in the context of the Hamiltonian, Dirichlet energy function, gradient Ricci curvature, and nonzero eigenvalue of the Laplacian of the warping functions.

1. Introduction

We can examine the energy, angles, and lengths of their second fundamental form using certain warped product manifolds. These manifolds are generalizations of Riemannian product manifolds and provide examples of manifolds with a strictly negative curvature from a mathematical standpoint. They can be usefully applied to models of spacetime around black holes and bodies with enormous gravitational fields from a mechanical standpoint. From the geometric standpoint of applied mathematics, their warping functions can solve numerous partial differential equations (see [1, 2]). Bishop and O'Neill [3] first proposed the concept of warped product manifolds in order to analyze negative curvature manifolds. Here's how they define it.

Definition 1. Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds, where $f : N_1 \rightarrow (0, \infty)$ and $\gamma_1 : N_1 \times N_2 \rightarrow N_1, \gamma_2 : N_1 \times N_2 \rightarrow N_2$; the orthogonal projection maps are defined as $\gamma_1(t, s) = t$ and $\gamma_2(t, s) = s$ for any $(t, s) \in N_1 \times N_2$. Then, the warped product $N_1 \times_f N_2$ is a product manifold that is $N_1 \times N_2$ associated with the Riemannian structure; in other words,

$$g(X, Y) = g_1(\gamma_1 * X, \gamma_1 * Y) + (f \circ \gamma_1)^2 g_2(\gamma_2 * X, \gamma_2 * Y), \quad (1)$$

for any $X, Y \in TM^n$, where the tangent map is denoted by $*$ and f represents a warping function of M^n .

The theory of slant submanifolds is currently under investigation; it was first established by Chen in [4] for nearly Hermitian manifolds. The almost complex (holomorphic) and entirely real submanifolds are specific examples among the classes of slant submanifolds. As a result, the warped product semi-slant submanifold is the most basic generalization of a CR-warped product submanifold. al-Solamy et al. [5] recently investigated a warped product semi-slant submanifold of a nearly Kaehler manifold, proving that no such warped product semi-slant submanifold exists of the form $M^n = N_\theta^{n_2} \times_f N_T^{n_1}$, where $N_\theta^{n_2}$ is a proper slant submanifold and $N_T^{n_1}$ is a complex submanifold.

The researchers next looked into warped products of the type $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ and came up with a number of fascinating conclusions, including characterizations and an inequality. We refer to [6] for a survey of warped product submanifolds. We first remark that the utilization of the

Codazzi equation in Chen's study [7] problem atomizes attempts to extend its results from the warped product semi-slant submanifold setting, due to the slant angle's involvement. We use a novel strategy in this work, substituting the Codazzi equation (used in [7]) with the Gauss equation. As a generalization of the contact CR-warped products, we construct a sharp general inequality for warped product semi-slant submanifolds isometrically immersed in a generalized space form. We also investigate nontrivial warped product semi-slant submanifolds of type $M^n = N_g^{n_2} \times_f N_T^{n_1}$ that are isometrically immersed in an arbitrary nearly Kaehler manifold; we obtain results (cf. Theorem 21) and consider interesting applications thereof (cf. Theorem 22).

Chen developed a general inequality for the CR-warped product of complex space forms in [7]. Furthermore, in [8–11], the classifications of contact CR-warped products in spheres that satisfy the equality cases similarly were given. The classifications of the totally geodesic and totally umbilical submanifolds are examples of how these relations might be used to classify equalities in the derived inequality. For various types of inequalities, several authors (in [12–17]) have presented thorough classifications of CR-warped products in complex projective space forms and Lagrangian submanifolds in complex space forms. Motivated by previous studies, we derived necessary and sufficient conditions to determine whether a compact oriented warped product semi-slant submanifold in a generalized complex space forms is trivial (cf. Theorems 24, 25, and 26, and Corollaries 27 and 28).

Calin and Chang presented a geometric approach to Riemannian manifolds in [18], identifying its applicability to partial differential equations that implement a Lagrangian formalism on Riemannian manifolds; for example, they considered its application to the energy-momentum tensor and conservation laws; the Hamiltonian formalism; Hamilton-Jacobi theory; harmonic functions, maps, and geodesics; and harmonic functions, maps, and geodesics. Let us note that the geometry of a Riemannian manifold can be thought of as a compact Riemannian submanifold with a boundary; in other words, $\partial M \neq \emptyset$. We considered the Euler-Lagrange equation, kinetic energy function, and Hamiltonian approach to warped product submanifolds for which the warping function plays an important role as a positive differential function for such identities because of the influence of the slant angle in a warped product semi-slant submanifold of a nearly Kaehler we provide (cf. Theorems 29, 30, and 32).

The effect of Ricci curvature on the structure of warped products is investigated. In Riemannian geometry, one important question arises: What is the geometric meaning of Ricci curvature? Answer: Ricci-flat manifolds require us to solve the Riemannian manifold's Einstein field equations with a vanishing cosmological constant geometrically.

We study the Ricci curvature on the structure of warped products. One fundamental question arises: What is the geometric meaning of Ricci curvature in Riemannian geometry? Answer: Geometrically, Ricci-flat manifolds require us to solve the Einstein field equations of the Riemannian manifold with a vanishing cosmological constant. In general relativity, the Ricci tensor corresponds to the universe's matter

content via Einstein field equations. The degree to which matter tends to converge or diverge over time is determined by this term of spacetime curvature. As a result, in physics, Ricci curvature is more essential than Riemannian curvature, and geometric obstacles of the Ricci curvature and Ricci tensor will be found in warped product manifolds (for further details, see [12, 19] and the references therein). Our next goal is to look into the physical implications of these issues in terms of warping functions. We propose our result (cf. Theorem 33) to enable our study to uncover the useful applications of the obtained inequality in physics. The work described in this paper will be combined with the singularity theory techniques presented in [20–24].

The following is a breakdown of the paper's structure: We review some basic formulas and definitions in Section 2 and give a quick overview of semi-slant submanifolds. In Section 3, we analyze warped product semi-slant submanifolds and prove an inequality for an intrinsic invariant in a nearly Kaehler manifold in terms of the second basic form, the squared norm of the warping function, and the Laplacian of the warping functions. The case of equality is also examined. In this section, we get the main result for warped product semi-slant submanifolds immersed isometrically in a nearly Kaehler manifold. In Section 4, we use boundary conditions to explain multiple classifications of such inequalities for Riemannian and compact Riemannian submanifolds. In Section 5, we strengthen the second fundamental form inequality in a virtually Kaehler manifold for warped product semi-slant submanifolds and CR-warped product submanifolds. We also show that the warped product semi-slant manifold in a nearly Kaehler manifold becomes a Riemannian product under a set of complicated requirements expressed in terms of the kinetic energy function and the Hamiltonian of the warping function. In Section 6, we prove that the compact warped product semi-slant submanifold of a virtually Kaehler manifold is either a CR-warped product manifold or a simple Riemannian product manifold in terms of the gradient Ricci curvature of warped functions.

2. Preliminaries

An almost Hermitian manifold (\tilde{M}, J, g) of a $2m$ -dimensional space, such that J is an almost complex structure and g is a Riemannian metric, satisfies

$$(a) J^2 = -I, (b) g(JX, JY) = g(X, Y), \quad (2)$$

for any X, Y on \tilde{M}^{2m} , where the identity map is denoted by I . Let $\Gamma(T\tilde{M}^{2m})$ denote the set of all vector fields tangent to \tilde{M}^{2m} ; the Levi-Civita connection defined on \tilde{M}^{2m} is denoted by $\tilde{\nabla}$. Then, a *Kaehler manifold* with an almost complex structure J satisfies

$$(\tilde{\nabla}_X J)Y = 0, \quad (3)$$

for any $X, Y \in \Gamma(T\tilde{M}^{2m})$ and \tilde{M}^{2m} . Moreover, if the almost complex structure J is such that

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \quad (4)$$

for any vector field X, Y tangent to \tilde{M}^{2m} , then the manifold \tilde{M} represents a *nearly Kaehler manifold* [25–27]. The above equation is similar to the following:

$$(\nabla_X J)X = 0, \quad X \in \Gamma(T\tilde{M}^{2m}). \quad (5)$$

Assume that \tilde{M}^n is a complex space form of constant holomorphic sectional curvature 4κ and it is denoted by $\tilde{M}^n(4\kappa)$. The curvature tensor \tilde{R} of $\tilde{M}^n(4\kappa)$ can be expressed as

$$\begin{aligned} \tilde{R}(W_1, W_2)W_3 = & \kappa\{g(W_2, W_3)W_1 - g(W_1, W_3)W_2 \\ & + g(W_3, JW_2)JW_1 - g(W_3, JW_1)JW_2 \\ & + 2g(W_1, JW_2)JW_3\}, \end{aligned} \quad (6)$$

for all $W_1, W_2, W_3 \in \Gamma(T\tilde{N})$. Based on the cases $\kappa < 0$, $\kappa = 0$, and $\kappa > 0$, $\tilde{N}^n(4\kappa)$ is the complex hyperbolic space $\mathbb{C}H^n$, complex Euclidean space \mathbb{C}^n , or the complex projective space $\mathbb{C}P^n$. Now, we consider the generalized complex space forms which are a natural generality of complex space forms and a special family of Hermitian manifolds. Actually, a generalized complex space form is a RK -manifold of constant type α with constant holomorphic sectional curvature κ . Moreover, it is denoted by $\tilde{M}^{2m}(\kappa, \alpha)$. Hence, the curvature tensor \tilde{R} for generalized complex space is given by

$$\begin{aligned} \tilde{R}(X, Y, Z, W) = & \frac{\kappa + 3\alpha}{4}\{g(Y, Z)g(X, W) - g(Y, W)g(X, Z)\} \\ & + \frac{\kappa - \alpha}{4}\{g(X, JZ)g(JY, W) - g(Y, JZ)g(JX, W) \\ & + 2g(X, JY)g(JZ, W)\}, \end{aligned} \quad (7)$$

for any $X, Y, Z, W \in \Gamma(T\tilde{M}^{2m})$. Thus, for the more classifications of generalized complex space forms, we refer to [10, 11, 28–32]. The curvature tensor \tilde{R} for a nearly Kaehler 6-sphere is given by

$$\tilde{R}(X, Y, Z, W) = g(Y, Z)g(X, W) - g(Y, W)g(X, Z), \quad (8)$$

for any $X, Y, Z, W \in \Gamma(T\tilde{M}(\mathbb{S}^6))$.

A submanifold is denoted by the M^n of an almost Hermitian manifold \tilde{M}^{2m} with an induced Riemannian metric g . However, ∇^\perp and ∇ represent the induced Riemannian connections on the normal bundle $T^\perp M^n$ and tangent bundle TM^n of M^n , respectively. Thus, the Gauss and Weingarten formulas are defined as

$$(i) \nabla_X^\perp Y = \nabla_X Y + h(X, Y), \quad (ii) \nabla_X^\perp N = -A_N X + \nabla_X^\perp N, \quad (9)$$

for every $N \in \Gamma(T^\perp M^n)$ and $X, Y \in \Gamma(TM^n)$, where A_N and h denote the shape operator and second fundamental form for an immersion of M into \tilde{M}^{2m} , respectively. Now, for any $N \in \Gamma(T^\perp M^n)$ and $X \in \Gamma(TM^n)$, we have

$$(i) JX = PX + FX, \quad (ii) JN = tN + fN, \quad (10)$$

where $FX(fN)$ and $PX(tN)$ are the normal and tangential components of $JN(JX)$, respectively. From (2), it can be clearly seen that, for each $X, Y \in \Gamma(TM^n)$, we have

$$(a) g(PX, Y) = -g(X, PY), \quad (b) \|P\|^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j), \quad (11)$$

for each $e_i, i = 1, \dots, n$ tangent to M^n . Assuming \tilde{M}^{2m} to be a Riemannian manifold and M^n a submanifold of \tilde{M}^{2m} , the Gauss equation can be defined as

$$\begin{aligned} \tilde{R}(X, Y, Z, W) = & R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) \\ & - g(h(X, W), h(Y, Z)), \end{aligned} \quad (12)$$

for any $X, Y, Z, W \in \Gamma(TM^n)$, where \tilde{R} and R represent the curvature tensors on \tilde{M}^{2m} and M^n , respectively. Furthermore, *totally umbilical* and *totally geodesic* submanifolds satisfy $h(X, Y) = g(X, Y)H$ and $h(X, Y) = 0$, respectively, for any $X, Y \in \Gamma(TM^n)$, where H is the mean curvature vector of M^n . If $H = 0$, then M^n is called a *minimal submanifold*. The mean curvature vector H is expressed in terms of $\{e_1, e_2, \dots, e_n\}$, which is the so-called orthonormal frame of the tangent space TM^n ; it is defined as

$$H = \frac{1}{n} \text{trace}(h) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i), \quad (13)$$

where $n = \dim M$. Moreover, we have

$$\|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)), \quad h_{ij}^r = g(h(e_i, e_j), e_r), \quad (14)$$

for which $\{e_i\}_{i=1, \dots, n}$ and $\{e_r\}_{r=n+1, \dots, 2m}$ are orthonormal frames tangent to M^n and normal to M^n , respectively. The scalar curvature τ for a submanifold M^n of an almost complex manifold \tilde{M}^{2m} is given by

$$\tau(TM^n) = \sum_{1 \leq i \neq j \leq n} K(e_i \wedge e_j). \quad (15)$$

In the above equation, e_i and e_j represent the span of the plane section, and its sectional curvature is denoted by $K(e_i \wedge e_j)$. Let G_r be an r -plane section on TM^n and $\{e_1, e_2, \dots, e_r\}$ be any orthonormal basis of G_r . Then, the scalar curvature $\tau(G_r)$ of G_r is defined as

$$\tau(G_r) = \sum_{1 \leq i \neq j \leq r} K(e_i \wedge e_j). \quad (16)$$

Let ϕ be a differential function defined on M^n . Thus, the gradient $\vec{\nabla}\phi$ is given as

$$g(\vec{\nabla}\phi, X) = Xf, \quad \vec{\nabla}\phi = \sum_{i=1}^n e_i(\phi)e_i. \quad (17)$$

Thus, from the above equation, the Hamiltonian in a local orthonormal frame is defined as

$$H(d\phi, x) = \frac{1}{2} \sum_{j=1}^n df(e_j)^2 = \frac{1}{2} \sum_{j=1}^n e_j(\phi)^2 = \frac{1}{2} \|\nabla\phi\|^2. \quad (18)$$

Moreover, the Laplacian Δf of f is also given by

$$\Delta\phi = \sum_{i=1}^n \{(\nabla_{e_i} e_i)\phi - e_i(e_i(\phi))\} = - \sum_{i=1}^n g(\nabla_{e_i} \text{grad } \phi, e_i). \quad (19)$$

Similarly, the Hessian tensor of function f is given by

$$\Delta\phi = -\text{Trace}H^\phi = - \sum_{i=1}^n \text{Hess}(\phi)(e_i, e_i), \quad (20)$$

where H^ϕ denotes the Hessian tensor. The compact manifold M^n is considered as being without a boundary; that is, $\partial M^n = \emptyset$. Thus, we have the following lemma.

Lemma 2 (see [18]; Hopf's lemma). *Let M^n be a connected and compact Riemannian manifold and ϕ a smooth function on M^n such that $\Delta\phi \geq 0$ ($\Delta\phi \leq 0$). Then, ϕ is a constant function on M^n .*

Moreover, the integration of the Laplacian of the smooth function, defined on a compact-orientated Riemannian manifold M^n without boundary, vanishes with respect to the volume element of such a manifold, and we obtain the following formula:

$$\int_{M^n} \Delta\phi dV = 0, \quad (21)$$

where dV denotes the volume of M^n (see [33]).

Theorem 3 (see [18]). *The Euler-Lagrange equation for the Lagrangian is*

$$\Delta\phi = 0. \quad (22)$$

Hopf's lemma becomes the uniqueness theorem for the Dirichlet problem if manifold M^n has a boundary.

Theorem 4 (see [18]). *Let M^n be a connected and compact manifold and f a positive differentiable function on M^n such that $\Delta\phi = 0$, $\phi = 0$, where ∂M^n is the boundary of M .*

Moreover, let M^n be a compact Riemannian manifold and f be a positive differentiable function on M^n . Then, the Dirichlet energy function is defined as described in [18]; that is,

$$E(\phi) = \frac{1}{2} \int_{M^n} \|\nabla\phi\|^2 dV. \quad (23)$$

If M^n is compact, then $0 \leq E(\phi) < \infty$. We provide the following definition of a slant submanifold.

Definition 5 (see [4]). Assume $T_x M^n - \{0\}$ to be a set containing all nonzero tangent vector fields of immersion M^n in an almost Hermitian manifold \tilde{M}^{2m} at a point $x \in M^n$. Then, for each vector $X \in (T_x M^n)$ at point $x \in M^n$, the angle between JX and the tangent space $T_x M$ is considered to be the Wirtinger angle of X at $x \in M^n$; this is denoted as $\vartheta(X)$. In this case, a submanifold M^n of \tilde{M}^{2m} is called a slant submanifold such that ϑ is a slant angle.

It is clear that the slant submanifolds include totally real and holomorphic submanifolds. However, Chen proved the following characterization theorem of slant submanifolds.

Theorem 6. *Let \tilde{M}^{2m} be an almost Hermitian manifold and M^n be a submanifold of \tilde{M}^{2m} . Then, M^n is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = -\lambda I, \quad (24)$$

where $\lambda = \cos^2 \vartheta$ for a slant angle ϑ defined on the tangent bundle TM^n of M^n .

Hence, we have the following consequences of Theorem 6:

$$\begin{aligned} g(PX, PY) &= \cos^2 \vartheta g(X, Y), \\ g(FX, FY) &= \sin^2 \vartheta g(X, Y), \end{aligned} \quad (25)$$

for any $X, Y \in \Gamma(TM^n)$.

In an essentially Hermitian manifold, another group of submanifolds known as semi-slant submanifolds exists as a natural generalization of slant submanifolds, CR-submanifolds, and holomorphic and antiholomorphic submanifolds. Papaghiuc researched and defined semi-slant submanifolds in [34] as a natural extension of CR-submanifolds of an almost Hermitian manifold. The following is the definition of a semi-slant submanifold.

Definition 7. A Riemannian submanifold M^n of an almost Hermitian manifold \tilde{M}^{2m} is defined as a semi-slant submanifold if there exist two complementary distributions \mathcal{D} and \mathcal{D}^ϑ such that

- (i) $TM = \mathcal{D} \oplus \mathcal{D}^\vartheta$
- (ii) \mathcal{D} is a holomorphic distribution; that is, $J(\mathcal{D}) = \mathcal{D}$
- (iii) \mathcal{D}^ϑ is called a slant distribution if slant angle $\vartheta \neq 0, \pi/2$

Remark 8. For a semi-slant submanifold, let us consider the dimensions of \mathcal{D} and \mathcal{D}^ϑ in d_1 and d_2 ; then, M^n is holomorphic if $d_2 = 0$ and slant if $d_1 = 0$. Furthermore, if $\vartheta = \pi/2$ and $d_1 = 0$, then M^n is represented as an antiholomorphic (totally real) submanifold. Moreover, M^n is referred to as a proper semi-slant submanifold if ϑ differs from 0 and $\pi/2$. We can also define M^n as proper if $d_1 \neq 0$ and $d_2 \neq 0$.

Remark 9. If ν is an invariant normal subspace under an almost complex structure J of the normal bundle $T^\perp M^n$, then this normal bundle is decomposed as

$$T^\perp M^n = F\mathcal{D}^\vartheta \oplus \nu. \quad (26)$$

3. Warped Product semi-slant Submanifolds of Nearly Kaehler Manifolds

We will go through some of the findings on warped product manifolds in this section. References [5, 6, 35–40] provide more information. We derive our main inequality for the squared norm of the second fundamental form in terms of constant holomorphic sectional curvature using numerous geometric conditions for the mean curvature of a warped product semi-slant submanifold.

In particular, a warped product manifold is classified to be *trivial* if the warping function is constant. In such cases, we refer to the warped product manifold as a Riemannian product manifold. It was proven in [3] that, for $X \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$, the following is satisfied:

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \quad (27)$$

where ∇ denotes the Levi-Civita connection on M^n . We recall the following lemma obtained in [3].

Lemma 10. A warped product manifold $M^n = N_1 \times_f N_2$. Thus,

- (i) $\nabla_X Y \in \Gamma(TN_1)$
- (ii) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$

for any $Z, W \in \Gamma(TN_2)$ and $X, Y \in \Gamma(TN_1)$, where ∇' is the Levi-Civita connection on N_2 .

Remark 11. If the warping function f is constant, then $M = N_1 \times_f N_2$ is a trivial warped product or a simple Riemannian product.

Remark 12. In a nontrivial warped product manifold $M^n = N_1 \times_f N_2$, the manifold N_1 is totally geodesic, and N_2 is a totally umbilical submanifold in M^n .

Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^{2m}$ be an isometric immersion of a warped product manifold $N_1 \times_f N_2$ in an arbitrary Riemannian manifold \tilde{M}^{2m} . Furthermore, let n_1, n_2 , and n represent the real dimensions of N_1, N_2 , and M^n , respectively. Then, for any unit tangent vectors Y and W on N_1 and N_2 , respectively, we have

$$K(Y \wedge W) = g(\nabla_W \nabla_Y Y - \nabla_Y \nabla_W Y, W) = \frac{1}{f} \{ (\nabla_Y Y)f - Y^2 f \}. \quad (28)$$

If we consider the local orthonormal frame $\{e_1, e_2, \dots, e_n\}$ such that the vectors e_1, e_2, \dots, e_{n_1} are tangential to N_1 and e_{n_1+1}, \dots, e_n are tangential to N_2 , then in view of the Gauss equation (12), we can deduce that

$$\tau(TM^n) = \tilde{\tau}(TM^n) + \sum_{r=1}^{2m} \sum_{1 \leq i \neq j \leq n} \left(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2 \right), \quad (29)$$

for each $j = n_1 + 1 \dots n$.

Hereafter, we will denote the corresponding dimensions as indices. Recall that [5] proved several results for both types of warped product semi-slant submanifolds in nearly Kaehler manifolds.

Theorem 13 (see [5]). *There does not exist a proper warped product submanifold of the form $M^n = N_1^{n_1} \times_f N_2^{n_2}$ in a nearly Kaehler manifold \tilde{M}^{2m} such that $N_1^{n_1}$ is a proper slant submanifold and $N_2^{n_2}$ is a holomorphic submanifold of \tilde{M}^{2m} .*

Lemma 14. For a nontrivial warped product semi-slant submanifold $M^n = N_1^{n_1} \times_f N_2^{n_2}$ in a nearly Kaehler manifold \tilde{M}^{2m} , we have the following equalities:

- (i) $g(h(JX, Z), FW) + g(h(JX, W), FZ) = 2(X \ln f)g(Z, W)$
- (ii) $g(h(X, Z), FW) + g(h(X, W), FZ) = -2(JX \ln f)g(Z, W)$

for any $X \in \Gamma(TN_1^{n_1})$ and $Z, W \in \Gamma(TN_2^{n_2})$.

Proof. For the first part of the proof, using (9) (i) and the orthogonality of vector fields, we establish that

$$g(h(JX, W), FZ) = -g(\nabla'_W JZ, JX) - g(\nabla'_W JX, FZ). \quad (30)$$

From the covariant derivative of the almost complex structure J and from (27), we derive that

$$\begin{aligned} g(h(JX, W), FZ) &= (X \ln f)g(Z, W) - g(JX, (\nabla_W^- J)Z) \\ &\quad - (JX \ln f)g(W, PZ). \end{aligned} \quad (31)$$

Using the structure equation (4) in the above equation and in (10) (i), we find that

$$\begin{aligned} g(h(JX, W), FZ) &= (X \ln f)g(Z, W) + g(\nabla_Z^- FW, JX) \\ &\quad - g(\nabla_Z^- JX, PW) + g(\nabla_Z^- X, W) \\ &\quad - (JX \ln f)g(W, PZ). \end{aligned} \quad (32)$$

Thus, from (27) and (10) (ii), we finally obtain

$$g(h(JX, W), FZ) + g(h(JX, Z), FW) = 2(X \ln f)g(Z, W), \quad (33)$$

which is (i). Replacing X with JX in (i) and using (2) (i), we obtain the required result (ii). The lemma is proven completely. \square

Lemma 15. Assume that $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ is a nontrivial warped product semi-slant submanifold in a nearly Kaehler manifold; thus, we have

$$\begin{aligned} (i) \quad &g(h(JX, PZ), FPW) + g(h(JX, PW), FPZ) = 2(X \ln f) \cos^2 \vartheta g(Z, W) \\ (ii) \quad &g(h(X, PZ), FPW) + g(h(X, PW), FPZ) = -2(JX \ln f) \cos^2 \vartheta g(Z, W) \end{aligned}$$

for any $X \in \Gamma(TN_T^{n_1})$ and $Z, W \in \Gamma(TN_\theta^{n_2})$.

Proof. Replacing Z and W by PZ and PW , respectively, and using (24) in Lemma 14 (i)–(ii), we directly obtain (i) and (ii), respectively. This completes the proof of the lemma. \square

Remark 16. In particular, if we substitute $Z = W$ in Lemma 10, then Lemma 10 coincides with Lemma 3.1 in [5].

Lemma 17. Let $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ be a warped product semi-slant submanifold of a nearly Kaehler manifold \tilde{M}^{2m} . Then,

$$\begin{aligned} (i) \quad &g(h(X, Z), FPW) + g(h(X, W), FPZ) = (2/3) \cos^2 \vartheta \\ &\quad (X \ln f) \|Z\|^2 \\ (ii) \quad &g(h(X, PZ), FW) + g(h(X, PW), FZ) = (2/3) \cos^2 \vartheta \\ &\quad (X \ln f) \|Z\|^2 \end{aligned}$$

for any $X \in \Gamma(TN_T^{n_1})$ and $Z, W \in \Gamma(TN_\theta^{n_2})$.

Proof. Replacing Z by $Z + W$ in Lemma 5.2 in [41], and using the linearity property of vector fields, we derive (i) and (ii). This completes the proof of the lemma. \square

Lemma 18. Assume that $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ is a nontrivial warped product semi-slant submanifold in a nearly Kaehler manifold. For any $X \in \Gamma(TN_T^{n_1})$ and $Z, W \in \Gamma(TN_\theta^{n_2})$, we have

$$\begin{aligned} (i) \quad &g(h(JX, Z), FPW) + g(h(JX, W), FPZ) = -(2/3) \cos^2 \vartheta (JX \ln f) \|Z\|^2 \\ (ii) \quad &g(h(JX, PZ), FW) + g(h(JX, PW), FZ) = -(2/3) \cos^2 \vartheta (JX \ln f) \|Z\|^2 \end{aligned}$$

Proof. Similarly, by replacing X with JX in Lemma 17 (i)–(ii), we arrive at our required results (i) and (ii) using (2). \square

To prove the general inequality, we require an orthonormal frame for orthonormal vector fields, as well as some preparatory results.

Lemma 19. Let M^n be a warped product semi-slant submanifold in a nearly Kaehler manifold \tilde{M}^{2m} . Thus,

$$\begin{aligned} (i) \quad &g(h(X, Y), FZ) = 0 \\ (ii) \quad &g(h(JX, JY), \xi) = -g(h(X, Y), \xi) \end{aligned}$$

for any $X, Y \in \Gamma(TN_T^{n_1})$, $Z \in \Gamma(TN_\theta^{n_2})$, and $\xi \in \Gamma(\nu)$.

Proof. The first part of the proof is trivial; the second part can be proved in a similar manner to Lemma 5.1 in [39]. \square

Lemma 20. Let $\varphi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \tilde{M}^{2m}$ be an isometric immersion of a warped product semi-slant submanifold in a nearly Kaehler manifold \tilde{M}^{2m} . Then, $N_T^{n_1}$ is a minimal submanifold of \tilde{M}^{2m} , and the squared norm of the mean curvature of M^n is given by

$$\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m} \left(h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r \right)^2, \quad (34)$$

where H denotes the mean curvature vector. Moreover, n_1 , n_2 , n , and $2m$ are dimensions of $N_T^{n_1}$, $N_\theta^{n_2}$, $N_T^{n_1} \times_f N_\theta^{n_2}$, and \tilde{M}^{2m} , respectively.

Proof. The above lemma can be readily proven in a similar manner to Lemma 5.2 in [39]. \square

Main inequality for warped product semi-slant submanifolds.

Theorem 21. Let $\varphi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \tilde{M}^{2m}$ be an isometric immersion of a warped product semi-slant submanifold M^n in a nearly Kaehler manifold \tilde{M}^{2m} . Thus,

- (i) The squared norm of the second fundamental form of M^n satisfies

$$\|h\|^2 \geq 2(n_2 \|\nabla \ln f\|^2 + \delta - n_2 \Delta(\ln f)), \quad (35)$$

where $\delta = \tilde{\tau}(TM^n) - \tilde{\tau}(TN_T^{n_1}) - \tilde{\tau}(TN_\theta^{n_2})$, n_2 is the dimension of the slant submanifold $N_\theta^{n_2}$, and Δ is the Laplacian operator of $N_T^{n_1}$.

- (ii) The equality holds in (35) if and only if $N_T^{n_1}$ is totally geodesic and $N_\theta^{n_2}$ is totally umbilical in \tilde{M}^{2m} and if M^n is a minimal submanifold of \tilde{M}^{2m} .

Proof. The proof proceeds in a similar manner to the proof of Theorem 29 [31] if we consider the nearly Kaehler manifold, instead of nearly trans-Sasakian manifold. \square

3.1. Applications of Theorem 21 to Generalized Complex Space Forms. In this section, we prove our main theorem using Theorem 21 for a generalized complex space form. Then, we give the following result.

Theorem 22. Assume that $\varphi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \tilde{M}^{2m}(\kappa, \alpha)$ be an isometric immersion of a warped product semi-slant $N_T^{n_1} \times_f N_\theta^{n_2}$ into generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure with constant holomorphic sectional curvature κ of constant type α . Then,

- (i) The squared norm of the second fundamental form of M^n satisfies

$$\|h\|^2 \geq 2n_2 \left\{ \left(\frac{\kappa + 3\alpha}{4} \right) n_1 - \Delta(\ln f) + \|\nabla \ln f\|^2 \right\}, \quad (36)$$

where $n_1 = \dim N_T$, $n_2 = \dim_{\mathbb{R}} N_\theta$, $n = \dim N_T \times_f N_\theta$ and $2m = \dim \tilde{M}^{2m}(\kappa, \alpha)$.

- (ii) The equality holds in (36), if and only if $N_\theta^{n_2}$ and $N_T^{n_1}$ are totally umbilical and totally geodesic submanifolds in \tilde{M} , respectively. Moreover, M^n is a minimal submanifold of $\tilde{M}^{2m}(\kappa, \alpha)$.

Proof. Letting us substitute $X = W = e_i$ and $Y = Z = e_j$ in (8), we get

$$\begin{aligned} \tilde{R}(e_i, e_j, e_j, e_i) &= \frac{\kappa + 3\alpha}{4} \{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)g(e_j, e_i)\} \\ &\quad + \frac{\kappa - \alpha}{4} \{g(e_i, Je_j)g(Je_j, e_i) \\ &\quad - g(e_i, Je_i)g(e_j, Je_j) + 2g^2(Je_j, e_i)\}. \end{aligned} \quad (37)$$

Taking the summation over the basis vector fields of $T M^n$ such that $1 \leq i \neq j \leq n$, one shows that

$$2\tilde{\tau}(TM^n) = \left(\frac{\kappa + 3\alpha}{4} \right) n(n-1) + 3 \left(\frac{\kappa - \alpha}{4} \right) \sum_{1 \leq i \neq j \leq n} g^2(Pe_i, e_j). \quad (38)$$

M^n is a warped product of holomorphic and proper slant submanifolds in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$. Thus, we set the following frame of orthonormal vector fields as

$$\begin{aligned} e_1, e_2 &= Je_1, \dots, e_{2d_1-1}, e_{2d_1} = Je_{2d_1-1}, e_{2d_1+1}, e_{2d_1+2} \\ &= \sec \vartheta Pe_{2d_1+1}, \dots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec \vartheta Pe_{d_1-1}. \end{aligned} \quad (39)$$

From using the above orthonormal frame, we obtain

$$\begin{aligned} g^2(Je_i, e_{i+1}) &= 1, \text{ for } i \in \{1 \dots n_1 - 1\} \\ &= \cos^2 \vartheta \text{ for } i \in \{n_1 + 1, \dots, n_1 + n_2 - 1\}. \end{aligned} \quad (40)$$

Thus, it is easily seen that

$$\sum_{i,j=1}^n g^2(Pe_i, e_j) = n_1 + n_2 \cos^2 \vartheta. \quad (41)$$

From (38) and (40), it follows that

$$2\tilde{\tau}(TM^n) = \left(\frac{\kappa + 3\alpha}{4} \right) n(n-1) + 3 \left(\frac{\kappa - \alpha}{4} \right) \{n_1 + n_2 \cos^2 \vartheta\}. \quad (42)$$

Similarly, for $TN_T^{n_1}$, we derive

$$2\tilde{\tau}(TN_T^{n_1}) = \left(\frac{\kappa + 3\alpha}{4} \right) n_1(n_1 - 1) + 3 \left(\frac{\kappa - \alpha}{4} \right) n_1. \quad (43)$$

Now using fact that $\|P\|^2 = n_2 \cos^2 \vartheta$, for slant bundle $TN_\theta^{n_2}$, one derives

$$2\tilde{\tau}(TN_\theta^{n_2}) = \left(\frac{\kappa + 3\alpha}{4} \right) n_2(n_2 - 1) + 3 \left(\frac{\kappa - \alpha}{4} \right) n_2 \cos^2 \vartheta. \quad (44)$$

Therefore, substituting (41), (42), and (40) in Theorem

21, we get the required result (29). The equality case follows from Theorem 21 (ii). Thus, the proof is complete. \square

Notably, the following corollary can be readily obtained in terms of the Hessian tensor of the warping function $\ln f$ for a warped product submanifold.

Corollary 23. *Let $\varphi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \tilde{M}^{2m}(\kappa, \alpha)$ be an isometric immersion of a warped product $N_T^{n_1} \times_f N_\theta^{n_2}$ into generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure. Then,*

$$\|h\|^2 \geq 2n_2 \left(\left(\frac{\kappa + 3\alpha}{4} \right) n_1 + \|\nabla \ln f\|^2 + \text{Trace}(H^{\ln f}) \right), \quad (45)$$

where $H^{\ln f}$ is the Hessian tensor of the warping function $\ln f$.

4. Compact-Orientated Warped Product semi-slant Submanifolds

In this section, we consider compact Riemannian manifolds without boundaries; that is, $\partial M^n = \emptyset$. Applying these to warped product semi-slant submanifolds, and using integration theory on the manifold, we obtain several characterizations.

Theorem 24. *Let $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ be a compact-orientated warped product semi-slant submanifold generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure. Then, M^n is a trivial warped product if and only if*

$$\|h\|^2 \geq 2 \left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1. \quad (46)$$

Proof. From Theorem 21, we have

$$\|h\|^2 \geq 2 \left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 - n_2 \Delta(\ln f) + n_2 \|\nabla \ln f\|^2. \quad (47)$$

This implies that

$$n_2 \|\nabla \ln f\|^2 + 2 \left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 - \|h\|^2 \leq n_2 \Delta(\ln f). \quad (48)$$

Applying integration theory on the compact-orientated Riemannian manifold M^n without boundary, and then using (21), we obtain

$$\int_{M^n} \left(2 \left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 + n_2 \|\nabla \ln f\|^2 - \|h\|^2 \right) dV \leq \int_{M^n} \Delta(\ln f) dV = 0. \quad (49)$$

Now, if the inequality (44) holds, then from (47), we find that

$$\int_{M^n} (\|\nabla \ln f\|^2) dV \leq 0, \quad (50)$$

which is impossible for a positive integrable function; hence, $\nabla \ln f = 0$; that is, f is a constant function on M^n . Thus, by Remark 11 on warped product manifolds, M^n is trivial. The converse proof is straightforward. \square

To prove the equality case, we must prove the following theorem for later use.

Theorem 25. *Let φ be a \mathcal{D}^θ -minimal isometric immersion of a warped product semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ in a nearly Kaehler manifold \tilde{M}^{2m} . If $N_\theta^{n_2}$ is totally umbilical in \tilde{M}^{2m} , then ϕ is $N_\theta^{n_2}$ -totally geodesic.*

Proof. Let us assume that the second fundamental forms of M^n and \tilde{M}^{2m} are denoted by h^* and \tilde{h} , respectively; we define

$$h(Z, W) + h^*(Z, W) = \tilde{h}(Z, W). \quad (51)$$

for any vector fields Z and W that are tangential to $N_\theta^{n_2}$. Thus, the above hypothesis and Remark 12 show that $N_\theta^{n_2}$ is totally umbilical in \tilde{M}^{2m} , owing to its being totally umbilical in M^n . Then, following Lemma 10 (ii), equation (47) can be written as

$$h(Z, W) = g(Z, W)(\xi + \nabla(\ln f)), \quad (52)$$

where the vector field ξ is normal to $\Gamma(TN_\theta)$ and is such that $\xi \in \Gamma(TM^n)$. Assuming that $\{e_1^*, \dots, e_{n_2}^*\}$ is an orthonormal frame of the slant submanifold $N_\theta^{n_2}$, then by taking a summation over the vector fields of $N_\theta^{n_2}$ in equation (49), we obtain

$$\sum_{i,j=1}^{n_2} h(e_i^*, e_j^*) = (\xi + \nabla(\ln f)) \sum_{i,j=1}^{n_2} g(e_i^*, e_j^*). \quad (53)$$

The left-hand side of the above equation identically vanishes due to the \mathcal{D}^θ -minimality of φ , such that $\sum_{i,j=1}^{n_2} h(e_i^*, e_j^*) = 0$. Then, equation (50) takes the following form:

$$n_2(\xi + \nabla(\ln f)) = 0. \quad (54)$$

This implies that $N_\theta^{n_2}$ is nonempty, such that

$$\xi = -\nabla(\ln f). \quad (55)$$

Thus, from (50) and (53), it follows that $h(Z, W) = 0$, for every $Z, W \in \Gamma(TN_\theta^{n_2})$. This means that φ is $N_\theta^{n_2}$ -totally geodesic. This completes the proof of the theorem. \square

Theorem 26. *Let $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ be a compact-orientated warped product semi-slant submanifold in a generalized*

complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure. Then, M^n is a trivial warped product if and only if

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2 = \left(\frac{\kappa + 3\alpha}{4}\right) n_2 n_1, \quad (56)$$

where $n_1 = \dim N_T$ and $n_2 = \dim_{\mathbb{R}} N_{\mathfrak{g}}$. Moreover, h_v is a component of h in $\Gamma(v)$.

