Journal of Function Spaces

Fractional Delay Differential Equations and their Numerical Solutions

Lead Guest Editor: Qifeng Zhang
Guest Editors: Ahmed S. Hendy and Mahmoud A. Zaky

Fractional Delay Differential Equations and their Numerical Solutions

Journal of Function Spaces

Fractional Delay Differential Equations and their Numerical Solutions

Lead Guest Editor: Qifeng Zhang
Guest Editors: Ahmed S. Hendy and Mahmoud A. Zaky

Contents

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 9820258, 10 pages
https://doi.org/10.1155/2022/9820258

Corrigendum

Corrigendum to "Fractional Crank-Nicolson-Galerkin Finite Element Methods for Nonlinear Time Fractional Parabolic Problems with Time Delay"

Lili Li,[1] **Mianfu She**,[1] **and Yuanling Niu**[2]

[1]School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China
[2]School of Mathematics and Statistics, Central South University, Changsha 410083, China

Correspondence should be addressed to Yuanling Niu; yuanlingniu@csu.edu.cn

Received 28 February 2022; Accepted 28 February 2022; Published 5 May 2022

In the article titled "Fractional Crank-Nicolson-Galerkin Finite Element Methods for Nonlinear Time Fractional Parabolic Problems with Time Delay" [1], there are a number of minor typographical errors introduced to the equations during the typesetting of the article. The corrected article is as follows.

Abstract
A linearized numerical scheme is proposed to solve the nonlinear time fractional parabolic problems with time delay. The scheme is based on the standard Galerkin finite element method in the spatial direction, the fractional Crank-Nicolson method, and extrapolation methods in the temporal direction. A novel discrete fractional Grönwall inequality is established. Thanks to the inequality, the error estimate of fully discrete scheme is obtained. Several numerical examples are provided to verify the effectiveness of the fully discrete numerical method.

1. Introduction

In this paper, we consider the linearized fractional Crank-Nicolson-Galerkin finite element method for solving the nonlinear time fractional parabolic problems with time delay

$$\begin{cases} {}^{R}D_{t}^{\alpha}u - \Delta u = f(t, u(x,t), u(x,t-\tau)), & \text{in } \Omega \times (0, T], \\ u(x,t) = \varphi(x,t), & \text{in } \Omega \times (-\tau, 0], \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, T], \end{cases} \tag{1}$$

where Ω is a bounded convex and convex polygon in R^2 (or polyhedron in R^3) and τ is the delay term. ${}^{R}D_{t}^{\alpha}u$ denotes the Riemann-Liouville fractional derivative, defined by

$${}^{R}D_{t}^{\alpha}u(\cdot, t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_{0}^{t}(t-s)^{-\alpha}u(\cdot, s)ds, \quad 0 < \alpha < 1. \tag{2}$$

The nonlinear fractional parabolic problems with time delay have attracted significant attention because of their widely range of applications in various fields, such as biology, physics, and engineering [1–9]. Recently, plenty of numerical methods were presented for solving the linear time fractional diffusion equations. For instance, Chen et al. [10] used finite difference methods and the Kansa method to approximate time and space derivatives, respectively. Dehghan et al. [11] presented a full discrete scheme based on the finite difference methods in time direction and the meshless Galerkin method in space direction and proved that the scheme was unconditionally stable and convergent. Murio [12] and Zhuang [13] proposed a fully implicit finite difference numerical scheme and obtained unconditionally stability. Jin et al. [14] derived the time fractional Crank-Nicolson scheme to approximate Riemann-Liouville fractional derivative. Li et al. [15] used a transformation to develop some new schemes for solving the time-fractional problems. The new schemes admit some advantages for both capturing the initial layer and solving the models with small parameter α. More studies can be found in [16–32].

Recently, it has been one of the hot spots in the investigations of different numerical methods for the nonlinear time fractional problems. For the analysis of the L1-type methods, we refer readers to the paper [33–40]. For the analysis of the convolution quadrature methods or the fractional Crank-Nicolson scheme, we refer to the recent papers [41–46]. The key role in the convergence analysis of the schemes is the fractional Grönwall type inequations. However, as pointed out in [47–49], the similar fractional Grönwall type inequations can not be directly applied to study the convergence of numerical schemes for the nonlinear time fractional problems with delay.

In this paper, we present a linearized numerical scheme for solving the nonlinear fractional parabolic problems with time delay. The time Riemann-Liouville fractional derivative is approximated by fractional Crank-Nicolson type time-stepping scheme, the spatial derivative is approximated by using the standard Galerkin finite element method, and the nonlinear term is approximated by the extrapolation method. To study the numerical behavior of the fully discrete scheme, we construct a novel discrete fractional type Grönwall inequality. With the inequality, we consider the convergence of the numerical methods for the nonlinear fractional parabolic problems with time delay.

The rest of this article is organized as follows. In Section 2, we present a linearized numerical scheme for the nonlinear time fractional parabolic problems with delay and main convergence results. In Section 3, we present a detailed proof of the main results. In Section 4, numerical examples are given to confirm the theoretical results. Finally, the conclusions are presented in Section 5.

2. Fractional Crank-Nicolson-Galerkin FEMs

Denote $\mathcal{T}_h$ is a shape regular, quasi-uniform triangulation of the Ω into d-simplexes. Let $h = \max_{K\in\mathcal{T}_h}\{\text{diam } K\}$. Let X_h be the finite-dimensional subspace of $H_0^1(\Omega)$ consisting of continuous piecewise function on $\mathcal{T}_h$. Let $\Delta t = \tau/m_\tau$ be the time step size, where m_τ is a positive integer. Denote $N = \lceil T/\Delta t\rceil$, $t_j = j\Delta t$, $j = -m_\tau, -m_\tau + 1, \cdots, 0, 1, 2, \cdots, N$.

The approximation to the Riemann-Liouville fractional derivative at point $t = t_{n-(\alpha/2)}$ is given by [14]:

$$\begin{aligned} {}^R D^{\alpha}_{t_{n-(\alpha/2)}} u(x,t) &= \Delta t^{-\alpha}\sum_{i=0}^{n}\omega^{(\alpha)}_{n-i} u(x,t_i) + \mathcal{O}\left(\Delta t^2\right) \\ &:= {}^R D^{\alpha}_{\Delta t} u^n + \mathcal{O}\left(\Delta t^2\right), \end{aligned} \tag{3}$$

where

$$\omega^{(\alpha)}_i = (-1)^i \frac{\Gamma(\alpha+1)}{\Gamma(i+1)\Gamma(\alpha-i+1)}. \tag{4}$$

For simplicity, denote $\|v\| = \left(\int_\Omega |v(x)|^2 dx\right)^{1/2}$, $\eta^{n,\alpha} = (1-(\alpha/2))\eta^n + (\alpha/2)\eta^{n-1}$, $\widehat{\eta}^{n,\alpha} = (2-(\alpha/2))\eta^{n-1} - (1-(\alpha/2))\eta^{n-2}$, $t_n^\alpha = (n\Delta t)^\alpha$.

With the notation, the fully discrete scheme is to find $U_h^n \in X_h$ such that

$$\begin{aligned} &\left\langle {}^R D^{\alpha}_{\Delta t} U_h^n, v\right\rangle + \left\langle \nabla U_h^{n,\alpha}, \nabla v\right\rangle \\ &= \left\langle f\left(t_{n-(\alpha/2)}, \widehat{U}_h^{n,\alpha}, U_h^{n-m_\tau,\alpha}\right), v\right\rangle, \forall v \in X_h, n = 1, 2, \cdots, N, \end{aligned} \tag{5}$$

and the initial condition

$$U_h^n = R_h \varphi(x, t_n), \quad n = -m_\tau, -m_\tau + 1, \cdots, 0, \tag{6}$$

where $R_h : H_0^1(\Omega) \longrightarrow X_h$ is Ritz projection operator which satisfies following equality [50]

$$\langle \nabla R_h u, \nabla v\rangle = \langle \nabla u, \nabla v\rangle, \forall u \in H_0^1(\Omega) \cap H^2(\Omega), v \in X_h. \tag{7}$$

We present the main convergence results here and leave its proof in the next section.

Theorem 1. Suppose the system (1) has a unique solution u satisfying

$$\begin{aligned} &\|u_0\|_{H^{r+1}} + \|u\|_{C\left([0,T];H^{r+1}\right)} + \|u_t\|_{C\left([0,T];H^{r+1}\right)} \\ &+ \|u_{tt}\|_{C\left([0,T];H^2\right)} + \left\|{}^R D^{\alpha}_{\Delta t} u\right\|_{C\left([0,T];H^{r+1}\right)} \le K, \end{aligned} \tag{8}$$

and the source term $f(t, u(x,t), u(x, t-\tau))$ satisfies the Lipschitz condition

$$\begin{aligned} &|f(t, u(x,t), u(x,t-\tau)) - f(t, v(x,t), v(x,t-\tau))| \\ &\le L_1|u(x,t) - v(x,t)| + L_2|u(x,t,\tau) - v(x,t,\tau)|, \end{aligned} \tag{9}$$

where K is a constant independent of n, h and Δt, L_1 and L_2 are given positive constants. Then, there exists a positive constant Δt^* such that for $\Delta t \le \Delta t^*$, the following estimate holds that

$$\|u^n - U_h^n\| \le C_1^*\left(\Delta t^2 + h^{r+1}\right), \quad n = 1, 2, \cdots, N, \tag{10}$$

where C_1^* is a positive constant independent of h and Δt.

Remark 2. The main contribution of the present study is that we obtain a discrete fractional Grönwall's Grönwall's inequality. Thanks to the inequality, the convergence of the fully discrete scheme for the nonlinear time fractional parabolic problems with delay can be obtained.

Remark 3. At present, the convergence of the proposed scheme is proved without considering the weak singularity of the solutions. In fact, if the initial layer of the problem is taken into account, some corrected terms are added at the beginning. Then, the scheme can be of order two in the temporal direction for nonsmooth initial data and some incompatible source term. However, we still have the difficulties to get the similar discrete fractional Grönwall's inequality. We hope to leave the challenging problems in future.

3. Proof of the Main Results

In this section, we will present a detailed proof of the main result.

3.1. Preliminaries and Discrete Fractional Grönwall Inequality

Firstly, we review the definition of weights $\omega_i^{(\alpha)}$, denote $g_n^{(\alpha)} = \sum_{i=0}^{n}\omega_i^{(\alpha)}$. Then, we can get

$$\begin{cases} \omega_0^{(\alpha)} = g_0^{(\alpha)}, \\ \omega_i^{(\alpha)} = g_i^{(\alpha)} - g_{i-1}^{(\alpha)}, \quad 1 \le i \le n. \end{cases} \tag{11}$$

Actually, it has been shown [51] that $\omega_i^{(\alpha)}$ and $g_n^{(\alpha)}$ process following properties.

(1) The weights $\omega_i^{(\alpha)}$ can be evaluated recursively, $\omega_i^{(\alpha)} = (1-((\alpha+1)/i))\omega_{i-1}^{(\alpha)}, i \ge 1, \omega_0^{(\alpha)} = 1$

(2) The sequence $\{\omega_i^{(\alpha)}\}_{i=0}^{\infty}$ are monotone increasing $-1 < \omega_i^{(\alpha)} < \omega_{i+1}^{(\alpha)} < 0, i \ge 1$

(3) The sequence $\{g_i^{(\alpha)}\}_{i=0}^{\infty}$ are monotone decreasing, $g_i^{(\alpha)} > g_{i+1}^{(\alpha)}$ for $i \ge 0$ and $g_0^{(\alpha)} = 1$

Noticing the definition of $g_i^{(\alpha)}$, ${}^{R}D_{\Delta t}^{\alpha}u^n$ can be rewritten as

$${}^{R}D_{\Delta t}^{\alpha}u^n = \Delta t^{-\alpha}\sum_{i=1}^{n}\left(g_i^{(\alpha)} - g_{i-1}^{(\alpha)}\right)u^{n-i} + \Delta t^{-\alpha}g_0^{(\alpha)}u^n. \tag{12}$$

In fact, rearranging this identity yields

$${}^{R}D_{\Delta t}^{\alpha}u^n = \Delta t^{-\alpha}\sum_{i=1}^{n}g_{n-i}^{(\alpha)}\delta_t u^i + \Delta t^{-\alpha}g_n^{(\alpha)}u^0, \tag{13}$$

where $\delta_t u^i = u^i - u^{i-1}$.

Lemma 4 (see [51]). Consider the sequence $\{\phi_n\}$ given by

$$\phi_0 = 1, \phi_n = \sum_{i=1}^{n}\left(g_{i-1}^{(\alpha)} - g_i^{(\alpha)}\right)\phi_{n-i}, \quad n \ge 1. \tag{14}$$

Then, $\{\phi_n\}$ satisfies the following properties

(i) $0 < \phi_n < 1, \sum_{i=j}^{n}\phi_{n-i}g_{i-j}^{(\alpha)} = 1, 1 \le j \le n$

(ii) $(1/\Gamma(\alpha))\sum_{i=1}^{n}\phi_{n-i} \le (n^{\alpha}/\Gamma(1+\alpha))$

(iii) $(1/(\Gamma(\alpha)\Gamma(1+(k-1)\alpha)))\sum_{i=1}^{n-1}\phi_{n-i}i^{(k-1)\alpha} \le (n^{k\alpha}/\Gamma(1+\alpha)), k = 1, 2\cdots$.

Lemma 5 (see [51]). Consider the matrix

$$W = 2\mu(\Delta t)^{\alpha}\begin{pmatrix} 0 & \phi_1 & \cdots & \phi_{n-2} & \phi_{n-1} \\ 0 & 0 & \cdots & \phi_{n-3} & \phi_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \phi_1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}. \tag{15}$$

Then, W satisfies the following properties

(i) $W^l = 0, l \ge n$

(ii) $W^k\vec{e} \le (1/\Gamma(1+k\alpha))[(2\Gamma(\alpha)\mu t_n^{\alpha})^k, (2\Gamma(\alpha)\mu t_{n-1}^{\alpha})^k, \cdots, (2\Gamma(\alpha)\mu t_1^{\alpha})^k]', k = 0, 1, 2, \cdots$

(iii) $\sum_{k=0}^{l}W^k\vec{e} = \sum_{k=0}^{n-1}W^k\vec{e} \le [E_{\alpha}(2\Gamma(\alpha)\mu t_n^{\alpha}), E_{\alpha}(2\Gamma(\alpha)\mu t_{n-1}^{\alpha}), \cdots, E_{\alpha}(2\Gamma(\alpha)\mu t_1^{\alpha})]', l \ge n$

where $\vec{e} = [1, 1, \cdots, 1]' \in \mathbb{R}^n$, μ is a constant.

Theorem 6. Assuming $\{u^n | n = -m, -m+1, \cdots, 0, 1, 2, \cdots\}$ and $\{f^n | n = 0, 1, 2, \cdots\}$ are nonnegative sequence, for $\lambda_i > 0$, $i = 1, 2, 3, 4, 5$, if

$${}^{R}D_{\Delta t}^{\alpha}u^j \le \lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1} + f^j, \quad j = 1, 2\cdots, \tag{16}$$

then there exists a positive constant Δt^*, for $\Delta t < \Delta t^*$, the following holds

$$u^n \le 2\left(\lambda_4\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \lambda_5\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \max_{1\le j\le n}f^j\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)} + 2M + \lambda_2 M\Delta t^{\alpha} + 2\lambda_3 M\Delta t^{\alpha}\right)E_{\alpha}(2\Gamma(\alpha)\lambda t_n^{\alpha}), \quad 1 \le n \le N, \tag{17}$$

where $\lambda = \lambda_1 + (1/(g_0^{(\alpha)} - g_1^{(\alpha)}))\lambda_2 + (1/(g_1^{(\alpha)} - g_2^{(\alpha)}))\lambda_3 + (1/(g_{m-1}^{(\alpha)} - g_m^{(\alpha)}))\lambda_4 + (1/(g_m^{(\alpha)} - g_{m+1}^{(\alpha)}))\lambda_5$, $E_{\alpha}(z) = \sum_{k=0}^{\infty}(z^k/\Gamma(1+k\alpha))$ is the Mittag-Leffler function, and $M = \max\{u^{-m}, u^{-m+1}, \cdots, u^0\}$.

Proof. By using the definition of ${}^{R}D^{\alpha}_{\Delta t}u^n$ in (13), we have

$$\sum_{k=1}^{j} g_{j-k}^{(\alpha)}\delta_t u^k + g_j^{(\alpha)}u^0 \le \Delta t^{\alpha}\left(\lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1}\right) + \Delta t^{\alpha} f^j. \quad (18)$$

Multiplying equation (18) by ϕ_{n-j} and summing the index j from 1 to n, we get

$$\sum_{j=1}^{n}\phi_{n-j}\sum_{k=1}^{j} g_{j-k}^{(\alpha)}\delta_t u^k \le \Delta t^{\alpha}\sum_{j=1}^{n}\phi_{n-j}\left(\lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1}\right) + \Delta t^{\alpha}\sum_{j=1}^{n}\phi_{n-j}f^j - \sum_{j=1}^{n}\phi_{n-j}g_j^{(\alpha)}u^0. \quad (19)$$

We change the order of summation and make use of the definition of ϕ_{n-j} to obtain

$$\sum_{j=1}^{n}\phi_{n-j}\sum_{k=1}^{j} g_{j-k}^{(\alpha)}\delta_t u^k = \sum_{k=1}^{n}\delta_t u^k\sum_{j=1}^{k}\phi_{n-j}g_{j-k}^{(\alpha)} = \sum_{k=1}^{n}\delta_t u^k = u^n - u^0, \quad (20)$$

and using Lemma 4, we have

$$\Delta t^{\alpha}\sum_{j=1}^{n}\phi_{n-j}f^j \le \Delta t^{\alpha}\max_{1\le j\le n} f^j\sum_{j=1}^{n}\phi_{n-j} \le \Delta t^{\alpha}\max_{1\le j\le n} f^j\frac{\Gamma(\alpha)n^{\alpha}}{\Gamma(1+\alpha)} = \max_{1\le j\le n} f^j\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}. \quad (21)$$

Noticing $g_j^{(\alpha)}$ is monotone decreasing, and using Lemma 4, we have

$$-\sum_{j=1}^{n}\phi_{n-j}g_j^{(\alpha)}u^0 \le \sum_{j=1}^{n}\phi_{n-j}g_j^{(\alpha)}u^0 \le u^0\sum_{j=1}^{n}\phi_{n-j}g_{j-1}^{(\alpha)} = u^0. \quad (22)$$

Substituting (20), (21), and (22) into (19), we can obtain

$$u^n \le \Delta t^{\alpha}\sum_{j=1}^{n}\phi_{n-j}\left(\lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1}\right) + 2u^0 + \max_{1\le j\le n} f^j\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}. \quad (23)$$

Applying Lemma 4, we have

$$\Delta t^{\alpha}\sum_{j=1}^{m}\phi_{n-j}u^{j-m} \le \frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M, \Delta t^{\alpha}\sum_{j=1}^{m+1}\phi_{n-j}u^{j-m-1} \le \frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M. \quad (24)$$

Therefore

$$\lambda_4\Delta t^{\alpha}\sum_{j=1}^{m}\phi_{n-j}u^{j-m} + \lambda_5\Delta t^{\alpha}\sum_{j=1}^{m+1}\phi_{n-j}u^{j-m-1} + 2u^0 + \lambda_2\Delta t^{\alpha}\phi_{n-1}u^0 + \lambda_3\Delta t^{\alpha}\left(\phi_{n-1}u^{-1} + \phi_{n-2}u^0\right) \le \lambda_4\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \lambda_5\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + 2M + \lambda_2 M\Delta t^{\alpha} + 2\lambda_3 M\Delta t^{\alpha}. \quad (25)$$

Denote

$$\Psi_n = \lambda_4\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \lambda_5\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)}M + \max_{1\le j\le n} f^j\frac{\Gamma(\alpha)t_n^{\alpha}}{\Gamma(1+\alpha)} + 2M + \lambda_2 M\Delta t^{\alpha} + 2\lambda_3 M\Delta t^{\alpha}. \quad (26)$$

(23) can be rewritten as

$$(1-\lambda_1\Delta t^{\alpha})u^n \le \lambda_1\Delta t^{\alpha}\sum_{j=1}^{n-1}\phi_{n-j}u^j + \lambda_2\Delta t^{\alpha}\sum_{j=2}^{n}\phi_{n-j}u^{j-1} + \lambda_3\Delta t^{\alpha}\sum_{j=3}^{n}\phi_{n-j}u^{j-2} + \lambda_4\Delta t^{\alpha}\sum_{j=m+1}^{n}\phi_{n-j}u^{j-m} + \lambda_5\Delta t^{\alpha}\sum_{j=m+2}^{n}\phi_{n-j}u^{j-m-1} + \Psi_n. \quad (27)$$

Let $\Delta t^* = \sqrt[\alpha]{1/(2\lambda_1)}$, when $\Delta t \le \Delta t^*$, we have

$$u^n \le 2\Psi_n + 2\Delta t^{\alpha}\left[\lambda_1\sum_{j=1}^{n-1}\phi_{n-j}u^j + \lambda_2\sum_{j=2}^{n}\phi_{n-j}u^{j-1} + \lambda_3\sum_{j=3}^{n}\phi_{n-j}u^{j-2} + \lambda_4\sum_{j=m+1}^{n}\phi_{n-j}u^{j-m} + \lambda_5\sum_{j=m+2}^{n}\phi_{n-j}u^{j-m-1}\right]. \quad (28)$$

Let $V = (u^n, u^{n-1}, \cdots, u^1)^T$, then (28) can be rewritten in the following matrix form

$$V \le 2\Psi_n\vec{e} + (\lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_3 + \lambda_4 W_4 + \lambda_5 W_5)V, \quad (29)$$

where

$$W_1 = 2(\Delta t)^{\alpha} \begin{pmatrix} 0 & \phi_1 & \phi_2 & \cdots & \phi_{n-2} & \phi_{n-1} \\ 0 & 0 & \phi_1 & \cdots & \phi_{n-3} & \phi_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_1 & \phi_2 \\ 0 & 0 & 0 & \cdots & 0 & \phi_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n},$$

$$W_2 = 2(\Delta t)^{\alpha} \begin{pmatrix} 0 & \phi_0 & \phi_1 & \cdots & \phi_{n-3} & \phi_{n-2} \\ 0 & 0 & \phi_0 & \cdots & \phi_{n-4} & \phi_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_0 & \phi_1 \\ 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n},$$

$$W_3 = 2(\Delta t)^{\alpha} \begin{pmatrix} 0 & 0 & \phi_0 & \cdots & \phi_{n-4} & \phi_{n-3} \\ 0 & 0 & 0 & \cdots & \phi_{n-5} & \phi_{n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{n\times n},$$

$$W_4 = 2(\Delta t)^{\alpha} \begin{pmatrix} 0 & \cdots & 0 & \phi_0 & \phi_1 & \cdots & \phi_{n-m-2} & \phi_{n-m-1} \\ 0 & \cdots & 0 & 0 & \phi_0 & \cdots & \phi_{n-m-3} & \phi_{n-m-2} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \phi_0 & \phi_1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n},$$

$$W_5 = 2(\Delta t)^{\alpha} \begin{pmatrix} 0 & \cdots & 0 & 0 & \phi_0 & \cdots & \phi_{n-m-3} & \phi_{n-m-2} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \phi_{n-m-4} & \phi_{n-m-3} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}. \tag{30}$$

Since the definition of ϕ_n, we have

$$\phi_{n-j} \leq \frac{1}{g_{j-1}^{(\alpha)} - g_j^{(\alpha)}} \phi_n. \tag{31}$$

Then,

$$\begin{aligned} W_2 V &\leq \frac{1}{g_0^{(\alpha)} - g_1^{(\alpha)}} W_1 V, \\ W_3 V &\leq \frac{1}{g_1^{(\alpha)} - g_2^{(\alpha)}} W_1 V, \\ W_4 V &\leq \frac{1}{g_{m-1}^{(\alpha)} - g_m^{(\alpha)}} W_1 V, \\ W_5 V &\leq \frac{1}{g_m^{(\alpha)} - g_{m+1}^{(\alpha)}} W_1 V. \end{aligned} \tag{32}$$

Hence, (29) can be shown as follows

$$\begin{aligned} V &\leq \left(\lambda_1 + \frac{1}{g_0^{(\alpha)} - g_1^{(\alpha)}} \lambda_2 + \frac{1}{g_1^{(\alpha)} - g_2^{(\alpha)}} \lambda_3 \right. \\ &\quad \left. + \frac{1}{g_{m-1}^{(\alpha)} - g_m^{(\alpha)}} \lambda_4 + \frac{1}{g_m^{(\alpha)} - g_{m+1}^{(\alpha)}} \lambda_5 \right) W_1 V + 2\Psi_n \vec{e} \\ &= WV + 2\Psi_n \vec{e}, \end{aligned} \tag{33}$$

where $W = \lambda W_1$.
Therefore,

$$\begin{aligned} V &\leq WV + 2\Psi_n \vec{e} \leq W\left(WV + 2\Psi_n \vec{e}\right) + 2\Psi_n \vec{e} \\ &= W^2 V + 2\Psi_n \sum_{j=0}^{1} W^j \vec{e} \leq \cdots \leq W^n V + 2\Psi_n \sum_{j=0}^{n-1} W^j \vec{e}. \end{aligned} \tag{34}$$

According to Lemma 5, the result can be proved.

Lemma 7 (see [51]). For any sequence $\{e^k\}_{k=0}^{N} \subset X_h$, the following inequality holds

$$\left\langle {}^R D_{\Delta t}^{\alpha} e^k, \left(1 - \frac{\alpha}{2}\right) e^k + \frac{\alpha}{2} e^{k-1} \right\rangle \geq \frac{1}{2} {}^R D_{\Delta t}^{\alpha} \left\| e^k \right\|^2, \quad 1 \leq k \leq N. \tag{35}$$

Lemma 8 (see [52]). There exists a positive constant C_{Ω}, independent of h, for any $v \in H^s(\Omega) \cap H_0^1(\Omega)$, such that

$$\|v - R_h v\|_{L^2} + h\|\nabla(v - R_h v)\|_{L^2} \leq C_{\Omega} h^s \|v\|_{H^s}, \quad 1 \leq s \leq r+1. \tag{36}$$

3.2. Proof of Theorem 1

Now, we are ready to prove our main results.

Proof. Taking $t = t_{n-(\alpha/2)}$ in the first equation (1), we can find that u^n satisfies the following equation

$$\left\langle {}^R D^\alpha_{\Delta t} u^n, v\right\rangle + \langle \nabla u^{n,\alpha}, \nabla v\rangle = \left\langle f\left(t_{n-(\alpha/2)}, \widehat{u}^{n,\alpha}, u^{n-m_\tau,\alpha}\right), v\right\rangle + \langle P^n, v\rangle, \tag{37}$$

for $n = 1, 2, 3, \cdots, N$ and $\forall v \in X_h$, where

$$\begin{aligned} P^n = {}& {}^R D^\alpha_{\Delta t} u^n - {}^R D^\alpha_{t_{n-(\alpha/2)}} u + \Delta u^{n-(\alpha/2)} - \Delta u^{n,\alpha} \\ & + f\left(t_{n-(\alpha/2)}, u^{n-(\alpha/2)}, u^{n-m_\tau-(\alpha/2)}\right) \\ & - f\left(t_{n-(\alpha/2)}, \widehat{u}^{n,\alpha}, u^{n-m_\tau,\alpha}\right). \end{aligned} \tag{38}$$

Now, we estimate the error of $\|P^n\|$. Actually, from the definition of $u^{n,\alpha}$ and $\widehat{u}^{n,\alpha}$ and the regularity of the exact solution (8), we can obtain that

$$\begin{aligned} & \left\| u^{n-(\alpha/2)} - u^{n,\alpha} \right\| \\ & = \left\| \left(1 - \frac{\alpha}{2}\right)\left(u^{n-(\alpha/2)} - u^n\right) + \frac{\alpha}{2}\left(u^{n-(\alpha/2)} - u^{n-1}\right) \right\| \\ & = \left\| -\left(1 - \frac{\alpha}{2}\right)\frac{\alpha}{2}\Delta t u'(\xi_1) + \left(1 - \frac{\alpha}{2}\right)\frac{\alpha}{2}\Delta t u'(\xi_2) \right\| \\ & = \left(1 - \frac{\alpha}{2}\right)\frac{\alpha}{2}\Delta t \left\| \left(u'(\xi_2) - u'(\xi_1)\right) \right\| \\ & \le \left(1 - \frac{\alpha}{2}\right)\frac{\alpha}{2}\Delta t \int_{t_{n-1}}^{t_n} \|u_{tt}(s)\| ds \le C_1 \Delta t^2, \end{aligned} \tag{39}$$

$$\begin{aligned} & \left\| u^{n-(\alpha/2)} - \widehat{u}^{n,\alpha} \right\| \\ & = \left\| u^{n-(\alpha/2)} - \left(2 - \frac{\alpha}{2}\right)u^{n-1} + \left(1 - \frac{\alpha}{2}\right)u^{n-2} \right\| \\ & = \left\| \left(2 - \frac{\alpha}{2}\right)u^{n-(\alpha/2)} - \left(2 - \frac{\alpha}{2}\right)u^{n-1} \right. \\ & \quad \left. + \left(1 - \frac{\alpha}{2}\right)u^{n-2} - \left(1 - \frac{\alpha}{2}\right)u^{n-(\alpha/2)} \right\| \\ & = \left\| \left(2 - \frac{\alpha}{2}\right)\left(u^{n-(\alpha/2)} - u^{n-1}\right) + \left(1 - \frac{\alpha}{2}\right)\left(u^{n-2} - u^{n-(\alpha/2)}\right) \right\| \\ & = \left\| \left(2 - \frac{\alpha}{2}\right)\left(1 - \frac{\alpha}{2}\right)\Delta t u'(\xi_3) - \left(2 - \frac{\alpha}{2}\right)\left(1 - \frac{\alpha}{2}\right)\Delta t u'(\xi_4) \right\| \\ & = \left(2 - \frac{\alpha}{2}\right)\left(1 - \frac{\alpha}{2}\right)\Delta t \left\| u'(\xi_3) - u'(\xi_4) \right\| \\ & \le \left(2 - \frac{\alpha}{2}\right)\left(1 - \frac{\alpha}{2}\right)\Delta t \int_{t_{n-2}}^{t_{n-1}} \|u_{tt}(s)\| ds \le C_2 \Delta t^2, \end{aligned} \tag{40}$$

where $\xi_1 \in (t_{n-(\alpha/2)}, t_n), \xi_2 \in (t_{n-1}, t_{n-(\alpha/2)}), \xi_3 \in (t_{n-(\alpha/2)}, t_{n-1}), \xi_4 \in (t_{n-2}, t_{n-(\alpha/2)})$, $C_1 = (1 - (\alpha/2))(\alpha/2)K$, $C_2 = (2 - (\alpha/2))(1 - (\alpha/2))K$ are constants.

Applying (39) and (40) and the Lipschitz condition

$$\begin{aligned} & \left\| f\left(t_{n-(\alpha/2)}, u^{n-(\alpha/2)}, u^{n-m_\tau-(\alpha/2)}\right) - f\left(t_{n-(\alpha/2)}, \widehat{u}^{n,\alpha}, u^{n-m_\tau,\alpha}\right) \right\| \\ & \quad \le (L_1 C_1 + L_2 C_2)\Delta t^2, \end{aligned} \tag{41}$$

$$\left\| \Delta\left(u^{n,\alpha} - u^{n-(\alpha/2)}\right) \right\| \le C_1 \Delta t^2, \tag{42}$$

which further implies that

$$\|P^n\| \le C_K (\Delta t)^2, n = 1, 2, 3, \cdots, N, \tag{43}$$

here $C_K = L_1 C_1 + L_2 C_2$.

Denote $\theta^n_h = R_h u^n - U^n_h, n = 0, 1, \cdots, N$.

Substituting fully scheme (5) from equation (37) and using the property in (7), we can get that

$$\left\langle {}^R D^\alpha_{\Delta t} \theta^n_h, v\right\rangle + \langle \nabla \theta^{n,\alpha}_h, v\rangle = \langle R^n_1, v\rangle + \langle P^n, v\rangle - \left\langle {}^R D^\alpha_{\Delta t}(u^n - R_h u^n), v\right\rangle, \tag{44}$$

where

$$R^n_1 = f\left(t_{n-(\alpha/2)}, \widehat{U}^{n,\alpha}_h, U^{n-m_\tau,\alpha}_h\right) - f\left(t_{n-(\alpha/2)}, \widehat{u}^{n,\alpha}, u^{n-m_\tau,\alpha}\right). \tag{45}$$

Setting $v = \theta^{n,\alpha}_h$ and applying Cauchy-Schwarz inequality, it holds that

$$\begin{aligned} & \left\langle {}^R D^\alpha_{\Delta t} \theta^n_h, \theta^{n,\alpha}_h \right\rangle + \|\nabla \theta^{n,\alpha}_h\|^2 \\ & \quad \le \|R^n_1\| \|\theta^{n,\alpha}_h\| + \|P^n\| \|\theta^{n,\alpha}_h\| + \left\| {}^R D^\alpha_{\Delta t}(u^n - R_h u^n) \right\| \|\theta^{n,\alpha}_h\|. \end{aligned} \tag{46}$$

Noticing the fact $ab \le 1/2(a^2 + b^2)$ and $\|\nabla \theta^{n,\alpha}_h\|^2 \ge 0$,

$$\begin{aligned} \left\langle {}^R D^\alpha_{\Delta t} \theta^n_h, \theta^{n,\alpha}_h \right\rangle \le {}& \frac{1}{2}\left(\|R^n_1\|^2 + \|P^n\|^2 + \left\| {}^R D^\alpha_{\Delta t}(u^n - R_h u^n) \right\|^2 \right) \\ & + \frac{3}{2}\|\theta^{n,\alpha}_h\|^2. \end{aligned} \tag{47}$$

Together with (9) and (36), we can arrive that

$$\left\| {}^R D^\alpha_{\Delta t}(u^n - R_h u^n) \right\| \le C_\Omega h^{r+1} \left\| {}^R D^\alpha_{\Delta t} u^n \right\|_{H^{r+1}} \le C_\Omega K h^{r+1},$$

$$\begin{aligned} & \|\widehat{u}^{n,\alpha} - R_h \widehat{u}^{n,\alpha}\| \\ & = \left\| \left(2 - \frac{\alpha}{2}\right)u^{n-1} - \left(1 - \frac{\alpha}{2}\right)u^{n-2} \right. \\ & \quad \left. - \left(2 - \frac{\alpha}{2}\right)R_h u^{n-1} + \left(1 - \frac{\alpha}{2}\right)R_h u^{n-2} \right\| \\ & \le \left(2 - \frac{\alpha}{2}\right)\left\| u^{n-1} - R_h u^{n-1} \right\| + \left(1 - \frac{\alpha}{2}\right)\left\| u^{n-2} - R_h u^{n-2} \right\| \\ & \le \left(2 - \frac{\alpha}{2}\right) C_\Omega h^{r+1} \|u^{n-1}\|_{H^{r+1}} + \left(1 - \frac{\alpha}{2}\right) C_\Omega h^{r+1} \|u^{n-2}\|_{H^{r+1}} \\ & \le \left(2 - \frac{\alpha}{2}\right) C_\Omega K h^{r+1} + \left(1 - \frac{\alpha}{2}\right) C_\Omega K h^{r+1} \le C_3 h^{r+1}. \end{aligned} \tag{48}$$

Similarly, we have

$$\begin{aligned}&\|u^{n-m_\tau,\alpha}-R_hu^{n-m_\tau,\alpha}\|\\&\quad=\left\|\left(1-\frac{\alpha}{2}\right)u^{n-m_\tau}+\frac{\alpha}{2}u^{n-m_\tau-1}\right.\\&\qquad\left.-\left(1-\frac{\alpha}{2}\right)R_hu^{n-m_\tau}-\frac{\alpha}{2}R_hu^{n-m_\tau-1}\right\|\\&\quad\le\left(1-\frac{\alpha}{2}\right)C_\Omega Kh^{r+1}+\frac{\alpha}{2}C_\Omega Kh^{r+1}\le C_4h^{r+1},\end{aligned}\tag{49}$$

where $C_3=2(2-(\alpha/2))C_\Omega K$, $C_4=2\max\{(1-(\alpha/2)),\alpha/2\}C_\Omega K$.

Therefore

$$\begin{aligned}\|R_1^n\|&=\left\|f\left(t_{n-(\alpha/2)},\widehat{u}^{n,\alpha},u^{n-m_\tau,\alpha}\right)-f\left(t_{n-(\alpha/2)},\widehat{U}_h^{n,\alpha},U_h^{n-m_\tau,\alpha}\right)\right\|\\&\le L_1\left\|\widehat{u}^{n,\alpha}-\widehat{U}_h^{n,\alpha}\right\|+L_2\|u^{n-m_\tau,\alpha}-U_h^{n-m_\tau,\alpha}\|\\&\le L_1\left\|\widehat{\theta}_h^{n,\alpha}\right\|+L_2\|\theta_h^{n-m_\tau,\alpha}\|+L_1\|\widehat{u}^{n,\alpha}-R_h\widehat{u}^{n,\alpha}\|\\&\quad+L_2\|u^{n-m_\tau,\alpha}-R_hu^{n-m_\tau,\alpha}\|\\&\le L_1\left\|\widehat{\theta}_h^{n,\alpha}\right\|+L_2\|\theta_h^{n-m_\tau,\alpha}\|+(L_1C_3+L_2C_4)h^{r+1}.\end{aligned}\tag{50}$$

Substituting (43), (48), and (50) into (47) and the fact $(a+b+c)^2\le 3a^2+3b^2+3c^2$, we can get

$$\begin{aligned}\left\langle {}^RD_{\Delta t}^\alpha\theta_h^n,\theta_h^{n,\alpha}\right\rangle&\le\frac{3}{2}\|\theta_h^{n,\alpha}\|^2+\frac{3L_1^2}{2}\left\|\widehat{\theta}_h^{n,\alpha}\right\|^2\\&\quad+\frac{3L_2^2}{2}\left\|\theta_h^{n-m_\tau,\alpha}\right\|^2+\frac{C_K^2}{2}(\Delta t)^4\\&\quad+\frac{1}{2}\left[3\left(L_1^2C_3^2+L_2^2C_4^2\right)+(C_KK)^2\right]h^{2(r+1)}\\&\le\frac{3}{2}\|\theta_h^{n,\alpha}\|^2+\frac{3L_1^2}{2}\left\|\widehat{\theta}_h^{n,\alpha}\right\|^2\\&\quad+\frac{3L_2^2}{2}\left\|\theta_h^{n-m_\tau,\alpha}\right\|^2+\frac{C_4}{2}\left(\Delta t^2+h^{r+1}\right)^2,\end{aligned}\tag{51}$$

where $C_4=\max\{C_K^2,3(L_1^2C_3^2+L_2^2C_4^2)+(C_KK)^2\}$.

Table 1: The errors and convergence orders in temporal direction by using Q-FEM.

M	$\alpha=0.4$ Errors	Orders	$\alpha=0.6$ Errors	Orders
5	1.6856e-03	*	5.3999e-03	*
10	2.9420e-04	2.5184	1.2503e-03	2.1106
20	5.9619e-05	2.3030	3.0266e-04	2.0465
40	1.3851e-05	2.1058	7.4700e-05	2.0185

Applying Lemma 7, we have

$$\begin{aligned}{}^RD_{\Delta t}^\alpha\|\theta_h^n\|^2&\le 3\|\theta_h^{n,\alpha}\|^2+3L_1^2\left\|\widehat{\theta}_h^{n,\alpha}\right\|^2\\&\quad+3L_2^2\left\|\theta_h^{n-m_\tau,\alpha}\right\|^2+C_4\left(\Delta t^2+h^{r+1}\right)^2.\end{aligned}\tag{52}$$

In terms of the definition of $\|\theta_h^{n,\alpha}\|$ and $\widehat{\theta}_h^{n,\alpha}$, we obtain

$$\begin{aligned}{}^RD_{\Delta t}^\alpha\|\theta_h^n\|^2&\le 3\left(1-\frac{\alpha}{2}\right)^2\|\theta_h^n\|^2+\left(3\left(\frac{\alpha}{2}\right)^2+3L_1^2\left(2-\frac{\alpha}{2}\right)^2\right)\|\theta_h^{n-1}\|^2\\&\quad+3L_1^2\left(1-\frac{\alpha}{2}\right)^2\|\theta_h^{n-2}\|^2+3L_2^2\left(1-\frac{\alpha}{2}\right)^2\|\theta_h^{n-m_\tau}\|^2\\&\quad+3L_2^2\left(\frac{\alpha}{2}\right)^2\left\|\theta_h^{n-m_\tau-1}\right\|^2+C_4\left(\Delta t^2+h^{r+1}\right)^2.\end{aligned}\tag{53}$$

Using Theorem 6, we can find a positive constant Δt^* such that $\Delta t\le\Delta t^*$, then

$$\|\theta_h^n\|^2\le C_5\left(\Delta t^2+h^{r+1}\right)^2,\tag{54}$$

where C_5 is a nonnegative constant which only dependents on $L_1, L_2, C_4, C_K, C_\Omega$. In terms of the definition of θ_h^n, we have

$$\|u^n-U_h^n\|\le\|u^n-R_hu^n\|+\|R_hu^n-U_h^n\|\le C_1^*\left(\Delta t^2+h^{r+1}\right).\tag{55}$$

Then, we complete the proof.

4. Numerical Examples

In this section, we give two examples to verify our theoretical results. The errors are all calculated in L2-norm.

Example 1. Consider the nonlinear time fractional Mackey-Glass-type equation

$$\begin{cases}{}^RD_{\Delta t}^\alpha u(x,y,t)=\Delta u(x,y,t)-2u(x,y,t)+\dfrac{u(x,y,t-0.1)}{1+u^2(x,y,t-0.1)}+f(x,y,t), (x,y)\in[0,1]^2, t\in[0,1],\\ u(x,y,t)=t^2\sin(\pi x)\sin(\pi y),\quad (x,y)\in[0,1]^2, t\in[-0.1,0],\end{cases}\tag{56}$$

Table 2: The errors and convergence orders in spatial direction by using L-FEM.

M	$\alpha = 0.4$ Errors	Orders	$\alpha = 0.6$ Errors	Orders
5	7.2603e-02	*	7.2065e-02	*
10	1.9449e-02	1.9003	1.9297e-02	1.9009
20	8.7594e-03	1.9673	8.6948e-03	1.9662
40	4.9508e-03	1.9834	4.9180e-03	1.9807

Table 3: The errors and convergence orders in spatial direction by using Q-FEM.

M	$\alpha = 0.4$ Errors	Orders	$\alpha = 0.6$ Errors	Orders
5	2.0750e-03	*	2.0746e-03	*
10	2.4888e-04	3.0596	2.5148e-04	3.0443
20	7.3251e-05	3.0165	7.5802e-05	2.9577
40	3.0946e-05	2.9952	3.4200e-05	2.7666

where

$$f(x,y,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \sin(\pi x)\sin(\pi y) + 2t^2\pi^2 \sin(\pi x)\sin(\pi y) - 2t^2 \sin(\pi x)\sin(\pi y) - \frac{(t-0.1)^2 \sin(\pi x)\sin(\pi y)}{1+\left[(t-0.1)^2 \sin(\pi x)\sin(\pi y)\right]^2}. \quad (57)$$

The exact solution is given as

$$u(x,t) = t^2 \sin(\pi x)\sin(\pi y). \quad (58)$$

Table 4: The errors and orders in temporal and spatial direction by using L-FEM.

M	$\alpha = 0.4$ Errors	Orders	$\alpha = 0.6$ Errors	Orders
5	8.3275e-02	*	8.3375e-02	*
10	2.2615e-02	1.8806	2.2732e-02	1.8749
20	5.8356e-03	1.9543	5.8662e-03	1.9542
40	1.4707e-03	1.9884	1.4784e-03	1.9884

Table 5: The errors and orders in temporal direction and spatial direction by using Q-FEM.

	M	$\alpha = 0.4$ Errors	Orders	$\alpha = 0.6$ Errors	Orders
	8	6.7379e-04	*	6.9141e-04	*
$N = M^{(3/2)}$	10	3.1416e-04	3.0459	3.4945e-04	3.0579
	12	1.9415e-04	3.0968	1.9787e-04	3.1196
	14	1.1891e-04	3.1806	1.1992e-04	3.2485

In order to test the convergence order in temporal direction, we fixed $M = 40$ for $\alpha = 0.4$, $\alpha = 0.6$ and different N. Similarly, to obtain the convergence order in spatial direction, we fixed $N = 100$ for $\alpha = 0.4$, $\alpha = 0.6$ and different M. Table 1 gives the errors and convergence orders in temporal direction by using the Q-FEM. Table 1 shows that the convergence order in temporal direction is 2. Similarly, Tables 2 and 3 give the errors and convergence orders in spatial direction by using the L-FEM and Q-FEM, respectively. These numerical results correspond to our theoretical convergence order.

Example 2. Consider the following nonlinear time fractional Nicholsons blowflies equation

$$\begin{cases} {}^{R}D_{\Delta t}^{\alpha} u(x,y,z,t) = \Delta u(x,y,z,t) - 2u(x,y,z,t) + u(x,y,z,t-0.1)\exp\{-u(x,y,z,t-0.1)\} \\ +f(x,y,z,t), \quad (x,y,z) \in [0,1]^3, \quad t \in [0,1], \\ u(x,y,z,t) = t^2 \sin(\pi x)\sin(\pi y)\sin(\pi z), \quad (x,y,z) \in [0,1]^3, \quad t \in [-0.1,0], \end{cases} \quad (59)$$

where

$$\begin{aligned} f(x,y,z,t) = {} & \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \sin(\pi x)\sin(\pi y)\sin(\pi z) \\ & + 2t^2(\pi^2-1)\sin(\pi x)\sin(\pi y)\sin(\pi z) \\ & - (t-0.1)^2 \sin(\pi x)\sin(\pi y)\sin(\pi z)\exp \\ & \cdot \left\{-(t-0.1)^2 \sin(\pi x)\sin(\pi y)\sin(\pi z)\right\}, \end{aligned} \quad (60)$$

the exact solution is given as

$$u(x,t) = t^2 \sin(\pi x)\sin(\pi y)\sin(\pi z). \quad (61)$$

In this example, in order to test the convergence order in temporal and spatial direction, we solve this problem by using the L-FEM with $M = N$ and the Q-FEM with $N = M^{(3/2)}$, respectively. Tables 4 and 5 show that the convergence orders in temporal and spatial direction are 2 and 3, respectively. The numerical results confirm our theoretical convergence order.

5. Conclusions

We proposed a linearized fractional Crank-Nicolson-Galerkin FEM for the nonlinear fractional parabolic equations with time delay. A novel fractional Grönwall type inequality is developed. With the help of the inequality, we prove convergence of the numerical scheme. Numerical examples confirm our theoretical results.

References

[1] F. Höfling, T. Franosch, *Anomalous transport in the crowded world of biological cells*, *Rep. Progr. Phys.* 76 (4) (2013) 046602.

[2] A. Arafa, S. Rida, M. Khalil, *Fractional modeling dynamics of HIV and CD4+ T-cells during primary infection*, *Nonlinear. Biomed. phys.* 6 (1) (2012) 1.

[3] R. L. Magin, *Fractional calculus in bioengineering*, *Begell House Redding*, 2006.

[4] N. Sebaa, Z. E. A. Fellah, W. Lauriks, C. Depollier, *Application of fractional calculus to ultrasonic wave propagation in human cancellousbone*, *Signal. Processing.* 86 (10) (2006) 2668-2677.

[5] A. Carpinteri, F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, Vol. 378, *Springer*, 2014.

[6] B. West, M. Bologna, P. Grigolini, *Physics of fractal operators*, *Springer Science amp; Business Media*, 2012.

[7] D. Li, C. Zhang, *Long time numerical behaviors of fractional pantograph equations*, *Math. Comput. Simul.* 172 (2020) 244–257.

[8] A. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204. *Elsevier Science Limited*. 2006.

[9]Q. Zhang, Y. Ren, X. Lin, Y. Xu, Uniform convergence of compact and BDF methods for the space fractional semilinear delay reaction-diffusion equations. Appl. Math. Comput. 358 (2019) 91-110.

[10] W. Chen, L. Ye, H. Sun, *Fractional diffusion equations by the Kansa method*, *Comput. Math. Appl.* 59 (5) (2010) 1614-1620.

[11] M. Dehghan, M. Abbaszadeh, A. Mohebbi, *Error estimate for the numerical solution of fractional reaction-subdiffusion process based on a meshless method*, *J. Comput. Appl. Math.* 280 (2015) 14-36.

[12] D. A. Murio, *Implicit finite difference approximation for time fractional diffusion equations*, *Comput. and Math. Appl.* 56 (2008) 1138-1145.

[13] P. Zhuang, F. Liu, *Implicit difference approximation for the time fractional diffusion equation*, *J. Appl. Math. Comput.* 22 (3) (2006) 87-99.

[14] B. Jin, B. Li, Z. Zhou, An analysis of the Crank-Nicolson method for subdiffusion, IMA. J. Numer. Anal. 38 (1) (2017) 518-541.

[15] D. Li, W. Sun, C. Wu, A novel numerical approach to time-fractional parabolic equations with nonsmooth solutions, Numer. Math. Theor. Meth. Appl. 14 (2021) 355-376.

[16] L. Li, D. Li, Exact solutions and numerical study of time fractional Burgersï¿½ï¿½ equations, Appl. Math. Lett. 100(2020) 106011.

[17] C. Li, W. Deng, *High order schemes for the tempered fractional diffusion equations*, *Adv. Comput. Math.* 42 (2016) 543-572.

[18] S.B. Yuste, *Weighted average finite difference methods for fractional diffusion equations*, *J. Comput. Phys.* 216 (2006) 264-274.

[19] S.B. Yuste, L Acedo, *An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations*, *SIAM J. Numer. Anal.* 42 (5) (2005) 1862-1874.

[20] C. Çelik, M Duman, *Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative*, *J. Comput. Phys.* 231 (4) (2012) 1743-1750.

[21] X. Lin and C. Xu, *Finite difference/spectral approximations for the time-fractional diffusion equation*, *J. Comput. Phys.* 225 (2007), 1533-1552.

[22] X. Chen, Y. Di, J. Duan, D. Li, *Linearized compact ADI schemes for nonlinear time-fractional Schrödinger equations*, *Appl. Math. Lett.* 84 (2018) 160-167.

[23] D. Li, J. Wang, J. Zhang, *Unconditionally convergent L1-Galerkin FEMs for nonlinear time-fractional Schrödinger equations*, *SIAM. J. Sci. Comput.* 39 (2017) A3067-A3088.

[24] Z. Sun, J. Zhang, Z. Zhang, *Optimal error estimates in numerical solution of time fractional Schrödinger equations on unbounded domains*, *East Asian J. Appl. Math.* 8 (2018) 634-655.

[25] M. Gunzburger, J. Wang, *A second-order Crank-Nicolson method for time-fractional PDEs*, *Int. J. Numer. Anal. Model.* 16 (2) (2019) 225-239.

[26] N. H. Sweilam, H. Moharram, N. K. A. Abdel Moniem, S. Ahmed, *A parallel Crank-Nicolson finite difference method for time-fractional parabolic equation*, *J. Numer. Math.* 22 (4) (2014) 363-382.

[27] N. H. Sweilam, M. M. Khader, A. M. Mahdy, *Crank-Nicolson finite difference method for solving time-fractional diffusion equation*, *J. Frac. Calcu. Appl.* 2 (2) (2012) 1-9.

[28] Q. Zhang, M. Ran, D. Xu, *Analysis of the compact difference scheme for the semilinear fractional partial differential equation with time delay*, *Appl. Anal.* 96 (11) (2016) 1867-1884.

[29] F.A. Rihan, *Computational methods for delay parabolic and time-fractional partial differential equations*, *Numer. Methods. P.D.E* 26 (2010) 1556-1571.

[30] M. Li, C. Huang, F. Jiang, *Galerkin finite element method for higher dimensional multi-term fractional diffusion equation on non-uniform meshes*, *Appl. Anal.* 96 (8) (2017) 1269-1284.

[31] J. Cao, C. Xu, *A high order schema for the numerical solution of the fractional ordinary differential equations*, *J. Comput. Phys.* 238 (2013) 154-168.

[32] M. Stynes, E. O'riordan, J. L. Gracia, *Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation*, *SIAM J. Numer. Anal.* 55 (2017) 1057-1079.

[33] D. Li, H. Liao, W. Sun, J. Wang, J. Zhang, *Analysis of L1-Galerkin FEMs for time-fractional nonlinear parabolic problems*, *Commun. Comput. Phys.* 24 (2018) 86-103.

[34] D. Li, J. Zhang, Z. Zhang, *Unconditionally optimal error estimates of a linearized galerkin method for nonlinear time fractional reaction-subdiffusion equations*, *J. Sci. Comput.* 76 (2) (2018) 848-866.

[35] R. Lin, F. Liu, *Fractional high order methods for the nonlinear fractional ordinary differential equation*, *Nonlinear. Anal. Theory. Methods. Appl.* 66 (4) (2007) 856-869.

[36] Y. Liu, Y. Du, H. Li, S. He, W. Gao, *Finite difference/finite element method for a nonlinear time-fractional fourth-order reaction-diffusion problem*, *Comput. Math. Appl.* 70(4) (2015) 573-591.

[37] C. Li, F. Zeng, *Finite difference methods for fractional differential equations*, *Internat. J. Bifur. Chaos.* 22 (4) (2012). 1230014.

[38] B. Jin, B. Li, Z. Zhou, *Numerical analysis of nonlinear subdiffusion equations*, *SIAM. J. Numer. Anal.* 56 (2018) 1-23.

[39] H. Liao, D. Li, J. Zhang, *Sharp error estimate of the nonuniform L1 formula for linear reaction-subdiffusion equations*, *SIAM J. Numer. Anal.* 56 (2018) 1112-1133.

[40] D. Li, C. Wu, Z. Zhang, *Linearized Galerkin FEMs for nonlinear time fractional parabolic problems with nonsmooth solutions in time direction*, J. Sci. Comput., 80 (2019) 403-419.

[41] Z. Wang, S. Vong, *Compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation*, *J. Comput. Phys.* 277 (2014)1-15.

[42] F. Zeng, C. Li, F. Liu, I. Turner, *Numerical algorithms for time-fractional subdiffusion equation with second-order accuracy*, *SIAM J. Sci. Comput.* 37(1) (2015) A55-A78.

[43] X. Zhao, Z. Sun, *Compact Crank-Nicolson Schemes for a class of fractional Cattaneo equation in inhomogeneous medium*, *J. Sci. Comput.* 62 (3) (2015) 747-771.

[44] C. Lubich, *Convolution quadrature and discretized operational calculus*, *I. Numer. Math.* 52 (1988) 129-145.

[45] B. Jin, B. Li, Z. Zhou, *Correction of high-order BDF convolution quadrature for fractional evolution equations*, *SIAM J. Sci. Comput.* 39 (6) (2017) A3129-A3152.

[46] N. Liu, Y. Liu, H. Li, J. Wang, *Time second-order finite difference/finite element algorithm for nonlinear time-fractional diffusion problem with fourth-order derivative term*, *Comput. Math. Appl.* 75 (10) (2018) 3521-3536.

[47] L. Li, B. Zhou, X. Chen, Z. Wang, *Convergence and stability of compact finite difference method for nonlinear time fractional reaction-diffusion equations with delay*, *Appl. Math. Comput.* 337 (2018) 144-152.

[48] A. S. Hendy, V. G. Pimenov, J. E. Macas-Daz, *Convergence and stability estimates in difference setting for time-fractional parabolic equations with functional delay*, *Numer. Methods. Partial. Differ. Equ.* (2019).

[49] A. S. Hendy, J. E. Macas-Daz, *A novel discrete Gronwall inequality in the analysis of difference schemes for time-fractional multi-delayed diffusion equations*, *Commun. Nonlinear. Sci. Numer. Simulat.* 73 (2019) 110-119.

[50] V. Thomée, *Galerkin finite element methods for parabolic problems*, Vol. 1054, *Springer,* 1984.

[51] D. Kumar, S. Chaudhary, V. Kumar, *Fractional Cank-Nicolson-Galerkin finite element scheme for the time-fractional nonlinear diffusion equation*, *Numer. Meth. Part. Differ. Eq.* 35 (6) (2019) 2056-2075.

[52] R. Rannacher, R. Scott, *Some optimal error estimates for piecewise linear finite element approximations*, *Math. Comp.* 38 (158) (1982) 437-445.

References

[1] L. Li, M. She, and Y. Niu, “Fractional Crank-Nicolson-Galerkin Finite Element Methods for Nonlinear Time Fractional Parabolic Problems with Time Delay,” *Journal of Function Spaces*, vol. 2021, Article ID 9981211, 10 pages, 2021.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 6665420, 11 pages
https://doi.org/10.1155/2021/6665420

Research Article

Interpolating Stabilized Element Free Galerkin Method for Neutral Delay Fractional Damped Diffusion-Wave Equation

Mostafa Abbaszadeh,[1] **Mehdi Dehghan**,[1] **Mahmoud A. Zaky**,[2,3] **and Ahmed S. Hendy**[4]

[1]*Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology, No. 424, Hafez Ave., 15914 Tehran, Iran*
[2]*College of Education, King Saud University, Saudi Arabia*
[3]*Department of Mathematics, Nazarbayev University, NurSultan, Kazakhstan*
[4]*Department of Computational Mathematics and Computer Science, Institute of Natural Sciences and Mathematics, Ural Federal University, ul. Mira. 19, Yekaterinburg 620002, Russia*

Correspondence should be addressed to Mostafa Abbaszadeh; m.abbaszadeh@aut.ac.ir

Received 26 December 2020; Revised 8 June 2021; Accepted 6 October 2021; Published 19 November 2021

Academic Editor: Alberto Lastra

A numerical solution for neutral delay fractional order partial differential equations involving the Caputo fractional derivative is constructed. In line with this goal, the drift term and the time Caputo fractional derivative are discretized by a finite difference approximation. The energy method is used to investigate the rate of convergence and unconditional stability of the temporal discretization. The interpolation of moving Kriging technique is then used to approximate the space derivative, yielding a meshless numerical formulation. We conclude with some numerical experiments that validate the theoretical findings.

1. Introduction

Partial differential equations (PDEs) with time delay play an important role in the mathematical modeling of complex phenomena and processes whose states depend not only on a given moment in time but also on one or more previous moments. We can mention a simple scenario involving the hemodynamic behavior of a person suffering from low or high glucose decompensation. This person can then be given intravenous insulin to compensate for the low level. Because the drug must be introduced into the bloodstream for it to take effect, the preceding scenario can be interpreted as a delay problem. As a result, there is a growing interest in studying biological and physical models with delay. The solutions of delay PDEs may represent voltage, concentrations, temperature, or various particle densities such as bacteria, cells, animals, and chemicals [1–3].

Delay PDEs with fractional derivatives have recently been studied using various numerical and analytical techniques such as [4–8]. It was pointed out in [9] that the derivatives of the dependent variable in the neutral type delay differential equations are both with and without time delays. Delay differential equations of neutral type appear in a variety of new phenomena, and its theory is even more complicated than the theory of nonneutral delay differential equations. From both a theoretical and practical standpoint, the oscillatory behavior of neutral system solutions is important. For some applications, such as the population growth, motion of radiating electrons, and the spread of epidemics in networks with lossless transmission lines, we refer the interested reader to [9–14].

A consideration of the following fractional PDE with a constant delay is the goal of this paper. For that end, we introduce

$$\begin{aligned} {}_0^C D_t^{\nu}\Psi(t,x) + \Psi_t(t,x) = \Delta\Psi_t(t,x) + \Delta\Psi(t,x) + \Delta\Psi(t-s,x) \\ + f(t,x), t \in (0,T), x \in \Lambda, \end{aligned} \tag{1}$$

with initial and boundary conditions

$$\Psi(0,x)=\psi(x),\left.\frac{\partial\Psi(t,x)}{\partial t}\right|_{t=0}=\varphi(x), x\in\Lambda,$$
$$\Psi(t,x)=g(t,x), x\in\partial\Lambda, t\in[0,T], \tag{2}$$
$$\Psi(t,x)=\varphi^*(t,x),(t,x)\in(-s,0]\times\Lambda,$$

where ${}_0^CD_t^\nu$ is the Caputo fractional derivative which is defined by

$$ {}_0^CD_t^\nu\Psi(t,x)=\frac{\partial^\nu\Psi(t,x)}{\partial t^\nu}=\frac{1}{\Gamma(2-\nu)}\int_0^t\frac{\partial^2\Psi(s,x)}{\partial s^2}\cdot(t-s)^{1-\nu}ds, \nu\in(1,2). \tag{3}$$

A novel interpolating element-free Galerkin approach to approximate the solution of the two-dimensional elastoplasticity problems was constructed in [15] using the interpolating moving least squares scheme for obtaining the shape function. Moreover, an improved element-free Galerkin scheme to solve nonlinear elastic large deformation problems was considered in [16]. The interpolating moving least squares approach using a nonsingular weight function is employed in [17] to approximate the solution of the problem of inhomogeneous swelling of polymer gels, and also the penalty scheme is used to enforce the displacement boundary condition; thus, an improved element-free Galerkin approach was constructed.

The interpolating element free Galerkin method has been developed to solve a variety of problems, including two-dimensional elastoplasticity problems [15, 18], two-dimensional potential problems [19], two- and three-dimensional Stokes flow problems [20], two-dimensional large deformation problems [21], incompressible Navier-Stokes equation [22], steady heat conduction problems [23], two-dimensional transient heat conduction problems [24], three-dimensional wave equations [25], two-dimensional Schrödinger equation [26], two-dimensional large deformation of inhomogeneous swelling of gels [27], biological populations [28], two-dimensional elastodynamics problems [29], and two-dimensional unsteady state heat conduction problems [30]. The theoretical analysis for the complex moving least squares approximation, the properties of its shape function, and its stability was analyzed in [31]. In [32], a variational multiscale interpolating element-free Galerkin scheme was established for solving the Darcy flow. For the numerical solution of generalised Oseen problems, a novel variational multiscale interpolating element-free Galerkin scheme was developed in [33] based on moving Kriging interpolation for obtaining shape functions using the Kronecker delta function. Zaky and Hendy [34] constructed a finite difference/Galerkin spectral approach for solving the Higgs boson equation in the de Sitter spacetime universe, which can inherit the discrete energy dissipation property. A high-order efficient difference/Galerkin spectral approach was proposed in [35] for solving the time-space fractional Ginzburg-Landau equation. Hendy and Zaky [36] proposed a finite difference/spectral method based on the $L1$ formula on nonuniform meshes for time stepping and the Legendre-Galerkin spectral approach for solving a coupled system of nonlinear multiterm time-space fractional diffusion equations.

This paper is built up as follows. In Section 2, the temporal discretization is discussed. The analysis of the temporal discretization scheme is constructed in Section 3. The moving Kriging technique and its implementation are demonstrated in Section 4. Finally, numerical experiments are presented in Section 5 to illustrate the analysis of the obtained scheme.

2. Temporal Discretization

Assume that $\tau=s/m$ such that m is a positive integer. Take $N=\lceil T/\tau\rceil$ and $t_n=n\tau, \forall n\in\mathbb{N}^+\cup\{0\}$. Also, to make $t=s$, $2s,\cdots$ being grid points, the time-variable step size should be surrounded by $s=m\tau$ instead of $\tau=T/N_1$ for $N_1\in\mathbb{Z}^+$. Thus, $t_n=n\tau$ for $n=-m,-m+1,\cdots,0$. Here, we present a time-discrete scheme for Equation (1). For any function $\xi^n=\xi(x,y,t_n)$, we set

$$\xi^{n-(1/2)}=\frac{1}{2}\left(\xi^n+\xi^{n-1}\right),$$
$$\delta_t\xi^{n-(1/2)}=\frac{1}{\tau}\left(\xi^n-\xi^{n-1}\right). \tag{4}$$

Lemma 1 (see [37]). *Assume $\phi(t)\in C^2[0,t_n]$ and $\nu\in(1,2)$. Then*

$$\left|\int_0^{t_n}\frac{\phi'(t)}{(t_n-t)^{\nu-1}}dt-\mathscr{B}_C^\nu(\phi(t_n),\phi(t_0))\right|\le C_\nu\max_{0\le t\le t_n}\left|\phi''(t)\right|\tau^{3-\nu}, \tag{5}$$

in which

$$\mathscr{B}_C^\nu(\phi(t_n),\phi(t_0))=\frac{1}{\tau}\left[\lambda_0\phi(t_n)-\sum_{k=1}^{n-1}(\lambda_{n-k-1}-\lambda_{n-k})\phi(t_k)-\lambda_{n-1}\phi(t_0)\right],$$
$$\lambda_k=\frac{\tau^{2-\nu}}{2-\nu}\left[(k+1)^{2-\nu}-k^{2-\nu}\right]. \tag{6}$$

Let Ψ be the exact solution of (1) and

$$w(t,x,y,z)=\frac{1}{\Gamma(2-\nu)}\int_0^t(t-s)^{1-\nu}\frac{\partial v(s,x,y,z)}{\partial s}ds, \tag{7}$$

where $v(t,x,y,z)=\partial\Psi(t,x,y,z)/\partial t$. Thus, Equation (1) at (t_n,x,y,z) can be rewritten as

$$w\left(t_{n-(1/2)},\mathbf{x}\right)+v\left(t_{n-(1/2)},\mathbf{x}\right)=\Delta v\left(t_{n-(1/2)},\mathbf{x}\right)+\Delta\Psi\left(t_{n-(1/2)},\mathbf{x}\right)+\Delta\Psi\left(t_{n-(1/2)-m},\mathbf{x}\right)+f\left(t_{n-(1/2)},\mathbf{x}\right), n\ge0. \tag{8}$$

Making use of Taylor expansion yields

$$v^{n-12} = \delta_t \Psi^{n-12} + (e_1)^{n-12}, \quad (9)$$

$$w^{n-(1/2)} + v^{n-(1/2)} = \Delta v^{n-(1/2)} + \Delta \Psi^{n-(1/2)} + \Delta \Psi^{n-(1/2)-s} + f^{n-(1/2)} + (e_2)^{n-(1/2)}. \quad (10)$$

Employing Lemma 1 and putting $v^0 = v(x,0) = \varphi(x) = \varphi$ give

$$w^{n-12} = \frac{1}{\Gamma(2-\nu)\tau} \mathscr{B}_C^{\nu}\left(v^{n-12}, \varphi\right) + (e_3)^{n-12}. \quad (11)$$

Furthermore, there is a constant $c > 0$ that

$$\left|(e_1)^{n-12}\right| \le c\tau^2, \ \left|(e_2)^{n-12}\right| \le c\tau^2, \ \left|(e_3)^{n-12}\right| \le c\tau^{3-\nu}. \quad (12)$$

Substituting the above result into (10) arrives at

$$\begin{aligned} &\frac{1}{\Gamma(2-\nu)\tau} \mathscr{B}_C^{\nu}\left(\delta_t \Psi^{n-(1/2)}, \varphi\right) + \delta_t \Psi^{n-(1/2)} \\ &\quad = \Delta \delta_t \Psi^{n-(1/2)} + \Delta \Psi^{n-(1/2)} + \Delta \Psi^{n-(1/2)-m} + f^{n-(1/2)} \\ &\qquad + \mathscr{E}_\nu^{n-(1/2)}, n \ge 1, \end{aligned} \quad (13)$$

in which there exists $C \in \mathbb{R}^+$ such that

$$\left|E_\nu^{n-(1/2)}\right| \le C\tau^{3-\nu}. \quad (14)$$

Removing $\mathscr{E}_\nu^{n-(1/2)}$ yields

$$\begin{aligned} &\frac{1}{\Gamma(2-\nu)\tau} \mathscr{B}_C^{\nu}\left(\delta_t U^{n-(1/2)}, \varphi\right) + \delta_t U^{n-(1/2)} \\ &\quad = \Delta \delta_t U^{n-(1/2)} + \Delta U^{n-(1/2)} + \Delta U^{n-(1/2)-m} + f^{n-(1/2)}, n \ge 1. \end{aligned} \quad (15)$$

In the current paper, U^n is an approximation of exact solution Ψ^n.

3. Analysis of the Temporal Discretization

In the current section, we check the stability of the numerical procedure.

Lemma 2 (see [38]). *Let ω_s be a nonnegative sequence, and the sequence χ_s satisfies*

$$\begin{cases} \chi_0 \le a_0, \\ \chi_s \le a_0 + \sum_{r=0}^{s-1} b_r + \sum_{r=0}^{s-1} \omega_r \chi_r, \quad s \ge 1, \end{cases} \quad (16)$$

Then, for $a_0 \ge 0$ and $b_0 \ge 0$, we have

$$\chi_s \le \left(a_0 + \sum_{r=0}^{s-1} b_r\right) \exp\left(\sum_{r=0}^{s-1} \omega_r\right), s \ge 1. \quad (17)$$

Lemma 3 (see [37]). *For any $\mathscr{K} = \{\mathscr{K}_1, \mathscr{K}_2, \cdots\}$ and $\mathbb{Q}$, we obtain*

$$\begin{aligned} &\sum_{r=1}^{N}\left[\lambda_0 \mathscr{K}_r - \sum_{s=1}^{r-1}(\lambda_{r-s-1} - \lambda_{r-s})\mathscr{K}_s - \lambda_{r-1}\mathbb{Q}\right]\mathscr{K}_r \\ &\quad \ge \frac{t_N^{1-\nu}}{2}\tau \sum_{r=1}^{N} \mathscr{K}_r^2 - \frac{t_N^{2-\nu}}{2(2-\nu)}\mathbb{Q}^2. \end{aligned} \quad (18)$$

Theorem 4. *Let $\Psi^s \in H_0^1(\Lambda)$; then scheme (15) is unconditionally stable.*

Proof. We define $\mathscr{W}^s = \Psi^s - U^s$. Now, we have

$$\begin{aligned} &\frac{1}{\tau\Gamma(2-\nu)}\left\{\lambda_0 \delta_t \mathscr{W}^{s-(1/2)} - \sum_{k=1}^{s-1}(\lambda_{s-k-1} - \lambda_{s-k})\delta_t \mathscr{W}^{k-(1/2)}\right\} + \delta_t \mathscr{W}^{s-(1/2)} \\ &\quad = \Delta \mathscr{W}^{s-(1/2)} + \Delta \mathscr{W}^{s-(1/2)-m}. \end{aligned} \quad (19)$$

Multiplying relation (19) by $\tau\delta_t \mathscr{W}^{s-(1/2)}$, integrating over Λ and then summing from $s = 1$ to $\mathscr{M}$ give

$$\begin{aligned} &\frac{1}{\Gamma(2-\nu)} \sum_{s=1}^{\mathscr{M}} \left\{\lambda_0\left(\delta_t \mathscr{W}^{s-(1/2)}, \delta_t \mathscr{W}^{s-(1/2)}\right) \right. \\ &\quad \left. - \sum_{k=1}^{s-1}(\lambda_{s-k-1} - \lambda_{s-k})\left(\delta_t \mathscr{W}^{k-(1/2)}, \delta_t \mathscr{W}^{s-(1/2)}\right)\right\} \\ &\quad + \tau \sum_{s=1}^{\mathscr{M}}\left(\delta_t \mathscr{W}^{s-(1/2)}, \delta_t \mathscr{W}^{s-(1/2)}\right) \\ &\quad = \tau \sum_{s=1}^{M}\left(\Delta \delta_t \mathscr{W}^{s-(1/2)}, \delta_t \mathscr{W}^{s-(1/2)}\right) + \tau \sum_{s=1}^{M}\left(\Delta W^{s-(1/2)}, \delta_t W^{s-(1/2)}\right) \\ &\quad + \tau \sum_{s=1}^{M}\left(\Delta W^{s-(1/2)-m}, \delta_t W^{s-(1/2)}\right). \end{aligned} \quad (20)$$

Recalling the left hand side of the above relation, invoking Schwartz inequality and Lemma 3 yields

$$\begin{aligned} &\frac{1}{\Gamma(2-\nu)} \sum_{s=1}^{\mathscr{M}} \left\{\lambda_0 \left\|\delta_t \mathscr{W}^{s-(1/2)}\right\|_{L^2(\Lambda)}^2 - \sum_{k=1}^{s-1}(\lambda_{s-k-1} - \lambda_{s-k}) \right. \\ &\quad \left. \cdot\left(\delta_t \mathscr{W}^{k-(1/2)}, \delta_t \mathscr{W}^{s-(1/2)}\right)\right\} \\ &\quad \ge \frac{1}{\Gamma(2-\nu)} \sum_{s=1}^{\mathscr{M}} \left\{\lambda_0 \left\|\delta_t \mathscr{W}^{s-(1/2)}\right\|_{L^2(\Lambda)}^2 - \sum_{k=1}^{s-1}(\lambda_{s-k-1} - \lambda_{s-k}) \right. \\ &\quad \left. \cdot\left\|\delta_t \mathscr{W}^{k-(1/2)}\right\|_{L^2(\Lambda)}\left\|\delta_t \mathscr{W}^{s-(1/2)}\right\|_{L^2(\Lambda)}\right\} \\ &\quad = \frac{1}{\Gamma(2-\nu)} \sum_{s=1}^{\mathscr{M}} B_C^{\nu}\left(\left\|\delta_t W^{s-(1/2)}\right\|_{L^2(\Lambda)}, 0\right)\left\|\delta_t \mathscr{W}^{s-(1/2)}\right\|_{L^2(\Lambda)} \\ &\quad \ge \frac{\tau t_{\mathscr{M}}^{1-\nu}}{2\Gamma(2-\nu)} \sum_{s=1}^{\mathscr{M}} \left\|\delta_t \mathscr{W}^{s-(1/2)}\right\|_{L^2(\Lambda)}^2. \end{aligned} \quad (21)$$

Moreover, for the first term in the right hand side of Equation (20), we have

$$\tau \sum_{s=1}^{\mathscr{M}} \left(\Delta\delta_t \mathscr{W}^{s-12}, \delta_t \mathscr{W}^{s-12}\right) = -\tau \sum_{s=1}^{\mathscr{M}} \left(\nabla\delta_t \mathscr{W}^{s-12}, \nabla\delta_t \mathscr{W}^{s-12}\right) = -\tau \sum_{s=1}^{\mathscr{M}} \left\|\nabla\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)}^2. \quad (22)$$

On the other hand, according to some simple mathematical actions, we have

$$\begin{aligned}
\tau \sum_{s=1}^{\mathscr{M}} \left(\Delta\mathscr{W}^{s-(1/2)}, \delta_t \mathscr{W}^{s-(1/2)}\right) &= -\tau \sum_{s=1}^{\mathscr{M}} \left(\nabla\mathscr{W}^{s-(1/2)}, \nabla\delta_t \mathscr{W}^{s-(1/2)}\right) \\
&= -\tau \sum_{s=1}^{\mathscr{M}} \int_\Lambda \left(\frac{\nabla\mathscr{W}^s + \nabla\mathscr{W}^{s-1}}{2}\right)\left(\frac{\nabla\mathscr{W}^s - \nabla\mathscr{W}^{s-1}}{\tau}\right) d\Lambda \\
&= -\frac{1}{2}\sum_{s=1}^{\mathscr{M}} \left\{\int_\Lambda \left[\left(\nabla\mathscr{W}^s\right)^2 - \left(\nabla\mathscr{W}^{s-1}\right)^2\right] d\Lambda\right\} \\
&= \frac{1}{2}\sum_{s=1}^{\mathscr{M}} \left\{\int_\Lambda \left(\nabla\mathscr{W}^{s-1}\right)^2 d\Lambda - \int_\Lambda \left(\nabla\mathscr{W}^s\right)^2 d\Lambda\right\} \\
&= \frac{1}{2}\sum_{s=1}^{\mathscr{M}} \left\{\left\|\nabla\mathscr{W}^{s-1}\right\|_{L^2(\Lambda)}^2 - \left\|\nabla\mathscr{W}^s\right\|_{L^2(\Lambda)}^2\right\} \\
&= \frac{1}{2}\left\{\left\|\nabla\mathscr{W}^0\right\|_{L^2(\Lambda)}^2 - \left\|\nabla\mathscr{W}^{\mathscr{M}}\right\|_{L^2(\Lambda)}^2\right\}.
\end{aligned} \quad (23)$$

Also, for the delay term, we arrive at

$$\begin{aligned}
\tau \sum_{s=1}^{\mathscr{M}} \left(\Delta\mathscr{W}^{s-m-12}, \delta_t \mathscr{W}^{s-12}\right) &= -\tau \sum_{s=1}^{\mathscr{M}} \left(\nabla\mathscr{W}^{s-m-12}, \nabla\delta_t \mathscr{W}^{s-12}\right) \\
&\le \tau \sum_{s=1}^{\mathscr{M}} \left\|\nabla\mathscr{W}^{s-m-12}\right\|_{L^2(\Lambda)} \left\|\nabla\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)} \\
&\le \frac{\tau}{2}\sum_{s=1}^{\mathscr{M}} \left\|\nabla\mathscr{W}^{s-m-(1/2)}\right\|_{L^2(\Lambda)}^2 + \frac{\tau}{2}\sum_{s=1}^{\mathscr{M}} \cdot \left\|\nabla\delta_t \mathscr{W}^{s-(1/2)}\right\|_{L^2(\Lambda)}^2.
\end{aligned} \quad (24)$$

Replacing the above relations in Equation (20) yields

$$\begin{aligned}
&\frac{\tau t_M^{1-\nu}}{2\Gamma(2-\nu)} \sum_{s=1}^{\mathscr{M}} \left\|\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)}^2 + \tau \sum_{s=1}^{\mathscr{M}} \left\|\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)}^2 \\
&\quad \le -\tau \sum_{s=1}^{\mathscr{M}} \left\|\nabla\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)}^2 + \frac{1}{2}\left\{\left\|\nabla\mathscr{W}^0\right\|_{L^2(\Lambda)}^2 - \left\|\nabla\mathscr{W}^M\right\|_{L^2(\Lambda)}^2\right\} \\
&\qquad + \frac{\tau}{2}\sum_{s=1}^{\mathscr{M}} \left\|\nabla\mathscr{W}^{s-m-12}\right\|_{L^2(\Lambda)}^2 + \frac{\tau}{2}\sum_{s=1}^{\mathscr{M}} \left\|\nabla\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)}^2,
\end{aligned} \quad (25)$$

or

$$\begin{aligned}
&\frac{\tau t_M^{1-\nu}}{\Gamma(2-\nu)} \sum_{s=1}^{\mathscr{M}} \left\|\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)}^2 + 2\tau \sum_{s=1}^{\mathscr{M}} \left\|\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)}^2 \\
&\quad \le -2\tau \sum_{s=1}^{\mathscr{M}} \left\|\nabla\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)}^2 + \left\{\left\|\nabla\mathscr{W}^0\right\|_{L^2(\Lambda)}^2 - \left\|\nabla\mathscr{W}^{\mathscr{M}}\right\|_{L^2(\Lambda)}^2\right\} \\
&\qquad + \tau \sum_{s=1}^{\mathscr{M}} \left\|\nabla\mathscr{W}^{s-m-12}\right\|_{L^2(\Lambda)}^2 + \tau \sum_{s=1}^{\mathscr{M}} \left\|\nabla\delta_t \mathscr{W}^{s-12}\right\|_{L^2(\Lambda)}^2.
\end{aligned} \quad (26)$$

Now, Equation (26) can be simplified as

$$\left\|\nabla\mathscr{W}^{\mathscr{M}}\right\|_{L^2(\Lambda)}^2 \le \left\|\nabla\mathscr{W}^0\right\|_{L^2(\Lambda)}^2 + \tau \sum_{s=1}^{\mathscr{M}} \left\|\nabla\mathscr{W}^{s-m-12}\right\|_{L^2(\Lambda)}^2. \quad (27)$$

Changing index from $\mathscr{M}$ to s arrives at

$$\begin{aligned}
\left\|\nabla\mathscr{W}^s\right\|_{L^2(\Lambda)} &\le \left\|\nabla\mathscr{W}^0\right\|_{L^2(\Lambda)} + 2\tau \sum_{k=1}^{s} \left\|\nabla\mathscr{W}^{k-m}\right\|_{L^2(\Lambda)} \\
&= \left\|\nabla\mathscr{W}^0\right\|_{L^2(\Lambda)} + 2\tau \sum_{k=1}^{s} \left\|\nabla(\varphi^*)^{k-m}\right\|_{L^2(\Lambda)}.
\end{aligned} \quad (28)$$

The use of Equation (29) and Lemma 2 yields

$$\begin{aligned}
\left\|\nabla\mathscr{W}^s\right\|_{L^2(\Lambda)} &\le \left[\left\|\nabla\mathscr{W}^0\right\|_{L^2(\Lambda)} + 2\tau \sum_{k=1}^{s} \left\|\nabla(\varphi^*)^{k-m}\right\|_{L^2(\Lambda)}\right] \exp(2s\tau) \\
&\le \left[\left\|\nabla\mathscr{W}^0\right\|_{L^2(\Lambda)} + 2\tau \sum_{k=1}^{s} \left\|\nabla(\varphi^*)^{k-m}\right\|_{L^2(\Lambda)}\right] \exp(2T).
\end{aligned} \quad (29)$$

Thus, there exists $C \in \mathbb{R}^+$ that

$$\left\|\nabla\mathscr{W}^s\right\|_{L^2(\Lambda)} \le C\left[\left\|\nabla\mathscr{W}^0\right\|_{L^2(\Lambda)} + 2\tau \sum_{k=1}^{s} \left\|\nabla(\varphi^*)^{k-m}\right\|_{L^2(\Lambda)}\right]. \quad (30)$$

□

4. Moving Kriging Interpolation and Its Implementation

Following [39, 40], we will invoke the technique of moving Kriging. Up to our knowledge and armed by the fact of the advantage of less CPU time consuming needed for the element free Galerkin approach based on the shape functions of moving Kriging than what needed for the element free Galerkin approach based on the shape functions of moving least squares approximation. In the meantime, the shape functions of moving Kriging interpolation can be deduced, which is analogous to moving least squares approximation over subdomain $\Lambda_1 \subset \Lambda$. Let $\Psi_h(\mathbf{x})$ is the approximate solution of $\Psi(\mathbf{x})$ on Λ. The local approximation is formulated for any subdomain as

$$\Psi^h(x) = \sum_{r=1}^{m} q_r(x)a_r + S(x) = \mathbf{q}^T(\mathbf{x})\mathbf{a} + S(\mathbf{x}), \tag{31}$$

such that q_r and a_r are monomial basis functions and monomial coefficients, respectively. Also, $S(\mathbf{x})$ be the realization of a stochastic process. The covariance matrix of $S(\mathbf{x})$ is given as

$$\operatorname{cov}\{S(\mathbf{x}_i), S(\mathbf{x}_r)\} = \sigma^2\mathbf{E}[E(\mathbf{x}_i, \mathbf{x}_r)], \tag{32}$$

in which

(i) $E[E(\mathbf{x}_i, \mathbf{x}_r)]$ is the correlation matrix

(ii) $E(\mathbf{x}_i, \mathbf{x}_r)$ is the correlation function between any pair of nodes located at $\mathbf{x}_i$ and $\mathbf{x}_r$

The correlation function is defined as [39, 40]

$$E(\mathbf{x}_i, \mathbf{x}_r) = \exp\left(-\theta r_{ir}^2\right), r_{ir} = \|\mathbf{x}_i - \mathbf{x}_r\|, \tag{33}$$

such that $\theta > 0$ is a value of the correlation parameter [39, 40]. Using the best linear unbiased (BLUP) [39], we can write Equation (31) as follows [39, 40]

$$\Psi^h(\mathbf{x}) = \mathbf{q}^T(\mathbf{x})\eta + \mathbf{r}^T(\mathbf{x})\mathbf{E}^{-1}(\mathbf{u} - \mathbf{Q}\eta), \tag{34}$$

in which

$$\eta = \left(\mathbf{Q}^T\mathbf{E}^{-1}\mathbf{Q}\right)^{-1}\mathbf{Q}^T\mathbf{E}^{-1}\mathbf{u}. \tag{35}$$

We will introduce some notations. The vector of known m functions can be written as follows [39, 40]

$$\mathbf{q}(\mathbf{x}) = \begin{bmatrix} q_1(\mathbf{x}) \\ q_2(\mathbf{x}) \\ \vdots \\ q_m(\mathbf{x}) \end{bmatrix}_{1\times m}, \tag{36}$$

and the matrix of defined function values at the nodes $x_1, x_2, \cdots, x_n$ has the following representation [39, 40]

$$\mathbf{Q} = \begin{bmatrix} q_1(x_1) & q_2(x_1) & \cdots & q_m(x_1) \\ q_1(x_2) & q_2(x_2) & \cdots & q_m(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ q_1(x_n) & q_2(x_n) & \cdots & q_m(x_n) \end{bmatrix}_{n\times m}. \tag{37}$$

The correlation matrix is given as [39, 40]

$$\mathbf{E}[E(\mathbf{x}_i, \mathbf{x}_r)] = \begin{bmatrix} 1 & E(\mathbf{x}_1, \mathbf{x}_2) & \cdots & E(\mathbf{x}_1, \mathbf{x}_n) \\ E(\mathbf{x}_2, \mathbf{x}_1) & 1 & \cdots & E(\mathbf{x}_2, \mathbf{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ E(\mathbf{x}_n, \mathbf{x}_1) & E(\mathbf{x}_n, \mathbf{x}_2) & \cdots & 1 \end{bmatrix}_{n\times n}. \tag{38}$$

The correlation vector at the nodes $x_1, x_2, \cdots, x_n$ has the following form

$$\mathbf{r}(\mathbf{x}) = \begin{bmatrix} E(\mathbf{x}_1, \mathbf{x}) \\ E(\mathbf{x}_2, \mathbf{x}) \\ \vdots \\ E(\mathbf{x}_n, \mathbf{x}) \end{bmatrix}. \tag{39}$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given as

$$\begin{aligned} \mathbf{A} &= \left(\mathbf{Q}^T\mathbf{E}^{-1}\mathbf{Q}\right)^{-1}\mathbf{Q}^T\mathbf{E}^{-1}, \\ \mathbf{B} &= \mathbf{E}^{-1}(\mathbf{I} - \mathbf{Q}\mathbf{A}), \end{aligned} \tag{40}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Accordingly, Equation (34) can be written as follows [39, 40]

$$\Psi^h(\mathbf{x}) = \mathbf{q}^T(\mathbf{x})\mathbf{A}\mathbf{u} + \mathbf{r}^T(\mathbf{x})\mathbf{B}\mathbf{u}, \tag{41}$$

or

$$\Psi^h(\mathbf{x}) = \left[\mathbf{q}^T(\mathbf{x})\mathbf{A} + \mathbf{r}^T(\mathbf{x})\mathbf{B}\right]\mathbf{u} = \sum_{k=1}^{n} \phi_k(\mathbf{x})\Psi_k = \varphi(\mathbf{x})\mathbf{u}, \tag{42}$$

where the moving Kriging approach's shape functions are as follows [39, 40]:

$$\varphi(\mathbf{x}) = \left(\mathbf{q}^T(\mathbf{x})\mathbf{A} + \mathbf{r}^T(\mathbf{x})\mathbf{B}\right)_r = [\phi_1, \phi_2, \cdots, \phi_n]^T. \tag{43}$$

Now, we are ready to implement that kind of interpolation to the problem under consideration. Let the approximation solution of this equation be

$$U^n(x, y, z) = \sum_{j=1}^{N} \varpi_j^n \phi_j(x, y, z), \tag{44}$$

in which $\phi_j(\mathbf{x})$ are shape functions of moving Kriging approximation. Substituting Equation (44) in relation (15) gives

Table 1: Results obtained with 500 collocation points for Example 1.

τ	$\nu=1.1$		$\nu=1.2$		$\nu=1.9$		CPU time(s)
	L_∞	C_τ-order	L_∞	C_τ-order	L_∞	C_τ-order	
0.1	1.20×10^{-1}	—	1.04×10^{-1}	—	7.74×10^{-1}	—	0.5
0.05	3.00×10^{-2}	2.00	2.69×10^{-2}	1.94	6.02×10^{-1}	0.36	0.9
0.025	7.51×10^{-3}	1.99	7.20×10^{-3}	1.90	3.31×10^{-1}	0.86	3.2
0.0125	1.88×10^{-3}	1.99	1.97×10^{-3}	1.87	1.61×10^{-1}	1.04	10.4
0.00625	4.70×10^{-4}	1.99	5.46×10^{-4}	1.85	7.37×10^{-2}	1.13	37.1
0.003125	1.17×10^{-4}	1.99	1.53×10^{-4}	1.83	3.29×10^{-2}	1.16	87.5
0.0015625	2.94×10^{-5}	1.99	4.36×10^{-5}	1.81	1.45×10^{-2}	1.18	163.4
0.00078125	7.38×10^{-6}	1.99	1.25×10^{-5}	1.80	6.35×10^{-3}	1.19	277.3
TO		1.9		1.8		1.1	

$$\begin{aligned}&\frac{1}{\Gamma(2-\nu)\tau}\mathscr{B}_C^\nu\left(\delta_t\sum_{j=1}^N\varpi_j^{n-(1/2)}\phi_j(x,y,z),\varphi\right)+\sum_{j=1}^N\delta_t\varpi_j^{n-(1/2)}\phi_j(x,y,z)\\&\quad=\sum_{j=1}^N\delta_t\varpi_j^{n-(1/2)}\Delta\phi_j(x,y,z)+\sum_{j=1}^N\varpi_j^{n-(1/2)}\Delta\phi_j(x,y,z)\\&\qquad+\sum_{j=1}^N\delta_t\varpi_j^{n-(1/2)-m}\Delta\phi_j(x,y,z)+f^{n-(1/2)}.\end{aligned}\tag{45}$$

By collocating a set of arbitrary distributed nodes $\{(x_i,y_i,z_i)\}_{i=1}^N$ in the computational domain Λ concludes

$$\begin{aligned}&\frac{1}{\Gamma(2-\nu)\tau}\mathscr{B}_C^\nu\left(\delta_t\sum_{j=1}^N\varpi_j^{n-(1/2)}\phi_j(x_i,y_i,z_i),\varphi\right)+\sum_{j=1}^N\delta_t\varpi_j^{n-(1/2)}\phi_j(x_i,y_i,z_i)\\&\quad=\sum_{j=1}^N\delta_t\varpi_j^{n-(1/2)}\Delta\phi_j(x_i,y_i,z_i)+\sum_{j=1}^N\varpi_j^{n-(1/2)}\Delta\phi_j(x_i,y_i,z_i)\\&\quad+\sum_{j=1}^N\delta_t\varpi_j^{n-(1/2)-m}\Delta\phi_j(x_i,y_i,z_i)+f^{n-(1/2)}(x_i,y_i,z_i).\end{aligned}\tag{46}$$

After doing some simplifications, we have

$$\begin{aligned}&(\mu\lambda_0+1)\sum_{j=1}^N\varpi_j^n\phi_j(x_i,y_i,z_i)-\left(1+\frac{\tau}{2}\right)\sum_{j=1}^N\varpi_j^n\Delta\phi_j(x_i,y_i,z_i)\\&\quad=(\mu\lambda_0+1)\sum_{j=1}^N\varpi_j^{n-1}\phi_j(x_i,y_i,z_i)(x,y,z)\\&\qquad+\left(-1+\frac{\tau}{2}\right)\sum_{j=1}^N\varpi_j^{n-1}\Delta\phi_j(x_i,y_i,z_i)+\mu\tau\lambda_{n-1}\varphi\\&\qquad+\mu\tau\sum_{k=1}^{n-1}(\lambda_{n-k-1}-\lambda_{n-k})\left(\sum_{j=1}^N\varpi_j^k\phi_j(x_i,y_i,z_i)-\sum_{j=1}^N\varpi_j^{k-1}\phi_j(x_i,y_i,z_i)\right)\\&\qquad+\frac{\tau}{2}\left(\sum_{j=1}^N\varpi_j^{n-m}\phi_j(x_i,y_i,z_i)-\sum_{j=1}^N\varpi_j^{n-m-1}\phi_j(x_i,y_i,z_i)\right)+\tau f^{n-(1/2)},\end{aligned}\tag{47}$$

where $\mu=1/\Gamma(2-\nu)\tau$. Now, the above formulation yields the following system of equations

$$\mathbf{A}\Lambda^n=\mathbf{F^n},\tag{48}$$

in which

$$\begin{aligned}A_{ij}&=(\mu\lambda_0+1)\phi_j(x_i,y_i,z_i)-\left(1+\frac{\tau}{2}\right)\Delta\phi_j(x_i,y_i,z_i),\\F_i&=(\mu\lambda_0+1)\sum_{j=1}^N\varpi_j^{n-1}\phi_j(x,y,z)+\left(-1+\frac{\tau}{2}\right)\sum_{j=1}^N\varpi_j^{n-1}\Delta\phi_j(x,y,z)\\&\quad+\mu\tau\sum_{k=1}^{n-1}(\lambda_{n-k-1}-\lambda_{n-k})\left(\sum_{j=1}^N\varpi_j^k\phi_j(x,y,z)-\sum_{j=1}^N\varpi_j^{k-1}\phi_j(x,y,z)\right)\\&\quad+\mu\tau\lambda_{n-1}\varphi+\frac{\tau}{2}\left(\sum_{j=1}^N\varpi_j^{n-m}\phi_j(x,y,z)-\sum_{j=1}^N\varpi_j^{n-m-1}\phi_j(x,y,z)\right)\\&\quad+\tau f^{n-(1/2)}.\end{aligned}\tag{49}$$

5. Numerical Verification

In the current section, we investigate the convergence, capability, and stability of the developed numerical procedure. Also, the computational rate is calculated by

$$C-\text{order}=(\log(2))^{-1}\times\log\left(\frac{E(h,2\tau)}{E(h,\tau)}\right).\tag{50}$$

We consider the following problem

$$\begin{aligned}&\frac{1}{\Gamma(2-\nu)}\int_0^t\frac{\partial^2\Psi(x,y,s)}{\partial s^2}\frac{ds}{(t-s)^{\nu-1}}+\frac{\partial\Psi(x,y,t)}{\partial t}\\&\quad=\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi(x,y,t)}{\partial x^2}+\frac{\partial^2\Psi(x,y,t)}{\partial y^2}\right)+\frac{\partial^2\Psi(x,y,t)}{\partial x^2}\\&\qquad+\frac{\partial^2\Psi(x,y,t)}{\partial y^2}+\frac{\partial^2\Psi(x,y,t-s)}{\partial x^2}+\frac{\partial^2\Psi(x,y,t-s)}{\partial y^2}+f(x,y,t),\end{aligned}\tag{51}$$

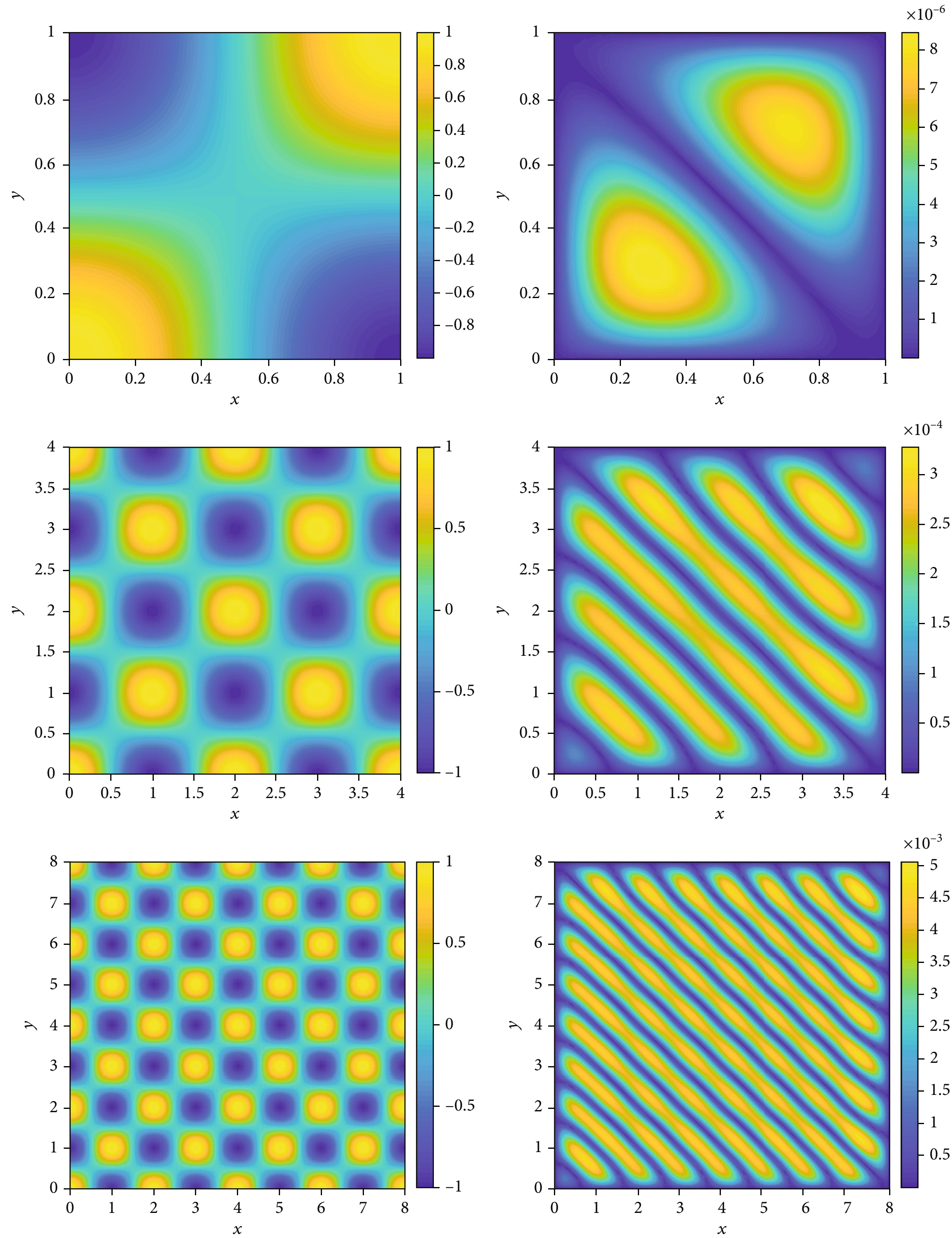


Figure 1: Approximate solution and its absolute error on square domains.

in which

$$f(x, y, t) = \cos(\pi x)\cos(\pi y)\left[2t + 2\pi^2(s-t)^2 + 4\pi^2 t + 2\pi^2 t^2\right], \tag{52}$$

the initial conditions are

$$\Psi(x, y, t)|_{t=0} = 0, \left.\frac{\partial \Psi(x, y, t)}{\partial t}\right|_{t=0} = 0, (x, y) \in \Lambda, \tag{53}$$

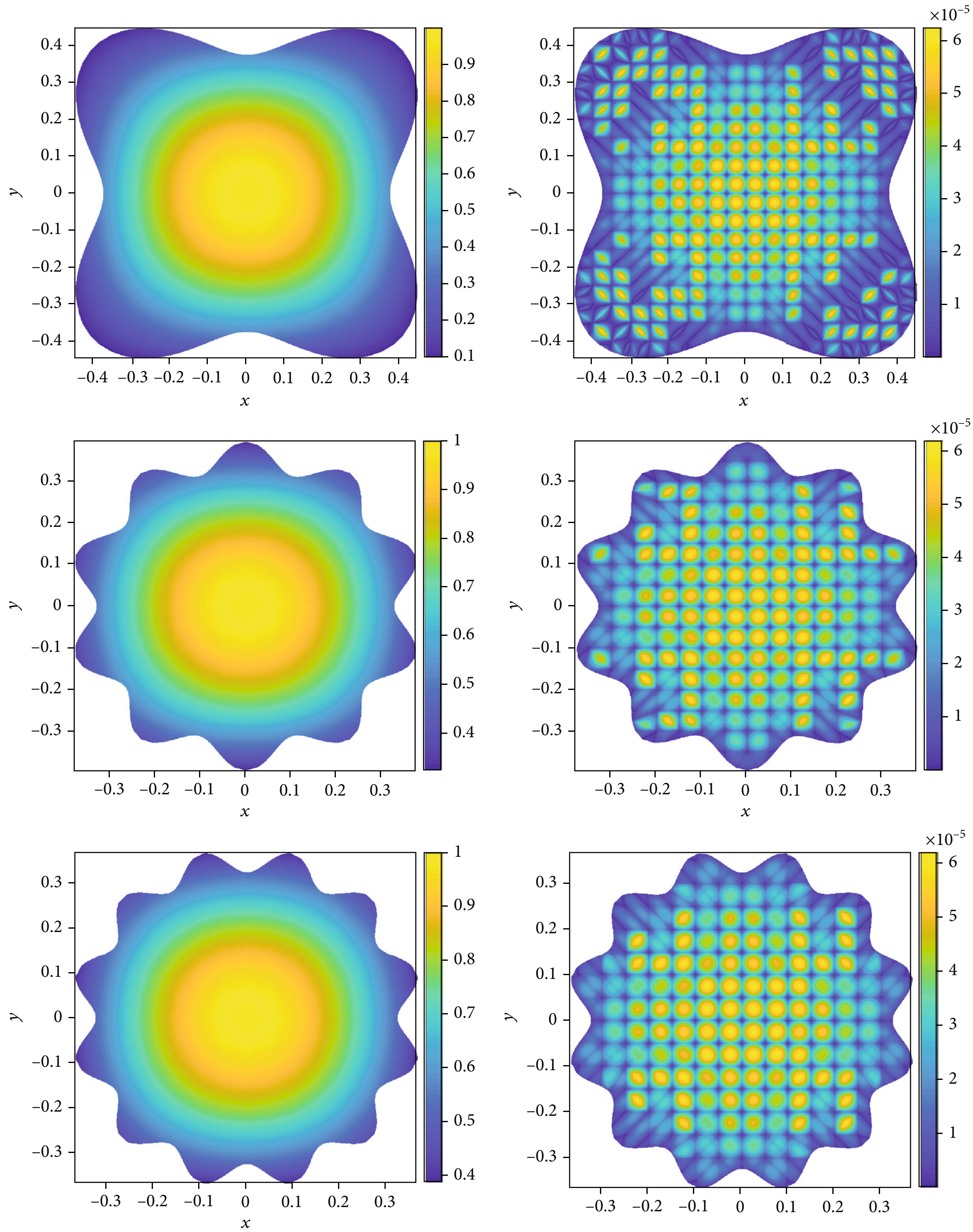


FIGURE 2: Approximate solution and its absolute error on irregular domains.

and also

$$\Psi(x, y, t) = t^2 \cos(\pi x) \cos(\pi y), (x, y, t) \in \Lambda \times (-s, 0], \quad (54)$$

with no-flux boundary condition. The exact solution is $\Psi(x, y, t) = t^2 \cos(\pi x) \cos(\pi y)$.

