Journal of Function Spaces

Approximation Methods: Theory and Applications

Lead Guest Editor: Tuncer Acar
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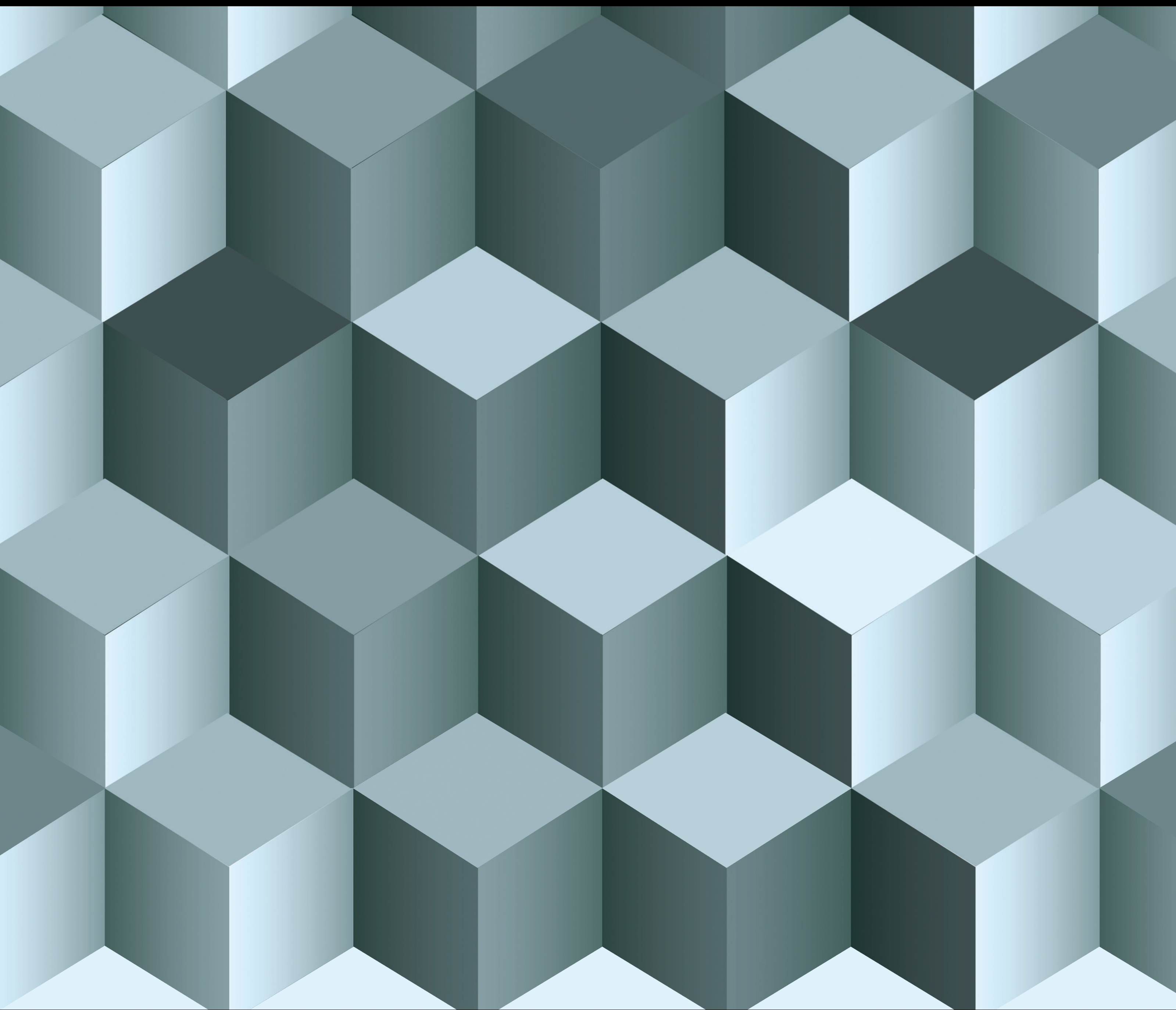

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Contents

Approximation by Bézier Variant of Baskakov-Durrmeyer-Type Hybrid Operators
Lahsen Aharouch, Khursheed J. Ansari, and M. Mursaleen
Research Article (9 pages), Article ID 6673663, Volume 2021 (2021)

Approximation Theorem for New Modification of q-Bernstein Operators on (0,1)
Yun-Shun Wu, Wen-Tao Cheng, Feng-Lin Chen, and Yong-Hui Zhou
Research Article (9 pages), Article ID 6694032, Volume 2021 (2021)

The Approximation of Laplace-Stieltjes Transforms Concerning Sun's Type Function
Xia Shen and Hong Yan Xu
Research Article (10 pages), Article ID 9970287, Volume 2021 (2021)

A New Family of Degenerate Poly-Genocchi Polynomials with Its Certain Properties
Waseem A. Khan, Rifaqat Ali, Khaled Ahmad Hassan Alzobydi, and Naeem Ahmed
Research Article (8 pages), Article ID 6660517, Volume 2021 (2021)

Coincidence Point Results on Relation Theoretic $(F_w, \mathscr{R})_g$-Contractions and Applications
Muhammad Aslam, Hassen Aydi, Samina Batul, and Amna Naz
Research Article (10 pages), Article ID 9937318, Volume 2021 (2021)

Approximation Properties of New Modified Gamma Operators
Yun-Shun Wu, Wen-Tao Cheng, Wei-Ping Zhou, and Lun-Zhi Deng
Research Article (10 pages), Article ID 6696979, Volume 2021 (2021)

Approximation by One and Two Variables of the Bernstein-Schurer-Type Operators and Associated GBS Operators on Symmetrical Mobile Interval
Reşat Aslan and Aydın İzgi
Research Article (12 pages), Article ID 9979286, Volume 2021 (2021)

On Controlled Rectangular Metric Spaces and an Application
Nayab Alamgir, Quanita Kiran, Hassen Aydi, and Yaé Ulrich Gaba
Research Article (9 pages), Article ID 5564324, Volume 2021 (2021)

Some Novel Sixth-Order Iteration Schemes for Computing Zeros of Nonlinear Scalar Equations and Their Applications in Engineering
M. A. Rehman, Amir Naseem, and Thabet Abdeljawad
Research Article (11 pages), Article ID 5566379, Volume 2021 (2021)

On the Porous-Elastic System with Thermoelasticity of Type III and Distributed Delay: Well-Posedness and Stability
Djamel Ouchenane, Abdelbaki Choucha, Mohamed Abdalla, Salah Mahmoud Boulaaras, and Bahri Belkacem Cherif
Research Article (12 pages), Article ID 9948143, Volume 2021 (2021)

Nonunique Coincidence Point Results via Admissible Mappings
Erdal Karapınar, Chi-Ming Chen, and Andreea Fulga
Research Article (10 pages), Article ID 5538833, Volume 2021 (2021)

Differences of Positive Linear Operators on Simplices
Ana-Maria Acu, Gülen Başcanbaz-Tunca, and Ioan Rasa
Research Article (11 pages), Article ID 5531577, Volume 2021 (2021)

Approximation of Functions by Dunkl-Type Generalization of Szász-Durrmeyer Operators Based on (p, q)-Integers
Abdullah Alotaibi
Research Article (8 pages), Article ID 5511610, Volume 2021 (2021)

An Extension of the Picard Theorem to Fractional Differential Equations with a Caputo-Fabrizio Derivative
H. R. Marasi, A. Soltani Joujehi, and H. Aydi
Research Article (6 pages), Article ID 6624861, Volume 2021 (2021)

Soft Fixed Point Theorems for the Soft Comparable Contractions
Chi-Ming Chen, Zhi-Hao Xu, and Erdal Karapinar
Research Article (8 pages), Article ID 5554510, Volume 2021 (2021)

Some Fixed-Point Results via Mix-Type Contractive Condition
Özlem Acar
Research Article (7 pages), Article ID 5512254, Volume 2021 (2021)

Global Well-Posedness for Coupled System of mKdV Equations in Analytic Spaces
Khaled Zennir, Aissa Boukarou, and Rehab Nasser Alkhudhayr
Research Article (11 pages), Article ID 6614375, Volume 2021 (2021)

Phillips-Type q-Bernstein Operators on Triangles
Asif Khan, M. S. Mansoori, Khalid Khan, and M. Mursaleen
Research Article (13 pages), Article ID 6637893, Volume 2021 (2021)

A Novel Value for the Parameter in the Dai-Liao-Type Conjugate Gradient Method
Branislav Ivanov, Predrag S. Stanimirović, Bilall I. Shaini, Hijaz Ahmad, and Miao-Kun Wang
Research Article (10 pages), Article ID 6693401, Volume 2021 (2021)

A Note on the Górnicki-Proinov Type Contraction
Erdal Karapınar, Manuel De La Sen, and Andreea Fulga
Research Article (8 pages), Article ID 6686644, Volume 2021 (2021)

Contents

Quantum Integral Inequalities with Respect to Raina's Function via Coordinated Generalized Ψ-Convex Functions with Applications
Saima Rashid, Saad Ihsan Butt, Shazia Kanwal, Hijaz Ahmad, and Miao-Kun Wang
Research Article (16 pages), Article ID 6631474, Volume 2021 (2021)

New Generalizations of Set Valued Interpolative Hardy-Rogers Type Contractions in b-Metric Spaces
Muhammad Usman Ali, Hassen Aydi, and Monairah Alansari
Research Article (8 pages), Article ID 6641342, Volume 2021 (2021)

On Stancu-Type Generalization of Modified (p, q)-Szász-Mirakjan-Kantorovich Operators
Yong-Mo Hu, Wen-Tao Cheng, Chun-Yan Gui, and Wen-Hui Zhang
Research Article (11 pages), Article ID 6683004, Volume 2021 (2021)

Unified Framework of Approximating and Interpolatory Subdivision Schemes for Construction of Class of Binary Subdivision Schemes
Pakeeza Ashraf, Ghulam Mustafa, Abdul Ghaffar, Rida Zahra, Kottakkaran Sooppy Nisar, Emad E. Mahmoud, and Wedad R. Alharbi
Research Article (12 pages), Article ID 6677778, Volume 2020 (2020)

Some Identities Involving Derangement Polynomials and Numbers and Moments of Gamma Random Variables
Lee-Chae Jang, Dae San Kim, Taekyun Kim, and Hyunseok Lee
Research Article (9 pages), Article ID 6624006, Volume 2020 (2020)

Fixed Point Results via G-Function over the Complete Partial b-Metric Space
Sara Salem Alzaid, Andreea Fulga, and Badr Alqahtani
Research Article (7 pages), Article ID 6666229, Volume 2020 (2020)

A Nonlinear Integral Equation Related to Infectious Diseases
Mohamed Jleli and Bessem Samet
Research Article (7 pages), Article ID 6633708, Volume 2020 (2020)

On $\mathfrak{R}$-Partial b-Metric Spaces and Related Fixed Point Results with Applications
Muhammad Usman Ali, Yajing Guo, Fahim Uddin, Hassen Aydi, Khalil Javed, and Zhenhua Ma
Research Article (8 pages), Article ID 6671828, Volume 2020 (2020)

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Research Article

Approximation by Bézier Variant of Baskakov-Durrmeyer-Type Hybrid Operators

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We give a Bézier variant of Baskakov-Durrmeyer-type hybrid operators in the present article. First, we obtain the rate of convergence by using Ditzian-Totik modulus of smoothness and also for a class of Lipschitz function. Then, weighted modulus of continuity is investigated too. We study the rate of point-wise convergence for the functions having a derivative of bounded variation. Furthermore, we establish the quantitative Voronovskaja-type formula in terms of Ditzian-Totik modulus of smoothness at the end.

1. Introduction

To approximate continuous functions, many approximating operators have been introduced under certain conditions and with different parameters too. Many researchers have later generalized and modified these introduced operators and discussed various approximating properties of these operators. In 1957, Baskakov [1] introduced and studied such a class of positive linear operators, called Baskakov operators defined on the positive semiaxis. For $f \in \mathscr{C}[0,\infty)$, the sequence of Baskakov operators is given as

$$\mathscr{B}_n(f\,;y)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}y^k(1+y)^{-n-k}f\left(\frac{k}{n}\right), \tag{1}$$

for $y\in[0,\infty)$ and $n\in\mathbb{N}$. Later on, many authors have been considering the Baskakov operators; for instance, Aral in [2] defines the parametric generalization of Baskakov operators as

$$\mathscr{B}_n^{\nu}(f\,;x)=\sum_{k=0}^{\infty}\mathscr{P}_{n,k}^{\nu}(x)f\left(\frac{k}{n}\right), \tag{2}$$

where

$$\mathscr{P}_{n,k}^{\nu}(x)=\frac{x^{k-1}}{(1+x)^{n+k-1}}\left[\frac{\nu x}{1+x}\binom{n+k-1}{k}-(1-\nu)(1+x)\cdot\binom{n+k-3}{k-2}+(1-\nu)x\binom{n+k-1}{k}\right], \tag{3}$$

with $\binom{n+k-1}{k-2}=0$ if $k=0,1$.

Among interesting studies realized in this context, we cite those based on the Baskakov-Kantorovich-type operators in the generalized form (the original operator given by Kantorovich in [3]) defined as, for $f\in L^1([0,1])$ (the class of Lebesgue integrable functions on $[0,1]$),

$$\mathscr{B}K_n(f\,;x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k(1+x)^{-n-k}\int_0^1\chi_{n,k}(t)f(t)dt, \tag{4}$$

where $\chi_{n,k}$ is the characteristic function of the interval $[k/n, k+1/n]$.

It is well known that Bézier curves are the mathematically defined curves successively used in computer-aided geometric design (CAGD), image processing, and curve fitting. The miscellaneous Bézier variant of operators is crucial subject matter in approximation theory. In 1983, Chang [4] pioneered the Bernstein-Bézier operators. Afterwards, several researchers established the Bézier variant of various operators (c.f. [5, 6]). For more details on the approximation by Durrmeyer-type and Baskakov-Durrmeyer-type operators, one can refer to [7, 8], respectively. For more about Bézier variant of operators, one can refer to [9, 10].

We will be mainly interested to the Bézier variant operator type based on those of Baskakov-Durrmeyer defined as follows:

$$\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x)=\sum_{k=1}^{\infty}\mathscr{X}_{n,k}^{\nu,\theta}(x)\int_0^{\infty}\mathscr{J}_{n,k}^{\rho}(t)f(t)dt+\mathscr{X}_{n,0}^{\nu,\theta}(x)f(0), \tag{5}$$

where

$$\mathscr{X}_{n,k}^{\nu,\theta}(x)=\left[\xi_{n,k}^{\nu}(x)\right]^{\theta}-\left[\xi_{n,k+1}^{\nu}(x)\right]^{\theta},\xi_{n,k}^{\nu}(x)=\sum_{j=k}^{\infty}\mathscr{P}_{n,j}^{\nu}(x)\ k=0,1,2,\cdots$$
$$\mathscr{J}_{n,k}^{\rho}(t)=n\rho e^{-n\rho t}\frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}. \tag{6}$$

If we take $\theta=1$, then operator (5) reduces to the following operator studied by [11].

$$\mathscr{B}_{n,\rho}^{\nu}(f;x)=\sum_{k=1}^{\infty}\mathscr{P}_{n,k}^{\nu}(x)\int_0^{\infty}\mathscr{J}_{n,k}^{\rho}(t)f(t)dt+\mathscr{P}_{n,0}^{\nu}(x)f(0). \tag{7}$$

Let us briefly summarize the outline of the paper. Next section is devoted to the computation of some auxiliary results which we need to prove our theorems in coming sections. In Section 3, we will prove some approximations of functions using Ditzian-Totik modulus and then we will deal to functions lie in the Lipschitz spaces. We treat in Section 4 the rate of convergence in the context of suitable weighted spaces and functions having a derivative of bounded variation. Finally, in Section 5, we state and prove the quantitative Voronovskaja-type theorem.

2. Preliminary Results

Lemma 1. *$\xi_{n,k}^{\nu}(x)$ satisfies the following important properties:*

(1) $\xi_{n,k}^{\nu}(x)-\xi_{n,k+1}^{\nu}(x)=\mathscr{P}_{n,k}^{\nu}(x)\ k=0,1,2,\cdots$

(2) $\xi_{n,0}^{\nu}(x)>\xi_{n,1}^{\nu}(x)>.\cdots\xi_{n,k}^{\nu}(x)>\xi_{n,k+1}^{\nu}(x)>\cdots$

(3) $[\xi_{n,k}^{\nu}(x)]^{\theta}-[\xi_{n,k+1}^{\nu}(x)]^{\theta}\le\begin{cases}\theta\mathscr{P}_{n,k}^{\nu}(x) & \text{if } \theta\ge 1\\ (\mathscr{P}_{n,k}^{\nu}(x))^{\theta} & \text{if } \theta\le 1\end{cases}$

Proof. Since (1) and (2) are evident, we prove only the assertion (3).

If $\theta\ge 1$, it suffices to remark that by the mean value theorem, we have

$$b^{\theta}-a^{\theta}\le\theta(b-a)\text{ for every }0<a<b<1. \tag{8}$$

If $\theta<1$, we shall prove that

$$b^{\theta}-a^{\theta}\le(b-a)^{\theta}\text{ for every }0<a<b. \tag{9}$$

Dividing this inequality by a^{θ}, it is equivalent to prove that

$$f(r)=(r-1)^{\theta}-r^{\theta}+1\ge 0\text{ for every }r>1. \tag{10}$$

We have $f'(r)=(\theta/(r-1))e^{\theta\ln(r-1)}-(\theta/r)e^{\theta\ln(r)}$; then,

$$f'(r)>0\text{ if and only if }\ln\left(\frac{r}{r-1}\right)>\ln\left(\frac{e^{\theta\ln(r)}}{e^{\theta\ln(r-1)}}\right), \tag{11}$$

and this is true as $\theta<1$.

We proved then f is increasing, so $f(r)>f(s)$ for all $r>s>1$, letting s to 1, and we deduce that $f(r)\ge 0$. ☐

Remark 2. The operators $\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x)$ have the integral representation

$$\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x)=\int_0^{\infty}\mathscr{K}_{n,k}^{\nu,\theta}(x,u)f(u)du, \tag{12}$$

where $\mathscr{K}_{n,k}^{\nu,\theta}(x,u)$ is the kernal defined by

$$\mathscr{K}_{n,k}^{\nu,\theta}(x,u)=\sum_{k=1}^{\infty}\mathscr{X}_{n,k}^{\nu,\theta}(x)\mathscr{J}_{n,k}^{\rho}(u)+\mathscr{X}_{n,0}^{\nu,\theta}(x)\delta(u). \tag{13}$$

$\delta(u)$ is the Dirac-delta function.

Lemma 3. *Let $e_m(t)=t^m$ and $\varphi(t)=1/(1+t)^{n+2}$. For the operator $\mathscr{B}_{n,\rho}^{\nu}(f;x)$, we have*

(1) $\mathscr{B}_{n,\rho}^{\nu}(e_0;x)=\sum_{k=0}^{\infty}\mathscr{P}_{n,k}^{\nu}(x)=\sum_{k=0}^{\infty}\mathscr{P}_{n,k}^{0}(x)=\sum_{k=0}^{\infty}(-1)^k\varphi^{(k)}(x)/k!=\varphi(0)=1$

(2) $\mathscr{B}_{n,\rho}^{\nu}(e_m;x)=\sum_{k=1}^{\infty}\mathscr{P}_{n,k}^{\nu}(x).(k\rho+m-1).(k\rho+m-2)\cdots(k\rho)/(n\rho)^m,\ m=1,2,3,\cdots$

As an easy consequence of last lemma, we will prove the following result.

Lemma 4. *We have the following moments:*

(1) $\mathscr{B}^{\nu}_{n,\rho}(t\,;x) = x + 2x(\nu-1)/n$

(2) $\mathscr{B}^{\nu}_{n,\rho}(t^2\,;x) = x^2 + x^2(4\nu-3)/n + x(-2+n+2\nu+n\rho + 4\rho(\nu-1))/n^2\rho$

(3) $\mathscr{B}^{\nu}_{n,\rho}(t-x\,;x) = 2x(\nu-1)/n$

(4) $\mathscr{B}^{\nu}_{n,\rho}((t-x)^2\,;x) = x^2/n + (x/n^2\rho)n(1+\rho) + 2(\nu-1)(1+2\rho)$

(5) $n^2\mathscr{B}^{\nu}_{n,\rho}((t-x)^4\,;x) = (x^4/n^3)\alpha_1 + (6x^3/n^3\rho)\alpha_2 + (x^2(1+\rho)/n^3\rho^2)\alpha_3 + (x(1+\rho)/n^4\rho^3)\alpha_4$

where

$$\begin{aligned}\alpha_1 &= 3n + 16\nu - 10,\\ \alpha_2 &= n + 6\nu - 4 + \rho(n + 8\nu - 6),\\ \alpha_3 &= 3n(1+\rho) + 4\nu(7+8\rho) - 25\rho - 17,\end{aligned} \tag{14}$$

$$\alpha_4 = n(1+\rho)(3+\rho) + 4(\nu-1)(3+4\rho(2+\rho)). \tag{15}$$

Remark 5. We have

(1) $\lim_{n\longrightarrow\infty} n\mathscr{B}^{\nu}_{n,\rho}(t-x\,;x) = 2x(\nu-1)$

(2) $\lim_{n\longrightarrow\infty} n\mathscr{B}^{\nu}_{n,\rho}((t-x)^2\,;x) = x(1+\rho+\rho x)/\rho$

(3) $\lim_{n\longrightarrow\infty} n^2\mathscr{B}^{\nu}_{n,\rho}((t-x)^4\,;x) = 3(x(1+\rho+\rho x))^2/\rho^2$

Remark 6. For n large enough, we have the following inequalities:

(1) $|\mathscr{B}^{\nu}_{n,\rho}((t-x)^2\,;x)| \le C_1(x(1+\rho+\rho x)/n\rho)$

(2) $|\mathscr{B}^{\nu}_{n,\rho}((t-x)^4\,;x)| \le C_2((x(1+\rho+\rho x))^2/(n\rho)^2)$

Throughout this article, let $\mathscr{C}_B(\mathbb{R}_0^+)$ denote the space of all functions f on $\mathbb{R}_0^+$ which are bounded and continuous. We endowed it by the norm $\|f\| = \sup_{x\in\mathbb{R}_0^+} |f(x)|$.

Lemma 7. *Let $f \in \mathscr{C}_B(\mathbb{R}_0^+)$, and we have*

(1) $\|\mathscr{G}^{\nu,\theta}_{n,\rho}(f\,;x)\| \le \mathscr{G}^{\nu,\theta}_{n,\rho}(e_0\,;x)\|f\|$ *and* $\mathscr{G}^{\nu,\theta}_{n,\rho}(e_0\,;x) = 1$

(2) $\mathscr{G}^{\nu,\theta}_{n,\rho}(f\,;x) \le \theta\mathscr{B}^{\nu}_{n,\rho}(f\,;x) \le \theta\|f\|$

Proof.

(1) On the one hand, we have

$$\begin{aligned}\left|\mathscr{G}^{\nu,\theta}_{n,\rho}(f\,;x)\right| &= \left|\sum_{k=1}^{\infty}\mathscr{X}^{\nu,\theta}_{n,k}(x)\int_0^{\infty}\mathscr{J}^{\rho}_{n,k}(t)f(t)dt + \mathscr{X}^{\nu,\theta}_{n,0}(x)f(0)\right|\\ &\le \left|\sum_{k=1}^{\infty}\mathscr{X}^{\nu,\theta}_{n,k}(x)\int_0^{\infty}\mathscr{J}^{\rho}_{n,k}(t)dt + \mathscr{X}^{\nu,\theta}_{n,0}(x)\right|.\|f\|\\ &\le \mathscr{G}^{\nu,\theta}_{n,\rho}(e_0\,;x)\|f\|.\end{aligned} \tag{16}$$

On the other hand,

$$\mathscr{G}^{\nu,\theta}_{n,\rho}(e_0\,;x) = \sum_{k=0}^{\infty}\mathscr{X}^{\nu,\theta}_{n,k}(x) = \left(\xi^{\nu}_{n,0}(x)\right)^{\theta} = \left(\sum_{k=0}^{\infty}\mathscr{P}^{\nu}_{n,k}(x)\right)^{\theta} = 1^{\theta} = 1. \tag{17}$$

(2) We have

$$\begin{aligned}\mathscr{G}^{\nu,\theta}_{n,\rho}(f\,;x) &= \sum_{k=1}^{\infty}\mathscr{X}^{\nu,\theta}_{n,k}(x)\int_0^{\infty}\mathscr{J}^{\rho}_{n,k}(t)f(t)dt + \mathscr{X}^{\nu,\theta}_{n,0}(x)f(0)\\ &= \sum_{k=1}^{\infty}\left(\left[\xi^{\nu}_{n,k}(x)\right]^{\theta} - \left[\xi^{\nu}_{n,k+1}(x)\right]^{\theta}\right)\int_0^{\infty}\mathscr{J}^{\rho}_{n,k}(t)f(t)dt\\ &\quad + \left(\left[\xi^{\nu}_{n,0}(x)\right]^{\theta} - \left[\xi^{\nu}_{n,1}(x)\right]^{\theta}\right)f(0).\end{aligned} \tag{18}$$

Using Lemma 1, it is easy to see that

$$\begin{aligned}\mathscr{G}^{\nu,\theta}_{n,\rho}(f\,;x) &\le \theta\sum_{k=1}^{\infty}\mathscr{P}^{\nu}_{n,k}(x)\int_0^{\infty}\mathscr{J}^{\rho}_{n,k}(t)f(t)dt + \theta\mathscr{P}^{\nu}_{n,0}(x)f(0)\\ &\le \theta\mathscr{B}^{\nu}_{n,\rho}(f\,;x).\end{aligned} \tag{19}$$

□

3. Direct Approximation

Before we discuss the different approximations, we need some definitions. First, we recall the definition of the well-known Ditizian-Totik modulus of smoothness $w_{\varphi^\tau}(.,.)$ and Peetre's K-functional [12].

Definition 8. Let $\varphi(x) = \sqrt{x}$ and $f \in \mathscr{C}_B(\mathbb{R}_0^+)$. For $0 \le \tau \le 1$, we define

$$w_{\varphi^\tau}(f,\delta) = \sup_{0\le h\le\delta}\ \sup_{x\pm h\varphi^\tau(x)/2\in\mathbb{R}_0^+}\left|f\left(x + \frac{h\varphi^\tau(x)}{2}\right) - f\left(x - \frac{h\varphi^\tau(x)}{2}\right)\right|, \tag{20}$$

and the K-functional

$$K_{\varphi^\tau}(f,\delta)=\inf_{g\in W_\tau}\left\{\|f-g\|+\delta\left\|\varphi^\tau g'\right\|\right\}, \tag{21}$$

where

$$W_\tau=\left\{g\in \mathrm{AC}_{loc}:\left\|\varphi^\tau g'\right\|<\infty\right\}, \tag{22}$$

with AC_{loc} is the set of all absolutely continuous function on every finite subinterval of $\mathbb{R}_0^+$.

Remark 9. $w_{\varphi^\tau}(f,\delta)$ and $K_{\varphi^\tau}(f,\delta)$ are equivalent, that is, there exists a constant $C>0$ such that

$$C^{-1}w_{\varphi^\tau}(f,\delta)\leq K_{\varphi^\tau}(f,\delta)\leq Cw_{\varphi^\tau}(f,\delta). \tag{23}$$

In the next definition, we cite Lipschitz-type functions:

Definition 10 [13]. For $a\geq 0$, $b>0$ to be fixed, the class of two parametric Lipschitz-type functions is defined as

$$\mathrm{Lip}_M^{a,b}(\beta)=\left\{g\in\mathscr{C}_B(\mathbb{R}_0^+):\ |f(y)-f(x)|\leq M\frac{|y-x|^\beta}{(y+ax^2+bx)^{\beta/2}},x,y>0\right\}, \tag{24}$$

where M is any positive constant and $0<\beta\leq 1$.

The space $\mathrm{Lip}_M^{0,1}(\beta)$ is the space $\mathrm{Lip}_M^*(\beta)$ given by Szász [14].

We now proceed with the approximation results.

Theorem 11. *For $f\in\mathscr{C}_B(\mathbb{R}_0^+)$, we have*

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)\right|\leq Cw_{\varphi^\tau}\left(f,\frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}}\right), \tag{25}$$

where w_{φ^τ} is given by (20) and C is a constant free from the choice of n and x.

For the proof of this theorem, we use the following lemma proved in [15].

Lemma 12. *Let $\varphi(x)=\sqrt{x}$ and $0\leq\tau\leq 1$; then, for $f\in W_\tau$ and $x,y>0$, we have*

$$\left|\int_x^y f'(u)du\right|\leq 2^\tau x^{-\tau/2}\|x-y\|\left\|\varphi^\tau f'\right\|. \tag{26}$$

Proof (Theorem 11). Let $g\in W_\tau$. Using Lemma 7, we have

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)\right|&=\left|\mathscr{G}_{n,\rho}^{v,\theta}(f-g;x)\right|+|f(x)-g(x)|\\&\quad+\left|\mathscr{G}_{n,\rho}^{v,\theta}(g;x)-g(x)\right|\\&\leq(1+\theta)\|f(x)-g(x)\|\\&\quad+\left|\mathscr{G}_{n,\rho}^{v,\theta}(g;x)-g(x)\right|.\end{aligned} \tag{27}$$

Since $g(y)=g(x)+\int_x^y g'(u)du$ and $\mathscr{G}_{n,\rho}^{v,\theta}(1;x)=1$, we conclude that

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(g;x)-g(x)\right|=\left|\mathscr{G}_{n,\rho}^{v,\theta}\left(\int_x^y g'(u)du;x\right)\right|. \tag{28}$$

Therefore, Lemma 12 implies

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(g;x)-g(x)\right|\leq 2^\tau x^{-\tau/2}\left\|\varphi^\tau g'\right\|\mathscr{G}_{n,\rho}^{v,\theta}(|x-y|;x). \tag{29}$$

By Cauchy-Schwarz inequality and Remark 6, it is easy to check that

$$\begin{aligned}\mathscr{G}_{n,\rho}^{v,\theta}(|x-y|;x)&=\sqrt{\mathscr{G}_{n,\rho}^{v,\theta}(1;x)}\sqrt{\mathscr{G}_{n,\rho}^{v,\theta}\left((x-y)^2;x\right)}\\&\leq\sqrt{\frac{C_1x\theta(1+\rho+\rho x)}{n\rho}}.\end{aligned} \tag{30}$$

Combining (27)-(30), we get

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)\right|&\leq(1+\theta)\|f(x)-g(x)\|\\&\quad+C_3\left\|\varphi^\tau g'\right\|\frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}}.\end{aligned} \tag{31}$$

Let now taking the infimum over $g\in W_\tau$, and we have

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)\right|\leq C_4K_{\varphi^\tau}\left(f,\frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}}\right). \tag{32}$$

We thank to (26).

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)\right|\leq Cw_{\varphi^\tau}\left(f,\frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}}\right). \tag{33}$$

□

Theorem 13. *For $f\in Lip_M^{a,b}(\beta)$, then for every $n\in\mathbb{N}$, $\rho>0$, $\theta\geq 1$ and $x\in(0,+\infty)$, we have*

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)\right|\leq M\left(\frac{\theta\mathscr{B}_{n,\rho}^{v}\left((y-x)^2;x\right)}{ax^2+bx}\right)^{\beta/2}, \tag{34}$$

where $\mathscr{B}_{n,\rho}^{v}((y-x)^2;x)$ is given in Lemma 4.

Proof. Let $f\in\mathrm{Lip}_M^{a,b}(\beta)$ and $x\in(0+\infty)$, and we have

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)\right|&=\left|\mathscr{G}_{n,\rho}^{v,\theta}(f(y)-f(x);x)\right|\\&\leq\mathscr{G}_{n,\rho}^{v,\theta}(|f(y)-f(x)|;x)\\&\leq\mathscr{G}_{n,\rho}^{v,\theta}\left(M\frac{|y-x|^\beta}{(y+ax^2+bx)^{\beta/2}};x\right).\end{aligned} \tag{35}$$

Let us consider the case $\beta = 1$. By the Cauchy-Schwarz inequality and the fact $\mathscr{G}_{n,\rho}^{\nu,\theta}(1;x) = 1$, we have immediately that

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x) - f(x)\right| &\le \frac{M}{\sqrt{ax^2+bx}}\left(\mathscr{G}_{n,\rho}^{\nu,\theta}\left((y-x)^2;x\right)\right)^{1/2} \\ &\le \frac{M}{\sqrt{ax^2+bx}}\left(\theta\mathscr{B}_{n,\rho}^{\nu}\left((y-x)^2;x\right)\right)^{1/2} \\ &\le M\left(\frac{\theta\mathscr{B}_{n,\rho}^{\nu}\left((y-x)^2;x\right)}{ax^2+bx}\right)^{1/2}.\end{aligned} \tag{36}$$

This proves the result for $\beta = 1$.