Proof. We assume that the equality sign holds in (36); then, we have

$$\begin{aligned} \left(\frac{\kappa + 3\alpha}{4}\right) n_2 n_1 + 2n_2 \|\nabla \ln f\|^2 &= \|h(\mathcal{D}, \mathcal{D})\|^2 + \|h(\mathcal{D}^{\mathfrak{g}}, \mathcal{D}^{\mathfrak{g}})\|^2 \\ &\quad + 2\|h(\mathcal{D}, \mathcal{D}^{\mathfrak{g}})\|^2 + 2n_2 \Delta(\ln f). \end{aligned} \quad (57)$$

However, the equality case of inequality (36) implies that $N_T^{n_1}$ is totally geodesic in a nearly Kaehler manifold; this means that $h(e_i, e_j) = 0$, for any $1 \leq i, j \leq 2d_1$. Moreover, $N_{\mathfrak{g}}^{n_2}$ is totally umbilical and can be written as

$$h(e_t^*, e_s^*) = g(e_t^*, e_s^*)H, \quad (58)$$

for any $1 \leq t, s \leq 2d_2$. Furthermore, M^n is a minimal submanifold of a nearly Kaehler manifold; thus, its mean curvature vector H should be zero; that is, $H = 0$; hence, $h(e_t^*, e_s^*) = 0$, for every $1 \leq t, s \leq 2d_2$ through Theorem 25

$$\left(\frac{\kappa + 3\alpha}{4}\right) n_2 n_1 = n_2 \Delta(\ln f) + \|h(\mathcal{D}, \mathcal{D}^{\mathfrak{g}})\|^2 - n_2 \|\nabla \ln f\|^2. \quad (59)$$

We assume that M^n is a compact submanifold; thus, M^n is closed and bounded; hence, by integrating the above equation over the volume element dV of M^n and using (21), we find that

$$\int_{M^n} \left\{ \|h(\mathcal{D}, \mathcal{D}^{\mathfrak{g}})\|^2 - n_2 \|\nabla \ln f\|^2 \right\} dV = \int_{M^n} \left(\frac{\kappa + 3\alpha}{4}\right) n_2 n_1 dV. \quad (60)$$

Now, let $X = e_i$ and $Z = e_j$ for $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$, respectively; then, using (14) and expressing in terms of the orthonormal frame, we have

$$\|h(\mathcal{D}, \mathcal{D}^{\mathfrak{g}})\|^2 = \sum_{k=1}^{2m} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(h(e_i, e_j^*), e_k)^2. \quad (61)$$

In the above equation, the first term on the right-hand side is the $F\mathcal{D}^{\mathfrak{g}}$ -component and the second term is the v -component. Let us suppose that $M^n = N_T^{n_1} \times_f N_{\mathfrak{g}}^{n_2}$ is a warped product semi-slant submanifold of an n -dimension in a nearly Kaehler manifold \tilde{M}^{2m} of $2m$ dimensions,

such that $\dim N_T = n_1 = 2d_1$ and $\dim_{\mathbb{R}} N_{\mathfrak{g}} = n_2 = 2d_2$. We assume that the tangent spaces of $N_T^{n_1}$ and $N_{\mathfrak{g}}^{n_2}$ are \mathcal{D} and $\mathcal{D}^{\mathfrak{g}}$, respectively. We further assume that $\{e_1, e_2, \dots, e_{d_1}, e_{d_1+1} = Je_1, \dots, e_{2d_1} = Je_{d_1}\}$ is a local orthonormal frame of $TN_T^{n_1}$ and that $\{e_{2d_1+1} = e_1^*, \dots, e_{2d_1+d_2} = e_{d_2}^*, e_{2d_1+d_2+1} = e_{d_2+1}^* = \sec \vartheta Pe_1^*, \dots, e_{n_1+n_2} = e_{n_2}^* = \sec \vartheta Pe_{d_2}^*\}$ is a local orthonormal frame of $TN_{\mathfrak{g}}^{n_2}$. Thus, the orthonormal frames of the normal subbundles $F\mathcal{D}^{\mathfrak{g}}$ and v are $\{e_{n+1} = \tilde{e}_1 = \csc \vartheta F e_1^*, \dots, e_{n+d_2} = \tilde{e}_{d_2} = \csc \vartheta F e_{d_2}^*, e_{n+d_2+1} = \tilde{e}_{d_2+1} = \csc \vartheta \sec \vartheta F P e_1^*, \dots, e_{n+2d_2} = \tilde{e}_{2d_2} = \csc \vartheta \sec \vartheta F P e_{d_2}^*\}$ and $\{e_{n+2d_2+1}, \dots, e_{2m}\}$, respectively. Taking a summation over the vector fields on $N_T^{n_1}$ and $N_{\mathfrak{g}}^{n_2}$ and using adapted frame fields, we obtain

$$\begin{aligned} \|h(\mathcal{D}, \mathcal{D}^{\mathfrak{g}})\|^2 &= \csc^2 \vartheta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(e_i, e_j^*), F e_k^*)^2 \\ &\quad + \csc^2 \vartheta \sec^2 \vartheta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(e_i, P e_j^*), F e_k^*)^2 \\ &\quad + \csc^2 \vartheta \sec^2 \vartheta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(J e_i, e_j^*), F P e_k^*)^2 \\ &\quad + \csc^2 \vartheta \sec^2 \vartheta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(J e_i, e_j^*), F P e_k^*)^2 \\ &\quad + \csc^2 \vartheta \sec^4 \vartheta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(J e_i, P e_j^*), F P e_k^*)^2 \\ &\quad + \csc^2 \vartheta \sec^2 \vartheta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(J e_i, P e_j^*), F e_k^*)^2 \\ &\quad + \csc^2 \vartheta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(J e_i, e_j^*), F e_k^*)^2 \\ &\quad + \csc^2 \vartheta \sec^4 \vartheta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(e_i, P e_j^*), F P e_k^*)^2 \\ &\quad + \sum_{r=n+n_2+1}^{2m} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(h(e_i, e_j), e_r)^2. \end{aligned} \quad (62)$$

Then, using Lemmas 14 to 18 in the above equations, we derive

$$\begin{aligned} \|h(\mathcal{D}, \mathcal{D}^{\mathfrak{g}})\|^2 &= 2 \left(\csc^2 \vartheta + \frac{1}{9} \cot^2 \vartheta \right) \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} (e_i \ln f)^2 g(e_j^*, e_j^*)^2 \\ &\quad + 2 \left(\csc^2 \vartheta + \frac{1}{9} \cot^2 \vartheta \right) \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} (J e_i \ln f)^2 g(e_j^*, e_j^*)^2 \\ &\quad + \sum_{r=n+n_2+1}^{2m} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(h(e_i, e_j), e_r)^2. \end{aligned} \quad (63)$$

From (17), the last equation can be expressed as

$$\begin{aligned} \|h(\mathcal{D}, \mathcal{D}^\vartheta)\|^2 &= 2 \left(\csc^2 \vartheta + \frac{1}{9} \cot^2 \vartheta \right) \|\nabla \ln f\|^2 \sum_{j=1}^{d_2} g(e_j^*, e_j^*)^2 \\ &+ \sum_{r=n+n_2+1}^{2m} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(h(e_i, e_j), e_r)^2, \end{aligned} \quad (64)$$

which implies that

$$\|h(\mathcal{D}, \mathcal{D}^\vartheta)\|^2 = n_2 \left(1 + \frac{10}{9} \cot^2 \vartheta \right) \|\nabla \ln f\|^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2. \quad (65)$$

Then, from (58) and (63), it follows that

$$\int_{M^n} \left\{ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2 + \frac{10}{9} n_2 \cot^2 \vartheta \|\nabla \ln f\|^2 \right\} dV = \int_{M^n} \left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 dV. \quad (66)$$

If (54) holds identically, then from (66), we find that either f is constant on M^n or $\cot \vartheta = 0$. However, M^n is a proper semi-slant submanifold; thus, M^n is a simple Riemannian product. The converse proof follows immediately from (66). Hence, the theorem is proven completely. \square

Corollary 27. Assume that $M^n = N_T^{n_1} \times_f N_\vartheta^{n_2}$ is a warped product semi-slant submanifold in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure. Suppose that $N_T^{n_1}$ is a compact invariant submanifold and λ_T is a nonzero eigenvalue of the Laplacian on $N_T^{n_1}$. Then,

$$\int_{N_T^{n_1}} \|h\|^2 dV_T \geq \int_{N_T^{n_1}} \left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 dV + 2n_2 \lambda_T \int_{N_T^{n_1}} (\ln f)^2 dV_T, \quad (67)$$

where dV_T is the volume element on $N_T^{n_1}$. The equality sign holds if and only if the following are satisfied:

- (i) $\nabla \ln f = \lambda_T \ln f$
- (ii) A warped product semi-slant submanifold $M^n = N_T^{n_1} \times_f N_\vartheta^{n_2}$ is both $N_T^{n_1}$ - and $N_\vartheta^{n_2}$ -totally geodesic

Proof. Thus, using the minimum principle property, we have

$$\int_{N_T^{n_1}} \|\nabla \ln f\|^2 dV_T \geq \lambda_T \int_{N_T^{n_1}} (\ln f)^2 dV_T, \quad (68)$$

The equality holds if and only if one has $\nabla \ln f = \lambda_T \ln f$. Thus, from (44) and (66), we require the result (65). Similarly, it is clear that the equality sign of (65) holds identically if and only if the warped product is both $N_T^{n_1}$ - and $N_\vartheta^{n_2}$ -

-totally geodesic. This completes the proof of the corollary. \square

Corollary 28. Let $M^n = N_T^{n_1} \times_f N_\vartheta^{n_2}$ be a warped product semi-slant submanifold in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure such that $N_T^{n_1}$ is compact, and let λ_T be a nonzero eigenvalue of the Laplacian on $N_T^{n_1}$. Then,

$$\begin{aligned} \int_{N_T^{n_1}} \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2 \right) dV_T &\geq \int_{N_T^{n_1}} 2 \left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 dV \\ &+ \frac{10}{9} n_2 \cot^2 \vartheta \lambda_T \int_{N_T^{n_1}} (\ln f)^2 dV_T. \end{aligned} \quad (69)$$

The equality sign holds if and only if the following are satisfied:

- (a) $\nabla \ln f = \lambda_T \ln f$
- (b) A warped product semi-slant submanifold $M^n = N_T^{n_1} \times_f N_\vartheta^{n_2}$ is both $N_T^{n_1}$ - and $N_\vartheta^{n_2}$ -totally geodesic

Proof. The proof follows from (67) and (66). This completes the proof of the theorem. \square

5. Applications to Dirichlet Energy Functions and Hamiltonian

We discuss connected, compact Riemannian manifolds with borders in this section; that is, $\partial M \neq \emptyset$. Using Hopf's lemma, we apply these to warped product submanifolds. To determine whether nontrivial warped products become trivial warped product submanifolds of nearly Kaehler manifolds, we obtain necessary and sufficient conditions in terms of Dirichlet energy (analogous to kinetic energy) and Hamiltonian of warping functions.

Theorem 29. Assume that $\varphi : M^n = N_T^{n_1} \times_f N_\vartheta^{n_2}$ is an isometric immersion of a warped product semi-slant in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure. A connected and compact warped product $N_T^{n_1} \times_f N_\vartheta^{n_2}$ is trivial if and only if the Dirichlet energy function satisfies

$$E(\ln f) = \frac{9}{20n_2} \tan^2 \vartheta \int_{M^n} \left\{ \left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2 \right\} dV, \quad (70)$$

where $E(\ln f)$ represents the Dirichlet energy of the warping function $\ln f$ and dV is the volume element on M^n .

Proof. Combining equations (57) and (63), we obtain

$$\left(\frac{\kappa+3\alpha}{4}\right)n_2n_1 = n_2\Delta(\ln f) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2 + n_2 \frac{10}{9} \cot^2 \vartheta \|\nabla \ln f\|^2. \quad (71)$$

Taking an integration on M^n over the volume element dV with a nonempty boundary in the above equation, we find that

$$\begin{aligned} \int_{M^n} \left(\frac{\kappa+3\alpha}{4}\right)n_2n_1 dV &= n_2 \int_{M^n} (\Delta(\ln f)) dV \\ &+ \int_{M^n} \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2 \right) dV \\ &+ \frac{10}{9} n_2 \cot^2 \vartheta \int_{M^n} (\|\nabla \ln f\|^2) dV. \end{aligned} \quad (72)$$

Then, from (23) and (70), it follows that

$$\begin{aligned} \frac{1}{n_2} \int_{M^n} \left(\frac{\kappa+3\alpha}{4}\right)n_2n_1 dV &= \int_{M^n} \Delta(\ln f) dV \\ &+ \frac{1}{n_2} \int_{M^n} \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2 \right) dV \\ &+ \frac{20}{9} \cot^2 \vartheta E(\ln f). \end{aligned} \quad (73)$$

Equality (68) is satisfied if and only if we obtain from (71) the condition that $\int_{M^n} \Delta(\ln f) dV = 0$, which implies that $\Delta(\ln f) = 0$. The theorem hypothesizes M^n as a connected, compact warped product semi-slant submanifold; thus, Theorem 4 implies that $\ln f = 0 \implies f = 1$, which means that f is constant on M^n . Thus, the theorem is proven completely. \square

In a similar manner, we derive several characterizations in terms of the Hamiltonian.

Theorem 30. Let $M^n = N_T^{n_1} \times_f N_{\vartheta}^{n_2}$ be a connected and compact warped product semi-slant submanifold in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure. Then, M^n is a trivial warped product submanifold of $N_T^{n_1}$ and $N_{\vartheta}^{n_2}$ if and only if the Hamiltonian of the warping function satisfies the following equality:

$$H(d(\ln f), x) = \frac{9}{20n_2} \tan^2 \vartheta \left(\left(\frac{\kappa+3\alpha}{4}\right)n_2n_1 - \frac{1}{n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2 \right). \quad (74)$$

Proof. Using (15) in (69), we derive

$$\frac{20}{9} \cot^2 \vartheta H(d(\ln f), x) + \Delta(\ln f) = \left(\frac{\kappa+3\alpha}{4}\right)n_1 - \frac{1}{n_2} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2. \quad (75)$$

Equation (72) is obtained if and only if $\Delta(\ln f) = 0$ on M^n ; thus, by Theorem 4, the warped product submanifold M^n is trivial. This completes the proof of the theorem. \square

The analogy of Theorem 3 for this case is classified below.

Theorem 31. Assume that $\varphi : M^n = N_T^{n_1} \times_f N_{\vartheta}^{n_2}$ is an isometric immersion of a compact warped product semi-slant submanifold in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure. Let the warping function be the solution of the Euler-Lagrange equation; then, M^n is necessarily a trivial warped product if

$$\|h\|^2 \geq 2 \left(\frac{\kappa+3\alpha}{4}\right)n_2n_1. \quad (76)$$

Proof. If the warping function satisfies the conditions of the Euler-Lagrange equation, then from Theorem 3, we obtain

$$\Delta(\ln f) = 0. \quad (77)$$

Thus, from (44) and (75), we derive

$$\|h\|^2 \geq 2 \left(\frac{\kappa+3\alpha}{4}\right)n_2n_1 + n_2 \|\nabla \ln f\|^2. \quad (78)$$

Suppose that inequality (74) holds; then, (76) implies that the warping function must be constant on M^n . This completes the proof of the theorem. \square

Theorem 32. Assume that $\varphi : M^n = N_T^{n_1} \times_f N_{\vartheta}^{n_2}$ is an isometric immersion of a compact warped product semi-slant submanifold in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure and that the warping function is a solution of the Euler-Lagrange equation. Then, the necessary and sufficient conditions for the warped product $N_T^{n_1} \times_f N_{\vartheta}^{n_2}$ being trivial are as follows:

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j)\|^2 = \left(\frac{\kappa+3\alpha}{4}\right)n_2n_1. \quad (79)$$

Proof. The proof of the above theorem is the same as that of Theorem 31; it uses (57), (63), and Theorem 3. This completes the proof of the theorem. \square

6. Classification of Ricci Curvature and Divergence of the Hessian Tensor

In this section, we study several applications of the derived inequality by considering equality cases. Let us identify any $(0, 2)$ -tensor T on M with a $(1, 1)$ -tensor via

$$g(T(Z), Y) = T(Z, Y), \quad (80)$$

for all $Y, Z \in \Gamma(TM)$. Thus, we obtain

$$\operatorname{div}(\phi T) = \phi \operatorname{div} T + T(\nabla \phi, \cdot), \quad \nabla(\phi T) = \phi \nabla T + d\phi \otimes T, \quad (81)$$

for all $\phi \in C^\infty(M)$. In particular, we have $\operatorname{div}(\phi g) = d\phi$. Moreover, the following general facts are well established in the literature:

$$(1) \operatorname{div} \nabla^2 \phi = \operatorname{Ric}(\nabla \phi, \cdot) + d\Delta \phi, \quad (ii) \frac{1}{2} d\|\nabla\|^2 = \nabla^2 \phi(\nabla, \cdot). \quad (82)$$

We consider M^n to be a compact Riemannian manifold with boundaries and obtain the following classification results.

Theorem 33. Assume that $\varphi : M^n = N_T^{n_1} \times_f N_9^{n_2}$ is an isometric immersion of a compact warped product semi-slant submanifold $N_T^{n_1} \times_f N_9^{n_2}$ in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure. If

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_v(e_i, e_j^*)\|^2 = \left(\frac{\kappa + 3\alpha}{4}\right) n_2 n_1 + n_2 \int_{M^n} \operatorname{Ric}(\nabla \ln f, \cdot) dV, \quad (83)$$

is satisfied for the warped product submanifold M^n , then at least one of the following statements is true for M^n :

- (i) The warped product semi-slant submanifold $N_T^{n_1} \times_f N_9^{n_2}$ becomes a CR-warped product that is isometrically immersed in a nearly Kaehler manifold
- (ii) The nontrivial warped product semi-slant submanifold $N_T^{n_1} \times_f N_9^{n_2}$ in a nearly Kaehler manifold is a simple Riemannian product of $N_T^{n_1}$ and $N_9^{n_2}$

Proof. Using the first identity of (79) and setting $\phi = \ln f$, we derive

$$\operatorname{div} \nabla^2 \ln f = \operatorname{Ric}(\nabla \ln f, \cdot) + d\Delta(\ln f), \quad (84)$$

from the hypothesis of the theorem; assuming that M^n is a compact warped product submanifold with a boundary and integrating along the volume element dV , we obtain

$$\Delta(\ln f) = \int_M (\operatorname{div} \nabla^2 \ln f) dV - \int_M \operatorname{Ric}(\nabla \ln f, \cdot) dV. \quad (85)$$

We use the Green theorem on the compact manifold M^n ; given a smooth function $f : M \rightarrow \mathbb{R}$, we have $\int_M \Delta f dV = 0$. We can apply the results of Yano and Kon (see [33]) immediately as $\Delta f = -\operatorname{div}(\nabla f)$. From the Green lemma, $\int_M \operatorname{div}(X) dV = 0$ for any arbitrary vector field X on M^n . Thus, we obtain $\int_M (\operatorname{div} \nabla^2 \ln f) dV = 0$; $\nabla^2 \ln f$ is the Hessian tensor of the warped function (or the Laplacian of $\ln f$); hence, (82) implies that

$$\Delta(\ln f) = - \int_M \operatorname{Ric}(\nabla \ln f, \cdot) dV. \quad (86)$$

Meanwhile, if we assume that the equality holds in the inequality (36), then from (57) and (63), we have

$$n_2 \Delta(\ln f) + \frac{10}{9} n_2 \cot^2 \vartheta \|\nabla \ln f\|^2 = \left(\frac{\kappa + 3\alpha}{4}\right) n_2 n_1 - \sum_{i=1}^{2d_1} \sum_{j=1}^{2d_2} \|h_v(e_i, e_j^*)\|^2. \quad (87)$$

From (83) and (84), we find the following equation:

$$\frac{10}{9} \cot^2 \vartheta \|\nabla \ln f\|^2 = n_1 - \frac{1}{n_2} \sum_{i=1}^{2d_1} \sum_{j=1}^{2d_2} \|h_v(e_i, e_j^*)\|^2 + \int_M \operatorname{Ric}(\nabla \ln f, \cdot) dV. \quad (88)$$

Further simplifications give

$$\begin{aligned} \int_{M^n} \operatorname{Ric}(\nabla \ln f, \cdot) dV + \left(\frac{\kappa + 3\alpha}{4}\right) n_1 &= \frac{1}{n_2} \sum_{i=1}^{2d_1} \sum_{j=1}^{2d_2} \|h_v(e_i, e_j^*)\|^2 \\ &+ \frac{10}{9} \cot^2 \vartheta \|\nabla \ln f\|^2. \end{aligned} \quad (89)$$

If the equality (80) is satisfied, then from (86), we obtain the following condition:

$$\frac{10}{9} \cot^2 \vartheta \|\nabla \ln f\|^2 = 0. \quad (90)$$

Therefore, from the above equation, we derive two cases such that

$$\cot^2 \vartheta = 0, \text{ or } \|\nabla \ln f\|^2 = 0. \quad (91)$$

Case I: We consider $\cot^2 \vartheta = 0 (\cos^2 \vartheta / \sin^2 \vartheta) = 0$, which implies that $\cos \vartheta = 0 \Rightarrow \vartheta = \pi/2$. From Remark 8, we conclude that $N_9^{n_2}$ becomes a totally real submanifold; hence, M^n becomes a CR-warped product submanifold of a nearly

Kaehler manifold. This completes the proof of (i) from Theorem 33.

Case II: We assume that $\|\nabla \ln f\|^2 = 0$, which means that $\nabla \ln f = 0$ and $\text{grad} \ln f = 0$. This implies that f is a constant function on M^n . Hence, from Remark 11, we conclude that M^n is a trivial warped product semi-slant submanifold of a nearly Kaehler manifold. This is the second part (ii) of Theorem 33. \square

Theorem 34. Let $\varphi : M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ be an isometric immersion of a compact warped product semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure, such that the warping function $\ln f$ is the first eigenfunction of the Laplacian of $N_T^{n_1}$; this is associated with the first eigenvalue λ_1 , which satisfies the following:

$$\begin{aligned} & \int \|\text{Hess} \ln f\|^2 dV + \int \text{Ric}(\nabla \ln f, \nabla \ln f) dV \\ &= \frac{9\lambda_1 \tan^2 \theta}{10n_2} \int \left(\left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\nu(e_i, e_j^*)\|^2 \right) dV. \end{aligned} \quad (92)$$

Proof. $\ln f$ is the first eigenfunction of the Laplacian of M^n and is associated with the first eigenvalue λ_1 ; that is, $\Delta \ln f = \lambda_1 \ln f$. Thus, we recall the Bochner formula (see, e.g., [42]), which states that for a differentiable function f defined on a Riemannian manifold, the following relation holds:

$$\frac{1}{2} \Delta \|\nabla \ln f\|^2 = \|\text{Hess} \ln f\|^2 + \text{Ric}(\nabla \ln f, \nabla \ln f) + g(\nabla \ln f, \nabla(\Delta \ln f)). \quad (93)$$

Integrating the above equation with the help of the Stokes theorem, we obtain

$$\int \|\text{Hess} \ln f\|^2 dV + \int \text{Ric}(\nabla \ln f, \nabla \ln f) dV + \int g(\nabla \ln f, \nabla(\Delta \ln f)) dV = 0. \quad (94)$$

Now, by using $\Delta \ln f = \lambda_1 \ln f$ and rearranging the above equation, we derive

$$\int \|\nabla \ln f\|^2 dV = -\frac{1}{\lambda_1} \left(\int \|\text{Hess} \ln f\|^2 dV + \int \text{Ric}(\nabla \ln f, \nabla \ln f) dV \right). \quad (95)$$

Integrating equation (84), we obtain

$$\frac{10n_2}{9} \cot^2 \theta \int \|\nabla \ln f\|^2 dV = \int \left(\left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 - \sum_{i=1}^{2d_1} \sum_{j=1}^{2d_2} \|h_\nu(e_i, e_j^*)\|^2 \right) dV. \quad (96)$$

It follows from (92) and (93) that

$$\begin{aligned} & \frac{10n_2 \cot^2 \theta}{9\lambda_1} \left(\int \|\text{Hess} \ln f\|^2 dV + \int \text{Ric}(\nabla \ln f, \nabla \ln f) dV \right) \\ &= \int \left(\left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 - \sum_{i=1}^{2d_1} \sum_{j=1}^{2d_2} \|h_\nu(e_i, e_j^*)\|^2 \right) dV. \end{aligned} \quad (97)$$

The above equation and (75) imply that $\cot^2 \theta = 0$ ($\cos^2 \theta / \sin^2 \theta = 0$), which implies that $\cos \theta = 0 \Rightarrow \theta = \pi/2$. Again, from Remark 8, we conclude that $N_\theta^{n_2}$ becomes a totally real submanifold; by using the statement of Theorem 34, we obtain our desired result. \square

Riemannian manifolds with no Ricci curvature are known as Ricci-flat manifolds. Ricci-flat manifolds are Einstein manifolds that do not require the cosmological constant to vanish. In a Ricci-flat manifold (particularly in Euclidean space), a circle, for example, can be deformed into an ellipse of equal area. We get the following result after taking into account the fact that warped product submanifolds are Ricci-flat.

Theorem 35. Let $\varphi : M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ be an isometric immersion of a compact warped product semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ in a generalized complex space form $\tilde{M}^{2m}(\kappa, \alpha)$ admitting a nearly Kaehler structure, such that the warping function $\ln f$ is the first eigenfunction of the Laplacian of $N_T^{n_1}$ and is associated with the first eigenvalue λ_1 ; then, $N_T^{n_1}$ is Ricci-flat if and only if

$$\int \|\text{Hess} \ln f\|^2 dV = \frac{9\lambda_1 \tan^2 \theta}{10n_2} \int \left(\left(\frac{\kappa + 3\alpha}{4} \right) n_2 n_1 - \sum_{i=1}^{2d_1} \sum_{j=1}^{2d_2} \|h_\nu(e_i, e_j^*)\|^2 \right) dV. \quad (98)$$

Proof. Thus, from (94) and (95), we obtain $\int \text{Ric}(\nabla \ln f, \nabla \ln f) dV = 0$. This means that $N_T^{n_1}$ is Ricci-flat. The converse proof is straightforward. This completes the proof of the theorem. \square

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors declare no competing of interest.

Authors' Contributions

All authors contributed equally to this work. All authors finalized the manuscript.

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Research Article

Characteristic Properties of Type-2 Smarandache Ruled Surfaces According to the Type-2 Bishop Frame in E^3

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In this paper, we define and investigate a special kind of ruled surfaces called type-2 Smarandache ruled surfaces related to the type-2 Bishop frame in E^3 . From this point and depending on the type-2 Bishop curvature, we provide the necessary and sufficient conditions that allow these surfaces to be developable in a minimal amount of time. Furthermore, an example is given to clear the results.

1. Introduction

In the classical differential geometry, the theory of ruled surfaces is one of its branches which has been developed by several researchers. A ruled surface is generally defined as the set of a family of straight lines that depend on a parameter that is mentioned as the ruled surface's rulings. A ruled surface's parametric representation is $Y(\sigma, v) = c(\sigma) + vX(\sigma)$ where $c(\sigma)$ is the base curve of $Y(\sigma, v)$ and $X(\sigma)$ define the ruling directions [1, 2]. Surfaces' developability and minimalist notions are two of their most important properties. One of the most interesting points is the study of ruled surfaces with different moving frames, as seen in this example [3–7].

The Smarandache curve in Euclidean and Minkowski spaces is the curve whose position vector is made by Frenet frame vectors on another regular curve [8–11]. Several researchers [12–20] have recently studied Smarandache curves in Minkowski and the Euclidean spaces.

In this work, in E^3 , we introduce the definitions of type-2 Smarandache ruled surfaces using the type-2 Bishop frame, namely, $\mu_1\mu_2$, μ_1B , and μ_2B type-2 Smarandache ruled surfaces. Our main results are presented in theorems that look into the necessary and sufficient conditions for those surfaces to be developable and minimal. Throughout the response, an example with illustrations is created.

2. Preliminaries

Let E^3 be a 3-dimensional Euclidean space provided with the metric

$$\langle \cdot, \cdot \rangle = du_1^2 + du_2^2 + du_3^2, \quad (1)$$

where (u_1, u_2, u_3) is the rectangular coordinate system of E^3 .

Representing the moving Frenet frame along its regular curve ψ by $\{T, N, B\}$ in conjunction with curvature functions κ and τ in E^3 , the Frenet formula is given as follows [1]:

$$\frac{d}{d\sigma} \begin{pmatrix} T(\sigma) \\ N(\sigma) \\ B(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(\sigma) & 0 \\ -\kappa(\sigma) & 0 & \tau(\sigma) \\ 0 & -\tau(\sigma) & 0 \end{pmatrix} \begin{pmatrix} T(\sigma) \\ N(\sigma) \\ B(\sigma) \end{pmatrix}, \quad (2)$$

where $\langle T, T \rangle = \langle N, N \rangle = \langle B, B \rangle = 1$ and $\langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0$.

For any arbitrary curve ψ with $\tau \neq 0$ in \mathbf{E}^3 , the type-2 Bishop frame of ψ is given as follows [21]:

$$\frac{d}{d\sigma} \begin{pmatrix} \mu_1(\sigma) \\ \mu_2(\sigma) \\ B(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -k_1(\sigma) \\ 0 & 0 & -k_2(\sigma) \\ k_1(\sigma) & k_2(\sigma) & 0 \end{pmatrix} \begin{pmatrix} \mu_1(\sigma) \\ \mu_2(\sigma) \\ B(\sigma) \end{pmatrix}, \quad (3)$$

where k_1 and k_2 are the type-2 Bishop curvatures and satisfying

$$\begin{pmatrix} T(\sigma) \\ N(\sigma) \\ B(\sigma) \end{pmatrix} = \begin{pmatrix} \sin \theta(\sigma) & -\cos \theta(\sigma) & 0 \\ \cos \theta(\sigma) & \sin \theta(\sigma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1(\sigma) \\ \mu_2(\sigma) \\ B(\sigma) \end{pmatrix}, \quad (4)$$

where $\theta(\sigma) = \arctan(k_2/k_1)$ and

$$\begin{aligned} k_1 &= -\tau \cos \theta(\sigma), \\ k_2 &= -\tau \sin \theta(\sigma). \end{aligned} \quad (5)$$

Definition 1. [21]. $\mu_1\mu_2$ type-2 Smarandache curves of the curve $\psi(\sigma)$ via $\{\mu_1, \mu_2, B\}$ are given as

$$\beta(\sigma^*(\sigma)) = \frac{1}{\sqrt{2}}(\mu_1(\sigma) + \mu_2(\sigma)). \quad (6)$$

Definition 2. [21]. μ_1B type-2 Smarandache curves of the curve $\psi(\sigma)$ via $\{\mu_1, \mu_2, B\}$ are given as

$$\gamma(\sigma^*(\sigma)) = \frac{1}{\sqrt{2}}(\mu_1(\sigma) + B(\sigma)). \quad (7)$$

Definition 3. [21]. μ_2B type-2 Smarandache curves of the curve $\psi(\sigma)$ via $\{\mu_1, \mu_2, B\}$ are given as

$$\delta(\sigma^*(\sigma)) = \frac{1}{\sqrt{2}}(\mu_2(\sigma) + B(\sigma)). \quad (8)$$

A ruled surface v in \mathbf{E}^3 can be reparametrized as

$$v(\sigma, v) = \psi(\sigma) + v\chi(\sigma), \quad (9)$$

where $\psi(\sigma)$ is really the base curve and $\chi(\sigma)$ is its unit which defines a space curve that characterizes the straight line's direction [22].

v 's unit normal vector N is given as follows [23]:

$$\mathbf{U} = \frac{v_\sigma \times v_v}{\|v_\sigma \times v_v\|}, \quad (10)$$

where $v_\sigma = \partial v / \partial \sigma$ and $v_v = \partial v / \partial v$. The Gaussian curvature K and the mean curvature H are given as follows [23]:

$$\begin{aligned} K &= \frac{\ell n - m^2}{EG - F^2}, \\ H &= \frac{En + G\ell - 2mF}{2(EG - F^2)}, \end{aligned} \quad (11)$$

where $E = \|Y_\sigma\|^2$, $F = \langle Y_\sigma, Y_v \rangle$, $G = \|Y_v\|^2$, $\ell = \langle Y_{\sigma\sigma}, \mathbf{U} \rangle$, $m = \langle Y_{\sigma v}, \mathbf{U} \rangle$, and $n = \langle Y_{vv}, \mathbf{U} \rangle$. The normal curvature, geodesic curvature, and geodesic torsion that connects the curve $\psi(\sigma)$ on Y are computed as follows:

$$\kappa_n = \langle \psi'', \mathbf{U} \rangle, \kappa_g = \langle \mathbf{U} \times T, T' \rangle, \tau_g = \langle \mathbf{U} \times \mathbf{U}', T' \rangle. \quad (12)$$

Definition 4. A ruled surface is developable if and only if $K = 0$ and minimal if and only if $H = 0$.

3. Main Results

In this part, we define the type-2 Smarandache ruled surfaces within Euclidean 3-space \mathbf{E}^3 referring to the frame $\{\mu_1, \mu_2, B\}$. Furthermore, we evaluate the sufficient and necessary conditions that enable these surfaces to be developable and minimal.

3.1. $\mu_1\mu_2$ Type-2 Smarandache Ruled Surface

Definition 5. For a regular curve $\psi = \psi(\sigma)$ in \mathbf{E}^3 related to the frame $\{\mu_1, \mu_2, B\}$, the $\mu_1\mu_2$ type-2 Smarandache ruled surface is given as

$$\Omega = \Omega(\sigma, v) = \frac{1}{\sqrt{2}}(\mu_1(\sigma) + \mu_2(\sigma)) + vB(\sigma). \quad (13)$$

Theorem 6. Let $\Omega = \Omega(\sigma, v)$ be the $\mu_1\mu_2$ type-2 Smarandache ruled surface in \mathbf{E}^3 defined by (13). Then, we have

- (1) Ω is a developable surface with asymptotic base curve $\psi(\sigma)$
- (2) Ω is a minimal surface if and only if the type-2 Bishop curvatures satisfy the following equation

$$k_1 = k_2 e^{\sigma+c}, \quad (14)$$

where c is real constant.

Proof. Considering that the $\mu_1\mu_2$ type-2 Smarandache ruled surface given by (13), then, the velocity vectors of Ω are given as follows:

$$\begin{aligned} \Omega_\sigma &= vk_1\mu_1 + vk_2\mu_2 - \left(\frac{k_1 + k_2}{\sqrt{2}}\right)B, \\ \Omega_v &= B. \end{aligned} \quad (15)$$

From equation (15), we can obtain that the Ω 's quantities of fundamental forms are

$$\begin{aligned} E &= v^2 \tau^2 + \frac{1}{2} (k_1 + k_2)^2, \\ F &= \frac{-1}{\sqrt{2}} (k_1 + k_2), \\ G &= 1, \\ \ell &= \left(\frac{k_2}{\tau} \right) \left[vk_1' - \frac{k_1(k_1 + k_2)}{\sqrt{2}} \right] - \left(\frac{k_1}{\tau} \right) \left[vk_2' - \frac{k_1(k_1 + k_2)}{\sqrt{2}} \right], \\ m &= 0, \\ n &= 0. \end{aligned} \quad (16)$$

Consequently, from the above data, we obtain K_Ω and H_Ω of the $\mu_1 \mu_2$ type-2 Smarandache ruled surface given as follows:

$$\begin{aligned} K_\Omega &= 0, \\ H_\Omega &= \frac{k_1' k_2 - k_1 k_2'}{2v\tau^3}. \end{aligned} \quad (17)$$

Also, we use (12) to get the normal curvature, the geodesic curvature, and the geodesic torsion that associate $\psi(\sigma)$ on Ω as the following:

$$\begin{aligned} \kappa_n &= 0, \\ \kappa_g &= -\frac{k_1^2}{\tau}, \\ \tau_g &= \frac{1}{\tau^3} \left[\tau^2 \left(\frac{k_1 k_2}{\tau} \right)' - k_1 k_2 \tau' \right]. \end{aligned} \quad (18)$$

So, the proof ended. \square

3.2. $\mu_1 B$ Type-2 Smarandache Ruled Surface

Definition 7. For a regular curve $\psi = \psi(\sigma)$ in \mathbb{E}^3 related to the frame $\{\mu_1, \mu_2, B\}$, the $\mu_1 B$ type-2 Smarandache ruled surface is given as

$$\Phi = \Phi(\sigma, v) = \frac{1}{\sqrt{2}} (\mu_1(\sigma) + B(\sigma)) + v \mu_2(\sigma). \quad (19)$$

Theorem 8. Let $\Phi = \Phi(\sigma, v)$ be the $\mu_1 B$ type-2 Smarandache ruled surface in \mathbb{E}^3 defined by (19). Then, we have

- (1) If $k_1 k_2 = 0$, then, Φ is a developable surface with the geodesic base curve
- (2) Φ is a minimal surface with the geodesic base curve if and only if the type-2 Bishop curvatures satisfy the following differential equation

$$\begin{aligned} & \left(k_1 + \sqrt{2} v k_2 \right) \left[k_1' - k_1 \left(k_1 + \sqrt{2} v k_2 \right) \right] \\ & - k_1 \left(\tau^2 + k_1' + \sqrt{2} v k_2' \right) - 2 k_1 k_2^2 = 0. \end{aligned} \quad (20)$$

Proof. Considering the $\mu_1 B$ type-2 Smarandache ruled surface given by (19), then, the velocity vectors of Φ are given as follows:

$$\Phi_\sigma = \left(\frac{k_1}{\sqrt{2}} \right) \mu_1 + \left(\frac{k_2}{\sqrt{2}} \right) \mu_2 - \left(\frac{k_1 + \sqrt{2} v k_2}{\sqrt{2}} \right) B, \quad (21)$$

$$\Phi_v = \mu_2.$$

From equation (21), the Φ 's quantities of fundamental forms are

$$\begin{aligned} E &= \frac{1}{2} \left[\tau^2 + \left(k_1 + \sqrt{2} v k_2 \right)^2 \right], \\ F &= \frac{k_2}{\sqrt{2}}, \\ G &= 1, \\ \ell &= \frac{\left(k_1 + \sqrt{2} v k_2 \right) \left[k_1' - k_1 \left(k_1 + \sqrt{2} v k_2 \right) \right] - k_1 \left[\tau^2 + k_1' + \sqrt{2} v k_2' \right]}{\sqrt{2} \sqrt{k_1^2 + \left(k_1 + \sqrt{2} v k_2 \right)^2}}, \\ m &= -\frac{k_1 k_2}{\sqrt{k_1^2 + \left(k_1 + \sqrt{2} v k_2 \right)^2}}, \\ n &= 0. \end{aligned} \quad (22)$$

Then, K_Φ and H_Φ of the $\mu_1 B$ type-2 Smarandache ruled surface is given as follows:

$$\begin{aligned} K_\Phi &= -\frac{2k_1^2 k_2^2}{\left[k_1^2 + \left(k_1 + \sqrt{2} v k_2 \right)^2 \right] \left[\tau^2 - k_2^2 + \left(k_1 + \sqrt{2} v k_2 \right)^2 \right]}, \\ H_\Phi &= \frac{\left(k_1 + \sqrt{2} v k_2 \right) \left[k_1' - k_1 \left(k_1 + \sqrt{2} v k_2 \right) \right] - k_1 \left[\tau^2 + k_1' + \sqrt{2} v k_2' \right] + 2k_1 k_2^2}{\sqrt{2} \sqrt{k_1^2 + \left(k_1 + \sqrt{2} v k_2 \right)^2}}. \end{aligned} \quad (23)$$

Furthermore, from (12), we have

$$\begin{aligned}\kappa_n &= \frac{(k'_1 - k_1^2)(k_1 + \sqrt{2}vk_2) - k_1(\tau^2 + k_1')}{\sqrt{2}\sqrt{k_1^2 + (k_1 + \sqrt{2}vk_2)^2}}, \\ \kappa_g &= 0, \\ \tau_g &= -\frac{k_1^2 k_2 (k_1 + \sqrt{2}vk_2)}{k_1^2 + (k_1 + \sqrt{2}vk_2)^2},\end{aligned}\quad (24)$$

which replies to the above theorem.

3.3. $\mu_2 B$ Type-2 Smarandache Ruled Surface

Definition 9. For a regular curve $\psi = \psi(\sigma)$ in \mathbf{E}^3 related to the frame $\{\mu_1, \mu_2, B\}$, the $\mu_2 B$ type-2 Smarandache ruled surface is given as

$$\Psi = \Psi(\sigma, v) = \frac{1}{\sqrt{2}}(\mu_2(\sigma) + B(\sigma)) + v\mu_1(\sigma). \quad (25)$$

Theorem 10. Let $\Psi = \Psi(\sigma, v)$ be the $\mu_2 B$ type-2 Smarandache ruled surface in \mathbf{E}^3 defined by (25). Then, we have

- (1) If $k_1 k_2 = 0$, then, Ψ is a developable surface with the principal base curve
- (2) Ψ is a minimal surface if and only if the type-2 Bishop curvatures satisfy the following differential equation

$$\begin{aligned}k_2(\tau^2 + k_2' + \sqrt{2}vk_1') - 2k_1^2 k_2 - (k_2 + \sqrt{2}vk_1) \\ \cdot [k_2' - k_2(k_2 + \sqrt{2}vk_1)] = 0.\end{aligned}\quad (26)$$

Proof. Considering the $\mu_2 B$ type-2 Smarandache ruled surface given by (25), then, the velocity vectors of Ψ are given as follows:

$$\begin{aligned}\Psi_\sigma &= \left(\frac{k_1}{\sqrt{2}}\right)\mu_1 + \left(\frac{k_2}{\sqrt{2}}\right)\mu_2 - \left(\frac{k_2 + \sqrt{2}vk_2}{\sqrt{2}}\right)B, \\ \Psi_v &= \mu_1.\end{aligned}\quad (27)$$

From equation (27), the Ψ 's quantities of fundamental forms are

$$\begin{aligned}E &= \frac{1}{2}[\tau^2 + (k_2 + \sqrt{2}vk_1)^2], \\ F &= \frac{k_1}{\sqrt{2}}, \\ G &= 1, \\ e &= \frac{k_2[\tau^2 + k_2' + \sqrt{2}vk_1'] - (k_2 + \sqrt{2}vk_1)[k_2' - k_2(k_2 + \sqrt{2}vk_1)]}{\sqrt{2}\sqrt{k_2^2 + (k_2 + \sqrt{2}vk_1)^2}}, \\ m &= \frac{k_1 k_2}{\sqrt{k_2^2 + (k_2 + \sqrt{2}vk_1)^2}}, \\ n &= 0.\end{aligned}\quad (28)$$

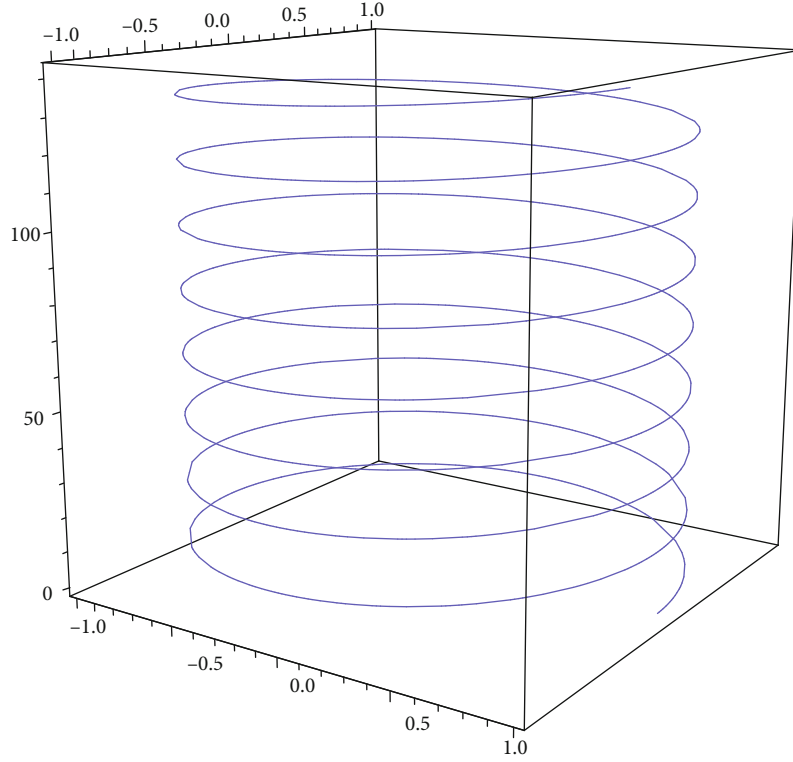
The K_Ψ and H_Ψ of the $\mu_2 B$ type-2 Smarandache ruled surface given as follows:

$$\begin{aligned}K_\Psi &= -\frac{2k_1^2 k_2^2}{\left[k_2^2 + (k_2 + \sqrt{2}vk_1)^2\right] \left[\tau^2 - k_1^2 + (k_2 + \sqrt{2}vk_1)^2\right]}, \\ H_\Psi &= \frac{k_2(\tau^2 + k_2' + \sqrt{2}vk_1') - 2k_1^2 k_2 - (k_2 + \sqrt{2}vk_1)[k_2' - k_2(k_2 + \sqrt{2}vk_1)]}{\sqrt{2}\sqrt{k_2^2 + (k_2 + \sqrt{2}vk_1)^2}}.\end{aligned}\quad (29)$$

So, the proof ended. \square

Also, from (12), we have

$$\begin{aligned}\kappa_n &= \frac{k_2(\tau^2 + k_2') - (k_2' - k_1^2)(k_2 + \sqrt{2}vk_1)}{\sqrt{2}\sqrt{k_2^2 + (k_2 + \sqrt{2}vk_1)^2}}, \\ \tau_g &= \frac{k_1^2 k_2 (k_2 + \sqrt{2}vk_1)}{k_2^2 + (k_2 + \sqrt{2}vk_1)^2}.\end{aligned}\quad (30)$$

FIGURE 1: Curve $\psi = \psi(\sigma)$.