Table 1 shows the results obtained based on the 500 collocation points, $\nu = 1.1$, $\nu = 1.2$, $\nu = 1.9$, and different values of τ. Table 1 confirms that the theoretical order (TO) in temporal direction is near to the computational order, i.e., $3 - \nu$. Figure 1 demonstrates the approximate solutions (a) and its absolute errors (b) on square domains $[0, 1] \times [0, 1]$ (top figures), $[0, 4] \times [0, 4]$ (middle figures), and $[0, 8] \times [0, 8]$ (bottom figures) with $\tau = 0.001$, $\nu = 1.5$, and also 1000

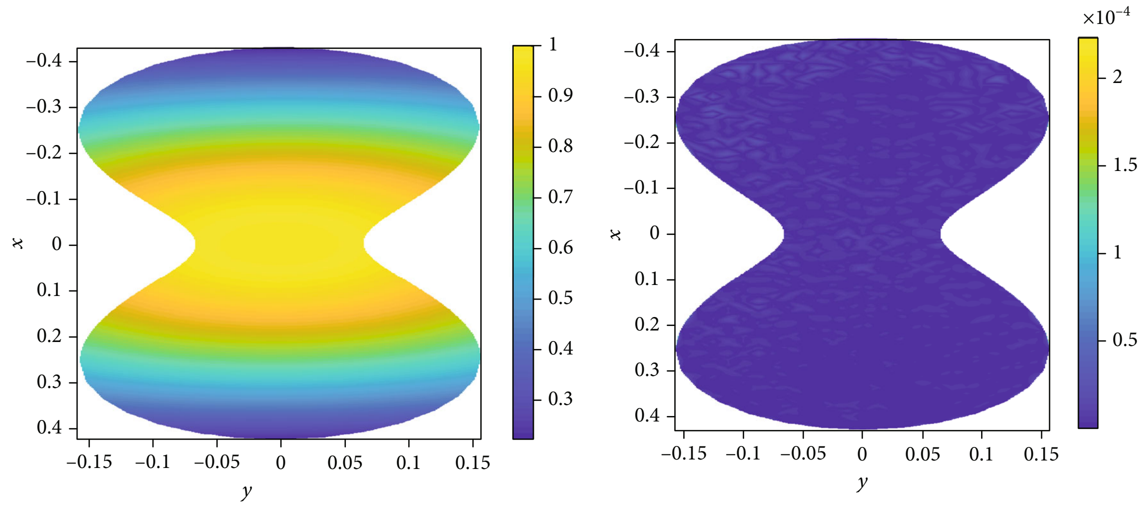


Figure 3: Approximate solution and its absolute error on irregular domains.

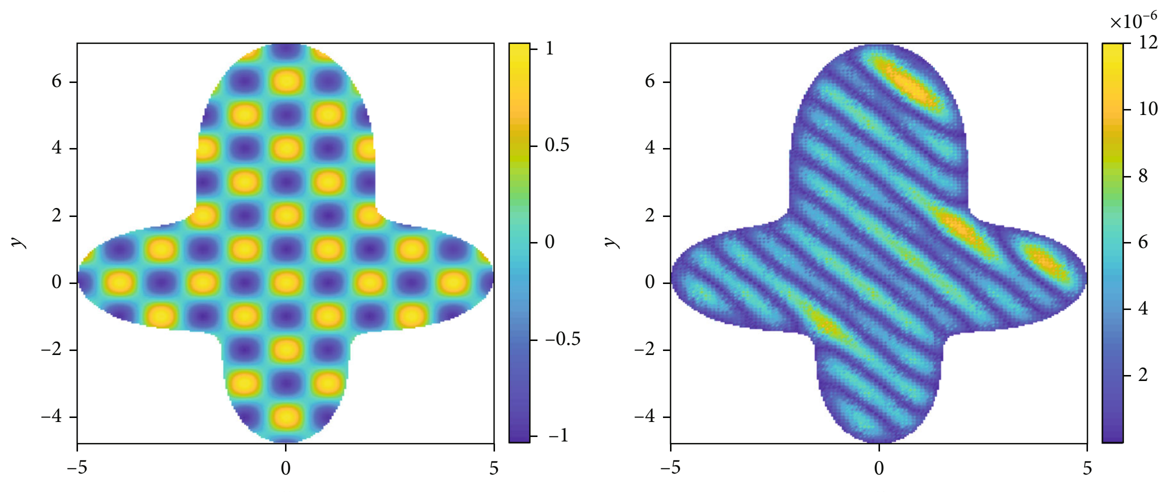


Figure 4: Approximate solution and its absolute error on irregular domains.

collocation points. Figure 2 illustrates the approximate solutions (a) and its absolute errors (b) on irregular domains

$$r(\theta) = \frac{3}{10n^2}\left(1 + 2n + n.^2 - (n+1)\cos(n\theta)\right), \tag{55}$$

where $n = 4$ (top figures), $n = 8$ (middle figures), and $n = 12$ (bottom figures) with $\tau = 0.001$, $\nu = 1.5$, and also 1000 collocation points. Figure 3 presents the approximate solutions (a) and its absolute errors (b) on irregular domains

$$r(\theta) = \frac{3}{10n^2}\left(1 + 2n + n.^2 - (n+1)\cos(n\theta)\right), \tag{56}$$

with $\tau = 0.001$, $\nu = 1.5$, and also 1000 collocation points. Figure 4 presents the approximate solutions (a) and its absolute errors (b) on irregular domains

$$r(\theta) = 0.3\sqrt{\cos(2\theta) + \mathrm{sqrt}\left(\sqrt{1.1 - \sin(\theta)^2}\right)}, \tag{57}$$

with $\tau = 0.001$, $\nu = 1.5$, and also 1000 collocation points.

6. Conclusion

The current paper presented a new numerical procedure for solving fractional damped diffusion-wave equations with delay. In this process, the time derivative is discretized by a finite difference scheme, and we constructed a time-discrete scheme. The stability and convergence of the proposed numerical formulation are studied, analytically and numerically. Then, the moving Kriging interpolation technique, as a meshless method, is used to get a fully discrete scheme. The proposed numerical method is flexible to simulate a wide range of PDEs including delay PDEs on irregular computational domains. Finally, an example is provided to demonstrate the stability and convergence of the new technique.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

M. Abbaszadeh contributed to the conceptualization, writing—original draft, data curation, figure preparation, and methodology. M. Dehghan contributed to the supervision and formal analysis and provided expert advice and validation. M. A. Zaky contributed to the writing—review and editing and carried out the experiments and formal analysis. A.S. Hendy contributed to the writing—review and editing and carried out the experiments and formal analysis. All coauthors contributed to editing the manuscript. All authors contributed significantly in writing this article. All authors read and approved the final manuscript.

References

[1] R. Bellman and K. Cooke, *Differential Difference Equations*, Academic Press, New York-London, 1963.

[2] R. D. Driver, *Ordinary and Delay Differential Equations*, Springer Verlag, New York, NY, USA, 1977.

[3] K. Schmitt, *Delay and Functional Differential Equations and Their Applications*, Elsevier, 2014.

[4] M. A. Abdelkawy and S. A. Alyami, "A spectral collocation technique for Riesz fractional Chen-Lee-Liu equation," *Journal of Function Spaces*, vol. 2021, Article ID 5567970, 9 pages, 2021.

[5] L. Li, M. She, and Y. Niu, "Fractional Crank-Nicolson-Galerkin finite element methods for nonlinear time fractional parabolic problems with time delay," *Journal of Function Spaces*, vol. 2021, Article ID 9981211, 10 pages, 2021.

[6] B. Zhu and M. Zhu, "Existence theorems for fractional semilinear integrodifferential equations with noninstantaneous impulses and delay," *Journal of Function Spaces*, vol. 2020, Article ID 2914269, 9 pages, 2020.

[7] J. Singh, B. Ganbari, D. Kumar, and D. Baleanu, "Analysis of fractional model of guava for biological pest control with memory effect," *Journal of Advanced Research*, vol. 32, pp. 99–108, 2021.

[8] H. Singh, D. Baleanu, J. Singh, and H. Dutta, "Computational study of fractional order smoking model," *Chaos, Solitons & Fractals*, vol. 142, article 110440, 2021.

[9] H. Peics and J. Karsai, "Positive solutions of neutral delay differential equation," *Novi Sad Journal of Mathematics*, vol. 32, no. 2, pp. 95–108, 2002.

[10] R. D. Driver, "A mixed neutral system," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 8, no. 2, pp. 155–158, 1984.

[11] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press-Oxford, 1991.

[12] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag New York, 1977.

[13] T. Krisztin and J. Wu, "Asymptotic behaviors of solutions of scalar neutral functional differential equations," *Differential Equations and Dynamical Systems*, vol. 4, no. 3/4, pp. 351–366, 1996.

[14] Z. Jackiewicz and B. Zubik-Kowal, "Discrete variable methods for delay-differential equations with threshold-type delays," *Journal of Computational and Applied Mathematics*, vol. 228, no. 2, pp. 514–523, 2009.

[15] Y. M. Cheng, F. Bai, and M. Peng, "A novel interpolating element-free Galerkin (IEFG) method for two-dimensional elastoplasticity," *Applied Mathematical Modelling*, vol. 38, no. 21-22, pp. 5187–5197, 2014.

[16] Y. M. Cheng, F. Bai, C. Liu, and M. Peng, "Analyzing nonlinear large deformation with an improved element-free Galerkin method via the interpolating moving least-squares method," *International Journal of Computational Materials Science and Engineering*, vol. 5, no. 4, article 1650023, 2016.

[17] F. Liu and Y. M. Cheng, "The improved element-free Galerkin method based on the nonsingular weight functions for inhomogeneous swelling of polymer gels," *International Journal of Applied Mechanics*, vol. 10, no. 4, article 1850047, 2018.

[18] F. X. Sun, J. F. Wang, and Y. M. Cheng, "An improved interpolating element-free Galerkin method for elasticity," *Chinese Physics B*, vol. 22, no. 12, article 120203, 2013.

[19] H. Ren and Y. Cheng, "The interpolating element-free Galerkin (IEFG) method for two-dimensional potential problems," *Engineering Analysis with Boundary Elements*, vol. 36, no. 5, pp. 873–880, 2012.

[20] X. Li, "A meshless interpolating Galerkin boundary node method for Stokes flows," *Engineering Analysis with Boundary Elements*, vol. 51, pp. 112–122, 2015.

[21] D. Li, F. Bai, Y. M. Cheng, and K. M. Liew, "A novel complex variable element-free Galerkin method for two-dimensional large deformation problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 233, pp. 1–10, 2012.

[22] M. Dehghan and M. Abbaszadeh, "Proper orthogonal decomposition variational multiscale element free Galerkin (POD-VMEFG) meshless method for solving incompressible Navier-Stokes equation," *Computer Methods in Applied Mechanics and Engineering*, vol. 311, pp. 856–888, 2016.

[23] Q. Li, S. Chen, and X. Luo, "Steady heat conduction analyses using an interpolating element-free Galerkin scaled boundary method," *Applied Mathematics and Computation*, vol. 300, pp. 103–115, 2017.

[24] N. Zhao and H. Ren, "The interpolating element-free Galerkin method for 2D transient heat conduction problems," *Mathematical Problems in Engineering*, vol. 2014, Article ID 712834, 9 pages, 2014.

[25] L. Zhang, D. Huang, and K. M. Liew, "An element-free IMLS-Ritz method for numerical solution of three-dimensional wave equations," *Computer Methods in Applied Mechanics and Engineering*, vol. 297, pp. 116–139, 2015.

[26] L. Zhang, Y. Deng, K. M. Liew, and Y. Cheng, "The improved complex variable element-free Galerkin method for two-dimensional Schrodinger equation," *Computers & Mathematics with Applications*, vol. 68, no. 10, pp. 1093–1106, 2014.

[27] D. Li, Z. Zhang, and K. M. Liew, "A numerical framework for two-dimensional large deformation of inhomogeneous swelling of gels using the improved complex variable element-free

Galerkin method," *Computer Methods in Applied Mechanics and Engineering*, vol. 274, pp. 84–102, 2014.

[28] L. Zhang, Y. Deng, and K. M. Liew, "An improved element-free Galerkin method for numerical modeling of the biological population problems," *Engineering Analysis with Boundary Elements*, vol. 40, pp. 181–188, 2014.

[29] Z. Zhang, S. Hao, K. M. Liew, and Y. Cheng, "The improved element-free Galerkin method for two-dimensional elastodynamics problems," *Engineering Analysis with Boundary Elements*, vol. 37, no. 12, pp. 1576–1584, 2013.

[30] B. D. Dai, B. Zheng, Q. Liang, and L. Wang, "Numerical solution of transient heat conduction problems using improved meshless local Petrov-Galerkin method," *Applied Mathematics and Computation*, vol. 219, no. 19, pp. 10044–10052, 2013.

[31] X. Li and S. Li, "Analysis of the complex moving least squares approximation and the associated element-free Galerkin method," *Applied Mathematical Modelling*, vol. 47, pp. 45–62, 2017.

[32] T. Zhang and X. Li, "Meshless analysis of Darcy flow with a variational multiscale interpolating element-free Galerkin method," *Engineering Analysis with Boundary Elements*, vol. 100, pp. 237–245, 2019.

[33] T. Zhang and X. Li, "A novel variational multiscale interpolating element-free Galerkin method for generalized Oseen problems," *Computers & Structures*, vol. 209, pp. 14–29, 2018.

[34] M. A. Zaky and A. S. Hendy, "An efficient dissipation-preserving Legendre-Galerkin spectral method for the Higgs boson equation in the de Sitter spacetime universe," *Applied Numerical Mathematics*, vol. 160, pp. 281–295, 2021.

[35] M. A. Zaky, A. S. Hendy, and J. E. Macias-Diaz, "High-order finite difference/spectral-Galerkin approximations for the nonlinear time-space fractional Ginzburg-Landau equation," *Numerical Methods for Partial Differential Equations*, 2020.

[36] A. S. Hendy and M. A. Zaky, "Graded mesh discretization for coupled system of nonlinear multi-term time-space fractional diffusion equations," *Engineering with Computers*, 2020.

[37] Z. Z. Sun and X. Wu, "A fully discrete difference scheme for a diffusion-wave system," *Applied Numerical Mathematics*, vol. 56, no. 2, pp. 193–209, 2006.

[38] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations, vol. 23*, Springer Science & Business Media, 2008.

[39] L. Gu, "Moving kriging interpolation and element-free Galerkin method," *International Journal for Numerical Methods in Engineering*, vol. 56, no. 1, pp. 1–11, 2003.

[40] B. Zheng and B. D. Dai, "A meshless local moving kriging method for two-dimensional solids," *Applied Mathematics and Computation*, vol. 218, no. 2, pp. 563–573, 2011.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9945364, 12 pages
https://doi.org/10.1155/2021/9945364

Research Article

Numerical Analysis of Time-Fractional Diffusion Equations via a Novel Approach

Nehad Ali Shah,[1] **S. Saleem**,[2] **Ali Akgül**,[3] **Kamsing Nonlaopon**,[4] **and Jae Dong Chung**[1]

[1]*Department of Mechanical Engineering, Sejong University, Seoul 05006, Republic of Korea*
[2]*Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia*
[3]*Siirt University, Art and Science Faculty, Department of Mathematics, 56100 Siirt, Turkey*
[4]*Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*

Correspondence should be addressed to Kamsing Nonlaopon; nkamsi@kku.ac.th

Received 15 March 2021; Revised 9 April 2021; Accepted 16 April 2021; Published 27 May 2021

Academic Editor: Qifeng Zhang

The aim of this paper is a new semianalytical technique called the variational iteration transform method for solving fractional-order diffusion equations. In the variational iteration technique, identifying of the Lagrange multiplier is an essential rule, and variational theory is commonly used for this purpose. The current technique has the edge over other methods as it does not need extra parameters and polynomials. The validity of the proposed method is verified by considering some numerical problems. The solution achieved has shown that the better accuracy of the proposed technique. This paper proposes a simpler method to calculate the multiplier using the Shehu transformation, making a valuable technique to researchers dealing with various linear and nonlinear problems.

1. Introduction

In the last decade, significant achievements have been made to applying and the theory of fractional differential equations (FDEs). These problems are increasingly implemented to model equations in research fields as diverse as mechanical schemes, dynamical schemes, chaos, continuous-time random walks, control, chaos synchronization, subdiffusive systems and anomalous diffusive, wave propagation phenomena and unification of diffusion, and so on. The benefits of the fractional-order scheme are that it allows more significant degrees of freedom in the problem. An integer-order differential operator (DO) is a local operator, while a fractional-order DO is a nonlocal operator, taking into account that a potential state depends not only on the current state but also on the past of all its previous instances. Fractional-order schemes have become famous for this valuable property. Another explanation for applying fractional-order derivatives is that they are naturally linked to memory structures that prevail in most physical and scientific structure models. The book by Spanier and Oldham [1] continued to play an essential role in the improvement of the subject. A few other primary results connected to the solution of FDEs can be identified in the books of Ross and Miller [2], Podlubny [3], and Kilbas et al. [4].

Adolf Fick introduces the laws of diffusion of Fick in 1885. After that, the second law of Fick became identified as the diffusion equation. Diffusion is the mesh atom's movement from a high chemical potential or higher concentration field to a lower concentration or low chemical potential field. Investigators have investigated classical wave and diffusion equations to many physical schemes, such as classical diffusion, slow diffusion, diffusion-wave combination, and classical wave equation. Many diffusion equation implementations, such as phase transformation, electrochemistry, magnetic fields, filtration, microbiology, acoustics, astrophysics, and biochemical group structures. Diffusion is determined by the gradient of the potential energy of the diffusing form. The gradient is the difference in the value of a number, e.g., concentration, strain, or temperature, with differences in one or more variables is often differentiated. Researchers have been seeking to recognize and reduce manufacturing systems problems to reach better productivity. In applied science schemes, there are various causes for entropy production. In heat engines, heat transfer, the primary source of entropy production is a mass transfer, the coupling between heat, entropy generation and chemical reaction, electrical conduction, as described in the seminal sequence of publications by Bejan et al. [5, 6]. Scholars have utilized different methods for the analysis of diffusion equations such as Chebyshev collocation technique [7], finite difference technique [8], finite volume element technique [9], variational iteration technique [10], two-step Adomian decomposition technique [11], finite volume technique [12], and Laplace transform [10].

In this paper, we implemented the variational iteration transform method to solve the fractional-order diffusion equations.

The fractional-order two-dimensional diffusion equation is given as

$$\frac{\partial^{\delta}\mu}{\partial\eta^{\delta}} = \frac{\partial^{2}\mu}{\partial\zeta^{2}} + \frac{\partial^{2}\mu}{\partial\psi^{2}}, 0 < \delta \leq 1, \eta \geq 0, \tag{1}$$

with initial condition

$$\mu(\zeta, \psi, 0) = g(\zeta, \psi). \tag{2}$$

The fractional-order three-dimensional diffusion equation

$$\frac{\partial^{\delta}\mu}{\partial\eta^{\delta}} = \frac{\partial^{2}\mu}{\partial\zeta^{2}} + \frac{\partial^{2}\mu}{\partial\psi^{2}} + \frac{\partial^{2}\mu}{\partial\mathfrak{I}^{2}}, 0 < \delta \leq 1, \eta \geq 0, \tag{3}$$

with initial condition

$$\mu(\zeta, \psi, \mathfrak{I}, 0) = g(\zeta, \psi, \mathfrak{I}). \tag{4}$$

A Lagrange multiplier technique has been widely utilized to solve different types of nonlinear equations [13]. This occurs in mathematics and physics or certain related fields but has been established as a basic analytical method, i.e., a variational iteration method (VIM) to model differential equations [14]. The VIM was first recommended by He [15] and was implemented effectively to address the heat transformation problem [15–17]. Recently, several researchers have widely used this method to solve linear and nonlinear equations. The approach offers a consistent and efficient mechanism for a wide variety of applications in engineering and science. It is based on a specific Lagrange multiplier and has the merits of simplicity and easy implementation. Unlike conventional numerical approaches, VIM does not require linearization, discretion, or perturbation. The successive approximation provides quick convergence for the exact result [18–21]. The variational iteration method was introduced in 2010 using the modified Riemann-Liouville derivative [22]. Recently, a procedure combining in this sense Laplace transformation and VIM was proposed [23, 24], and Wu developed a modification via fractional calculus and Laplace transformation [25]. LVIM for solving nonlinear PDEs [26] and system of fractional PDEs [27].

2. Basic Definitions

2.1. Definition. The fractional-order Riemann-Liouville integral is defined as [28, 29]

$$I_0^{\delta}h(\eta) = \frac{1}{\Gamma(\delta)}\int_0^{\eta}(\eta - s)^{\delta-1}h(s)ds. \tag{5}$$

2.2. Definition. The fractional-order Caputo's derivative of $f(\eta)$ is given as [28, 29]

$$\begin{aligned} &D_{\eta}^{\delta}f(\eta) = I^{J-\delta}f^{j}, J-1 < \delta < J, J \in \mathbb{N}; \\ &\frac{d^{J}}{d\eta^{J}}h(\eta), \delta = J, J \in \mathbb{N}. \end{aligned} \tag{6}$$

2.3. Definition. Shehu transformation is new and identical to other integral transformations defined for exponential order functions. In Set A, the function is defined by [30–32]

$$A = \{v(\eta): \exists, \rho_1, \rho_2 > 0, |v(\eta)| < Me^{\frac{|\eta|}{\rho_i}}, \text{ if } \eta \in [0,\infty). \tag{7}$$

The Shehu transform which is described as $S(.)$ for a function $v(\eta)$ is defined as

$$S\{v(\eta)\} = V(s, u) = \int_0^{\infty} e^{\frac{-s\eta}{u}}v(\eta)d\eta, \eta > 0, s > 0. \tag{8}$$

The Shehu transform of a function $v(\eta)$ is $V(s, \mu)$: then, $v(\eta)$ is called the inverse of $V(s, \mu)$ which is given as

$$S^{-1}\{V(s, u)\} = v(\eta), \text{for } \eta \geq 0, S^{-1} \text{is inverse Shehu transform.} \tag{9}$$

2.4. Definition. Shehu transform for nth derivatives is given as [30–32]

$$S\left\{v^{(J)}(\eta)\right\} = \frac{s^{J}}{u^{J}}V(s, u) - \sum_{k=0}^{J-1}\left(\frac{s}{u}\right)^{J-k-1}v^{(k)}(0). \tag{10}$$

2.5. Definition. The fractional-order derivatives of Shehu transformation are defined as [30–32]

$$S\left\{v^{(\delta)}(\eta)\right\} = \frac{s^{\delta}}{u^{\delta}}V(s, u) - \sum_{k=0}^{J-1}\left(\frac{s}{u}\right)^{\delta-k-1}v^{(k)}(0), 0 < \beta \leq n. \tag{11}$$

2.6. Definition. The Mittag-Leffler function, $E_{\delta}(z)$ for $\delta > 0$, is given as

$$E_{\delta}(z) = \sum_{J=0}^{\infty}\frac{z^{J}}{\Gamma(\delta J + 1)}, \delta > 0, z \in \mathbb{C}. \tag{12}$$

3. The Methodology of VITM

This section introduces the general producer of VITM to solve time-fractional partial differential equation [23].

$$D_{\eta}^{\delta}\mu(\zeta,\eta)+\mathscr{M}(\zeta,\eta)+\mathscr{N}(\zeta,\eta)-\mathscr{K}(\zeta,\eta)=0, 0<\delta\leq 1, \tag{13}$$

with the initial sources

$$\mu(\zeta,0)=g_1(\zeta), \tag{14}$$

where$D_{\eta}^{\delta}=\partial^{\delta}/(\partial\eta^{\delta})$is the fractional-order Caputo derivative andδ,$\mathscr{M}$, and$\mathscr{N}$, are linear and nonlinear functions, respectively, and sources function$\mathscr{K}$.

The implementation of Shehu transformation to Eq. (13)

$$S\left[D_{\eta}^{\delta}\mu(\zeta,\eta)\right]+S[\mathscr{M}(\zeta,\eta)+\mathscr{N}(\zeta,\eta)-\mathscr{K}(\zeta,\eta)]=0. \tag{15}$$

Using the differentiation property of Shehu transformation, we get

$$S[\mu(\zeta,\eta)]-\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\zeta,\eta)}{\partial^k\eta}\bigg|_{\eta=0}=-S[\mathscr{M}(\zeta,\eta)+\mathscr{N}(\zeta,\eta)-\mathscr{K}(\zeta,\eta)]. \tag{16}$$

The Lagrange multiplier of the iterative system as

$$\begin{aligned}S[\mu_{m+1}(\zeta,\eta)]=S[\mu_m(\zeta,\eta)]+\lambda(s)\left[\frac{s^{\delta}}{u^{\delta}}\mu_m(\zeta,\eta)-\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\zeta,\eta)}{\partial^k\eta}\bigg|_{\eta=0}\right.\\ \left.-S[\mathscr{K}(\zeta,\eta)]-S\{\mathscr{M}(\zeta,\eta)+\mathscr{N}(\zeta,\eta)\}\right].\end{aligned} \tag{17}$$

A Lagrange multiplier as

$$\lambda(s)=-\frac{u^{\delta}}{s^{\delta}}. \tag{18}$$

Applying inverse Shehu transform S^{-1}, Eq. (17) can be defined as

$$\begin{aligned}\mu_{m+1}(\zeta,\eta)=\mu_m(\zeta,\eta)-S^{-1}\left[\frac{u^{\delta}}{s^{\delta}}\left[\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\zeta,\eta)}{\partial^k\eta}\bigg|_{\eta=0}\right.\right.\\ \left.\left.-S[\mathscr{K}(\zeta,\eta)]-S\{\mathscr{M}(\zeta,\eta)+\mathscr{N}(\zeta,\eta)\}\right]\right],\end{aligned} \tag{19}$$

the initial value can be described as

$$\mu_0(\zeta,\eta)=S^{-1}\left[\frac{u^{\delta}}{s^{\delta}}\left\{\sum_{k=0}^{m-1}\frac{s^{\delta-k-1}}{u^{\delta-k}}\frac{\partial^k\mu(\zeta,\eta)}{\partial^k\eta}\bigg|_{\eta=0}\right\}\right]. \tag{20}$$

4. Implementation of VITM

4.1. Problem. Consider the fractional-order diffusion equation [11]

$$\frac{\partial^{\delta}\mu}{\partial\eta^{\delta}}=\frac{\partial^2\mu}{\partial\zeta^2}-\frac{\partial\mu}{\partial\zeta}+\mu\frac{\partial^2\mu}{\partial\zeta^2}-\mu^2+\mu\, 0<\alpha\leq 1, \tag{21}$$

with the initial condition

$$\mu(\zeta,0)=e^{\zeta}. \tag{22}$$

Applying VITM on equation (21), we have

$$\begin{aligned}\mu_{m+1}(\zeta,\eta)=S^{-1}\left[\frac{\mu_m(\zeta,\eta)}{s}\right]+S^{-1}\\ \cdot\left[\lambda(s)S\left\{\frac{\partial^2\mu_m}{\partial\zeta^2}-\frac{\partial\mu_m}{\partial\zeta}+\mu_m\frac{\partial^2\mu_m}{\partial\zeta^2}-\mu_m^2+\mu_m\right\}\right],\end{aligned} \tag{23}$$

where the Lagrange multiplier is $\lambda(s)$

$$\begin{aligned}\lambda(s)=-\frac{u^{\delta}}{s^{\delta}},\\ \mu_{m+1}(\zeta,\eta)=S^{-1}\left[\frac{\mu_m(\zeta,\eta)}{s}\right]-S^{-1}\\ \cdot\left[\frac{u^{\delta}}{s^{\delta}}S\left\{\frac{\partial^2\mu_m}{\partial\zeta^2}-\frac{\partial\mu_m}{\partial\zeta}+\mu_m\frac{\partial^2\mu_m}{\partial\zeta^2}-\mu_m^2+\mu_m\right\}\right].\end{aligned} \tag{24}$$

Now take,

$$\mu_0(\zeta,\eta)=e^{\zeta}, \tag{25}$$

consequently, we get

$$\begin{aligned}&m=0,1,2,3\cdots\\ &\mu_1(\zeta,\eta)=S^{-1}\left[\frac{\mu_0(\zeta,\eta)}{s}\right]-S^{-1}\left[\frac{u^{\delta}}{s^{\delta}}S\left\{\frac{\partial^2\mu_0}{\partial\zeta^2}-\frac{\partial\mu_0}{\partial\zeta}+\mu_0\frac{\partial^2\mu_0}{\partial\zeta^2}-\mu_0^2+\mu_0\right\}\right],\\ &\mu_1(\zeta,\eta)=e^{\zeta}+\frac{e^{\zeta}\eta^{\delta}}{\Gamma(\delta+1)},\\ &\mu_2(\zeta,\eta)=S^{-1}\left[\frac{\mu_1(\zeta,\eta)}{s}\right]-S^{-1}\left[\frac{u^{\delta}}{s^{\delta}}S\left\{\frac{\partial^2\mu_1}{\partial\zeta^2}-\frac{\partial\mu_1}{\partial\zeta}+\mu_1\frac{\partial^2\mu_1}{\partial\zeta^2}-\mu_1^2+\mu_1\right\}\right],\\ &\mu_2(\zeta,\eta)=e^{\zeta}+\frac{e^{\zeta}\eta^{\delta}}{\Gamma(\delta+1)}+\frac{e^{\zeta}\eta^{2\delta}}{\Gamma(2\delta+1)},\\ &\mu_3(\zeta,\eta)=S^{-1}\left[\frac{\mu_2(\zeta,\eta)}{s}\right]-S^{-1}\left[\frac{u^{\delta}}{s^{\delta}}S\left\{\frac{\partial^2\mu_2}{\partial\zeta^2}-\frac{\partial\mu_2}{\partial\zeta}+\mu_2\frac{\partial^2\mu_2}{\partial\zeta^2}-\mu_2^2+\mu_2\right\}\right],\\ &\mu_3(\zeta,\eta)=e^{\zeta}+\frac{e^{\zeta}\eta^{\delta}}{\Gamma(\delta+1)}+\frac{e^{\zeta}\eta^{2\delta}}{\Gamma(2\delta+1)}+\frac{e^{\zeta}\eta^{3\delta}}{\Gamma(3\delta+1)}.\end{aligned} \tag{26}$$

The approximate result of equation (21) can be achieved as

$$\mu(\zeta,\eta)=e^{\zeta}+\frac{e^{\zeta}\eta^{\delta}}{\Gamma(\delta+1)}+\frac{e^{\zeta}\eta^{2\delta}}{\Gamma(2\delta+1)}+\frac{e^{\zeta}\eta^{3\delta}}{\Gamma(3\delta+1)}+\cdots+\frac{e^{\zeta}\eta^{m\delta}}{\Gamma(m\delta+1)},$$
$$\mu(\zeta,\eta)=e^{\zeta}\sum_{m=0}^{\infty}\frac{\left(\eta^{\delta}\right)^{m}}{\Gamma(m\delta+1)}=e^{\zeta}E_{\delta}\left(\eta^{\delta}\right). \tag{27}$$

The exact result of equation (21)

$$\mu(\zeta,\eta)=e^{(\zeta+\eta)}. \tag{28}$$

In Figure 1, the exact and the VITM solutions of problem 1 at $\delta=1$ are show by subgraphs, respectively. From the given figures, it can be seen that both the VITM and exact results are in close contact with each other. Also, in Figure 2, the VITM results of problem 1 are investigated at different fractional-order $\delta=0.8$ and 0.6 of 3D and 2D. It is analyzed that in Table 1, the time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order.

4.2. Problem. Consider the two-dimensional fractional-order diffusion equation [11]

$$\frac{\partial^{\delta}\mu}{\partial\eta^{\delta}}=\frac{\partial^{2}\mu}{\partial\zeta^{2}}+\frac{\partial^{2}\mu}{\partial\psi^{2}}0<\delta\leq 1, \tag{29}$$

with the initial condition

$$\mu(\zeta,\psi,0)=(1-\psi)e^{\zeta}. \tag{30}$$

Applying VITM on equation (38), we have

$$\mu_{m+1}(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_m(\zeta,\psi,\eta)}{s}\right]+S^{-1}\left[\lambda(s)S\left\{\frac{\partial^{2}\mu_m}{\partial\zeta^{2}}+\frac{\partial^{2}\mu_m}{\partial\psi^{2}}\right\}\right], \tag{31}$$

where the Lagrange multiplier $\lambda(s)$ is

$$\lambda(s)=-\frac{u^{\delta}}{s^{\delta}},$$
$$\mu_{m+1}(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_m(\zeta,\psi,\eta)}{s}\right]-S^{-1}\left[\frac{u^{\delta}}{s^{\delta}}S\left\{\frac{\partial^{2}\mu_m}{\partial\zeta^{2}}+\frac{\partial^{2}\mu_m}{\partial\psi^{2}}\right\}\right]. \tag{32}$$

Now take,

$$\mu_0(\zeta,\psi,\eta)=(1-\psi)e^{\zeta}, \tag{33}$$

consequently, we get

$$m=0,1,2,3\cdots$$
$$\mu_1(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_0(\zeta,\psi,\eta)}{s}\right]-S^{-1}\left[\frac{u^{\delta}}{s^{\delta}}S\left\{\frac{\partial^{2}\mu_0}{\partial\zeta^{2}}+\frac{\partial^{2}\mu_0}{\partial\psi^{2}}\right\}\right],$$
$$\mu_1(\zeta,\psi,\eta)=(1-\psi)e^{\zeta}+(1-\psi)e^{\zeta}\frac{\eta^{\delta}}{\Gamma(\delta+1)},$$
$$\mu_2(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_1(\zeta,\psi,\eta)}{s}\right]-S^{-1}\left[\frac{u^{\delta}}{s^{\delta}}S\left\{\frac{\partial^{2}\mu_1}{\partial\zeta^{2}}+\frac{\partial^{2}\mu_1}{\partial\psi^{2}}\right\}\right],$$
$$\mu_2(\zeta,\psi,\eta)=(1-\psi)e^{\zeta}+(1-\psi)e^{\zeta}\frac{\eta^{\delta}}{\Gamma(\delta+1)}+(1-\psi)e^{\zeta}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)},$$
$$\mu_3(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_2(\zeta,\psi,\eta)}{s}\right]-S^{-1}\left[\frac{u^{\delta}}{s^{\delta}}S\left\{\frac{\partial^{2}\mu_2}{\partial\zeta^{2}}+\frac{\partial^{2}\mu_2}{\partial\psi^{2}}\right\}\right],$$
$$\mu_3(\zeta,\psi,\eta)=(1-\psi)e^{\zeta}+(1-\psi)e^{\zeta}\frac{\eta^{\delta}}{\Gamma(\delta+1)}+(1-\psi)e^{\zeta}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)}+(1-\psi)e^{\zeta}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)}.$$
$$\vdots \tag{34}$$

The approximate result of equation (38) can be achieved as

$$\mu(\zeta,\psi,\eta)=(1-\psi)e^{\zeta}+(1-\psi)e^{\zeta}\frac{\eta^{\delta}}{\Gamma(\delta+1)}+(1-\psi)e^{\zeta}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)}+(1-\psi)e^{\zeta}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)}+\cdots+(1-\psi)e^{\zeta}\frac{\eta^{m\delta}}{\Gamma(m\delta+1)}. \tag{35}$$

When $\alpha=1$, the VITM solution is

$$\mu(\zeta,\psi,\eta)=(1-\psi)e^{\zeta}\sum_{m=0}^{\infty}\frac{(\eta)^{m}}{m!}. \tag{36}$$

The exact solution in closed form is

$$\mu(\zeta,\psi,\eta)=(1-\psi)e^{(\zeta+\eta)}. \tag{37}$$

In Figure 3, the exact and the VITM solutions of problem 2 at $\delta=1$ are shown by subgraphs, respectively. From the given figures, it can be seen that both the VITM and exact results are in close contact with each other. Also, in Figure 4, the VITM results of problem 2 are investigated at different fractional-order $\delta=0.8$ and 0.6 of 3D and 2D. It is analyzed that in Table 2, the time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order.

4.3. Problem. Consider the two-dimensional fractional-order diffusion equation [11]

$$\frac{\partial^{\delta}\mu}{\partial\eta^{\delta}}=\frac{\partial^{2}\mu}{\partial\zeta^{2}}+\frac{\partial^{2}\mu}{\partial\psi^{2}}0<\delta\leq 1, \tag{38}$$

with the initial condition

$$\mu(\zeta,\psi,0)=e^{(\zeta+\psi)}. \tag{39}$$

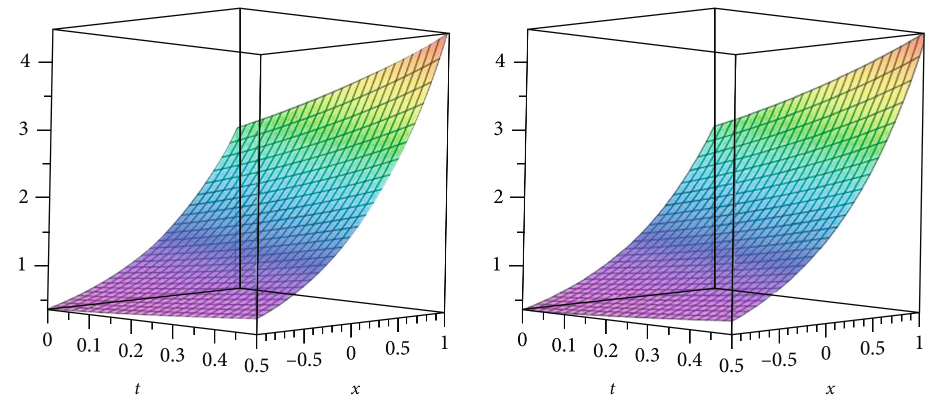


FIGURE 1: Graph of exact and analytical results of Problem 3.1.

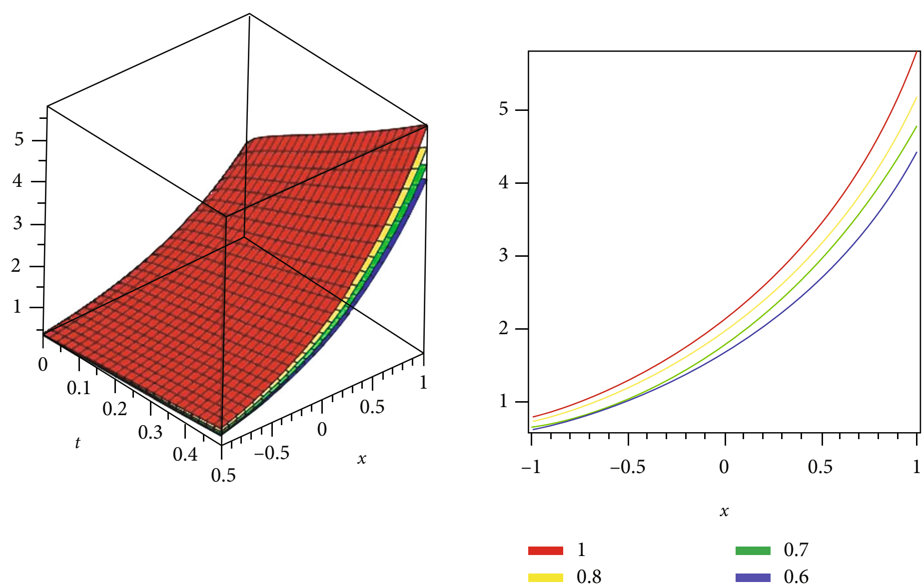


FIGURE 2: The different fractional-order graphs of Problem 3.1.

Applying VITM on equation (38), we have

$$\mu_{m+1}(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_m(\zeta,\psi,\eta)}{s}\right]+S^{-1}\left[\lambda(s)S\left\{\frac{\partial^2\mu_m}{\partial\zeta^2}+\frac{\partial^2\mu_m}{\partial\psi^2}\right\}\right], \tag{40}$$

where the Lagrange multiplier $\lambda(s)$ is

$$\lambda(s)=-\frac{u^\delta}{s^\delta},$$
$$\mu_{m+1}(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_m(\zeta,\psi,\eta)}{s}\right]-S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{\partial^2\mu_m}{\partial\zeta^2}+\frac{\partial^2\mu_m}{\partial\psi^2}\right\}\right]. \tag{41}$$

Now take,

$$\mu_0(\zeta,\psi,\eta)=e^{(\zeta+\psi)}, \tag{42}$$

consequently, we get

$$m=0,1,2,3\cdots$$
$$\mu_1(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_0(\zeta,\psi,\eta)}{s}\right]-S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{\partial^2\mu_0}{\partial\zeta^2}+\frac{\partial^2\mu_0}{\partial\psi^2}\right\}\right],$$
$$\mu_1(\zeta,\psi,\eta)=e^{(\zeta+\psi)}+2e^{(\zeta+\psi)}\frac{\eta^\delta}{\Gamma(\delta+1)},$$
$$\mu_2(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_1(\zeta,\psi,\eta)}{s}\right]-S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{\partial^2\mu_1}{\partial\zeta^2}+\frac{\partial^2\mu_1}{\partial\psi^2}\right\}\right],$$
$$\mu_2(\zeta,\psi,\eta)=e^{(\zeta+\psi)}+2e^{(\zeta+\psi)}\frac{\eta^\delta}{\Gamma(\delta+1)}+4e^{(\zeta+\psi)}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)},$$
$$\mu_3(\zeta,\psi,\eta)=S^{-1}\left[\frac{\mu_2(\zeta,\psi,\eta)}{s}\right]-S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{\partial^2\mu_2}{\partial\zeta^2}+\frac{\partial^2\mu_2}{\partial\psi^2}\right\}\right],$$
$$\mu_3(\zeta,\psi,\eta)=e^{(\zeta+\psi)}+2e^{(\zeta+\psi)}\frac{\eta^\delta}{\Gamma(\delta+1)}+4e^{(\zeta+\psi)}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)}$$
$$+16e^{(\zeta+\psi)}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)},$$
$$\vdots \tag{43}$$

Table 1: VITM at fractional-order value $\delta = 0.8$ and absolute error $\delta = 1$ of example 1.

η	ζ	$\delta = 0.75$	$\delta = 1$	Exact	AE($\delta = 1$)
	1	0.853687662	0.7165300518	0.7165306597	7.56E-08
	2	0.383587401	0.3331299364	0.3331301601	3.37E-08
0.5	3	0.210647180	0.0930849163	0.0930849986	9.35E-09
	4	0.048026028	0.0411973531	0.0411973834	4.24E-09
	5	0.024621114	0.0222089854	0.0222089965	2.10E-09
	1	1.389724464	0.9988367591	1.0000000000	9.35E-06
	2	0.580784320	0.4778488184	0.4778794412	4.42E-06
1	3	0.283191872	0.2453240178	0.2453352832	2.24E-06
	4	0.074713729	0.0587829240	0.0587870683	5.45E-07
	5	0.034438971	0.0193141142	0.0193156388	2.32E-07

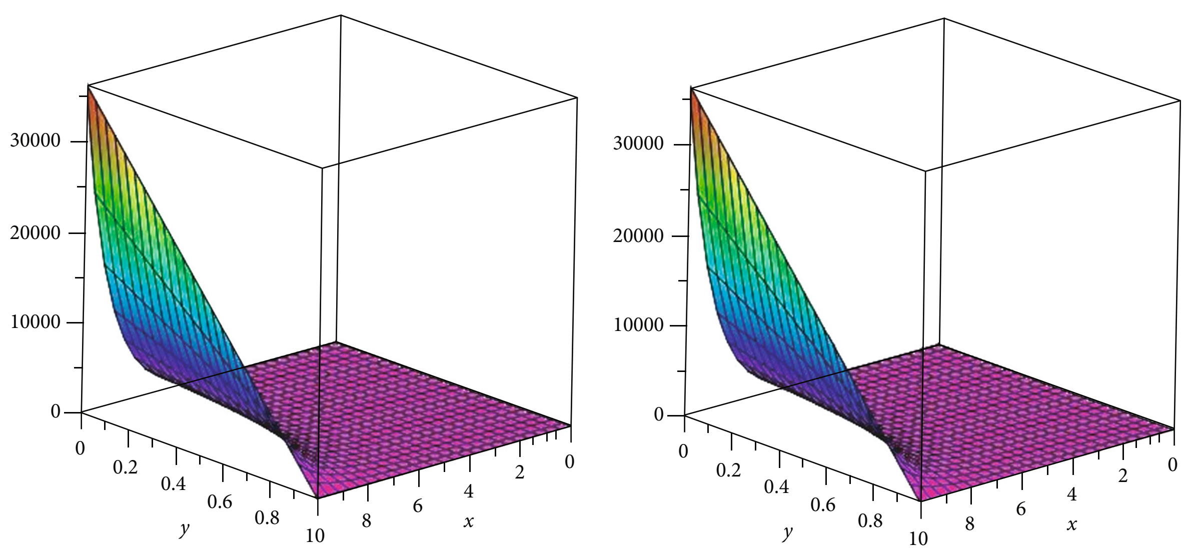


Figure 3: Graph of exact and approximate solutions of Problem 3.2.

The approximate result of equation (38) can be achieved as

$$\mu(\zeta,\psi,\eta) = e^{(\zeta+\psi)} + 2e^{(\zeta+\psi)}\frac{\eta^{\delta}}{\Gamma(\delta+1)} + 4e^{(\zeta+\psi)}\frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + 16e^{(\zeta+\psi)}\frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \cdots + (2)^m e^{(\zeta+\psi)}\frac{\eta^{m\delta}}{\Gamma(m\delta+1)},$$

$$\mu(\zeta,\psi,\eta) = e^{(\zeta+\psi)}\left(1 + \frac{2\eta^{\delta}}{\Gamma(\delta+1)} + \frac{(2\eta^{\delta})^2}{\Gamma(2\delta+1)} + \frac{(2\eta^{\delta})^3}{\Gamma(3\delta+1)} + \frac{(2\eta^{\delta})^4}{\Gamma(4\delta+1)} + \cdots + \frac{(2\eta^{\delta})^m}{\Gamma(m\delta+1)}\right),$$

$$\mu(\zeta,\psi,\eta) = e^{(\zeta+\psi)}\sum_{m=0}^{\infty}\frac{(\eta^{\delta})^m}{\Gamma(m\delta+1)} = (1-\psi)e^{\zeta}E_{\delta}\left(\eta^{\delta}\right). \quad (44)$$

When $\delta = 1$, then the VITM solution is

$$\mu(\zeta,\psi,\eta) = e^{(\zeta+\psi)}\sum_{m=0}^{\infty}\frac{(\eta)^m}{m!}. \quad (45)$$

The exact solution in closed form is

$$\mu(\zeta,\psi,\eta) = e^{(\zeta+\psi+\eta)}. \quad (46)$$

In Figure 5, the exact and the VITM solutions of problem 3 at $\delta = 1$ are shown by subgraphs, respectively. From the given figures, it can be seen that both the VITM and exact results are in close contact with each other. Also, in Figure 6, the VITM results of problem 3 are investigated at different fractional-order $\delta = 0.8$ and 0.6 of 3D and 2D. It is analyzed that time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order.

4.4. Problem. Consider the three-dimensional fractional-order diffusion equation [11]

$$\frac{\partial^{\delta}\mu}{\partial\eta^{\delta}} = \frac{\partial^2\mu}{\partial\zeta^2} + \frac{\partial^2\mu}{\partial\psi^2} + \frac{\partial^2\mu}{\partial\mathfrak{T}^2}, 0 < \delta \le 1, \eta \ge 0, \quad (47)$$

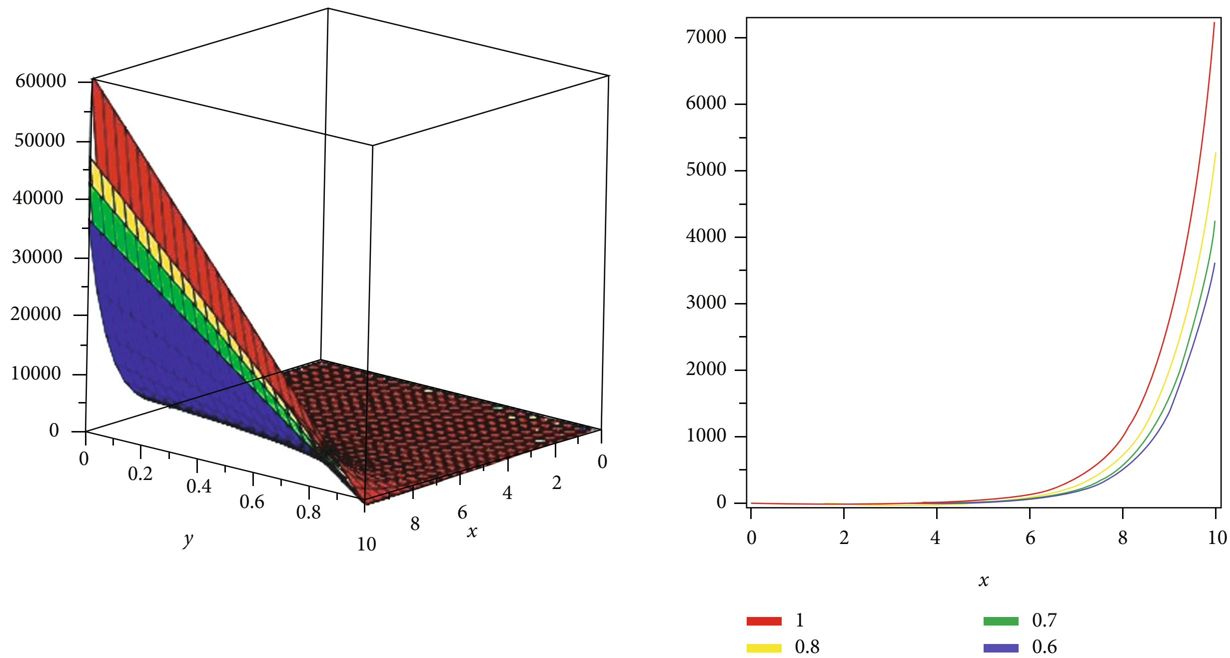


FIGURE 4: The different fractional-order δ graphs of Problem 3.2.

TABLE 2: VITM at fractional-order value $\delta = 0.8$ and absolute error $\delta = 1$ of example 2.

η	ζ	$\delta = 0.75$	$\delta = 1$	Exact	AE($\delta = 1$)
	1	0.6873975264	0.4262266609	0.326227766	2.23E-07
	2	0.0687397526	0.0426226660	0.042622776	2.34E-08
0.5	3	0.0067739752	0.0042622666	0.004262277	0.21E-09
	4	0.0006773975	0.0004362266	0.000436227	1.12E-10
	5	0.0000687397	0.0000446226	0.000044622	2.14E-10
	1	2.7771190120	0.9984599728	1.000000000	4.56E-05
	2	0.3567119012	0.0998953372	0.100000000	5.89E-07
1	3	0.0376711901	0.0098746437	0.010000000	7.12E-07
	4	0.0042671190	0.0009988544	0.001000000	7.89E-08
	5	0.0004567119	0.0000998459	0.000100000	7.22E-09

with the initial condition

$$\mu(\zeta, \psi, \mathfrak{T}, 0) = \sin \zeta \sin \psi \sin \mathfrak{T}. \tag{48}$$

Applying VITM on equation (47), we have

$$\mu_{m+1}(\zeta, \psi, \mathfrak{T}, \eta) = S^{-1}\left[\frac{\mu_m(\zeta, \psi, \mathfrak{T}, \eta)}{s}\right] + S^{-1}\left[\lambda(s) S\left\{\frac{\partial^2 \mu_m}{\partial \zeta^2} + \frac{\partial^2 \mu_m}{\partial \psi^2} + \frac{\partial^2 \mu_m}{\partial \mathfrak{T}^2}\right\}\right], \tag{49}$$

where the Lagrange multiplier $\lambda(s)$ is

$$\lambda(s) = -\frac{u^\delta}{s^\delta},$$
$$\mu_{m+1}(\zeta, \psi, \mathfrak{T}, \eta) = S^{-1}\left[\frac{\mu_m(\zeta, \psi, \mathfrak{T}, \eta)}{s}\right] - S^{-1}\left[\frac{u^\delta}{s^\delta} S\left\{\frac{\partial^2 \mu_m}{\partial \zeta^2} + \frac{\partial^2 \mu_m}{\partial \psi^2} + \frac{\partial^2 \mu_m}{\partial \mathfrak{T}^2}\right\}\right]. \tag{50}$$

Now take,

$$\mu_0(\zeta, \psi, \mathfrak{T}, \eta) = \sin \zeta \sin \psi \sin \mathfrak{T}, \tag{51}$$

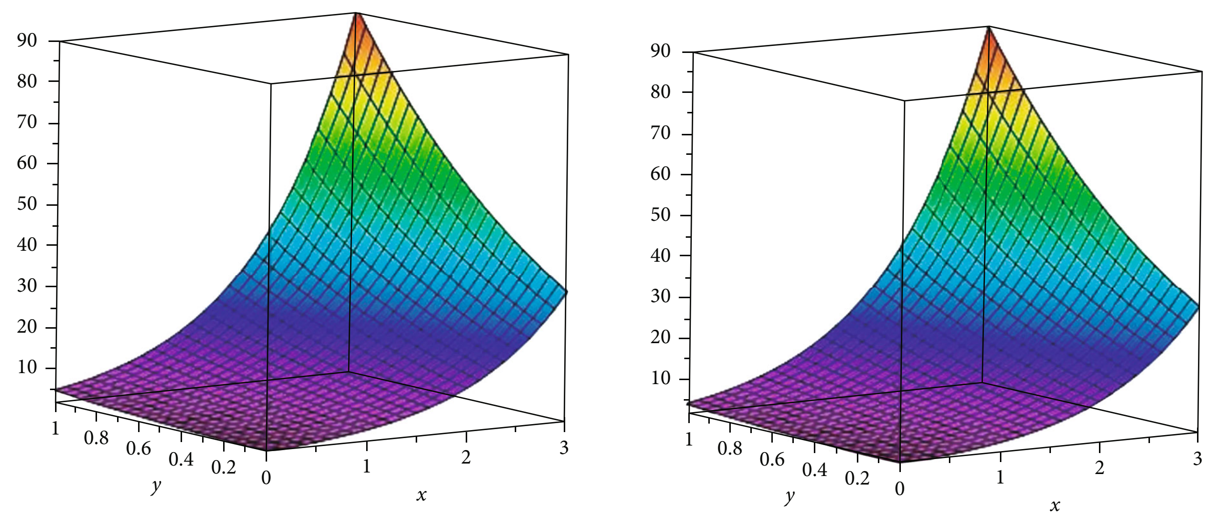


Figure 5: Graph of exact and approximate solutions of Problem 3.3.

consequently, we get

$$m = 0, 1, 2, 3 \cdots$$

$$\mu_1(\zeta, \psi, \mathfrak{T}, \eta) = S^{-1}\left[\frac{\mu_0(\zeta, \psi, \mathfrak{T}, \eta)}{s}\right] - S^{-1}\left[\frac{u^\delta}{s^\delta} S\left\{\frac{\partial^2 \mu_0}{\partial \zeta^2} + \frac{\partial^2 \mu_0}{\partial \psi^2} + \frac{\partial^2 \mu_0}{\partial \mathfrak{T}^2}\right\}\right],$$

$$\mu_1(\zeta, \psi, \mathfrak{T}, \eta) = \sin\zeta \sin\psi \sin\mathfrak{T}\left(1 - 3\frac{\eta^\delta}{\Gamma(\delta+1)}\right),$$

$$\mu_2(\zeta, \psi, \mathfrak{T}, \eta) = S^{-1}\left[\frac{\mu_1(\zeta, \psi, \mathfrak{T}, \eta)}{s}\right] - S^{-1}\left[\frac{u^\delta}{s^\delta} S\left\{\frac{\partial^2 \mu_1}{\partial \zeta^2} + \frac{\partial^2 \mu_1}{\partial \psi^2} + \frac{\partial^2 \mu_1}{\partial \mathfrak{T}^2}\right\}\right],$$

$$\mu_2(\zeta, \psi, \mathfrak{T}, \eta) = \sin\zeta \sin\psi \sin\mathfrak{T}\left(1 - 3\frac{\eta^\delta}{\Gamma(\delta+1)} + (-3)^2\frac{\eta^{2\delta}}{\Gamma(2\delta+1)}\right),$$

$$\mu_3(\zeta, \psi, \mathfrak{T}, \eta) = S^{-1}\left[\frac{\mu_2(\zeta, \psi, \mathfrak{T}, \eta)}{s}\right] - S^{-1}\left[\frac{u^\delta}{s^\delta} S\left\{\frac{\partial^2 \mu_2}{\partial \zeta^2} + \frac{\partial^2 \mu_2}{\partial \psi^2} + \frac{\partial^2 \mu_2}{\partial \mathfrak{T}^2}\right\}\right],$$

$$\mu_3(\zeta, \psi, \mathfrak{T}, \eta) = \sin\zeta \sin\psi \sin\mathfrak{T}\left(1 - 3\frac{\eta^\delta}{\Gamma(\delta+1)} + (-3)^2\frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + (-3)^3\frac{\eta^{3\delta}}{\Gamma(3\delta+1)}\right),$$
$$\vdots \tag{52}$$

The approximate result of equation (47) can be achieved as

$$u(\zeta, \psi, \mathfrak{T}, \eta) = \sin\zeta \sin\psi \sin\mathfrak{T} \cdot \left(1 - 3\frac{\eta^\delta}{\Gamma(\delta+1)} + (-3)^2\frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + (-3)^3\frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \cdots + (-3)^m\frac{\eta^{m\delta}}{\Gamma(m\delta+1)}\right). \tag{53}$$

When $\delta = 1$, then the VITM solution in a closed form is

$$\mu(\zeta, \psi, \mathfrak{T}, \eta) = \sin\zeta \sin\psi \sin\mathfrak{T} \cdot \left(1 - 3\eta + \frac{(-3\eta)^2}{2!} + \frac{(-3\eta)^3}{3!} + \frac{(-3\eta)^4}{4!} + \cdots\right). \tag{54}$$

The exact solution in closed form is

$$\mu(\zeta, \psi, \mathfrak{T}, \eta) = e^{-3\eta}\sin\zeta \sin\psi \sin\mathfrak{T}. \tag{55}$$

In Figure 7, the exact and the VITM solutions of problem 4 at $\delta = 1$ are shown by subgraphs, respectively. From the given figures, it can be seen that both the VITM and exact results are in close contact with each other. Also, in Figure 8, the VITM results of problem 4 are investigated at different fractional-order $\delta = 0.8$ and 0.6. It is analyzed that time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order.

4.5. Example. Consider the fractional-order nonlinear convection-diffusion equation

$$\frac{\partial^\delta \mu}{\partial \eta^\delta} = \frac{\partial^2 \mu}{\partial \zeta^2} - \frac{\partial \mu}{\partial \zeta} + \frac{\partial}{\partial \eta}\left(\mu\frac{\partial^2 \mu}{\partial \zeta^2}\right) - 2\zeta, 0 < \zeta \le 1, 0 < \delta \le 1, \eta > 0, \tag{56}$$

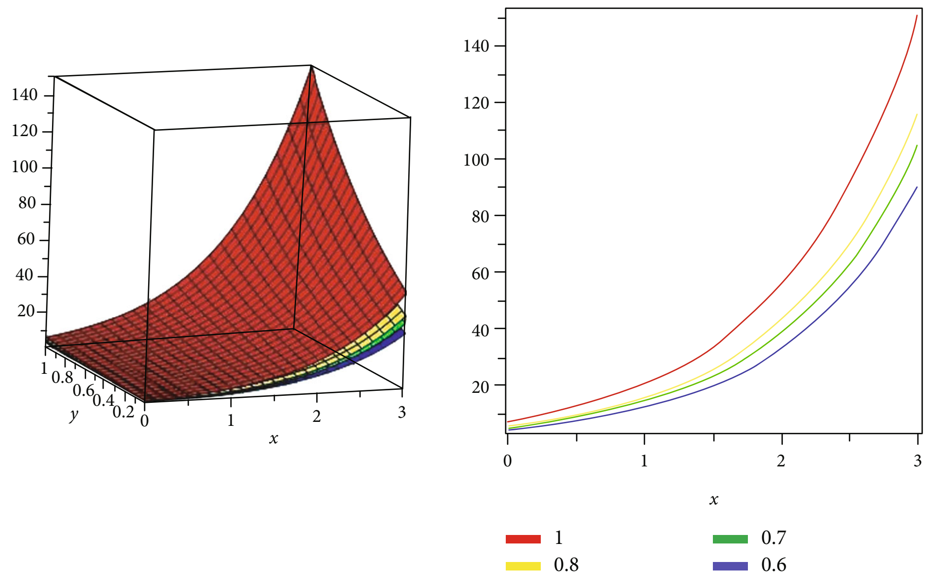


Figure 6: The different fractional-order graphs of δ Problem 3.3.

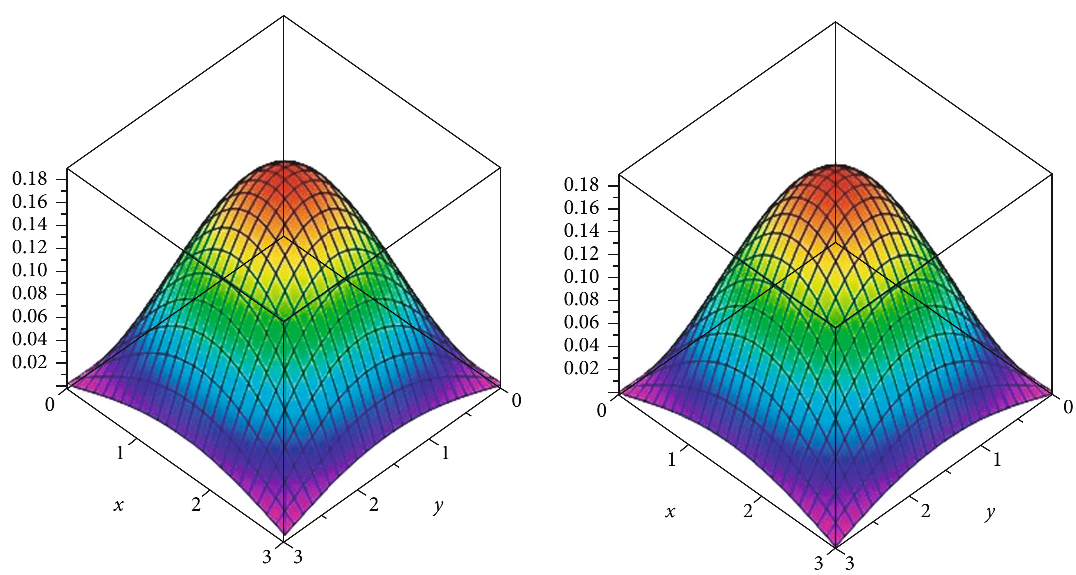


Figure 7: Graph of exact and approximate solutions of Problem 3.4.

with the boundary conditions

$$\mu(0,\eta)=2\eta, \mu(1,\eta)=1+2\eta, \tag{57}$$

and initial condition

$$\mu(\zeta,0)=\zeta^2, \tag{58}$$

Applying VITM on equation (56), we have

$$\mu_{m+1}(\zeta,\eta)=S^{-1}\left[\frac{\mu_m(\zeta,\eta)}{s}\right] + S^{-1}\left[\lambda(s)S\left\{\frac{\partial^2\mu_m}{\partial\zeta^2}-\frac{\partial\mu_m}{\partial\zeta}+\frac{\partial}{\partial\eta}\left(\mu_m\frac{\partial^2\mu_m}{\partial\zeta^2}\right)-2\zeta\right\}\right], \tag{59}$$

where the Lagrange multiplier is $\lambda(s)$.

$$\lambda(s)=-\frac{u^\delta}{s^\delta},$$

$$\mu_{m+1}(\zeta,\eta)=S^{-1}\left[\frac{\mu_m(\zeta,\eta)}{s}\right] - S^{-1}\left[\frac{u^\delta}{s^\delta}S\left\{\frac{\partial^2\mu_m}{\partial\zeta^2}-\frac{\partial\mu_m}{\partial\zeta}+\frac{\partial}{\partial\eta}\left(\mu_m\frac{\partial^2\mu_m}{\partial\zeta^2}\right)-2\zeta\right\}\right]. \tag{60}$$

Now take,

$$\mu_0(\zeta,\eta)=\zeta^2, \tag{61}$$

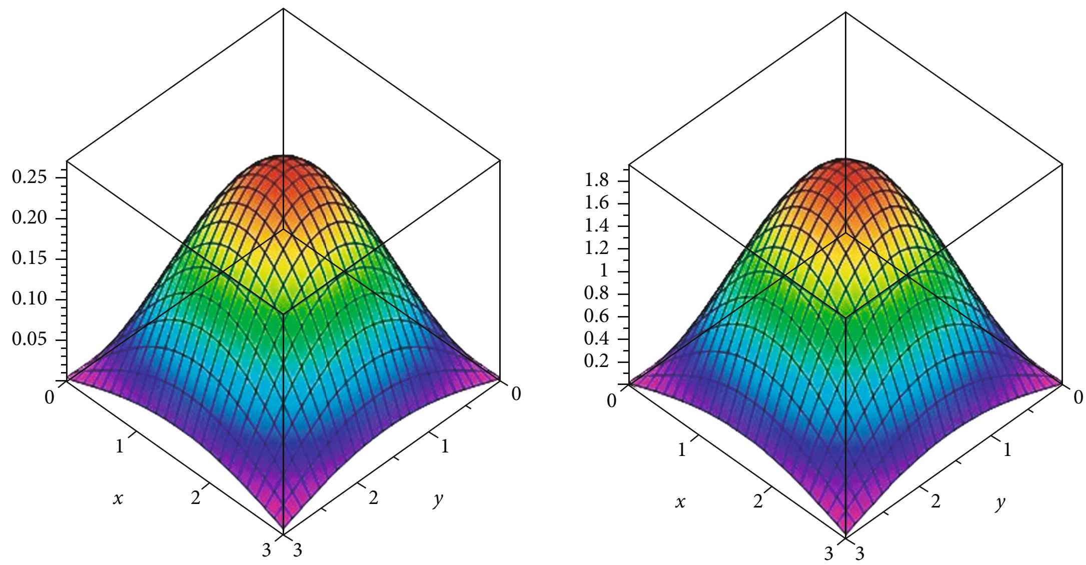


Figure 8: The fractional-order of $\delta = 0.8$ and 0.6 of Problem 3.4.

consequently, we get

$$\begin{aligned}
&m = 0, 1, 2, 3 \cdots \\
&\mu_1(\zeta, \eta) = S^{-1}\left[\frac{\mu_0(\zeta, \eta)}{s}\right] \\
&\qquad - S^{-1}\left[\frac{u^\delta}{s^\delta} S\left\{\frac{\partial^2 \mu_0}{\partial \zeta^2} - \frac{\partial \mu_0}{\partial \zeta} + \frac{\partial}{\partial \eta}\left(\mu_0 \frac{\partial^2 \mu_0}{\partial \zeta^2}\right) - 2\zeta\right\}\right], \\
&\mu_1(\zeta, \eta) = \zeta^2 + (2 - 4\zeta)\frac{\eta^\delta}{\Gamma(\delta + 1)}, \\
&\mu_2(\zeta, \eta) = S^{-1}\left[\frac{\mu_1(\zeta, \eta)}{s}\right] \\
&\qquad - S^{-1}\left[\frac{u^\delta}{s^\delta} S\left\{\frac{\partial^2 \mu_1}{\partial \zeta^2} - \frac{\partial \mu_1}{\partial \zeta} + \frac{\partial}{\partial \eta}\left(\mu_1 \frac{\partial^2 \mu_1}{\partial \zeta^2}\right) - 2\zeta\right\}\right], \\
&\mu_2(\zeta, \eta) = \zeta^2 + (2 - 4\zeta)\frac{\eta^\delta}{\Gamma(\delta + 1)} + 4\frac{\eta^{2\delta}}{\Gamma(2\delta + 1)} \\
&\qquad - 4\zeta(3\zeta - 1)\frac{\eta^{2\delta-1}}{\Gamma(2\delta)}, \\
&\mu_3(\zeta, \eta) = S^{-1}\left[\frac{\mu_2(\zeta, \eta)}{s}\right] \\
&\qquad - S^{-1}\left[\frac{u^\delta}{s^\delta} S\left\{\frac{\partial^2 \mu_2}{\partial \zeta^2} - \frac{\partial \mu_2}{\partial \zeta} + \frac{\partial}{\partial \eta}\left(\mu_2 \frac{\partial^2 \mu_2}{\partial \zeta^2}\right) - 2\zeta\right\}\right], \\
&\mu_3(\zeta, \eta) = \zeta^2 + (2 - 4\zeta)\frac{\eta^\delta}{\Gamma(\delta + 1)} + 4\frac{\eta^{2\delta}}{\Gamma(2\delta + 1)} \\
&\qquad - 4\zeta(3\zeta - 1)\frac{\eta^{2\delta-1}}{\Gamma(2\delta)} - 24\frac{\eta^{3\delta-1}}{\Gamma(3\delta)} - 4(6\zeta + 1)\frac{\eta^{3\delta-1}}{\Gamma(3\delta)} \\
&\qquad - 4\zeta^2(6\zeta - 1)\frac{\eta^{3\delta-2}}{\Gamma(3\delta - 1)} - 8(1 - 2\zeta)\frac{\Gamma(2\delta + 1)\eta^{3\delta-1}}{\Gamma(3\delta)(\Gamma(\delta + 1))^2} \\
&\qquad + 8\zeta\frac{\eta^{3\delta-1}}{\Gamma(3\delta)} + 8\zeta^2(1 - 3\zeta)\frac{\eta^{3\delta-2}}{\Gamma(3\delta - 1)}.
\end{aligned} \tag{62}$$

The approximate result of equation (56) can be achieved as

$$\begin{aligned}
\mu(\zeta, \eta) = {}& \zeta^2 + (2 - 4\zeta)\frac{\eta^\delta}{\Gamma(\delta + 1)} + 4\frac{\eta^{2\delta}}{\Gamma(2\delta + 1)} \\
& - 4\zeta(3\zeta - 1)\frac{\eta^{2\delta-1}}{\Gamma(2\delta)} - 24\frac{\eta^{3\delta-1}}{\Gamma(3\delta)} \\
& - 4(6\zeta + 1)\frac{\eta^{3\delta-1}}{\Gamma(3\delta)} - 4\zeta^2(6\zeta - 1)\frac{\eta^{3\delta-2}}{\Gamma(3\delta - 1)} \\
& - 8(1 - 2\zeta)\frac{\Gamma(2\delta + 1)\eta^{3\delta-1}}{\Gamma(3\delta)(\Gamma(\delta + 1))^2} + 8\zeta\frac{\eta^{3\delta-1}}{\Gamma(3\delta)} \\
& + 8\zeta^2(1 - 3\zeta)\frac{\eta^{3\delta-2}}{\Gamma(3\delta - 1)} + \cdots.
\end{aligned} \tag{63}$$

The exact result of equation (56) is

$$\mu(\zeta, \eta) = \zeta^2 + 2\eta. \tag{64}$$

5. Conclusion

In this article, an extended variational iteration transform method is implemented to achieve the analytical result of time-fractional diffusion equations. The suggested method is an effective and simple tool to solve fractional-order partial differential equations, because it applies the Lagrange multiplier directly to solve fractional-order partial differential equations. In conclusion, the current technique has the small number of calculations and straightforward implementation and therefore can be applied to other fractional-order partial differential equation, which frequently arises in applied science.

Abbreviations

VITM: Variational iteration transform method
ST: Shehu transform
DE: Diffusion equation
FC: Fractional calculus
PDEs: Partial differential equations
ADM: Adomian decomposition method.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Nehad Ali Shah and Jae Dong Chung contributed equally to this work and are first coauthors.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a research group program under Grant No. RGP.2/38/42. This work was supported by Korea Institute of Energy Technology Evaluation and Planning (KETEP) grant funded by the Korea government (MOTIE) (20202020900060, The Development and Application of Operational Technology in Smart Farm Utilizing Waste Heat from Particulates Reduced Smokestack)

References

[1] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, 2003.

[3] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[4] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[5] A. Bejan, "Second-law analysis in heat transfer and thermal design," in *Advances in Heat Transfers*, vol. 15, pp. 1–58, Academic Press Inc., 1982.

[6] A. Bejan, "A study of entropy generation in fundamental convective heat transfer," *Journal of Heat Transfer*, vol. 101, no. 4, pp. 718–725, 1979.

[7] P. Agarwal and A. A. El-Sayed, "Non-standard finite difference and Chebyshev collocation methods for solving fractional diffusion equation," *Physica A: Statistical Mechanics and its Applications*, vol. 500, pp. 40–49, 2018.

[8] L. Li, Z. Jiang, and Z. Yin, "Fourth-order compact finite difference method for solving two-dimensional convection-diffusion equation," *Advances in Difference Equations*, vol. 2018, no. 1, 24 pages, 2018.

[9] M. Badr, A. Yazdani, and H. Jafari, "Stability of a finite volume element method for the time-fractional advection diffusion equation," *Numerical Methods for Partial Differential Equations*, vol. 34, no. 5, pp. 1459–1471, 2018.

[10] S. Das, "Analytical solution of a fractional diffusion equation by variational iteration method," *Computers & Mathematics with Applications*, vol. 57, no. 3, pp. 483–487, 2009.

[11] S. S. Ray, "Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 1295–1306, 2009.

[12] F. Liu, P. Zhuang, I. Turner, K. Burrage, and V. Anh, "A new fractional finite volume method for solving the fractional diffusion equation," *Applied Mathematical Modelling*, vol. 38, no. 15-16, pp. 3871–3878, 2014.

[13] M. Inokuti, H. Sekine, and T. Mura, "General use of the Lagrange multiplier in nonlinear mathematical physics," in *Variational method in the mechanics of solids*, vol. 33, pp. 156–162, Elsevier, 1978.

[14] J.-H. He, "Approximate analytical solution for seepage flow with fractional derivatives in porous media," *Computer Methods in Applied Mechanics and Engineering*, vol. 167, no. 1-2, pp. 57–68, 1998.

[15] J.-H. He, "Variational iteration method - a kind of non-linear analytical technique: some examples," *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.

[16] J. Hristov, "An exercise with the He's variation iteration method to a fractional Bernoulli equation arising in transient conduction with non-linear heat flux at the boundary," *International Review of Chemical Engineering*, vol. 4, no. 5, pp. 489–497, 2012.

[17] E. Hetmaniok, K. Kaczmarek, D. Słota, R. Wituła, and A. Zielonka, "Application of the variational iteration method for determining the temperature in the heterogeneous casting-mould system," *International Review of Chemical Engineering*, vol. 4, no. 5, pp. 511–515, 2012.

[18] M. A. Abdou and A. A. Soliman, "Variational iteration method for solving Burger's and coupled Burger's equations," *Journal of Computational and Applied Mathematics*, vol. 181, no. 2, pp. 245–251, 2005.

[19] A. M. Wazwaz, "The variational iteration method: a reliable analytic tool for solving linear and nonlinear wave equations," *Computers and Mathematics with Applications*, vol. 54, no. 7-8, pp. 926–932, 2007.

[20] M. Inc, "Numerical simulation of KdV and mKdV equations with initial conditions by the variational iteration method," *Chaos, Solitons and Fractals*, vol. 34, no. 4, pp. 1075–1081, 2007.

[21] J. H. He and X. H. Wu, "Variational iteration method: new development and applications," *Computers and Mathematics with Applications*, vol. 54, no. 7-8, pp. 881–894, 2007.

[22] G. C. Wu and E. W. M. Lee, "Fractional variational iteration method and its application," *Physics Letters A*, vol. 374, no. 25, pp. 2506–2509, 2010.

[23] E. Hesameddini and H. Latifizadeh, "Reconstruction of variational iteration algorithms using the Laplace transform," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 11-12, pp. 1377–1382, 2009.

[24] S. A. Khuri and A. Sayfy, "A Laplace variational iteration strategy for the solution of differential equations," *Applied Mathematics Letters*, vol. 25, no. 12, pp. 2298–2305, 2012.

[25] G. C. Wu and D. Baleanu, “Variational iteration method for fractional calculus an universal approach by Laplace transform,” *Advances in Difference Equations*, vol. 2013, 9 pages, 2013.

[26] H. Jafari and H. K. Jassim, “Local fractional Laplace variational iteration method for solving nonlinear partial differential equations on Cantor sets within local fractional operators,” *Journal of Zankoy Sulaimani - Part A*, vol. 16, pp. 49–57, 2014.

[27] H. F. Ahmed, M. S. M. Bahgat, and M. Zaki, “Numerical approaches to system of fractional partial differential equations,” *Journal of the Egyptian Mathematical Society*, vol. 25, no. 2, pp. 141–150, 2017.

[28] J. A. T. Machado, D. Baleanu, W. Chen, and J. Sabatier, “New trends in fractional dynamics,” *Journal of Vibration and Control*, vol. 20, no. 7, pp. 963–963, 2014.

[29] D. Baleanu, Z. B. Güvenç, and J. A. Tenreiro Machado, Eds., *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.

[30] S. Maitama and W. Zhao, “New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations,” 2019, https://arxiv.org/abs/1904.11370.

[31] A. Bokhari, D. Baleanu, and R. Belgacem, “Application of Shehu transform to Atangana-Baleanu derivatives,” *Journal of Mathematics and Computer Science*, vol. 20, no. 2, pp. 101–107, 2019.

[32] R. Belgacem, D. Baleanu, and A. Bokhari, “Shehu transform and applications to Caputo-fractional differential equations,” *International Journal of Analysis and Applications*, vol. 17, no. 6, pp. 917–927, 2019.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 6650783, 11 pages
https://doi.org/10.1155/2021/6650783

Research Article

Qualitative Analysis of a Three-Species Reaction-Diffusion Model with Modified Leslie-Gower Scheme

Xiaoni Wang,[1] Gaihui Guo ,[2] Jian Li ,[1,2] and Mengmeng Du [2]

[1] *School of Electrical and Control Engineering, Shaanxi University of Science and Technology, Xi'an 710021, China*
[2] *School of Arts and Sciences, Shaanxi University of Science and Technology, Xi'an 710021, China*

Correspondence should be addressed to Gaihui Guo; guogaihui@sust.edu.cn and Jian Li; jianli@sust.edu.cn

Received 10 December 2020; Revised 27 March 2021; Accepted 5 April 2021; Published 4 May 2021

Academic Editor: Qifeng Zhang

The qualitative analysis of a three-species reaction-diffusion model with a modified Leslie-Gower scheme under the Neumann boundary condition is obtained. The existence and the stability of the constant solutions for the ODE system and PDE system are discussed, respectively. And then, the priori estimates of positive steady states are given by the maximum principle and Harnack inequality. Moreover, the nonexistence of nonconstant positive steady states is derived by using Poincaré inequality. Finally, the existence of nonconstant positive steady states is established based on the Leray-Schauder degree theory.

1. Introduction

Three-species reaction-diffusion models with Holling-type II functional response have been a familiar subject for the analysis. Taking more practical factors into consideration, a model with a modified Leslie-Gower scheme is worthy to explore. Leslie-Gower's scheme indicates that the carrying capacity of the predator is proportional to the population size of the prey. The existing works [1–3] are all about models with this scheme. As a matter of fact, predators prefer to prey on other prey in the event of a shortage of favorite prey, so the research of the modified Leslie-Gower model springs up. Aziz-Alaoui and Okiye [4] focused on a two-dimensional continuous time dynamical system modeling a predator-prey food chain and gave the main result of the boundedness of solutions, the existence of an attracting set, and the global stability of the coexisting interior equilibrium, which was based on a modified version of the Leslie-Gower scheme and Holling-type II scheme. Singh and Gakkhar [5] investigated the stabilization problem of the modified Leslie-Gower type prey-predator model with the Holling-type II functional response. The analysis of models with a modified Leslie-Gower scheme can be also found in [6–10].

Nonconstant positive steady states have received increasing attention in recent years, see [11–18] and references therein. Ko and Ryu [19] showed that the predator-prey model with Leslie-Gower functional response had no nonconstant positive solution in homogeneous environment, but the system with a general functional response might have at least one nonconstant positive steady state under some conditions. Zhang and Zhao [20] analyzed a diffusive predator-prey model with toxins under the homogeneous Neumann boundary condition, including the existence and nonexistence of nonconstant positive steady states of this model by considering the effect of large diffusivity. Shen and Wei [21] considered a reaction-diffusion mussel-algae model with state-dependent mussel mortality which involved a positive feedback scheme. Wang and his partners [22] considered a tumor-immune model with diffusion and nonlinear functional response and investigated the effect of diffusion on the existence of nonconstant positive steady states and the steady-state bifurcations. Hu and Li [23] were concerned about a strongly coupled diffusive predator-prey system with a modified Leslie-Gower scheme and established the existence of nonconstant positive steady states. Qiu and Guo [24] analyzed a stationary Leslie-Gower model with diffusion and advection.

Motivated by the mentioned above, we consider a three-species reaction-diffusion model with a modified Leslie-Gower and Holling-type II scheme under the homogeneous Neumann boundary condition as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u\left(A - A_1 u - B_1 v - \dfrac{C_1 w}{1 + D_1 u}\right), & t > 0, \\ \dfrac{\partial v}{\partial t} = v(B - A_2 v - B_2 u), & t > 0, \\ \dfrac{\partial w}{\partial t} = w\left(C - \dfrac{C_2 w}{1 + D_2 u}\right), & t > 0, \end{cases} \tag{1}$$

where u and v represent the density of two competitors, respectively, while w stands for the density of the predator who preys on u. A, B, and C are all positive as the intrinsic growth rates, A_1 and A_2 regard as influencing factors within diverse populations themselves while B_1 and B_2 are influencing factors between different populations. All of them are nonnegative. $C_1 w/(1 + D_1 u)$ and $C_2 w/(1 + D_2 u)$ are the modified Leslie-Gower scheme, and C_1, C_2, D_1, and D_2 are positive. Applying the following scaling to (1), as well as assuming $C_1 D_2/D_1 C_2 = 1$ for simplicity of calculation:

$$m_1 = \frac{D_1}{C_1} u, m_2 = \frac{A_2 D_1}{A_1 C_1} v, m_3 = \frac{D_1}{A_1 C_1} w, s = \frac{A_1 C_1}{D_1} t, \tag{2}$$

still using u, v, w, t replace m_1, m_2, m_3, s, the following ODE system can be logically obtained:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u\left(a - u - \alpha_1 v - \dfrac{w}{\beta_1 + u}\right), & t > 0, \\ \dfrac{\partial v}{\partial t} = v(b - v - \alpha_2 u), & t > 0, \\ \dfrac{\partial w}{\partial t} = w\left(c - \dfrac{w}{\beta_2 + u}\right), & t > 0, \\ u(0) \geq 0, v(0) \geq 0, w(0) \geq 0, \end{cases} \tag{3}$$

where $a = AD_1/A_1 C_1, b = BD_1/A_1 C_1, c = CD_1/A_1 C_1, \alpha_1 = B_1/A_2, \alpha_2 = B_2/A_1, \beta_1 = 1/C_1, \beta_2 = 1/C_2$.

It is clear that $(0,0,0), (a,0,0), (0,b,0), (0,0,c\beta_2)$, and $(0,b,c\beta_2)$ are nonnegative constant solutions of system (3). $((a - \alpha_1 b)/(1 - \alpha_1 \alpha_2), (b - \alpha_2 a)/(1 - \alpha_1 \alpha_2), 0)$ is a semitrivial solution when it satisfies $(a - \alpha_1 b)(b - \alpha_2 a) > 0$. When $a\beta_1 > c\beta_2$, $(\dot{u}, 0, \dot{w})$ is a semitrivial solution where

$$\dot{u} = \frac{a - \beta_1 - c + \sqrt{(\beta_1 - a + c)^2 - 4c\beta_2 + 4a\beta_1}}{2}, \tag{4}$$

$$\dot{w} = c(\beta_2 + u_0). \tag{5}$$

System (3) yields that

$$(\alpha_1\alpha_2 - 1)u^2 + (a - \beta_1 - \alpha_1 b + \alpha_1\alpha_2\beta_1 - c)u + \beta_1 a - \alpha_1\beta_1 b - \beta_2 c = 0. \tag{6}$$

If the following alternative conditions hold:

$$(i)\ \alpha_1\alpha_2 > 1 \text{ and } a < \frac{b}{\alpha_2}, \tag{7}$$

$$(\text{ii})\ \alpha_1 b + c\frac{\beta_2}{\beta_1} < a < \frac{b}{\alpha_2}, \tag{8}$$

there exists the unique positive equilibrium (u^*, v^*, w^*) as

$$\begin{aligned} u^* &= \frac{-a + \beta_1 + \alpha_1 b - \alpha_1\alpha_2\beta_1 + c + \sqrt{\Delta}}{2(\alpha_1\alpha_2 - 1)}, \\ v^* &= b - \alpha_2 u^*, \\ w^* &= c(\beta_2 + u^*), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Delta &= (a - \beta_1 - \alpha_1 b + \alpha_1\alpha_2\beta_1 - c)^2 \\ &\quad - 4(\alpha_1\alpha_2 - 1)(\beta_1 a - \alpha_1\beta_1 b - \beta_2 c). \end{aligned} \tag{10}$$

Taking the diffusion into account, the corresponding PDE system can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = u\left(a - u - \alpha_1 v - \dfrac{w}{\beta_1 + u}\right), & t > 0, x \in \Omega, \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = v(b - v - \alpha_2 u), & t > 0, x \in \Omega, \\ \dfrac{\partial w}{\partial t} - d_3 \Delta w = w\left(c - \dfrac{w}{\beta_2 + u}\right), & t > 0, x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = \dfrac{\partial w}{\partial n} = 0, & t > 0, x \in \partial\Omega, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \bar{\Omega}, \end{cases} \tag{11}$$

where $\Omega \subset R^N$ is a smooth bounded domain, n is the outward unit normal vector on $\partial\Omega$, Δ is the Laplace operator, and diffusion coefficients are $d_1, d_2, d_3 > 0$.

The rest of this paper is arranged as follows. In Section 2, the stability of constant solutions for the ODE system is discussed. In Section 3, the stability of constant solutions for the PDE system is studied. In Section 4, we focus on the priori estimates of positive steady states. In the last two sections, we have a discussion about the nonexistence and existence of nonconstant positive steady states under different conditions.

2. Stability of Constant Solutions for the ODE System

In this section, we discuss the stability of constant solutions with the condition of their existence for the ODE system.

Theorem 1. *For the ODE system (3), let $\Gamma = \{a, b, c, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ and $1/(\beta_1 + u^*) \triangleq B$.*

(i) $(0,0,0), (a,0,0), (0,b,0), (0,0,c\beta_2)$ and $((a - \alpha_1 b)/(1 - \alpha_1\alpha_2), (b - \alpha_2 a)/(1 - \alpha_1\alpha_2), 0)$ are all unconditionally unstable

(ii) *If Γ satisfies $a < b/\alpha_2$, then $(\dot{u}, 0, \dot{w})$ is unstable; if $b < \alpha_2(a - \beta_1 - c)$ holds, $(\dot{u}, 0, \dot{w})$ is local asymptotically stable*

(iii) *If Γ satisfies $a > \alpha_1 b + c(\beta_2/\beta_1)$, then $(0, b, c\beta_2)$ is unstable; if $a < \alpha_1 b + c(\beta_2/\beta_1)$ holds, $(0, b, c\beta_2)$ is local asymptotically stable*

(iv) *If Γ and B satisfy $2u^* - a + (\alpha_1 + 1)(b - \alpha_2 u^*) + \beta_1 c B^2(\beta_2 + u^*) + c < 0$, then (u^*, v^*, w^*) is unstable; if $2u^* - a - c \geq 0$ and $c - \alpha_1\alpha_2 u^* \geq 0$ holds, (u^*, v^*, w^*) is local asymptotically stable*

Proof. The Jacobian matrix of the ODE system (3) is

$$J = \begin{pmatrix} a - 2u - \alpha_1 v - \dfrac{\beta_1 w}{(\beta_1 + u)^2} & -\alpha_1 u & -\dfrac{u}{\beta_1 + u} \\ -\alpha_2 v & b - \alpha_2 u - 2v & 0 \\ \dfrac{w^2}{(\beta_2 + u)^2} & 0 & c - \dfrac{2w}{\beta_2 + u} \end{pmatrix}. \tag{12}$$

Obviously, we can obtain

$$J = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \tag{13}$$

at $(0, 0, 0)$ and its corresponding characteristic polynomial is

$$\varphi(\lambda) = (\lambda - a)(\lambda - b)(\lambda - c) = 0, \tag{14}$$

so its eigenvalues are $\lambda_1 = a > 0$, $\lambda_2 = b > 0$, and $\lambda_3 = c > 0$. Therefore, $(0, 0, 0)$ is unstable to system (3).

By the same manner, we know that $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c\beta_2)$, and $((a - \alpha_1 b)/(1 - \alpha_1 \alpha_2), (b - \alpha_2 a)/(1 - \alpha_1 \alpha_2), 0)$ are all unstable to ODE system (3).

The Jacobian matrix of the ODE system at $(\dot{u}, 0, \dot{w})$ is

$$J = \begin{pmatrix} -\dot{u} + \dfrac{\dot{u}(a - \dot{u})}{\beta_1 + \dot{u}} & -\alpha_1 \dot{u} & -\dfrac{\dot{u}}{\beta_1 + \dot{u}} \\ 0 & b - \alpha_2 \dot{u} & 0 \\ c^2 & 0 & -c \end{pmatrix}. \tag{15}$$

The characteristic polynomial is

$$[\lambda - (b - \alpha_2 \dot{u})]\left[\left(\lambda + \dot{u} - \frac{\dot{u}(a - \dot{u})}{\beta_1 + \dot{u}}\right)(\lambda + c) + \frac{c^2 \dot{u}}{\beta_1 + \dot{u}}\right] = 0. \tag{16}$$

When the eigenvalue satisfies $\lambda_1 = b - \alpha_2 \dot{u} > 0$, it deduces that $a < b/\alpha_2$, so we can see that $(\dot{u}, 0, \dot{w})$ is unstable to ODE system (3). When $\lambda_1 = b - \alpha_2 \dot{u} < 0$, we consider that

$$\lambda^2 + \left(\dot{u} - \frac{\dot{u}(a - \dot{u})}{\beta_1 + \dot{u}} + c\right)\lambda + c\dot{u} - \frac{c\dot{u}(a - \dot{u})}{\beta_1 + \dot{u}} + \frac{c^2 \dot{u}}{\beta_1 + \dot{u}} = 0. \tag{17}$$

Let$p_1 = \dot{u} - (\dot{u}(a - \dot{u}))/(\beta_1 + \dot{u}) + c$, $p_2 = c\dot{u} - (c\dot{u}(a - \dot{u})/\beta_1 + \dot{u}) + (c^2\dot{u}/\beta_1 + \dot{u})$and take value for $\dot{u}, \dot{w}$ as (4) and (5), we know that $p_1 = (\dot{u}^2 + a\beta_1 - c\beta_2 + c\beta_1)/(\beta_1 + \dot{u})$, $p_2 = ((c\dot{u}(2\dot{u} + \beta_1 + c - a))/(\beta_1 + \dot{u})) > 0$. With the existence condition $a\beta_1 > c\beta_2$, $p_1 > 0$ and $p_2 > 0$ hold, such that equation (17) has two solutions with negative real parts.

Because of $a\beta_1 > c\beta_2$,

$$\begin{aligned} \lambda_1 = b - \alpha_2 \dot{u} &= b - \alpha_2 \frac{a - \beta_1 - c + \sqrt{(\beta_1 - a + c)^2 - 4c\beta_2 + 4a\beta_1}}{2} \\ &< b - \alpha_2 \frac{a - \beta_1 - c + \sqrt{(\beta_1 - a + c)^2}}{2} \end{aligned} \tag{18}$$

holds, then $\lambda_1 < 0$ if $b - \alpha_2(a - \beta_1 - c) < 0$. So we can conclude that when $b < \alpha_2(a - \beta_1 - c)$, $(\dot{u}, 0, \dot{w})$ is local asymptotically stable to ODE system (3).

The Jacobian matrix of the ODE system at $(0, b, c\beta_2)$ is

$$J = \begin{pmatrix} a - \alpha_1 b - c\dfrac{\beta_2}{\beta_1} & 0 & 0 \\ -\alpha_2 b & -b & 0 \\ c^2 & 0 & -c \end{pmatrix}. \tag{19}$$

The characteristic polynomial is

$$\left(\lambda - a + \alpha_1 b + c\frac{\beta_2}{\beta_1}\right)(\lambda + b)(\lambda + c) = 0. \tag{20}$$

The corresponding eigenvalues are $\lambda_1 = a - \alpha_1 b - c(\beta_2/\beta_1)$, $\lambda_2 = -b < 0$, $\lambda_3 = -c < 0$. If $a > \alpha_1 b + c(\beta_2/\beta_1)$, $(0, b, c\beta_2)$ is unstable. Otherwise, $a < \alpha_1 b + c(\beta_2/\beta_1)$, $(0, b, c\beta_2)$ is local asymptotically stable to ODE system (3).

The Jacobian matrix of the ODE system at (u^*, v^*, w^*) is

$$\begin{aligned} J &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a - 2u^* - \alpha_1(b - \alpha_2 u^*) - \beta_1 c B^2(\beta_2 + u^*) & -\alpha_1 u^* & -u^* B \\ -\alpha_2(b - \alpha_2 u^*) & -b + \alpha_2 u^* & 0 \\ c^2 & 0 & -c \end{pmatrix}. \end{aligned} \tag{21}$$

The corresponding characteristic polynomial is $\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0$, where

$$\begin{cases} A_1 = & 2u^* - a + (\alpha_1 + 1)(b - \alpha_2 u^*) + \beta_1 c B^2(\beta_2 + u^*) + c, \\ A_2 = & \left[a - 2u^* - \alpha_1(b - \alpha_2 u^*) - \beta_1 c B^2(\beta_2 + u^*)\right](\alpha_2 u^* - b - c) + c^2 u^* B + (\alpha_1\alpha_2 u^* - c)(\alpha_2 u^* - b), \\ A_3 = & c(\alpha_2 u^* - b)\left[a - 2u^* - \alpha_1(b - 2\alpha_2 u^*) - \beta_1 c B^2(\beta_2 + u^*) - cu^* B\right]. \end{cases} \quad (22)$$

When Γ satisfies $2u^* - a + (\alpha_1 + 1)(b - \alpha_2 u^*) + \beta_1 c B^2(\beta_2 + u^*) + c < 0$, then $A_1 < 0$, (u^*, v^*, w^*) is unstable applying the Hurwitz criterion [25]. When $2u^* - a - c \geq 0, c - \alpha_1\alpha_2 u^* \geq 0$, we can find $A_1 > 0, A_2 > 0, A_3 > 0, A_1A_2 - A_3 > 0$. So (u^*, v^*, w^*) is local asymptotically stable to ODE system (3).

The proof is complete.

3. Stability of Constant Solutions for the PDE System

In this section, the stability of the constant solutions with the condition of their existence for the PDE system is discussed.

Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots$ as the eigenvalues of the operator $-\Delta$ over Ω under the homogeneous Neumann boundary condition and $E(\mu_i)$ be the corresponding eigenspace while $\{\varphi_{ij} \mid j = 1, 2, \cdots, \dim E(\mu_i)\}$ is a set of the orthogonal basis of $E(\mu_i)$, $\mathbf{X} = \{U \in C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \mid \partial_n U = 0, x \in \partial\Omega\}$, and $\mathbf{X}_{ij} = \{\mathbf{c}\varphi_{ij} \mid \mathbf{c} \in R^3\}$. Then, $\mathbf{X} = \oplus_{i=o}^{\infty} \oplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}$.

Theorem 2. *For the PDE system (11), let $\Gamma = \{a, b, c, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ and $1/(\beta_1 + u^*) \triangleq B$.*

(i) $(0,0,0), (a,0,0), (0,b,0), (0,0,c\beta_2)$ and $((a - \alpha_1 b)/(1 - \alpha_1\alpha_2), (b - \alpha_2 a)/(1 - \alpha_1\alpha_2), 0)$ are all unconditionally unstable

(ii) If Γ satisfies $a < b/\alpha_2$, then $(\dot{u}, 0, \dot{w})$ is unstable; if $b < \alpha_2(a - \beta_1 - c)$ and $d_1/d_3 > (a + \beta_2)/\beta_1$ holds, $(\dot{u}, 0, \dot{w})$ is uniformly asymptotically stable

(iii) If Γ satisfies $a > \alpha_1 b + c(\beta_2/\beta_1)$, then $(0, b, c\beta_2)$ is unstable; if $a < \alpha_1 b + c(\beta_2/\beta_1)$ holds, $(0, b, c\beta_2)$ is uniformly asymptotically stable

(iv) If Γ and B satisfy $2u^* - a + (\alpha_1 + 1)(b - \alpha_2 u^*) + \beta_1 c B^2(\beta_2 + u^*) + c < 0$, then (u^*, v^*, w^*) is unstable; if $2u^* - a - c \geq 0$ and $c - \alpha_1\alpha_2 u^* \geq 0$ holds, (u^*, v^*, w^*) is uniformly asymptotically stable

Proof. The linearization of (11) at the positive constant solution U^* can be expressed by $U_t = (D\Delta + G_U(U^*))U$ where $U = (u, v, w)^T$, $U^* = (u^*, v^*, w^*)^T$, $D = \operatorname{diag}(d_1, d_2, d_3)$ and $G_U(U^*)$ is the Jacobian matrix at U^*. For each $i \geq 0$, $\oplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}$ is invariant under the operator $D\Delta + G_U(U^*)$. And λ is an eigenvalue of $D\Delta + G_U(U^*)$ on $\oplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}$ if and only if λ is an eigenvalue of the matrix $-\mu_i D + G_U(U^*)$.