If $0 < \beta < 1$, Holder's inequality with exponents $p = 1/\beta$ and $p' = 1/1-\beta$, we get

$$\left|\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x) - f(x)\right| \le \frac{M}{(ax^2+bx)^{\beta/2}}\left(\mathscr{G}_{n,\rho}^{\nu,\theta}(|y-x|;x)\right)^{\beta}. \tag{37}$$

Using again the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x) - f(x)\right| &\le \frac{M}{(ax^2+bx)^{\beta/2}}\left(\mathscr{G}_{n,\rho}^{\nu,\theta}\left((y-x)^2;x\right)\right)^{\beta/2} \\ &\le \frac{M}{(ax^2+bx)^{\beta/2}}\left(\theta\mathscr{B}_{n,\rho}^{\nu}\left((y-x)^2;x\right)\right)^{\beta/2} \\ &\le M\left(\frac{\theta\mathscr{B}_{n,\rho}^{\nu,\theta}\left((y-x)^2;x\right)}{ax^2+bx}\right)^{\beta/2},\end{aligned} \tag{38}$$

and this gives the result. □

4. Rate of Convergence in Weighted Spaces

In this section, we focus about the rate of convergence of operators (5) in the context of suitable weighted function spaces and functions having a derivative of bounded variation. We will use the following spaces:

$$\mathscr{B}_2(\mathbb{R}_0^+) = \left\{f : |f(x)| \le M_f\left(1+x^2\right), M_f \text{ is a constant depend on } f\right\}. \tag{39}$$

Introduce also

$$\begin{aligned}&\mathscr{C}_2(\mathbb{R}_0^+) = \{f \in \mathscr{B}_2(\mathbb{R}_0^+) : f \text{ is continuous}\}, \\ &\mathscr{C}_2^*(\mathbb{R}_0^+) = \left\{f \in \mathscr{C}_2(\mathbb{R}_0^+) : \exists \lim_{x\longrightarrow\infty}\frac{|f(x)|}{1+x^2} < \infty\right\}.\end{aligned} \tag{40}$$

These spaces are endowed with the norm

$$\|f\|_2 = \sup_{x\in\mathbb{R}_0^+}\frac{|f(x)|}{1+x^2}. \tag{41}$$

The weighted modulus of continuity is defined as (see [16])

$$\Omega(f,\delta) = \sup_{x\ge 0}\sup_{|t|<\delta}\frac{|f(x+t)-f(x)|}{1+(x+t)^2}. \tag{42}$$

Theorem 14. *Let $f \in \mathscr{C}_2^*(\mathbb{R}_0^+)$. Then, for $x \in \mathbb{R}_0^+, \rho, \delta > 0$, $\theta \ge 1$ and for large enough n, we have*

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x) - f(x)\right| &\le 2\left(1+x^2\right)\Omega\left(f, \frac{1}{\sqrt{n}}\right) \\ &\times\left[1 + \theta C_1\frac{x(1+\rho+x\rho)}{n\rho} + \sqrt{\theta C_1}\left(\frac{x(1+\rho+x\rho)}{\rho}\right)^{1/2}\right. \\ &\left.\times\left(1+\sqrt{\theta C_2}\frac{x(1+\rho+x\rho)}{n\rho}\right)\right],\end{aligned} \tag{43}$$

where $C_1, C_2 > 1$ are constants independent of x and n.

Proof. Let $u, x \in \mathbb{R}_0^+, \delta > 0$. An immediate consequence of the definition of weighted modulus of continuity is

$$|f(u)-f(x)| \le 2\left(1+x^2\right)\left(1+(u-x)^2\right)\left(1+\frac{|u-x|}{\delta}\right)\Omega(f,\delta). \tag{44}$$

Since $\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x)$ is linear and increasing, we have from (44)

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{\nu,\theta}(|f(u)-f(x)|;x)\right| &\le 2\left(1+x^2\right)\Omega(f,\delta) \\ &\cdot\left[\mathscr{G}_{n,\rho}^{\nu,\theta}\left(\left(1+(u-x)^2\right);x\right) + \mathscr{G}_{n,\rho}^{\nu,\theta}\left(\frac{\left(1+(u-x)^2\right)|u-x|}{\delta};x\right)\right].\end{aligned} \tag{45}$$

Cauchy-Schwarz inequality was applied in the last term, and it gives us

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{\nu,\theta}(|f(u)-f(x)|;x)\right| &\le 2\left(1+x^2\right)\Omega(f,\delta) \\ &\cdot\left[1 + \mathscr{G}_{n,\rho}^{\nu,\theta}\left((u-x)^2;x\right)\frac{1}{\delta}\left(\mathscr{G}_{n,\rho}^{\nu,\theta}\left((u-x)^4;x\right)\right)^{1/2}\right. \\ &\left.\cdot\left(\mathscr{G}_{n,\rho}^{\nu,\theta}\left((u-x)^2;x\right)\right)^{1/2} + \frac{1}{\delta}\left(\mathscr{G}_{n,\rho}^{\nu,\theta}\left((u-x)^2;x\right)\right)^{1/2}\right].\end{aligned} \tag{46}$$

Choosing $\delta = 1/\sqrt{n}$, we get the required result in virtue of Remark 6. □

5. Rate of Convergence for Functions of Bounded Variation

Let $\mathrm{DBV}(\mathbb{R}_0^+)$ be the space of functions on $\mathbb{R}_0^+$ having a derivative of bounded variation on every finite subinterval of $\mathbb{R}_0^+$. Consider the space

$$\mathrm{DBV}_2(\mathbb{R}_0^+) = \left\{ f \in \mathrm{DBV}(\mathbb{R}_0^+) : |f(x)| \le M_f\left(1 + x^2\right) \text{ for some constant } M_f > 0 \right\}. \tag{47}$$

It is known that every function f in $\mathrm{DBV}_2(\mathbb{R}_0^+)$ has a representation of the form

$$f(x) = \int_0^x g(u)\,du + f(0), \tag{48}$$

where g is a function of bounded variation on each finite subinterval of $\mathbb{R}_0^+$.

Lemma 15. *Let $x \in \mathbb{R}_0^+$, and let $\mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)$ be the kernel defined by (13). Then, for $C_1 > 1$ and for n large enough, we have*

(1) $\xi_{n,\rho}^{\nu,\theta}(x;y) = \int_0^y \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)\,du \le \theta C_1 (x(1+\rho+x\rho)/n\rho)\, 1/(x-y)^2,\ 0 \le y < x$

(2) $1 - \xi_{n,\rho}^{\nu,\theta}(x;z) = \int_z^\infty \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)\,du \le \theta C_1 (x(1+\rho+x\rho)/n\rho)\, 1/(z-x)^2,\ x < z < \infty$

Proof. Using Remark 6, we get

$$\begin{aligned}
\xi_{n,\rho}^{\nu,\theta}(x;y) &= \int_0^y \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)\,du \le \int_0^y \left(\frac{u-y}{x-y}\right)^2 \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)\,du \\
&\le \frac{1}{(x-y)^2}\,\mathscr{G}_{n,\rho}^{\nu,\theta}\left((u-x)^2, x\right) \\
&\le \theta C_1 \frac{x(1+\rho+x\rho)}{n\rho}\,\frac{1}{(x-y)^2}.
\end{aligned} \tag{49}$$

Similarly, we can show the second part; hence, the proof is omitted. □

Theorem 16. *Let $f \in DBV_2(\mathbb{R}_0^+)$, and for every $x \in (0,\infty)$, consider the function f_x' defined by*

$$f_x'(u) = \begin{cases} f'(u) - f'(x^-), & \text{if } 0 \le u < x, \\ 0, & \text{if } u = x, \\ f'(u) - f'(x^+), & \text{if } x < u < \infty. \end{cases} \tag{50}$$

Let us denote by $\bigvee_c^d f_x'$ the total variation of f_x' on $[c,d] \subset \mathbb{R}_0^+$. Then, for every $x \in (0,\infty)$ and large n,

$$\begin{aligned}
&\left|\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x) - f(x)\right| \\
&\quad\le \frac{\sqrt{\theta}}{1+\theta}\left|f'(x^+) + \theta f'(x^-)\right|\left(\frac{C_1 x(1+\rho+x\rho)}{n\rho}\right)^{1/2} \\
&\qquad+ \frac{\theta^{3/2}}{1+\theta}\left|f'(x^+) + \theta f'(x^-)\right|\left(\frac{C_1 x(1+\rho+x\rho)}{n\rho}\right)^{1/2} \\
&\qquad+ \theta\frac{C_1(1+\rho+x\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f_x'\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'\right) \\
&\qquad+ \frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f_x'\right) + \theta\frac{C_1(1+\rho+x\rho)}{n\rho}\sum_{0}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{k}} f_x'\right).
\end{aligned} \tag{51}$$

Proof. For any $f \in \mathrm{DBV}_2(\mathbb{R}_0^+)$, from the definition of $f_x'(u)$, we can write

$$\begin{aligned}
f'(u) &= \frac{1}{1+\theta}\left(f'(x^+) + \theta f'(x^-)\right) + \delta_x(u) \\
&\quad\cdot\left(f'(x) - \frac{1}{2}\left(f'(x^+) + f'(x^-)\right)\right) \\
&\quad+ f_x'(u) + \frac{1}{2}\left(f'(x^+) - f'(x^-)\right)\left(\operatorname{sgn}\ (u-x) + \frac{\theta-1}{1+\theta}\right),
\end{aligned} \tag{52}$$

where

$$\delta_x(u) = \begin{cases} 1, & \text{if } u = x, \\ 0, & \text{if } u \ne x. \end{cases} \tag{53}$$

By the fact that $\mathscr{G}_{n,\rho}^{\nu,\theta}(1;x) = 1$, we have

$$\begin{aligned}
\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x) - f(x) &= \mathscr{G}_{n,\rho}^{\nu,\theta}(f(u) - f(x);x) \\
&= \int_0^\infty \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)(f(u) - f(x))\,du \\
&= \int_0^\infty \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)\left(\int_x^u f'(v)\,dv\right)du.
\end{aligned} \tag{54}$$

From (52), we obtain

$$\begin{aligned}
\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x) - f(x) &= \int_0^\infty \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)\left(\int_x^u\left\{\frac{1}{1+\theta}\left(f'(x^+) + \theta f'(x^-)\right)\right\}dv\right)du \\
&\quad+ \int_0^\infty \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)\left(\int_x^u\left\{\frac{1}{2}\left(f'(x^+) - f'(x^-)\right) \times \left(\operatorname{sgn}\ (v-x) + \frac{\theta-1}{1+\theta}\right)\right\}dv\right)du \\
&\quad+ \int_0^\infty \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)\left(\int_x^u \delta_x(v)\left(f'(x) - \frac{1}{2}\left(f'(x^+) + f'(x^-)\right)\right)dv\right)du \\
&\quad+ \int_0^\infty \mathscr{K}_{n,\rho}^{\nu,\theta}(x,u)\left(\int_x^u f_x'(v)\,dv\right)du.
\end{aligned} \tag{55}$$

From the definition of $\delta_x(v)$, it is clear that

$$\int_0^\infty \mathscr{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u \delta_x(v)\left(f'(x)-\frac{1}{2}\left(f'(x^+)+f'(x^-)\right)\right)dv\right)du=0. \tag{56}$$

The first integral on the right hand side of (55) can be estimated as follows:

$$\begin{aligned}&\left|\int_0^\infty \mathscr{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u\left\{\frac{1}{1+\theta}\left(f'(x)++\theta f'(x^-)\right)\right\}dv\right)du\right|\\&\quad\le\frac{1}{1+\theta}\left|f'(x^+)+\theta f'(x^-)\right|\int_0^\infty \mathscr{K}_{n,\rho}^{v,\theta}(x,u)|u-x|du.\end{aligned} \tag{57}$$

Applying the Cauchy-Schwarz inequality and Remark 2, we have, for n large enough,

$$\begin{aligned}&\left|\int_0^\infty \mathscr{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u\left\{\frac{1}{1+\theta}\left(f'(x^+)+\theta f'(x^-)\right)\right\}dv\right)du\right|\\&\quad\le\frac{1}{1+\theta}\left|f'(x^+)+\theta f'(x^-)\right|\sqrt{\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right)}\\&\quad\le\frac{1}{1+\theta}\left|f'(x^+)+\theta f'(x^-)\right|\left(\frac{C_1\theta x(1+\rho+x\rho)}{n\rho}\right)^{1/2}\\&\quad\le\frac{\sqrt{\theta}}{1+\theta}\left|f'(x^+)+\theta f'(x^-)\right|\left(\frac{C_1 x(1+\rho+x\rho)}{n\rho}\right)^{1/2}.\end{aligned} \tag{58}$$

Similarly, it is easy to find

$$\begin{aligned}&\left|\int_0^\infty \mathscr{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u\left\{\frac{1}{2}\left(f'(x^+)-f'(x^-)\right)\left(\operatorname{sgn}\ (v-x)+\frac{\theta-1}{1+\theta}\right)\right\}dv\right)du\right|\\&\quad\le\frac{\theta^{3/2}}{1+\theta}\left|f'(x^+)+\theta f'(x^-)\right|\left(\frac{C_1x(1+\rho+x\rho)}{n\rho}\right)^{1/2}.\end{aligned} \tag{59}$$

Write the last term of (55) as

$$\int_0^\infty \mathscr{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u f'_x(v)dv\right)du=\mathscr{A}_{n,\rho}^{v,\theta}\left(f'_x;x\right)+\mathscr{B}_{n,\rho}^{v,\theta}\left(f'_x;x\right), \tag{60}$$

where

$$\begin{aligned}\mathscr{A}_{n,\rho}^{v,\theta}\left(f'_x;x\right)&=\int_0^x \mathscr{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u f'_x(v)dv\right)du,\\\mathscr{B}_{n,\rho}^{v,\theta}\left(f'_x;x\right)&=\int_x^\infty \mathscr{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u f'_x(v)dv\right)du.\end{aligned} \tag{61}$$

Now, we estimate the terms $\mathscr{A}_{n,\rho}^{v,\theta}(f'_x;x)$ and $\mathscr{B}_{n,\rho}^{v,\theta}(f'_x;x)$.

Using the definition of $\xi_{n,\rho}^{v,\theta}(.;.)$ given in Lemma 15 and integrating by parts, we can write

$$\begin{aligned}\mathscr{A}_{n,\rho}^{v,\theta}\left(f'_x;x\right)&=\int_0^x\left(\int_x^u f'_x(v)dv\right)\frac{\partial\xi_{n,\rho}^{v,\theta}(x;u)}{\partial u}du\\&=\int_0^x f'_x(u)\xi_{n,\rho}^{v,\theta}(x;u)du.\end{aligned} \tag{62}$$

Thus,

$$\begin{aligned}\left|\mathscr{A}_{n,\rho}^{v,\theta}\left(f'_x;x\right)\right|\le&\int_0^{x-x/\sqrt{n}}\left|f'_x(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du\\&+\int_{x-x/\sqrt{n}}^x\left|f'_x(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du.\end{aligned} \tag{63}$$

Since $f'_x(x)=0$ and $\xi_{n,\rho}^{v,\theta}(x;u)\le 1$, we get

$$\begin{aligned}&\int_{x-x/\sqrt{n}}^x\left|f'_x(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du\\&\quad=\int_{x-x/\sqrt{n}}^x\left|f'_x(u)-f'_x(x)\right|\xi_{n,\rho}^{v,\theta}(x;u)du\\&\quad\le\int_{x-x/\sqrt{n}}^x\left|f'_x(u)-f'_x(x)\right|du\le\int_{x-x/\sqrt{n}}^x\left(\bigvee_u^x f'_x\right)du\\&\quad\le\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).\end{aligned} \tag{64}$$

Concerning the first integral on the right hand side of (63), using Lemma 15, we have

$$\begin{aligned}&\int_0^{x-x/\sqrt{n}}\left|f'_x(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du\\&\quad\le\theta\frac{C_1x(1+\rho+x\rho)}{n\rho}\int_0^{x-x/\sqrt{n}}\frac{\left|f'_x(u)\right|}{(x-y)^2}du\\&\quad=\theta\frac{C_1x(1+\rho+x\rho)}{n\rho}\int_0^{x-x/\sqrt{n}}\frac{\left|f'_x(u)-f'_x(x)\right|}{(x-y)^2}du\\&\quad=\theta\frac{C_1x(1+\rho+x\rho)}{n\rho}\int_0^{x-x/\sqrt{n}}\left(\bigvee_u^x f'_x\right)\frac{du}{(x-y)^2}.\end{aligned} \tag{65}$$

By changing of variable $u=x-x/v$, we deduce that

$$\begin{aligned}\int_0^{x-x/\sqrt{n}}\left|f'_x(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du&\le\theta\frac{C_1(1+\rho+x\rho)}{n\rho}\int_1^{\sqrt{n}}\left(\bigvee_{x-\frac{x}{v}}^x f'_x\right)dv\\&\le\theta\frac{C_1(1+\rho+x\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^x f'_x\right).\end{aligned} \tag{66}$$

Therefore,

$$\left|\mathscr{A}_{n,\rho}^{\nu,\theta}\left(f'_x;x\right)\right| \le \theta\frac{C_1(1+\rho+x\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right). \tag{67}$$

What concerns the second term of the right hand side of (60), integrating by parts and Lemma 15 with $z = x + x/\sqrt{n}$, we can write

$$\begin{aligned}\left|\mathscr{B}_{n,\rho}^{\nu,\theta}\left(f'_x;x\right)\right| &\le \left|\int_x^z f'_x(u)\left(1-\xi_{n,\rho}^{\nu,\theta}(x;u)\right)du\right| \\ &\quad+\left|\int_z^\infty f'_x(u)\left(1-\xi_{n,\rho}^{\nu,\theta}(x;u)\right)du\right| \\ &\le \int_x^z \bigvee_x^u f'_x du+\theta\frac{C_1 x(1+\rho+x\rho)}{n\rho}\int_z^\infty \bigvee_x^u f'_x\frac{1}{(u-x)^2}du \\ &\le \frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x\right)+\theta\frac{C_1 x(1+\rho+x\rho)}{n\rho} \\ &\quad\cdot\int_{x+x/\sqrt{n}}^\infty \bigvee_x^u f'_x(u-x)^{-2}du.\end{aligned} \tag{68}$$

Putting $u = x + x/v$, we get

$$\begin{aligned}&\theta\frac{C_1 x(1+\rho+x\rho)}{n\rho}\int_{x+x/\sqrt{n}}^\infty \bigvee_x^u f'_x(u-x)^{-2}du \\ &\quad\le\theta\frac{C_1(1+\rho+x\rho)}{n\rho}\int_0^{\sqrt{n}}\bigvee_x^{x+\frac{x}{v}} f'_x dv \\ &\quad\le\theta\frac{C_1(1+\rho+x\rho)}{n\rho}\sum_0^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}} f'_x\right).\end{aligned} \tag{69}$$

Combining (68) and (69), we have

$$\left|\mathscr{B}_{n,\rho}^{\nu,\theta}\left(f'_x;x\right)\right| \le \frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x\right)+\theta\frac{C_1(1+\rho+x\rho)}{n\rho}\sum_0^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}} f'_x\right). \tag{70}$$

Finally, by combining (52)-(70), we get (51). □

6. Quantitative Voronovskaja-Type Asymptotic Formula

In this last section, we deal with the Voronovskaja-type asymptotic theorem for $\mathscr{G}_{n,\rho}^{\nu,\theta}$. More precisely we will prove the following result:

Theorem 17. *For $f\in\mathscr{C}_B(\mathbb{R}_0^+)$ such that $f', f''\in\mathscr{C}_B(\mathbb{R}_0^+)$. Then,*

$$\begin{aligned}&\left|n\left\{\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x)-f(x)-f'(x)\mathscr{G}_{n,\rho}^{\nu,\theta}(u-x;x)\right.\right. \\ &\quad\left.\left.-\frac{1}{2}f''(x)\mathscr{G}_{n,\rho}^{\nu,\theta}\left((u-x)^2;x\right)\right\}\right| \\ &\quad\le C\theta\frac{x(1+\rho+x\rho)}{\rho}w_{\varphi^\tau}\left(f,\frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}}\right),\end{aligned} \tag{71}$$

where C is independent of n and x.

Proof. By Taylor's formula, we write

$$f(u)=f(x)+(u-x)f'(x)+\int_x^u(u-v)f''(v)dv. \tag{72}$$

It is clear that

$$\begin{aligned}&f(u)-f(x)-(u-x)f'(x)-\frac{1}{2}(u-x)^2f''(x) \\ &\quad=\int_x^u(u-v)\left(f''(v)-f''(x)\right)dv.\end{aligned} \tag{73}$$

On the one hand, we apply $\mathscr{G}_{n,\rho}^{\nu,\theta}(.;x)$ to both sides of the above equality, and we get

$$\begin{aligned}&\left|\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x)-f(x)-f'(x)\mathscr{G}_{n,\rho}^{\nu,\theta}(u-x;x)\right. \\ &\quad\left.-\frac{1}{2}f''(x)\mathscr{G}_{n,\rho}^{\nu,\theta}\left((u-x)^2;x\right)\right| \\ &\quad=\left|\mathscr{G}_{n,\rho}^{\nu,\theta}\left(\int_x^u(u-v)\left(f''(v)-f''(x)\right)dv;x\right)\right| \\ &\quad\le\mathscr{G}_{n,\rho}^{\nu,\theta}\left(\left|\int_x^u(u-v)\left(f''(v)-f''(x)\right)dv\right|;x\right).\end{aligned} \tag{74}$$

On the other hand, for $g\in W_\tau$, we have

$$\begin{aligned}&\left|\int_x^u(u-v)\left(f''(v)-f''(x)\right)dv\right| \\ &\quad\le\left\|f''-g\right\|(u-x)^2+2^\tau\varphi^{-\tau}\left\|\varphi^\tau g'\right\||u-x|^3,\end{aligned} \tag{75}$$

which implies, by (74),

$$\begin{aligned}&\left|\mathscr{G}_{n,\rho}^{\nu,\theta}(f;x)-f(x)-f'(x)\mathscr{G}_{n,\rho}^{\nu,\theta}(u-x;x)\right. \\ &\quad\left.-\frac{1}{2}f''(x)\mathscr{G}_{n,\rho}^{\nu,\theta}\left((u-x)^2;x\right)\right| \\ &\quad\le\left\|f''-g\right\|\mathscr{G}_{n,\rho}^{\nu,\theta}\left((u-x)^2;x\right) \\ &\quad\quad+2^\tau\varphi^{-\tau}\left\|\varphi^\tau g'\right\|\mathscr{G}_{n,\rho}^{\nu,\theta}\left(|u-x|^3;x\right).\end{aligned} \tag{76}$$

After using the Cauchy-Schwarz inequality in the last term, we obtain

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)-f'(x)\mathscr{G}_{n,\rho}^{v,\theta}(u-x;x)-\frac{1}{2}f''(x)\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right)\right| \le 2^\tau\varphi^{-\tau}\left\|\varphi^\tau g'\right\|\left(\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^4;x\right)\right)^{1/2}\left(\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right)\right)^{1/2} + \left\|f''-g\right\|\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right). \tag{77}$$

In view of Remark 6, we have

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)-f'(x)\mathscr{G}_{n,\rho}^{v,\theta}(u-x;x)-\frac{1}{2}f''(x)\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right)\right| &\le 2^\tau\varphi^{-\tau}\left\|\varphi^\tau g'\right\|\left(C_1\theta\frac{x(1+\rho+x\rho)}{n\rho}\right)^{1/2}\left(C_2\theta\left(\frac{x(1+\rho+x\rho)}{n\rho}\right)^2\right)^{1/2} \\ &\quad + \left\|f''-g\right\|C_1\theta\frac{x(1+\rho+x\rho)}{n\rho} \\ &\le C_1\theta\frac{x(1+\rho+x\rho)}{n\rho}\left\{\left\|f''-g\right\|+M^*\frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}}\left\|\varphi^\tau g'\right\|\right\}.\end{aligned} \tag{78}$$

Taking the infimum on the right-hand side of the above inequality over $g \in W_\tau$, we get

$$\begin{aligned}&\left|n\left\{\mathscr{G}_{n,\rho}^{v,\theta}(f;x)-f(x)-f'(x)\mathscr{G}_{n,\rho}^{v,\theta}(u-x;x) - \frac{1}{2}f''(x)\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right)\right\}\right| \\ &\quad\le C\theta\frac{x(1+\rho+x\rho)}{\rho}K_{\varphi^\tau}\left(f,M^*\frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}}\right).\end{aligned} \tag{79}$$

Recalling (23), the theorem is proved. □

Data Availability

We do not have any data supporting our results.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Research Article

Approximation Theorem for New Modification of q-Bernstein Operators on (0,1)

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In this work, we extend the works of F. Usta and construct new modified q-Bernstein operators using the second central moment of the q-Bernstein operators defined by G. M. Phillips. The moments and central moment computation formulas and their quantitative properties are discussed. Also, the Korovkin-type approximation theorem of these operators and the Voronovskaja-type asymptotic formula are investigated. Then, two local approximation theorems using Peetre's **K**-functional and Steklov mean and in terms of modulus of smoothness are obtained. Finally, the rate of convergence by means of modulus of continuity and three different Lipschitz classes for these operators are studied, and some graphs and numerical examples are shown by using Matlab algorithms.

1. Introduction

In [1], Phillips introduced q-analogue of Bernstein operators as follows:

$$B_l^q(\zeta;z)=\sum_{i=0}^{l}\zeta\left(\frac{[i]_q}{[l]_q}\right)p_{l,i}^q(z), z\in[0,1], \tag{1}$$

where $p_{l,i}^q(z)=\begin{pmatrix} l \\ i \end{pmatrix}_q z^i(1-z)_q^{l-i}, i=0,1,\cdots,l,$ and $\zeta\in C[0,1]$. Later, generalizations of q-Bernstein operators (1) attracted a lot of interest and were constructed and researched widely by a number of researchers. For instance, in [2], Mahmudov and Sabancigil introduced q-Bernstein-Kantorovich operators and studied local and global approximation properties. In [3], Acu et al. defined modified q-Bernstein-Kantorovich operators and established the shape-preserving properties of these operators, e.g., monotonicity and convexity. Some other papers also mention Bernstein operators with parameter(s) and their modification: λ-Bernstein operators [4], α-Bernstein operators [5–8], (α,q)-Bernstein operators [9], (p,q)-Bernstein operators [10], (p,q)-Bernstein-Stancu operators [11], (p,q)-Bernstein-Kantorovich operators [12], generalized Bernstein operators [13], and so on.

In this article, we consider q-analogue of the following new Bernstein operators constructed by the second central moment of the classic Bernstein operators which was given in [14].

$$B_l^*(\zeta;z)=\frac{1}{l}\sum_{i=0}^{l}\begin{pmatrix} l \\ i \end{pmatrix}(i-lz)^2z^{i-1}(1-z)^{l-i-1}\zeta\left(\frac{i}{l}\right). \tag{2}$$

There are many papers about the research and application of q-operators, and we mention some of them: q-Bleimann-Butzer-Hahn operators [15], Bivariater q-Meyer-König-Zeller operators [16], q-Baskakov operators [17, 18], q-Meyer-König-Zeller-Durrmeyer operators [19], q-Phillips operators [20, 21], q-Szász operators [22], q-Bernstein operators [23], and so on. All this achievement motivates us to construct the q-analogue of the operators (2). Before continuing further, let us recall some useful concepts and notations from

q-calculus, which can be found in [24]. For nonnegative integer l, the q-integer $[l]_q$, q-factorial $[l]_q!$, and q-binomial coefficients $\begin{pmatrix} l \\ i \end{pmatrix}_q$ are defined by

$$[l]_q = 1 + q + \cdots + q^{l-1} = \begin{cases} \dfrac{1-q^l}{1-q}, & q \neq 1, \\ l, & q = 1, \end{cases}$$
$$[l]_q! = \begin{cases} [1]_q[2]_q \cdots [l]_q, & l \geq 1, \\ 1, & l = 0, \end{cases} \tag{3}$$
$$\begin{pmatrix} l \\ i \end{pmatrix}_q = \frac{[l]_q!}{[i]_q![l-i]_q!}, i = 0, 1, \cdots, l.$$

Further, q-power basis can be defined by

$$(x-y)_q^l = \begin{cases} (x-y)(x-qy)\cdots\left(x-q^{l-1}y\right), & l = 1, 2, \cdots, \\ 1, & l = 0, \\ \left(x-q^{-1}y\right)\cdots\left(x-q^l y\right), & l = -1, -2, \cdots. \end{cases} \tag{4}$$

The q-derivative $D_q\zeta$ of a function ζ can be defined by

$$\left(D_q\zeta\right)(z) = \frac{\zeta(z) - \zeta(qz)}{(1-q)z}, \text{if } z \neq 0, \tag{5}$$

and $(D_q\zeta)(0) = \zeta'(0)$ provided $\zeta'(0)$ exists. High-order q-derivatives can be defined by $D_q^0\zeta = \zeta$, $D_q^n\zeta = D_q(D_q^{n-1}\zeta)$, $n = 1, 2, \cdots$. The formula for the q-derivative of a product is

$$D_q(\zeta(x)\eta(x)) = D_q(\zeta(x))\eta(x) + D_q(\eta(x))\zeta(qx). \tag{6}$$

The q-analogue of new-Bernstein operators (2) on $(0, 1)$ is defined by the following:

$$\mathfrak{B}_l^q(\zeta; z) = \frac{1}{[l]_q}\sum_{i=0}^{l}\begin{pmatrix} l \\ i \end{pmatrix}_q z^{i-1}(1-qz)_q^{l-i-1}\left([i]_q - [l]_q z\right)^2 \zeta\left(\frac{[i]_q}{[l]_q}\right), \tag{7}$$

where $q \in (0, 1]$, $l \in \mathbb{N}_+ := \{1, 2, \cdots\}$, $z \in (0, 1)$, and $\zeta \in C[0, 1]$.

The rest of the paper is organized as follows: In Section 2, we get the basic results by the moment computation formulas. And the first, second, fourth, and sixth order central moment computation formulas and limit equalities are also computed. In Sections 3 and 4, we investigate the Korovkin-type approximation theorem and the Voronovskaja-type asymptotic formula for the operators (7). In Section 5, we obtain two local approximation theorems using Peetre's $\mathbf{K}$-functional and Steklov mean and in terms of modulus of smoothness. In Section 6, we study the rate of convergence by means of modulus of continuity and three different Lipschitz classes for these operators (7). In Section 7, we show some graphs and numerical examples to analyze the theoretical results by using Matlab algorithms.

2. Auxiliary Lemmas

In this section, we present certain auxiliary results which will be used to prove our main theorems for the operators (7). Using the lemma in ([25], Lemma 2), we have the moment computation formulas for the operators (1):

Lemma 1. *If we define $T_{l,m}^q(z) = B_l^q(u^m; z)$, then there holds the following relation:*

$$T_{l,0}^q(z) = 1, T_{l,1}^q(z) = z, T_{l,2}^q(z) = z^2 + \frac{z(1-z)}{[l]_q}, \tag{8}$$

$$[l]_q T_{l,m+1}^q(z) = z(1-z)D_q\left(T_{l,m}^q(z)\right) + [l]_q z T_{l,m}^q(z), m = 0, 1, 2, \cdots. \tag{9}$$

Now, we give the moment relation between the operators (1) and the operators (7) as follows:

Lemma 2. *If we define $\mathfrak{T}_{l,m}^q(z) = \mathfrak{B}_l^q(u^m; z)$, then there holds the following relation:*

$$\frac{z(1-z)}{[l]_q}\mathfrak{T}_{l,m}^q(z) = T_{l,m+2}^q(z) - 2zT_{l,m+1}^q(z) + z^2T_{l,m}^q(z), \tag{10}$$

$$\mathfrak{T}_{l,m}^q(z) = D_q\left(T_{l,m+1}^q(z)\right) - zD_q\left(T_{l,m}^q(z)\right), \tag{11}$$

$$\mathfrak{T}_{l,m}^q(z) = \frac{1}{[l]_q}D_q\left(z(1-z)D_q\left(T_{l,m}^q(z)\right)\right) + T_{l,m}^q(qz), \tag{12}$$

where $q \in (0, 1]$, $l \in \mathbb{N}_+ := \{1, 2, \cdots\}$, $m \in \mathbb{N} := \{0, 1, 2, \cdots\}$, and $z \in (0, 1)$.