Then, equations (29) and (30) complete the proof.

3.4. *Example.* Let ψ be a circular helix parameterized as $\psi(\sigma) = (\cos(\sigma/3), \sin(\sigma/3), (2\sqrt{2}\sigma/3))$ (see Figure 1). Then, we have

$$\begin{aligned} T(\sigma) &= \left(-\frac{1}{3} \sin\left(\frac{\sigma}{3}\right), \frac{1}{3} \cos\left(\frac{\sigma}{3}\right), \frac{2\sqrt{2}}{3} \right), \\ N(\sigma) &= \left(-\cos\left(\frac{\sigma}{3}\right), -\sin\left(\frac{\sigma}{3}\right), 0 \right), \\ B(\sigma) &= \left(\frac{2\sqrt{2}}{3} \sin\left(\frac{\sigma}{3}\right), -\frac{2\sqrt{2}}{3} \cos\left(\frac{\sigma}{3}\right), \frac{1}{3} \right). \end{aligned} \quad (31)$$

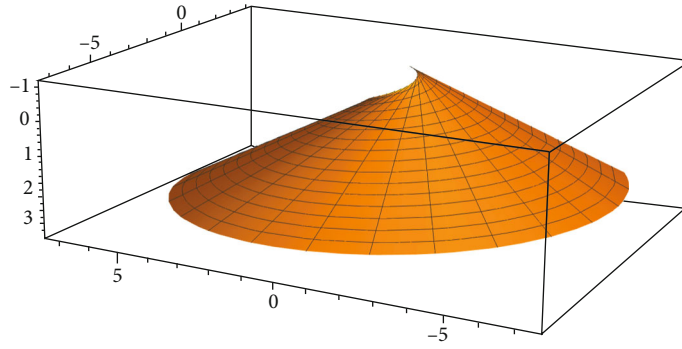
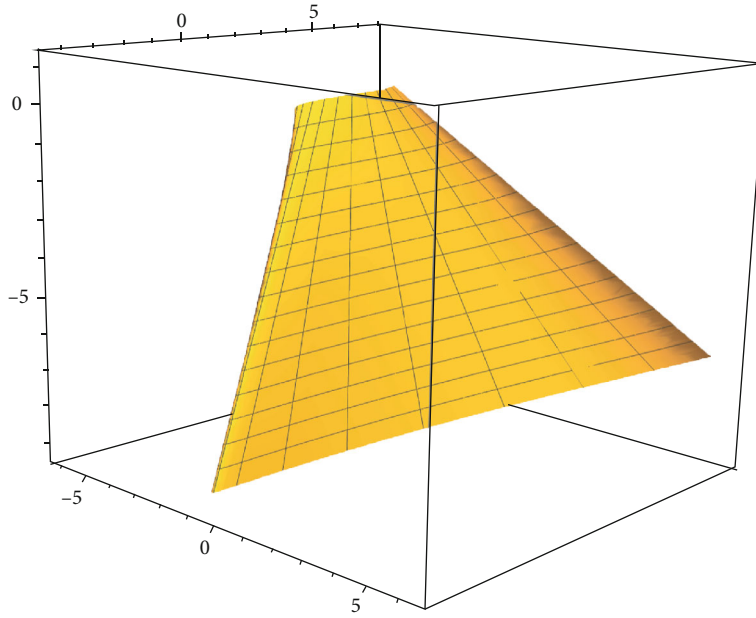
Then, $\tau = 2\sqrt{2}/9 \neq 0$ and $\theta(\sigma) = \int_0^\sigma (1/9) d\sigma = \sigma/9$. From (4), we get $\kappa_1(\sigma) = -(2\sqrt{2}/9) \cos(\sigma/9)$, $\kappa_2(\sigma) = -(2\sqrt{2}/9) \sin(\sigma/9)$. Also, we have

$$\begin{aligned} \mu_1(\sigma) &= \left(-\cos\left(\frac{\sigma}{9}\right) \cos\left(\frac{\sigma}{3}\right) - \frac{1}{3} \sin\left(\frac{\sigma}{9}\right) \right. \\ &\quad \cdot \sin\left(\frac{\sigma}{3}\right), \frac{1}{3} \cos\left(\frac{\sigma}{3}\right) \sin\left(\frac{\sigma}{9}\right) \\ &\quad \left. - \cos\left(\frac{\sigma}{9}\right) \sin\left(\frac{\sigma}{3}\right), \frac{2\sqrt{2}}{9} \sin\left(\frac{\sigma}{9}\right) \right), \end{aligned}$$

$$\begin{aligned} \mu_2(\sigma) &= \left(-\cos\left(\frac{\sigma}{3}\right) \sin\left(\frac{\sigma}{9}\right) + \frac{1}{3} \cos\left(\frac{\sigma}{9}\right) \sin\left(\frac{\sigma}{3}\right), \right. \\ &\quad \left. -\frac{1}{3} \cos\left(\frac{\sigma}{9}\right) \cos\left(\frac{\sigma}{3}\right) - \sin\left(\frac{\sigma}{9}\right) \sin\left(\frac{\sigma}{3}\right), \right. \\ &\quad \left. -\frac{2\sqrt{2}}{9} \cos\left(\frac{\sigma}{9}\right) \right). \end{aligned} \quad (32)$$

The $\mu_1\mu_2$ type-2 Smarandache ruled surface $\Omega(\sigma, v)$ is (see Figure 2)

$$\begin{aligned} \Omega(\sigma, v) &= \left(\frac{1}{\sqrt{2}} \left\{ \frac{1}{3} \sin\left(\frac{\sigma}{3}\right) \left(\cos\left(\frac{\sigma}{9}\right) - \sin\left(\frac{\sigma}{9}\right) \right) \right. \right. \\ &\quad \left. \left. - \cos\left(\frac{\sigma}{3}\right) \left(\sin\left(\frac{\sigma}{9}\right) + \cos\left(\frac{\sigma}{9}\right) \right) \right\} \right. \\ &\quad \left. + \frac{2}{3} v \sin\left(\frac{\sigma}{3}\right), \frac{1}{\sqrt{2}} \left\{ \frac{1}{3} \cos\left(\frac{\sigma}{3}\right) \left(\sin\left(\frac{\sigma}{9}\right) \right. \right. \right. \\ &\quad \left. \left. - \cos\left(\frac{\sigma}{9}\right) \right) - \sin\left(\frac{\sigma}{3}\right) \left(\cos\left(\frac{\sigma}{9}\right) \right. \right. \right. \\ &\quad \left. \left. + \sin\left(\frac{\sigma}{9}\right) \right) \right\} - \frac{2}{3} v \cos\left(\frac{\sigma}{3}\right), \frac{2}{3} \right. \\ &\quad \left. \cdot \left(\sin\left(\frac{\sigma}{9}\right) - \cos\left(\frac{\sigma}{9}\right) + \frac{v}{3} \right) \right). \end{aligned} \quad (33)$$

FIGURE 2: $\mu_1\mu_2$ type-2 Smarandache ruled surface $\Omega(\sigma, v)$.FIGURE 3: $\mu_1 B$ type-2 Smarandache ruled surface $\Phi(\sigma, v)$.

The $\mu_1 B$ type-2 Smarandache ruled surface $\Phi(\sigma, v)$ is (see Figure 3)

$$\begin{aligned} \Phi(\sigma, v) = & \left(\frac{1}{3} \sin \left(\frac{\sigma}{3} \right) \left(v \cos \left(\frac{\sigma}{9} \right) - \frac{1}{\sqrt{2}} \sin \left(\frac{\sigma}{9} \right) \right) \right. \\ & - \cos \left(\frac{\sigma}{3} \right) \left(v \sin \left(\frac{\sigma}{9} \right) + \frac{1}{\sqrt{2}} \cos \left(\frac{\sigma}{9} \right) \right) \\ & + \frac{2}{3} \sin \left(\frac{\sigma}{3} \right), \frac{1}{3} \cos \left(\frac{\sigma}{3} \right) \left(\frac{1}{\sqrt{2}} \sin \left(\frac{\sigma}{3} \right) \right. \\ & - v \cos \left(\frac{\sigma}{3} \right) \left. \right) - \sin \left(\frac{\sigma}{3} \right) \left(v \sin \left(\frac{\sigma}{3} \right) \right. \\ & + \frac{1}{\sqrt{2}} \cos \left(\frac{\sigma}{3} \right) \left. \right) - \frac{2}{3} \cos \left(\frac{\sigma}{3} \right), \frac{2}{3} \\ & \cdot \left(\sin \left(\frac{\sigma}{9} \right) - \sqrt{2} v \cos \left(\frac{\sigma}{9} \right) + \frac{1}{2\sqrt{2}} \right) \Bigg). \end{aligned} \quad (34)$$

The $\mu_2 B$ type-2 Smarandache ruled surface $\Psi(\sigma, v)$ is (see Figure 4)

$$\begin{aligned} \Psi(\sigma, v) = & \left(\frac{1}{3} \sin \left(\frac{\sigma}{3} \right) \left(\frac{1}{\sqrt{2}} \cos \left(\frac{\sigma}{9} \right) - v \sin \left(\frac{\sigma}{9} \right) \right) \right. \\ & - \cos \left(\frac{\sigma}{3} \right) \left(\frac{1}{\sqrt{2}} \sin \left(\frac{\sigma}{9} \right) + v \cos \left(\frac{\sigma}{9} \right) \right) \\ & + \frac{2}{3} \sin \left(\frac{\sigma}{3} \right), \frac{1}{3} \cos \left(\frac{\sigma}{3} \right) \left(v \sin \left(\frac{\sigma}{3} \right) \right. \\ & - \frac{1}{\sqrt{2}} \cos \left(\frac{\sigma}{3} \right) \left. \right) - \sin \left(\frac{\sigma}{3} \right) \left(\frac{1}{\sqrt{2}} \sin \left(\frac{\sigma}{3} \right) \right. \\ & + v \cos \left(\frac{\sigma}{3} \right) \left. \right) - \frac{2}{3} \cos \left(\frac{\sigma}{3} \right), \frac{2}{3} \\ & \cdot \left(\sqrt{2} v \sin \left(\frac{\sigma}{9} \right) - \cos \left(\frac{\sigma}{9} \right) + \frac{1}{2\sqrt{2}} \right) \Bigg). \end{aligned} \quad (35)$$

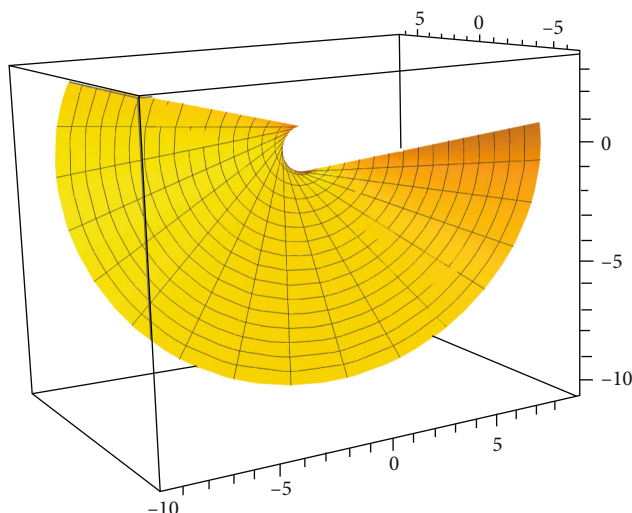


FIGURE 4: $\mu_2 B$ type-2 Smarandache ruled surface $\Psi(\sigma, \nu)$.

4. Conclusion

The study of ruled surfaces with different moving frames is one of the most interesting points of this paper. The researchers found that these surfaces could be developed in a minimal amount of time. In this work, we describe and study type-2 Smarandache ruled surfaces, which are a specific form of ruled surfaces. We create the essential and adequate circumstances for these surfaces to be developable in a minimal amount of time.

Data Availability

No data is used in this study.

Conflicts of Interest

The authors declare no competing interest.

Authors' Contributions

All authors have equal contributions and finalized the study.

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Research Article

Chen-Ricci Inequalities with a Quarter Symmetric Connection in Generalized Space Forms

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In this article, we obtain improved Chen-Ricci inequalities for submanifolds of generalized space forms with quarter-symmetric metric connection, with the help of which we completely characterized the Lagrangian submanifold in generalized complex space form and a Legendrian submanifold in a generalized Sasakian space form. We also discuss some geometric applications of the obtained results.

1. Introduction

One of the most basic problems in submanifold theory is to develop a simple relationship between the extrinsic invariants and the intrinsic invariants. The sectional curvature, the scalar curvature, and the Ricci curvature are the main intrinsic invariants while the squared mean curvature is the main extrinsic invariant.

Chen obtained the following important bound of the Ricci curvature Ric in terms of the mean curvature \mathcal{H} for Lagrangian submanifolds in complex space forms [1]:

$$\text{Ric} \leq (m-1)c + \frac{m^2}{4} \|\mathcal{H}\|^2, \quad (1)$$

where c is the constant holomorphic sectional curvature of the complex space form.

Further, he discussed the geometry of a Lagrangian submanifold satisfying the equality case of the inequality under the condition that the dimension of the kernel of the second fundamental form is constant. The inequality (1) is known as the Chen-Ricci inequality. This inequality attracted many researchers due to its geometric importance [2–12].

Deng [13] improved the above inequality as

$$\text{Ric}(U) \leq \frac{m-1}{4} (c + m \|\mathcal{H}\|^2). \quad (2)$$

In [14], Deng further extended his result for Lagrangian submanifolds in quaternion space forms. In [15], Tripathi improved the inequality in the case of curvature-like tensors. In [6], Mihai and Radulescu obtained the same relation in Sasakian space forms using semisymmetric connection as

$$\text{Ric}(U) + (m-2)\alpha(U, U) + \text{tr}\alpha \leq \frac{m-1}{4} (c + 3 + m \|\mathcal{H}\|^2). \quad (3)$$

As the curvature invariants are of great interest in theoretical physics (see [16]), the above studies motivate us to obtain a complete characterization of Lagrangian submanifold in generalized complex space form and a Legendrian submanifold in a generalized Sasakian space form.

2. Preliminaries

Let N be a Riemannian manifold and $\bar{\nabla}$ be a linear connection on N . Then, $\bar{\nabla}$ is said to be a semisymmetric connection if its torsion tensor T satisfies

$$T(U, V) = \pi(V)U - \pi(U)V, \quad (4)$$

for a 1-form π , then the connection $\bar{\nabla}$ is called a semisymmetric connection [17]. Let g be a Riemannian metric on N . If $\bar{\nabla}g = 0$, then $\bar{\nabla}$ is called a semisymmetric metric connection on N . The semisymmetric metric connection $\bar{\nabla}$ on N is given by

$$\bar{\nabla}_U V = \bar{\nabla}_U V + \pi(V)U - g(U, V)\Gamma, \quad (5)$$

for any U, V on N , where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to Riemannian metric g and Γ is a vector field. Further, $\bar{\nabla}$ is said to be a semisymmetric nonmetric connection if it satisfies

$$\bar{\nabla}_U V = \bar{\nabla}_U V + \pi(V)U. \quad (6)$$

Moreover, the linear connection $\bar{\nabla}$ on a Riemannian manifold N with Riemannian metric g is said to be a quarter-symmetric connection if its torsion tensor T is given by

$$T(U, V) = \bar{\nabla}_U V - \bar{\nabla}_V U - [U, V], \quad (7)$$

which satisfies

$$T(U, V) = \pi(V)\phi U - \pi(U)\phi V, \quad (8)$$

such that π is a 1-form given by

$$\pi(U) = g(U, \Gamma), \quad (9)$$

where Γ is a vector field and ϕ is a (1,1) tensor field.

Then, we can define a special quarter-symmetric connection by

$$\bar{\nabla}_U V = \bar{\nabla}_U V + \psi_1 \pi(V)U - \psi_2 g(U, V)\Gamma, \quad (10)$$

where ψ_1 and ψ_2 are real constants.

Remark 1. We notice from (5) that [18]

- (1) if $\psi_1 = \psi_2 = 1$, then a quarter symmetric connection becomes a semisymmetric metric connection
- (2) if $\psi_1 = 1$ and $\psi_2 = 0$, then a quarter-symmetric connection becomes a semisymmetric nonmetric connection

Remark 2. It is also worthy to mention here that the quarter symmetric connections generalized several well-known connections.

The curvature tensor \bar{R} with respect to $\bar{\nabla}$ is

$$\bar{R}(U, V)Z = \bar{\nabla}_U \bar{\nabla}_V Z - \bar{\nabla}_V \bar{\nabla}_U Z - \bar{\nabla}_{[U, V]} Z. \quad (11)$$

In the same way, we can also define the curvature tensor $\tilde{\bar{R}}$.

Let

$$\begin{aligned} \beta_1(U, V) &= (\bar{\nabla}_U \pi)(V) - \psi_1 \pi(U)\pi(V) + \frac{\psi_2}{2} g(U, V)\pi(\Gamma), \\ \beta_2(U, V) &= \frac{\pi(\Gamma)}{2} g(U, V) + \pi(U)\pi(V), \end{aligned} \quad (12)$$

are (0, 2) tensors. Then, the curvature tensor of N is given by [19]

$$\begin{aligned} \bar{R}(U, V, Z, W) &= \tilde{\bar{R}}(U, V, Z, W) + \psi_1 \beta_1(U, Z)g(V, W) \\ &\quad - \psi_1 \beta_1(V, Z)g(U, W) + \psi_2 \beta_1(V, W)g(U, Z) \\ &\quad - \psi_2 \beta_1(U, W)g(V, Z) + \psi_2 (\psi_1 - \psi_2) g(U, Z)\beta_2(V, W) \\ &\quad - \psi_2 (\psi_1 - \psi_2) g(V, Z)\beta_2(U, W). \end{aligned} \quad (13)$$

Let \mathcal{M} be an m -dimensional submanifold in a Riemannian manifold N . Let ∇ and $\tilde{\nabla}$ be the induced quarter symmetric-metric connection and Levi-Civita connection, respectively, on \mathcal{M} . Then, the Gauss formulas are

$$\begin{aligned} \bar{\nabla}_U V &= \nabla_U V + \zeta(U, V), \quad U, V \in \Gamma(T\mathcal{M}), \\ \bar{\nabla}_U V &= \nabla_U V + \tilde{\zeta}(U, V), \quad U, V \in \Gamma(T\mathcal{M}), \end{aligned} \quad (14)$$

where $\tilde{\zeta}$ is the second fundamental form that satisfies the relation

$$\zeta(U, V) = \tilde{\zeta}(U, V) - \psi_2 g(U, V)\Gamma^\perp, \quad (15)$$

where Γ^\perp is the normal component of the vector field Γ on \mathcal{M} .

Moreover, the equation of Gauss is defined by [19]

$$\begin{aligned} \bar{R}(U, V, Z, W) &= R(U, V, Z, W) - g(\zeta(U, W), \zeta(V, Z)) \\ &\quad + g(\zeta(V, W), \zeta(U, Z)) \\ &\quad + (\psi_1 - \psi_2) g(\zeta(V, Z), \Gamma^\perp) g(U, W) \\ &\quad + (\psi_2 - \psi_1) g(\zeta(U, Z), \Gamma^\perp) g(V, W). \end{aligned} \quad (16)$$

3. Characterization of Lagrangian Submanifold in Generalized Complex Space Form

A smooth manifold N endowed with an almost complex structure J and a Riemannian metric g that is compatible with J is called an almost Hermitian manifold. Further, for

the Levi-Civita connection ∇ if $\nabla J = 0$, then an almost Hermitian manifold is said to be a Kaehler manifold. A Kaehler manifold of constant holomorphic curvature is called a complex space form. The curvature tensor of a complex space form is given by

$$\begin{aligned}\tilde{R}(U, V, Z, W) = & \frac{c}{4} \{g(V, Z)g(U, W) - g(U, Z)g(V, W) \\ & + g(U, JZ)g(JV, W) - g(V, JZ)g(JU, W) \\ & + 2g(U, JV)g(JZ, W)\}.\end{aligned}\quad (17)$$

However, an almost Hermitian manifold N is called a generalized complex space form [20–22], denoted by $N(f_1, f_2)$, if for all vector fields U, V , and Z on N , the Riemannian curvature tensor \tilde{R} satisfies

$$\begin{aligned}\tilde{R}(U, V, Z, W) = & f_1 \{g(V, Z)g(U, W) - g(U, Z)g(V, W)\} \\ & + f_2 \{g(U, JZ)g(JV, W) - g(V, JZ)g(JU, W) \\ & + 2g(U, JV)g(JZ, W)\},\end{aligned}\quad (18)$$

where f_1 and f_2 are smooth functions on N .

In fact, we have following fundamental result from Tricerri and Vanhecke [20].

Theorem 3 (see [20]). *Let N be a connected almost Hermitian manifold with real dimension $2m > 6$ and Riemannian curvature \tilde{R} is of the form (18) such that f_2 is not identically zero. Then, N is a complex space form.*

Remark 4. From (18), we notice that if $f_1 = f_2 = c/4$, then we recover the complex space form.

From (13) and (18), we have

$$\begin{aligned}\bar{R}(U, V, Z, W) = & f_1 \{g(V, Z)g(U, W) - g(U, Z)g(V, W)\} \\ & + f_2 \{g(U, JZ)g(JV, W) - g(V, JZ)g(JU, W) \\ & + 2g(U, JV)g(JZ, W)\} + \psi_1 \beta_1(U, Z)g(V, W) \\ & - \psi_1 \beta_1(V, Z)g(U, W) + \psi_2 \beta_1(V, W)g(U, Z) \\ & - \psi_2 \beta_1(U, W)g(V, Z) + \psi_2(\psi_1 - \psi_2)g(U, Z)\beta_2(V, W) \\ & - \psi_2(\psi_1 - \psi_2)g(V, Z)\beta_2(U, W).\end{aligned}\quad (19)$$

Lemma 5 (see [13]). *Let $f_1(u_1, u_2, \dots, u_m)$ be a function on \mathbb{R}^m defined by*

$$f_1(u_1, u_2, \dots, u_m) = u_1 \sum_{j=2}^m u_j - \sum_{j=2}^m u_j^2. \quad (20)$$

If $u_1 + u_2 + \dots + u_m = 2ma$, then

$$f_1(u_1, u_2, \dots, u_m) \leq \frac{m-1}{4m} (u_1 + u_2 + \dots + u_m)^2, \quad (21)$$

and the equality holds if and only if $(1/(m+1))u_1 = u_2 = \dots = u_m = a$, where a is a constant.

Lemma 6 (see [13]). *Let $f_2(u_1, u_2, \dots, u_m)$ be a function on \mathbb{R}^m defined by*

$$f_2(u_1, u_2, \dots, u_m) = u_1 \sum_{j=2}^m u_j - u_1^2. \quad (22)$$

If $u_1 + u_2 + \dots + u_m = 4a$, then

$$f_2(u_1, u_2, \dots, u_m) \leq \frac{1}{8} (u_1 + u_2 + \dots + u_m)^2, \quad (23)$$

and the equality holds if and only if $u_1 = a$ and $u_2 + \dots + u_m = 3a$, where a is a constant.

Let M^m be an m -dimensional submanifold of an almost Hermitian manifold N . Then, M^m is said to be totally real if

$$J(T_p M^m) \subset T_p^\perp M^m. \quad (24)$$

Then, we have the following relations [23]:

$$\tilde{A}_{JU} V = \tilde{A}_{JV} U, \quad U, V \in T_p M, \quad (25)$$

or equivalently,

$$\tilde{\zeta}_{ij}^k = \tilde{A}_{ik}^j = \tilde{\zeta}_{jk}^i, \quad \forall i, j, k = 1, \dots, m, \quad (26)$$

where \tilde{A}^k is the shape operator with respect to $\bar{\nabla}$ and

$$\tilde{\zeta}_{ij}^k = g(\tilde{\zeta}(e_i, e_j), J e_k), \quad i, j, k = 1, \dots, m. \quad (27)$$

Remark 7. A totally real submanifold which is of maximal dimension is known as the Lagrangian submanifold [24].

Definition 8 (see [25]). A nontotally geodesic Lagrangian submanifold M^m of a complex space form $N^{2m}(4c)$ is called H -umbilical if its second fundamental form satisfies

$$\begin{aligned}h(e_1, e_i) &= \mu J e_n, h(e_i, e_m) = \mu J e_i, \quad i = 1 \dots m-1, \\ h(e_m, e_m) &= \lambda J e_m, h(e_i, e_j) = 0, \quad 1 \leq i \neq j \leq m-1,\end{aligned}\quad (28)$$

for some functions μ and λ with respect to an orthonormal frame $\{e_1, \dots, e_m\}$, where J is the complex structure of $N^{2m}(4c)$.

Theorem 9. *Let M^m be a totally real submanifold of maximal dimension $m(m \geq 2)$ in a connected complex space form $N(f_1, f_2)$ of dimension $2m$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m .*

Then, for any unit tangent vector U to M^m

$$\begin{aligned} \frac{m(m-1)}{4} \|\mathcal{H}\|^2 &\geq \text{Ric}(U) - f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U, U) \\ &\quad + \psi_2 \text{trace}\beta_1 - \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + \text{trace}\beta_2] \\ &\quad - (m-1)(\psi_1 - \psi_2)\pi(\zeta), \end{aligned} \quad (29)$$

and the equality holds in (29) identically if and only if either

- (1) M^m is totally geodesic, provided that $m > 2$, or
- (2) $m = 2$ and M^2 is a \mathcal{H} -umbilical Lagrangian surface with $\lambda = 3\mu$

Proof. As Γ is tangent to M^m , we have

$$\zeta = \tilde{\zeta}, \quad \mathcal{H} = \tilde{\mathcal{H}}. \quad (30)$$

Let us assume an orthonormal basis $\{e_1 = U, e_2, \dots, e_m\}$ $\subset T_p M^m$ and $\{e_{m+1} = Je_1, \dots, e_{2m} = Je_m\} \subset T_p^\perp M^m$ at point $p \in M^m$ with unit vector $U \in T_p M^m$. Then, by combining (16) and (19) and substituting $U = W = e_j$ and $V = Z = e_1$, for $j = 2, \dots, m$, we get

$$\begin{aligned} R(e_j, e_1, e_1, e_j) &= f_1 \{g(e_j, e_1)g(e_j, e_j) - g(e_j, e_1)g(e_1, e_j)\} \\ &\quad + f_2 \{g(e_j, Je_1)g(Je_1, e_j) - g(e_1, Je_1)g(Je_j, e_j) \\ &\quad + 2g(e_j, Je_1)g(Je_1, e_j)\} + \psi_1 \beta_1(e_j, e_1)g(e_1, e_j) \\ &\quad - \psi_1 \beta_1(e_1, e_1)g(e_j, e_j) + \psi_2 \beta_1(e_1, e_j)g(e_j, e_1) \\ &\quad - \psi_2 \beta_1(e_j, e_j)g(e_1, e_1) + \psi_2(\psi_1 - \psi_2)g(e_j, e_1)\beta_2(e_1, e_j) \\ &\quad - \psi_2(\psi_1 - \psi_2)g(e_1, e_1)\beta_2(e_j, e_j) + g(\zeta(e_j, e_j), \zeta(e_1, e_1)) \\ &\quad - g(\zeta(e_1, e_j), \zeta(e_j, e_1)) - (\psi_1 - \psi_2)g(\zeta(e_1, e_1), \Gamma^\perp)g(e_j, e_j) \\ &\quad - (\psi_2 - \psi_1)g(\zeta(e_j, e_1), \Gamma^\perp)g(e_1, e_j). \end{aligned} \quad (31)$$

Taking the summation over $j = 2, \dots, m$, we find

$$\begin{aligned} \text{Ric}(U) &= f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U, U) - \psi_2 \text{trace}\beta_1 \\ &\quad + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) - \text{trace}\beta_2] \\ &\quad - (m-1)(\psi_1 - \psi_2)\pi(\zeta) + \sum_{s=1}^m \sum_{j=2}^m \left(\zeta_{11}^s \zeta_{jj}^s - (\zeta_{1j}^s)^2 \right), \end{aligned} \quad (32)$$

which implies

$$\begin{aligned} \text{Ric}(U) - f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U, U) + \psi_2 \text{trace}\beta_1 \\ - \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) - \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ = \sum_{s=1}^m \sum_{j=2}^m \left(\zeta_{11}^s \zeta_{jj}^s - (\zeta_{1j}^s)^2 \right) \leq \sum_{s=1}^m \sum_{j=2}^m \zeta_{11}^s \zeta_{jj}^s - \sum_{j=2}^m (\zeta_{1j}^1)^2 - \sum_{j=2}^m (\zeta_{1j}^j)^2. \end{aligned} \quad (33)$$

From the above equation and (26), it is easy to see that

$$\begin{aligned} \text{Ric}(U) - f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U, U) + \psi_2 \text{trace}\beta_1 \\ - \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) - \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ \leq \sum_{s=1}^m \sum_{j=2}^m \zeta_{11}^s \zeta_{jj}^s - \sum_{j=2}^m (\zeta_{1j}^j)^2 - \sum_{j=2}^m (\zeta_{1j}^1)^2. \end{aligned} \quad (34)$$

Putting

$$\begin{aligned} f_1(\zeta_{11}^1, \zeta_{22}^1, \dots, \zeta_{mm}^1) &= \zeta_{11}^1 \sum_{j=2}^m \zeta_{jj}^1 - \sum_{j=2}^m (\zeta_{jj}^1)^2, \\ f_s(\zeta_{11}^s, \zeta_{22}^s, \dots, \zeta_{mm}^s) &= \zeta_{11}^s \sum_{j=2}^m \zeta_{jj}^s - (\zeta_{11}^s)^2, \quad \forall s = 2, \dots, m, \end{aligned} \quad (35)$$

and combining Lemma 5 with the relation $mH^1 = \zeta_{11}^1 + \zeta_{22}^1 + \dots + \zeta_{mm}^1$, we obtain

$$f_1(\zeta_{11}^1, \zeta_{22}^1, \dots, \zeta_{mm}^1) \leq \frac{m-1}{4m} (m\mathcal{H}^1)^2 = \frac{m(m-1)}{4} (\mathcal{H}^1)^2. \quad (36)$$

Then, by Lemma 6 for $s = 2, \dots, m$, we get

$$f_s(\zeta_{11}^s, \zeta_{22}^s, \dots, \zeta_{mm}^s) \leq \frac{1}{8} (m\mathcal{H}^s)^2 = \frac{m^2}{8} (\mathcal{H}^s)^2 \leq \frac{m(m-1)}{4} (\mathcal{H}^s)^2. \quad (37)$$

We find from (34), (36), and (37) that

$$\begin{aligned} \text{Ric}(U) &\leq f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U, U) - \psi_2 \text{trace}\beta_1 \\ &\quad + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) - \text{trace}\beta_2] \\ &\quad - (m-1)(\psi_1 - \psi_2)\pi(\zeta) + \frac{m(m-1)}{4} \|\mathcal{H}\|^2, \end{aligned} \quad (38)$$

which is the desired inequality (29). \square

Now, we discuss the equality cases.

Case 1. For $m > 2$, if $Je_1 \perp \mathcal{H}$, then

$$\mathcal{H}^s = 0, \quad (39)$$

for all $s > 1$. Therefore, using Lemma 6, we derive

$$\begin{aligned} \zeta_{1j}^1 &= \zeta_{11}^j = \frac{m\mathcal{H}^j}{4} = 0, \quad \text{for all } j > 1, \\ \zeta_{jk}^1 &= 0, \quad \text{for all } j, k > 1, j \neq k. \end{aligned} \quad (40)$$

Further, Lemma 5 yields

$$\zeta_{11}^1 = (m+1)\frac{H^1}{2}, \quad \zeta_{jj}^1 = \frac{H^1}{2}, \quad \text{for all } j > 1. \quad (41)$$

In (33), we see that $\text{Ric}(U) = \text{Ric}(e_1)$. In the same way, by deriving $\text{Ric}(e_2)$ and making use of the equality, we conclude that

$$\zeta_{2j}^s = \zeta_{js}^2 = 0, \quad \text{for all } s \neq 2, j \neq 2, s \neq j. \quad (42)$$

In consequence, we find

$$\frac{\zeta_{11}^2}{n+1} = \zeta_{22}^2 = \dots = \zeta_{mm}^2 = \frac{\mathcal{H}^2}{2} = 0. \quad (43)$$

We see that the equality holds for every unit tangent vectors. The above conclusion is also valid for (ζ_{jk}^s) . Thus,

$$\zeta_{2j}^2 = \zeta_{22}^j = \frac{\mathcal{H}^j}{2} = 0, \quad \forall j \geq 3. \quad (44)$$

Then, the only possible nonzero entries for (ζ_{jk}^2) (resp., for (ζ_{jk}^s)) are

$$\zeta_{12}^2 = \zeta_{21}^2 = \zeta_{22}^1 = \frac{\mathcal{H}^1}{2} \left(\text{respectively } \zeta_{1s}^s = \zeta_{s1}^s = \zeta_{ss}^1 = \frac{\mathcal{H}^1}{2}, \forall s \geq 3 \right). \quad (45)$$

Substituting $U = Z = e_2$ and $V = W = e_j, j = 2, \dots, m$ in (16), we derive

$$\tilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \left(\frac{\mathcal{H}^1}{2}\right)^2, \quad \forall j \geq 3. \quad (46)$$

On the other hand, if we substitute $U = Z = e_2$ and $V = W = e_1$ in (16), we get

$$\tilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - (m+1)\left(\frac{\mathcal{H}^1}{2}\right)^2 + \left(\frac{\mathcal{H}^1}{2}\right)^2. \quad (47)$$

Using (46) and (47), we find

$$\begin{aligned} \text{Ric}(e_2) - f_1(n-1) - [\psi_1 + \psi_2 - m\psi_1]\beta_1(e_2, e_2) + \psi_1 \text{trace}\beta_1 \\ - \psi_2(\psi_1 - \psi_2)[m\beta_2(e_2, e_2) - \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ = 2(m-1)\left(\frac{\mathcal{H}^1}{2}\right)^2. \end{aligned} \quad (48)$$

Moreover, the equality case of (29) implies that

$$\begin{aligned} \text{Ric}(e_2) - f_1(m-1) - [\psi_1 + \psi_2 - m\psi_1]\beta_1(U, U) + \psi_1 \text{trace}\beta_1 \\ - \psi_2(\psi_1 - \psi_2)[m\beta_2(U, U) - \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ = m(m-1)\left(\frac{\mathcal{H}^1}{2}\right)^2. \end{aligned} \quad (49)$$

Using the fact $m \neq 1, 2$, by (48) and (49), it is easy to see that $\mathcal{H}^1 = 0$. This implies that M^m is a totally geodesic in $N^{2m}(4c)$.

Case 2. In case $m = 2$, M^2 is nontotally geodesic, then $\zeta(e_1, e_1) = \lambda e_3, \zeta(e_2, e_2) = \mu e_3, \zeta(e_1, e_2) = \mu e_4$, together with $\lambda = 3\mu$. This proves that M^2 is \mathcal{H} -umbilical surface.

The above theorem gives the following results.

Corollary 10. Let M^m be a totally real submanifold of maximal dimension $m(m \geq 2)$ in a connected complex space form $N(f_1, f_2)$ of dimension $2m$ with a semisymmetric metric connection such that the vector field Γ is tangent to M^m . Then, for any unit tangent vector U to M^m

$$\text{Ric}(U) \leq \frac{m(m-1)}{4} \|\mathcal{H}\|^2 + f_1(m-1) + (2-m)\beta_1(U, U) - \text{trace}\beta_1, \quad (50)$$

and the equality holds in (50) identically if and only if either

- (1) M^m is totally geodesic, provided $m > 2$, or
- (2) $m = 2$ and M^2 is a \mathcal{H} -umbilical Lagrangian surface with $\lambda = 3\mu$

Proof. Using the fact $\psi_1 = \psi_2 = 1$ together with Theorem 9, the result directly follows. \square

Remark 11. It is worthy to mention here that Corollary 10 together with Remark 4 is the main result for complex case of the paper [26].

Corollary 12. Let M^m be a totally real submanifold of maximal dimension $m(m \geq 2)$ in a connected complex space form $N(f_1, f_2)$ of dimension $2m$ with a semisymmetric nonmetric connection such that the vector field Γ is tangent to M^m . Then, for any unit tangent vector U to M^m

$$\text{Ric}(U) \leq \frac{m(m-1)}{4} \|\mathcal{H}\|^2 + f_1(m-1) + (1-m)\beta_1(U, U) + (m-1)\pi(\zeta), \quad (51)$$

and the equality holds in (51) identically if and only if either

- (1) M^m is totally geodesic, provided $n > 2$, or
- (2) M^2 is a \mathcal{H} -umbilical Lagrangian surface with $\lambda = 3\mu$

Proof. Using the fact $\psi_1 = 1$ and $\psi_2 = 0$ together with Theorem 9, the result directly follows. \square

4. Characterization of Legendrian Submanifold in Generalized Sasakian Space Form

Let a $(2m+1)$ -dimensional almost contact metric manifold N^{2m+1} furnished with the almost complex structure (φ, ξ, η, g) , where φ is a $(1,1)$ tensor field, ξ is the structure vector field, η , the 1-form, and g is the Riemannian metric on N^{2m+1} . Then, following relations hold good:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V). \quad (52)$$

It also follows from the above relations that

$$\varphi\xi = 0, \quad \eta(\varphi U) = 0, \quad \eta(U) = g(U, \xi), \quad g(\varphi U, V) + g(U, \varphi V) = 0, \quad (53)$$

for all vector fields U, V on N .

Let $(N, \varphi, \xi, \eta, g)$ be an almost contact metric manifold whose curvature tensor satisfies [27]

$$\begin{aligned} \tilde{R}(U, V)Z &= f_1\{g(V, Z)U - g(U, Z)V\} + f_2\{g(U, \varphi Z)\varphi V \\ &\quad - g(V, \varphi Z)\varphi U + 2g(U, \varphi V)\varphi Z\} + f_3\{\eta(U)\eta(Z)V \\ &\quad - \eta(V)\eta(Z)U + g(U, Z)\eta(V)\xi - g(V, Z)\eta(U)\xi\}, \end{aligned} \quad (54)$$

for all vector fields U, V, Z on N , where f_1, f_2, f_3 are differentiable functions on N . Then, $N(f_1, f_2, f_3)$ is said to be a generalized Sasakian space form.

Remark 13. The generalized Sasakian space forms are [27]

- (1) Sasakian space forms if $f_1 = (c+3)/4, f_2 = f_3 = (c-1)/4$
- (2) Kenmotsu space forms if $f_1 = (c-3)/4$ and $f_2 = f_3 = (c+1)/4$
- (3) cosymplectic space forms if $f_1 = f_2 = f_3 = c/4$

From (13) and (54), we have

$$\begin{aligned} \tilde{R}(U, V)Z &= f_1\{g(V, Z)U - g(U, Z)V\} + f_2\{g(U, \varphi Z)\varphi V \\ &\quad - g(V, \varphi Z)\varphi U + 2g(U, \varphi V)\varphi Z\} + f_3\{\eta(U)\eta(Z)V \\ &\quad - \eta(V)\eta(Z)U + g(U, Z)\eta(V)\xi - g(V, Z)\eta(U)\xi\} \\ &\quad + \psi_1\beta_1(U, Z)g(V, W) - \psi_1\beta_1(V, Z)g(U, W) \\ &\quad + \psi_2\beta_1(V, W)g(U, Z) - \psi_2\beta_1(U, W)g(V, Z) \\ &\quad + \psi_2(\psi_1 - \psi_2)g(U, Z)\beta_2(V, W) - \psi_2(\psi_1 - \psi_2)g(V, Z)\beta_2(U, W). \end{aligned} \quad (55)$$

A submanifold M^m of an almost contact manifold N^{2n+1} normal to ξ is called a C-totally real submanifold. On such a submanifold, φ maps any tangent vector to M^m at $p \in M^m$

into the normal space $T_p^\perp M^m$. In particular, if $n = m$, i.e., M^m has maximum dimension, then it is a Legendrian submanifold. For a Legendrian submanifold M^m , if $\{e_1, \dots, e_m\}$ and $\{e_{m+1} = \varphi e_1, \dots, e_{2m} = \varphi e_m, e_{2m+1} = \xi\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on M^m . One has

$$\tilde{A}_{JU}V = \tilde{A}_{JV}U, \quad U, V \in T_p M, \quad (56)$$

or equivalently,

$$\tilde{\zeta}_{ij}^k = \tilde{A}_{ik}^j = \tilde{\zeta}_{jk}^i, \quad \forall i, j, k = 1, \dots, m, \quad (57)$$

where \tilde{A}^k is the shape operator with respect to \bar{V} and

$$\tilde{\zeta}_{ij}^k = g(\tilde{\zeta}(e_i, e_j), J e_k), \quad i, j, k = 1, \dots, m. \quad (58)$$

Definition 14 (see [28]). A nontotally geodesic Legendrian submanifold M^m of a Sasakian space form $N^{2m+1}(4c)$ is called H -umbilical if its second fundamental form satisfies

$$\begin{aligned} h(e_1, e_1) &= \lambda \phi e_1, \quad h(e_2, e_2) = \dots = h(e_m, e_m) = \mu \phi e_1, \\ h(e_1, e_j) &= \mu \phi e_j, \quad h(e_j, e_k) = 0, \quad j \neq k, j, k = 2, \dots, m, \end{aligned} \quad (59)$$

for some functions μ and λ with respect to an orthonormal frame $\{e_1, \dots, e_m\}$, where ϕ is the contact structure of $N^{2m+1}(4c)$.