The Jacobian matrix of PDE system (11) is

$$J = \begin{pmatrix} a - 2u - \alpha_1 v - \dfrac{\beta_1 w}{(\beta_1 + u)^2} - d_1\mu_i & -\alpha_1 u & -\dfrac{u}{\beta_1 + u} \\ -\alpha_2 v & b - \alpha_2 u - 2v - d_2\mu_i & 0 \\ \dfrac{w^2}{(\beta_1 + u)^2} & 0 & c - \dfrac{2w}{\beta_2 + u} - d_3\mu_i \end{pmatrix}. \quad (23)$$

According to the Theorem 1, $(0,0,0), (a,0,0), (0,b,0), (0,0,c\beta_2), ((a - \alpha_1 b)/(1 - \alpha_1\alpha_2), (b - \alpha_2 a)/(1 - \alpha_1\alpha_2), 0)$ are all unstable to ODE system (3). Hence, there exist the eigenvalue with positive real parts in the PDE system. It means that $(0,0,0), (a,0,0), (0,b,0), (0,0,c\beta_2), ((a - \alpha_1 b)/(1 - \alpha_1\alpha_2), (b - \alpha_2 a)/(1 - \alpha_1\alpha_2), 0)$ are all unstable to PDE system (11).

The Jacobian matrix of the PDE system at $(\dot{u}, 0, \dot{w})$ is

$$J = \begin{pmatrix} -\dot{u} + \dfrac{\dot{u}(a - \dot{u})}{\beta_1 + \dot{u}} - d_1\mu_i & -\alpha_1\dot{u} & -\dfrac{\dot{u}}{\beta_1 + \dot{u}} \\ 0 & b - \alpha_2\dot{u} - d_2\mu_i & 0 \\ c^2 & 0 & -c - d_3\mu_i \end{pmatrix}. \quad (24)$$

The characteristic polynomial is

$$\left[\lambda - (b - \alpha_2\dot{u} - d_2\mu_i)\right]\left[\left(\lambda + \dot{u} - \frac{\dot{u}(a - \dot{u})}{\beta_1 + \dot{u}} + d_1\mu_i\right) \cdot (\lambda + c + d_3\mu_i) + \frac{c^2\dot{u}}{\beta_1 + \dot{u}}\right] = 0. \quad (25)$$

When the eigenvalue satisfies $\lambda_1 = b - \alpha_2\dot{u} > 0$, it deduces that $a < b/\alpha_2$, there exists an eigenvalue with positive real part, and $(\dot{u}, 0, \dot{w})$ is unstable to PDE system (11).

It is clear that eigenvalue $\lambda_{1\mu_i} = b - \alpha_2\dot{u} - d_2\mu_i < 0$ as $\lambda_1 = b - \alpha_2\dot{u} < 0$. Then, we discuss the following equation emphatically:

$$\lambda^2 + \left(\dot{u} - \frac{\dot{u}(a-\dot{u})}{\beta_1 + \dot{u}} + d_1\mu_i + c + d_3\mu_i\right)\lambda + \left(\dot{u} - \frac{\dot{u}(a-\dot{u})}{\beta_1 + \dot{u}} + d_1\mu_i\right)(c + d_3\mu_i) + \frac{c^2\dot{u}}{\beta_1 + \dot{u}} = 0. \tag{26}$$

Let

$$\begin{aligned} p_3 &= \dot{u} - \frac{\dot{u}(a-\dot{u})}{\beta_1 + \dot{u}} + d_1\mu_i + c + d_3\mu_i = (d_1 + d_3)\mu_i + p_1, \\ p_4 &= \left(\dot{u} - \frac{\dot{u}(a-\dot{u})}{\beta_1 + \dot{u}} + d_1\mu_i\right)(c + d_3\mu_i) + \frac{c^2\dot{u}}{\beta_1 + \dot{u}} \\ &= d_1 d_3\mu_i^2 + \left\{cd_1 + \left[\dot{u} - \frac{\dot{u}(a-\dot{u})}{\beta_1 + \dot{u}}\right]d_3\right\}\mu_i + p_2. \end{aligned} \tag{27}$$

It shows that $p_3 > 0$ on account of $p_1 > 0$. When $cd_1 + [\dot{u} - ((\dot{u}(a-\dot{u}))/(\beta_1 + \dot{u}))]d_3 > 0$, we know $p_4 > 0$ holds. So the eigenvalues all have negative real parts.

The Jacobian matrix of PDE system (11) at $(0, b, c\beta_2)$ can be written as

$$\begin{pmatrix} a - \alpha_1 b - c\dfrac{\beta_2}{\beta_1} - d_1\mu_i & 0 & 0 \\ -\alpha_2 b & -b - d_2\mu_i & 0 \\ c^2 & 0 & -c - d_3\mu_i \end{pmatrix}. \tag{28}$$

The characteristic polynomial is

$$\left(\lambda - a + \alpha_1 b + c\frac{\beta_2}{\beta_1} + d_1\mu_i\right)(\lambda + b + d_2\mu_i)(\lambda + c + d_3\mu_i) = 0. \tag{29}$$

The corresponding eigenvalues are $\lambda_{1\mu_i} = a - \alpha_1 b - c(\beta_2/\beta_1) - d_1\mu_i$, $\lambda_{2\mu_i} = -b - d_2\mu_i < 0$ and $\lambda_{3\mu_i} = -c - d_3\mu_i < 0$. If $a > \alpha_1 b + c(\beta_2/\beta_1)$, there exists an eigenvalue with positive real part; $(0, b, c\beta_2)$ is unstable to PDE system (11). On the contrary, if $a < \alpha_1 b + c(\beta_2/\beta_1)$, the eigenvalues all have negative real parts.

The Jacobian matrix of the PDE system at (u^*, v^*, w^*) is

$$G_U(U^*) = \begin{pmatrix} a - 2u^* - \alpha_1(b - \alpha_2 u^*) - \beta_1 cB^2(\beta_2 + u^*) - d_1\mu_i & -\alpha_1 u^* & -u^* B \\ -\alpha_2(b - \alpha_2 u^*) & -b + \alpha_2 u^* - d_2\mu_i & 0 \\ c^2 & 0 & -c - d_3\mu_i \end{pmatrix}. \tag{30}$$

Its characteristic polynomial is $\lambda^3 + A_{1\mu_i}\lambda^2 + A_{2\mu_i}\lambda + A_{3\mu_i} = 0$, where

$$\begin{cases} A_{1\mu_i} = (d_1 + d_2 + d_3)\mu_i + A_1, \\ A_{2\mu_i} = (d_1d_2 + d_1d_3 + d_2d_3)\mu_i^2 - (d_1a_{22} + d_2a_{11} + d_1a_{33} + d_3a_{11} + d_2a_{33} + d_3a_{22})\mu_i + A_2, \\ A_{3\mu_i} = d_1d_2d_3\mu_i^3 - (d_1d_2a_{33} + d_1d_3a_{22} + d_2d_3a_{11})\mu_i^2 + (d_3a_{11}a_{22} + d_2a_{11}a_{33} + d_1a_{22}a_{33} - d_3a_{12}a_{21} - d_2a_{13}a_{31})\mu_i + A_3. \end{cases} \tag{31}$$

When $2u^* - a + (\alpha_1 + 1)(b - \alpha_2 u^*) + \beta_1 cB^2(\beta_2 + u^*) + c < 0$, there exists an eigenvalue with positive real part; (u^*, v^*, w^*) is unstable to PDE system (11).

When $A_1 > 0$ and $d_1, d_2, d_3 > 0$, $A_{1\mu_i} > 0$ holds. Similarly, $A_{2\mu_i} > 0$ since $A_2 > 0$ and $d_1, d_2, d_3 > 0$. If $2u^* - a - c \geq 0$, $c - \alpha_1\alpha_2 u^* \geq 0$, we have $d_3a_{11}a_{22} + d_2a_{11}a_{33} + d_1a_{22}a_{33} - d_3a_{12}a_{21} - d_2a_{13}a_{31} > 0$ and $d_1d_2a_{33} + d_1d_3a_{22} + d_2d_3a_{11} < 0$. As a result of $A_3 > 0$ and $d_1, d_2, d_3 > 0$, $A_{3\mu_i} > 0$ can be obtained. What is more, $A_1A_2 - A_3 > 0$ leads to $A_{1\mu_i}A_{2\mu_i} - A_{3\mu_i} > 0$. Thus, the eigenvalues all have negative real parts.

In the following, we shall prove that there exists a positive constant κ when the corresponding eigenvalues all have negative real parts, such that

$$\mathrm{Re}\left(\lambda_{1\mu_i}\right), \mathrm{Re}\left(\lambda_{2\mu_i}\right), \mathrm{Re}\left(\lambda_{3\mu_i}\right) < -\kappa, \quad \text{for all } i \geq 1. \tag{32}$$

Let $\lambda = \mu_i\zeta$, then

$$\psi_i(\lambda) = \mu_i^3\zeta^3 + A_{1\mu_i}\mu_i^2\zeta^2 + A_{2\mu_i}\mu_i\zeta + A_{3\mu_i} \triangleq \tilde{\psi}_i(\zeta). \tag{33}$$

Since $\mu_i \to \infty$ as $i \to \infty$, it follows that

$$\lim_{i\to\infty}\left\{\frac{\tilde{\psi}_i(\zeta)}{\mu_i^3}\right\} = \zeta^3 + (d_2 + d_2 + d_3)\zeta^2 + (d_1d_2 + d_1d_3 + d_2d_3)\zeta + d_1d_2d_3 \triangleq \bar{\psi}(\zeta). \tag{34}$$

Applying the Hurwitz criterion, the three roots $\zeta_1, \zeta_2, \zeta_3$ of $\bar{\psi}(\zeta) = 0$ all have negative real parts. Thus, there exists a positive constant κ' such that Re (ζ_1), Re (ζ_1), Re $(\zeta_1) \le -\kappa'$. By continuity, there exists i_0 such that the three roots ζ_{i1}, ζ_{i2}, ζ_{i3} of $\bar{\psi}(\zeta) = 0$ satisfy

$$\text{Re }\{\zeta_{i1}\}, \text{Re }\{\zeta_{i2}\}, \text{Re }\{\zeta_{i3}\} \le -\frac{\kappa'}{2}, \quad \text{for all } i \ge i_0. \tag{35}$$

Hence, Re $(\lambda_{1\mu_i})$, Re $(\lambda_{2\mu_i})$, Re $(\lambda_{3\mu_i}) \le -\mu_i\kappa'/2 \le -\kappa'/2$ for all $i \ge i_0$.

Let $-\kappa'' = \max_{1\le i\le i_0}\{\text{Re }(\lambda_{1\mu_i}), \text{Re }(\lambda_{2\mu_i}), \text{Re }(\lambda_{3\mu_i})\}, \kappa = \min\{\kappa', \kappa''\}$. Then, for $i \ge 1$,

$$\text{Re }\left(\lambda_{1\mu_i}\right), \text{Re }\left(\lambda_{2\mu_i}\right), \text{Re }\left(\lambda_{3\mu_i}\right) < -\kappa. \tag{36}$$

Therefore, the constant solutions are uniformly asymptotically stable when the corresponding eigenvalues all have negative real parts.

The proof is complete.

4. A Priori Estimates of Positive Steady States

The corresponding steady-state problem of system (11) is

$$\begin{cases} -d_1\Delta u = u\left(a - u - \alpha_1 v - \dfrac{w}{\beta_1 + u}\right), & x \in \Omega, \\ -d_2\Delta v = v(b - v - \alpha_2 u), & x \in \Omega, \\ -d_3\Delta w = w\left(c - \dfrac{w}{\beta_2 + u}\right), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{37}$$

Two lemmas are listed here for the preliminary.

Lemma 3. *(Harnack inequality [26]).*

Let $\omega(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive solution to $\Delta\omega + c(x)\omega = 0, x \in \Omega$, where $c(x) \in C(\bar{\Omega})$, satisfying the homogeneous Neumann boundary condition. Then, there exists a positive constant $C_* = C_*(N, \Omega, \|c\|_\infty)$ such that

$$\max_{\bar{\Omega}}\omega \le C_* \min_{\bar{\Omega}}\omega. \tag{38}$$

Lemma 4. *(maximum principle [27]).*

Suppose that $g \in C(\Omega \times R^1)$ and $b_j \in C(\bar{\Omega}), j = 1, 2, \cdots, N$.

(i) if $\omega(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies

$$\begin{gathered} \Delta\omega + \sum_{j=1}^N b_j(x)\omega_{x_j} + g(x, \omega(x)) \ge 0, \quad x \in \Omega, \\ \frac{\partial\omega}{\partial\nu} \le 0, \quad x \in \partial\Omega, \end{gathered} \tag{39}$$

and $\omega(x_0) = \max_{\bar{\Omega}}\omega(x)$, then $g(x_0, \omega(x_0)) \ge 0$.

(ii) if $\omega(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies

$$\begin{gathered} \Delta\omega + \sum_{j=1}^N b_j(x)\omega_{x_j} + g(x, \omega(x)) \le 0, \quad x \in \Omega, \\ \frac{\partial\omega}{\partial\nu} \ge 0, \quad x \in \partial\Omega, \end{gathered} \tag{40}$$

and $\omega(x_0) = \min_{\bar{\Omega}}\omega(x)$, then $g(x_0, \omega(x_0)) \le 0$.

The results of upper and lower bounds can be stated as follows.

Theorem 5. *(upper bounds).*

Assuming that (u, v, w) is a positive solution of system (37), we get

$$\max_{\bar{\Omega}} u \le a, \tag{41}$$

$$\max_{\bar{\Omega}} v \le b, \tag{42}$$

$$\max_{\bar{\Omega}} w \le c(\beta_2 + a). \tag{43}$$

Proof. Since $u(a - u - \alpha_1 v - w/(\beta_1 + u)) \le u(a - u)$ and $v(b - v - \alpha_2 u) \le v(b - v)$, such that $\max_{\bar{\Omega}} u \le a, \max_{\bar{\Omega}} v \le b$ according to Lemma 4. Because of $\max_{\bar{\Omega}} u \le a$, it is evident that $\max_{\bar{\Omega}} w \le c(\beta_2 + a)$.

The proof is complete.

Theorem 6. *(lower bounds).*

Fix Γ and $\underline{d}_1, \underline{d}_2, \underline{d}_3$ as positive constants. Assume that

$$(d_1, d_2, d_3) \in [\underline{d}_1, \infty) \times [\underline{d}_2, \infty) \times [\underline{d}_3, \infty), \tag{44}$$

then there exists a positive constant $\underline{C} = \underline{C}(\Gamma, \Omega, N, \underline{d}_1, \underline{d}_2, \underline{d}_3)$ who can make every positive solution (u, v, w) of system (37) satisfy

$$\min_{\bar{\Omega}} u(x) > \underline{C}, \tag{45}$$

$$\min_{\bar{\Omega}} v(x) > \underline{C}, \tag{46}$$

$$\min_{\bar{\Omega}} w(x) > \underline{C}. \tag{47}$$

Proof. Let

$$\begin{cases} c_1(x) = d_1^{-1}\left(a - u - \alpha_1 v - \dfrac{w}{\beta_1 + u}\right), \\ c_2(x) = d_2^{-1}(b - v - \alpha_2 u), \\ c_3(x) = d_3^{-1}\left(c - \dfrac{w}{\beta_2 + u}\right). \end{cases} \tag{48}$$

In view of (41), (42), and (43), a positive constant $\bar{C} = \bar{C}(\Omega, N, \bar{D}, \Gamma)$ can be easily found, such that

$$\|c_1(x)\|_\infty, \|c_2(x)\|_\infty, \|c_3(x)\|_\infty \le \bar{C}, \tag{49}$$

where $d_1, d_2, d_3 > \bar{D}$. Thus, u, v, and w satisfy that

$$\begin{gathered} \Delta u + c_1(x)u = 0, \quad x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \\ \Delta v + c_2(x)v = 0, \quad x \in \Omega, \\ \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega, \\ \Delta w + c_3(x)w = 0, \quad x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega. \end{gathered} \tag{50}$$

According to the Harnack inequality in Lemma 3, there must be a positive constant $C_* = C_*(\Omega, N, \bar{D}, \Gamma)$, such that

$$\begin{gathered} \max_{\bar{\Omega}} u \le C_* \min_{\bar{\Omega}} u, \\ \max_{\bar{\Omega}} v \le C_* \min_{\bar{\Omega}} v, \\ \max_{\bar{\Omega}} w \le C_* \min_{\bar{\Omega}} w. \end{gathered} \tag{51}$$

Suppose that (45), (46), and (47) hold of no account.

There must be a sequence $\{(d_{1i}, d_{2i}, d_{3i})\}_{i=1}^{\infty}$ with $(d_{1i}, d_{2i}, d_{3i}) \in [\underline{d}_1,\infty) \times [\underline{d}_2,\infty) \times [\underline{d}_3,\infty)$, such that the corresponding positive solutions (u_i, v_i, w_i) of system (37) reach the qualification

$$\max_{\bar{\Omega}} u_i \to 0, \text{or } \max_{\bar{\Omega}} v_i \to 0 \text{ or } \max_{\bar{\Omega}} w_i \to 0 (i \to \infty). \tag{52}$$

Then, we apply (u_i, v_i, w_i) to the system of (37) and integrate by parts, so we obtain that

$$\begin{gathered} \int_\Omega u_i\left(a - u_i - \alpha_1 v_i - \frac{w_i}{\beta_1 + u_i}\right)dx = 0, \\ \int_\Omega v_i(b - v_i - \alpha_2 u_i)dx = 0, \\ \int_\Omega w_i\left(c - \frac{w_i}{\beta_2 + u_i}\right)dx = 0. \end{gathered} \tag{53}$$

There exists a subsequence of $\{(d_{1i}, d_{2i}, d_{3i})\}_{i=1}^{\infty}$ according to the L^p-regularity theory and Sobolev embedding theorem, but we still use $\{(d_{1i}, d_{2i}, d_{3i})\}_{i=1}^{\infty}$ to represent for convenience. So there must be u^*, v^*, w^* and $(\bar{d}_1, \bar{d}_2, \bar{d}_3)$ as the limiting of (u_i, v_i, w_i) and (d_{1i}, d_{2i}, d_{3i}) when $i \to \infty$. They can be written as follows:

$$\begin{gathered} (u_i, v_i, w_i) \to (u^*, v^*, w^*) \in C^2(\Omega) \times C^2(\Omega) \times C^2(\Omega), \\ (d_{1i}, d_{2i}, d_{3i}) \to (\bar{d}_1, \bar{d}_2, \bar{d}_3) \in [\underline{d}_1,\infty) \times [\underline{d}_2,\infty) \times [\underline{d}_3,\infty). \end{gathered} \tag{54}$$

Let $i \to \infty$, we get that

$$\begin{gathered} \int_\Omega u_*\left(a - u_* - \alpha_1 v_* - \frac{w_*}{\beta_1 + u_*}\right)dx = 0, \\ \int_\Omega v_*(b - v_* - \alpha_2 u_*)dx = 0, \\ \int_\Omega w_*\left(c - \frac{w_*}{\beta_2 + u_*}\right)dx = 0. \end{gathered} \tag{55}$$

We now discuss the following three cases.

Case 1. $u^* \equiv 0$. Since $v_i \to v^*$ as $i \to \infty$, $b - v_i - \alpha_2 u_i > 0, x \in \bar{\Omega}$ holds for every $i \gg 1$, so that

$$\int_\Omega v_i(b - v_i - \alpha_2 u_i)dx > 0, \tag{56}$$

which contradicts with (55).

Case 2. $v^* \equiv 0, u^* \neq 0$. Since $u_i \to u^*$ as $i \to \infty$, $a - u_i - \alpha_1 v_i - w_i/(\beta_1 + u_i) > 0, x \in \bar{\Omega}$ holds for every $i \gg 1$, so that

$$\int_\Omega u_i\left(a - u_i - \alpha_1 v_i - \frac{w_i}{\beta_1 + u_i}\right)dx > 0, \tag{57}$$

which contradicts with (55).

Case 3. $w^* \equiv 0, u^* \neq 0, v^* \neq 0$. Since $w_i \to w^*$ as $i \to \infty$, $c - w_i/(\beta_2 + u_i) > 0, x \in \bar{\Omega}$ holds for every $i \gg 1$, so that

$$\int_\Omega w_i\left(c - \frac{w_i}{\beta_2 + u_i}\right)dx > 0, \tag{58}$$

which contradicts with (55).

The proof is complete.

5. Nonexistence of Nonconstant Positive Steady States

We prove the nonexistence of nonconstant positive steady states of system (37) in this section.

Theorem 7. *Let μ_1 is the smallest positive eigenvalue of operator $-\Delta$ over Ω under the homogeneous Neumann boundary conditions and fixed positive constants d_2^*, d_3^* satisfy $\mu_1 d_2^* > b$ and $\mu_1 d_3^* > c+1$, then there exists a positive constant $D_1 = D_1(\Gamma, d_2^*, d_3^*)$ such that when $d_1 > D_1$, $d_2 \geq d_2^*$ and $d_3 \geq d_3^*$, system (37) has no nonconstant positive steady states.*

Proof. Assume that (u, v, w) is the positive solution of (37). For any $\phi \in L^1(\Omega)$, let $\bar{\phi} = (1/|\Omega|)\int_\Omega \phi dx$. The differential equation (37) multiplies $u - \bar{u}, v - \bar{v}, w - \bar{w}$ and integrates by parts over Ω to get

$$\begin{aligned}
\int_\Omega d_1 |\nabla u|^2 dx
&= \int_\Omega \left[u\left(a - u - \alpha_1 v - \frac{w}{\beta_1 + u}\right) \right.\\
&\quad \left. -u\left(a - u - \alpha_1 v - \frac{w}{\beta_1 + u}\right)\right](u - u)dx\\
&= \int_\Omega \left[a(u-u)^2 - (u+u)(u-u)^2 \right.\\
&\quad - \alpha_1(uv - uv)(u-u)\\
&\quad \left. - \frac{uw(\beta_1 + u)(u-u) - uw(\beta_1+u)(u-u)}{(\beta_1+u)(\beta_1+u)}\right]dx,
\end{aligned} \tag{59}$$

$$\begin{aligned}
\int_\Omega d_2 |\nabla v|^2 dx
&= \int_\Omega [v(b - v - \alpha_2 u) - v(b - v - \alpha_2 u)]\\
&\quad \cdot (v - v)dx\\
&= \int_\Omega \left[b(v-v)^2 - \alpha_2(uv - uv)(v-v)\right.\\
&\quad \left. -(v+v)(v-v)^2\right]dx,
\end{aligned} \tag{60}$$

$$\begin{aligned}
\int_\Omega d_3 |\nabla w|^2 dx
&= \int_\Omega \left[\int_\Omega \left[w\left(c - \frac{w}{\beta_2 + u}\right) - w\left(c - \frac{w}{\beta_2+u}\right)\right]\right.\\
&\quad \cdot (w - w)dx\\
&= \int_\Omega \left[c(w-w)^2\right.\\
&\quad \left. - \frac{w^2(\beta_2+u)(w-w) - w^{\,2}(\beta_2+u)(w-w)}{(\beta_2+u)(\beta_2+u)}\right]dx.
\end{aligned} \tag{61}$$

Combine (59), (60), and (61), we have

$$\begin{aligned}
&\int_\Omega \left(d_1|\nabla u|^2 + d_2|\nabla v|^2 + d_3|\nabla w|^2\right)dx\\
&\leq \int_\Omega \left[a(u-u)^2 + b(v-v)^2 + c(w-w)^2\right.\\
&\quad + (\alpha_1 u + \alpha_2 v)|u-u||v-v|\\
&\quad \left. + \left(1 + \frac{w^{\,2}}{\beta_2^2}\right)|u-u||w-w|\right]dx\\
&\leq \int_\Omega \left[a + \frac{\alpha_1 a + \alpha_2 b}{2\varepsilon_1} + 1 + \frac{c^2(\beta_2+a)^2}{2\varepsilon_2\beta_2^2}\right]\\
&\quad \cdot (u-u)^2 dx\\
&\quad + \int_\Omega \left[b + \frac{\varepsilon_1(\alpha_1 a + \alpha_2 b)}{2}\right](v-v)^2 dx\\
&\quad + \int_\Omega \left[c + 1 + \frac{\varepsilon_2 c^2(\beta_2+a)^2}{2\beta_2^2}\right](w-w)^2 dx,
\end{aligned} \tag{62}$$

where $\varepsilon_1, \varepsilon_2$ are the arbitrary small positive constants arising from Young inequality. Meanwhile, applying the Poincaré inequality $\mu_1 \int_\Omega (f - \bar{f})^2 dx \leq \int_\Omega |\nabla f|^2 dx$, we gain that

$$\begin{aligned}
&\mu_1 \int_\Omega \left[d_1(u-\bar{u})^2 + d_2(v-\bar{v})^2 + d_3(w-\bar{w})^2\right]dx\\
&\leq \int_\Omega [a + 1 + C_1^*(\varepsilon_1, \varepsilon_2)](u-\bar{u})^2 dx\\
&\quad + \int_\Omega \left[b + \frac{\varepsilon_1(\alpha_1 a + \alpha_2 b)}{2}\right](v-\bar{v})^2 dx\\
&\quad + \int_\Omega \left[c + 1 + \frac{\varepsilon_2 c^2(\beta_2+a)^2}{2\beta_2^2}\right](w-\bar{w})^2 dx
\end{aligned} \tag{63}$$

for some positive constants $C_1^*(\varepsilon_1, \varepsilon_2)$. Choose $\varepsilon_1, \varepsilon_2 > 0$ very small such that

$$\mu_1 d_2^* \geq b + \frac{\varepsilon_1(\alpha_1 a + \alpha_2 b)}{2}, \tag{64}$$

$$\mu_1 d_3^* \geq c + 1 + \frac{\varepsilon_2 c^2(\beta_2+a)^2}{2\beta_2^2}. \tag{65}$$

Hence, (65) implies that $v = \bar{v} =$ constant, $w = \bar{w} =$ constant, and $u = \bar{u} =$ constant if $d_1 > D_1 \triangleq \mu_1^{-1}[a + 1 + C_1^*(\varepsilon_1, \varepsilon_2)]$.

The proof is complete.

6. Existence of Nonconstant Positive Steady States

In this part, we discuss the existence of nonconstant positive solutions of (37) by using the degree theorem.

Fix the Γ, d_1, d_3 still as positive number and define $\mathbf{X}^+ = \{U \in \mathbf{X} \mid U > 0, x \in \bar{\Omega}, i = 1, 2, 3\}$, $B(l) = \{U \in \mathbf{X} \mid l^{-1} < u, v, w < l, x \in \bar{\Omega}\}$, $l > 0$. Then, (37) can be noted as

$$\begin{cases} -D\Delta U = G(U), & x \in \Omega, \\ \dfrac{\partial U}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{66}$$

So U is a positive solution to (37) if and only if

$$\mathbf{F}(U) \triangleq U - (\mathbf{I} - \Delta)^{-1}\left\{D^{-1}G(U) + U\right\} = 0, U \in \mathbf{X}^{+}, \tag{67}$$

where $(I - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ in $\mathbf{X}$ under the homogeneous Neumann boundary condition. And if $\mathbf{F}(U) \neq 0$ on ∂B, the Leray-Schauder degree $\deg(\mathbf{F}(\cdot), 0, B)$ can be well defined. Besides, we note that

$$D_U\mathbf{F}(U^*) \triangleq \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\left\{D^{-1}G_U(U^*) + \mathbf{I}\right\}. \tag{68}$$

The index of $\mathbf{F}(U)$ at U^* can be either 1 or -1 if $D_U\mathbf{F}(U^*)$ is invertible, which is defined as $\mathrm{index}(\mathbf{F}(\cdot), U^*) = (-1)^r$, where r is the total number of eigenvalues with negative real parts of $D_U\mathbf{F}(U^*)$.

Let λ be an eigenvalue of $D_U\mathbf{F}(U^*)$ on X_{ij} for each integer $i \geq 1$ and each integer $1 \leq j \leq \dim E(\mu_i)$, if and only if it is an eigenvalue of the matrix

$$\mathbf{I} - \frac{1}{1+\mu_i}\left[D^{-1}G_U(U^*) + \mathbf{I}\right] = \frac{1}{1+\mu_i}\left[\mu_i\mathbf{I} - D^{-1}G_U(U^*)\right]. \tag{69}$$

Hence, $D_U\mathbf{F}(U^*)$ is invertible if and only if, for all $i \geq 1$, $i \in Z$, the matrix $\mathbf{I} - (1/(1+\mu_i))[D^{-1}G_U(U^*) + \mathbf{I}]$ is nonsingular. Let

$$\begin{aligned} H(\mu) = H(U^*; \mu) &= \det\left\{\mu\mathbf{I} - D^{-1}G_U(U^*)\right\} \\ &= \frac{1}{d_1d_2d_3}\det\left\{\mu D - G_U(U^*)\right\}. \end{aligned} \tag{70}$$

We can know that if $H(\mu_i) \neq 0$, the number of negative eigenvalues of $D_U\mathbf{F}(U^*)$ on X_{ij} is odd if and only if $H(\mu_i) < 0$ for every $1 \leq j \leq \dim E(\mu_i)$. According to this, we can form the following result.

Proposition 8. *Assume that the matrix $\mu_i\mathbf{I} - D^{-1}G_U(U^*)$ is nonsingular for all $i \geq 1$, then*

$$\mathit{index}(\mathbf{F}(\cdot), U^*) = (-1)^{\sigma}, \tag{71}$$

where $\sigma = \sum_{i\geq 1, H(\mu_i)<0} \dim E(\mu_i)$.

For calculating the sign of $H(\mu_i)$, we firstly consider the index of $(\mathbf{F}(\cdot), U^*)$. The calculation shows that

$$\begin{aligned} \det\{\mu D - G_U(U^*)\} &= \Phi_3(d_2)\mu^3 + \Phi_2(d_2)\mu^2 \\ &\quad + \Phi_1(d_2)\mu - \det\{G_U(U^*)\} \\ &\triangleq \Phi(d_2; \mu), \end{aligned} \tag{72}$$

with

$$\begin{aligned} \Phi_1(d_2) &= d_3a_{11}a_{22} + d_2a_{11}a_{33} + d_1a_{22}a_{33} - d_3a_{12}a_{21} - d_2a_{13}a_{31}, \\ \Phi_2(d_2) &= -(d_2d_3a_{11} + d_1d_3a_{22} + d_1d_2a_{33}), \\ \Phi_3(d_2) &= d_1d_2d_3, \end{aligned} \tag{73}$$

where a_{ij} are shown as (21).

Consider the dependence of Φ on d_2. Let $\bar{\mu}_1(d_2), \bar{\mu}_2(d_2)$, and $\bar{\mu}_3(d_2)$ be the three roots of $\Phi(d_2; \mu) = 0$, so that $\bar{\mu}_1(d_2)\bar{\mu}_2(d_2)\bar{\mu}_3(d_2) = \det\{G_U(U^*)\}/(\Phi_3(d_2))$. The computation leads to $\det\{G_U(U^*)\} < 0$. Therefore, one of $\bar{\mu}_1(d_2), \bar{\mu}_2(d_2), \bar{\mu}_3(d_2)$ is real and negative, and the product of the other two is positive.

Considering the following limits:

$$\begin{aligned} \lim_{d_2\to\infty}\frac{\Phi_3(d_2)}{d_2} &= d_1d_3, \\ \lim_{d_2\to\infty}\frac{\Phi_2(d_2)}{d_2} &= -(d_1a_{33} + d_3a_{11}), \\ \lim_{d_2\to\infty}\frac{\Phi_1(d_2)}{d_2} &= a_{11}a_{33} - a_{13}a_{31}, \\ \lim_{d_2\to\infty}\frac{\Phi(d_2)}{d_2} &= \mu\left[d_1d_3\mu^2 - (d_1a_{33} + d_3a_{11})\mu + a_{11}a_{33} - a_{13}a_{31}\right]. \end{aligned} \tag{74}$$

We establish the following result.

Proposition 9. *Assume the parameters satisfy (7) or (8) and satisfy $2u^* - a + (\alpha_1 + 1)(b - \alpha_2u^*) + \beta_1cB^2(\beta_2 + u^*) + c < 0$. If $a_{11}a_{33} - a_{13}a_{31} < 0$, there is a positive constant D_2, such that when $d_2 \geq D_2$, the three roots $\bar{\mu}_1(d_2), \bar{\mu}_2(d_2), \bar{\mu}_3(d_2)$ of $\Phi(d_2; \mu) = 0$ are all real and satisfy*

$$\begin{aligned} \lim_{d_2\to\infty}\bar{\mu}_1(d_2) &= \frac{(d_1a_{33} + d_3a_{11}) - \sqrt{(d_1a_{33} + d_3a_{11})^2 + 4d_1d_3(a_{13}a_{31} - a_{11}a_{33})}}{2d_1d_3} < 0, \\ \lim_{d_2\to\infty}\bar{\mu}_2(d_2) &= 0, \\ \lim_{d_2\to\infty}\bar{\mu}_3(d_2) &= \frac{(d_1a_{33} + d_3a_{11}) + \sqrt{(d_1a_{33} + d_3a_{11})^2 + 4d_1d_3(a_{13}a_{31} - a_{11}a_{33})}}{2d_1d_3} \triangleq \tilde{\mu} > 0, \end{aligned} \tag{75}$$

$$-\infty < \bar{\mu}_1(d_2) < 0 < \bar{\mu}_2(d_2) < \bar{\mu}_3(d_2), \tag{76}$$

$$\Phi(d_2 ; \mu) < 0, \mu \in (-\infty, \bar{\mu}_1(d_2)) \cup (\bar{\mu}_2(d_2), \bar{\mu}_3(d_2)), \tag{77}$$

$$\Phi(d_2 ; \mu) > 0, \mu \in (\bar{\mu}_1(d_2), \bar{\mu}_2(d_2)) \cup (\bar{\mu}_3(d_2), +\infty). \tag{78}$$

Now, we prove the existence of nonconstant positive solutions of (37) when d_2 is sufficiently large.

Theorem 10. *Let the parameters d_1, d_3 are fixed, Γ satisfies (7) or (8), and satisfies $2u^* - a + (\alpha_1 + 1)(b - \alpha_2 u^*) + \beta_1 c B^2(\beta_2 + u^*) + c < 0$. If $a_{11}a_{33} - a_{13}a_{31} < 0$, $\tilde{\mu} \in (\mu_n, \mu_{n+1})$ for some $n \geq 1$, and the sum $\sigma_n = \sum_{i=1}^{n} \dim E(\mu_i)$ is odd. Then, D_2 must be as a positive constant such that (37) has one nonconstant positive solution at least if $d_2 \geq D_2$.*

Proof. There exists a positive constant D_2 by Proposition 9, such that for $d_2 \geq D_2$, (76), (77), and (78) hold and

$$0 = \mu_0 < \bar{\mu}_2(d_2) < \mu_1, \bar{\mu}_3(d_2) \in (\mu_n, \mu_{n+1}). \tag{79}$$

We will testify that for any $d_2 \geq D_2$, system (37) has at least one nonconstant positive solution and the proof is proved by contradiction. Assume on the contrary that the statement is not true for some $\bar{d}_2 \geq D_2$. Afterwards, we fix $d_2 = \bar{d}_2$, $d_1^* = C_1^*/\mu_{1i}$, $d_2^* = C_2^*/\mu_{1i}$, $d_3^* = C_3^*/\mu_{1i}$, and

$$\begin{aligned} \hat{d}_1 &\geq \max\{d_1^*, d_1\}, \\ \hat{d}_2 &\geq d_2^*, \\ \hat{d}_3 &\geq \max\{d_3^*, d_3\}. \end{aligned} \tag{80}$$

As for $t \in [0,1]$, make $D(t) = \operatorname{diag}(d_1(t), d_2(t), d_3(t))$ with $d_i(t) = t d_i + (1-t)\hat{d}_i$, $i = 1, 2, 3$ and think about the problem

$$\begin{cases} -D(t)\Delta U = G(U), & x \in \Omega, \\ \dfrac{\partial U}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{81}$$

U is a nonconstant positive solution of (37) if and only if it is a positive solution of (81) when $t = 1$. Obviously for any $0 \leq t \leq 1$, U^* is the unique constant positive solution of (81). U is a positive solution of (81) if and only if

$$\mathbf{F}(t ; U) \triangleq U - (\mathbf{I} - \Delta)^{-1}\{D^{-1}(t)G(U) + U\} = 0, \quad U \in \mathbf{X}^+. \tag{82}$$

It is evident that $\mathbf{F}(1 ; U) = \mathbf{F}(U)$. $\mathbf{F}(0 ; U) = 0$ has been shown in Theorem 7, which has only positive solution U^* in $\mathbf{X}^+$. After computing, we get that

$$D_U\mathbf{F}(t ; U^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{D^{-1}(t)G_U(U^*) + \mathbf{I}\}. \tag{83}$$

Specifically,

$$\begin{aligned} D_U\mathbf{F}(0 ; U^*) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{D\wedge^{-1}G_U(U^*) + \mathbf{I}\}, \\ D_U\mathbf{F}(1 ; U^*) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{D^{-1}G_U(U^*) + \mathbf{I}\} = D_U\mathbf{F}(U^*), \end{aligned} \tag{84}$$

where $\hat{D} = \operatorname{diag}(\hat{d}_1, \hat{d}_2, \hat{d}_3)$. From (70) and (72), we know that

$$H(\mu) = \frac{1}{d_1 d_2 d_3}\Phi(d_2 ; \mu). \tag{85}$$

In view of (76) - (79), and (85), it follows that

$$\begin{aligned} &H(\mu_0) = H(0) > 0, \\ &H(\mu_i) < 0, \quad 1 \leq i \leq n, \\ &H(\mu_i) > 0, \quad i \geq n+1. \end{aligned} \tag{86}$$

Thus, 0 is not an eigenvalue of the matrix $\mu_i\mathbf{I} - D^{-1}\mathbf{G}_U(U^*)$ for any $i \geq 1$, and

$$\sum_{i \geq 0, H(\mu_i) < 0} \dim E(\mu_i) = \sum_{i=1}^{n} \dim E(\mu_i), \tag{87}$$

which is odd. Because of Proposition 8, it can be true that

$$\operatorname{index}(\mathbf{F}(1 ; \cdot), U^*) = (-1)^r = (-1)^{\sigma_n} = -1. \tag{88}$$

The same method is available to index $((\mathbf{F}(0 ; \cdot), U^*)) = (-1)^0 = 1$.

According to Theorems 5 and 6, we can find a positive constant C, such that the positive solutions of (81) can meet the demand $C^{-1} < u, v, w < C$ for all $\forall 0 \leq t \leq 1$. So, $F(t ; U) \neq 0$ on $\partial B(C)$. By using the homotopy invariance of the topological degree, it is clear that

$$\deg(\mathbf{F}(1 ; \cdot), 0, B(C)) = \deg(\mathbf{F}(0 ; \cdot), 0, B(C)). \tag{89}$$

Moreover, by our assumption, both equations $\mathbf{F}(1 ; \cdot) = 0$ and $\mathbf{F}(0 ; \cdot) = 0$ have only the positive solution U^* in $B(C)$, so

$$\begin{aligned} &\deg(\mathbf{F}(0 ; \cdot), 0, B(C) = \operatorname{index}(\mathbf{F}(0 ; \cdot), U^*) = 1, \\ &\deg(\mathbf{F}(1 ; \cdot), 0, B(C) = \operatorname{index}(\mathbf{F}(1 ; \cdot), U^*) = -1, \end{aligned} \tag{90}$$

which is contradictory with (89).
The proof is complete.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 61872227 and 11771259).

References

[1] A. Korobeinikov, “A Lyapunov function for Leslie-Gower predator-prey models,” *Applied Mathematics Letters*, vol. 14, no. 6, pp. 697–699, 2001.

[2] M. A. Aziz-Alaoui, “Study of a Leslie-Gower-type tritrophic population model,” *Chaos, Solitons and Fractals*, vol. 14, no. 8, pp. 1275–1293, 2002.

[3] Y. Li and D. Xiao, “Bifurcations of a predator-prey system of Holling and Leslie types,” *Chaos, Solitons and Fractals*, vol. 34, no. 2, pp. 606–620, 2007.

[4] M. A. Aziz-Alaoui and M. D. Okiye, “Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes,” *Applied Mathematics Letters*, vol. 16, no. 7, pp. 1069–1075, 2003.

[5] A. Singh and S. Gakkhar, “Stabilization of modified Leslie-Gower prey-predator model,” *Differential Equations and Dynamical Systems*, vol. 22, no. 3, pp. 239–249, 2014.

[6] A. F. Nindjin, M. A. Aziz-Alaoui, and M. Cadivel, “Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay,” *Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 1104–1118, 2006.

[7] R. Yafia, F. E. Adnani, and H. T. Alaoui, “Limit cycle and numerical similations for small and large delays in a predator-prey model with modified Leslie-Gower and Holling-type II schemes,” *Nonlinear Analysis: Real World Applications*, vol. 9, no. 5, pp. 2055–2067, 2008.

[8] H. Guo and X. Song, “An impulsive predator-prey system with modified Leslie-Gower and Holling type II schemes,” *Chaos, Solitons and Fractals*, vol. 36, no. 5, pp. 1320–1331, 2008.

[9] W. Yang and Y. Li, “Dynamics of a diffusive predator-prey model with modified Leslie-Gower and Holling-type III schemes,” *Computers and Mathematics with Applications*, vol. 65, no. 11, pp. 1727–1737, 2013.

[10] L. Yang and S. Zhong, “Dynamics of a diffusive predator-prey model with modified Leslie-Gower schemes and additive Allee effect,” *Computational and Applied Mathematics*, vol. 34, no. 2, pp. 671–690, 2015.

[11] X. Li, Y. Cai, K. Wang, S. Fu, and W. Wang, “Non-constant positive steady states of a host-parasite model with frequency- and density-dependent transmissions,” *Journal of the Franklin Institute*, vol. 357, no. 7, pp. 4392–4413, 2020.

[12] D. Tong, Y. Cai, B. Wang, and W. Wang, “Bifurcation structure of nonconstant positive steady states for a diffusive predator-prey model,” *Mathematical Biosciences and Engineering*, vol. 16, no. 5, pp. 3988–4006, 2019.

[13] H. Shi and S. Ruan, “Spatial, temporal and spatiotemporal patterns of diffusive predator-prey models with mutual interference,” *IMA Journal of Applied Mathematics*, vol. 80, no. 5, pp. 1534–1568, 2015.

[14] Z. Du, X. Zhang, and H. Zhu, “Dynamics of nonconstant steady states of the Sel’kov model with saturation effect,” *Journal of Nonlinear Science*, vol. 30, no. 4, pp. 1553–1577, 2020.

[15] L. Hei and Y. Yu, “Non-constant positive steady state of one resource and two consumers model with diffusion,” *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 566–581, 2008.

[16] W. Ni and M. Wang, “Dynamics and patterns of a diffusive Leslie-Gower prey-predator model with strong Allee effect in prey,” *Journal of Differential Equations*, vol. 261, no. 7, pp. 4244–4274, 2016.

[17] H. Yin, J. Zhou, X. Xiao, and X. Wen, “Analysis of a diffusive Leslie-Gower predator-prey model with nonmonotonic functional response,” *Chaos, Solitons and Fractals*, vol. 65, pp. 51–61, 2014.

[18] P. Y. H. Pang and M. Wang, “Non-constant positive steady states of a predator-prey system with non-monotonic functional response and diffusion,” *Proceedings of the London Mathematical Society*, vol. 88, no. 1, pp. 135–157, 2004.

[19] W. Ko and K. Ryu, “Non-constant positive steady-states of a diffusive predator-prey system in homogeneous environment,” *Journal of Mathematical Analysis and Applications*, vol. 327, no. 1, pp. 539–549, 2007.

[20] X. Zhang and H. Zhao, “Dynamics and pattern formation of a diffusive predator-prey model in the presence of toxicity,” *Nonlinear Dynamics*, vol. 95, no. 3, pp. 2163–2179, 2019.

[21] Z. Shen and J. Wei, “Stationary pattern of a reaction-diffusion mussel-algae model,” *Bulletin of Mathematical Biology*, vol. 82, no. 4, pp. 1–31, 2020.

[22] J. Wang, H. Zheng, and Y. Jia, “Existence and bifurcation of non-constant positive steady states for a tumor-immune model,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 71, no. 5, p. 170, 2020.

[23] G. Hu and X. Li, “Turing patterns of a predator-prey model with a modified Leslie-Gower term and cross-diffusions,” *International Journal of Biomathematics*, vol. 5, no. 6, article 1250060, 2012.

[24] H. Qiu and S. Guo, “Steady-states of a Leslie-Gower model with diffusion and advection,” *Applied Mathematics and Computation*, vol. 346, no. 1, pp. 695–709, 2019.

[25] W. Liu, “Criterion of Hopf bifurcations without using eigenvalues,” *Journal of Mathematical Analysis and Applications*, vol. 182, no. 1, pp. 250–256, 1994.

[26] C. Lin, W. Ni, and I. Takagi, “Large amplitude stationary solutions to a chemotaxis system,” *Journal of Differential Equations*, vol. 72, no. 1, pp. 1–27, 1988.

[27] Y. Lou and W. M. Ni, “Diffusion, self-diffusion and cross-diffusion,” *Journal of Differential Equations*, vol. 131, no. 1, pp. 79–131, 1996.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9918955, 9 pages
https://doi.org/10.1155/2021/9918955

Research Article

Fast High-Order Difference Scheme for the Modified Anomalous Subdiffusion Equation Based on Fast Discrete Sine Transform

Lijuan Nong and An Chen

College of Science, Guilin University of Technology, Guilin 541004, China

Correspondence should be addressed to An Chen; chena@glut.edu.cn

Received 12 March 2021; Revised 11 April 2021; Accepted 15 April 2021; Published 27 April 2021

Academic Editor: Qifeng Zhang

The modified anomalous subdiffusion equation plays an important role in the modeling of the processes that become less anomalous as time evolves. In this paper, we consider the efficient difference scheme for solving such time-fractional equation in two space dimensions. By using the modified L1 method and the compact difference operator with fast discrete sine transform technique, we develop a fast Crank-Nicolson compact difference scheme which is proved to be stable with the accuracy of $O(\tau^{\min(1+\alpha,1+\beta)} + \mathbf{h}^4)$. Here, α and β are the fractional orders which both range from 0 to 1, and τ and $\mathbf{h}$ are, respectively, the temporal and spatial stepsizes. We also consider the method of adding correction terms to efficiently deal with the nonsmooth problems. Numerical examples are provided to verify the effectiveness of the proposed scheme.

1. Introduction

In this paper, we focus on the numerical method for the time-fractional modified subdiffusion equation [1]:

$$\partial_t u(\mathbf{x}, t) = \left(\kappa_{1RL} D_{0,t}^{1-\alpha} + \kappa_{2RL} D_{0,t}^{1-\beta}\right) \Delta u + f(\mathbf{x}, t), (\mathbf{x}, t) \in \Omega \times (0, T], \quad (1)$$

with the initial condition $u(\mathbf{x}, 0) = v(\mathbf{x})$ and the homogeneous Dirichlet boundary condition. Here, $\mathbf{x} = (x_1, x_2)$, Ω is the rectangle domain, $T > 0$, and Δ is the Laplacian defined by $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. The parameters κ_1 and κ_2 are some fixed positive constants, f and v are two given functions, $0 < \alpha, \beta < 1$, and ${}_{RL}D_{0,t}^{\gamma}$ is the Riemann-Liouville derivative of order γ given by:

$${}_{RL}D_{0,t}^{\gamma} u(\cdot, t) = \frac{1}{\Gamma(n-\gamma)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\gamma-1} u(\cdot, s) ds, n-1 < \gamma < n, n \in \mathbb{N}, \quad (2)$$

where $\Gamma(\cdot)$ is the Gamma function.

Anomalous diffusion is ubiquitous in nature and it can be characterized by the method of mean square particle displacement at the microscopic level. When the mean square displacement (MSD) is linear with time, the particle is precisely in classical Brownian motion. If the MSD grows either sublinearly or superlinearly with time, then this phenomenon is regarded as the subdiffusion and superdiffusion, respectively. Numerous experimental studies have shown that the anomalous diffusion may adequately describe a number of physical processes in recent decades [2, 3]. Equation (1) is an important class of anomalous diffusion models in which the physical processes are observed to become less anomalous as time evolves [4].

In [4], the author presented the solution of a one-dimensional modified anomalous subdiffusion equation on an infinite domain. The analytical solution the author obtained contains the infinite series of Fox special functions, which is of complex form that makes it difficult to apply to practical numerical simulations. So, one needs to resort to the numerical methods for efficiently solving equation (1). Many efficient numerical methods for solving fractional models have emerged in recent years, see the book [5] and the two review papers [2, 6]. For equation (1), some

Table 1: The L^2-norm errors in time for nonsmooth case in Example 1 with $h = 1/64$.

(α, β)	n_T	$m = 0$		$m = 1$		$m = 3$	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
(0.3,0.8)	80	3.58E-03	—	1.71E-03	—	3.53E-04	—
	160	2.24E-03	0.67	1.03E-03	0.73	2.07E-04	0.77
	320	1.42E-03	0.66	6.28E-04	0.72	1.22E-04	0.76
	640	9.06E-04	0.65	3.83E-04	0.71	7.31E-05	0.74
(0.5,0.6)	80	1.45E-03	—	7.86E-04	—	2.93E-04	—
	160	8.52E-04	0.76	4.11E-04	0.94	1.54E-04	0.93
	320	5.24E-04	0.70	2.15E-04	0.93	8.05E-05	0.93
	640	3.36E-04	0.64	1.13E-04	0.93	4.21E-05	0.94
(0.7,0.4)	80	2.04E-03	—	1.05E-03	—	3.14E-04	—
	160	1.22E-03	0.73	5.86E-04	0.85	1.72E-04	0.87
	320	7.54E-04	0.70	3.28E-04	0.84	9.43E-05	0.87
	640	4.76E-04	0.66	1.84E-04	0.83	5.20E-05	0.86

numerical schemes have been developed. Ding and Li applied two kinds of high-order compact finite difference methods to construct efficient numerical schemes. The stability and convergence analysis are proved by the Fourier method [7]. In [8], the authors developed the compact difference scheme based on the second-order compact approximation formula of the first-order derivative. The two papers mentioned above both focus on the one-dimensional case.

For the two-dimensional case, Chen and Li employed the modified L1 method and compact difference method and proposed a compact alternating direction implicit scheme. By utilizing the energy method, they proved that their scheme is stable with an accuracy of $O(\tau^{2\min(\alpha,\beta)} + \mathbf{h}^4)$ in the new defined norm which is equivalent with H^1-norm, under the assumption that the solution is sufficiently smooth [1]. Such assumption may be too restrictive to limit the scope of application of their scheme. To address this issue, Chen proposed two robust fully discrete finite element methods by convolution quadrature in time. He proved that the schemes are convergent under data regularity without relying on the assumption of the solution regularity. In addition, he also proposed a Crank-Nicolson finite element scheme to numerically compare and verify the robustness of the convolution-based schemes in solving nonsmooth solution problems, but no detailed theoretical analysis of the scheme was given [9]. It seems that the numerical methods for equation (1) have not been sufficiently studied. This motivates us to design efficient numerical schemes for (1), especially for high-dimensional problems where the solutions are not sufficiently smooth.

As the further work on the high-dimensional equation (1), we focus on designing numerical schemes that are computationally efficient and can handle the nonsmooth solution case. In [10], Li et al. implemented the fourth-order compact difference operator by a fast discrete sine transform (DST) via the FFT algorithm, which greatly reduces the computational cost and storage requirement. Notice that the DST technology can avoid solving matrix inversion directly and has been successfully applied in the discretization of classical models, such as Poisson equation [11] and general order semilinear evolution equations [12], just to name a few. On the other hand, the weak singularity of the fractional model has gradually attracted the attention of scholars in the fractional community, and some kinds of methods have been proposed to resolve this issue, such as nonuniform meshes [13–16] and convolution quadrature [9, 17, 18]. The method of adding correction terms is also an efficient way of dealing with nonsmooth solutions problems. However, this method is generally not very stable as the starting weights need to be obtained through a linear system which involves the ill-conditioned exponential Vandermonde matrix. To resolve this issue, Zeng et al. theoretically and numerically shown that the accuracy of numerical solution can be efficiently improved with only a few correction terms [19]. Since then, a variety of numerical schemes based on the addition of correction terms have emerged for fractional problems with nonsmooth solutions, see [3, 20, 21]. To the best of our knowledge, it seems that the method of adding correction terms with DST for solving equation (1) has not been considered in the existing literatures yet.

The contributions of this paper are as follows. First, we apply the modified L1 method to discrete the Riemann-Liouville derivative and compact difference operator with DST to discrete the Laplacian and then naturally obtain the fast Crank-Nicolson compact difference scheme for the two-dimensional problem (1), see (7). Second, the stability and error estimate are rigorously proved by the energy method, see Theorems 2 and 3. Specially, we improve the convergence results in [1] and guarantee computational efficiency but without sacrificing the accuracy of the scheme. Note that the small term added during the construction of the ADI scheme in [1] destroys the accuracy of their original scheme. Third, by using the method of adding correction terms, we successfully improve the accuracy of the proposed scheme in solving the nonsmooth problem with no impact on the stability of the original scheme, see (9). Finally, we provide concrete numerical tests to show the effectiveness of the scheme in solving the high-dimensional problem with nonsmooth solution, see Table 1 and Figures 1–3.

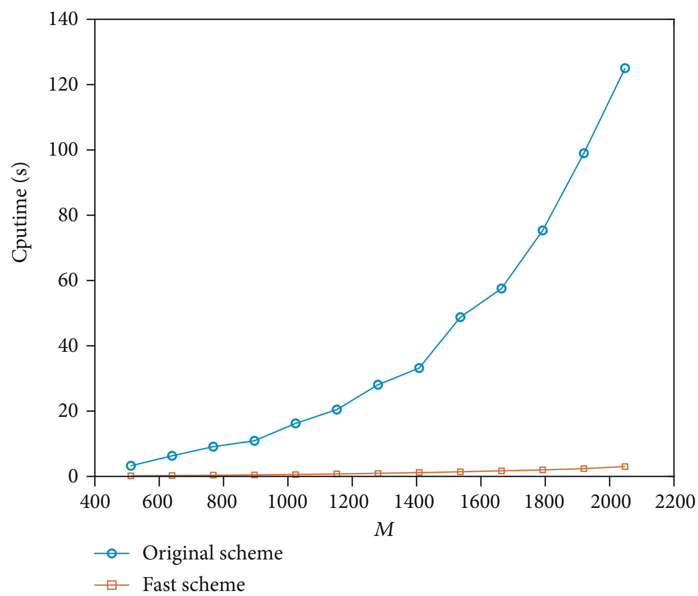


Figure 1: Comparison of CPU execution time between original and fast schemes with fixed $\alpha = 0.3$ and $\beta = 0.8$.

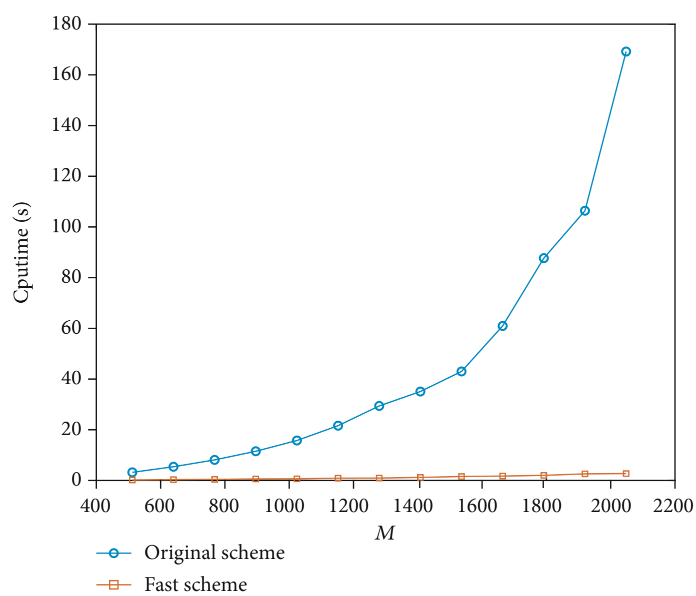


Figure 2: Comparison of CPU execution time between original and fast schemes with fixed $\alpha = 0.5$ and $\beta = 0.6$

The rest of the paper is organized as follows. In Section 2, we derive the fast Crank-Nicolson compact difference scheme by the modified L1 method and the compact difference operator with DST. In Section 3, we prove that the proposed numerical scheme is stable with an accuracy of $O(\tau^{\min(1+\alpha,1+\beta)} + \mathbf{h}^4)$ under the assumption that the solution is sufficiently smooth. To weaken the assumption and make the scheme more robust in solving nonsmooth solution problems, we present the improved version in Section 4 with the method of adding correction terms. Numerical examples are given in Section 5 to confirm the effectiveness of the proposed scheme. Finally, we present the conclusions of this paper in Section 6.

Throughout this paper, we shall let the symbol c (with or without subscript) be a positive constant which may vary at different locations but is always independent of the temporal and spatial stepsizes.

2. The Compact Difference Scheme with Fast Solver

In this section, we derive the fast compact difference scheme for (1).

We first introduce the temporal discretization. The time stepsize τ is given by $\tau = T/n_T$ with the positive integer n_T. The grid point is denoted by $t_n = n\tau$ for $n \geq 0$. Let $t_{n+1/2} =$

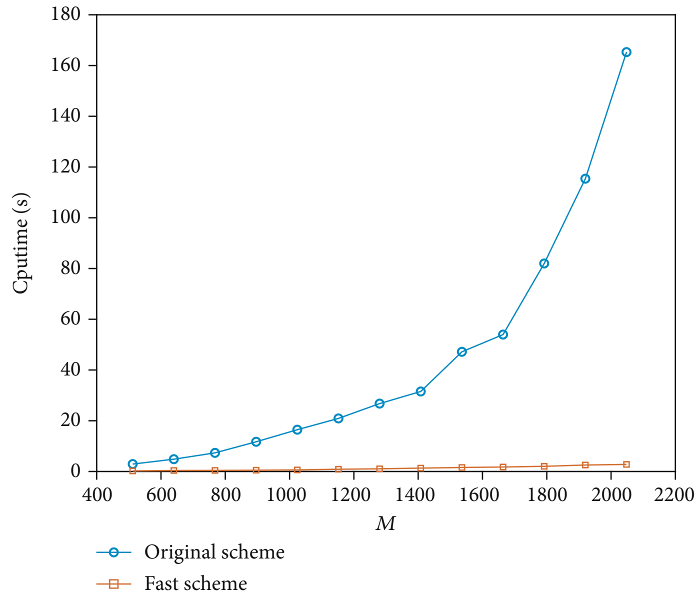


Figure 3: Comparison of CPU execution time between original and fast schemes with fixed $\alpha = 0.7$ and $\beta = 0.4$

$(t_n + t_{n+1})/2$. For $g(t) \in C^2[0, T]$, the modified L1 method for the approximation of Riemann-Liouville derivative ${}_{RL}D_{0,t}^{\gamma} g(t)$ with $\gamma \in (0, 1)$ at $t = t_{n+1/2}$ is described as:

$$ {}_{RL}D_{0,t}^{\gamma} g\left(t_{n+\frac{1}{2}}\right) = \bar{D}_{\tau}^{\gamma} g^{n+1/2} + R^{n+1/2}, \quad (3) $$

where $|R^{n+1/2}| \leq c\tau^{2-\gamma} \max_{t\in[0,T]} |g''(t)|$ ([22], Lemma 1). The operator $\bar{D}_{\tau}^{\gamma}$ in (2) is given by

$$ \begin{aligned} \bar{D}_{\tau}^{\gamma} g^{n+1/2} = {} & b_0^{(\gamma)} g\left(t_{n+\frac{1}{2}}\right) - \sum_{k=1}^{n} \left(b_{n-k}^{(\gamma)} - b_{n-k+1}^{(\gamma)}\right) g\left(t_{k-\frac{1}{2}}\right) \\ & - \left(b_n^{(\gamma)} - B_n^{(\gamma)}\right) g\left(t_{\frac{1}{2}}\right) - A_n^{(\gamma)} g(t_0), \end{aligned} \quad (4) $$

where

$$ \begin{cases} b_k^{(\gamma)} = \dfrac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left((k+1)^{1-\gamma} - k^{1-\gamma}\right), \\ B_k^{(\gamma)} = \dfrac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left(\left(k + \dfrac{1}{2}\right)^{1-\gamma} - k^{1-\gamma}\right), \end{cases} \quad (5) $$
$$ A_n^{(\gamma)} = B_n^{(\gamma)} - \frac{(1-\gamma)\tau^{-\gamma}}{\Gamma(2-\gamma)} \left(n + \frac{1}{2}\right)^{-\gamma}. $$

So, applying the difference discretization $\partial_t u(x, t_{n+1/2}) = \delta_t u^{n+1/2} + O(\tau^2)$ with $\delta_t u^{n+1/2} = (u^{n+1} - u^n)/\tau$, we derive that

$$ \delta_t u^{n+1/2} = \left(\kappa_1 \bar{D}_{\tau}^{1-\alpha} + \kappa_2 \bar{D}_{\tau}^{1-\beta}\right) \Delta u^{n+1/2} + f^{n+1/2} + R_{\mathbf{x}}^{n+1/2}, \quad (6) $$

where the local truncation error $R_{\mathbf{x}}^{n+1/2} = O(\tau^{\min(1+\alpha, 1+\beta)})$ and $u^{n+1/2} = (u^{n+1} + u^n)/2$.

Next, we consider the spatial discretization for (4). In order to make our discussion more general, we follow the notations presented in [10] and always set the symbol $d = 2$ unless otherwise noted. Denote the domain $\Omega = [x_1^L, x_1^R] \times [x_2^L, x_2^R] \times \cdots \times [x_d^L, x_d^R]$. Let $M_k (1 \leq k \leq d)$ be a positive integer. The spatial stepsize is then denoted as $h_k = (x_k^R - x_k^L)/M_k$ and $x_{k,j_k} = x_k^L + j_k h_k$ for $j_k = 0, 1, \cdots, M_k$. The fully discrete grids in space are denoted by $\bar{\Omega}_h = \{x_h = (x_{1,j_1}, x_{2,j_2}, \cdots, x_{d,j_d}) \mid 0 \leq j_k \leq M_k, 1 \leq k \leq d\}$. We further denote that $\Omega_h = \bar{\Omega}_h \cap \Omega$ and the boundary $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$. The space of grid function is denoted as $V_h = \{v \mid v = (v_h)_{\mathbf{x}_h} \text{ and } v_h = 0 \text{ for } \mathbf{x}_h \in \partial\Omega_h\}$. For the grid function $v_h = v(\mathbf{x}_h)$ with the index vector $h = (i_1, i_2, \cdots, i_d)$ at kth position, we denote the compact difference operator as $\bar{\Delta}_k v_{i_k} = \delta_k^2 / \mathscr{H}_k v_{i_k}$, with the difference operator $\mathscr{H}_k v_{i_k} := (I + h_k^2/12\delta_k^2) v_{i_k}$. Here, I is the identity operator, $\delta_k^2 v_{i_k} = (\delta_k v_{i_k+1/2} - \delta_k v_{i_k-1/2})/h_k$ and $\delta_k v_{i_k+1/2} := (v_{i_k+1} - v_{i_k})/h_k$. So, the fourth-order spatial approximation of $\Delta v(\mathbf{x}_h)$ for $\mathbf{x}_h \in \Omega_h$ is given by $\bar{\Delta}_h v_h := \sum_k \bar{\Delta}_k v_h$.

Combining the compact difference approximation in space with (4), we have

$$ \delta_t u(\mathbf{x}_h)^{n+1/2} = \left(\kappa_1 \bar{D}_{\tau}^{1-\alpha} + \kappa_2 \bar{D}_{\tau}^{1-\beta}\right) \bar{\Delta}_h u(\mathbf{x}_h)^{n+1/2} + f^{n+1/2} + R_{\mathbf{x}t}^{n+1/2}, \quad (7) $$

where the local truncation error is given by $R_{\mathbf{x}t}^{n+1/2} = O(\tau^{\min(1+\alpha, 1+\beta)} + \mathbf{h}^4)$. Here, $\mathbf{h}^4 = h_1^4 + h_2^4 + \cdots + h_d^4$. Omitting the small term $R_{\mathbf{x}t}^{n+1/2}$, we obtain the following Crank-Nicolson compact difference scheme for (1): find u_h^n of $u(\mathbf{x}_h, t_n)$ for $n \geq 1$, such that

$$ \delta_t u_h^{n+1/2} = \left(\kappa_1 \bar{D}_{\tau}^{1-\alpha} + \kappa_2 \bar{D}_{\tau}^{1-\beta}\right) \bar{\Delta}_h u_h^{n+1/2} + f^{n+1/2}, \quad (8) $$

where $u_h^0 = v(\mathbf{x}_h)$ and $u(\mathbf{x}_h)|_{\mathbf{x}_h \in \partial\Omega_h} = 0$.

If we solve the discretized system (6) directly, the computational cost will be $O((M_1M_2\cdots M_d)^2)$ on each time level due to the calculation of matrix inversion. Next, we employ the fast discrete sine transform based on FFT to reduce the computational cost to $O((M_1M_2\cdots M_d)\log(M_1M_2\cdots M_d))$ [11], which greatly improves the computational performance. Since the discrete sine transform of the grid function v_h at the kth position is provided by $v_{i_k}=\sum_{j_k=1}^{M_k-1}\widehat{v_{j_k}}\sin(i_kj_k\pi/M_k)$, it follows from the definition of the compact difference operator $\bar{\Delta}_k$ that

$$\widehat{v'_{j_k}}\approx\widehat{v_{j_k}}\frac{12}{h_k^2}\cdot\frac{s_{j_k}-1}{s_{j_k}+5}=\widehat{v_{j_k}}\lambda^{(j_k,M_k)},\tag{9}$$

where $s_{j_k}=\cos(j_k\pi/M_k)$ and $1\le j_k\le M_k-1$. One can refer to [11] or [10] for the derivation. Denote the index set $v=\{(j_1,j_2,\cdots,j_d)\mid 1\le j_k\le M_k-1, 1\le k\le d\}$. Therefore, the scheme (6) is equivalent to

$$\delta_t\widehat{u}_v^{n+1/2}=\left(\kappa_1\bar{D}_\tau^{1-\alpha}+\kappa_2\bar{D}_\tau^{1-\beta}\right)\left(\sum_{k=1}^{d}\lambda^{(j_k,M_k)}\right)\widehat{u}_v^{n+1/2}+f\wedge^{n+1/2}.\tag{10}$$

The computational procedure is described as follows:

(a) For $n\ge0$, we first computed $\widehat{u}_v^n$ and $f\wedge^{n+1/2}$ from u_h^n and $f^{n+1/2}$ by means of DST

(b) And then we solve equation (10) from which the numerical solution u_h^n is obtained from $\widehat{u}_v^n$ by the inverse of DST.

3. Stability and Error Estimates

In this part, we demonstrate the stability and error estimates for the compact difference scheme (6).

We first introduce some useful notations. For any grid function $v\in V_h$, the discrete L^2-norm is given by $\|v\|=\sqrt{(v,v)_h}$ with the discrete inner product $(u,v)_h=(\prod_{k=1}^d h_k)\sum_{\mathbf{x}_h\in\Omega_h}u_hv_h$. The discrete H^1 seminorm and H^1 norm are denoted as $|v|_1=\sqrt{\|\nabla_hv_h\|^2}=\sqrt{\sum_{k=1}^d\|\delta_kv_h\|^2}$ and $\|v\|_1=\sqrt{\|v\|^2+|v|_1^2}$. Here, $\nabla_h=(\delta_1,\delta_2,\cdots,\delta_d)$. One can readily have the equivalence of $|v|_1$ and $\|v\|_1$ for any $v\in V_h$ in view of the embedding theorem.

We shall first need the following lemma.

Lemma 1. *The operator $\bar{D}_\tau^\alpha$ given by (3) satisfies the inequality:*

$$-2\left(\bar{D}_\tau^\alpha v^{n+1/2},v^{n+1/2}\right)_h\le\sum_{k=1}^{n}b_{n-k}^{(\gamma)}\left\|v^{k-1/2}\right\|^2-\sum_{k=1}^{n+1}b_{n+1-k}^{(\gamma)}\left\|v^{k-1/2}\right\|^2+A_n^{(\gamma)}\left\|v^0\right\|^2,\tag{11}$$

where $v^n\in V_h$, $n\ge0$.

Proof. The proof of the lemma can be obtained in view of Lemma 4.2 in [22] or Lemma 4.4 in [1], thus, the details are omitted here.

We are ready to present the stability of the scheme (6).

Theorem 2. *The Crank-Nicolson compact difference scheme (6) is stable in the sense that*

$$\left\|u_h^{n+1}\right\|^2\le c\left(\left\|u_h^0\right\|^2+\left(c_1\tau^\alpha+c_2\tau^\beta\right)\left\|\nabla_hu_h^0\right\|^2+\tau\sum_{k=1}^{n+1}\left\|f^{k-1/2}\right\|^2\right).\tag{12}$$

Proof. By taking the discrete inner product on both sides of (6) with $2\tau u_h^{n+1/2}$, we get

$$2\tau\left(\delta_tu_h^{n+1/2},u_h^{n+1/2}\right)_h=2\tau\left(\left(\kappa_1\bar{D}_\tau^{1-\alpha}+\kappa_2\bar{D}_\tau^{1-\beta}\right)\bar{\Delta}_hu_h^{n+1/2},u_h^{n+1/2}\right)_h+2\tau\left(f^{n+1/2},u_h^{n+1/2}\right)_h.\tag{13}$$

Notice that the difference operator $\bar{\Delta}_h$ is bounded in discrete inner product ([10], Theorem 2):

$$\frac{3}{2}\left(\Delta_hu_h^{n+1/2},u_h^{n+1/2}\right)_h<\left(\bar{\Delta}_hu_h^{n+1/2},u_h^{n+1/2}\right)_h<\left(\Delta_hu_h^{n+1/2},u_h^{n+1/2}\right)_h,\tag{14}$$

with the notation $\Delta_hu_h^n=\sum_{k=1}^d\delta_k^2u_h^n$. By the identity $(\Delta_hu_h^{n+1/2},u_h^{n+1/2})_h=-(\nabla_hu_h^{n+1/2},\nabla_hu_h^{n+1/2})_h$, the Lemma 1 yields

$$\left\|u_h^{n+1}\right\|^2-\left\|u_h^n\right\|^2\le\tau\left(\sum_{k=1}^{n}b_{n-k}\left\|\nabla_hu_h^{k-1/2}\right\|^2-\sum_{k=1}^{n+1}b_{n+1-k}\left\|\nabla_hu_h^{k-1/2}\right\|^2+A_n\left\|\nabla_hu_h^0\right\|^2\right)+2\tau\left(f^{n+1/2},u_h^{n+1/2}\right)_h,\tag{15}$$

where $b_k=b_k^{(1-\alpha)}+b_k^{(1-\beta)}$, $B_k=B_k^{(1-\alpha)}+B_k^{(1-\beta)}$, and $A_n=A_n^{(1-\alpha)}+A_n^{(1-\beta)}$. With $G^n=\|u_h^n\|^2+\tau\sum_{k=1}^nb_{n-k}\|\nabla_hu_h^{k-1/2}\|^2$, we write the above inequality as:

$$G^{n+1}\le G^n+\tau A_n\left\|\nabla_hu_h^0\right\|^2+2\tau\left(f^{n+1/2},u_h^{n+1/2}\right)_h.\tag{16}$$

We sum up n from 1 to m and replace m with n to get

$$G^{n+1}\le G^1+\tau\sum_{k=1}^{n}A_k\left\|\nabla_hu_h^0\right\|^2+2\tau\sum_{k=1}^{n}\left(f^{k+1/2},u_h^{k+1/2}\right)_h.\tag{17}$$

By the Cauchy-Schwarz inequality and the inequality $\|v_h\|\le c\|v_h\|_1$ with the equivalence of $\|v_h\|_1$ and $\|\nabla_hv_h\|$, we obtain the estimate:

$$2\tau\sum_{k=1}^{n}\left(f^{k+1/2},u_h^{k+1/2}\right)_h \le \tau\sum_{k=1}^{n+1} b_{n+1-k}\left\|\nabla_h u_h^{k-1/2}\right\|^2 + \tau\sum_{k=2}^{n+1}\frac{1}{b_{n+1-k}}\left\|f^{k-1/2}\right\|^2, \tag{18}$$

from which we derive that

$$\|u_h^{n+1}\|^2 \le G^1 + \tau\sum_{k=1}^{n} A_k\left\|\nabla u_h^0\right\|^2 + \tau\sum_{k=2}^{n+1}\frac{1}{b_{n+1-k}}\left\|f^{k-1/2}\right\|^2. \tag{19}$$

Next, we consider the case $n=0$ for the scheme (6) for the estimate of G^1. By a similar procedure, we take the discrete inner product for (6) with $2\tau u_h^{1/2}$ when $n=0$, we have

$$\begin{aligned}2\tau\left(\delta_t u_h^{1/2}, u_h^{1/2}\right)_h &= 2\tau\left(\left(\kappa_1\bar{D}_\tau^{1-\alpha}+\kappa_2\bar{D}_\tau^{1-\beta}\right)\bar{\Delta}_h u_h^{1/2}, u_h^{1/2}\right)_h \\ &\quad + 2\tau\left(f^{1/2}, u_h^{1/2}\right)_h,\end{aligned} \tag{20}$$

from which we have

$$\begin{aligned}\left\|u_h^1\right\|^2 + 2\tau B_0\left\|\nabla_h u_h^{1/2}\right\|^2 &= \left\|u_h^0\right\|^2 + 2A_0\tau\left(\nabla_h u_h^0, \nabla_h u_h^{1/2}\right)_h \\ &\quad + 2\tau\left(f^{1/2}, u_h^{1/2}\right)_h,\end{aligned} \tag{21}$$

where $B_0=\kappa_1 B_0^{(1-\alpha)}+\kappa_2 B_0^{(1-\beta)}$ and $A_0=\kappa_1 A_0^{(1-\alpha)}+\kappa_2 A_0^{(1-\beta)}$. Utilizing the Cauchy-Schwarz inequality again, we arrive at the estimate for the last two terms on the right-hand of the above inequality:

$$\begin{aligned}&2A_0\tau\left(\nabla_h u_h^0, \nabla_h u_h^{1/2}\right)_h + 2\tau\left(f^{1/2}, u_h^{1/2}\right)_h \\ &\quad\le 2A_0\tau\left(\frac{1}{4\varepsilon_1}\left\|\nabla_h u_h^0\right\|^2 + \varepsilon_1\left\|\nabla_h u_h^{1/2}\right\|^2\right) \\ &\quad\quad + 2\tau\left(\frac{1}{4\varepsilon_2}\left\|f^{1/2}\right\|^2 + \varepsilon_2\left\|u_h^{1/2}\right\|^2\right).\end{aligned} \tag{22}$$

By letting the constants $\varepsilon_1 = B_0/(4A_0)$ and $\varepsilon_2 = B_0/4$ and the equivalence of the $\|v_h\|_1$ and $|v_h|_1$, we further get

$$\begin{aligned}&2A_0\tau\left(\nabla_h u_h^0, \nabla_h u_h^{1/2}\right)_h + 2\tau\left(f^{1/2}, u_h^{1/2}\right)_h \\ &\quad\le 2\tau\frac{A_0^2}{B_0}\left\|\nabla_h u_h^0\right\|^2 + B_0\tau\left\|\nabla_h u_h^{1/2}\right\|^2 + 2\tau\frac{1}{B_0}\left\|f^{1/2}\right\|^2,\end{aligned} \tag{23}$$

which implies that the G^1 has the following estimate:

$$\begin{aligned}G^1 &= \left\|u_h^1\right\|^2 + \tau b_0\left\|u_h^{1/2}\right\|^2 \le \left\|u_h^1\right\|^2 + \tau B_0\left\|u_h^{1/2}\right\|^2 \\ &\le \left\|u_h^0\right\|^2 + 2\tau\frac{A_0^2}{B_0}\left\|\nabla_h u_h^0\right\|^2 + 2\tau\frac{1}{B_0}\left\|f^{1/2}\right\|^2.\end{aligned} \tag{24}$$

Therefore, the inequality (8) yields

$$\begin{aligned}\left\|u_h^{n+1}\right\|^2 &\le \left\|u_h^0\right\|^2 + 2\tau\frac{A_0^2}{B_0}\left\|\nabla_h u_h^0\right\|^2 + 2\tau\frac{1}{B_0}\left\|f^{1/2}\right\|^2 \\ &\quad + \tau\sum_{k=1}^{n} A_k\left\|\nabla u_h^0\right\|^2 + \tau\sum_{k=2}^{n+1}\frac{1}{b_{n+1-k}}\left\|f^{1/2}\right\|^2.\end{aligned} \tag{25}$$

By the mean value theorem, one can readily check that the coefficients appearing in the above inequality are all bounded, that is, we formally have $A_0^2/B_0 \le B_0 = c_1\tau^{\alpha-1} + c_2\tau^{\beta-1}$, $1/B_0 \le c_3\tau^{\min(1-\alpha,1-\beta)}$, $\tau\sum_{k=1}^{n}A_k \le c_4\tau^\alpha + c_5\tau^\beta$, and $\sum_{k=2}^{n+1}1/b_{n+1-k} \le c_5\max(T^{1-\alpha}, T^{1-\beta})$. Thus, the proof is completed.

By means of the error equation and the stability conclusion, we have the following convergence result.

Theorem 3. *Suppose that $u\in C^2(0,T;C^6(\Omega))$, then we have the discrete L^2 -norm error estimate: For $n\ge 1$,*

$$\|u(t_n) - u_h^n\| \le c\left(\tau^{\min(1+\alpha,1+\beta)} + \mathbf{h}^4\right). \tag{26}$$

Proof. The error equation can be obtained by subtracting (6) from (5), that is, by letting the error $e_h^n = u(\mathbf{x}_h, t_n) - u_h^n$ for $\mathbf{x}_h\in\Omega_h$, we have

$$\delta_t e_h^{n+1/2} = \left(\kappa_1\bar{D}_\tau^{1-\alpha} + \kappa_2\bar{D}_\tau^{1-\beta}\right)\bar{\Delta}_h e_h^{n+1/2} + R_{\mathbf{x}t}^{n+1/2}. \tag{27}$$

It follows from Theorem 2 that

$$\begin{aligned}\|e_h^n\|^2 &\le c\left(\left\|e_h^0\right\|^2 + \left(c_1\tau^\alpha + c_2\tau^\beta\right)\left\|\nabla_h e_h^0\right\|^2 + \tau\sum_{k=1}^{n}\left\|R_{\mathbf{x}t}^{k-1/2}\right\|^2\right) \\ &\le c\left(\tau^{\min(1+\alpha,1+\beta)} + \mathbf{h}^4\right)^2,\end{aligned} \tag{28}$$

which leads to the desired convergence result.

4. Numerical Implementation for Nonsmooth Problems

In general, the solution of equation (1) may not have the regularity required in Theorem 3. If the nonsmooth solution problems are directly solved by the fast Crank-Nicolson compact difference scheme (7), unsatisfactory accuracy may be obtained. In this part, we apply the method of adding suitable correction terms when dealing with such nonsmooth issue.

Following the idea presented in [3], we, respectively, take the numerical approximations of ${}_{RL}D_{0,t}^{\gamma}g(t)$ and the first-order time derivative $dg(t)/dt$ at $t=t_{n+1/2}$ as follows:

$$
\begin{aligned}
{}_{RL}D_{0,t}^{\gamma} g(t)\big|_{t=t_{n+1/2}} &\approx \bar{D}_{\tau}^{\gamma} g(t_{n+1/2})\text{red} + \sum_{k=1}^{m} w_{n,k}^{(\gamma)}(g(t_k)-g(0)),\\
\left.\frac{dg(t)}{dt}\right|_{t=t_{n+1/2}} &\approx \delta_t g(t_{n+1/2})\text{red} + \sum_{k=1}^{m} w_{n,k}^{(1)}(g(t_k)-g(0)).
\end{aligned}
\tag{29}
$$

Here, $w_{n,k}^{(\gamma)}$ and $w_{n,k}^{(1)}$ are the starting weights which are chosen such that the above schemes are exact for some power functions $g(t)=t^{\zeta_j}$ with $0<\zeta_j<\zeta_{j+1}$ and $0\le j\le m$, that is, they can be determined by the two linear systems:

$$
\begin{aligned}
\sum_{k=1}^{m} w_{n,k}^{(\gamma)} t_k^{\zeta_j} &= \frac{\Gamma(1+\zeta_j)}{\Gamma(1+\zeta_j-\gamma)} t_{n+1/2}^{\zeta_j-\gamma} - \bar{D}_{\tau}^{\gamma} t_{n+1/2}^{\zeta_j},\\
\sum_{k=1}^{m} w_{n,k}^{(1)} t_k^{\zeta_j} &= \zeta_j t_{n+\frac{1}{2}}^{\zeta_j-1} - \frac{t_{n+1}^{\zeta_j}-t_n^{\zeta_j}}{\tau},
\end{aligned}
\tag{30}
$$

respectively. So, we have the following fast Crank-Nicolson compact difference scheme with correction terms: for $n\ge 0$,

$$
\begin{aligned}
\delta_t \widehat{u}_{\nu}^{n+1/2} + \sum_{k=1}^{m} w_{n,k}^{(1)} \widehat{u}_{\nu}^{k} &= \left(\kappa_1 \bar{D}_{\tau}^{1-\alpha} + \kappa_2 \bar{D}_{\tau}^{1-\beta}\right)\left(\sum_{k=1}^{d} \lambda^{(j_k,M_k)}\right)\widehat{u}_{\nu}^{n+1/2}\\
&\quad + \sum_{k=1}^{m}\left(w_{n,k}^{(1-\alpha)} + w_{n,k}^{(1-\beta)}\right)\left(\sum_{k=1}^{d}\lambda^{(j_k,M_k)}\right)\widehat{u}_{\nu}^{k}\\
&\quad + f\wedge^{n+1/2}.
\end{aligned}
\tag{31}
$$

The execution procedure of the above scheme is similar to that of (7). We can observe that the scheme (9) is stable and effective in solving nonsmooth problems, which will be verified by numerical examples in the next section. We remark that the method of adding correction terms is based on the assumption that the problem solution can be divided into two terms: low regularity and high regularity terms (with respect to time). Such assumption is valid for equation (1) in view of the solution formulation discussed in [4]. By using the starting weights in the correction terms, one can improve the accuracy of the proposed scheme for dealing with the nonsmooth solution problem. For further details about the parameters m and ζ_j, one may refer to [19].

5. Numerical Examples

In this part, we present two numerical examples to verify the accuracy and effectiveness of the scheme (9). The L^2-norm error at $t=t_n$ is obtained by $e(n,h)=\|u(\mathbf{x}_h,t_n)-u_h^n\|$, and the convergence orders in time and in space are calculated by $\log(e(n,h)/e(2n,h))$ and $\log(e(n,h)/e(n,h/2))$, respectively. For simplicity, we set the parameters κ_1 and κ_2 in (1) to be one and restrict the computational domain to be $\Omega=(0,1)^2$. We remark that the numerical tests in this paper are implemented by MATLAB software (R2020a) on an Apple OS platform with a quad-core 2.3 GHz processor and 8 GB of memory.

Example 1. (Accuracy). Consider the following problem with zero Dirichlet boundary conditions:

$$
\begin{cases}
\partial_t u(x,y,t) = \left({}_{RL}D_{0,t}^{1-\alpha} + {}_{RL}D_{0,t}^{1-\beta}\right)\Delta u(x,y,t) + f(x,y,t), & (x,y)\in\Omega,\\
u(x,y,0) = c\sin(\pi x)\sin(\pi y),
\end{cases}
\tag{32}
$$

where

$$
\begin{aligned}
f(x,y,t) = \sin(\pi x)\sin(\pi y)\Bigg(&\gamma t^{\gamma-1} + 2\pi^2 c\frac{t^{\alpha-1}}{\Gamma(\alpha)} + 2\pi^2 c\frac{t^{\beta-1}}{\Gamma(\beta)}\\
&+ 2\pi^2\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha)} t^{\gamma+\alpha-1} + 2\pi^2\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\beta)} t^{\gamma+\beta-1}\Bigg).
\end{aligned}
\tag{33}
$$

The exact solution is $u=\sin(\pi x)\sin(\pi y)(c+t^{\gamma})$ with the two given nonnegative parameters c and γ.

We verify the accuracy of the proposed scheme (9) using two cases: the smooth and nonsmooth solutions. We first let $c=1$ and $\gamma=2.1$. The numerical results are obtained at $T=1$ by fast Crank-Nicolson compact difference scheme (9) with no correction terms and demonstrated in Tables 2 and 3. One can observe that accuracy of the scheme is $O(\tau^{\min(1+\alpha,1+\beta)}+\mathbf{h}^4)$ for different fractional orders α and β, which is in agreement with the theoretical analysis.

Next, for the nonsmooth case, we let $c=0$ and $\gamma=0.4$. One can see that the first-order partial derivative of u with respect to t is $\partial_t u(x,y,t)=\gamma t^{\gamma-1}\sin(\pi x)\sin(\pi y)$, which is unbounded at $t=0$ when $\gamma=0.4$. By using the fast Crank-Nicolson compact difference scheme with correction terms (9), we compute the L^2-norm errors at $T=0.5$ and present the results in Tables 1 and 4. We can see from Table 1 that when $m=0$, that is, no correction term is added to the scheme, the accuracy of the numerical solution suffers from the low regularity of the analytic solution. In contrast, when m is greater than 0, the accuracy of the numerical solution seems to be improved to some extents. Similar phenomenon is also observed in Table 4. This suggests that adding a small number of correction terms does improve the accuracy of the numerical solution in nonsmooth problems. Thus, the fast Crank-Nicolson compact difference scheme with correction terms (9) is valid for solving non-smooth solution problems.

Example 2. (Computational efficiency). In this example, we investigate the computational efficiency of the fast Crank-Nicolson compact difference scheme (7). So, we consider the comparison between results from the schemes with fast solver and the direct solver, that is, fast scheme (7) and original scheme (6). We separately solve the smooth solution case in Example 1 with the two numerical schemes and report the numerical results obtained in Figures 1–3. For the given fractional orders α and β, by fixing the time stepsize $\tau=1/4$ and

Table 2: The L^2-norm errors in time for smooth case in Example 1 with $h = 1/64$.

n_T	(0.3,0.8)		(0.5,0.6)		(0.7,0.4)	
	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
20	3.87E-03	—	1.95E-03	—	2.55E-03	—
40	1.63E-03	1.25	7.32E-04	1.41	1.00E-03	1.34
80	6.74E-04	1.27	2.67E-04	1.45	3.87E-04	1.37
160	2.76E-04	1.29	9.59E-05	1.48	1.48E-04	1.39

Table 3: The L^2-norm errors in space for smooth case in Example 1 with $\tau = T/8000$.

M	(0.3,0.8)		(0.5,0.6)		(0.7,0.4)	
	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
4	1.57E-03	—	1.58E-03	—	1.58E-03	—
8	9.82E-05	4.00	9.73E-05	4.02	9.75E-05	4.02
16	7.72E-06	3.67	6.30E-06	3.95	6.63E-06	3.88

Table 4: The L^2-norm errors in space for nonsmooth case in Example 1 with $\tau = T/8000$.

α	M	$m = 0$		$m = 1$		$m = 2$	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
(0.3,0.8)	4	4.01E-04	—	5.32E-04	—	5.79E-04	—
	8	1.59E-04	1.34	2.76E-05	4.27	1.91E-05	4.92
	16	1.93E-04	-0.28	6.19E-05	-1.16	1.52E-05	0.33
(0.5,0.6)	4	5.08E-04	—	5.87E-04	—	5.93E-04	—
	8	5.28E-05	3.27	2.59E-05	4.50	3.16E-05	4.23
	16	8.72E-05	-0.72	8.49E-06	1.61	2.80E-06	3.50
(0.7,0.4)	4	4.86E-04	—	5.74E-04	—	5.89E-04	—
	8	7.49E-05	2.70	1.34E-05	5.42	2.81E-05	4.39
	16	1.09E-04	-0.54	2.09E-05	-0.64	6.22E-06	2.18

varying the stepsize in each spatial direction simultaneously, we obtain the CPU execution time at $T = 1$ for both schemes. The comparison shows that the execution time spent using the direct solver in numerical scheme is more expensive than that using the DST technique, especially when the spatial stepsize is getting smaller. This is due to the fact that the direct solver requires to solve matrix inversion on each time level, and such operation would be extremely inefficient when the size of the matrix is large. It is clear that the DST technique can speed up the computational efficiency, thus, the proposed scheme (7) has more potential than the direct solver (6) in high-dimensional problems.

6. Conclusions

In this paper, we propose the efficient compact difference scheme for solving the modified anomalous subdiffusion equation based on the modified L1 method in time and compact difference operator in space. By combining the DST technology, we improve the effectiveness of the scheme for the two-dimensional problem. The stability and error estimate of the scheme are provided rigorously. We also improve the accuracy of the scheme for the nonsmooth solution problems with the method of adding correction terms. Numerical examples illustrate the effectiveness and accuracy of the proposed scheme.

The results of this paper can be readily generalized to three-dimensional problems. In addition, for inhomogeneous boundary conditions, one can convert them into homogeneous boundary condition problems by variable substitution. For other types of boundary condition problems, such as Neumann, Robin, or other combinations of boundary conditions, we do not discuss them in this paper. In [23], the authors introduced the augmented matched interface and boundary (AMIB) method to efficiently solving the Poisson equation via the FFT. The authors also pointed out that the AMIB method can easily handle different types of boundary conditions. So, it may be possible to combine this method with the correction terms to rapidly solve high-dimensional problems with complex boundary conditions, and this is the possible one of the future research directions.

Data Availability

The data of numerical simulation used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

This research was funded by Guangxi Natural Science Foundation (grant numbers 2018GXNSFBA281020 and 2018GXNSFAA138121).

References

[1] A. Chen and C. Li, "A novel compact ADI scheme for the time-fractional subdiffusion equation in two space dimensions," *International Journal of Computer Mathematics*, vol. 93, no. 6, pp. 889–914, 2015.

[2] B. Jin, R. Lazarov, and Z. Zhou, "Numerical methods for time-fractional evolution equations with nonsmooth data: a concise overview," *Computer Methods in Applied Mechanics and Engineering*, vol. 346, pp. 332–358, 2019.

[3] L. Feng, I. Turner, P. Perré, and K. Burrage, "An investigation of nonlinear time-fractional anomalous diffusion models for simulating transport processes in heterogeneous binary media," *Communications in Nonlinear Science and Numerical Simulation*, vol. 92, p. 105454, 2020.

[4] T. Langlands, "Solution of a modified fractional diffusion equation," *Physica A: Statistical Mechanics and its Applications*, vol. 367, pp. 136–144, 2006.

[5] C. Li and F. Zeng, *Numerical Methods for Fractional Calculus*, Chapman and Hall/CRC, Boca Raton, 2015.

[6] C. Li and A. Chen, "Numerical methods for fractional partial differential equations," *International Journal of Computer Mathematics*, vol. 95, no. 6-7, pp. 1048–1099, 2018.

[7] H. Ding and C. Li, "High-order compact difference schemes for the modified anomalous subdiffusion equation," *Numerical Methods for Partial Differential Equations*, vol. 32, no. 1, pp. 213–242, 2015.

[8] Y. Chen and C. Chen, "Numerical simulation with the second order compact approximation of first order derivative for the modified fractional diffusion equation," *Applied Mathematics and Computation*, vol. 320, pp. 319–330, 2018.

[9] A. Chen, "Two efficient Galerkin finite element methods for the modified anomalous subdiffusion equation," *International Journal of Computer Mathematics*, pp. 1–18, 2020.

[10] X. Li, H. Liao, and L. Zhang, "A second-order fast compact scheme with unequal time-steps for subdiffusion problems," *Numerical Algorithms*, vol. 86, no. 3, pp. 1011–1039, 2021.

[11] H. Wang, Y. Zhang, X. Ma, J. Qiu, and Y. Liang, "An efficient implementation of fourth-order compact finite difference scheme for Poisson equation with Dirichlet boundary conditions," *Computers and Mathematics with Applications*, vol. 71, no. 9, pp. 1843–1860, 2016.

[12] J. Huang, L. Ju, and B. Wu, "A fast compact time integrator method for a family of general order semilinear evolution equations," *Journal of Computational Physics*, vol. 393, pp. 313–336, 2019.

[13] J. Ren, H. Liao, and Z. Zhang, "Superconvergence error estimate of a finite element method on nonuniform time meshes for reaction-subdiffusion equations," *Journal of Scientific Computing*, vol. 84, no. 2, p. 38, 2020.

[14] J. Ren, H. Chen, J. Zhang, and Z. Zhang, "Error analysis of a fully discrete scheme for a multi-term time fractional diffusion equation with initial singularity," *Scientia Sinica Mathematica*, vol. 50, no. 12, pp. 1–18, 2020.

[15] R. Zheng, F. Liu, and X. Jiang, "A Legendre spectral method on graded meshes for the two-dimensional multi-term time-fractional diffusion equation with non-smooth solutions," *Applied Mathematics Letters*, vol. 104, p. 106247, 2020.

[16] J. Ren, H. Liao, J. Zhang, and Z. Zhang, "Sharp H1-norm error estimates of two time-stepping schemes for reaction- subdiffusion problems," *Journal of Computational and Applied Mathematics*, vol. 389, article 113352, 2021.

[17] B. Jin, R. Lazarov, and Z. Zhou, "Two fully discrete schemes for fractional diffusion and diffusion-wave equations with non-smooth data," *SIAM Journal on Scientific Computing*, vol. 38, no. 1, pp. A146–A170, 2016.

[18] A. Chen and L. Nong, "Efficient Galerkin finite element methods for a time-fractional Cattaneo equation," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[19] F. Zeng, Z. Zhang, and G. E. Karniadakis, "Second-order numerical methods for multi-term fractional differential equations: smooth and non-smooth solutions," *Computer Methods in Applied Mechanics and Engineering*, vol. 327, pp. 478–502, 2017.

[20] H. Zhang, J. Jia, and X. Jiang, "An optimal error estimate for the two-dimensional nonlinear time fractional advection–diffusion equation with smooth and non-smooth solutions," *Computers and Mathematics with Applications*, vol. 79, no. 10, pp. 2819–2831, 2020.

[21] B. Yin, Y. Liu, and H. Li, "A class of shifted high-order numerical methods for the fractional mobile/immobile transport equations," *Applied Mathematics and Computation*, vol. 368, p. 124799, 2020.

[22] F. Zeng and C. Li, "A new Crank-Nicolson finite element method for the time-fractional subdiffusion equation," *Applied Numerical Mathematics*, vol. 121, pp. 82–95, 2017.

[23] H. Feng and S. Zhao, "FFT-based high order central difference schemes for three-dimensional Poisson's equation with various types of boundary conditions," *Journal of Computational Physics*, vol. 410, p. 109391, 2020.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9979791, 11 pages
https://doi.org/10.1155/2021/9979791

Research Article

A Newton Linearized Crank-Nicolson Method for the Nonlinear Space Fractional Sobolev Equation

Yifan Qin,[1] Xiaocheng Yang,[2] Yunzhu Ren,[1] Yinghong Xu,[1] and Wahidullah Niazi[1]

[1]*Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China*
[2]*School of Information Science and Technology, Zhejiang Sci-Tech University, Hangzhou 310018, China*

Correspondence should be addressed to Xiaocheng Yang; yangxiaoch209@163.com

Received 16 March 2021; Accepted 10 April 2021; Published 27 April 2021

Academic Editor: Maria Alessandra Ragusa

In this paper, one class of finite difference scheme is proposed to solve nonlinear space fractional Sobolev equation based on the Crank-Nicolson (CN) method. Firstly, a fractional centered finite difference method in space and the CN method in time are utilized to discretize the original equation. Next, the existence, uniqueness, stability, and convergence of the numerical method are analyzed at length, and the convergence orders are proved to be $O(\tau^2 + h^2)$ in the sense of l^2-norm, $H^{\alpha/2}$-norm, and l^∞-norm. Finally, the extensive numerical examples are carried out to verify our theoretical results and show the effectiveness of our algorithm in simulating spatial fractional Sobolev equation.

1. Introduction

The main propose of this paper is to construct one class of the Newton linearized finite difference method based on CN discretization in temporal direction to efficiently solve the following spatial fractional Sobolev equation:

$$\begin{cases} \partial_t u - \mu \partial_x^\alpha \partial_t u = \kappa \partial_x^\beta u + f(u), & \text{in } \mathbb{R} \times (0, T], \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R} \times \{0\}. \end{cases} \tag{1}$$

where $1 < \alpha, \beta \leqslant 2$, μ and κ are given positive constants, $u_0(x)$ and $f(u)$ are known sufficiently smooth functions. ∂_x^α in (1) denotes the Riesz fractional derivative operator for $1 < \alpha \leqslant 2$ and is defined in [1] as follows:

$$\partial_x^\alpha u(x, t) = -\frac{1}{2\cos(\pi\alpha/2)\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} |x - \xi|^{1-\alpha} u(\xi, t) d\xi. \tag{2}$$

This type of equation is widely used as a mathematical model for fluid flow through thermodynamics [2], shear in second-order fluids [3], consolidation of clay [4], and so on. Note that some special forms of equation (1) are frequently encountered in many fields. For example, taking $\alpha, \beta = 2$, (1) reduces to a one-dimensional integral-order Sobolev equation in the bounded domain [5]. When $f(u) = \sum_{i=1}^{p} \gamma_i u^p$ with integer p and given constants $\gamma_i (i = 1, 2, \cdots, p)$, then the equation is called a semiconductor equation [6]. When $f(u) = 0$, it is reduced to a homogeneous space fractional Sobolev equation. When $\mu = 0$, (1) is reduced to the classical nonlinear reaction-diffusion equations. Recently, many scholars are dedicated to the numerical investigation on fractional diffusion equations and Sobolev equations based on finite difference or finite element methods in the literature. For example, Çelik and Duman [7] investigated the CN method to approximate the fractional diffusion equation with the Riesz fractional derivative in a finite domain. Wang et al. [8] studied the finite difference method for the space fractional Schrödinger equations under the framework of the fractional Sobolev space. Ran and He [9] investigated the nonlinear multidelay fractional diffusion equation based on the CN method in time and the fractional centered difference in space. Chen et al. [5] proposed a Newton linearized compact finite difference scheme to numerically solve a class of Sobolev equations based on the CN method and proved the unique solvability, convergence, and stability of the proposed scheme. Wang and Huang [10] constructed a conservative linearized difference scheme for the nonlinear fractional Schrödinger equation. Zhang et al. [11] established the numerical asymptotic stability result of the compact θ -method for the generalized delay diffusion equation. More researches on delay fractional problems can be referred to [12, 13] and the references therein.

The main work in this paper is to develop an efficient Newton linearized CN method to solve the nonlinear space fractional Sobolev problem (1). The existence, uniqueness, stability, and convergence of the proposed numerical scheme are demonstrated, and the convergent orders are obtained in the sense of l^2-norm, $H^{\alpha/2}$-norm, and l^∞-norm. Besides, we also prove that the convergence orders of the constructed linearized numerical scheme are $O(\tau^2+h^2)$ under three types of norms. The extensive numerical examples are proposed to argue a second-order accuracy in both temporal and spatial dimensions.

The organization of this paper is as follows. In Section 2, we define the fractional Sobolev norm and introduce the second-order centered finite difference approximation for the space Riesz derivative. In Section 3, we construct a CN finite difference scheme for the space fractional Sobolev equation. The existence, uniqueness, stability, and convergence of the proposed scheme in three classes of conventional norms are proved. Finally, the theoretical results are verified by several numerical examples.