Proof. By the definition of B_l^q and $\mathfrak{B}_l^q$, we can obtain

$$\begin{aligned} \frac{z(1-z)}{[l]_q}\mathfrak{T}_{l,m}^q(z) &= \frac{z(1-z)}{[l]_q^2}\sum_{i=0}^{l}\begin{pmatrix} l \\ i \end{pmatrix}_q z^{i-1}(1-qz)_q^{l-i-1} \\ &\quad\cdot\left([i]_q - [l]_q z\right)^2\left(\frac{[i]_q}{[l]_q}\right)^m \\ &= \sum_{i=0}^{l}\begin{pmatrix} l \\ i \end{pmatrix}_q z^i(1-z)_q^{l-i}\left(\frac{[i]_q}{[l]_q} - z\right)^2\left(\frac{[i]_q}{[l]_q}\right)^m \\ &= T_{l,m+2}^q(z) - 2zT_{l,m+1}^q(z) + z^2T_{l,m}^q(z). \end{aligned} \tag{13}$$

Next, by (9) and (10), we can obtain

$$\begin{aligned} z(1-z)\mathfrak{T}^{q}_{l,m}(z) &= [l]_q T^{q}_{l,m+2}(z) - 2[l]_q z T^{q}_{l,m+1}(z) + [l]_q z^2 T^{q}_{l,m}(z) \\ &= z(1-z)D_q\left(T^{q}_{l,m+1}(z)\right) - [l]_q z T^{q}_{l,m+1}(z) \\ &\quad + [l]_q z^2 T^{q}_{l,m}(z) = z(1-z)D_q\left(T^{q}_{l,m+1}(z)\right) \\ &\quad - z\left([l]_q T^{q}_{l,m+1}(z) + [l]_q z T^{q}_{l,m}(z)\right) \\ &= z(1-z)D_q\left(T^{q}_{l,m+1}(z)\right) - z^2(1-z)D_q\left(T^{q}_{l,m}(z)\right). \end{aligned} \tag{14}$$

Thus, we complete the proof of (11). Finally, by (6), (9), and (11), we can obtain

$$\begin{aligned} \mathfrak{T}^{q}_{l,m}(z) &= D_q\left(T^{q}_{l,m+1}(z)\right) - zD_q\left(T^{q}_{l,m}(z)\right) \\ &= D_q\left(\frac{z(1-z)}{[l]_q}D_q\left(T^{q}_{l,m}(z)\right) + zT^{q}_{l,m}(z)\right) - zD_q\left(T^{q}_{l,m}(z)\right) \\ &= \frac{1}{[l]_q}D_q\left(z(1-z)D_q\left(T^{q}_{l,m}(z)\right)\right) + D_q\left(zT^{q}_{l,m}(z)\right) - zD_q\left(T^{q}_{l,m}(z)\right) \\ &= \frac{1}{[l]_q}D_q\left(z(1-z)D_q\left(T^{q}_{l,m}(z)\right)\right) + T^{q}_{l,m}(qz). \end{aligned} \tag{15}$$

We complete the proof of the Lemma 2. □

Then, the following lemma can be obtain immediately:

Lemma 3. *For $q \in (0,1]$, $l \in \mathbb{N}_+$, and $z \in (0,1)$, we have*

$$\mathfrak{T}^{q}_{l,0}(z) = 1, \mathfrak{T}^{q}_{l,1}(z) = \left(q - \frac{[2]_q}{[l]_q}\right)z + \frac{1}{[l]_q},$$

$$\begin{aligned} \mathfrak{T}^{q}_{l,2}(z) &= \left(q^2 - \frac{q^2 + [2]_q[3]_q}{[l]_q} + \frac{[2]_q[3]_q}{[l]_q^2}\right)z^2 \\ &\quad + \left(\frac{q + [2]_q^2}{[l]_q} - \frac{[2]_q^2 + [2]_q}{[l]_q^2}\right)z + \frac{1}{[l]_q^2}, \end{aligned}$$

$$\alpha^{q}_{l}(z) := \mathfrak{B}^{q}_{l}(u - z; z) = (q-1)z + \frac{1 - [2]_q z}{[l]_q},$$

$$\begin{aligned} \beta^{q}_{l}(z) := \mathfrak{B}^{q}_{l}\left((u-z)^2; z\right) &= \left((q-1)^2 - \frac{q^3 + 3q^2 - 1}{[l]_q} + \frac{[2]_q[3]_q}{[l]_q^2}\right)z^2 \\ &\quad + \left(\frac{q^2 + 3q - 1}{[l]} - \frac{[2]_q^2 + [2]_q}{[l]_q^2}\right)z + \frac{1}{[l]_q^2}. \end{aligned} \tag{16}$$

Lemma 4. *The sequence (q_l) satisfies $q_l \in (0,1]$, such that $q_l \longrightarrow 1$ and $q_l^l \longrightarrow k \in [0,1]$ as $l \longrightarrow \infty$; then, for any $z \in (0,1)$, we have*

$$\lim_{l\longrightarrow\infty} [l]_{q_l}\alpha^{q_l}_{l}(z) = 1 + (k-3)z, \tag{17}$$

$$\lim_{l\longrightarrow\infty} [l]_{q_l}\beta^{q_l}_{l}(z) = 3z(1-z), \tag{18}$$

$$\lim_{l\longrightarrow\infty} [l]_{q_l}\mathfrak{B}^{q_l}_{l}\left((u-z)^4; z\right) = 0. \tag{19}$$

Proof. First, we prove the limit $\lim_{l\longrightarrow\infty} [l]_{q_l} = +\infty$. In fact, for any $\mathbf{q} \in (0,1)$, such sufficiently large $l_0 \in \mathbb{N}_+$ that $[l_0]_{\mathbf{q}} > 1/2(1-\mathbf{q})$. But for $l > l_0$ such that $q_l > \mathbf{q}$, we easily have $[l]_{q_l} > [l_0]_{\mathbf{q}}$. Applying Lemma 3, we can directly obtain (17) and (18). As $l \longrightarrow \infty$, using Lemma 1, we can rewrite

$$\begin{aligned} T^{q_l}_{l,3}(z) &= z^3 + \frac{[2]_{q_l} + 1}{[l]_{q_l}}z^2(1-z) + o\left(\frac{1}{[l]_{q_l}}\right), \\ T^{q_l}_{l,4}(z) &= z^4 + \frac{[3]_{q_l} + [2]_{q_l} + 1}{[l]_{q_l}}z^3(1-z) + o\left(\frac{1}{[l]_{q_l}}\right). \end{aligned} \tag{20}$$

Applying (12), we can obtain

$$\mathfrak{T}^{q_l}_{l,2}(z) = \left(q_l^2 - \frac{q_l^2 + [2]_{q_l}[3]_{q_l}}{[l]_{q_l}}\right)z^2 + \frac{q_l + [2]_{q_l}^2}{[l]_{q_l}}z + o\left(\frac{1}{[l]_{q_l}}\right),$$

$$\begin{aligned} \mathfrak{T}^{q_l}_{l,3}(z) &= \left(q_l^3 - \frac{q_l^3\left(1 + [2]_{q_l}\right) + [3]_{q_l}[4]_{q_l}}{[l]_{q_l}}\right)z^3 \\ &\quad + \frac{[3]_{q_l}^2 + q_l^2\left(1 + [2]_{q_l}\right)}{[l]_{q_l}}z^2 + o\left(\frac{1}{[l]_{q_l}}\right), \end{aligned}$$

$$\begin{aligned} \mathfrak{T}^{q_l}_{l,4}(z) &= \left(q_l^4 - \frac{q_l^4\left(1 + [2]_{q_l} + [3]_{q_l}\right) + [4]_{q_l}[5]_{q_l}}{[l]_{q_l}}\right)z^4 \\ &\quad + \frac{[4]_{q_l}^2 + q_l^3\left(1 + [2]_{q_l} + [3]_{q_l}\right)}{[l]_{q_l}}z^3 + o\left(\frac{1}{[l]_{q_l}}\right). \end{aligned} \tag{21}$$

Combining Lemma 3 and $\mathfrak{B}^{q_l}_{l}\left((u-z)^4; z\right) = \sum_{m=0}^{4}\begin{pmatrix}4\\ m\end{pmatrix}(-1)^{4-m}\mathfrak{B}^{q_l}_{l}(u^{4-m}; z)z^m$, we can obtain

$$\begin{aligned} \mathfrak{B}^{q_l}_{l}\left((u-z)^4; z\right) &= (q_l - 1)^4 z^4 - \frac{A_4^{q_l} - 4A_3^{q_l} + 6A_2^{q_l} - 4A_1^{q_l}}{[l]_{q_l}}z^4 \\ &\quad + \frac{B_4^{q_l} - 4B_3^{q_l} + 6B_2^{q_l} - 4B_1^{q_l}}{[l]_{q_l}}z^3 + o\left(\frac{1}{[l]_{q_l}}\right), \end{aligned} \tag{22}$$

where $A_4^{q_l} = q_l^4(1 + [2]_{q_l} + [3]_{q_l}) + [4]_{q_l}[5]_{q_l}$, $A_3^{q_l} = q_l^3(1 + [2]_{q_l}) + [3]_{q_l}[4]_{q_l}$, $A_2^{q_l} = q_l^2 + [2]_{q_l}[3]_{q_l}$, $A_1^{q_l} = [2]_{q_l}$, $B_4^{q_l} = [4]_{q_l}^2 + q_l^3(1 + [2]_{q_l} + [3]_{q_l})$, $B_3^{q_l} = [3]_{q_l}^2 + q_l^2(1 + [2]_{q_l})$, $B_2^{q_l} = [2]_{q_l}^2 + q_l$, and $B_1^{q_l} = 1$. Hence, $A_4^{q_l} \sim 26$, $A_3^{q_l} \sim 15$, $A_2^{q_l} \sim 7$, $A_1^{q_l} \sim 2$, $B_4^{q_l} \sim 22$, $B_3^{q_l} \sim 12$, $B_2^{q_l} \sim 5$, and $B_1^{q_l} \sim 1$ as $l \longrightarrow \infty$, we have

$$\lim_{l\longrightarrow\infty}\left(A_4^{q_l}-4A_3^{q_l}+6A_2^{q_l}-4A_1^{q_l}\right)=26-4\times 15+6\times 7-4\times 2=0,$$
$$\lim_{l\longrightarrow\infty}\left(B_4^{q_l}-4B_3^{q_l}+6B_2^{q_l}-4B_1^{q_l}\right)=22-4\times 12+6\times 5-4\times 1=0. \tag{23}$$

Combining $\lim_{l\longrightarrow\infty}[l]_{q_l}(q_l-1)^4=0$, we can obtain (19). □

Lemma 5. *For $q\in(0,1]$, $l\in\mathbb{N}_+$, and $\zeta\in C_B(0,1)$, we can have $\|\mathfrak{B}_l^{q_l}(\zeta;z)\|\le\|\zeta\|$, where $C_B(0,1)$ denotes the set of all real-valued bounded and continuous functions defined on $(0,1)$, endowed with the norm $\|\zeta\|=\sup_{z\in(0,1)}|\zeta(z)|$.*

Proof. In view of (7) and Lemma 3, for any $z\in(0,1)$, we have

$$\left|\mathfrak{B}_l^{q_l}(\zeta;z)\right|\le\mathfrak{T}_{l,0}^{q}(z)\|\zeta\|=\|\zeta\|. \tag{24}$$

Taking supremum over all $z\in(0,1)$, we obtain the required result. □

3. Korovkin Approximation Theorem

Theorem 6. *Let the sequence (q_l) satisfy $q_l\in(0,1]$, for any $\zeta\in C_B(0,1)$; then, the sequence $\{\mathfrak{B}_l^{q_l}(\zeta;z)\}$ converges uniformly to ζ on $(0,1)$ if and only if $q_l\longrightarrow 1$ as $l\longrightarrow\infty$.*

Proof. Let $q_l\in(0,1]$ and $q_l\longrightarrow 1$ as $l\longrightarrow\infty$; then, we have $[l]_{q_l}\longrightarrow\infty$ as $l\longrightarrow\infty$. By the Korovkin theorem ([26], p. 8, Theorem 6), it is sufficient to prove the three following limit equalities:

$$\lim_{l\longrightarrow\infty}\left\|\mathfrak{B}_l^{q_l}(u^m;z)-z^m\right\|=0, m=0,1,2. \tag{25}$$

We can obtain these three limit equalities easily by Lemma 3. Thus, we get that the sequence $\{\mathfrak{B}_l^{q_l}(\zeta;z)\}$ converges uniformly to ζ on $(0,1)$.

We prove the converse result by contradiction. Assume that $\{q_l\}$ does not tend to 1 as $l\longrightarrow\infty$; then, it must contain a subsequence $\{q_{l_m}\}\subset(0,1]$ with $l_m\ge 1$, such that $q_{l_m}\longrightarrow\mathbf{q}\in[0,1)$. Thus,

$$\lim_{m\longrightarrow\infty}\frac{1}{[l_m]_{q_{l_m}}}=\lim_{m\longrightarrow\infty}\frac{1-q_{l_m}}{1-q_{l_m}^{l_m}}=1-\mathbf{q}. \tag{26}$$

Taking $l=l_m$ and $q=q_{l_m}$ in $\mathfrak{B}_l^{q}(u;z)$ and using Lemma 3, we can obtain

$$\mathfrak{B}_{l_m}^{q_{l_m}}(u;z)=\mathfrak{T}_{l_m,1}^{q_{l_m}}(z)\longrightarrow(1-2(1-\mathbf{q}))z+(1-\mathbf{q})\ne z, \tag{27}$$

as $m\longrightarrow\infty$. This leads to a contradiction; hence, $q_l\longrightarrow 1$ as $l\longrightarrow\infty$. The proof is completed. □

4. Voronovskaja-Type Theorem

In this section, we give a Voronovskaja-type asymptotic formula for the operators (7) by means of the first, second, and fourth central moments.

Theorem 7. *Under the condition of Lemma 4 and $\zeta\in C_B(0,1)$. Suppose that $\zeta''(z)$ exists at a point $z\in(0,1)$; then, we have*

$$\lim_{l\longrightarrow\infty}[l]_{q_l}\left(\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right)=(1+(k-3)z)\zeta'(z)+\frac{3}{2}z(1-z)\zeta''(z). \tag{28}$$

Proof. Applying the Taylor's expansion formula for ζ, we have

$$\zeta(u)=\zeta(z)+\zeta'(z)(u-z)+\frac{1}{2}\zeta''(z)(u-z)^2+\Theta(u;z)(u-z)^2, \tag{29}$$

where

$$\Theta(u;z)=\begin{cases}\dfrac{\zeta(u)-\zeta(z)-\zeta'(z)(u-z)-1/2\zeta''(z)(u-z)^2}{(u-z)^2}, & u\ne z,\\ 0, & u=z.\end{cases} \tag{30}$$

Using the L'Hospital's Rule,

$$\lim_{u\longrightarrow z}\Theta(u;z)=\frac{1}{2}\lim_{u\longrightarrow z}\frac{\zeta'(u)-\zeta'(z)}{u-z}-\frac{1}{2}\zeta''(z)=0. \tag{31}$$

Thus, $\Theta(.;z)\in C_B(0,1)$. Applying $\mathfrak{B}_l^{q_l}$ to both sides of (29) and using Lemma 3, we have

$$\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)=\zeta'(z)\alpha_l^{q_l}(z)+\frac{1}{2}\zeta''(z)\beta_l^{q_l}(z)+\mathfrak{B}_l^{q_l}\left(\Theta(u;z)(u-z)^2;z\right). \tag{32}$$

Applying the Cauchy-Schwarz inequality, we have

$$\left|\mathfrak{B}_l^{q_l}\left(\Theta(u;z)(u-z)^2;z\right)\right|\le\sqrt{\mathfrak{B}_l^{q_l}\left(\Theta^2(u;z);z\right)}\sqrt{\mathfrak{B}_l^{q_l}\left((u-z)^4;z\right)}. \tag{33}$$

By Theorem 6, we can obtain

$$\lim_{l\longrightarrow\infty}\Theta^2(u;z)=\Theta^2(z;z)=0. \tag{34}$$

From (19), (33), and (34), we have

$$\lim_{l\longrightarrow\infty}[l]_{q_l}\mathfrak{B}_l^{q_l}\left(\Theta(u;z)(u-z)^2;z\right)=0. \tag{35}$$

Combining (17), (18), and (35), we complete the proof of Theorem 7. □

Corollary 8. *Under the condition of Lemma 4 and $\zeta'' \in C_B(0,1)$, then*

$$\lim_{l\longrightarrow\infty} [l]_{q_l}\left(\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right)=(1+(k-3)z)\zeta'(z)+\frac{3}{2}z(1-z)\zeta''(z), \tag{36}$$

uniformly in $z \in (0,1)$.

5. Local Approximation

Firstly, we recall the first and second order modulus of continuity of $\zeta \in C_B(0,1)$ are defined, respectively, by

$$\begin{aligned}\omega(\zeta;\delta) &= \sup_{0<t\le\delta z, z+t\in(0,1)} \sup |\zeta(z+t)-\zeta(z)|,\\ \omega_2\left(\zeta;\sqrt{\delta}\right) &= \sup_{0<t\le\delta z, z+2t\in(0,1)} \sup |\zeta(z+2t)-2\zeta(z+t)+\zeta(z)|.\end{aligned} \tag{37}$$

The Peetre's $\mathbf{K}$-functional is defined by

$$\mathbf{K}(\zeta;\delta)=\inf_{\eta,\eta''\in C_B(0,1)}\left\{\|\zeta-\eta\|+\delta\|\eta''\|\right\}. \tag{38}$$

By ([26], p. 177, Theorem 2.4), there exists an absolute constant $C>0$ depending only on ζ such that

$$\mathbf{K}(\zeta;\delta)\le C\omega_2\left(\zeta;\sqrt{\delta}\right). \tag{39}$$

Theorem 9. *Under the condition of Lemma 4, then for all $\zeta \in C_B(0,1)$ and $z \in (0,1)$, there exists an absolute positive constant $C_1 = 4C$ such that*

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right| &\le C_1\omega_2\left(\zeta;\sqrt{\left(\alpha_l^{q_l}(z)\right)^2+\left|\beta_l^{q_l}(z)\right|}\right)\\ &\quad+\omega\left(\zeta;\left|\alpha_l^{q_l}(z)\right|\right).\end{aligned} \tag{40}$$

Proof. For $\zeta \in C_B(0,1)$, we define new operators by

$$\bar{\mathfrak{B}}_l^{q_l}(\zeta;z)=\mathfrak{B}_l^{q_l}(\zeta;z)+\zeta(z)-\zeta\left(\alpha_l^{q_l}(z)+z\right). \tag{41}$$

By Lemma 3, we can obtain immediately

$$\begin{aligned}\bar{\mathfrak{B}}_l^{q_l}(1;z)&=\mathfrak{B}_l^{q_l}(1;z)=\mathfrak{T}_{l,0}^{q_l}(z)=1,\\ \bar{\mathfrak{B}}_l^{q_l}(1;z)&=\mathfrak{B}_l^{q_l}(u;z)+z-\left(\alpha_l^{q_l}(z)+z\right)=\mathfrak{T}_{l,1}^{q_l}(z)-\alpha_l^{q_l}(z)=z.\end{aligned} \tag{42}$$

For any $\eta, \eta'' \in C_B(0,1)$ and $u, z \in (0,1)$, by the Taylor's expansion formula, we can obtain

$$\eta(u)=\eta(z)+\eta'(z)(u-z)+\int_z^u (u-v)\eta''(v)dv. \tag{43}$$

Applying the operators (7) to both sides of the above equation, we have

$$\begin{aligned}\bar{\mathfrak{B}}_l^{q_l}(\eta;z)&=\eta(z)+\bar{\mathfrak{B}}_l^{q_l}\left(\int_z^u (u-v)\eta''(v)dv;z\right)\\ &=\eta(z)+\mathfrak{B}_l^{q_l}\left(\int_z^u (u-v)\eta''(v)dv;z\right)\\ &\quad-\int_z^{\alpha_l^{q_l}(z)+z}\left(\alpha_l^{q_l}(z)+z-v\right)\eta''(v)dv.\end{aligned} \tag{44}$$

Hence,

$$\begin{aligned}\left|\bar{\mathfrak{B}}_l^{q_l}(\eta;z)-\eta(z)\right| &= \left|\bar{\mathfrak{B}}_l^{q_l}\left(\int_z^u (u-v)\eta''(v)dv;z\right)\right|\\ &\le \left|\mathfrak{B}_l^{q_l}\left(\left|\int_z^u |u-v|\eta''(v)dv\right|;z\right)\right|\\ &\quad+\left|\int_z^{\alpha_l^{q_l}(z)+z}\left|\alpha_l^{q_l}(z)+z-v\right|\left|\eta''(v)\right|dv\right|\\ &\le \left\{\left|\beta_l^{q_l}(z)\right|+\left(\alpha_l^{q_l}(z)\right)^2\right\}\|\eta''\|.\end{aligned} \tag{45}$$

By Lemma 5, we have

$$\begin{aligned}\left|\bar{\mathfrak{B}}_l^{q_l}(\zeta;z)\right| &\le \left|\mathfrak{B}_l^{q_l}(\zeta;z)\right|+|\zeta(z)|+\left|\zeta\left(\alpha_l^{q_l}(z)+z\right)\right|\\ &\le 3\|\zeta\|, \forall \zeta \in C_B(0,1).\end{aligned} \tag{46}$$

For any $\zeta \in C_B(0,1)$ and $\eta, \eta'' \in C_B(0,1)$, combining (45) and (46), we obtain

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right| &= \left|\bar{\mathfrak{B}}_l^{q_l}(\zeta;z)-\zeta(z)+\zeta\left(\alpha_l^{q_l}(z)+z\right)-\zeta(z)\right|\\ &\le \left|\bar{\mathfrak{B}}_l^{q_l}(\zeta-\eta;z)\right|+\left|\bar{\mathfrak{B}}_l^{q_l}(\eta;z)-\eta(z)\right|\\ &\quad+|\eta(z)-\zeta(z)|+\left|\zeta\left(\alpha_l^{q_l}(z)+z\right)-\zeta(z)\right|\\ &\le 4\|\zeta-\eta\|+\left\{\left|\beta_l^{q_l}(z)\right|+\left(\alpha_l^{q_l}(z)\right)^2\right\}\|\eta''\|\\ &\quad+\omega\left(\zeta;\left|\alpha_l^{q_l}(z)\right|\right).\end{aligned} \tag{47}$$

Taking infinum on the right hand side over all $\eta'' \in C_B(0,1)$, using (39), we obtain the desired assertion. □

Now, we discuss local approximation theorems for the operators (7) by Steklov mean. For any $\zeta \in C_B(0,1)$ and $h>0$, the Steklov mean is defined by

$$\zeta_h(z)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}(2\zeta(z+u+v)-\zeta(z+2u+2v))dudv. \tag{48}$$

In direct computation, it is proved that

$$\|\zeta_h-\zeta\|\le\omega_2(\zeta;h), \tag{49}$$

$$\zeta_h', \zeta_h'' \in C_B(0,1), \left\|\zeta_h'\right\| \le \frac{5}{h}\omega(\zeta\,;h), \left\|\zeta''{}_h\right\| \le \frac{9}{h^2}\omega_2(\zeta\,;h), \tag{50}$$

while $\zeta \in C_B(0,1)$.

Theorem 10. *Under the condition of Lemma 4, then for all $\zeta \in C_B(0,1)$ and $z \in (0,1)$,*

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta\,;z)-\zeta(z)\right| &\le 5\sqrt{[l]_{q_l}}\left|\alpha_l^{q_l}(z)\right|\omega\left(\zeta\,;\frac{1}{[l]_{q_l}}\right)\\ &\quad+\left(\frac{9}{2}[l]_{q_l}\left|\beta_l^{q_l}(z)\right|+2\right)\omega_2\left(\zeta\,;\frac{1}{[l]_{q_l}}\right).\end{aligned} \tag{51}$$

Proof. For $z, h, z+2h \in (0,1)$, by the definition of the Steklov mean, we can write

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta\,;z)-\zeta(z)\right| &\le \mathfrak{B}_l^{q_l}(|\zeta-\zeta_h|\,;z)+\left|\mathfrak{B}_l^{q_l}(\zeta_h\,;z)-\zeta_h(z)\right|\\ &\quad+|\zeta_h(z)-\zeta(z)|.\end{aligned} \tag{52}$$

By Lemma 5 and (49), we have

$$\mathfrak{B}_l^{q_l}(|\zeta-\zeta_h|\,;z) \le \left\|\mathfrak{B}_l^{q_l}(|\zeta-\zeta_h|\,;z)\right\| \le \|\zeta-\zeta_h\| \le \omega_2(\zeta\,;h). \tag{53}$$

Now, by the Taylor's expansion formula for ζ_h, we have

$$\zeta_h(u)=\zeta_h(z)+\zeta_h'(z)(u-z)+\int_z^u (u-v)\zeta''{}_h(v)dv. \tag{54}$$

Combining (49) and (50), we have

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta_h\,;z)-\zeta_h(z)\right| &\le \left|\mathfrak{B}_l^{q_l}\left(\zeta_h'(z)(u-z)\,;z\right)\right|+\left|\mathfrak{B}_l^{q_l}\left(\int_z^u (u-v)\zeta''{}_h(v)\mathrm{d}v\,;z\right)\right|\\ &\le \|\zeta_h'\|\left|\mathfrak{B}_l^{q_l}((u-z)\,;z)\right|+\|\zeta_h''\|\left|\mathfrak{B}_l^{q_l}\left(\int_z^u |u-v|dv\,;z\right)\right|\\ &\le \|\zeta_h'\|\left|\alpha_l^{q_l}(z)\right|+\frac{1}{2}\|\zeta_h''\|\left|\beta_l^{q_l}(z)\right|\\ &\le \frac{5}{h}\left|\alpha_l^{q_l}(z)\right|\omega(\zeta\,;h)+\frac{9}{2h^2}\left|\beta_l^{q_l}(z)\right|\omega_2(\zeta\,;h).\end{aligned} \tag{55}$$

Hence,

$$\left|\mathfrak{B}_l^{q_l}(\zeta\,;z)-\zeta(z)\right| \le \frac{5}{h}\omega(\zeta\,;h)\left|\alpha_l^{q_l}(z)\right|+\left(\frac{9}{2h^2}\left|\beta_l^{q_l}(z)\right|+2\right)\omega_2(\zeta\,;h), \tag{56}$$

for $z \in (0,1)$. Setting $h=1/\sqrt{[l]_{q_l}}$, we obtain the desired result. □

6. Rate of Convergence

First, we discuss the rate of convergence of the operators (7) by means of the modulus of continuity $\omega(\zeta\,;\delta)$.

Theorem 11. *Under the condition of Lemma 4, then for all $\zeta \in C(0,1)$ and $z \in (0,1)$,*

$$\left|\mathfrak{B}_l^{q_l}(\zeta\,;z)-\zeta(z)\right| \le 2\omega\left(\zeta\,;\sqrt{\left|\beta_l^{q_l}(z)\right|}\right). \tag{57}$$

Proof. Using ([26], p. 41, (6.5)), for all $\zeta \in C(0,1)$ and $\delta > 0$, we have

$$|\zeta(u)-\zeta(z)| \le \omega(\zeta\,;\delta)\left(1+\frac{|u-z|}{\delta}\right). \tag{58}$$

Applying the monotonicity and the linearity of $\mathfrak{B}_l^{q_l}$ and Cauchy-Schwarz inequality, for any $\delta > 0$, we have

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta\,;z)-\zeta(z)\right| &\le \mathfrak{B}_l^{q_l}(|\zeta(u)-\zeta(z)|\,;z)\\ &\le \mathfrak{B}_l^{q_l}(\omega(\zeta\,;|u-z|)\,;z)\\ &\le \omega(\zeta\,;\delta)\left(1+\frac{\mathfrak{B}_l^{q_l}(|u-z|\,;z)}{\delta}\right)\\ &\le \omega(\zeta\,;\delta)\left(1+\frac{\sqrt{\mathfrak{B}_l^{q_l}((u-z)^2\,;z)}}{\delta}\right)\\ &\le \omega(\zeta\,;\delta)\left(1+\frac{\sqrt{\left|\beta_l^{q_l}(z)\right|}}{\delta}\right),\end{aligned} \tag{59}$$

by taking $\delta=\sqrt{\left|\beta_l^{q_l}(z)\right|}$. We complete the proof of Theorem 11. □

Theorem 12. *Under the condition of Lemma 4, then for all $\zeta' \in C(0,1)$ and $z \in (0,1)$,*

$$\left|\mathfrak{B}_l^{q_l}(\zeta\,;z)-\zeta(z)\right| \le \left|\alpha_l^{q_l}(z)\right|+2\sqrt{\left|\beta_l^{q_l}(z)\right|}\omega\left(\zeta'\,;\sqrt{\left|\beta_l^{q_l}(z)\right|}\right). \tag{60}$$

Proof. Applying $\mathfrak{B}_l^{q_l}$ to both sides of $\zeta(u)=\zeta(z)+\zeta'(z)(u-z)+\zeta(u)-\zeta(z)-\zeta'(z)(u-z)$, we have

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right| &\le \left|\zeta'(z)\right|\left|\mathfrak{B}_l^{q_l}(u-z;z)\right| + \mathfrak{B}_l^{q_l} \\ &\quad \cdot\left(\left|\zeta(u)-\zeta(z)-\zeta'(z)(u-z)\right|;z\right) \\ &\le \left|\alpha_l^{q_l}(z)\right|\left|\zeta'(z)\right| + \mathfrak{B}_l^{q_l} \\ &\quad \cdot\left(|u-z|\left(1+\frac{|u-z|}{\delta}\right);z\right)\omega\left(\zeta';\delta\right) \\ &\le \left|\alpha_l^{q_l}(z)\right|\left|\zeta'(z)\right| + \sqrt{\mathfrak{B}_l^{q_l}\left((u-z)^2;z\right)} \\ &\quad \cdot\left(1+\frac{\sqrt{\mathfrak{B}_l^{q_l}\left((u-z)^2;z\right)}}{\delta}\right)\omega\left(\zeta';\delta\right),\end{aligned} \tag{61}$$

with the help of Cauchy-Schwartz inequality and mean value theorem. Taking $\delta=\sqrt{\mathfrak{B}_l^{q_l}((u-z)^2;z)}$ and by Lemma 3, we can get the desired result. □

Next, we discuss the rate of convergence of the operators (7) by means of three Lipschitz classes: $\mathrm{Lip}_M\gamma$, $\mathrm{Lip}_M^{s,t}\gamma$, and Li $\mathrm{p}_M(\gamma,I)$. A function $\zeta\in C(0,1)$ belongs to $\mathrm{Lip}_M\gamma(\gamma\in(0,1])$, if the condition

$$|\zeta(u)-\zeta(z)|\le M|u-z|^{\gamma},\ u,z\in(0,1), \tag{62}$$

is satisfied, where M is a positive constant depending only on γ and ζ.

Theorem 13. *Under the condition of Lemma 4, then for all $\zeta\in Lip_M\gamma$ and $z\in(0,1)$,*

$$\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right|\le M\left|\beta_l^{q_l}(z)\right|^{\gamma/2}. \tag{63}$$

Proof. According to the monotonicity and the linearity of the operators (7) and taking into account that $\zeta\in\mathrm{Lip}_M\gamma$, we can obtain

$$\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right|\le\mathfrak{B}_l^{q_l}(|\zeta(u)-\zeta(z)|;z)\le M\mathfrak{B}_l^{q_l}(|u-z|^{\gamma};z). \tag{64}$$

Applying well-known Hölder's inequality with $t_1=2/\gamma$ and $t_2=2/2-\gamma$, we can get

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right| &\le M\mathfrak{B}_l^{q_l}(|u-z|^{\gamma};z) \\ &\le M\left(\mathfrak{B}_l^{q_l}\left(|u-z|^{t_1\gamma};x\right)\right)^{1/t_1}\left(\mathfrak{B}_l^{q_l}\left(1^{t_2};x\right)\right)^{1/t_2} \\ &= M\left(\mathfrak{B}_l^{q_l}\left(|u-t|^2;x\right)\right)^{\gamma/2}=M\left|\beta_l^{q_l}(z)\right|^{\gamma/2}.\end{aligned} \tag{65}$$

We obtain the required result. □

In [27], Özarslan and Aktuğlu constructed the following Lipschitz-type space $\mathrm{Lip}_M^{s,t}\gamma$ with two distinct parameters $s,t>0$ as follows:

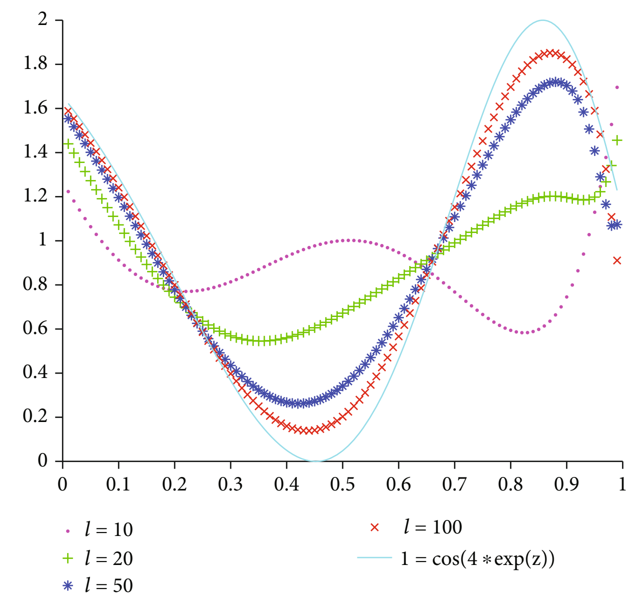

Figure 1: Approximation process by $\mathfrak{B}_l^{q_l}$.

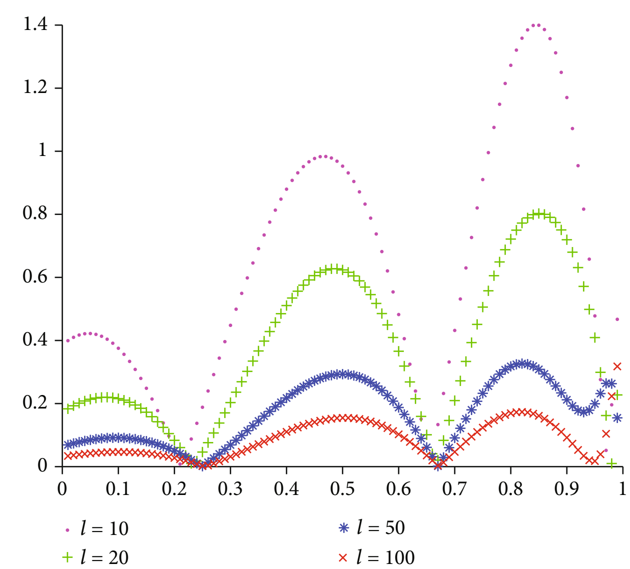

Figure 2: Error of approximation $|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)|$.

$$\mathrm{Lip}_M^{s,t}\gamma:=\left\{\zeta\in C(0,1)\colon |\zeta(u)-\zeta(z)|\le M\frac{|u-z|^{\gamma}}{u+sz+tz^2},\ u,z\in(0,1)\right\}, \tag{66}$$

where $\gamma\in(0,1]$ and M is a positive constant depending only on γ, s, t, and ζ.