Theorem 15. Let M^m be a totally real submanifold of maximal dimension $m(m \geq 2)$ in a generalized Sasakian space form $N(f_1, f_2, f_3)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m . Then, for any unit tangent vector U to M^m

$$\begin{aligned} \text{Ric}(U) &\leq \frac{m(m-1)}{4} \|\mathcal{H}\|^2 + f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U, U) \\ &\quad - \psi_2 \text{trace} \beta_1 + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + \text{trace} \beta_2] \\ &\quad + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \end{aligned} \quad (60)$$

and the equality holds in (60) identically if and only if either

- (1) M^m is totally geodesic, provided $m > 2$, or
- (2) $m = 2$ and M^2 is a \mathcal{H} -umbilical Legendrian surface with $\lambda = 3\mu$

Proof. As Γ is tangent to M^m , we have

$$\zeta = \tilde{\zeta}, \quad \mathcal{H} = \tilde{\mathcal{H}}. \quad (61)$$

Let us assume an orthonormal basis $\{e_1 = U, e_2, \dots, e_m\} \subset T_p M^m$ and $\{e_{m+1} = \varphi e_1, \dots, e_{2m} = \varphi e_m, e_{2m+1} = \xi\} \subset T_p^\perp M^m$ at point $p \in M^m$ with unit vector $U \in T_p M^m$. Then, by

combining (16) and (55) and substituting $U = W = e_j$ and $V = Z = e_1$ and summing over $j = 2, \dots, m$, we compute

$$\begin{aligned} \text{Ric}(U) &= f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U, U) - \psi_2 \text{trace}\beta_1 \\ &\quad + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) - \text{trace}\beta_2] \\ &\quad - (m-1)(\psi_1 - \psi_2)\pi(\zeta) + \sum_{s=1}^m \sum_{j=2}^m \left(\zeta_{11}^s \zeta_{jj}^s - (\zeta_{ij}^s)^2 \right). \end{aligned} \quad (62)$$

From (62) and (57), we deduce

$$\begin{aligned} \text{Ric}(U) &- f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U, U) + \psi_2 \text{trace}\beta_1 \\ &- \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) - \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ &\leq \sum_{s=1}^m \sum_{j=2}^m \zeta_{11}^s \zeta_{jj}^s - \sum_{j=2}^m (\zeta_{11}^j)^2 - \sum_{j=2}^m (\zeta_{jj}^1)^2. \end{aligned} \quad (63)$$

Putting

$$\begin{aligned} f_1(\zeta_{11}^1, \zeta_{22}^1, \dots, \zeta_{mm}^1) &= \zeta_{11}^1 \sum_{j=2}^m \zeta_{jj}^1 - \sum_{j=2}^m (\zeta_{jj}^1)^2, \\ f_s(\zeta_{11}^s, \zeta_{22}^s, \dots, \zeta_{mm}^s) &= \zeta_{11}^s \sum_{j=2}^m \zeta_{jj}^s - (\zeta_{11}^s)^2, \quad \forall s = 2, \dots, m, \end{aligned} \quad (64)$$

and by using the fact $m\mathcal{H}^1 = \zeta_{11}^1 + \zeta_{22}^1 + \dots + \zeta_{mm}^1$ together with the Lemma 5, we see that

$$f_1(\zeta_{11}^1, \zeta_{22}^1, \dots, \zeta_{mm}^1) \leq \frac{m-1}{4m} (m\mathcal{H}^1)^2 = \frac{m(m-1)}{4} (\mathcal{H}^1)^2. \quad (65)$$

Application of Lemma 6 for $s = 2, \dots, m$, gives

$$f_s(\zeta_{11}^s, \zeta_{22}^s, \dots, \zeta_{mm}^s) \leq \frac{1}{8} (m\mathcal{H}^s)^2 = \frac{m^2}{8} (\mathcal{H}^s)^2 \leq \frac{m(m-1)}{4} (\mathcal{H}^s)^2. \quad (66)$$

Equations (65), (67), and (68) yield the following relation

$$\begin{aligned} \text{Ric}(U) &- f_1(m-1) - [\psi_2 + \psi_1(1-m)]\beta_1(U, U) + \psi_2 \text{trace}\beta_1 \\ &- \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) - \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ &\leq \frac{m(m-1)}{4} \sum_{s=1}^m (\mathcal{H}^s)^2 = \frac{m(m-1)}{4} \|\mathcal{H}\|^2. \end{aligned} \quad (67)$$

Thus, we derive

$$\begin{aligned} \text{Ric}(U) &\leq f_1(m-1) + [\psi_2 + \psi_1(1-m)]\beta_1(U, U) - \psi_2 \text{trace}\beta_1 \\ &\quad + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) - \text{trace}\beta_2] \\ &\quad + (m-1)(\psi_1 - \psi_2)\pi(\zeta) + \frac{m(m-1)}{4} \|\mathcal{H}\|^2, \end{aligned} \quad (68)$$

which is the desired inequality (60). \square

Now, we discuss the equality cases.

Case 1. For $m > 2$, if $Je_1 \parallel \mathcal{H}$. Then,

$$\mathcal{H}^s = 0, \quad (69)$$

for all $s > 1$. Therefore, using Lemma 6, we derive

$$\zeta_{1j}^1 = \zeta_{11}^j = \frac{m\mathcal{H}^j}{4} = 0, \quad \text{for all } j > 1, \quad (70)$$

$$\zeta_{jk}^1 = 0, \quad \text{for all } j, k > 1, j \neq k. \quad (71)$$

Further, Lemma 5 yields

$$\zeta_{11}^1 = (m+1)\frac{H^1}{2}, \quad \zeta_{jj}^1 = \frac{H^1}{2}, \quad \text{for all } j > 1. \quad (72)$$

In (63), we see that $\text{Ric}(U) = \text{Ric}(e_1)$. In the same way, by deriving $\text{Ric}(e_2)$ and making use of the equality, we conclude that

$$\zeta_{2j}^s = \zeta_{js}^2 = 0, \quad \text{for all } s \neq 2, j \neq 2, s \neq j. \quad (73)$$

In consequence, we find

$$\frac{\zeta_{11}^2}{m+1} = \zeta_{22}^2 = \dots = \zeta_{mm}^2 = \frac{\mathcal{H}^2}{2} = 0. \quad (74)$$

We see that the equality holds for every unit tangent vectors. The above conclusion is also valid for (ζ_{jk}^s) . Thus,

$$\zeta_{2j}^2 = \zeta_{22}^j = \frac{\mathcal{H}^j}{2} = 0, \quad \forall j \geq 3. \quad (75)$$

Then, the only possible nonzero entries for (ζ_{jk}^2) (resp., for (ζ_{jk}^s)) are

$$\zeta_{12}^2 = \zeta_{21}^2 = \zeta_{22}^1 = \frac{\mathcal{H}^1}{2} \left(\text{resp. } \zeta_{1s}^s = \zeta_{s1}^s = \zeta_{ss}^1 = \frac{\mathcal{H}^1}{2}, \forall s \geq 3 \right). \quad (76)$$

Substituting $U = Z = e_2$ and $V = W = e_j, j = 2, \dots, m$ in

(16), we obtain

$$\tilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \left(\frac{\mathcal{H}^1}{2}\right)^2, \quad \forall j \geq 3. \quad (77)$$

On the other hand, if we put $U = Z = e_2$ and $V = W = e_1$ in (16), we get

$$\tilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - (m+1) \left(\frac{\mathcal{H}^1}{2}\right)^2 + \left(\frac{\mathcal{H}^1}{2}\right)^2. \quad (78)$$

From (77) and (78), it follows that

$$\begin{aligned} \text{Ric}(e_2) - f_1(m-1) - [\psi_1 + \psi_2 - m\psi_1]\beta_1(e_2, e_2) + \psi_1 \text{trace}\beta_1 \\ - \psi_2(\psi_1 - \psi_2)[m\beta_2(e_2, e_2) - \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ = 2(m-1) \left(\frac{\mathcal{H}^1}{2}\right)^2. \end{aligned} \quad (79)$$

Moreover, using the equality case of (29), we see that

$$\begin{aligned} \text{Ric}(e_2) - f_1(m-1) - [\psi_1 + \psi_2 - m\psi_1]\beta_1(U, U) + \psi_1 \text{trace}\beta_1 \\ - \psi_2(\psi_1 - \psi_2)[m\beta_2(U, U) - \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta) \\ = m(m-1) \left(\frac{\mathcal{H}^1}{2}\right)^2. \end{aligned} \quad (80)$$

Indeed $m \neq 1, 2$, with (81) and (84), we find $\mathcal{H}^1 = 0$. This implies that M^m is a totally geodesic in $N^{2m+1}(c)$.

Case 2. In the case $m = 2$, M^2 is nontotally geodesic, then $\zeta(e_1, e_1) = \lambda e_3$, $\zeta(e_2, e_2) = \mu e_3$, $\zeta(e_1, e_2) = \mu e_4$ together with $\lambda = 3\mu$. This proves that M^2 is \mathcal{H} -umbilical surface.

Remark 16. If we consider the structure vector field ξ tangential to the submanifold M , then we have the following result.

Theorem 17. Let M^{m+1} be a totally real submanifold of maximal dimension $m+1$ ($m \geq 2$) in a generalized Sasakian space form $N(f_1, f_2, f_3)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^{m+1} . Then, for any unit tangent vector U to M^{m+1} ,

$$\begin{aligned} \text{Ric}(U) \leq \frac{m(m+1)}{4} \|\mathcal{H}\|^2 + f_1 m - \{(m-1)\eta^2(e_1) + 1\}f_3 \\ + [\psi_2 + \psi_1 m]\beta_1(U, U) \\ - \psi_2 \text{trace}\beta_1 + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + \text{trace}\beta_2] \\ + m(\psi_1 - \psi_2)\pi(\zeta), \end{aligned} \quad (81)$$

and the equality holds in (81) identically if and only if either

- (1) M^{m+1} is totally geodesic, provided $m > 2$, or
- (2) $m = 2$ and M^2 is a \mathcal{H} -umbilical Legendrian surface with $\lambda = 3\mu$

Proof. We obtain the proof on the same lines of the proof for Theorem 15 additionally assuming an orthonormal basis

$$\{e_1 = U, e_2, \dots, e_m, e_{m+1}, e_{m+2}, \dots, e_{2m+1}\}, \quad (82)$$

such that $e_1, e_2, \dots, e_m, e_{m+1} \in T_p M$.

As a consequence of Theorem 15, we obtain the following results. \square

Corollary 18. Let M^m be a totally real submanifold of maximal dimension m ($m \geq 2$) in a Sasakian space form $N(c)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m . Then, for any unit tangent vector U to M^m

$$\begin{aligned} \text{Ric}(U) \leq \frac{m(m-1)}{4} \|\mathcal{H}\|^2 + \frac{c+3}{4}(m-1) \\ + [\psi_2 + \psi_1(1-m)]\beta_1(U, U) - \psi_2 \text{trace}\beta_1 \\ + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta), \end{aligned} \quad (83)$$

and the equality holds in (83) identically if and only if either

- (1) M^m is totally geodesic, provided $m > 2$, or
- (2) $m = 2$ and M^2 is a \mathcal{H} -umbilical Legendrian surface with $\lambda = 3\mu$

Proof. The proof follows immediately from Theorem 15 by putting $f_1 = (c+3)/4$, $f_2 = f_3 = (c-1)/4$. \square

Corollary 19. Let M^m be a totally real submanifold of maximal dimension m ($m \geq 2$) in a Kenmotsu space form $N(c)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m . Then, for any unit tangent vector U to M^m

$$\begin{aligned} \text{Ric}(U) \leq \frac{m(m-1)}{4} \|\mathcal{H}\|^2 + \frac{c-3}{4}(m-1) \\ + [\psi_2 + \psi_1(1-m)]\beta_1(U, U) - \psi_2 \text{trace}\beta_1 \\ + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + \text{trace}\beta_2] \\ + (m-1)(\psi_1 - \psi_2)\pi(\zeta), \end{aligned} \quad (84)$$

and the equality holds in (84) identically if and only if either

- (1) M^m is totally geodesic, provided $m > 2$, or
- (2) $m = 2$ and M^2 is a \mathcal{H} -umbilical Legendrian surface with $\lambda = 3\mu$

Proof. The proof follows immediately from Theorem 15 by replacing $f_1 = (c-3)/4$ and $f_2 = f_3 = (c+1)/4$. \square

Corollary 20. Let M^m be a totally real submanifold of maximal dimension $m(m \geq 2)$ in a cosymplectic space form $N(c)$ of dimension $2m+1$ with a quarter-symmetric metric connection such that the vector field Γ is tangent to M^m . Then, for any unit tangent vector U to M^m

$$\begin{aligned} \text{Ric}(U) \leq & \frac{m(m-1)}{4} \|\mathcal{R}\|^2 + \frac{c-3}{4}(m-1) \\ & + [\psi_2 + \psi_1(1-m)]\beta_1(U, U) - \psi_2 \text{trace}\beta_1 \\ & + \psi_2(\psi_1 - \psi_2)[\beta_2(U, U) + \text{trace}\beta_2] + (m-1)(\psi_1 - \psi_2)\pi(\zeta), \end{aligned} \quad (85)$$

and the equality holds in (85) identically if and only if either

- (1) M^m is totally geodesic, provided $m > 2$, or
- (2) $m = 2$ and M^2 is a \mathcal{H} -umbilical Legendrian surface with $\lambda = 3\mu$

Proof. The proof follows immediately from Theorem 15 by substituting $f_1 = f_2 = f_3 = c/4$. \square

Corollary 21. Let M^m be a totally real submanifold of maximal dimension $m(m \geq 2)$ in a generalized Sasakian space form $N(f_1, f_2, f_3)$ of dimension $2m+1$ with a semisymmetric metric connection such that the vector field Γ is tangent to M^m . Then, for any unit tangent vector U to M^m

$$\text{Ric}(U) \leq \frac{m(m-1)}{4} \|\mathcal{R}\|^2 + f_1(m-1) + (2-m)\beta_1(U, U) - \text{trace}\beta_1, \quad (86)$$

and the equality in (60) holds identically if and only if either

- (1) M^m is totally geodesic, provided $m > 2$, or
- (2) $m = 2$ and M^2 is a \mathcal{H} -umbilical Legendrian surface with $\lambda = 3\mu$

Proof. Using the fact that $\psi_1 = \psi_2 = 1$ together with Theorem 15, the result directly follows. \square

Remark 22. It is worthy to mention here that Corollary 21 together with Remark 13 (1) is the main result of the paper [6].

Corollary 23. Let M^m be a totally real submanifold of maximal dimension $m(m \geq 2)$ in a generalized Sasakian space form $N(f_1, f_2, f_3)$ of dimension $2m+1$ with a semisymmetric nonmetric connection such that the vector field Γ is tangent to

M^m . Then, for any unit tangent vector U to M^m

$$\text{Ric}(U) \leq \frac{m(m-1)}{4} \|\mathcal{R}\|^2 + f_1(m-1) + (1-m)\beta_1(U, U) + (m-1)\pi(\zeta), \quad (87)$$

and the equality in (87) holds identically if and only if either

- (1) M^m is totally geodesic, provided $m > 2$, or
- (2) $m = 2$ and M^2 is a \mathcal{H} -umbilical Legendrian surface with $\lambda = 3\mu$

Proof. Using the fact that $\psi_1 = 1$ and $\psi_2 = 0$ together with Theorem 15, the result directly follows. \square

Remark 24. All the above cases for Theorem 15 can be seen in the case of Theorem 17 as well.

Remark 25. Examples of totally geodesic submanifolds, H -umbilical Lagrangian submanifolds, and H -umbilical Legendrian submanifolds, i.e., examples of submanifolds attaining the equality case of the inequalities stated in this article, can be found in [25, 29, 30].

Data Availability

No data is used for the research

Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Research Article

A Study of Doubly Warped Product Immersions in a Nearly Trans-Sasakian Manifold with Slant Factor

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In this article, we discuss the de Rham cohomology class for bislant submanifolds in nearly trans-Sasakian manifolds. Moreover, we give a classification of warped product bislant submanifolds in nearly trans-Sasakian manifolds with some nontrivial examples in the support. Next, it is of great interest to prove that there does not exist any doubly warped product bislant submanifolds other than warped product bislant submanifolds in nearly trans-Sasakian manifolds. Some immediate consequences are also obtained.

1. Introduction and Motivations

The most inventive topic in the field of differential geometry currently is the theory of warped product manifolds. These manifolds are the most fruitful and natural generalization of Riemannian product manifolds. Due to the important roles of the warped product in mathematical physics and geometry, it has become the most active and interesting topic for researchers, and many nice results are available in the literature (see [1–3]).

Chen [4, 5] initiates the concept of warped product submanifolds by proving the nonexistence result of warped product CR-submanifolds of type $\mathcal{N}_\perp \times_f \mathcal{N}_T$ in Kähler manifolds, where \mathcal{N}_\perp and \mathcal{N}_T are anti-invariant and invariant submanifolds, respectively. Moreover, he considers warped product CR-submanifolds of type $\mathcal{N}_T \times_f \mathcal{N}_\perp$ and gives an inequality involving a warping function f and the squared norm of the second fundamental form $\|h\|^2$.

On the other hand, the concept of ordinary warped products can be extended to doubly warped products. By using this generalization, Sahin [6] shows that there exist no doubly warped product CR-submanifolds in Kähler manifolds other than warped product CR-submanifolds. He also investigates the existence of doubly twisted product CR-submanifolds in the same ambient. Many geometers have

obtained several results on warped products and doubly warped products [7–12].

The concept of bislant submanifolds is defined by Cabrerizo et al. [13] as the natural generalization of contact CR-, slant, and semislant submanifolds. Such submanifolds generalize invariant, anti-invariant, and pseudoslant submanifolds as well. Recently, the warped product bislant submanifolds in nearly trans-Sasakian manifolds is studied by Siddiqui et al. in [1]. They obtain several inequalities for the squared norm of the second fundamental form in terms of a warping function f .

In this paper, firstly, we discuss the de Rham cohomology class for closed bislant submanifolds in a nearly trans-Sasakian manifold. Secondly, in view of embedding theorem of Nash [14], we study an isometric immersion of a warped product bislant submanifold into an arbitrary nearly trans-Sasakian manifold. Then, we investigate the existence of doubly warped products in the same ambient.

2. Nearly Trans-Sasakian Manifolds and their Submanifolds

Definition 1 (see [15]). A $(2m+1)$ -dimensional differentiable manifold \mathcal{N} is said to have an almost contact structure (ϕ, ξ, η, g) if there exists on \mathcal{N} , where

- (i) a tensor field ϕ of type $(1, 1)$
- (ii) a vector field ξ
- (iii) a 1-form η
- (iv) a Riemannian metric g

such that

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi, \phi\xi = 0, \eta(\xi) = 1, \eta \circ \phi = 0, \eta(X) = g(X, \xi), \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), g(\phi X, Y) + g(X, \phi Y) = 0, \end{aligned} \quad (1)$$

for any $X, Y \in T\bar{\mathcal{N}}$.

The covariant derivative of the tensor field ϕ is given by

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \quad (2)$$

for any $X, Y \in T\bar{\mathcal{N}}$.

In 2000, Gherghe introduced a notion of nearly trans-Sasakian structure of type (α, β) , which generalizes the trans-Sasakian structure. A nearly trans-Sasakian structure of type (α, β) is called nearly α -Sasakian (resp. nearly β -Kenmotsu) if $\beta = 0$ (resp. $\alpha = 0$).

Definition 2 (see [16]). An almost contact metric structure (ϕ, ξ, η, g) on $\bar{\mathcal{N}}$ is called a nearly trans-Sasakian structure if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y), \quad (3)$$

for any $X, Y \in T\bar{\mathcal{N}}$.

Remark 3.

- (i) A nearly trans-Sasakian structure of type (α, β) is

- (a) *nearly Sasakian* if $\beta = 0, \alpha = 1$ [17]
- (b) *nearly Kenmotsu* if $\alpha = 0, \beta = 1$ [18]
- (c) *nearly cosymplectic* if $\alpha = \beta = 0$ [19]

- (ii) Remark that every Kenmotsu manifold is a nearly Kenmotsu manifold but the converse is not true. Also, a nearly Kenmotsu manifold is not a Sasakian manifold. On another hand, every nearly Sasakian manifold of dimension greater than five is a Sasakian manifold.

We put $\dim \mathcal{N} = n$ and $\dim \bar{\mathcal{N}} = 2m + 1$. The Riemannian metric for \mathcal{N} and $\bar{\mathcal{N}}$ is denoted by the same symbol g . Let $T\mathcal{N}$ and $T^\perp \mathcal{N}$ denote the Lie algebra of the vector field

and set of all normal vector fields on \mathcal{N} , respectively. The operator of covariant differentiation with respect to the Levi-Civita connection in \mathcal{N} and $\bar{\mathcal{N}}$ is denoted by ∇ and $\bar{\nabla}$, respectively. The Gauss and Weingarten formulae are respectively given as [15]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (4)$$

$$\bar{\nabla}_X \mathcal{V} = -A_{\mathcal{V}}(X) + \nabla_X^\perp Y, \quad (5)$$

for any $X, Y \in T\mathcal{N}$ and $\mathcal{V} \in T^\perp \mathcal{N}$. Here, h is the second fundamental form, A is the shape operator, and ∇^\perp is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T^\perp \mathcal{N}$.

The second fundamental form and the shape operator are related as [15]

$$g(h(X, Y), \mathcal{V}) = g(A_{\mathcal{V}}(X), Y), \quad (6)$$

for any $X, Y \in T\mathcal{N}$ and $\mathcal{V} \in T^\perp \mathcal{N}$. Here, g denote the induced metric on \mathcal{N} as well as the Riemannian metric on $\bar{\mathcal{N}}$.

Let $x \in \mathcal{N}$ and $\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$ be a local orthonormal frame of $T_x \mathcal{N}$ and $\{\mathcal{E}_{n+1}, \dots, \mathcal{E}_{2m+1}\}$ be a local orthonormal frame of $T_x^\perp \mathcal{N}$. The mean curvature vector \mathcal{H} of a submanifold \mathcal{N} at x is given by [15]

$$\mathcal{H} = \frac{1}{n} \sum_{i=1}^n h(\mathcal{E}_i, \mathcal{E}_i). \quad (7)$$

A submanifold \mathcal{N} of $\bar{\mathcal{N}}$ is said to be [15]

- (i) *totally umbilical* if $h(X, Y) = g(X, Y)\mathcal{H}$, for any $X, Y \in T\mathcal{N}$
- (ii) *totally geodesic* if $h(X, Y) = 0$, for any $X, Y \in T\mathcal{N}$
- (iii) *minimal* if $\mathcal{H} = 0$, that is, $\text{trace } h \equiv 0$

For any $X \in T\mathcal{N}$, we put [15]

$$\phi X = PX + FX, \quad (8)$$

where $PX = \text{tangent}(\phi X)$ and $FX = \text{normal}(\phi X)$. Then P is an endomorphism of $T\mathcal{N}$, and F is the normal bundle valued 1-form on $T\mathcal{N}$. In the same way, for any $\mathcal{V} \in T^\perp \mathcal{N}$, we put [15]

$$\phi \mathcal{V} = \mathbf{B}\mathcal{V} + \mathcal{C}\mathcal{V}, \quad (9)$$

where $\mathbf{B}\mathcal{V} = \text{tangent}(\phi \mathcal{V})$ and $\mathcal{C}\mathcal{V} = \text{normal}(\phi \mathcal{V})$. It is easy to see that P and \mathcal{C} are skew-symmetric and

$$g(FX, \mathcal{V}) = -g(X, \mathbf{B}\mathcal{V}), \quad (10)$$

for any $X \in T\mathcal{N}$ and $\mathcal{V} \in T^\perp \mathcal{N}$.

Definition 4. A submanifold \mathcal{N} of an almost contact metric manifold $\bar{\mathcal{N}}$ is said to be invariant if $F \equiv 0$, that is, $\phi X \in T$

\mathcal{N} , and anti-invariant if $P \equiv 0$, that is, $\phi X \in T^\perp \mathcal{N}$, for any $X \in T\mathcal{N}$.

In contact geometry, Lotta introduced slant immersions as follows [20].

Definition 5. Let \mathcal{N} be a submanifold of an almost contact metric manifold $\bar{\mathcal{N}}$. For each nonzero vector $X \in T_x \mathcal{N} - \{\xi_x\}$ and $x \in \mathcal{N}$, the angle $\theta(p) \in [0, \pi/2]$ between ϕX and PX is called slant angle of \mathcal{N} . If slant angle is constant for each $X \in T_x \mathcal{N} - \{\xi_x\}$, then the submanifold is called the slant submanifold.

For slant submanifolds, the following facts are known:

$$\begin{aligned} P^2(X) &= \cos^2 \theta (-X + \eta(X)\xi), \\ g(PX, PY) &= \cos^2 \theta (g(X, Y) - \eta(Y)\eta(X)), \end{aligned} \quad (11)$$

$$g(FX, FY) = \sin^2 \theta (g(X, Y) - \eta(Y)\eta(X)), \quad (12)$$

for any $X, Y \in T\mathcal{N}$. Here, θ is slant angle of \mathcal{N} .

Remark 6. If we assume

- (i) $\theta = 0$, then \mathcal{N} is an *invariant submanifold*
- (ii) $\theta = \pi/2$, then \mathcal{N} is an *anti-invariant submanifold*
- (iii) $\theta(p) \in (0, \pi/2)$, then \mathcal{N} is a *proper slant submanifold*

There are some other important classes of submanifolds which are determined by the behavior of tangent bundle of the submanifold under the action of an almost contact metric structure ϕ of $\bar{\mathcal{N}}$ [1]:

- (i) A submanifold \mathcal{N} of $\bar{\mathcal{N}}$ is called a *contact CR-submanifold* of $\bar{\mathcal{N}}$ if there exists a differentiable distribution D on \mathcal{N} whose orthogonal complementary distribution D^\perp is anti-invariant
- (ii) A submanifold \mathcal{N} of $\bar{\mathcal{N}}$ is called a *semislant submanifold* of $\bar{\mathcal{N}}$ if there exists a pair of orthogonal distributions D and D_θ such that D is invariant and D_θ is proper slant
- (iii) A submanifold \mathcal{N} of $\bar{\mathcal{N}}$ is called *pseudoslant submanifold* of $\bar{\mathcal{N}}$ if there exists a pair of orthogonal distributions D^\perp and D_θ such that D^\perp is anti-invariant and D_θ is proper slant

Definition 7 (see [13]). A submanifold \mathcal{N} of an almost contact metric manifold $\bar{\mathcal{M}}$ is said to be a bislant submanifold if there exists a pair of orthogonal distributions D_{θ_1} and D_{θ_2} of

\mathcal{N} such that

$$T\mathcal{N} = D_{\theta_1} \oplus D_{\theta_2} \oplus \{\xi\}. \quad (13)$$

- (i) $PD_{\theta_1} \perp D_{\theta_2}$ and $PD_{\theta_2} \perp D_{\theta_1}$
- (ii) Each distribution D_{θ_i} is slant with slant angle θ_i for $i = 1, 2$

Remark 8. If we assume

- (i) $\theta_1 = 0$ and $\theta_2 = \pi/2$, then \mathcal{N} is a *CR-submanifold*
- (ii) $\theta_1 = 0$ and $\theta_2 \neq 0, \pi/2$, then \mathcal{N} is a *semislant submanifold*
- (iii) $\theta_1 = \pi/2$ and $\theta_2 \neq 0, \pi/2$, then \mathcal{N} is a *pseudoslant submanifold*
- (iv) $\theta_1, \theta_2 \in (0, \pi/2)$, then \mathcal{N} is a *proper bislant submanifold*

For a bislant submanifold \mathcal{N} of an almost contact metric manifold, the normal bundle of \mathcal{N} is decomposed as

$$T^\perp \mathcal{N} = FD_{\theta_1} \oplus FD_{\theta_2} \oplus \mu, \quad (14)$$

where μ is a ϕ -invariant normal subbundle of \mathcal{N} .

3. Cohomology Class for Bislant Submanifolds of Nearly Trans-Sasakian Manifolds

Chen [21] introduces a canonical de Rham cohomology class for a closed CR-submanifold in a Kähler manifold. So, in this section, we discuss the de Rham cohomology class for a closed bislant submanifold of a nearly trans-Sasakian manifold $(\bar{\mathcal{N}}, \phi, \xi, \eta, g)$ with minimal horizontal distribution $(D_{\theta_1} \oplus \{\xi\})$. We put $\dim(\mathcal{N}) = m$ and $\dim(D_{\theta_1} \oplus \{\xi\}) = 2a + 1$. Let us assume the following:

- (i) $\{\mathcal{E}_1, \dots, \mathcal{E}_a, \mathcal{E}_{a+1} = \sec \theta_1 P\mathcal{E}_1, \dots, \mathcal{E}_{2a} = \sec \theta_1 P\mathcal{E}_a, \mathcal{E}_{2a+1} = \xi, \mathcal{E}_{2a+2} = \mathcal{E}_1^*, \dots, \mathcal{E}_{2a+b+1} = \mathcal{E}_b^*, \mathcal{E}_{2a+b+2} = \mathcal{E}_{b+1}^* = \sec \theta_2 P\mathcal{E}_1^*, \dots, \mathcal{E}_m = \mathcal{E}_{2a+2b+1} = \mathcal{E}_{2b}^* = \sec \theta_2 P\mathcal{E}_b^*\}$ is a local orthonormal frame of \mathcal{N}
- (ii) $\{\mathcal{E}_1, \dots, \mathcal{E}_a, \mathcal{E}_{a+1} = \sec \theta_1 P\mathcal{E}_1, \dots, \mathcal{E}_{2a} = \sec \theta_1 P\mathcal{E}_a, \mathcal{E}_{2a+1} = \xi\}$ is a local orthonormal frame of $(D_{\theta_1} \oplus \{\xi\})$
- (iii) $\{\mathcal{E}_{2a+2} = \mathcal{E}_1^*, \dots, \mathcal{E}_{2a+b+1} = \mathcal{E}_b^*, \mathcal{E}_{2a+b+2} = \mathcal{E}_{b+1}^* = \sec \theta_2 P\mathcal{E}_1^*, \dots, \mathcal{E}_m = \mathcal{E}_{2a+2b+1} = \mathcal{E}_{2b}^* = \sec \theta_2 P\mathcal{E}_b^*\}$ is a local orthonormal frame of D_{θ_2}

We choose $\zeta^1, \dots, \zeta^{2a+1}, \zeta^{2a+2}, \dots, \zeta^m$ as the dual frame of 1-forms to the above local orthonormal frame. Then, we define a $(2a + 1)$ -form $\bar{\omega}$ on \mathcal{N} by $\bar{\omega} = \zeta^1 \wedge \zeta^2 \wedge \dots \wedge \zeta^{2a+1}$. It is globally defined on \mathcal{N} . In the same way, we again define

a $(m - 2a - 1)$ -form Ω on \mathcal{N} by $\Omega = \zeta^{2a+2} \wedge \zeta^{2a+3} \wedge \cdots \wedge \zeta^m$, which is globally defined on \mathcal{N} .

We prepare some preliminary lemmas.

Lemma 9. *Let \mathcal{N} be a submanifold of an arbitrary nearly trans-Sasakian manifold $\bar{\mathcal{N}}$, then*

$$\begin{aligned} \nabla_X PY - A_{FY}X - P\nabla_Y X - 2Bh(X, Y) + \nabla_Y PX - A_{FX}Y - P\nabla_X Y \\ = \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)PX + \eta(X)PY), \end{aligned} \quad (15)$$

$$\begin{aligned} h(X, PY) + \nabla_X^\perp FY - F\nabla_X Y - 2\mathcal{C}h(X, Y) + h(Y, PX) + \nabla_Y^\perp FX - F\nabla_Y X \\ = -\beta(\eta(Y)FX + \eta(X)FY), \end{aligned} \quad (16)$$

for any $X, Y \in T\mathcal{N}$.

Proof. For any vector fields $X, Y \in T\mathcal{N}$, making use of the structure equation and (2), we obtain

$$\begin{aligned} \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y + \bar{\nabla}_Y \phi X - \phi \bar{\nabla}_Y X = \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) \\ - \beta(\eta(Y)\phi X + \eta(X)\phi Y), \end{aligned} \quad (17)$$

which gives

$$\begin{aligned} \nabla_X PY + h(PY, X) - A_{FY}X + \nabla_X^\perp FY - P\nabla_X Y - 2Bh(X, Y) \\ - F\nabla_X Y - 2\mathcal{C}h(X, Y) + \nabla_X PY + h(PX, Y) - A_{FX}Y \\ + \nabla_Y^\perp FX - P\nabla_Y X - F\nabla_Y X \\ = \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) \\ - \beta(\eta(Y)PX + \eta(X)PY + \eta(Y)FX + \eta(X)FY). \end{aligned} \quad (18)$$

Comparing the tangential and normal components of the above equation, we get the desired relations (15) and (16).

The next lemma gives the integrability condition of slant distribution D_{θ_2} . \square

Lemma 10. *Let \mathcal{N} be a bislant submanifold of an arbitrary nearly trans-Sasakian manifold $(\bar{\mathcal{N}}, \phi, \xi, \eta, g)$. Then, slant distribution D_{θ_2} is integrable if and only if*

$$-2F\nabla_Y X + h(X, PY) + h(Y, PX) - 2\mathcal{C}h(X, Y) + \nabla_X^\perp FY + \nabla_Y^\perp FX \in FD_{\theta_2}, \quad (19)$$

for any $X, Y \in D_{\theta_2}$.

Proof. Making use of Lemma 9, we obtain

$$\begin{aligned} g(F[X, Y], FZ) = -2\{g(F\nabla_Y X, FZ) + g(h(X, PY), FZ) \\ + g(h(Y, PX), FZ) - g(2\mathcal{C}h(X, Y), FZ) \\ + g(\nabla_X^\perp FY, FZ) + g(\nabla_Y^\perp FX, FZ)\}, \end{aligned} \quad (20)$$

for any $Z \in (D_{\theta_1} \oplus \{\xi\})$. Thus, the assertion follows from the fact that FD_{θ_1} and FD_{θ_2} are mutually perpendicular. In this way, we proved the integrability condition of slant distribution D_{θ_2} . \square

We prove the following.

Theorem 11. *For any closed bislant submanifold \mathcal{N} of an arbitrary nearly trans-Sasakian manifold $(\bar{\mathcal{N}}, \phi, \xi, \eta, g)$ with minimal $(D_{\theta_1} \oplus \{\xi\})$ and*

$$-2F\nabla_Y X + h(X, PY) + h(Y, PX) - 2\mathcal{C}h(X, Y) + \nabla_X^\perp FY + \nabla_Y^\perp FX \in FD_{\theta_2}, \quad (21)$$

for any $X, Y \in D_{\theta_2}$, the $(2a + 1)$ -form $\bar{\omega}$ is closed and defines a canonical de Rham cohomology class $[\bar{\omega}] \in H^{2a+1}(\mathcal{N}, \mathbb{R})$, where $\dim(D_{\theta_1} \oplus \{\xi\}) = 2a + 1$.

Moreover, the cohomology group $H^{2a+1}(\mathcal{N}, \mathbb{R})$ is nontrivial if D_{θ_2} is minimal and $(D_{\theta_1} \oplus \{\xi\})$ is integrable.

Proof. From the definition of $\bar{\omega}$, we have $d\bar{\omega} = \sum_{i=1}^{2a+1} (-1)^{i-1} \zeta^1 \wedge \cdots \wedge d\zeta^i \wedge \cdots \wedge \zeta^{2a+1}$, which implies that $d\bar{\omega} = 0$ if and only if

$$d\bar{\omega}(X_2, Y_2, X_1, \dots, X_{2a}) = 0, \quad (22)$$

$$d\bar{\omega}(X_2, X_1, \dots, X_{2a+1}) = 0, \quad (23)$$

for any $X_2, Y_2 \in D_{\theta_2}$ and $X_1, \dots, X_{2a+1} \in (D_{\theta_1} \oplus \{\xi\})$. Thus, by simple computation, we find that (22) is satisfied if and only if D_{θ_2} is integrable. On the other hand, (23) is satisfied if and only if $(D_{\theta_1} \oplus \{\xi\})$ is minimal. However, the integrability condition of D_{θ_2} holds due to Lemma 10, and by the hypothesis of the theorem, we have $(D_{\theta_1} \oplus \{\xi\})$ is minimal. Hence, the form $\bar{\omega}$ is closed. It defines a canonical de Rham cohomology class $[\bar{\omega}] \in H^{2a+1}(\mathcal{N}, \mathbb{R})$.

Next, we prove that the cohomology class $[\bar{\omega}]$ is nontrivial. Since D_{θ_2} is minimal and $(D_{\theta_1} \oplus \{\xi\})$ is integrable, then in this case, we need to show that $\bar{\omega}$ is harmonic. By definition of Ω and the similar argument for $\bar{\omega}$, we see that $d\Omega = 0$, that is, Ω is closed, if $(D_{\theta_1} \oplus \{\xi\})$ is integrable and D_{θ_2} is minimal. This further proves that $\delta\bar{\omega} = 0$, that is, $\bar{\omega}$ is coclosed. From $d\bar{\omega} = 0$, $\delta\bar{\omega} = 0$, and \mathcal{N} is a closed submanifold, we deduce that $\bar{\omega}$ is harmonic $(2a + 1)$ -form. Hence, the cohomology group $H^{2a+1}(\mathcal{N}, \mathbb{R})$ is nontrivial if D_{θ_2} is minimal and $(D_{\theta_1} \oplus \{\xi\})$ is integrable. \square

4. Warped Product Bislant Submanifolds

Definition 12 (see [22]). Let (\mathcal{N}_1, g_1) and (\mathcal{N}_2, g_2) be two Riemannian manifolds and $f > 0$ be a differentiable function on \mathcal{N}_1 . Consider two projections on $\mathcal{N}_1 \times \mathcal{N}_2$, $\rho : \mathcal{N}_1 \times \mathcal{N}_2 \longrightarrow \mathcal{N}_1$ and $\delta : \mathcal{N}_1 \times \mathcal{N}_2 \longrightarrow \mathcal{N}_2$. The projection maps given by $\rho(p, q) = p$ and $\delta(p, q) = q$ for $(p, q) \in \mathcal{N}_1 \times \mathcal{N}_2$. Then, the warped product $\mathcal{N} = \mathcal{N}_1 \times_f \mathcal{N}_2$ is the product

manifold $\mathcal{N}_1 \times \mathcal{N}_2$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\rho_* X, \rho_* Y) + (f \circ \rho)^2 g_2(\delta_* X, \delta_* Y), \quad (24)$$

for any $X, Y \in T\mathcal{N}$, where $*$ is the symbol for the tangent maps. The function f is called the *warping function* of \mathcal{N} .

Example 13. A surface of revolution is a warped product manifold.

Example 14. The standard space-time models of the universe are warped products as the simplest models of neighbourhoods of stars and black holes.

Remark 15. In particular, a warped product manifold is said to be trivial if its warping function is constant. In such a case, we call the warped product manifold a Riemannian product manifold. If $\mathcal{N} = \mathcal{N}_1 \times_f \mathcal{N}_2$ is a warped product manifold, then \mathcal{N}_1 is totally geodesic and \mathcal{N}_2 is totally umbilical submanifold of \mathcal{N} [22].

Let $\mathcal{N} = \mathcal{N}_1 \times_f \mathcal{N}_2$ be a warped product manifold with a warping function f . Then,

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \quad (25)$$

for any $X \in T\mathcal{N}_1$ and $Z \in T\mathcal{N}_2$, where $\nabla \ln f$ is the gradient of $\ln f$ and ∇ and $\nabla^{\mathcal{N}_2}$ denote the Levi-Civita connections on \mathcal{N} and \mathcal{N}_2 , respectively.

The definition of warped product bislant submanifolds in a nearly trans-Sasakian manifold is as follows.

Definition 16. A warped product $\mathcal{N}_1 \times_f \mathcal{N}_2$ of two slant submanifolds \mathcal{N}_1 and \mathcal{N}_2 of a nearly trans-Sasakian manifold $\bar{\mathcal{N}}$ is called a warped product bislant submanifold.

Remark 17. A warped product bislant submanifold $\mathcal{N}_1 \times_f \mathcal{N}_2$ is called proper if \mathcal{N}_1 and \mathcal{N}_2 are proper slant in $\bar{\mathcal{N}}$. Otherwise, the warped product bislant submanifold $\mathcal{N}_1 \times_f \mathcal{N}_2$ is called nonproper.

For a warped product bislant submanifold in a nearly trans-Sasakian manifold such that $\xi \in T\mathcal{N}_1$, we have the following result.

Theorem 18. Let $\mathcal{N} = \mathcal{N}_1 \times_f \mathcal{N}_2$ be a warped product bislant submanifold with bislant angles $\{\theta_1, \theta_2\}$ in a nearly trans-Sasakian manifold $\bar{\mathcal{N}}$ such that $\xi \in T\mathcal{N}_1$. If, for any $X_1 \in T\mathcal{N}_1$ and $X_2, Y_2 \in T\mathcal{N}_2$,

$$g(h(X_1, X_2), FY_2) = g(h(X_1, Y_2), FX_2), \quad (26)$$

holds, then one of the following cases must occur:

(i) \mathcal{N} is a warped product pseudoslant submanifold such that \mathcal{N}_2 is a totally real submanifold \mathcal{N}^\perp of $\bar{\mathcal{N}}$

(ii) If $\bar{\mathcal{N}}$ is nearly Sasakian manifold, that is, $\beta = 0$, then \mathcal{N} is a Riemannian product

(iii) If $\beta \neq 0$, then $\beta\eta(X_1) = -(X_1 \ln f)$

Proof. For any vector fields $X_1 \in T\mathcal{N}_1$ and $X_2, Y_2 \in T\mathcal{N}_2$, we have

$$\begin{aligned} g(h(X_1, X_2), FY_2) &= g(\bar{\nabla}_{X_1} X_2, \phi Y_2) - g(\nabla_{X_1} X_2, PY_2) \\ &= g((\bar{\nabla}_{X_1} \phi)X_2, Y_2) - g(\bar{\nabla}_{X_1} \phi X_2, Y_2) \\ &\quad - (X_1 \ln f)g(X_2, PY_2). \end{aligned} \quad (27)$$

On the other hand, we have

$$\begin{aligned} g(h(X_1, X_2), FY_2) &= g(\bar{\nabla}_{X_2} X_1, \phi Y_2) - g(\nabla_{X_2} X_1, PY_2) \\ &= g((\bar{\nabla}_{X_2} \phi)X_1, Y_2) - g(\bar{\nabla}_{X_2} \phi X_1, Y_2) \\ &\quad - (X_1 \ln f)g(X_2, PY_2). \end{aligned} \quad (28)$$

By adding (27) and (28), we get

$$\begin{aligned} 2g(h(X_1, X_2), FY_2) &= g(h(X_1, Y_2), FX_2) + g(h(X_2, Y_2), FX_1) \\ &\quad - (P \ln f)g(X_2, Y_2) - (X_1 \ln f)g(X_2, PY_2) \\ &\quad - \alpha\eta(X_1)g(X_2, Y_2) + \beta\eta(X_1)g(PX_2, Y_2). \end{aligned} \quad (29)$$

Interchanging X_2 by Y_2 in (29), we find

$$\begin{aligned} 2g(h(X_1, Y_2), FX_2) &= g(h(X_1, X_2), FY_2) + g(h(X_2, Y_2), FX_1) \\ &\quad - (P \ln f)g(X_2, Y_2) - (X_1 \ln f)g(Y_2, PX_2) \\ &\quad - \alpha\eta(X_1)g(X_2, Y_2) + \beta\eta(X_1)g(PY_2, X_2). \end{aligned} \quad (30)$$

By subtracting (30) from (29) and by applying our assumption, we obtain

$$g(PX_2, Y_2)[(X_1 \ln f) + \beta\eta(X_1)] = 0. \quad (31)$$

For $Y_2 = PY_2$, we get

$$\cos^2 \theta_2 g(Y_2, X_2)[(X_1 \ln f) + \beta\eta(X_1)] = 0. \quad (32)$$

From the last expression, any one of the following holds: if $\beta = 0$, then f is constant, or if $\beta \neq 0$, then $\beta\eta(X_1) = -(X_1 \ln f)$ or $\theta_2 = \pi/2$. Thus, our assertions follow.