2. Preliminaries

Firstly, we present some notations and lemmas which will be used to construct and analyze our numerical scheme.

2.1. Fractional Sobolev Norm. Firstly, we define the fractional Sobolev norm (cf. [14]). Let $h\mathbb{Z}$ be denoted by the infinite grid with grid points $x_j = jh$ $(j \in \mathbb{Z})$. For arbitrary grid functions $u=\{u_j\}$, $v=\{v_j\}$ on $h\mathbb{Z}$, we define the discrete inner products and the corresponding l^2-norm and l^∞-norm

$$(u, v) = h\sum_{j\in\mathbb{Z}} u_j v_j, \|u\|^2 = (u, u), \|u\|_{l^\infty} = \sup_{j\in\mathbb{Z}} |u_j|. \tag{3}$$

Denote $l^2 := \{u \mid u=\{u_j\}, \|u\|^2 < +\infty\}$. For $u \in l^2$, the semidiscrete Fourier transformation $\hat{u}$ is written as

$$\hat{u}(k) := \frac{1}{\sqrt{2\pi}} h \sum_{j\in\mathbb{Z}} u_j e^{-ikx_j}. \tag{4}$$

It is easy to get $\hat{u} \in L^2[-\pi/h, \pi/h]$ due to $u \in l^2$. The inversion formula is defined by

$$u_j = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \hat{u}(k) e^{ikx_j} dk, \tag{5}$$

then we can easily check that Parseval's equality

$$(u, v) = \int_{-\pi/h}^{\pi/h} \hat{u}(k)\hat{v}(k) dk, \tag{6}$$

holds. Moreover, For the given constant $0 \leqslant \sigma \leqslant 1$, the fractional Sobolev norm $\|\cdot\|_{H^\sigma}$ and seminorm $|\cdot|_{H^\sigma}$ are defined as follows:

$$\|u\|_{H^\sigma}^2 = \int_{-\pi/h}^{\pi/h} \left(1+|k|^{2\sigma}\right) |u\wedge(k)|^2 dk, |u|_{H^\sigma}^2 = \int_{-\pi/h}^{\pi/h} |k|^{2\sigma} |u\wedge(k)|^2 dk. \tag{7}$$

Obviously, $\|u\|_{H^\sigma}^2 = \|u\| + |u|_{H^\sigma}^2$.

2.2. Second-Order Approximation of Spatial Riesz Fractional Derivative. In this section, we will review a second-order approximation for the Riesz fractional derivative. Introduce

$$\mathscr{C}^{n+\alpha}(R) = \left\{ f \left| \int_{-\infty}^{\infty} (1+|\omega|)^{n+\alpha} |\hat{f}(\omega)| d\omega < \infty, f \in L^1(\mathbb{R}) \right. \right\}, \tag{8}$$

where $\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt$ denotes the Fourier transformation of $f(x)$.

Lemma 1. *(cf. [7]). Suppose the function $f(\cdot) \in \mathscr{C}^{2+\alpha}(\mathbb{R})$ and the fractional central difference is defined as follows:*

$$\delta_x^\alpha f(x) = -h^{-\alpha} \sum_{k=-\infty}^{+\infty} g_k^{(\alpha)} f(x-kh). \tag{9}$$

Then, it holds

$$\delta_x^\alpha f(x) = \partial_x^\alpha f(x) + O\left(h^2\right). \tag{10}$$

$g_k^{(\alpha)}$ is defined as

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)}, k \in \mathbb{Z}. \tag{11}$$

This is consistently established for arbitrary $x \in \mathbb{R}$.

Remark 2. (cf. [15, 16]). *If we define f^* by*

$$f^*(x) = \begin{cases} f(x), & x \in [a, b], \\ 0, & x \notin [a, b], \end{cases} \tag{12}$$

such that $f^*(x) \in \mathscr{C}^{2+\alpha}(\mathbb{R})$. We get

$$\delta_x^\alpha f(x) = -h^{-\alpha} \sum_{k=-[(b-x)/h]}^{[(x-a)/h]} g_k^{(\alpha)} f(x-kh) + O\left(h^2\right). \tag{13}$$

For any $t \in [0, T]$, we define

$$u^*(x) = \begin{cases} u(x), & x \in [a, b], \\ 0, & x \notin [a, b], \end{cases} \tag{14}$$

and suppose $u^*(x) \in \mathscr{C}^{2+\alpha}(\mathbb{R})$.

3. Second-Order CN Method and Theoretical Analysis

In this section, we are concentrated on the derivation and theoretical analysis of the finite different scheme. In practical computation, it is necessary to truncate the whole space problem onto a finite interval (boundaries are usually chosen sufficient large such that the truncation error is negligible or the exact solution has compact support in the bounded domain [17]). Here, we will truncate (1) on the interval $\Omega = (a, b)$ as follows:

$$\partial_t u - \mu \partial_x^\alpha \partial_t u = \kappa \partial_x^\beta u + f(u), \text{ in } \Omega \times (0, T], \tag{15}$$

$$u(x, 0) = u_0(x), \text{ in } \mathbb{R} \times \{0\}, \tag{16}$$

$$u(x, t) = 0, \text{ on } \mathbb{R} \setminus \Omega \times [0, T]. \tag{17}$$

3.1. The Derivation of the Linearized Numerical Scheme. Take positive integers M, N and let $\tau = T/N$, $h = (b-a)/M$ be the temporal and spatial step sizes, respectively. Denote $x_i = a + ih$, $0 \leqslant i \leqslant M$; $t_k = k\tau$, $0 \leqslant k \leqslant N$; $t_{k+1/2} = (k + 1/2)\tau$, $0 \leqslant k \leqslant N-1$; $\Omega_h = \{x_i \mid 0 \leqslant i \leqslant M\}$, $\Omega_\tau = \{t_k \mid 0 \leqslant k \leqslant N\}$. Define $\bar{\omega} = \{j \mid j = 0, 1, \cdots, M\}$, $\omega = \{j \mid j = 1, 2, \cdots, M-1\}$, $\partial\omega = \bar{\omega} \setminus \omega$. Let $V_h = \{u \mid u = u_i^k \mid 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N, u_0^k = u_M^k = 0\}$ be grid function space defined on $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Then, for a given grid function $u \in V_h$, we introduce the following notations:

$$u_i^{k+(1/2)} = \frac{1}{2}\left(u_i^{k+1} + u_i^k\right), \delta_t u_i^{k+(1/2)} = \frac{1}{\tau}\left(u_i^{k+1} - u_i^k\right). \tag{18}$$

Define the grid function

$$U_i^k = u(x_i, t_k), i \in \bar{\omega}, 0 \leqslant k \leqslant N. \tag{19}$$

Then, we consider (15) at the point $(x_i, t_{k+(1/2)})$ and have

$$\begin{aligned}&\partial_t u\left(x_i, t_{k+(1/2)}\right) - \mu \partial_x^\alpha \partial_t u\left(x_i, t_{k+(1/2)}\right) \\ &\quad = \kappa \partial_x^\beta u\left(x_i, t_{k+(1/2)}\right) + f\left(u\left(x_i, t_{k+(1/2)}\right)\right), i \in \omega, 0 \leqslant k \leqslant N-1.\end{aligned} \tag{20}$$

Utilizing the Taylor expansion, the first term on the left hand side (LHS) in (20) can be estimated as

$$\partial_t u\left(x_i, t_{k+(1/2)}\right) = \delta_t U_i^{k+(1/2)} + O\left(\tau^2\right). \tag{21}$$

Noticing Lemma 1, for the second term on LHS in (20), we have

$$\begin{aligned}\partial_x^\alpha \partial_t u\left(x_i, t_{k+(1/2)}\right) &= -\frac{1}{h^\alpha}\sum_{j=0}^{M} g_{i-j}^{(\alpha)} \delta_t U_j^{k+(1/2)} + O\left(\tau^2 + h^2\right) \\ &= \delta_t \delta_x^\alpha U_i^{k+(1/2)} + O\left(\tau^2 + h^2\right).\end{aligned} \tag{22}$$

For the first term on the right hand side (RHS) in (20), it yields

$$\begin{aligned}\partial_x^\beta u\left(x_i, t_{k+(1/2)}\right) &= -\frac{1}{h^\beta}\sum_{j=0}^{M} g_{i-j}^{(\beta)} U_j^{k+(1/2)} + O\left(\tau^2 + h^2\right) \\ &= \delta_x^\beta U_i^{k+(1/2)} + O\left(\tau^2 + h^2\right).\end{aligned} \tag{23}$$

Moreover, we have

$$\begin{aligned}u\left(x_i, t_{k+(1/2)}\right) &= \frac{1}{2}\left(U_i^{k+1} + U_i^k\right) + O\left(\tau^2\right), \\ u(x_i, t_{k+1}) - u(x_i, t_k) &= O(\tau) \leqslant c_0 \tau,\end{aligned} \tag{24}$$

where c_0 is a positive constant.

Applying the Newton linearized method to the nonlinear term f on RHS in (20) and using Taylor expansion at the point U_i^k, it yields

$$\begin{aligned}f\left(u\left(x_i, t_{k+(1/2)}\right)\right) &= f\left(U_i^k\right) + \left(U_i^{k+(1/2)} - U_i^k\right) f'\left(U_i^k\right) + O\left(\tau^2\right) \\ &= f\left(U_i^k\right) + \frac{1}{2}\left(U_i^{k+1} - U_i^k\right) f'\left(U_i^k\right) + O\left(\tau^2\right),\end{aligned} \tag{25}$$

where $f'(U_i^k) = \partial_U f|_{U=U_i^k}$. Plugging (21)–(23) and substituting (25) into (20), we have

$$\begin{aligned}\delta_t U_i^{k+(1/2)} - \mu \delta_t \delta_x^\alpha U_i^{k+(1/2)} &= \kappa \delta_x^\beta U_i^{k+(1/2)} + f\left(U_i^k\right) \\ &\quad + \frac{1}{2} f'\left(U_i^k\right)\left(U_i^{k+1} - U_i^k\right) + R_i^k, i \in \omega, 0 \leqslant k \leqslant N-1.\end{aligned} \tag{26}$$

There exists a positive constant $c_1 > 0$ such that

$$\left|R_i^k\right| \leqslant c_1\left(\tau^2 + h^2\right), i \in \omega, 0 \leqslant k \leqslant N-1. \tag{27}$$

Omitting R_i^k in (26), replacing $U_i^{k+(1/2)}$ with $u_i^{k+(1/2)}$ in (26), then the finite difference scheme reads

$$\begin{aligned}\delta_t u_i^{k+(1/2)} - \mu \delta_t \delta_x^\alpha u_i^{k+(1/2)} &= \kappa \delta_x^\beta u_i^{k+(1/2)} + f\left(u_i^k\right) + \frac{1}{2} f'\left(u_i^k\right) \\ &\quad \cdot \left(u_i^{k+1} - u_i^k\right), i \in \omega, 0 \leqslant k \leqslant N-1,\end{aligned} \tag{28}$$

$$u_i^0 = u_0(x_i), i \in \bar{\omega}, \tag{29}$$

$$u_i^k = 0, i \in \partial\omega, 1 \leqslant k \leqslant N. \tag{30}$$

3.2. The Unique Solvability of Finite Difference Scheme. This section is concerned with the solvability of scheme (28)–(30). Now, we give some lemmas which will be used in the demonstration of solvability.

Lemma 3. *(cf. [7]). Let*

$$A^{\alpha}=\begin{pmatrix} g_0^{(\alpha)} & g_{-1}^{(\alpha)} & \cdots & g_{3-M}^{(\alpha)} & g_{2-M}^{(\alpha)} \\ g_1^{(\alpha)} & g_0^{(\alpha)} & g_{-1}^{(\alpha)} & \cdots & g_{3-M}^{(\alpha)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ g_{M-3}^{(\alpha)} & \cdots & g_1^{(\alpha)} & g_0^{(\alpha)} & g_{-1}^{(\alpha)} \\ g_{M-2}^{(\alpha)} & g_{M-3}^{(\alpha)} & \cdots & g_1^{(\alpha)} & g_0^{(\alpha)} \end{pmatrix}. \tag{31}$$

It holds

$$g_0^{(\alpha)}=\frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)}\leqslant 0,\ \sum_{j=-\infty}^{+\infty} g_j^{(\alpha)}=0,\ g_j^{(\alpha)}=g_{-j}^{(\alpha)}\leqslant 0, \tag{32}$$

where for any $|j|\leqslant 1$, *and* $0<\lambda_i<2g_0^{(\alpha)}\,(i\in\omega)$, λ_i *is the ith the eigenvalue of matrix* A^{α}. A^{β} *is given in a similar way. It implies that the matrices* A^{α} *and* A^{β} *are real symmetric positive definite matrices.*

Lemma 4. *(discrete Sobolev inequality (cf. [14])) For every* $1/2<\sigma\leqslant 1$, *there exists a constant* $C_\sigma=C(\sigma)>0$, *independent of* $h>0$, *such that*

$$\|v\|_{l^\infty}\leqslant C_\sigma\|v\|_{H^\sigma}. \tag{33}$$

Lemma 5. *(cf. [8]). For any* $1<\alpha\leqslant 2$ *and any grid function* $v\in V_h$, *we have*

$$C_\alpha|v|^2_{H^{\alpha/2}}\leqslant(-\delta_x^\alpha v,v)\leqslant|v|^2_{H^{\alpha/2}}, \tag{34}$$

where $C_\alpha=(\pi/2)^\alpha$.

Lemma 6. *(cf. [17]). For any grid function* $v\in V_h$, *there exists a fractional symmetric positive quotient operator* $\delta_x^{\alpha/2}$, *such that*

$$(-\delta_x^\alpha v,v)=\left(\delta_x^{\alpha/2}v,\delta_x^{\alpha/2}v\right). \tag{35}$$

Lemma 7. *(cf. [18]) (discrete uniform Sobolev inequality). For every* $1/2<\sigma\leqslant 1$, *there exists a constant* $C_\sigma=C(\sigma)>0$ *independent of* $h>0$ *such that*

$$\|u\|_{l^\infty}\leqslant\|u\|_{H^\sigma}. \tag{36}$$

Lemma 8. *(cf. [19]). Suppose* $\{F^k\}_{k=0}^{\infty}$ *be nonnegative sequence and satisfy*

$$F^k\leqslant c\tau\sum_{l=0}^{k-1}F^l+g,\ k=0,1,2,\cdots, \tag{37}$$

Then, we have

$$F^k\leqslant ge^{ck\tau},\ k=0,1,2,\cdots, \tag{38}$$

where c and g are nonnegative constants.

Theorem 9. *The linearized finite difference scheme (28)–(30) is uniquely solvable.*

Proof. Denote $u^k=(u_1^k,u_2^k,\cdots,u_{M-1}^k)^T$. We will prove the above result by the mathematical induction. Obviously, (29) is true for $k=0$. Now, we suppose $u^l(0\leqslant k\leqslant l\leqslant N-1)$ has been uniquely determined; then, we only need to prove that u^{l+1} is uniquely determined by (28). We can rewrite (28) in the following matrix form

$$\begin{aligned}&\left(I+\frac{\mu}{h^\alpha}A^\alpha+\frac{\tau\kappa}{2h^\beta}A^\beta-\frac{\tau}{2}\operatorname{diag}\left(f'\left(u^l\right)\right)\right)u^{l+1}\\&=\left(I+\frac{\mu}{h^\alpha}A^\alpha-\frac{\tau\kappa}{2h^\beta}A^\beta-\frac{\tau}{2}\operatorname{diag}\left(f'\left(u^l\right)\right)\right)u^l\\&\quad+\tau f\left(u^l\right)+\tilde{G}^{l+1},\end{aligned} \tag{39}$$

where $\tilde{G}^{l+1}$ is a vector which depends only on the boundary value. By using Lemma 3, when τ is sufficiently small, it is easy to verify that the coefficient matrix of (39) is strictly diagonally dominant, which implies that there exists a unique solution u^{l+1}. This completes the proof.

3.3. The Convergence and Stability of the Finite Difference Scheme. Firstly, we easily have the estimation of the local truncation error, according to (27).

Lemma 10. *Let* $u(x,\cdot)\in\mathscr{C}^{(2+\alpha)}(x,\cdot)$ *be the solution of the problem (15)–(17). Then, we have*

$$\|R^k\|^2\leqslant(b-a)c_1^2\left(\tau^2+h^2\right)^2,\ 0\leqslant k\leqslant N-1, \tag{40}$$

where c_1 *is a positive constant independent of* τ *and h.*

Denote

$$e_i^k=U_i^k-u_i^k,\ i\in\omega,\ 0\leqslant k\leqslant N. \tag{41}$$

We will obtain the main convergence result.

Theorem 11. *Let* $u(x,\cdot)\in\mathscr{C}^{(2+\alpha)}(x,\cdot)$ *be the solution of the problem (15)–(17). Then, there exist positive constants* τ_0 *and* h_0, *when* $\tau<\tau_0$ *and* $h<h_0$, *for* $0\leqslant k\leqslant N$, *we have*

$$\|e^k\|\leqslant C_1\left(\tau^2+h^2\right),\ \left|e^k\right|_{H^{\alpha/2}}\leqslant C_2\left(\tau^2+h^2\right),\|e^k\|_{l^\infty}\leqslant C_3\left(\tau^2+h^2\right), \tag{42}$$

where $C_1,C_2,C_3>0$ *are positive constants independent of* τ *and h.*

Proof. The mathematical induction will be employed. Firstly, it is obvious (42) is true for $k=0$, via (29). Then, it assumes that (42) is true for $1 \leqslant k \leqslant m \leqslant N-1$. We will discuss that (42) holds for $k=m+1$. According to the hypothesis, we can obtain the following estimation:

$$\|u^k\|_{l^\infty} \leqslant \|e^k\|_{l^\infty} + \|U^k\|_{l^\infty} \leqslant C_3\left(\tau^2+h^2\right) + \tilde{c}_0 \leqslant 1+\tilde{c}_0,\ 1 \leqslant k \leqslant m, \tag{43}$$

where $\tau < \tau_0 = (2C_3)^{-1/2}$, $h < h_0 = (2C_3)^{-1/2}$, and $\tilde{c}_0 = \max_{(x,t)\in\Omega\times[0,T]} |U(x,t)|$.

In the view of Lipschitz condition, we have

$$|f\left(U_i^k\right) - f\left(u_i^k\right)| \leqslant c_2\left|e_i^k\right|,\ i \in \bar{\omega},\ 0 \leqslant k \leqslant N, \tag{44}$$

$$|f'\left(U_i^k\right) - f'\left(u_i^k\right)| \leqslant c_3\left|e_i^k\right|,\ i \in \bar{\omega},\ 0 \leqslant k \leqslant N, \tag{45}$$

$$\left|f'\left(u_i^k\right)\right| \leqslant c_4,\ i \in \bar{\omega},\ 0 \leqslant k \leqslant N, \tag{46}$$

where c_2, c_3, and c_4 are positive constants independent of τ and h.

Now, subtracting (28) from (26), we can obtain the error equation

$$\delta_t e_i^{k+(1/2)} - \mu\delta_t\delta_x^\alpha e_i^{k+(1/2)} = \kappa\delta_x^\beta e_i^{k+(1/2)} + P_i^k + \mathrm{R}_i^k,\ i \in \omega,\ 0 \leqslant k \leqslant N-1, \tag{47}$$

where

$$P_i^k = f\left(U_i^k\right) - f\left(u_i^k\right) + \frac{1}{2}\left(f'\left(U_i^k\right)\left(U_i^{k+1} - U_i^k\right) - f'\left(u_i^k\right)\left(u_i^{k+1} - u_i^k\right)\right). \tag{48}$$

Firstly, we establish l^2-error estimation. Taking the discrete inner product of (47) with $e^{k+(1/2)}$, we have

$$\begin{aligned}&\left(\delta_t \mathrm{e}^{k+(1/2)}, e^{k+(1/2)}\right) - \mu\left(\delta_t\delta_x^\alpha e^{k+(1/2)}, e^{k+(1/2)}\right)\\&\quad = \kappa\left(\delta_x^\beta e^{k+(1/2)}, e^{k+(1/2)}\right) + \left(P^k, e^{k+(1/2)}\right) + \left(R^k, e^{k+(1/2)}\right).\end{aligned} \tag{49}$$

Now, we estimate each term in (49). The first term on LHS in (49) can be estimated as

$$\left(\delta_t e^{k+(1/2)}, e^{k+(1/2)}\right) = \frac{1}{2\tau}\left(\|e^{k+1}\|^2 - \|e^k\|^2\right). \tag{50}$$

Noticing Lemma 6, for the second term on the LHS in (49), we have

$$\begin{aligned}\left(\delta_t\delta_x^\alpha e^{k+(1/2)}, e^{k+(1/2)}\right) &= -\left(\delta_x^{\alpha/2}\left(\frac{e^{k+1}-e^k}{\tau}\right), \delta_x^{\alpha/2}\left(\frac{e^{k+1}+e^k}{2}\right)\right)\\&= -\frac{1}{2\tau}\left(\|\delta_x^{\alpha/2}e^{k+1}\|^2 - \|\delta_x^{\alpha/2}e^k\|^2\right).\end{aligned} \tag{51}$$

Similarly, the first term on RHS in (49) can be obtained by

$$\begin{aligned}\left(\delta_x^\beta e^{k+(1/2)}, e^{k+(1/2)}\right) &= -\left(\delta_x^{\beta/2}\left(\frac{e^{k+1}+e^k}{2}\right), \delta_x^{\beta/2}\left(\frac{e^{k+1}+e^k}{2}\right)\right)\\&= -\frac{1}{4}\|\delta_x^{\beta/2}\left(e^{k+1}+e^k\right)\|^2.\end{aligned} \tag{52}$$

According to (44)–(46), we have

$$\begin{aligned}|P_i^k| &= \left|f\left(U_i^k\right) - f\left(u_i^k\right) + \frac{1}{2}\left(\left(f'\left(U_i^k\right) - f'\left(u_i^k\right)\right)\right.\right.\\&\quad\left.\left.\cdot\left(U_i^{k+1} - U_i^k\right) + f'\left(u_i^k\right)\left(e_i^{k+1} - e_i^k\right)\right)\right|\\&\leqslant c_2|e_i^k| + \frac{1}{2}\left(c_3c_0\tau\,|\,e_i^k| + c_4\,|\,e_i^{k+1} - e_i^k\,|\right).\end{aligned} \tag{53}$$

Using the Cauchy-Schwarz inequality and Young inequality, the second term on the RHS in (49) becomes

$$\begin{aligned}\left(P^k, e^{k+(1/2)}\right) &\leqslant \|P^k\|\|e^{k+(1/2)}\| \leqslant \frac{3}{4}\|P^k\|^2 + \frac{1}{3}\|e^{k+(1/2)}\|^2\\&\leqslant \frac{9}{4}\left(c_2^2\|e^k\|^2 + \frac{1}{4}c_3^2c_0^2\tau^2\|e^k\|^2 + \frac{1}{4}c_4^2\|e^{k+1} - e^k\|^2\right)\\&\quad + \frac{1}{6}\left(\|e^{k+1}\|^2 + \|e^k\|^2\right)\\&\leqslant \left(\frac{9c_2^2}{4} + \frac{9c_3^2c_0^2\tau^2}{16} + \frac{9}{8}c_4^2 + \frac{1}{6}\right)\|e^k\|^2\\&\quad + \left(\frac{9c_4^2}{8} + \frac{1}{6}\right)\|e^{k+1}\|^2.\end{aligned} \tag{54}$$

The last term of RHS in (49) is estimated as

$$\begin{aligned}\left(R^k, e^{k+(1/2)}\right) &\leqslant \|R^k\|\|e^{k+(1/2)}\| \leqslant \frac{3}{4}\|R^k\|^2 + \frac{1}{3}\|e^{k+(1/2)}\|^2\\&\leqslant \frac{3}{4}(b-a)c_1^2\left(\tau^2+h^2\right)^2 + \frac{1}{6}\left(\|e^{k+1}\|^2 + \|e^k\|^2\right).\end{aligned} \tag{55}$$

Substituting (50)–(55) into (49), we get

$$\begin{aligned}&\frac{\|e^{k+1}\|^2 - \|e^k\|^2}{2\tau} + \mu\frac{\|\delta_x^{\alpha/2}e^{k+1}\|^2 - \|\delta_x^{\alpha/2}e^k\|^2}{2\tau} + \kappa\frac{\|\delta_x^{\beta/2}\left(e^{k+1}+e^k\right)\|^2}{4}\\&\quad \leqslant c_5\left(\|e^{k+1}\|^2 + \|e^k\|^2\right) + \frac{3}{4}(b-a)c_1^2\left(\tau^2+h^2\right)^2,\end{aligned} \tag{56}$$

where $c_5 = (9c_2^2/4) + (9c_3^2c_0^2\tau^2/16) + ((9/8)9/8c_4^2) + (1/3)$.

Summing for k from 0 to m, we have

$$\frac{\|e^{m+1}\|^2-\|e^0\|^2}{2\tau}+\mu\frac{\|\delta_x^{\alpha/2}e^{m+1}\|^2-\|\delta_x^{\alpha/2}e^0\|^2}{2\tau}+\frac{\kappa}{4}\sum_{k=0}^{m}\|\delta_x^{\beta/2}\left(e^{k+1}+e^k\right)\|^2 \leqslant c_5\sum_{k=0}^{m}\left(\|e^{k+1}\|^2+\|e^k\|^2\right)+\frac{3}{4}(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2. \tag{57}$$

Noticing that $e^0=0$ and $\kappa>0$, we have

$$\|e^{m+1}\|^2+\mu\|\delta_x^{\alpha/2}e^{m+1}\|^2 \leqslant 4\tau c_5\sum_{k=0}^{m}\|e^k\|^2+2\tau c_5\|e^{m+1}\|^2 + \frac{3\tau}{2}(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2. \tag{58}$$

Let $F^m=\|e^m\|^2+\mu\|\delta_x^{\alpha/2}e^m\|^2$, we have

$$F^{m+1}\leqslant 4c_5\tau\sum_{k=0}^{m}F^k+2c_5\tau F^{m+1}+\frac{3\tau}{2}(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2. \tag{59}$$

It implies when $\tau\leqslant\tau_0=1/3c_5$, we have

$$F^{m+1}\leqslant 12\tau c_5\sum_{k=0}^{m}F^k+\frac{9}{2}\tau(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2. \tag{60}$$

Using Gronwall Lemma 8, we have

$$F^{m+1}\leqslant \exp\left(12c_5m\tau\right)\left(\frac{9}{2}\tau(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2\right). \tag{61}$$

Therefore, we have

$$\|e^{m+1}\|\leqslant C_1\left(\tau^2+h^2\right), \tag{62}$$

where $C_1:=\sqrt{(9(b-a)c_1^2T\exp\left(12c_5T\right))/2}$.

Similarly, applying Lemma 5 yields

$$\left|e^{m+1}\right|_{H^{\alpha/2}}\leqslant C_2\left(\tau^2+h^2\right), \tag{63}$$

where $C_2:=\sqrt{(9(b-a)c_1^2T\exp\left(12c_5T\right))/2C_\alpha\mu}$.

Finally, we can establish l^∞-error estimate by combining (62) with (63). Denoting $C_3:=C_\sigma\sqrt{C_1^2+C_2^2}$, it follows from Lemma 7 that

$$\|e^{m+1}\|_{l^\infty}\leqslant C_3\left(\tau^2+h^2\right). \tag{64}$$

We complete the proof.

Next, we will analyze the stability of the scheme (28)–(30). Let $\{v_i^k\mid 0\leqslant i\leqslant M, 0\leqslant k\leqslant N\}$ be the solution of the fractional Sobolev equation

$$\delta_t v_i^{k+(1/2)}-\mu\delta_t\delta_x^\alpha v_i^{k+(1/2)}=\kappa\delta_x^\beta v_i^{k+(1/2)}+f\left(v_i^k\right)+\frac{1}{2}f'\left(v_i^k\right)\cdot\left(v_i^{k+1}-v_i^k\right), i\in\omega, 0\leqslant k\leqslant N-1, \tag{65}$$

$$v_i^0=u_0(x_i)+\phi_i^0, i\in\bar{\omega}, \tag{66}$$

$$v_i^k=0, i\in\partial\omega, 1\leqslant k\leqslant N, \tag{67}$$

where ϕ_i^0 is the perturbation of the initial value. Subtracting (65)–(67) from (28)–(30) and denoting $\rho_i^k=v_i^k-u_i^k$, we have

$$\delta_t\rho_i^{k+(1/2)}-\mu\delta_t\delta_x^\alpha\rho_i^{k+(1/2)}=\kappa\delta_x^\beta\rho_i^{k+(1/2)}+f\left(v_i^k\right)-f\left(u_i^k\right)+\frac{1}{2}\cdot\left[f'\left(v_i^k\right)\left(v_i^{k+1}-v_i^k\right)-f'\left(u_i^k\right)\left(u_i^{k+1}-u_i^k\right)\right],$$

$$i\in\omega, 0\leqslant k\leqslant N-1,$$

$$\rho_i^0=\phi_i^0, i\in\bar{\omega},$$

$$\rho_i^k=0, i\in\partial\omega, 1\leqslant k\leqslant N. \tag{68}$$

Similar to the proof of Theorem 11, we have the following result.

Theorem 12. *Denote $\rho_i^k=v_i^k-u_i^k, i\in\bar{\omega}, 0\leqslant k\leqslant N$. Then, there exist positive constants τ_0 and h_0, when $\tau<\tau_0$ and $h<h_0$, we have*

$$\|\rho^k\|\leqslant C_4\|\rho^0\|, \left|\rho^k\right|_{H^{\alpha/2}}\leqslant C_5\left|\rho^0\right|_{H^{\alpha/2}}, \|\rho^k\|_{l^\infty}\leqslant C_6\|\rho^0\|_{l^\infty}, \tag{69}$$

where $C_4, C_5, C_6>0$ are positive constants independent of τ and h.

4. Numerical Examples

In this section, we will provide extensive numerical examples to testify the theoretical results. we will define the discrete l^2-norm and l^∞-norm separately and the corresponding convergence orders are defined as follows:

$$E(h,\tau)=\sqrt{h\sum_{i=0}^{M}\left(U_i^N-u_i^N\right)^2}, E_\infty(h,\tau)=\max_{0\leqslant i\leqslant M, 0\leqslant k\leqslant N}\left|U_i^k-u_i^k\right|,$$

$$\mathrm{Ord}_2=\log_2\left(\frac{\|E(h,\tau)\|}{\|E(h/2,\tau/2)\|}\right),$$

$$\mathrm{Ord}_\infty=\log_2\left(\frac{\|E(h,\tau)\|_\infty}{\|E(h/2,\tau/2)\|_\infty}\right). \tag{70}$$

Table 1: l^2- and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha < 2$ in the spatial direction for (72) with fixed time step $\tau = 1/2000$ for Example 1.

(α, β)	h	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/10	$7.5025e-4$	–	$1.0862e-3$	–
(1.2,1.8)	1/20	$1.7359e-4$	2.1117	$2.5686e-4$	2.0802
	1/40	$4.0650e-5$	2.0702	$6.1167e-5$	2.0702
	1/10	$7.7557e-4$	–	$1.1391e-3$	–
(1.5,1.5)	1/20	$1.7445e-4$	2.1524	$2.6326e-4$	2.1133
	1/40	$3.9827e-5$	2.1310	$6.1202e-5$	2.1048
	1/10	$1.1777e-3$	–	$1.6525e-3$	–
(1.8,1.2)	1/20	$2.7908e-4$	2.0773	$3.9673e-4$	2.0584
	1/40	$6.6203e-5$	2.0757	$9.5326e-5$	2.0572

Table 2: l^2- and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha < 2$ in the temporal direction for (72) with fixed spatial step $h = 1/2000$ for Example 1.

(α, β)	τ	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/10	$1.9481e-4$	–	$3.1163e-4$	–
(1.2,1.8)	1/20	$4.8708e-5$	1.9998	$7.7916e-5$	1.9998
	1/40	$1.2169e-5$	2.0010	$1.9465e-5$	2.0010
	1/10	$1.5223e-4$	–	$2.4163e-4$	–
(1.5,1.5)	1/20	$3.8057e-5$	2.0000	$6.0406e-5$	2.0001
	1/40	$9.5069e-6$	2.0011	$1.5088e-5$	2.0013
	1/10	$1.2416e-4$	–	$1.9507e-4$	–
(1.8,1.2)	1/20	$3.1025e-5$	2.0006	$4.8747e-5$	2.0006
	1/40	$7.7412e-6$	2.0028	$1.2163e-5$	2.0028

Table 3: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha < 2$ in the spatial direction for (73) with $\tau = 1/2000$ for Example 2.

(α, β)	h	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/100	$1.3736e-4$	–	$2.9683e-4$	–
(1.1,1.9)	1/200	$4.9959e-5$	1.4591	$9.9718e-5$	1.5737
	1/400	$1.6497e-5$	1.5985	$3.1155e-5$	1.6784
	1/100	$4.6568e-4$	–	$6.5750e-4$	–
(1.5,1.5)	1/200	$1.3983e-4$	1.7357	$2.0089e-4$	1.7106
	1/400	$4.0066e-5$	1.8032	$5.8340e-5$	1.7839
	1/100	$1.7483e-4$	–	$2.1547e-4$	–
(1.9,1.1)	1/200	$4.6475e-5$	1.9115	$5.7682e-5$	1.9013
	1/400	$1.2052e-5$	1.9471	$1.5024e-5$	1.9408

Example 1. We firstly consider the following fractional Sobolev equation as

$$\partial_t u - \mu \partial_x^\alpha \partial_t u = \kappa \partial_x^\beta u + \sin(u) + g(x,t), (x,t) \in (0,1) \times (0,1], \tag{71}$$

The exact solution is

$$u(x,t) = t^3 x^2 (1-x)^2. \tag{72}$$

The initial boundary conditions and $g(x,t)$ are determined by (72).

Table 4: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1<\alpha<2$ in the temporal direction for (73) with $h=1/1000$ for Example 2.

(α,β)	τ	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/100	$2.0697e-2$	–	$5.6643e-2$	–
(1.1,1.9)	1/200	$8.1358e-3$	1.3471	$1.3366e-2$	2.0833
	1/400	$2.0509e-3$	1.9880	$2.9698e-3$	2.1701
	1/100	$2.0526e-2$	–	$5.6522e-2$	–
(1.5,1.5)	1/200	$8.1446e-3$	1.3336	$1.3303e-2$	2.0871
	1/400	$2.0523e-3$	1.9886	$2.9423e-3$	2.1767
	1/100	$2.0734e-2$	–	$5.6310e-2$	–
(1.9,1.1)	1/200	$8.1558e-3$	1.3461	$1.3202e-2$	2.0927
	1/400	$2.0536e-3$	1.9897	$2.9033e-3$	2.1850

Table 5: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1<\alpha<2$ in the spatial direction for (75) with $\tau=1/1000$ for Example 3.

(α,β)	h	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/10	$1.4761e-2$	–	$1.6303e-2$	–
(1.3,1.7)	1/20	$3.3659e-3$	2.1327	$3.8460e-3$	2.0837
	1/40	$8.0373e-4$	2.0662	$9.0995e-4$	2.0795
	1/10	$1.6167e-2$	–	$1.8043e-2$	–
(1.5,1.5)	1/20	$3.6642e-3$	2.1414	$4.2293e-3$	2.0930
	1/40	$8.7053e-4$	2.0735	$9.9262e-4$	2.0911
	1/10	$2.0933e-2$	–	$2.2257e-2$	–
(1.7,1.3)	1/20	$4.8190e-3$	2.1190	$5.3128e-3$	2.0667
	1/40	$1.1227e-3$	2.1018	$1.2648e-3$	2.0705

Table 6: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1<\alpha<2$ in the temporal direction for (75) with $h=1/1000$ for Example 3.

(α,β)	τ	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/10	$9.1725e-3$	–	$1.0470e-2$	–
(1.3,1.7)	1/20	$2.5218e-3$	1.8629	$2.8729e-3$	1.8657
	1/40	$6.6380e-4$	1.9256	$7.5586e-4$	1.9263
	1/10	$8.7813e-3$	–	$9.8734e-3$	–
(1.5,1.5)	1/20	$2.4228e-3$	1.8578	$2.7208e-3$	1.8595
	1/40	$6.3844e-4$	1.9240	$7.1689e-4$	1.9242
	1/10	$8.2127e-3$	–	$9.1056e-3$	–
(1.7,1.3)	1/20	$2.2771e-3$	1.8507	$2.5228e-3$	1.8517
	1/40	$6.0134e-4$	1.9209	$6.6620e-4$	1.9210

Taking $\mu=1$, $\kappa=1$, the linearized numerical scheme (28)–(30) with $\tau=h$ is applied to solve the above Sobolev equation. The global numerical errors and convergence orders with respect to different α and β are listed in the following tables. Table 1 lists the l^2-norm and l^∞-norm errors and spatial convergence orders with fixed time step $\tau=1/2000$. Table 2 tests the temporal convergence orders with fixed spatial step $h=1/2000$. It demonstrates that the convergence orders of the scheme (28)–(30) is second-order accurate in both spatial and temporal directions which is consistent with Theorem 11.

All the data are referred to MATLAB codes in Example 1 in the supplementary files.

Table 7: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha \leqslant 2$ in the spatial direction with $\tau = 1/1000$ for Example 4.

(α, β)	h	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/40	$1.7548e-2$	–	$1.1092e-2$	–
(1.2,1.8)	1/80	$3.9216e-3$	2.1618	$2.3542e-3$	2.2362
	1/160	$9.5790e-4$	2.0558	$5.6623e-4$	2.0558
	1/40	$1.2035e-2$	–	$7.0740e-3$	–
(1.5,1.5)	1/80	$2.5833e-3$	2.2199	$1.4349e-3$	2.3016
	1/160	$6.2934e-4$	2.0373	$3.4473e-4$	2.0574
	1/40	$7.8751e-3$	–	$3.5547e-3$	–
(1.8,1.2)	1/80	$1.6022e-3$	2.2972	$8.7242e-4$	2.0266
	1/160	$3.9110e-4$	2.0345	$2.1876e-4$	1.9957
	1/40	$1.4624e-2$	–	$8.1939e-3$	–
(2.0,2.0)	1/80	$3.1088e-3$	2.2339	$1.5922e-3$	2.3635
	1/160	$7.5817e-4$	2.0358	$3.8083e-4$	2.0638

Table 8: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha \leqslant 2$ in the temporal direction with $h = 1/1000$ for Example 4.

(α, β)	τ	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/40	$9.2300e-9$	–	$5.5579e-9$	–
(1.2,1.8)	1/80	$2.3078e-9$	1.9998	$1.3897e-9$	1.9998
	1/160	$5.7564e-10$	2.0033	$3.4662e-10$	2.0034
	1/40	$7.5499e-9$	—	$3.2739e-9$	–
(1.5,1.5)	1/80	$1.8868e-9$	2.0005	$8.1803e-10$	2.0008
	1/160	$4.7057e-10$	2.0034	$2.0378e-10$	2.0051
	1/40	$6.7080e-9$	–	$2.7409e-9$	–
(1.8,1.2)	1/80	$1.6769e-9$	2.0001	$6.8522e-10$	2.0000
	1/160	$4.1960e-10$	1.9987	$1.7116e-10$	2.0012
	1/40	$9.2576e-9$	–	$4.4312e-9$	–
(2.0,2.0)	1/80	$2.3143e-9$	2.0001	$1.1077e-9$	2.0001
	1/160	$5.7857e-10$	2.0000	$2.7694e-10$	2.0000

Example 2. Next, we consider the nonlinear fractional Sobolev equation as

The exact solution

$$u(x, t) = \sin (t + 1)(2 + x)^2(2 - x)^2 \tag{73}$$

is oscillatory along with the temporal direction, where $\mu = 1$ and $\kappa = 1$. And the initial boundary conditions and $g(x, t)$ are determined by (73).

In this example, we examine the spatial convergence orders with the fixed time step $\tau = 1/2000$ and the temporal convergence orders with the fixed spatial step $h = 1/1000$ in l^2-norm and l^∞-norm errors, respectively. All the numerical results in the example are listed in Tables 3 and 4. Similar results are observed. All the data are referred to MATLAB codes in Example 2 in the supplementary files.

Example 3. Then, we calculate the nonlinear fractional Sobolev equation as

$$\partial_t u - \partial_x^\alpha \partial_t u = \partial_x^\beta u + u^2 - u^4 + g(x, t), (x, t) \in (-1, 1) \times (0, 1], \tag{74}$$

We choose the exact solution

$$u(x, t) = \left(t + t^3\right)(1 + x)^2(1 - x)^2. \tag{75}$$

The initial boundary conditions and $g(x, t)$ are determined by (75).

Similar to above example, Tables 5 and 6 list the l^2-norm and l^∞-norm errors and corresponding spatial and temporal convergence orders of (28)–(30), respectively. To testify the spatial convergence orders, we fixed the time step $\tau = 1/1000$. Similarly, we take the fixed spatial step $h = 1/1000$ to obtain the temporal convergence orders. The numerical

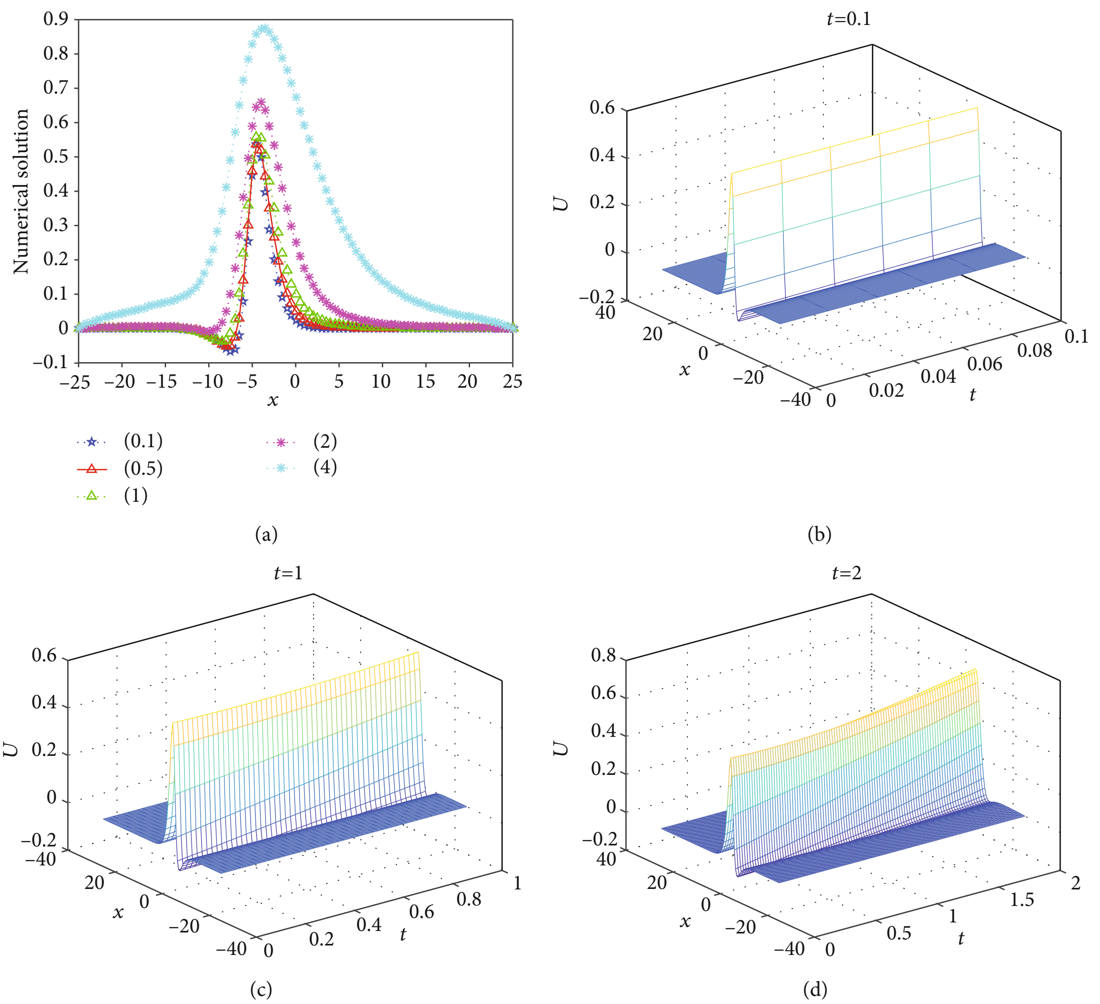


FIGURE 1: Using scheme (28)–(30), curves of $u(x,t)$ with respect to x at different time with $h = 0.5$, $\tau = 0.02$ (a) and the evolutionary surfaces of $u(x,t)$ at $T = 0.1$ (b), $T = 1$ (c), and $T = 2$ (d) in Example 4.

results show that (28)–(30) is close to second-order accurate in spatial and temporal directions.

All the data are referred to MATLAB codes in Example 3 in the supplementary files.

In the following model, the exact solution is unknown, we test convergence orders using the posterior error estimation

$$\mathrm{Ord}_2 = \log_2\left(\frac{\|u(h,\tau) - u(h,\tau/2)\|}{\|u(h,\tau/2) - u(h,\tau/2)\|}\right), \mathrm{Ord}_\infty = \log_2\left(\frac{\|u(h,\tau) - u(h,\tau/2)\|_\infty}{\|u(h,\tau/2) - u(h,\tau/4)\|_\infty}\right). \quad (76)$$

Example 4. We consider the following equation:

$$\partial_t u - \partial_x^\alpha \partial_t u = \partial_x^\beta u + u - u^2, \quad (x,t) \in (-25,25) \times (0,0.1],$$

$$u(x,0) = \sqrt{2}sech(x+5) \cos(4/x), x \in [-25,25],$$

$$u(-25,t) = u(25,t) = 0, t \in [0,0.1], \quad (77)$$

with the exact solution is unknown.

In the computation, we take different spatial and temporal step sizes. The l^2-norm, l^∞-norm errors, and their convergence orders of (28)–(30) are listed in Table 7 with the fixed temporal step size $\tau = 1/1000$. Similarly, the spatial step size fixed at $h = 1/1000$ in Table 8. Tables 7 and 8 show that the numerical results have second-order accurate in spatial and temporal directions. Figure 1 presents curves of $u(x,t)$ with respect to x at different time with the step sizes $h = 0.5$ and $\tau = 0.02$. All the data are referred to MATLAB codes in Example 4 in the supplementary files.

5. Conclusion

In the article, we establish an efficient finite difference scheme for nonlinear spatial fractional Sobolev equation based on Newton linearized technique. We have proved that the numerical solution of the scheme is unique solvable, stable, and convergent. The pointwise error estimate is proved with the convergence order $O(\tau^2 + h^2)$. Extensive numerical examples are carried out to testify the numerical theoretical results. Extending the current work to high dimensional cases is possible, which will leave as our future work.

Data Availability

All the data are available and referred to the supplementary file.

Conflicts of Interest

The authors declare that they have no competing interests.

Acknowledgments

This work was supported by Zhejiang Provincial Natural Science Foundation of China under Grant LY18D060009.

References

[1] R. Gorenflo and F. Mainardi, "Random walk models for space-fractional diffusion processes," *Fractional Calculus and Applied Analysis*, vol. 1, no. 2, pp. 167–191, 1998.

[2] R. E. Ewing, "A coupled non-linear hyperbolic-Sobolev system," *Annali di Matematica Pura ed Applicata*, vol. 114, no. 1, pp. 331–349, 1977.

[3] T. W. Ting, "A cooling process according to two-temperature theory of heat conduction," *Journal of Mathematical Analysis and Applications*, vol. 45, no. 1, pp. 23–31, 1974.

[4] D. W. Taylor, *Research on Consolidation of Clays*, Massachusetts Institute of Technology Press, Cambridge, 1942.

[5] X. Chen, J. Duan, and D. Li, "A Newton linearized compact finite difference scheme for one class of Sobolev equations," *Numerical Methods for Partial Differential Equations*, vol. 34, no. 3, pp. 1093–1112, 2018.

[6] M. O. Korpusov and A. G. Sveshnikov, "Blow-up of solutions of abstract Cauchy problems for nonlinear operator differential equations," *Doklady Akademii Nauk*, vol. 195, pp. 12–15, 2005.

[7] C. Çelik and M. Duman, "Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative," *Journal of Computational Physics*, vol. 231, no. 4, pp. 1743–1750, 2012.

[8] D. Wang, A. Xiao, and W. Yang, "Maximum-norm error analysis of a difference scheme for the space fractional CNLS," *Applied Mathematics and Computation*, vol. 257, pp. 241–251, 2015.

[9] M. Ran and Y. He, "Linearized Crank-Nicolson method for solving the nonlinear fractional diffusion equation with multi-delay," *International Journal of Computer Mathematics*, vol. 95, pp. 1–14, 2017.

[10] P. Wang and C. Huang, "A conservative linearized difference scheme for the nonlinear fractional Schrödinger equation," *Numerical Algorithms*, vol. 69, no. 3, pp. 625–641, 2015.

[11] Q. Zhang, M. Chen, Y. Xu, and D. Xu, "Compact θ-method for the generalized delay diffusion equation," *Applied Mathematics and Computation*, vol. 316, pp. 357–369, 2018.

[12] Q. Zhang, X. Lin, K. Pan, and Y. Ren, "Linearized ADI schemes for two-dimensional space-fractional nonlinear Ginzburg-Landau equation," *Computers and Mathematics with Applications*, vol. 80, no. 5, pp. 1201–1220, 2020.

[13] Q. Zhang and T. Li, "Asymptotic stability of compact and linear $$\theta $$-methods for space fractional delay generalized diffusion equation," *Journal of Scientific Computing*, vol. 81, no. 3, pp. 2413–2446, 2019.

[14] K. Kirkpatrick, E. Lenzmann, and G. Staffilani, "On the continuum limit for discrete NLS with long-range lattice interactions," *Communications in Mathematical Physics*, vol. 317, no. 3, pp. 563–591, 2013.

[15] H. Sun, Z. Sun, and G. Gao, "Some high order difference schemes for the space and time fractional Bloch-Torrey equations," *Applied Mathematics and Computation*, vol. 281, pp. 356–380, 2016.

[16] Q. Zhang, Y. Ren, X. Lin, and Y. Xu, "Uniform convergence of compact and BDF methods for the space fractional semilinear delay reaction-diffusion equations," *Applied Mathematics and Computation*, vol. 358, pp. 91–110, 2019.

[17] P. Wang and C. Huang, "An energy conservative difference scheme for the nonlinear fractional Schrödinger equations," *Journal of Computational Physics*, vol. 293, pp. 238–251, 2015.

[18] P. Wang, C. Huang, and L. Zhao, "Point-wise error estimate of a conservative difference scheme for the fractional Schrödinger equation," *Journal of Computational and Applied Mathematics*, vol. 306, pp. 231–247, 2016.

[19] Q. Zhang, M. Ran, and D. Xu, "Analysis of the compact difference scheme for the semilinear fractional partial differential equation with time delay," *Applicable Analysis*, vol. 96, no. 11, pp. 1867–1884, 2017.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 5588601, 13 pages
https://doi.org/10.1155/2021/5588601

Research Article

The New Semianalytical Technique for the Solution of Fractional-Order Navier-Stokes Equation

Nehad Ali Shah,[1] **Mounirah Areshi**,[2] **Jae Dong Chung**,[1] **and Kamsing Nonlaopon**[3]

[1]*Department of Mechanical Engineering, Sejong University, Seoul 05006, Republic of Korea*
[2]*Mathematics Department, College of Science, University of Tabuk, Tabuk, Saudi Arabia*
[3]*Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*

Correspondence should be addressed to Kamsing Nonlaopon; nkamsi@kku.ac.th

Received 1 March 2021; Revised 29 March 2021; Accepted 3 April 2021; Published 20 April 2021

Academic Editor: Qifeng Zhang

In this paper, we introduce a modified method which is constructed by mixing the residual power series method and the Elzaki transformation. Precisely, we provide the details of implementing the suggested technique to investigate the fractional-order nonlinear models. Second, we test the efficiency and the validity of the technique on the fractional-order Navier-Stokes models. Then, we apply this new method to analyze the fractional-order nonlinear system of Navier-Stokes models. Finally, we provide 3-D graphical plots to support the impact of the fractional derivative acting on the behavior of the obtained profile solutions to the suggested models.

1. Introduction

The fractional-order Navier-Stokes equation (NSE) has been extensively analyzed. These equations model the fluid motion defined by several physical processes, such as the movement of blood, the ocean's current, the flow of liquid in vessels, and the airflow around an aircraft's arms [1–3]. The classical NSEs were generalized by El-Shahed and Salem [4] by replacing the first time derivative with a Caputo fractional derivative of order α, where $0 < \alpha \leq 1$. Using Hankel transform, Fourier sine transform, and Laplace transform, the researchers achieved the exact solution for three different equations. In 2006, Momani and Odibat [5] solve fractional-order NSEs using the Adomian decomposition method. Ganji et al. [6] applied an analytical technique, the homotopy perturbation method, for solving the fractional-order NSEs in polar coordinates, and the results achieved were expressed in a closed form. Singh and Kumar [7] solved the fractional-order reduced differential transformation method (FRDTM) to achieve an approximated analytical result of fractional-order multidimensional NSE. Oliveira and Oliveira [8] analyzed the residual power series method (RPSM) to find the result of the nonlinear fractional-order two-dimensional NSEs. Zhang and Wang [9] suggested numerical analysis for a class of NSEs with fractional-order derivatives; Ravindran, the exact boundary controllability of Galerkin approximations of a Navier-Stokes system for soret convection [10]; and Cibik and Yilmaz, the Brezzi-Pitkaranta stabilization and a priori error analysis for the Stokes control [11].

Some researchers mix two powerful techniques to achieve another result technique to solve equations and systems of fractional-order NSEs. Below, we define some of these combinations: a combination of the Laplace transformation and Adomian decomposition method; Kumar et al. [12] introduced the homotopy perturbation transform method (HPTM), combined Laplace transformation with the homotopy perturbation method, and solved fractional-order NSEs in a tube. Jena and Chakraverty [13] implemented the homotopy perturbation transformation method (HPETM), and this technique consists in the mixture of Elzaki transformation technique and homotopy perturbation technique; Prakash et. al [1] suggested q-homotopy analysis transformation technique to achieve a result of coupled fractional-order NSEs. This technique mixture of the Laplace transformation and residual power series method is defined:

$$\begin{cases} D_\tau^\alpha u + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u, & 0 < \alpha \leq 1, \\ \nabla u = 0, \end{cases} \quad (1)$$

where $D_\tau^\alpha u$ is the Caputo derivative of order α, u is the velocity vector, τ is the time, ν is kinematics viscosity, p is the pressure, and ρ is the density.

In this work, we consider two special cases. First, we consider unsteady, one-dimensional motion of a viscous fluid in a tube. The fractional-order Navier-Stokes equations in cylindrical coordinates that governs the flow field in the tube are given by

$$D_\tau^\alpha u + P + \nu \left(\frac{\partial^2 u}{\partial \psi^2} + \frac{1}{\psi} \frac{\partial u}{\partial \psi} \right), \quad 0 < \alpha \leq 1, \quad (2)$$

with initial condition

$$u(\psi, 0) = g(\psi), \quad (3)$$

where $P = -1/\rho \partial p / \partial z$ and $g(\psi)$ is a function depending only on ψ.

Consider that the fractional-order two-dimensional Navier-Stokes equations is defined as

$$\begin{aligned} D_\tau^\alpha u &= \rho_0 \left(\frac{\partial^2}{\partial \psi^2} u + \frac{\partial^2}{\partial \varphi^2} u \right) - u \frac{\partial}{\partial \psi} u - \nu \frac{\partial}{\partial \varphi} u + g, \\ D_\tau^\alpha \nu &= \rho_0 \left(\frac{\partial^2}{\partial \psi^2} \nu + \frac{\partial^2}{\partial \varphi^2} \nu \right) - u \frac{\partial}{\partial \psi} \nu - \nu \frac{\partial}{\partial \varphi} \nu - g, \end{aligned} \quad (4)$$

with initial conditions

$$\begin{aligned} u(\psi, \varphi, \tau) &= f(\psi, \varphi), \\ \nu(\psi, \varphi, \tau) &= g(\psi, \varphi), \end{aligned} \quad (5)$$

where $u = u(\psi, \varphi, \tau)$, $\nu = \nu(\psi, \varphi, \tau)$, ρ, τ, p denote as constant density, time, and pressure, respectively. ψ, φ are the spatial components, and $f(\psi, \varphi)$ and $g(\psi, \varphi)$ are two functions depending only on ψ and φ.

The residual power series method (RPSM) is a simple and efficient technique for constructing a power series result for extremely linear and nonlinear equations without perturbation, linearization, and discretization. Unlike the classical power series technique, the RPS approach does not need to compare the coefficients of the corresponding terms and a recursion relation is not required. This approach calculates the power series coefficients by a series of algebraic equations of one or more variables, and its reliance on derivation, which is much simpler and more precise than integration, which is the basis of most other solution approaches, is the main advantage of this methodology. This method is, in effect, an alternative strategy for obtaining theoretical results for the fractional-order partial differential equations [14].

The RPSM was introduced as an essential tool for assessing the power series solution's values for the first and second-order fuzzy DEs [15]. It has been successfully implemented in the approximate result of the generalized Lane-Emden equation [16], which is a highly nonlinear singular DE, in the inaccurate work of higher-order regular DEs [17], in the solution of composite and noncomposite fractional-order DEs [18], in predicting and showing the diversity of results to the fractional-order boundary value equations [19], and in the numerical development of the nonlinear fractional-order KdV and Burgers equation [20], in addition to some other implementations [21–23], and recently, it has been applied to investigate the approximate result of a fractional-order two-component evolutionary scheme [24].

This paper introduces the modified analytical technique: the residual power series transform method (RPSTM) is implemented to investigate the fractional-order NS equations. The result of certain illustrative cases is discussed to explain the feasibility of the suggested method. The results of fractional-order models and integral-order models are defined by using the current techniques. The new approach has lower computing costs and higher rate convergence. The suggested method is also constructive for addressing other fractional orders of linear and nonlinear PDEs.

2. Preliminaries

Definition 1. The Abel-Riemann of fractional operator D^α of order α is given as [25–27]

$$D^\alpha \nu(\zeta) = \begin{cases} \frac{d^j}{d\zeta^j} \nu(\zeta), & \alpha = j, \\ \frac{1}{\Gamma(j-\alpha)} \frac{d}{d\zeta^j} \int_0^\zeta \frac{\nu(\zeta)}{(\zeta - \psi)^{\alpha - j + 1}} d\psi, & j - 1 < \alpha < j, \end{cases} \quad (6)$$

where $j \in \mathbb{Z}^+$, $\alpha \in \mathbb{R}^+$ and

$$D^{-\alpha} \nu(\zeta) = \frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \psi)^{\alpha - 1} \nu(\psi) d\psi, \quad 0 < \alpha \leq 1. \quad (7)$$

Definition 2. The fractional-order Abel-Riemann integration operator J^α is defined as [25–27]

$$J^\alpha \nu(\zeta) = \frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \psi)^{\alpha - 1} \nu(\zeta) d\zeta, \quad \zeta > 0, \alpha > 0. \quad (8)$$

The operator of basic properties

$$\begin{aligned} J^\alpha \zeta^j &= \frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)} \zeta^{j+\psi}, \\ D^\alpha \zeta^j &= \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} \zeta^{j-\psi}. \end{aligned} \quad (9)$$

Definition 3. The Caputo fractional operator ${}^C D^\alpha$ of α is defined as [25–27]

$$ {}^{C}D^{\alpha}\nu(\zeta)=\begin{cases}\dfrac{1}{\Gamma(j-\alpha)}\displaystyle\int_0^{\zeta}\dfrac{\nu^{j}(\psi)}{(\zeta-\psi)^{\alpha-j+1}}d\psi, & j-1<\alpha<j,\\ \dfrac{d^{j}}{d\zeta^{j}}\nu(\zeta), & j=\alpha.\end{cases} \tag{10} $$

Definition 4. The fractional-order Caputo operator of Elzaki transform is given as [25–27]

$$ E\left[D_{\zeta}^{\alpha}g(\zeta)\right]=s^{-\alpha}E[g(\zeta)]-\sum_{k=0}^{j-1}s^{2-\alpha+k}g^{(k)}(0),\ \text{where}\ j-1<\alpha<j. \tag{11} $$

Definition 5. A power series definition of the form [14]

$$ \sum_{m=0}^{\infty}P_m(\tau-\psi)^{m\alpha}=P_0+P_1(\tau-\psi)+P_2(\tau-\psi)^{2\alpha}+\cdots, \tag{12} $$

where $0\le m-1<\alpha\le m$ and $\tau\le\psi$ is called fractional power series (FPS) about ψ, where P_m are the constants called the coefficients of the series. If $\psi=0$, then the fractional power series will be reduced to the fractional Maclaurin series.

Theorem 6. *Assume that f_o has a fractional power series representation at $\tau=\psi$ of the form [14]*

$$ f_o(\tau)=\sum_{m=0}^{\infty}P_m(\tau-\psi)^{m\alpha}. \tag{13} $$

For $m\in NU\{0\}$, if $D^{m\alpha}f_o(\tau)$ are continuous on $(\psi,\psi+R_o)$, then the coefficients P_m can be written as

$$ P_m=\frac{D^{m\alpha}f_o(\psi)}{\Gamma(1+m\alpha)}, \tag{14} $$

where $\psi\le\tau<\psi+R_o$, and R_o is the radius of convergence.

Definition 7. The expansion of power series of the form [14]

$$ \sum_{m=0}^{\infty}G_m(\phi)(\tau-\psi)^{m\alpha} \tag{15} $$

is said to be multifractional power series at $\tau=\psi$, where $G_m(\phi)$ are the coefficients of multifractional power series.

Theorem 8. *Let us assume that $u_o(\phi,\tau)$ which has the multifractional power representation at $\tau=\psi$ can be written as [14]*

$$ u_o(\phi,\tau)=\sum_{m=0}^{\infty}G_m(x)(\tau-\psi)^{m\alpha}. \tag{16} $$

For $m\in NU\{0\}$, if $D_{\tau}^{m\alpha}u_o(\phi,\tau)$ are the continuous on $I_o\times(\psi,\psi+R_o)$, then the coefficients G_m are given by

$$ G_m(x)=\frac{D_{\tau}^{m\alpha}u_o(\phi,\tau)}{\Gamma(1+m\alpha)}, \tag{17} $$

where $\psi\in I_o$ and $\psi\le\tau<\psi+R_o$.

So, we can write the fractional power expansion of $u_o(\phi,\tau)$ of the form

$$ \mu(\phi,\tau)=\sum_{m=0}^{\infty}\frac{D_{\tau}^{m\alpha}\mu(\phi,\tau)}{\Gamma(1+m\alpha)}(\tau-\psi)^{m\alpha}, \tag{18} $$

which is the generalized Taylor expansion. If we consider $\alpha=1$, then the generalized Taylor formula will be converted to classical Taylor series.

Corollary 9. *Let us assume that $\mu(\phi,\varphi,\tau)$ has a multifractional power series representation about $\tau=\psi$ as [14]*

$$ u_o(\phi,\varphi,\tau)=\sum_{m=0}^{\infty}G_m(\phi,\varphi)(\tau-\psi)^{m\alpha}. \tag{19} $$

For $m\in NU\{0\}$ if $D_{\tau}^{m\alpha}\mu(\phi,\varphi,\tau)$ are continuous on $I_1\times I_2\times(\psi,\psi+R_o)$, then

$$ G_m(\psi,\varphi)=\frac{D_{\tau}^{m\alpha}\mu(\phi,\varphi,\tau)}{\Gamma(1+m\alpha)}, \tag{20} $$

where $(\phi,\varphi)\in I_1\times I_2,\psi\le\tau<\psi+R_o$.

3. The Procedure of RPSTM

In this section, we explain the steps of RPSTM for solving the fractional-order partial differential equation

$$ D_{\tau}^{\alpha}u(\psi,\tau)=aD_{\psi}^{2}u(\psi,\tau)+bu(\psi,\tau)-cu^{q}(\psi,\tau), \tag{21} $$

with initial condition

$$ u(\psi,0)=f_0(\psi). \tag{22} $$

First, we use the Elzaki transform to (21); we get

$$ \mathscr{E}[D_{\tau}^{\alpha}u(\psi,\tau)]=a\mathscr{E}\left[D_{\psi}^{2}u(\psi,\tau)\right]+b\mathscr{E}[u(\psi,\tau)]-c\mathscr{E}[u^{q}(\psi,\tau)]. \tag{23} $$

By the fact that $\mathscr{E}[D_{\tau}^{\alpha}u(\psi,\tau)]=1/s^{\alpha}\mathscr{E}[u(\psi,\tau)]-s^{1-\alpha}u(x,0)$ and using the initial condition (22), we can write (23) as

$$ \begin{aligned}U(\psi,s)={}&s^{2}f_0(\psi)+s^{\alpha}aD_x^{2}U(\psi,s)+bs^{\alpha}U(\psi,s)\\&-cs^{\alpha}\mathscr{E}^{-1}[(\mathscr{E}[U(\psi,s)])^{q}],\end{aligned} \tag{24} $$

where $U(\psi,s)=\mathscr{E}[u(\psi,\tau)]$.

Second, we define the transform function $U(\psi,s)$ as the following formula:

$$ U(\psi,s)=\sum_{n=0}^{\infty}s^{n\alpha+1}f_n(x). \tag{25} $$

We write the kth truncated series of (25) as

$$U(\psi,s)=\sum_{n=0}^{\infty}s^{n\alpha+1}f_n(x)=s^2f_0(x)+\sum_{n=1}^{\infty}s^{n\alpha+1}f_n(x). \quad (26)$$

As stated in [25], the definition of Elzaki residual function to (25) is

$$\mathscr{E}\mathrm{Res}(\psi,s)=U(\psi,s)-f_0(x)s^2-as^{\alpha}D_x^2U(\psi,s)-bs^{\alpha}U(\psi,s)+cs^{\alpha}\mathscr{E}^{-1}[(\mathscr{E}[U(\psi,s)])^q], \quad (27)$$

and the kth Elzaki residual function of (27) is

$$\mathscr{E}\mathrm{Res}(\psi,s)=U_k(\psi,s)-f_0(x)s^2-as^{\alpha}D_x^2U_k(\psi,s)-bs^{\alpha}U_k(\psi,s)+cs^{\alpha}\mathscr{E}\left[\left(\mathscr{E}^{-1}[U_k(\psi,s)]\right)^q\right]. \quad (28)$$

Third, we expand a few of the properties arising in the basic RPSM to find out certain facts:

(i) $\mathscr{E}\mathrm{Res}(\psi,s)=0$ and $\lim_{k\to\infty}\mathscr{E}\ \mathrm{Re}\ s_k(\psi,s)=\mathscr{E}\mathrm{Res}(\psi,s)$ for each $s>0$

$$\lim_{k\to\infty}\mathscr{E}\mathrm{Res}(\psi,s)=0\Rightarrow\lim_{k\to\infty}s\mathscr{E}\ \mathrm{Re}\ s_k(\psi,s)=0 \quad (29)$$

(ii) $\lim_{k\to\infty}s^{k\alpha+1}\mathscr{E}\mathrm{Res}(\psi,s)=\lim_{k\to\infty}s^{k\alpha+1}\mathscr{E}\ \mathrm{Re}\ s_k(\psi,s)=0,\ 0<\alpha\le 1,\ k=1,2,3,\cdots$

Furthermore, to evaluate the coefficient functions $f_n(\psi)$, we can recursively solve the following scheme

$$\lim_{s\to\infty}\left(s^{k\alpha+1}\mathscr{E}\ \mathrm{Re}\ s_k(\psi,s)\right)=0,\quad 0<\alpha\le 1, k=1,2,3,\cdots. \quad (30)$$

Finally, we implemented the Elzaki inverse to $U_k(\psi,s)$ to achieve the kth approximate supportive solution $u_k(\psi,\tau)$.

4. Numerical Results

Example 1. Consider the time-fractional-order one-dimensional NS equation of the form

$$D_\tau^{\alpha}u(\psi,\tau)=P+\frac{\partial^2u}{\partial\psi^2}+\frac{1}{\psi}\frac{\partial u}{\partial\psi},\quad 0<\alpha\le 1. \quad (31)$$

Subject to the initial condition

$$u(\psi,0)=1-\psi^2. \quad (32)$$

Applying Elzaki transform to (31) and using the initial condition given in (32), we get

$$U(\psi,s)=s^2\left(1-\psi^2\right)+s^{\alpha}\mathscr{E}_\tau[P]+s^{\alpha}\mathscr{E}_\tau\cdot\left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}U(\psi,s)\right\}\right]+s^{\alpha}\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{\frac{1}{\psi}\frac{\partial}{\partial\psi}U(\psi,s)\right\}\right]. \quad (33)$$

The kth truncated term series of (33) is

$$U_k(\psi,s)=s^2\left(1-\psi^2\right)+\sum_{n=1}^{k}s^{n\alpha+2}f_n(\psi), \quad (34)$$

and the kth Elzaki residual function is

$$\mathscr{E}_\tau\ \mathrm{Re}\ s_k=U_k(\psi,s)-s^2\left(1-\psi^2\right)-s^{\alpha+2}P-s^{\alpha}\mathscr{E}_\tau\cdot\left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}U_k(\psi,s)\right\}\right]-s^{\alpha}\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{\frac{1}{\psi}\frac{\partial}{\partial x}U_k(\psi,s)\right\}\right]. \quad (35)$$

Now, to determine $f_k(\psi)$, $k=1,2,3,\cdots$, we substitute the kth truncated series (34) into the kth Elzaki residual function (35), multiply the resulting equation by $s^{k\alpha+2}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\alpha+2}\ \mathrm{Re}\ s_k(\psi,s)]=0$, $k=1,2,3,\cdots$, for $f_k(\psi)$. The following are the first several components of the series $f_k(\psi,\varphi)$:

$$\begin{aligned}f_1(\psi)&=p-4,\\ f_2(\psi)&=0,\\ f_3(\psi)&=0,\\ &\vdots.\end{aligned} \quad (36)$$

Putting the values of $f_n(\psi)(n\ge 1)$ in (34), we get

$$U(\psi,s)=s^2\left(1-\psi^2\right)+s^{\alpha+2}f_1(\psi)+s^{2\alpha+2}f_2(\psi)+s^{3\alpha+2}f_3(\psi)+\cdots,$$
$$U(\psi,s)=s^2\left(1-\psi^2\right)+s^{\alpha+2}(P-4)+s^{2\alpha+2}(0)+s^{3\alpha+2}(0)+\cdots, \quad (37)$$

$$U(\psi,s)=s^2\left(1-\psi^2\right)+s^{\alpha+2}(P-4). \quad (38)$$

Using inverse Elzaki transform to (38), we get

$$u(\psi,\tau)=1-\psi^2+\frac{(P-4)\tau^{\alpha}}{\Gamma(\alpha+2)}. \quad (39)$$

Putting $\alpha=1$, we have

$$u(\psi,\tau)=1-\psi^2+(P-4)\tau. \quad (40)$$

In Figure 1, the RPSTM and the exact results of Example 1 at $\alpha=1$ are shown by plots (a) and (b), respectively. From the given figures, it can be seen that both the exact and the EDM results are in close contact with each other. Also, in the Figure 2 subgraph, the RPSTM results of Example 1 are calculated at different fractional-order $\alpha=0.8$ and 0.6. It is investigated that fractional-order problem results are

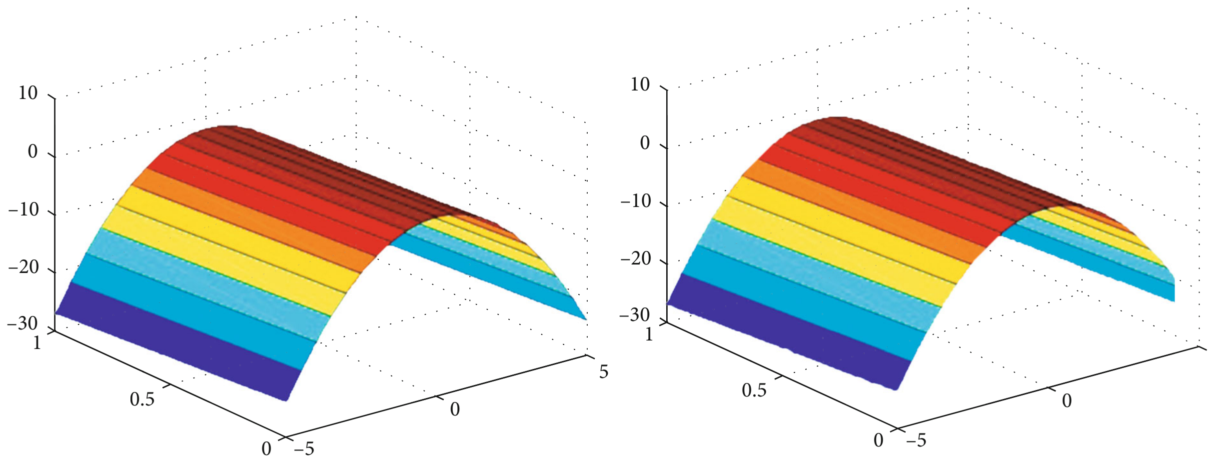


Figure 1: Graph of exact and analytical results of Problem 1.

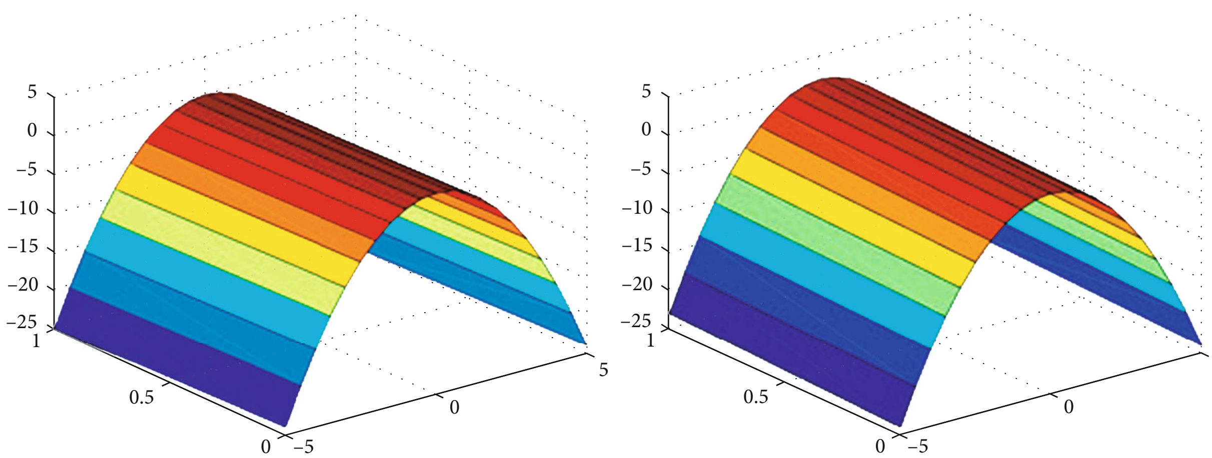


Figure 2: The fractional order of $\alpha = 0.8$ and 0.6 of Problem 1.

convergent to an integer-order effect as fractional-order analysis to integer-order. The same phenomenon of convergence of fractional-order solutions towards integral-order solutions is observed.

Example 2. Consider the fractional-order one-dimensional NS equation of the form

$$D_\tau^\alpha u(\psi,\tau) = \frac{\partial^2 u}{\partial \psi^2} + \frac{1}{\psi}\frac{\partial u}{\partial \psi}, \quad 0 < \alpha \le 1. \tag{41}$$

Subject to the initial condition,

$$u(\psi, 0) = \psi. \tag{42}$$

Applying Elzaki transform to (41) and using the initial condition given in (42), we get

$$U(\psi,s) = s^2(\psi) + s^\alpha \mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}U(\psi,s)\right\}\right] + s^\alpha \mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{\frac{1}{\psi}\frac{\partial}{\partial x}U(\psi,s)\right\}\right]. \tag{43}$$

The kth truncated term series of (43) is

$$U_k(\psi,s) = s^2(\psi) + \sum_{n=1}^{k} s^{n\alpha+2} f_n(\psi), \tag{44}$$

and the kth Elzaki residual function is

$$\mathscr{E}_\tau \operatorname{Re} s_k = U_k(\psi,s) - s^2(\psi) - s^\alpha \mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}U_k(\psi,s)\right\}\right] - s^\alpha \mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{\frac{1}{\psi}\frac{\partial}{\partial x}U_k(\psi,s)\right\}\right]. \tag{45}$$

Now, to determine $f_k(\psi)$, $k = 1, 2, 3, \cdots$, we substitute the kth truncated series (44) into the kth Elzaki residual function (45), multiply the resulting equation by $s^{k\alpha+2}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\alpha+2} \operatorname{Re} s_k(\psi,s)] = 0$, $k = 1, 2, 3, \cdots$, for $f_k(\psi)$. The following are the first several components of the series $f_k(\psi,\varphi)$:

$$\begin{aligned} f_1(\psi) &= \frac{1}{\psi}, \\ f_2(\psi) &= \frac{1}{\psi^3}, \\ f_3(\psi) &= 3^3\frac{1}{\psi^5}, \\ f_4(\psi) &= 5^2\frac{1}{\psi^7}, \\ &\vdots . \end{aligned} \tag{46}$$

Putting the values of $f_n(\psi)(n \geq 1)$ in (44), we get

$$U(\psi, s) = s^2(\psi) + s^{\alpha+2}f_1(\psi) + s^{2\alpha+2}f_2(\psi) + s^{3\alpha+2}f_3(\psi) + \cdots, \tag{47}$$

$$U(\psi, s) = s^2(\psi) + \frac{s^{\alpha+2}}{\psi} + \frac{s^{2\alpha+2}}{\psi^3} + \frac{3^2 s^{3\alpha+2}}{\psi^5} + \cdots. \tag{48}$$

Using inverse Elzaki transform to (48), we get

$$u(\psi, \tau) = \psi + \frac{1}{\psi}\frac{\tau^\alpha}{\Gamma(\alpha+2)} + \frac{1}{\psi^3}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+2)} + \frac{3^2}{\psi^5}\frac{\tau^{3\alpha}}{\Gamma(3\alpha+2)} + \cdots. \tag{49}$$

Putting $\alpha = 1$, we have

$$\begin{aligned} u(\psi, \tau) &= \psi + \frac{1}{\psi}t + \frac{1}{\psi^3}\frac{\tau^2}{2!} + \frac{3^2}{\psi^5}\frac{\tau^3}{3!} + \cdots, \\ u(\psi, \tau) &= \psi + \sum_{n=1}^{\infty}\frac{1^2 \times 3^2 \times 5^3 \times \cdots \times (2n-3)^2}{r^{2n-1}}\frac{\tau^n}{n!}. \end{aligned} \tag{50}$$

In Figure 3, the RPSTM and the exact results of Example 2 at $\alpha = 1$ are shown by graphs, respectively. From the given figures, it can be seen that both the exact and the EDM results are in close contact with each other. Also, in the Figure 4 subgraph, the RPSTM results of Example 2 are calculated at different fractional-order $\alpha = 0.8$ and 0.6. It is investigated that fractional-order problem results are convergent to an integer-order effect as fractional-order analysis to integer-order. The same phenomenon of convergence of fractional-order solutions towards integral-order solutions is observed.

Example 3. Consider the fractional-order two-dimensional NS equation of the form

$$\begin{aligned} D_\tau^\alpha u &= \rho_0\left(\frac{\partial^2}{\partial\psi^2}u + \frac{\partial^2}{\partial\varphi^2}u\right) - u\frac{\partial}{\partial\psi}u - v\frac{\partial}{\partial\varphi}u + g, \\ D_\tau^\alpha v &= \rho_0\left(\frac{\partial^2}{\partial\psi^2}v + \frac{\partial^2}{\partial\varphi^2}v\right) - u\frac{\partial}{\partial\psi}v - v\frac{\partial}{\partial\varphi}v - g, \end{aligned} \tag{51}$$

with initial condition

$$\begin{aligned} u(\psi, \varphi, 0) &= -\sin(\psi + \varphi), \\ v(\psi, \varphi, 0) &= \sin(\psi + \varphi). \end{aligned} \tag{52}$$

Applying Elzaki transform to (51) and using (52), we get

$$\begin{aligned} U(\psi, \varphi, s) &= s^2(-\sin(\psi+\varphi)) + \rho_0 s^\alpha \mathscr{E}_\tau \\ &\cdot \left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}U_k(\psi, \varphi, s) + \frac{\partial^2}{\partial\varphi^2}U_k(\psi, \varphi, s)\right\}\right] \\ &- s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{U(\psi, \varphi, s)\frac{\partial}{\partial\psi}U(\psi, \varphi, s)\right\}\right] \\ &- s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{v(\psi, \varphi, s)\frac{\partial}{\partial\varphi}U(\psi, \varphi, s)\right\}\right] + s^\alpha\mathscr{E}_\tau[g], \end{aligned}$$

$$\begin{aligned} V(\psi, \varphi, s) &= s^2(\sin(\psi+\varphi)) + \rho_0 s^\alpha \mathscr{E}_\tau \\ &\cdot \left[\mathscr{E}_\tau^{-1}\left(\frac{\partial^2}{\partial\psi^2}V(\psi, \varphi, s) + \frac{\partial^2}{\partial\varphi^2}V(\psi, \varphi, s)\right)\right] \\ &- s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{U(\psi, \varphi, s)\frac{\partial}{\partial\psi}V(\psi, \varphi, s)\right\}\right] \\ &- s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left(V(\psi, \varphi, s)\frac{\partial}{\partial\varphi}V(\psi, \varphi, s)\right)\right] - s^\alpha\mathscr{E}_\tau[g]. \end{aligned} \tag{53}$$

The kth truncated term series of (53) is

$$\begin{aligned} U_k(\psi, \varphi, s) &= -\sin(\psi+\varphi)s^2 + \sum_{n=1}^{k} s^{n\alpha+2}f_n(\psi, \varphi), \\ V_k(\psi, \varphi, s) &= \sin(\psi+\varphi)s^2 + \sum_{n=1}^{k} s^{n\alpha+2}g_n(\psi, \varphi), \end{aligned} \tag{54}$$

and the kth Elzaki residual function is

$$\begin{aligned} \mathscr{E}_\tau \operatorname{Re} s_k(\psi, \varphi, s) &= U_k(\psi, \varphi, s) - (-\sin(\psi+\varphi))s^2 - \rho_0 s^\alpha\mathscr{E}_\tau \\ &\cdot \left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}U_k(\psi, \varphi, s) + \frac{\partial^2}{\partial\varphi^2}U_k(\psi, \varphi, s)\right\}\right] \\ &+ s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{U_k(\psi, \varphi, s)\frac{\partial}{\partial\psi}U_k(\psi, \varphi, s)\right\}\right] \\ &+ s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{V_k(\psi, \varphi, s)\frac{\partial}{\partial\varphi}U_k(\psi, \varphi, s)\right\}\right] \\ &- gs^{\alpha+2}, \end{aligned}$$

$$\begin{aligned} \mathscr{E}_\tau \operatorname{Re} s_k(\psi, \varphi, s) &= V_k(\psi, \varphi, s) - \sin(\psi+\varphi)s^2 - \rho_0 s^\alpha\mathscr{E}_\tau \\ &\cdot \left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}V_k(\psi, \varphi, s) + \frac{\partial^2}{\partial\varphi^2}V_k(\psi, \varphi, s)\right\}\right] \\ &+ s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{U_k(\psi, \varphi, s)\frac{\partial}{\partial\psi}V_k(\psi, \varphi, s)\right\}\right] \\ &+ s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{V_k(\psi, \varphi, s)\frac{\partial}{\partial\varphi}V_k(\psi, \varphi, s)\right\}\right] \\ &+ gs^{\alpha+2}. \end{aligned} \tag{55}$$

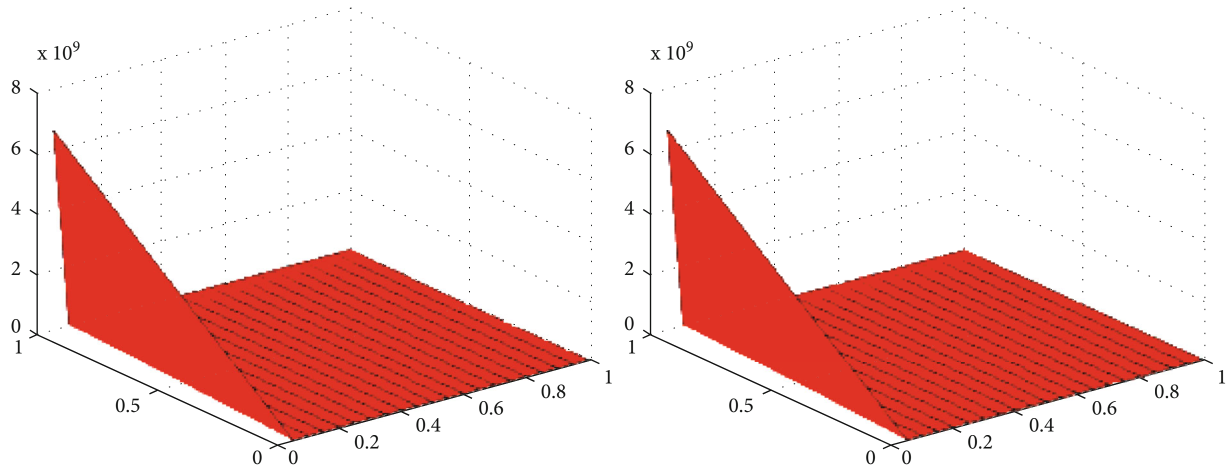


Figure 3: Graph of exact and analytical results of Problem 2.

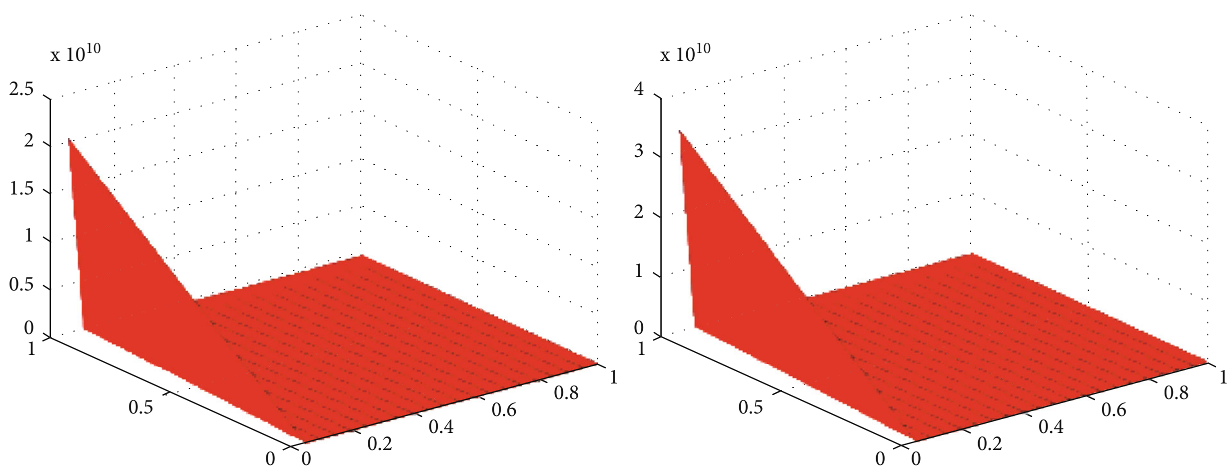


Figure 4: The fractional order of $\alpha = 0.8$ and 0.6 of Problem 2.

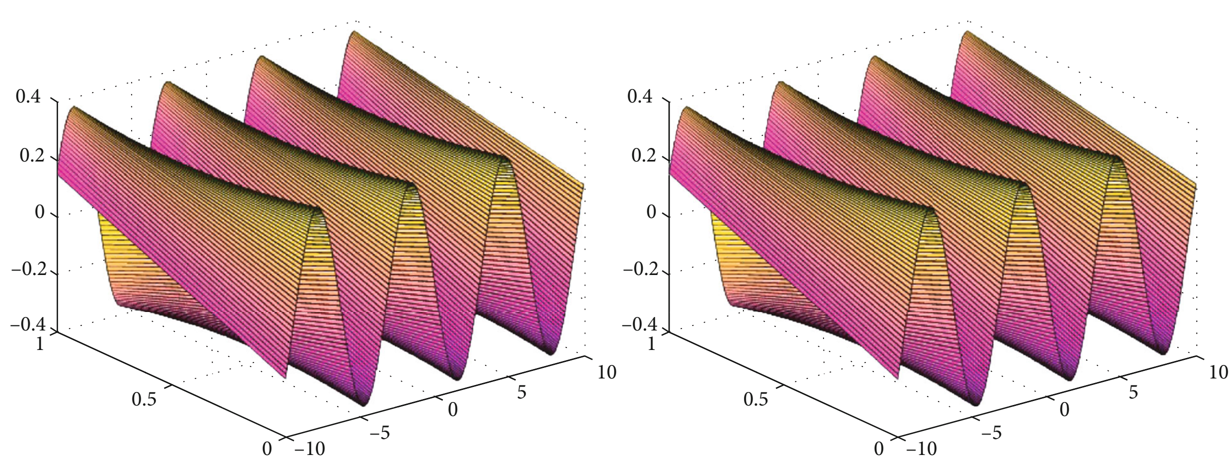


Figure 5: Graph of exact and analytical results of Problem 3.

Now, to determine $f_k(\psi, \varphi)$ and $g_k(\psi, \varphi), k = 1, 2, 3, \cdots,$ we substitute the kth truncated series (54) into the kth Elzaki residual function (55), multiply the resulting equation by $s^{k\alpha+2}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\alpha+2} \operatorname{Re} s_k(\psi, \varphi, s)] = 0, k = 1, 2, 3, \cdots,$ for f_k and g_k. The following are the first several components of the series $f_k(\psi, \varphi)$ and $g_k(\psi, \varphi)$:

$$\begin{aligned}
f_1(\psi,\varphi) &= 2\rho_0 \sin(\psi+\varphi) + g,\\
g_1(\psi,\varphi) &= -2\rho_0 \sin(\psi+\varphi) - g,\\
f_2(\psi,\varphi) &= -(2\rho_0)^2 \sin(\psi+\varphi),\\
g_2(\psi,\varphi) &= (2\rho_0)^2 \sin(\psi+\varphi),\\
f_3(\psi,\varphi) &= (2\rho_0)^3 \sin(\psi+\varphi),\\
g_3(\psi,\varphi) &= -(2\rho_0)^3 \sin(\psi+\varphi),\\
&\vdots .
\end{aligned} \tag{56}$$

Putting the values of $f_n(\psi,\varphi)$ and $g_n(\psi,\varphi)(n \geq 1)$ in (54), we have

$$\begin{aligned}
U(\psi,\varphi,s) &= -\sin(\psi+\varphi)s^2 + f_1(\psi,\varphi)s^{\alpha+2} + f_2(\psi,\varphi)s^{2\alpha+2}\\
&\quad + f_3(\psi,\varphi)s^{3\alpha+2} + \cdots,\\
V(\psi,\varphi,s) &= \sin(\psi+\varphi)s^2 + g_1(\psi,\varphi)s^{\alpha+2} + g_2(\psi,\varphi)s^{2\alpha+2}\\
&\quad + g_3(\psi,\varphi)s^{3\alpha+2} + \cdots,\\
U(\psi,\varphi,s) &= -\sin(\psi+\varphi)s^2 + (2\rho_0 \sin(\psi+\varphi) + g)s^{\alpha+2}\\
&\quad - \left((2\rho_0)^2 \sin(\psi+\varphi)\right)s^{2\alpha+2}\\
&\quad + \left((2\rho_0)^3 \sin(\psi+\varphi)\right)s^{3\alpha+2} + \cdots,\\
V(\psi,\varphi,s) &= \sin(\psi+\varphi)s^2 - (2\rho_0 \sin(\psi+\varphi) - g)s^{\alpha+2}\\
&\quad + \left((2\rho_0)^2 \sin(\psi+\varphi))\right)s^{2\alpha+2}\\
&\quad - \left((2\rho_0)^3 \sin(\psi+\varphi))\right)s^{3\alpha+2} + \cdots,\\
U(\psi,\varphi,s) &= -\sin(\psi+\varphi)\left[s^2 - 2\rho_0 s^{\alpha+2} + (2\rho_0)^2 s^{2\alpha+2}\right.\\
&\quad \left. - (2\rho_0)^3 s^{3\alpha+2} + \cdots\right] + g s^{\alpha+2},\\
V(\psi,\varphi,s) &= \sin(\psi+\varphi)\left[s^2 - 2\rho_0 s^{\alpha+2} + (2\rho_0)^2 s^{2\alpha+2}\right.\\
&\quad \left. - (2\rho_0)^3 s^{3\alpha+2} + \cdots\right] - g s^{\alpha+2}.
\end{aligned} \tag{57}$$

Using inverse Elzaki transform, we get

$$\begin{aligned}
u(\psi,\varphi,\tau) &= -\sin(\psi+\varphi)\left[1 - \frac{2\rho_0\tau^\alpha}{\Gamma(\alpha+2)} + \frac{(2\rho_0)^2\tau^{2\alpha}}{\Gamma(2\alpha+2)} - \frac{(2\rho_0)^3\tau^{3\alpha}}{\Gamma(3\alpha+2)} + \cdots\right]\\
&\quad + g\frac{\tau^\alpha}{\Gamma(\alpha+2)},\\
v(\psi,\varphi,\tau) &= \sin(\psi+\varphi)\left[1 - \frac{2\rho_0\tau^\alpha}{\Gamma(\alpha+2)} + \frac{(2\rho_0)^2\tau^{2\alpha}}{\Gamma(2\alpha+2)} - \frac{(2\rho_0)^3\tau^{3\alpha}}{\Gamma(3\alpha+2)} + \cdots\right]\\
&\quad - g\frac{\tau^\alpha}{\Gamma(\alpha+2)}.
\end{aligned} \tag{58}$$

Putting $\alpha = 1$, we get the solution in closed form

$$\begin{aligned}
u(\psi,\varphi,\tau) &= -\sin(\psi+\varphi)e^{-2\rho_0\tau} + g,\\
v(\psi,\varphi,\tau) &= \sin(\psi+\varphi)e^{-2\rho_0\tau} - g.
\end{aligned} \tag{59}$$

In Figures 5 and 6, the RPSTM and the exact results of Example 3 at $\alpha = 1$ are shown by graphs, respectively. From the given figures, it can be seen that both the exact and the RPSTM results are in close contact with each other. Also, in the Figure 7 and 8 subgraph, the RPSTM results of Example 3 are calculated at different fractional-order $\alpha = 0.8$ and 0.6. It is investigated that fractional-order problem results are convergent to an integer-order effect as fractional-order analysis to integer-order. The same phenomenon of convergence of fractional-order solutions towards integral-order solutions is observed.

Example 4. Consider the fractional-order two-dimensional NS equation as

$$\begin{aligned}
D_\tau^\alpha u &= \rho_0\left(\frac{\partial^2}{\partial\psi^2}u + \frac{\partial^2}{\partial\varphi^2}u\right) - u\frac{\partial}{\partial\psi}u - v\frac{\partial}{\partial\varphi}u + g,\\
D_\tau^\alpha v &= \rho_0\left(\frac{\partial^2}{\partial\psi^2}v + \frac{\partial^2}{\partial\varphi^2}v\right) - u\frac{\partial}{\partial\psi}v - v\frac{\partial}{\partial\varphi}v - g,
\end{aligned} \tag{60}$$

with initial condition

$$\begin{aligned}
u(\psi,\varphi,0) &= -e^{\psi+\varphi},\\
v(\psi,\varphi,0) &= e^{\psi+\varphi}.
\end{aligned} \tag{61}$$

Applying Elzaki transform to (60) and using (61), we get

$$\begin{aligned}
U(\psi,\varphi,s) &= -e^{\psi+\varphi}s^2 + \rho_0 s^\alpha \mathscr{E}_\tau\\
&\quad \cdot \left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}U_k(\psi,\varphi,s) + \partial^2\partial\varphi^2 U_k(\psi,\varphi,s)\right\}\right]\\
&\quad - s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{U(\psi,\varphi,s)\frac{\partial}{\partial\psi}U(\psi,\varphi,s)\right\}\right]\\
&\quad - s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{v(\psi,\varphi,s)\frac{\partial}{\partial\varphi}U(\psi,\varphi,s)\right\}\right]\\
&\quad + s^\alpha\mathscr{E}_\tau[g],\\
V(\psi,\varphi,s) &= e^{\psi+\varphi}s^2 + \rho_0 s^\alpha \mathscr{E}_\tau\\
&\quad \cdot \left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}V(\psi,\varphi,s) + \frac{\partial^2}{\partial\varphi^2}V(\psi,\varphi,s)\right\}\right]\\
&\quad - s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{U(\psi,\varphi,s)\frac{\partial}{\partial\psi}V(\psi,\varphi,s)\right\}\right]\\
&\quad - s^\alpha\mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{V(\psi,\varphi,s)\frac{\partial}{\partial\varphi}V(\psi,\varphi,s)\right\}\right]\\
&\quad - s^\alpha\mathscr{E}_\tau[g].
\end{aligned} \tag{62}$$

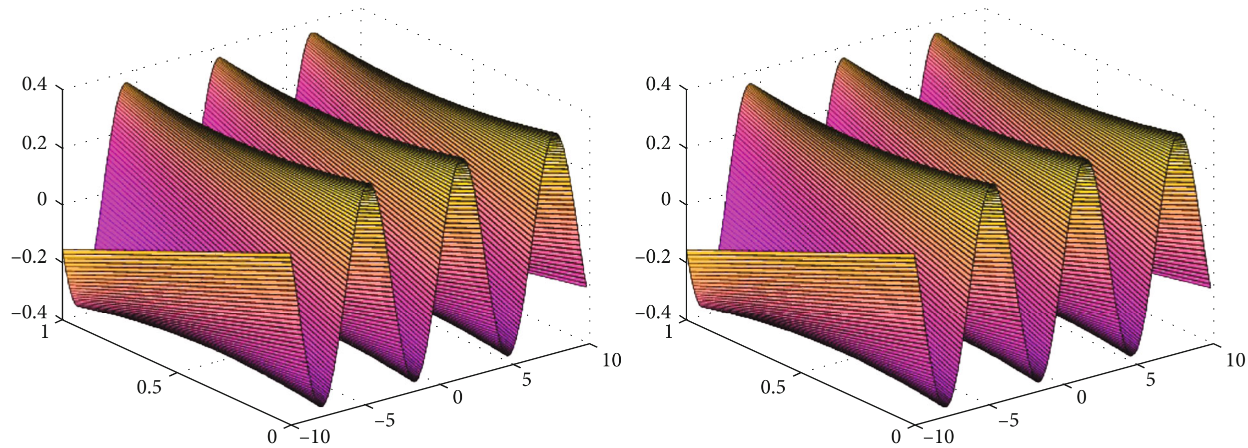


Figure 6: Graph of exact and analytical results of Problem 3.

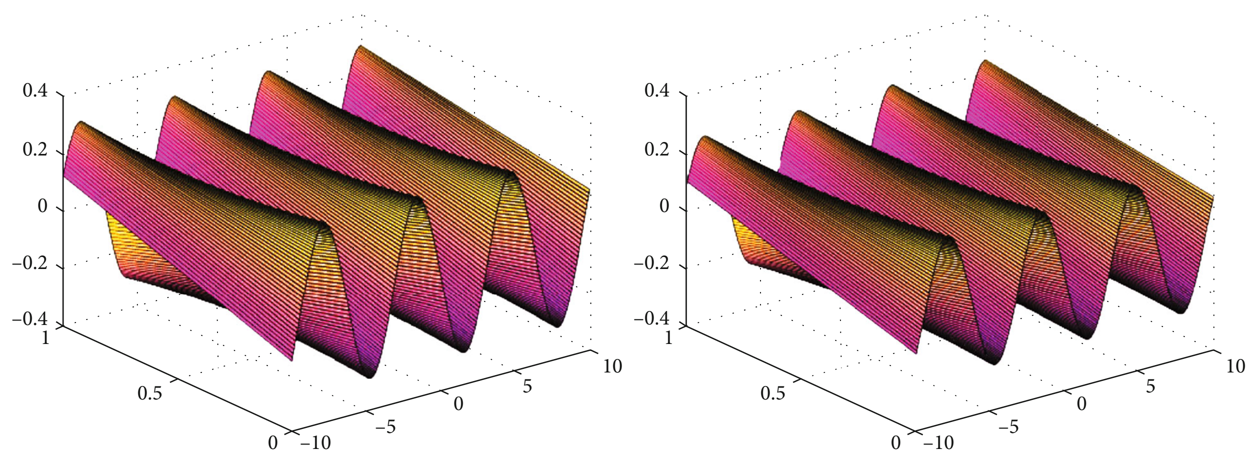


Figure 7: The fractional order of $\alpha = 0.8$ and 0.6 of Problem 3.