Theorem 14. *Under the condition of Lemma 4, then for all $\zeta\in Lip_M^{s,t}\gamma$ and $z\in(0,1)$,*

$$\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right|\le M\left(\frac{\left|\beta_l^{q_l}(z)\right|}{sz+tz^2}\right)^{\gamma/2}. \tag{67}$$

Table 1: Error of approximation $|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)|$ for $l=10,20,50,100$.

z	$\left|\mathfrak{B}_{10}^{q_{10}}(\zeta;z)-\zeta(z)\right|$	$\left|\mathfrak{B}_{20}^{q_{20}}(\zeta;z)-\zeta(z)\right|$	$\left|\mathfrak{B}_{50}^{q_{50}}(\zeta;z)-\zeta(z)\right|$	$\left|\mathfrak{B}_{100}^{q_{100}}(\zeta;z)-\zeta(z)\right|$
0.1	0.3753	0.2156	0.0916	0.0465
0.2	0.0536	0.0834	0.0496	0.0277
0.3	0.4481	0.2030	0.0688	0.0316
0.4	0.8797	0.5107	0.2177	0.1104
0.5	0.9527	0.6237	0.2932	0.1546
0.6	0.4822	0.3658	0.1875	0.1025
0.7	0.4319	0.2094	0.0911	0.0467
0.8	1.2725	0.7215	0.3178	0.1693
0.9	1.1703	0.7194	0.2119	0.0921

Proof. Applying the well-known Hölder inequality with $k_1=2/\gamma$ and $k_2=2/2-\gamma$, we have

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right| &\le \left|\mathfrak{B}_l^{q_l}(|\zeta(u)-\zeta(z)|;z)\right|\\ &\le \left|\mathfrak{B}_l^{q_l}\left(M\frac{|u-z|^\gamma}{u+sz+tz^2};z\right)\right|\\ &\le \frac{M}{sz+tz^2}\mathfrak{B}_l^{q_l}(|u-z|^\gamma;z)\\ &\le \frac{M}{sz+tz^2}\left(\mathfrak{B}_l^{q_l}\left(|u-z|^{\gamma k_1};z\right)\right)^{1/k_1}\\ &\quad\cdot\left(\mathfrak{B}_l^{q_l}\left(1^{k_2};z\right)\right)^{1/k_2}=M\left(\frac{\left|\beta_l^{q_l}(z)\right|}{sz+tz^2}\right)^{\gamma/2}.\end{aligned}\tag{68}$$

Thus, the proof of Theorem 14 is completed. □

A function $\zeta\in C(0,1)$ belongs to $\mathrm{Lip}_M(\gamma,I)(\gamma\in(0,1],I\subset(0,1))$, if the condition

$$|\zeta(u)-\zeta(z)|\le M|u-z|^\gamma,\ u\in I,z\in(0,1),\tag{69}$$

is satisfied, where M is a positive constant depending only on γ and ζ.

Theorem 15. *Under the condition of Lemma 4, then for all $\zeta\in C(0,1)\bigcap Lip_M(\gamma,I)$ and $z\in(0,1)$,*

$$\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right|\le M\left(\left|\beta_l^{q_l}(z)\right|^{\gamma/2}+2d^\gamma(z;I)\right),\tag{70}$$

where $d(z;I)=\inf\{|u-z|: u\in I\}$ denotes the distance between z and I.

Proof. Let $\bar{I}$ be the closure of I. Using the properties of infimum, there is at least a point $u_0\in\bar{I}$ such that $d(z;I)=|z-u_0|$. By the triangle inequality

$$|\zeta(u)-\zeta(z)|\le|\zeta(u)-\zeta(u_0)|+|\zeta(z)-\zeta(u_0)|,\tag{71}$$

we have

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right| &\le \mathfrak{B}_l^{q_l}(|\zeta(u)-\zeta(u_0)|;z)+\mathfrak{B}_l^{q_l}(|\zeta(z)-\zeta(u_0)|;z)\\ &\le M\left\{\mathfrak{B}_l^{q_l}(|u-u_0|^\gamma;z)+|z-u_0|^\gamma\right\}\\ &\le M\left\{\mathfrak{B}_l^{q_l}(|u-z|^\gamma+|z-u_0|^\gamma;z)+|z-u_0|^\gamma\right\}\\ &\le M\left\{\mathfrak{B}_l^{q_l}(|u-z|^\gamma;z)+2|z-u_0|^\gamma\right\}.\end{aligned}\tag{72}$$

Choosing $k_1=2/\gamma$ and $k_2=2/2-\gamma$ and using the well-known Hölder inequality, we have

$$\begin{aligned}\left|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)\right| &\le M\left\{\left(\mathfrak{B}_l^{q_l}\left(|u-z|^{\gamma k_1};z\right)\right)^{1/k_1}\right.\\ &\quad\left.\cdot\left(\mathfrak{B}_l^{q_l}\left(1^{k_2};z\right)\right)^{1/k_2}+2d^\gamma(z;I)\right\}\\ &\le M\left\{\left(\mathfrak{B}_l^{q_l}\left((u-z)^2;z\right)\right)^{1/k_1}+2d^\gamma(z;I)\right\}\\ &\le M\left(\left|\beta_l^{q_l}(z)\right|^{\gamma/2}+2d^\gamma(z;I)\right).\end{aligned}\tag{73}$$

This completes the proof. □

7. Numerical Examples

In this section, we will analyze the theoretical results presented in the previous sections by numerical examples.

Let $\zeta(z)=1-\cos(4e^z)$, $z\in[0.01,0.99]$, $q_l=1-1/l^2$, and $l\in\{10,20,50,100\}$. The convergence of the operators $\mathfrak{B}_l^{q_l}$ to function ζ is shown in Figure 1. The error of approximation $|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)|$ is given in Figure 2. Meantime, we compute the error of approximation $|\mathfrak{B}_l^{q_l}(\zeta;z)-\zeta(z)|$ while $l=10,20,50,100$ at points $\{0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9\}$ in Table 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

The Approximation of Laplace-Stieltjes Transforms Concerning Sun's Type Function

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The main aim of this paper is to establish some theorems concerning the error $E_n(F, \beta)$, the Sun's type function $U(r)$, and $M_u(\sigma, F)$ of entire functions defined by Laplace-Stieltjes transforms with infinite order converge in the whole complex plane. Our results exhibit the growth of Laplace-Stieltjes transforms from the point of view of approximation.

1. Introduction and Main Results

In 1946, Widder [1] considered the convergence of the following form

$$G(s) = \int_0^{+\infty} e^{-sx} d\alpha(x), s = \sigma + it, \tag{1}$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0, Y](0 < Y < +\infty)$, and σ and t are real variables and obtained the following theorem.

Theorem 1 (see ([1], Theorem 1, Page 36)). *If*

$$\sup_{0\le u<\infty} \left| \int_0^u e^{-s_0 t} d\alpha(t) \right| = M < \infty, \tag{2}$$

then (1) converges for every s for which $\sigma > \sigma_0$, and

$$\int_0^{+\infty} e^{-sx} d\alpha(x) = (s-s_0) \int_0^{+\infty} e^{-(s-s_0)t} \beta(t) dt, \tag{3}$$

where $\beta(u) = \int_0^u e^{-s_0 t} d\alpha(t), (u \ge 0)$.

As we know, (1) can be called as Laplace-Stieltjes transform, which is an integral transform similar to the Laplace transform, named for Pierre-Simon Laplace and Thomas Joannes Stieltjes. Moreover, it can be used in many fields of mathematics, such as functional analysis, and certain areas of theoretical and applied probability.

In view of Ref. [1], $G(s)$ can become the classical Laplace integral form

$$G(s) = \int_0^{\infty} e^{-st} \varphi(t) dt, \tag{4}$$

when $\alpha(t)$ is absolutely continuous. Moreover, if $\alpha(t)$ is a step function, choosing a sequence $\{\lambda_n\}_0^\infty$ such that

$$0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \longrightarrow \infty \text{as}\, n \longrightarrow \infty, \tag{5}$$

$$\alpha(x) = \begin{cases} a_1 + a_2 + \cdots + a_n, & \lambda_n \le x < \lambda_{n+1}, \\ 0, & x = 0, \\ \dfrac{\alpha(x+) + \alpha(x-)}{2}, & x > 0, \end{cases} \tag{6}$$

then we can conclude from Theorem 1 that $G(s)$ becomes a Dirichlet series

$$G(s) = f(s) \coloneqq \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, s = \sigma + it, \tag{7}$$

where σ, t is real variables, a_n is nonzero complex numbers. For Dirichlet series (7), it can become a Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ if $\lambda_n = n$ and $e^s = z$, and it further can also become a classical Dirichlet series $f(s) = \sum_{n=0}^{\infty}(a_n/n^s)$ if $e^{\lambda_n} = 1/n$, which is important in the fields of number theory. Hence, we can say that Laplace-Stieltjes transform is a general form of Dirichlet series. Under some conditions related to a_n, λ_n, and n, the series (7) can converge in the whole plane or the half plane; that is, $f(s)$ is analytic in the whole plane or the half plane.

In the past several decades, the problem on the growth and value distribution of analytic functions has been an important and interesting subject in the fields of complex analysis. Moreover, considerable attention has been paid to the growth and the value distribution of analytic functions defined by Dirichlet series and Laplace-Stieltjes transforms, and a great deal of interesting results focusing on the growth and value distribution of such functions can be found in (see [2–17]). For example, Yu [18] in 1963 first proved a series of theorems about the Valiron-Knopp-Bohr formula of the associated abscissas of bounded convergence, absolute convergence and uniform convergence of Laplace-Stieltjes transforms, the maximal molecule $M_u(\sigma, G)$, the maximal term $\mu(\sigma, G)$, the Borel line and the order of entire functions represented by Laplace-Stieltjes transforms convergent in the complex plane. Batty, Sheremeta, Kong, and Sun investigated the growth of analytic functions with kinds of order defined by Laplace-Stieltjes transforms (see [19–25]), and Shang, Gao, Zhang, and Xu investigated the value distribution of such functions (see [26–28]).

In 2012,Luo and Kong [29] studied the following form, is differ from (1), of Laplace-Stieltjes transform

$$F(s) = \int_0^{+\infty} e^{sx} d\alpha(x), s = \sigma + it, \tag{8}$$

where $\alpha(x)$ is stated as in (1), and $\{\lambda_n\}$ satisfies (5) and

$$\limsup_{n\to+\infty}(\lambda_{n+1} - \lambda_n) = h < +\infty, \tag{9}$$

$$\limsup_{n\longrightarrow+\infty} \frac{\log n}{\lambda_n} = D < \infty. \tag{10}$$

Set

$$A_n^* = \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{ity} d\alpha(y) \right|. \tag{11}$$

By using the same argument as in [18], we can get the similar result about the abscissa of uniformly convergent of $F(s)$ easily. If

$$\limsup_{n\longrightarrow+\infty} \frac{\log A_n^*}{\lambda_n} = -\infty, \tag{12}$$

by (5), (9)-(12), and Ref.[18], one can get that $\sigma_u^F = +\infty$, i.e., $F(s)$ is entire in the whole plane.

Set

$$\begin{aligned} \mu(\sigma, F) &= \max_{n\in N}\left\{A_n^* e^{\lambda_n\sigma}\right\}(\sigma < +\infty), M(\sigma, F) \\ &= \sup_{-\infty<t<+\infty} |F(\sigma + it)|, \\ M_u(\sigma, F) &= \sup_{0<x<+\infty, -\infty<t<+\infty} \left| \int_0^x e^{(\sigma+it)y} d\alpha(y) \right|. \end{aligned} \tag{13}$$

Definition 2 (see [30]). If Laplace-Stieltjes transform (8) satisfies $\sigma_u^F = +\infty$ (the sequence $\{\lambda_n\}$ satisfy (5) and (9)-(12)), we define the order and the lower order of $F(s)$ by

$$\begin{aligned} \limsup_{\sigma\longrightarrow+\infty} \frac{\log^+ \log^+ M_u(\sigma, F)}{\sigma} &= \rho, \\ \liminf_{\sigma\longrightarrow+\infty} \frac{\log^+ \log^+ M_u(\sigma, F)}{\sigma} &= \lambda, \end{aligned} \tag{14}$$

respectively, where $\log^+ x = \max\{\log x, 0\}$.

Remark 3. If $\rho = 0$, $\rho \in (0, +\infty)$ and $\rho = \infty$, we say that $F(s)$ is an entire function of zero order, finite order, and infinite order in the whole plane, respectively.

Definition 4. If Laplace-Stieltjes transform (8) satisfies $\sigma_u^F = +\infty$ (the sequence $\{\lambda_n\}$ satisfy (5) and (9)-(12)) and is of order $\rho(0 < \rho < \infty)$, then we define

$$T = \limsup_{\sigma\longrightarrow+\infty} \frac{\log^+ M_u(\sigma, F)}{e^{\sigma\rho}}, \tag{15}$$

which is called the type of Laplace-Stieltjes transform $F(s)$.

In 2012 and 2014, Luo and Kong [29, 30] studied the growth of Laplace-Stieltjes transform of finite order and obtained the following theorem.

Theorem 5 (see [29, 30]). *If Laplace-Stieltjes transform (8) satisfies $\sigma_u^F = +\infty$ (the sequence $\{\lambda_n\}$ satisfy (5) and (9)-(12)), and is of order $\rho(0 < \rho < \infty)$ and of type T, then*

$$\rho = \limsup_{n\longrightarrow+\infty} \frac{\lambda_n \log \lambda_n}{-\log A_n^*}, T = \limsup_{n\longrightarrow+\infty} \frac{\lambda_n}{\rho e}(A_n^*)^{\rho/\lambda_n}. \tag{16}$$

In order to state our main results of this paper, we also introduce some definitions and notations below. We denote by $\bar{L}_\beta$ the set of all the functions $F(s)$ of the form (8) which are analytic in the half plane $Res < \beta(-\infty < \beta < \infty)$ and the sequence $\{\lambda_n\}$ satisfy (5), (9), and (10) and by L_∞ the set of all the functions $F(s)$ of the form (8) which are analytic in the half plane $\Re s < +\infty$ and the sequence $\{\lambda_n\}$ satisfy (5) and (9)-(12). Obviously, if $-\infty < \beta < +\infty$ and $F(s) \in \bar{L}_\beta$, then $F(s) \in L_\infty$. If (8) satisfies $A_n^* = 0$ for $n \ge k+1$, and $A_k^* \ne 0$, then we say that $F(s)$ is an exponential polynomial of degree k usually denoted by p_k, i.e., $p_k(s) = \int_0^{\lambda_k} \exp(sy) d\alpha(y)$. If we choose a suitable function $\alpha(y)$, the function $p_k(s)$ may be reduced to a polynomial in terms of $\exp(s\lambda_i)$, that is, $\sum_{i=1}^{k} b_i \exp(s\lambda_i)$.

We denote Π_n to be the class of all exponential polynomial of degree almost n, that is,

$$\Pi_n = \left\{ \sum_{j=1}^{n} b_j e^{\lambda_j s} : (b_1, b_2, \cdots, b_n) \in \mathbb{C}^n \right\}. \tag{17}$$

For $F(s) \in \bar{L}_\beta, -\infty < \beta < +\infty$, we use $E_n(F, \beta)$ to denote the error in approximating the function $F(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(F, \beta) = \inf_{p \in \Pi_n} \|F - p\|_\beta, n = 1, 2, \cdots, \tag{18}$$

where

$$\|F - p\|_\beta = \max_{-\infty < t < +\infty} |F(\beta + it) - p(\beta + it)|. \tag{19}$$

Around 2017, Singhal and Srivastava [31, 32] studied the approximation of entire functions represented by Laplace-Stieltjes transforms (8) of finite order and obtained the following result.

Theorem 6 (see [32]). *If Laplace-Stieltjes transform $F(s) \in L_\infty$ and is of order $\rho (0 < \rho < \infty)$ and of type T, then for any real number $-\infty < \beta < +\infty$, we have*

$$\begin{aligned} \rho &= \limsup_{n \longrightarrow +\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F, \beta) \exp(-\beta \lambda_n)} \\ &= \limsup_{n \longrightarrow +\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F, \beta)}, \\ T &= \limsup_{n \longrightarrow +\infty} \frac{\lambda_n}{\rho e} (E_{n-1}(F, \beta) \exp(-\beta \lambda_n))^{\rho/\lambda_n} \\ &= \limsup_{n \longrightarrow +\infty} \frac{\lambda_n}{\rho \exp(\rho\beta + 1)} (E_{n-1}(F, \beta))^{\rho/\lambda_n}. \end{aligned} \tag{20}$$

In the same year, the authors [33] further the approximation on the entire function represented by Laplace-Stieltjes transforms with irregular growth and obtained.

Theorem 7 (see ([33], Theorem 6)). *If the Laplace-Stieltjes transform $F(s) \in L_\infty$ and is of the lower order $\lambda (0 \leq \lambda \neq \rho < \infty)$, if $\lambda_n \sim \lambda_{n+1}$, then for any real number $-\infty < \beta < +\infty$, we have*

$$\tau_\lambda \geq \liminf_{n \longrightarrow \infty} \left(\frac{\lambda_n}{e \lambda} \right) (E_{n-1}(F, \beta) \exp(-\beta \lambda_n))^{\lambda/\lambda_n}, (0 \leq \tau_\lambda \leq \infty). \tag{21}$$

Furthermore, there exists a positive integer n_0 such that

$$\psi_1(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n} \tag{22}$$

forms a nondecreasing function of n for $n > n_0$, and then we have

$$\tau_\lambda = \liminf_{n \longrightarrow \infty} \left(\frac{\lambda_n}{e \lambda} \right) (E_{n-1}(F, \beta) \exp(-\beta \lambda_n))^{\lambda/\lambda_n}, (0 \leq \tau_\lambda \leq \infty). \tag{23}$$

As far as we know, there are few papers focusing on the approximation of Laplace-Stieltjes transform of infinite order. Inspired by this issue, our main purpose of this paper is to deal with the approximation of Laplace-Stieltjes transforms of infinite order $\rho(F) = \infty$ with the help of the type function given by Sun. In 1986, Sun [34] studied the existence of type function of the complex function of infinite order and established a new type function which is more precise than Xiong's.

Theorem 8 (see [34]). *If $S(r)$ is a continuous function in $[a, +\infty)$ and*

$$\limsup_{r \longrightarrow +\infty} \frac{\log^+ S(r)}{\log r} = +\infty, \tag{24}$$

then we say that $U(r)$ is the type function of $S(r)$, if there exist two continuous and differential functions $\rho(r)$ and $U(r)$ satisfying

$\rho(r)$ monotonous, decreasing and trend to 0, $\rho'(r)$ monotonous, increasing

(ii) $\lim_{r \longrightarrow +\infty} r \rho'(r) \log r \log \log r = 0$

(iii) For sufficient large r, $S(r) \ll U(r) := r^{\exp(1/\rho(r))}$

(iv) $U(R) < (1 + o(1)) U(r)$

where $R = r + (r \log r / \log U(r) \log^2 \log U(r))$ and $S(r) \ll U(r)$ mean that $S(r) \leq U(r)$ and $\exists$ a sequence $\{r_n\} \uparrow +\infty$ such that $S(r_n) = U(r_n)$.

Remark 9. If $F(s) \in L_\infty$ and is of infinite order $\rho = +\infty$, then in view of Theorem 7, there exists a type function $U(r)$ such that

$$\limsup_{\sigma \longrightarrow +\infty} \frac{S(r)}{U(r)} = 1, \tag{25}$$

where $S(r) := \log \mu(\sigma, F)$ and $r = e^\sigma$.

The main theorems of this article are listed as follows.

Theorem 10. *Let $F(s)$ be of infinite order, and the sequence $\{\lambda_n\}$ satisfies (5), (9), (12) and*

$$\limsup_{n \longrightarrow +\infty} \frac{loglogn}{\log \lambda_n} = d < 1, \tag{26}$$

and then for any real number $-\infty < \beta < +\infty$, we have

$$\begin{aligned}&\limsup_{\sigma\longrightarrow+\infty}\frac{\log M_u(\sigma,F)}{U(r)}\\&\quad=1\le Q_1:=\limsup_{\sigma\longrightarrow+\infty}\frac{-\log\left[E_{n-1}(F,\beta)\exp\left(-\beta\lambda_n\right)\right]}{U(\psi_n)},\end{aligned}\tag{27}$$

where $\psi_n=[E_{n-1}(F,\beta)\exp(-\beta\lambda_n)]^{-1/\lambda_n}$, $r=e^{\sigma}$, and $U(r)$ are stated as in Theorem 10.

Remark 11. We can easily get (10) from (26), thus $F(s)\in L_{\infty}$ when the sequence $\{\lambda_n\}$ satisfies (5), (9), (12), and (26). That is to say, our condition in our theorem is better than the previous results.

Theorem 12. *Under the assumptions of Theorem 10, then*

$$\limsup_{\sigma\longrightarrow+\infty}\frac{\log M_u(\sigma,F)}{U(r)}=1\ge Q_2:=\limsup_{\sigma\longrightarrow+\infty}\frac{\lambda_n}{U_1(\psi_n)},\tag{28}$$

where

$$U_1(\bullet)=U(\bullet)\log U(\bullet)\log^2\log U(\bullet).\tag{29}$$

Let $U_2(\bullet)=U(\bullet)\log^2 U(\bullet)$, and then it follows $U_1(\psi_n)\le U_2(\psi_n)$ for any positive integer n and any real number β. Hence, we get the following corollary.

Corollary 13. *Under the assumptions of Theorem 10, then*

$$\limsup_{\sigma\longrightarrow+\infty}\frac{\log M_u(\sigma,F)}{U(r)}=1\ge Q_3:=\limsup_{\sigma\longrightarrow+\infty}\frac{\lambda_n}{U_2(\psi_n)}.\tag{30}$$

Theorem 14. *Under the assumptions of Theorem 10, then we have*

$$\limsup_{\sigma\longrightarrow+\infty}\frac{loglogM_u(\sigma,F)}{\log U(r)}=1\Leftrightarrow\limsup_{\sigma\longrightarrow+\infty}\frac{\log\lambda_n}{\log U(\psi_n)}=1.\tag{31}$$

2. The Proof of Theorem 6

To prove Theorem 10, we require the following lemma.

Lemma 15 (see ([1], Theorem 6b)). *If $f(x)$ and $\varphi(x)$ are continuous and $\alpha(x)$ is of bounded variation in $a\le x\le b$, and if*

$$\gamma(x)=\int_c^x\varphi(x)d\alpha(t),\ (a\le x\le b, a\le c\le b),\tag{32}$$

then

$$\int_a^b f(x)d\gamma(x)=\int_a^b f(x)\varphi(x)d\alpha(x).\tag{33}$$

Lemma 16. *If Laplace-stieltjes transform $F(s)$ is of infinite order, and the sequence $\{\lambda_n\}$ satisfies (5), (9), (12), and (26), then*

$$\limsup_{\sigma\longrightarrow+\infty}\frac{\log M_u(\sigma,F)}{U(r)}=1\Longleftrightarrow\limsup_{\sigma\longrightarrow+\infty}\frac{\log\mu(\sigma,F)}{U(r)}=1,\tag{34}$$

where $r=e^{\sigma}$, $S(r)=\log\mu(\sigma,F)$, and $U(r)$ are stated as in Theorem 7.

Proof. The idea of the proof of this lemma come from Ref. [22]. Next, we will show the completely details.

Set

$$I_n(x;\sigma+it)=\int_{\lambda_n}^x\exp\{(\sigma+it)y\}d\alpha(y).\tag{35}$$

From (9), there exists $\eta>0$ satisfying $0<\lambda_{n+1}-\lambda_n\le\eta$ $(n=1,2,3,\cdots)$, and then it follows $e^{-\eta\sigma}<1$, as $\sigma>0$. Thus, for $x>\lambda_n$ and by Theorem 1 and Lemma 15, we deduce

$$\begin{aligned}\int_{\lambda_n}^x\exp\{ity\}d\alpha(y)&=\int_{\lambda_n}^x e^{-\sigma y}d_yI_n(y;\sigma+it)=I(y;\sigma+it)e^{-\sigma y}\Big|_{\lambda_n}^x\\&\quad+\sigma\int_{\lambda_n}^x e^{-\sigma y}I_n(y;\sigma+it)dy;\end{aligned}\tag{36}$$

that is,

$$\begin{aligned}\left|\int_{\lambda_n}^x\exp\{ity\}d\alpha(y)\right|&\le M_u(\sigma,F)\left[\left|e^{-\sigma x}+e^{-\sigma\lambda_n}\right|\right.\\&\quad\left.+\left|e^{-\sigma x}-e^{-\sigma\lambda_n}\right|\right]\le 2M_u(\sigma,F)e^{-\sigma x}.\end{aligned}\tag{37}$$

Hence, for any $\sigma>0$ and any $x\in(\lambda_n,\lambda_{n+1}]$, it follows

$$\left|\int_{\lambda_n}^x\exp\{ity\}d\alpha(y)\right|\le 2M_u(\sigma,F)e^{-\sigma\lambda_n}e^{-\sigma\eta}\le 2M_u(\sigma,F)e^{-\sigma\lambda_n};\tag{38}$$

that is,

$$\mu(\sigma,F)\le 2M_u(\sigma,F).\tag{39}$$

Therefore, we can conclude from (39) that

$$\limsup_{\sigma\longrightarrow+\infty}\frac{\log M_u(\sigma,F)}{U(r)}\ge\limsup_{\sigma\longrightarrow+\infty}\frac{\log\mu(\sigma,F)}{U(r)}.\tag{40}$$

On the other hand, assume that

$$\limsup_{\sigma\longrightarrow+\infty}\frac{\log\mu(\sigma,F)}{U(r)}=1, \tag{41}$$

then for any fixed $\varepsilon\in(0,1/2)$ and sufficiently large σ, it follows

$$\log\mu(\sigma,F)\le(1+\varepsilon)U(r). \tag{42}$$

For any positive real number x, in view of (5), there exists a positive integer n such that $\lambda_n<x\le\lambda_{n+1}$; thus, it yields

$$\begin{aligned}
&\int_0^x \exp\{(\sigma+\mathrm{i}t)y\}d\alpha(y)\\
&=\sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}}\exp\{(\sigma+\mathrm{i}t)y\}d\alpha(y)+\int_{\lambda_n}^{x}\exp\{(\sigma+\mathrm{i}t)y\}d\alpha(y)\\
&=\sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}}\exp\{\sigma y\}d_yI_k(y;it)+\int_{\lambda_n}^{x}\exp\{\sigma y\}d_yI_n(y;\mathrm{i}t)\\
&=\sum_{k=1}^{n-1}\left[\exp(\lambda_{k+1}\sigma)I_k(\lambda_{k+1};\mathrm{i}t)-\sigma\int_{\lambda_k}^{\lambda_{k+1}}\exp\{\sigma y\}I_k(y;\mathrm{i}t)dy\right]\\
&\quad+\exp(x\sigma)I_n(x;\mathrm{i}t)-\sigma\int_{\lambda_n}^{x}\exp\{\sigma y\}I_n(y;\mathrm{i}t)dy,
\end{aligned} \tag{43}$$

where $I_k(x;it)=\int_{\lambda_k}^{x}\exp\{\mathrm{i}ty\}d\alpha(y)$. Similar to the argument as in (38), it follows

$$\left|\int_0^x \exp\{(\sigma+\mathrm{i}t)y\}d\alpha(y)\right|\le 2\sum_{k=1}^{n}A_k^{*}e^{\lambda_k\sigma}. \tag{44}$$

Set $Q=\sigma+\log(1+(\log r/\log U(r)\log^2\log U(r)))=\log R$, where $r=e^{\sigma}$, and then from (44), we have

$$\begin{aligned}
M_u(\sigma,F)&\le 2\sum_{n=1}^{+\infty}A_n^{*}\exp\left\{\lambda_nQ-\lambda_n\log\left(1+\frac{\log r}{\log U(r)\log^2\log U(r)}\right)\right\}\\
&\le 2\mu(Q,F)\sum_{n=1}^{+\infty}e^{-\lambda_nE},
\end{aligned} \tag{45}$$

where

$$E:=\log\left(1+\frac{\log r}{\log U(r)\log^2\log U(r)}\right). \tag{46}$$

Thus, for any real number $\delta\in(0,1-d)$, in view of (26), there exists a positive integer $N\in\mathbb{N}_+$ such that

$$\lambda_n>(\log n)^{1/d+\delta}>logn, asn>N. \tag{47}$$

Hence, we can conclude from (45) and (47) that

$$\begin{aligned}
M_u(\sigma,F)&\le 2\mu(Q,F)\sum_{n=1}^{+\infty}e^{-\lambda_nE}\\
&\le 2\mu(Q,F)\left(N+\sum_{n=N}^{+\infty}e^{-E(\log n)^{1/d+\delta}}\right).
\end{aligned} \tag{48}$$

Set $T=[\exp\{(2/E)^{d+\delta/1-d-\delta}\}]+1$, where $[x]$ is an integral function. Then, it follows $E(\log n)^{1-d-\delta/d+\delta}>2$ as $n>T$. So, from (48), we can deduce

$$\begin{aligned}
M_u(\sigma,F)&\le 2\mu(Q,F)\left(N+\sum_{n=N}^{T}e^{-E\log n}+\sum_{n=T+1}^{+\infty}e^{-2\log n}\right)\\
&\le 2\mu(Q,F)\left(N+\int_1^T t^{-E}dt\right)\\
&=2\mu(Q,F)\left(N+\frac{1}{1-E}T^{1-E}\right).
\end{aligned} \tag{49}$$

Hence, from (39), (49), and by Theorem 8, it becomes

$$\begin{aligned}
\log M_u(\sigma,F)&\le\log\mu(Q,F)+\log 2+\log N\\
&\quad+\log\frac{1}{1-E}+(1-E)\log T\\
&\le(1+\varepsilon)U(R)+K_1+\left(\frac{2}{E}\right)^{d+\delta/1-d-\delta}\\
&\le(1+2\varepsilon)U(r),
\end{aligned} \tag{50}$$

where K_1 is a finite constant. Since ε is arbitrary and $\delta\in(0,1-d)$, then we conclude

$$\limsup_{\sigma\longrightarrow+\infty}\frac{\log M_u(\sigma,F)}{U(r)}\le\limsup_{\sigma\longrightarrow+\infty}\frac{\log\mu(\sigma,F)}{U(r)}=1. \tag{51}$$

Therefore, this completes the proof of Lemma 2 from (40) and (51).

Proof of Theorem 10. In view of

$$Q_1:=\limsup_{\sigma\longrightarrow+\infty}\frac{-\log[E_{n-1}(F,\beta)\exp(-\beta\lambda_n)]}{U(\psi_n)}, \tag{52}$$

and then it is obvious that the conclusion of Theorem 10 holds as $Q_1=+\infty$. Next, we will prove that the conclusion of Theorem 10 holds for $Q_1\in(0,+\infty)$.

If $Q_1\in(0,+\infty)$, then for any fixed real number $\varepsilon\in(0,1/3)$, there exists a positive integer $N\in\mathbb{N}_+$ such that

$$-\log[E_{n-1}(F,\beta)\exp(-\beta\lambda_n)]<(Q_1+\varepsilon)U(\psi_n), forn>N. \tag{53}$$

Let $V:=V(\sigma)=U(e^{\sigma})/\sigma$ and W be the inverse function of V, and then we know in view of Theorem 8 that V is an increasing function for $\sigma(\sigma\to+\infty)$. Thus, from the above inequality, we can deduce

$$\frac{\lambda_n}{Q_1+\varepsilon} \le \frac{\lambda_n}{-\log\left[E_{n-1}\exp\left(-\beta\lambda_n\right)\right]} U(\psi_n); \tag{54}$$

that is,

$$\log\left[E_{n-1}\exp\left((\sigma-\beta)\lambda_n\right)\right] \le -\lambda_n\left[W\left(\frac{\lambda_n}{Q_1+\varepsilon}\right)-\sigma\right]. \tag{55}$$

For sufficiently large σ, set

$$\begin{aligned} H &:= H(\sigma) = (Q_1+\varepsilon)U\left(r+\frac{r\log r}{\log U(r)\log^2\log U(r)}\right) \\ &= (Q_1+\varepsilon)U(R), \end{aligned} \tag{56}$$

and then it follows

$$\frac{H}{(Q_1+\varepsilon)Q} = \frac{U(R)}{Q}, W\left[\frac{H}{(Q_1+\varepsilon)Q}\right] = X. \tag{57}$$

If $\lambda_n\sigma \le H$, it yields from (55) and (57) that

$$\begin{aligned} \log\left[E_{n-1}\exp\left((\sigma-\beta)\lambda_n\right)\right] &\le -\lambda_n\left[W\left(\frac{\lambda_n}{Q_1+\varepsilon}\right)-\sigma\right] < \lambda_n\sigma \\ &\le H \le (Q_1+2\varepsilon)U(r). \end{aligned} \tag{58}$$

If $\lambda_n\sigma > H$, that is, $\lambda_n > H/\sigma$, it yields from (55) and (57) that

$$\begin{aligned} \log\left[E_{n-1}\exp\left((\sigma-\beta)\lambda_n\right)\right] &< -\lambda_n\left[W\left(\frac{\lambda_n}{Q_1+\varepsilon}\right)-\sigma\right] \\ &< -\lambda_n\left[W\left(\frac{H}{(Q_1+\varepsilon)\sigma}\right)-\sigma\right] = 0. \end{aligned} \tag{59}$$

Hence, we can conclude from (58) and (59) that

$$\log\left[E_{n-1}\exp\left((\sigma-\beta)\lambda_n\right)\right] \le (Q_1+2\varepsilon)U(r), for\, n > N. \tag{60}$$

For any $\beta < +\infty$, it follows

$$\begin{aligned} &A_n^*\exp\{\beta\lambda_n\} \\ &\quad= \sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}\exp\{ity\}d\alpha(y)\right|\exp\{\beta\lambda_n\} \\ &\quad\le \sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}\exp\{(\beta+it)y\}d\alpha(y)\right| \\ &\quad\le \sup_{-\infty<t<+\infty}\left|\int_{\lambda_n}^{\infty}\exp\{(\beta+it)y\}d\alpha(y)\right|, \end{aligned} \tag{61}$$

and thus for any $p\in\Pi_{n-1}$, we can deduce by combining the above inequalities that

$$A_n^*\exp\{\beta\lambda_n\} \le |F(\beta+it)-p(\beta+it)| \le \|F-p\|_\beta. \tag{62}$$

On the other hand, there exists $p_1\in\Pi_{n-1}$ such that

$$\|F-p_1\| \le 2E_{n-1}(F,\beta). \tag{63}$$

Hence, from (60)–(63) and for any $\beta<+\infty$, it follows

$$\begin{aligned} \log\left(A_n^*\exp\{\sigma\lambda_n\}\right) &\le \log 2\left[E_{n-1}\exp\left((\sigma-\beta)\lambda_n\right)\right] \\ &\le (Q_1+3\varepsilon)U(r), \text{for } n > N; \end{aligned} \tag{64}$$

that is,

$\log\mu(\sigma,F) \le (Q_1+3\varepsilon)U(r)$, for $\sigma\longrightarrow+\infty$.