Now, we have the following theorem for a warped product bislant submanifold in a nearly trans-Sasakian manifold such that $\xi \in T\mathcal{N}_2$. \square

Theorem 19. Let $\mathcal{N} = \mathcal{N}_1 \times_f \mathcal{N}_2$ be a warped product bislant submanifold with bislant angles $\{\theta_1, \theta_2\}$ in a nearly trans-Sasakian manifold $\bar{\mathcal{N}}$ such that $\xi \in T\mathcal{N}_2$. If, for any $X_1 \in T\mathcal{N}_1$ and $X_2, Y_2 \in T\mathcal{N}_2$,

$$g(h(X_1, X_2), FY_2) = g(h(X_1, Y_2), FX_2), \quad (33)$$

holds, then one of the following cases must occur:

- (i) \mathcal{N} is a warped product pseudoslant submanifold such that \mathcal{N}_2 is a totally real submanifold \mathcal{N}^\perp of $\bar{\mathcal{N}}$
- (ii) \mathcal{N} is a Riemannian product

Proof. For any vector fields $X_1 \in T\mathcal{N}_1$ and $X_2, Y_2 \in T\mathcal{N}_2$, we have

$$\begin{aligned} g(h(X_1, X_2), FY_2) &= g(\bar{\nabla}_{X_1} X_2, \phi Y_2) - g(\nabla_{X_1} X_2, PY_2) \\ &= g((\bar{\nabla}_{X_1} \phi)X_2, Y_2) - g(\bar{\nabla}_{X_1} \phi X_2, Y_2). \end{aligned} \quad (34)$$

On the other hand, we have

$$\begin{aligned} g(h(X_1, X_2), FY_2) &= g(\bar{\nabla}_{X_2} X_1, \phi Y_2) - g(\nabla_{X_2} X_1, PY_2) \\ &= g((\bar{\nabla}_{X_2} \phi)X_1, Y_2) - g(\bar{\nabla}_{X_2} \phi X_1, Y_2). \end{aligned} \quad (35)$$

By adding (34) and (35), we get

$$\begin{aligned} 2g(h(X_1, X_2), FY_2) &= g(h(X_1, Y_2), FX_2) + g(h(X_2, Y_2), FX_1) \\ &\quad - (\text{Plnf})g(X_2, Y_2) - (X_1 \ln f)g(X_2, PY_2). \end{aligned} \quad (36)$$

Interchanging X_2 by Y_2 in (36), we find

$$\begin{aligned} 2g(h(X_1, Y_2), FX_2) &= g(h(X_1, X_2), FY_2) + g(h(X_2, Y_2), FX_1) \\ &\quad - (\text{Plnf})g(X_2, Y_2) - (X_1 \ln f)g(Y_2, PX_2). \end{aligned} \quad (37)$$

By subtracting (37) from (36) and by applying our assumption, we obtain

$$(X_1 \ln f)g(PX_2, Y_2) = 0. \quad (38)$$

For $Y_2 = PY_2$, we get

$$\cos^2 \theta_2 (X_1 \ln f)[g(Y_2, X_2) - \eta(X_2)\eta(Y_2)] = 0. \quad (39)$$

Therefore, either f is constant or $\cos \theta_2 = 0$ holds. Consequently, either \mathcal{N} is a Riemannian product or $\theta_2 = \pi/2$. In the latter case, \mathcal{N} is a warped product pseudoslant submanifold. \square

We give some nontrivial examples of warped product bislant submanifold of the form $\mathcal{N} = \mathcal{N}_\theta \times_f \mathcal{N}_\perp$ whose

bislant angles $\theta_1 \neq 0, \pi/2$ and $\theta_2 = \pi/2$. Such warped product bislant submanifolds are called pseudoslant submanifolds.

Example 20. Let \mathbb{C}^4 be the complex Euclidean space with its usual Kähler structure and the real global coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$ and $\bar{\mathcal{N}} = \mathbb{R} \times_f \mathbb{C}^4$ be a warped product manifold between the product real line of \mathbb{R} and the complex space \mathbb{C}^4 . Let \langle, \rangle be the Euclidean metric tensor of \mathbb{R}^9 . An almost contact structure ϕ of $\bar{\mathcal{N}}$ is defined by

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \phi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \phi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \leq i, j \leq 4 \quad (40)$$

such that

$$\xi = e^t \left(\frac{\partial}{\partial t}\right), \eta = e^t dt, g = e^t \langle, \rangle. \quad (41)$$

On the other hand, we define a submanifold \mathcal{N} by immersion g as follows:

$$g(u, v, w, s, t) = (u, v, 0, 0, v \cos r, v \sin r, s \cos w, s \sin w, t). \quad (42)$$

Therefore, it is easy to choose tangent bundle of \mathcal{N} which is spanned by the following:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1}, X_2 = \cos r \frac{\partial}{\partial y_1} + \sin r \frac{\partial}{\partial y_2}, \\ X_3 &= \cos w \frac{\partial}{\partial y_3} + \sin w \frac{\partial}{\partial y_4}, X_5 = \frac{\partial}{\partial z}. \end{aligned} \quad (43)$$

Thus, $D_{\theta_1} = \text{Span}\{X_1, X_2\}$ is a slant distribution with slant angle $\pi/4$. Also, it is easy to verify that $D_{\theta_2} = \text{Span}\{X_3, X_4\}$ is a totally real distribution. Hence, the submanifold \mathcal{N} defined by f is a bislant submanifold, which is tangent to the structure vector ξ and whose bislant angles satisfy $\theta_1 \neq 0, \pi/2$ and $\theta_2 = \pi/2$. It is easy to check that the distributions D_{θ_1} and D_{θ_2} are integrable. Then, it can be verified that $\mathcal{N} = \mathcal{N}_\theta \times_f \mathcal{N}_\perp$ is a warped product bislant submanifold of $\bar{\mathcal{N}}$ with warping function $f = e^t, t \in \mathbb{R}$.

Example 21. We consider any submanifold \mathcal{N} in a nearly trans-Sasakian manifold \mathbb{R}^7

$$\tilde{f}(u, v, w, q) = (u \cos v, w \cos v, u \sin v, w \sin v, w - u, w + u, q). \quad (44)$$

The tangent bundle of \mathcal{N} is spanned by

$$\begin{aligned}\mathcal{E}_1 &= \cos v \frac{\partial}{\partial x_1} + \sin v \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3}, \\ \mathcal{E}_2 &= -u \sin v \frac{\partial}{\partial x_1} + u \cos v \frac{\partial}{\partial x_2} - w \sin v \frac{\partial}{\partial y_1} + w \cos v \frac{\partial}{\partial y_2}, \\ \mathcal{E}_3 &= \frac{\partial}{\partial x_3} + \cos v \frac{\partial}{\partial y_1} + \sin v \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}, \\ \mathcal{E}_4 &= \frac{\partial}{\partial q}.\end{aligned}\quad (45)$$

Furthermore, we have

$$\begin{aligned}\phi \mathcal{E}_1 &= \cos v \frac{\partial}{\partial y_1} + \sin v \frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} - \frac{\partial}{\partial x_3}, \\ \phi \mathcal{E}_2 &= -u \sin v \frac{\partial}{\partial y_1} + u \cos v \frac{\partial}{\partial y_2} + w \sin v \frac{\partial}{\partial x_1} - w \cos v \frac{\partial}{\partial x_2}, \\ \phi \mathcal{E}_3 &= \frac{\partial}{\partial y_3} - \cos v \frac{\partial}{\partial x_1} - \sin v \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}, \\ \phi \mathcal{E}_4 &= 0.\end{aligned}\quad (46)$$

It is easy to check that $\phi \mathcal{E}_2$ is orthogonal to $T\mathcal{N}$. Then, the proper slant and anti-invariant distributions of \mathcal{N} are respectively defined by $D_\theta = \text{Span}\{\mathcal{E}_1, \mathcal{E}_3\}$ with slant angle $\theta = \arccos(1/3)$ and $D_\perp = \text{Span}\{\mathcal{E}_2\}$. Also, $\mathcal{E}_4 = \xi$ is tangent to D_θ . Hence, \mathfrak{f} defines a proper 4-dimensional pseudoslant submanifold (bislant submanifold with bislant angles $\{\arccos(1/3), \pi/2\}$) \mathcal{N} in \mathbb{R}^7 . It is easy to check that the distributions $D_\theta \oplus \{\xi\}$ and D_\perp are integrable.

Now, we assume that \mathcal{N}_θ and \mathcal{N}_\perp are the integral manifolds of D_θ and D_\perp , respectively. Then, it follows from Definition 12 and (44) that the induced metric tensor g of \mathcal{N} is given by

$$\begin{aligned}g &= (\cos^2 v + \sin^2 v + 2)du^2 + (u^2 \sin^2 v + u^2 \cos^2 v + w^2 \sin^2 v + w^2 \cos^2 v)dv^2 \\ &\quad + (\cos^2 v + \sin^2 v + 2)dw^2 + dq^2 = 3(du^2 + dw^2) + dq^2 + (u^2 + w^2)dv^2 \\ &= g_1 + g_2,\end{aligned}\quad (47)$$

where $g_1 = 3(du^2 + dw^2) + dq^2$ and $g_2 = (u^2 + w^2)dv^2$ are respectively the metric tensors of \mathcal{N}_θ and \mathcal{N}_\perp . As a consequence, $\mathcal{N} = \mathcal{N}_\theta \times_{\mathfrak{f}} \mathcal{N}_\perp$ is a warped product pseudoslant submanifold of \mathbb{R}^7 with a warping function, that is, $f = \sqrt{u^2 + w^2}$ such that ξ is tangent to \mathcal{N}_θ .

5. Doubly Warped Product Bislant Submanifolds

In general, doubly warped products can be considered as a generalization of warped products.

Definition 22 (see [23, 24]). Let (\mathcal{N}_1, g_1) and (\mathcal{N}_2, g_2) be Riemannian manifolds. A doubly warped product (\mathcal{N}, g) is a product manifold which is of the form $\mathcal{N} = {}_{f_2}\mathcal{N}_1 \times_{f_1}\mathcal{N}_2$ with the metric $g = f_1^2 g_1 \oplus f_2^2 g_2$, where $f_1 : \mathcal{N}_1 \times \mathcal{N}_2 \rightarrow (0, \infty)$ and $f_2 : \mathcal{N}_1 \times \mathcal{N}_2 \rightarrow (0, \infty)$ are smooth maps. More precisely, if $\rho : \mathcal{N}_1 \times \mathcal{N}_2 \rightarrow \mathcal{N}_1$ and $\delta : \mathcal{N}_1 \times \mathcal{N}_2 \rightarrow \mathcal{N}_2$ are natural projections, the metric g is defined by

$$g(X, Y) = (f_2 \circ \delta)^2 g_1(\rho_* X, \rho_* Y) + (f_1 \circ \rho)^2 g_2(\delta_* X, \delta_* Y), \quad (48)$$

for any $X, Y \in T\mathcal{N}$, where $*$ is the symbol for the tangent maps. The functions f_1 and f_2 are called the *warping functions* of \mathcal{N} .

Remark 23. If we assume

- (i) either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we obtain a *warped product*
- (ii) both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a *product manifold*
- (iii) neither f_1 nor f_2 is constant, then we have a non-trivial *doubly warped product*

For doubly warped product manifold $\mathcal{N} = {}_{f_2}\mathcal{N}_1 \times_{f_1}\mathcal{N}_2$ with warping functions f and g , we have the following:

$$\nabla_Y X = \nabla_X Y = (Y \ln f_1)X + (X \ln f_2)Y, \quad (49)$$

for any $X \in T\mathcal{N}_1$ and $Y \in T\mathcal{N}_2$.

Now, we define the notion of doubly warped product bislant submanifolds in nearly trans-Sasakian manifolds as follows.

Definition 24. The doubly warped product of two slant submanifolds, ${}_{f_2}\mathcal{N}_1 \times_{f_1}\mathcal{N}_2$, is called the doubly warped product bislant submanifold of slant submanifolds \mathcal{N}_1 and \mathcal{N}_2 with slant angles θ_1 and θ_2 , respectively, of a nearly trans-Sasakian manifold with warping functions f_1 and f_2 if only depend on the points of \mathcal{N}_1 and \mathcal{N}_2 , respectively.

First we have the following theorem for doubly warped product submanifolds $\mathcal{N} = {}_{f_2}\mathcal{N}_1 \times_{f_1}\mathcal{N}_2$ in nearly trans-Sasakian manifolds such that $\xi \in T\mathcal{N}_1$.

Theorem 25. Let $\mathcal{N} = {}_{f_2}\mathcal{N}_1 \times_{f_1}\mathcal{N}_2$ be a doubly warped product submanifold in a nearly trans-Sasakian manifold $\bar{\mathcal{N}}$, where \mathcal{N}_1 and \mathcal{N}_2 are Riemannian submanifolds of $\bar{\mathcal{N}}$ and $\xi \in T\mathcal{N}_1$. Then, \mathcal{N} is a warped product bislant submanifold of type $\mathcal{N}_1 \times_{f_1}\mathcal{N}_2$ if and only if

$$g(h(X, Y), FX) = g(h(X, X), FY), \quad (50)$$

for any $X \in T\mathcal{N}_1$ and $Y \in T\mathcal{N}_2$.

Proof. From Lemma 9, we get

$$\begin{aligned} \nabla_X PY - A_{FY}X - P\nabla_Y X - 2\mathbf{B}h(X, Y) + \nabla_Y PX - A_{FX}Y - P\nabla_X Y \\ = -\alpha\eta(X)Y - \beta\eta(X)PY, \end{aligned} \quad (51)$$

for any $X \in T\mathcal{N}_1$ and $Y \in T\mathcal{N}_2$. Applying (49), we derive

$$\begin{aligned} (PY \ln f_2)X - (Y \ln f_2)PX - (X \ln f_1)PY + (PX \ln f_1)Y \\ - A_{FY}X - 2\mathbf{B}h(X, Y) - A_{FX}Y = -\alpha\eta(X)Y - \beta\eta(X)PY. \end{aligned} \quad (52)$$

Taking the inner product with $X \in T\mathcal{N}_1$, we obtain

$$(PY \ln f_2)\|X\|^2 - g(h(X, X), FY) - 2g(\mathbf{B}h(X, Y), X) - g(h(Y, X), FX) = 0. \quad (53)$$

Using relation (10) in the above equation, we get

$$(PY \ln f_2)\|X\|^2 = g(h(X, X), FY) - g(h(Y, X), FX) = 0. \quad (54)$$

Thus, from (54), we conclude that $(PY \ln f_2) = 0$ if and only if

$$g(h(X, Y), FX) = g(h(X, X), FY), \quad (55)$$

for any $X \in T\mathcal{N}_1$ and $Y \in T\mathcal{N}_2$. $(PY \ln f_2) = 0$ shows that f_2 is constant, that is, f_2 depends only on the points of \mathcal{N}_1 . Thus, it follows that \mathcal{N} is a warped product bislant submanifold of type $\mathcal{N}_1 \times_{f_1} \mathcal{N}_2$. This proves the theorem completely. \square

Secondly, we prove the following theorem for doubly warped product bislant submanifolds $\mathcal{N} =_{f_2} \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ in nearly trans-Sasakian manifolds such that $\xi \in T\mathcal{N}_2$.

Theorem 26. Let $\mathcal{N} =_{f_2} \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ be a doubly warped product bislant submanifold in a nearly trans-Sasakian manifold $\bar{\mathcal{N}}$, where \mathcal{N}_1 and \mathcal{N}_2 are proper slant submanifolds with respect to θ_1 and θ_2 , respectively, and $\xi \in T\mathcal{N}_2$. Then, \mathcal{N} is a warped product bislant submanifold of type $\mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ if and only if

$$g(h(X, Y), FY) = g(h(Y, Y), FX), \quad (56)$$

for any $X \in T\mathcal{N}_2$ and $Y \in T\mathcal{N}_1$.

Proof. For any vector fields $X \in T\mathcal{N}_2$ and $Y \in T\mathcal{N}_1$, we have

$$\begin{aligned} g(h(PX, Y), FY) &= g(\bar{\nabla}_Y PX, \phi Y) = -g(\phi \bar{\nabla}_Y PX, Y) \\ &= g((\bar{\nabla}_Y \phi)PX, Y) - g(\bar{\nabla}_Y \phi PX, Y) \\ &= -g((\bar{\nabla}_Y \phi)Y, PX) - g(\bar{\nabla}_Y P^2 X, Y) - g(\bar{\nabla}_Y \mathcal{F}PX, Y) \\ &= \cos^2 \theta_1 g(\nabla_Y X, Y) + g(h(Y, Y), \mathcal{F}PX) \\ &= \cos^2 \theta_1 (X \ln f_2)\|Y\|^2 + g(h(Y, Y), \mathcal{F}PX). \end{aligned} \quad (57)$$

Replacing X by PX in the last relation, we obtain

$$(PX \ln f_2)\|Y\|^2 = g(h(Y, Y), FX) - g(h(X, Y), FY). \quad (58)$$

Thus, from (54), we conclude that $(PX \ln f_2) = 0$ if and only if

$$g(h(Y, Y), FX) = g(h(X, Y), FY), \quad (59)$$

for any $X \in T\mathcal{N}_2$ and $Y \in T\mathcal{N}_1$.

$(PX \ln f_2) = 0$ implies that f_2 is constant, that is, f_2 depends only on the points of \mathcal{N}_1 . Hence, \mathcal{N} is a warped product bislant submanifold of type $\mathcal{N}_1 \times_{f_1} \mathcal{N}_2$. This proves the theorem completely. \square

6. Conclusion

From Theorems 25 and 26, we conclude that there exist no doubly warped product bislant submanifolds in nearly trans-Sasakian manifolds, other than warped product bislant submanifolds, under some additional conditions.

7. Some Applications of Theorem 25 for Different Kinds of Ambient Manifolds

Let $\mathcal{N} =_{f_2} \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ be a doubly warped product submanifold, where \mathcal{N}_1 and \mathcal{N}_2 are Riemannian submanifolds of $\bar{\mathcal{N}}$ and $\xi \in T\mathcal{N}_1$. The following corollaries are the immediate consequences of Theorem 25.

Corollary 27. There does not exist any doubly warped product submanifold $\mathcal{N} =_{f_2} \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ in a nearly Sasakian manifold $\bar{\mathcal{N}}$, other than the warped product bislant submanifold, if and only if (50) holds.

Corollary 28. There does not exist a doubly warped product submanifold $\mathcal{N} =_{f_2} \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ in a nearly Kenmotsu manifold $\bar{\mathcal{N}}$, other than the warped product bislant submanifold, if and only if (50) holds.

Corollary 29. There does not exist a doubly warped product submanifold $\mathcal{N} =_{f_2} \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ in a nearly cosymplectic manifold $\bar{\mathcal{N}}$, other than the warped product bislant submanifold, if and only if (50) holds.

8. Some Applications of Theorem 26 for Different Kinds of Ambient Manifolds

Let $\mathcal{N} = f_2 \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ be a doubly warped product bislant submanifold, where \mathcal{N}_1 and \mathcal{N}_2 are proper slant submanifolds with respect to θ_1 and θ_2 , respectively, and $\xi \in T\mathcal{N}_2$. The following corollaries are the immediate consequences of Theorem 26.

Corollary 30. *There is no doubly warped product bislant submanifold $\mathcal{N} = f_2 \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ in a nearly Sasakian manifold $\bar{\mathcal{N}}$, other than the warped product bislant submanifold, if and only if (56) holds.*

Corollary 31. *There is no doubly warped product bislant submanifold $\mathcal{N} = f_2 \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ in a nearly Kenmotsu manifold $\bar{\mathcal{N}}$, other than the warped product bislant submanifold, if and only if (56) holds.*

Corollary 32. *There is no doubly warped product bislant submanifold $\mathcal{N} = f_2 \mathcal{N}_1 \times_{f_1} \mathcal{N}_2$ in a nearly cosymplectic manifold $\bar{\mathcal{N}}$, other than the warped product bislant submanifold, if and only if (56) holds.*

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors declare no competing of interest.

Authors' Contributions

All authors have equal contribution and finalized.

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Research Article

Characterization of Lagrangian Submanifolds by Geometric Inequalities in Complex Space Forms

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In this paper, we give an estimate of the first eigenvalue of the Laplace operator on a Lagrangian submanifold M^n minimally immersed in a complex space form. We provide sufficient conditions for a Lagrangian minimal submanifold in a complex space form with Ricci curvature bound to be isometric to a standard sphere S^n . We also obtain Simons-type inequality for same ambient space form.

1. Introduction

In the last few years, there has been attention to the classification of Lagrangian submanifolds. Lagrangian submanifolds give an impression being of foliations in the cotangent bundle, and Hamilton-Jacobi type leads to the classification via partial differential equation. In differential geometry of submanifolds, theorems which connect the intrinsic and extrinsic curvatures have significant role in physics [1]. Moreover, the notion of second order differential equations (PDEs) has built essential contribution in the analyze problems in fluid mechanics, heat conduction in solids, diffusive transport of chemicals in porous media, and wave propagation in strings and in mechanics or solids. The eigenvalue problems are trying to obtain all possible real λ such that there exists a nontrivial solution φ to the second order partial differential equation (PDEs) $\Delta\varphi + \lambda\varphi = 0$ [2, 3]. On the other hand, the Ricci tensor is involving in the curvature space-time, which finds the degree where matter will incline to converge or diverge in time (via the Raychaudhuri equation). By means of the Einstein field equation, it is also correlated to the matter content of the universe. In Riemannian geometry, on a Riemannian manifold, lower bounds of the Ricci tensor grant one to right geometric and topological understanding with the notion of a constant curvature space form. In Einstein manifold, the Ricci tensor verifies the vacuum Einstein

equation, which have been broadly studied in [4]. In this relation, the Ricci flow equation supervises the working out of a given metric to an Einstein metric. Similarly, the eigenvalue problems are fascinating topics in differential geometry which has physical background. Therefore, a distinguished problem in Riemannian geometry is to find isometrics on a given manifold. One of the most interesting geometries of Riemannian manifolds is to characterize complex space form in the framework of Lagrangian submanifold geometry among the classes of compact, connected Riemannian manifolds. Beginning from the originate work of Obata [5], differential equation has become an influential tool in the investigation of geometric analysis. Obata [5] tested characterizing theorem for the standard sphere. A complete manifold (M^n, g) yields function φ which is nonconstant and gratifying the ordinary differential equation

$$\nabla^2\varphi + \varphi g = 0, \quad (1)$$

if and only if (M^n, g) is isometric the sphere S^n . A large scale of observations has been dedicated to this subject, and therefore, characterization of spaces, the Euclidean space \mathbb{R}^n , the Euclidean sphere S^n , and the complex projective space $\mathbb{C}P^n$ are esteemed fields in differential geometry and are studied by a number of authors [2, 6–26]. Similarly, Tashiro [27] has proved that the Euclidean space \mathbb{R}^n is designated through

the differential equation $\nabla^2 \varphi = c\varphi$, where c is a positive constant. In [28], Lichnerowicz has been classified that the first nonzero eigenvalue μ_1 of the Laplacian on a compact manifold (M^n, g) with $\text{Ric} \geq n-1$ is not less than n , while $\mu_1 = n$, then (M^n, g) is isometric to the sphere \mathbb{S}^n . This means that the Obata's rigidity theorem could be used to analyze the equality case of Lichnerowicz's eigenvalue estimates in [28]. Motivated from previously studied and historical development on such characterizations, we give our first result as the following.

Theorem 1. *Let $\Psi : M^n \longrightarrow \tilde{M}^n(4c)$ be a minimal immersion of a compact Lagrangian submanifold into complex space form $\tilde{M}^n(4c)$. If the Laplacian of M^n endowed to the first eigenvalue μ_1 corresponding eigenfunction φ , then the following inequality holds*

$$\int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV + c \int_M |\nabla \varphi|^2 dV \geq \int_M |\nabla^2 \varphi|^2 dV, \quad (2)$$

where $|\nabla^2 \varphi|^2$ denotes the norm of the Hessian of φ and $\{e_1, \dots, e_n\}$ is frame on M^n which is orthonormal. The equality holds if and only if $\mu_1 = nc$. Besides, if the inequality holds

$$\int_M |\nabla^2 \varphi|^2 dV \geq \int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV. \quad (3)$$

Then, we have $\mu_1 \geq c(n-1)$. In particular, if the following inequality satisfying

$$\int_M |\nabla \varphi|^2 dV \geq \frac{nc}{\mu_1} \int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV. \quad (4)$$

Then, eigenvalue is satisfied $\mu_1 \geq c(n-1)$.

By considered that compact submanifold M^n immersed in the Euclidean sphere \mathbb{S}^{n+p} or Euclidean space \mathbb{R}^{n+p} , Jiancheng and Zhang [29] derived the Simons-type [30] inequalities about the first eigenvalue μ_1 and the squared norm of the second fundamental form S without using the condition that submanifold M is minimal. They also established a lower bound for S if it is constant. Similar results can be found in [4, 31]. Simon's inequalities and its corollary motivate the mathematicians try to improve the estimate the upper bound of S and study the rigidity of associated submanifolds. As a generalization of Euclidean sphere and Euclidean spaces, we consider a Lagrangian submanifold which minimally immersed into complex space form with constant holomorphic sectional curvature $4c$; we obtain our next result as the following.

Theorem 2. *Let $\Psi : M^n \longrightarrow \tilde{M}^n(4c)$ be a minimal immersion of a compact Lagrangian submanifold into the complex space form $\tilde{M}^n(4c)$. If $\dim \text{Ker } h = k$, then we have*

$$\int_M S |\nabla^2 \varphi|^2 dV \geq \left\{ \frac{(n-k)(nc-1)(nc-\mu_1)}{(n-k-1)nc} \right\} \int_M |\nabla \varphi|^2 dV. \quad (5)$$

In circumstantial, if S is constant, then it is equal to

$$S \geq \frac{(n-k)(nc-1)}{nc(n-k-1)(nc-\mu_1)}. \quad (6)$$

A greatly motivated idea of Obata is associated to characterizing sphere $\mathbb{S}^n(c)$ through the second-order differential equation (1). By using the techniques of conformal vector field which have prominent appearance in deriving characterizations of spaces but also have high-level geometry in the theory of relativity and mechanics, Deshmukh and Al-Solamy [32] proved that an n -dimensional compact connected Riemannian manifold whose Ricci curvature satisfied the bound $0 < \text{Ric} \leq (n-1)(2 - nc/\mu_1)c$ for a constant c and μ_1 is the first nonzero eigenvalue of the Laplace operator; then, M^n is isometric to $\mathbb{S}^n(c)$ if M^n admitted a nonzero conformal gradient vector field. They also proved that if M^n is Einstein manifold such that Einstein constant $\mu = (n-1)c$, then M^n is isometric to $\mathbb{S}^n(c)$ with $c > 0$ if it is admitted conformal gradient vector field. Taking account of Obata equation (1), Barros et al. [20] shows that a compact gradient almost Ricci soliton $(M^n, g, \nabla \varphi, \lambda)$ is isometric to a Euclidean sphere whose Ricci tensor is Codazzi and has constant sectional curvature. For more terminology of Obata equation, see [14]. In the sequel, inspired by ideas are developed in [4, 29, 30]. So we give our result.

Theorem 3. *Let $\Psi : M^n \longrightarrow \tilde{M}^n(4c)$ be a minimal immersion of a compact Lagrangian submanifold into the complex space form $\tilde{M}^n(4c)$ and φ a first eigenfunction associated to the Laplacian of M^n . Then, we have the following:*

- (i) *If $\nabla \varphi$ on $\text{Ker } h$, then $\Psi(M^n)$ is locally geodesic sphere \mathbb{S}^n , or $\Psi(M^n)$ is isometric to standard sphere \mathbb{S}^n*
- (ii) *If $\text{Ric}_{M^n}(\nabla \varphi, \nabla \varphi) \geq c(n-1)|\nabla \varphi|^2$, then $\Psi(M^n)$ is isometric to a sphere \mathbb{S}^n*

The paper is organize as follows: In Section 2, we recall some preliminary formulas related to our study. Moreover, we also prove a proposition in this section which helps to derive our main results. In Section 3, we give the proofs of our theorems which we proposed in the first section. Finally, in Section 4, we provided some consequences of main results.

2. Preliminaries and Notations

Let $\tilde{M}(4c)$ be a complex space form of constant holomorphic sectional curvature $4c$ and of complex dimension m . Then, the curvature tensor R of $\tilde{M}^m(4c)$ can be expressed as:

$$R(U, V)Z = c(g(U, Z)V - g(V, Z)U + g(U, JZ)JV - g(V, JZ)U + 2g(U, JV)JZ), \quad (7)$$

for any $U, V, Z \in \Gamma(T\tilde{M})$ [7, 33]. An n -dimensional Riemannian submanifold M^n of $\tilde{M}(4c)$ is classified as totally real if the standard complex structure J of $\tilde{M}(4c)$ maps any tangent space of M^n into the corresponding normal space [34]. In particular, a totally real submanifold is said to be a Lagrangian submanifold if $n = m$ (maximum dimension). Let $\{e_1, \dots, e_{n+p}\}$ becoming an orthogonal frame to M^n ; the second fundamental form h to M^n is given by

$$h(e_i, e_j) = \sum_{\alpha=1}^n \sigma_{ij}^\alpha e_\alpha, \quad (8)$$

where $\sigma_{ij}^\alpha = \langle A_{\alpha e_i}, e_j \rangle$ and A_α denote the shape operator. The Gauss equation for Lagrangian submanifold M^n in a complex space form $\tilde{M}^{n+p}(4c)$ in the form of local coordinates is given by

$$R_{jkl}^i = (\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji})c + \sum_{\alpha=1}^n (\sigma_{ik}^\alpha \sigma_{jl}^\alpha - \sigma_{il}^\alpha \sigma_{jk}^\alpha). \quad (9)$$

Then, for Ricci curvature

$$R_{jij}^i = (\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji})c + \sum_{\alpha=1}^p (\sigma_{ii}^\alpha \sigma_{jj}^\alpha - \sigma_{ij}^\alpha \sigma_{ji}^\alpha). \quad (10)$$

As we assumed that Ψ is an immersion which is minimal, (10) yields

$$\text{Ric}(e_i, e_j) = (n-1)c\delta_{ij} - \sum_{\alpha=1}^n \sigma_{ik}^\alpha \sigma_{jk}^\alpha. \quad (11)$$

Let a function $\varphi : M^n \longrightarrow \mathbb{R}$ established on a Riemannian manifold, then the Bochner formula (see, e.g., [2]) given as:

$$\frac{1}{2} \Delta |\nabla \varphi|^2 = |\nabla^2 \varphi|^2 + \text{Ric}_{M^n}(\nabla \varphi, \nabla \varphi) + g(\nabla \varphi, \nabla(\Delta \varphi)), \quad (12)$$

where Hessian is denoted by $\nabla^2 \varphi$ and Ric denotes the Ricci curvature of M^n .

Now, we prove a proposition which authorizes to construct the proof of Theorems 1 and 2, that is the following:

Proposition 4. *Let $\Psi : M^n \longrightarrow \tilde{M}^{n+p}(4c)$ be an immersion of a compact Lagrangian submanifold into the complex space form $\tilde{M}^{n+p}(4c)$. Let φ be a first eigenfunction endowed to the Laplacian of M^n and Φ is minimal, then*

$$(nc - \mu_1) \int_M |\nabla \varphi|^2 dV = \int_M \sum_{i=1}^n |B(\nabla \varphi, e_i)|^2 dV + c \int_M |\nabla \varphi|^2 dV - \int_M |\nabla^2 \varphi|^2 dV. \quad (13)$$

For exceptional, we have

$$\int_M \sum_{i=1}^n |B(\nabla \varphi, e_i)|^2 dV = \int_M \left| \nabla^2 \varphi + \frac{\mu_1}{n} \varphi I \right|^2 dV + \left\{ \frac{(n-1)(nc - \mu_1)}{n} \right\} \int_M |\nabla \varphi|^2 dV, \quad (14)$$

for any orthonormal frame $\{e_1, \dots, e_n\}$ tangent to M^n .

Proof. If the identity operator on TM is denoted by I , then we have

$$|\nabla^2 \varphi - t\varphi I|^2 = |\nabla^2 \varphi|^2 - 2t\varphi \Delta \varphi + nt^2 \varphi^2. \quad (15)$$

Therefore, if $\Delta \varphi + \mu \varphi = 0$, we obtain for any $t \in \mathbb{R}$. The norm of an operator which is given by $|I|^2 = \text{tr}(II^*)$. Taking integration in the above equation (15) and from Stokes theorem, we have

$$\int_M |\nabla^2 \varphi - t\varphi I|^2 dV = \int_M |\nabla^2 \varphi|^2 dV + \left(2t + \frac{n}{\mu_1} t^2 \right) \int_M |\nabla \varphi|^2 dV. \quad (16)$$

We setting $t = -\mu_1/n$ in (16), we get

$$\int_M |\nabla^2 \varphi|^2 dV = \int_M \left| \nabla^2 \varphi + \frac{\mu_1}{n} \varphi I \right|^2 dV + \frac{\mu_1}{n} \int_M |\nabla \varphi|^2 dV. \quad (17)$$

On other hand, (11) yields

$$\text{Ric}(\varphi_i e_i, \varphi_j e_j) = (n-1)c\delta_{ij}\varphi_i \varphi_j - \sum_{\alpha=1}^p \sum_{k=1}^n \sigma_{ik}^\alpha \sigma_{jk}^\alpha \varphi_i \varphi_j. \quad (18)$$

Tracing the above equation, we obtain

$$\text{Ric}(\nabla \varphi, \nabla \varphi) = c(n-1)|\nabla \varphi|^2 - \sum_{i=1}^n |B(\nabla \varphi, e_i)|^2. \quad (19)$$

Let us assume that $\Delta \varphi = -\mu_1 \varphi$. Taking integration in Bochner formula and using Stokes theorem, we get

$$\int_M |\nabla^2 \varphi|^2 dV + \int_M \text{Ric}(\nabla \varphi, \nabla \varphi) dV = \mu_1 \int_M |\nabla \varphi|^2 dV. \quad (20)$$

From (19), we conclude

$$(cn - \mu_1) \int_M |\nabla \varphi|^2 dV = \int_M \sum_{i=1}^n |B(\nabla \varphi, e_i)|^2 dV + c \int_M |\nabla \varphi|^2 dV - \int_M |\nabla^2 \varphi|^2 dV. \quad (21)$$

This is the first result of the proposition. On the other hand, using (17) in the last equality, we obtain

$$(cn - \mu_1) \int_M |\nabla \varphi|^2 dV = \int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV + c \int_M |\nabla \varphi|^2 dV - \int_M \left| \nabla^2 \varphi + \frac{\mu_1}{n} \varphi I \right|^2 dV - \frac{\mu_1}{n} \int_M |\nabla \varphi|^2 dV. \quad (22)$$

After some computation, we get

$$\int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV = \int_M \left| \nabla^2 \varphi + \frac{\mu_1}{n} \varphi I \right|^2 dV + \left\{ \frac{(n-1)(cn - \mu_1)}{n} \right\} \int_M |\nabla \varphi|^2 dV. \quad (23)$$

Now, we have reached the proof of the proposition. \square

Recall the following lemma which set up to eliminate the proof of Theorem 2.

Lemma 5 [4]. *Let a valid symmetric linear operator $T : V \rightarrow V$ which trace-less defined over a finite dimensional vector space V . If it is diagonalized T , i.e., $Te_i = \mu_i e_i$ and $\dim \text{Ker} T = k$, they for any j we have*

$$\mu_j^2 \leq \frac{(n-k-1)|T|^2}{(n-k)}, \quad (24)$$

for any integer k and for an orthonormal basis $\{e_1, \dots, e_n\}$.

3. Proof of Main Theorems

We are in the position to prove our main results.

3.1. Proof of Theorem 1. Let us consider

$$nc \geq \mu_1. \quad (25)$$

Then, we noticed that left-hand side of (13) of Proposition 4 is different from negative. Therefore, the other side also non-negative, we get

$$\int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV + c \int_M |\nabla \varphi|^2 dV \geq \int_M |\nabla^2 \varphi|^2 dV. \quad (26)$$

Additionally, the equality holds if and only if the following holds

$$\mu_1 = nc. \quad (27)$$

Moreover, we expressed the first equation of Proposition 4 in a new form

$$\int_M |\nabla^2 \varphi|^2 dV - \int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV = \{\mu_1 - c(n-1)\} \int_M |\nabla \varphi|^2 dV. \quad (28)$$

If we consider the following

$$\int_M |\nabla^2 \varphi|^2 dV \geq \int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV. \quad (29)$$

Then, equation (28) implies that

$$\{\mu_1 - c(n-1)\} \geq 0. \quad (30)$$

If we notice that

$$\int_M |\nabla \varphi|^2 dV \geq \frac{nc}{\mu_1} \int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV. \quad (31)$$

This implies that

$$\int_M |\nabla^2 \varphi|^2 dV \geq \int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV. \quad (32)$$

This completes the proof of Theorem 1.

3.2. Proof of Theorem 2. Let the second fundamental form T which diagonalized via an orthogonal frame $\{e_1, \dots, e_n\}$, i.e., $Te_i = k_i e_i$, and the angle is denoted by θ_i between $\nabla \varphi$ and e_i . Thus, we find that

$$|h(\nabla \varphi, e_i)|^2 = g(T\nabla \varphi, e_i)^2 = g(\nabla \varphi, Te_i)^2 = k_i^2 \cos^2 \theta_i |\nabla \varphi|^2. \quad (33)$$

From the virtue of (13) of Proposition 4, we construct

$$\begin{aligned} & \int_M \left(\sum_{i=1}^n k_i^2 \cos^2 \theta_i \right) |\nabla \varphi|^2 dV \\ &= \int_M |\nabla^2 \varphi|^2 dV + \{(n-1)c - \mu_1\} \int_M |\nabla \varphi|^2 dV. \end{aligned} \quad (34)$$

Implementation Lemma 5 to the previous equation to establish

$$\begin{aligned} & \left(\frac{n-k-1}{n-k} \right) \int_M S |\nabla \varphi|^2 dV \\ & \geq \int_M |\nabla^2 \varphi|^2 dV + \{(n-1)c - \mu_1\} \int_M |\nabla \varphi|^2 dV. \end{aligned} \quad (35)$$

Let us assume the following inequality

$$\int_M |\nabla^2 \varphi|^2 dV \geq \frac{\mu_1}{nc} \int_M |\nabla \varphi|^2 dV. \quad (36)$$

Plugging above equation into (35), we arrive at

$$\begin{aligned} & \left(\frac{n-k-1}{n-k} \right) \int_M S |\nabla \varphi|^2 dV \\ & \geq \left(\frac{n^2 c^2 - nc\mu_1 - nc^2 + \mu_1}{nc} \right) \int_M |\nabla \varphi|^2 dV. \end{aligned} \quad (37)$$

After some computations, finally, we get

$$\int_M S |\nabla^2 \varphi|^2 dV \geq \left\{ \frac{(n-k)(nc-1)(nc-\mu_1)}{(n-k-1)nc} \right\} \int_M |\nabla \varphi|^2 dV. \quad (38)$$

This completes the proof of Theorem 2.

3.3. Proof of Theorem 3. As recognize a theorem as a result of Obata in [5], a differentiable function φ on Riemannian manifold M^n is satisfied the following ordinary differential equation

$$\nabla^2 \varphi = -\varphi, \quad (39)$$

if and only if M^n is isometric to a unit sphere \mathbb{S}^n , where $\nabla^2 \varphi$ is two time derivatives of φ and is called Hessian of φ . As we assumed that

$$\nabla \varphi \in \ker h(\nabla \varphi, e_i) = 0, \quad 1 \leq i \leq n. \quad (40)$$

Then, using equation (14), we get

$$\frac{(n-1)(\mu_1 - nc)}{n} \int_M |\nabla \varphi|^2 dV = \int_M \left| \nabla^2 \varphi + \frac{\mu_1}{n} \varphi \right|^2 dV. \quad (41)$$

The left-hand side of this previous equation is not negative; we summarize that

$$\mu_1 = nc. \quad (42)$$

Therefore, we have

$$\nabla^2 \varphi = -\varphi. \quad (43)$$

Now using Obata theorem [5], we conclude that $\Psi(M^n)$ is isometric to a unit sphere \mathbb{S}^n . This completes the proof of first part of Theorem 3.

On the other case, if we consider that

$$\text{Ric}(\nabla \varphi, \nabla \varphi) \geq c(n-1) |\nabla \varphi|^2. \quad (44)$$

Follows the equation (19), we find that

$$\begin{aligned} \int_M (n-1)c |\nabla \varphi|^2 dV & \geq \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV \\ & + (n-1)c \int_M |\nabla \varphi|^2 dV, \end{aligned} \quad (45)$$

which implies that

$$\sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV \leq 0. \quad (46)$$

From where we conclude that $h(\nabla \varphi, e_i) = 0$, this means that $\nabla \varphi \in \ker B$. Now, we invoke the first case (i) of Theorem 3 we get required result. The proof of Theorem 3 is completed.