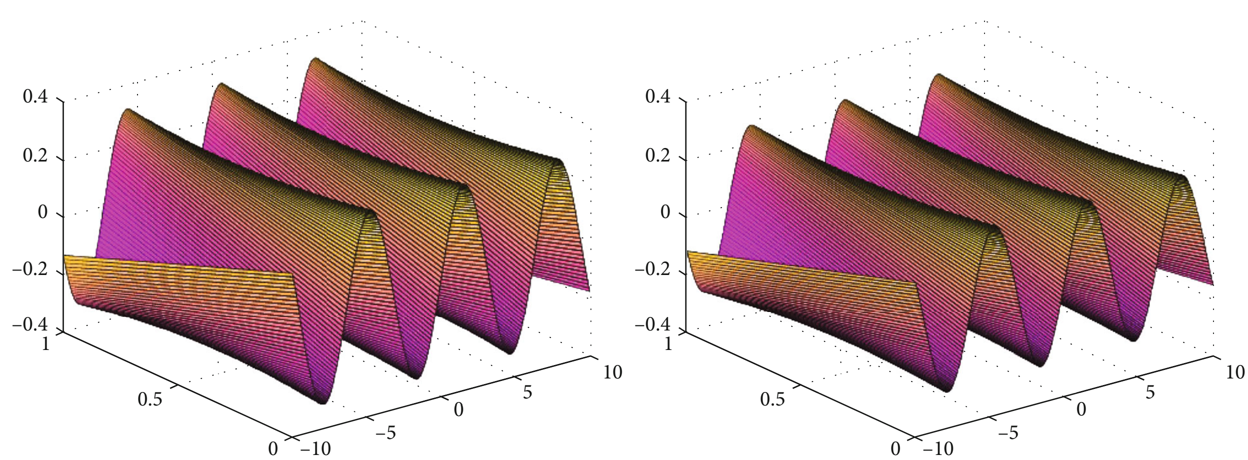


Figure 8: The fractional order of $\alpha = 0.8$ and 0.6 of Problem 3.

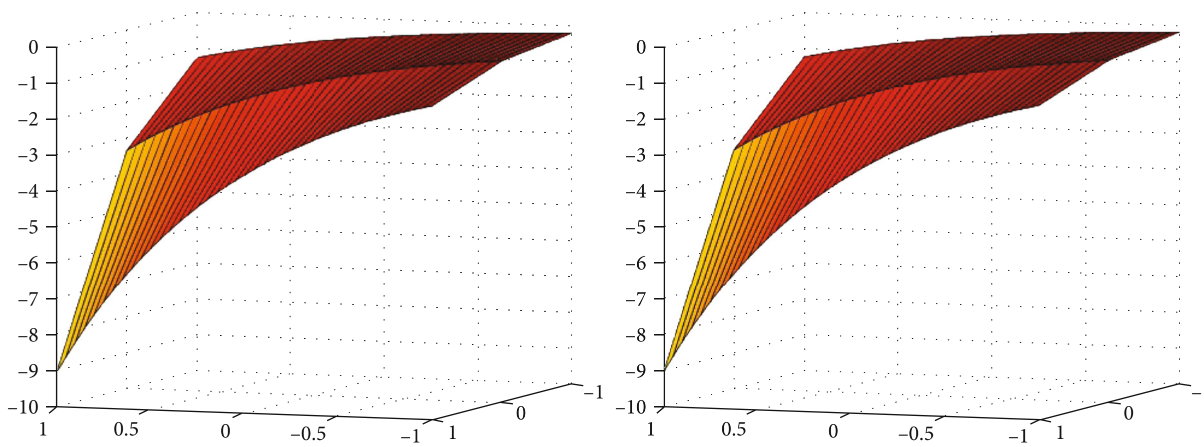


Figure 9: Graph of exact and analytical results of Problem 4.

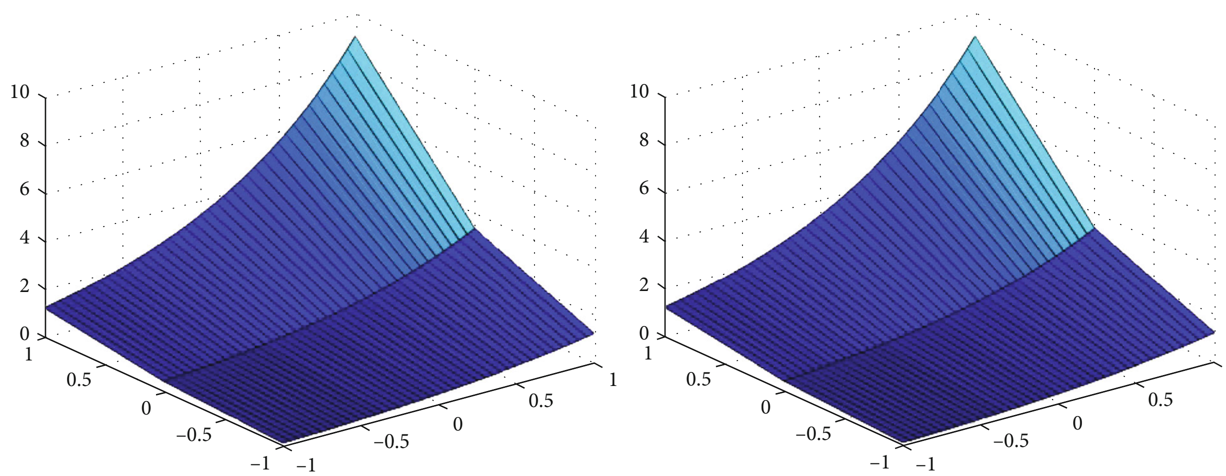


Figure 10: Graph of exact and analytical results of Problem 4.

The kth truncated term series of (62) is

$$\begin{aligned} U_k(\psi,\varphi,s) &= -e^{\psi+\varphi}s^2 + \sum_{n=1}^{k} s^{n\alpha+2} f_n(\psi,\varphi), \\ V_k(\psi,\varphi,s) &= e^{\psi+\varphi}s^2 + \sum_{n=1}^{k} s^{n\alpha+2} g_n(\psi,\varphi), \end{aligned} \tag{63}$$

and the kth Elzaki residual function is

$$\begin{aligned} \mathscr{E}_\tau \operatorname{Re} s_k(\psi,\varphi,s) = {} & U_k(\psi,\varphi,s) - -e^{\psi+\varphi}s^2 - \rho_0 s^\alpha \mathscr{E}_\tau \\ & \cdot \left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}U_k(\psi,\varphi,s) + \frac{\partial^2}{\partial\varphi^2}U_k(\psi,\varphi,s)\right\}\right] \\ & + s^\alpha \mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{U_k(\psi,\varphi,s)\frac{\partial}{\partial\psi}U_k(\psi,\varphi,s)\right\}\right] \\ & + s^\alpha \mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{V_k(\psi,\varphi,s)\frac{\partial}{\partial\varphi}U_k(\psi,\varphi,s)\right\}\right] \\ & - g\frac{1}{s^{\alpha+2p}}, \end{aligned}$$

$$\begin{aligned} \mathscr{E}_\tau \operatorname{Re} s_k(\psi,\varphi,s) = {} & V_k(\psi,\varphi,s) - e^{\psi+\varphi}s^2 - \rho_0 s^\alpha \mathscr{E}_\tau \\ & \cdot \left[\mathscr{E}_\tau^{-1}\left\{\frac{\partial^2}{\partial\psi^2}V_k(\psi,\varphi,s) + \frac{\partial^2}{\partial\varphi^2}V_k(\psi,\varphi,s)\right\}\right] \\ & + s^\alpha \mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{U_k(\psi,\varphi,s)\frac{\partial}{\partial\psi}V_k(\psi,\varphi,s)\right\}\right] \\ & + s^\alpha \mathscr{E}_\tau\left[\mathscr{E}_\tau^{-1}\left\{V_k(\psi,\varphi,s)\frac{\partial}{\partial\varphi}V_k(\psi,\varphi,s)\right\}\right] \\ & + g\frac{1}{s^{\alpha+2}}. \end{aligned} \tag{64}$$

Now, to determine $f_k(\psi,\varphi)$ and $g_k(\psi,\varphi)$, $k = 1, 2, 3, \cdots$, we substitute the kth truncated series (63) into the kth Elzaki residual function (64), multiply the resulting equation by $s^{k\alpha+2}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\alpha+2} \operatorname{Re} s_k(\psi,\varphi,s)] = 0$, $k = 1, 2, 3, \cdots$, for f_k and g_k. The following are the first several components of the series $f_k(\psi,\varphi)$ and $g_k(\psi,\varphi)$:

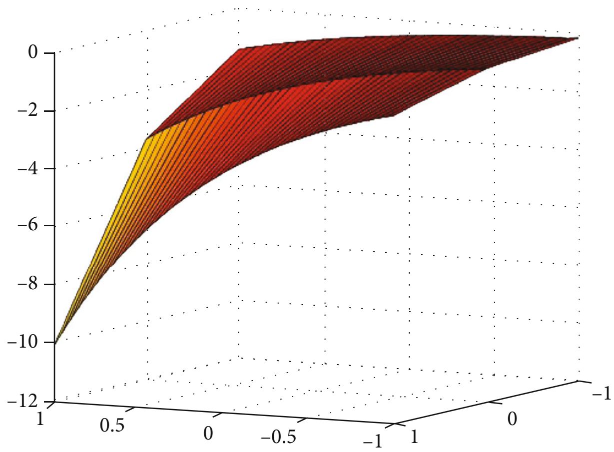


Figure 11: The fractional order of $\alpha = 0.8$ and 0.6 of Problem 4.

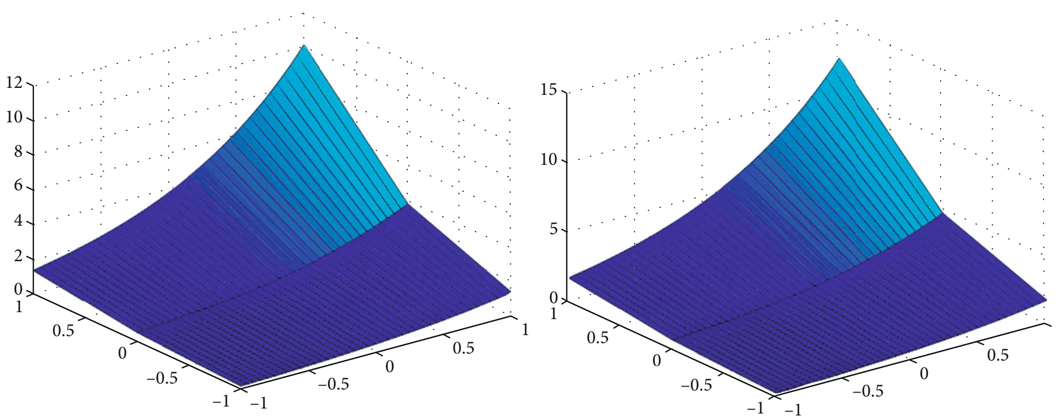


Figure 12: The fractional order of $\alpha = 0.8$ and 0.6 of Problem 4.

$$\begin{aligned}
f_1(\psi,\varphi) &= -2\rho_0 e^{\psi+\varphi} + g,\\
g_1(\psi,\varphi) &= 2\rho_0 e^{\psi+\varphi} - g,\\
f_2(\psi,\varphi) &= -(2\rho_0)^2 e^{\psi+\varphi},\\
g_2(\psi,\varphi) &= (2\rho_0)^2 e^{\psi+\varphi},\\
f_3(\psi,\varphi) &= -(2\rho_0)^3 e^{\psi+\varphi},\\
g_3(\psi,\varphi) &= (2\rho_0)^3 e^{\psi+\varphi},\\
&\vdots .
\end{aligned} \tag{65}$$

Putting the values of $f_n(\psi,\varphi)$ and $g_n(\psi,\varphi) (n \geq 1)$ in (63), we have

$$\begin{aligned}
U(\psi,\varphi,s) = -e^{\psi+\varphi}s^2 + f_1(\psi,\varphi)s^{\alpha+2} + f_2(\psi,\varphi)s^{2\alpha+2}\\
+ f_3(\psi,\varphi)s^{3\alpha+2} + \cdots,
\end{aligned}$$

$$\begin{aligned}
V(\psi,\varphi,s) = e^{\psi+\varphi}s^2 + g_1(\psi,\varphi)s^{\alpha+2} + g_2(\psi,\varphi)s^{2\alpha+2}\\
+ g_3(\psi,\varphi)s^{3\alpha+2} + \cdots,
\end{aligned}$$

$$\begin{aligned}
U(\psi,\varphi,s) = -e^{\psi+\varphi}s^2 - 2\rho_0 e^{\psi+\varphi} + g s^{\alpha+2} - (2\rho_0)^2 e^{\psi+\varphi}s^{2\alpha+2}\\
- (2\rho_0)^3 e^{\psi+\varphi}s^{3\alpha+2} + \cdots,
\end{aligned}$$

$$\begin{aligned}
V(\psi,\varphi,s) = e^{\psi+\varphi}s^2 + 2\rho_0 e^{\psi+\varphi} - g s^{\alpha+2} + (2\rho_0)^2 e^{\psi+\varphi}s^{2\alpha+2}\\
+ (2\rho_0)^3 e^{\psi+\varphi}s^{3\alpha+2} + \cdots,
\end{aligned}$$

$$\begin{aligned}
U(\psi,\varphi,s) &= -e^{\psi+\varphi}\left[s^2 + 2\rho_0 s^{\alpha+2} + (2\rho_0)^2 s^{2\alpha+2} + (2\rho_0)^3 s^{3\alpha+2} + \cdots\right] + g s^{\alpha+2},\\
V(\psi,\varphi,s) &= e^{\psi+\varphi}\left[s^2 + 2\rho_0 s^{\alpha+2} + (2\rho_0)^2 s^{2\alpha+2} + (2\rho_0)^3 s^{3\alpha+2} + \cdots\right] - g s^{\alpha+2}.
\end{aligned} \tag{66}$$

Using inverse Elzaki transform, we get

$$\begin{aligned}
u(\psi,\varphi,\tau) = -e^{\psi+\varphi}\left[1 + \frac{2\rho_0\tau^{\alpha}}{\Gamma(\alpha+2)} + \frac{(2\rho_0)^2\tau^{2\alpha}}{\Gamma(2\alpha+2)} + \frac{(2\rho_0)^3\tau^{3\alpha}}{\Gamma(3\alpha+2)} + \cdots\right]\\
+ g\frac{\tau^{\alpha}}{\Gamma(\alpha+2)},
\end{aligned}$$

$$v(\psi, \varphi, \tau) = e^{\psi+\varphi}\left[1 + \frac{2\rho_0\tau^{\alpha}}{\Gamma(\alpha+2)} + \frac{(2\rho_0)^2\tau^{2\alpha}}{\Gamma(2\alpha+2)} + \frac{(2\rho_0)^3\tau^{3\alpha}}{\Gamma(3\alpha+2)} + \cdots\right] - g\frac{\tau^{\alpha}}{\Gamma(\alpha+2)}. \quad (67)$$

Putting $\alpha = 1$, we get the solution in closed form

$$\begin{aligned} u(\psi, \varphi, \tau) &= -e^{\psi+\varphi+2\rho_0\tau} + g, \\ v(\psi, \varphi, \tau) &= e^{\psi+\varphi+2\rho_0\tau} - g. \end{aligned} \quad (68)$$

In Figures 9 and 10, the RPSTM and the exact results of Example 4 at $\alpha = 1$ are shown by graphs, respectively. From the given figures, it can be seen that both the exact and the RPSTM results are in close contact with each other. Also, in the Figure 11 and 12 subgraph, the RPSTM results of Example 4 are calculated at different fractional-order $\alpha = 0.8$ and 0.6. It is investigated that fractional-order problem results are convergent to an integer-order effect as fractional-order analysis to integer-order. The same phenomenon of convergence of fractional-order solutions towards integral-order solutions is observed.

5. Conclusions

In this article, a modified method constructed by a mixture of the residual power series and Elzaki transformation operator is presented to solve fractional-order Navier-Stokes models. The merit of the modified technique is to reduce the size of computational work needed to find the result in a power series form whose coefficient to be calculated is in successive algebraic steps. The technique gives a series form of results that converges very fast in physical models. It is predicted that this article achieved results which will be useful for further analysis of the complicated nonlinear physical problems. The calculations of this technique are very straightforward and simple. Thus, we deduce that this technique can be implemented to solve several schemes of nonlinear fractional-order partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

This work was supported by the Korea Institute of Energy Technology Evaluation and Planning (KETEP) grant funded by the Korea government (MOTIE) (20202020900060, The Development and Application of Operational Technology in Smart Farm Utilizing Waste Heat from Particulates Reduced Smokestack).

References

[1] A. Prakash, P. Veeresha, D. G. Prakasha, and M. Goyal, “A new efficient technique for solving fractional coupled Navier-Stokes equations using q-homotopy analysis transform method,” *Pramana*, vol. 93, no. 1, pp. 1–10, 2019.

[2] A. Rauf, Y. Mahsud, I. A. Mirza, and Q. Rubbab, “Multi-layer flows of immiscible fractional Maxwell fluids with generalized thermal flux,” *Chinese Journal of Physics*, vol. 62, pp. 313–334, 2019.

[3] A. Shah, N. A. Nehad, T. Elnaqeeb, and M. M. Rashidi, “Magnetohydrodynamic free convection flows with thermal memory over a moving vertical plate in porous medium,” *Journal of Applied and Computational Mechanics*, vol. 5, no. 1, pp. 150–161, 2019.

[4] M. El-Shahed and A. Salem, “On the generalized Navier-Stokes equations,” *Applied Mathematics and Computation*, vol. 156, no. 1, pp. 287–293, 2004.

[5] S. Momani and Z. Odibat, “Analytical solution of a time-fractional Navier-Stokes equation by Adomian decomposition method,” *Applied Mathematics and Computation*, vol. 177, no. 2, pp. 488–494, 2006.

[6] Z. Z. Ganji, D. D. Ganji, A. D. Ganji, and M. Rostamian, “Analytical solution of time-fractional Navier-Stokes equation in polar coordinate by homotopy perturbation method,” *Numerical Methods for Partial Differential Equations*, vol. 26, no. 1, pp. 117–124, 2010.

[7] B. K. Singh and P. Kumar, “FRDTM for numerical simulation of multi-dimensional, time-fractional model of Navier-Stokes equation,” *Ain Shams Engineering Journal*, vol. 9, no. 4, pp. 827–834, 2018.

[8] D. S. Oliveira and E. C. de Oliveira, “Analytical solutions for Navier-Stokes equations with Caputo fractional derivative,” *SeMA Journal*, vol. 78, no. 1, pp. 137–154, 2021.

[9] J. Zhang and J. R. Wang, “Numerical analysis for Navier-Stokes equations with time fractional derivatives,” *Applied Mathematics and Computation*, vol. 336, pp. 481–489, 2018.

[10] S. S. Ravindran, “Exact boundary controllability of Galerkin approximations of Navier-Stokes system for soret convection,” *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, vol. 5, no. 2, pp. 41–49, 2015.

[11] A. Cibik and F. Yilmaz, “Brezzi-Pitkaranta stabilization and a priori error analysis for the Stokes control,” *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, vol. 7, no. 1, pp. 75–82, 2017.

[12] D. Kumar, J. Singh, and S. Kumar, “A fractional model of Navier-Stokes equation arising in unsteady flow of a viscous fluid,” *Journal of the Association of Arab Universities for Basic and Applied Sciences*, vol. 17, no. 1, pp. 14–19, 2015.

[13] R. M. Jena and S. Chakraverty, “Solving time-fractional Navier-Stokes equations using homotopy perturbation Elzaki transform,” *SN Applied Sciences*, vol. 1, no. 1, pp. 1–13, 2019.

[14] M. Alquran, M. Ali, M. Alsukhour, and I. Jaradat, “Promoted residual power series technique with Laplace transform to solve some time-fractional problems arising in physics,” *Results in Physics*, vol. 19, p. 103667, 2020.

[15] A. Çibik, “The effect of a sparse grad-div stabilization on control of stationary Navier-Stokes equations,” *Journal of*

Mathematical Analysis and Applications, vol. 437, no. 1, pp. 613–628, 2016.

[16] O. A. Arqub, A. El-Ajou, A. S. Bataineh, and I. Hashim, "A representation of the exact solution of generalized Lane-Emden equations using a new analytical method," *Abstract and Applied Analysis*, vol. 2013, Article ID 378593, 10 pages, 2013.

[17] A. Arqub, Z. A.-H. Omar, R. Al-Badarneh, and S. Momani, "A reliable analytical method for solving higher-order initial value problems," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 673829, 12 pages, 2013.

[18] A. El-Ajou, O. A. Arqub, Z. Al Zhour, and S. Momani, "New results on fractional power series: theories and applications," *Entropy*, vol. 15, no. 12, pp. 5305–5323, 2013.

[19] O. A. Arqub, A. El-Ajou, Z. Al Zhour, and S. Momani, "Multiple solutions of nonlinear boundary value problems of fractional order: a new analytic iterative technique," *Entropy*, vol. 16, no. 1, pp. 471–493, 2014.

[20] A. El-Ajou, O. A. Arqub, and S. Momani, "Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: a new iterative algorithm," *Journal of Computational Physics*, vol. 293, pp. 81–95, 2015.

[21] O. A. Arqub, A. El-Ajou, and S. Momani, "Constructing and predicting solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations," *Journal of Computational Physics*, vol. 293, pp. 385–399, 2015.

[22] M. Alquran, "Analytical solutions of fractional foam drainage equation by residual power series method," *Mathematical Sciences*, vol. 8, no. 4, pp. 153–160, 2014.

[23] S.-D. Lin and L. Chia-Hung, "Laplace transform for solving some families of fractional differential equations and its applications," *Advances in Difference Equations*, vol. 2013, no. 1, p. 9, 2013.

[24] M. Alquran, "Analytical solution of time-fractional two-component evolutionary system of order 2 by residual power series method," *Journal of Applied Analysis & Computation*, vol. 5, no. 4, pp. 589–599, 2015.

[25] T. M. Elzaki, "The new integral transform Elzaki transform," *Global Journal of Pure and Applied Mathematics*, vol. 7, no. 1, pp. 57–64, 2011.

[26] T. M. Elzaki, "On the connections between Laplace and Elzaki transforms," *Advances in Theoretical and Applied Mathematics*, vol. 6, no. 1, pp. 1–11, 2011.

[27] T. M. Elzaki, "On the new integral transform"Elzaki transform"fundamental properties investigations and applications," *Global Journal of Mathematical Sciences: Theory and Practical*, vol. 4, no. 1, pp. 1–13, 2012.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 5567970, 9 pages
https://doi.org/10.1155/2021/5567970

Research Article

A Spectral Collocation Technique for Riesz Fractional Chen-Lee-Liu Equation

M. A. Abdelkawy [1,2] and S. A. Alyami [1]

[1] *Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Beni-Suef University, Beni Suef, Egypt*

Correspondence should be addressed to M. A. Abdelkawy; melkawy@yahoo.com

Received 5 January 2021; Revised 1 February 2021; Accepted 24 March 2021; Published 14 April 2021

Academic Editor: Ahmed S. Hendy

This paper discusses the study of optical solitons that are modeled by Riesz fractional Chen-Lee-Liu model, one of the versions of the famous nonlinear Schrödinger equation. This model is solved by the assistance of consecutive spectral collocation technique with two independent approaches. The first is the approach of the spatial variable, while the other is the approach of the temporal variable. It is concluded that the method of the current paper is far more efficient and credible for the proposed problem. Numerical results illustrate the performance efficiency of the algorithm. The results also point out that the scheme can lead to spectral accuracy of the studied model.

1. Introduction

Several numerical methods, including local and global methods, have been listed as approximation techniques for treating the differential equations. The local methods listed the approximate solution at specific points, while the global methods give the approximate solution in whole the mentioned interval. The numerical approximations for differential equations [1–4] are listed at specific points using finite difference methods. While the finite element methods subdivide the whole interval into subintervals and give the approximate solution in them. The finite element methods are used for various types of differential equation; see for example [5–7].

Recently, there are more interests of appointing the spectral methods to treat with various kinds of differential and integral equations [8, 9], due to their applicability to bounded and unbounded domains [10, 11]. The convergence speed is one of the major advantages of spectral method. Spectral methods are promising candidates for solving fractional differential equations since their global nature fits well with the nonlocal definition of fractional operators. They have gained new popularity in automatic computations for a wide class of different problems which included linear and nonlinear differential equation of integer or fractional (fixed, variable, Riesz, tempered, and distributed orders); see [12, 13]. Also, they are more reliable to treat the integral and integro-differential equations. Spectral methods have exponential convergence rates as well as a high accuracy level. The spectral method has been classified into four classes, collocation [14], tau [15], Galerkin [16], and Petrov-Galerkin [17] methods.

The theory of optical solitons [18–21] is mainly governed by the well-known nonlinear Schrödinger equation (NLSE) [22–25]. However, there exists a wide variety of its manifestations and modifications that also govern pulse transfer across the globe through optical fibers, PCF, metamaterials, and couplers. A few such models are Schrödinger-Hirota equation [26], Manakov equation, complex Ginzburg-Landau equation, Fokas-Lenells equation, Gabitov-Turitsyn equation, and many others. These models are considered under different circumstances such as dispersive solitons, differential group delay, and dispersion-managed solitons. Besides these familiar models, there is another class of versions of NLSE that is referred to as derivative NLSE (DNLSE) [27–29] that appears in three forms. One such form is the Chen-Lee-Liu equation [30–32] that incorporates higher order perturbations from optics and is going to be the focus of today's paper. While a plethora of preexisting work has been already reported in regard to this model, today's focus is going to be handling the model by the aid of fully shifted Legendre collocation method.

Shifted Legendre collocation schemes are used to numerically solve the Riesz fractional Chen-Lee-Liu model. The solution $\Theta(\xi,\tau)$ is firstly placed in its real $\mathscr{U}(\xi,\tau)$ and imaginary $\mathscr{V}(\xi,\tau)$ parts. Accordingly, the real $\mathscr{U}(\xi,\tau)$ and imaginary $\mathscr{V}(\xi,\tau)$ parts of such equation are approximated as $\mathscr{U}_{\mathscr{N},\mathscr{M}}(\xi,\tau)$ and $\mathscr{U}_{\mathscr{N},\mathscr{M}}(\xi,\tau)$, respectively, which can be expressed as a finite expansion of shifted Legendre polynomials for spatial variable. Subsequently, the Chen-Lee-Liu equation with boundary conditions is reduced to temporal differential system with initial conditions. Then, the shifted Legendre-Gauss-Radau collocation is assigned for temporal discretization, which is more reliable for treating with such problems. Substituting these discretizations in the mentioned equation gets a nonlinear system of algebraic equations which solved numerically using the Newton-Raphson approach.

This paper is arranged as follows. In Section 1, some properties of Riemann-Liouville fractional derivatives, shifted Legendre polynomials, and shifted Chebyshev polynomials are listed. The mentioned scheme is implemented for the Chen-Lee-Liu equation with initial-boundary conditions in Section 2. In Section 3, two test examples are discussed. The competence of our numerical approach is exhibited by diverse examples in Section 4. Few remarks are mentioned in the last section (Section 5).

2. Riemann-Liouville Fractional Derivative

The fractional integration of order $\mu > 0$ exists in different formulas [33]. Riemann-Liouville formula, the most common and widely used, is defined as follows:

$$J^{\mu} f(\zeta) = \frac{1}{\Gamma(\mu)} \int_0^{\zeta} (\zeta - \tau)^{\mu-1} f(\tau) d\tau, \quad \mu > 0, \zeta > 0, \qquad J^0 f(\zeta) = f(\zeta). \tag{1}$$

Here, we introduce some properties of the fractional operators. The left-sided and the right-sided fractional derivatives of Riemann-Liouville type of order $\beta\,(n-1<\beta<n)$ are defined as follows:

$$\begin{aligned} {}_{-\infty}D_{\xi}^{\beta}\psi(\xi,\tau) &= \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial \xi^n} \int_{-\infty}^{\xi} (\xi - z)^{n-1-\beta} \psi(z,\tau) dz, \\ {}_{\xi}D_{+\infty}^{\beta}\psi(\xi,\tau) &= \frac{(-1)^n}{\Gamma(n-\beta)} \frac{\partial^n}{\partial \xi^n} \int_{\xi}^{+\infty} (z - \xi)^{n-1-\beta} \psi(z,\tau) dz. \end{aligned} \tag{2}$$

The Riesz fractional derivative is defined as follows:

$$\frac{\partial^{\beta}}{\partial|\xi|^{\beta}}\psi(\xi,\tau) = -(-\nabla)^{\beta/2}\psi(\xi,\tau) = c_{\beta}\left[{}_{-\infty}D_{\xi}^{\beta}\psi(\xi,\tau) + {}_{\xi}D_{+\infty}^{\beta}\psi(\xi,\tau)\right], \tag{3}$$

where $c_{\beta} = -1/2 \cos(\pi\beta/2)$. The fractional Laplacian operator in Equation (3) can be represented in the following equivalent Fourier form on the spatial variable ξ:

$$-(-\nabla)^{\beta/2}\psi(\xi,\tau) = -\mathscr{F}^{-1}(|\xi|\mathscr{F}(\psi(\xi,\tau))). \tag{4}$$

If ψ is defined on $[\mathscr{A},\mathscr{B}]$ and satisfies $\psi(\mathscr{A},\tau) = \psi(\mathscr{B},\tau) = 0$, then the function can be extended by taking $\psi(\xi,\tau) \equiv 0$ for $x \ll a$ and $x \gg b$. Moreover, as shown in [34], if $\psi_{\xi}(\mathscr{A},\tau) = \psi_{\xi}(\mathscr{B},\tau) = 0$, then the Riesz fractional derivative can be written as follows:

$$\frac{\partial^{\beta}}{\partial|\xi|^{\beta}}\psi(\xi,\tau) = -(-\nabla)^{\beta/2}\psi(\xi,\tau) = -\frac{1}{2\cos(\pi\beta/2)}\left[{}_{a}D_{\xi}^{\beta}\psi(\xi,\tau) + {}_{\xi}D_{b}^{\beta}\psi(\xi,\tau)\right]. \tag{5}$$

The left and right RL-FDs of the Legendre polynomial are given by the following:

$$\begin{aligned} {}_{-1}D_{\xi}^{\mu}P_k(\xi) &= \sum_{k=0}^{j} \frac{(-1)^{k+j}\Gamma(k+j+1)}{(j-k)!\Gamma(k+1)2^k\Gamma(k-\mu+1)}(\xi+1)^{k-\mu}, \\ {}_{\xi}D_{1}^{\mu}P_k(\xi) &= \sum_{k=0}^{j} \frac{(-1)^{k}\Gamma(k+j+1)}{(j-k)!\Gamma(k+1)2^k\Gamma(k-\mu+1)}(1-\xi)^{k-\mu}. \end{aligned} \tag{6}$$

3. Chen-Lee-Liu Equation

In this section, we treat the next nonlinear Riesz space Chen-Lee-Liu equation

$$i\frac{\partial\Theta(\xi,\tau)}{\partial\tau} + \frac{\partial^{\mu}\Theta(\xi,\tau)}{\partial|\xi|^{\mu}} + i\gamma|\Theta(\xi,\tau)|^2\frac{\partial\Theta(\xi,\tau)}{\partial\xi} = \Delta(\xi,\tau), \quad (\xi,\tau) \in [0,\xi_{\text{end}}] \times [0,\tau_{\text{end}}], \tag{7}$$

with the following conditions:

$$\begin{aligned} &\Theta(0,\tau_{\text{end}}) = \chi_1(\tau), \Theta(\xi_{\text{end}},\tau) = \chi_2(\tau), \quad t \in [0,\tau_{\text{end}}], \\ &\Theta(x,0) = \phi_1(x), \quad x \in [0,\xi_{\text{end}}]. \end{aligned} \tag{8}$$

We now split the complex function $\Theta(\xi,\tau)$ into two real functions $\mathscr{U}(\xi,\tau)$ and $\mathscr{V}(\xi,\tau)$ as follows:

$$\begin{aligned} &\Theta(\xi,\tau) = \mathscr{U}(\xi,\tau) + i\mathscr{V}(\xi,\tau), \Delta(\xi,\tau) = \Delta(\xi,\tau) + i\Delta_2(\xi,\tau), \\ &\chi_1(\tau) = \eta_1(\tau) + i\eta_3(\tau), \chi_2(\tau) = \eta_2(\tau) + i\eta_4(\tau), \phi_1(x) = \varphi_1(x) + i\varphi_2(x), \end{aligned} \tag{9}$$

where $\mathscr{U}(\xi,\tau), \mathscr{V}(\xi,\tau), \Delta_1(\xi,\tau)$, Delta$_2(\xi,\tau), \eta_1(\tau), \eta_3(\tau), \eta$

$2(\tau), \eta_4(\tau), \varphi_1(x)$, and $\varphi_2(x)$ are the real functions. Thereafter,

$$\frac{\partial \mathscr{U}(\xi,\tau)}{\partial \tau} + \frac{\partial^\mu \mathscr{V}(\xi,\tau)}{\partial |\xi|^\mu} + \gamma\left(u^2(\xi,\tau)+v^2(\xi,\tau)\right)\frac{\partial \mathscr{U}(\xi,\tau)}{\partial \xi} = \Delta_1(\xi,\tau),$$
$$\frac{\partial \mathscr{V}(\xi,\tau)}{\partial \tau} + \frac{\partial^\mu \mathscr{U}(\xi,\tau)}{\partial |\xi|^\mu} + \gamma\left(u^2(\xi,\tau)+v^2(\xi,\tau)\right)\frac{\partial \mathscr{V}(\xi,\tau)}{\partial \xi} = \Delta_2(\xi,\tau), \tag{10}$$

with the next conditions:

$$\mathscr{U}(0,\tau_{\text{end}}) = \eta_1(\tau), \mathscr{U}(\xi_{\text{end}},\tau) = \eta_2(\tau), \quad t \in [0,\tau_{\text{end}}], \tag{11}$$

$$\mathscr{V}(0,\tau_{\text{end}}) = \eta_3(\tau), \mathscr{V}(\xi_{\text{end}},\tau) = \eta_4(\tau), \quad t \in [0,\tau_{\text{end}}], \tag{12}$$

$$\mathscr{U}(x,0) = \eta_5(x), \mathscr{V}(x,0) = \eta_6(x), \quad x \in [0,\xi_{\text{end}}]. \tag{13}$$

3.1. Spatial Discretization. The distribution of shifted Legendre-Gauss-Lobatto nodes in $[0,\xi_{\text{end}}]$ is the major feature of considering them in our discretization. Here, we list the basic main of implementing our Legendre-Gauss-Lobatto collocation scheme for converting the nonlinear system (Equations (10) and (11)) into temporal ordinary differential system.

The spectral approximation of $\mathscr{P}(\xi,\tau)$ and $\mathscr{Q}(\xi,\tau)$ is given as follows:

$$\mathscr{U}_{\mathscr{N}}(\xi,\tau) = \sum_{j=0}^{\mathscr{N}} \varepsilon_j(\tau)\mathscr{P}_{\xi_{\text{end}},j}(x),$$
$$\mathscr{V}_{\mathscr{N}}(\xi,\tau) = \sum_{j=0}^{\mathscr{N}} \varepsilon_j(\tau)\mathscr{P}_{\xi_{\text{end}},j}(x), \tag{14}$$

where the orthogonal property and discrete inner product permit the following:

$$\varepsilon_j(\tau) = \frac{1}{h_{\xi_{\text{end}},j}}\sum_{i=0}^{\mathscr{N}} P_j\left(\xi_{\xi_{\text{end}},\mathscr{N},i}\right)\varpi_{\xi_{\text{end}},\mathscr{N},i}\mathscr{U}\left(\xi_{\xi_{\text{end}},\mathscr{N},i}\tau\right),$$
$$\varepsilon_j(\tau) = \frac{1}{h_{\xi_{\text{end}},j}}\sum_{i=0}^{\mathscr{N}} P_j\left(\xi_{\xi_{\text{end}},\mathscr{N},i}\right)\varpi_{\xi_{\text{end}},\mathscr{N},i}\mathscr{V}\left(\xi_{\xi_{\text{end}},\mathscr{N},i}\tau\right). \tag{15}$$

In that case, Equation (14) takes the form:

$$\mathscr{U}(\xi,\tau) = \sum_{i=0}^{\mathscr{N}}\left(\sum_{j=0}^{\mathscr{N}}\frac{1}{h_{\xi_{\text{end}},j}}\mathscr{P}_{\xi_{\text{end}},j}\left(\xi_{\xi_{\text{end}},\mathscr{N},i}\right)\mathscr{P}_{\xi_{\text{end}},j}(x)\varpi_{\xi_{\text{end}},\mathscr{N},i}\right)\mathscr{U}\left(\xi_{\xi_{\text{end}},\mathscr{N},i},\tau\right),$$
$$\mathscr{V}(\xi,\tau) = \sum_{i=0}^{\mathscr{N}}\left(\sum_{j=0}^{\mathscr{N}}\frac{1}{h_{\xi_{\text{end}},j}}\mathscr{P}_{\xi_{\text{end}},j}\left(\xi_{\xi_{\text{end}},\mathscr{N},i}\right)\mathscr{P}_{\xi_{\text{end}},j}(x)\varpi_{\xi_{\text{end}},\mathscr{N},i}\right)\mathscr{V}\left(\xi_{\xi_{\text{end}},\mathscr{N},i},\tau\right). \tag{16}$$

Over and above that, the partial derivative of first order in space evaluated at shifted Legendre-Gauss-Lobatto collocation is as follows:

Table 1: Maximum absolute errors of Equation (32).

$(\mathscr{N},\mathscr{M})$	$\mathscr{M}_{\mathscr{U}\mathscr{N},\mathscr{M}}$	$\mathscr{M}_{V\mathscr{N},\mathscr{M}}$	$M_{N,M}$
(2, 2)	1.5625×10^{-2}	7.39136×10^{-3}	1.5625×10^{-2}
(4, 4)	7.01531×10^{-3}	2.43449×10^{-3}	7.01531×10^{-3}
(6, 6)	1.26263×10^{-3}	4.44263×10^{-4}	1.26263×10^{-3}
(8, 8)	6.75387×10^{-13}	1.47693×10^{-12}	1.50175×10^{-12}
(10, 10)	4.35416×10^{-16}	9.29812×10^{-16}	9.56769×10^{-16}
(12, 12)	5.73001×10^{-17}	2.48174×10^{-16}	2.54703×10^{-16}

$$\frac{\partial \mathscr{U}\left(\xi_{\xi_{\text{end}},\mathscr{N},n},\tau\right)}{\partial \xi} = \sum_{i=0}^{\mathscr{N}}\rho_{n,i}\mathscr{U}\left(\xi_{\xi_{\text{end}},\mathscr{N},i},\tau\right),$$
$$\frac{\partial \mathscr{V}\left(\xi_{\xi_{\text{end}},\mathscr{N},n},\tau\right)}{\partial \xi} = \sum_{i=0}^{\mathscr{N}}\rho_{n,i}\mathscr{V}\left(\xi_{\xi_{\text{end}},\mathscr{N},i},\tau\right), \quad n=0,1,\cdots,N, \tag{17}$$

where

$$\rho_{n,i} = \sum_{j=0}^{N}\frac{\varpi_{\xi_{\text{end}},\mathscr{N},i}}{h_{\xi_{\text{end}},j}}\mathscr{P}_{\xi_{\text{end}},j}\left(\xi_{\xi_{\text{end}},\mathscr{N},i}\right)\left(\frac{\partial \mathscr{P}_{\xi_{\text{end}},j}(x)}{\partial \xi}\right)\Bigg|_{x=\xi_{\xi_{\text{end}},\mathscr{N},n}}. \tag{18}$$

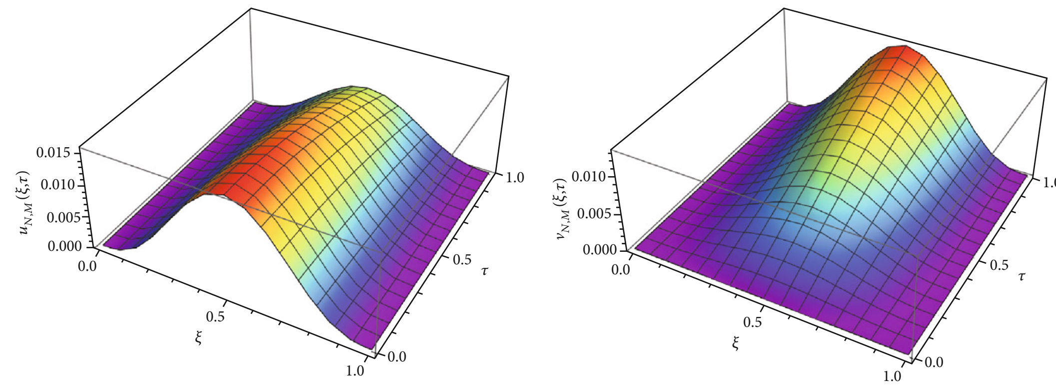
Comparable procedure can be performed to the Riesz fractional derivative $\partial^\mu\phi_{\mathscr{N},\mathscr{M}}(\xi,\eta,\tau)/\partial|\xi|^\mu$ for space variable to get

$$\frac{\partial^\mu \mathscr{U}\left(\xi_{\xi_{\text{end}},\mathscr{N},n},\tau\right)}{\partial |\xi|^\mu} = \sum_{i=0}^{\mathscr{N}}\lambda_{n,i}\mathscr{U}\left(\xi_{\xi_{\text{end}},\mathscr{N},i},\tau\right),$$
$$\frac{\partial^\mu \mathscr{V}\left(\xi_{\xi_{\text{end}},\mathscr{N},n},\tau\right)}{\partial |\xi|^\mu} = \sum_{i=0}^{\mathscr{N}}\lambda_{n,i}\mathscr{V}\left(\xi_{\xi_{\text{end}},\mathscr{N},i},\tau\right), \quad n=0,1,\cdots,N, \tag{19}$$

where

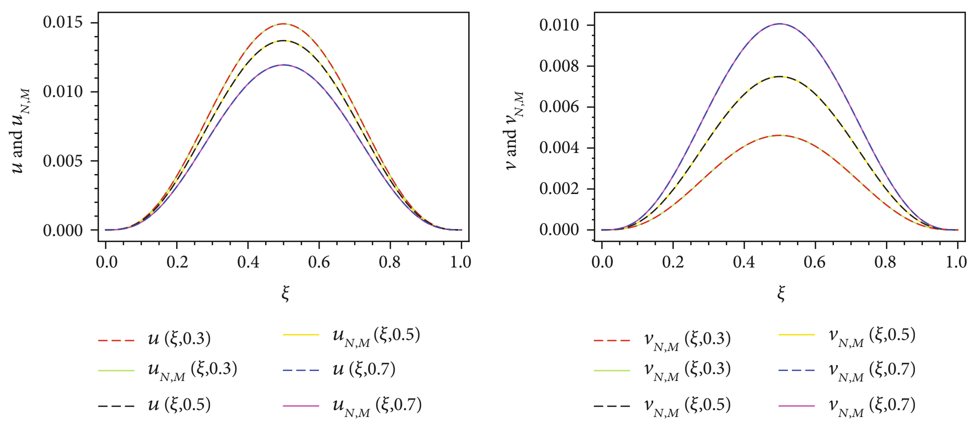
$$\lambda_{n,i} = \sum_{j=0}^{\mathscr{N}}\frac{\varpi_{\xi_{\text{end}},\mathscr{N},i}}{h_{\xi_{\text{end}},j}}\mathscr{P}_{\xi_{\text{end}},j}\left(\xi_{\xi_{\text{end}},\mathscr{N},i}\right)\left(\frac{\partial^\mu \mathscr{P}_{\xi_{\text{end}},j}(x)}{\partial |\xi|^\mu}\right)\Bigg|_{x=\xi_{\xi_{\text{end}},\mathscr{N},n}}. \tag{20}$$

Combining the boundary conditions with the abovementioned equations and equalizing the residual of Equation (7)



(a) Real part (b) Imaginary part

FIGURE 1: Space graphs of real and imaginary parts of the numerical solution of Equation (32).



(a) Real part (b) Imaginary part

FIGURE 2: ξ-direction curves for the approximate and exact solutions of real and imaginary parts of Equation (32).

by zero give us the following:

$$\begin{aligned}\dot{\mathscr{U}}_n(\tau) &= \Delta_{1,n}(\tau) - \sum_{i=1}^{\mathscr{N}-1} \lambda_{n,i}\mathscr{V}_i(\tau) - \gamma\left(\mathscr{U}_n^2(\tau) + \mathscr{V}_n^2(\tau)\right)\\ &\quad\cdot\left(\rho_{n,0}\eta_1(\tau) + \rho_{n,\mathscr{N}}\eta_1(\tau) + \sum_{i=1}^{\mathscr{N}-1}\rho_{n,i}\mathscr{U}_i(\tau)\right)\\ &\quad - \lambda_{n,0}\eta_3(\tau) - \lambda_{n,\mathscr{N}}\eta_4(\tau),\\ \dot{\mathscr{V}}_n(\tau) &= \Delta_{2,n}(\tau) + \sum_{i=1}^{\mathscr{N}-1}\lambda_{n,i}\mathscr{U}_i(\tau) - \gamma\left(\mathscr{U}_n^2(\tau) + \mathscr{V}_n^2(\tau)\right)\\ &\quad\cdot\left(\rho_{n,0}\eta_3(\tau) + \rho_{n,\mathscr{N}}\eta_4(\tau) + \sum_{i=1}^{\mathscr{N}-1}\rho_{n,i}\mathscr{V}_i(\tau)\right)\\ &\quad + \lambda_{n,0}\eta_1(\tau) - \lambda_{n,\mathscr{N}}\eta_2(\tau), \quad n = 1, 2, \cdots, \mathscr{N}-1,\end{aligned} \tag{21}$$

with initial values

$$\mathscr{U}_n(0) = \eta_5(0), \mathscr{V}_n(0) = \eta_6(0), \quad n = 1, \cdots, N-1, \tag{22}$$

where

$$\begin{aligned}\mathscr{U}_k(\tau) &= \mathscr{U}\left(\xi_{\xi\text{end},\mathscr{N},k}, \tau\right), \mathscr{V}_k(\tau) = \mathscr{V}\left(\xi_{\xi\text{end},\mathscr{N},k}, \tau\right), \Delta_{r,n}\\ &= \Delta_r\left(\xi_{\xi\text{end},N,k}, \tau\right), \quad k = 1, \cdots, \mathscr{N}-1, r = 1, 2.\end{aligned} \tag{23}$$

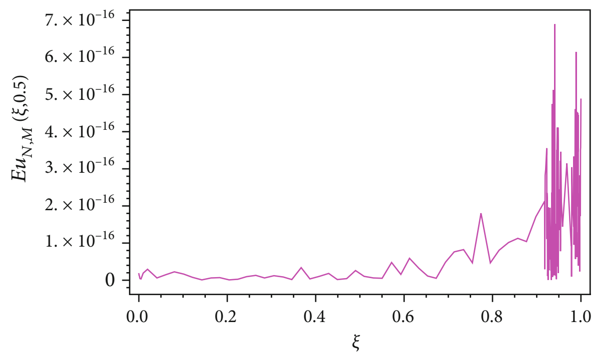
The numerical approach of such system will be listed in Subsection 3.2.

3.2. Temporal Discretization. Here, we numerically treat the temporal differential system with initial conditions:

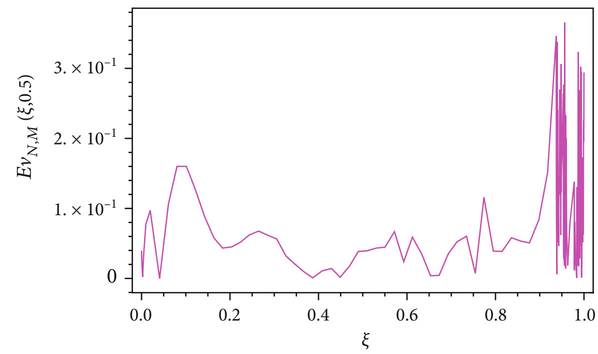
$$\dot{\mathscr{W}}_r(\tau) = \mathscr{G}_r(t, \mathscr{W}_1(\tau), \cdots, \mathscr{W}_{\mathscr{R}}(\tau)), 0 < \alpha < 1, \quad r = 1, \cdots, \mathscr{R}, t \in [0, \tau_{\text{end}}], \tag{24}$$

$$\mathscr{W}_r(0) = \tau_r, \quad r = 1, \cdots, \mathscr{R}, \tag{25}$$

where $\mathscr{G}_r(t, \mathscr{W}_1(\tau), \cdots, \mathscr{W}_{\mathscr{R}}(\tau)), r = 1, \cdots, \mathscr{R}$ are given functions. Shifted Legendre-Gauss-Radau collocation is assigned for temporal discretization, which is more reliable for



(a) Real part



(b) Imaginary part

FIGURE 3: ξ-direction curves of real and imaginary parts of the absolute error of Equation (32).

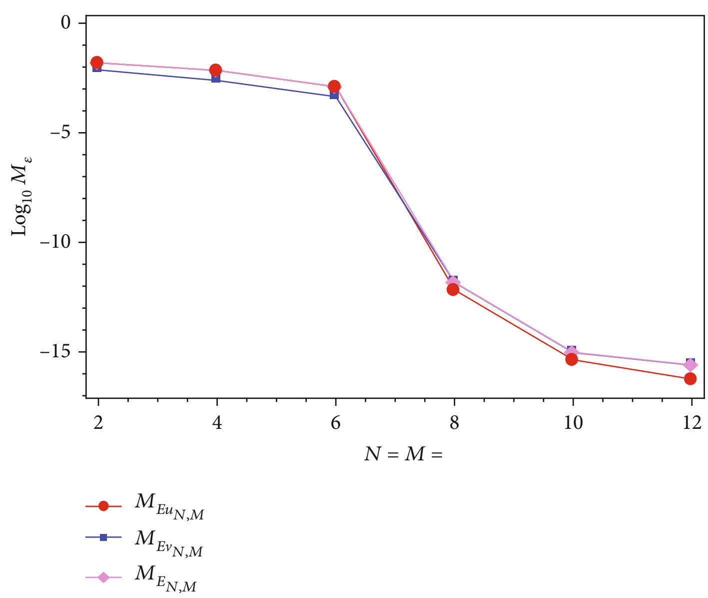


FIGURE 4: M_E convergence of Equation (32).

TABLE 2: Maximum absolute errors of Equation (35).

$(\mathscr{N}, \mathscr{M})$	$\mathscr{M}_{\mathscr{U}\mathscr{N},\mathscr{M}}$	$\mathscr{M}_{V\mathscr{N},\mathscr{M}}$	$M_{N,M}$
(2, 2)	3.90625×10^{-3}	3.67244×10^{-3}	3.90625×10^{-3}
(4, 4)	2.11826×10^{-3}	1.98249×10^{-3}	2.11826×10^{-3}
(6, 6)	6.60009×10^{-4}	6.04953×10^{-4}	6.60009×10^{-4}
(8, 8)	8.74126×10^{-5}	7.91024×10^{-5}	8.74126×10^{-5}
(10, 10)	3.1572×10^{-16}	2.1453×10^{-16}	3.53179×10^{-16}
(12, 12)	4.75375×10^{-16}	2.13208×10^{-16}	4.86791×10^{-16}

treating with such problems. We approximate $\mathscr{W}_r(\tau)$ as follows:

$$\mathscr{W}_r(\tau) = \sum_{j=0}^{\mathscr{K}} a_{r,j} \mathscr{C}_{\tau_{\text{end}},j}(\tau), \quad r = 1, \cdots, \mathscr{R}. \tag{26}$$

The temporal derivative $\dot{\mathscr{W}}_r(\tau)$ is evaluated as follows:

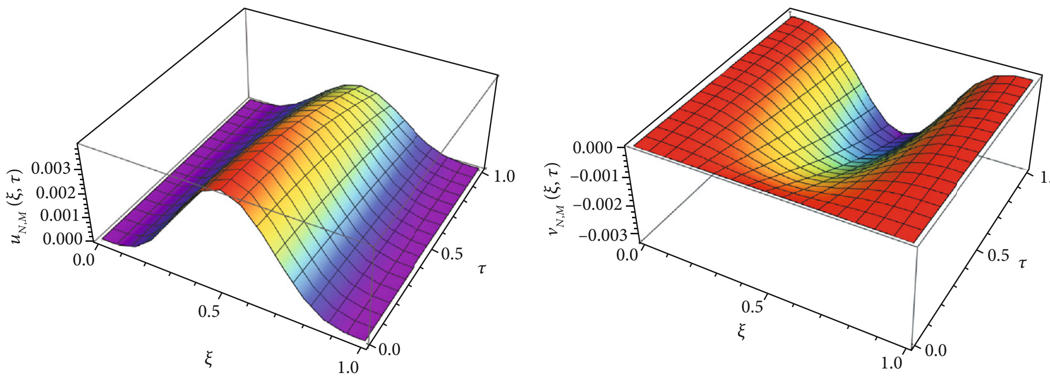
$$\dot{\mathscr{W}}_r(\tau) = \sum_{j=0}^{\mathscr{K}} a_{r,j} \frac{d}{d\tau}\left(\mathscr{C}_{\tau_{\text{end}},j}(\tau)\right) = \sum_{j=0}^{\mathscr{K}} a_{r,j} \mathscr{C}^{(1)}_{\tau_{\text{end}},j}(\tau), \quad r = 1, \cdots, \mathscr{R}. \tag{27}$$

Thus, we get the following:

$$\sum_{j=0}^{\mathscr{K}} a_{r,j} \mathscr{C}^{(1)}_{\tau_{\text{end}},j}(\tau) = \mathscr{G}_r\left(t, \sum_{j=0}^{\mathscr{K}} a_{1,j} \mathscr{C}_{\tau_{\text{end}},j}(\tau), \cdots, \sum_{j=0}^{\mathscr{K}} a_{\mathscr{K},j} \mathscr{C}_{\tau_{\text{end}},j}(\tau)\right), \quad r = 1, \cdots, \mathscr{R}, t \in [0, \tau_{\text{end}}],$$

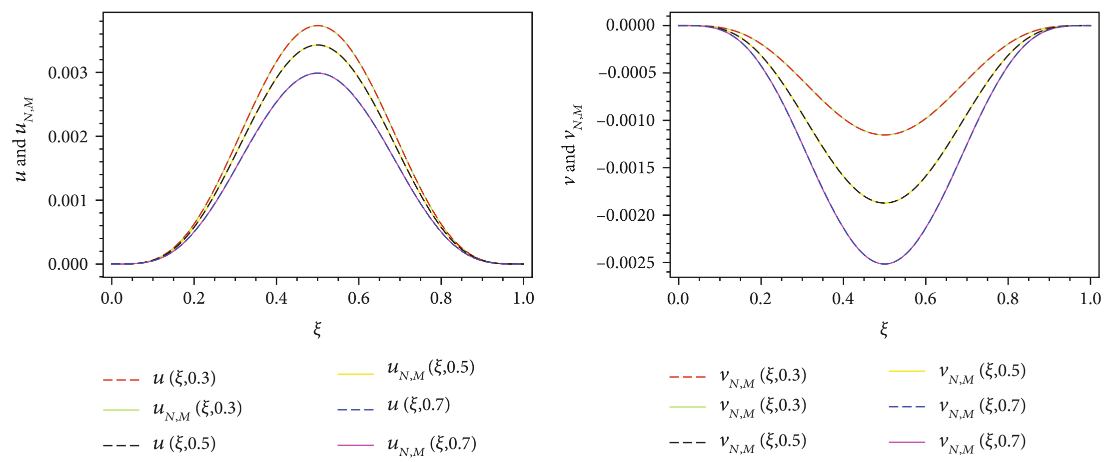
$$\sum_{j=0}^{\mathscr{K}} a_{r,j} \mathscr{C}_{\tau_{\text{end}},j}(0) = \tau_r, \quad r = 1, \cdots, \mathscr{R}. \tag{28}$$

Combining the initial conditions with the abovementioned equations and equalizing the residual of Equation



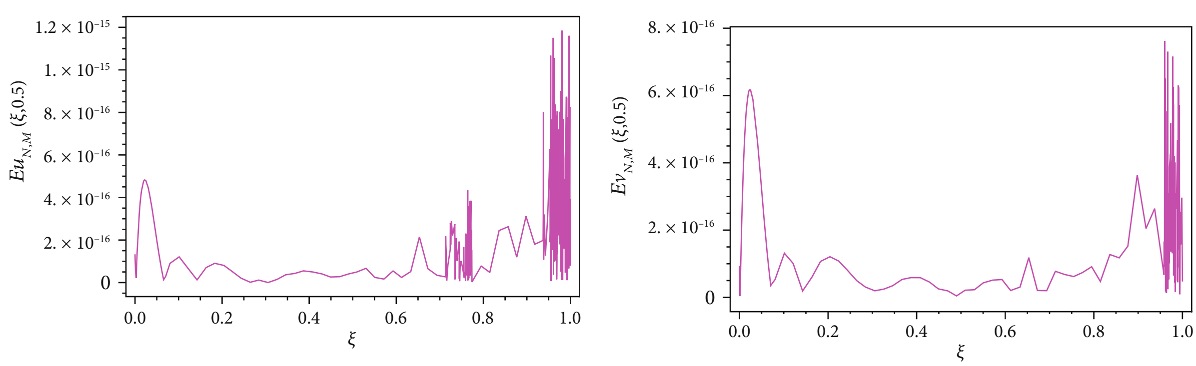
(a) Real part (b) Imaginary part

FIGURE 5: Space graphs of real and imaginary parts of the numerical solution of Equation (35).



(a) Real part (b) Imaginary part

FIGURE 6: ξ-direction curves for the approximate and exact solutions of real and imaginary parts of Equation (35).



(a) Real part (b) Imaginary part

FIGURE 7: ξ-direction curves of real and imaginary parts of the absolute error of Equation (35).

(24) by zero at $(\mathscr{R}\mathscr{K})$ shifted Legendre-Gauss-Radau collocation points give us the following:

$$\sum_{j=0}^{\mathscr{K}} a_{r,j}\mathscr{C}^{(1)}_{\tau_{\text{end}},j} = \mathscr{G}_r\left(t, \sum_{j=0}^{\mathscr{K}} a_{1,j}\mathscr{C}_{\tau_{\text{end}},j}\left(\tau_{\tau_{\text{end}}}\mathscr{K}, s\right), \cdots, \sum_{j=0}^{\mathscr{K}} a_{\mathscr{K},j}\mathscr{C}_{\tau_{\text{end}},j}\left(\tau_{\tau_{\text{end}}}, \mathscr{K}, s\right)\right), \quad r = 1, \cdots, \mathscr{R}, s = 1, \cdots, \mathscr{K}, \tag{29}$$

where the rest $(\mathscr{R})$ algebraic equations are outputted by the initial conditions as follows:

$$\sum_{j=0}^{\mathscr{K}} a_{r,j}\mathscr{C}_{\tau_{\text{end}},j}(0) = \tau_r, \quad r = 1, \cdots, \mathscr{R}. \tag{30}$$

Finally, we have $(\mathscr{R}(\mathscr{K}+1))$ algebraic equations as follows:

$$\begin{aligned}&\sum_{j=0}^{\mathscr{K}} a_{r,j}\mathscr{C}^{(1)}_{\tau_{\text{end}},j}(\tau_{\tau_{\text{end}}},\mathscr{K},s)\\&=\mathscr{G}_r\left(t,\sum_{j=0}^{\mathscr{K}} a_{1,j}\mathscr{C}_{\tau_{\text{end}},j}(\tau_{\tau_{\text{end}}},\mathscr{K},s),\cdots,\sum_{j=0}^{\mathscr{K}} a_{\mathscr{K},j}\mathscr{C}_{\tau_{\text{end}},j}(\tau_{\tau_{\text{end}}},\mathscr{K},s)\right),\quad r=1,\\&\cdots,\mathscr{R}, s=1,\cdots,\mathscr{K},\\&\sum_{j=0}^{\mathscr{K}} a_{r,j}\mathscr{C}_{\tau_{\text{end}},j}(0)=\tau_r,\quad r=1,\cdots,\mathscr{R}.\end{aligned}\tag{31}$$

The numerical approach of the previous system will be acquired by using Newton's iterative method.

4. Applications and Numerical Results

Here, the adequacy of the spectral collocation algorithms is verified by the obtained results. Problems including initial-boundary conditions are examined. Mathematica version 10 is utilized to carry out the code.

Example 1. We test the next problem:

$$i\frac{\partial\Theta}{\partial\tau}+\frac{\partial^{\mu}\Theta}{\partial|\xi|^{\mu}}+i|\Theta|^2\frac{\partial\Theta}{\partial\tau}=\Delta(\xi,\tau),(\xi,\tau)\in[0,1]\times[0,1],\tag{32}$$

where the function $\Delta(\xi,\tau)$, initial condition, and the boundary conditions are given such as the continuous problem has the next exact solution:

$$\Theta(\xi,\tau)=e^{i\tau}\xi^3(1-\xi)^3.\tag{33}$$

In Table 1, the numerical results based on the maximum absolute errors of Equation (32) obtained using the previous algorithms are listed, where

$$\begin{aligned}E_{\mathscr{U}_{\mathscr{N},\mathscr{M}}}(\xi,\tau)&=\left|\mathscr{U}_{\mathscr{N},\mathscr{M}}(\xi,\tau)-\mathscr{U}(\xi,\tau)\right|,(\xi,\tau)\in[0,\xi_{\text{end}}],\\E_{\mathscr{V}_{\mathscr{N},\mathscr{M}}}(\xi,\tau)&=\left|\mathscr{V}_{\mathscr{N},\mathscr{M}}(\xi,\tau)-\mathscr{V}(\xi,\tau)\right|,(\xi,\tau)\in[0,\xi],\\E_{\mathscr{V}_{\mathscr{N},\mathscr{M}}}(\xi,\tau)&=\sqrt{\left(E_{\mathscr{U}_{\mathscr{N},\mathscr{M}}}(\xi,\tau)\right)^2+\left(E_{\mathscr{V}_{\mathscr{N},\mathscr{M}}}(\xi,\tau)\right)^2},(\xi,\tau)\in[0,\xi],\\M_{\mathscr{U}_{\mathscr{N},\mathscr{M}}}(\xi,\tau)&=\text{Max}\left\{E_{\mathscr{U}_{\mathscr{N},\mathscr{M}}}(\xi,\tau),\quad\forall(\xi,\tau)\in[0,\xi]\right\},\\M_{\mathscr{V}_{\mathscr{N},\mathscr{M}}}(\xi,\tau)&=\text{Max}\left\{E_{\mathscr{V}_{\mathscr{N},\mathscr{M}}}(\xi,\tau),\quad\forall(\xi,\tau)\in[0,\xi]\right\},\\M_{\mathscr{N},\mathscr{M}}(\xi,\tau)&=\text{Max}\left\{E_{\mathscr{N},\mathscr{M}}(\xi,\tau),\quad\forall(\xi,\tau)\in[0,\xi]\right\}.\end{aligned}\tag{34}$$

Space graphs of real and imaginary parts of the numerical solution of Equation (32) are shown in Figures 1(a) and 1(b), respectively, where $\mathscr{N}=\mathscr{M}=12$. While in Figures 2(a) and 2(b), we recognize the outright matching of numerical and exact solutions in its real and imaginary parts of Equation (32), where $N=M=12$. Also, ξ-direction curves for real and imaginary parts of the absolute errors of Equation (32) are plotted in Figures 3(a) and 3(b), respectively, where $\tau=0.5$, $\mathscr{N}=\mathscr{M}=12$. Moreover, we sketched in Figure 4 the logarithmic graphs of M_{ε} (i.e., $\log_{10}M_{\varepsilon}$) of Equation (32) obtained by the present method with different values of ($\mathscr{N}=\mathscr{M}=2,4,6,\cdots,12$).

Example 2. Now, consider the following:

$$i\frac{\partial\Theta}{\partial\tau}+\frac{\partial^{\mu}\Theta}{\partial|\xi|^{\mu}}+i|\Theta|^2\frac{\partial\Theta}{\partial\tau}=\Delta(\xi,\tau),\quad(\xi,\tau)\in[0,1]\times[0,1],\tag{35}$$

where the function $\Delta(\xi,\tau)$, initial condition, and the boundary conditions are given such as the continuous problem has the next exact solution:

$$\Theta(\xi,\tau)=e^{i\tau}\xi^4(1-\xi)^4.\tag{36}$$

In Table 2, the numerical results based on the maximum absolute errors of Equation (35) obtained using the previous algorithms are listed. Space graphs of real and imaginary parts of the numerical solution of Equation (35) are shown in Figures 5(a) and 5(b), respectively, where $\mathscr{N}=\mathscr{M}=12$. While in Figures 6(a) and 6(b), we recognize the outright matching of numerical and exact solutions in its real and imaginary parts, respectively, where $\mathscr{N}=\mathscr{M}=12$. Also, ξ -direction curves for real and imaginary parts of absolute errors of Equation (35) are plotted in Figures 7(a) and 7(b), respectively, where $\tau=0.5$, $\mathscr{N}=\mathscr{M}=12$. Even though few values of N and M, the accurate results have been spotted in these tables. This is consistent with which was predicted in case of using a spectral collocation method. Likewise, these results bring to light the responsibility convergence of the shifted Legendre collocation method for such problems.

5. Conclusions

This paper adopted fully collocation method to study Riesz fractional Chen-Lee-Liu equation that discusses soliton propagation down the optical fibers with perturbation terms incorporated into the waveguides. The powerful numerical scheme gave way to a number of impressive numerical results that prove high efficiency of the algorithm. The study was carried out with initial-boundary conditions.

The results of the algorithm pave way to conduct further additional research in this field to display additional results in future. One avenue is to consider Riesz fractional Chen-Lee-Liu equation with differential group delay and then further along study the model with additional optoelectronic devices such as in magneto-optic waveguides. Subsequently, this model will be treated with the same algorithm for dense wavelength division multiplexing (DWDM) topology.

Thus, a lot lies in the bucket list!

Data Availability

There is no data used for this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] F. S. Sousa, C. F. Lages, J. L. Ansoni, A. Castelo, and A. Simao, "A finite difference method with meshless interpolation for incompressible flows in non-graded tree-based grids," *Journal of Computational Physics*, vol. 396, pp. 848–866, 2019.

[2] N. A. Mbroh and J. B. Munyakazi, "A fitted operator finite difference method of lines for singularly perturbed parabolic convection–diffusion problems," *Mathematics and Computers in Simulation*, vol. 165, pp. 156–171, 2019.

[3] H. M. Patil and R. Maniyeri, "Finite difference method based analysis of bio-heat transfer in human breast cyst," *Thermal Science and Engineering Progress*, vol. 10, pp. 42–47, 2019.

[4] P.-W. Li, F. Zhuo-Jia, G. Yan, and L. Song, "The generalized finite difference method for the inverse Cauchy problem in two-dimensional isotropic linear elasticity," *International Journal of Solids and Structures*, vol. 174, pp. 69–84, 2019.

[5] S. Yu, J. Li, and C. Zhang, "A local and parallel Uzawa finite element method for the generalized Navier–Stokes equations," *Applied Mathematics and Computation*, vol. 387, p. 124671, 2020.

[6] C. Wang and J. Wang, "Primal–dual weak Galerkin finite element methods for elliptic Cauchy problems," *Computers & Mathematics with Applications*, vol. 79, no. 3, pp. 746–763, 2020.

[7] X. Xiao, Z. Dai, and X. Feng, "A positivity preserving characteristic finite element method for solving the transport and convection–diffusion–reaction equations on general surfaces," *Computer Physics Communications*, vol. 247, p. 106941, 2020.

[8] E. H. Doha, M. A. Abdelkawy, A. Z. Amin, and A. M. Lopes, "Shifted Jacobi-Gauss-collocation with convergence analysis for fractional integro-differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 72, pp. 342–359, 2019.

[9] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, "A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order," *Computers & Mathematics with Applications*, vol. 62, no. 5, pp. 2364–2373, 2011.

[10] H. Ali, "A review of operational matrices and spectral techniques for fractional calculus," *Nonlinear Dynamics*, vol. 81, no. 3, pp. 1023–1052, 2015.

[11] A. H. Bhrawy and M. A. Zaky, "A method based on the Jacobi tau approximation for solving multi-term time–space fractional partial differential equations," *Journal of Computational Physics*, vol. 281, pp. 876–895, 2015.

[12] R. M. Hafez and Y. H. Youssri, "Jacobi collocation scheme for variable-order fractional reaction-subdiffusion equation," *Computational and Applied Mathematics*, vol. 37, no. 4, pp. 5315–5333, 2018.

[13] R. M. Hafez and Y. H. Youssri, "Shifted Jacobi collocation scheme for multidimensional time-fractional order telegraph equation," *Iranian Journal of Numerical Analysis and Optimization*, vol. 10, no. 1, pp. 195–223, 2020.

[14] A. H. Bhrawy and M. A. Abdelkawy, "A fully spectral collocation approximation for multi-dimensional fractional Schrodinger equations," *Journal of Computational Physics*, vol. 294, pp. 462–483, 2015.

[15] A. H. Bhrawy, E. H. Doha, D. Baleanu, and S. S. Ezz-Eldien, "A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations," *Journal of Computational Physics*, vol. 293, pp. 142–156, 2015.

[16] R. M. Hafez, M. A. Zaky, and M. A. Abdelkawy, "Jacobi spectral Galerkin method for distributed-order fractional Rayleigh-Stokes problem for a generalized second grade fluid," *Frontiers in Physics*, vol. 7, 2020.

[17] E. H. Doha and W. M. Abd-Elhameed, "Efficient spectral ultraspherical-dual-Petrov–Galerkin algorithms for the direct solution of (2n+1) th-order linear differential equations," *Mathematics and Computers in Simulation*, vol. 79, no. 11, pp. 3221–3242, 2009.

[18] A.-M. Wazwaz and L. Kaur, "Optical solitons and peregrine solitons for nonlinear Schrödinger equation by variational iteration method," *Optik*, vol. 179, pp. 804–809, 2019.

[19] H. Triki and A.-M. Wazwaz, "Combined optical solitary waves of the Fokas—Lenells equation," *Waves in Random and Complex Media*, vol. 27, no. 4, pp. 587–593, 2017.

[20] A.-M. Wazwaz and L. Kaur, "Complex simplified Hirota's forms and Lie symmetry analysis for multiple real and complex soliton solutions of the modified KdV–Sine-Gordon equation," *Nonlinear Dynamics*, vol. 95, no. 3, pp. 2209–2215, 2019.

[21] A.-M. Wazwaz, "Two-mode fifth-order KdV equations: necessary conditions for multiple-soliton solutions to exist," *Nonlinear Dynamics*, vol. 87, no. 3, pp. 1685–1691, 2017.

[22] E. H. Doha, A. H. Bhrawy, M. A. Abdelkawy, and R. A. Van Gorder, "Jacobi–Gauss–Lobatto collocation method for the numerical solution of 1+1 nonlinear Schrödinger equations," *Journal of Computational Physics*, vol. 261, pp. 244–255, 2014.

[23] H. Wang, "Numerical studies on the split-step finite difference method for nonlinear Schrödinger equations," *Applied Mathematics and Computation*, vol. 170, no. 1, pp. 17–35, 2005.

[24] M. Dehghan and A. Taleei, "Numerical solution of nonlinear Schrödinger equation by using time-space pseudo-spectral method," *Numerical Methods for Partial Differential Equations: An International Journal*, vol. 26, no. 4, pp. 979–992, 2010.

[25] M. Dehghan and A. Taleei, "A Chebyshev pseudospectral multi-domain method for the soliton solution of coupled nonlinear Schrödinger equations," *Computer Physics Communications*, vol. 182, no. 12, pp. 2519–2529, 2011.

[26] L. Kaur and A.-M. Wazwaz, "Bright–dark optical solitons for Schrödinger-Hirota equation with variable coefficients," *Optik*, vol. 179, pp. 479–484, 2019.

[27] H. H. Chen, Y. C. Lee, and C. S. Liu, "Integrability of nonlinear Hamiltonian systems by inverse scattering method," *Physica Scripta*, vol. 20, no. 3-4, p. 490, 1979.

[28] E. Fan, "Integrable systems of derivative nonlinear Schrödinger type and their multi-Hamiltonian structure," *Journal of Physics A: Mathematical and General*, vol. 34, no. 3, p. 513, 2001.

[29] J. Zhang, W. Liu, D. Qiu, Y. Zhang, K. Porsezian, and J. He, "Rogue wave solutions of a higher-order Chen–Lee–Liu equation," *Physica Scripta*, vol. 90, no. 5, article 055207, 2015.

[30] A. J.'a. M. Jawad, A. Biswas, Q. Zhou, M. Alfiras, S. P. Moshokoa, and M. Belic, "Chirped singular and combo optical solitons for Chen–Lee–Liu equation with three forms of integration architecture," *Optik*, vol. 178, pp. 172–177, 2019.

[31] A. Biswas, M. Ekici, A. Sonmezoglu et al., "Chirped optical solitons of Chen–Lee–Liu equation by extended trial equation scheme," *Optik*, vol. 156, pp. 999–1006, 2018.

[32] Y. Yıldırım, "Optical solitons to Chen–Lee–Liu model with modified simple equation approach," *Optik*, vol. 183, pp. 792–796, 2019.

[33] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, 1993.

[34] P. Zhuang, F. Liu, V. Anh, and I. Turner, "Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term," *SIAM Journal on Numerical Analysis*, vol. 47, no. 3, pp. 1760–1781, 2009.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9974034, 16 pages
https://doi.org/10.1155/2021/9974034

Research Article

Well-Posedness and Stability Result of the Nonlinear Thermodiffusion Full von Kármán Beam with Thermal Effect and Time-Varying Delay

Abdelbaki Choucha,[1] Djamel Ouchenane,[2] Salah Mahmoud Boulaaras,[3,4] Bahri Belkacem Cherif,[3,5] and Mohamed Abdalla[6,7]

[1]*Laboratory of Operator Theory and PDEs: Foundations and Applications, Department of Mathematics, Faculty of Exact Sciences, University of El Oued, Algeria*
[2]*Laboratory of Pure and Applied Mathematics, Amar Telidji Laghouat University, Algeria*
[3]*Department of Mathematics, College of Sciences and Arts, Qassim University, Ar Rass, Saudi Arabia*
[4]*Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Algeria*
[5]*Preparatory Institute for Engineering Studies in Sfax, Tunisia*
[6]*Mathematics Department, College of Science, King Khalid University, Abha 61413, Saudi Arabia*
[7]*Mathematics Department, Faculty of Science, South Valley University, Qena 83523, Egypt*

Correspondence should be addressed to Salah Mahmoud Boulaaras; s.boularas@qu.edu.sa,
Bahri Belkacem Cherif; bahi1968@yahoo.com, and Mohamed Abdalla; moabdalla@kku.edu.sa

Received 5 March 2021; Revised 19 March 2021; Accepted 22 March 2021; Published 5 April 2021

Academic Editor: Qifeng Zhang

In this work, we consider a new full von Kármán beam model with thermal and mass diffusion effects according to the Gurtin-Pinkin model combined with time-varying delay. Heat and mass exchange with the environment during thermodiffusion in the von Kármán beam. We establish the well-posedness and the exponential stability of the system by the energy method under suitable conditions.

1. Introduction and Preliminaries

In this paper, we are concerned with the following problem:

$$\begin{cases} w_{tt} - d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right)w_x\right]_x + d_2 w_{xxxx} + \mu_1 w_t + \mu_2 w_t(x, t-\tau(t)) = 0, \\ u_{tt} - d_1\left[u_x + \frac{1}{2}(w_x)^2\right]_x - \delta_1\theta_x - \delta_2 P_x = 0, \\ c\theta_t + dP_t - \int_0^\infty \beta_1(\sigma)\theta_{xx}(t-\sigma)d\sigma - \delta_1 u_{tx} = 0, \\ d\theta_t + rP_t - \int_0^\infty \beta_2(\sigma)P_{xx}(t-\sigma)d\sigma - \delta_2 u_{tx} = 0, \end{cases} \tag{1}$$

where

$$(x, \sigma, t) \in (0, L) \times \mathbb{R}_+ \times (0,\infty). \tag{2}$$

Here, $\tau(t) > 0$ represents the time-varying delay, and d_1, d_2, δ_1, δ_2, c, d, r, and μ_1 are positive constants; μ_2 is a real number, and β_1 and β_2 are the relaxation functions, with the initial data

$$w(x, 0) = w_0(x),$$

$$w_t(x, 0) = w_1(x),$$

$$u(x, 0) = u_0(x),$$

$$u_t(x,0) = u_1(x),$$

$$\theta(x,0) = \theta_0(x),$$

$$P(x,0) = P_0(x),$$

$$w_t(x, t-\tau(0)) = f_0(x, t-\tau(0)), \tag{3}$$

where

$$(x,t) \in (0,L) \times (0,\tau(0)), \tag{4}$$

and Neumann-Dirichlet boundary conditions

$$\begin{aligned} &w(x,t) = u(x,t) = P(x,t) = 0, \quad x = 0, L, \forall t \geq 0, \\ &w_x(x,t) = \theta(x,t) = 0, \quad x = 0, L, \forall t \geq 0. \end{aligned} \tag{5}$$

The case of time-varying delay in the wave equation has been studied recently by Nicaise et al. [1]; they proved the exponential stability under the condition

$$\mu_2 < \sqrt{1-d}\mu_1, \tag{6}$$

where d is a constant that satisfies

$$\tau'(t) \leq d < 1, \quad \forall t > 0. \tag{7}$$

For the wave equation with a time-varying delay, in [1], the authors consider the system

$$\begin{cases} u_{tt} - \Delta u = 0, \\ u(x,t) = 0, \\ \dfrac{du}{dv}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)), \end{cases} \tag{8}$$

where the time-varying delay $\tau(t) > 0$ satisfies

$$0 \leq \tau(t) \leq \overline{\tau}, \quad \forall t > 0, \tag{9}$$

$$\tau'(t) \leq 1, \quad \forall t > 0, \tag{10}$$

$$\tau(t) \in W^{2,\infty}([0,T]), \quad \forall T > 0. \tag{11}$$

They proved the exponential stability under suitable conditions.

The purpose of this work is to study problem (1)–(5), with a delay term appearing in the control term at the first equation, introducing the time-varying delay term $\beta_2 w_t(x, t-\tau(t))$; thermal and mass diffusion effects make the problem different from those considered in the literature (see [2–30]).

This paper is organized as follows: in the rest of this section, we put the preliminaries necessary for problem (1); in Section 2, we establish the well-posedness. As for Section 3, we prove the exponential stability result by the energy method and Lyapunov function.

In order to prove the existence of a unique solution of problem (1)–(5), we introduce the new variable

$$z(x,\rho,t) = w_t(x, t-\tau(t)\rho). \tag{12}$$

Then, we obtain

$$\begin{cases} \tau(t) z_t(x,\rho,t) + \left(1 - \tau'(t)\rho\right) z_\rho(x,\rho,t) = 0, \\ z(x,0,t) = w_t(x,t). \end{cases} \tag{13}$$

And it is more convenient to work in the history space setting by introducing the so-called summed past history of θ and P defined by (see [31–36])

$$\begin{cases} \eta^t(\sigma) = \displaystyle\int_0^\sigma \theta(t-\zeta)\, d\zeta, \\ \nu^t(\sigma) = \displaystyle\int_0^\sigma P(t-\zeta)\, d\zeta, \quad (t,\sigma) \in [0,\infty) \times \mathbb{R}_+. \end{cases} \tag{14}$$

Differentiating $(14)_1$ and $(14)_2$, we get

$$\begin{cases} \eta_t^t(\sigma) + \eta_\sigma^t(\sigma) = \theta(t), \\ \nu_t^t(\sigma) + \nu_\sigma^t(\sigma) = P(t), \end{cases} \tag{15}$$

with the boundary and initial conditions

$$\begin{aligned} &\eta^t(0) = \nu^t(0) = 0, \quad t \geq 0, \\ &\eta^0(\sigma) = \eta_0(\sigma), \nu^0(\sigma) = \nu_0(\sigma), \quad \sigma \geq 0. \end{aligned} \tag{16}$$

We set

$$\begin{cases} \eta_0(\sigma) = \displaystyle\int_0^\sigma \overline{\theta}_0(\tau)\, d\tau, \\ \nu_0(\sigma) = \displaystyle\int_0^\sigma \overline{P}_0(\tau)\, d\tau, \sigma \in \mathbb{R}_+. \end{cases} \tag{17}$$

Concerning the memory kernels β_1 and β_2, we set

$$\begin{aligned} \beta(\sigma) &= -\beta_1{}'(\sigma), \\ \lambda(\sigma) &= -\beta_2{}'(\sigma). \end{aligned} \tag{18}$$

Assuming $\beta_1(\infty) = \beta_2(\infty) = 0$, then from (14), we infer

$$\begin{cases} \displaystyle\int_0^\infty \beta_1(\sigma)\theta(t-\sigma)\, d\sigma = -\int_0^\infty \beta_1{}'(\sigma)\eta^t(\sigma)\, d\sigma, \\ \displaystyle\int_0^\infty \beta_2(\sigma)P(t-\sigma)\, d\sigma = -\int_0^\infty \beta_2{}'(\sigma)\nu^t(\sigma)\, d\sigma, \end{cases} \tag{19}$$

and therefore,

$$\begin{cases} \int_0^\infty \beta_1(\sigma)\theta_{xx}(t-\sigma)d\sigma = \int_0^\infty \beta(\sigma)\eta_{xx}^t(\sigma)d\sigma, \\ \int_0^\infty \beta_2(\sigma)P_{xx}(t-\sigma)d\sigma = \int_0^\infty \lambda(\sigma)\nu^t(\sigma)d\sigma. \end{cases} \tag{20}$$

Consequently, the problem is equivalent to

$$\begin{cases} w_{tt} - d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right)w_x\right]_x + d_2 w_{xxxx} + \mu_1 w_t + \mu_2 z(x,1,t) = 0, \\ u_{tt} - d_1\left[u_x + \frac{1}{2}(w_x)^2\right]_x - \delta_1\theta_x - \delta_2 P_x = 0, \\ c\theta_t + dP_t - \int_0^\infty \beta(\sigma)\eta_{xx}^t(\sigma)d\sigma - \delta_1 u_{tx} = 0, \\ d\theta_t + rP_t - \int_0^\infty \lambda(\sigma)\nu_{xx}^t(\sigma)d\sigma - \delta_2 u_{tx} = 0, \\ \eta_t^t + \eta_\sigma^t = \theta, \\ \nu_t^t + \nu_\sigma^t = P, \\ \tau(t)z_t(x,\rho,t) + \left(1-\tau'(t)\rho\right)z_\rho(x,\rho,t) = 0, \end{cases} \tag{21}$$

where

$$(x,\sigma,\rho,t) \in (0,L)\times\mathbb{R}_+\times(0,1)\times(0,\infty), \tag{22}$$

with the initial and boundary conditions

$$\begin{cases} w(x,t) = w_x(x,t) = u(x,t) = P(x,t) = \theta(x,t) = 0, \quad x = 0, L, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ \theta(x,0) = \theta_0(x), P(x,0) = P_0(x), \\ z(x,\rho,0) = f_0(x,-\rho\tau(0)), \\ \eta^0(x,\sigma) = \eta_0(x,\sigma), \nu^0(x,\sigma) = \nu_0(x,\sigma), \quad (x,\sigma) \in (0,1)\times\mathbb{R}_+, \end{cases} \tag{23}$$

$$\forall (x,\rho,\sigma,t) \in (0,L)\times(0,1)\times(0,\infty)\times(0,\infty), \tag{24}$$

where the function $\tau(t)$ satisfies (7), (11), and the condition

$$0 < \tau_0 < \tau(t) < \bar{\tau}, \quad \forall t > 0. \tag{25}$$

In this paper, we establish the well-posedness and prove the exponential stability by using the variable of Kato under some restrictions and assumptions:

(H1).

$$|\mu_2| \le \sqrt{1-d}\mu_1. \tag{26}$$

(H2). The symmetric matrix Λ is positive definite, where

$$\Lambda = \begin{pmatrix} c\, d \\ d\, r \end{pmatrix}. \tag{27}$$

That is, $|\Lambda| = cr - d^2 > 0$ implies that

$$c\int_0^L \theta^2 dx + 2d\int_0^L \theta P dx + r\int_0^L P^2 dx > 0. \tag{28}$$

Condition (28) is needed to stabilize the system when diffusion effects are added to thermal effects (see, e.g., [31–38] for more information on this). By virtue of $cr > d^2$, we deduce that $d/c < r/d$. Let, then, ζ be a number chosen in such a way that

$$\frac{d}{c} < \zeta < \frac{r}{d}. \tag{29}$$

Thus, Young's inequality leads to

$$2d\int_0^L \theta P dx \le \frac{d}{\zeta}\int_0^L \theta^2 dx + d\zeta\int_0^L P^2 dx. \tag{30}$$

(H3). We assume the following set of hypotheses on μ and λ:

$$\begin{gathered} \beta, \lambda \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+), \\ \beta(\sigma), \lambda(\sigma) \ge 0, \beta'(\sigma), \lambda'(\sigma) \le 0, \quad \forall \sigma \in \mathbb{R}_+, \\ \beta'(\sigma) + \alpha_1\beta(\sigma) \le 0, \lambda'(\sigma) + \alpha_2\lambda(\sigma) \le 0, \quad \text{for some } \alpha_1, \alpha_2 > 0, \forall \sigma \in \mathbb{R}_+, \end{gathered} \tag{31}$$

$$\begin{cases} \beta(0) = \int_0^\infty \beta(\sigma)d\sigma := \beta_0 > 0, \\ \lambda(0) = \int_0^\infty \lambda(\sigma)d\sigma := \lambda_0 > 0. \end{cases} \tag{32}$$

Let f be a memory kernel satisfying the assumptions (31) and (32).

Now, we consider the weighted Hilbert spaces

$$\begin{aligned} \mathscr{M}_f &= L^2\left(\mathbb{R}_+, H_0^1(0,L)\right) \\ &= \left\{\Phi : \mathbb{R}_+ \to H_0^1(0,L) / \int_0^L\int_0^\infty f(\sigma)\Phi_x^2(\sigma)d\sigma dx < \infty\right\}, \end{aligned} \tag{33}$$

equipped with the inner product

$$\langle \Phi, \Psi \rangle_{\mathscr{M}_f} = \int_0^L\int_0^\infty f(\sigma)\Phi_x(\sigma)\Psi_x(\sigma)d\sigma dx, \tag{34}$$

and the norm

$$\|\Phi\|^2_{\mathscr{M}_f} = <\Phi, \Phi>_{\mathscr{M}_f} = \int_0^L \int_0^\infty f(\sigma)\Phi_x^2(\sigma)d\sigma dx. \quad (35)$$

We also introduce the linear operator T on $\mathscr{M}_f$ defined by

$$T_\Phi = -\Phi_\sigma, \quad \Phi \in \mathscr{D}(T), \quad (36)$$

with

$$\mathscr{D}(T) = \left\{\Phi \in \mathscr{M}_f / \Phi_\sigma \in \mathscr{M}_f, \Phi(0) = 0\right\}, \quad (37)$$

where Φ_σ is the distributional derivative of Φ with respect to the internal variable σ, and then, the operator T is the infinitesimal generator of a C_0-semigroup of contractions. Following Ref. [39], there holds

$$\begin{aligned} <T\Phi, \Phi>_{\mathscr{M}_f} &= <-\Phi_\sigma, \Phi>_{\mathscr{M}_f} \\ &= -\frac{1}{2}\int_0^\infty f(\sigma)\frac{d}{d\sigma}\int_0^L \Phi_x^2(\sigma)dxd\sigma, \quad \forall \Phi \in \mathscr{D}(T). \end{aligned} \quad (38)$$

Integration by parts yields

$$\begin{aligned} &\int_0^\infty f(\sigma)\frac{d}{d\sigma}\int_0^L \Phi_x^2(\sigma)dxd\sigma \\ &= f(\sigma)\int_0^L \Phi_x^2(\sigma)dx\bigg|_0^\infty - \int_0^\infty f'(\sigma)\int_0^L \Phi_x^2(\sigma)dxd\sigma. \end{aligned} \quad (39)$$

Hence, from (31), we obtain

$$<T\Phi, \Phi>_{\mathscr{M}_f} = \frac{1}{2}\int_0^\infty f'(\sigma)\int_0^L \Phi_x^2(\sigma)dxd\sigma \le 0. \quad (40)$$

As a direct consequence, we deduce from (32) and (40) that

$$\begin{aligned} <T\eta, \eta>_{\mathscr{M}_\beta} &= \frac{1}{2}\int_0^\infty \beta'(\sigma)\int_0^L \eta_x^2(\sigma)dxd\sigma \\ &\le -\frac{\alpha_1}{2}\int_0^\infty \beta(\sigma)\int_0^L \eta_x^2(\sigma)dxd\sigma = -\frac{\alpha_1}{2}\|\eta_x\|^2_{\mathscr{M}_\beta}, \\ <T\nu, \nu>_{\mathscr{M}_\lambda} &= \frac{1}{2}\int_0^\infty \lambda'(\sigma)\int_0^L \nu_x^2(\sigma)dxd\sigma \\ &\le -\frac{\alpha_2}{2}\int_0^\infty \lambda(\sigma)\int_0^L \nu_x^2(\sigma)dxd\sigma = -\frac{\alpha_2}{2}\|\nu_x\|^2_{\mathscr{M}_\lambda}, \end{aligned} \quad (41)$$

for all $\eta, \nu \in \mathscr{D}(T)$. Finally, we define the operator $\mathbb{L}_f : \mathscr{D}(\mathbb{L}_f) \to L^2(0, L)$ by

$$\mathbb{L}_f\Phi = \int_0^\infty f(\sigma)\Phi_{xx}(\sigma)d\sigma, \quad (42)$$

with the domain

$$\mathscr{D}(\mathbb{L}_f) = \left\{\Phi \in \mathscr{M}_f / \int_0^\infty f(\sigma)\Phi_{xx}(\sigma)d\sigma \in L^2(0, L), \Phi(0) = 0\right\}. \quad (43)$$

2. Well-Posedness

In this section, we give sufficient conditions that guarantee the well-posedness of this problem. Let

$$U = \left(w, w_t, u, u_t, \theta, \eta^t, P, \nu^t, z\right)^T. \quad (44)$$

For the sake of simplicity, we write $\eta = \eta^t(\sigma)$ and $\nu = \nu^t(\sigma)$ and the new dependent variables $\varphi = \omega_t$ and $\psi = u_t$; then, (21)–(23) can be written as

$$\begin{cases} U' = \mathscr{A}(t)U + \mathscr{F}(U), \\ U(0) = (w_0, w_1, u_0, u_1, \theta_0, \eta_0, P_0, \nu_0, f_0(., -\rho\tau(0)))^T, \end{cases} \quad (45)$$

with the linear problem

$$\begin{cases} U' = \mathscr{A}(t)U, \\ U(0) = (w_0, w_1, u_0, u_1, \theta_0, \eta_0, P_0, \nu_0, f_0(., -\rho\tau(0)))^T, \end{cases} \quad (46)$$

where the time-varying operator $\mathscr{A}$ is defined by

$$\mathscr{A}(t)\begin{pmatrix} w \\ \varphi \\ u \\ \psi \\ \theta \\ \eta \\ P \\ \nu \\ z \end{pmatrix} = \begin{pmatrix} \varphi \\ -d_2 w_{xxxx} - \mu_1\varphi - \mu_2 z(x, 1, t) \\ \psi \\ d_1 u_{xx} + \delta_1\theta_x + \delta_2 P_x \\ -\frac{1}{\alpha_1}\left[(d\delta_2 - r\delta_1)\psi_x - r\mathbb{L}_\beta\eta + d\mathbb{L}_\lambda\nu\right] \\ \theta + T\eta \\ -\frac{1}{\alpha_2}\left[(d\delta_1 - c\delta_2)\psi_x + d\mathbb{L}_\beta\eta - c\mathbb{L}_\lambda\nu\right] \\ P + T\nu \\ \dfrac{\left(\tau'(t)\rho - 1\right)}{\tau(t)} z_\rho \end{pmatrix}, \quad (47)$$

$$\mathcal{F}(U)=\begin{pmatrix}0\\ d_1\left[u_x+\frac{1}{2}(w_x)^2\right]_x\\ 0\\ \frac{d_1}{2}(w_x)^2\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}. \tag{48}$$

The energy space $\mathcal{H}$ is defined as

$$\begin{aligned}\mathcal{H}=&\left[H^4(0,L)\cap H_0^2(0,L)\right]\times H_0^1(0,L)\\ &\times\left[H^2(0,L)\cap H_0^2(0,L)\right]\times H_0^1(0,L)\times L^2(0,L)\\ &\times\mathcal{M}_\beta\times L^2(0,L)\times\mathcal{M}_\lambda\times L^2((0,L),(0,1)),\end{aligned} \tag{49}$$

and the domain of $\mathcal{A}$ is

$$\begin{aligned}\mathcal{D}(\mathcal{A}(t))=\{&U\in\mathcal{H}/\varphi=z(.,0),\theta,P\in H_0^1(0,L),\mathbb{L}_\beta\eta,\mathbb{L}_\lambda\nu\\ &\in L^2(0,L),\eta,\nu\in\mathcal{D}(T)\}.\end{aligned} \tag{50}$$

We equip $\mathcal{H}$ with the inner product

$$\begin{aligned}<U,\bar{U}>_{\mathcal{H}}=&\int_0^L\{\varphi\bar{\varphi}+d_1u_x\bar{u}_x+\psi\bar{\psi}+d_2w_{xx}\bar{w}_{xx}\}dx\\ &+\int_0^L\int_0^1 z(x,\rho,t)\bar{z}(x,\rho,t)d\rho dx\\ &+<\Lambda(\theta,P)^T,(\bar{\theta},\bar{P})^T>+<\eta,\bar{\eta}>_{\mathcal{M}_\beta}+<\nu,\bar{\nu}>_{\mathcal{M}_\lambda},\end{aligned} \tag{51}$$

with the existence and the uniqueness in the following result.

Theorem 1. *Let (7), (11), and (25) be satisfied and assume that (26)–(31) hold. Then, for all $U_0\in\mathcal{D}(\mathcal{A}(0))$, there exists a unique solution U of problem (21)–(23) satisfying*

$$U\in C([0,+\infty),\mathcal{D}(\mathcal{A}(0))\cap C^1([0,+\infty),\mathcal{H}). \tag{52}$$

In order to prove Theorem 1, we will use the variable norm technique developed by Kato in [40]. The following theorem is proved in [40].

Theorem 2. *Assume that*

(1) $\mathcal{D}(\mathcal{A}(0))$ is a dense subset of $\mathcal{H}$

(2) $\mathcal{D}(\mathcal{A}(t))=\mathcal{D}(\mathcal{A}(0)),\forall t>0$

(3) For all $t\in[0,T]$, $\mathcal{A}(t)$ generates a strongly continuous semigroup on $\mathcal{H}$ and the family $\mathcal{A}=\{\mathcal{A}(t): t\in[0,T]\}$ is stable with stability constants C and m independent of t; i.e., the semigroup $(S_t(s))_{s\geq 0}$ generated by $\mathcal{A}(t)$ satisfies

$$\|S_t(s)(u)\|_{\mathcal{H}}\leq Ce^{ms}\|u\|_{\mathcal{H}},\quad \forall u\in\mathcal{H},s\geq 0. \tag{53}$$

(4) $d_t\mathcal{A}(t)\in L_*^\infty([0,T],B(\mathcal{D}(\mathcal{A}(0)),\mathcal{H}))$, where $L_*^\infty([0,T],B(\mathcal{D}(\mathcal{A}(0)),\mathcal{H}))$ is the space of equivalent classes of essentially bounded, strongly measurable functions from $[0,T]$ into the set $B(\mathcal{D}(\mathcal{A}(0)),\mathcal{H})$ of bounded operators from $\mathcal{D}(\mathcal{A}(0))$ into $\mathcal{H}$

Then, problem (46) has a unique solution

$$U\in C([0,T],\mathcal{D}(\mathcal{A}(0)))\cap C^1([0,T],\mathcal{H}), \tag{54}$$

for any initial datum in $\mathcal{D}(\mathcal{A}(0))$.

Proof. To prove Theorem 1, we use the method in [1] with the necessary modification.

(1) First, we show that $\mathcal{D}(\mathcal{A}(0))$ is dense in $\mathcal{H}$

Let $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9)\in\mathcal{H}$ be orthogonal to all elements of $\mathcal{D}(\mathcal{A}(0))$ with respect to the inner product $\langle .,.\rangle_{\mathcal{H}}$:

$$\begin{aligned}0=<U,F>_{\mathcal{H}}=&\int_0^L\{\varphi f_2+\psi f_4+d_2w_{xx}f_{1xx}+d_1u_xf_{3x}\}dx\\ &+\int_0^L\int_0^1 z(x,\rho,t)f_9d\rho dx+<\Lambda(\theta,P)^T,(f_5,f_7)^T>\\ &+<\eta,f_6>_{\mathcal{M}_\beta}+<\nu,f_8>_{\mathcal{M}_\lambda}.\end{aligned} \tag{55}$$

For all $U=(w,\varphi,u,\psi,\theta,\eta,P,\nu,z)^T\in\mathcal{D}(\mathcal{A}(0))$, our goal is to prove that $f_i=0,\forall i=1,\cdots,9$. Let us first take $z\in\mathcal{D}((0,L)\times(0,1))$ and $w=\varphi=\psi=u=\theta=q=\phi=0$, so the vector $U=(0,0,0,0,0,0,0,0,z)^T\in\mathcal{D}(\mathcal{A}(0))$, and therefore, from (55), we deduce that

$$\int_0^L\int_0^1 z(x,\rho)f_7d\rho dx=0. \tag{56}$$

Since $\mathcal{D}((0,L)\times(0,1))$ is dense in $L^2((0,L)\times(0,1))$, it follows then that $f_9=0$.

Similarly, let $\varphi\in H_0^1(0,L)$; then, $U=(0,\varphi,0,0,0,0,0,0,0)^T\in\mathcal{D}(\mathcal{A}(0))$, which implies from (55) that

$$\int_0^L\varphi f_2dx=0. \tag{57}$$

So, as above, $f_2=0$.

And let $U=(w,0,0,0,0,0,0,0,0)^T$; then, we obtain from (55) that

$$\int_0^L w_{xx} f_{1xx} dx = 0. \quad (58)$$

It is obvious that $U=(w,0,0,0,0,0,0,0,0)^T \in \mathscr{D}(\mathscr{A}(0))$ only if $w \in H^4(0,L) \cap H_0^2(0,L)$ is dense in $H_0^2(0,L)$, with respect to the inner product

$$<g,h>_{H_0^2(0,L)} \int_0^L g_{xx} h_{xx} dx. \quad (59)$$

We get $f_1 = 0$. By the same ideas as above, we can also show that $f_3 = 0$.

For $u \in \mathscr{D}(\mathscr{A}(t))$, we get from (55) that

$$\int_0^L u_x f_{3x} dx = 0, \quad (60)$$

and by the density of $\mathscr{D}(\mathscr{A}(t))$ in $H_0^1(0,L)$, we obtain $f_3 = 0$.

For $\psi \in \mathscr{D}(\mathscr{A}(t))$, we get from (55) that

$$\int_0^L \psi f_4 dx = 0, \quad (61)$$

and by the density of $\mathscr{D}(\mathscr{A}(t))$ in $H^1(0,L)$, we obtain $f_4 = 0$.