Since ε is arbitrary and by Lemma 16, we get

$\limsup_{\sigma\longrightarrow+\infty}\log M_u(\sigma,F)/U(r) = 1 \le Q_1$.

We therefore completes the Proof of Theorem 6.

3. Proofs of Theorems 7 and 8

3.1. The Proof of Theorem 7. Here, we will adopt the reduction to absurdity to prove Theorem 7. Suppose that

$$\limsup_{\sigma\longrightarrow+\infty}\frac{\log M_u(\sigma,F)}{U(r)} = 1 < Q_2. \tag{65}$$

If $Q_2\in(1,+\infty)$, set $t\in(0,(Q_2-1/7)]\cap(0,1)$. From (65), for any small t and any positive integer n, we have

$$\log M_u(\sigma,F) \le (1+t)U(r). \tag{66}$$

If $F(s)\in L_\infty$ and $\beta(-\infty<\beta<+\infty)$, we have $F(s)\in\bar{L}_\beta$. Moreover, for $\beta<\sigma<+\infty$ and $p_n\in\Pi_n$, it follows

$$\begin{aligned} E_n(F,\beta) &\le \|F-p_n\|_\beta \le |F(\beta+it)-p_n(\beta+\mathrm{it})| \\ &\le \left|\int_0^{+\infty}\exp\{(\beta+\mathrm{it})y\}d\alpha(y)\right. \\ &\qquad \left.-\int_0^{\lambda_n}\exp\{(\beta+\mathrm{it})y\}d\alpha(y)\right| \\ &= \left|\int_{\lambda_n}^{\infty}\exp\{(\beta+\mathrm{it})y\}d\alpha(y)\right|. \end{aligned} \tag{67}$$

Similar to (44), we have

$$\left|\int_{\lambda_k}^{\infty}\exp\{(\beta+\mathrm{it})y\}d\alpha(y)\right| \le 2\sum_{n=k}^{+\infty}A_n^*\exp\{\beta\lambda_{n+1}\}. \tag{68}$$

Thus, for any $\sigma(\beta < \sigma < +\infty)$, we can conclude from (39), (67), and (68) that

$$\begin{aligned} E_n(F,\beta) &\le 2\sum_{k=n+1}^{\infty} A_{k-1}^{*} \exp\{\beta\lambda_k\} \\ &\le 4M_u(\sigma,F)\sum_{k=n+1}^{\infty} \exp\{(\beta-\sigma)\lambda_k\}. \end{aligned} \tag{69}$$

From (9), there exists $h'(0<h'<h)$ such that $(\lambda_{n+1}-\lambda_n)\ge h'$ for $n\ge 0$. Then, for $\sigma\ge\beta+1$, it follows from (69) that

$$\begin{aligned} E_n(F,\beta) &\le 4M_u(\sigma,F)\exp\{\lambda_{n+1}(\beta-\sigma)\} \\ &\quad\cdot\sum_{k=n+1}^{\infty}\exp\{(\lambda_k-\lambda_{n+1})(\beta-\sigma)\} \\ &\le 4M_u(\sigma,F)\exp\{\lambda_{n+1}(\beta-\sigma)\} \\ &\quad\cdot\exp\left\{h'(n+1)\right\}\sum_{k=n+1}^{\infty}\left(\exp\left\{-h'k\right\}\right) \\ &= 4M_u(\sigma,F)\exp\{\lambda_{n+1}(\beta-\sigma)\} \\ &\quad\cdot\left(1-\exp\left\{h'\right\}\right)^{-1}; \end{aligned} \tag{70}$$

that is,

$$E_{n-1}(F,\beta)\le K_1M_u(\sigma,F)\exp\{\lambda_n(\beta-\sigma)\}, \tag{71}$$

where K_1 is a constant. Hence, we conclude from (66) and (71) that

$$\begin{aligned} &\log\left[E_{n-1}\exp\left((\sigma-\beta)\lambda_n\right)\right] \\ &\quad\le\log K_1M_u(\sigma,F)\le(1+2t)U(r),\ \text{for}\ \sigma\longrightarrow+\infty. \end{aligned} \tag{72}$$

On the other hand, in view of Q_2 and t, there exists a subsequence $\{n(p)\}$ such that

$$\lambda_{n(p)} > (Q_2-t)U_1\left(\psi_{n(p)}\right)\ge(1+6t)U_1\left(\psi_{n(p)}\right). \tag{73}$$

We choose a sequence $\{\sigma_{n(p)}\}$ such that

$$\begin{aligned} &(1+2t)U\left(r_{n(p)}\right) \\ &\quad=\lambda_{n(p)}\log\left[1+\frac{1}{\log U\left(\psi_{n(p)}\right)\log^2\log U\left(\psi_{n(p)}\right)}\right], \end{aligned} \tag{74}$$

and then it follows from (72) and (74) that

$$\sigma_{n(p)}\le\log\psi_{n(p)}+\log\left[1+\frac{1}{\log U\left(\psi_{n(p)}\right)\log^2\log U\left(\psi_{n(p)}\right)}\right]. \tag{75}$$

Hence,

$$\begin{aligned} U\left(r_{n(p)}\right) &\le U\left\{\psi_{n(p)}\left[1+\frac{1}{\log U\left(\psi_{n(p)}\right)\log^2\log U\left(\psi_{n(p)}\right)}\right]\right\} \\ &\le U\left\{\psi_{n(p)}\left[1+\frac{\log\psi_{n(p)}}{\log U\left(\psi_{n(p)}\right)\log^2\log U\left(\psi_{n(p)}\right)}\right]\right\} \\ &\le(1+t)U\left(\psi_{n(p)}\right). \end{aligned} \tag{76}$$

In view of $t\in(0,(Q_2-1/7)]\cap(0,1)$, then it follows $(1+2t)(1+t)<1+5t$. Thus, by combining (74) and (76), we deduce

$$\begin{aligned} \lambda_{n(p)} &= (1+2t)U\left(r_{n(p)}\right) \\ &\quad\cdot\left\{\log\left[1+\frac{1}{\log U\left(\psi_{n(p)}\right)\log^2\log U\left(\psi_{n(p)}\right)}\right]\right\}^{-1} \\ &\le(1+2t)(1+t)(1+o(1))U_1\left(\psi_{n(p)}\right) \\ &\le(1+5t)U_1\left(\psi_{n(p)}\right), \end{aligned} \tag{77}$$

which is a contradiction with (73).

If $Q_2=+\infty$, we choose $t=1$, and by using the same argument as above, we also get a contradiction.

Therefore, this completes the Proof of Theorem 7.

3.2. The Proof of Theorem 8. From Lemma 15, it is easy to get the following lemma.

Lemma 17. *If Laplace-Stieltjes transform $F(s)$ is of infinite order, and the sequence $\{\lambda_n\}$ satisfied (5), (9), (12), and (26), then*

$$\begin{aligned} &\limsup_{\sigma\longrightarrow+\infty}\frac{\log\ \log M_u(\sigma,F)}{\log U(r)} \\ &\quad= 1\Longleftrightarrow\limsup_{\sigma\longrightarrow+\infty}\frac{\log\ \log\mu(\sigma,F)}{\log U(r)}=1. \end{aligned} \tag{78}$$

Proof of Theorem 8. If $Q_3=+\infty$, then it follows

$$\limsup_{\sigma\longrightarrow+\infty}\frac{\lambda_n}{U(\psi_n)\log^2U(\psi_n)}=+\infty. \tag{79}$$

By combining Corollary 13, it yields $\limsup_{\sigma\longrightarrow+\infty}\log M_u(\sigma,F)/U(r)=+\infty$, *which is a contradiction with the properties of $U(r)$. Hence, $Q_3<+\infty$. Thus, for any fixed $\varepsilon>0$ and sufficiently large n, we have*

$$\lambda_n < U^{Q_3+\varepsilon}(\psi_n), \log V_1\left((\lambda_n)^{1/Q_3+\varepsilon}\right) \leq \frac{-\log\left[E_{n-1}(F,\beta)\exp(-\beta\lambda_n)\right]}{\lambda_n}, \tag{80}$$

where $r = V_1(x)$ *and* $x = U(r)$ *are two reciprocally inverse functions; that is,*

$$\log\left[E_{n-1}(F,\beta)\exp((\sigma-\beta)\lambda_n)\right] \leq -\lambda_n\left[\log V_1\left((\lambda_n)^{1/Q_3+\varepsilon}\right) - \sigma\right]. \tag{81}$$

For any fixed sufficiently large σ, take

$$I := I(\sigma) = \sigma U^{Q_3+\varepsilon}\left(r + \frac{r\log r}{\log U(r)\log^2\log U(r)}\right), \tag{82}$$

and then it yields

$$\log V_1\left[\left(\frac{I}{\sigma}\right)^{1/Q_3+\varepsilon}\right] = \sigma + \log\left(1 + \frac{\log r}{\log U(r)\log^2\log U(r)}\right). \tag{83}$$

If $\lambda_n\sigma \leq I$, *from (81) and (83), it follows*

$$\begin{aligned}\log\left[E_{n-1}(F,\beta)\exp((\sigma-\beta)\lambda_n)\right] &\leq -\lambda_n\left[\log V_1\left((\lambda_n)^{1/Q_3+\varepsilon}\right) - \sigma\right] < \sigma\lambda_n \\ &\leq I \leq U^{Q_3+3\varepsilon}(r).\end{aligned} \tag{84}$$

If $\lambda_n\sigma > I$, *that is,* $\lambda_n > I/\sigma$, *thus in view of (81) and (83), it yields*

$$\begin{aligned}\log\left[E_{n-1}(F,\beta)\exp((\sigma-\beta)\lambda_n)\right] &\leq -\lambda_n\left[\log V_1\left((\lambda_n)^{1/Q_3+\varepsilon}\right) - \sigma\right] \\ &< -\lambda_n\left\{\log V_1\left[\left(\frac{I}{\sigma}\right)^{1/Q_3+\varepsilon}\right] - \sigma\right\} < 0.\end{aligned} \tag{85}$$

Hence, from (62), (63), (84), and (85), we deduce

$$\log\ \log\mu(\sigma,F) \leq (Q_3+4\varepsilon)\log U(r), \text{for } n > N, \sigma \longrightarrow +\infty. \tag{86}$$

Let $\varepsilon \longrightarrow 0$ *and by Lemma 17, and it leads to*

$$\limsup_{\sigma\longrightarrow+\infty}\frac{\log\ \log M_u(\sigma,F)}{\log U(r)} \leq Q_3. \tag{87}$$

Suppose that

$$1 = \limsup_{\sigma\longrightarrow+\infty}\frac{\log\ \log M_u(\sigma,F)}{\log U(r)} < Q_3. \tag{88}$$

Set $t \in (0, (Q_3 - 1/7)]$, *and then for any positive integer n and sufficiently large σ, from (71), we have*

$$\begin{aligned}&\log\left[E_{n-1}(F,\beta)\exp((\sigma-\beta)\lambda_n)\right] \\ &\quad\leq \log M_u(\sigma,F) + \log K_1 \leq U^{1+2t}(r),\end{aligned} \tag{89}$$

where K_1 *is a constant. Since* $1+6t \leq Q_3 - t$, *then there exists a subsequence* $\{n(p)\}$ *such that*

$$\lambda_{n(p)} \geq U^{Q_3-t}\left(\psi_{n(p)}\right) > U^{1+6t}\left(\psi_{n(p)}\right). \tag{90}$$

We choose a sequence $\{\sigma_{n(p)}\}$ *such that*

$$U^{1+2t}\left(r_{n(p)}\right) = \lambda_{n(p)}\log\left[1 + \frac{1}{\log^2 U\left(\psi_{n(p)}\right)}\right]. \tag{91}$$

Thus, it follows from (89) and (91) that

$$e^{\sigma_{n(p)}} \leq \psi_{n(p)}\left[1 + \frac{1}{\log^2 U\left(\psi_{n(p)}\right)}\right]. \tag{92}$$

That is,

$$\begin{aligned}U\left(r_{n(p)}\right) &\leq U\left\{\psi_{n(p)}\left[1 + \frac{1}{\log^2 U\left(\psi_{n(p)}\right)}\right]\right\} \\ &\leq (1+t)U\left(\psi_{n(p)}\right).\end{aligned} \tag{93}$$

We therefore can conclude from (91) that

$$\begin{aligned}\lambda_{n(p)} &= U^{1+2t}\left(r_{n(p)}\right)\left\{\log\left[1 + \frac{1}{\log^2 U\left(\psi_{n(p)}\right)}\right]\right\}^{-1} \\ &\leq 2U^{1+2t}\left(r_{n(p)}\right)\log^2 U\left(\psi_{n(p)}\right) \\ &\leq 2(1+t)^{1+2t}U^{1+2t}\left(\psi_{n(p)}\right)\log^2 U\left(\psi_{n(p)}\right) \\ &\leq U^{1+5t}\left(\psi_{n(p)}\right),\end{aligned} \tag{94}$$

which is a contradiction with the inequality (90).

Therefore, this completes the Proof of Theorem 8.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that none of the authors have any competing interests in the manuscript.

Authors' Contributions

H. Y. Xu performed the conceptualization. H. Y. Xu and X. Shen performed the writing-original draft preparation. H. Y. Xu performed the writing review and editing. H. Y. Xu and X. Shen contributed to the funding acquisition.

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Research Article

A New Family of Degenerate Poly-Genocchi Polynomials with Its Certain Properties

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In this paper, we introduce a new type of degenerate Genocchi polynomials and numbers, which are called degenerate poly-Genocchi polynomials and numbers, by using the degenerate polylogarithm function, and we derive several properties of these polynomials systematically. Then, we also consider the degenerate unipoly-Genocchi polynomials attached to an arithmetic function, by using the degenerate polylogarithm function, and investigate some identities of those polynomials. In particular, we give some new explicit expressions and identities of degenerate unipoly polynomials related to special numbers and polynomials.

1. Introduction

In [1, 2], Carlitz initiated a study of degenerate versions of some special polynomials and numbers, namely, the degenerate Bernoulli and Euler polynomials and numbers. Kim et al. [3–5] have studied the degenerate versions of special numbers and polynomials actively. These ideas provide a powerful tool in order to define special numbers and polynomials of their degenerate versions. The notion of degenerate version forms a special class of polynomials because of their great applicability. Despite the applicability of special functions in classical analysis and statistics, they also arise in communication systems, quantum mechanics, nonlinear wave propagation, electric circuit theory, electromagnetic theory, etc. In particular, Genocchi numbers have been extensively studied in many different contexts in such branches of mathematics as, for instance, elementary number theory, complex analytic number theory, differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic number theory (p-adic L-functions), and quantum physics (quantum groups). The works of Genocchi numbers and their combinatorial relations have received much attention [6–11]. In the paper, we focus on a new type of degenerate poly-Genocchi polynomial and numbers.

The aim of this paper is to introduce a degenerate version of the poly-Genocchi polynomials and numbers, the so-called new type of degenerate poly-Genocchi polynomials and numbers, constructing from the degenerate polylogarithm function. We derive some explicit expressions and identities for those numbers and polynomials.

The classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are, respectively, defined by the following generating functions (see [12–22]):

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad |t|<\pi, \tag{1}$$

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad |t| < \pi. \tag{2}$$

In the case when $x = 0$, $E_n(0) := E_n$ and $G_n(0) := G_n$ are, respectively, called the Euler numbers and Genocchi numbers.

The degenerate exponential function [23, 24] is defined by

$$\begin{aligned} &e_\lambda^x(t) = (1+\lambda t)^{x/\lambda}, \\ &e_\lambda^1(t) = e_\lambda(t)\ (\lambda \in \mathbb{R}). \end{aligned} \tag{3}$$

Note that

$$\lim_{\lambda \longrightarrow 0} (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} = e^{xt}. \tag{4}$$

In [1, 2], Carlitz introduced the degenerate Bernoulli and degenerate Euler polynomials defined by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \frac{t}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \beta_n(x;\lambda)\frac{t^n}{n!}, \tag{5}$$

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t) = \frac{2}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathfrak{E}_n(x;\lambda)\frac{t^n}{n!}. \tag{6}$$

In the case when $x = 0$, $B_{n,\lambda}(0) := B_{n,\lambda}$ are called the degenerate Bernoulli numbers and $E_{n,\lambda}(0) := E_{n,\lambda}$ are called the degenerate Euler numbers.

Let $(x)_{n,\lambda}$ be the degenerate falling factorial sequence given by

$$(x)_{n,\lambda} := x(x-\lambda)\cdots(x-(n-1)\lambda)\ (n \geq 1), \tag{7}$$

with the assumption $(x)_{0,\lambda} = 1$.

In [5], Kim et al. considered the degenerate Genocchi polynomials given by

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!}. \tag{8}$$

In the case when $x = 0$, $G_{n,\lambda} := G_{n,\lambda}(0)$ are called the degenerate Genocchi numbers.

For $k \in \mathbb{Z}$, the polylogarithm function is defined by a power series in t, which is also a Dirichlet series in k (see [25, 26]):

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k} = t + \frac{t^2}{2^k} + \frac{t^3}{3^k} + \cdots (|t| < 1). \tag{9}$$

This definition is valid for arbitrary complex order k and for all complex arguments t with $|t| < 1$: it can be extended to $|t| \geq 1$ by analytic continuation.

It is noticed that

$$\mathrm{Li}_1(t) = \sum_{n=1}^{\infty} \frac{t^n}{n} = -\log(1-t). \tag{10}$$

For $\lambda \in \mathbb{R}$, Kim and Kim [3] defined the degenerate version of the logarithm function, denoted by $\log_\lambda(1+t)$, as follows (see [4]):

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!}, \tag{11}$$

being the inverse of the degenerate version of the exponential function $e_\lambda(t)$ as has been shown below:

$$e_\lambda(\log_\lambda(t)) = \log_\lambda(e_\lambda(t)) = t. \tag{12}$$

It is noteworthy to mention that

$$\lim_{\lambda \longrightarrow 0} \log_\lambda(1+t) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{t^n}{n!} = \log(1+t). \tag{13}$$

The degenerate polylogarithm function [3] is defined by Kim and Kim to be

$$l_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}x^n\ (k \in \mathbb{Z}, |x| < 1). \tag{14}$$

It is clear that (see [27, 28])

$$\lim_{\lambda \longrightarrow 0} l_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \mathrm{Li}_k(x). \tag{15}$$

From (11) and (14), we get

$$l_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{n!}x^n = -\log_\lambda(1-x). \tag{16}$$

Very recently, Kim and Kim [3] introduced the new type of degenerate version of the Bernoulli polynomials and numbers, by using the degenerate polylogarithm function as follows:

$$\frac{l_{k,\lambda}(1-e_\lambda(-t))}{1-e_\lambda(-t)}e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{17}$$

When $x = 0$, $\beta_{j,\lambda}^{(k)} := \beta_{j,\lambda}^{(k)}(0)$ are called the new type of degenerate poly-Bernoulli numbers.

The degenerate Stirling numbers of the first kind [24] are defined by

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}\ (k \geq 0). \tag{18}$$

It is clear that

$$\lim_{\lambda \longrightarrow 0} S_{1,\lambda}(n,k) \coloneqq S_1(n,k), \tag{19}$$

calling the Stirling numbers of the first kind given by (see [29, 30])

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} \; (k \geq 0). \tag{20}$$

The degenerate Stirling numbers of the second kind [31] are given by (see [2, 13–22, 25–32])

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(j,k)\frac{t^n}{n!} \; (k \geq 0). \tag{21}$$

Note here that

$$\lim_{\lambda \longrightarrow 0} S_{2,\lambda}(n,k) \coloneqq S_2(n,k), \tag{22}$$

standing for the Stirling numbers of the second kind given by means of the following generating function (see [1–8, 12–38]):

$$\frac{1}{k!}\left(e^t-1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} \; (k \geq 0). \tag{23}$$

This paper is organized as follows. In Section 1, we recall some necessary stuffs that are needed throughout this paper. These include the degenerate exponential functions, the degenerate Genocchi polynomials, the degenerate Euler polynomials, and the degenerate Stirling numbers of the first and second kinds. In Section 2, we introduce the new type of degenerate poly-Genocchi polynomials by making use of the degenerate polylogarithm function. We express those polynomials in terms of the degenerate Genocchi polynomials and the degenerate Stirling numbers of the first kind and also of the degenerate Euler polynomials and the Stirling numbers of the first kind. We represent the generating function of the degenerate poly-Genocchi numbers by iterated integrals from which we obtain an expression of those numbers in terms of the degenerate Bernoulli numbers of the second kind. In Section 3, we introduce the new type of degenerate unipoly-Genocchi polynomials by making use of the degenerate polylogarithm function. We express those polynomials in terms of the degenerate Genocchi polynomials and the degenerate Stirling numbers of the first kind and also of the degenerate Euler polynomials and the Stirling numbers of the first kind and second kind.

2. New Type of Degenerate Genocchi Numbers and Polynomials

In this section, we define the new type of degenerate Genocchi numbers and polynomials by using the degenerate polylogarithm function which is called the degenerate poly-Genocchi polynomials as follows.

For $k \in \mathbb{Z}$, we define the new type of degenerate Genocchi numbers, which are called the degenerate poly-Genocchi numbers, as

$$\frac{2}{e_\lambda(t)+1} l_{k,\lambda}(1-e_\lambda(-t)) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}\frac{t^n}{n!}. \tag{24}$$

Note that

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{(1)}\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+1} l_{1,\lambda}(1-e_\lambda(-t)) = \frac{2t}{e_\lambda(t)+1} = \sum_{n=0}^{\infty} G_{n,\lambda}\frac{t^n}{n!}. \tag{25}$$

Thus, we have (see [6])

$$G_{n,\lambda}^{(1)} = G_{n,\lambda} \; (n \geq 0). \tag{26}$$

Now, we consider the new type of degenerate Genocchi polynomials which are called the degenerate poly-Genocchi polynomials defined by

$$\frac{2l_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{27}$$

In the case when $x=0$, $G_{n,\lambda}^{(k)} \coloneqq G_{n,\lambda}^{(k)}(0)$. Using equation (27), we see

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{2l_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)+1} e_\lambda^x(t) \\ &= \sum_{m=0}^{\infty} G_{m,\lambda}^{(k)}\frac{t^m}{m!}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(n/m)G_{m,\lambda}^{(k)}(x)_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{28}$$

Therefore, by equation (28), we obtain the following theorem.

Theorem 1. *Let n be a nonnegative integer. Then,*

$$G_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n}(n/m)G_{m,\lambda}^{(k)}(x)_{n-m,\lambda}. \tag{29}$$

From (27), we note that

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} = \frac{2t}{e_\lambda(t)+1} e_\lambda^x(t)\frac{1}{t} l_{k,\lambda}(1-e_\lambda(-t)), \tag{30}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)\frac{1}{t}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^k}(1-e_\lambda(-t))^m \\
&= \frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)\frac{1}{t}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^{k-1}} \\
&\cdot \sum_{l=m}^{\infty}(-1)^{l-m}S_{2,\lambda}(l,m)\frac{t^l}{l!} = \frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)\frac{1}{t} \\
&\cdot \sum_{l=1}^{\infty}\left(\sum_{m=1}^{l}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1}}{m^{k-1}}S_{2,\lambda}(l,m)\right)\frac{t^l}{l!} \\
&= \left(\sum_{n=0}^{\infty}G_{n,\lambda}(x)\frac{t^n}{n!}\right)\frac{1}{t}\left(\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l}}{m^{k-1}}S_{2,\lambda}(l+1,m)\right)\frac{t^{l+1}}{(l+1)!}\right) \\
&= \left(\sum_{n=0}^{\infty}G_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l}}{m^{k-1}}\frac{S_{2,\lambda}(l+1,m)}{l+1}\right)\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\sum_{m=1}^{l+1}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l}}{m^{k-1}}\frac{S_{2,\lambda}(l+1,m)}{l+1}G_{n-l,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{31}$$

Therefore, by equations (30) and (31), we get the following theorem.

Theorem 2. *Let n be a nonnegative integer. Then,*

$$G_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}\sum_{m=1}^{l+1}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l}}{m^{k-1}}\frac{S_{2,\lambda}(l+1,m)}{l+1}G_{n-l,\lambda}(x). \tag{32}$$

Using equations (27) and (6), we see

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+1}e_\lambda^x(t)l_{k,\lambda}(1-e_\lambda(-t)), \tag{33}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^k}(1-e_\lambda(-t))^m \\
&= \frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^{k-1}} \\
&\cdot \sum_{l=m}^{\infty}(-1)^{l-m}S_{2,\lambda}(l,m)\frac{t^l}{l!} = \frac{2}{e_\lambda(t)+1}e_\lambda^x(t) \\
&\cdot \sum_{l=1}^{\infty}\left(\sum_{m=1}^{l}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1}}{m^{k-1}}S_{2,\lambda}(l,m)\right)\frac{t^l}{l!} \\
&= \left(\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{l=1}^{\infty}\left(\sum_{m=1}^{l}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1}}{m^{k-1}}S_{2,\lambda}(l,m)\right)\frac{t^l}{l!}\right) \\
&= \sum_{n=1}^{\infty}\left(\sum_{l=1}^{n}\binom{n}{l}\sum_{m=1}^{l}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1}}{m^{k-1}}S_{2,\lambda}(l,m)E_{n-l,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{34}$$

By equations (33) and (34), we obtain the following theorem.

Theorem 3. *Let n be a nonnegative integer. Then,*

$$G_{n,\lambda}^{(k)}(x) = \sum_{l=1}^{n}\binom{n}{l}\sum_{m=1}^{l}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1}}{m^{k-1}}S_{2,\lambda}(l,m)E_{n-l,\lambda}(x). \tag{35}$$

From (27), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}\frac{x^n}{n!} &= \frac{2}{e_\lambda(x)+1}l_{k,\lambda}(1-e_\lambda(-x)) = \frac{2}{e_\lambda(x)+1} \\
&\cdot \underbrace{\int_0^x\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\int_0^t\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\cdots\int_0^t\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}}_{(k-2)-times}tdtdt\cdots dt\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}\frac{x^n}{n!} \\
&= \frac{2}{e_\lambda(x)+1}\underbrace{\int_0^x\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\int_0^t\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\cdots\int_0^t\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}tdtdt\cdots dt}_{(k-2)-times}.
\end{aligned} \tag{36}$$

For k = 2 in (36) and using [3] (Eq. (27)), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{(2)}\frac{x^n}{n!} &= \frac{2}{e_\lambda(x)+1}\int_0^x\frac{t}{1-e_\lambda(-t)}e_\lambda^{1-\lambda}(-t)dt \\
&= \frac{2}{e_\lambda(x)+1}\int_0^x\sum_{j=0}^{\infty}\beta_{j,\lambda}(1-\lambda)(-1)^j\frac{t^j}{j!}dt \\
&= \frac{2x}{e_\lambda(x)+1}\sum_{j=0}^{\infty}\frac{\beta_{j,\lambda}(1-\lambda)}{j+1}(-1)^j\frac{x^j}{j!} \\
&= \sum_{n=0}^{\infty}G_{n,\lambda}\frac{x^n}{n!}\sum_{j=0}^{\infty}\frac{\beta_{j,\lambda}(1-\lambda)}{j+1}(-1)^j\frac{x^j}{j!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}(n/j)(-1)^jG_{n-j,\lambda}\frac{\beta_{j,\lambda}(1-\lambda)}{j+1}\right)\frac{x^n}{n!}.
\end{aligned} \tag{37}$$

Therefore, by equation (37), we get the following theorem.

Theorem 4. *Let n be a nonnegative integer. Then,*

$$G_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n}(n/m)G_{m,\lambda}^{(k)}(x)_{n-m,\lambda}. \tag{38}$$

In general, by equation (37), we see

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}\frac{x^n}{n!} &= \frac{2}{e_\lambda(x)+1}\int_0^x\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\int_0^t\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\cdots \\
&\cdot \int_0^t\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}tdtdt\cdots dt \\
&= \sum_{n_1,n_2,\cdots,n_{k-1}=n}^{\infty}\frac{1}{n_1!n_2!\cdots n_{k-1}!}\frac{\beta_{n_1,\lambda}(1-\lambda)}{n_1+1}\frac{\beta_{n_2,\lambda}(1-\lambda)}{n_1+n_2+1} \\
&\times\cdots\frac{\beta_{n_{k-1},\lambda}(1-\lambda)}{n_1+\cdots+n_{k-1}+1}(-x)^{n_1,n_2,\cdots,n_{k-1}}\frac{2x}{e_\lambda(x)+1} \\
&= \sum_{n=0}^{\infty}(-1)^n\sum_{n_1,n_2,\cdots,n_k=n}\binom{n}{n_1,n_2,\cdots,n_k}\frac{\beta_{n_1,\lambda}(1-\lambda)}{n_1+1} \\
&\cdot\frac{\beta_{n_2,\lambda}(1-\lambda)}{n_1+n_2+1}\cdots\frac{\beta_{n_{k-1},\lambda}(1-\lambda)}{n_1+\cdots+n_{k-1}+1}G_{n,\lambda}\frac{x^n}{n!}.
\end{aligned} \tag{39}$$

By equation (39), we obtain the following theorem.

Theorem 5. *Let $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$G_{n,\lambda}^{(k)} = (-1)^n \sum_{n_1,n_2,\cdots,n_k=n} \binom{n}{n_1, n_2, \cdots, n_k} \frac{\beta_{n_1,\lambda}(1-\lambda)}{n_1+1} \cdot \frac{\beta_{n_2,\lambda}(1-\lambda)}{n_1+n_2+1} \cdots \frac{\beta_{n_{k-1},\lambda}(1-\lambda)}{n_1+\cdots+n_{k-1}+1} G_{n,\lambda}. \tag{40}$$

From (27), we observe that

$$\begin{aligned} 2l_{k,l}(1-e_\lambda(-t)) &= (1+e_\lambda(t)) \sum_{m=0}^{\infty} G_{m,\lambda}^{(k)} \frac{t^m}{m!} \\ &= \sum_{j=1}^{\infty} \left(G_{j,\lambda}^{(k)} + G_{j,\lambda}^{(k)}(1) \right) \frac{t^j}{j!}. \end{aligned} \tag{41}$$

On the other hand,

$$\begin{aligned} 2l_{k,l}(1-e_\lambda(-t)) &= 2\sum_{r=1}^{\infty} \frac{(-\lambda)^{r-1}(1)_{r,1/\lambda}}{(r-1)!r^k} (1-e_\lambda(-t))^r \\ &= 2\sum_{m=1}^{\infty} \frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^{k-1}} \frac{1}{m!} (1-e_\lambda(-t))^m \\ &= 2\sum_{r=1}^{\infty} \frac{(-\lambda)^{r-1}(1)_{r,1/\lambda}}{r^{k-1}} \sum_{j=r}^{\infty} S_{2,\lambda}(j,r)(-1)^{j-r} \frac{t^j}{j!} \\ &= 2\sum_{j=1}^{\infty} \left(\sum_{r=1}^{j} \frac{(-1)^{j-1}(1)_{r,1/\lambda}}{r^{k-1}} \lambda^{r-1} S_{2,\lambda}(j,r) \right) \frac{t^j}{j!}. \end{aligned} \tag{42}$$

Therefore, by equations (41) and (42), we get the following theorem.

Theorem 6. *Let $k \in \mathbb{Z}$ and $j \geq 1$. Then,*

$$\frac{1}{2}\left[G_{j,\lambda}^{(k)} + G_{j,\lambda}^{(k)}(1) \right] = (-1)^{j-1} \sum_{r=1}^{j} \frac{(1)_{r,1/\lambda}}{r^{k-1}} \lambda^{r-1} S_{2,\lambda}(j,r). \tag{43}$$

From equations (27) and (14), we see

$$\begin{aligned} 2t = 2l_{1,l}(1-e_\lambda(-t)) &= 2\sum_{m=1}^{\infty} \frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^k} (1-e_\lambda(-t))^m \\ &= 2\sum_{m=1}^{\infty} \frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m!} (1-e_\lambda(-t))^m \\ &= 2\sum_{m=1}^{\infty} (-\lambda)^{m-1}(1)_{m,1/\lambda} \sum_{n=m}^{\infty} S_{2,\lambda}(n,m)(-1)^{n-m} \frac{t^n}{n!} \\ &= 2\sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} (-1)^{n-1}(1)_{m,1/\lambda} \lambda^{m-1} S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{44}$$

By comparing the coefficients on both sides of (44), we obtain the following theorem.

Theorem 7. *For $n \in \mathbb{N}$, we have*

$$\sum_{m=1}^{n} (-1)^{n-1}(1)_{m,\frac{1}{\lambda}} \lambda^{m-1} S_{2,\lambda}(n,m) = \delta_{n,1}, \tag{45}$$

where $\delta_{n,k}$ is Kronecker's symbol.