4. Some Applications

It is renowned that the complex Euclidean space \mathbb{C}^n , the complex projective n -space $\mathbb{CP}^n(4)$, and complex hyperbolic n -space $\mathbb{CH}^n(-4)$ are special cases of a complex space form $\tilde{M}^n(4c)$ with $c = 0, 1$ and $c = -1$, respectively. Therefore, we define following corollaries for complex projective spaces as consequences of Theorems 1, 2, and 3.

Corollary 6. Let $\Psi : M^n \longrightarrow \mathbb{CP}^n(4)$ be an immersion of a compact Lagrangian submanifold into complex projective space $\mathbb{CP}^n(4)$. If the Laplacian of M^n endowed to the first eigenvalue μ_1 corresponding eigenfunction φ and Ψ is minimal, then the following inequality holds

$$\int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV + \int_M |\nabla \varphi|^2 dV \geq \int_M |\nabla^2 \varphi|^2 dV. \quad (47)$$

The equality holds if and only if $\mu_1 = n$. Furthermore, if the following inequality holds

$$\int_M |\nabla^2 \varphi|^2 dV \geq \int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV. \quad (48)$$

Then, we have $\mu_1 \geq (n-1)$. In particular, if the following inequality satisfying

$$\int_M |\nabla \varphi|^2 dV \geq \frac{n}{\mu_1} \int_M \sum_{i=1}^n |h(\nabla \varphi, e_i)|^2 dV. \quad (49)$$

Then, eigenvalue is satisfied $\mu_1 \geq (n-1)$.

Corollary 7. Let $\Psi : M^n \longrightarrow \mathbb{CP}^n(4)$ be an immersion of a compact Lagrangian submanifold into the complex projective space $\mathbb{CP}^n(4)$. If Ψ is minimal and $\dim \text{Ker} h = k$, then we have

$$\int_M S |\nabla^2 \varphi|^2 dV \geq \left\{ \frac{(n - \mu_1)(n - 1)(n - k)}{(n - k - 1)n} \right\} \int_M |\nabla \varphi|^2 dV. \quad (50)$$

In especial case, if S is constant, then we define

$$S \geq \frac{(n - 1)(n - k)}{n(n - \mu_1)(n - k - 1)}. \quad (51)$$

From Theorem 3, we have the following:

Corollary 8. Let $\Psi : M^n \longrightarrow \mathbb{CP}^n(4)$ be an immersion of a compact Lagrangian submanifold into the complex projective space $\mathbb{CP}^n(4)$. Assuming that Ψ is minimal and φ be a first eigenfunction endowed to the Laplacian of M^n . Then, we get following:

- (i) If $\nabla \varphi$ on Kerh , then $\Psi(M^n)$ is locally geodesic sphere \mathbb{S}^n , or $\Psi(M^n)$ is isometric to standard sphere \mathbb{S}^n
- (ii) If $\text{Ric}_{M^n}(\nabla \varphi, \nabla \varphi) \geq (n - 1)|\nabla \varphi|^2$, then $\Psi(M^n)$ is isometric to sphere \mathbb{S}^n

Data Availability

No datasets were generated or analyzed during the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Rigidity of Complete Gradient Shrinkers with Pointwise Pinching Riemannian Curvature

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Let (M^n, g, f) be a complete gradient shrinking Ricci soliton of dimension $n \geq 3$. In this paper, we study the rigidity of (M^n, g, f) with pointwise pinching curvature and obtain some rigidity results. In particular, we prove that every n -dimensional gradient shrinking Ricci soliton (M^n, g, f) is isometric to \mathbb{R}^n or a finite quotient of \mathbb{S}^n under some pointwise pinching curvature condition. The arguments mainly rely on algebraic curvature estimates and several analysis tools on (M^n, g, f) , such as the property of f -parabolic and a Liouville type theorem.

1. Introduction

An n -dimensional ($n \geq 3$) Riemannian manifold (M^n, g) is called a *Ricci soliton* if there exist a smooth vector field X and a constant $\lambda \in \mathbb{R}$ on M^n such that

$$Rc + \frac{1}{2} \mathcal{L}_X g = \lambda g, \quad (1)$$

where Rc and $\mathcal{L}_X g$ denote the Ricci tensor and the Lie derivative of g in the direction of X , respectively, and λ is sometimes called the *soliton constant*. The soliton is shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. When X is a gradient of a smooth function f on M^n , the soliton is called a *gradient Ricci soliton* and (1) becomes

$$Rc + \text{Hess } f = \lambda g. \quad (2)$$

Note that when X or ∇f is a Killing vector field, equations (1) and (2) reduce to the Einstein equation. Thus, Ricci solitons are natural generalizations of Einstein manifolds. In particular, when $X = 0$ or f is a constant, the soliton is *trivial*.

In recent decades, increasing investigations have been done to the rigidity of gradient shrinking Ricci solitons (gradient shrinker for short). In dimension 2, Hamilton [1]

showed that a gradient shrinker is isometric to \mathbb{R}^2 or to a quotient of \mathbb{S}^2 . The first rigidity theorem in dimension 3 was proved by Ivey [2] saying that a 3-dimensional compact gradient shrinker is a quotient of \mathbb{S}^3 . In the noncompact case, the relevant rigidity result was showed by Perelman [3] with noncollapsing assumption, which was removed by Naber [4] later. Adopting different arguments, Ni and Wallach [5] and Cao et al. [6] obtained the full classification; they proved that any 3-dimensional gradient shrinker must be isometric to \mathbb{R}^3 or to a quotient of $\mathbb{R} \times \mathbb{S}^2$ or \mathbb{S}^3 . Some relevant conclusions can be found in [4, 7, 8].

When $n \geq 4$, under the assumption of nonnegative curvature operator or vanishing Weyl tensor, Naber [4], Ni and Wallach [5], Petersen and Wylie [8], and Zhang [9] proved corresponding rigidity results on gradient shrinkers, which were improved by Catino [10] using a general pointwise pinching condition on the Weyl tensor.

On the other hand, Munteanu and Wang [11] investigated the curvature behavior of 4-dimensional gradient shrinker and proved that there exists a constant $C > 0$ for 4-dimensional gradient shrinkers with bounded scalar curvature R so that

$$|Rm| \leq CR, \quad (3)$$

which along with the fact $|Rm|^2 = |R^\circ m|^2 + (R^2/6)$ implies

$$|\mathring{R}m| \leq \left(C^2 - \frac{1}{6}\right)R^2, \quad (4)$$

and $C \geq (\sqrt{6}/6)$. Here, $R^\circ m$ is the trace-free curvature tensor.

In [12], the authors established the following rigidity theorem under pointwise pinching condition of $R^\circ m$:

Theorem 1 (Theorem 1.1 in [12]). *Let (M^n, g, f) be an n -dimensional ($n \geq 3$) complete gradient shrinker. If*

$$|\mathring{R}m| \leq \frac{1}{C(n)} \left(\lambda - \frac{n-2}{n(n-1)}R \right), \quad (5)$$

then (M^n, g) is isometric to \mathbb{R}^n or a finite quotient of \mathbb{S}^n .

In this paper, we will restrict our attention to the rigidity of gradient shrinkers with pointwise pinched conditions associated with $R^\circ m$ and the traceless Ricci tensor $R^\circ c = Rc - (R/n)g$. By establishing f -parabolic and algebraic curvature estimates, we prove two rigidity results for gradient shrinkers. More precisely, setting $C(n) = ((n-2)/(\sqrt{n(n-1)})) + ((n^2 - n - 4)/(2\sqrt{(n-2)(n-1)n(n+1)}))$, which is defined in Lemma 10, we have the following Theorem 2.

Theorem 2. *Assume that (M^n, g, f) is a complete gradient shrinker of dimension $n \geq 3$. If*

$$|\mathring{R}m|^2 \leq \frac{1}{C^2(n)} \left(\lambda - \frac{n-2}{n(n-1)}R \right)^2 + \frac{4|\mathring{R}c|^2}{n}, \quad (6)$$

then (M^n, g) is isometric to \mathbb{R}^n or a finite quotient of \mathbb{S}^n . Moreover, when the pinching condition in the right hand of (6) is weakened to

$$|\mathring{R}m|^2 < \frac{2(n-1)}{n-2} \left(\lambda - \frac{n-2}{n(n-1)}R \right)^2 + \frac{4|\mathring{R}c|^2}{n}, \quad (7)$$

then (M^n, g) is Einstein.

Remark 3. When

$$|\mathring{R}m| \leq \frac{1}{C(n)} \left(\lambda - \frac{n-2}{n(n-1)}R \right), \quad (8)$$

by equation (6) and Theorem 2, we see that (M^n, g) is isometric to \mathbb{R}^n or a finite quotient of \mathbb{S}^n . Therefore, Theorem 2 can be seen as a generalization of Theorem 1.

Theorem 4. *Let (M^n, g, f) be a complete gradient shrinker of dimension $n \geq 3$ with nonnegative Ricci curvature. If*

$$|\mathring{R}m| \leq \frac{1}{C(n)} \left(\frac{R}{n(n-1)} + \frac{|\mathring{R}c|^2}{R} \right), \quad (9)$$

then (M^n, g) is isometric to \mathbb{R}^n or a finite quotient of \mathbb{S}^n .

Remark 5. As is shown in the proof, the condition of non-negative Ricci curvature in Theorem 4 can be relaxed to that $|Rc| \leq cR^{1+\alpha}$ for some constants $c > 0$ and $\alpha \geq 0$ satisfying $cR^\alpha \geq \sqrt{(n-2)/(n(n-1))}$.

Remark 6. Since any three-dimensional gradient shrinker must have nonnegative sectional curvature (cf. Corollary 2.4 of [13]), we see that the condition on Ricci curvature in Theorem 4 is not needed.

2. Preliminaries of Curvature Estimates

Let (M^n, g) be a connected Riemannian manifold of dimension $n \geq 3$. In local coordinates, denoting by R_{ijkl} , W_{ijkl} , and $R^\circ_{jk} = R_{jk} - (R/n)g_{jk}$ the components of the curvature tensor Rm , the Weyl tensor W , and the traceless Ricci tensor $R^\circ c$, respectively, we have the well-known orthogonal decomposition of Rm (see e.g., [14]).

$$\begin{aligned} R_{ijkl} = & W_{ijkl} + \frac{R}{n(n-1)} (g_{il}g_{jk} - g_{ik}g_{jl}) \\ & + \frac{1}{n-2} (\mathring{R}_{il}g_{jk} + \mathring{R}_{jk}g_{il} - \mathring{R}_{ik}g_{jl} - \mathring{R}_{jl}g_{ik}). \end{aligned} \quad (10)$$

Correspondingly, the soliton equation (2) is rewritten as

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}. \quad (11)$$

Taking the trace in equation (11) gives

$$R + \Delta f = n\lambda. \quad (12)$$

Writing $R^\circ m = \{R^\circ_{ijkl}\} = \{R_{ijkl} - (R/n(n-1))(g_{il}g_{jk} - g_{ik}g_{jl})\}$ and using the properties of Rm , one can easily derive the following equalities:

$$g^{il}\mathring{R}_{ijkl} = \mathring{R}_{jk}, \quad (13)$$

$$\mathring{R}_{ijkl} + \mathring{R}_{iljk} + \mathring{R}_{iklj} = 0, \quad (14)$$

$$\mathring{R}_{ijkl} = \mathring{R}_{klij} = -\mathring{R}_{jikl} = -\mathring{R}_{iljk}, \quad (15)$$

$$|\mathring{R}m|^2 = |W|^2 + \frac{4}{n-2} |\mathring{R}c|^2, \quad (16)$$

where the norm of a $(0, 4)$ -type tensor T is defined by

$$|T|^2 = |T_{ijkl}|^2 = g^{im} g^{jn} g^{ks} g^{lt} T_{ijkl} T_{mnst} \triangleq T^{ijkl} T_{ijkl}. \quad (17)$$

Here and subsequently, the notations $u_* = \Delta \inf_{M^n} u$ as well as $u^* = \Delta \sup_{M^n} u$ for a function u on M^n and Einstein summation convention are always adopted.

Recall the f -Laplacian Δ_f , which is sometimes called the drifted Laplacian or Witten-Laplacian and is defined on a function $u \in \text{Lip}_{\text{loc}}(M^n)$ by

$$\Delta_f u = \Delta u - g(\nabla f, \nabla u) = e^f \operatorname{div} (e^{-f} \nabla u), \quad (18)$$

in the weak sense, which is a self-adjoint operator on the space of square integrable functions on (M^n, g, f) with respect to weighted volume form $e^{-f} dV_g$. That is,

$$\int_{M^n} \nabla \varphi \cdot \nabla \psi e^{-f} dV_g = - \int_{M^n} (\Delta_f \varphi) \psi e^{-f} dV_g, \quad (19)$$

for any $\varphi, \psi \in C_0^\infty(M^n)$, where dV_g is the volume element induced by the metric g .

First of all, we will compute the f -Laplacian of the norm square of $R^\circ m$, by which we will establish the key estimate for any gradient Ricci soliton of dimension $n \geq 3$ in Lemma 10. We start from Lemma 7.

Lemma 7. *For any gradient Ricci soliton of dimension $n \geq 3$, we have*

$$\begin{aligned} 2\dot{R}^{\circ ijkl} \nabla_l \nabla^p R_{ijkp} - \frac{1}{2} \nabla \left| \dot{R} m \right|^2 \cdot \nabla f &= 2\lambda \left| \dot{R} m \right|^2 - 2\dot{R}^{\circ ijkl} \dot{R}_{ijk}^{\circ p} \dot{R}_{lp} \\ &\quad - \frac{2R}{n} \left| \dot{R} m \right|^2 - \frac{4R}{n(n-1)} \left| \dot{R} c \right|^2. \end{aligned} \quad (20)$$

Proof. For convenience, we set

$$\begin{aligned} A &= 2\dot{R}^{\circ ijkl} \nabla_l \nabla^p R_{ijkp} - \frac{1}{2} \nabla \left| \dot{R} m \right|^2 \cdot \nabla f \\ &= 2\dot{R}^{\circ ijkl} \nabla_l \nabla^p R_{ijkp} - \dot{R}^{\circ ijkl} (\nabla_p R_{ijkl}) \nabla^p f. \end{aligned} \quad (21)$$

□

On the one hand, by the second Bianchi identity, we get

$$\begin{aligned} -\dot{R}^{\circ ijkl} (\nabla_p R_{ijkl}) \nabla^p f &= -\dot{R}^{\circ ijkl} (\nabla_l R_{ijkp} - \nabla_k R_{ijlp}) \nabla^p f \\ &= -2\dot{R}^{\circ ijkl} (\nabla_l R_{ijkp}) \nabla^p f. \end{aligned} \quad (22)$$

On the other hand, by the Ricci identity and the equation (11), we deduce that

$$\begin{aligned} 2\dot{R}^{\circ ijkl} \nabla_l \nabla^p R_{ijkp} &= 2\dot{R}^{\circ ijkl} \nabla_l (\nabla_i R_{jk} - \nabla_j R_{ik}) \\ &= 2\dot{R}^{\circ ijkl} \nabla_l (\nabla_j \nabla_i \nabla_k f - \nabla_i \nabla_j \nabla_k f) \\ &= 2\dot{R}^{\circ ijkl} (\nabla_l R_{ijkp}) \nabla^p f + 2\dot{R}^{\circ ijkl} R_{ijkp} (\lambda g_l^p - R_l^p). \end{aligned} \quad (23)$$

Combining the facts $R_{ijkp} = R^\circ_{ijkp} + (R/n(n-1))(g_{ip}g_{jk} - g_{ik}g_{jp})$ and $R_{ij} = R^\circ_{ij} + (R/n)g_{ij}$ with (23) yields

$$\begin{aligned} A &= 2\lambda \left| \dot{R} m \right|^2 - 2\dot{R}^{\circ ijkl} \left(\dot{R}_{ijk}^{\circ p} + \frac{R}{n(n-1)} (g_i^p g_{jk} - g_{ik} g_j^p) \right) \left(\dot{R}_{lp} + \frac{R}{n} g_{lp} \right) \\ &= 2\lambda \left| \dot{R} m \right|^2 - 2\dot{R}^{\circ ijkl} \dot{R}_{ijk}^{\circ p} \dot{R}_{lp} - \frac{2R}{n} \left| \dot{R} m \right|^2 - \frac{4R}{n(n-1)} \left| \dot{R} c \right|^2. \end{aligned} \quad (24)$$

Our next step is to compute the Laplacian of $\left| R^\circ m \right|^2$ for all Riemannian manifolds.

Lemma 8. *Let (M^n, g) be an n -dimensional ($n \geq 3$) complete Riemannian manifold. Then,*

$$\begin{aligned} \frac{1}{2} \Delta \left| \dot{R}_{ijkl} \right|^2 &= \left| \nabla \dot{R} m \right|^2 - 2\dot{R}^{\circ ijkl} \left(2\dot{R}_{il}^{\circ h} \dot{R}_{hjkp} - \frac{1}{2} \dot{R}_{ijhp} \dot{R}_{kl}^{\circ hp} - \dot{R}_{ijk}^{\circ h} \dot{R}_{lh} \right) \\ &\quad + 2\dot{R}^{\circ ijkl} \nabla_l \nabla^p R_{ijkp} + \frac{2R}{n} \left| \dot{R} m \right|^2 - \frac{4R}{n(n-1)} \left| \dot{R} c \right|^2. \end{aligned} \quad (25)$$

Proof. By the definitions of $R^\circ m$, $\left| R^\circ m \right|^2$, and (13), we have

$$\frac{1}{2} \Delta \left| \dot{R}_{ijkl} \right|^2 = \left| \nabla \dot{R} m \right|^2 + \dot{R}^{\circ ijkl} \nabla^p \nabla_p \dot{R}_{ijkl} = \left| \nabla \dot{R} m \right|^2 + \dot{R}^{\circ ijkl} \nabla^p \nabla_p R_{ijkl}. \quad (26)$$

□

Employing Bianchi's second identity, we obtain

$$\begin{aligned} \dot{R}^{\circ ijkl} \nabla^p \nabla_p R_{ijkl} &= \dot{R}^{\circ ijkl} \nabla^p (\nabla_l R_{ijkp} - \nabla_k R_{ijlp}) \\ &= 2\dot{R}^{\circ ijkl} \nabla^p \nabla_l R_{ijkp} = 2\dot{R}^{\circ ijkl} \nabla^p \nabla_l \left(\dot{R}_{ijkp} + \frac{2R}{n(n-1)} g_{ip} g_{jk} \right) \\ &= 2\dot{R}^{\circ ijkl} \nabla^p \nabla_l \dot{R}_{ijkp} + \frac{4R}{n(n-1)} \nabla_p \nabla_l R, \end{aligned} \quad (27)$$

which together with (26) implies

$$\frac{1}{2} \Delta \left| \dot{R}_{ijkl} \right|^2 = \left| \nabla \dot{R} m \right|^2 + 2\dot{R}^{\circ ijkl} \nabla^p \nabla_l \dot{R}_{ijkp} + \frac{4R}{n(n-1)} \nabla_p \nabla_l R. \quad (28)$$

Making use of the Ricci identity and (13), we have

$$\begin{aligned}
2\mathring{R}^{\circ ijkl} \nabla^p \nabla_l \mathring{R}_{ijkp} &= 2\mathring{R}^{\circ ijkl} \left(\nabla_l \nabla^p \mathring{R}_{ijkp} - R^p_{li} \mathring{R}^h_{hjkp} \right. \\
&\quad \left. + R^p_{lj} \mathring{R}^h_{hikp} - R^p_{lk} \mathring{R}^h_{ijhp} - R^p_{lp} \mathring{R}^h_{ijkh} \right) \\
&= 2\mathring{R}^{\circ ijkl} \left(\nabla_l \nabla^p \mathring{R}_{ijkp} - \mathring{R}^{\circ h}_{il} \mathring{R}^p_{hjkp} + \mathring{R}^{\circ h}_{jl} \mathring{R}^p_{hikp} - \mathring{R}^{\circ h}_{kl} \mathring{R}^p_{ijhp} \right) \\
&\quad + 2\mathring{R}^{\circ ijkl} \mathring{R}_{ijk}^{\circ h} \left(\mathring{R}_{lh} + \frac{R}{n} g_{lh} \right) - \frac{2R}{n(n-1)} \mathring{R}^{\circ ijkl} \\
&\quad \cdot \left(\mathring{R}_{hjkp} (g^{hp} g_{il} - g^h_l g^p_i) - \mathring{R}_{hikp} (g^{hp} g_{jl} - g^h_l g^p_j) \right. \\
&\quad \left. + \mathring{R}_{ijhp} (g^{hp} g_{kl} - g^h_k g^p_l) \right) \\
&= 2\mathring{R}^{\circ ijkl} \left(\nabla_l \nabla^p \mathring{R}_{ijkp} - 2\mathring{R}^{\circ h}_{il} \mathring{R}^p_{hjkp} - \mathring{R}^{\circ h}_{kl} \mathring{R}^p_{ijhp} \right) \\
&\quad + 2\mathring{R}^{\circ ijkl} \mathring{R}_{ijk}^{\circ h} \mathring{R}_{lh} + \frac{2R}{n} |\mathring{R}m|^2 - \frac{2R}{n(n-1)} \mathring{R}^{\circ ijkl} \\
&\quad \cdot \left(-\mathring{R}_{ljk i} - \mathring{R}_{ilkj} - \mathring{R}_{ijlk} + \mathring{R}_{jk} g_{li} - \mathring{R}_{ik} g_{lj} \right). \tag{29}
\end{aligned}$$

Combining (13) and (14) with (29), we get

$$\begin{aligned}
2\mathring{R}^{\circ ijkl} \nabla^p \nabla_l \mathring{R}_{ijkp} &= -2\mathring{R}^{\circ ijkl} \left(2\mathring{R}^{\circ h}_{il} \mathring{R}^p_{hjkp} + \mathring{R}^{\circ h}_{kl} \mathring{R}^p_{ijhp} - \mathring{R}^{\circ h}_{ijk} \mathring{R}_{lh} \right) \\
&\quad + 2\mathring{R}^{\circ ijkl} \nabla_l \nabla^p \mathring{R}_{ijkp} + \frac{2R}{n} |\mathring{R}m|^2 - \frac{4R}{n(n-1)} |\mathring{R}c|^2, \tag{30}
\end{aligned}$$

where

$$\mathring{R}^{\circ ijkl} \mathring{R}^{\circ h}_{kl} \mathring{R}_{ijhp} = \mathring{R}^{\circ ijkl} \mathring{R}^{\circ h}_{ijhp} \left(-\mathring{R}^{\circ h}_{kl} - \mathring{R}^h_{l k} \right) = -\frac{1}{2} \mathring{R}^{\circ ijkl} \mathring{R}^{\circ h}_{ijhp} \mathring{R}^{\circ h}_{kl}. \tag{31}$$

Substituting (30) and (31) into (28), we obtain

$$\begin{aligned}
\frac{1}{2} \Delta_f |\mathring{R}_{ijkl}|^2 &= |\nabla \mathring{R}m|^2 - 2\mathring{R}^{\circ ijkl} \left(2\mathring{R}^{\circ h}_{il} \mathring{R}^p_{hjkp} - \frac{1}{2} \mathring{R}^{\circ h}_{ijhp} \mathring{R}^{\circ h}_{kl} - \mathring{R}^{\circ h}_{ijk} \mathring{R}_{lh} \right) \\
&\quad + 2\mathring{R}^{\circ ijkl} \nabla_l \nabla^p \mathring{R}_{ijkp} + \frac{2R}{n} |\mathring{R}m|^2 - \frac{4R}{n(n-1)} |\mathring{R}c|^2, \tag{32}
\end{aligned}$$

where the formula

$$\mathring{R}^{\circ ijkl} \nabla_l \nabla^p \mathring{R}_{ijkp} + \frac{2\mathring{R}^{\circ pl}}{n(n-1)} \nabla_p \nabla_l \mathring{R} = \mathring{R}^{\circ ijkl} \nabla_l \nabla^p \mathring{R}_{ijkp} \tag{33}$$

is used in (32).

By Lemmas 7 and 8 and the fact that

$$\frac{1}{2} \Delta_f |\mathring{R}_{ijkl}|^2 = \frac{1}{2} \Delta |\mathring{R}_{ijkl}|^2 - \mathring{R}^{\circ ijkl} (\nabla_p \mathring{R}_{ijkl}) \nabla^p f, \tag{34}$$

we now arrive at the f -Laplacian formula of $|R^\circ m|^2$ for all gradient Ricci solitons.

Lemma 9. *Let (M^n, g, f) be an n -dimensional ($n \geq 3$) gradient Ricci soliton. Then,*

$$\begin{aligned}
\frac{1}{2} \Delta_f |\mathring{R}_{ijkl}|^2 &= |\nabla \mathring{R}m|^2 - 2\mathring{R}^{\circ ijkl} \left(2\mathring{R}^{\circ h}_{il} \mathring{R}^p_{hjkp} - \frac{1}{2} \mathring{R}^{\circ h}_{ijhp} \mathring{R}^{\circ h}_{kl} \right) \\
&\quad + 2\lambda |\mathring{R}m|^2 - \frac{8R}{n(n-1)} |\mathring{R}c|^2. \tag{35}
\end{aligned}$$

Consequently, we conclude from (35) and (16) that

$$\begin{aligned}
\frac{1}{2} \Delta_f |\mathring{R}_{ijkl}|^2 &= |\nabla \mathring{R}m|^2 - 2\mathring{R}^{\circ ijkl} \left(2\mathring{R}^{\circ h}_{il} \mathring{R}^p_{hjkp} - \frac{1}{2} \mathring{R}^{\circ h}_{ijhp} \mathring{R}^{\circ h}_{kl} \right) \\
&\quad + 2\lambda |\mathring{R}m|^2 - \frac{2(n-2)R}{n(n-1)} |\mathring{R}m|^2 \\
&\quad + \frac{2(n-2)R}{n(n-1)} |W|^2, \tag{36}
\end{aligned}$$

for any gradient Ricci soliton.

Utilizing the inequalities proved by Li and Zhao [15] and Huiskens [16] (see also [17]), we have

$$2 \left| \mathring{R}^{\circ ijkl} \mathring{R}^{\circ h}_{il} \mathring{R}^p_{hjkp} \right| \leq \frac{n-2}{\sqrt{n(n-1)}} |\mathring{R}m|^3, \tag{37}$$

$$\left| \mathring{R}^{\circ ijkl} \mathring{R}^{\circ h}_{ijhp} \mathring{R}^{\circ h}_{kl} \right| \leq \frac{n^2 - n - 4}{\sqrt{(n-2)(n-1)n(n+1)}} |\mathring{R}m|^3. \tag{38}$$

Combining (37) and (38) with (36) gives

Lemma 10. *Let (M^n, g, f) be an n -dimensional ($n \geq 3$) gradient Ricci soliton. Then,*

$$\begin{aligned}
\frac{1}{2} \Delta_f |\mathring{R}m|^2 &\geq |\nabla \mathring{R}m|^2 \\
&\quad - 2 \left(\frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2 - n - 4}{2\sqrt{(n-2)(n-1)n(n+1)}} \right) |\mathring{R}m|^3 \\
&\quad + 2\lambda |\mathring{R}m|^2 - \frac{2(n-2)R}{n(n-1)} |\mathring{R}m|^2 \\
&\quad + \frac{2(n-2)R}{n(n-1)} |W|^2 \triangleq |\nabla \mathring{R}m|^2 - 2C(n) |\mathring{R}m|^3 \\
&\quad + 2 \left(\lambda - \frac{(n-2)R}{n(n-1)} \right) |\mathring{R}m|^2 + \frac{2(n-2)R}{n(n-1)} |W|^2. \tag{39}
\end{aligned}$$

Remark 11. Inequality (39) can also be derived by setting $X = \nabla f$ in Lemma 2.5 of [12]; here, we give its proof for the sake of completeness.

Correspondingly, the f -Laplacian of $|\mathring{R}c|^2$ is (see e.g. Lemma 2.1 of [18])

Lemma 12. *Let (M^n, g, f) be a gradient Ricci soliton of dimension $n \geq 3$. Then,*

$$\begin{aligned} \frac{1}{2} \Delta_f |\mathring{R}c|^2 &= |\nabla \mathring{R}c|^2 + 2 \left(\lambda - \frac{(n-2)R}{n(n-1)} \right) |\mathring{R}c|^2 \\ &\quad + \frac{4}{n-2} \mathring{R}_{ij} \mathring{R}^{jk} \mathring{R}_k^i - 2W_{ijkl} \mathring{R}^{il} \mathring{R}^{jk}. \end{aligned} \quad (40)$$

Employ the following curvature inequality.

Lemma 13. (Proposition 2.1 of [19]). *Let (M^n, g) be an n -dimensional ($n \geq 3$) Riemannian manifold. Then,*

$$\begin{aligned} \left| -W_{ijkl} \mathring{R}^{il} \mathring{R}^{jk} + \frac{2}{n-2} \mathring{R}^{ij} \mathring{R}_i^p \mathring{R}_{pj} \right| \\ \leq \sqrt{\frac{n-2}{2(n-1)}} \left(|W|^2 + \frac{8}{n(n-2)} |\mathring{R}c|^2 \right)^{\frac{1}{2}} |\mathring{R}c|^2. \end{aligned} \quad (41)$$

We deduce from Lemma 12 the following.

Lemma 14. *Let (M^n, g) be an n -dimensional ($n \geq 3$) Riemannian manifold. Then,*

$$\begin{aligned} \frac{1}{2} \Delta_f |\mathring{R}c|^2 &\geq |\nabla \mathring{R}c|^2 + 2 \left(\lambda - \frac{(n-2)R}{n(n-1)} - \sqrt{\frac{n-2}{2(n-1)}} \right. \\ &\quad \cdot \left. \left(|W|^2 + \frac{8}{n(n-2)} |\mathring{R}c|^2 \right)^{\frac{1}{2}} \right) |\mathring{R}c|^2. \end{aligned} \quad (42)$$

Since the scalar curvature of nonflat Ricci shrinker is positive, by Proposition 2.7 of [12], we get the following curvature inequality.

Lemma 15. *Let (M^n, g, f) be an n -dimensional ($n \geq 3$) complete nonflat gradient shrinker. Then,*

$$\begin{aligned} \Delta_{f-2\nabla \log R} \left(\frac{|\mathring{R}m|^2}{R^2} \right) &\geq \frac{2}{R^3} \left(\sqrt{R} |\nabla \mathring{R}m| - \frac{|\mathring{R}m| |\nabla R|}{\sqrt{R}} \right)^2 \\ &\quad + \frac{4(n-2)|W|^2}{n(n-1)R} + \frac{4|\mathring{R}m|^2}{R^3} \\ &\quad \cdot \left(\frac{R^2}{n(n-1)} + |\mathring{R}c|^2 - C(n)R |\mathring{R}m| \right). \end{aligned} \quad (43)$$

For the sake of the proofs of our main theorems, we recall the following results due to Catino [10], Pigola et al. [20], and Petersen and Wylie [8] (see also [4]).

Lemma 16 (Proposition 1 of [10]). *Let (M^n, g, f) be a complete gradient nonflat shrinker of dimension $n \geq 3$. Then,*

$$\begin{aligned} \Delta_{f-2\nabla \log R} \left(\frac{|\mathring{R}c|^2}{R^2} \right) &\geq \frac{2}{R^4} |R \nabla_j R_{ik} - \nabla_j R R_{ik}|^2 \\ &\quad + \frac{4}{R^3} \left[|\mathring{R}c|^2 \left(|\mathring{R}c| - \frac{R}{\sqrt{n(n-1)}} \right)^2 \right. \\ &\quad \left. - RW_{ijkl} \mathring{R}^{il} \mathring{R}^{jk} \right]. \end{aligned} \quad (44)$$

Lemma 17 (Theorem 3 of [20]). *Let (M^n, g, f) be a complete gradient shrinker of dimension $n \geq 3$. Then, $0 \leq R_* \leq n\lambda$. Moreover, $R_* < n\lambda$ unless (M, g) is Einstein and the soliton is trivial, and $R_* > 0$ unless $R \equiv 0$ and (M, g) is isometric to \mathbb{R}^n .*

Lemma 18 (Theorem 22 of [20]). *Any complete gradient shrinker (M^n, g, f) is f -parabolic, namely, every solution of $\Delta_f u \geq 0$ satisfying $u^* < +\infty$ must be a constant.*

Lemma 19 (Lemma 4.2 of [8]). *Assume that (M^n, g) is an n -dimensional manifold with finite w -volume, i.e., $\int_M e^{-w} dV_g < +\infty$. If a smooth function $u \in L^2(e^{-w} dV_g)$ is bounded below such that $\Delta_w u \geq 0$, then u is a constant.*

3. Proofs of Main Theorems

We are now in a position to give the proofs of our main theorems.

Proof of Theorem 2. Using (16), we see that pinching conditions (6) and (7) in Theorem 2 are equivalent to the following inequality, respectively:

$$|W|^2 + \frac{8}{n(n-2)} |\mathring{R}c|^2 \leq \frac{1}{C^2(n)} \left(\lambda - \frac{n-2}{n(n-1)} R \right)^2, \quad (45)$$

$$|W|^2 + \frac{8}{n(n-2)} |\mathring{R}c|^2 < \frac{2(n-1)}{n-2} \left(\lambda - \frac{n-2}{n(n-1)} R \right)^2. \quad (46)$$

By Lemma 14 and (46), we have

$$\begin{aligned} \frac{1}{2} \Delta_f |\dot{R}c|^2 &\geq |\nabla \dot{R}c|^2 + 2 \left(\lambda - \frac{(n-2)R}{n(n-1)} - \sqrt{\frac{n-2}{2(n-1)}} \right. \\ &\quad \cdot \left(|W|^2 + \frac{8}{n(n-2)} |\dot{R}c|^2 \right)^{1/2} \Big) |\dot{R}c|^2 \\ &\geq 2 \left(\lambda - \frac{(n-2)R}{n(n-1)} - \sqrt{\frac{n-2}{2(n-1)}} \right. \\ &\quad \cdot \left(|W|^2 + \frac{8}{n(n-2)} |\dot{R}c|^2 \right)^{1/2} \Big) |\dot{R}c|^2 \geq 0, \end{aligned} \quad (47)$$

which along with Lemma 18 and (46) yields $|R^\circ c| = 0$ and therefore (M^n, g) is Einstein.

On the other hand, if (45) holds, it is easily seen that (46) also holds. Indeed, when $n = 3$, clearly $1/C^2(3) = 2/3 < 4$. When $n \geq 4$, we see from the fact

$$C(n) = \frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2 - n - 4}{2\sqrt{(n-2)(n-1)n(n+1)}} > \frac{n-2}{\sqrt{n(n-1)}} \quad (48)$$

that

$$\frac{1}{C^2(n)} < \frac{n(n-1)}{(n-2)^2} \leq \frac{2(n-1)}{n-2}. \quad (49)$$

It follows from Lemma 18 and (45) that $|R^\circ c| = 0$, which together with (16) implies

$$|\dot{R}m|^2 = |W|^2 \leq \frac{1}{C^2(n)} \left(\lambda - \frac{n-2}{n(n-1)} R \right)^2. \quad (50)$$

By Lemma 10 and (50) we know that

$$\begin{aligned} \frac{1}{2} \Delta_f |\dot{R}m|^2 &\geq |\nabla \dot{R}m|^2 + 2 \left(\lambda - \frac{(n-2)R}{n(n-1)} - C(n) |\dot{R}m| \right) |\dot{R}m|^2 \\ &\quad + \frac{2(n-2)R}{n(n-1)} |W|^2 \\ &\geq 2 \left(\lambda - \frac{(n-2)R}{n(n-1)} - C(n) |\dot{R}m| \right) |\dot{R}m|^2 \geq 0, \end{aligned} \quad (51)$$

where the fact that $R \geq 0$ for shrinking solitons (see Lemma 17 or Corollary 2.5 of [13]) is used in the second inequality in (51). It follows from Lemma 18 and (50) that $|R^\circ m|$ is a constant and therefore all equalities in (51) hold.

If there exists $x_0 \in M^n$ such $R(x_0) = 0$, then, we see from Lemma 17 that M^n is isometric to \mathbb{R}^n .

Otherwise, the facts $R > 0$, $|R^\circ c| = 0$ and the equalities of (51) imply that $R^\circ m = W = 0$. Hence, we know that (M^n, g) has constant sectional curvature when $R > 0$; it follows from

the Myers theorem and the condition $R > 0$ that (M^n, g) is compact and therefore is a finite quotient of \mathbb{S}^n .

Proof of Theorem 4. It is well known that $R \geq 0$ for shrinking solitons. When R achieves its infimum 0, Lemma 17 says that (M^n, g, f) is flat and therefore is isometric to \mathbb{R}^n .

In the rest of the proofs of Theorem 4, we assume that $R > 0$. By (43) and (9), we see that

$$\begin{aligned} \Delta_{f-2\nabla \log R} \left(\frac{|\dot{R}m|^2}{R^2} \right) &\geq \frac{2}{R^3} \left(\sqrt{R} |\nabla \dot{R}m| - \frac{|\dot{R}m| |\nabla R|}{\sqrt{R}} \right)^2 \\ &\quad + \frac{4(n-2)|W|^2}{n(n-1)R} + \frac{4|\dot{R}m|^2}{R^3} \\ &\quad \cdot \left(\frac{R^2}{n(n-1)} + |\dot{R}c|^2 - C(n) |\dot{R}m| \right) \\ &\geq \frac{4|\dot{R}m|^2}{R^3} \left(\frac{R^2}{n(n-1)} + |\dot{R}c|^2 - C(n)R |\dot{R}m| \right) \\ &\geq 0. \end{aligned} \quad (52)$$

Set $w = f - \nabla \log(R^2)$ and $u = (|R^\circ m|^2)/R^2$. In order to apply Lemma 19 to (52), we need to verify

$$\int_{M^n} e^{-w} dV_g = \int_{M^n} R^2 e^{-f} dV_g < +\infty, \quad (53)$$

$$\int_{M^n} \frac{|\dot{R}m|^4}{R^4} e^{-w} dV_g = \int_{M^n} \frac{|\dot{R}m|^4}{R^2} e^{-f} dV_g < +\infty. \quad (54)$$

In fact, (53) follows from the result that $R \in L^p(e^{-f} dV_g)$ for $1 \leq p < +\infty$ (see e.g., [10]) for all gradient shrinkers.

Under the assumption that (M^n, g, f) has nonnegative Ricci curvature and (9), we get $|Rc| \leq R$. Thus,

$$\begin{aligned} |\dot{R}m| &\leq \frac{1}{C(n)} \left(R - \frac{(n-2)R}{n(n-1)} \right) = \frac{n^2 - 2n + 2}{n(n-1)C(n)} R, \\ \frac{|\dot{R}m|^4}{R^2} &\leq \left(\frac{n^2 - 2n + 2}{n(n-1)C(n)} \right)^4 R^2 \in L(e^{-f} dV_g). \end{aligned} \quad (55)$$

Furthermore, as observed in Remark 5, if we relax the condition of negative Ricci curvature to that $|Rc| \leq cR^{1+\alpha}$ for some constants $c > 0$ and $\alpha \geq 0$ satisfying $cR^\alpha \geq \sqrt{(n-2)/(n(n-1))}$, then

$$|\dot{R}m| \leq \frac{R}{C(n)} \left(c^2 R^{2\alpha} - \frac{n-2}{n(n-1)} \right). \quad (56)$$

It follows from the result $R \in L^p(e^{-f} dV_g)$ for $1 \leq p < +\infty$ that

$$\frac{|\mathring{R}m|^4}{R^2} \in L\left(e^{-f} dV_g\right). \quad (57)$$

These together with (52) and Lemma 19 yield $R^\circ m = 0$ or $W = 0$ and $|R^\circ m| = 1/C(n)((R/(n(n-1))) + (|R^\circ c|^2/R))$.

What we need to prove now is that $R^\circ c = 0$ in the latter case. In fact, by (16), the facts $W = 0$ and $|R^\circ m| = 1/C(n)((R/(n(n-1))) + (|R^\circ c|^2/R))$, we derive

$$|\mathring{R}c| = \frac{\sqrt{n-2}}{2} |\mathring{R}m| = \frac{\sqrt{n-2}}{2C(n)} \left(\frac{R}{n(n-1)} + \frac{|\mathring{R}c|^2}{R} \right). \quad (58)$$

It is easy to check from the definition of $C(n)$ that the different two solutions of equation (58) satisfy

$$|\mathring{R}c|_1 = \frac{C(n)R}{\sqrt{n-2}} + R\sqrt{\frac{C^2(n)}{n-2} - \frac{1}{n(n-1)}} > \frac{\sqrt{n-2}}{\sqrt{n(n-1)}} R, \quad (59)$$

$$\begin{aligned} |\mathring{R}c|_2 &= \frac{(1/(n(n-1)))R^2}{\left(C(n)R/\sqrt{n-2} + R\sqrt{(C^2(n)/n-2) - (1/(n(n-1)))}\right)} \\ &< \frac{R}{\sqrt{(n-2)(n-1)n}}. \end{aligned} \quad (60)$$

Combining (59) and (60) and the fact $W = 0$ with Lemma 16 gives

$$\begin{aligned} \Delta_{f-2\nabla\log R} \left(\frac{|\mathring{R}c|^2}{R^2} \right) &\geq \frac{2}{R^4} |\nabla_j R \nabla_j R_{ik} - \nabla_j R R_{ik}|^2 + \frac{4|\mathring{R}c|^2}{R^3} \\ &\quad \cdot \left(|\mathring{R}c| - \frac{R}{\sqrt{n(n-1)}} \right)^2 \\ &\geq \frac{4|\mathring{R}c|^2}{R^3} \left(|\mathring{R}c| - \frac{R}{\sqrt{n(n-1)}} \right)^2 \\ &\geq 0. \end{aligned} \quad (61)$$

By a similar argument, we conclude from Lemma 19 and the assumption on the Ricci curvature that $|R^\circ c|^2/R$ is a constant and $R^\circ c = 0$ since $|R^\circ c|_{1,2} \neq R/\sqrt{n(n-1)}$. This concludes the proof of Theorem 4.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A General Inequality for CR-Warped Products in Generalized Sasakian Space Form and Its Applications

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In the present paper, by considering the Gauss equation in place of the Codazzi equation, we derive new optimal inequality for the second fundamental form of CR-warped product submanifolds into a generalized Sasakian space form. Moreover, the inequality generalizes some inequalities for various ambient space forms.