Next, let $U=(0,0,0,0,\theta,0,0,0,0)^T$; then, we obtain from (55) that

$$\int_0^L \theta f_5 dx = 0. \quad (62)$$

It is obvious that $U=(0,0,0,0,\theta,0,0,0,0)^T \in \mathscr{D}(\mathscr{A}(0))$ only if $\theta \in L^2(0,L)$ is dense in $L^2(0,L)$; we get $f_5 = 0$; for $\eta \in \mathscr{M}_\beta$, we get from (55) that

$$\int_0^L \int_0^\infty \beta(\sigma) \eta_x f_{6x} d\sigma dx = 0, \quad (63)$$

which gives $f_6 = 0$. Similarly, for P and ν. This completes the proof of (1).

(2) With our choice, $\mathscr{D}(\mathscr{A}(t))$ is independent of t; consequently,

$$\mathscr{D}(\mathscr{A}(t)) = \mathscr{D}(\mathscr{A}(0)), \quad \forall t > 0. \quad (64)$$

(3) Now, we show that the operator $\mathscr{A}(t)$ generates a C_0-semigroup in $\mathscr{H}$ for a fixed t. We define the time-dependent inner product on $\mathscr{H}$:

$$\begin{aligned} <U,\bar{U}>_t &= \int_0^L \{\varphi\bar{\varphi} + d_1 u_x \bar{u}_x + \psi\bar{\psi} + d_2 w_{xx}\bar{w}_{xx}\} dx \\ &+ \xi\tau(t) \int_0^L \int_0^1 z(x,\rho,t)\bar{z}(x,\rho,t) d\rho dx \\ &+ <\Lambda(\theta,P)^T, (\bar{\theta},\bar{P})^T> + <\eta,\bar{\eta}>_{\mathscr{M}_\beta} + <\nu,\bar{\nu}>_{\mathscr{M}_\lambda}, \end{aligned} \quad (65)$$

where ξ satisfies

$$\frac{|\mu_2|}{\sqrt{1-d}} \le \xi \le \left(2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}}\right), \quad (66)$$

thanks to hypothesis (26).

Let us set

$$\kappa(t) = \frac{\left(\tau'(t)^2 + 1\right)^{1/2}}{2\tau(t)}. \quad (67)$$

In this step, we prove the dissipativity of the operator $\bar{\mathscr{A}}(t) = \mathscr{A}(t) - \tau(t)I$.

For a fixed t and $U=(w,\varphi,u,\psi,\theta,\eta,P,\nu,z)^T \in \mathscr{D}(\mathscr{A}(t))$, we have

$$\begin{aligned} <\mathscr{A}(t)U,U>_t &= -\mu_1 \int_0^L \varphi^2 dx - \mu_2 \int_0^L \varphi z(x,1,t) dx \\ &+ <T\eta,\eta>_{\mathscr{M}_\beta} + <T\nu,\eta>_{\mathscr{M}_\lambda} \\ &- \xi \int_0^L \int_0^1 \left(1-\tau'(t)\rho\right) z(x,\rho,t) z_\rho(x,\rho,t) d\rho dx. \end{aligned} \quad (68)$$

Observe that

$$\begin{aligned} &\int_0^L \int_0^1 \left(1-\tau'(t)\rho\right) z(x,\rho,t) z_\rho(x,\rho,t) d\rho dx \\ &= \frac{1}{2} \int_0^L \int_0^1 \left(1-\tau'(t)\rho\right) \frac{d}{d\rho} z^2 d\rho dx \\ &= \frac{\tau'(t)}{2} \int_0^L \int_0^1 z^2(x,\rho,t) d\rho dx \\ &+ \frac{1}{2} \int_0^L \left\{ z^2(x,1,t)\left(1-\tau'(t)\right) - z^2(x,0,t) \right\} d\rho dx, \\ &<T\eta,\eta>_{\mathscr{M}_\beta} + <T\nu,\eta>_{\mathscr{M}_\lambda} \\ &= +\frac{1}{2} \int_0^\infty \beta'(\sigma) \int_0^L \eta_x^2(\sigma) dx d\sigma + \frac{1}{2} \int_0^\infty \lambda'(\sigma) \int_0^L \nu_x^2(\sigma) dx d\sigma \\ &\le -\frac{\alpha_1}{2} \|\eta(\sigma)\|_{\mathscr{M}_\beta}^2 - \frac{\alpha_2}{2} \|\nu(\sigma)\|_{\mathscr{M}_\lambda}^2, \end{aligned} \quad (69)$$

whereupon

$$\begin{aligned}<\mathscr{A}(t)U, U>_t = & -\mu_1 \int_0^L \varphi^2 dx - \mu_2 \int_0^L \varphi z(x,1,t) dx \\ & - \frac{\alpha_1}{2} \|\eta(\sigma)\|^2_{\mathscr{M}_\beta} - \frac{\alpha_2}{2} \|\nu(\sigma)\|^2_{\mathscr{M}_\lambda} \\ & - \frac{\xi\tau'(t)}{2} \int_0^L \int_0^1 z^2(x,\rho,t) d\rho dx \\ & - \frac{\xi}{2} \int_0^L z^2(x,1,t)\left(1-\tau'(t)\right) dx + \frac{\xi}{2} \int_0^L \varphi^2 dx. \end{aligned} \tag{70}$$

By using Young's inequality and (7), we get

$$\begin{aligned}<\mathscr{A}(t)U, U>_t \le & \left(-\mu_1 + \frac{|\mu_2|}{2\sqrt{1-d}} + \frac{\xi}{2}\right) \int_0^L \varphi^2 dx \\ & + \left(\frac{|\mu_2|\sqrt{1-d}}{2} - \xi\frac{(1-d)}{2}\right) \int_0^L z^2(x,1,t) dx \\ & - \frac{\alpha_1}{2} \|\eta(\sigma)\|^2_{\mathscr{M}_\beta} - \frac{\alpha_2}{2} \|\nu(\sigma)\|^2_{\mathscr{M}_\lambda} + \kappa(t) < U, U>_t, \end{aligned} \tag{71}$$

under condition (66) which allows to write

$$\begin{aligned} -\mu_1 + \frac{|\mu_2|}{2\sqrt{1-d}} + \frac{\xi}{2} \le 0, \\ \frac{|\mu_2|\sqrt{1-d}}{2} - \xi\frac{(1-d)}{2} \le 0. \end{aligned} \tag{72}$$

Consequently, the operator $\bar{\mathscr{A}}(t) = \mathscr{A}(t) - \kappa(t)I$ is dissipative.

Now, we prove the subjectivity of the operator $I - \mathscr{A}(t)$ for fixed $t > 0$.

Let $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathscr{H}$; we seek $U = (w, \varphi, u, \psi, \theta, \eta, P, \nu, z)^T \in \mathscr{D}(\mathscr{A}(t))$ solution of the following system:

$$\begin{cases} w - \varphi = f_1, \\ \varphi + d_2 w_{xxxx} + \mu_1\varphi + \mu_2 z(.,1,t) = f_2, \\ u - \psi = f_3, \\ \psi - d_1 u_{xx} - \delta_1\theta_x - \delta_2 P_x = f_4, \\ \alpha_1\theta + (d\delta_2 - r\delta_1)\psi_x - r\mathbb{L}_\beta\eta + d\mathbb{L}_\lambda\nu = \alpha_1 f_5, \\ \eta - \theta - T\eta = f_6, \\ \alpha_2 P + (d\delta_1 - c\delta_2)\psi_x + d\mathbb{L}_\beta\eta + -c\mathbb{L}_\lambda\nu = \alpha_2 f_7, \\ \nu - P - T\nu = f_8, \\ z - \dfrac{\left(\tau'(t)\rho - 1\right)}{\tau(t)} z_\rho = f_9. \end{cases} \tag{73}$$

Suppose that we have found w and u. Then,

$$\begin{cases} w - \varphi = f_1, \\ u - \psi = f_3. \end{cases} \tag{74}$$

Furthermore, by (73), we can find z as

$$z(x,0) = \varphi(x), \quad x \in (0, L). \tag{75}$$

Following the same approach as in [1], we obtain, by using the last equation in (73),

$$\begin{cases} z(x,\rho) = \varphi(x)e^{-\rho\tau(t)} + \tau(t)e^{-\rho\tau(t)} \displaystyle\int_0^1 f_9(x,y)e^{y\tau(t)} dy, & \text{if } \tau'(t) = 0, \\ z(x,\rho) = \varphi(x)e^{\eta_\rho(t)} + e^{\eta_\rho(t)} \displaystyle\int_0^1 \frac{\tau(t)}{1-\tau'(t)y} f_9(x,y)e^{-\eta_y(t)} dy, & \text{if } \tau'(t) \ne 0, \end{cases} \tag{76}$$

where $\eta_y(t) = (\tau(t)/\tau'(t)) \ln(1 - \tau'(t)\rho)$. Whereupon, from (74), we obtain

$$\begin{cases} z(x,\rho) = \varphi(x)e^{-\rho\tau(t)} - f_1 e^{-\rho\tau(t)} + \tau(t)e^{-\rho\tau(t)} \displaystyle\int_0^1 f_9(x,y)e^{y\tau(t)} dy, & \text{if } \tau'(t) = 0, \\ z(x,\rho) = \varphi(x)e^{\eta_\rho(t)} - f_1 e^{\eta_\rho(t)} + e^{\eta_\rho(t)} \displaystyle\int_0^1 \frac{\tau(t)}{1-\tau'(t)y} f_9(x,y)e^{-\eta_y(t)} dy, & \text{if } \tau'(t) \ne 0. \end{cases} \tag{77}$$

Integrating $(73)_6$ and $(73)_8$ with $\eta(0) = \nu(0) = 0$, we have

$$\begin{cases} \eta(\sigma) = (1 - e^{-\sigma})\theta + \int_0^\sigma e^{s-\sigma} f_6(s)ds, \\ \nu(\sigma) = (1 - e^{-\sigma})P + \int_0^\sigma e^{s-\sigma} f_8(s)ds. \end{cases} \tag{78}$$

Substituting $(73)_{1,3,6,8,9}$ into the others, we obtain the following system. Now, we have to find w, u, θ, and P as solutions of the equations:

$$\begin{cases} w + d_2 w_{xxxx} + \mu_1\varphi + \mu_2 z(.,1,t) = f_2 + f_1 + \beta_1 f_1, \\ u - d_1 u_{xx} - \delta_1\theta_x - \delta_2 P_x = f_4 + f_3, \\ \alpha_1\theta - rC_\beta\theta_{xx} + dC_\lambda P_{xx} + (d\delta_2 - r\delta_1)u_x = h_3, \\ \alpha_2 P + dC_\beta\theta_{xx} - cC_\lambda P_{xx} + (d\delta_1 - c\delta_1)u_x = h_4. \end{cases} \tag{79}$$

Solving (79), we get

$$\begin{cases} \mu_3 w + d_2 w_{xxxx} = h_1, \\ u - d_1 u_{xx} - \delta_1\theta_x - \delta_2 P_x = h_2, \\ \alpha_1\theta - rC_\beta\theta_{xx} + dC_\lambda P_{xx} + (d\delta_2 - r\delta_1)u_x = h_3, \\ \alpha_2 P + dC_\beta\theta_{xx} - cC_\lambda P_{xx} + (d\delta_1 - c\delta_1)u_x = h_4, \end{cases} \tag{80}$$

where

$$\begin{cases} \mu_3 = 1 + \mu_1 + e^{-\tau(t)}, \\ h_1 = f_2 + (1 + \mu_1) f_2 - \mu_2 z_0, \\ h_2 = f_4 + f_3, \\ h_3 = \alpha_1 f_5 + (d\delta_2 - r\delta_1) f_{3x} + r\int_0^\infty \beta(\sigma)\int_0^\sigma e^{s-\sigma} f_{6xx}(s)dsd\sigma - d\int_0^\infty \lambda(\sigma)\int_0^\sigma e^{s-\sigma} f_{8xx}(s)dsd\sigma, \\ h_4 = \alpha_2 f_7 + (d\delta_1 - c\delta_2) f_{5x} - d\int_0^\infty \beta(\sigma)\int_0^\sigma e^{s-\sigma} f_{6xx}(s)dsd\sigma + c\int_0^\infty \lambda(\sigma)\int_0^\sigma e^{s-\sigma} f_{8xx}(s)dsd\sigma. \end{cases} \tag{81}$$

From (77), we have

$$z(x,1) = \begin{cases} w(x)e^{-\tau(t)} + z_0(x), & \text{if } \tau'(t) = 0, \\ w(x)e^{\eta_\rho(t)} + z_0(x), & \text{if } \tau'(t) \neq 0, \end{cases} \tag{82}$$

where $x \in (0, L)$ and

$$z_0(x) = \begin{cases} -f_1 e^{-\rho\tau(t)} + \tau(t)e^{-\rho\tau(t)}\int_0^1 f_9(x,y)e^{y\tau(t)}dy, & \text{if } \tau'(t) = 0, \\ -f_1 e^{\eta_\rho(t)} + e^{\eta_\rho(t)}\int_0^1 \dfrac{\tau(t)}{1 - \tau'(t)y} f_9(x,y)e^{-\eta_y(t)}dy, & \text{if } \tau'(t) \neq 0. \end{cases} \tag{83}$$

It is clear from the above formula that z_0 depends only on f_1, f_9. Consequently, problem (80) is equivalent to

$$\zeta\left((w,u,\theta,P), \left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right) = \Gamma\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right), \tag{84}$$

where the bilinear form $\zeta : [H_0^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L^2(0,L)]^2 \to \mathbb{R}$ and the linear form $\Gamma : [H_0^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L^2(0,L)] \to \mathbb{R}$ are defined by

$$\begin{aligned} \zeta&\left((w,u,\theta,P), \left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right) \\ &= \int_0^L (\mu_3 w\widehat{w} + d_2 w_{xx}\widehat{w}_{xx} + u\widehat{u} + d_1 u_x\widehat{u}_x)dx + \alpha_1\int_0^L \theta\widehat{\theta}dx \\ &\quad + \alpha_2\int_0^L P\widehat{P}dx + rC_\beta\int_0^L \theta_x\widehat{\theta}_x dx + cC_\lambda\int_0^L P_x\widehat{P}_x dx \\ &\quad - dC_\beta\int_0^L \theta_x\widehat{P}_x dx - dC_\lambda\int_0^L P_x\widehat{\theta}_x dx \\ &\quad + (d\delta_2 - r\delta_1)\int_0^L u_x\widehat{\theta}dx + (d\delta_1 - c\delta_2)\int_0^L u_x\widehat{P}dx \\ &\quad + \int_0^L (\delta_1\theta + \delta_2 P)\widehat{u}_x dx, \end{aligned}$$

$$\Gamma\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right) = \int_0^L h_1\widehat{w}dx + \int_0^L h_2\widehat{u}dx + \int_0^L h_3\widehat{\theta}dx + \int_0^L h_4\widehat{P}dx. \tag{85}$$

Now, for $\mathscr{H}_1 = H_0^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L^2(0,L)$, equipped with the norm

$$\|(w,u,\theta,P)\|_{\mathscr{H}_1}^2 = \|w\|_2^2 + \|w_{xx}\|_2^2 + \|u\|_2^2 + \|u_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2 + \|P_x\|_2^2 + \|P\|_2^2, \tag{86}$$

then, we have

$$\begin{aligned} B((w,u,\theta,P),(w,u,\theta,P)) &= \mu_3 \int_0^L w^2 dx + d_2 \int_0^L w_{xx}^2 dx + \int_0^L u^2 dx + d_1 \int_0^L u_x^2 dx \\ &+ \alpha_1 \int_0^L \theta^2 dx + \alpha_2 \int_0^L P^2 dx + rC_\beta \int_0^L \theta_x^2 dx + cC_\lambda \int_0^L P_x^2 dx \\ &- \left(dC_\beta + dC_\lambda\right) \int_0^L P_x \theta_x dx + (d\delta_2 - r\delta_1) \int_0^L u_x \theta dx \\ &+ (d\delta_1 - c\delta_2) \int_0^L u_x P dx + \int_0^L (\delta_1 \theta + \delta_2 P) u_x dx. \end{aligned} \tag{87}$$

Then, for some $M_0 > 0$,

$$B((w,u,\theta,P),(w,u,\theta,P)) \geq M_0 \|(w,u,\theta,P)\|_{\mathscr{H}_1}^2. \tag{88}$$

Thus, B is coercive.

By Cauchy-Schwarz's and Poincaré's inequalities, we obtain

$$B\left((w,u,\theta,P),\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right) \leq M_1 \|(w,u,\theta,P)\|_{\mathscr{H}_1}^2 \left\|\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right\|_{\mathscr{H}_1}^2. \tag{89}$$

Similarly, we get

$$\Gamma\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right) \leq M_2 \left\|\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right\|_{\mathscr{H}_1}^2. \tag{90}$$

Consequently, applying the Lax-Milgram theorem, problem (84) admits a unique solution $(w,u,\theta,P) \in \mathscr{H}_1$, for all $(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}) \in \mathscr{H}_1$. Applying the classical elliptic regularity, it follows from (80) that $(w,u,\theta,P) \in \mathscr{H}_1$.

Therefore, the operator $I - \mathscr{A}(t)$ is surjective for any fixed $t > 0$. Since $\kappa(t) > 0$ and

$$I - \bar{\mathscr{A}}(t) = (1 + \kappa(t)) I - \mathscr{A}(t), \tag{91}$$

we deduce that the operator $I - \bar{\mathscr{A}}(t)$ is also surjective for any $t > 0$.

To complete the proof of (3), it suffices to show that

$$\frac{\|U\|_t}{\|U\|_s} \leq e^{(c/2\tau_0)|t-s|}, \quad \forall t, s \in [0,T], \tag{92}$$

where $U = (w,\varphi,u,\psi,\theta,\eta,P,\nu,z)^T$ and $\|.\|_t$ is the norm associated with the inner product (56).

For $t, s \in [0,T]$, we have from (56) that

$$\begin{aligned} \|U\|_t^2 - \|U\|_s^2 e^{(c/\tau_0)|t-s|} &= \left(1 - e^{(c/\tau_0)|t-s|}\right) \int_0^L \left\{\varphi^2 + d_2 w_{xx}^2 + d_1 u_x^2 + \psi^2\right\} dx \\ &+ \left(1 - e^{(c/\tau_0)|t-s|}\right) < \Lambda(\theta,P)^T, (\theta,P)^T > \\ &+ \left(1 - e^{(c/\tau_0)|t-s|}\right) \left\{\|\eta\|_{\mathscr{M}_\beta}^2 + \|\nu\|_{\mathscr{M}_\lambda}^2\right\} \\ &+ \xi\left(\tau(t) - \tau(s) e^{(c/\tau_0)|t-s|}\right) \int_0^L \int_0^1 z^2(x,\rho,t) d\rho dx. \end{aligned} \tag{93}$$

It is clear that $(1 - e^{(c/\tau_0)|t-s|}) \leq 0$. Now, we will prove that $(\tau(t) - \tau(s) e^{(c/\tau_0)|t-s|}) \leq 0$ for $c > 0$. To do this, we have

$$\tau(t) = \tau(s) + \tau'(a)(t-s), \tag{94}$$

where $a \in (s,t)$, which implies

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\tau(a)|}{\tau(s)} |t-s|. \tag{95}$$

By using (11), we deduce that

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0} |t-s| \leq e^{(c/\tau_0)|t-s|}, \tag{96}$$

which proves (92); therefore, this completes the proof of (3).

(4) It is clear that

$$\frac{d}{dt} \mathscr{A}(t) U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{\left(\tau''(t)\tau'(t)\rho - \tau'(t)\left(\tau'(t)\rho - 1\right)\right)}{\tau^2(t)} z_\rho \end{pmatrix}. \tag{97}$$

Then, by (11) and (25), (4) holds exactly as in [1]. Consequently, from the above analysis, we deduce that the problem

$$\begin{cases} \bar{U}_t = \bar{\mathscr{A}}(t) \bar{U}_t, \\ \bar{U}_t(0) = U_0, \end{cases} \tag{98}$$

has a solution $\bar{U} \in C([0,\infty),\mathscr{H})$, and if $U_0 \in \mathscr{D}(\mathscr{A}(0))$, then

$$\bar{U} \in C([0,\infty),\mathscr{D}(\mathscr{A}(0))) \cap C^1([0,\infty),\mathscr{H}). \tag{99}$$

Now, let

$$U(t) = e^{\vartheta(t)}\bar{U}(t), \tag{100}$$

with $\vartheta(t) = \int_0^t \kappa(s)ds$; then, by using (98), we have

$$\begin{aligned} U_t(t) &= \kappa(t)e^{\vartheta(t)}\bar{U}(t) + e^{\vartheta(t)}\bar{U}_t(t) \\ &= \kappa(t)e^{\vartheta(t)}\bar{U}(t) + e^{\vartheta(t)}\bar{\mathscr{A}}(t)\bar{U}(t) \\ &= e^{\vartheta(t)}\left(\kappa(t)\bar{U}(t) + \bar{\mathscr{A}}(t)\bar{U}(t)\right) \\ &= e^{\vartheta(t)}\left(\mathscr{A}(t)\bar{U}(t)\right) = \mathscr{A}(t)U(t)). \end{aligned} \tag{101}$$

Consequently, $U(t)$ is the unique solution of (46).

It remains to prove that the operator $\mathscr{F}$ defined in (48) is locally Lipschitz in $\mathscr{H}$.

Let $U_1 = (w_1,\varphi_1,u_1,\psi_1,\theta_1,\eta_1,P_1,\nu_1,z_1)^T \in \mathscr{H}$ and $U_2 = (w_2,\varphi_2,u_2,\psi_2,\theta_2,\eta_2,P_2,\nu_2,z_2)^T \in \mathscr{H}$. Then, we have

$$\|\mathscr{F}(U_1) - \mathscr{F}(U_2)\| = d_1\left(|R|^2 + |K|^2\right), \tag{102}$$

where

$$\begin{aligned} R &= \left[\left(u_{1x} + \frac{1}{2}w_{1x}^2\right)w_{1x} - \left(u_{2x} + \frac{1}{2}w_{2x}^2\right)w_{2x}\right], \\ K &= \frac{1}{2}\left(w_{1x}^2 - w_{2x}^2\right). \end{aligned} \tag{103}$$

Adding and subtracting the term $(u_{1x} + (1/2)w_{1x}^2)w_{2x}$ inside the norm $|R|$, we find

$$\begin{aligned} |R| \le{}& \|w_{1x} - w_{2x}\|_{L^\infty(0,L)}\left|u_{1x} + \frac{1}{2}w_{1x}^2\right| + \|w_{2x}\|_{L^\infty}|u_{1x} - u_{2x}| \\ &+ \frac{1}{2}\|w_{2x}\|_{L^\infty}|w_{1x} + w_{2x}|\|w_{1x} - w_{2x}\|_{L^\infty(0,L)}. \end{aligned} \tag{104}$$

Using the embedding of $H^1(0,L)$ into $L^\infty(0,L)$, from (104), one has

$$|R| \le k_1(\|U_1\|_{\mathscr{H}},\|U_2\|_{\mathscr{H}})\|U_1 - U_2\|. \tag{105}$$

Using once again the embedding of $H^1(0,L)$ into $L^\infty(0,L)$, one also sees that

$$|K| \le k_2(\|U_1\|_{\mathscr{H}},\|U_2\|_{\mathscr{H}})\|U_1 - U_2\|. \tag{106}$$

Combining (102), (105), and (106), consequently, $\mathscr{F}(U)$ is locally Lipschitz continuous in $\mathscr{H}$. This ends the proof of Theorem 1.

3. General Decay

In this section, we state and prove the stability of system (21)–(23) using the multiplier technique under the assumptions (26)–(31).

We define the energy functional E by

$$\begin{aligned} E(t) ={}& \frac{1}{2}\int_0^L\left\{w_t^2 + u_t^2 + d_2w_{xx}^2 + d_1\left(u_x + \frac{1}{2}w_x^2\right)^2 + c\theta^2 + rP^2\right\}dx \\ &+ d < \theta, P > + \frac{1}{2}\|\eta\|_{\mathscr{M}_\beta}^2 + \frac{1}{2}\|\nu\|_{\mathscr{M}_\lambda}^2 \\ &+ \frac{\xi}{2}\int_0^L\int_0^1 \tau(t)z^2(x,\rho,t)d\rho dx, \end{aligned} \tag{107}$$

where

$$\frac{|\mu_2|}{\sqrt{1-d}} \le \xi \le \left(2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}}\right). \tag{108}$$

The following lemma shows that the energy is decreasing.

Lemma 3. *Assume that (26)–(31) hold and the hypotheses (7), (11), and (25) are satisfied. Then, for $\forall C \ge 0$,*

$$\begin{aligned} E'(t) \le{}& -C\left(\int_0^L w_t^2dx + \int_0^L z^2(x,1,t)dx\right) - \frac{\alpha_1}{4}\|\eta\|_{\mathscr{M}_\beta}^2 \\ &+ \frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2d\sigma - \frac{\alpha_2}{4}\|\nu\|_{\mathscr{M}_\lambda}^2 \\ &+ \frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2d\sigma \le 0. \end{aligned} \tag{109}$$

Proof. Multiplying the equations of (21) by w_t, u_t, θ, η, P, ν, and ξz, respectively, then by integration by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L\left\{w_t^2 + u_t^2 + d_2w_{xx}^2 + d_1\left(u_x + \frac{1}{2}w_x^2\right)^2 + c\theta^2 + rP^2\right\}dx \\ &\quad + \frac{d}{dt}\left\{d < \theta, P > + \frac{1}{2}\|\eta\|_{\mathscr{M}_\beta}^2 + \frac{1}{2}\|\nu\|_{\mathscr{M}_\lambda}^2\right\} \\ &\quad + \frac{\xi}{2}\frac{d}{dt}\int_0^L\int_0^1 \tau(t)z^2(x,\rho,t)d\rho dx \\ &= -\mu_1\int_0^1 w_t^2dx - \mu_2\int_0^L w_tz(x,1,t)dx \\ &\quad + \frac{1}{2}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2d\sigma + \frac{1}{2}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2d\sigma \\ &\quad + \frac{\xi}{2}\int_0^L\int_0^1 \tau'(t)z^2(x,\rho,t)d\rho dx \\ &\quad - \xi\int_0^L\int_0^1\left(1 - \tau'(t)\rho\right)z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx \\ &\le -\mu_1\int_0^1 w_t^2dx - \mu_2\int_0^L w_tz(x,1,t)dx - \frac{\alpha_1}{4}\|\eta\|_{\mathscr{M}_\beta}^2 - \frac{\alpha_2}{4}\|\nu\|_{\mathscr{M}_\lambda}^2 \end{aligned}$$

$$+\frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma+\frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma$$
$$-\frac{\xi}{2}\int_0^L\int_0^1 \frac{d}{d\rho}\left(\left(1-\tau'(t)\rho\right)z^2(x,\rho,t)\right)d\rho dx$$
$$=-\mu_1\int_0^1 \omega_t^2 dx-\mu_2\int_0^L w_t z(x,1,t)dx-\frac{\alpha_1}{4}\|\eta\|^2_{\mathcal{M}_\beta}-\frac{\alpha_2}{4}\|\nu\|^2_{\mathcal{M}_\lambda}$$
$$+\frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma+\frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma$$
$$+\frac{\xi}{2}\int_0^L\left(z^2(x,0,t)-z^2(x,1,t)\right)dx$$
$$+\frac{\xi\tau'(t)}{2}\int_0^L z^2(x,1,t))dx. \tag{110}$$

From (110), we find

$$\begin{aligned}E'(t)\le&-\left(\mu_1-\frac{\xi}{2}\right)\int_0^L w_t^2 dx+\left(\frac{\xi\tau'(t)}{2}-\frac{\xi}{2}\right)\int_0^L z^2(x,1,t)dx\\&-\mu_2\int_0^L w_t z(x,1,t)dx-\frac{\alpha_1}{4}\|\eta\|^2_{\mathcal{M}_\beta}-\frac{\alpha_2}{4}\|\nu\|^2_{\mathcal{M}_\lambda}\\&+\frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma+\frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma.\end{aligned}\tag{111}$$

Using Young's inequality, we have

$$\begin{aligned}-\mu_2\int_0^L w_t z(x,1,t)dx\le&\frac{|\mu_2|}{2\sqrt{1-d}}\int_0^L w_t^2 dx\\&+\frac{|\mu_2|\sqrt{1-d}}{2}\int_0^L z^2(x,1,t)dx.\end{aligned}\tag{112}$$

Inserting (112) into (111), we get

$$\begin{aligned}E'(t)\le&-\left(\mu_1-\frac{\xi}{2}-\frac{|\mu_2|}{2\sqrt{1-d}}\right)\int_0^L w_t^2 dx\\&+\left(\frac{\xi}{2}\left(\tau'(t)-1\right)+\frac{|\mu_2|\sqrt{1-d}}{2}\right)\int_0^L z^2(x,1,t)dx\\&-\frac{\alpha_1}{4}\|\eta\|^2_{\mathcal{M}_\beta}+\frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma-\frac{\alpha_2}{4}\|\nu\|^2_{\mathcal{M}_\lambda}\\&+\frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma.\end{aligned}\tag{113}$$

Then, by using (7), (28)–(31), and (108), we obtain (109).

In the following, we state and prove our stability result; we introduce and prove several lemmas.

Lemma 4. *The functional*

$$F_1(t):=\int_0^L\left(u_t u+\frac{1}{2}w_t w+\frac{\beta_1}{4}w^2\right)dx,\tag{114}$$

satisfies, for any $\varepsilon_1>0$,

$$\begin{aligned}F_1'(t)\le&-d_1\int_0^L\left(u_x+\frac{1}{2}w_x^2\right)^2 dx-\frac{d_2}{4}\int_0^L w_{xx}^2 dx+\int_0^L u_t^2 dx\\&+\frac{1}{2}\int_0^L w_t^2 dx+2\varepsilon_1\int_0^L u_x^2 dx+\frac{\delta_1^2}{4\varepsilon_1}\int_0^L\theta^2+\frac{\delta_2^2}{4\varepsilon_1}\int_0^L P^2\\&+c\int_0^L z^2(x,1,t)dx.\end{aligned}\tag{115}$$

Proof. By differentiating F_1, then by integration by parts, we obtain

$$\begin{aligned}F_1'(t)=&\int_0^L u_t^2 dx+\frac{1}{2}\int_0^L w_t^2 dx-\frac{1}{2}d_1\int_0^L\left(u_x+\frac{1}{2}w_x^2\right)w_x^2 dx\\&-d_1\int_0^L u_x\left(u_x+\frac{1}{2}w_x^2\right)dx-\frac{\mu_2}{2}\int_0^L wz(x,1,t)dx\\&-\frac{d_2}{2}\int_0^L w_{xx}^2 dx+\delta_1\int_0^L\theta u_x dx+\delta_2\int_0^L Pu_x dx.\end{aligned}\tag{116}$$

In what follows, using Young's and Poincaré's inequalities, we obtain (115).

Then, we have the following lemma.

Lemma 5. *The functional*

$$F_2(t):=\int_0^L u_t\Phi dx,\tag{117}$$

where $-\delta_1\Phi_x=c\theta+dP$, with $\Phi(0)=\Phi(L)=0$, satisfies

$$\begin{aligned}F_2'(t)\le&-\int_0^L u_t^2 dx+\varepsilon_2\int_0^L\left(u_x+\frac{1}{2}w_x^2\right)^2 dx+c\|\eta\|^2_{\mathcal{M}\mu}\\&+c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^L\theta^2 dx+c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^L P^2 dx.\end{aligned}\tag{118}$$

Proof. For direct computations, we have

$$F_2'(t)=\underbrace{\int_0^L u_{tt}\Phi dx}_{f}$$
$$1(t)+\underbrace{\int_0^L u_t\Phi_t dx}_{f_2(t)}.\tag{119}$$

Using Young's inequality and integrating by parts, we obtain

$$f_1(t) \le \varepsilon_2 \int_0^L \left(u_x + \frac{1}{2}w_x^2\right)^2 dx + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^L \theta^2 dx + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^L P^2 dx, \tag{120}$$

$$\begin{aligned} f_2(t) &= -\frac{1}{\delta_1}\int_0^L u_t \partial_x^{-1}\left(\int_0^\infty \beta(\sigma)\eta_{xx}(\sigma)d\sigma + \delta_1 u_{tx}\right)dx \\ &= -\frac{1}{\delta_1}\int_0^L u_t\left(\int_0^\infty \beta(\sigma)\eta_x(\sigma)d\sigma + \delta_1 u_t\right)dx \\ &\le -\int_0^L u_t^2 dx + c\|\eta\|_{\mathcal{M}_\beta}^2. \end{aligned} \tag{121}$$

From (120) and (121), we obtain (118).

Lemma 6. *Assuming that assumptions (31) and (32) hold, the functional*

$$F_3(t) := \underbrace{-\int_0^\infty \beta(\sigma)\int_0^L (c\theta + dP)\eta dx d\sigma}_{\mathcal{G}_1} \underbrace{-\int_0^\infty \lambda(\sigma)\int_0^L (d\theta + rP)v dx d\sigma}_{\mathcal{G}_2}, \tag{122}$$

satisfies

$$\begin{aligned} F_3'(t) \le{}& -\hat{c}\int_0^L \theta^2 dx - \hat{r}\int_0^L P^2 dx + \beta_0\|\eta\|_{\mathcal{M}_\beta}^2 + \lambda_0\|v\|_{\mathcal{M}_\lambda}^2 \\ &+ \frac{c}{\varepsilon_3}\int_0^L u_t^2 dx - C_{\beta_0}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2(\sigma)d\sigma \\ &- C_{\lambda_0}\int_0^\infty \lambda'(\sigma)\|v_x(\sigma)\|^2(\sigma)d\sigma, \end{aligned} \tag{123}$$

where

$$\begin{aligned} \hat{c} &= \frac{1}{2}\left(\beta_0 c - (\beta_0 + \lambda_0)\frac{d}{\zeta}\right), \\ \hat{r} &= \frac{1}{2}(\lambda_0 r - (\mu_0 + \lambda_0)d\zeta), \end{aligned} \tag{124}$$

and $\zeta > 0$ satisfies (29).

Proof. We take the derivative of $F_3 = \mathcal{G}_1 + \mathcal{G}_2$, which gives

$$\begin{aligned} \mathcal{G}_1'(t) ={}& -\int_0^\infty \beta(\sigma)\int_0^L (c\theta + dP)_t \eta dx d\sigma \\ &- \int_0^\infty \beta(\sigma)\int_0^L (c\theta + dP)\eta_t dx d\sigma \\ ={}& -\int_0^\infty \beta(\sigma)\int_0^L (c\theta_t + dP_t)\eta dx d\sigma + c\int_0^\infty \beta(\sigma)\int_0^L \theta\eta_\sigma dx d\sigma \\ &+ d\int_0^\infty \beta(\sigma)\int_0^L P\eta_\sigma dx d\sigma - c\beta_0\int_0^L \theta^2 dx - d\int_0^\infty \beta(\sigma)\int_0^L P\theta dx d\sigma. \end{aligned} \tag{125}$$

The first term on the right-hand side of (125) is

$$\begin{aligned} &-\int_0^\infty \beta(\sigma)\int_0^L (c\theta + dP)_t \eta dx d\sigma \\ &\quad = -\delta_1\int_0^\infty \beta(\sigma)\int_0^L u_{tx}\eta dx d\sigma \\ &\qquad - \int_0^L \left(\int_0^\infty \beta(\sigma)\eta_{xx} d\sigma\right)\left(\int_0^\infty \beta(\sigma)\eta d\sigma\right)dx, \end{aligned} \tag{126}$$

and can be controlled in the following way:

$$\left|-\delta_1\int_0^\infty \beta(\sigma)\int_0^L u_{tx}\eta dx d\sigma\right| \le C(\varepsilon_3)\|\eta\|_{\mathcal{M}_\beta}^2 + \frac{c}{\varepsilon_3}\int_0^L u_t^2 dx, \tag{127}$$

$$-\int_0^L \left(\int_0^\infty \beta(\sigma)\eta_{xx} d\sigma\right)\left(\int_0^\infty \beta(\sigma)\eta d\sigma\right)dx \le \beta_0\|\eta\|_{\mathcal{M}_\beta}^2. \tag{128}$$

Moreover, by integration by parts, we get

$$\begin{aligned} \left|c\int_0^\infty \beta(\sigma)\int_0^L \theta\eta_\sigma dx d\sigma\right| &= c\left|-\int_0^\infty \beta'(\sigma)\int_0^L \theta\eta dx d\sigma\right| \\ &\le \frac{c\mu_0}{4}\int_0^L \theta^2 dx - C_{\beta_0}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma, \end{aligned} \tag{129}$$

where $C_{\beta_0} > 0$. Similarly, we obtain

$$\begin{aligned} \left|d\int_0^\infty \beta(\sigma)\int_0^L P\eta_\sigma dx d\sigma\right| &= c\left|-\int_0^\infty \beta'(\sigma)\int_0^L P\eta dx d\sigma\right| \\ &\le \frac{r\lambda_0}{4}\int_0^L P^2 dx - C_{\beta_0}'\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma, \end{aligned} \tag{130}$$

where $C'_{\beta_0} > 0$. Using (29), we get

$$-d\int_0^\infty \beta(\sigma)\left(\int_0^L \theta P dx\right)d\sigma \le \beta_0\frac{d}{2\zeta}\int_0^L \theta^2 dx + \beta_0\frac{d\zeta}{2}\int_0^L P^2 dx. \tag{131}$$

Then, we obtain

$$\begin{aligned} \mathcal{G}_1'(t) \le{}& \frac{\beta_0}{2}\left(\frac{d}{\zeta} - \frac{3c}{2}\right)\int_0^L \theta^2 dx + \frac{1}{2}\left(\beta_0 d\zeta + \frac{r\lambda_0}{2}\right)\int_0^L P^2 dx \\ &+ \frac{c}{\varepsilon_3}\int_0^L u_t^2 dx - \mathcal{C}_{\beta_0}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma \\ &+ (\beta_0 + C(\varepsilon_3)\|\eta\|_{\mathcal{M}_\beta}^2, \end{aligned} \tag{132}$$

where $\mathscr{C}_{\beta_0} = C_{\beta_0} + C'_{\beta_0}$. Then, using the same arguments, we find

$$\begin{aligned}\mathscr{G}_2'(t) \le& \frac{1}{2}\left(\lambda_0 \frac{d}{\zeta} + \frac{\beta_0 c}{2}\right)\int_0^L \theta^2 dx + \frac{\lambda_0}{2}\left(d\zeta - \frac{3r}{2}\right)\int_0^L P^2 dx \\ &+ \frac{c}{\varepsilon_3}\int_0^L u_t^2 dx - \mathscr{C}_{\lambda_0}\int_0^\infty \lambda'(\sigma)\|v_x(\sigma)\|^2 d\sigma \\ &+ (\lambda_0 + C(\varepsilon_3)\|v\|^2_{\mathscr{M}_\lambda}.\end{aligned} \tag{133}$$

Adding (127) and (133), we obtain (123).

We choose ζ in such a way that

$$\begin{aligned}\hat{c} &= \frac{1}{2}\left(\beta_0 c - (\beta_0 + \lambda_0)\frac{d}{\zeta}\right) > 0, \\ \hat{r} &= \frac{1}{2}(\lambda_0 r - (\beta_0 + \lambda_0) d\zeta) > 0,\end{aligned} \tag{134}$$

which implies

$$\frac{d}{c} < \frac{\beta_0 + \lambda_0}{\beta_0}\frac{d}{c} < \zeta < \frac{\lambda_0}{\beta_0 + \lambda_0}\frac{r}{d} < \frac{r}{d}. \tag{135}$$

Then, ζ satisfies (29).

Now, let us introduce the following functional.

Lemma 7. *The functional*

$$F_4(t) := \xi\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho} z^2(x,\rho,t)\, d\rho\, dx, \tag{136}$$

satisfies

$$F_4'(t) \le -2F_4(t) - \eta_1\int_0^L z^2(x,1,t)dx + \xi\int_0^L w_t^2 dx, \tag{137}$$

where η_1 is a positive constant.

Proof. By differentiating F_4, with respect to t, we have

$$\begin{aligned}F_4'(t) =& \xi\tau'(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho} z^2(x,\rho,t)\, d\rho\, dx \\ &+ \xi\tau(t)\int_0^L\int_0^1\left\{-2\tau'(t)\rho e^{-2\tau(t)\rho} z^2 + e^{-2\tau(t)\rho} z_t z\right\} d\rho\, dx.\end{aligned} \tag{138}$$

By using the last equation of (21), we have

$$\begin{aligned}&\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho} z_t z\, d\rho\, dx \\ &= \int_0^L\int_0^1\left(\tau'(t)\rho - 1\right)e^{-2\tau(t)\rho} z_\rho z\, d\rho\, dx \\ &= \frac{1}{2}\int_0^L\int_0^1 \frac{d}{d\rho}\left\{\left(\tau'(t)\rho - 1\right)e^{-2\tau(t)\rho} z^2\right\} d\rho\, dx \\ &\quad + \tau(t)\int_0^L\int_0^1\left(\tau'(t)\rho - 1\right)e^{-2\tau(t)\rho} z^2\, d\rho\, dx \\ &\quad - \frac{\tau'(t)}{2}\int_0^L\int_0^1 e^{-2\tau(t)\rho} z^2 dx.\end{aligned} \tag{139}$$

Using (137)–(139), we get

$$\begin{aligned}F_4'(t) =& -2\xi\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho} z^2(x,\rho,t)\, d\rho\, dx + \xi\int_0^L z^2(x,0,t)dx \\ &- \xi\left(1 - \tau'(t)\right)e^{-2\tau(t)}\int_0^L z^2(x,1,t)dx.\end{aligned} \tag{140}$$

Then, by using (7), (25), and the fact that $z(x,0,t) = w_t(x,t)$ and setting $\eta_1 = \xi(1-d)e^{-2\bar{\tau}}$, we obtain (137).

We are now ready to prove the following result.

Theorem 8. *Assume (26)–(31) hold; there exist positive constants C_1 and C_2 such that the energy functional given by (107) satisfies*

$$E(t) \le C_2 e^{-C_1 t}, \quad \forall t \ge 0. \tag{141}$$

Proof. We define a Lyapunov functional

$$\mathscr{L}(t) := NE(t) + \sum_{i=1}^{i=3} N_i F_i(t) + F_4(t), \tag{142}$$

where N and N_i, $i = 1, 2, 3$, are positive constants to be selected later.

By differentiating (142) and using (109), (115), (118), (123), and (137), including the relation

$$\begin{aligned}\int_0^L u_x^2 dx &= \int_0^L\left(u_x^2 + \frac{1}{2}w_x^2 - \frac{1}{2}w_x^2\right)dx \\ &\le 2\int_0^L\left(u_x + \frac{1}{2}w_x^2\right)^2 dx - \frac{1}{2}\int_0^L w_x^4 dx \\ &\le 2\int_0^L\left(u_x + \frac{1}{2}w_x^2\right)^2 dx - \frac{L}{4}\int_0^L w_{xx}^2 dx,\end{aligned} \tag{143}$$

we get

$$\begin{aligned}
\mathscr{L}'(t) \leq & -[(d_1 - 2\varepsilon_1)N_1 - \varepsilon_2 N_2]\int_0^L \left(u_x + \frac{1}{2}w_x^2\right)^2 dx \\
& - \left[N_2 - N_1 - \frac{c}{\varepsilon_3}N_3\right]\int_0^L u_t^2 dx \\
& - \left[\left(\frac{d_2}{4} - \frac{L}{2}\varepsilon_1\right)N_1\right]\int_0^L w_{xx}^2 dx \\
& - \left[CN - \frac{1}{2}N_1 - \xi\right]\int_0^L \omega_t^2 dx \\
& - \left[\widehat{c}N_3 - \frac{\delta_1^2}{4\varepsilon_1}N_1 - c\left(1 + \frac{1}{\varepsilon_2}\right)N_2\right]\int_0^L \theta^2 dx \\
& - \left[\widehat{r}N_3 - \frac{\delta_2^2}{4\varepsilon_1}N_1 - c\left(1 + \frac{1}{\varepsilon_2}\right)N_2\right]\int_0^L P^2 dx \\
& - [CN - cN_1 + \eta_1]\int_0^L z^2(x,1,t)dx - 2F_4(t) \\
& - \left[\frac{\alpha_1}{4}N - cN_2 - \mu_0 N_3\right]\|\eta\|^2_{\mathscr{M}_\beta} \\
& - \left[\frac{\alpha_2}{4}N - \lambda_0 N_3\right]\|\nu\|^2_{\mathscr{M}_\lambda} \\
& + \left[\frac{1}{4}N - C_{\mu_0}N_3\right]\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma \\
& + \left[\frac{1}{4}N - C_{\lambda_0}N_3\right]\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma.
\end{aligned} \tag{144}$$

First, we choose ε_1 small enough such that

$$\begin{aligned}
& d_1 - 2\varepsilon_1 > 0, \\
& \frac{d_2}{4} - \frac{L}{2}\varepsilon_1 > 0.
\end{aligned} \tag{145}$$

By setting

$$\begin{aligned}
\varepsilon_2 &= \frac{(d_1 - 2\varepsilon_1)N_1}{2N_2}, \\
\varepsilon_3 &= \frac{2cN_3}{N_2},
\end{aligned} \tag{146}$$

we obtain

$$\begin{aligned}
\mathscr{L}'(t) \leq & -\left[\frac{1}{2}(d_1 - 2\varepsilon_1)N_1\right]\int_0^L \left(u_x + \frac{1}{2}w_x^2\right)^2 dx \\
& - \left[\frac{1}{2}N_2 - N_1\right]\int_0^L u_t^2 dx - \left[\left(\frac{d_2}{4} - \frac{L}{2}\varepsilon_1\right)N_1\right]\int_0^L w_{xx}^2 dx \\
& - \left[CN - \frac{1}{2}N_1 - \xi\right]\int_0^L \omega_t^2 dx \\
& - \left[\widehat{c}N_3 - \frac{\delta_1^2}{4\varepsilon_1}N_1 - c\left(1 + \frac{N_2}{N_1}\right)N_2\right]\int_0^L \theta^2 d \\
& - \left[\widehat{r}N_3 - \frac{\delta_2^2}{4\varepsilon_1}N_1 - c\left(1 + \frac{N_2}{N_1}\right)N_2\right]\int_0^L P^2 dx \\
& - [CN - cN_1 + \eta_1]\int_0^L z^2(x,1,t)dx - 2F_4(t) \\
& - \left[\frac{\alpha_1}{4}N - cN_2 - \mu_0 N_3\right]\|\eta\|^2_{\mathscr{M}_\beta} \\
& - \left[\frac{\alpha_2}{4}N - \lambda_0 N_3\right]\|\nu\|^2_{\mathscr{M}_\lambda} \\
& + \left[\frac{1}{4}N - C_{\mu_0}N_3\right]\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma \\
& + \left[\frac{1}{4}N - C_{\lambda_0}N_3\right]\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma.
\end{aligned} \tag{147}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We choose N_2 large enough such that

$$k_1 = \frac{1}{2}N_2 - N_1 > 0. \tag{148}$$

Then, we choose N_3 large enough such that

$$\begin{aligned}
k_2 &= \widehat{r}N_3 - \frac{\delta_2^2}{4\varepsilon_1}N_1 - c\left(1 + \frac{N_2}{N_1}\right)N_2 > 0, \\
k_3 &= \widehat{r}N_3 - \frac{\delta_2^2}{4\varepsilon_1}N_1 - c\left(1 + \frac{N_2}{N_1}\right)N_2 > 0.
\end{aligned} \tag{149}$$

Thus, we arrive at

$$\begin{aligned}
\mathscr{L}'(t) \leq & -k_0\int_0^L \left(u_x + \frac{1}{2}w_x^2\right)^2 dx - k_1\int_0^L u_t^2 dx - k_4\int_0^L w_{xx}^2 dx \\
& - (CN - c)\int_0^L \omega_t^2 dx - k_2\int_0^L \theta^2 dx - k_3\int_0^L P^2 dx \\
& - (CN - c)\int_0^L z^2(x,1,t)dx - 2F_4(t) \\
& - \left(\frac{\alpha_1}{4}N - c\right)\|\eta\|^2_{\mathscr{M}_\beta} - \left(\frac{\alpha_2}{4}N - c\right)\|\nu\|^2_{\mathscr{M}_\lambda} \\
& + \left(\frac{1}{4}N - c\right)\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma \\
& + \left(\frac{1}{4}N - c\right)\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma,
\end{aligned} \tag{150}$$

where $k_0 = (1/2)(d_1 - 2\varepsilon_1)N_1$ and $k_4 = ((d_2/4) - (L/2)\varepsilon_1)N_1$.

On the other hand, we let

$$\mathfrak{T}(t) = \sum_{i=1}^{i=3} N_i F_i(t) + F_4(t). \tag{151}$$

Exploiting Young's, Cauchy-Schwarz's, and Poincaré's inequalities, we get

$$|\mathfrak{T}(t)| \le c\int_0^L \left(\omega_t^2 + u_t^2 + \left(u_x + \frac{1}{2}\omega_x^2\right)^2 + \omega_{xx}^2 + \theta^2 + P^2\right)dx + c\|\eta\|^2_{\mathscr{M}_\beta} + c\|\nu\|^2_{\mathscr{M}_\lambda} + c\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx. \quad (152)$$

Then,

$$|\mathfrak{T}(t)| \le cE(t). \quad (153)$$

Consequently, we obtain

$$|\mathfrak{T}(t)| = |\mathscr{L}(t) - NE(t)| \le cE(t), \quad (154)$$

that is,

$$(N-c)E(t) \le \mathscr{L}(t) \le (N+c)E(t). \quad (155)$$

Now, we choose N large enough such that

$$\begin{aligned} N - c &> 0, \\ \frac{\alpha_1}{4}N - c &> 0, \\ \frac{\alpha_2}{4}N - c &> 0, \\ \frac{1}{4}N - c &> 0 > 0, \\ CN - c &> 0. \end{aligned} \quad (156)$$

Exploiting (107), estimates (150) and (155), respectively, give

$$\mathscr{L}'(t) \le -a_1E(t), \quad (157)$$

for some $a_1 > 0$, and

$$c_1E(t) \le \mathscr{L}(t) \le c_2E(t), \quad \forall t \ge 0, \quad (158)$$

for some $c_1, c_2 > 0$; we have

$$\mathscr{L}(t) \sim E(t). \quad (159)$$

A combination with (157) and (158) gives

$$\mathscr{L}'(t) \le -C_1\mathscr{L}(t), \quad \forall t \ge 0, \quad (160)$$

where $C_1 = a_1/c_2$.

Finally, by simple integration of (159) and (160), we obtain the result (141).

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

Acknowledgments

The fifth author extends their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through General Research Project under Grant No. G.R.P-10/42.

References

[1] A. S. Nicaise, C. Pignotti, and J. Valein, "Exponential stability of the wave equation with boundary time-varying delay," *Discrete Contin. Dyn. Syst. Ser. S*, vol. 2, pp. 559–581, 2009.

[2] A. Benabdallah and I. Lasiecka, "Exponential decay rates for a full von Kármán system of dynamic thermoelasticity," *Journal of Differential Equations*, vol. 160, pp. 51–93, 2000.

[3] A. Benabdallah and D. Teniou, "Exponential stability of a von Kármán model with thermal effects," *Electron. J. Differ. Equations*, vol. 7, p. 13, 1998.

[4] S. Boulaaras and N. Doudi, "Global existence and exponential stability of coupled Lamesystem with distributed delay and source term without memory term," *Boundary Value Problems*, vol. 2020, no. 1, 2020.

[5] S. M. Boulaaras, A. Choucha, A. Zara, M. Abdalla, and B. B. Cheri, "Global existence and decay estimates of energy of solutions for a new class of -Laplacian heat equations with logarithmic nonlinearity," *Journal of Function Spaces*, vol. 2021, Article ID 5558818, 11 pages, 2021.

[6] L. Bouzettouta and A. Djebabla, "Exponential stabilization of the full von Kármán beam by a thermal effect and a frictional damping and distributed delay," *Journal of Mathematical Physics*, vol. 60, article 041506, 2019.

[7] A. Choucha and D. Ouchenane, "Local existence and blow up of solutions to a logarithmic nonlinear wave equation with time-varying delay," Studia. UBB. Math, 2020.

[8] A. Choucha, S. M. Boulaaras, D. Ouchenane, B. B. Cherif, and M. Abdalla, "Exponential stability of swelling porous elastic with a viscoelastic damping and distributed delay term," *Journal of Function Spaces*, vol. 2021, Article ID 5581634, 8 pages, 2021.

[9] A. Choucha, D. Ouchenane, and S. Boulaaras, "Well posedness and stability result for a thermoelastic laminated Timoshenko beam with distributed delay term," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 17, pp. 9983–10004, 2020.

[10] I. Chueshov and I. Lasiecka, *Von Kármán Evolution Equations, Springer Monographs in Mathematics*, Springer, New York, 2010.

[11] C. M. Dafermos, "Asymptotic stability in viscoelasticity," *Archive for Rational Mechanics and Analysis*, vol. 37, no. 4, pp. 297–308, 1970.

[12] A. Djebabla and N. Tatar, "Exponential stabilization of the full von Kármán beam by a thermal effect and a frictional damping," *Georgian Mathematical Journal*, vol. 20, pp. 427–438, 2013.

[13] A. Favini, M. Ann Horn, I. Lasiecka, and D. Tataru, "Global existence, uniqueness and regularity of solutions to a von Kármán system with nonlinear boundary dissipation," *Differential and Integral Equations*, vol. 9, pp. 267–294, 1996.

[14] T. Fastovska, "Upper semicontinuous attractor for 2D Mindlin-Timoshenko thermoelastic model with memory," *Communications on Pure & Applied Analysis*, vol. 6, no. 1, pp. 83–101, 2007.

[15] A. E. Green and P. M. Naghdi, "On undamped heat waves in an elastic solid," *Journal of Thermal Stresses*, vol. 15, pp. 253–264, 1992.

[16] M. A. Horn and I. Lasiecka, "Global stabilization of a dynamic von Kármán plate with nonlinear boundary feedback," *Applied Mathematics and Optimization*, vol. 31, pp. 57–84, 1995.

[17] M. Ann Horn and I. Lasiecka, "Uniform decay of weak solutions to a von Kármán plate with nonlinear boundary dissipation," *Differential and Integral equations*, vol. 7, pp. 885–908, 1994.

[18] J. U. Kim and Y. Renardy, "Boundary control of the Timoshenko beam," *SIAM Journal on Control and Optimization*, vol. 25, no. 6, pp. 1417–1429, 1987.

[19] M. Kirane, B. Said-houari, and M. N. Anwar, "Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks," *Pure and Applied Analysis*, vol. 10, no. 2, pp. 667–686, 2011.

[20] J. E. Lagnese, "Modelling and stabilization of nonlinear plates," in *Estimation and Control of Distributed Parameter Systems. International Series of Numerical Mathematics / Internationale Schriftenreihe zur Numerischen Mathematik / Série Internationale d'Analyse Numérique*, W. Desch, F. Kappel, and K. Kunisch, Eds., vol. 100, pp. 247–264, Birkhäuser, Basel, 1991.

[21] I. Lasiecka, "Uniform stabilizability of a full von Kármán system with nonlinear boundary feedback," *SIAM Journal on Control and Optimization*, vol. 36, pp. 1376–1422, 1998.

[22] I. Lasiecka, "Uniform decay rates for full von Kármán system of dynamic thermoelasticity with free boundary conditions and partial boundary dissipation," *Communications in Partial Differential Equations*, vol. 24, pp. 1801–1847, 1999.

[23] G. Perla Menzala, A. F. Pazoto, and E. Zuazua, "Stabilization of Berger-Timoshenko's equation as limit of the uniform stabilization of the von Kármán system of beams and plates," *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 36, pp. 657–691, 2002.

[24] G. P. Menzala and E. Zuazua, "Explicit exponential decay rates for solutions of von Kármán's system of thermoelastic plates," *Comptes Rendus de l'Academie des Sciences-Serie I-Mathematique*, vol. 324, pp. 49–54, 1997.

[25] J. P. Puel and M. Tucsnak, "Boundary stabilization for the von Kármán equations," *SIAM Journal on Control and Optimization*, vol. 33, pp. 255–273, 1995.

[26] Z. Quanxin, "Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control," *IEEE Transactions on Automatic Control*, vol. 64, no. 9, pp. 3764–3771, 2019.

[27] Z. Quanxin and H. Tingwen, "Stability analysis for a class of stochastic delay nonlinear systems driven by G-Brownian motion," *Systems & Control Letters*, vol. 140, article 104699, 2020.

[28] M. A. Ragusa and A. Tachikawa, "Partial regularity of the minimizers of quadratic functionals with VMO coefficients," *Journal of the London Mathematical Society*, vol. 72, no. 3, pp. 609–620, 2005.

[29] M. Reissig and Y. G. Wang, "Cauchy problems for linear thermoelastic systems of type III in one space variable," *Mathematicsl Methods in the Applied Sciences*, vol. 28, no. 11, pp. 1359–1381, 2005.

[30] S. Timoshenko, "On the correction for shear of the differential equation for transverse vibrations of prismaticbars," *Philosophical Magazine*, vol. 41, pp. 744–746, 1921.

[31] A. Aouadi and A. Miranville, "Quasi-stability and global attractor in nonlinear thermoelastic diffusion plate with memory," *Evolution Equations & Control Theory*, vol. 4, pp. 241–263, 2015.

[32] A. Aouadi and A. Miranville, "Smooth attractor for a nonlinear thermoelastic diffusion thin plate based on Gurtin-Pipkinâ€™s model," *Asymptotic Analysis*, vol. 95, pp. 129–160, 2015.

[33] M. Aouadi and A. Castejon, "Properties of global and exponential attractors for nonlinear thermo-diffusion Timoshenko system," *Journal of Mathematical Physics*, vol. 60, article 081503, 2019.

[34] S. Boulaaras, A. Choucha, B. Cherif et al., "Blow up of solutions for a system of two singular nonlocal viscoelastic equations with damping, general source terms and a wide class of relaxation functions," *AIMS Mathematics*, vol. 6, no. 5, pp. 4664–4676, 2021.

[35] A. Choucha, S. Boulaaras, D. Ouchenane, S. Alkhalaf, I. Mekawy, and M. Abdalla, "On the system of coupled nondegenerate Kirchhoff equations with distributed delay: global existence and exponential decay," *Journal of Function Spaces*, vol. 2021, Article ID 5577277, 13 pages, 2021.

[36] A. Menaceur, S. Boulaaras, A. Makhlouf, K. Rajagobal, and M. Abdalla, "Limit cycles of a class of perturbed differential systems via the first-order averaging method," *Complexity*, vol. 2021, Article ID 5581423, 6 pages, 2021.

[37] M. Aouadi, M. Campo, M. I. M. Copetti, and J. R. Fernández, "Existence, stability and numerical results for a Timoshenko beam with thermodiffusion effects," *Zeitschrift für angewandte Mathematik und Physik*, vol. 70, article 117, 2019.

[38] Y. Qin and X. Pan, "Global existence, asymptotic behavior and uniform attractors for a non- autonomous Timoshenko system of thermoelasticity of type III with a time- varying delay," *Journal of Mathematical Analysis and Applications*, vol. 484, no. 1, article 123672, 2020.

[39] C. Giorgi, M. G. Naso, and V. Pata, "Exponential stability in linear heat conduction with memory: a semigroup approach," *Communications in Applied Analysis*, vol. 5, pp. 121–134, 2001.

[40] T. Kato, "Linear and quasilinear equations of evolution of hyperbolic type," C.I.M.E., II, 1976.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9981211, 10 pages
https://doi.org/10.1155/2021/9981211

Research Article

Fractional Crank-Nicolson-Galerkin Finite Element Methods for Nonlinear Time Fractional Parabolic Problems with Time Delay

Lili Li,[1] **Mianfu She**,[1] **and Yuanling Niu**[2]

[1]*School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China*
[2]*School of Mathematics and Statistics, Central South University, Changsha 410083, China*

Correspondence should be addressed to Yuanling Niu; yuanlingniu@csu.edu.cn

Received 11 March 2021; Accepted 18 March 2021; Published 31 March 2021

Academic Editor: Qifeng Zhang

A linearized numerical scheme is proposed to solve the nonlinear time-fractional parabolic problems with time delay. The scheme is based on the standard Galerkin finite element method in the spatial direction, the fractional Crank-Nicolson method, and extrapolation methods in the temporal direction. A novel discrete fractional Grönwall inequality is established. Thanks to the inequality, the error estimate of a fully discrete scheme is obtained. Several numerical examples are provided to verify the effectiveness of the fully discrete numerical method.

1. Introduction

In this paper, we consider the linearized fractional Crank-Nicolson-Galerkin finite element method for solving the nonlinear time-fractional parabolic problems with time delay

$$\begin{cases} {}^{R}D_t^{\alpha} u - \Delta u = f(t, u(x,t), u(x,t-\tau)), & \text{in } \Omega \times (0,T], \\ u(x,t) = \varphi(x,t), & \text{in } \Omega \times (-\tau, 0], \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \end{cases} \tag{1}$$

where Ω is a bounded convex and convex polygon in R^2 (or polyhedron in R^3) and τ is the delay term. ${}^{R}D_t^{\alpha} u$ denotes the Riemann-Liouville fractional derivative, defined by

$${}^{R}D_t^{\alpha} u(\cdot,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(\cdot,s) ds, \quad 0 < \alpha < 1. \tag{2}$$

The nonlinear fractional parabolic problems with time delay have attracted significant attention because of their wide range of applications in various fields, such as biology, physics, and engineering [1–9]. Recently, plenty of numerical methods were presented for solving the linear time-fractional diffusion equations. For instance, Chen et al. [10] used finite difference methods and the Kansa method to approximate time and space derivatives, respectively. Dehghan et al. [11] presented a fully discrete scheme based on the finite difference methods in time direction and the meshless Galerkin method in space direction and proved the scheme was unconditionally stable and convergent. Murio [12] and Zhuang and Liu [13] proposed a fully implicit finite difference numerical scheme and obtained unconditionally stability. Jin et al. [14] derived the time-fractional Crank-Nicolson scheme to approximate Riemann-Liouville fractional derivative. Li et al. [15] used a transformation to develop some new schemes for solving the time-fractional problems. The new schemes admit some advantages for both capturing the initial layer and solving the models with small parameter α. More studies can be found in [16–32].

Recently, it has been one of the hot spots in the investigations of different numerical methods for the nonlinear time-fractional problems. For the analysis of the L1-type methods, we refer readers to the paper [33–40]. For the analysis of the convolution quadrature methods or the fractional Crank-Nicolson scheme, we refer to the recent papers [41–46]. The key role in the convergence analysis of the schemes is the fractional Grönwall-type inequations. However, as pointed out in [47–49], the similar fractional Grönwall-type inequations can not be directly applied to study the convergence of numerical schemes for the nonlinear time-fractional problems with delay.

In this paper, we present a linearized numerical scheme for solving the nonlinear fractional parabolic problems with time delay. The time Riemann-Liouville fractional derivative is approximated by the fractional Crank-Nicolson-type time-stepping scheme, the spatial derivative is approximated by using the standard Galerkin finite element method, and the nonlinear term is approximated by the extrapolation method. To study the numerical behavior of the fully discrete scheme, we construct a novel discrete fractional type Grönwall inequality. With the inequality, we consider the convergence of the numerical methods for the nonlinear fractional parabolic problems with time delay.

The rest of this article is organized as follows. In Section 2, we present a linearized numerical scheme for the nonlinear time-fractional parabolic problems with delay and main convergence results. In Section 3, we present a detailed proof of the main results. In Section 4, numerical examples are given to confirm the theoretical results. Finally, the conclusions are presented in Section 5.

2. Fractional Crank-Nicolson-Galerkin FEMs

Denote $\mathcal{T}_h$ is a shape regular, quasiuniform triangulation of the Ω into d-simplexes. Let $h=\max_{K\varepsilon\mathcal{T}_h}\{\text{diam } K\}$. Let X_h be the finite-dimensional subspace of $H_0^1(\Omega)$ consisting of continuous piecewise function on $\mathcal{T}_h$. Let $\Delta t=\tau/m_\tau$ be the time step size, where m_τ is a positive integer. Denote $N=\lceil T/\Delta t\rceil$, $t_j=j\Delta t$, $j=-m_\tau,-m_\tau+1,\cdots,0,1,2,\cdots,N$.

The approximation to the Riemann-Liouville fractional derivative at point $t=t_{n-(\alpha/2)}$ is given by [14]

$$ {}^RD^\alpha_{t_{n-(\alpha/2)}}u(x,t)=\Delta t^{-\alpha}\sum_{i=0}^{n}\omega_{n-i}^{(\alpha)}u(x,t_i)+\mathcal{O}\left(\Delta t^2\right):={}^RD^\alpha_{t_{\Delta t}}u^n+\mathcal{O}\left(\Delta t^2\right), \tag{3}$$

where

$$\omega_i^{(\alpha)}=(-1)^i\frac{\Gamma(\alpha+1)}{\Gamma(i+1)\Gamma(\alpha-i+1)}. \tag{4}$$

For simplicity, denote $\|v\|=\left(\int_\Omega|v(x)|^2dx\right)^{1/2}$, $\eta^{n,\alpha}=(1-(\alpha/2))\eta^n+(\alpha/2)\eta^{n-1}$, $\eta\wedge^{n,\alpha}=(2-(\alpha/2))\eta^{n-1}-(1-(\alpha/2))\eta^{n-2}$, $t_n^\alpha=(n\Delta t)^\alpha$.

With the notation, the fully discrete scheme is to find $U_h^n\varepsilon X_h$ such that

$$\begin{aligned}&\left\langle {}^RD^\alpha_{\Delta t}U_h^n,v\right\rangle+\left\langle\nabla U_h^{n,\alpha},\nabla v\right\rangle\\&\quad=\left\langle f\left(t_{n-(\alpha/2)},\widehat{U}_h^{n,\alpha},U_h^{n-m_\tau,\alpha}\right),v\right\rangle,\quad\forall v\varepsilon X_h,n=1,2,\cdots,N,\end{aligned} \tag{5}$$

and the initial condition

$$U_h^n=R_h\varphi(x,t_n),\quad n=-m_\tau,-m_\tau+1,\cdots,0, \tag{6}$$

where $R_h:H_0^1(\Omega)\to X_h$ is Ritz projection operator which satisfies the following equality [50]

$$\langle\nabla R_hu,\nabla v\rangle=\langle\nabla u,\nabla v\rangle,\quad\forall u\varepsilon H_0^1(\Omega)\cap H^2(\Omega),v\varepsilon X_h. \tag{7}$$

We present the main convergence results here and leave their proof in the next section.

Theorem 1. *Suppose the system (1) has a unique solution u satisfying*

$$\begin{aligned}&\|u_0\|_{H^{r+1}}+\|u\|_{C\left([0,T];H^{r+1}\right)}+\|u_t\|_{C\left([0,T];H^{r+1}\right)}+\|u_{tt}\|_{C\left([0,T];H^2\right)}\\&\quad+\left\|{}^RD^\alpha_{\Delta t}u\right\|_{C\left([0,T];H^{r+1}\right)}\le K,\end{aligned} \tag{8}$$

and the source term $f(t,u(x,t),u(x,t-\tau))$ satisfies the Lipschitz condition

$$\begin{aligned}&|f(t,u(x,t),u(x,t-\tau))-f(t,v(x,t),v(x,t-\tau))|\\&\quad\le L_1|u(x,t)-v(x,t)|+L_2|u(x,t,\tau)-v(x,t,\tau)|,\end{aligned} \tag{9}$$

where K is a constant independent of n, h, and Δt and L_1 and L_2 are given positive constants. Then, there exists a positive constant Δt^* such that for $\Delta t\le\Delta t^*$, the following estimate holds that

$$\|u^n-U_h^n\|\le C_1^*\left(\Delta t^2+h^{r+1}\right),\quad n=1,2,\cdots,N, \tag{10}$$

where C_1^* is a positive constant independent of h and Δt.

Remark 2. The main contribution of the present study is that we obtain a discrete fractional Grönwall's inequality. Thanks to the inequality, the convergence of the fully discrete scheme for the nonlinear time-fractional parabolic problems with delay can be obtained.

Remark 3. At present, the convergence of the proposed scheme is proved without considering the weak singularity of the solutions. In fact, if the initial layer of the problem is taken into account, there are some corrected terms at the beginning. Then, the scheme can be of order two in the temporal direction for nonsmooth initial data and some incompatible source terms. However, we still have the difficulties to get the similar discrete fractional Grönwall's inequality. We hope to leave the challenging problems in the future.

3. Proof of the Main Results

In this section, we will present a detailed proof of the main result.

3.1. Preliminaries and Discrete Fractional Grönwall Inequality. Firstly, we review the definition of weights $\omega_i^{(\alpha)}$ and denote $g_n^{(\alpha)} = \sum_{i=0}^{n} \omega_i^{(\alpha)}$. Then, we can get

$$\begin{cases} \omega_0^{(\alpha)} = g_0^{(\alpha)}, \\ \omega_i^{(\alpha)} = g_i^{(\alpha)} - g_{i-1}^{(\alpha)}, \quad 1 \le i \le n. \end{cases} \tag{11}$$

Actually, it has been shown [51] that $\omega_i^{(\alpha)}$ and $g_n^{(\alpha)}$ process following properties:

(1) The weights $\omega_i^{(\alpha)}$ can be evaluated recursively, $\omega_i^{(\alpha)} = (1 - ((\alpha+1)/i))\,\omega_{i-1}^{(\alpha)}, i \ge 1, \omega_0^{(\alpha)} = 1$

(2) The sequence $\{\omega_i^{(\alpha)}\}_{i=0}^{\infty}$ are monotone increasing $-1 < \omega_i^{(\alpha)} < \omega_{i+1}^{(\alpha)} < 0, i \ge 0$

(3) The sequence $\{g_i^{(\alpha)}\}_{i=0}^{\infty}$ are monotone decreasing, $g_i^{(\alpha)} > g_{i+1}^{(\alpha)}$ for $i \ge 0$ and $g_0^{(\alpha)} = 1$

Noticing the definition of $g_i^{(\alpha)}$, ${}^{R}D_{\Delta t}^{\alpha} u^n$ can be rewritten as

$${}^{R}D_{\Delta t}^{\alpha} u^n = \Delta t^{-\alpha} \sum_{i=1}^{n} \left(g_i^{(\alpha)} - g_{i-1}^{(\alpha)} \right) u^{n-i} + \Delta t^{-\alpha} g_0^{(\alpha)} u^n. \tag{12}$$

In fact, rearranging this identity yields

$${}^{R}D_{\Delta t}^{\alpha} u^n = \Delta t^{-\alpha} \sum_{i=1}^{n} g_{n-i}^{(\alpha)} \delta_t u^i + \Delta t^{-\alpha} g_n^{(\alpha)} u^0, \tag{13}$$

where $\delta_t u^i = u^i - u^{i-1}$.

Lemma 4 (see [51]). *Consider the sequence $\{\phi_n\}$ given by*

$$\phi_0 = 1, \phi_n = \sum_{i=1}^{n} \left(g_{i-1}^{(\alpha)} - g_i^{(\alpha)} \right) \phi_{n-i}, \quad n \ge 1. \tag{14}$$

Then, $\{\phi_n\}$ satisfies the following properties:

(i) $0 < \phi_n < 1, \sum_{i=j}^{n} \phi_{n-i} g_{i-j}^{(\alpha)} = 1, 1 \le j \le n$

(ii) $1/(\Gamma(\alpha)) \sum_{i=1}^{n} \phi_{n-i} \le n^{\alpha}/\Gamma(1+\alpha)$

(iii) $1/(\Gamma(\alpha)\Gamma(1+(k-1)\alpha)) \sum_{i=1}^{n-1} \phi_{n-i} i^{(k-1)\alpha} \le n^{k\alpha}/\Gamma(1+\alpha), k = 1, 2 \cdots$

Lemma 5 (see [51]). *Consider the matrix*

$$W = 2\mu(\Delta t)^{\alpha} \begin{pmatrix} 0 & \phi_1 & \cdots & \phi_{n-2} & \phi_{n-1} \\ 0 & 0 & \cdots & \phi_{n-3} & \phi_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \phi_1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}. \tag{15}$$

Then, W satisfies the following properties:

(i) $W^l = 0, l \ge n$

(ii) $W^k \overleftarrow{e} \le (1/(\Gamma(1+k\alpha)))[(2\Gamma(\alpha)\mu t_n^{\alpha})^k, (2\Gamma(\alpha)\mu t_{n-1}^{\alpha})^k, \cdots, (2\Gamma(\alpha)\mu t_1^{\alpha})^k]', k = 0, 1, 2, \cdots$

(iii) $\sum_{k=0}^{l} W^k \overleftarrow{e} = \sum_{k=0}^{n-1} W^k \overleftarrow{e} \le [E_{\alpha}(2\Gamma(\alpha)\mu t_n^{\alpha}), E_{\alpha}(2\Gamma(\alpha)\mu t_{n-1}^{\alpha}), \cdots, E_{\alpha}(2\Gamma(\alpha)\mu t_1^{\alpha})]', l \ge n$

where $\overleftarrow{e} = [1, 1, \cdots, 1]' \in \mathbb{R}^n$, μ is a constant.

Theorem 6. *Assuming $\{u^n \mid n = -m, -m+1, \cdots, 0, 1, 2, \cdots\}$ and $\{f^n \mid n = 0, 1, 2, \cdots\}$ are nonnegative sequence, for $\lambda_i > 0$, $i = 1, 2, 3, 4, 5$, if*

$${}^{R}D_{\Delta t}^{\alpha} u^j \le \lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1} + f^j, \quad j = 1, 2 \cdots, \tag{16}$$

then, there exists a positive constant Δt^*, for $\Delta t < \Delta t^*$, the following holds

$$u^n \le 2\left(\lambda_4 \frac{\Gamma(\alpha) t_n^{\alpha}}{\Gamma(1+\alpha)} M + \lambda_5 \frac{\Gamma(\alpha) t_n^{\alpha}}{\Gamma(1+\alpha)} M + \max_{1 \le j \le n} f^j \frac{\Gamma(\alpha) t_n^{\alpha}}{\Gamma(1+\alpha)} + 2M + \lambda_2 M \Delta t^{\alpha} + 2\lambda_3 M \Delta t^{\alpha} \right) E_{\alpha}(2\Gamma(\alpha)\lambda t_n^{\alpha}), \quad 1 \le n \le N, \tag{17}$$

where $\lambda = \lambda_1 + (1/(g_0^{(\alpha)} - g_1^{(\alpha)}) g_0^{(\alpha)} - g_1^{(\alpha)})\lambda_2 + (1/(g_1^{(\alpha)} - g_2^{(\alpha)}))\lambda_3 + (1/(g_{m-1}^{(\alpha)} - g_m^{(\alpha)}))\lambda_4 + (1/(g_m^{(\alpha)} - g_{m+1}^{(\alpha)}))\lambda_5$, $E_{\alpha}(z) = \sum_{k=0}^{\infty} (z^k/(\Gamma(a+k\alpha)))$ *is the Mittag-Leffler function, and* $M = \max\{u^{-m}, u^{-m+1}, \cdots, u^0\}$.

Proof. By using the definition of ${}^{R}D_{\Delta t}^{\alpha} u^n$ in (13), we have

$$\sum_{k=1}^{j} g_{j-k}^{(\alpha)} \delta_t u^k + g_j^{(\alpha)} u^0 \le \Delta t^{\alpha} \left(\lambda_1 u^j + \lambda_2 u^{j-1} + \lambda_3 u^{j-2} + \lambda_4 u^{j-m} + \lambda_5 u^{j-m-1} \right) + \Delta t^{\alpha} f^j. \tag{18}$$

Multiplying equation (18) by ϕ_{n-j} and summing the index j from 1 to n, we get

$$\sum_{j=1}^{n}\phi_{n-j}\sum_{k=1}^{j}g_{j-k}^{(\alpha)}\delta_t u^k \le \Delta t^\alpha \sum_{j=1}^{n}\phi_{n-j}\left(\lambda_1 u^j+\lambda_2 u^{j-1}+\lambda_3 u^{j-2}+\lambda_4 u^{j-m}+\lambda_5 u^{j-m-1}\right) + \Delta t^\alpha \sum_{j=1}^{n}\phi_{n-j}f^j - \sum_{j=1}^{n}\phi_{n-j}g_j^{(\alpha)}u^0. \tag{19}$$

We change the order of summation and make use of the definition of ϕ_{n-j} to obtain

$$\sum_{j=1}^{n}\phi_{n-j}\sum_{k=1}^{j}g_{j-k}^{(\alpha)}\delta_t u^k = \sum_{k=1}^{n}\delta_t u^k \sum_{j=1}^{k}\phi_{n-j}g_{j-k}^{(\alpha)} = \sum_{k=1}^{n}\delta_t u^k = u^n - u^0, \tag{20}$$

and using Lemma 4, we have

$$\begin{aligned}\Delta t^\alpha \sum_{j=1}^{n}\phi_{n-j}f^j &\le \Delta t^\alpha \max_{1\le j\le n} f^j \sum_{j=1}^{n}\phi_{n-j} \le \Delta t^\alpha \max_{1\le j\le n} f^j \frac{\Gamma(\alpha)n^\alpha}{\Gamma(1+\alpha)} \\ &= \max_{1\le j\le n} f^j \frac{\Gamma(\alpha)t_n^\alpha}{\Gamma(1+\alpha)}.\end{aligned} \tag{21}$$

Noticing $g_j^{(\alpha)}$ is monotone decreasing and using Lemma 4, we have

$$-\sum_{j=1}^{n}\phi_{n-j}g_j^{(\alpha)}u^0 \le \sum_{j=1}^{n}\phi_{n-j}g_j^{(\alpha)}u^0 \le u^0\sum_{j=1}^{n}\phi_{n-j}g_{j-1}^{(\alpha)} = u^0. \tag{22}$$

Substituting (20), (21), and (22) into (19), we can obtain

$$u^n \le \Delta t^\alpha \sum_{j=1}^{n}\phi_{n-j}\left(\lambda_1 u^j+\lambda_2 u^{j-1}+\lambda_3 u^{j-2}+\lambda_4 u^{j-m}+\lambda_5 u^{j-m-1}\right) + 2u^0 + \max_{1\le j\le n} f^j \frac{\Gamma(\alpha)t_n^\alpha}{\Gamma(1+\alpha)}. \tag{23}$$

Applying Lemma 4, we have

$$\begin{aligned}\Delta t^\alpha \sum_{j=1}^{m}\phi_{n-j}u^{j-m} &\le \frac{\Gamma(\alpha)t_n^\alpha}{\Gamma(1+\alpha)}M, \\ \Delta t^\alpha \sum_{j=1}^{m+1}\phi_{n-j}u^{j-m-1} &\le \frac{\Gamma(\alpha)t_n^\alpha}{\Gamma(1+\alpha)}M.\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}&\lambda_4\Delta t^\alpha \sum_{j=1}^{m}\phi_{n-j}u^{j-m} + \lambda_5\Delta t^\alpha \sum_{j=1}^{m+1}\phi_{n-j}u^{j-m-1} + 2u^0 + \lambda_2\Delta t^\alpha\phi_{n-1}u^0 \\ &\quad + \lambda_3\Delta t^\alpha\left(\phi_{n-1}u^{-1}+\phi_{n-2}u^0\right) \le \lambda_4\frac{\Gamma(\alpha)t_n^\alpha}{\Gamma(1+\alpha)}M \\ &\quad + \lambda_5\frac{\Gamma(\alpha)t_n^\alpha}{\Gamma(1+\alpha)}M + 2M + \lambda_2 M\Delta t^\alpha + 2\lambda_3 M\Delta t^\alpha.\end{aligned} \tag{25}$$

Denote

$$\Psi_n = \lambda_4\frac{\Gamma(\alpha)t_n^\alpha}{\Gamma(1+\alpha)}M + \lambda_5\frac{\Gamma(\alpha)t_n^\alpha}{\Gamma(1+\alpha)}M + \max_{1\le j\le n} f^j \frac{\Gamma(\alpha)t_n^\alpha}{\Gamma(1+\alpha)} + 2M + \lambda_2 M\Delta t^\alpha + 2\lambda_3 M\Delta t^\alpha, \tag{26}$$

Equation (23) can be rewritten as

$$\begin{aligned}(1-\lambda_1\Delta t^\alpha)u^n &\le \lambda_1\Delta t^\alpha\sum_{j=1}^{n-1}\phi_{n-j}u^j + \lambda_2\Delta t^\alpha\sum_{j=2}^{n}\phi_{n-j}u^{j-1} \\ &\quad + \lambda_3\Delta t^\alpha\sum_{j=3}^{n}\phi_{n-j}u^{j-2} + \lambda_4\Delta t^\alpha\sum_{j=m+1}^{n}\phi_{n-j}u^{j-m} \\ &\quad + \lambda_5\Delta t^\alpha\sum_{j=m+2}^{n}\phi_{n-j}u^{j-m-1} + \Psi_n.\end{aligned} \tag{27}$$

Let $\Delta t^* = \sqrt[\alpha]{1/(2\lambda_1)}$, when $\Delta t \le \Delta t^*$, we have

$$\begin{aligned}u^n \le 2\Psi_n + 2\Delta t^\alpha\Bigg[&\lambda_1\sum_{j=1}^{n-1}\phi_{n-j}u^j + \lambda_2\sum_{j=2}^{n}\phi_{n-j}u^{j-1} + \lambda_3\sum_{j=3}^{n}\phi_{n-j}u^{j-2} \\ &+ \lambda_4\sum_{j=m+1}^{n}\phi_{n-j}u^{j-m} + \lambda_5\sum_{j=m+2}^{n}\phi_{n-j}u^{j-m-1}\Bigg].\end{aligned} \tag{28}$$

Let $V = (u^n, u^{n-1},\cdots,u^1)^T$, then (28) can be rewritten in the following matrix form:

$$V \le 2\Psi_n \overleftarrow{e} + (\lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_3 + \lambda_4 W_4 + \lambda_5 W_5)V, \tag{29}$$

where

$$W_1 = 2(\Delta t)^\alpha \begin{pmatrix} 0 & \phi_1 & \phi_2 & \cdots & \phi_{n-2} & \phi_{n-1} \\ 0 & 0 & \phi_1 & \cdots & \phi_{n-3} & \phi_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_1 & \phi_2 \\ 0 & 0 & 0 & \cdots & 0 & \phi_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n},$$

$$W_2 = 2(\Delta t)^{\alpha} \begin{pmatrix} 0 & \phi_0 & \phi_1 & \cdots & \phi_{n-3} & \phi_{n-2} \\ 0 & 0 & \phi_0 & \cdots & \phi_{n-4} & \phi_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_0 & \phi_1 \\ 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n},$$

$$W_3 = 2(\Delta t)^{\alpha} \begin{pmatrix} 0 & 0 & \phi_0 & \cdots & \phi_{n-4} & \phi_{n-3} \\ 0 & 0 & 0 & \cdots & \phi_{n-5} & \phi_{n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{n\times n},$$

$$W_4 = 2(\Delta t)^{\alpha} \begin{pmatrix} 0 & \cdots & 0 & \phi_0 & \phi_1 & \cdots & \phi_{n-m-2} & \phi_{n-m-1} \\ 0 & \cdots & 0 & 0 & \phi_0 & \cdots & \phi_{n-m-3} & \phi_{n-m-2} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \phi_0 & \phi_1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n},$$

$$W_5 = 2(\Delta t)^{\alpha} \begin{pmatrix} 0 & \cdots & 0 & 0 & \phi_0 & \cdots & \phi_{n-m-3} & \phi_{n-m-2} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \phi_{n-m-4} & \phi_{n-m-3} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \phi_0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}. \tag{30}$$

Since the definition of ϕ_n, we have

$$\phi_{n-j} \le \frac{1}{g_{j-1}^{(\alpha)} - g_j^{(\alpha)}} \phi_n. \tag{31}$$

Then,

$$\begin{aligned} W_2 V &\le \frac{1}{g_0^{(\alpha)} - g_1^{(\alpha)}} W_1 V, \\ W_3 V &\le \frac{1}{g_1^{(\alpha)} - g_2^{(\alpha)}} W_1 V, \\ W_4 V &\le \frac{1}{g_{m-1}^{(\alpha)} - g_m^{(\alpha)}} W_1 V, \\ W_5 V &\le \frac{1}{g_m^{(\alpha)} - g_{m+1}^{(\alpha)}} W_1 V. \end{aligned} \tag{32}$$

Hence, (29) can be shown as follows

$$\begin{aligned} V \le \Bigg(& \lambda_1 + \frac{1}{g_0^{(\alpha)} - g_1^{(\alpha)}} \lambda_2 + \frac{1}{g_1^{(\alpha)} - g_2^{(\alpha)}} \lambda_3 + \frac{1}{g_{m-1}^{(\alpha)} - g_m^{(\alpha)}} \lambda_4 \\ & + \frac{1}{g_m^{(\alpha)} - g_{m+1}^{(\alpha)}} \lambda_5 \Bigg) W_1 V + 2\Psi_n \overleftarrow{e} = WV + 2\Psi_n \overleftarrow{e}, \end{aligned} \tag{33}$$

where $W = \lambda W_1$.

Therefore,

$$\begin{aligned} V &\le WV + 2\Psi_n \overleftarrow{e} \le W\left(WV + 2\Psi_n \overleftarrow{e}\right) + 2\Psi_n \overleftarrow{e} \\ &= W^2 V + 2\Psi_n \sum_{j=0}^{1} W^j \overleftarrow{e} \le \cdots \le W^n V + 2\Psi_n \sum_{j=0}^{n-1} W^j \overleftarrow{e}. \end{aligned} \tag{34}$$

According to Lemma 5, the result can be proved.

Lemma 7 (see [51]). *For any sequence $\{e^k\}_{k=0}^N \subset X_h$, the following inequality holds:*

$$\left\langle {}^R D_{\Delta t}^{\alpha} e^k, \left(1 - \frac{\alpha}{2}\right) e^k + \frac{\alpha}{2} e^{k-1} \right\rangle \ge \frac{1}{2} {}^R D_{\Delta t}^{\alpha} \|e^k\|^2, \quad 1 \le k \le N. \tag{35}$$

Lemma 8 (see [52]). *There exists a positive constant C_Ω, independent of h, for any $v \in H^s(\Omega) \cap H_0^1(\Omega)$, such that*

$$\|v - R_h v\|_{L^2} + h \|\nabla (v - R_h v)\|_{L^2} \le C_\Omega h^s \|v\|_{H^s}, \quad 1 \le s \le r + 1. \tag{36}$$

3.2. Proof of Theorem 1. Now, we are ready to prove our main results.

Proof. Taking $t = t_{n-(\alpha/2)}$ in the first equation (1), we can find that u^n satisfies the following equation:

$$\left\langle {}^R D_{\Delta t}^{\alpha} u^n, v \right\rangle + \left\langle \nabla u^{n,\alpha}, \nabla v \right\rangle = \left\langle f\left(t_{n-(\alpha/2)}, u\wedge^{n,\alpha}, u^{n-m_\tau,\alpha}\right), v \right\rangle + \left\langle P^n, v \right\rangle, \tag{37}$$

for $n = 1, 2, 3, \cdots, N$ and $\forall v \in X_h$, where

$$\begin{aligned} P^n = {}^R D^{\alpha}_{\Delta t} u^n - {}^R D^{\alpha}_{t_{n-(\alpha/2)}} u + \Delta u^{n-(\alpha/2)} - \Delta u^{n,\alpha} \\ + f\left(t_{n-(\alpha/2)}, u^{n-(\alpha/2)}, u^{n-m_\tau-(\alpha/2)}\right) \\ - f\left(t_{n-(\alpha/2)}, u\wedge^{n,\alpha}, u^{n-m_\tau,\alpha}\right). \end{aligned} \tag{38}$$

Now, we estimate the error of $\|P^n\|$. Actually, from the definition of $u^{n,\alpha}$ and $u\wedge^{n,\alpha}$ and the regularity of the exact solution (8), we can obtain that

$$\begin{aligned} \left\|u^{n-(\alpha/2)} - u^{n,\alpha}\right\| &= \left\|\left(1-\frac{\alpha}{2}\right)u^{n-(\alpha/2)} + \frac{\alpha}{2}u^{n-(\alpha/2)} - \left(1-\frac{\alpha}{2}\right)u^{n-(\alpha/2)}u^{n-1}\right\| \\ &= \left\|\left(1-\frac{\alpha}{2}\right)\left(u^{n-(\alpha/2)-u^n}\right) + \frac{\alpha}{2}\left(u^{n-(\alpha/2)-u^{n-1}}\right)\right\| \\ &= \left\|-\left(1-\frac{\alpha}{2}\right) + \frac{\alpha}{2}\Delta t u'(\xi_1) + \left(1-\frac{\alpha}{2}\right) + \frac{\alpha}{2}\Delta t u'(\xi_2)\right\| \\ &= \left(1-\frac{\alpha}{2}\right) + \frac{\alpha}{2}\Delta t\left\|u'(\xi_2) - u'(\xi_1)\right\| \\ &\le \left(1-\frac{\alpha}{2}\right) + \frac{\alpha}{2}\Delta t\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|ds \le C_1\Delta t^2, \end{aligned} \tag{39}$$

$$\begin{aligned} \|u^{n-(\alpha/2)} - u\wedge^{n,\alpha}\| &= \|u^{n-(\alpha/2)} - \left(2-\frac{\alpha}{2}\right)u^{n-1} + \left(1-\frac{\alpha}{2}\right)u^{n-2}\| \\ &= \|\left(2-\frac{\alpha}{2}\right)u^{n-(\alpha/2)} - \left(2-\frac{\alpha}{2}\right)u^{n-1} \\ &\quad + \left(1-\frac{\alpha}{2}\right)u^{n-2} - \left(1-\frac{\alpha}{2}\right)u^{n-(\alpha/2)}\| \\ &= \|\left(2-\frac{\alpha}{2}\right)\left(u^{n-(\alpha/2)} - u^{n-1}\right) \\ &\quad + \left(1-\frac{\alpha}{2}\right)\left(u^{n-2} - u^{n-(\alpha/2)}\right)\| \\ &= \|\left(2-\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{2}\right)\Delta t u'(\xi_3) \\ &\quad - \left(2-\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{2}\right)\Delta t u'(\xi_4)\| \\ &= \left(2-\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{2}\right)\Delta t\|u'(\xi_3) - u'(\xi_4)\| \\ &\le \left(2-\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{2}\right)\Delta t\int_{t_{n-2}}^{t_{n-1}}\|u_{tt}(s)\|ds \le C_2\Delta t^2, \end{aligned} \tag{40}$$

where $\xi_1 \in (t_{n-(\alpha/2)}, t_n), \xi_2 \in (t_{n-1}, t_{n-(\alpha/2)}), \xi_3 \in (t_{n-(\alpha/2)}, t_{n-1})$ $, \xi_4 \in (t_{n-2}, t_{n-(\alpha/2)})$, $C_1 = (1-(\alpha/2))(\alpha/2)K$, $C_2 = (2-(\alpha/2))$ $(1-(\alpha/2))K$ are constants.

Applying (39) and (40) and the Lipschitz condition,

$$\begin{aligned} &\|f\left(t_{n-(\alpha/2)}, u^{n-(\alpha/2)}, u^{n-m_\tau-(\alpha/2)}\right) - f\left(t_{n-(\alpha/2)}, u\wedge^{n,\alpha}, u^{n-m_\tau,\alpha}\right)\| \\ &\quad \le (L_1C_1 + L_2C_2)\Delta t^2, \end{aligned}$$

$$\|\Delta\left(u^{n,\alpha} - u^{n-(\alpha/2)}\right)\| \le C_1\Delta t^2, \tag{41}$$

which further implies that

$$\|P^n\| \le C_K(\Delta t)^2, \quad n = 1, 2, 3, \cdots, N, \tag{42}$$

here $C_K = L_1C_1 + L_2C_2$.

Denote $\theta_h^n = R_h u^n - U_h^n$, $n = 0, 1, \cdots, N$.

Substituting fully scheme (5) from equation (37) and using the property in (7), we can get that

$$\left\langle {}^R D^{\alpha}_{\Delta t}\theta_h^n, v\right\rangle + \left\langle\nabla\theta_h^{n,\alpha}, v\right\rangle = \left\langle R_1^n, v\right\rangle + \left\langle P^n, v\right\rangle - \left\langle {}^R D^{\alpha}_{\Delta t}(u^n - R_h u^n), v\right\rangle, \tag{43}$$

where

$$R_1^n = f\left(t_{n-(\alpha/2)}, \widehat{U}_h^{n,\alpha}, U_h^{n-m_\tau,\alpha}\right) - f\left(t_{n-(\alpha/2)}, u\wedge^{n,\alpha}, u^{n-m_\tau,\alpha}\right). \tag{44}$$

Setting $v = \theta_h^{n,\alpha}$ and applying the Cauchy-Schwarz inequality, it holds that

$$\begin{aligned} \left\langle {}^R D^{\alpha}_{\Delta t}\theta_h^n, \theta_h^{n,\alpha}\right\rangle + \|\nabla\theta_h^{n,\alpha}\|^2 \le \|R_1^n\|\|\theta_h^{n,\alpha}\| + \|P^n\|\|\theta_h^{n,\alpha}\| \\ + \|{}^R D^{\alpha}_{\Delta t}(u^n - R_h u^n)\|\|\theta_h^{n,\alpha}\|. \end{aligned} \tag{45}$$

Noticing the fact $ab \le 1/2(a^2 + b^2)$ and $\|\nabla\theta_h^{n,\alpha}\|^2 \ge 0$,

$$\begin{aligned} \left\langle {}^R D^{\alpha}_{\Delta t}\theta_h^n, \theta_h^{n,\alpha}\right\rangle \le \frac{1}{2}\left(\|R_1^n\|^2 + \|P^n\|^2 + \|{}^R D^{\alpha}_{\Delta t}(u^n - R_h u^n)\|^2\right) \\ + \frac{3}{2}\|\theta_h^{n,\alpha}\|^2. \end{aligned} \tag{46}$$

Together with (9) and (36), we can arrive that

$$\|{}^R D^{\alpha}_{\Delta t}(u^n - R_h u^n)\| \le C_\Omega h^{r+1}\|{}^R D^{\alpha}_{\Delta t}u^n\|_{H^{r+1}} \le C_\Omega K h^{r+1}. \tag{47}$$

$$\begin{aligned} \|u\wedge^{n,\alpha} - R_h u\wedge^{n,\alpha}\| &= \|\left(2-\frac{\alpha}{2}\right)u^{n-1} - \left(1-\frac{\alpha}{2}\right)u^{n-2} \\ &\quad - \left(2-\frac{\alpha}{2}\right)R_h u^{n-1} + \left(1-\frac{\alpha}{2}\right)R_h u^{n-2}\| \\ &\le \left(2-\frac{\alpha}{2}\right)\|u^{n-1} - R_h u^{n-1}\| + \left(1-\frac{\alpha}{2}\right)\|u^{n-2} - R_h u^{n-2}\| \\ &\le \left(2-\frac{\alpha}{2}\right)C_\Omega h^{r+1}\|u^{n-1}\|_{H^{r+1}} \\ &\quad + \left(1-\frac{\alpha}{2}\right)C_\Omega h^{r+1}\|u^{n-2}\|_{H^{r+1}} \\ &\le \left(2-\frac{\alpha}{2}\right)C_\Omega K h^{r+1} + \left(1-\frac{\alpha}{2}\right)C_\Omega K h^{r+1} \le C_3 h^{r+1}. \end{aligned} \tag{48}$$

Similarly, we have

$$\begin{aligned}\|u^{n-m_\tau,\alpha} - R_h u^{n-m_\tau,\alpha}\| &= \left\|\left(1-\frac{\alpha}{2}\right)u^{n-m_\tau} + \frac{\alpha}{2}u^{n-m_\tau-1} - \left(1-\frac{\alpha}{2}\right)R_h u^{n-m_\tau} - \frac{\alpha}{2}R_h u^{n-m_\tau-1}\right\| \\ &\le \left(1-\frac{\alpha}{2}\right)C_\Omega K h^{r+1} + \frac{\alpha}{2}C_\Omega K h^{r+1} \\ &\le C_4 h^{r+1},\end{aligned} \tag{49}$$

where $C_3 = 2(2-(\alpha/2))C_\Omega K$, $C_4 = 2\max\{(1-(\alpha/2)), (\alpha/2)\}C_\Omega K$.

Therefore,

$$\begin{aligned}\|R_1^n\| &= \left\|f\left(t_{n-(\alpha/2)}, u\wedge^{n,\alpha}, u^{n-m_\tau,\alpha}\right) - f\left(t_{n-(\alpha/2)}, \widehat{U}_h^{n,\alpha}, U_h^{n-m_\tau,\alpha}\right)\right\| \\ &\le L_1\|u\wedge^{n,\alpha} - \widehat{U}_h^{n,\alpha}\| + L_2\|u^{n-m_\tau,\alpha} - U_h^{n-m_\tau,\alpha}\| \\ &\le L_1\|\widehat{\theta}_h^{n,\alpha}\| + L_2\|\theta_h^{n-m_\tau,\alpha}\| + L_1\|u\wedge^{n,\alpha} - R_h u\wedge^{n,\alpha}\| \\ &\quad + L_2\|u^{n-m_\tau,\alpha} - R_h u^{n-m_\tau,\alpha}\| \le L_1\|\widehat{\theta}_h^{n,\alpha}\| + L_2\|\theta_h^{n-m_\tau,\alpha}\| \\ &\quad + (L_1C_3 + L_2C_4)h^{r+1}.\end{aligned} \tag{50}$$

Substituting (42), (47), and (50) into (46) and the fact $(a+b+c)^2 \le 3a^2 + 3b^2 + 3c^2$, we can get

$$\begin{aligned}\left\langle {}^R D_{\Delta t}^\alpha \theta_h^n, \theta_h^{n,\alpha}\right\rangle &\le \frac{3}{2}\|\theta_h^{n,\alpha}\|^2 + \frac{3L_1^2}{2}\|\widehat{\theta}_h^{n,\alpha}\|^2 + \frac{3L_2^2}{2}\|\theta_h^{n-m_\tau,\alpha}\|^2 \\ &\quad + \frac{C_K^2}{2}(\Delta t)^4 + \frac{1}{2}\left[3\left(L_1^2C_3^2 + L_2^2C_4^2\right) + (C_K K)^2\right]h^{2(r+1)} \\ &\le \frac{3}{2}\|\theta_h^{n,\alpha}\|^2 + \frac{3L_1^2}{2}\|\widehat{\theta}_h^{n,\alpha}\|^2 + \frac{3L_2^2}{2}\|\theta_h^{n-m_\tau,\alpha}\|^2 \\ &\quad + \frac{C_4}{2}\left(\Delta t^2 + h^{r+1}\right)^2,\end{aligned} \tag{51}$$

where $C_4 = \max\{C_K^2, 3(L_1^2C_3^2 + L_2^2C_4^2) + (C_K K)^2\}$.

Applying Lemma 7, we have

$${}^R D_{\Delta t}^\alpha \|\theta_h^n\|^2 \le 3\|\theta_h^{n,\alpha}\|^2 + 3L_1^2\|\widehat{\theta}_h^{n,\alpha}\|^2 + 3L_2^2\|\theta_h^{n-m_\tau,\alpha}\|^2 + C_4\left(\Delta t^2 + h^{r+1}\right)^2. \tag{52}$$

Table 1: The errors and convergence orders in temporal direction by using Q-FEM.