Note that

$$\lim_{\lambda \longrightarrow 0} G_{n,\lambda}^{(1)} = G_n, \lim_{\lambda \longrightarrow 0} G_{n,\lambda}^{(1)}(x) = G_n(x). \tag{46}$$

3. Degenerate Unipoly-Genocchi Numbers and Polynomials

Let p be any arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. Kim and Kim [29] defined the unipoly function attached to polynomials $p(x)$ by

$$u_k(x \mid p) = \sum_{n=1}^{\infty} \frac{p(n)}{n^k} x^n \ (k \in \mathbb{Z}). \tag{47}$$

Moreover (see [25]),

$$u_k(x \mid 1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \mathrm{Li}_k(x) \tag{48}$$

is the ordinary polylogarithm function.

In [8], Lee and Kim defined the degenerate unipoly function attached to polynomials $p(x)$ as follows:

$$u_{k,\lambda}(x \mid p) = \sum_{i=1}^{\infty} p(i) \frac{(-\lambda)^{i-1}(1)_{i,1/\lambda}}{i^k} x^i. \tag{49}$$

It is worthy to note that

$$u_{k,\lambda}\left(x \mid \frac{1}{\Gamma}\right) = l_{k,\lambda}(x) \tag{50}$$

is the degenerate polylogarithm function.

Now, we define the degenerate unipoly-Genocchi polynomials attached to polynomials $p(x)$ by

$$\frac{2u_{k,\lambda}(1-e_\lambda(-t) \mid p)}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!}. \tag{51}$$

In the case when $x=0$, $G_{n,\lambda,p}^{(k)} := G_{n,\lambda,p}^{(k)}(0)$ are called the degenerate unipoly-Genocchi numbers attached to p.

From (51), we see

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,\lambda,1/\Gamma}^{(k)} \frac{t^n}{n!} &= \frac{2}{e_\lambda(t)+1} u_{k,\lambda}\left(1-e_\lambda(-t) \mid \frac{1}{\Gamma}\right) \\ &= \frac{2}{e_\lambda(t)+1} \sum_{r=1}^{\infty} \frac{(-\lambda)^{r-1}(1)_{r,1/\lambda}(1-e_\lambda(-t))^r}{r^k (r-1)!} \\ &= \frac{2}{e_\lambda(t)+1} l_{k,\lambda}(1-e_\lambda(-t)) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(k)} \frac{t^n}{n!}.\end{aligned} \tag{52}$$

Thus, by (52), we have

$$G_{n,\lambda,\frac{1}{\Gamma}}^{(k)} = G_{n,\lambda}^{(k)}. \tag{53}$$

From (51), we have

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!} &= \frac{2e_\lambda^x(t)}{e_\lambda(t)+1} u_{k,\lambda}(1-e_\lambda(-t) \mid p) \\ &= \frac{2e_\lambda^x(t)}{e_\lambda(t)+1} \frac{1}{t} \sum_{m=1}^{\infty} \frac{p(m)(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^k} (1-e_\lambda(-t))^m \\ &= \frac{2t}{e_\lambda(t)+1} e_\lambda^x(t) \frac{1}{t} \sum_{m=1}^{\infty} \frac{p(m)(-\lambda)^{m-1}(1)_{m,1/\lambda} m!}{m^k} \\ &\quad \cdot \sum_{l=m}^{\infty} (-1)^{l-m} S_{2,\lambda}(l,m) \frac{t^l}{l!} = \frac{2t}{e_\lambda(t)+1} e_\lambda^x(t) \frac{1}{t} \sum_{l=1}^{\infty} \\ &\quad \cdot \left(\sum_{m=1}^{l} \frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1} m!}{m^k} S_{2,\lambda}(l,m) \right) \frac{t^l}{l!} = \left(\sum_{n=0}^{\infty} G_{n,\lambda}(x) \frac{t^n}{n!} \right) \frac{1}{t} \\ &\quad \cdot \left(\sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^l m!}{m^k} S_{2,\lambda}(l+1,m) \right) \frac{t^{l+1}}{(l+1)!} \right) \\ &= \left(\sum_{n=0}^{\infty} G_{n,\lambda}(x) \frac{t^n}{n!} \right) \left(\sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^l m!}{m^k} \frac{S_{2,\lambda}(l+1,m)}{l+1} \right) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \sum_{m=1}^{l+1} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^l m!}{m^k} \frac{S_{2,\lambda}(l+1,m)}{l+1} G_{n-l,\lambda}(x) \right) \frac{t^n}{n!}.\end{aligned} \tag{54}$$

Therefore, by equation (54), we get the following theorem.

Theorem 8. *Let n be a nonnegative integer. Then,*

$$G_{n,\lambda,p}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=1}^{l+1} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^l m!}{m^k} \frac{S_{2,\lambda}(l+1,m)}{l+1} G_{n-l,\lambda}(x). \tag{55}$$

Using equations (49) and (51), we see

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} &= \frac{2}{e_\lambda(t)+1} e_\lambda^x(t) \sum_{m=1}^{\infty} \frac{p(m)(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^k} (1-e_\lambda(-t))^m \\ &= \frac{2}{e_\lambda(t)+1} e_\lambda^x(t) \sum_{m=1}^{\infty} \frac{p(m)(-\lambda)^{m-1}(1)_{m,1/\lambda} m!}{m^k} \\ &\quad \cdot \sum_{l=m}^{\infty} (-1)^{l-m} S_{2,\lambda}(l,m) \frac{t^l}{l!} = \frac{2}{e_\lambda(t)+1} e_\lambda^x(t) \\ &\quad \cdot \sum_{l=1}^{\infty} \left(\sum_{m=1}^{l} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1} m!}{m^k} S_{2,\lambda}(l,m) \right) \frac{t^l}{l!} \\ &= \left(\sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!} \right) \left(\sum_{l=1}^{\infty} \left(\sum_{m=1}^{l} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1} m!}{m^k} S_{2,\lambda}(l,m) \right) \frac{t^l}{l!} \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \binom{j}{l} \sum_{m=1}^{l} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1} m!}{m^k} S_{2,\lambda}(l,m) E_{n-l,\lambda}(x) \right) \frac{t^n}{n!}.\end{aligned} \tag{56}$$

By, equations (51) and (56), we obtain the following theorem.

Theorem 9. *Let n be a nonnegative integer. Then,*

$$G_{n,\lambda,p}^{(k)}(x) = \sum_{l=1}^{n} \binom{n}{l} \sum_{m=1}^{l} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{l-1} m!}{m^k} S_{2,\lambda}(l,m) E_{n-l,\lambda}(x). \tag{57}$$

From (6), (49), and (51), we get

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!} &= \frac{2e_\lambda^x(t)}{e_\lambda(t)+1} u_{k,\lambda}(1-e_\lambda(-t) \mid p) = \frac{2t}{e_\lambda(t)+1} \frac{e_\lambda(t)-1}{e_\lambda(t)-1} e_\lambda^x(t) \frac{1}{t} \\ &\quad \cdot \sum_{m=1}^{\infty} \frac{p(m)(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^k} (1-e_\lambda(-t))^m \\ &= \frac{2te_\lambda^x(t)}{e_\lambda^2(t)-1} (e_\lambda(t)-1) \frac{1}{t} \sum_{m=1}^{\infty} \frac{p(m)(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^k} (1-e_\lambda(-t))^m \\ &= \frac{2te_{\lambda/2}^{x/2}(2t)}{e_{\lambda/2}(2t)-1} (e_\lambda(t)-1) \sum_{l=0}^{\infty} \sum_{m=1}^{l+1} \\ &\quad \cdot \frac{p(m)(-1)^l(\lambda)^{m-1}(1)_{m,1/\lambda} m!}{m^k} S_{2,\lambda}(l+1,m) \\ &\quad \cdot \frac{t^l}{(l+1)!} = \left(\sum_{n=0}^{\infty} \beta_{n,\lambda/2}\left(\frac{x}{2}\right) \frac{2^n t^n}{n!} \right) \left(\sum_{i=1}^{\infty} (1)_{i,\lambda} \frac{t^i}{i!} \right) \\ &\quad \cdot \left(\sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^l m!}{m^k} \frac{S_{2,\lambda}(l+1,m)}{l+1} \right) \frac{t^l}{l!} \right) \\ &= \left(\sum_{n=0}^{\infty} \beta_{n,\lambda/2}\left(\frac{x}{2}\right) \frac{2^n t^n}{n!} \right) \left(\sum_{i=0}^{\infty} \frac{(1)_{i+1,\lambda}}{i+1} \frac{t^i}{i!} \right) \\ &\quad \cdot \left(\sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^l m!}{m^k} \frac{S_{2,\lambda}(l+1,m)}{l+1} \right) \frac{t^l}{l!} \right) \\ &\quad \cdot \left(\sum_{n=0}^{\infty} \beta_{n,\lambda/2}\left(\frac{x}{2}\right) \frac{2^n t^n}{n!} \right) \left(\sum_{i=0}^{\infty} \left(\sum_{l=0}^{i} \binom{i}{l} \frac{(1)_{i-l+1,\lambda}}{i-l+1} \right.\right. \\ &\quad \left.\left. \cdot \sum_{m=1}^{l+1} \frac{p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^l m!}{m^k} \frac{S_{2,\lambda}(l+1,m)}{l+1} \right) \frac{t^i}{i!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{l=0}^{i} \binom{n}{i} \binom{i}{l} \sum_{m=1}^{l+1} \right. \\ &\quad \cdot \frac{(1)_{i-l+1,\lambda} p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^l m!}{(i-l+1)m^k} \frac{S_{2,\lambda}(l+1,m)}{l+1} \\ &\quad \left. \cdot 2^{n-i} \beta_{n-i,\lambda/2}\left(\frac{x}{2}\right) \right) \frac{t^n}{n!}.\end{aligned} \tag{58}$$

Therefore, by (58), we obtain the following theorem.

Theorem 10. *Let n be a nonnegative integer and $k \in \mathbb{Z}$. Then,*

$$\begin{aligned}G_{n,\lambda,p}^{(k)}(x) &= \sum_{i=0}^{n} \sum_{l=0}^{i} \binom{n}{i} \binom{i}{l} \sum_{m=1}^{l+1} \frac{(1)_{i-l+1,\lambda} p(m)\lambda^{m-1}(1)_{m,1/\lambda}(-1)^l m!}{(i-l+1)m^k} \\ &\quad \cdot \frac{S_{2,\lambda}(l+1,m)}{l+1} \times 2^{n-i} \beta_{n-i,\lambda/2}\left(\frac{x}{2}\right).\end{aligned} \tag{59}$$

From (51), we have

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!} &= \frac{2}{e_\lambda(t)+1}u_{k,\lambda}(1-e_\lambda(-t)\mid p)(e_\lambda(t)-1+1)^x \\ &= \frac{2u_{k,\lambda}(1-e_\lambda(-t)\mid p)}{e_\lambda(t)+1}\sum_{i=0}^{\infty}(x)_i\frac{(e_\lambda(t)-1)^i}{i!} \\ &= \sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}\frac{t^n}{n!}\sum_{i=0}^{\infty}(x)_i\sum_{l=i}^{\infty}S_{2,\lambda}(l,i)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}\frac{t^n}{n!}\sum_{i=0}^{\infty}\sum_{i=0}^{l}(x)_i S_{2,\lambda}(l,i)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{i=0}^{l}\binom{n}{l}(x)_i S_{2,\lambda}(l,i)G_{n-l,\lambda,p}^{(k)}\right)\frac{t^n}{n!}.\end{aligned} \tag{60}$$

By equation (60), we get the following theorem.

Theorem 11. *Let n be a nonnegative integer and $k\in\mathbb{Z}$. Then,*

$$G_{n,\lambda,p}^{(k)}(x) = \sum_{l=0}^{n}\sum_{i=0}^{l}\binom{n}{l}(x)_i S_{2,\lambda}(l,i)G_{n-l,\lambda,p}^{(k)}. \tag{61}$$

4. Conclusion

In this article, we introduced degenerate poly-Genocchi polynomials and numbers by using the degenerate polylogarithm function and derived several properties on the degenerate poly-Genocchi numbers. We represented the generating function of the degenerate poly-Genocchi numbers by iterated integrals in Theorems 4–6 and explicit degenerate poly-Genocchi polynomials in terms of the Euler polynomials and degenerate Stirling numbers of the second kind in Theorem 3. We also represented those numbers in terms of the degenerate Stirling numbers of the second kind in Theorem 7. In the last section, we defined the degenerate unipoly-Genocchi polynomials by using degenerate polylogarithm function and obtained the identity degenerate unipoly-Genocchi polynomials in terms of the degenerate Genocchi polynomials and degenerate Stirling numbers of the second kind in Theorem 8, the degenerate Euler polynomials and the degenerate Stirling numbers of the second kind in Theorem 9, the degenerate Bernoulli and degenerate Stirling numbers of the second kind in Theorem 10, and the degenerate unipoly-Genocchi numbers and Stirling numbers of the second kind in Theorem 11. It is important that the study of the degenerate version is widely applied not only to numerical theory and combinatorial theory but also to symmetric identity, differential equations, and probability theory. In particular, many symmetric identities have been studied for degenerate versions of many special polynomials [1, 3, 12, 23, 29–32]. Genocchi numbers have been also extensively studied in many different branches of mathematics. The works of Genocchi numbers and their combinatorial relations have received much attention [6–9]. With this in mind, as a future project, we would like to continue to study degenerate versions of certain special polynomials and numbers and their applications to physics, economics, and engineering as well as mathematics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Research Article

Coincidence Point Results on Relation Theoretic $(F_w, \mathscr{R})_g$ -Contractions and Applications

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Motivated by the ideas of F-weak contractions and $(F, \mathscr{R})_g$-contractions, the notion of $(F_w, \mathscr{R})_g$-contractions is introduced and studied in the present paper. The idea is to establish some interesting results for the existence and uniqueness of a coincidence point for these contractions. Further, using an additional condition of weakly compatible mappings, a common fixed-point theorem and a fixed-point result are proved for $(F_w, \mathscr{R})_g$-contractions in metric spaces equipped with a transitive binary relation. The results are elaborated by illustrative examples. Some consequences of these results are also deduced in ordered metric spaces and metric spaces endowed with graph. Finally, as an application, the existence of the solution of certain Voltera type integral equations is investigated.

1. Introduction and Preliminaries

In the development of the metric fixed-point theory, one of the main pillar is the Banach contraction principle [1], which states that every contraction on a complete metric space has a unique fixed point. Due to its extensive application potential, this concept has been observed in various forms over the years (see [2–9]).

The concept of F-contractions was introduced by Wardowski [10]. He proved some new fixed-point results for such kind of contractions. He built these results in a different way rather than traditional ways as done by many authors. Later on, fixed points for F-contractions were proved by Secelean [11] using an iterated function. Abbas et al. [12] extended the work of Wardowski and established various results of fixed points using F-contraction mappings. For further related works on F-contractions, see [13–16].

The idea of $(F, \mathscr{R})$-contractions was established by Sawangsup et al. [17]. They used this idea to demonstrate some fixed-point consequences using a binary relation. It is further investigated by Imdad et al. [18]. In present paper, we study the results presented by Alfaqih et al. [19] and we define $(F_w, \mathscr{R})_g$-contractions. We also prove similar results for $(F_w, \mathscr{R})_g$-contractions.

Recall that a binary relation $\mathscr{R}$ on nonempty set X is said to be a partial order if it is reflexive, antisymmetric, and transitive. Moreover, the inverse or transpose or dual relation of $\mathscr{R}$, denoted by $\mathscr{R}^{-1}$, is defined by

$$\mathscr{R}^{-1} = \left\{ (x, y) \in X^2 : (y, x) \in \mathscr{R} \right\}. \tag{1}$$

The symmetric closure of $\mathscr{R}$, denoted by $\mathscr{R}^s$, is defined as the set $\mathscr{R} \cup \mathscr{R}^{-1}$, that is, $\mathscr{R}^s := \mathscr{R} \cup \mathscr{R}^{-1}$. In fact, $\mathscr{R}^s$ is the smallest symmetric relation on X containing $\mathscr{R}$.

Notice that there is another binary relation $\mathscr{R}^{\sharp} \subseteq \mathscr{R}$ on X, which is defined by $k\mathscr{R}^{\sharp}\ell$, whenever $k\mathscr{R}\ell$ and $k \neq \ell$.

Definition 1 [10]. Let $\mathbb{F}$ be the set of functions $F : (0,\infty) \longrightarrow \mathbb{R}$ such that

(F_1) F is strictly increasing;

(F_2) For every sequence $\{\beta_n\} \subset (0,\infty)$, $\lim_{n\longrightarrow\infty}\beta_n = 0$ iff$\lim_{n\longrightarrow\infty}F(\beta_n) = -\infty$;

(F_3) There is $k \in (0, 1)$ so that $\lim_{\beta\longrightarrow 0^+}\beta^k F(\beta) = 0$.

The following functions are in $\mathbb{F}$:

$$\begin{aligned} F(\beta) &= \ln(\beta), \\ F(\beta) &= \ln(\beta) + \beta, \\ F(\beta) &= \ln\left(\beta^2 + \beta\right), \\ F(\beta) &= -\left(\frac{1}{\sqrt{\beta}}\right). \end{aligned} \tag{2}$$

Many papers in literature deal with the concept of F-contractions (see [20–22]). Throughout this work, the set of all continuous functions verifying (F_2) is denoted by $\mathscr{F}$.

Definition 2. Let $X \neq \phi$ and $\mathscr{R}$ be a binary relation on X. A sequence $\{\varsigma_n\} \subseteq X$ is such that $\varsigma_n\mathscr{R}\varsigma_{n+1}$ for all $n \in \mathbb{N}_0$, then it is called an $\mathscr{R}$ preserving sequence.

Definition 3. Consider a metric space (X, d) with a binary relation $\mathscr{R}$. Then, X is called $\mathscr{R}$ complete if each $\mathscr{R}$ preserving Cauchy sequence is convergent in X.

Definition 4 [23]. Let (X, d) be a metric space and $\mathscr{R}$ be a binary relation on X, $T : X \longrightarrow X$ and $x \in X$. We say that T is $\mathscr{R}$-continuous at x if for each $\mathscr{R}$-preserving sequence $\{\varsigma_n\} \subseteq X$ so that $\varsigma_n \longrightarrow x$, we have $T\varsigma_n \longrightarrow Tx$. Also, T is named to be $\mathscr{R}$-continuous if it is $\mathscr{R}$-continuous at any element of X.

Definition 5 [23]. Let (X, d) be a metric space and $\mathscr{R}$ be a binary relation on X and $T, g : X \longrightarrow X$ and $x \in X$. We say that T is $(g, \mathscr{R})$-continuous at x if for each sequence $\{\varsigma_n\} \subseteq X$ so that $\{g\varsigma_n\}$ is $\mathscr{R}$-preserving and $g\varsigma_n \longrightarrow gx$, we have $T\varsigma_n \longrightarrow Tx$. Also, T is named to be $(g, \mathscr{R})$-continuous if it is $(g, \mathscr{R})$-continuous at any element of X.

Definition 6 [24]. For $x, y \in X$, a path of length $p(p \in \mathbb{N})$ in $\mathscr{R}$ from x to y is a finite sequence $\{u_0, u_1, \cdots\cdots, u_p\} \subseteq X$ such that $u_0 = x$, $u_p = y$, and $(u_i, u_{i+1}) \in \mathscr{R}$ for every $i \in \{0, 1, .. \cdots, p-1\}$. Also, a subset $L \subseteq X$ is called $\mathscr{R}$ connected if for any two elements $x, y \in L$, there is a path from x to y in $\mathscr{R}$.

Definition 7 [23]. Let (X, d) be a metric space and $\mathscr{R}$ be a binary relation on X and $T, g : X \longrightarrow X$. The pair (T, g) is $\mathscr{R}$-compatible if for each sequence $\{\varsigma_n\} \subseteq X$ so that $\{T\varsigma_n\}$ and $\{g\varsigma_n\}$ are $\mathscr{R}$-preserving and $\lim_{n\longrightarrow\infty}g\varsigma_n = \lim_{n\longrightarrow\infty}T\varsigma_n = x \in X$,

$$\lim_{n\longrightarrow\infty} d(gT\varsigma_n, Tg\varsigma_n) = 0. \tag{3}$$

Definition 8. Let f and g be self-maps of a set X. If $x = fx = gx$ for some $x \in X$, then x is said to a common fixed point of f and g.

Definition 9 [25]. Let $f, g : X \longrightarrow X$. If $w = fx = gx$ for some $x \in X$, then x is said to be a coincidence point of f and g, and w is said to be a point of coincidence of f and g.

f and g are said to be weakly compatible if they commute at their coincidence point, i.e., if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Definition 10 [26]. Let (M, d) be a metric space endowed with a binary relation $\mathscr{R}$. Such a $\mathscr{R}$ is named to be d-self closed if for each $\mathscr{R}$-preserving sequence $\{\varsigma_n\} \subseteq M$ so that $\{\varsigma_n\} \longrightarrow x$, there is $\{\varsigma_{n_k}\}$ of $\{\varsigma_n\}$ so that $[\varsigma_{n_k}, x] \in \mathscr{R}\forall k \in \mathbb{N}_0$.

Definition 11 [23]. Let M be a nonempty set and $T, g : M \longrightarrow M$. A binary relation $\mathscr{R}$ on M is called (T, g) closed if for any $x, y \in M$, $gx\mathscr{R}gy$ yields that $Tx\mathscr{R}Ty$.

Lemma 12 [27, 28]. *Consider a metric space (X, d) and a sequence $\{k_m\}$ in X. If $\{k_m\}$ is not Cauchy in X, then are $\varepsilon > 0$ and $\{k_{m(j)}\}$ and $\{k_{t(j)}\}$ of $\{k_m\}$ so that*

$$j \leq m(j) \leq t(j), d\left(k_{m(j)}, k_{t(j)-1}\right) \leq \varepsilon < d\left(k_{m(j)}, k_{t(j)}\right) \forall j \in N_0. \tag{4}$$

Moreover, if $\{k_m\}$ is so that $\lim_{m\longrightarrow\infty}d(k_m, k_{m+1}) = 0$, then

$$\lim_{j\longrightarrow\infty} d\left(k_{m(j)}, k_{t(j)}\right) = \lim_{j\longrightarrow\infty} d\left(k_{m(j)-1}, k_{t(j)-1)}\right) = \varepsilon. \tag{5}$$

Lemma 13 [29]. *Let X be a nonempty set and $g : X \longrightarrow X$. Then, there is a subset $E \subseteq X$ so that $g(E) = g(X)$ and $g : E \longrightarrow E$ is one to one.*

2. Main Results

We begin this section by introducing the idea of $(F_w, \mathscr{R})_g$-contractions as follows.

Definition 14. Consider a metric space (X, d) endowed with a transitive binary relation $\mathscr{R}$ on X and Q, $g : X \longrightarrow X$. Then, T is called an $(F_w, \mathscr{R})_g$-contractions if there exist $F \in \mathscr{F}$ and $\tau > 0$ such that

$$\begin{aligned} &\tau + F((d(Qk, Q\ell)) \\ &\quad \leq F\left(\max\left\{d(gk, g\ell), d(gk, Qk), d(g\ell, Q\ell), \frac{d(gk, Q\ell) + d(g\ell, Qk)}{2}\right\}\right), \end{aligned} \tag{6}$$

for all $k, l \in X$ with $gk\mathscr{R}^{\sharp}g\ell$ and $Qk\mathscr{R}^{\sharp}Q\ell$.

Remark 15. Every $(F, \mathscr{R})_g$ contraction is an $(F_w, \mathscr{R})_g$ contraction, but the converse of statement is not true.

The following result is easy to prove. We omit it.

Proposition 16. *Let (X, d) be a metric space endowed with a transitive binary relation $\mathscr{R}$. Given $Q, g : X \longrightarrow X$. Then, for each $F \in \mathscr{F}$, we have equivalence of the two following statements:*

(a) *$\forall k, l \in X$ so that $(gk, g\ell) \in \mathscr{R}^\sharp$ and $(Qk, Q\ell) \in \mathscr{R}^\sharp$*

$$\tau + F(d(Qk, Q\ell)) \le F \cdot \left(\max\left\{d(gk, g\ell), d(gk, Qk), (g\ell, Q\ell), \frac{d(gk, Q\ell) + d(g\ell, Qk)}{2}\right\}\right). \tag{7}$$

(b) *$\forall k, l \in X$ such that either $(gk, g\ell), (Qk, Q\ell) \in \mathscr{R}^\sharp$ or $(g\ell, gk), (Q\ell, Qk) \in \mathscr{R}^\sharp$*

$$\tau + F(d(Qk, Q\ell)) \le F \cdot \left(\max\left\{d(gk, g\ell), d(gk, Qk), (g\ell, Q\ell), \frac{d(gk, Q\ell) + d(g\ell, Qk)}{2}\right\}\right). \tag{8}$$

Theorem 17. *Consider a metric space (X, d) equipped with $\mathscr{R}$ (where $\mathscr{R}$ is a transitive binary relation) and $Q, g : X \longrightarrow X$. Assume that:*

(1) *there exists $k_0 \in X$ such that $gk_0 \mathscr{R} Qk_0$*

(2) *$\mathscr{R}$ is (Q, g)-closed*

(3) *Q is an $(F_w, \mathscr{R})_g$-contraction*

(4)

  (a) *A subset K of X exists such that $Q(X) \subseteq K \subseteq g(X)$ and K is $\mathscr{R}$-complete*

  (b) *One of the subsequent conditions is fulfilled:*

    (i) *Q is $(g, \mathscr{R})$-continuous, or*

    (ii) *Q and g are continuous, or*

    (iii) *$\mathscr{R} \mid K$ is d-self closed in condition that (6) holds for all $k, l \in X$ with $gk\mathscr{R}g\ell$ and $Qk\mathscr{R}^\sharp Q\ell$*

or on the other hand:

(α) (α_1) $\exists$ a subset L of X such that $Q(X) \subseteq g(X) \subseteq L$ and L is $\mathscr{R}$- complete,

(α_2) (Q, g) is an $\mathscr{R}$-compatible pair,

(α_3) Q and g are $\mathscr{R}$-continuous.

Then, the pair (Q, g) admits a coincidence point.

Proof. In the above two cases (11) and (α), note that $Q(X) \subseteq g(X)$. Using assumption (6), we get $gk_0\mathscr{R}Qk_0$. If $Qk_0 = gk_0$, then a coincidence point of (Q, g) is k_0. This completes the proof. Suppose that $Qk_0 \ne gk_0$. Since $Q(X) \subseteq g(X)$, there must exist $k_1 \in X$ such that $gk_1 = Qk_0$. Similarly, there is $k_2 \in X$ such that $gk_2 = Qk_1$. Proceeding in this way, we can construct a sequence $\{k_m\} \subseteq X$ such that

$$gk_{m+1} = Qk_m \forall m \in \mathbb{N}_0. \tag{9}$$

Now, we will prove $\{gk_m\}$ is an $\mathscr{R}$-preserving sequence, that is,

$$gk_m \mathscr{R} gk_{m+1} \forall m \in \mathbb{N}_0. \tag{10}$$

By using induction, we will prove this claim. If we put $m = 0$ in (9) and use condition (6), we get $gk_o\mathscr{R}gk_1$. This implies that the above statement holds for $m = 0$. Suppose that (10) is accurate for $m = j \ge 1$, that is, $gk_j\mathscr{R}gk_{j+1}$. Since $\mathscr{R}$ is (Q, g)-closed, we get $Qk_j\mathscr{R}Qk_{j+1}$, and so $gk_{j+1}\mathscr{R}gk_{j+2}$.

Hence, our claim is true for all $m \in \mathbb{N}_0$. By using (9) and (10), we can conclude that $\{Qk_m\}$ is also an $\mathscr{R}$-preserving sequence, that is,

$$Qk_m \mathscr{R} Qk_{m+1} \forall m \in \mathbb{N}_0. \tag{11}$$

If $Qk_{m_0} = Qk_{m_{0+1}}$ for some $m_0 \in \mathbb{N}_0$, then k_{m_0} is a coincidence point of (Q, g).

Suppose on the contrary that $Qk_m \ne Qk_{m+1}$ for al $m \in \mathbb{N}_0$. With the help of (9), (10), (11), and condition (10), we can see that

$$\begin{aligned}
\tau + F(d(Qk_{m-1}, Qk_m)) &\le F \\
&\cdot \Bigg(\max\Bigg\{d(gk_{m-1}, gk_m), d(gk_{m-1}, Qk_{m-1}), d(gk_m, Qk_m), \\
&\cdot \frac{d(gk_{m-1}, Qk_m) + d(gk_m, Qk_{m-1})}{2}\Bigg\}\Bigg) \forall m \in \mathbb{N}_0. = F \\
&\cdot \left(\max\left\{d(gk_{m-1}, Qk_{m-1}), d(gk_m, Qk_m), \frac{d(gk_{m-1}, Qk_m)}{2}\right\}\right). \le F \\
&\cdot \Bigg(\max\Bigg\{d(gk_{m-1}, Qk_{m-1}), d(gk_m, Qk_m), \\
&\cdot \frac{d(gk_{m-1}, gk_m) + d(gk_m, Qk_m)}{2}\Bigg\}\Bigg). = F \\
&\cdot (\max\{d(gk_{m-1}, gk_m), d(gk_m, Qk_m)\}).
\end{aligned} \tag{12}$$

Now, $\max\{d(gk_{m-1}, gk_m), d(gk_m, Qk_m)\}$ cannot be $d(gk_m, Qk_m)$. Otherwise,

$$\tau + F(d(gk_m, gk_{m+1})) \le d(gk_m, gk_{m+1}), \tag{13}$$

which is a contradiction. Hence, $\max\{d(gk_{m-1}, gk_m), d(gk_m, Qk_m)\} = d(gk_{m-1}, gk_m)$. Therefore,

$$\begin{aligned}
\tau + F(d(Qk_{m-1}, Qk_m)) \le F(d(gk_{m-1}, gk_m)) \Rightarrow F(d(Qk_{m-1}, Qk_m)) \\
\le F(d(gk_{m-1}, gk_m))) - \tau.
\end{aligned} \tag{14}$$

Take $\gamma_m = d(gk_m, gk_{m+1})$. With the help of above condition, we obtain

$$F(\gamma_m) \leq F(\gamma_{m-1}) - \tau \leq F(\gamma_{m-2}) - 2\tau \ldots \cdots \leq F(\gamma_0) - m\tau (\forall m \in \mathbb{N}). \tag{15}$$

By using (F_2) and taking $m \longrightarrow \infty$ in above inequality, we obtain

$$\lim_{m \longrightarrow \infty} F(\gamma_m) = -\infty. \tag{16}$$

This together with (F_2) imply that

$$\lim_{m \longrightarrow \infty} \gamma_m = d(gk_m, gk_{m+1}) = 0. \tag{17}$$

Now, we will show that $\{gk_m\}$ is a Cauchy sequence. We argue by contradiction. In this case, Lemma 12 guarantees the existence of $\varepsilon > 0$ and two subsequences $\{gk_{m_j}\}$ and $\{gk_{t_j}\}$ of $\{gk_m\}$ such that

$$d\left(gk_{m(j)}, gk_{t(j-1)}\right) \leq \varepsilon < d\left(gk_{m(j)}, gk_{t(j)}\right), \tag{18}$$

with

$$j \leq m(j) \leq t(j), \quad \forall j \in \mathbb{N}_0, \tag{19}$$

$$\lim_{j \longrightarrow \infty} d\left(gk_{m(j)}, gk_{t(j)}\right) = d\left(gk_{m(j)-1}, gk_{t(j)-1}\right) = \varepsilon. \tag{20}$$

This implies that there is $j_0 \in \mathbb{N}_0$ so that $d(gk_{m(j)-1}, g k_{t(j)-1}) > 0 \forall j \geq j_0$.

Since $\mathscr{R}$ is transitive, one writes

$$gk_{m(j)-1} \mathscr{R}^{\#} gk_{t(j)-1} \text{ and } Qk_{m(j)-1} \mathscr{R}^{\#} Qk_{t(j)-1} \forall j \geq j_0. \tag{21}$$

Using condition (10), we have for all $j \geq j_0$,

$$\begin{aligned} \tau + F\left(d\left(Qk_{m(j)-1}, Qk_{t(j)-1}\right)\right) &\leq F \max \\ &\cdot \left(d\left(gk_{m(j)-1}, gk_{t(j)-1}\right), d\left(gk_{m(j)-1}, Qk_{m(j)-1}\right), d\left(gk_{t(j)-1}, Qk_{t(j)-1}\right), \right. \\ &\left. \cdot \frac{d\left(gk_{m(j)-1}, Qk_{t(j)-1}\right) + d\left(gk_{t(j)-1}, Qk_{m(j)-1}\right)}{2} \right). \end{aligned} \tag{22}$$

Denote

$$\begin{aligned} \max &\left\{ d\left(gk_{m(j)-1}, gk_{t(j)-1}\right), d\left(gk_{m(j)-1}, Qk_{m(j)-1}\right), d\left(gk_{t(j)-1}, Qk_{t(j)-1}\right), \right. \\ &\left. \cdot \frac{d\left(gk_{m(j)-1}, Qk_{t(j)-1}\right) + d\left(gk_{t(j)-1}, Qk_{m(j)-1}\right)}{2} \right\} = \mathscr{D}\left(k_{m(j)-1}, k_{t(j)-1}\right). \end{aligned} \tag{23}$$

If $\mathscr{D}(k_{m(j)-1}, k_{t(j)-1}) = d(gk_{m(j)-1}, gk_{t(j)-1})$ or it is equal to $(d(gk_{m(j)-1}, Qk_{t(j)-1}) + d(gk_{t(j)-1}, Qk_{m(j)-1}))/2$ then taking $j \longrightarrow \infty$ and using (20), we get

$$\lim_{j \longrightarrow \infty} \mathscr{D}\left(k_{m(j)-1}, k_{t(j)-1}\right) = \varepsilon. \tag{24}$$

Since F is continuous, letting $m \longrightarrow \infty$ in (22) and using (20) and (24), we get

$$\tau + F(\varepsilon) \leq F(\varepsilon), \tag{25}$$

which is a contradiction. On the other hand, if $\mathscr{D}(k_{m(j)-1}, k_{t(j)-1}) = d(gk_{m(j)-1}, Qk_{m(j)-1})$ or it is equal to $d(gk_{t(j)-1}, Q k_{t(j)-1})$ then letting $m \longrightarrow \infty$ in (22), using continuity of F and (20) together with condition F_2, we get $\tau + F(\varepsilon) \leq -\infty$, which is again a contradiction. Thus, $\{gk_m\}$ is a Cauchy sequence.