1. Introduction

The fundamental idea of warped product manifolds was first initiated in [1] with manifolds of negative curvature. Let N_1 and N_2 be two Riemannian manifolds endowed with Riemannian matrices g_1 and g_2 , respectively, such that $f : N_1 \rightarrow (0, \infty)$ is a positive smooth function on N_1 . Then, the warped product $M = N_1 \times_f N_2$ is characterized as the product manifold $N_1 \times N_2$ with the equipped metric $g = g_1 + f^2 g_2$. In particular, if $f = \text{constant}$, then M turned to be a Riemannian product manifold; otherwise, M is called a nontrivial warped product manifold. Let $M = N_1 \times_f N_2$ be a nontrivial warped product manifold. Then,

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \quad (1)$$

for any vector fields $X, Y \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$. If we consider a local orthonormal frame $\{e_1, e_2, \dots, e_n\}$ such that $\{e_i\}_{1 \leq i \leq n_1} \in N_1$ and $\{e_j\}_{n_1+1 \leq j \leq n} \in N_2$, we have

$$\sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq j \leq n} K(e_i \wedge e_j) = \frac{n_2 \Delta f}{f}. \quad (2)$$

In [2], Chen established the inequality for the squared

norm of the mean curvature and the warping function f of a CR-warped product $N_T \times_f N_\perp$, where N_\perp is a totally real submanifold and N_T is a holomorphic submanifold, isometrically immersed in a complex space form as follows.

Theorem 1 (see [2]). *$N_T^{n_1} \times_f N_\perp^{n_2}$ be a CR-warped product into a complex space form $\tilde{M}(4c)$ with constant sectional curvature c . Then,*

$$\|h\|^2 \geq 2n_2 \{ \|\nabla \ln f\|^2 + \Delta(\ln f) + 2n_1 c \}, \quad (3)$$

where Δ is the Laplacian operator of N_T . Moreover, the equality holds if and only if N_T is totally geodesic and N_\perp is totally umbilical in $\tilde{M}(4c)$.

Moreover, Theorem 1 is extended to CR-warped product submanifolds in a generalized Sasakian space form by using the same technique.

Theorem 2 (see [3]). *Let $N_T^{n_1} \times_f N_\perp^{n_2}$ be a contact CR-warped product submanifold of a generalized Sasakian space form $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ such that the structure vector field ξ is tangent to base manifold. Then, the following inequality is satisfied:*

$$\|h\|^2 \geq 2n_2(\|\nabla(\ln f)\|^2 - \Delta(\ln f) + 1) + 4n_1n_2(\lambda_1 + 1), \quad (4)$$

where Δ denotes the Laplace operator on $N_T^{n_1}$. The equality holds if and only if $N_T^{n_1}$ is a totally geodesic submanifold of $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$; in this case, $N_T^{n_1}$ is a generalized Sasakian space form of $(\lambda_1 + 3\lambda_3)$.

Furthermore, Mustafa et al. [4] recalled some fundamental problems of CR-warped products in Kenmotsu space forms as to simple relationships between the second fundamental form and the main intrinsic invariants by using the Gauss equation. In [5–7], some sharp inequalities are established for the sectional curvature of warped product pointwise semislant submanifolds in various space forms such as a Sasakian space form, a cosymplectic space form, a Kenmotsu space form, and a complex space form in terms of the Laplacian and the squared norm of a warping function with pointwise slant immersions. Afterward, several geometers [1, 2, 4, 8–18] obtained similar inequalities for different types of warped products in different kinds of structures.

Al-Ghefari et al. [3] proved the existence of CR-warped product submanifolds of type $N_T \times_f N_\perp$ in trans-Sasakian manifolds. They obtained an inequality for the second fundamental form with constant sectional curvature in terms of a warping function. Moreover, the nonexistence of CR-warped products of the form $N_\perp \times_f N_T$ in a generalized Sasakian space form was proved in [19].

In this paper, we shall establish a Chen-type inequality for CR-warped product submanifolds in a generalized Sasakian space form by considering the nontrivial case $N_T \times_f N_\perp$. We also find some applications of the inequality in the compact Riemannian manifold by using integration theory on manifolds. Our future work then is combining the work done in this paper with the techniques of singularity theory presented in [20–23] to explore new results on manifolds.

2. Preliminaries

An almost contact metric manifold $(\tilde{M}, g, \varphi, \eta, \xi)$ is an odd-dimensional manifold \tilde{M} , endowed with a field φ of an endomorphism on the tangent space, the Reeb vector field ξ , a 1-form η and admits Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (5)$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad (6)$$

$$\eta(U) = g(U, \xi), \quad (7)$$

for any $U, V \in \Gamma(T\tilde{M})$. An almost contact metric manifold $(\tilde{M}, g, \varphi, \eta, \xi)$ is said to be trans-Sasakian manifold (cf. [12, 13]) if

$$(\nabla_U \varphi)V = \alpha(g(U, V)\xi - \eta(U)V) - \beta(g(\varphi U, V) - \eta(V)\varphi U), \quad (8)$$

for any $U, V \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Riemannian

connection on (\tilde{M}, g) . If we replace $U = \xi$ and $V = \xi$ in (8), we find that $(\nabla_\xi \varphi)\xi = 0$, which implies that $\nabla_\xi \xi = 0$. For a trans-Sasakian manifold, (8) implies

$$\nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi). \quad (9)$$

Remark 3. We classify a trans-Sasakian manifold in the following way:

- (a) If $\alpha = 0$ and $\beta = 0$ in (8), a trans-Sasakian manifold becomes a cosymplectic manifold [7]
- (b) If $\alpha = 1$ and $\beta = 0$ in (8), it is a Sasakian manifold [5]
- (c) If $\alpha = 0$ and $\beta = 1$ in (8), it is a Kenmotsu manifold [6]
- (d) α -Sasakian manifold and β -Kenmotsu manifold can be derived from the trans-Sasakian manifold when $\beta = 0$ and $\alpha = 0$ in (8), respectively

Given an almost contact metric manifold $(\tilde{M}, \varphi, \eta, \xi)$, it is said to be a generalized Sasakian space form $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ if there exist three functions λ_1, λ_2 , and λ_3 on \tilde{M} such that the curvature tensor \tilde{R} is

$$\begin{aligned} \tilde{R}(X, Y, Z, W) = & \lambda_1(g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ & + \lambda_2(g(X, \varphi Z)g(\varphi Y, W) - g(Y, \varphi Z)g(\varphi X, W) \\ & + 2g(X, \varphi Y)g(\varphi Z, W)) + \lambda_3(\eta(X)\eta(Z)g(Y, W) \\ & - \eta(Y)\eta(Z)g(X, W) + g(X, Z)\eta(Y)\eta(W) \\ & - g(Y, Z)\eta(X)\eta(W)), \end{aligned} \quad (10)$$

for any $X, Y, Z, W \in \Gamma(T\tilde{M})$ [24].

Remark 4. The characteristics are as follows:

- (a) If $\lambda_1 = c + 3/4$ and $\lambda_2 = \lambda_3 = (c - 1)/4$, then \tilde{M} is a Sasakian space form [25]
- (b) If $\lambda_1 = (c - 3)/4$ and $\lambda_2 = \lambda_3 = (c + 1)/4$, then \tilde{M} is a Kenmotsu space form [6]
- (c) If $\lambda_1 = \lambda_2 = \lambda_3 = c/4$, then \tilde{M} is a cosymplectic space form [26]

Let ∇ and ∇^\perp be the induced Riemannian connections on the tangent bundle TM and the normal bundle $T^\perp M$ of a submanifold M of an almost contact metric manifold $(\tilde{M}, \varphi, \eta, \xi)$ with the induced metric g . Then, the Gauss and Weingarten formulas are given by

$$(i) \nabla_U V = \nabla_U V + h(U, V), (ii) \nabla_U N = -A_N U + \nabla_U^\perp N, \quad (11)$$

for $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator on

M . We have the relation:

$$g(h(U, V), N) = g(A_N U, V), \quad (12)$$

for $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$. For any tangent vector $U \in \Gamma(TM)$ and normal vector $N \in \Gamma(T^\perp M)$, we have

$$\begin{aligned} \text{(i)} \quad \varphi U &= TU + FU, \\ \text{(ii)} \quad \varphi N &= tN + fN, \end{aligned} \quad (13)$$

where $TU(tN)$ and $FU(fN)$ are tangential and normal components of $\varphi U(\varphi N)$, respectively. If T is identically zero, then a submanifold M is called a *totally real submanifold*. The Gauss equation with curvature tensors \tilde{R} and R on \tilde{M} and M , respectively, is defined by

$$\begin{aligned} \tilde{R}(U, V, Z, W) &= R(U, V, Z, W) + g(h(U, Z), h(V, W)) \\ &\quad - g(h(U, W), h(V, Z)), \end{aligned} \quad (14)$$

for any $U, V, Z, W \in \Gamma(TM)$. The mean curvature vector H for a local frame $\{e_1, e_2, \dots, e_n\}$ of the tangent space T on M is defined by

$$\begin{aligned} \text{(i)} \quad H &= \frac{1}{n} \text{trace}(h) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i), \\ \text{(ii)} \quad \|T\|^2 &= \sum_{i,j=1}^n g^2(\varphi e_i, e_j). \end{aligned} \quad (15)$$

The scalar curvature τ for a Riemannian submanifold M is given by

$$\tau = \sum_{1 \leq i=j \leq n} K(e_i \wedge e_j), \quad (16)$$

where $K(e_i \wedge e_j)$ is the sectional curvature of section plane and spanned by e_i and e_j . Let G_r be an r -plane section on TM and let $\{e_1, e_2, \dots, e_r\}$ be an orthonormal basis of G_r . Then, the scalar curvature $\tau(G_r)$ of G_r is given by

$$\tau(G_r) = \sum_{1 \leq i=j \leq r} K(e_i \wedge e_j). \quad (17)$$

Similarly, we classify a Riemannian submanifold M said to be *totally umbilical* and *totally geodesic* if $h(U, V) = g(U, V)H$ and $h(U, V) = 0$, respectively, for any $U, V \in \Gamma(TM)$.

Furthermore, if $H = 0$, then M is *minimal* in $(\tilde{M}, \varphi, \eta, \xi)$. If φ preserves any tangent space of M tangent to the structure vector field ξ , i.e., $\varphi(T_p M) \subseteq T_p M$, for each $p \in M$; then, M is called an *invariant submanifold*. Similarly, M is called an *anti-invariant submanifold* tangent to the Reeb vector field ξ if $\varphi(T_p M) \subseteq T^\perp M$, for each $p \in M$. To generalize these definitions, we give the following definition.

Definition 5. A submanifold M including the structure vector field ξ of an almost contact metric manifold $(\tilde{M}, \varphi, \eta, \xi)$ is characterized to be a contact CR-submanifold if the pair of orthogonal distributions \mathcal{D} and \mathcal{D}^\perp exists such that

- (i) $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle \xi \rangle$, where $\langle \xi \rangle$ is 1-dimensional distribution spanned by ξ
- (ii) the distribution \mathcal{D} is invariant, i.e., $\varphi(\mathcal{D}) \subseteq \mathcal{D}$
- (iii) the distribution \mathcal{D}^\perp is anti-invariant, i.e., $\varphi\mathcal{D}^\perp \subseteq (T^\perp M)$

If the dimensions of *invariant* distribution \mathcal{D} and *anti-invariant* distribution \mathcal{D}^\perp of a contact CR-submanifold of $(\tilde{M}, \varphi, \eta, \xi)$ are d_1 and d_2 , respectively, such that $d_2 = 0$, then M is invariant and anti-invariant if $d_1 = 0$. It is called a *proper contact CR-submanifold* if neither $d_1 = 0$ nor $d_2 = 0$. The normal bundle $T^\perp M$ of a contact CR-submanifold with an invariant subspace μ under φ can be decomposed as

$$T^\perp M = \varphi\mathcal{D}^\perp \oplus \mu. \quad (18)$$

M is a compact orientable Riemannian submanifold without boundary. Thus, we have

$$\int_M \Delta f dV = 0, \quad (19)$$

where dV is the volume element of M [27].

3. Main Inequalities of CR-Warped Products

We are mentioning that in the following study, we shall consider the structure field ξ tangent to the base manifold of warped product manifold. In this main section, we classify the contact CR-warped product submanifolds in a trans-Sasakian manifold.

Lemma 6. Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold in a trans-Sasakian manifold. Then,

$$\begin{aligned} g(h(\varphi X, Y), \varphi Z) &= g(h(X, Y), \varphi Z) = 0, \\ g(h(X, X), \beta) &= -g(h(\varphi X, \varphi X), \beta), \end{aligned} \quad (20)$$

for $X, Y \in \Gamma(TN_T)$, $Z, W \in \Gamma(TN_\perp)$, and $\beta \in \Gamma(\mu)$.

Proof. From (11)(i), (8), and (5), we obtain

$$g(h(\varphi X, Y), \varphi Z) = g(\bar{\nabla}_Y \varphi X, \varphi Z) = g(\bar{\nabla}_Y X, Z) - \eta(\bar{\nabla}_Y X)\eta(Z). \quad (21)$$

Since N_T is totally geodesic in M with $\xi \in \Gamma(TN_T)$, (9) implies the results. \square

Lemma 7. Let $\ell : M = N_T \times_f N_\perp \longrightarrow (\tilde{M}, \varphi, \eta, \xi)$ be an isometric immersion from an n -dimensional contact CR-

warped product submanifold into a trans-Sasakian manifold $(\tilde{M}, \varphi, \eta, \xi)$ such that N_T is invariant submanifold of dimension $n_1 = 2d_1 + 1$ tangent to ξ . Then, N_T is always ℓ -minimal submanifold of \tilde{M} .

Proof. We skip the proof of the above lemma due to the similar proof of Theorem 4.2 in [4]. \square

By helping the above lemma, the following result can be obtained as follows.

Proposition 8. Assume that $\ell : M = N_T \times_f N_\perp \longrightarrow \tilde{M}$ is an isometric immersion of an n -dimensional contact CR-warped product submanifold M into a trans-Sasakian manifold \tilde{M} . Thus,

- (i) the squared norm of the second fundamental form of M is satisfied:

$$\|h\|^2 \geq 2(n_2 \|\nabla \ln f\|^2 + \tilde{\tau}(TM) - \tilde{\tau}(TN_T) - \tilde{\tau}(TN_\perp) - n_2 \Delta(\ln f)), \quad (22)$$

where n_2 is the dimension of anti-invariant submanifold N_\perp and Δ is the Laplacian operator of N_T

- (ii) the equality holds in (22) if and only if N_T is totally geodesic and N_\perp is totally umbilical in \tilde{M} . Moreover, M is minimal submanifold of \tilde{M}

Proof. It can be easily proven as the proof of Theorem 4.4 in [4] if we consider a Riemannian submanifold as a CR-warped product submanifold, and the base manifold is a trans-Sasakian manifold instead of a Kenmotsu manifold.

Now, we prove our main theorem using Proposition 8 for a generalized Sasakian space form. \square

Theorem 9. Let $\ell : M = N_T \times_f N_\perp \longrightarrow \tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ be an isometric immersion from an n -dimensional contact CR-warped product submanifold of a generalized Sasakian space form $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$. Then, the second fundamental form is given by

$$\|h\|^2 \geq 2n_2 \left\{ \|\nabla \ln f\|^2 + \lambda_1 n_1 + \frac{3}{2} \lambda_2 - \lambda_3 - \Delta(\ln f) \right\}, \quad (23)$$

where $n_1 = \dim N_T$, $n_2 = \dim N_\perp$, and Δ is the Laplacian operator on N_T . The equality holds in (23) if and only if N_T and N_\perp are totally geodesic and totally umbilical submanifolds in $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$, respectively, and hence, M is a minimal submanifold of $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$.

Proof. Substituting $X = W = e_i$ and $Y = Z = e_j$ in (10), we get

$$\begin{aligned} \tilde{R}(e_i, e_j, e_j, e_i) = & \lambda_1 \{ g(e_i, e_i) g(e_j, e_j) - g(e_i, e_j) g(e_j, e_i) \} \\ & + \lambda_2 \{ g(e_i, \varphi e_j) g(\varphi e_j, e_i) - g(e_i, \varphi e_i) g(e_j, \varphi e_j) \\ & + 2g^2(\varphi e_j, e_i) \} + \lambda_3 \{ g(e_i, e_i) \eta(e_j) \eta(e_j) \\ & - g(e_i, e_j) \eta(e_i) \eta(e_i) + g(e_j, e_j) \eta(e_i) \eta(e_i) \\ & \cdot g(e_j, e_i) (e_i) \eta(e_j) \}. \end{aligned} \quad (24)$$

Summing up along the orthonormal vector fields of M , it can be derived from the above as

$$2\tilde{\tau}(TM) = \lambda_1 n(n-1) + 3\lambda_2 \sum_{1 \leq i=j \leq n} g^2(\varphi e_i, e_j) - 2\lambda_3(n-1). \quad (25)$$

As for an n -dimensional CR-warped product submanifold tangent ξ , one can derive $\|T\|^2 = n-1$ from (15)(ii); we obtain

$$2\tilde{\tau}(TM) = \lambda_1 n(n-1) + 3(n-1)\lambda_2 - 2\lambda_3(n-1). \quad (26)$$

On the other hand, by helping the frame field of TN_\perp , we have

$$2\tilde{\tau}(TN_\perp) = \lambda_1 n_2(n_2-1). \quad (27)$$

Similarly, we considered that ξ is tangent to invariant submanifold N_T . Then, using the frame vector fields of TN_T , we get from (24)

$$2\tilde{\tau}(TN_T) = \lambda_1 n_1(n_1-1) + 3\lambda_2(n_1-1) - 2\lambda_3(n_1-1). \quad (28)$$

Therefore, using (26), (27), and (28) in Proposition 8, we get the required result. The equality case follows from Proposition 8. Thus, the proof is completed. \square

4. Geometric Applications

Remark 10. Consider $\lambda_1 = (c-3)/4$ and $\lambda_2 = \lambda_3 = (c+1)/4$ in Theorem 9. It is the generalization of Theorem 4.6 in [4] for the result of contact CR-warped products in Kenmotsu space forms.

Remark 11. If we put $\lambda_1 = (c+3)/4$ and $\lambda_2 = \lambda_3 = (c-1)/4$ in Theorem 9, then it generalizes Corollary 4.6 in [5].

Remark 12. If $\lambda_1 = \lambda_2 = \lambda_3 = c/4$ in Theorem 9, then Theorem 9 coincides with Theorem 1.2 in [26].

Corollary 13. Let $\ln f$ be a harmonic function on N_T . Then there does not exist any CR-warped product submanifold $N_T \times_f N_\perp$ into a generalized Sasakian space form $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ with $c \leq -\lambda_1$.

Corollary 14. Assume that $\ln f$ is a nonnegative eigenfunction on N_T with the corresponding nonzero positive eigenvalue. Then, there does not exist any CR-warped product submanifold $N_T \times_f N_\perp$ into a generalized Sasakian space form $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ with $c \leq -\lambda_1$.

Theorem 15. Let $M = N_T \times_f N_\perp$ be a compact orientated CR-warped product into a generalized Sasakian space form $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$. Then, M is a simply Riemannian product if

$$\|h\|^2 \geq 2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_2, \quad (29)$$

where $n_1 = \dim N_T$ and $n_2 = \dim N_\perp$.

Proof. From Theorem 9, we get

$$\begin{aligned} \|h\|^2 &\geq 2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_3 - n_2 \Delta(\ln f) + n_2 \|\nabla \ln f\|^2, \\ n_2 \|\nabla \ln f\|^2 + 2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_3 - \|h\|^2 &\leq n_2 \Delta(\ln f). \end{aligned} \quad (30)$$

We obtain

$$\begin{aligned} \int_{N_T \times q} (2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_2 + n_2 \|\nabla \ln f\|^2 - \|h\|^2) dV \\ \leq n_2 \int_{N_T \times q} \Delta(\ln f) dV = 0. \end{aligned} \quad (31)$$

Now, if

$$\|h\|^2 \geq 2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_2. \quad (32)$$

Then, from (31), we find

$$\int_{N_T \times q} (\|\nabla \ln f\|^2) dV \leq 0, \quad (33)$$

which is impossible for a positive integral function, and hence, $\nabla \ln f = 0$, i.e., f is a constant function on N_T . Thus, by the definition of a warped product manifold, M is trivial. The converse part is straightforward. \square

Corollary 16. Assume that $M = N_T \times_f N_\perp$ is a CR-warped product submanifold in a generalized Sasakian space form $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$. Let N_T be a compact invariant submanifold and γ be nonzero eigenvalue of the Laplacian on N_T . Then,

$$\begin{aligned} \int_{N_T \times q} \|h\|^2 dV_T &\geq (2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_2) \text{Vol}(N_T) \\ &+ 2n_2 \gamma \int_{N_T \times q} (\ln f)^2 dV_T. \end{aligned} \quad (34)$$

Proof. From the minimum principle property, we obtain

$$\int_{N_T} \|\nabla \ln f\|^2 dV_T \geq \gamma \int_{N_T} (\ln f)^2 dV_T. \quad (35)$$

From (23) and (35), we get the required result (34). \square

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors declare no competing interest.

Authors' Contributions

All authors have equal contribution and finalized.

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Research Article

Study of Differential Equations on Warped Product Semi-Invariant Submanifolds of the Generalized Sasakian Space Forms

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The purpose of the present paper is to study the applications of Ricci curvature inequalities of warped product semi-invariant product submanifolds in terms of some differential equations. More precisely, by analyzing Bochner's formula on these inequalities, we demonstrate that, under certain conditions, the base of these submanifolds is isometric to Euclidean space. We also look at the effects of certain differential equations on warped product semi-invariant product submanifolds and show that the base is isometric to a special type of warped product under some geometric conditions.

1. Introduction

Bishop and O'Neill [1] evaluated the geometry of manifolds having negative curvature and noticed that Riemannian product manifolds do have nonnegative curvature. As a result, they came up with the recommendation of warped product manifolds, which are described as follows.

Consider two Riemannian manifolds (L_1, g_1) and (L_2, g_2) with corresponding Riemannian metrics g_1 and g_2 and $\psi : L_1 \rightarrow R$ be a positive differentiable function. If x and y are projection maps such that $x : L_1 \times L_2 \rightarrow L_1$ and $y : L_1 \times L_2 \rightarrow L_2$, which are defined as $x(m, n) = m$ and $y(m, n) = n \forall (m, n) \in L_1 \times L_2$, then, $\bar{L} = L_1 \times L_2$ is called warped product manifold if the Riemannian structure on L satisfies

$$g(\bar{E}, \bar{F}) = g_1(x_*\bar{E}, x_*\bar{F}) + (\psi \circ x)^2 g_2(y_*\bar{E}, y_*\bar{F}), \quad (1)$$

for all $\bar{E}, \bar{F} \in T\bar{L}$. The function ψ represents the warping function of $L_1^n \times L_2$. The Riemannian product manifold is a special case of warped product manifold in which the warping function $\psi = 1$. The study of Bishop and O'Neill [1] revealed that these types of manifolds have a wide range of applications in physics and theory of relativity. It is well

known that the warping function is the solution of some partial differential equations; for example, the Einstein field equation can be solved by the approach of warped product [2]. The warped product is also applicable in the study of space time near black holes [3].

On the other hand, the analysis of differential equation on Riemannian manifolds yields some important geometric and isometric intrinsic properties. It is well known that categorization of differential equation has a major influence on the global analysis of Riemannian manifolds. Tanno [4] explored various aspects of differential equations on Riemannian manifolds in 1978. The approach of differential equations was used by the authors [5, 6] to describe the Euclidean sphere. These calculations demonstrated that a nonconstant function λ on a complete Riemannian manifold (U^n, g) satisfies the differential equation as follows:

$$\nabla^2 \lambda + k g = 0, \quad (2)$$

if and only if (U^n, g) is congruent to Euclidean space R^n , where k is constant.

Furthermore, under some geometric conditions, Garcia-Rio et al. [6] proved that the Riemannian manifold is isometric to the warped product $U \times_f R$, where U is a complete

Riemannian manifold, R is the Euclidean line, and f is the warping function. Moreover, warping function f is the solution of the following differential equation:

$$\frac{d^2 f}{dt^2} + \mu_1 f = 0, \quad (3)$$

if and only if there exists a nonconstant function $\phi : U^n \rightarrow R$ with an eigenvalue $\lambda_1 < 0$, which satisfies the following differential equation:

$$\Delta \phi + \mu_1 \phi = 0. \quad (4)$$

The categorization of differential equations on Riemannian manifolds turns into an attractive research subject that has been explored by various researchers, for example, [7–11].

Al-Dayel et al. [7] recently investigated the effect of the differential equation (3) on the Riemannian manifold (L^n, g) using the concircular vector field, showing that the Riemannian manifold (L^n, g) is isometric to the Euclidean manifold R^n . By using the gradient conformal vector field, Chen et al. [12] discovered that the Riemannian manifold (L^n, g) is isometric to the Euclidean space R^n . However, it has been shown in [13] that the complete totally real submanifold in CP^n (complex projective space) with bounded Ricci curvature satisfying (4) is isometric to a special form of hyperbolic space.

Latterly, Ali et al. [8] characterized warped product submanifolds in Sasakian space form by the approach of differential equation. The purpose of this paper is to study the impact of differential equation on warped product semi-invariant product submanifolds in the framework of generalized Sasakian space form.

2. Preliminaries

A $(2k+1)$ -dimensional C^∞ -manifold \bar{L} is said to have an almost contact structure if there exists on \bar{L} a tensor field ϕ of the type $(1, 1)$, a vector field χ , and a 1-form η satisfying

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \chi, \\ \phi \chi &= 0, \\ \eta \circ \phi &= 0, \\ \eta(\chi) &= 1. \end{aligned} \quad (5)$$

On an almost contact metric manifold \bar{L} , there is always a Riemannian metric g that meets the following requirements:

$$\begin{aligned} \eta(E) &= g(E, \chi), \\ g(\phi E, \phi F) &= g(E, F) - \eta(E)\eta(F), \end{aligned} \quad (6)$$

for all $E, F \in T\bar{L}$.

An almost contact metric manifold is said to be nearly Sasakian manifold, if

$$(\bar{\nabla}_E \phi)F + (\bar{\nabla}_F \phi)E = -2g(E, F)\chi + \eta(F)E + \eta(E)F, \quad (7)$$

for all $E, F \in T\bar{L}$.

In [14], Alegre et al. gave the concept of generalized Sasakian space form as that an almost contact metric manifold $(\bar{L}, f, \chi, \eta, g)$ whose curvature tensor \bar{R} satisfies

$$\begin{aligned} \bar{R}(E, F, G, W) &= \phi_1[g(F, G)g(E, W) - g(E, G)g(F, W)] \\ &\quad - \phi_2[g(\phi E, G)g(\phi F, W) - g(\phi E, W)g(\phi F, G) \\ &\quad + 2g(\phi E, F)g(\phi G, W)] - \phi_3[\eta(G)\{\eta(F)g(E, W) \\ &\quad - \eta(E)g(G, W)\} + \eta(W)\{\eta(E)g(F, G) - \eta(F)g(E, G)\}], \end{aligned} \quad (8)$$

for any vector fields E, F, G, W and certain differentiable functions ϕ_1, ϕ_2, ϕ_3 on \bar{L} . A generalized Sasakian space form with functions ϕ_1, ϕ_2, ϕ_3 is denoted by $\bar{L}(\phi_1, \phi_2, \phi_3)$. If $\phi_1 = (c+3)/4$, $\phi_2 = \phi_3 = (c-1)/4$, then $\bar{M}(\phi_1, \phi_2, \phi_3)$ is a Sasakian space form $\bar{M}(c)$ [14]. If $\phi_1 = (c-3)/4$, $\phi_2 = \phi_3 = (c+1)/4$, then $\bar{M}(\phi_1, \phi_2, \phi_3)$ is a Kenmotsu space form $\bar{M}(c)$ [14], and if $\phi_1 = \phi_2 = \phi_3 = c/4$, then $\bar{M}(\phi_1, \phi_2, \phi_3)$ is a cosymplectic space form $\bar{M}(c)$ [14].

A submanifold L of an almost contact metric manifold \bar{L} is called semi-invariant submanifolds (contact CR-submanifolds) if there exist two orthogonal complementary distributions D and D^\perp satisfying the following conditions:

- (i) $TL = D \oplus D^\perp \oplus \langle \xi \rangle$, where $\langle \xi \rangle$ is the distribution spanned by the structure vector field ξ
- (ii) D is invariant distribution, i.e., $\phi D \subset TL$
- (iii) D^\perp is anti-invariant, i.e., $\phi D^\perp \subseteq T^\perp L$

Recently, we [15] studied warped product semi-invariant product submanifolds of the type $L^n = L_T^{n_1} \times_f L_\perp^{n_2}$ isometrically immersed in the generalized Sasakian space form admitting a nearly Sasakian structure, where $L_T^{n_1}$ is an invariant submanifold of dimension n_1 and $L_\perp^{n_2}$ is a totally real submanifold of dimension n_2 . More precisely, the computed Ricci curvature inequalities for these submanifolds are as follows:

Theorem 1. *Let $L^n = L_T^{n_1} \times_f L_\perp^{n_2}$ be a warped product semi-invariant submanifold isometrically immersed in a generalized Sasakian space form $\bar{L}^m(\phi_1, \phi_2, \phi_3)$ with nearly Sasakian structure. Then, for each orthogonal unit vector field $\xi \in T_x M$ orthogonal to χ , either tangent to $N_T^{n_1}$ or $N_\perp^{n_2}$, the Ricci curvature satisfies the following inequalities:*

- (i) If $\xi \in TL_T^{n_1}$, then

$$\begin{aligned} Ric(\xi) \leq & \frac{1}{4}n^2\|H\|^2 - n_2\Delta \ln f + n_2\|\nabla \ln f\|^2 + \frac{3\phi_2}{2} \\ & + (n + n_1n_2 - 1)\phi_1 - (n_2 + 1)\phi_3 \end{aligned} \quad (9)$$

(ii) If $\xi \in TL_{\perp}^{n_2}$, then

$$\begin{aligned} Ric(\xi) \leq & \frac{1}{4}n^2\|H\|^2 - n_2\Delta \ln f + n_2\|\nabla \ln f\|^2 \\ & + (n + n_1n_2 - 1)\phi_1 - (n_2 + 1)\phi_3 \end{aligned} \quad (10)$$

The equality cases can be seen in [15].

Let f be a real-valued differential function on a Riemannian manifold L^n , then the Bochner formula [16] is stated as

$$\frac{1}{2}\Delta|\nabla f|^2 = R^L(\nabla f, \nabla f) + |H(f)|^2 + g(\nabla \Delta f, \nabla f), \quad (11)$$

where R^L denotes Ricci tensor and $H(f)$ is the Hessian of the function f .

3. Main Results

In this section, we obtain some characterization by the application of Bochner's formula.

Theorem 2. Let $L^n = L_T^{n_1} \times_f L_{\perp}^{n_2}$ be a n -dimensional warped product semi-invariant product submanifold in a generalized Sasakian space form $\bar{L}^m(c)$, where $L_T^{n_1}$ is a n -dimensional invariant submanifold and $L_{\perp}^{n_2}$ is an anti-invariant submanifold. Such that Ricci curvature $R^L(\xi) \geq b$, $b > 0$. If $\xi \in TL_T^{n_1}$ and satisfying the following equality:

$$(\lambda_1 + n_2)b = \lambda_1 \left[\frac{n_2}{n} + \frac{n^2}{4}\|H\|^2 - (n + n_1n_2 - 1)\phi_1 - \frac{3\phi_2}{2} + (n_2 + 1)\phi_3 \right], \quad (12)$$

then, the base submanifold $L_T^{n_1}$ is isometric to R^{n_1} (Euclidean space).

Proof. Since $\xi \in TN_T^{n_1}$, by equation (9)

$$\begin{aligned} R^L(\xi) + n_2\Delta \ln f \leq & \frac{1}{4}n^2\|H\|^2 + n_2\|\nabla \ln f\|^2 + (n + n_1n_2 - 1)\phi_1 \\ & + \frac{3\phi_2}{2} - (n_2 + 1)\phi_3. \end{aligned} \quad (13)$$

By the assumption that $R_L(\xi) \geq b$, we have

$$\begin{aligned} b + n_2\Delta \ln f \leq & \frac{1}{4}n^2\|H\|^2 + n_2\|\nabla \ln f\|^2 + (n + n_1n_2 - 1)\phi_1 \\ & + \frac{3\phi_2}{2} - (n_2 + 1)\phi_3. \end{aligned} \quad (14)$$

Since the Ricci curvature $R^L(\xi)$ is bounded below by $b \geq 0$, then by virtue of theorem of Myers [17], the base manifold $L_T^{n_1}$ is compact. On integrating (9) and using Green's theorem, we have

$$\begin{aligned} \text{Vol}(L_T^{n_1})b \leq & \frac{n^2}{4} \int_{L_T^{n_1} \times \{q\}} \|H\|^2 dV + n_2 \int_{L_T^{n_1} \times \{q\}} \|\nabla \ln f\|^2 dV \\ & + \int_{L_T^{n_1} \times \{q\}} \left[(n + n_1n_2 - 1)\phi_1 + \frac{3\phi_2}{2} - (n_2 + 1)\phi_3 \right] dV, \end{aligned} \quad (15)$$

or

$$\begin{aligned} \int_{L_T^{n_1} \times \{q\}} \|\nabla \ln f\|^2 dV \geq & \frac{b}{n_2} \text{Vol}(L_T^{n_1}) - \frac{n^2}{4n_2} \int_{L_T^{n_1} \times \{q\}} \|H\|^2 dV - \frac{1}{n_2} \\ & \cdot \int_{L_T^{n_1} \times \{q\}} \left[(n + n_1n_2 - 1)\phi_1 + \frac{3\phi_2}{2} - (n_2 + 1)\phi_3 \right] dV. \end{aligned} \quad (16)$$

Suppose $H(\ln f)$ denotes the Hessian of the warping function $\ln f$, then we have

$$|H(\ln f) - nI|^2 = |H(\ln f)|^2 + n^2|I|^2 - 2ng(I, H(\ln f)), \quad (17)$$

after some calculations, the above formula turns to

$$|H(\ln f) - tI|^2 = 2t\Delta(\ln f) + t^2(n_1) + |H(\ln f)|^2. \quad (18)$$

Putting $t = \lambda_1/n_1$ and integrating the last equation with respect to dV (volume element), we get

$$\int_{L_T^{n_1} \times \{q\}} \left| H(\ln f) - \frac{\lambda_1}{n_1} I \right|^2 dV = \int_{L_T^{n_1} \times \{q\}} |H(\ln f)|^2 dV + \int_{L_T^{n_1} \times \{q\}} \frac{\lambda_1^2}{n_1} dV, \quad (19)$$

using (11), with the fact $\Delta \ln f = \lambda_1 \ln f$, we have

$$\int_{L_T^{n_1} \times \{q\}} |H(\ln f)|^2 dV = -\lambda_1 \int_{L_T^{n_1} \times \{q\}} |\nabla \ln f|^2 dV - \int_{L_T^{n_1} \times \{q\}} R^L(\nabla \ln f, \nabla \ln f). \quad (20)$$

Merging (19) and (20), we derive

$$\begin{aligned} \int_{L_T^{n_1} \times \{q\}} \left| H(\ln f) - \frac{\lambda_1}{n_1} I \right|^2 dV = & \int_{L_T^{n_1} \times \{q\}} \frac{\lambda_1^2}{n_1} dV - \lambda_1 \int_{L_T^{n_1} \times \{q\}} |\nabla \ln f|^2 dV \\ & - \int_{L_T^{n_1} \times \{q\}} R^L(\nabla f, \nabla f) dV. \end{aligned} \quad (21)$$

By the assumption $R^L(\nabla f, \nabla f) \geq b$, the above equation yields

$$\int_{L_T^{n_1} \times \{q\}} \left| H(\ln f) - \frac{\lambda_1}{n_1} I \right|^2 dV \leq \int_{L_T^{n_1} \times \{q\}} \frac{\lambda_1^2}{n_1} dV - b \text{Vol}(L_T^{n_1}) - \lambda_1 \int_{L_T^{n_1} \times \{q\}} |\nabla \ln f|^2 dV. \quad (22)$$

Using (16), the last inequality leads to

$$\begin{aligned} \int_{L_T^{n_1} \times \{q\}} \left| H(\ln f) - \frac{\lambda_1}{n_1} I \right|^2 dV &\leq \int_{L_T^{n_1} \times \{q\}} \frac{\lambda_1^2}{n_1} dV - \int_{L_T^{n_1} \times \{q\}} \left(\frac{\lambda_1 b}{n_2} + b \right) dV - \frac{\lambda_1}{n_2} \\ &\quad \cdot \int_{L_T^{n_1} \times \{q\}} \left[(n + n_1 n_2 - 1) \phi_1 + \frac{3\phi_2}{2} - (n_2 + 1) \phi_3 \right] dV \\ &\quad - \frac{\lambda_1 n^2}{4n_2} \int_{L_T^{n_1} \times \{q\}} \|H\|^2 dV. \end{aligned} \quad (23)$$

If (12) holds, then the above inequality produces

$$\left| H(\ln f) - \frac{\lambda_1}{n_1} I \right|^2 = 0. \quad (24)$$

Therefore, we have $H(\ln f)(X, X) = \lambda_1/n_1$. Hence, by the application of the result of Tashiro [18], the fibre $L_T^{n_1}$ is isometric to R^{n_1} (Euclidean space).

If we consider the unit vector field $\xi \in TL_{\perp}^{n_2}$, then we have the following results which can be proved by adopting similar steps in Theorem 2. \square

Theorem 3. Let $L^n = L_T^{n_1} \times_f L_{\perp}^{n_2}$ be a n -dimensional warped product semi-invariant product submanifold in a generalized Sasakian space form $\bar{L}^m(c)$, where $L_T^{n_1}$ is a n -dimensional invariant submanifold and $L_{\perp}^{n_2}$ is an anti-invariant submanifold. Such that Ricci curvature $R^L(\xi) \geq b, b > 0$. If $\xi \in TL_T^{n_1}$ and satisfying the following equality:

$$(\lambda_1 + n_2)b = \lambda_1 \left[\frac{n_2}{n} + \frac{n^2}{4} \|H\|^2 - (n + n_1 n_2 - 1) \phi_1 + (n_2 + 1) \phi_3 \right], \quad (25)$$

then, the base submanifold $L_T^{n_1}$ is isometric to R^{n_1} (Euclidean space).

Now, we have the next result which is based on the study of Garcia-Rio et al. [6].

Theorem 4. Let $L^n = L_T^{n_1} \times_f L_{\perp}^{n_2}$ be a warped product semi-invariant product submanifold in a generalized Sasakian space form admitting the nearly Sasakian structure $\bar{L}^m(c)$. Such that Ricci curvature $R^L(\xi) > b, b > 0$. If $\xi \in TN_T^{n_1}$ and satisfying the following relation:

$$\begin{aligned} n^2 \|H\|^2 + \frac{4(n_1 n_2)}{\lambda_1} |H(\ln f)|^2 \\ = \frac{4n_1 n_2}{\lambda_1} \left(b - (n + n_1 n_2 - 1) \phi_1 - \frac{3\phi_2}{2} + (n_2 + 1) \phi_3 \right), \end{aligned} \quad (26)$$

for $\lambda_1 < 0$, then $L_T^{n_1}$ is isometric to warped product of the type $R \times_{\theta} U$ with the warping function θ , which satisfies the differential equation $d\theta^2/dt^2 + \lambda_1 \theta = 0$.