M	$\alpha = 0.4$		$\alpha = 0.6$	
	Errors	Orders	Errors	Orders
5	$1.6856e-03$	*	$5.3999e-03$	*
10	$2.9420e-04$	2.5184	$1.2503e-03$	2.1106
20	$5.9619e-05$	2.3030	$3.0266e-04$	2.0465
40	$1.3851e-05$	2.1058	$7.4700e-05$	2.0185

In terms of the definition of $\|\theta_h^{n,\alpha}\|$ and $\widehat{\theta}_h^{n,\alpha}$, we obtain

$$\begin{aligned}{}^R D_{\Delta t}^\alpha \|\theta_h^n\|^2 &\le 3\left(1-\frac{\alpha}{2}\right)^2\|\theta_h^n\|^2 + \left(3\left(\frac{\alpha}{2}\right)^2 + 3L_1^2\left(2-\frac{\alpha}{2}\right)^2\right)\|\theta_h^{n-1}\|^2 \\ &\quad + 3L_1^2\left(1-\frac{\alpha}{2}\right)^2\|\theta_h^{n-2}\|^2 + 3L_2^2\left(1-\frac{\alpha}{2}\right)^2\|\theta_h^{n-m_\tau}\|^2 \\ &\quad + 3L_2^2\left(\frac{\alpha}{2}\right)^2\|\theta_h^{n-m_\tau-1}\|^2 + C_4\left(\Delta t^2 + h^{r+1}\right)^2.\end{aligned} \tag{53}$$

Using Theorem 6, we can find a positive constant Δt^* such that $\Delta t \le \Delta t^*$, then

$$\|\theta_h^n\|^2 \le C_5\left(\Delta t^2 + h^{r+1}\right)^2, \tag{54}$$

where C_5 is a nonnegative constant which only depends on $L_1, L_2, C_4, C_K, C_\Omega$. In terms of the definition of θ_h^n, we have

$$\|u^n - U_h^n\| \le \|u^n - R_h u^n\| + \|R_h u^n - U_h^n\| \le C_1^*\left(\Delta t^2 + h^{r+1}\right). \tag{55}$$

Then, we complete the proof.

4. Numerical Examples

In this section, we give two examples to verify our theoretical results. The errors are all calculated in L2-norm.

Example 1. Consider the nonlinear time-fractional Mackey-Glass-type equation

$$\begin{cases} {}^R D_t^\alpha u(x,y,t) = \Delta u(x,y,t) - 2u(x,y,t) + \dfrac{u(x,y,t-0.1)}{1+u^2(x,y,t-0.1)} + f(x,y,t), & (x,y) \in [0,1]^2, t \in [0,1], \\ u(x,y,t) = t^2 \sin(\pi x)\sin(\pi y), & (x,y) \in [0,1]^2, t \in [-0.1,0], \end{cases} \tag{56}$$

Table 2: The errors and convergence orders in spatial direction by using L-FEM.

M	$\alpha=0.4$ Errors	Orders	$\alpha=0.6$ Errors	Orders
5	$7.2603e-02$	*	$7.2065e-02$	*
10	$1.9449e-02$	1.9003	$1.9297e-02$	1.9009
20	$8.7594e-03$	1.9673	$8.6948e-03$	1.9662
40	$4.9508e-03$	1.9834	$4.9180e-03$	1.9807

Table 3: The errors and convergence orders in spatial direction by using Q-FEM.

M	$\alpha=0.4$ Errors	Orders	$\alpha=0.6$ Errors	Orders
5	$2.0750e-03$	*	$2.0746e-03$	*
10	$2.4888e-04$	3.0596	$2.5148e-04$	3.0443
20	$7.3251e-05$	3.0165	$7.5802e-05$	2.9577
40	$3.0946e-05$	2.9952	$3.4200e-05$	2.7666

where

$$f(x,y,t)=\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\sin(\pi x)\sin(\pi y)+2t^2\pi^2\sin(\pi x)\sin(\pi y)-2t^2\sin(\pi x)\sin(\pi y)-\frac{(t-0.1)^2\sin(\pi x)\sin(\pi y)}{1+\left[(t-0.1)^2\sin(\pi x)\sin(\pi y)\right]^2}. \tag{57}$$

The exact solution is given as

$$u(x,t)=t^2\sin(\pi x)\sin(\pi y). \tag{58}$$

Table 4: The errors and orders in temporal and spatial direction by using L-FEM.

M	$\alpha=0.4$ Errors	Orders	$\alpha=0.6$ Errors	Orders
5	$8.3275e-02$	*	$8.3375e-02$	*
10	$2.2615e-02$	1.8806	$2.2732e-02$	1.8749
20	$5.8356e-03$	1.9543	$5.8662e-03$	1.9542
40	$1.4707e-03$	1.9884	$1.4784e-03$	1.9884

Table 5: The errors and orders in temporal direction and spatial direction by using Q-FEM.

	M	$\alpha=0.4$ Errors	Orders	$\alpha=0.6$ Errors	Orders
$N=M^{(3/2)}$	8	$6.7379e-04$	*	$6.9141e-04$	*
	10	$3.1416e-04$	3.0459	$3.4945e-04$	3.0579
	12	$1.9415e-04$	3.0968	$1.9787e-04$	3.1196
	14	$1.1891e-04$	3.1806	$1.1992e-04$	3.2485

In order to test the convergence order in temporal direction, we fixed $M=40$ for $\alpha=0.4, \alpha=0.6$ and different N. Similarly, to obtain the convergence order in spatial direction, we fixed $N=100$ for $\alpha=0.4, \alpha=0.6$, and different M. Table 1 gives the errors and convergence orders in temporal direction by using the Q-FEM. Table 1 shows that the convergence order in temporal direction is 2. Similarly, Tables 2 and 3 give the errors and convergence orders in spatial direction by using the L-FEM and Q-FEM, respectively. These numerical results correspond to our theoretical convergence order.

Example 2. Consider the following nonlinear time-fractional Nicholson's blowflies equation

$$\begin{cases} {}^{R}D_t^{\alpha}u(x,y,z,t)=\Delta u(x,y,z,t)-2u(x,y,z,t)+u(x,y,z,t-0.1)\exp\{-u(x,y,z,t-0.1)\}+f(x,y,z,t), & (x,y,z)\in[0,1]^3, t\in[0,1],\\ u(x,y,z,t)=t^2\sin(\pi x)\sin(\pi y)\sin(\pi z), & (x,y,z)\in[0,1]^3, t\in[-0.1,0], \end{cases} \tag{59}$$

where

$$f(x,y,z,t)=\left(2t^{2-\alpha}/\Gamma(3-\alpha)\right)\sin(\pi x)\sin(\pi y)\sin(\pi z)+2t^2\left(\pi^2-1\right)\sin(\pi x)\sin(\pi y)\sin(\pi z)-(-0.1)^2\sin(\pi x)\sin(\pi y)\sin(\pi z)\exp\cdot\left\{-(t-0.1)^2\sin(\pi x)\sin(\pi y)\sin(\pi z)\right\}, \tag{60}$$

the exact solution is given as

$$u(x,t)=t^2\sin(\pi x)\sin(\pi y)\sin(\pi z). \tag{61}$$

In this example, in order to test the convergence order in temporal and spatial direction, we solve this problem by using the L-FEM with $M=N$ and the Q-FEM with $N=M^{(3/2)}$, respectively. Tables 4 and 5 show that the convergence orders in temporal and spatial direction are 2 and 3, respectively. The numerical results confirm our theoretical convergence order.

5. Conclusions

We proposed a linearized fractional Crank-Nicolson-Galerkin FEM for the nonlinear fractional parabolic equations with time delay. A novel fractional Grönwall-type inequality is developed. With the help of the inequality, we prove convergence of the numerical scheme. Numerical examples confirm our theoretical results.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by the Natural Science Foundation of Hunan Province (Grant No. 2018JJ3628) and the National Natural Science Foundation of China (Grant Nos.12071488 and 11971488).

References

[1] F. Höfling and T. Franosch, "Anomalous transport in the crowded world of biological cells," *Reports on Progress in Physics*, vol. 76, no. 4, article 046602, 2013.

[2] A. Arafa, S. Rida, and M. Khalil, "Fractional modeling dynamics of HIV and CD4+ T-cells during primary infection," *Nonlinear Biomedical Physics*, vol. 6, no. 1, p. 1, 2012.

[3] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Redding, 2006.

[4] N. Sebaa, Z. E. A. Fellah, W. Lauriks, and C. Depollier, "Application of fractional calculus to ultrasonic wave propagation in human cancellous bone," *Signal Processing*, vol. 86, no. 10, pp. 2668–2677, 2006.

[5] A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, vol. 378, Springer, 2014.

[6] B. West, M. Bologna, and P. Grigolini, *Physics of Fractal Operators, Springer Science*, Business Media, 2012.

[7] D. Li and C. Zhang, "Long time numerical behaviors of fractional pantograph equations," *Mathematics and Computers in Simulation*, vol. 172, pp. 244–257, 2020.

[8] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204, Elsevier Science Limited, 2006.

[9] Q. Zhang, Y. Ren, X. Lin, and Y. Xu, "Uniform convergence of compact and BDF methods for the space fractional semilinear delay reaction-diffusion equations," *Applied Mathematics and Computation*, vol. 358, pp. 91–110, 2019.

[10] W. Chen, L. Ye, and H. Sun, "Fractional diffusion equations by the Kansa method," *Computers & Mathematcs with Applications*, vol. 59, no. 5, pp. 1614–1620, 2010.

[11] M. Dehghan, M. Abbaszadeh, and A. Mohebbi, "Error estimate for the numerical solution of fractional reaction-subdiffusion process based on a meshless method," *Journal of Computational and Applied Mathematics*, vol. 280, pp. 14–36, 2015.

[12] D. A. Murio, "Implicit finite difference approximation for time fractional diffusion equations," *Computers & Mathematics with Applications*, vol. 56, no. 4, pp. 1138–1145, 2008.

[13] P. Zhuang and F. Liu, "Implicit difference approximation for the time fractional diffusion equation," *Journal of Applied Mathematics and Computing*, vol. 22, no. 3, pp. 87–99, 2006.

[14] B. Jin, B. Li, and Z. Zhou, "An analysis of the Crank-Nicolson method for subdiffusion," *IMA Journal of Numerical Analysis*, vol. 38, no. 1, pp. 518–541, 2018.

[15] D. Li, W. Sun, and C. Wu, "A novel numerical approach to time-fractional parabolic equations with nonsmooth solutions," *Numerical Mathematics: Theory, Methods and Applications*, vol. 14, no. 2, pp. 355–376, 2021.

[16] L. Li and D. Li, "Exact solutions and numerical study of time fractional Burgers' equations," *Applied Mathematics Letters*, vol. 100, p. 106011, 2020.

[17] C. Li and W. Deng, "High order schemes for the tempered fractional diffusion equations," *Advances in Computational Mathematics*, vol. 42, no. 3, pp. 543–572, 2016.

[18] S. B. Yuste, "Weighted average finite difference methods for fractional diffusion equations," *Journal of Computational Physics*, vol. 216, no. 1, pp. 264–274, 2006.

[19] S. B. Yuste and L. Acedo, "An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations," *SIAM Journal on Numerical Analysis*, vol. 42, no. 5, pp. 1862–1874, 2005.

[20] C. Çelik and I. Duman, "Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative," *Journal of Computational Physics*, vol. 231, no. 4, pp. 1743–1750, 2012.

[21] X. Lin and C. Xu, "Finite difference/spectral approximations for the time-fractional diffusion equation," *Journal of Computational Physics*, vol. 225, no. 2, pp. 1533–1552, 2007.

[22] X. Chen, Y. Di, J. Duan, and D. Li, "Linearized compact ADI schemes for nonlinear time-fractional Schrodinger equations," *Applied Mathematics Letters*, vol. 84, pp. 160–167, 2018.

[23] D. Li, J. Wang, and J. Zhang, "Unconditionally convergent $L1$-Galerkin FEMs for nonlinear time-fractional Schrödinger equations," *SIAM Journal on Scientific Computing*, vol. 39, no. 6, pp. A3067–A3088, 2017.

[24] Z. Sun, J. Zhang, and Z. Zhang, "Optimal error estimates in numerical solution of time fractional Schrödinger equations on unbounded domains," *East Asian Journal on Applied Mathematics*, vol. 8, no. 4, pp. 634–655, 2019.

[25] M. Gunzburger and J. Wang, "A second-order Crank-Nicolson method for time-fractional PDEs," *International Journal of Numerical Analysis and Modeling*, vol. 16, no. 2, pp. 225–239, 2019.

[26] N. H. Sweilam, H. Moharram, N. K. A. Abdel Moniem, and S. Ahmed, "A parallel Crank-Nicolson finite difference method for time-fractional parabolic equation," *Journal of Numerical Mathematics*, vol. 22, no. 4, pp. 363–382, 2014.

[27] N. H. Sweilam, M. M. Khader, and A. M. Mahdy, "Crank-Nicolson finite difference method for solving time-fractional diffusion equation," *Journal of Fractional Calculus and Applications*, vol. 2, no. 2, pp. 1–9, 2012.

[28] Q. Zhang, M. Ran, and D. Xu, "Analysis of the compact difference scheme for the semilinear fractional partial differential equation with time delay," *Applicable Analysis*, vol. 96, no. 11, pp. 1867–1884, 2016.

[29] F. A. Rihan, "Computational methods for delay parabolic and time-fractional partial differential equations," *Numerical Methods for Partial Differential Equations*, vol. 26, no. 6, pp. 1556–1571, 2010.

[30] M. Li, C. Huang, and F. Jiang, "Galerkin finite element method for higher dimensional multi-term fractional diffusion equation on non-uniform meshes," *Applicable Analysis*, vol. 96, no. 8, pp. 1269–1284, 2016.

[31] J. Cao and C. Xu, "A high order schema for the numerical solution of the fractional ordinary differential equations," *Journal of Computational Physics*, vol. 238, pp. 154–168, 2013.

[32] M. Stynes, E. O'riordan, and J. L. Gracia, "Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation," *SIAM Journal on Numerical Analysis*, vol. 55, no. 2, pp. 1057–1079, 2017.

[33] D. Li, H.-L. Liao, W. Sun, J. Wang, and J. Zhang, "Analysis of L1-Galerkin FEMs for time-fractional nonlinear parabolic problems," *Communications in Computational Physics*, vol. 24, pp. 86–103, 2018.

[34] D. Li, J. Zhang, and Z. Zhang, "Unconditionally optimal error estimates of a linearized Galerkin method for nonlinear time fractional reaction-subdiffusion equations," *Journal of Scientific Computing*, vol. 76, no. 2, pp. 848–866, 2018.

[35] R. Lin and F. Liu, "Fractional high order methods for the nonlinear fractional ordinary differential equation," *Nonlinear Analysis: Theory Methods & Applications*, vol. 66, no. 4, pp. 856–869, 2007.

[36] Y. Liu, Y. Du, H. Li, S. He, and W. Gao, "Finite difference/finite element method for a nonlinear time-fractional fourth-order reaction-diffusion problem," *Computers & Mathematatcs with Applications*, vol. 70, no. 4, pp. 573–591, 2015.

[37] C. Li and F. Zeng, "Finite difference methods for fractional differential equations," *International Journal of Bifurcation and Chaos*, vol. 22, no. 4, p. 1230014, 2012.

[38] B. Jin, B. Li, and Z. Zhou, "Numerical analysis of nonlinear subdiffusion equations," *SIAM Journal on Numerical Analysis*, vol. 56, no. 1, pp. 1–23, 2018.

[39] H. Liao, D. Li, and J. Zhang, "Sharp error estimate of the nonuniform L1 formula for linear reaction-subdiffusion equations," *SIAM Journal on Numerical Analysis*, vol. 56, no. 2, pp. 1112–1133, 2018.

[40] D. Li, C. Wu, and Z. Zhang, "Linearized Galerkin FEMs for nonlinear time fractional parabolic problems with nonsmooth solutions in time direction," *Journal of Scientific Computing*, vol. 80, no. 1, pp. 403–419, 2019.

[41] Z. Wang and S. Vong, "Compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation," *Journal of Computational Physics*, vol. 277, pp. 1–15, 2014.

[42] F. Zeng, C. Li, F. Liu, and I. Turner, "Numerical algorithms for time-fractional subdiffusion equation with second-order accuracy," *SIAM Journal on Scientific Computing*, vol. 37, no. 1, pp. A55–A78, 2015.

[43] X. Zhao and Z. Sun, "Compact Crank-Nicolson schemes for a class of fractional Cattaneo equation in inhomogeneous medium," *Journal of Scientific Computing*, vol. 62, no. 3, pp. 747–771, 2015.

[44] C. Lubich, "Convolution quadrature and discretized operational calculus. I," *Numerische Mathematik*, vol. 52, no. 2, pp. 129–145, 1988.

[45] B. Jin, B. Li, and Z. Zhou, "Correction of high-order BDF convolution quadrature for fractional evolution equations," *SIAM Journal on Scientific Computing*, vol. 39, no. 6, pp. A3129–A3152, 2017.

[46] N. Liu, Y. Liu, H. Li, and J. Wang, "Time second-order finite difference/finite element algorithm for nonlinear time-fractional diffusion problem with fourth-order derivative term," *Computers & Mathematcs with Applications*, vol. 75, no. 10, pp. 3521–3536, 2018.

[47] L. Li, B. Zhou, X. Chen, and Z. Wang, "Convergence and stability of compact finite difference method for nonlinear time fractional reaction-diffusion equations with delay," *Applied Mathematics and Computation*, vol. 337, pp. 144–152, 2018.

[48] A. S. Hendy, V. G. Pimenov, and J. E. Macías-Díaz, "Convergence and stability estimates in difference setting for time-fractional parabolic equations with functional delay," *Numerical Methods for Partial Differential Equations*, vol. 36, no. 1, pp. 118–132, 2019.

[49] A. S. Hendy and J. E. Macías-Díaz, "A novel discrete Gronwall inequality in the analysis of difference schemes for time-fractional multi-delayed diffusion equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 73, pp. 110–119, 2019.

[50] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, vol. 1054, Springer, 1984.

[51] D. Kumar, S. Chaudhary, and V. V. K. S. Kumar, "Fractional Crank–Nicolson–Galerkin finite element scheme for the time-fractional nonlinear diffusion equation," *Numerical Methods for Partial Differential Equations*, vol. 35, no. 6, pp. 2056–2075, 2019.

[52] R. Rannacher and R. Scott, "Some optimal error estimates for piecewise linear finite element approximations," *Mathematics of Computation*, vol. 38, no. 158, pp. 437–445, 1982.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 5581102, 11 pages
https://doi.org/10.1155/2021/5581102

Research Article

$L1$-Multiscale Galerkin's Scheme with Multilevel Augmentation Algorithm for Solving Time Fractional Burgers' Equation

Jian Chen,[1] **Yong Huang**,[1] **and Taishan Zeng**[2,3]

[1]*Department of Mathematics, Foshan University, Foshan 528000, China*
[2]*School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China*
[3]*Guangdong Key Laboratory of Big Data Analysis and Processing, Guangzhou 510006, China*

Correspondence should be addressed to Jian Chen; chenjian704@163.com and Taishan Zeng; zengtsh@m.scnu.edu.cn

Received 23 January 2021; Accepted 11 February 2021; Published 2 March 2021

Academic Editor: Qifeng Zhang

In this paper, we consider the initial boundary value problem of the time fractional Burgers equation. A fully discrete scheme is proposed for the time fractional nonlinear Burgers equation with time discretized by $L1$-type formula and space discretized by the multiscale Galerkin method. The optimal convergence orders reach $\mathcal{O}(\tau^{2-\alpha}+h^r)$ in the L^2 norm and $\mathcal{O}(\tau^{2-\alpha}+h^{r-1})$ in the H^1 norm, respectively, in which τ is the time step size, h is the space step size, and r is the order of piecewise polynomial space. Then, a fast multilevel augmentation method (MAM) is developed for solving the nonlinear algebraic equations resulting from the fully discrete scheme at each time step. We show that the MAM preserves the optimal convergence orders, and the computational cost is greatly reduced. Numerical experiments are presented to verify the theoretical analysis, and comparisons between MAM and Newton's method show the efficiency of our algorithm.

1. Introduction

In this paper, we consider the following time fractional Burgers equation [1–7]:

$$ {}_0^cD_t^\alpha u(x,t)+u(x,t)u_x(x,t)-u_{xx}(x,t)=f(x,t),(x,t)\in\Omega, \tag{1}$$

with the initial and boundary conditions, given by

$$\begin{aligned} u(x,0)&=u_0(x), && 0\le x\le 1,\\ u(0,t)&=u(1,t)=0, && 0<t\le T,\end{aligned} \tag{2}$$

where $0<\alpha<1,\Omega=\{(x,t)\mid 0\le x\le 1, 0<t\le T\},u_0(x)$ and $f(x,t)$ are given functions, and the notation ${}_0^cD_t^\alpha$ denotes the Caputo fractional partial derivative of order α, defined by

$$ {}_0^cD_t^\alpha u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{1}{(t-s)^\alpha}\frac{\partial u(x,s)}{\partial s}\mathrm{d}s, \alpha\in(0,1), \tag{3}$$

in which $\Gamma(\cdot)$ represents the Gamma fuction.

The time fractional Burgers equation is a kind of nonlinear subdiffusion convection equation occurring in several physical problems such as unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe, propagation of weak shock, compressible turbulence shallow-water waves, shock waves in a viscous medium, waves in bubbly liquids, and electromagnetic waves [1, 4, 8]. Till now, there have been several analytical techniques developed to solve the time fractional Burgers equation. These methods include the Cole-Hopf transformation, Laplace transform, variable separation method [8], Adomian decomposition method [4], homotopy analysis method [6], and so on.

However, even if some fractional differential equations can be solved, the expressions of their exact solutions are often expressed by special functions, which are difficult to apply in practice. Moreover, due to the nonlocality of the fractional operators, analytical methods do not always work well on most fractional differential equations in real applications. Hence, it is of great importance to develop reliable and efficient numerical methods for solving fractional differential equations. Nowadays, the numerical methods cover the quadratic B-spline Galerkin method [2], cubic B-spline finite element method [3], finite difference methods [1, 9–13], and Fourier pseudospectral schemes [14]. There are also some other numerical methods (see, for example, [8, 15–17]).

In this paper, we first present a fully discrete scheme for solving the time fractional Burgers equation with the time approximated by the $L1$-type formula and the space discretization based on the multiscale Galerkin method. We give rigorous convergence analysis for the fully discrete scheme, which shows that the scheme enjoys the optimal convergence order $\mathcal{O}(\tau^{2-\alpha}+h^r)$ in the L^2 norm and $\mathcal{O}(\tau^{2-\alpha}+h^{r-1})$ in the H^1 norm, respectively, where τ,h, and r are the time step size, space step size, and the order of piecewise polynomial space, respectively. Since the time fractional Burgers equation is a nonlinear differential equation, the fully discrete scheme results in a system of nonlinear algebraic equation at each time step. Iteration methods such as the Newton iteration method and the quasi-Newton iteration method are often employed to solve these nonlinear equations. In this case, a large amount of computational effort is demanded to compute and update the Jacobian matrix in each iteration process. The higher accuracy of the approximate solution is required, the larger dimension of the subspace is needed, and the longer computational time is consumed. To overcome this problem, we develop the multilevel augmentation method for solving the fully discrete scheme. The MAM solves a nonlinear equation at a high level consisting of two parts: solving the nonlinear equation only in a fixed initial subspace with the dimension much lower than that of the whole approximate subspace; compensating the error by matrix-vector multiplications at the high level. The MAM reduces the computational costs significantly and leads to a fast solution for the fully discrete scheme. We prove that the MAM preserves the same optimal convergence order as the original fully discrete scheme. The idea of MAM was first introduced in [18] for solving the linear Fredholm integral equations of the second kind. The theoretical setting of MAM was established by Chen et al. in [19] for solving operator equations covering both first kind and second kind equations; they further develop MAM for solving the nonlinear Hammerstein integral equation in [20]. We modified the framework and extended the idea of MAM to solve general nonlinear operator equations of the second kind and applied it to the Sine-Gordon equation in [21]. Readers are referred to [22–27] and the references therein for more applications of MAM.

This paper is organized in seven sections. In "Preliminaries," some necessary notations, multiscale orthonormal bases in Sobolev space, and useful lemmas are introduced. In "$L1$ Scheme for Discretization of Caputo Derivative in Time," we introduce the $L1$-formula for time discretization. In "Fully Discrete Scheme and Convergence," a fully discrete scheme for time fractional Burgers equation is established, and the convergence analysis are given. The MAM and its convergence analysis are developed in "Multilevel Augmentation Method for Solving the Fully Discrete Scheme." The numerical experiments are provided in "Numerical Experiments" to verify the theoretical estimates. Finally, a conclusion is included in "Conclusion."

2. Preliminaries

Denote $I=[0,1]$. Let $(\cdot,\cdot)$ stand for the inner product on the space $L^2(I)$ with the L^2 norm $\|\cdot\|_2$. We denote by $H_0^1(I)$ the Sobolev space of elements u satisfying the homogeneous boundary conditions that $u(0)=u(1)=0$. The inner product and norm of $H_0^1(I)$ are defined by

$$\begin{aligned}\langle u,v\rangle &:= \left(u',v'\right)=\int_0^1 u'(x)v'(x)\mathrm{d}x,\ u,v\in H_0^1(I),\\ |u|_1 &:= \sqrt{\langle u,u\rangle},\ u\in H_0^1(I),\end{aligned} \tag{4}$$

respectively. Let n be a positive integer, we denote by $\mathbb{X}_n$ the subspace of $H_0^1(I)$ whose elements are the piecewise polynomials of order r with knots $j/2^n$, $j-1\in\mathbb{Z}_{2^n-1}$, where the notation $\mathbb{Z}_n:=\{0,1,2\cdots,n-1\}$. Obviously, the sequence of $\mathbb{X}_n$ is nested, that is

$$\mathbb{X}_n\subset\mathbb{X}_{n+1},\ n\in\mathbb{N}_0:=\{0,1,2\cdots\}, \tag{5}$$

which yields the following decomposition:

$$\mathbb{X}_n=\mathbb{X}_{n-1}\oplus^{\perp}\mathbb{W}_n=\mathbb{X}_0\oplus^{\perp}\mathbb{W}_1\oplus^{\perp}\mathbb{W}_2\oplus^{\perp}\cdots\oplus^{\perp}\mathbb{W}_n, \tag{6}$$

where $\mathbb{W}_n$ is the orthogonal complement of $\mathbb{X}_{n-1}$ in $\mathbb{X}_n$.

It is easily concluded from the definition of $\mathbb{X}_n$ and $\mathbb{W}_n$ that the dimensions of $\mathbb{X}_n$ and $\mathbb{W}_n$ are given by $x(n):=\dim(\mathbb{X}_n)=(r-1)2^n-1$ and $w(n):=\dim(\mathbb{W}_n)=x(n)-x(n-1)=(r-1)2^{n-1}$,respectively.

Define two affine mappings on the interval I by $\varphi_0(x)=x/2$ and $\varphi_1(x)=x+1/2, x\in I$, which map the interval $[0,1]$ into $[0,1/2]$ and $[1/2,1]$, respectively. Associated with the two mappings, we introduce two linear operators as follows:

$$\begin{aligned}(\mathcal{T}_0u)(x) &:= \frac{\sqrt{2}}{2}u\left(\varphi_0^{-1}(x)\right)=\frac{\sqrt{2}}{2}u(2x),\\ (\mathcal{T}_1u)(x) &:= \frac{\sqrt{2}}{2}u\left(\varphi_1^{-1}(x)\right)=\frac{\sqrt{2}}{2}u(2x-1).\end{aligned} \tag{7}$$

Lemma 1 (see [26]). *Let w_{ij},$j\in\mathbb{Z}_{w(i)}$, be an orthonormal basis of $\mathbb{W}_i$,$i\geq 1$. Then the functions $\{\mathcal{T}_0w_{ij},\mathcal{T}_1w_{ij}: j\in\mathbb{Z}_{w(i)}\}$ form an orthonormal basis for $\mathbb{W}_{i+1}$.*

Lemma 1 shows that the space $\mathbb{W}_n$ can be recursively constructed by the linear operators $\mathcal{T}_0$ and $\mathcal{T}_1$ once $\mathbb{W}_1$ has been given. Therefore, the basis of the space $\mathbb{X}_n$ can be

Let k, m be two fixed positive integers and $n := k + m$.
Step 1: obtain the approximation of initial value function
$u_n^0 := \mathscr{P}_n u_0(x) = \sum_{(i,j)\in J_n} c_{ij} w_{ij}(x)$,
where $c_{ij} = \langle u_0(x), w_{ij}\rangle_1, (i,j) \in J_n$.
Step 2: for $i = 1 : N$ $(T = N\tau)$, do the following:
(I): Solve $u_k^i \in \mathbb{X}_k$ from (33) with $n := k$. Set $u_{k,0}^i := u_k^i, l = 1$
(II): Compute $u_{k,l}^{i,H} := (\mathscr{P}_{k+l} - \mathscr{P}_k)(\tilde{f}^i - \mathscr{K} u_{k,l-1}^i)$, that is
$u_{k,l}^{i,H} = \sum_{(i,j)\in J_{k,m}} c_{ij} w_{ij}(x)$,
where $J_{k,m} = J_{k+m} \setminus J_k, c_{ij} = \langle \tilde{f}^i - \mathscr{K} u_{k,l-1}^i, w_{ij}\rangle_1$ with
$\langle \tilde{f}^i, w_{ij}\rangle_1 := (f^i, w_{ij}) + \mu(r_{k,m}^i, w_{ij})$.
(III): solve $u_{k,l}^{i,L} \in \mathbb{X}_k$ from the following equation
$\mathscr{P}_k(\mathscr{I} + \mathscr{K})(u_{k,l}^{i,L} + u_{k,l}^{i,H}) = \mathscr{P}_k \tilde{f}^i$,
that is
$\langle (\mathscr{I} + \mathscr{K})(u_{k,l}^{i,L} + u_{k,l}^{i,H}), w_{ij}\rangle_1 = \langle \tilde{f}^i, w_{ij}\rangle_1$, for all $(i,j) \in J_k$.
(IV): Let $u_{k,l}^i := u_{k,l}^{i,L} + u_{k,l}^{i,H}$. Set $l \leftarrow l + 1$ and go back to (III) until $l = m$..$\cdots$

ALGORITHM 1: (MAM for time fractional Burgers equation).

constructed by Lemma 1 step by step. For the details of the construction and more, the readers can refer to [26].

Let $\mathscr{P}_n$ be an orthogonal projection operator from $H_0^1(I)$ into $\mathbb{X}_n$ with respect to the inner product $\langle \cdot, \cdot \rangle$, that is, for all $u \in H_0^1(I)$,

$$\langle \mathscr{P}_n u, v\rangle = \langle u, v\rangle, v \in \mathbb{X}_n, \tag{8}$$

or

$$(\partial_x \mathscr{P}_n u, \partial_x v) = (\partial_x u, \partial_x v), v \in \mathbb{X}_n. \tag{9}$$

The following approximation results on the operator $\mathscr{P}_n$ will be used later. Throughout this paper, unless stated otherwise, c denotes a generic positive constant whose value may differ in different occurrences.

Lemma 2 (see [28]). *If $u \in H^r(I) \cap H_0^1(I)$, then*

$$\begin{aligned} \|u - \mathscr{P}_n u\|_2 &\le c h^r \|u\|_r, \\ |u - \mathscr{P}_n u|_1 &\le c h^{r-1} \|u\|_r, \end{aligned} \tag{10}$$

where $h := 2^{-n}$.

3. $L1$ Scheme for Discretization of Caputo Derivative in Time

For a positive integer N, let $\tau = T/N$ be the time step size and $t_i = i\tau$ for $i = 0, 1, \cdots, N$. Let u^i be the solution of $u(x,t)$ on $t = t_i$.

Define

$$u^{i+\frac{1}{2}} := \frac{u^{i+1} + u^i}{2} \text{ and } \delta_t u^{i+\frac{1}{2}} := \frac{u^{i+1} - u^i}{\tau}. \tag{11}$$

For the approximation of fractional derivative ${}_0^cD_t^\alpha g(t_i)$, we use the following $L1$ scheme [29, 30]:

$${}_0^cD_t^\alpha g(t_i) \simeq D_\tau^\alpha g(t_i) := \mu\left[a_0 g(t_i) - \sum_{k=1}^{i-1}(a_{i-k-1} - a_{i-k}) g(t_k) - a_{i-1} g(t_0)\right], \tag{12}$$

where $0 < \alpha < 1, \mu = \tau^{-\alpha}/\Gamma(2-\alpha)$ and $a_k = (k+1)^{1-\alpha} - k^{1-\alpha}$.

Lemma 3 (see [30]). *If $0 < \alpha < 1$ and $a_k = (k+1)^{1-\alpha} - k^{1-\alpha}$, $k \in \mathbb{N}_0$, then*

$$\begin{gathered} 1 = a_0 > a_1 > a_2 > \cdots > a_i > \cdots \longrightarrow 0, \text{as } k \longrightarrow \infty, \\ (1-\alpha)(k+1)^{-\alpha} < a_k < (1-\alpha) k^{-\alpha}. \end{gathered} \tag{13}$$

Lemma 4 (see [30]). *Suppose $0 < \alpha < 1, g(t) \in C^2[0, t_i]$. Let*

$$R(g(t_i)) := {}_0^cD_t^\alpha g(t_i) - D_\tau^\alpha g(t_i). \tag{14}$$

Then

$$|R(g(t_i))| \le \frac{1}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha})\right] \max_{0\le t\le t_i} |g''(t)| \tau^{2-\alpha}. \tag{15}$$

4. Fully Discrete Scheme and Convergence

In this section, we present a fully discrete scheme for the time fractional Burgers equation (1), and we derive the error estimates and convergence of the proposed fully discrete scheme. The Galerkin method associated with the multiscale basis introduced in “Preliminaries” is employed to discretize the spatial variable. The fully discrete scheme in weak formulation

Table 1: Errors and convergent orders of MAM in temporal direction for Example 1.

α	τ	Linear basis ($k=3, m=7$) $\lVert u^* - u_{3,7}\rVert_2$	Rate	Quadratic basis ($k=2, m=5$) $\lVert u^* - u_{2,5}\rVert_2$	Rate
1/3	1/4	3.8885e-4		3.8871e-4	
	1/8	1.3023e-4	1.5781	1.3012e-4	1.5789
	1/16	4.2962e-5	1.5999	4.2893e-5	1.6010
	1/32	1.3946e-5	1.6233	1.3985e-5	1.6168
	1/64	4.2759e-6	1.7055	4.5237e-6	1.6283
	1/128	1.3118e-6	1.7047	1.4543e-6	1.6372
1/2	1/4	7.8755e-4		7.8742e-4	
	1/8	2.8852e-4	1.4487	2.8850e-4	1.4486
	1/16	1.0421e-4	1.4692	1.0448e-4	1.4654
	1/32	3.7337e-5	1.4808	3.7550e-5	1.4763
	1/64	1.3340e-5	1.4848	1.3427e-5	1.4837
	1/128	4.5749e-6	1.5439	4.7858e-6	1.4883
3/4	1/4	1.8737e-3		1.8737e-3	
	1/8	7.9862e-4	1.2303	7.9879e-4	1.2300
	1/16	3.3825e-4	1.2394	3.3833e-4	1.2394
	1/32	1.4287e-4	1.2434	1.4285e-4	1.2440
	1/64	6.0343e-5	1.2434	6.0206e-5	1.2465
	1/128	2.5613e-5	1.2363	2.5352e-5	1.2478

for (1) reads as follows: for each $t=t_i, i=1,2\cdots n$, find $u_n^i \in \mathbb{X}_n$ such that

$$\begin{cases} \left(D_\tau^\alpha u_n^i, v_n\right) + \left(u_{nx}^i, v_{nx}\right) + \left(u_n^i u_{nx}^i, v_n\right) = \left(f^i, v_n\right), v_n \in \mathbb{X}_n, \\ u_n^0 = \Pi_n u_0, \end{cases} \tag{16}$$

where $f^i = f(x,t)|_{t=t_i}$ and Π_n denotes the interpolation operator.

We present an optimal error estimate of the fully discrete scheme (16) in the following theorem.

Theorem 5. *Suppose that the problem (1)–(2) has a unique solution $u \in C^2([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^r(\Omega))$. Then*

$$\left\lVert u^i - u_n^i\right\rVert_2 \le c\left(\tau^{2-\alpha} + h^r\right), \tag{17}$$

where $u^i = u(x,t)|_{t=t_i}$.

Proof. Denote $e_n^i := \mathscr{P}_n u^i - u_n^i, i=0,1,2\cdots N$. We conclude from (1) and (16) that e_n^i satisfies

$$\begin{aligned} \left(D_\tau^\alpha e_n^i, v_n\right) + \left(e_{nx}^i, v_{nx}\right) &= \left(D_\tau^\alpha\left(\mathscr{P}_n u^i - u^i\right), v_n\right) \\ &+ \left(D_\tau^\alpha u^i - {}_0^c D_{t_i}^\alpha u, v_n\right) + \left(u^i u_x^i - u_n^i u_{nx}^i, v_n\right). \end{aligned} \tag{18}$$

Taking $v_n = e_n^i$ in (18), we have

$$\begin{aligned} \left(D_\tau^\alpha e_n^i, e_n^i\right) + \left|e_n^i\right|_1^2 &= \left(\left(\mathscr{P}_n - \mathscr{I}\right) D_\tau^\alpha u^i, e_n^i\right) + \left(D_\tau^\alpha u^i - {}_0^c D_{t_i}^\alpha u, e_n^i\right) \\ &+ \left(u^i u_x^i - u_n^i u_{nx}^i, e_n^i\right). \end{aligned} \tag{19}$$

We estimate the terms of the right-hand side of (19) one by one. For the first term in right-hand side of (19), using the Cauchy-Schwarz inequality and Lemma 2, we have

$$\left|\left(\left(\mathscr{P}_n - \mathscr{I}\right) D_\tau^\alpha u^i, e_n^i\right)\right| \le ch^r \left\lVert D_\tau^\alpha u^i\right\rVert_2 \left\lVert e_n^i\right\rVert_2 \le ch^{2r}\left\lVert D_\tau^\alpha u^i\right\rVert_2^2 + \frac{1}{2}\left\lVert e_n^i\right\rVert_2^2. \tag{20}$$

To estimate the second term in the right-hand side of (19), we conclude from Lemma 4 that

$$\left|\left(D_\tau^\alpha u^i - {}_0^c D_{t_i}^\alpha u, e_n^i\right)\right| \le \left\lVert D_\tau^\alpha u^i - {}_0^c D_{t_i}^\alpha u\right\rVert_2 \left\lVert e_n^i\right\rVert_{2 \le c\tau^{4-2r} + \frac{1}{2}\left\lVert e_n^i\right\rVert_2^2}. \tag{21}$$

For the last term in the right-hand side of (19), using integration by parts and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left|\left(u^i u_x^i - u_n^i u_{nx}^i, e_n^i\right)\right| &= \frac{1}{2}\left|\int_I \left(u^i - u_n^i\right)\left(u^i + u_n^i\right) e_{nx}^i \mathrm{d}x\right| \\ &\le M\left\lVert u^i - u_n^i\right\rVert_2 \left|e_n^i\right|_1 \le \frac{M}{2}\left\lVert u^i - u_n^i\right\rVert_2^2 + \left|e_n^i\right|_1^2 \\ &\le \frac{M}{2}\left\lVert u^i - \mathscr{P}_n u^i\right\rVert_2^2 + \frac{M}{2}\left\lVert e_n^i\right\rVert_2^2 + \left|e_n^i\right|_1^2 \le ch^{2r} \\ &+ \frac{M}{2}\left\lVert e_n^i\right\rVert_2^2 + \left|e_n^i\right|_1^2, \end{aligned} \tag{22}$$

where $\lVert u^i\rVert \le M$ and $\lVert u_n^i\rVert \le M$, due to the smoothness of u and the approximation $u_n \in \mathbb{X}_n \subset H_0^1(I)$.

On the other hand

$$\begin{aligned} \left(D_\tau^\alpha e_n^i, e_n^i\right) &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(a_0 e_n^i - \sum_{k=1}^{i-1}\left(a_{i-k-1} - a_{i-k}\right) e_n^k - a_{i-1} e_n^0, e_n^i\right), \\ &= \frac{a_0 \tau^{-\alpha}}{\Gamma(2-\alpha)}\left\lVert e_n^i\right\rVert_2^2 - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{k=1}^{i-1}\left(a_{i-k-1} - a_{i-k}\right) e_n^k + a_{i-1} e_n^0, e_n^i\right). \end{aligned} \tag{23}$$

Substituting (20)–(23) into (19) and noting that $1 = a_0 > a_1 > \cdots > a_i > \cdots$, we deduce that

Table 2: Errors and convergent orders of MAM and DNM via linear basis in spatial direction with $k = 3, \tau = 0.001$, and $\alpha = 0.25$ for Example 1.

m	$x(n)$	MAM				DNM			
		$\lvert u^* - u_{3,m} \rvert_1$	Rate	$\lVert u^* - u_{3,m} \rVert_2$	Rate	$\lvert u^* - u_n \rvert_1$	Rate	$\lVert u^* - u_n \rVert_2$	Rate
0	7	3.1264e-2		1.2161e-3		3.1264e-2		1.2161e-3	
1	15	1.7726e-2	0.8186	3.1564e-4	1.9461	1.6011e-2	0.9654	3.1282e-4	1.9590
2	31	8.9080e-3	0.9927	7.7686e-5	2.0225	8.0530e-3	0.9915	7.8743e-5	1.9901
3	63	4.4596e-3	0.9982	1.9377e-5	2.0033	4.0324e-3	0.9979	1.9714e-5	1.9979
4	127	2.2305e-3	0.9996	4.8450e-6	1.9998	2.0169e-3	0.9995	4.9263e-6	2.0006
5	255	1.1153e-3	0.9999	1.2132e-6	1.9976	1.0086e-3	0.9999	1.2342e-6	1.9970
6	511	5.5768e-4	1.0000	2.8891e-7	2.0702	5.0429e-4	1.0000	3.3678e-7	1.8737

Table 3: Comparison of CPU time between MAM and DNM via linear basis with $k = 3, \tau = 0.001$, and $\alpha = 0.25$ for Example 1.

$n = k + m$	$x(n)$	T_{MAM}	T_{DNM}
$3 = 3 + 0$	7	1.33	1.28
$4 = 3 + 1$	15	3.21	3.70
$5 = 3 + 2$	31	7.09	12.22
$6 = 3 + 3$	63	15.45	41.96
$7 = 3 + 4$	127	41.47	152.71
$8 = 3 + 5$	255	112.52	570.63
$9 = 3 + 6$	511	340.87	4581.64

$$\begin{aligned}\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left\|e_n^i\right\|_2^2 \le{}& \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{k=1}^{i-1}(a_{i-k-1}-a_{i-k})\frac{\|e_n^k\|_2^2+\|e_n^i\|_2^2}{2}+a_{i-1}\frac{\|e_n^0\|_2^2+\|e_n^i\|_2^2}{2}\right)\\ &+\left(\frac{M}{2}+1\right)\left\|e_n^i\right\|_2^2+c\left(\tau^{2-\alpha}+h^r\right)^2\\ ={}& \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\left(\left\|e_n^i\right\|_2^2+\sum_{k=1}^{i-1}\left\|e_n^k\right\|_2^2+\left\|e_n^0\right\|_2^2\right)\\ &+\left(\frac{M}{2}+1\right)\left\|e_n^i\right\|_2^2+c\left(\tau^{2-\alpha}+h^r\right)^2.\end{aligned} \tag{24}$$

Choose τ such that $\tau^{-\alpha}/2\Gamma(2-\alpha) > ((M/2)+1)$, and denote $\sigma = \tau^{-\alpha}/2\Gamma(2-\alpha), \lambda = \tau^{-\alpha}/2\Gamma(2-\alpha) - ((M/2)+1)$; then, we have

$$\left\|e_n^i\right\|_2^2 \le \frac{\sigma}{\lambda}\sum_{k=1}^{i-1}\left\|e_n^k\right\|_2^2 + \frac{\sigma}{\lambda}\left\|e_n^0\right\|_2^2 + c\left(\tau^{2-\alpha}+h^r\right)^2. \tag{25}$$

By Gronwall's inequality, we have

$$\left\|e_n^i\right\|_2 \le c\left\|e_n^0\right\|_2^2 + c\left(\tau^{2-\alpha}+h^r\right), \tag{26}$$

which, together with Lemma 2 and the initial error estimate, yields that

$$\left\|u^i - u_n^i\right\|_2 \le \left\|u^i - \mathscr{P}_n u^i\right\|_2 + \left\|e_n^i\right\|_2 \le c\left(\tau^{2-\alpha}+h^r\right). \tag{27}$$

This completes the proof.

Remark 6. If we choose $v_n = D_\tau^\alpha e_n^i$ in (19) and make a similar analysis as the above Theorem 5, we can obtain the optimal convergence order in H^1 norm

$$\left|u^i - u_n^i\right|_1 \le c\left(\tau^{2-\alpha}+h^{r-1}\right). \tag{28}$$

5. Multilevel Augmentation Method for Solving the Fully Discrete Scheme

At each time step, the fully discrete scheme (16) leads to a nonlinear system, which makes the computational cost expensive. We present a fast multilevel augmentation method in this section to solve these nonlinear systems. To this end, we rewrite (16) into

$$\left(u_{nx}^i, v_{nx}\right) + \left(\mu u_n^i + u_n^i u_{nx}^i, v_n\right) = \left(f^i + \mu r_n^i, v_n\right), \tag{29}$$

where $r_n^i = \sum_{k=1}^{i-1}(a_{i-k-1}-a_{i-k})u_n^k + a_{i-1}u_n^0$ and $\mu = \tau^{-\alpha}/\Gamma(2-\alpha)$.

Define a nonlinear operator $\mathscr{K}: \mathbb{X} \longrightarrow \mathbb{X}$ as follows:

$$\langle \mathscr{K}(u), v\rangle_1 := (\mu u + u u_x, v), v \in \mathbb{X}. \tag{30}$$

Similar to the proof of Lemma 3 in [23], we applied the Riesz representation theorem to the right-hand side of (29); there exists a element $\tilde{f}^i \in \mathbb{X}$, such that

$$\left\langle \tilde{f}^i, v\right\rangle_1 = \left(f^i + \mu r_n^i, v\right), \forall v \in \mathbb{X}_n. \tag{31}$$

Then, Equation (29) can be reformulated as

$$\left\langle u_n^i, v_n\right\rangle_1 + \left\langle \mathscr{K} u_n^i, v_n\right\rangle_1 = \left\langle \tilde{f}^i, v_n\right\rangle, v_n \in \mathbb{X}_n, \tag{32}$$

or equivalently

$$(\mathscr{I} + \mathscr{P}_n\mathscr{K})u_n^i = \mathscr{P}_n\tilde{f}^i. \tag{33}$$

Since Equation (16) has been reformulated as a nonlinear operator equation of the second kind (33), and $\mathscr{K}$ has the properties (P1) and (P2) described in [23], then MAM developed in [21] is applicable.

Table 4: Errors and convergent orders of MAM and DNM via quadratic basis in spatial direction with $k=2,\tau=0.001$, and $\alpha=0.1$ for Example 1.

m	$x(n)$	MAM				DNM			
		$\vert u^*-u_{2,m}\vert_1$	Rate	$\Vert u^*-u_{2,m}\Vert_2$	Rate	$\vert u^*-u_n\vert_1$	Rate	$\Vert u^*-u_n\Vert_2$	Rate
0	7	1.5670e-2		6.0316e-4		1.5670e-2		6.0316e-4	
1	15	4.0566e-3	1.9496	8.1280e-5	2.8916	4.0054e-3	1.9679	7.7216e-5	2.9656
2	31	1.0196e-3	1.9922	1.0218e-5	2.9918	1.0068e-3	1.9922	9.7080e-6	2.9917
3	63	2.5528e-4	1.9979	1.2796e-6	2.9973	2.5203e-4	1.9981	1.2151e-6	2.9980
4	127	6.3843e-5	1.9995	1.5919e-7	3.0070	6.3030e-5	1.9995	1.5096e-7	3.0089
5	255	1.5962e-5	1.9999	2.1784e-8	2.8694	1.5759e-5	1.9999	1.7198e-8	3.1339

Table 5: Comparison of CPU time between MAM and DNM via quadratic basis with $k=2,\tau=0.001$, and $\alpha=0.1$ for Example 1.

$n=k+m$	$x(n)$	T_{MAM}	T_{DNM}
$2=2+0$	7	2.44	2.48
$3=2+1$	15	5.18	8.91
$4=2+2$	31	8.49	31.60
$5=2+3$	63	14.04	128.97
$6=2+4$	127	23.16	518.63
$7=2+5$	255	43.77	2506.75

We now briefly describe the MAM for solving (33). As we presented in "Preliminaries," the approximation subspace sequence is nested, for a fixed positive integer $k,n:=k+m$, m is any nonnegative integer, and we have the following decomposition:

$$\mathbb{X}_{k+m}:=\mathbb{X}_k\oplus^{\perp}\mathbb{W}_{k,m},\ \text{with}\ \mathbb{W}_{k,m}:=\mathbb{W}_{k+1}\oplus^{\perp}\mathbb{W}_{k+2}\oplus^{\perp}\cdots\oplus^{\perp}\mathbb{W}_{k+m}. \tag{34}$$

Now, we are in a position to solve (33) with $n:=k+m$, and k is fixed and smaller than n. Firstly, we solve (33) with $n:=k$ exactly and obtain u_k^i. Next, we obtain an approximation of u_{k+1}^i of (33) with $n:=k+1$. To this end, we decompose $u_{k+1}^i=u_{k+1}^{i,L}+u_{k+1}^{i,H}$, with $u_{k+1}^{i,L}\in\mathbb{X}_k$, $u_{k+1}^{i,H}\in\mathbb{W}_{k+1}$.

With the help of (34), Equation (33) with $n:=k+1$ can be rewritten as an equivalent form as

$$(\mathscr{P}_{k+1}-\mathscr{P}_k)\left(u_{k+1}^{i,L}+u_{k+1}^{i,H}\right)-(\mathscr{P}_{k+1}-\mathscr{P}_k)\mathscr{K}u_{k+1}^i=(\mathscr{P}_{k+1}-\mathscr{P}_k)\tilde{f}^i, \tag{35}$$

$$\mathscr{P}_k(\mathscr{I}-\mathscr{K})\left(u_{k+1}^{i,L}+u_{k+1}^{i,H}\right)=\mathscr{P}_k\tilde{f}^i. \tag{36}$$

Note that

$$(\mathscr{P}_{k+1}-\mathscr{P}_k)\left(u_{k+1}^{i,L}+u_{k+1}^{i,H}\right)=u_{k+1}^{i,H}. \tag{37}$$

Equation (35) becomes

$$u_{k+1}^{i,H}=(\mathscr{P}_{k+1}-\mathscr{P}_k)\left(\tilde{f}^i+\mathscr{K}u_{k+1}^i\right). \tag{38}$$

The u_{k+1}^i in the right-hand side can be approximated by the previous level solution $u_{k,0}^i:=u_k^i$. We compute

$$u_{k,1}^{i,H}=(\mathscr{P}_{k+1}-\mathscr{P}_k)\left(\tilde{f}^i+\mathscr{K}u_{k,0}^i\right)(\in\mathbb{W}_{k+1}). \tag{39}$$

Replace $u_{k+1}^{i,H}$ in (36) by $u_{k,1}^{i,H}$, and solve $u_{k,1}^{i,L}\in\mathbb{X}_k$ from

$$\mathscr{P}_k(\mathscr{I}-\mathscr{K})\left(u_{k,1}^{i,L}+u_{k,1}^{i,H}\right)=\mathscr{P}_k\tilde{f}^i. \tag{40}$$

Let

$$u_{k,1}^i:=u_{k,1}^{i,L}+u_{k,1}^{i,H}, \tag{41}$$

which is an approximation to the solution u_{k+1}^i.

This procedure is repeated m times to obtain an approximation $u_{k,m}^i$ of the solution u_{k+m}^i of (33) with $n=k+m$. The solution $u_{k,m}^i$ is called a multilevel augmentation solution. Since at any step $l=0,1,\cdots m$, we only need to invert the same nonlinear operator $\mathscr{P}_k(\mathscr{I}-\mathscr{K})$ with a fixed small k instead of the nonlinear operator $\mathscr{P}_{k+l}(\mathscr{I}-\mathscr{K})$. This means the algorithm has a high computational efficiency. At every time step, the fully discrete scheme (33) is solved by the MAM, and the whole process can be summarized as the following algorithm:

Theorem 7. *Let u be the exact solution of (1) and $u_{k,m}^i$ be the approximation solution obtained by Algorithm 1. Suppose that the solution of Equation (33) belongs to $H^r(I)$ for $i=1,2,\cdots T/\tau$. Then, there exist a positive integer N such that for all $k\geq N$ and $m\in\mathbb{N}$*

$$\left\|u^i-u_{k,m}^i\right\|_2\leq C\left(\tau^{2-\alpha}+h^r\right). \tag{42}$$

Proof. As stated in [20, 21, 23], $u_{k,m}^i$ is the solution of the equation:

$$(\mathscr{I}+\mathscr{P}_k\mathscr{K})u_{k,m}^i=\mathscr{P}_{k+m}\tilde{f}^i-(\mathscr{P}_{k+m}-\mathscr{P}_k)\mathscr{K}u_{k,l-1}^i, \tag{43}$$

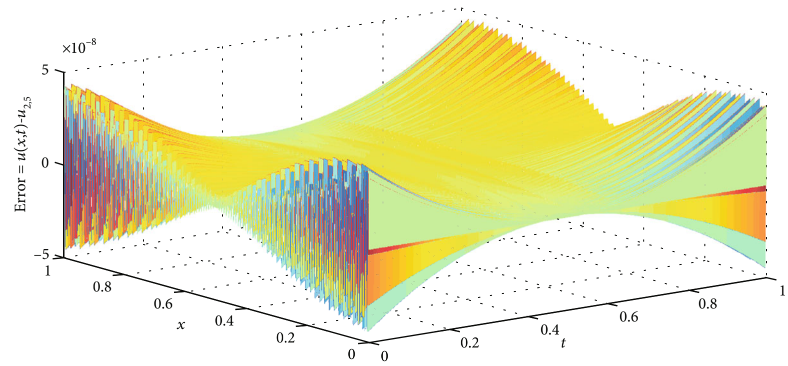


FIGURE 1: The graph of error function $u(x,t) - u_{2,5}(x,t)$ for Example 1.

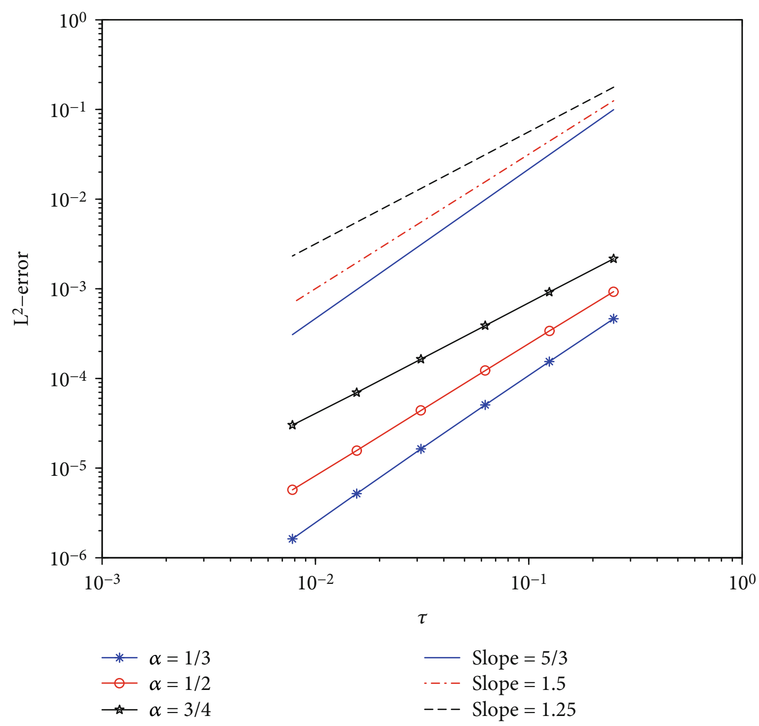


FIGURE 2: The temporal convergence order of MAM for Example 2.

TABLE 6: Errors and convergent orders of MAM and DNM via linear basis in spatial direction with $k = 3, \tau = 0.005$, and $\alpha = 0.25$ for Example 2.

m	$x(n)$	MAM				DNM			
		$\lvert u^* - u_{3,m}\rvert_1$	Rate	$\lVert u^* - u_{3,m}\rVert_2$	Rate	$\lvert u^* - u_n\rvert_1$	Rate	$\lVert u^* - u_n\rVert_2$	Rate
0	7	9.9718e-1		3.8317e-2		9.9718e-1		3.8317e-2	
1	15	5.1444e-1	0.9548	7.8333e-3	2.2903	5.0239e-1	0.9891	9.6573e-3	1.9883
2	31	2.5795e-1	0.9959	1.9554e-3	2.0021	2.5167e-1	0.9973	2.4190e-3	1.9972
3	63	1.2907e-1	0.9990	4.8547e-4	2.0100	1.2589e-1	0.9993	6.0479e-4	1.9999
4	127	6.4546e-2	0.9998	1.2093e-4	2.0052	6.2955e-2	0.9998	1.5096e-4	2.0022
5	255	3.2274e-2	0.9999	2.9983e-5	2.0120	3.1478e-2	1.0000	3.7490e-5	2.0096
6	511	1.6137e-2	1.0000	7.2066e-6	2.0568	1.5739e-2	1.0000	9.1214e-6	2.0392
7	1023	8.0687e-3	1.0000	1.5895e-6	2.1808	7.8696e-3	1.0000	2.0338e-6	2.1651

TABLE 7: Comparison of CPU time between MAM and DNM via linear basis with $k = 3, \tau = 0.005$, and $\alpha = 0.25$ for Example 2.

$n = k + m$	$x(n)$	T_{MAM}	T_{DNM}
$3 = 3 + 0$	7	0.29	0.51
$4 = 3 + 1$	15	0.78	1.08
$5 = 3 + 2$	31	1.67	3.75
$6 = 3 + 3$	63	3.86	13.18
$7 = 3 + 4$	127	9.82	48.16
$8 = 3 + 5$	255	27.78	179.33
$9 = 3 + 6$	511	87.87	716.02
$10 = 3 + 7$	1023	289.36	7004.77

which is equivalent to the following discrete form:

$$\begin{aligned}\left(u^i_{k,m,x}, v_x\right) + \left(\mu u^i_{k,m} + u^i_{k,m}u^i_{k,m,x}, v\right) &= \mu\left(u^i_{k,m} - u^i_{k,m-1}, v_2\right)\\ &\quad + \left(f^i, v\right) + \mu\left(r^i_{k,m}, v\right) + \left(u^i_{k,m}u^i_{k,m,x} - u^i_{k,m-1}u^i_{k,m-1,x}, v_2\right),\end{aligned} \tag{44}$$

where $v \in \mathbb{X}_{k+m}, v_2 = (\mathscr{P}_{k+m} - \mathscr{P}_k)v$. Rearranging the terms, we have

$$\begin{aligned}\left(D^\alpha_\tau u^i_{k,m}, v\right) + \left(u^i_{k,m,x}, v_x\right) &= -\left(u^i_{k,m}u^i_{k,m,x}, v\right)\\ &\quad + \mu\left(u^i_{k,m} - u^i_{k,m-1}, v_2\right) + \left(u^i_{k,m}u^i_{k,m,x} - u^i_{k,m-1}u^i_{k,m-1,x}, v_2\right)\\ &\quad + \left(f^i, v\right).\end{aligned} \tag{45}$$

Noting that the exact solution u at $t = t_i$ satisfies

$$\left(D^\alpha_\tau u^i, v\right) + \left(u^i_x, v_x\right) = -\left(u^i u^i_x, v\right) + \left(f^i, v\right) + \left(D^\alpha_\tau u^i - {}^c_0D^\alpha_{t_i}u, v\right). \tag{46}$$

Subtracting (45) from (46), we obtain that for all $v \in \mathbb{X}_{k+m}, v_2 = (\mathscr{P}_{k+m} - \mathscr{P}_k)v$

$$\begin{aligned}\left(D^\alpha_\tau\left(u^i - u^i_{k,m}\right), v\right) + \left(u^i_x - u^i_{k,m,x}, v_x\right) &= -\left(u^i u^i_x - u^i_{k,m}u^i_{k,m,x}, v\right)\\ &\quad + \left(D^\alpha_\tau u^i - {}^c_0D^\alpha_{t_i}u, v\right) - \mu\left(u^i_{k,m} - u^i_{k,m-1}, v_2\right)\\ &\quad - \left(u^i_{k,m}u^i_{k,m,x} - u^i_{k,m-1}u^i_{k,m-1,x}, v_2\right).\end{aligned} \tag{47}$$

Denote $\rho^i \coloneqq u^i - \mathscr{P}_{k+m}u^i$ and $e^i \coloneqq \mathscr{P}_{k+m}u^i - u^i_{k,m}$, then $u^i - u^i_{k,m} = \rho^i + e^i$. Using these notations and noting that $(\rho^i_x, v_x) = 0$, we derive the error equation as follows:

$$\begin{aligned}\left(D^\alpha_\tau e^i, v\right) + \left(e^i_x, v_x\right) &= \left(D^\alpha_\tau \rho^i, v\right) + \left(D^\alpha_\tau u^i - {}^c_0D^\alpha_{t_i}u, v\right)\\ &\quad - \left(u^i u^i_x - u^i_{k,m}u^i_{k,m,x}, v\right) - \mu\left(u^i_{k,m} - u^i_{k,m-1}, v_2\right)\\ &\quad - \left(u^i_{k,m}u^i_{k,m,x} - u^i_{k,m-1}u^i_{k,m-1,x}, v_2\right).\end{aligned} \tag{48}$$

Let $M_1 = \max\{\|u^i_{k,m}\|_\infty, i \in \mathbb{Z}_N\}$ and $M' = \max\{M_1, M\}$, where M is the positive constant appearing in (22); then, $\|u^i\|_\infty \le M', \|u^i_{k,m}\|_\infty \le M'$. We take $v = e^i$ in (48) and estimate the terms in the right-hand side of (48).

For the first three terms in the right-hand side of (48), similar to the analysis of (20)–(22), we have

$$\left|\left(D^\alpha_\tau \rho^i, e^i\right)\right| = \left|\left((\mathscr{P}_{k+m} - \mathscr{I})D^\alpha_\tau u^i, e^i\right)\right| \le ch^{2r}\left\|D^\alpha_\tau u^i\right\|_2^2 + \frac{1}{2}\left\|e^i\right\|_2^2, \tag{49}$$

$$\left|\left(D^\alpha_\tau u^i - {}^c_0D^\alpha_{t_i}u, e^i\right)\right| \le c\tau^{4-2r} + \frac{1}{2}\left\|e^i\right\|_2^2, \tag{50}$$

$$\left|\left(u^i u^i_x - u^i_{k,m}u^i_{k,m,x}, e^i\right)\right| \le ch^{2r} + M'\left\|e^i\right\|_2^2 + \frac{1}{2}\left|e^i\right|_1^2. \tag{51}$$

By the Cauchy-Schwartz inequality, Young's inequality, and noting that $\|v_2\|_2 \le \|v\|_2$, we have

$$\left|-\mu\left(u^i_{k,m} - u^i_{k,m-1}, v_2\right)\right| \le \frac{\mu^2}{\sigma_1}\left\|u^i_{k,m} - u^i_{k,m-1}\right\|_2^2 + \frac{\sigma_1}{4}\left\|e^i\right\|_2^2. \tag{52}$$

For the last term in the right-hand side of (48), it follows from integration by parts, the Cauchy-Schwartz inequality, and the Young inequality that

$$\begin{aligned}\left|\left(u^i_{k,m}u^i_{k,m,x} - u^i_{k,m-1}u^i_{k,m-1,x}, v_2\right)\right| &\le 2M'\left\|u^i_{k,m} - u^i_{k,m-1}\right\|_2\left|e^i\right|_1\\ &\le M'^2\left\|u^i_{k,m} - u^i_{k,m-1}\right\|_2^2 + \frac{1}{2}\left|e^i\right|_1^2.\end{aligned} \tag{53}$$

On the other hand side, as presented in (23), we have

$$\left(D^\alpha_\tau e^i, e^i\right) \le \frac{\tau^\alpha}{\Gamma(2-\alpha)}\left\|e^i\right\|_2^2 - \frac{\tau^\alpha}{\Gamma(2-\alpha)}\left(\sum_{k=1}^{i-1}(a_{i-k-1} - a_{i-k})e^k + a_{i-1}e^0, e^i\right). \tag{54}$$

Combining (49)–(54) and $(e^i_x, e^i_x) = |e^i|_1^2$, we have

$$\begin{aligned}\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left\|e^i\right\|_2^2 &\le \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{k=1}^{i-1}(a_{i-k-1} - a_{i-k})\frac{\|e^k\|_2^2 + \|e^i\|_2^2}{2} + a_{i-1}\frac{\|e^0\|_2^2 + \|e^i\|_2^2}{2}\right)\\ &\quad + \left(1 + M' + \frac{\sigma_1}{4}\right)\left\|e^i\right\|_2^2 + \left(\frac{\mu^2}{\sigma_1}\right)\left\|u^i_{k,m} - u^i_{k,m-1}\right\|_2^2 + c\left(\tau^{2-\alpha} + h^r\right)^2\\ &= \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\left(\left\|e^i\right\|_2^2 + \sum_{k=1}^{i-1}\left\|e^k\right\|_2^2 + \left\|e^0\right\|_2^2\right)\\ &\quad + \left(1 + M' + \frac{\sigma_1}{4}\right)\left\|e^i\right\|_2^2 + \left(\frac{\mu^2}{\sigma_1}\right)\left\|u^i_{k,m} - u^i_{k,m-1}\right\|_2^2 + c\left(\tau^{2-\alpha} + h^r\right)^2.\end{aligned} \tag{55}$$

Choose τ such that $\tau^{-\alpha}/2\Gamma(2-\alpha) > 1 + M' + (\sigma_1/4)$, and denote $\bar{\sigma} = \tau^{-\alpha}/2\Gamma(2-\alpha), \bar{\mu} \coloneqq \bar{\sigma} - (1 + M' + (\sigma_1/4))$. Then, we have

Table 8: Errors and convergent orders of MAM and DNM via quadratic basis in spatial direction with $k = 2, \tau = 0.005$, and $\alpha = 0.1$ for Example 2.

m	$x(n)$	MAM				DNM			
		$\lvert u^* - u_{2,m}\rvert_1$	Rate	$\lVert u^* - u_{2,m}\rVert_2$	Rate	$\lvert u^* - u_n\rvert_1$	Rate	$\lVert u^* - u_n\rVert_2$	Rate
0	7	3.9445e-1		1.5119e-2		3.9445e-1		1.5119e-2	
1	15	1.0367e-1	1.9279	2.0779e-3	2.8631	1.0124e-1	1.9620	1.9499e-3	2.9549
2	31	2.9991e-2	1.7893	3.8616e-4	2.4279	2.5478e-2	1.9905	2.4561e-4	2.9889
3	63	7.5607e-3	1.9879	5.0534e-5	2.9339	6.3800e-3	1.9976	3.0758e-5	2.9973
4	127	1.8938e-3	1.9972	6.3432e-6	2.9940	1.5957e-3	1.9994	3.8482e-6	2.9987
5	255	4.7367e-4	1.9993	8.1030e-7	2.9687	3.9896e-4	1.9999	5.0488e-7	2.9301

Table 9: Comparison of CPU time between MAM and DNM with $k = 2, \tau = 0.005$, and $\alpha = 0.1$ for Example 2.

$n = k + m$	$x(n)$	T_{MAM}	T_{DNM}
$2 = 2 + 0$	7	0.57	0.57
$3 = 2 + 1$	15	1.33	2.02
$4 = 2 + 2$	31	2.16	7.64
$5 = 2 + 3$	63	3.43	32.32
$6 = 2 + 4$	127	5.89	129.80
$7 = 2 + 5$	255	10.34	543.35

$$\|e^i\|_2^2 \le \frac{\bar{\sigma}}{\bar{\mu}}\left(\sum_{k=1}^{i-1}\left\|e^k\right\|_2^2 + \|e^0\|_2^2\right) + \left(\frac{\tau^{4-2\alpha}}{4\bar{\mu}\Gamma^2(2-\alpha)} + \frac{2M'^2}{\bar{\mu}}\right)\|u_{k,m}^i - u_{k,m-1}^i\|_2^2 + c\left(\tau^{2-\alpha} + h^r\right)^2. \quad (56)$$

When the exact solution of (33) belongs to $H^r(I)$, there exists a positive integer N, for all $k \ge N$ and any $m \in \mathbb{N}$ (see [21, 23]):

$$\|u_{k,m}^i - u_{k,m-1}^i\| \le ch^r. \quad (57)$$

Combining (56) and (57), we conclude from Gronwall's inequality that

$$\begin{aligned}\|e^i\|_2 &\le \|e^0\|_2 + c\left(\tau^{2-\alpha}h^r + h^r\right) + c\left(\tau^{2-\alpha} + h^r\right)\\ &\le \|e^0\|_2 + c\left(\tau^{2-\alpha} + h^r\right).\end{aligned} \quad (58)$$

Noting that $\|e^0\|_2 \le ch^r$, then

$$\|u^i - u_{k,m}^i\|_2 \le \|u^i - \mathscr{P}_{k+m}u^i\|_2 + \|e^i\|_2 \le C\left(\tau^{2-\alpha} + h^r\right). \quad (59)$$

This completes the proof.

Remark 8. If we choose $v = D_\tau^\alpha e^i$ in (48) and make a similar analysis as the above Theorem 7, we can obtain the optimal convergence order in H^1 norm:

$$\left|u^i - u_{k,m}^i\right|_1 \le c\left(\tau^{2-\alpha} + h^{r-1}\right). \quad (60)$$

6. Numerical Experiments

We present in this section numerical examples to illustrate the efficiency and accuracy of our proposed method. The computer programs are run on a personal computer with 2.5G CPU and 8G memory.

Example 1. We consider the time fractional Burgers equation (1) with the exact solution:

$$u(x,t) = \left(4t^2 - 4t + 1\right)x^2(x-1)^2. \quad (61)$$

The corresponding initial condition and forcing term are

$$\begin{aligned} u(x,0) &= u_0(x) = x^2(x-1)^2\\ f(x,t) &= \left(\frac{8t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{4t^{1-\alpha}}{\Gamma(2-\alpha)}\right)x^2(x-1)^2\\ &\quad + 2\left(4t^2 - 4t + 1\right)^2 x^3(x-1)^3(2x-1)\\ &\quad - 2\left(4t^2 - 4t + 1\right)\left(6x^2 - 6x + 1\right). \end{aligned} \quad (62)$$

Both piecewise linear ($r = 2$) and quadratic ($r = 3$) multiscale orthonormal bases introduced in "Preliminaries" are employed in our numerical approximation. The numerical results are reported in Tables 1–5. k and m stand for the numbers of initial level and augmentation level used in the MAM, respectively. $x(n)$ denotes the dimension of approximation subspace $\mathbb{X}_n$ with $n = k + m$. Table 1 shows the L^2 errors and temporal convergence rates for different α using the MAM with $(k,m) = (3,7)$ for linear basis and $(k,m) = (2,5)$ for quadratic basis. It is seen that our numerical scheme has an accuracy of $2 - \alpha$, which is in agreement with our theoretical analysis. In the spatial direction, we illustrate the accuracy, convergence order, and computational efficiency of the MAM, with a comparison to those of the direct Newton's method (DNM) for solving the fully discrete scheme (16). The numerical results listed in Tables 2 and 3 are linear basis cases, and Tables 4 and 5 are quadratic basis cases. We can easily see from these tables that both MAM and DNM have the optimal convergence orders in the H^1 norm (1 for the linear case and 2 for the quadratic case) and in the L^2 norm (2 for the linear case and 3 for the quadratic case). We also observe that MAM and DNM have nearly the same accuracy, while MAM takes significantly less time than DNM. To intuitively show the approximation effect, we plot in Figure 1 the absolute error surface of the approximation solution $u_{2,5}$ obtained by MAM.

Example 2. We consider the time fractional Burgers equation (1) with initial condition:

$$u_0(x) = 0,$$
$$f(x,t) = \frac{2t^{2-\alpha} \sin(2\pi x)}{\Gamma(3-\alpha)} + \pi t^4 \sin(4\pi x) + 4\pi^2 t^2 \sin(2\pi x). \tag{63}$$

The exact solution of this problem is

$$u(x,t) = t^2 \sin(2\pi x). \tag{64}$$

The numerical results are presented in Figure 2 and Tables 6–9, where Figure 2 displays the convergence orders in temporal direction with different α, and Tables 6–9 show the accuracy, convergence order, and computing time for the spatial direction. All the numerical results verify our theoretical analysis and also show the efficiency of the proposed algorithm.

7. Conclusion

In this article, the $L1$-discretization formula and the multiscale Galerkin method are adopted to discrete the Caputo fractional derivative and spatial variable, respectively, and the multilevel augmentation algorithm is proposed for solving the resulting fully discrete scheme which is a nonlinear system at each time step. The MAM only needs to solve nonlinear systems in a fixed subspace with much lower dimension than that for the whole approximation subspace and compensate the error by multiplications of matrices and vectors at the high level. Therefore, the computational cost is greatly reduced. Numerical experiments are presented to confirm our theoretical results. Compared with the DNM, the proposed MAM has substantial advantages in computing time and is suitable for solving large-scale and high-accuracy problems.

Data Availability

The data of numerical simulation used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work has been supported in part by the Natural Science Foundation of China under grant 11501106, 12071160, and 62001115, by the Natural Science Foundation of Guangdong Province under grant 2016A030313835, 2018A0303130067, 2018A030313258, and 2019A1515110136, and by the Opening Project of Guangdong Key Laboratory of Big Data Analysis and Processing under grant 202101.

References

[1] W. Qiu, H. Chen, and X. Zheng, “An implicit difference scheme and algorithm implementation for the one- dimensional time-fractional Burgers equations,” *Mathematics and Computers in Simulation*, vol. 166, pp. 298–314, 2019.

[2] A. Esen and O. Tasbozan, “Numerical solution of time fractional Burgers equation,” *Acta Universitatis Sapientiae, Mathematica*, vol. 7, pp. 167–185, 2015.

[3] A. Esen and O. Tasbozan, “Numerical solution of time fractional Burgers equation by cubic B-spline finite elements,” *Mediterranean Journal of Mathematics*, vol. 13, no. 3, pp. 1325–1337, 2016.

[4] S. Momani, “Non-perturbative analytical solutions of the space- and time-fractional Burgers equations,” *Chaos, Solitons and Fractals*, vol. 28, no. 4, pp. 930–937, 2006.

[5] N. Sugimoto, “Burgers equation with a fractional derivative; hereditary effects on non-linear acoustic waves,” *Journal of Fluid Mechanics*, vol. 225, pp. 31–53, 1991.

[6] A. Yildirim and S. T. Mohyud-Din, “Analytical approach to space- and time-fractional Burgers equations,” *Chinese Physics Letters*, vol. 27, no. 9, article 090501, 2010.

[7] A. Yokus and D. Kaya, “Numerical and exact solutions for time fractional Burgers’ equation,” *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 7, pp. 3419–3428, 2017.

[8] L. Li and D. Li, “Exact solutions and numerical study of time fractional Burgers’ equations,” *Applied Mathematics Letters*, vol. 100, p. 106011, 2020.

[9] J. Liu, Z. Ma, and Z. Zhou, “Explicit and implicit TVD schemes for conservation laws with Caputo derivatives,” *Journal of Scientific Computing*, vol. 72, no. 1, pp. 291–313, 2017.

[10] D. Li, C. Zhang, and M. Ran, “A linear finite difference scheme for generalized time fractional Burgers equation,” *Applied Mathematical Modelling*, vol. 40, no. 11-12, pp. 6069–6081, 2016.

[11] S. Vong and P. Lyu, “Unconditional convergence in maximum-norm of a second-order linearized scheme for a time-fractional Burgers-type equation,” *Journal of Scientific Computing*, vol. 76, no. 2, pp. 1252–1273, 2018.

[12] Q. Zhang, X. Wang, and Z. Sun, “The pointwise estimates of a conservative difference scheme for Burgers’ equation,” *Numerical Methods for Partial Differential Equations*, vol. 36, no. 6, pp. 1611–1628, 2020.

[13] Q. Zhang, Y. Qin, X. Wang, and Z. Sun, “The study of exact and numerical solutions of the generalized viscous Burgers’ equation,” *Applied Mathematics Letters*, vol. 112, p. 106719, 2021.

[14] Z. Asgari and S. M. Hosseini, “Efficient numerical schemes for the solution of generalized time fractional Burgers type equations,” *Numerical Algorithms*, vol. 77, no. 3, pp. 763–792, 2018.

[15] T. S. El-Danaf and A. R. Hadhoud, “Parametric spline functions for the solution of the one time fractional Burgers’ equation,” *Applied Mathematical Modelling*, vol. 36, pp. 4557–4564, 2012.

[16] H. Hassani and E. Naraghirad, "A new computational method based on optimization scheme for solving variable- order time fractional Burgers' equation," *Mathematics and Computers in Simulation*, vol. 162, pp. 1–17, 2019.

[17] B. Jin, B. Li, and Z. Zhou, "Correction of high-order BDF convolution quadrature for fractional evolution equations," *SIAM Journal on Scientific Computing*, vol. 39, no. 6, pp. A3129–A3152, 2017.

[18] Z. Chen, C. A. Micchelli, and Y. Xu, "A multilevel method for solving operator equations," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 2, pp. 688–699, 2001.

[19] Z. Chen, B. Wu, and Y. Xu, "Multilevel augmentation methods for solve operator equations," *Numerical Mathematics*, vol. 14, pp. 31–55, 2005.

[20] Z. Chen, B. Wu, and Y. Xu, "Fast multilevel augmentation methods for solving Hammerstein equations," *SIAM Journal on Numerical Analysis*, vol. 47, no. 3, pp. 2321–2346, 2009.

[21] J. Chen, Z. Chen, and S. Cheng, "Multilevel augmentation methods for solving the sine-Gordon equation," *Journal of Mathematical Analysis and Applications*, vol. 375, no. 2, pp. 706–724, 2011.

[22] J. Chen, "Fast multilevel augmentation method for nonlinear integral equations," *International Journal of Computer Mathematics*, vol. 89, no. 1, pp. 80–89, 2012.

[23] J. Chen, Z. Chen, S. Cheng, and J. Zhan, "Multilevel augmentation methods for solving the Burgers' equation," *Numerical Methods for Partial Differential Equations*, vol. 31, no. 5, pp. 1665–1691, 2015.

[24] J. Chen, M. He, and T. Zeng, "A multiscale Galerkin method for second-order boundary value problems of Fredholm integro-differential equation II: efficient algorithm for the discrete linear system," *Journal of Visual Communication and Image Representation*, vol. 58, pp. 112–118, 2019.

[25] X. Chen, Z. Chen, B. Wu, and Y. Xu, "Fast multilevel augmentation methods for nonlinear boundary integral equations," *SIAM Journal on Numerical Analysis*, vol. 49, no. 6, pp. 2231–2255, 2011.

[26] Z. Chen, B. Wu, and Y. Xu, "Multilevel augmentation methods for differential equations," *Advances in computational mathematics*, vol. 24, no. 1-4, pp. 213–238, 2006.

[27] Z. Chen, Y. Xu, and H. Yang, "A multilevel augmentation method for solving ill-posed operator equations," *Inverse Problems*, vol. 22, no. 1, pp. 155–174, 2006.

[28] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, 1997.

[29] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1999.

[30] Z. Sun and X. Wu, "A fully discrete difference scheme for a diffusion-wave system," *Applied Numerical Mathematics*, vol. 56, no. 2, pp. 193–209, 2006.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 5581634, 8 pages
https://doi.org/10.1155/2021/5581634

Research Article

Exponential Stability of Swelling Porous Elastic with a Viscoelastic Damping and Distributed Delay Term

Abdelbaki Choucha,[1] Salah Mahmoud Boulaaras ,[2,3] Djamel Ouchenane,[4] Bahri Belkacem Cherif ,[5] and Mohamed Abdalla[6,7]

[1]*Laboratory of Operator Theory and PDEs: Foundations and Applications, Department of Mathematics, Faculty of Exact Sciences, University of El Oued, B.P. 789, El Oued 39000, Algeria*
[2]*Department of Mathematics, College of Sciences and Arts, Ar Rass, Qassim University, Saudi Arabia*
[3]*Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Algeria*
[4]*Laboratory of Pure and Applied Mathematics, Amar Teledji Laghouat University, Algeria*
[5]*Preparatory Institute for Engineering Studies in Sfax, Tunisia*
[6]*Mathematics Department, College of Science, King Khalid University, Abha 61413, Saudi Arabia*
[7]*Mathematics Department, Faculty of Science, South Valley University, Qena 83523, Egypt*

Correspondence should be addressed to Bahri Belkacem Cherif; bahi1968@yahoo.com

Received 3 February 2021; Revised 13 February 2021; Accepted 18 February 2021; Published 28 February 2021

Academic Editor: Qifeng Zhang

In this paper, we consider a swelling porous elastic system with a viscoelastic damping and distributed delay terms in the second equation. The coupling gives new contributions to the theory associated with asymptotic behaviors of swelling porous elastic soils. The general decay result is established by the multiplier method.

1. Introduction and Preliminaries

In the late 19th century, Eringen [1] proposed a theory in which he presented a mixture of viscous liquids and elastic solids in addition to gas. And he also studied the equilibrium laws for all components of this mixture, and finally, you get the field equations for a heat conductive mixture (for more details, see [2]). In [3], the author has classified expansive (swelling) soils under the classification of porous media theory.

On the other hand, it contains clay minerals that attract and absorb water, which leads to an increase in pressure [4], and this is considered a harmful and dangerous problem in architecture and civil engineering in most countries of the world, especially in foundations, which leads to cracks in buildings and ripples in sidewalks and roads (see [5–8]). From there, studies began to eliminate or reduce the damage, as in ([9–13]), where the basic field equations of the linear theory of swelling porous elastic soils were presented by

$$\rho_u u_{tt} = P_{1x} + G_1 + H_1, \tag{1}$$

$$\rho_\phi \phi_{tt} = P_{2x} + G_2 + H_2, \tag{2}$$

where u, ϕ are the displacement of the fluid and the elastic solid material. And $\rho_u, \rho_\phi > 0$ are the densities of each constituent. The functions (P_1, G_1, H_1) represent the partial tension, internal body forces, and eternal forces acting on the displacement, respectively. Similarly (P_2, G_2, H_2), it works on the elastic solid. In addition, the constitutive equations of partial tensions are given by

$$\begin{pmatrix} P_1 \\ P_1 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1, a_2 \\ a_2, a_3 \end{pmatrix}}_{A} \cdot \begin{pmatrix} u_x \\ \phi_x \end{pmatrix}, \tag{3}$$

where $a_1, a_3 > 0$ and $a_2 \neq 0$ is a real number. A is a matrix positive definite in the sense that $a_1 a_3 > a_2^2$.

Quintanilla [10] investigated (1) by taking

$$G_1 = G_2 = \xi(u_t - \phi_t),$$
$$H_1 = a_3 u_{xxt},\tag{4}$$
$$H_2 = 0,$$

where $\xi > 0$; they obtained that the stability is exponential. Similarly, in [14], the authors considered (1) with different conditions

$$G_1 = G_2 = 0,$$
$$H_1 = -\rho_u \gamma(x) u_t,$$
$$H_2 = 0,\tag{5}$$

where $\gamma(x)$ is an internal viscous damping function with a positive mean. They established the exponential stability result (see ([10–20]) for some other interesting results on the swelling porous system).

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences.

In recent years, the control of PDEs with time delay effects has become an active area of research (see, for example, [15, 20–27]). In many cases, it was shown that delay is a source of instability unless additional condition or control terms are used; the stability issue of systems with delay is of theoretical and practical great importance.

A complement to these works, and by introducing the terms of memory and distributed delay, forms a new problem different from previous studies. Under appropriate assumptions and by using the energy method, we prove the stability results.

In this paper, we are interested in problem (1) with null internal body forces, but the eternal force acting only on the elastic solid is in the form of viscoelastic damping and distributed delay terms, that is,

$$G_1 = G_2 = H_1 = 0,$$
$$H_2 = -\int_0^t g(t-s)\phi_{xx}(x,s)ds - \beta_1\phi_t - \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\phi_t(x, t-\sigma)d\sigma.\tag{6}$$

Remark 1. Regarding the problems of swelling porous elastic, we believe that there are no studies of viscoelasticity (the memory) and the distributed delay conditions that act as a simultaneous dissipation mechanism, and hence, our coupling constitutes a new contribution.

Thus, we are interested in the following problem:

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2 \phi_{xx} = 0, \\ \rho_\phi \phi_{tt} - a_3 \phi_{xx} - a_2 u_{xx} + \int_0^t g(t-s)\phi_{xx}(x,s)ds + \beta_1\phi_t + \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\phi_t(x,t-\sigma)d\sigma = 0, \end{cases}\tag{7}$$

where

$$(x, \sigma, t) \in \mathscr{H} = (0,1) \times (\tau_1, \tau_2) \times (0, \infty),\tag{8}$$

under the initial and boundary conditions

$$\begin{gathered} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), x \in (0,1), \\ \phi_t(x,-t) = f_0(x,t), x \in (0,1) \times (0,\tau_2), \\ u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0, t \geq 0. \end{gathered}\tag{9}$$

First, as in [27], taking the following new variable

$$\mathscr{Y}(x,\rho,\sigma,t) = \phi_t(x, t-\sigma\rho),\tag{10}$$

then we obtain

$$\begin{cases} \sigma\mathscr{Y}_t(x,\rho,\sigma,t) + \mathscr{Y}_\rho(x,\rho,\sigma,t) = 0, \\ \mathscr{Y}(x,0,\sigma,t) = \phi_t(x,t). \end{cases}\tag{11}$$

Consequently, the problem is equivalent to

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2 \phi_{xx} = 0, \\ \rho_\phi \phi_{tt} - a_3 \phi_{xx} - a_2 u_{xx} + \int_0^t g(t-s)\phi_{xx}(x,s)ds + \beta_1\phi_t + \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathscr{Y}(x,1,\sigma,t)d\sigma = 0, \\ \sigma\mathscr{Y}_t(x,\rho,\sigma,t) + \mathscr{Y}_\rho(x,\rho,\sigma,t) = 0, \end{cases}\tag{12}$$

where

$$(x, \rho, \sigma, t) \in (0, 1) \times \mathscr{H}, \tag{13}$$

with the initial data

$$\begin{cases} u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \\ \phi(x, 0) = \phi_0(x), \phi_t(x, 0) = \phi_1(x), x \in (0, 1), \\ \mathscr{Y}(x, \rho, \sigma, 0) = f_0(x, \rho\sigma), (x, \rho, \sigma) \in (0, 1) \times (0, 1) \times (0, \tau_2), \end{cases} \tag{14}$$

and the boundary conditions

$$u(0, t) = u(1, t) = \phi(0, t) = \phi(1, t) = 0, t \geq 0. \tag{15}$$

Here, $\rho_u, \rho_\phi, a_1, a_3, \beta_1$ are positive constants and a_2 is a real number, with a_1, a_2, a_3 satisfying $a = a_3 - a_2^2/a_1 > 0$. The integrals represent the memory and the distributed delay terms with $\tau_1, \tau_2 > 0$ are a time delay, β_2 is an L^∞ function, and the kernel g is the relaxation function, under the following assumptions.

(H1) $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is a nonincreasing function satisfying

$$g(0) > 0, \ a - \int_0^\infty g(s)ds = l > 0, \tag{16}$$

where $a = a_3 - a_2^2/a_1 > 0$.

(H2) There exists a $\vartheta \in (\mathbb{R}_+, \mathbb{R}_+)$ positive nonincreasing differentiable function, such that

$$g'(t) \leq -\vartheta(t)g(t), \quad t \geq 0. \tag{17}$$

(H3) $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| d\sigma < \beta_1. \tag{18}$$

Remark 2. The results that we obtained in this work are also correct with other conditions, including

$$\begin{aligned} &u_x(0, t) = u_x(1, t) = \phi_x(0, t) = \phi_x(1, t) = 0, \quad t \geq 0, \\ &u(0, t) = u_x(1, t) = \phi(0, t) = \phi_x(1, t) = 0, \quad t \geq 0, \\ &u_x(0, t) = u(1, t) = \phi_x(0, t) = \phi(1, t) = 0, \quad t \geq 0. \end{aligned} \tag{19}$$

Of course, there can be some difficulties with regard to the following boundary conditions:

$$\begin{aligned} &u_x(0, t) = u_x(1, t) = \phi(0, t) = \phi(1, t) = 0, \quad t \geq 0, \\ &u(0, t) = u(1, t) = \phi_x(0, t) = \phi_x(1, t) = 0, \quad t \geq 0, \end{aligned} \tag{20}$$

unless we assume

$$\int_0^1 u_0(x)dx = 0, \int_0^1 \phi_0(x)dx = 0, \tag{21}$$

respectively.

In this paper, we consider $(u, \phi, \mathscr{Y})$ to be a solution of system (12)–(15) with the regularity needed to justify the calculations. In Section 2, we proved our decay result. And we symbolize that c is a positive constant.

2. Main Result

In this section, we prove our stability result for the energy of system (12)–(15).

We need the following lemmas.

Lemma 3. *The energy functional E, defined by*

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 &\left[\rho_u u_t^2 + a_1 u_x^2 + \rho_\phi \phi_t^2 \right. \\ &\left. + \left(a_3 - \int_0^t g(s)ds\right)\phi_x^2 + 2a_2 u_x\phi_x\right] dx + \frac{1}{2} g \circ \phi_x \\ &+ \frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathscr{Y}^2(x, \rho, \sigma, t)d\sigma d\rho dx, \end{aligned} \tag{22}$$

satisfies

$$\begin{aligned} E'(t) &\leq \frac{1}{2}g' \circ \phi_x - \frac{1}{2}g(t)\int_0^1 \phi_x^2 dx \\ &\quad - \left(\beta_1 - \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| d\sigma\right)\int_0^1 \phi_t^2 dx \\ &\leq \frac{1}{2}g' \circ \phi_x - \eta_0\int_0^1 \phi_t^2 dx \leq 0, \end{aligned} \tag{23}$$

where $\eta_0 = \beta_1 - \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|d\sigma > 0$ and

$$(g \circ v_x)(t) = \int_0^1 \int_0^t g(t-s)(v_x(t) - v_x(s))^2 ds dx. \tag{24}$$

Proof. Multiplying $(12)_{1,2}$ by u_t and ϕ_t, then integration by parts over $(0, 1)$, with (15), gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_u u_t^2 + a_1 u_x^2 + \rho_\phi \phi_t^2 + a_3\phi_x^2 + 2a_2 u_x\phi_x\right] dx \\ &\quad + \beta_1\int_0^1 \phi_t^2 dx + \int_0^1 \phi_t\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathscr{Y}(x, 1, \sigma, t)d\sigma dx \\ &\quad - \int_0^1 \phi_{xt}\int_0^t g(t-s)\phi_x(s)ds dx = 0. \end{aligned} \tag{25}$$

The estimate of the last term in the LHS of (25) is as follows:

$$\begin{aligned}
&-\int_0^1 \phi_{xt}\int_0^t g(t-s)\phi_x(s)dsdx \\
&\quad= \int_0^1 \phi_{xt}\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))dsdx \\
&\quad\quad-\int_0^t g(s)ds\int_0^1 \phi_{xt}\phi_x dx \\
&\quad= \frac{1}{2}\frac{d}{dt}g\circ\phi_x - \frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\int_0^1 \phi_x^2 dx \\
&\quad\quad-\frac{1}{2}g'\circ\phi_x + \frac{1}{2}g(t)\int_0^1 \phi_x^2 dx.
\end{aligned} \tag{26}$$

Now, multiplying $((12))_3$ by $\mathcal{Y} \mid \beta_2(\sigma) \mid$, and by integration over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx \\
&\quad= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathcal{Y}\mathcal{Y}_\rho(x,\rho,\sigma,t)d\sigma d\rho dx \\
&\quad= -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\frac{d}{d\rho}\mathcal{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx \\
&\quad= \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\left(\mathcal{Y}^2(x,0,\sigma,t)-\mathcal{Y}^2(x,1,\sigma,t)\right)d\sigma dx \\
&\quad= \frac{1}{2}\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|d\sigma\int_0^1 \phi_t^2 dx - \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\sigma_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned} \tag{27}$$

Now, by substituting (26) into (25), and using Young's inequality, we have

$$\begin{aligned}
E'(t) &\le \frac{1}{2}g'\circ\phi_x - \frac{1}{2}g(t)\int_0^1 \phi_x^2 dx \\
&\quad-\left(\beta_1 - \int_{\tau_1}^{\tau_2} \mid \beta_2(\sigma) \mid d\sigma\right)\int_0^1 \phi_t^2 d \\
&\le \frac{1}{2}g'\circ\phi_x - \left(\beta_1 - \int_{\tau_1}^{\tau_2} \mid \beta_2(\sigma) \mid d\sigma\right)\int_0^1 \phi_t^2 dx,
\end{aligned} \tag{28}$$

then, by (18), there exists $\eta_0 > 0$ so that

$$E'(t) \le \frac{1}{2}g'\circ\phi_x - \eta_0\int_0^1 \phi_t^2 dx, \tag{29}$$

then we obtain (22) and (23) (E is a nonincreasing function).

Lemma 4. *The functional*

$$D_1(t) := \rho_\phi\int_0^1 \phi_t\phi dx - \frac{a_2}{a_1}\rho_u\int_0^1 \phi u_t dx + \frac{\beta_1}{2}\int_0^1 \phi^2 dx \tag{30}$$

satisfies

$$\begin{aligned}
D_1'(t) &\le -\frac{a_0}{2}\int_0^1 \phi_x^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx \\
&\quad+ cg\circ\phi_x + c\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned} \tag{31}$$

Proof. Direct computation using integration by parts and Young's inequality, for $\varepsilon_1 > 0$, yields

$$\begin{aligned}
D_1'(t) &= -a_3\int_0^1 \phi_x^2 dx + \rho_\phi\int_0^1 \phi_t^2 dx + \frac{a_2^2}{a_1}\int_0^1 \phi_x^2 dx \\
&\quad+\frac{a_2}{a_1}\rho_u\int_0^1 \phi_t u_t dx + \int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx \\
&\quad-\int_0^1 \phi\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathcal{Y}(x,1,\sigma,t)d\sigma dx \\
&\le -\left(a_3 - \frac{a_2^2}{a_1}\right)\int_0^1 \phi_x^2 dx + \rho_\phi\int_0^1 \phi_t^2 dx \\
&\quad+\frac{a_2}{a_1}\rho_u\int_0^1 \phi_t u_t d - \int_0^1 \phi\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathcal{Y}(x,1,\sigma,t)d\sigma dx \\
&\quad+\int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx.
\end{aligned} \tag{32}$$

The estimate of the two last terms in the RHS of (32) is as follows:

$$\begin{aligned}
&\int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx \\
&\quad= \int_0^t g(s)ds\int_0^1 \phi_x^2 dx - \int_0^1 \phi_x\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))dsdx \\
&\quad\le \left(\delta_1 + \int_0^t g(s)ds\right)\int_0^1 \phi_x^2 dx + \frac{1}{4\delta_1}\left(\int_0^t g(s)ds\right)g\circ\phi_x,
\end{aligned} \tag{33}$$

$$\begin{aligned}
&-\int_0^1 \phi\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathcal{Y}(x,1,\sigma,t)d\sigma dx \\
&\quad\le c\delta_2\left(\int_{\tau_1}^{\tau_2} \mid \beta_2(\sigma) \mid d\sigma\right)\int_0^1 \phi_x^2 dx \\
&\quad\quad+\frac{1}{4\delta_2}\int_0^t\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)d\sigma dx,
\end{aligned} \tag{34}$$

where we have used Cauchy-Schwartz, Young's, and Poincaré's inequalities, for $\delta_1, \delta_2 > 0$, and (18).

By substituting (33) and (34) into (32), we find

$$\begin{aligned} D_1'(t) \le& -\left(a_3 - \frac{a_2^2}{a_1} - \int_0^t g(s)ds - \delta_1 - \beta_1 c\delta_2\right)\int_0^1 \phi_x^2 dx \\ &+ \varepsilon_1 \int_0^1 u_t^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx \\ &+ \frac{1}{4\delta_1}\left(\int_0^t g(s)ds\right) g \circ \phi_x \\ &+ \frac{1}{4\delta_2}\int_0^t \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2(x,1,\sigma,t)d\sigma dx. \end{aligned} \tag{35}$$

Bearing in mind that $a = a_3 - a_2^2/a_1 > 0$ and using (16), we get

$$\int_0^t g(s)ds < \int_0^\infty g(s)ds < a, \tag{36}$$

let $a_0 = (a_3 - a_2^2/a_1) - \int_0^t g(s)ds > 0$, and letting $\delta_1 = a_0/4$, $\delta_2 = a_0/4c\mu_1$, gives (31).

Lemma 5. *Assume that (16) hold. Then, the functional*

$$D_2(t) := a_2\left(\int_0^1 \phi_t u dx - \int_0^1 \phi u_t dx\right) \tag{37}$$

satisfies,

$$\begin{aligned} D_2'(t) \le& -\frac{a_2^2}{2\rho_\phi}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx + c\int_0^1 \phi_t^2 dx \\ &+ c g \circ \phi_x + c\int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2(x,1,\sigma,t)d\sigma dx. \end{aligned} \tag{38}$$

Proof. By differentiating D_2, then using (12), integration by parts, and (15), we find

$$\begin{aligned} D_2'(t) =& -\frac{a_2^2}{\rho_\phi}\int_0^1 u_x^2 dx + \frac{a_2^2}{\rho_u}\int_0^1 \phi_x^2 dx \\ &- \left(\frac{a_2 a_3}{\rho_\phi} - \frac{a_1 a_2}{\rho_u}\right)\int_0^1 \phi_x u_x dx - \frac{a_2\beta_1}{\rho_\phi}\int_0^1 u\phi_t dx \\ &+ \frac{a_2}{\rho_\phi}\int_0^1 u_x \int_0^t g(t-s)\phi_x(s)dsd \\ &- \frac{a_2}{\rho_\phi}\int_0^1 u\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}(x,1,\sigma,t)d\sigma dx. \end{aligned} \tag{39}$$

In what follows, we estimate the different terms in the RHS of (39); we use Young's, Cauchy-Schwartz, and Poincaré's inequalities. For $\delta_3, \delta_4, \delta_5 > 0$, we have

$$\begin{aligned} &-\left(\frac{a_2 a_3}{\rho_\phi} - \frac{a_1 a_2}{\rho_u}\right)\int_0^1 \phi_x u_x dx \\ &\quad \le \delta_3 \int_0^1 u_x^2 dx + \left(\frac{a_2 a_3}{\rho_\phi} - \frac{a_1 a_2}{\rho_u}\right)^2 \frac{1}{4\delta_3}\int_0^1 \phi^2 dx, \\ &\frac{a_2}{\rho_\phi}\int_0^1 u_x \int_0^t g(t-s)\phi_x(s)dsdx \\ &\quad \le 2\delta_4 \int_0^1 u_x^2 dx + \frac{c}{4\delta_4}\int_0^1 \phi_x^2 dx + \frac{c}{\delta_4} g \circ \phi_x, \\ &-\frac{a_2\beta_1}{\rho_\phi}\int_0^1 u\phi_t dx \le c\delta_5 \int_0^1 u_x^2 dx + \frac{c}{4\delta_5}\int_0^1 \phi_t^2 dx, \\ &-\frac{a_2}{\rho_\phi}\int_0^1 u\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}(x,1,\sigma,t)d\sigma dx \\ &\quad \le c\delta_6 \int_0^1 u_x^2 dx - \frac{c}{4\delta_6}\int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2(x,1,\sigma,t)d\sigma dx. \end{aligned} \tag{40}$$

By letting $\delta_3 = a_2/8\rho_\phi, \delta_4 = a_2/16\rho_\phi, \delta_5 = \delta_6 = a_2/8c\rho_\phi$ and substituting into (39), we get (38).