Let the condition (11) hold. With the help of (9), we obtain $gk_m \subseteq Q(X)$. Therefore, $\{gk_m\}$ is $\mathscr{R}$-preserving Cauchy in K. By utilizing $\mathscr{R}$-completeness of K, there is $l \in K$ so that $gk_m \longrightarrow l$. As $K \subseteq g(X)$, there is $v \in X$ so that $l = gv$. Hence, by using (2),

$$\lim_{m \longrightarrow \infty} gk_m = \lim_{m \longrightarrow \infty} Qk_m = gv. \tag{26}$$

In order to prove that v is coincidence point of (Q, g), we will use three different cases of condition (b). First of all, suppose that Q is $(g, \mathscr{R})$-continuous. By utilizing (10) and (26), we get

$$\lim_{m \longrightarrow \infty} Qk_m = Qv. \tag{27}$$

By utilizing (26) and (27), we get $Qv = gv$. This shows that v is a coincidence point of (Q, g).

Now, suppose the second case of (b), that is, Q and g are continuous. Since $X \neq \phi$ and $g: X \longrightarrow X$, by using Lemma 13, there is $B \subseteq X$ so that $g(B) = g(X)$ and $g: B \longrightarrow B$ is one-one. Define a mapping $f: g(B) \longrightarrow g(X)$ by

$$f(gb) = Q(b) \forall gb \in g(B) \text{ where } b \in B. \tag{28}$$

Recall that g is one-one and $Q(X) \subseteq g(X)$, so f is well-defined mapping. As Q and g are continuous, f is also continuous. Now, utilizing the fact that $g(X) = g(B)$, we can rewrite condition (a) as $Q(X) \subseteq K \subseteq g(B)$, so that, without loss of generality, we can select a sequence $\{k_m\}$ in B and $v \in B$. By using (26), (28), and continuity of f, we have

$$Qv = f(gv) = f\left(\lim_{m \longrightarrow \infty} gk_m\right) = \lim_{m \longrightarrow \infty} f(gk_m) = \lim_{m \longrightarrow \infty} Qk_m = gv. \tag{29}$$

Finally, assume that condition (iii) of (b) holds, which implies that $\mathscr{R} \mid K$ is d-self closed and (2.1) detain $\forall k, l \in X$, with $gk \mathscr{R} g\ell$ and $Qk \mathscr{R}^{\#} Q\ell$. As $\{gk_m\} \subseteq K$, $\{gk_m\}$ is $\mathscr{R} \mid K$ preserving due to (10) and with the help of (26) $gk_m \longrightarrow gv$. So, there is a subsequence $\{gk_{m_j}\} \subseteq \{gk_m\}$ such that

$$\left[gk_{m_j}, gv\right] \in \mathscr{R}|K \subseteq \mathscr{R} \forall j \in \mathbb{N}_0. \tag{30}$$

Utilizing condition (b) and (30), one writes

$$\left[Qk_{m_j}, Qv\right] \in \mathscr{R}|K \subseteq \mathscr{R} \forall j \in \mathbb{N}_0. \tag{31}$$

Now, let $q = \{j \in \mathbb{N} : Qk_{m_j} = Qv\}$. If the set q is infinite, then $\{Qk_{m_j}\}$ has a subsequence $\{Qk_{m_{j_p}}\}$, such that $Qk_{m_{j_p}} = Qv$. This implies that $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv \forall p \in \mathbb{N}$. By using (26), we have $\lim_{m\longrightarrow\infty} Qk_m = gv$. So we obtain $Qv = gv$.

If the set q is finite, then $\{Qk_{m_j}\}$ has a subsequence $\{Qk_{m_{j_p}}\}$ such that $Qk_{m_{j_p}} \neq Qv \forall p \in \mathbb{N}$. Next, we will show that $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv$. With the help of (30), (31) and $k_{m_{j_p}} \neq Qv \forall p \in \mathbb{N}$, we have

$$\left[gk_{m_{j_p}}, gv\right] \in \mathscr{R}|K \subseteq \mathscr{R} \forall p \in \mathbb{N}_0, \tag{32}$$

$$\left[Qk_{m_{j_p}}, Qv\right] \in \mathscr{R}|K \subseteq \mathscr{R} \text{ and } Qk_{m_{j_p}} \neq Qv \forall p \in \mathbb{N}_0. \tag{33}$$

Now, with the help of (32), (33), Proposition 16 and the fact that (2.1) is satisfied, we get

$$F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(\max\left\{d\left(gk_{m_{j_p}}, gv\right), d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right), d(gv, Qv), \cdot \frac{d\left(gk_{m_{j_p}}, Qv\right) + d\left(gv, Qk_{m_{j_p}}\right)}{2}\right\}\right) - \tau. \tag{34}$$

Denote

$$\max\left\{d\left(gk_{m_{j_p}}, gv\right), d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right), d(gv, Qv), \cdot \frac{d\left(gk_{m_{j_p}}, Qv\right) + d\left(gv, Qk_{m_{j_p}}\right)}{2}\right\} = \mathscr{D}\left(k_{m_{j_p}}, v\right). \tag{35}$$

If $\mathscr{D}(k_{m_{j_p}}, v) = d(gk_{m_{j_p}}, gv)$then, we have

$$\tau + F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, gv\right)\right) \Rightarrow F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, gv\right)\right) - \tau. \tag{36}$$

By using (26), (F_2) and taking $p \longrightarrow \infty$, we get $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv$. If $\mathscr{D}(k_{m_{j_p}}, v) = d(gk_{m_{j_p}}, gv)$ then, we have

$$\tau + F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, gv\right)\right) \Rightarrow F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, gv\right)\right) + Ld\left(gk_{m_{j_p}}, gk_{m_{j_{p-1}}}\right) - \tau. \tag{37}$$

By using (26), (F_2) and taking $p \longrightarrow \infty$, we get $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv$. Now, if $\mathscr{D}(k_{m_{j_p}}, v) = d(gv, Qv)$, then

$$F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right)\right) - \tau. \tag{38}$$

By using (26), (F_2) and taking $p \longrightarrow \infty$, we get $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv$. If $\mathscr{D}(k_{m_{j_p}}, v) = (d(gk_{m_{j_p}}, Qv) + d(gv, Qk_{m_{j_p}}))/2$, then, we have

$$\begin{aligned} \tau + F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) &\leq F\left(\frac{d\left(gk_{m_{j_p}}, Qv\right) + d\left(gv, Qk_{m_{j_p}}\right)}{2}\right) \\ &\Rightarrow \lim_{p\longrightarrow\infty} F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \\ &\leq \lim_{p\longrightarrow\infty} F\left(\frac{d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right) + d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right)}{2}\right) - \tau \\ &= \lim_{p\longrightarrow\infty} F\left(\frac{d\left(gk_{m_{j_p}}, gk_{m_{j_{p-1}}}\right) + d\left(gk_{m_{j_p}}, gk_{m_{j_{p-1}}}\right)}{2}\right) - \tau. \end{aligned} \tag{39}$$

By using (26), (F_2) and taking $p \longrightarrow \infty$, we get

$$\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv. \tag{40}$$

From (26) and (40), we obtain $Qv = gv$. Hence, when the set q is finite or infinite, v is a coincidence point of Q and g. Now, if (α) holds, then $gk_m \subseteq L$, and hence $\{gk_m\}$ is an $\mathscr{R}$-preserving Cauchy sequence in L. Since L is $\mathscr{R}$-complete, there is $u \in L$ so that

$$\lim_{m\longrightarrow\infty} gk_m = u. \tag{41}$$

Using Equations (9) and (41), one gets

$$\lim_{m\longrightarrow\infty} Qk_m = u. \tag{42}$$

Now, with the help of (10), (41), and continuity of g, we have

$$\lim_{m\longrightarrow\infty} g(gk_m) = g\left(\lim_{m\longrightarrow\infty} gk_m\right) = gu. \tag{43}$$

Utilizing (11), (42) and continuity of g to find

$$\lim_{m\longrightarrow\infty} g(Qk_m) = g\left(\lim_{m\longrightarrow\infty} Qk_m\right) = gu. \tag{44}$$

As Qk_m and gk_m are $\mathscr{R}$-preserving due to (10), (11) and

$$\lim_{m\longrightarrow\infty} Qk_m = \lim_{m\longrightarrow\infty} gk_m = u. \tag{45}$$

Now, using (41), (42), and condition (α_2),

$$\lim_{m\longrightarrow\infty} d(gQk_m, Qgk_m) = 0. \tag{46}$$

Next, we will demonstrate that u is a coincidence point of (Q, g). Making use of (10), (41) and the $\mathscr{R}$-continuity of Q, we get

$$\lim_{m\longrightarrow\infty} Q(gk_m) = Q\left(\lim_{m\longrightarrow\infty} gk_m\right) = Qu. \tag{47}$$

With the use of (44), (46), and (47), we get

$$\begin{aligned} d(gu, Qu) &= d\left(\lim_{m\longrightarrow\infty} gQk_m, \lim_{m\longrightarrow\infty} Qgk_m\right) \\ &= \lim_{m\longrightarrow\infty} d(gQk_m, Qgk_m) = 0 \Rightarrow Qu = gu. \end{aligned} \tag{48}$$

This implies that u is a coincidence point of (Q, g). □

Theorem 17 does not guarantee the uniqueness of a coincidence point. The following theorem guarantees that coincidence point is unique.

Theorem 18. *Suppose all hypothesis of Theorem 17 are true except* (α) *and assume that* gu *and* gv *are* $\mathscr{R}$*-comparable for all* $u \neq v \in$ *coin* (Q, g)*, and one of* Q *or* g *is one-one, then there is a unique coincidence point of* (Q, g).

Proof. The set $\mathrm{coin}(Q, g)$ is nonempty, because of Theorem 17. Consider two elements $u, v \in \mathrm{coin}(Q, g)$, then by definition of $\mathrm{coin}(Q, g)$, we have $[gv, gu] \in \mathscr{R}$ and $Qu = gu$, $Qv = gv$. This implies $[Qu, Qv] \in \mathscr{R}$.

Now, if $gu = gv$, we obtain $Qv = gv = gu = Qu$, and hence, $v = u$, because one of Q and g is one-one.

If $gu \neq gv$, then by utilizing condition (10) and Proposition 16, we get

$$\begin{aligned} &\tau + F(d(Qu, Qv)) \leq F \\ &\cdot\left(d(gu, gv), d(gu, Qv), d(gv, Qv), \frac{d(gu, Qv) + d(gv, Qv)}{2}\right) \\ &= F(d(Qu, Qv)). \end{aligned} \tag{49}$$

Since $\tau > 0$, our assumption is false. Therefore, a unique coincidence point of (Q, g) exists. □

Theorem 19. *Consider above theorem and add a condition that* (Q, g) *is a weakly compatible pair, then a unique common fixed point of* (Q, g) *exists.*

Proof. Above theorem assures that the pair (Q, g) has a unique coincidence point. Let v be the common coincidence point and suppose $z \in X$ be such that

$$z = Qv = gv. \tag{50}$$

The weak compatibility of Q and g leads to $Qz = Qgv = gQv = gz$. That is, z is a coincidence point of Q and g. Since v is unique, one writes $z = v$. That is, the uniqueness of a common fixed point. Since all the assumptions of Theorem 18 are true, the set $\mathrm{coin}(Q, g)$ is nonempty. □

Example 1. Let $X = [0,\infty)$ and define $d : X \times X \longrightarrow \mathbb{R}$ *by* $d(x, y) = |x - y|$. Then, (X, d) is a complete metric space.

Consider the sequence $\{\varsigma_n\} \subseteq X$ which is defined by $\{\varsigma_n = (n(n+1)(4n-1))/3, n \geq 1\}$.

Define the binary relation $\mathscr{R}$ on X by

$$\mathscr{R} = \{(\varsigma_i, \varsigma_i), (\varsigma_i, \varsigma_{i+1}) \text{ such that } \varsigma_i \leq \varsigma_{i+1}\}. \tag{51}$$

Define $Q, g : X \longrightarrow X$ by

$$Qx = \begin{cases} x, & \text{if } 0 \leq x \leq \varsigma_1, \\ \varsigma_2, & \text{if } \varsigma_1 < x \leq \varsigma_2, \\ \varsigma_i + \dfrac{\varsigma_{i+1} - \varsigma_i}{\varsigma_{i+2} - \varsigma_{i+1}}(x - \varsigma_{i+1}), & \text{if } \varsigma_{i+1} \leq x \leq \varsigma_{i+2}, \end{cases} \tag{52}$$

and

$$gx = \left\{\varsigma_i + \frac{\varsigma_{i+1} - \varsigma_i}{\varsigma_{i+2} - \varsigma_{i+1}}(x - \varsigma_i), \quad \text{if } \varsigma_i \leq x \leq \varsigma_{i+1}, i = 1, 2, \cdots.\right. \tag{53}$$

Observe that if $gx\mathscr{R}^{\sharp}gy$ and $Qx\mathscr{R}^{\sharp}Qy$, then $x = \varsigma_i$ and $y = \varsigma_{i+1}$ for $i \in \mathbb{N} - 1$. Further, by choosing $F(\alpha) = \ln \alpha$ and $\alpha \in (0,+\infty)$, we have

$$\begin{aligned} &F(d(Q\varsigma_i, Q\varsigma_{i+1})) = F(|\varsigma_{i-1} - \varsigma_i|) = F(|\varsigma_i - \varsigma_{i-1}|) = \ln|\varsigma_i - \varsigma_{i-1}|, \\ &F\left(\max\left\{d(g\varsigma_i, g\varsigma_{i+1}), d(g\varsigma_i, Q\varsigma_i), d(g\varsigma_{i+1}, Q\varsigma_{i+1}),\right.\right. \\ &\left.\left.\cdot \frac{d(g\varsigma_i, Q\varsigma_{i+1}) + d(Q\varsigma_i, g\varsigma_{i+1})}{2}\right\}\right) \\ &\quad = F\left(\max\left\{|\varsigma_{i+1} - \varsigma_i|, |\varsigma_i - \varsigma_{i-1}|, |\varsigma_{i+1} - \varsigma_i|, \frac{|\varsigma_{i+1} - \varsigma_{i-1}|}{2}\right\}\right) \\ &\quad = F(\varsigma_{i+1} - \varsigma_i) = \ln d(g\varsigma_{i+1}, Qxi + 1). \end{aligned} \tag{54}$$

Now, for $n = 2, 3, \cdots$ and for $\tau = \ln 3$, we have

$$\tau + \ln|(\varsigma_i - \varsigma_{i-1}| \leq |\varsigma_{i+1} - \varsigma_i|. \tag{55}$$

Therefore,

$$\begin{aligned} &\ln(3) + F(d(Q\varsigma_i, Q\varsigma_{i+1})) \leq Fd(g\varsigma_{i+1}, Qxi + 1), \\ &\quad \forall x, y \in X \text{ such that } gx\mathscr{R}^{\sharp}gy \text{ and } Qx\mathscr{R}^{\sharp}Qy. \end{aligned} \tag{56}$$

Moreover, all the assumptions of Theorem 19 are true, and ς_1 is the unique common fixed point of (Q, g).

On setting $g = I$ in Theorem 19, we obtain the following result.

Theorem 20. *Consider a self-mapping* $Q : X \longrightarrow X$ *and let* (X, d) *be a metric space with a transitive binary relation* $\mathscr{R}$*. Assume that:*

(1) $\exists k_0 \in X$ such that $k_0 \mathscr{R} Q k_0$

(2) $\mathscr{R}$ is Q-closed

(3) Q is an $(F_w, \mathscr{R})$-contraction

(4) (α) ∃ a subset K of X such that $Q(X) \subseteq K$ and K is $\mathscr{R}$ -complete,

(η) one of these conditions hold:

(i) Q is $\mathscr{R}$-continuous, or

(ii) $\mathscr{R} | K$ is d-self closed on condition that (1.1) with binary relation holds $\forall k, l \in X$ with $k\mathscr{R}l$ and $Qk\mathscr{R}^{\sharp} Q\ell$

Then, a fixed point of Q exists. Furthermore, if
(e) $[u, v] \in Fix(Q) \Rightarrow [u, v] \in \mathscr{R}$.
Then, such fixed point of Q is unique.

Theorem 21. *Replace condition (e) of above theorem by:*
(e ∗) Fix(Q) is $\mathscr{R}^s$-connected,
then Q has a unique fixed point.

Proof. Assume on contrary that Q has more than one fixed point, say u and v with $u \neq v$. Then, there exists a path $\mathscr{R}^s \subseteq Fix(Q)$. As it is from v to u of length q, let us denote the path by $\{v_0, \cdots v_q\}$ such that $v_p \neq v_{p+1}$ for each p where $0 \leq p \leq j-1$. If $v = u$, it is a contradiction. Hence,

$$v_0 = v, v_q = u \text{ and } [v_p, v_{p+1}] \in \mathscr{R} \text{ for each } p (0 \leq p \leq q-1). \tag{57}$$

As $v_p \in Fix(Q)$, so $Q(v_p) = v_p$ for each $p \in \{0, 1, \cdots, q\}$. With the help of condition (c), we obtain

$$\tau + F(d(v_p, v_{p+1})) \leq F\left(\max\left\{d(v_p, v_{p+1}), d(v_p, v_{p+1}), d(v_{p+1}, v_{p+1}), \cdot \frac{d(v_p, v_{p+1}) + d(v_{p+1}, v_p)}{2}\right\}\right). \tag{58}$$

That is,

$$\tau + F(d(v_p, v_{p+1})) \leq F(d(v_p, v_{p+1})). \tag{59}$$

Since $\tau > 0$, our supposition is not true. Hence, Q has a unique fixed point. □

In the next section, we are presenting a significance of our results in ordered metric spaces.

3. Some Consequences in Ordered Metric Spaces

Definition 22. Let (X, d) be a metric space and $(X, \preccurlyeq)$ be an ordered set, then the triplet $(X, d, \preccurlyeq)$ is known as an ordered metric space.

Definition 23. Consider self-mappings $Q, g : X \longrightarrow X$ and an ordered set $(X, \preccurlyeq)$. If, for any $k, l \in X$, $gk \preccurlyeq g\ell$ implies that $Q k \preccurlyeq Q\ell$. Then, Q is g-increasing.

Remark 24. Notice that the notion of Q is g-increasing is equal to say that $\preccurlyeq$ is (Q, g)-closed.

Taking $\mathscr{R} = \preccurlyeq$ in Theorem 17 to 19 and with the help of Remark 24, we state the following result.

Corollary 25. *Consider self-mappings $Q, g : X \longrightarrow X$ and an ordered metric space $(X, d, \preccurlyeq)$. Assume that:*

(a) $\exists k_0 \in X$ such that $gk_0 \preccurlyeq Qk_0$

(b) Qisg-increasing

(c) There are $\tau > 0$ and $F \in \mathscr{F}$ so that

$$\tau + F(d(Qk, Q\ell)) \leq F\left(\max\left\{(d(gk, g\ell), d(gk, Qk), d(g\ell, Q\ell), \cdot \frac{d(gk, Q\ell) + d(g\ell, Qk)}{2}\right\}\right), \tag{60}$$

(d) ∃ a subset K of X such that $Q(X) \subseteq K \subseteq g(X)$ and K is $\preccurlyeq$ -complete

(e) Either Q and g are continuous, or Q is $(g, \preccurlyeq)$-continuous. Then, a coincidence point of (Q, g) exists. Additionally, we suppose that

(f) Qu and gv are $\preccurlyeq$-comparable for all distinct coincidence points $u, v \in coin(Q, g)$, then pair (Q, g) has a unique coincidence point

Furthermore, if Q and g are weakly compatible, then (Q , g) has a unique common fixed point.
Taking $\mathscr{R} = \preccurlyeq$ in Theorem 20 and with the help of Remark 24, we conclude the result given below.

Corollary 26. *Consider an ordered metric space $(X, d, \preccurlyeq)$ and mapping $Q : X \longrightarrow X$. Suppose the that conditions given below are fulfilled:*

(a) $\exists k_0 \in X$ such that $k_0 \preccurlyeq Qk_0$

(b) Q is $\preccurlyeq$-increasing

(c) $\exists \tau > 0$ and $F \in \mathscr{F}$ such that

$$\tau + F(d(Qk, Q\ell)) \leq F\left(\max\left\{(d(k, l), d(k, Qk), (l, Q\ell), \cdot \frac{d(k, Q\ell) + d(l, Qk)}{2}\right\}\right), \tag{61}$$

(d) A subset K of X exists such that $Q(X) \subseteq K$ and K is $\preccurlyeq$ -complete

(e) *Q is $\preccurlyeq$-continuous. Then a fixed point of Q exists. Furthermore,*

(f) *if for any two fixed points $u, v \in Q$ we have $[u, v] \in \preccurlyeq$, then Q has a unique fixed point*

4. Applications to Metric Spaces Endowed with a Graph

Jachymski [30] in 2008 has instituted the idea of metric spaces endowed with a graph in order to generalize the idea of a partial ordering and specified the Banach contraction principle in metric spaces and partially ordered metric spaces. In this section, we are going to present an application of our results in the situating of complete metric spaces endowed with a graph.

Corollary 27. *Consider self-mappings $Q, g : X \longrightarrow X$ on a metric space (X, d) endowed with a graph $G = (V(G), E(G))$. Define $\preccurlyeq$ on X as $u \preccurlyeq v$ if and only if there is an edge between u and v. Assume that all the conditions given in Corollary 25 are satisfied. Then a coincidence point of (Q, g) exists. Further, if we suppose that Qu and gv are comparable on edges for all distinct coincidence points $u, v \in coin(Q, g)$, then the pair (Q, g) has a unique coincidence point.*

Furthermore, a unique common fixed point of (Q, g) exists if Q and g are weakly compatible.

Corollary 28. *Consider a metric space (X, d) endowed with a graph G and a mapping $Q : X \longrightarrow X$. Define $\preccurlyeq$ on X as $u \preccurlyeq v$ if and only if there is an edge between u and v. Suppose that conditions given in Corollary 27 are fulfilled. Then, a fixed point of Q exists. Furthermore, if $u, v \in Fix(Q)$ are such that there is an edge between u and v, then a unique fixed point of Q exists.*

5. Applications to Integral Equations

In this section, we present an application of Theorem 21 by finding a solution of the integral equation of Volterra type given below:

$$u(t) = \int_0^t K(t, s, u(s))ds + h(t), \qquad t \in [0, 1]. \tag{62}$$

Here, $K : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h : [0, 1] \longrightarrow \mathbb{R}$.

Let X be the Banach space of all continuous functions $u : [0, 1] \longrightarrow \mathbb{R}$. Define a norm on X as follows.

$\|u\| = \max_{t\in[0,1]} |u(t)|$. Then, the metric d on X is defined as $d(u, v) = \|u - v\| \forall u, v \in X$.

Definition 29. A function $\alpha \in X$ such that

$$\alpha(t) \le \int_0^t K(t, s, \alpha(s))ds + h(t), \qquad t \in [0, 1], \tag{63}$$

is called a lower solution for (62).

Definition 30. A function $\beta \in X$ such that

$$\beta(t) \ge \int_0^t K(t, s, \beta(s))ds + h(t), \qquad t \in [0, 1], \tag{64}$$

is called an upper solution for (62).

Now, we have enough material to prove the following results.

Theorem 31. *Assume that in third variable K is nondecreasing and there is $\tau > 0$ such that*

$$|K(t, s, u) - K(t, s, v)| \le \frac{|u - v|}{\tau \mathscr{D}(u, v) + 1}, \tag{65}$$

for all $t, s \in [0, 1]$ and $u, v \in X$, where $\mathscr{D}(u, v) = \max\{d(u, v), d(u, Qu), (v, Qv), ((d(u, Qv) + d(v, Qu))/2)\}$. Then, the existence of a unique solution of the integral Equation (62) follows from the existence of lower solution of (62).

Proof. Let $Q(u(t)) = \int_0^t K(t, s, u(s))ds + h(t)$ for all $u \in X$, be a self operator on X. It is clear that u is a fixed point of the operator Q if and only if it is solution of the Equation (62).

Let $\mathscr{R}$ be the binary relation on X defined by

$$\mathscr{R} = \{(u, v) \in X \times X : u(t) \le v(t) \text{ for all } t \in [0, 1]\}. \tag{66}$$

Now, for any $u, v \in \mathscr{R}$ and for all $t \in [0, 1]$

$$\begin{aligned} Q(u(t)) &= \int_0^t K(t, s, u(s))ds + h(t) \le \int_0^t K(t, s, v(s))ds + h(t) \\ &= Q(v(t)). \end{aligned} \tag{67}$$

This implies that $(Qu, Qv) \in \mathscr{R}$. That is, $\mathscr{R}$ is Q closed.

Now, let $(u, v) \in \mathscr{R}$ and consider

$$\begin{aligned} |Q(u(t)) - Q(v(t))| &= \left| \int_0^t (K(t, s, u(s)) - K(t, s, v(s)))ds \right| \\ &\le \int_0^t |K(t, s, u(s)) - K(t, s, v(s))| ds \\ &\le \int_0^t \frac{|u - v|}{\tau\mathscr{D}(u, v) + 1} ds \le \frac{1}{\tau\mathscr{D}(u, v) + 1} \\ &\quad \cdot \int_0^t \max_{t\in[0,1]} |u(t) - v(t)| \Big) ds \\ &= \frac{1}{\tau\mathscr{D}(u, v) + 1} \int_0^t d(u, v) ds \le \frac{1}{\tau\mathscr{D}(u, v) + 1} \\ &\quad \cdot \int_0^t \mathscr{D}(u, v) ds = \frac{\mathscr{D}(u, v)}{\tau\mathscr{D}(u, v) + 1} t, \\ &\le \frac{\mathscr{D}(u, v)}{\tau\mathscr{D}(u, v) + 1} \text{ since } t \in [0, 1]. \end{aligned} \tag{68}$$

Therefore, we have

$$|Q(u(t)) - Q(v(t))| \leq \frac{\mathscr{D}(u,v)}{\tau\mathscr{D}(u,v)+1}, \quad \forall t \in [0,1]. \tag{69}$$

On taking supremum on both sides of above inequality, we obtain

$$\|Q(u) - Q(v)\| \leq \frac{\mathscr{D}(u,v)}{\tau\mathscr{D}(u,v)+1}. \tag{70}$$

It yields that

$$\tau - \frac{1}{\|Q(u)-Q(v)\|} \leq \frac{-1}{\mathscr{D}(u,v)}, \tag{71}$$

or

$$\tau - \frac{1}{d(Q(u),Q(v))} \leq \frac{-1}{\mathscr{D}(u,v)}. \tag{72}$$

By choosing $F(\mu) = -1/\mu$, $\mu > 0$, from the above inequality, we get

$$\tau + F(d(Q(u),Q(v))) \leq F(\mathscr{D}(u,v)). \tag{73}$$

Hence, inequality (6) is satisfied. We have defined binary relation $\mathscr{R}$ on X by $u\mathscr{R}v$ if and only if $u(t) \leq v(t)$ for all $t \in [0,1]$. Now, consider an $\mathscr{R}$-preserving sequence $\{u_n\}$ in $C[0,1]$ which converges to $u \in X$. Then, we have

$$u_0(t) \leq u_1(t) \leq \cdots u_n(t) \leq u_{t+1}(s) \leq \cdots, \tag{74}$$

which gives us $u_n(t) \leq u(t) \forall t \in [0,1]$. Therefore, $\mathscr{R}$ is d self closed on X. To show that $\mathrm{Fix}(Q)$ is $\mathscr{R}^s$-connected, if $u, v \in \mathrm{Fix}(Q)$, then $w = \max\{u, v\} \in C[0,1]$. Since $u \leq w$ and $v \leq w$, thus $u\mathscr{R}w$ and $v\mathscr{R}w$. Therefore, all conditions of Theorem 21 are true. Hence, the conclusion holds. ☐

Now, in the situation where upper solution is presented, we have the following result.

Theorem 32. *Consider that in third variable K is nonincreasing and there is $\tau > 0$ such that*

$$|K(t,s,u) - K(t,s,v)| \leq \frac{|u(t)-v(t)|}{\tau\mathscr{D}(u,v)+1}, \tag{75}$$

for all $t, s \in [0,1]$ and $u, v \in X$, where $\mathscr{D}(u,v) = \max\{(d(u,v), d(u,Qu), d(v,Qv), ((d(u,Qv)+d(v,Qu))/2)\}$. Then, the existence of a unique solution of the integral Equation (62) follows from the existence of an upper solution of (62).

Proof. Let the binary relation on X be defined by

$$\mathscr{R} = \{(u,v) \in X \times X : u(s) \geq v(s) \text{ for all } t \in [0,1]\}. \tag{76}$$

Now, proceeding as in Theorem 31, we can conclude that all the assumptions of Theorem 21 are satisfied and it guarantees the existence of a unique solution of (62). ☐

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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Research Article

Approximation Properties of New Modified Gamma Operators

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This paper is aimed at constructing new modified Gamma operators using the second central moment of the classic Gamma operators. And we will compute the first, second, fourth, and sixth order central moments by the moment computation formulas, and their quantitative properties are researched. Then, the global results are established in certain weighted spaces and the direct results including the Voronovskaya-type asymptotic formula, and point-wise estimates are investigated. Also, weighted approximation of these operators is discussed. Finally, the quantitative Voronovskaya-type asymptotic formula and Grüss Voronovskaya-type approximation are presented.

1. Introduction

Recently, Karsli et al. [1] constructed and estimated the rate of convergence for functions with derivatives of bounded variation on $\mathbb{R}_+ := (0,\infty)$ of new Gamma type operators preserving z^2 as (see also [2])

$$(\Phi_l\lambda(t))(z) = \frac{(2l+3)!z^{l+3}}{l!(l+2)!}\int_0^\infty \frac{t^l}{(z+t)^{2l+4}}\lambda(t)\mathrm{d}t, z \in \mathbb{R}_+. \tag{1}$$

In [3], Karsli et al. used analysis methods to obtain the rate of point-wise convergence for the operators (1). In [4], Karsli and Özarslan obtained some direct local and global approximation results for the operators (1). In [5], İzgi studied some direct results in asymptotic approximation about the operators (1). In [6], Krech gave a note about the results of İzgi in [5] and obtained an error estimate for the operators (1). In [7], Krech gave direct approximation theorems for the operators (1) in certain weighted spaces. In [8], Cai and Zeng constructed q-Gamma operators and gave their approximation properties. In [9], Zhao et al. extended the works of Cai and Zeng and considered the stancu generalization of q-Gamma operators. Recently, Cheng et al. constructed (p,q)-Gamma operators using (p,q)-Beta function of the second kind and discussed their approximation properties in [10]. In [11], Zhou et al. extended the works of Cheng et al. in [10] and constructed (p,q)-Gamma-Stancu operators. There are many papers about the research and application of other Gamma-type operators, and we mention some of them [12–17].

In this paper, we construct new modified Gamma operators using the second central moment of the operators (1) as follows:

Definition 1. For $l = 1, 2, \cdots$ and $\lambda : \mathbb{R}_+ \longrightarrow \mathbb{R}$, we construct new modified Gamma operators by

$$(Y_l\lambda(t))(z) = \int_0^\infty K_l(t,z)\lambda(t)\mathrm{d}t, z \in \mathbb{R}_+, \tag{2}$$

where

$$K_l(t,z) = \frac{2^l(2l+3)!!z^{l+1}}{l!}\frac{t^l(z-t)^2}{(z+t)^{2l+4}}, t, z \in \mathbb{R}_+. \tag{3}$$

The paper is organized as follows: In Section 1, we introduce the history of Gamma operators and construct new modified Gamma operators using the second central moment. In Section 2, we obtain the basic results by the moment computation formulas. And the first, second, fourth, and sixth order central moment computation formulas and limit equalities are also obtained. In Section 3, we establish the global approximation results for the operators (2) in certain weighted spaces. In Sections 4 and 5, we investigate the direct results including the Voronovskaya-type asymptotic formula and point-wise estimates in three different Lipschitz classes and discuss weighted approximation. In Section 6, we present a quantitative Voronovskaya-type asymptotic formula and a Grüss Voronovskaya-type approximation (for the quantitative Voronovskaya type theorem and Grüss-Voronovskaya theorem for the other operators, see also [18–24]).

2. Basic Results

In this section, we present certain auxiliary results which will be used to prove our main theorems for the operators (2).

Lemma 2 (see [1]). *For any $l \in \mathbb{N}_+$, $p = 0, 1, 2, \cdots, l+2$, we have*

$$\varphi_l(p) \coloneqq (\Phi_l t^p)(z) = \frac{(l+p)!(l+2-p)!}{l!(l+2)!} z^p, z \in \mathbb{R}_+. \quad (4)$$

Lemma 3. *If we define $\phi_l(p) \coloneqq (Y_l t^p)(z)$, then there holds the following relation*

$$\frac{2z^2}{l+2}\phi_l(p) = \varphi_l(p+2) - 2\varphi_l(p+1)z + \varphi_l(p)z^2, \quad (5)$$

where $p = 0, 1, \cdots, l$, $z \in \mathbb{R}_+$.