Proof. For the warping function $\ln f$, defining the following equation on $L_T^{n_1}$:

$$|b \ln f I + H(\ln f)|^2 = b^2 (\ln f)^2 |I|^2 + |H(\ln f)|^2 + 2b (\ln f) g(I, H(\ln f)). \quad (27)$$

But we know that $|I|^2 = \text{tr}(II^*) = n_1$ and $g(H(\ln f), I^*) = \text{tr}(I^* H(\ln f)) = \text{tr}(H(\ln f))$; using these facts, the above equation leads to

$$|b \ln f I + H(\ln f)|^2 = |H(\ln f)|^2 + n_1 b^2 (\ln f)^2 - 2b \ln f \Delta \ln f. \quad (28)$$

Let $\ln f$ is an eigenfunction corresponding to the eigenvalue λ_1 satisfying $\Delta \ln f = \lambda_1 \ln f$, we have

$$|b \ln f I + H(\ln f)|^2 = |H(\ln f)|^2 + (n_1 b^2 - 2b \lambda_1) (\ln f)^2. \quad (29)$$

Further, using $\Delta \ln f = \lambda_1 \ln f$, it is easy to see that

$$\nabla \frac{(\ln f)^2}{2} = \ln f \lambda_1 \ln f - |\nabla \ln f|^2, \quad (30)$$

which on integrating provides

$$\int_{L_T^{n_1} \times \{q\}} (\ln f)^2 dV = \frac{1}{\lambda_1} \int_{L_T^{n_1} \times \{q\}} |\nabla \ln f|^2. \quad (31)$$

Thus, we have

$$\begin{aligned} \int_{L_T^{n_1} \times \{q\}} |H(\ln f) + b \ln f I|^2 dV &= \int_{L_T^{n_1} \times \{q\}} |H(\ln f)|^2 dV \\ &\quad + \left(\frac{n_1 b^2}{\lambda_1} - 2b \right) \int_{L_T^{n_1} \times \{q\}} |\nabla \ln f|^2 dV. \end{aligned} \quad (32)$$

Choosing $b = \lambda_1/n_1$ in (32), we have

$$\int_{L_T^{n_1} \times \{q\}} \left| H(\ln f) + \frac{\lambda_1}{n_1} \ln f I \right|^2 dV = \int_{L_T^{n_1} \times \{q\}} |H(\ln f)|^2 dV - \frac{\lambda_1}{n_1} \cdot \int_{L_T^{n_1} \times \{q\}} |\nabla \ln f|^2 dV. \quad (33)$$

Further, integrating (9) and applying Green's lemma, we find

$$\begin{aligned} \int_{L_T^{n_1} \times \{q\}} R^L(\xi) dV &\leq \frac{n^2}{4} \int_{L_T^{n_1} \times \{q\}} \|H\|^2 dV + n_2 \int_{L_T^{n_1} \times \{q\}} \|\nabla \ln f\|^2 dV \\ &\quad + \int_{L_T^{n_1} \times \{q\}} \left[(n + n_1 n_2 - 1)\phi_1 + \frac{3\phi_2}{2} - (n_2 + 1)\phi_3 \right] dV. \end{aligned} \quad (34)$$

From the above two expressions, we have

$$\begin{aligned} \frac{1}{n_2} \int_{L_T^{n_1} \times \{q\}} R^L(\xi) dV &\leq \frac{n^2}{4n_2} \int_{L_T^{n_1} \times \{q\}} \|H\|^2 dV + \frac{n_1}{\lambda_1} \int_{L_T^{n_1} \times \{q\}} |H(\ln f)|^2 dV \\ &\quad - \frac{n_1}{\lambda_1} \int_{L_T^{n_1} \times \{q\}} \left| H(\ln f) + \frac{\lambda_1}{n_1} \ln f I \right|^2 dV + \int_{L_T^{n_1} \times \{q\}} \\ &\quad \cdot \left[(n + n_1 n_2 - 1)\phi_1 + \frac{3\phi_2}{2} - (n_2 + 1)\phi_3 \right] dV. \end{aligned} \quad (35)$$

On using the assumption that $R^L(\xi) \geq b$, for $b > 0$,

$$\begin{aligned} \int_{L_T^{n_1} \times \{q\}} \left| H(\ln f) + \frac{\lambda_1}{n_1} \ln f I \right|^2 dV &\leq \frac{n^2 \lambda_1}{4n_1 n_2} \int_{L_T^{n_1} \times \{q\}} \|H\|^2 dV + \int_{L_T^{n_1} \times \{q\}} |H(\ln f)|^2 dV \\ &\quad - \frac{\lambda_1}{n_1 n_2} \int_{L_T^{n_1} \times \{q\}} b dV + \frac{\lambda_1}{n_1} \int_{L_T^{n_1} \times \{q\}} \\ &\quad \cdot \left[(n + n_1 n_2 - 1)\phi_1 + \frac{3\phi_2}{2} - (n_2 + 1)\phi_3 \right] dV, \end{aligned} \quad (36)$$

equivalently,

$$\begin{aligned} \int_{L_T^{n_1} \times \{q\}} \left| H(\ln f) + \frac{\lambda_1}{n_1} \ln f I \right|^2 dV &\leq \int_{L_T^{n_1} \times \{q\}} \\ &\cdot \left\{ \frac{\lambda_1}{n_1} \left(\frac{n^2}{4n_2} \|H\|^2 + \left((n + n_1 n_2 - 1)\phi_1 + \frac{3\phi_2}{2} - (n_2 + 1)\phi_3 \right) \right) \right. \\ &\quad \left. + \frac{b}{n_2} + |H(\ln f)|^2 \right\} dV. \end{aligned} \quad (37)$$

By assumption (26), we have

$$\left| H(\ln f) + \frac{\lambda_1}{n_1} \ln f I \right|^2 \leq 0, \quad (38)$$

which is not possible; therefore,

$$H(\ln f) + \frac{\lambda_1}{n_1} \ln f I = 0. \quad (39)$$

By taking trace of the above equation, we get

$$\Delta \ln f + \lambda_1 \ln f = 0. \quad (40)$$

Now, applying the result proved in [6], together with the fact that $L^t = L_T^{n_1} \times_f L_\perp^{n_2}$ is nontrivial, we deduced that $L_T^{n_1}$ is isometric to a warped product of the form $R \times_\theta U$, where U is complete Riemannian manifold. Moreover, the warping function θ is the solution of the differential equation $d\theta^2/dt^2 + \lambda_1 \theta = 0$. Hence, the proof is completed. \square

Similarly, we can prove the following theorems by taking the unit vector field ξ tangent to $L_\perp^{n_2}$.

Theorem 5. Let $L^n = L_T^{n_1} \times_f L_\perp^{n_2}$ be a warped product semi-invariant product submanifold in a generalized Sasakian space form admitting the nearly Sasakian structure $\bar{L}^m(c)$. Such that Ricci curvature $R^L(\xi) > b$, $b > 0$. If $\xi \in TN_\perp^{n_2}$ and satisfying the following relation:

$$\begin{aligned} n^2 \|H\|^2 + \frac{4(n_1 n_2)}{\lambda_1} |H(\ln f)|^2 &= \frac{4n_1 n_2}{\lambda_1} \\ &\cdot (b - (n + n_1 n_2 - 1)\phi_1 + (n_2 + 1)\phi_3), \end{aligned} \quad (41)$$

for $\lambda_1 < 0$, then, $L_T^{n_1}$ is isometric to warped product of the type $R \times_\theta U$ with the warping function θ , which satisfies the differential equation $d\theta^2/dt^2 + \lambda_1 \theta = 0$.

4. Conclusions

This paper studies the geometric behavior of ordinary differential equations on the warped product semi-invariant product submanifolds. More precisely, we obtain characterizing theorems for warped product semi-invariant product submanifolds of generalized Sasakian space forms via differential and integral theory on Riemannian manifolds. Therefore, the present article provides a wonderful correlation of the theory of differential equations with the warped product submanifolds.

Data Availability

No data were used to support this study.

Conflicts of Interest





The author declares that there is no conflict of interest regarding the publication of this paper.

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Research Article

UV Index for Public Health Awareness Based on OMI/NASA Satellite Data at King Abdulaziz University, Saudi Arabia

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Exposure to ultraviolet radiation (UV) is essential for good health and formation of vitamin D while overexposure poses a risk to public health. Therefore, it is important to provide information to the public about the level of solar UV radiation. The ultraviolet index (UVI) is used to help avoid the negative effects of ultraviolet (UV) radiation on humans and to optimize individual exposure. There is limited ground measurement of solar UV radiation, but satellite Ozone Monitoring Instrument (OMIs) satellite products with a spatial resolution of $1^\circ \times 1^\circ$ can be used to create UV index climatology at local noon time. In this study, we utilize OMI satellite products collected over the campus of King Abdulaziz University (KAU) (21.5° North and 39.1° East), Jeddah, Saudi Arabia, to estimate changes in exposure to UV over a period of 15 years (2004–2020). The results indicate a significantly increasing trend in UV index over this period. Between 2004 and 2020, daily “extreme” UV ($UVI > 11$, as defined by the World Health Organization (WHO)) occurred on 46.60% of days. The frequency of low UVI ($UVI < 2$) was only about 0.06%. These results imply dangerous exposure levels to solar UV radiation on the KAU campus and call for safety measures to increase awareness and decrease direct exposure; for example, by implementing the United States Environmental Protection Agency (EPA) general guidelines.

1. Introduction

Most people love the sun and spend significant amounts of time outdoors, often in clothing that exposes their skin to direct ultraviolet (UV) radiation (UVR). Medical research has shown that some exposure to sunlight can be beneficial; for example, increasing the production of vitamin D [1–3]. However, overexposure to UVR can contribute to serious health problems, ranging from sunburn (erythema) and skin damage to skin cancer [4–8]. The effects of sunburn worsen

with cumulative exposure, and a higher number of incidents of severe sunburn, especially during childhood, increase the risk of developing skin cancer [9–11]. UVR exposure also places eyes at risk of photokeratitis, photo conjunctivitis, and cataracts [12–17].

1.1. Skin Cancers. Skin cancers present the greatest risk to fair skinned people because they have less protective melanin than people with darker skin [18]. Keratinocyte carcinomas, basal cell carcinoma, and squamous cell carcinoma are the

most common [19]. The third type of skin cancer, melanoma, is less common than the other skin cancers, but it accounts for most skin cancer deaths [4, 20, 21]: 95% of such deaths worldwide [22]. It is concerning that, in fair skinned populations, the incidence of melanoma is rising more quickly than it is for most other types of cancer [23]. The number of cases is expected to continue to rise [24], due in part to increased recreational and intentional UVR exposure, particularly in younger individuals [25]. The trends in the incidence of keratinocyte cancers are difficult to establish because of unreliable data [26].

1.2. Ocular Disease. Solar UVR causes various ocular diseases including eyelid malignancies and cortical cataracts [13]. UVR is linked to cataract induction [27], which is the most common cause of blindness globally [28, 29]. Cortical cataracts are more prevalent at lower geographical latitudes where UVR is abundant [30]. Studies have shown that even a single exposure of the cornea to UV light can lead to detectable changes [31]. While there are different risk factors associated with cataracts, sunlight is estimated to be the cause of 10–20% of cataracts [32, 33]. While there is increased awareness of the link between exposure to direct sun and skin cancer, fewer people are aware that the risks extend beyond skin cancer [34].

In response to increased awareness of the risks associated with exposure to direct UVR, the UV index (UVI) was developed by the World Health Organization (WHO) in collaboration with the United Nations Environment Programme (UNEP), the International Commission on Non-Ionizing Radiation Protection (ICNIRP), the World Meteorological Organization (WMO), and the German Federal Office for Radiation Protection (Bundesamt für Strahlenschutz, BfS). The higher the UVI, the greater the risk of skin or eye damage, and the less time it takes for damage to occur.

The UVI is important in raising public awareness of the risks involved in high exposure to UVR and in alerting people to the need for protective measures. Encouraging people to reduce exposure can limit the harmful effects and related healthcare costs. The equation to derive the UVI is as follows [35–37]:

$$\text{UV Index} = 40 \int E(\lambda) S_{\text{er}}(\lambda) d\lambda, \quad (1)$$

where λ is the wavelength in nm, $E(\lambda)$ is the irradiance in W/m^2 , and $S_{\text{er}}(\lambda)$ is the erythral weighting function, which is defined as

$$\begin{aligned} S_{\text{er}}(\lambda) &= 1.0 \text{ for } 250 < \lambda \leq 298, \\ \text{nm} S_{\text{er}}(\lambda) &= 10^{0.094(298-\lambda)} \text{ for } 298, \\ < \lambda \leq 328 \text{ nm} S_{\text{er}}(\lambda) &= 10^{0.015(139-\lambda)} \text{ for } 328 < \lambda \leq 400 \text{ nm}, \\ S_{\text{er}}(\lambda) &= 0.0 \text{ for } \lambda > 400 \text{ nm}. \end{aligned} \quad (2)$$

The UVI is derived from the erythral irradiance, integrating UV irradiance at ground level with weighting deter-

mined by the Commission International de l' Eclairage (CIE). The CIE weighting function is based on the susceptibility of the Caucasian skin to sunburn [38, 39].

Information is provided in terms of the UVI scale, which was adopted in 1994 [35] and revised in 2002 to improve its usefulness in raising public awareness [37]. In 2009, the International Agency for Research on Cancer (IARC) stated that UVR is carcinogenic to humans [40, 41]. The WHO promotes messages alerting the public to recommended prevention measures (such as the use of hats or sunscreen) for different UVI values [26].

Most commonly, the UVI is derived from modeling UV irradiance, while taking into account key atmospheric metrics (aerosol optical properties, cloud cover, and ozone levels). Predictive UVI models vary in complexity and accuracy, but all require good information on actual atmospheric parameters, which is often difficult to obtain [42–44].

Heckman reviewed 20 years of research into awareness of the UVI and its impact and found that awareness varies from country to country, with low levels of comprehension and use of the UVI as a means of informing safe behaviors [45]. It is evident that more research is needed to determine best practices in increasing public awareness of the potential of the UVI to aid effective skin protection [46, 47] and improve public health.

It is especially important to protect the most vulnerable population groups, given that more than 90% of nonmelanoma skin cancers affect skin types I and II [48, 49]; the key messages associated with the UVI need to be focused on people who are at higher risk of sunburn [47, 50]. Children are especially sensitive to UVR and need special protection. Further, even though the occurrence of skin cancer is lower in dark skinned people, they are still susceptible to damage to the eyes and immune system [5, 7, 51, 52]. Messaging at the national and local levels is needed to focus on the needs of vulnerable subgroups of the population. Differences in climate and culture and perceptions of risks also need to be considered [37]. The United States Environmental Protection Agency (EPA) has devised guidelines for using the UVI [53].

In this study, all sky conditions, UV index, climatology over the campus of King Abdulaziz University (KAU), Jeddah, Saudi Arabia, (21.5° North and 39.1° East) based on OMI satellite data with high spatial resolution accuracy of 1° lat. \times 1° Lon., and since there are only limited ground measurements, for the period from October 2004 to December 2020, was studied. In the last 15 years, the daily UVI studded to raise awareness among university students and staff of the UVI and its potential value. A description of the methods used is followed by the results and discussion and conclusions.

2. Methods

2.1. Questionnaire on the Awareness of the Effects of UVR. Established in 1967, King Abdulaziz University Campus (KAU) occupies an area of 2,224 acres (9 km²) in Jeddah, Saudi Arabia, (between 21°29'N and 21°30'N latitude and 39°14'E and 39°16'E longitude). KAU's vision, mission, and goals include protecting the health of students and staff.

TABLE 1: Analyses of the percentage of correct answers to the five statements in KAU during the study.

Statement	Agreement	Disagreement	Percent of correct responses
Fire burns are more dangerous than sunburn	228	314	42.07
Sunscreen only needs to be applied at resorts in summer	352	190	64.94
Damage from exposure to direct sun can occur on a day with moderate temperatures	327	215	60.33
You cannot get sunstroke on a cloudy day	142	400	26.20
Walking in the shade reduces damage from the sun rays	66	476	12.18

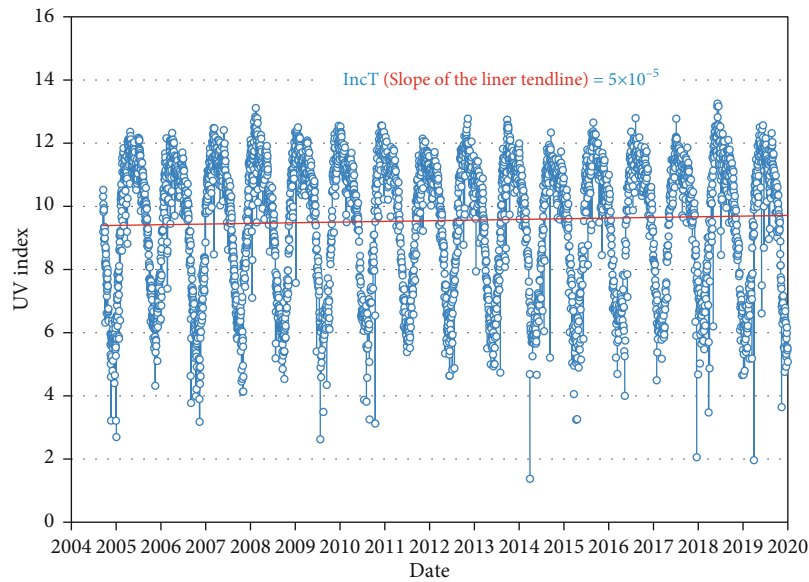


FIGURE 1: Time series variations in daily UVI at noon time on the campus of KAU during the study period.

Recognizing the significant damage and health risks resulting from exposure to direct solar radiation due to incautious behavior, the university aims to increase awareness of the damaging impact of UVR and address misconceptions. To support this effort, we developed a semistructured questionnaire which was distributed among all university staff and students, spanning all ages and educational groups.

The questionnaire was designed to measure awareness of UVR and its damaging impact and of mitigation measures that could be taken. Participants were asked to express their understanding based on a set of five statements, based on [37]:

- (i) Statement 1: fire burns are more dangerous than sunburn
- (ii) Statement 2: sunscreen only needs to be applied at resorts in summer
- (iii) Statement 3: damage from exposure to direct sun can occur on a day with moderate temperatures
- (iv) Statement 4: you cannot get sunstroke on a cloudy day
- (v) Statement 5: walking in the shade reduces damage from the sun rays

TABLE 2: Number and ratio of UVI in KAU during the study.

UVI category	UVI value	N	%	%
Low	1,2	3	0.06	
Moderate	3,4,5	127	2.73	23.53
High	6,7	964	20.73	
Very high	8,9,10	1389	29.87	29.87
Extreme	11 ⁺	2167	46.60	46.60
Total		4650		100

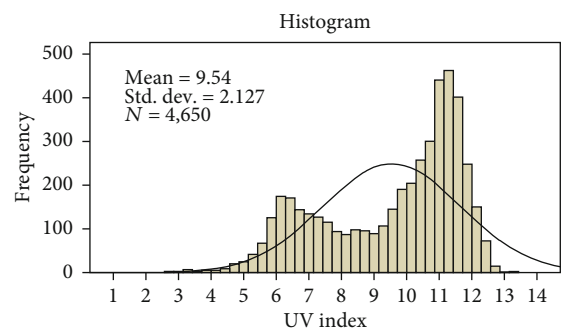


FIGURE 2: Histogram with normal distribution of daily UVI on the campus of KAU during the study period.

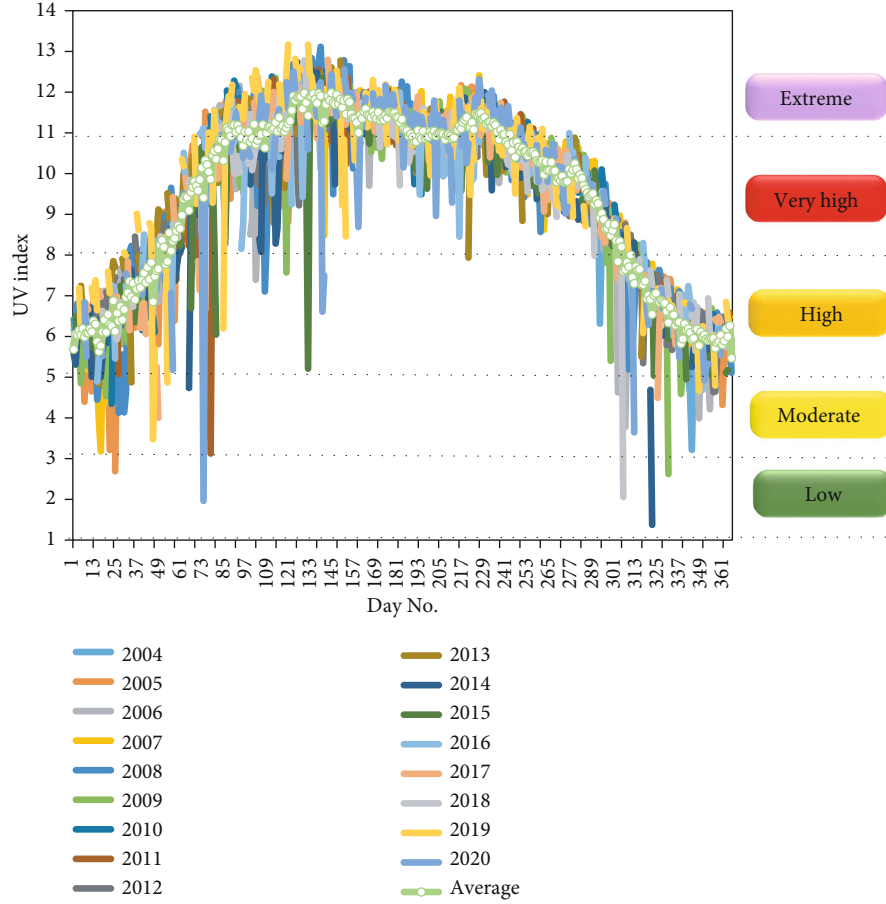


FIGURE 3: Daily average variation in UVI at KAS during the study period.

2.2. Ozone Monitoring Instruments. We also used data collected by the Ozone Monitoring Instrument (OMI) on board the EOS Aura rocket (launched in 2004) [54]. OMIs are derived from NASA's Total Ozone Mapping Spectrometer (TOMS) and the European Space Agency's Global Ozone Monitoring Experiment (GOME). An OMI can provide many more measures than TOMS and better ground resolution than GOME (13 km \times 25 km for OMI vs. 40 km \times 320 km for GOME) [55]. For more information about OMI instruments and products, refer to Data User's Guide [56].

The Giovanni product by GES DISC is a web application with a simple, intuitive way of visualizing and analyzing earth science remote sensing data, especially from satellites. Giovanni includes data on atmospheric chemistry and temperatures and rainfall. Giovanni is available at [57] and also includes output from models for atmospheric, land surface, and oceanographic parameters.

3. Results and Discussion

3.1. Data Analysis. First, we analyzed the percentage of correct answers to the five statements. Table 1 shows the distribution of the agreement and disagreement with these statements.

The results show that the majority of participants (61.5%) answered at least one of the statements incorrectly

compared to 38.5% who answered all statements correctly. The majority of participants who answered all the questions correctly were females who were 35 or older (26.5%). On the other hand, among males, the age group that showed the largest proportion of correct answers was age 30 and younger (11.8%).

When asked if they followed the weather forecast, 46.5% of all participants said they were interested only in knowing the forecast daily temperature, while 22.4% said they did not follow the weather forecast. On the other hand, 31.1% expressed an interest in knowing both the forecast temperature and the UVI. Furthermore, 10.2% of the participants said that they were unaware of the existence of the UVI. However, 60.8% of the participants expressed an interest in learning more about the index; 29% were already aware of the index and followed it.

When asked about protection measures, 22% of participants stated that they did not take any steps to protect themselves from UVR; although, the majority of participants (78%) stated that they did adopt measures, including the use of sunglasses and sunblock with a head cover (24.5%); 11.4% stated that they only used a head cover, 22.4% only used sunglasses, and 19.6% used both. Also, 72.7% of the participants did not have a mobile application or any other means of obtaining the UVI, while 20.8% expressed an interest in knowing about the index and its application, and 89.8%

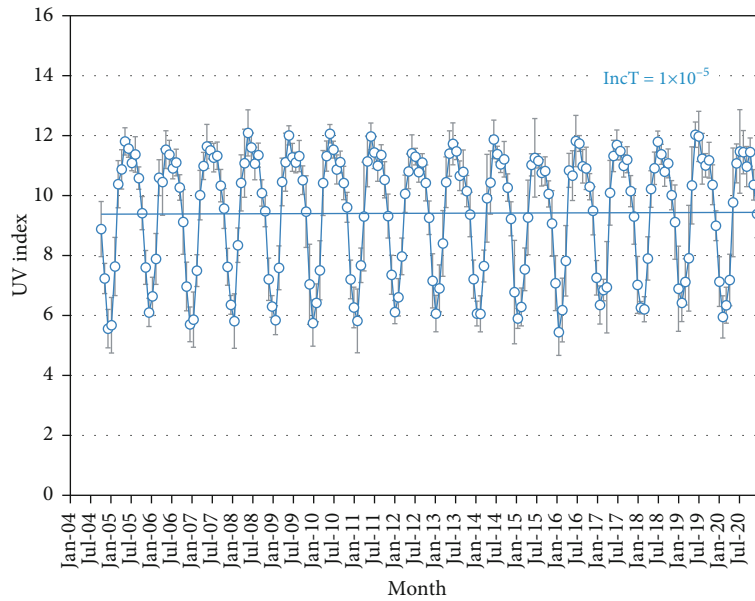


FIGURE 4: Time series variations in the monthly UVI (with standard deviation error bars) on the campus of KAU during the study period.

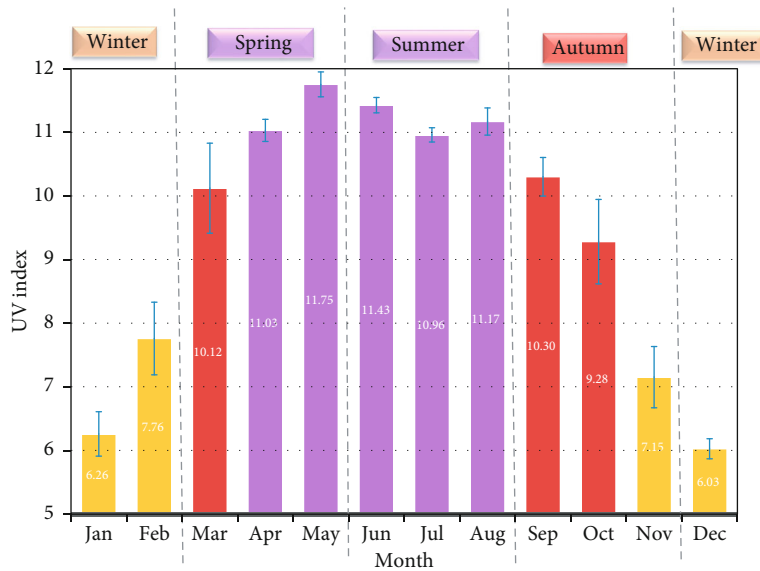


FIGURE 5: Monthly mean UVI on the campus of KAU during the study period.

wished that the UVI was advertised in public places to raise awareness.

3.2. Time-Series Variation in the UVI. Analysis of the daily variation in the UVI at midday (local time) over the KAU campus suggests a slightly increasing trend (IncT) over time (indicated by IncT of 0.5×10^{-4} UVI) (Figure 1) [58]. Variation in the UVI shows a clear annual waveform. Starting in the winter months (i.e., January), UVI increases until it reaches its maximum in the summer (i.e., June) and decreases again to a minimum in the fall (i.e., November).

The highest UVI values were recorded on May 2, 2019, (13.25), and the lowest value was recorded on November 18, 2014 (1.38). According to the categories of UVI (Table 2), 46.60% of the days (4,650) had an “Extreme” UVI, 29.87% of the days had a “Very High” UVI, and 23.53% of the days had a “Low”, “Moderate,” or “High” UVI. Given the EPA guidelines in Table [53], extra care is needed outdoors for much of the year.

Figure 2 provides a histogram with normal distribution and shows that the average value was 9.54 ± 2.127 . Figure 3 shows the daily average variation in the UVI in KAU during

TABLE 3: Analysis of the mean monthly UVI during the study period at KAU.

	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
<i>N</i>	31	29	31	30	31	30	31	31	30	31	30	31
Mean	6.26	7.76	10.12	11.03	11.75	11.43	10.96	11.17	10.30	9.28	7.15	6.03
Std. error of mean	.0627	.1060	.1270	.0318	.0352	.0222	.0202	.0386	.0552	.1191	.0879	.0284
Median	6.177 ^a	7.660 ^a	9.960 ^a	11.03 ^a	11.81 ^a	11.43 ^a	10.95 ^a	11.21 ^a	10.30 ^a	9.41 ^a	7.11 ^a	6.05 ^a
Mode	6.04 ^b	6.85 ^b	9.56 ^b	11.01	11.83	11.40 ^b	10.95	11.27 ^b	10.41	9.03 ^b	6.82 ^b	5.83 ^b
Std. deviation	.3493	.5707	.7073	.1742	.1961	.1218	.1123	.2147	.3025	.6633	.4812	.1581
Variance	.122	.326	.500	.030	.038	.015	.013	.046	.092	.440	.232	.025
Skewness	.529	.311	-.153	.049	-1.033	.869	1.026	-.515	-.231	-.458	.243	.106
Std. error of skewness	.421	.434	.421	.427	.421	.427	.421	.421	.427	.421	.427	.421
Kurtosis	.147	-.956	-.834	-.989	.531	3.566	1.138	-.434	-1.190	-.548	-.876	-.808
Std. error of kurtosis	.821	.845	.821	.833	.821	.833	.821	.821	.833	.821	.833	.821
Range	1.44	2.01	2.60	.61	.81	.65	.48	.81	.99	2.65	1.91	.57
Minimum	5.63	6.85	8.62	10.72	11.25	11.18	10.78	10.72	9.79	7.66	6.26	5.76
Maximum	7.07	8.86	11.22	11.33	12.06	11.83	11.26	11.53	10.78	10.31	8.17	6.33
Sum	194.1	225.1	313.9	330.9	364.4	342.9	339.8	346.1	309.0	287.8	214.6	186.9
Percentiles	25	6.065	7.350	9.570	10.89	11.70	11.39	10.90	11.02	10.08	8.715	5.913
	50	6.177	7.660	9.960	11.02	11.81	11.43	10.95	11.21	10.30	9.410	6.047
	75	6.431	8.272	10.75	11.18	11.88	11.48	10.98	11.30	10.55	9.883	6.142

^aCalculated from grouped data. ^bMultiple modes exist. The smallest value is shown.

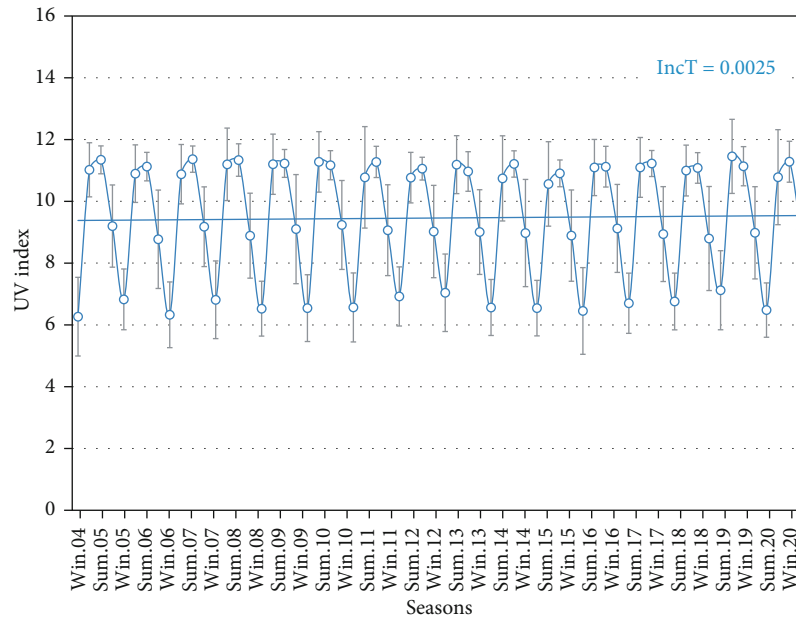


FIGURE 6: Time series variations in the seasonal UVI (with standard deviation error bars) on the campus of KAU during the study period.

the period studied. UVI typically has high values during spring and summer and low values during winter.

3.3. *Monthly Variation in the UVI.* Examination of the monthly variation in the UVI suggests a moderately increas-

ing trend over time (indicated by IncT of 0.1×10^{-4} UVI) (Figure 4). Figure 5 presents the monthly mean UVI on the campus of KAU during the study period, and Table 3 shows an analysis of the mean monthly UVI during the study period. Several points emerge as follows:

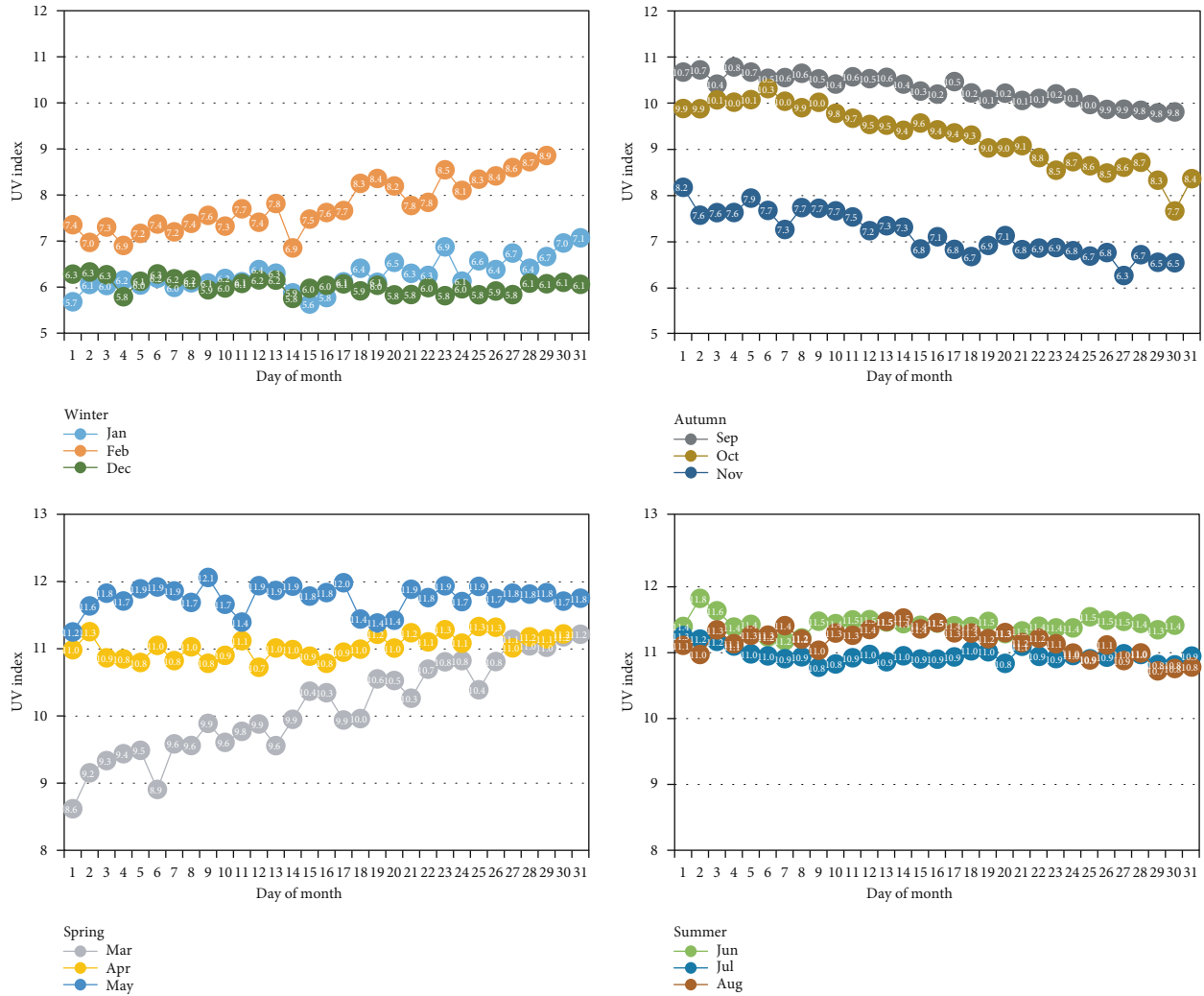


FIGURE 7: Seasonal variations in the UVI on the campus of KAU during the study period.

- (1) The monthly average values of the UVI are characterized by relatively high values in the spring and summer months (March to August) compared with the corresponding values in the winter and fall months (September to February)
- (2) The maximum average value of the UVI recorded was 11.75 ± 0.196 in May, and the minimum value was recorded in December, 6.02 ± 0.158
- (3) The standard deviation of the monthly average values is high March (± 0.7), October (± 0.7), February (± 0.6), and November (± 0.5) compared to the summer months; this can be attributed to climatological and synoptical conditions during the spring and winter months

3.4. Seasonally Variation of UV Index. Analysis of seasonal variations in the UVI suggests a moderately increasing trend over time ($\text{IncT} = 25 \times 10^{-4}$) (Figure 6). As illustrated in Figure 7, seasonal UVI trends in Winter (Dec.-Jan.-Feb.), Spring (Mar.-Apr.-May), Summer (Jun.-Jul.-Aug.), and

Autumn (Sep.-Oct.-Nov), the UVI generally decreases in the winter (December and January) and increases in February. In the spring, which is characterized by weather fluctuations (i.e., rainfall and temperature) [59–61], there is a notable difference between the three months of March, April, and May, with May showing the highest values. In the summer, climatic conditions are more stable, and the UVI is almost constant for the whole season. In the fall, September has the highest UVI, as weather fluctuations begin, with further fluctuations in October and November, as can be seen clearly in standard deviation.

3.5. The Box-Whisker Plots for UV Index. Boxplot or box-whisker plots is a method for graphically depicting groups of numerical data through their quartiles. It has also lines extending from the boxes (whiskers) indicating variability outside the upper and lower quartiles. The spacings between the different parts of the box indicate the degree of dispersion (spread) and skewness in the data and show outliers. In addition to the points themselves, they allow one to visually estimate the mean value. Seasonal values of UV index at midday

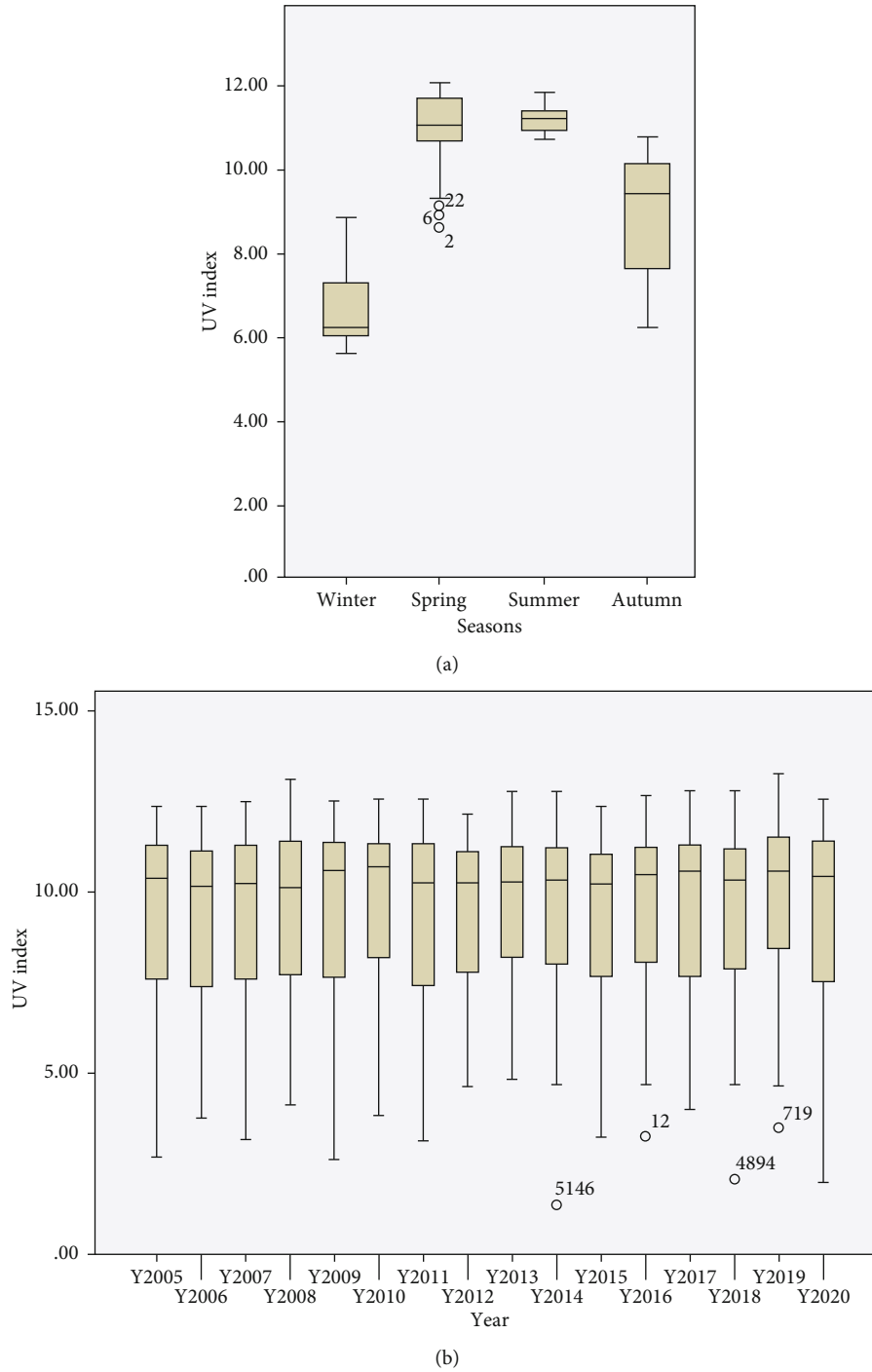


FIGURE 8: The box-whisker plots for UV index measured by OMI, the 5th and 95th percentiles (whisker), the interquartile range (box), and the mean value (dash in the Box Medill): (a) indicates for each season statistics, and (b) indicates for each year statistics on the campus of KAU during the study period.

(local time) over the KAU campus by box and whiskers diagrams were shown in Figure 8(a). Here, we notice a significant increasing trend from autumn to summer during spring and decreasing again to winter with low degree of dispersion in summer and high degree of dispersion in autumn. In Figure 8(b), it is clear to note that annual boxplot of UV index from year to another was approximately constant, the mean value ($UVI \cong 10$).

4. Conclusions

OMI (Ozone Monitoring Instruments) satellite products collected over the campus of King Abdulaziz University (KAU), Jeddah, Saudi Arabia, could estimate changes in UV index over a period of 15 years (Oct.2004-Dec. 2020), since there are only limited ground measurements, until now, in order to optimize individual exposure to solar UV radiation, public

health awareness, behavior in outdoor, and planning to sun safety by taken UV index (UVI) into account. UVI values at midday (local time) over the KAU campus suggest a slightly increasing trend (IncT) over time (indicated by an IncT of 0.5×10^{-4} UVI). Variability of the UVI shows an annual waveform. Starting in the winter months, the UVI increases until it reaches its maximum in summer, decreasing again to a minimum in the fall. According to the categories of UVI, 46.60% of the days (4,650) showed “Extreme” UVI while 29.87% showed “Very High” UVI; 23.53% of the days were characterized as “Low”, “Moderate,” or “High” UVI. From these findings, in the KAU campus, we recommended that

-Future studies should examine the effects of different parameters (such as surface ozone, aerosol, albedo, and cloud) on the UVR reaching the Earth’s surface

-The UVI should be reported along with the weather forecast in newspapers, on TV, the radio, and on webpages in affected countries and in different languages

We hope that this study will open the way to greater public awareness in KAU and the Kingdom to encourage safe exposure to the sun by alerting people to the health implications.

Data Availability

The data presented in this study are available on request from the corresponding author. The data are not publicly available due to privacy reasons.

Consent

Informed consent was obtained from all subjects involved in the study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors’ Contributions

AA, EE, and SA contributed to the conceptualization. AA, AM, MR, SA, and EE contributed to the methodology. AA, AM, MR, SA, and EE contributed to the formal analysis. AA, AM, MR, SA, and EE contributed to the investigation. AA and A.M. contributed to the resources. AA, AM, MR, SA, and EE contributed to the data curation. AA, EE, and SA performed the writing—original draft preparation. AA, AM, MR, SA, and EE performed the writing—review and editing. AA, AM, MR, SA, and EE performed the visualization. AA contributed to the supervision. AA contributed to the project administration. All authors have read and agreed to the published version of the manuscript.

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