Lemma 6. *The functional*

$$D_3(t) := -\rho_u \int_0^1 u_t u dx \tag{41}$$

satisfies

$$D_3'(t) \le -\rho_u \int_0^1 u_t^2 dx + 2a_1 \int_0^1 u_x^2 dx + \frac{a_3}{4}\int_0^1 \phi_x^2 dx. \tag{42}$$

Proof. Direct computations give

$$D_3'(t) = -\rho_u \int_0^1 u_t^2 dx + a_1 \int_0^1 u_x^2 dx + a_2 \int_0^1 u_x \phi_x dx. \tag{43}$$

Estimate (42) easily follows by using Young's inequality and $a_1 a_3 > a_2^2$.

Now, let us introduce the following functional used.

Lemma 7. *The functional*

$$D_4(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho} |\beta_2(\sigma)| \mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx \tag{44}$$

satisfies

$$D_4'(t) \leq -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx + \beta_1 \int_0^1 \phi_t^2 dx - \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)d\sigma dx, \tag{45}$$

where $\eta_1 > 0$.

Proof. By differentiating D_4, with respect to t and using the last equation in (12), we have

$$\begin{aligned} D_4'(t) &= -2\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\sigma\rho}|\beta_2(\sigma)|\mathcal{Y}\mathcal{Y}_\rho(x,\rho,\sigma,t)d\sigma d\rho dx \\ &= -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho}|\beta_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\left[e^{-\sigma}\mathcal{Y}^2(x,1,\sigma,t) - \mathcal{Y}^2(x,0,\sigma,t)\right]d\sigma dx. \end{aligned} \tag{46}$$

By using $\mathcal{Y}(x,0,\sigma,t) = \phi_t(x,t)$, and $e^{-\sigma} \leq e^{-\sigma\rho} \leq 1$, for all $0 < \rho < 1$, we find

$$\begin{aligned} D_4'(t) &= -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\sigma}|\beta_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)d\sigma dx \\ &\quad + \left(\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\, d\sigma\right)\int_0^1 \phi_t^2 dx. \end{aligned} \tag{47}$$

Because $-e^{-\sigma}$ is an increasing function, we have $-e^{-\sigma} \leq -e^{-\tau_2}$, for all $\sigma \in [\tau_1, \tau_2]$.

Finally, setting $\eta_1 = e^{-\tau_2}$ and recalling (18) give (45). We are now ready to prove the main result.

Theorem 8. *Assume (16)–(18) hold.*

Then, $\forall t_0 > 0$*, there exist* $\lambda_1, \lambda_2 > 0$ *such that the energy functional given by (22) satisfies*

$$E(t) \leq \lambda_1 e^{-\lambda_2 \int_{t_0}^t \vartheta(s)ds}, \quad \forall t \geq t_0. \tag{48}$$

Proof. We define the functional of Lyapunov

$$\mathcal{L}(t) := NE(t) + N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t), \tag{49}$$

where $N, N_1, N_2, N_4 > 0$ to be selected later.

By differentiating (49) and using (22), (31), (38), (42), and (45), we have

$$\begin{aligned} \mathcal{L}'(t) &\leq -\left[\frac{a_0 N_1}{2} - cN_2 - \frac{a_3}{4}\right]\int_0^1 \phi_x^2 dx \\ &\quad - [\rho_u - N_1\varepsilon_1]\int_0^1 u_t^2 dx - \left[\frac{a_2^2 N_2}{2\rho_\phi} - 2a_1\right]\int_0^1 u_x^2 dx \\ &\quad + c[N_1 + N_2]g \circ \phi_x + \frac{N}{2} g' \circ \phi_x \\ &\quad - \left[\eta_0 N - cN_1\left(1 + \frac{1}{\varepsilon_1}\right) - N_2 c - \beta_1 N_4\right]\int_0^1 \phi_t^2 dx \\ &\quad - [N_4\eta_1 - cN_1 - cN_2]\int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathcal{Y}^2 \\ &\quad \cdot (x,1,\sigma,t)d\sigma dx - N_4\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathcal{Y}^2 \\ &\quad \cdot (x,\rho,\sigma,t)d\sigma d\rho dx. \end{aligned} \tag{50}$$

By setting

$$\varepsilon_1 = \frac{\rho_u}{2N_1}, \tag{51}$$

we obtain

$$\begin{aligned} \mathcal{L}'(t) &\leq -\left[\frac{a_0 N_1}{2} - cN_2 - \frac{a_3}{4}\right]\int_0^1 \phi_x^2 dx - \left[\frac{\rho_u}{2}\right]\int_0^1 u_t^2 dx \\ &\quad - \left[\frac{a_2^2 N_2}{2\rho_\phi} - 2a_1\right]\int_0^1 u_x^2 dx + c[N_1 + N_2]g \circ \phi_x \\ &\quad + \frac{N}{2} g' \circ \phi_x - [\eta_0 N - cN_1 - N_2 c - \beta_1 N_4]\int_0^1 \phi_t^2 dx \\ &\quad - [N_4\eta_1 - cN_1 - cN_2]\int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathcal{Y}^2 \\ &\quad \cdot (x,1,\sigma,t)d\sigma dx - N_4\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathcal{Y}^2 \\ &\quad \cdot (x,\rho,\sigma,t)d\sigma d\rho dx. \end{aligned} \tag{52}$$

At this stage, we choose our different constants.
First, choosing N_2 large enough such that

$$\alpha_1 = \frac{a_2^2 N_2}{2\rho_\phi} - 2a_1 > 0, \tag{53}$$

then we pick N_1 large enough such that

$$\alpha_2 = \frac{a_0 N_1}{2} - cN_2 - \frac{a_3}{4} > 0, \tag{54}$$

then we select N_4 large enough such that

$$\alpha_3 = N_4\eta_1 - cN_1 - cN_2 > 0. \tag{55}$$

Thus, we arrive at

$$\begin{aligned}\mathscr{L}'(t) \le & -\alpha_2\int_0^1 \phi_x^2 dx - \frac{\rho_u}{2}\int_0^1 u_t^2 dx - \alpha_1\int_0^1 u_x^2 dx \\ & - [\eta_0 N - c]\int_0^1 \phi_t^2 dx + \alpha_6 g' \circ \phi_x + \alpha_7 g \circ \phi_x \\ & - \alpha_3 \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2(x,1,\sigma,t) d\sigma d \\ & - \alpha_5 \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t) d\sigma d\rho dx,\end{aligned} \tag{56}$$

where $\alpha_5 = \eta_1 N_4$, $\alpha_6 = N/2$, $\alpha_7 = c[N_1 + N_2]$.

On the other hand, if we let

$$\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t), \tag{57}$$

then

$$\begin{aligned}|\mathfrak{L}(t)| \le & N_1 \rho_\phi \int_0^1 |\phi\phi_t| dx + N_1 \frac{a_2}{a_1}\rho_u \int_0^1 |\phi u_t| dx \\ & + N_1 \frac{\mu_1}{2}\int_0^1 \phi^2 dx + N_2 a_2 \int_0^1 |\phi u_t - u\phi_t| dx \\ & + \rho_u \int_0^1 |u_t u| \mathrm{d}x + N_4 \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho}|\beta_2(\sigma)|\mathscr{Y}^2 \\ & \cdot (x,\rho,\sigma,t) d\sigma d\rho dx.\end{aligned} \tag{58}$$

Exploiting Young's, Cauchy-Schwartz, and Poincaré inequalities, we obtain

$$\begin{aligned}|\mathfrak{L}(t)| \le & c\int_0^1 \left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2\right) dx + c g \circ \phi_x \\ & + c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t) d\sigma d\rho.\end{aligned} \tag{59}$$

On the other hand, from (22), we can write

$$\begin{aligned}a_1 u_x^2 + 2a_2\phi_x u_x + a_4\phi_x^2 = & \frac{1}{2}\left[a_1\left(u_x + \frac{a_2}{a_1}\phi_x\right)^2 + a_4\left(\phi_x + \frac{a_2}{a_4}u_x\right)^2 \right. \\ & \left. + \left(a_1 - \frac{a_2^2}{a_4}\right)u_x^2 + \left(a_4 - \frac{a_2^2}{a_1}\right)\phi_x^2\right],\end{aligned} \tag{60}$$

where

$$a_4 = a_3 - \int_0^t g(s) ds. \tag{61}$$

Since $a_1 a_3 > a_2^2$ and (16), we deduce that

$$a_1 u_x^2 + 2a_2\phi_x u_x + a_4\phi_x^2 > \frac{1}{2}\left[\left(a_1 - \frac{a_2^2}{a_4}\right)u_x^2 + \left(a_4 - \frac{a_2^2}{a_1}\right)\phi_x^2\right]. \tag{62}$$

Consequently, we find

$$|\mathfrak{L}(t)| = |\mathscr{L}(t) - NE(t)| \le cE(t), \tag{63}$$

that is,

$$(N - c)E(t) \le \mathscr{L}(t) \le (N + c)E(t). \tag{64}$$

At this point, we choose N large enough such that

$$N - c > 0, N\eta_0 - c > 0, \tag{65}$$

and exploiting (22), estimates (56) and (64), respectively, leads to

$$c_2 E(t) \le \mathscr{L}(t) \le c_3 E(t), \quad \forall t \ge 0, \tag{66}$$

$$\mathscr{L}'(t) \le -k_1 E(t) + k_2 g \circ \phi_x, \forall t \ge t_0, \tag{67}$$

for some $k_1, k_2, c_2, c_3 > 0$.

By multiplying (67) by $\vartheta(t)$, we get

$$\vartheta(t)\mathscr{L}'(t) \le -k_1\vartheta(t)E(t) + k_2\vartheta(t) g\circ\phi_x, \forall t \ge t_0. \tag{68}$$

Now, by using (17), we have the following estimate:

$$\begin{aligned}\vartheta(t) g\circ\phi_x & = \vartheta(t)\int_0^1\int_0^t g(t-s)(\phi_x(t) - \phi_x(s))^2 ds dx \\ & \le \int_0^1\int_0^t \vartheta(\mathrm{t}-s) g(t-s)(\phi_x(t) - \phi_x(s))^2 ds d \\ & \le -\int_0^1\int_0^t g'(t-s)(\phi_x(t) - \phi_x(s))^2 ds dx \\ & = -g'\circ\phi_x \le -2E'(t).\end{aligned} \tag{69}$$

Thus, (68) becomes

$$\vartheta(t)\mathscr{L}'(t) \le -k_1\vartheta(t)E(t) - 2k_2 E'(t), \quad \forall t \ge t_0, \tag{70}$$

which can be rewritten as

$$(\vartheta(t)\mathscr{L}(t) + 2k_2 E(t))' - \vartheta'(t)\mathscr{L}(t) \le -k_1\vartheta(t)E(t), \quad \forall t \ge t_0. \tag{71}$$

By using $\vartheta'(t) \le 0, \forall t \ge 0$, we have

$$(\vartheta(t)\mathscr{L}(t) + 2k_2 E(t))' \le -k_1\vartheta(t)E(t), \quad \forall t \ge t_0. \tag{72}$$

By exploiting (66), we notice that

$$\mathscr{K}(t) = \vartheta(t)\mathscr{L}(t) + 2k_2 E(t) \sim E(t). \tag{73}$$

Consequently, for $\kappa > 0$, we get

$$\mathscr{K}'(t) \le -\kappa \mathscr{K}(t)\vartheta(t), \quad \forall t \ge t_0. \tag{74}$$

Integrating (74) over (t_0, t) gives

$$\mathscr{K}(t) \le \mathscr{K}(t_0) e^{-\kappa \int_{t_0}^{t} \vartheta(s) ds}, \quad \forall t \ge t_0. \tag{75}$$

Consequently, (48) is established by virtue of (66) and (75).

Remark 9. The estimate (48) also remains valid for $t \in [0, t_0]$, thanks to the boundedness and continuity of E and ϑ.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

Acknowledgments

The fifth author extends their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through General Research Project under Grant No. (G.R.P-2/42).

References

[1] A. C. Eringen, "A continuum theory of swelling porous elastic soils," *International Journal of Engineering Science*, vol. 32, no. 8, pp. 1337–1349, 1994.

[2] A. Bedford and D. S. Drumheller, "Theories of immiscible and structured mixtures," *International Journal of Engineering Science*, vol. 21, no. 8, pp. 863–960, 1983.

[3] T. K. Karalis, "On the elastic deformation of non-saturated swelling soils," *Acta Mechanica*, vol. 84, no. 1-4, pp. 19–45, 1990.

[4] R. L. Handy, "A stress path model for collapsible loess," in *Genesis and Properties of Collapsible Soils*, pp. 33–47, Springer, Dordrecht, 1995.

[5] J. E. Bowels, *Foundation Design and Analysis*, McGraw Hill Inc., New York, U.S.A, 1988.

[6] V. Q. Hung, *Hidden Disaster*, University of Saska Techwan, Saskatoon, Canada, University News, 2003.

[7] L. D. Jones and I. Jefferson, *Expansive Soils*, ICE Publishing, London, 2012.

[8] B. Kalantari, "Engineering significant of swelling soils," *Research Journal of Applied Sciences, Engineering and Technology*, vol. 4, no. 17, pp. 2874–2878, 2012.

[9] D. Iesan, "On the theory of mixtures of thermoelastic solids," *Journal of Thermal Stresses*, vol. 14, no. 4, pp. 389–408, 1991.

[10] R. Quintanilla, "Exponential stability for one-dimensional problem of swelling porous elastic soils with fluid saturation," *Journal of Computational and Applied Mathematics*, vol. 145, no. 2, pp. 525–533, 2002.

[11] R. Quintanilla, "Existence and exponential decay in the linear theory of viscoelastic mixtures," *European Journal of Mechanics - A/Solids*, vol. 24, no. 2, pp. 311–324, 2005.

[12] R. Quintanilla, "Exponential stability of solutions of swelling porous elastic soils," *Meccanica*, vol. 39, no. 2, pp. 139–145, 2004.

[13] R. Quintanilla, "On the linear problem of swelling porous elastic soils with incompressible fluid," *International Journal of Engineering Science*, vol. 40, no. 13, pp. 1485–1494, 2002.

[14] J. M. Wang and B. Z. Guo, "On the stability of swelling porous elastic soils with fluid saturation by one internal damping," *IMA Journal of Applied Mathematics*, vol. 71, no. 4, pp. 565–582, 2006.

[15] T. A. Apalara, "General stability result of swelling porous elastic soils with a viscoelastic damping," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 71, no. 6, p. 200, 2020.

[16] F. Bofill and R. Quintanilla, "Anti-plane shear deformations of swelling porous elastic soils," *International Journal of Engineering Science*, vol. 41, no. 8, pp. 801–816, 2003.

[17] B. Feng and T. A. Apalara, "Optimal decay for a porous elasticity system with memory," *Journal of Mathematical Analysis and Applications*, vol. 470, no. 2, pp. 1108–1128, 2019.

[18] R. L. Leonard, *Expansive Soils Shallow Foundation*, Kansas, U.S.A, Regent Centre, University of Kansas, 1989.

[19] M. A. Murad and J. H. Cushman, "Thermomechanical theories for swelling porous media with microstructure," *International Journal of Engineering Science*, vol. 38, no. 5, pp. 517–564, 2000.

[20] C. A. Raposo, T. A. Apalara, and J. O. Ribeiro, "Analyticity to transmission problem with delay in porous-elasticity," *Journal of Mathematical Analysis and Applications*, vol. 466, no. 1, pp. 819–834, 2018.

[21] T. A. Apalara, "General decay of solution in one-dimensional porous-elastic system with memory," *Journal of Mathematical Analysis and Applications*, vol. 469, no. 2, pp. 457–471, 2017.

[22] T. A. Apalara, "On the stabilization of a memory-type porous thermoelastic system," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 2, pp. 1433–1448, 2020.

[23] T. A. Apalara, "Uniform decay in weakly dissipative Timoshenko system with internal distributed delay feedbacks," *Acta Mathematica Scientia*, vol. 36, no. 3, pp. 815–830, 2016.

[24] A. Choucha, D. Ouchenane, and S. Boulaaras, "Exponential decay of solutions for a viscoelastic coupled lame system with logarithmic source and distributed delay terms," *Mathematical Methods in the Applied Sciences*, vol. 2020, 2020.

[25] A. Choucha, S. Boulaaras, D. Ouchenane, S. Alkhalf, and B. Cherif, "Stability result and well posedness for Timoshenko's beam laminated with termoelastic and past history distributed delay term," *Fractals*, vol. 29, pp. 1–21, 2021.

[26] A. Choucha, D. Ouchenane, and S. Boulaaras, *Mathematical Methods in the Applied Sciences*, vol. 43, no. 17, 2020Well posedness and stability result for a thermoelastic laminated Timoshenko beam with distributed delay term, 2020.

[27] A. S. Nicaise and C. Pignotti, "Stabilization of the wave equation with boundary or internal distributed delay," *Differential and Integral Equations*, vol. 21, no. 9-10, pp. 935–958, 2008.

Hindawi
Journal of Function Spaces
Volume 2021, Article ID 6683465, 18 pages
https://doi.org/10.1155/2021/6683465

Research Article

Global Existence for Two Singular One-Dimensional Nonlinear Viscoelastic Equations with respect to Distributed Delay Term

Abdelbaki Choucha,[1] Salah Mahmoud Boulaaras ,[2,3] Djamel Ouchenane,[4] and Ali Allahem [5]

[1] *Laboratory of Operator Theory and PDEs: Foundations and Applications, Department of Mathematics, Faculty of Exact Sciences, University of El Oued, El Oued, Algeria*
[2] *Department of Mathematics, College of Sciences and Arts, ArRass, Qassim University, Saudi Arabia*
[3] *Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Algeria*
[4] *Laboratory of Pure and Applied Mathematics, Amar Telidji Laghouat University, Algeria*
[5] *Department of Mathematics, College of Sciences, Qassim University, Saudi Arabia*

Correspondence should be addressed to Salah Mahmoud Boulaaras; s.boularas@qu.edu.sa and Ali Allahem; aallahem@qu.edu.sa

Received 31 December 2020; Revised 24 January 2021; Accepted 2 February 2021; Published 16 February 2021

Academic Editor: Mahmoud A. Zaky

In this current work, we are interested in a system of two singular one-dimensional nonlinear equations with a viscoelastic, general source and distributed delay terms. The existence of a global solution is established by the theory of potential well, and by using the energy method with the function of Lyapunov, we prove the general decay result of our system.

1. Introduction

We are interested in the following system:

$$\begin{cases} u_{tt} - \dfrac{1}{x}(xu_x)_x + \displaystyle\int_0^t g_1(t-s)\dfrac{1}{x}(xu_x(x,s))_x ds + \mu_1 u_t \\ + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\rho)| u_t(x, t-\rho) d\rho = f_1(u,v), \text{ in } Q, \\ v_{tt} - \dfrac{1}{x}(xv_x)_x + \displaystyle\int_0^t g_2(t-s)\dfrac{1}{x}(xv_x(x,s))_x ds + \mu_3 v_t \\ + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_4(\rho)| v_t(x, t-\rho) d\rho = f_2(u,v), \text{ in } Q, \end{cases} \tag{1}$$

with

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,L), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in (0,L), \\ u_t(x,-t) = f_0(x,t), v_t(x,-t) = g_0(x,t), t \in (0,\tau_2), \\ u(L,t) = v(L,t) = 0, \displaystyle\int_0^L xu(x,t)dx = \int_0^L xv(x,t)dx = 0, \end{cases} \tag{2}$$

where $Q = (0,L) \times (0,T)$, $L < \infty$, $T < \infty$, $g_1(.)$, $g_2(.)$: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $\mu_1, \mu_3 > 0$, the second integral represents the distributed delay and $\mu_2, \mu_4 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ are bounded functions, where τ_1, τ_2 are two real numbers satisfying $0 \le \tau_1 < \tau_2$, and $f_1(.,.)$, $f_2(.,.)$: $\mathbb{R}^2 \longrightarrow \mathbb{R}$ are defined functions later.

Three decades ago, these problems that arise in one-dimensional elasticity have been studied and developed with regard to viscosity with long-term memory. And it has been studied in many fields of science, engineering, medical sciences, and chemistry, as well as population and other matters; see, for example, [1–24]. Recently, in the absence of delay ($\mu_i = 0, i = 1..4$), problem (1) was studied in [25], and also later in [26], the authors considered problem (1) with localized frictional damping term. We also know that delay, especially distributed delay, is a phenomenon in our life and is almost found in various fields, and its inclusion in any problem makes it more important. The distributed delay in many works has been studied and many authors have taken care of it, for example, [5, 9, 27, 28]. Based on all this and the results of the research papers [14, 15, 17, 28–30, 31], the introduction of the term distributed delay as a damping mechanism in problem (1) makes it a new problem from what has been previously studied.

And we have divided this paper into the following. We present in the second section the definitions, basics, and theories of function spaces that are required throughout the rest of the paper. In Section 3, we present the energy function while proving to be decreasing. And in the final section, the general decay is obtained by applying the energy method and the function of Lyapunov.

2. Preliminaries

Let $L_x^p = L_x^p((0, L))$ be the weighted Banach space equipped with the norm

$$\|u\|_{L_x^p} = \left(\int_0^L x|u|^p dx\right)^{1/p}, \tag{3}$$

$H = L_x^2((0, L))$ be the Hilbert space of square integral functions having the finite norm

$$\|u\|_H = \left(\int_0^L xu^2 dx\right)^{1/2}, \tag{4}$$

and $K = L_x^2((0, L) \times (0, 1) \times (\tau_1, \tau_2))$ be the Hilbert space equipped with the norm

$$\|z\|_{K,\mu_2} = \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| \|z\|_H d\rho d\rho. \tag{5}$$

$V = V_x^1$ is the Hilbert space equipped with the norm

$$\|u\|_V = \left(\|u\|_H^2 + \|u_x\|_H^2\right)^{1/2}, \tag{6}$$

$$V_0 = \{u \in V \text{ such that } u(L) = 0\}. \tag{7}$$

Theorem 1 [27]. *For $2 < p < 4$ and $\forall v$ in V_0, we have*

$$\int_0^L x|v|^p dx \le C_* \|v_x\|_{H=L_x^2(0,L)}^p, \tag{8}$$

where C_* is a constant depending on L and p only.

As in [18], introducing the new variables

$$\begin{cases} z(x, \rho, \varrho, t) = u_t(x, t - \varrho\rho), \\ y(x, \rho, \varrho, t) = v_t(x, t - \varrho\rho), \end{cases} \tag{9}$$

yields

$$\begin{aligned} &\begin{cases} \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \\ z(x, 0, \varrho, t) = u_t(x, t), \end{cases} \\ &\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \\ y(x, 0, \varrho, t) = v_t(x, t). \end{cases} \end{aligned} \tag{10}$$

Problem (1) arrives at

$$\begin{cases} u_{tt} - \dfrac{1}{x}(xu_x)_x + \displaystyle\int_0^t g_1(t-s)\dfrac{1}{x}(xu_x(x,s))_x ds + \mu_1 u_t \\ + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u_t(x, t-\varrho) d\varrho = f_1(u, v), \\ v_{tt} - \dfrac{1}{x}(xv_x)_x + \displaystyle\int_0^t g_2(t-s)\dfrac{1}{x}(xv_x(x,s))_x ds + \mu_3 v_t \\ + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v_t(x, t-\varrho) d\varrho = f_2(u, v), \\ \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \\ \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \end{cases} \tag{11}$$

where

$$(x, \rho, s, t) \in (0, L) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty). \tag{12}$$

With the initial data and boundary conditions

$$\begin{cases} (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \text{ in } (0, L), \\ (u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)), \text{ in } (0, L), \\ (u_t(x, -t), v_t(x, -t)) = (f_0(x, t), g_0(x, t)), \text{ in } (0, L) \times (0, \tau_2), \\ u(0, t) = u(L, t) = v(0, t) = v(L, t) = 0, \\ z(x, \rho, \varrho, 0) = f_0(x, \rho\varrho), \text{ in } (0, L) \times (0, 1) \times (0, \tau_2), \\ y(x, \rho, \varrho, 0) = g_0(x, \rho\varrho), \\ u(L, t) = v(L, t) = 0, \displaystyle\int_0^L xu(x, t) dx = \int_0^L xv(x, t) dx = 0. \end{cases} \tag{13}$$

We have the following assumptions:

(G1) $g_i(t)$: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are C^1, nonincreasing functions satisfying

$$\begin{cases} g_i(s) \ge 0, \quad g_i'(s) \le 0, \\ g_i(0) > 0, 1 - \displaystyle\int_0^\infty g_i(s) ds = l_i > 0, i = 1, 2, \end{cases} \tag{14}$$

(G2) $\exists \xi(t) > 0$ a differentiable function, such that

$$g_i'(t) \le -\xi(t) g_i^\sigma(t), i = 1, 2, t \ge 0, 1 \le \sigma < \frac{3}{2}, \tag{15}$$

and $\xi(t)$ satisfies for some $l < 1$

$$\xi'(t) \le 0, \left|\frac{\xi'(t)}{\xi(t)}\right| \le l, \int_0^\infty \xi(s) ds = +\infty, \forall t > 0. \tag{16}$$

And also, where $1 < \sigma < 3/2$, $\forall t_0 > 0$ fixed, $\exists C_\sigma(\sigma) > 0$, such that

$$\frac{t}{\left(1+\int_{t_0}^{t}\xi(s)ds\right)^{1/(2(\sigma-1))}} \le C_\sigma, \quad \forall t \ge t_0. \tag{17}$$

(G3) we take

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(r+1)}(u+v) + b|u|^r u|v|^{r+2}, \\ f_2(u,v) &= a|u+v|^{2(r+1)}(u+v) + b|v|^r v|u|^{r+2}, \end{aligned} \tag{18}$$

where $a, b > 0$ and $r > -1$.

We have

$$uf_1(u,v) + vf_2(u,v) = 2(r+2)F(u,v), \forall (u,v) \in \mathbb{R}^2, \tag{19}$$

where

$$F(u,v) = \frac{1}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]. \tag{20}$$

(G4) $\mu_2, \mu_4 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \mu_1, \tag{21}$$

$$\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| d\varrho < \mu_3. \tag{22}$$

Theorem 2. *Assume (14) and $p < 3$. Then, $\forall (u_0, v_0) \in V_0^2$, $(v_1, v_2) \in H^2$ and $(f_0, g_0) \in K^2$ problem (1) has a unique local solution*

$$(u,v,z,y) \in C\left(0,t_*; V_0^2 \times K^2\right) \cap C^1\left(0,t_*; H^2 \times K^2\right), \tag{23}$$

for $t_* > 0$ small enough.

Lemma 3. *For $r > -1$, $\exists \eta > 0$ such that $\forall u, v \in V \cap V_0(0,L)$, we have*

$$\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{(r+2)} \le \eta\left(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2\right)^{r+2}. \tag{24}$$

Proof. It is clear that by using the Minkowski inequality we get

$$\|u+v\|_{L_x^{2(r+2)}}^2 \le 2\left(\|u\|_{L_x^{2(r+2)}}^2 + \|v\|_{L_x^{2(r+2)}}^2\right). \tag{25}$$

Also, Hölder's and Young's inequalities give us

$$\|uv\|_{L_x^{(r+2)}}^{(r+2)} \le \|u\|_{L_x^{2(r+2)}}\|v\|_{L_x^{2(r+2)}} \tag{26}$$

$$\le c\left(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2\right). \tag{27}$$

By applying the embedding $V \cap V_0(0,L) \hookrightarrow L_x^{2(r+2)}(0,L)$ and (25), (27) gives (15).

Lemma 4. *$\exists \Lambda_1, \Lambda_2 > 0$ such that*

$$\int_0^L x|f_i(u,v)|^2 dx \le \Lambda_i\left(l_1\int_0^L xu_x^2 dx + l_2\int_0^L xv_x^2 dx\right)^{2r+3}, \forall x \in (0,L), i = 1,2. \tag{28}$$

Proof. We prove inequality for f_1 and the same result also holds for f_2.

It is clear that

$$\begin{aligned} |f_1(u,v)| &\le C\left(|u+v|^{2r+3} + |u|^{r+1}|v|^{r+2}\right) \\ &\le C\left[|u|^{2r+3} + |v|^{2r+3} + |u|^{r+1}|v|^{r+2}\right]. \end{aligned} \tag{29}$$

By Young's inequality, with

$$\begin{aligned} q &= \frac{2r+3}{r+1}, \\ q' &= \frac{2r+3}{r+2}, \end{aligned} \tag{30}$$

we get

$$|u|^{r+1}|v|^{r+2} \le c_1|u|^{2r+3} + c_2|v|^{2r+3}. \tag{31}$$

Therefore,

$$|f_1(u,v)| \le C\left[|u|^{2r+3} + |v|^{2r+3}\right]. \tag{32}$$

Hence, by Poincaré's inequality and (11), we obtain

$$\begin{aligned} \int_0^L x|f_i(u,v)|^2 dx &\le C\left(\|u_x\|_H^{2(2r+3)} + \|v_x\|_H^{2(2r+3)}\right) \\ &\le \Lambda_1\left(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2\right)^{(2r+3)}. \end{aligned} \tag{33}$$

The proof of lemma is complete.

The energy function (see, e.g., [8, 19] and reference therein) is defined by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_0^L xu_t^2 dx + \frac{1}{2}\int_0^L xv_t^2 dx + \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right) \\ & \cdot \int_0^L xu_x^2 dx + \frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right) \\ & \cdot \int_0^L xv_x^2(x,t)dx + \frac{1}{2}K(z,y) \\ & + \frac{1}{2}(g_1 \circ u_x)(t) + \frac{1}{2}(g_2 \circ v_x)(t) - \int_0^L F(u,v)dx, \end{aligned} \tag{34}$$

where

$$K(z,y)=\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)\mid z^2(x,\rho,\varrho,t)+|\mu_4(\varrho)\mid y^2(x,\rho,\varrho,t)\right)d\varrho d\rho dx,$$
$$(g\circ u_x)(t)=\int_0^L\int_0^t xg(t-s)|u_x(x,t)-u_x(x,s)|^2 ds dx. \tag{35}$$

Lemma 5. *Let (u,v,z,y) be the solution of system (11); then, $E(t)$ is a nonincreasing function, that is, $\forall t\ge 0$*

$$E'(t)\le -d_1\int_0^L xu_t^2dx-d_2\int_0^L xv_t^2dx+\frac{1}{2}\left(g_1'\circ u_x\right)(t)+\frac{1}{2}\left(g_2'\circ v_x\right)(t)$$
$$-\frac{1}{2}g_1(t)\int_0^L xu_x^2dx-\frac{1}{2}g_2(t)\int_0^L xv_x^2dx\le 0, \tag{36}$$

where

$$d_1=\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)>0,$$
$$d_2=\mu_3-\left(\int_{\tau_1}^{\tau_2}\mid\mu_4(\varrho)\mid d\varrho\right)>0. \tag{37}$$

Proof. Multiplying equation $(11)_{1,2}$ by xu_t, xv_t, and integrating over $(0,L)$, we find

$$\int_0^L xu_{tt}u_tdx-\int_0^L(xu_x)_xu_tdx+\mu_1\int_0^L xu_t^2dx$$
$$+\int_0^L xu_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z(x,1,\varrho,t)d\rho dx$$
$$+\int_0^L\int_0^t g_1(t-s)(xu_x(x,s))_xdsu_tdx$$
$$+\int_0^L xv_{tt}v_tdx-\int_0^L(xv_x)_xv_tdx+\mu_3\int_0^L xv_t^2dx$$
$$+\int_0^L xv_t\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y(x,1,\varrho,t)d\varrho dx$$
$$+\int_0^L\int_0^t g_2(t-s)(xv_x(x,s))_xdsv_tdx$$
$$=\int_0^L\left[a|u+v|^{2(r+1)}(u+v)+b|u|^ru|v|^{r+2}\right]xu_tdx$$
$$+\int_0^L\left[a|u+v|^{2(r+1)}(u+v)+b|v|^rv|u|^{r+2}\right]xv_tdx. \tag{38}$$

Using integration by parts, we get

$$\int_0^L xu_{tt}u_tdx=\frac{1}{2}\frac{d}{dt}\left[\int_0^L xu_t^2dx\right], \tag{39}$$

$$\int_0^L xv_{tt}v_tdx=\frac{1}{2}\frac{d}{dt}\left[\int_0^L xv_t^2dx\right], \tag{40}$$

$$-\int_0^L(xu_x)_xu_tdx=\frac{1}{2}\frac{d}{dt}\left[\int_0^L xu_x^2dx\right], \tag{41}$$

$$-\int_0^L(xv_x)_xv_tdx=\frac{1}{2}\frac{d}{dt}\left[\int_0^L xv_x^2dx\right], \tag{42}$$

$$\frac{1}{2(r+2)}\int_0^L xf_1(u,v)uu_tdx+\frac{1}{2(r+2)}\int_0^L xf_2(u,v)vv_tdx$$
$$=\frac{1}{2(r+2)}\frac{d}{dt}\int_0^L\left[a|u+v|^{2(r+2)}+2b|uv|^{r+2}\right]xdx, \tag{43}$$

$$\int_0^L\int_0^t g_1(t-s)(xu_x(s))_xdsu_t(t)dx$$
$$=\frac{1}{2}\frac{d}{dt}\left[(g_1\circ u_x)(t)-\int_0^t g_1(s)ds\int_0^L xu_x^2dx\right] \tag{44}$$

$$-\frac{1}{2}\left(g_1'\circ u_x\right)(t)+\frac{1}{2}g_1(t)\int_0^L xu_x^2dx, \tag{45}$$

$$\int_0^L\int_0^t g_2(t-s)(xv_x(s))_xdsv_t(t)dx$$
$$=\frac{1}{2}\frac{d}{dt}\left[(g_2\circ v_x)(t)-\int_0^t g_2(t)ds\int_0^L xv_x^2dx\right]$$
$$-\frac{1}{2}\left(g_2'\circ v_x\right)(t)+\frac{1}{2}g_2(t)\int_0^L xv_x^2dx. \tag{46}$$

Now, multiplying equation $(11)_3$ by $xz\mid\mu_2(\varrho)\mid$ and integrating over $(0,L)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\frac{d}{dt}\frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|xz^2d\varrho d\rho dx$$
$$=-\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|xzz_\rho d\varrho d\rho dx$$
$$=-\frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}x|\mu_2(\varrho)|\frac{d}{d\rho}z^2d\varrho d\rho dx$$
$$=\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}x|\mu_2(\varrho)|\left((z(x,0,\varrho,t))^2\right.$$
$$\left.-(z(x,1,\varrho,t))^2\right)d\varrho dx$$
$$=\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\int_0^L|xu_t|^2dx$$
$$-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}x|\mu_2(\varrho)|(z(x,1,\varrho,t))^2d\varrho dx. \tag{47}$$

Similarly, by multiplying equation $(11)_4$ by $xy\mid\mu_4(\varrho)\mid$

and integrating over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$, we get

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_4(\varrho)|xy^2 d\varrho d\rho dx \\ &= \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\right)\int_0^L xv_t^2 dx \\ &\quad - \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx. \end{aligned} \tag{48}$$

Using Young's and Cauchy-Schwartz inequalities, we have

$$\begin{aligned} &-\int_0^L xu_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z(x,1,\varrho,t)d\varrho dx \\ &\leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)\int_0^L xu_t^2 dx \\ &\quad + \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|xz^2(x,1,\varrho,t)d\varrho dx. \end{aligned} \tag{49}$$

Similarly, we get

$$\begin{aligned} &-\int_0^L xv_t\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y(x,1,\varrho,t)d\varrho dx \\ &\leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\right)\int_0^L xv_t^2 dx \\ &\quad + \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|xy^2(x,1,\varrho,t)d\varrho dx. \end{aligned} \tag{50}$$

By combining (39), (40), (41), (42), (43), (45), (46), (47), (48), (49), and (50) in (38), we get (34) and (36).

3. Global Existence

In this section, we showed the global existence of the solutions of the system (11).

First, introducing the following notation

$$\begin{aligned} I(t) &:= I(u(t),v(t)) = \left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2 dx + (g_1\circ u_x)(t) \\ &\quad + \left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2 dx + (g_2\circ v_x)(t) \\ &\quad + K(z,y) - 2(r+2)\int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx, \end{aligned} \tag{51}$$

$$\begin{aligned} J(t) &:= J(u(t),v(t)) = \frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2 dx \\ &\quad + \frac{1}{2}(g_1\circ u_x)(t) + \frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2 dx \\ &\quad + \frac{1}{2}(g_2\circ v_x)(t) + \frac{1}{2}K(z,y) - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx, \end{aligned} \tag{52}$$

note that

$$E(t) = J(t) + \frac{1}{2}\int_0^L xu_t^2 dx + \frac{1}{2}\int_0^L xv_t^2 dx. \tag{53}$$

Lemma 6. *Assume that (24), (14), (15), (16), (17), and (22) hold, and* $\forall (u_0, v_0) \in V_0^2$, $(u_1, v_1) \in H^2$ *and* $(f_0, g_0) \in L_x^2((0,L),(0,1),(\tau_1,\tau_2))$ *satisfying*

$$I(0) > 0, \beta := \eta\left(\frac{2(r+2)}{(r+1)}E(0)\right)^{r+1} < 1. \tag{54}$$

Then, $\exists t_* > 0$ such that

$$I(t) > 0, \forall t \in [0, t_*), \tag{55}$$

where

$$E(0) = J(0) + \frac{1}{2}\int_0^L xu_1^2 dx + \frac{1}{2}\int_0^L xv_1^2 dx. \tag{56}$$

Proof. As $I(0) > 0$, then by continuity of $I(t)$, $\exists T_m \leq t_*$ such that $I(t) \geq 0$, $\forall t \in [0, T_m)$; this implies that we have a maximum time value noting T_m such that

$$\{I(T_m) = 0 \text{ and } I(t) > 0, \quad \text{for all } 0 \leq t < T_m\}. \tag{57}$$

This, with (51), (52), and (14), we have

$$\begin{aligned} J(t) &= \frac{r+1}{2(r+2)}\left[\left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2 dx \right.\\ &\quad \left. + \left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2 dx\right] \\ &\quad + \frac{r+1}{2(r+2)}\left[(g_1\circ u_x)(t) + (g_2\circ v_x)(t) + K(z,y)\right] \\ &\quad + \frac{1}{2(r+2)}I(t) \\ &\geq \frac{r+1}{2(r+2)}\left[\left(l_1\int_0^L xu_x^2 dx + l_2\int_0^L xv_x^2 dx\right)\right. \\ &\quad \left. + (g_1\circ u_x)(t) + (g_2\circ v_x)(t) + K(z,y)\right]. \end{aligned} \tag{58}$$

Hence,

$$\begin{aligned} &l_1\int_0^L xu_x^2 dx + l_2\int_0^L xv_x^2 dx \\ &\leq \frac{2(r+2)}{r+1}J(t) \\ &\leq \frac{2(r+2)}{r+1}E(t) \\ &\leq \frac{2(r+2)}{r+1}E(0), \quad \forall t \in [0, T_m). \end{aligned} \tag{59}$$

By (24) and (54), we get

$$\begin{aligned}
2(r+2)\int_0^L F(u(T_m), v(T_m))dx &\le \eta\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right)^{r+2} \\
&\le \eta\left(\frac{2(r+2)}{r+1}E(0)\right)^{r+1}\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right) \\
&= \beta\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right) \\
&< \left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx \\
&\quad + \left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx \\
&\quad + (g_1\circ u_x)(t) + (g_2\circ v_x)(t) + K(z,y).
\end{aligned} \tag{60}$$

Hence,

$$\begin{aligned}
&\left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx + \left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx \\
&\quad + (g_1\circ u_x)(t) + (g_2\circ v_x)(t) + K(z,y) \\
&\quad - 2(r+2)\int_0^L xF(u,v)dx > 0.
\end{aligned} \tag{61}$$

This proves that $I(t) > 0, \forall t \in [0, T_m)$. By repeating the procedure, T_m is extended to t_*.

Theorem 7. *Let (14), (15), (16), (17), (22), and (24) hold. Then,* $\forall (u_0, v_0) \in V_0^2$, $(u_1, v_1) \in H^2$, *and* $(f_0, g_0) \in L_x^2((0,L), (0,1), (\tau_1, \tau_2))$ *satisfying (54) the solution of system (11) is bounded and global.*

Proof. To prove that $\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2 + \|z\|_{K,\mu_2}^2 + \|y\|_{K,\mu_4}^2$ is bounded independently of t, using (36) yields

$$E(0) \ge E(t). \tag{62}$$

Using (52), we find

$$\begin{aligned}
&-2(r+2)\int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\
&= I(t) - \left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx \\
&\quad - \left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx \\
&\quad - (g_1\circ u_x)(t) - (g_2\circ v_x)(t) - K(z,y).
\end{aligned} \tag{63}$$

By using (62) in (63), we get

$$\begin{aligned}
E(0) \ge E(t) &= \frac{1}{2}\int_0^L xu_t^2dx + \frac{1}{2}\int_0^L xv_t^2dx \\
&\quad + \frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx \\
&\quad + \frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2(x,t)dx \\
&\quad + \frac{1}{2}(g_1\circ u_x)(t) + \frac{1}{2}(g_2\circ v_x)(t) \\
&\quad + \frac{1}{2}K(z,y) + I(t),
\end{aligned} \tag{64}$$

and using (14), (15), and (54) in (64), we get

$$\begin{aligned}
E(0) \ge E(t) &\ge \frac{1}{2}\int_0^L xu_t^2dx + \frac{1}{2}\int_0^L xv_t^2dx \\
&\quad + \left(\frac{r+1}{2(r+2)}\right)\left\{l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx + K(z,y)\right\} \\
&\ge \mu_0\left(\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx\right. \\
&\quad \left. + \int_0^L xv_x^2dx + K(z,y)\right).
\end{aligned} \tag{65}$$

So

$$\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2 + \|z\|_{K,\mu_2}^2 + \|y\|_{K,\mu_4}^2 \le \mu E(0)/\mu := \frac{1}{\mu_0}, \tag{66}$$

where

$$\mu_0 := \min\left\{\frac{1}{2}, \frac{(r+1)}{2(r+2)}l_1, \frac{(r+1)}{2(r+2)}l_2, \frac{(r+1)}{2(r+2)}\right\}. \tag{67}$$

Hence, the solution of system (11) is bounded and global.

4. Decay of Solutions

In this section, the decay result is showed by using several lemmas.

As, we let

$$F(t) := E(t) + \varepsilon_1\Phi(t) + \varepsilon_2\chi(t) + \varepsilon_3\Psi(t), \tag{68}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, and

$$\Phi(t) := \xi(t)\int_0^L xu_t u dx + \xi(t)\int_0^L xv_t v dx, \tag{69}$$

$$\chi(t) := -\xi(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx - \xi(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx, \tag{70}$$

$$\Psi(z,y) := \xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\rho\varrho}\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|y^2\right)d\varrho d\rho dx. \tag{71}$$

Lemma 8. *There exist $\alpha_1, \alpha_2 > 0$, such that*

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t), \tag{72}$$

for ε_1, ε_2, and ε_3 small enough.

Proof. Using the inequality of Young and the Poincaré-type inequality and $0<\xi(t)\le\xi(0)$, we find

$$\varepsilon_1\xi(t)\int_0^L xu_t u dx \le \frac{\varepsilon_1}{2}\xi(0)\int_0^L xu_t^2dx + \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xu_x^2dx, \tag{73}$$

$$\varepsilon_1\xi(t)\int_0^L xv_t v dx \le \frac{\varepsilon_1}{2}\xi(0)\int_0^L xv_t^2dx + \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xv_x^2dx, \tag{74}$$

$$-\varepsilon_2\xi(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx \le \frac{\varepsilon_2}{2}\xi(0)\int_0^L xu_t^2dx + \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_1)(g_1\circ u_x)(t), \tag{75}$$

$$-\varepsilon_2\xi(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx \le \frac{\varepsilon_2}{2}\xi(0)\int_0^L xv_t^2dx + \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_2)(g_2\circ v_x)(t), \tag{76}$$

$$\Psi(z,y) \le \xi(0)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)\mid z^2+|\mu_4(\varrho)\mid y^2\right)d\varrho d\rho dx, \tag{77}$$

where $C_p>0$.

A combination of (73), (74), (75), (76), and (77) in (68) gives

$$\begin{aligned}F(t) \le{}& E(t) + \left(\frac{\varepsilon_{1+}\varepsilon_2}{2}\right)\xi(0)\int_0^L xu_t^2dx + \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\xi(0)\int_0^L xv_t^2dx \\ &+ \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xu_x^2dx + \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xv_x^2dx \\ &+ \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_1)(g_1\circ u_x)(t) \\ &+ \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_2)(g_2\circ v_x)(t) \\ &+ \varepsilon_3\xi(0)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\rho\left(|\mu_2(\varrho)\mid z^2+|\mu_4(\varrho)\mid y^2\right)d\varrho d\rho dx.\end{aligned} \tag{78}$$

Then, $\exists\alpha_1>0$, for ε_1, ε_2, and ε_3 small enough, such that

$$F(t) \le \frac{1}{\alpha_1}E(t). \tag{79}$$

Similarly, thanks to the inequalities of Young and Poincaré-type and using $0<\xi(t)\le\xi(0)$ gives

$$\varepsilon_1\xi(t)\int_0^L xu_t u dx \ge \frac{-\varepsilon_1}{2}\xi(0)\int_0^L xu_t^2dx - \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xu_x^2dx, \tag{80}$$

$$\varepsilon_1\xi(t)\int_0^L xv_t v dx \ge \frac{-\varepsilon_1}{2}\xi(0)\int_0^L xv_t^2dx - \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xv_x^2dx, \tag{81}$$

$$-\varepsilon_2\xi(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx \ge \frac{-\varepsilon_2}{2}\xi(0)\int_0^L xu_t^2dx - \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_1)(g_1\circ u_x)(t), \tag{82}$$

$$-\varepsilon_2\xi(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx \ge \frac{-\varepsilon_2}{2}\xi(0)\int_0^L xv_t^2dx - \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_2)(g_2\circ v_x)(t), \tag{83}$$

and

$$-\varepsilon_3\Psi(z,y) \ge -\varepsilon_3\xi(0)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)\mid z^2+|\mu_4(\rho\varrho)\mid y^2\right)d\varrho d\rho dx. \tag{84}$$

By combining (80), (81), (82), (83), and (84) in (68), we find

$$
\begin{aligned}
F(t) \geq E(t) &- \left(\frac{\varepsilon_{1+}\varepsilon_2}{2}\right)\xi(0)\int_0^L x u_t^2 dx - \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\xi(0)\int_0^L x v_t^2 dx \\
&- \frac{\varepsilon_1}{2} C_p \xi(0)\int_0^L x u_x^2 dx - \frac{\varepsilon_1}{2} C_p \xi(0)\int_0^L x v_x^2 dx \\
&- \frac{\varepsilon_2}{2} C_p \xi(0)(1-l_1)(g_1 \circ u_x)(t) \\
&- \frac{\varepsilon_2}{2} C_p \xi(0)(1-l_2)(g_2 \circ v_x)(t) \\
&- \varepsilon_3 \xi(0)\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} x\rho\left(|\mu_2(\varrho)\mid z^2 + |\mu_4(\varrho)\mid y^2\right) d\varrho d\rho dx.
\end{aligned} \tag{85}
$$

Then, $\exists \alpha_2 > 0$, for ε_1, ε_2, and ε_3 small enough, such that

$$
F(t) \geq \frac{1}{\alpha_2} E(t). \tag{86}
$$

This completes the proof.

Lemma 9. *For $\sigma > 1$ and $0 < \theta < 1$, we have*

$$
\begin{aligned}
&\int_0^t g(t-s)\|w(s)\|^2 ds \\
&\leq \left(\int_0^t g^{1-\theta}(t-s)\|w(s)\|^2 ds\right)^{1/\sigma} \\
&\quad \times \left(\int_0^t g^{(\sigma-1+\theta)/\sigma-1}(t-s)\|w(s)\|^2 ds\right)^{(\sigma-1)/\sigma},
\end{aligned} \tag{87}
$$

$\forall w \in H$.

Proof. It suffices to note that

$$
\begin{aligned}
\int_0^t g(t-s)\|w(s)\|^2 ds = \int_0^t & g^{(1-\theta)/r}(t-s)\|w(s)\|^{2/r} g^{(\sigma-1+\theta)/\sigma} \\
& \cdot (t-s)\|w(s)\|^{(2(\sigma-1))/\sigma} ds,
\end{aligned} \tag{88}
$$

using Hölder's inequality for

$$
\begin{aligned}
p &= \sigma, \\
q &= \frac{\sigma}{\sigma-1}, \quad r > 1.
\end{aligned} \tag{89}
$$

This completes the proof.

Lemma 10. *Let $v \in L^\infty((0,T);H)$ be such that $v_x \in L^\infty((0,t);H)$ and g be a continuous function on $[0,T]$ and suppose that $0 < \theta < 1$ and $\rho > 1$. Then, $\exists C > 0$ so that*

$$
\begin{aligned}
&\int_0^t g(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \\
&\leq C\left(\sup_{0<s<T}\|v(.,s)\|_H^2 \int_0^t g^{1-\theta}(s) ds\right)^{(\rho-1)/(\rho-1+\theta)} \\
&\quad \times \left(\int_0^t g^\rho(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{\theta/(\rho-1+\theta)}.
\end{aligned} \tag{90}
$$

Proof. By applying Lemma 8 with $\sigma = (\rho-1+\theta)/(\rho-1)$ gives

$$
\begin{aligned}
&\int_0^t g(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \\
&\leq \left(\int_0^t g^{1-\theta}(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{(\rho-1)/(\rho-1+\theta)} \\
&\quad \times \left(\int_0^t g^\rho(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2\right)^{\theta/(\rho-1+\theta)}.
\end{aligned} \tag{91}
$$

We also have

$$
\int_0^t g^{1-\theta}(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \leq C \sup_{0<s<T}\|v_x(.,s)\|_H^2 \int_0^t g^{1-\theta}(s) ds, \tag{92}
$$

by combining (82) and (83). This completes the proof.

Lemma 11. *Suppose that $v \in L^\infty((0,T);H)$ be such that $v_x \in L^\infty((0,T);H)$ and g be a continuous function on $[0,T]$ and assume $\rho > 1$. Then, $\exists C > 0$ so that*

$$
\begin{aligned}
&\int_0^t g(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \\
&\leq c\left(t\|v_x(.,t)\|_H^2 + \int_0^t \|v_x(.,s)\|_H^2 ds\right)^{(\rho-1)/\rho} \\
&\quad \times \left(\int_0^t g^\rho(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{1/\rho}.
\end{aligned} \tag{93}
$$

Proof. By using (82) for $\theta = 1$ gives

$$
\begin{aligned}
&\int_0^t g(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \\
&\leq \left(\int_0^t \|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{(\rho-1)/\rho} \\
&\quad \times \left(\int_0^t g^\rho(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{1/\rho},
\end{aligned} \tag{94}
$$

where

$$\int_0^t \|v_x(.,t)-v_x(.,s)\|_H^2 ds \le 2t\|v_x(.,t)\|_H^2+2\int_0^t \|v_x(.,s)\|_H^2 ds, \tag{95}$$

to obtain (93). Hence, this ends the proof.

Lemma 12. *Suppose that r satisfies (15) and (52) hold. Then, the functional $\Phi(t)$, given by (69), satisfies*

$$\begin{aligned}\Phi'(t) \le & \left(1+\frac{l}{2\delta}+\frac{\mu_1}{2\delta_1}\right)\xi(t)\int_0^L x u_t^2 dx \\ & + \left(1+\frac{l}{2\delta}+\frac{\mu_3}{2\delta_2}\right)\xi(t)\int_0^L x v_t^2 dx - \xi(t) \\ & \cdot \left(\frac{l_1 - C_p(\delta l - 2\delta_1\mu_1)}{2}\right)\int_0^L x u_x^2 dx - \xi(t) \\ & \cdot \left(\frac{l_2 - C_p(\delta l - 2\delta_2\mu_3)}{2}\right)\int_0^L x v_x^2 dx \\ & + \frac{\xi(t)}{2l_1}\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) + \frac{\xi(t)}{2l_2} \\ & \cdot \left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t) \\ & + \frac{\xi(t)}{2\delta_1}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)| z^2(x,1,\varrho,t)d\varrho dx \\ & + \frac{\xi(t)}{2\delta_2}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)| y^2(x,1,\varrho,t)d\varrho dx \\ & + \frac{\xi(t)}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx.\end{aligned} \tag{96}$$

For any $\delta, \delta_1, \delta_2 > 0$.

Proof. The derivation of (11) gives

$$\begin{aligned}\Phi'(t) = & \xi'(t)\int_0^L x u_t u dx + \xi(t)\int_0^L x u_t^2 dx + \xi(t)\int_0^L x u_{tt} u dx \\ & + \xi'(t)\int_0^L x v_t v dx + \xi(t)\int_0^L x v_t^2 dx + \xi(t)\int_0^L x v_{tt} v dx \\ = & \xi'(t)\int_0^L x u_t u dx + \xi(t)\int_0^L x u_t^2 dx - \xi(t)\int_0^L x u_x^2 dx \\ & - \xi(t)\mu_1\int_0^L x u u_t dx - \xi(t)\int_0^L x u \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| z^2(x,1,\varrho,t)d\varrho dx \\ & + \xi(t)\int_0^L x u_x \int_0^t g_1(t-s)u_x(s)ds dx + \xi'(t)\int_0^L x v_t v dx \\ & + \xi(t)\int_0^L x v_t^2 dx - \xi(t)\int_0^L x v_x^2 dx - \xi(t)\mu_3\int_0^L x v v_t dx \\ & - \xi(t)\int_0^L x v \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| y^2(x,1,\varrho,t)d\varrho dx \\ & + \xi(t)\int_0^L x v_x \int_0^t g_2(t-s)v_x(s)ds dx \\ & + \frac{\xi(t)}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx.\end{aligned} \tag{97}$$

By Young's and Poincaré inequalities and (14) and (15), we find

$$\begin{aligned}& \xi(t)\int_0^L x u_x(t)\left(\int_0^t g_1(t-s)u_x(s)ds\right)dx \\ & \le \frac{\xi(t)}{2}\int_0^L x u_x^2 dx + \frac{\xi(t)}{2}\int_0^L x\left(\int_0^t g_1(t-s)(|u_x(s) - u_x(t)| + |u_x(t)|)ds\right)^2 dx \\ & \le \frac{\xi(t)}{2}\int_0^L x u_x^2 dx + \frac{\xi(t)}{2}(1+\eta_1)(1-l_1)^2\int_0^L x u_x^2(t)dx \\ & \quad + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) \\ & = \xi(t)\left(\frac{1+(1+\eta_1)(1-l_1)^2}{2}\right)\int_0^L x u_x^2 dx \\ & \quad + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) \\ & \quad + \frac{\xi(t)}{r+2}\int_0^L \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx.\end{aligned} \tag{98}$$

Similarly, we get

$$\begin{aligned}& \int_0^L x v_x(t)\left(\int_0^t g_1(t-s)v_x(s)ds\right)dx \\ & \le \xi(t)\left(\frac{1+(1+\eta_2)(1-l_2)^2}{2}\right)\int_0^L x v_x^2 dx \\ & \quad + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_2}\right)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t).\end{aligned} \tag{99}$$

$\forall \eta_1, \eta_2 > 0$. As we have

$$\begin{aligned}\xi'(t)\int_0^L x u_t u dx & \le \frac{\xi(t)}{2}\left|\frac{\xi'(t)}{\xi(t)}\right|\left(C_p\delta\int_0^L x u_x^2 dx + \frac{1}{\delta}\int_0^L x u_t^2 dx\right) \\ & \le \frac{\xi(t)}{2}\left(C_p l\delta\int_0^L x u_x^2 dx + \frac{l}{\delta}\int_0^L x u_t^2 dx\right), \forall \delta > 0,\end{aligned} \tag{100}$$

and similarly, we find

$$\xi'(t)\int_0^L x v_t v dx \le \frac{\xi(t)}{2}\left(C_p l\delta\int_0^L x v_x^2 dx + \frac{l}{\delta}\int_0^L x v_t^2 dx\right). \tag{101}$$

By using Young's and Poincaré's inequalities and (22) gives

$$-\xi(t)\int_0^L xu\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \leq \frac{\xi(t)}{2}\left(C_p\delta_1\mu_1\int_0^L xu_x^2dx+\frac{1}{\delta_1}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\right), \tag{102}$$

$$-\xi(t)\int_0^L xv\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \leq \frac{\xi(t)}{2}\left(C_p\delta_2\mu_3\int_0^L xv_x^2dx+\frac{1}{\delta_2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\right). \tag{103}$$

Similarly, we get

$$\xi(t)\int_0^L xu_t u dx \leq \frac{\xi(t)}{2}\left(C_p\delta_1\mu_1\int_0^L xu_x^2dx+\frac{\mu_1}{\delta_1}\int_0^L xu_t^2dx\right)\xi(t)\int_0^L xv_t v dx \leq \frac{\xi(t)}{2}\left(C_p\delta_2\mu_3\int_0^L xv_x^2dx+\frac{\mu_3}{\delta_2}\int_0^L xv_t^2dx\right). \tag{104}$$

In a combination of (98), (99), (100), (101), (102), (103), and (104) in (97), we obtain

$$\begin{aligned}\Phi'(t) \leq{}& \left(1+\frac{l}{2\delta}+\frac{\mu_1}{2\delta_1}\right)\xi(t)\int_0^L xu_t^2dx \\ &+\left(1+\frac{l}{2\delta}+\frac{\mu_3}{2\delta_2}\right)\xi(t)\int_0^L xv_t^2dx \\ &-\frac{\xi(t)}{2}\left[1-(1+\eta_1)(1-l_1)^2-\delta C_p l-2\delta_1 C_p\mu_1\right] \\ &\cdot\int_0^L xu_x^2dx-\frac{\xi(t)}{2}\left[1-(1+\eta_2)(1-l_2)^2-\delta C_p l-2\delta_2 C_p\mu_3\right] \\ &\cdot\int_0^L xv_x^2dx+\frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t) \\ &+\frac{\xi(t)}{2}\left(1+\frac{1}{\eta_2}\right)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t) \\ &+\frac{\xi(t)}{2}\int_0^L\int_{\tau_1}^{\tau_2}\left(\frac{1}{\delta_1}|\mu_2(\varrho)|z^2(x,1,\varrho,t)\right. \\ &\left.+\frac{1}{\delta_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)\right)d\varrho dx \\ &+\frac{\xi(t)}{2(r+2)}\int_0^L\left[a|u+v|^{2(r+2)}+2b|uv|^{r+2}\right]dx,\end{aligned} \tag{105}$$

by choosing η_1, η_1, so that $\eta_1 = l_1/(1-l_1)$; hence, $(1/2)(-1+(1+\eta_1)(1-l_1)^2) = -l_1/2$ and $(1+(1/\eta_1)) = 1/l_1$,and $\eta_2 = l_2/(1-l_2)$; therefore, $(1/2)(-1+(1+\eta_2)(1-l_2)^2) = -l_2/2$ and $(1+(1/\eta_2)) = 1/l_2$.

Then, (96) is proved.

Lemma 13. *Assuming that r satisfies (15), (14), and (15) and (22) and (52) hold. Then, the functional $\chi(t)$ given by (70) satisfies along the solution of (11)*

$$\begin{aligned}\chi'(t) \leq{}& \xi(t)\theta\left[1+c_1+c_1'+2(1-l_1)^2\right]\left(\int_0^L xu_x^2dx\right) \\ &+\xi(t)\theta\left[1+c_2+c_2'+2(1-l_2)^2\right]\left(\int_0^L xv_x^2dx\right) \\ &+\xi(t)\left[\theta-\left(\int_0^t g_1(s)ds\right)+\theta l+\theta_1\mu_1\right]\left(\int_0^L xu_t^2dx\right) \\ &+\xi(t)\left[\theta-\left(\int_0^t g_2(s)ds\right)+\theta l+\theta_2\mu_3\right]\left(\int_0^L xv_t^2dx\right) \\ &+\left[\frac{1}{2\theta}+2\theta+\frac{\mu_1 C_p}{2\theta_1}+\frac{C_p(1+l)}{4\theta}\right] \\ &\times\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t) \\ &+\left[\frac{1}{2\theta}+2\theta+\frac{\mu_3 C_p}{2\theta_2}+\frac{C_p(1+l)}{4\theta}\right] \\ &\times\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t) \\ &-\frac{C_p}{4\theta}\xi(t)g_1(0)\left(g_1'\circ u_x\right)(t) \\ &-\frac{C_p}{4\theta}\xi(t)g_2(0)\left(g_2'\circ v_x\right)(t) \\ &+\xi(t)\int_0^L\int_{\tau_1}^{\tau_2}x\left(\theta_1|\mu_2(\varrho)|z^2(x,1,\varrho,t)\right. \\ &\left.+\theta_2|\mu_4(\varrho)|y^2(x,1,\varrho,t)\right)d\varrho dx,\end{aligned} \tag{106}$$

for any $\theta, \theta_1, \theta_2 > 0$.

Proof. Direct calculation gives

$$\begin{aligned}\chi'(t) ={}& -\xi'(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ &-\xi(t)\int_0^L xu_{tt}\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ &-\xi(t)\int_0^L xu_t\frac{d}{dt}\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\ &-\xi'(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx \\ &-\xi(t)\int_0^L xv_{tt}\int_0^t g_2(t-s)(v(t)-v(s))dsdx \\ &-\xi(t)\int_0^L xv_t\frac{d}{dt}\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx,\end{aligned} \tag{107}$$

by using

$$\frac{d}{dt}\left(\int_{\alpha(t)}^{\beta(t)} f(t,s)ds\right) = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(t,s)}{\partial t} ds + \frac{\partial \beta(t)}{\partial t} f(t,\beta(t)) - \frac{\partial \alpha(t)}{\partial t} f(t,\alpha(t)). \tag{108}$$

As we have (u, v, z, y) the solution of (11), we find

$$\begin{aligned}
\chi'(t) = &-\xi'(t)\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&+\xi(t)\int_0^L xu_x\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\
&-\xi(t)\int_0^L x\left(\int_0^t g_1(t-s)u_x(s)ds\right) \\
&\cdot\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\
&-\xi(t)\mu_1\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&-\xi(t)\int_0^L x\left(\int_{\tau_1}^{\tau_2} |\mu_2(\rho)| z^2(x,1,\rho,t)d\rho\right) \\
&\cdot\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx - \xi(t)\int_0^L xf_1(u,v) \\
&\cdot\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx - \xi(t)\int_0^L xu_t \\
&\cdot\left(\int_0^t g_1'(t-s)(u(t)-u(s))ds\right)dx - \xi(t) \\
&\cdot\left(\int_0^t g_1(s)ds\right)\int_0^L xu_t^2dx - \xi(t)\left(\int_0^t g_2(s)ds\right)\int_0^L xv_t^2dx \\
&-\xi'(t)\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
&+\xi(t)\int_0^L xv_x\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx \\
&-\xi(t)\int_0^L x\left(\int_0^t g_2(t-s)v_x(s)ds\right) \\
&\cdot\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx - \xi(t)\mu_3\int_0^L xv_t \\
&\cdot\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx - \xi(t)\int_0^L x \\
&\cdot\left(\int_{\tau_1}^{\tau_2} |\mu_4(\rho)| y^2(x,1,\rho,t)d\rho\right) \\
&\cdot\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
&-\xi(t)\int_0^L xf_2(u,v)\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
&-\xi(t)\int_0^L xv_t\left(\int_0^t g_2'(t-s)(v(t)-v(s))ds\right)dx.
\end{aligned} \tag{109}$$

By Young's inequality and (14) and (15), we arrive to

$$\begin{aligned}
&-\xi'(t)\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&\le \xi(t)\left|\frac{\xi'(t)}{\xi(t)}\right|\left[\theta\int_0^L xu_t^2dx + \frac{C_p}{4\theta}\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t)\right] \\
&\le \theta l\xi(t)\int_0^L xu_t^2dx + \frac{C_pl}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t),
\end{aligned} \tag{110}$$

$$\begin{aligned}
&\xi(t)\int_0^L xu_x\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\
&\le \theta\xi(t)\int_0^L xu_x^2dx + \frac{1}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t).
\end{aligned} \tag{111}$$

Similarly, we get

$$\begin{aligned}
&\xi(t)\mu_1\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&\le \theta_1\mu_1\xi(t)\int_0^L xu_t^2dx + \frac{1}{4\theta_1}C_p\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t),
\end{aligned} \tag{112}$$

$$\begin{aligned}
&\xi(t)\mu_3\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
&\le \theta_2\mu_3\xi(t)\int_0^L xv_t^2dx + \frac{1}{4\theta_2}C_p\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t),
\end{aligned} \tag{113}$$

with

$$\begin{aligned}
&-\xi(t)\int_0^L x\left(\int_0^t g_1(t-s)u_x(s)ds\right)\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\
&\le 2\theta(1-l_1)^2\xi(t)\int_0^L xu_x^2dx + \left(2\theta+\frac{1}{4\theta}\right)\xi(t) \\
&\quad\cdot\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t).
\end{aligned} \tag{114}$$

So

$$\begin{aligned}
&-\xi(t)\int_0^L xf_1(u,v)\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&\le \frac{C_p}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t) \\
&\quad + c_1\theta\xi(t)\int_0^L xu_x^2dx + c_2\theta\xi(t)\int_0^L xv_x^2dx,
\end{aligned} \tag{115}$$

where

$$\begin{cases} c_1 := \Lambda_1\left(\dfrac{2(r+2)}{(r+1)}E(0)\right)^{2(r+1)}, \\ c_2 := \Lambda_2\left(\dfrac{2(r+2)}{(r+1)}E(0)\right)^{2(r+1)}, \end{cases} \tag{116}$$

$$\begin{aligned} &-\xi(t)\int_0^L xu_t\left(\int_0^t g_1'(t-s)(u(t)-u(s))ds\right)dx \\ &\quad\le \theta\xi(t)\int_0^L xu_t^2dx - \frac{g_1(0)}{4\theta}C_p\xi(t)\left(g_1'\circ u_x\right)(t). \end{aligned} \tag{117}$$

Then,

$$\begin{aligned} &-\xi'(t)\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\ &\quad\le \theta l\xi(t)\int_0^L xv_t^2dx + \frac{C_pl}{4\theta}\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t), \end{aligned} \tag{118}$$

$$\begin{aligned} &\xi(t)\int_0^L xv_x\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx \\ &\quad\le \theta\xi(t)\int_0^L xv_x^2dx + \frac{1}{4\theta}\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t). \end{aligned} \tag{119}$$

Thus,

$$\begin{aligned} &-\xi(t)\int_0^L x\left(\int_0^t g_2(t-s)v_x(s)ds\right)\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx \\ &\le 2\theta(1-l_2)^2\xi(t)\int_0^L xv_x^2dx + \left(2\theta+\frac{1}{4\theta}\right)\xi(t) \\ &\quad\cdot\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t), \end{aligned} \tag{120}$$

$$\begin{aligned} &-\frac{\xi(t)}{2(r+2)}\int_0^L xf_2(u,v)\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\ &\le \frac{C_p}{4\theta}\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t) + c_1'\theta\xi(t) \\ &\quad\cdot\int_0^L xu_x^2dx + c_2'\theta\xi(t)\int_0^L xv_x^2dx, \end{aligned} \tag{121}$$

where

$$\begin{cases} c_1' := \Lambda_1'\left(\dfrac{2(r+2)}{r+1}E(0)\right)^{2(r+1)}, \\ c_2' := \Lambda_2'\left(\dfrac{2(r+2)}{r+1}E(0)\right)^{2(r+1)}, \end{cases} \tag{122}$$

$$\begin{aligned} &-\xi(t)\int_0^L xv_t\left(\int_0^t g_2'(t-s)(v(t)-v(s))ds\right)dx \\ &\quad\le \theta\xi(t)\int_0^L xv_t^2dx - \frac{g_2(0)}{4\theta}C_p\xi(t)\left(g_2'\circ v_x\right)(t). \end{aligned} \tag{123}$$

Similarly, we have

$$\begin{aligned} &-\xi(t)\int_0^L x\left(\int_{\tau_1}^{\tau_2} \mid \mu_2(\rho)\mid z^2(x,1,\rho,t)d\rho\right) \\ &\quad\cdot\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\theta_1\xi(t) \\ &\quad\cdot\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\rho)|z^2(x,1,\rho,t)d\rho dx \\ &\quad+\frac{1}{4\theta_1}\mu_1C_p\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t), \end{aligned} \tag{124}$$

$$\begin{aligned} &-\xi(t)\int_0^L x\left(\int_{\tau_1}^{\tau_2} \mid \mu_4(\varrho)\mid y^2(x,1,\varrho,t)d\varrho\right) \\ &\quad\cdot\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx\theta_2\xi(t) \\ &\quad\cdot\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ &\quad+\frac{1}{4\theta_2}\mu_3C_p\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t). \end{aligned} \tag{125}$$

A combination of (110), (111), (112), (113), (114), (115), (117), (118), (119), (120), (121), (123), (124), and (125) into (109) gives (106).

Lemma 14. *Let (u,v,z,y) be the solution of (11). Then, for $\eta_3>0$, the functional $\Psi(t)$ satisfies*

$$\begin{aligned} \Psi'(t) &\le -\xi(t)\eta_4\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)\mid z^2+|\mu_4(\varrho)\mid y^2\right)d\varrho d\rho dx \\ &\quad+\xi(t)\mu_1\int_0^L xu_t^2dx+\xi(t)\mu_3\int_0^L xv_t^2dx \\ &\quad-\xi(t)\eta_3\int_0^L\int_{\tau_1}^{\tau_2} x\left(|\mu_2(\varrho)\mid z^2(x,1,\varrho,t)\right. \\ &\quad\left.+|\mu_4(\varrho)\mid y^2(x,1,\varrho,t)\right)d\varrho dx, \end{aligned} \tag{126}$$

where $\eta_3>0$ and $\eta_4=\eta_3(1-l)>0>0$.

Proof. By differentiating $\Psi(t)$ and using equations $(11)_3$ and $(11)_4$, we get

$$\begin{aligned}\Psi'(t) &= \xi'(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\rho e^{-\varrho\rho}\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|z^2\right)d\varrho d\rho dx\\ &\quad - 2\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varrho\rho}|\mu_2(\varrho)|zz_\rho(x,\rho,\varrho,t)d\varrho d\rho dx\\ &\quad - 2\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varrho\rho}|\mu_4(\varrho)|yy_\rho(x,\rho,\varrho,t)d\varrho d\rho dx\\ &= \xi'(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|z^2\right)d\varrho d\rho dx\\ &\quad - \xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}|\mu_2(\varrho)|z^2 d\varrho d\rho dx\\ &\quad - \xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|\left[e^{-\varrho}z^2(x,1,\varrho,t)-z^2(x,0,\varrho,t)\right]d\varrho dx\\ &\quad - \xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}|\mu_4(p)|y^2 d\rho\varrho\rho dx\\ &\quad - \xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|\left[e^{-\varrho}y^2(x,1,\varrho,t)-y^2(x,0,\varrho,t)\right]d\varrho dx.\end{aligned} \tag{127}$$

Using the equality $z(x,0,\varrho,t)=u_t(x,t)$, $y(x,0,\varrho,t)=v_t(x,t)$, and $e^{-\varrho}\le e^{-\rho\varrho}\le 1$, for any $0<\rho<1$, we find

$$\begin{aligned}\Psi'(t) &\le \xi(t)l\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|z^2\right)d\varrho d\rho dx\\ &\quad - \xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|y^2\right)d\varrho d\rho dx\\ &\quad - \xi(t)\int_0^L\int_{\tau_1}^{\tau_2} xe^{-\varrho}\left(|\mu_2(\varrho)|z^2(x,1,\varrho,t)+|\mu_4(\varrho)|y^2(x,1,\varrho,t)\right)d\varrho dx\\ &\quad + \left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)\xi(t)\int_0^L xu_t^2dx\\ &\quad + \left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\right)\xi(t)\int_0^L xv_t^2dx.\end{aligned} \tag{128}$$

As $-e^{-\varrho}$ is an increasing function, we have $-e^{-\varrho}\le -e^{-\tau_2}$, for any $\rho\in[\tau_1,\tau_2]$.

Then, setting $\eta_3=e^{-\tau_2}$ and (22), we obtain (126).

Theorem 15. *Let $(u_0,v_0)\in V_0^2$, $(u_1,v_1)\in H^2$, and $(f_0,g_0)\in L_x^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$ be defined and satisfy (163). Assume that r satisfies (24), (14), (15), (16), (17), and (22) hold. Then, for each $t_0>0$, $\exists K$ and k such that the solution of (11) satisfies $\forall t\ge t_0$, we have the following inequality for the energy function*

$$E(t)\le\begin{cases}Ke^{-k\int_{t_0}^t\xi(s)ds}, & \sigma=1,\\ K\left(1+\int_{t_0}^t\xi(s)ds\right)^{-(1/(\sigma-1))}, & 1<\sigma<\dfrac{3}{2}.\end{cases} \tag{129}$$

Proof. As g_1, g_2 is continuous and $g_1(0), g_2(0)>0$, hence $\forall t_0>0$; we have

$$\begin{cases}\displaystyle\int_0^t g_1(s)ds\ge\int_{t_0}^t g_1(s)ds=g_{1,0}>0, & \forall t\ge t_0,\\ \displaystyle\int_0^t g_2(s)ds\ge\int_{t_0}^t g_2(s)ds=g_{2,0}>0, & \forall t\ge t_0.\end{cases} \tag{130}$$

By using (36), (96), (106), (126), and (130) and $0<\xi(t)\le\xi(0)$) (hence $(\xi(t)/\xi(0))<1$), we get

$$\begin{aligned}F'(t) &= E'(t)+\varepsilon_1\Phi'(t)+\varepsilon_2\chi'(t)+\varepsilon_3\Psi'(t)\\ &\le -\left[d_1-\varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_1}{2\delta_1}\right)\right.\\ &\quad\left.+\varepsilon_2\left(g_{1,0}-\theta-\theta l-\mu_1\theta_1\right)-\varepsilon_3\mu_1\right]\xi(t)\left(\int_0^L xu_t^2dx\right)\\ &\quad-\left[d_2-\varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_3}{2\delta_2}\right)+\varepsilon_2\left(g_{2,0}-\theta-\theta l-\mu_1\theta_1\right)\right.\\ &\quad\left.-\varepsilon_3\mu_3\right]\xi(t)\left(\int_0^L xv_t^2dx\right)+2\varepsilon_1\xi(t)\\ &\quad\cdot\int_0^L x\left[a|u+v|^{2(r+2)}+2b|uv|^{r+2}\right]dx\\ &\quad+\left(\frac{1}{2}-\frac{\varepsilon_2\xi(0)}{4\theta}C_pg_1(0)\right)\left(g_1'\circ u_x\right)(t)\\ &\quad+\left(\frac{1}{2}-\frac{\varepsilon_2\xi(0)}{4\theta}C_pg_2(0)\right)\left(g_2'\circ v_x\right)(t)\\ &\quad-\left[\frac{\varepsilon_1}{2}\left(l_1-\delta C_pl-2\delta_1\mu_1C_p\right)-\varepsilon_2\theta\right.\\ &\quad\left.\cdot\left(1+c_1+c_1'+2(1-l_1)^2\right)\right]\xi(t)\\ &\quad\cdot\left(\int_0^L xu_x^2dx\right)-\left[\frac{\varepsilon_1}{2}\left(l_2-\delta C_pl-2\delta_2\mu_3C_p\right)\right.\\ &\quad\left.-\varepsilon_2\theta\left(1+c_2+c_2'+2(1-l_2)^2\right)\right]\xi(t)\left(\int_0^L xv_x^2dx\right)\\ &\quad+\left[\frac{\varepsilon_1}{2l_1}+\varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{\mu_1C_p}{2\theta_1}+\frac{C_p+lC_p}{4\theta}\right)\right]\xi(t)\\ &\quad\cdot\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t)\\ &\quad+\left[\frac{\varepsilon_1}{2l_2}+\varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{\mu_1C_p}{2\theta_1}+\frac{C_p+lC_p}{4\theta}\right)\right]\xi(t)\\ &\quad\cdot\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t)\\ &\quad-\left[\varepsilon_3\eta_3-\varepsilon_1\frac{1}{2\delta_1}-\varepsilon_2\theta_1\right]\xi(t)\\ &\quad\cdot\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\\ &\quad-\left[\varepsilon_3\eta_3-\varepsilon_1\frac{1}{2\delta_2}-\varepsilon_2\theta_2\right]\xi(t)\\ &\quad\cdot\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\\ &\quad-\varepsilon_3\eta_4\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\rho\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|y^2\right)d\varrho d\rho dx.\end{aligned} \tag{131}$$

By choosing δ, δ_1, and δ_2 so small that

$$\begin{cases} \left(l_1 - \delta C_p l - 2\mu_1 \delta_1 C_p\right) > \dfrac{l_1}{2}, \\ \left(l_2 - \delta C_p l - 2\mu_3 \delta_2 C_p\right) > \dfrac{l_2}{2}. \end{cases} \tag{132}$$

Then,

$$\begin{aligned} \delta &< \frac{1}{4C_p l} \min\{l_1, l_2\}, \\ \delta_1 &< \frac{1}{8\mu_1 C_p} \min\{l_1, l_2\}, \\ \delta_2 &< \frac{1}{8\mu_3 C_p} \min\{l_1, l_2\}. \end{aligned} \tag{133}$$

At this point, we choose θ small enough, such that

$$\begin{aligned} k_3 &:= \frac{\varepsilon_1 l_1}{4} - \varepsilon_2 \theta \left(1 + c_1 + c_1' + 2(1 - l_1)^2\right) > 0, \\ k_4 &:= \frac{\varepsilon_1 l_2}{4} - \varepsilon_2 \theta \left(1 + c_2 + c_2' + 2(1 - l_2)^2\right) > 0. \end{aligned} \tag{134}$$

Then,

$$\theta < \min\left\{ \frac{\varepsilon_1 l_1}{4\varepsilon_2 \left(1 + c_1 + c_1' + 2(1 - l_1)^2\right)}, \frac{\varepsilon_1 l_2}{4\varepsilon_2 \left(1 + c_2 + c_2' + 2(1 - l_2)^2\right)} \right\}. \tag{135}$$

Now, $\delta, \delta_1, \delta_2$, and θ are fixed. Then, we select $\varepsilon_1, \varepsilon_2, \varepsilon_3$, θ_1, and θ_2 so small that (72) and (162) remain correct and

$$\begin{aligned} k_1 &:= \left[d_1 - \varepsilon_1 \left(1 + \frac{1}{2\delta} + \frac{\mu_1}{2\delta_1}\right) + \varepsilon_2 \left(g_{1,0} - \mu_1 \theta_1 - (1 + l)\theta\right) - \varepsilon_3 \mu_1\right] > 0, \\ k_2 &:= \left[d_2 - \varepsilon_1 \left(1 + \frac{1}{2\delta} + \frac{\mu_3}{2\delta_2}\right) + \varepsilon_2 \left(g_{2,0} - \mu_3 \theta_2 - (1 + l)\theta\right) - \varepsilon_3 \mu_3\right] > 0, \\ k_5 &:= \left(\frac{1}{2} - \frac{\varepsilon_2 \xi(0)}{4\theta} C_p g_1(0)\right) - \left\{ \left[\frac{\varepsilon_1}{2l_1} + \varepsilon_2 \left(\frac{1}{2\theta} + 2\theta + \frac{\mu_1 C_p}{2\theta_1} + \frac{C_p + lC_p}{4\theta}\right)\right] \left(\int_0^t g_1^{2-\sigma}(s) ds\right) \right\} > 0, \\ k_6 &:= \left(\frac{1}{2} - \frac{\varepsilon_2 \xi(0)}{4\theta} C_p g_2(0)\right) - \left\{ \left[\frac{\varepsilon_1}{2l_2} + \varepsilon_2 \left(\frac{1}{2\theta} + 2\theta + \frac{\mu_3 C_p}{2\theta_2} + \frac{C_p + lC_p}{4\theta}\right)\right] \left(\int_0^t g_2^{2-\sigma}(s) ds\right) \right\} > 0, \\ k_7 &:= \varepsilon_3 \eta_3 - \varepsilon_1 \frac{1}{2\delta_1} - \varepsilon_2 \theta_1 > 0, \\ k_8 &:= \varepsilon_3 \eta_3 - \varepsilon_1 \frac{1}{2\delta_2} - \varepsilon_2 \theta_2 > 0. \end{aligned} \tag{136}$$

Hence, by using (15) gives, for some $\sigma > 0$,

$$\begin{aligned} F'(\mathrm{t}) \leq -\sigma \xi(t) \Bigg[&\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx \\ &- \int_0^L x \left[a|u + v|^{2(r+2)} + 2b|uv|^{r+2}\right] dx \\ &+ \int_0^L x u_x^2 dx + \int_0^L x v_x^2 dx + (g_1^\sigma \circ u_x)(t) \\ &+ (g_2^\sigma \circ v_x)(t) + K(z, y) \Bigg]. \end{aligned} \tag{137}$$

We choose θ, θ_1, and θ_2 so small that

$$\begin{aligned} \left(g_{1,0} - \mu_1 \theta_1 - (1 + l)\theta\right) &> \frac{1}{2} g_{1,0}, \\ \left(g_{2,0} - \mu_3 \theta_2 - (1 + l)\theta\right) &> \frac{1}{2} g_{2,0}. \end{aligned} \tag{138}$$

By (134), we get

$$\begin{aligned} &\theta < \min\left\{ \frac{\varepsilon_1 l_1}{4\varepsilon_2 \left(1 + c_1 + c_1' + 2(1 - l_1)^2\right)}, \frac{\varepsilon_1 l_2}{4\varepsilon_2 \left(1 + c_2 + c_2' + 2(1 - l_2)^2\right)}, \frac{1}{4(1 + l)} g_{1,0}, \frac{1}{4(1 + l)} g_{2,0} \right\}, \\ &\theta_1 < \frac{1}{4\mu_1 (1 + l)} g_{1,0}, \theta_2 < \frac{1}{4\mu_3 (1 + l)} g_{2,0}, \\ &\frac{4\theta \left(1 + c_1 + c_1' + 2(1 - l_1)^2\right)}{l_1} < \frac{g_{1,0}}{2 + (l/\delta) + (\mu_1/\delta_1)}, \\ &\frac{4\theta \left(1 + c_2 + c_2' + 2(1 - l_2)^2\right)}{l_2} < \frac{g_{2,0}}{2 + (l/\delta) + (\mu_3/\delta_2)}. \end{aligned} \tag{139}$$

With $\theta, \theta_1, \theta_2$, and α' fixed, we pick $\varepsilon_1, \varepsilon_2$, and ε_3 such that

$$\max\left\{\frac{4\theta\left(1+c_1+c_1'+2(1-l_1)^2\right)}{l_1}, \frac{4\theta\left(1+c_2+c_2'+2(1-l_2)^2\right)}{l_2}\right\}\varepsilon_2 < \varepsilon_1 < \frac{1}{2+(l/\delta)+\min\left((\mu_1/\delta_1),(\mu_3/\delta_2)\right)}\left(\min\left(d_1,d_2\right) + \varepsilon_2 \min\left\{g_{1,0},_{2,0}\right\} + \varepsilon_3 \min\left(\mu_1+\mu_3\right)\right). \quad (140)$$

We will make

$$\begin{cases} k_1 := \left[d_1 - \varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_1}{2\delta_1}\right) + \varepsilon_2\left(g_{1,0}-\mu_1\theta_1-(1+l)\theta\right) - \varepsilon_3\mu_1\right] > 0, \\ k_2 := \left[-\varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_3}{2\delta_2}\right) + \varepsilon_2\left(g_{2,0}-\mu_3\theta_2-\theta-\theta l\right) - \varepsilon_3\mu_3\right] > 0, \\ k_3 := \frac{\varepsilon_1}{2}\left(l_1 - 2\mu_1\delta_1 C_p - \delta C_p l\right) - \varepsilon_2\theta\left(1+c_1+c_1'+2(1-l_1)^2\right) > 0, \\ k_4 := \frac{\varepsilon_1}{2}\left(l_2 - 2\mu_3\delta_2 C_p - \delta C_p l\right) - \varepsilon_2\theta\left(1+c_2+c_2'+2(1-l_2)^2\right) > 0, \\ k_7 := \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_1} - \varepsilon_2\theta_1 > 0, \\ k_8 := \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_2} - \varepsilon_2\theta_2 > 0. \end{cases} \quad (141)$$

Then, we select $\varepsilon_1, \varepsilon_2$, and ε_3 so small that (72) and (137) remain correct and

$$k_5 := \left(\frac{1}{2} - \frac{\varepsilon_2\xi(0)}{4\theta}C_p g_1(0)\right) - \left\{\left[\frac{\varepsilon_1}{2l_1} + \varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{\mu_1 C_p}{2\theta_1} + \frac{C_p+lC_p}{4\theta}\right)\right]\left(\int_0^t g_1^{2-\sigma}(s)ds\right)\right\} > 0,$$
$$k_6 := \left(\frac{1}{2} - \frac{\varepsilon_2\xi(0)}{4\theta}C_p g_2(0)\right) - \left\{\left[\frac{\varepsilon_1}{2l_2} + \varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{\mu_3 C_p}{2\theta_2} + \frac{C_p+lC_p}{4\theta}\right)\right]\left(\int_0^t g_2^{2-\sigma}(s)ds\right)\right\} > 0. \quad (142)$$

Next, as (137) is showed, according to the different ranges of r, we give the following two cases.

Case 1. $\sigma = 1$.

By choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3, \theta_1, \theta_2$, and θ, (137) gives, for $\gamma > 0$ is constant so that,

$$F'(t) \le -\gamma\xi(t)E(t), \quad \forall t \ge t_0. \quad (143)$$

Therefore, with the help of the LHS of (72) and (143), we obtain

$$F'(t) \le -\gamma\alpha_1\xi(t)F(t), \forall t \ge t_0. \quad (144)$$

By integration of (144) over (t_0, t) gives

$$F'(t) \le F(t_0)e^{(-\gamma\alpha_1)\int_{t_0}^t \xi(s)ds}, \quad \forall t \ge t_0. \quad (145)$$

Therefore, $(129)_1$ is proved by (72) as well.

Case 2. $1 < \sigma < 3/2$.

We use (11), which gives

$$g_1(t)^{1-\sigma} \ge (\sigma-1)\left(\int_{t_0}^t \xi(s)ds\right) + g_1(t_0)^{1-\sigma},$$
$$g_2(t)^{1-\sigma} \ge (\sigma-1)\left(\int_{t_0}^t \xi(s)ds\right) + g_2(t_0)^{1-\sigma}. \quad (146)$$

We have, for $0 < \tau < 1$,

$$\int_0^\infty g_1^{1-\tau}(s)ds \le \int_0^\infty \frac{1}{\left[(\sigma-1)\left(\int_{t_0}^t \xi(s)ds\right) + g_1(t_0)^{1-\sigma}\right]^{(1-\tau)/(\sigma-1)}}ds,$$
$$\int_0^\infty g_2^{1-\tau}(s)ds \le \int_0^\infty \frac{1}{\left[(\sigma-1)\left(\int_{t_0}^t \xi(s)ds\right) + g_2(t_0)^{1-\sigma}\right]^{(1-\tau)/(\sigma-1)}}ds. \quad (147)$$

For $0 < \tau < 2-\sigma < 1$, we have $(1-\tau)/(\sigma-1) > 1$ and (15), we find

$$\int_0^\infty g_1^{1-\tau}(s)ds < \infty, \quad \forall 0 < \tau < 2-\sigma,$$
$$\int_0^\infty g_2^{1-\tau}(s)ds < \infty, \quad \forall 0 < \tau < 2-\sigma. \quad (148)$$

From (72) (for $\theta = \tau$ and $\rho = \sigma$) and (55) gives

$$\left(g_1 \circ u_x\right)(t) \le C_1\left(E(0)\int_0^\infty g_1^{1-\tau}(s)ds\right)^{(\sigma-1)/(\sigma-1+\tau)} \cdot \left(\left(g_1^\sigma \circ u_x\right)(t)\right)^{\tau/(\sigma-1+\tau)} \le C_1'\left(\left(g_1^\sigma \circ v_x\right)(t)\right)^{\tau/(\sigma-1+\tau)}. \quad (149)$$

Similarly, we have

$$\left(g_2 \circ v_x\right)(t) \le C_2'\left(\left(g_2^\sigma \circ v_x\right)(t)\right)^{\tau/(\sigma-1+\tau)}, \quad (150)$$

for some $C_1', C_2' > 0$. Hence, $\forall \sigma_1 > 1$, we find

$$\begin{aligned} E^{\sigma_1}(t) &\le C''E^{\sigma_1-1}(0)\left(\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx \right. \\ &\quad \left. + \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx + K(z,y)\right) \\ &\quad + C_1'((g_1 \circ u_x)(t))^{\sigma_1} + C_2'((g_2 \circ v_x)(t))^{\sigma_1} \\ &\le C''E^{\sigma_1-1}(0)\left(\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx \right. \\ &\quad \left. + \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx + K(z,y)\right) \\ &\quad + C_1'((g_1^\sigma \circ u_x)(t))^{\tau\sigma_1/(\sigma-1+\tau)} + C_2'((g_2^\sigma \circ v_x)(t))^{\tau\sigma_1/(\sigma-1+\tau)}. \end{aligned} \tag{151}$$

We choose $\tau = 1/2$ and $\sigma_1 = 2\sigma - 1$ (therefore, $\tau\sigma_1/(\sigma - 1 + \tau) = 1$) and (144); we get, for some $\Gamma > 0$,

$$\begin{aligned} E^{\sigma_1}(t) &\le \Gamma\left[\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx + \int_0^L xv_x^2dx \right. \\ &\quad + K(z,y) - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\ &\quad \left. + (g_1^\sigma \circ u_x)(t) + (g_2^\sigma \circ v_x)(t)\right]. \end{aligned} \tag{152}$$

By combining (72), (137), and (151), we find

$$F'(t) \le -\frac{\sigma}{\Gamma}\xi(t)E^{\sigma_1}(t) \le -\frac{\sigma}{\Gamma}\alpha_1^{\sigma_1}F^{\sigma_1}(t)\xi(t), \forall t \ge t_0. \tag{153}$$

By integrating (153) gives

$$F(t) \le C_1^*\left(1 + \int_{t_0}^t \xi(s)ds\right)^{-(1/(\sigma_1-1))}, \quad \forall t \ge t_0. \tag{154}$$

Hence,

$$\int_{t_0}^\infty F(t)dt \le C_1^* \int_{t_0}^\infty \frac{1}{\left(1 + \int_{t_0}^t \xi(s)ds\right)^{1/(\sigma_1-1)}}dt. \tag{155}$$

From $(1/(\sigma_1 - 1)) > 0$ and $(1 + \int_{t_0}^t \xi(s)ds) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, we find

$$\int_{t_0}^\infty F(t)dt < \infty. \tag{156}$$

Also, we use (24), and we get

$$tF(t) \le \frac{C_1^* t}{\left(1 + \int_{t_0}^t \xi(s)ds\right)^{1/(\sigma_1-1)}} \le C_\sigma. \tag{157}$$

Hence, we find

$$\sup_{t \ge t_0} tF(t) < \infty. \tag{158}$$

From $E(t)$ which is bounded, using (72), (156), and (158) to get

$$\int_{t_0}^\infty F(t)dt + \sup_{t\ge 0}(tF(t)) < \infty. \tag{159}$$

Therefore, using (55) and Lemma 10 (for $\rho = \sigma$) gives

$$\begin{aligned} (g_1 \circ u_x)(t) &\le C_2^*\left(t\|u_x(x,t)\|_H^2 + \int_0^t \|u_x(x,s)\|_H^2 ds\right)^{(\sigma-1)/\sigma} \\ &\quad \times \left(\int_0^t g^\sigma(t-s)\|u_x(x,t) - u_x(x,s)\|_H^2 ds\right) \\ &\le C_2^*\left(tF(t) + \int_{t_0}^t F(s)ds\right)^{(\sigma-1)/\sigma} ((g_1^\sigma \circ u_x)(t))^{1/\sigma} \\ &\le C_3^*((g_1^\sigma \circ u_x)(t))^{1/\sigma}. \end{aligned} \tag{160}$$

This means

$$(g_1^\sigma \circ u_x)(t) \ge C_4((g_1 \circ u_x)(t))^\sigma, \tag{161}$$

$$(g_2^\sigma \circ v_x)(t) \ge C_5((g_2 \circ v_x)(t))^\sigma, \tag{162}$$

for some $C_4, C_5 > 0$.

Then, combining (137), (161), and (162) yields

$$\begin{aligned} F'(t) &\le -C_6\xi(t)\left\{\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx \right. \\ &\quad + \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\ &\quad \left. + K(z,y) + ((g_1 \circ u_x)(t))^\sigma + ((g_2 \circ v_x)(t))^\sigma\right\}, \end{aligned} \tag{163}$$

for some $C_6 > 0$.

As in [1], we obtain

$$\begin{aligned} E^\sigma(t) &\le C_7\xi(t)\left\{\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx \right. \\ &\quad + \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\ &\quad \left. + K(z,y) + ((g_1 \circ u_x t)(t))^\sigma + ((g_2 \circ v_x)(t))^\sigma\right\}, \end{aligned} \tag{164}$$

$\forall t \ge 0$ and some $C_7 > 0$.

Combining (163), (164), and (72), we find

$$F'(t) \le -C_8\xi(t)F^{\sigma}(t), \forall t \ge t_0, \quad (165)$$

for some $C_8 > 0$.

By integrating (163) over (t_0, t), we get

$$F(t) \le C_9\left(1 + \int_{t_0}^{t} \xi(s)ds\right)^{-(1/(\sigma-1))}, \quad \forall t \ge t_0. \quad (166)$$

Hence, $(129)_2$ is showed by (72) as well.

Data Availability

No data were used to support this study.

Conflicts of Interest

This work does not have any conflicts of interest.

Acknowledgments

The researchers would like to thank the Deanship of Scientific Research, Qassim University, for funding the publication of this project.

References

[1] B. Cahlon, D. M. Kulkarni, and P. Shi, "Stepwise stability for the heat equation with a nonlocal constraint," *SIAM Journal on Numerical Analysis*, vol. 32, no. 2, pp. 571–593, 1995.

[2] J. R. Cannon, "The solution of the heat equation subject to the specification of energy," *Quarterly of Applied Mathematics*, vol. 21, no. 2, pp. 155–160, 1963.

[3] V. Capasso and K. Kunisch, "A reaction-diffusion system arising in modelling man-environment diseases," *Quarterly of Applied Mathematics*, vol. 46, no. 3, pp. 431–450, 1988.

[4] Y. S. Choi and K. Y. Chan, "A parabolic equation with nonlocal boundary conditions arising from electrochemistry," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 18, no. 4, pp. 317–331, 1992.

[5] A. Choucha, D. Ouchenane, K. Zennir, and F. Feng, "Global well-posedness and exponential stability results of a class of Bresse-Timoshenko-type systems with distributed delay term," *Mathematical Methods in the Applied Sciences*, vol. 42, pp. 1–26, 2020.

[6] C. Mu and J. Ma, "On a system of nonlinear wave equations with Balakrishnan-Taylor damping," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 65, no. 1, pp. 91–113, 2014.

[7] R. E. Ewing and T. Lin, "A class of parameter estimation techniques for fluid flow in porous media," *Advances in Water Resources*, vol. 14, no. 2, pp. 89–97, 1991.

[8] M. A. Goodrich and M. A. Ragusa, "Hölder continuity of weak solutions of p-Laplacian PDEs with VMO coefficients," *Nonlinear Analysis-Theory Methods and Applications*, vol. 185, pp. 336–355, 2019.

[9] J. Hao and F. Wang, "Energy decay in a Timoshenko-type system for thermoelasticity of type III with distributed delay and past history," *Electronic Journal of Differential Equations*, vol. 2018, pp. 1–27, 2018.

[10] E. Pişkin and F. Ekinci, "General decay and blowup of solutions for coupled viscoelastic equation of Kirchhoff type with degenerate damping terms," *Mathematical Methods in the Applied Sciences*, vol. 46, pp. 5468–5488, 2019.

[11] N. I. Ionkin and E. I. Moiseev, "A problem for the heat conduction equation with two-point boundary condition," *Differentsial'nye Uravneniya*, vol. 15, pp. 1284–1295, 1979.

[12] N. I. Ionkin, "Solution of boundary value problem in heat conduction theory with nonclassical boundary conditions," *Differentsial'nye Uravneniya*, vol. 13, pp. 1177–1182, 1977.

[13] A. V. Kartynnik, "Three-point boundary value problem with an integral space-variable condition for a second order parabolic equation," *Differential Equations*, vol. 26, pp. 1160–1162, 1990.

[14] L. I. Kamynin, "A boundary-value problem in the theory of heat conduction with non-classical boundary conditions," *USSR Computational Mathematics and Mathematical Physics*, vol. 4, pp. 1006–1024, 1964.

[15] M. R. Li and L. Y. Tsai, "Existence and nonexistence of global solutions of some system of semilinear wave equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 54, no. 8, pp. 1397–1415, 2003.

[16] L. S. Pulkina, "A nonlocal problem with integral conditions for hyperbolic equations," *Electronic Journal of Differential Equations*, vol. 45, pp. 1–6, 1999.

[17] L. S. Pul'kina, "TheL^2 solvability of a nonlocal problem with integral conditions for a hyperbolic equation," *Differential Equations*, vol. 36, no. 2, pp. 316–318, 2000.

[18] A. S. Nicaise and C. Pignotti, "Stabilization of the wave equation with boundary or internal distributed delay," *Differential and Integral Equations*, vol. 21, no. 9-10, pp. 935–958, 2008.

[19] M. A. Ragusa and A. Tachikawa, "Partial regularity of the minimizers of quadratic functionals with *VMO* coefficients," *Journal of the London Mathematical Society*, vol. 72, no. 3, pp. 609–620, 2005.

[20] M. A. Ragusa and A. Tachikawa, "Regularity for minimizers for functionals of double phase with variable exponents," *Adv. Nonlinear Anal.*, vol. 9, no. 1, pp. 710–728, 2019.

[21] P. Shi and M. Shillor, "On design of contact patterns in one dimensional thermoelasticity," in *Theoretical Aspects of Industrial Design*, SIAM, Philadelphia, PA, USA, 1992.

[22] P. Shi, "Weak solution to an evolution problem with a nonlocal constraint," *SIAM Journal on Mathematical Analysis*, vol. 24, no. 1, pp. 46–58, 1993.

[23] W. Shuntang, "Blow-up of solutions for a singular nonlocal viscoelastic equation," *Journal of Partial Differential Equations*, vol. 24, pp. 140–149, 2018.

[24] N. I. Yurchuk, "Mixed problem with an integral condition for certain parabolic equations," *Differentsial'nye Uravneniya*, vol. 22, pp. 2117–2126, 1986.

[25] S. Boulaaras, R. Guefaifia, and N. Mezouar, "Global existence and decay for a system of two singular one-dimensional nonlinear viscoelastic equations with general source terms," *Applicable Analysis*, pp. 1–25, 2020.

[26] S. Boulaaras and N. Mezouar, "Global existence and decay of solutions of a singular nonlocal viscoelastic system with a nonlinear source term, nonlocal boundary condition, and localized damping term," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 10, pp. 6140–6164, 2020.

[27] T. A. Apalara, “Uniform decay in weakly dissipative Timoshenko system with internal distributed delay feedbacks,” *Acta Mathematica Scientia*, vol. 36, no. 3, pp. 815–830, 2016.

[28] S. Boulaaras, A. Choucha, D. Ouchenane, and B. Cherif, “Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms,” *Adv. Difference Equ.*, vol. 2020, no. 1, article 310, 2020.

[29] W. Liu, Y. Sun, and G. Li, “On decay and blow-up of solutions for a singular nonlocal viscoelastic problem with a nonlinear source term,” *Topological Methods in Nonlinear Analysis*, vol. 49, pp. 299–323, 2017.

[30] A. Draifia, A. Zarai, and S. Boulaaras, “Global existence and decay of solutions of a singular nonlocal viscoelastic system,” *Rendiconti del Circolo Matematico di Palermo Series 2*, vol. 69, no. 1, pp. 125–149, 2020.

[31] N. Mezouar and S. Boulaaras, “Global existence and decay of solutions of a singular nonlocal viscoelastic system with damping terms,” *Topological Methods in Nonlinear Analysis*, vol. 56, no. 1, pp. 283–312, 2020.