Then, the following lemma can be obtained immediately:

Lemma 4. *For any $l \in \mathbb{N}_+$, $z \in \mathbb{R}_+$, we have*

$$\phi_l(0) = 1\,;\, \phi_l(1) = \frac{l+3}{l} z\,;\, \phi_l(2) = \frac{(l+2)(l+9)}{l(l-1)} z^2, for\ l > 1\,; \quad (6)$$

$$\phi_l(3) = \frac{(l+2)(l+3)(l+19)}{l(l-1)(l-2)} z^3, for\ l > 2\,; \quad (7)$$

$$\phi_l(4) = \frac{(l+2)(l+3)(l+4)(l+33)}{l(l-1)(l-2)(l-3)} z^4, for\ l > 3\,; \quad (8)$$

$$\phi_l(p) = \frac{(l-p)!(l+p)!\left(l+2p^2+1\right)}{l!(l+1)!} z^p, for\ l \ge p\,; \quad (9)$$

$$A_l(z) \coloneqq (Y_l(t-z))(z) = \frac{3}{l} z\,; \quad (10)$$

$$B_l(z) \coloneqq \left(Y_l(t-z)^2\right)(z) = \frac{6l+24}{l(l-1)} z^2, for\ l > 1\,; \quad (11)$$

$$\left(Y_l(t-z)^3\right)(z) = \frac{90l+230}{l(l-1)(l-2)} z^3, for\ l > 2\,; \quad (12)$$

$$\left(Y_l(t-z)^4\right)(z) = \frac{60\left(l^2+23l+48\right)}{l(l-1)(l-2)(l-3)} z^4, for\ l > 3\,; \quad (13)$$

$$\left(Y_l(t-z)^6\right)(z) = \frac{840\left(l^3+69l^2+506l+768\right)}{l(l-1)(l-2)(l-3)(l-4)(l-5)} z^6, for\ l > 5\,; \quad (14)$$

$$\lim_{l\longrightarrow\infty} A_l(z) = 3z\,; \quad (15)$$

$$\lim_{l\longrightarrow\infty} B_l(z) = 6z^2\,; \quad (16)$$

$$\lim_{l\longrightarrow\infty} l^2\left(Y_l(t-z)^4\right)(z) = 60z^4\,; \quad (17)$$

$$\lim_{l\longrightarrow\infty} l^3\left(Y_l(t-z)^6\right)(z) = 840z^6. \quad (18)$$

By the classical Korovkin theorem, we easily obtain the following lemma:

Lemma 5. *For all $\lambda \in C_B(\mathbb{R}_+)$ and any finite interval $I \subset \mathbb{R}_+$, then the sequence $\{(Y_l\lambda(t))(z)\}$ converges to λ uniformly on I, where $C_B(\mathbb{R}_+)$ denotes the set of all real-valued bounded and continuous functions defined on $\mathbb{R}_+$, endowed with the norm $\|\lambda\| = \sup_{z\in\mathbb{R}_+}|\lambda(z)|$.*

3. Global Results

In this section, we establish some global results by using certain Lipschitz classes. We first recall some basic definitions. Let $r \in \mathbb{N} \coloneqq \{0, 1, 2, \cdots\}$ and define the weighted function w_r as follows:

$$w_0(z) \coloneqq 1 \text{ and } w_r(z) \coloneqq \frac{1}{1+z^r} \text{ for } z \in \mathbb{R}_+ \text{ and } r \in \mathbb{N}\setminus\{0\}. \quad (19)$$

Meantime, we consider the following subspace $S_r(\mathbb{R}_+)$ of $C(\mathbb{R}_+)$ generated by w_r:

$$S_r(\mathbb{R}_+) \coloneqq \{\lambda \in C(\mathbb{R}_+): w_r\lambda \text{ is uniformly continuous and bounded on } \mathbb{R}_+\} \quad (20)$$

endowed with the norm $\|\lambda\|_r \coloneqq \sup_{z\in\mathbb{R}_+} w_r(z)|\lambda(z)|$ for $\lambda \in S_r(\mathbb{R}_+)$. For every $\lambda \in S_r(\mathbb{R}_+)$, $\delta > 0$, and $\alpha \in (0, 2]$, the usual weighted modulus of continuity, the second-order weighted modulus of smoothness, and the corresponding Lipschitz classes are, respectively, defined as

$$\omega_r^1(\lambda\,;\delta) \coloneqq \sup\{w_r(z)|\lambda(y)-\lambda(z)|: |y-z| \le \delta, y, z \in \mathbb{R}_+\}\,;$$
$$\omega_r^2(\lambda\,;\delta) \coloneqq \sup_{t\in(0,\delta]} \|\lambda(z+2t) - 2\lambda(z+t) + \lambda(z)\|_r\,;$$
$$\mathrm{Lip}_r^2\alpha \coloneqq \left\{\lambda \in S_r(\mathbb{R}_+): \omega_r^2(\lambda\,;\delta) = O(\delta^\alpha) \text{ as } \delta \longrightarrow 0^+\right\}. \quad (21)$$

Theorem 6. *Let $r \in \{0, 1, \cdots, l\}$ be fixed. Then, there exists a positive constant C_r such that*

$$\left\|\left(Y_l\left(\frac{1}{w_r}\right)\right)\right\|_r \le C_r. \tag{22}$$

Furthermore, for all $\lambda \in S_r(\mathbb{R}_+)$, we have

$$\|Y_l\lambda\|_r \le C_r\|\lambda\|_r. \tag{23}$$

Thus, Y_l is a linear positive operator from $S_r(\mathbb{R}_+)$ to $S_r(\mathbb{R}_+)$ for any $r \in \{0, 1,\cdots,l\}$.

Proof. Inequality (22) is obvious for $r = 0$. Assume that $l \ge r \ge 1$, using (6), we have

$$\begin{aligned} w_r(z)\left(Y_l\left(\frac{1}{w_r(t)}\right)\right)(z) &= w_r(z)(Y_l(1+t^r))(z) \\ &= w_r(z)(Y_l(1))(z) + w_r(z)(Y_l(t^r))(z) \\ &= w_r(z) + w_r(z)\frac{(l-r)!(l+r)!\left(l+2r^2+1\right)}{l!(l+1)!}z^r \\ &\le C_r w_r(z)(1+z^r) \\ &= C_r, \end{aligned} \tag{24}$$

where $C_r = \max\{1, \sup_l((l-r)!(l+r)!(l+2r^2+1)/l!(l+1)!)\}$, and then we obtain (22). Moreover, for every $\lambda \in S_r(\mathbb{R}_+)$ and $z \in \mathbb{R}_+$, we have

$$\begin{aligned} w_r(z)|(Y_l\lambda(t))(z)| &\le w_r(z)\int_0^\infty K_l(t,z)|\lambda(t)|\mathrm{d}t \\ &= w_r(z)\int_0^\infty K_l(t,z)|\lambda(t)|\frac{w_r(t)}{w_r(t)}\mathrm{d}t \\ &\le \|\lambda\|_r w_r(z)\left(Y_l\left(\frac{1}{w_r(t)}\right)\right)(z) \\ &\le C_r\|\lambda\|_r. \end{aligned} \tag{25}$$

Taking the supremum over $z \in \mathbb{R}_+$, we obtain (23).

Theorem 7. *For any fixed $r \in \{0, 1,\cdots,l-2\}$, $l \ge 2$, there exists a positive constant C_r such that*

$$w_r(z)\left(Y_l\left(\frac{(t-z)^2}{w_r(t)}\right)\right)(z) \le C_r\frac{z^2}{l}. \tag{26}$$

Proof. The formula (11) implies (26) for $r = 0$. If $r = 1$, then we obtain

$$\begin{aligned} \left(Y_l\left((t-z)^2(1+t)\right)\right)(z) &= \left(Y_l\left((t-z)^2\right)\right)(z) + \left(Y_l\left((t-z)^2 t\right)\right)(z) \\ &= \left(Y_l\left((t-z)^3\right)\right)(z) + (1+z)\left(Y_l\left((t-z)^2\right)\right)(z), \end{aligned} \tag{27}$$

which by (11) and (12) yield (26) for $r = 1$. Assuming $l-2 \ge r \ge 2$ and using (11) and (6), we obtain

$$\begin{aligned} \left(Y_l\left(\frac{(t-z)^2}{w_r(t)}\right)\right)(z) &= \left(Y_l\left((t-z)^2\right)\right)(z) + \left(Y_l\left(t^{r+2}\right)\right)(z) \\ &\quad - 2z\left(Y_l\left(t^{r+1}\right)\right)(z) + z^2\left(Y_l(t^r)\right)(z) \\ &= \frac{6l+24}{l(l-1)}z^2 \\ &\quad + \frac{(l-r-2)!(l+r+2)!\left(l+2(r+2)^2+1\right)}{l!(l+1)!}z^{r+2} \\ &\quad - 2\frac{(l-r-1)!(l+r+1)!\left(l+2(r+1)^2+1\right)}{l!(l+1)!}z^{r+2} \\ &\quad + \frac{(l-r)!(l+r)!\left(l+2r^2+1\right)}{l!(l+1)!}z^{r+2} \\ &= \frac{z^2}{l}\left\{\frac{6l+24}{l-1} + \{(l+r+1)(l+r+2)\left(l+2(r+2)^2\right.\right. \\ &\quad \left.+1\right)2(l-r-1)(l+r+1)\left(l+2(r+1)^2+1\right) \\ &\quad \left.+(l-r-1)(l-r)\left(l+2r^2+1\right)\}\frac{(l-r-2)!(l+r)!}{(l-1)!(l+1)!}z^r\right\} \\ &\le \frac{z^2}{l}\left(\frac{6l+24}{l-1} + \frac{(6l^2+C_{r,1}l+C_{r,2})(l-r-2)!(l+r)!}{(l-1)!(l+1)!}z^r\right) \\ &\le C_r\frac{z^2}{l}(1+z^r), \end{aligned} \tag{28}$$

where $C_{r,1}$ and $C_{r,1}$ are two constants only depending on r. This completes the proof.

Now, for $r \in \{0, 1,\cdots,l\}$, we consider the two spaces $S_r^1(\mathbb{R}_+) \coloneqq \{\lambda \in S_r(\mathbb{R}_+)\colon \lambda' \in S_r(\mathbb{R}_+)\}$ and $S_r^2(\mathbb{R}_+) \coloneqq \{\lambda \in S_r(\mathbb{R}_+)\colon \lambda'' \in S_r(\mathbb{R}_+)\}$, and we have the three following theorems:

Theorem 8. *For any fixed r, if $\lambda \in S_r^1(\mathbb{R}_+)$, there exists a positive constant C_r such that*

$$w_r(z)|(Y_l\lambda(t))(z) - \lambda(z)| \le C_r\left\|\lambda'\right\|_r\frac{z}{\sqrt{l}} \tag{29}$$

for all $z \in \mathbb{R}_+$ and $l \ge r+2$.

Proof. Let $z \in \mathbb{R}_+$. By $\lambda(t) - \lambda(z) = \int_t^z \lambda'(u)\mathrm{d}u$, $t \in \mathbb{R}_+$, Lemma (11), and the linearity of Y_l, we obtain

$$(Y_l\lambda(t))(z) - \lambda(z) = \left(Y_l\int_z^t \lambda'(u)\mathrm{d}u\right)(z). \tag{30}$$

Using

$$\left|\int_t^z \lambda'(u)\mathrm{d}u\right| \le \left\|\lambda'\right\|_r\left|\int_z^t \frac{1}{w_r(u)}\mathrm{d}u\right| \le \left\|\lambda'\right\|_r\left(\frac{1}{w_r(t)} + \frac{1}{w_r(z)}\right)|t-z|. \tag{31}$$

Hence,

$$\begin{aligned} w_r(z)|(Y_l\lambda(t))(z) - \lambda(z)| &\le \left\|\lambda'\right\|_r \\ &\quad \cdot\left((Y_l|t-z|)(z) + w_r(z)\left(Y_l\left(\frac{|t-z|}{w_r(t)}\right)\right)(z)\right). \end{aligned} \tag{32}$$

Applying the well-known Cauchy-Schwarz inequality, we can obtain

$$(Y_l\,|\,t-z\,|\,)(z) \le \sqrt{\left(Y_l(t-z)^2\right)(z)},$$
$$\left(Y_l\left(\frac{|t-z|}{w_r(t)}\right)\right)(z) \le \sqrt{\left(Y_l\left(\frac{1}{w_r(t)}\right)\right)(z)}\sqrt{\left(Y_l\left(\frac{(t-z)^2}{w_r(t)}\right)\right)(z)}. \tag{33}$$

Combining (22) and (26), we can get the required result.

Theorem 9. *For any fixed r, if $\lambda \in S_r^l(\mathbb{R}_+)$, then there exists a positive constant C_r such that*

$$w_r(z)\left|(Y_l\lambda(t))(z)-\lambda(z)\right| \le C_r\omega_r^l\left(\lambda\,;\frac{z}{\sqrt{l}}\right) \tag{34}$$

for all $z \in \mathbb{R}_+$ and $l \ge r+2$.

Proof. Let $z \in \mathbb{R}_+$. We denote the Steklov means of λ by λ_s, $s \in \mathbb{R}_+$:

$$\lambda_s(z) = \frac{1}{s}\int_0^s \lambda(u+z)\mathrm{d}u, z, s \in \mathbb{R}_+. \tag{35}$$

It is obvious that

$$\lambda_s(z)-\lambda(z) = \frac{1}{s}\int_0^s (\lambda(u+z)-\lambda(z))\mathrm{d}u,$$
$$\lambda_s'(z) = \frac{1}{s}(\lambda(z+s)-\lambda(z)) \tag{36}$$

for $z, s \in \mathbb{R}_+$. Hence, if $\lambda \in S_r^1(\mathbb{R}_+)$, then $\lambda_s \in S_r^2(\mathbb{R}_+)$ for every fixed $s \in \mathbb{R}_+$. Furthermore, we have

$$\|\lambda_s-\lambda\|_r \le \omega_r^1(\lambda\,;s), \left\|\lambda_s'\right\|_r \le \frac{1}{s}\omega_r^1(\lambda\,;s). \tag{37}$$

By

$$w_r(z)\left|(Y_l\lambda(t))(z)-\lambda(z)\right| \le w_r(z)\left|(Y_l(\lambda(t)-\lambda_s(t)))(z)\right| + w_r(z)\left|(Y_l\lambda_s(t))(z)-\lambda_s(z)\right| + w_r(z)\left|\lambda(z)-\lambda_s(z)\right|. \tag{38}$$

Using (23) and (37), we have

$$w_r(z)\left|(Y_l(\lambda(t)-\lambda_s(t)))(z)\right| \le C_r\|\lambda-\lambda_s\|_r \le C_r\omega_r^1(\lambda\,;s) \tag{39}$$

for any $z, s \in \mathbb{R}_+$. From (29) and (37), we have

$$w_r(z)\left|(Y_l\lambda_s(t))(z)-\lambda_s(z)\right| \le C_r\|\lambda_{s'}\|_r\frac{z}{\sqrt{l}} \le C_r\frac{1}{s}\omega_r^1(\lambda\,;s)\frac{z}{\sqrt{l}}. \tag{40}$$

By (37), we have

$$w_r(z)\left|\lambda(z)-\lambda_s(z)\right| \le \|\lambda-\lambda_s\|_r \le \omega_r^1(\lambda\,;s) \tag{41}$$

for any $z, s \in \mathbb{R}_+$. Finally, we have

$$w_r(z)\left|(Y_l\lambda(t))(z)-\lambda(z)\right| \le \omega_r^1(\lambda\,;s)\left(C_r+\frac{1}{s}C_r\frac{z}{\sqrt{l}}+1\right) \tag{42}$$

for any $z, s \in \mathbb{R}_+$. Choosing $s = z/\sqrt{l}$, the proof is proved.

Theorem 10. *Defining a new operator,*

$$(Y_l^*\lambda(t))(z) = (Y_l\lambda(t))(z)-\lambda(z+A_l(z))+\lambda(z). \tag{43}$$

For any fixed r, if $\lambda \in S_r^2(\mathbb{R}_+)$, then there exists a positive constant C_r such that

$$w_r(z)\left|(Y_l^*\lambda(t))(z)-\lambda(z)\right| \le C_r\left\|\lambda''\right\|_r\frac{z^2}{l}, \tag{44}$$

for all $z \in \mathbb{R}_+$ and $l \ge r+2$.

Proof. Using Taylor's expansion, we have

$$\lambda(t)-\lambda(z) = (t-z)\lambda'(z)+\int_z^t (t-u)\lambda''(u)\mathrm{d}u, z, t \in \mathbb{R}_+. \tag{45}$$

By $(Y_l^*(t-z))(z) = 0$ and $(Y_l^*1)(z) = 1$, we have

$$\left|(Y_l^*\lambda(t))(z)-\lambda(z)\right| \le (Y_l^*(\lambda(t)-\lambda(z)))(z) \le \left(Y_l^*\left(\int_z^t (t-u)\lambda''(u)\mathrm{d}u\right)\right)(z)\left|\left(Y_l\left(\int_z^t (t-u)\lambda''(u)\mathrm{d}u\right)\right)(z) - \int_z^{z+A_l(z)}(z+A_l(z)-u)\lambda''(u)\mathrm{d}u\right|. \tag{46}$$

Since

$$\left|\int_z^t (t-u)\lambda''(u)\right| \le \frac{\left\|\lambda''\right\|_r(t-z)^2}{2}\left(\frac{1}{w_r(z)}+\frac{1}{w_r(t)}\right),$$
$$\left|\int_z^{z+A_l(z)}(z+A_l(z)-u)\lambda''(u)\mathrm{d}u\right| \le \frac{\left\|\lambda''\right\|_r}{2w_r(z)}(A_l(z))^2, \tag{47}$$

we have

$$w_r(z)|(Y_l^*\lambda(t))(z)-\lambda(z)| \le \frac{\|\lambda''\|_r}{2} \cdot \left(B_l(z)+w_r(z)\left(Y_l\left(\frac{(t-z)^2}{w_r(t)}\right)\right)(z)\right) + \frac{\|\lambda''\|_r}{2}(A_l(z))^2. \tag{48}$$

Combining Lemma 4 and (26), we have

$$w_r(z)|(Y_l^*\lambda(t))(z)-\lambda(z)| \le C_r\|\lambda''\|_r\frac{z^2}{l} \tag{49}$$

for all $z \in \mathbb{R}_+$ and $l \ge r+2$. The theorem is completed.

Theorem 11. *For any fixed r, if $\lambda \in S_r^2(\mathbb{R}_+)$, then there exists a positive constant C_r such that*

$$w_r(z)|(Y_l\lambda(t))(z)-\lambda(z)| \le C_r\omega_r^2\left(\lambda;\frac{z}{\sqrt{l}}\right)+\omega_r^1(\lambda;A_l(z)) \tag{50}$$

for all $z \in \mathbb{R}_+$ and $l \ge r+2$. In particular, if $\lambda \in Lip_r^2\alpha$ for some $\alpha \in (0,2]$, then

$$w_r(z)|(Y_l\lambda(t))(z)-\lambda(z)| \le C_r\left(\frac{z^2}{l}\right)^{\alpha/2}+\omega_r^1(\lambda;A_l(z)) \tag{51}$$

holds.

Proof. Let $\lambda \in S_r(\mathbb{R}_+)$, and the Steklov means $\tilde{\lambda}_s(z)$ of the second order of λ defined by

$$\tilde{\lambda}_s(z)=\frac{4}{s^2}\int_0^{s/2}\int_0^{s/2}(2\lambda(z+u+v)-\lambda(z+2u+2v))\mathrm{d}u\mathrm{d}v \tag{52}$$

for $z, s \in \mathbb{R}_+$. By simple computation, we have

$$\begin{aligned}\left\|\lambda-\tilde{\lambda}_s\right\|_r &\le \omega_r^2(\lambda;s),\\ \left\|\tilde{\lambda}_s''\right\|_r &\le \frac{9}{s^2}\omega_r^2(\lambda;s).\end{aligned} \tag{53}$$

Meantime, $\tilde{\lambda}_s \in S_r^2(\mathbb{R}_+)$ while $\lambda \in S_r(\mathbb{R}_+)$. Using the following inequality,

$$\begin{aligned}|(Y_l\lambda(t))(z)-\lambda(z)| \le{}& \left(Y_l^*\left|\lambda(t)-\tilde{\lambda}_s(t)\right|\right)(z)+\left|\lambda(z)-\tilde{\lambda}_s(z)\right|\\ &+\left|\left(Y_l^*\tilde{\lambda}_s(t)\right)(z)-\tilde{\lambda}_s(z)\right|\\ &+|\lambda(z+A_l(z))-\lambda(z)|.\end{aligned} \tag{54}$$

Combining (23) and (44), we have

$$\begin{aligned}w_r(z)|(Y_l\lambda(t))(z)-\lambda(z)| \le{}& (C_r+3)\left\|\lambda-\tilde{\lambda}_s\right\|_r\\ &+C_r\left\|\tilde{\lambda}_s''\right\|_r\frac{z^2}{l}w_r(z)|\lambda(z+A_l(z))-\lambda(z)|\\ \le{}& C_r\omega_r^2(\lambda;s)\left(1+\frac{1}{s^2}\frac{z^2}{l}\right)\\ &+\omega_r^1(\lambda;A_l(z)).\end{aligned} \tag{55}$$

Hence, choosing $s=z/\sqrt{l}$, the first part of the proof is proved. The second part of the proof can be directly observed from the definition of the space $\mathrm{Lip}_r^2\alpha$.

4. Direct Results

4.1. Voronovskaya-Type Theorem

Theorem 12. *If $\lambda \in C_B(\mathbb{R}_+)$ and λ'' exists at a point $z \in \mathbb{R}_+$, then*

$$\lim_{l\longrightarrow\infty} l((Y_l\lambda(t))(z)-\lambda(z))=3z\left(\lambda'(z)+z\lambda''(z)\right). \tag{56}$$

Proof. By the Taylor's expansion formula for λ, we have

$$\lambda(t)=\lambda(z)+\lambda'(z)(t-z)+\frac{1}{2}\lambda''(z)(t-z)^2+R(t;z)(t-z)^2, \tag{57}$$

where

$$R(t;z)=\begin{cases}\dfrac{\lambda(t)-\lambda(z)-\lambda'(z)(t-z)-1/2\lambda''(z)(t-z)^2}{(t-z)^2}, & t\neq z;\\ 0, & t=z.\end{cases} \tag{58}$$

Applying the L'Hospital's Rule,

$$\lim_{t\longrightarrow z}R(t;z)=\frac{1}{2}\lim_{t\longrightarrow z}\frac{\lambda'(t)-\lambda'(z)}{t-z}-\frac{1}{2}\lambda''(z)=0. \tag{59}$$

Thus, $R(.;z) \in C_B(\mathbb{R}_+)$. Consequently, we can write

$$\begin{aligned}(Y_l\lambda(t))(z)-\lambda(z)={}&A_l(z)\lambda'(z)+\frac{1}{2}B_l(z)\lambda''(z)\\ &+\left(Y_l\left(R(t;z)(t-z)^2\right)\right)(z).\end{aligned} \tag{60}$$

By the Cauchy-Schwarz inequality, we have

$$l\left(Y_l\left(R(t;z)(t-z)^2\right)\right)(z) \le \sqrt{(Y_l(R^2(t;z)))(z)}\sqrt{l^2\left(Y_l\left((t-z)^4\right)\right)(z)}. \tag{61}$$

We observe that $R^2(z;z)=0$ and $R^2(t;z) \in C_B(\mathbb{R}_+)$. Then, it follows in Lemma 5 that

$$\lim_{l\longrightarrow\infty}\left(Y_l\left(R^2(t;z)\right)\right)(z)=R^2(z;z)=0. \tag{62}$$

Hence, from (17), we can obtain

$$\lim_{l\longrightarrow\infty} l\left(Y_l\left(R(t;z)(t-z)^2\right)\right)(z)=0. \tag{63}$$

Combining (15) and (16), we complete the proof of Theorem 12.

Corollary 13. *If $\lambda, \lambda'' \in C_B(\mathbb{R}_+)$, then we have*

$$\lim_{l\longrightarrow\infty} l((Y_l\lambda(t))(z)-\lambda(z))=3z\left(\lambda'(z)+z\lambda''(z)\right), \tag{64}$$

uniformly with respect to any finite interval $I\subset\mathbb{R}_+$.

4.2. Point-Wise Estimates. In this subsection, we establish three point-wise estimates of the operators (2). First, we obtain the rate of convergence locally by using functions belonging to the Lipschitz class. We denote that $\lambda\in C_B(\mathbb{R}_+)$ is in $\mathrm{Lip}_M(\gamma, D)$, $\gamma\in(0,1]$, and $D\subset\mathbb{R}_+$ if it satisfies the following condition:

$$|\lambda(t)-\lambda(z)|\le M|t-z|^\gamma, t\in D, z\in\mathbb{R}_+, \tag{65}$$

where M is a positive constant depending only on γ and λ.

Theorem 14. *If $\lambda\in C_B(\mathbb{R}_+)\bigcap Lip_M(\gamma, D)$, then for any $z\in\mathbb{R}_+$, we have*

$$|(Y_l\lambda(t))(z)-\lambda(z)|\le M\left((B_l(z))^{\gamma/2}+2d(z;D)\right), \tag{66}$$

where $d(z;D)=\inf\{|t-z|: t\in D\}$ denotes the distance between z and D.

Proof. Let $\bar{D}$ be the closure of D. Using the properties of infimum, there is at least a point $t_0\in\bar{D}$ such that $\mathrm{d}(z;D)=|z-t_0|$. By the triangle inequality

$$|\lambda(t)-\lambda(z)|\le|\lambda(t)-\lambda(t_0)|+|\lambda(z)-\lambda(t_0)|, \tag{67}$$

we have

$$\begin{aligned}|(Y_l\lambda(t))(z)-\lambda(z)|&\le(Y_l|\lambda(t)-\lambda(t_0)|)(z)+(Y_l|\lambda(z)-\lambda(t_0)|)(z)\\&\le M\{(Y_l|t-t_0|^\gamma)(z)+|z-t_0|^\gamma\}\\&\le M\{(Y_l(|t-z|^\gamma+|z-t_0|^\gamma))(z)+|z-t_0|^\gamma\}\\&\le M\{(Y_l|t-z|^\gamma)(z)+2|z-t_0|^\gamma\}.\end{aligned} \tag{68}$$

Choosing $p=2/\gamma$ and $q=2/2-\gamma$ and using the well-known Hölder inequality, we have

$$\begin{aligned}&|(Y_l\lambda(t))(z)-\lambda(z)|\\&\quad\le M\left\{\left((Y_l|t-z|^{p\gamma})(z)\right)^{1/p}((Y_l1^q)(z))^{1/q}+2\mathrm{d}^\gamma(z;D)\right\}\\&\quad\le M\left\{\left((Y_l(t-z)^2)(z)\right)^{\gamma/2}+2\mathrm{d}^\gamma(z;D)\right\}\\&\quad\le M\left((B_l(z))^{\gamma/2}+2\mathrm{d}^\gamma(z;D)\right).\end{aligned} \tag{69}$$

Next, we obtain the local direct estimate of the operators (2), using the Lipcshitz type maximal function of the order γ introduced by Lenze [25] as

$$\tilde{\omega}_\gamma(\lambda;z)=\sup_{z,t\in\mathbb{R}_+,t\neq z}\frac{|\lambda(t)-\lambda(z)|}{|t-z|^\gamma}, \gamma\in(0,1]. \tag{70}$$

Theorem 15. *If $\lambda\in C_B(\mathbb{R}_+)$, then for any $z\in\mathbb{R}_+$, we have*

$$|(Y_l\lambda(t))(z)-\lambda(z)|\le\tilde{\omega}_\gamma(\lambda;z)(B_l(z))^{\gamma/2}. \tag{71}$$

Proof. From equation (70), we have

$$|(Y_l\lambda(t))(z)-\lambda(z)|\le\tilde{\omega}_\gamma(\lambda;z)(Y_l(t-z)^\gamma)(z). \tag{72}$$

Applying the well-known Hölder inequality, we have

$$\begin{aligned}|(Y_l\lambda(t))(z)-\lambda(z)|&\le\tilde{\omega}_\gamma(\lambda;z)\left((Y_l(t-z)^2)(z)\right)^{\gamma/2}\\&=\tilde{\omega}_\gamma(\lambda;z)(B_l(z))^{\gamma/2}.\end{aligned} \tag{73}$$

Finally, we establish point-wise estimate of the operators (2) in the following Lipschitz-type space (see [26]) with two distinct parameters $\mu_1, \mu_2\in\mathbb{R}_+$:

$$\mathrm{Lip}_M^{(\mu_1,\mu_2)}(\gamma):=\left\{\lambda\in C(\mathbb{R}_+): |\lambda(t)-\lambda(z)|\le M\frac{|t-z|^\gamma}{t+\mu_1z^2+\mu_2z}\right\}, t,z\in\mathbb{R}_+, \tag{74}$$

where $\gamma\in(0,1]$, M, is a positive constant depending only on γ, μ_1, μ_2 and λ.

Theorem 16. *If $\lambda\in Lip_M^{(\mu_1,\mu_2)}(\gamma)$, then for any $z\in\mathbb{R}_+$, we have*

$$|(Y_l\lambda(t))(z)-\lambda(z)|\le M\left(\frac{B_l(z)}{\mu_1z^2+\mu_2z}\right)^{\gamma/2}. \tag{75}$$

Proof. Applying the well-known Hölder inequality with $p=2/\gamma$ and $q=2/2-\gamma$, we have

$$
\begin{aligned}
|(Y_l\lambda(t))(z)-\lambda(z)| &\le (Y_l|\lambda(t)-\lambda(z)|)(z)\\
&\le \left(Y_l M\frac{|t-z|^\gamma}{t+\mu_1 z^2+\mu_2 z}\right)(z)\\
&\le \frac{M}{\mu_1 z^2+\mu_2 z}(Y_l|t-z|^\gamma)(z)\\
&\le \frac{M}{\mu_1 z^2+\mu_2 z}\left((Y_l|t-z|^{p\gamma})(z)\right)^{1/p}((Y_l 1^q)(z))^{1/q}\\
&= M\left(\frac{B_l(z)}{\mu_1 z^2+\mu_2 z}\right)^{\gamma/2}.
\end{aligned} \tag{76}
$$

Thus, the proof is completed.

5. Weighted Approximation

Let $B_2(\mathbb{R}_+)$ be the set of all functions λ defined on $\mathbb{R}_+$ satisfying the condition $|\lambda(z)| \le M_\lambda(1+z^2)$ with an absolute constant $M_\lambda > 0$ which depends only on λ. $C_2(\mathbb{R}_+)$ denotes the subspace of all continuous functions $\lambda \in B_2(\mathbb{R}_+)$ with the norm $\|\lambda\|_2 = \sup_{z\in\mathbb{R}_+}(|\lambda(z)|/1+z^2)$. By $C_2^0(\mathbb{R}_+)$, we denote the subspace of all functions $f \in C_2(\mathbb{R}_+)$ for which $\lim_{z\longrightarrow+\infty}|\lambda(z)|/1+z^2$ is finite.

Theorem 17. *If $\lambda \in C_2^0(\mathbb{R}_+)$ and $\kappa > 0$, we have*

$$\lim_{l\longrightarrow\infty}\sup_{z\in\mathbb{R}_+}\frac{|(Y_l\lambda(t))(z)-\lambda(z)|}{(1+z^2)^{1+\kappa}} = 0. \tag{77}$$

Proof. Let $z_0 \in \mathbb{R}_+$ be arbitrary but fixed.

$$
\begin{aligned}
\sup_{z\in\mathbb{R}_+}\frac{|(Y_l\lambda(t))(z)-\lambda(z)|}{(1+z^2)^{1+\kappa}} &\le \sup_{z\in(0,z_0)}\frac{|(Y_l\lambda(t))(z)-\lambda(z)|}{(1+z^2)^{1+\kappa}}\\
&\quad+\sup_{z\in[z_0,\infty)}\frac{|(Y_l\lambda(t))(z)-\lambda(z)|}{(1+z^2)^{1+\kappa}}\\
&\le \|(Y_l\lambda(t))(z)-\lambda\|_{(0,z_0)}\\
&\quad+\|\lambda\|_2\sup_{z\in[z_0,\infty)}\frac{\left(Y_l\left(1+t^2\right)\right)(z)}{(1+z^2)^{1+\kappa}}\\
&\quad+\sup_{z\in[z_0,\infty)}\frac{|\lambda(z)|}{(1+z^2)^{1+\kappa}} := I_1+I_2+I_3.
\end{aligned} \tag{78}
$$

Applying $|\lambda(z)| \le \|\lambda\|_2(1+z^2)$, we have

$$I_3 = \sup_{z\in[z_0,\infty)}\frac{|\lambda(z)|}{(1+z^2)^{1+\kappa}} \le \sup_{z\in[z_0,\infty)}\frac{\|\lambda\|_2\left(1+z^2\right)}{(1+z^2)^{1+\kappa}} \le \frac{\|\lambda\|_2}{\left(1+z_0^2\right)^\kappa}. \tag{79}$$

Let $\varepsilon > 0$. Since $\lim_{l\longrightarrow\infty}\sup_{z\in[z_0,\infty)}(Y_l(1+t^2))(z)/(1+z^2) = 1$, there exists $L_1 \in \mathbb{N}$, such that for all $l > L_1$,

$$
\begin{aligned}
\frac{\|\lambda\|_2\left(Y_l\left(1+t^2\right)\right)(z)}{(1+z^2)^{1+\kappa}} &\le \frac{\|\lambda\|_2}{(1+z^2)^{1+\kappa}}\left(\left(1+z^2\right)+\frac{\varepsilon}{3\|\lambda\|_2}\right)\\
&\le \frac{\|\lambda\|_2}{(1+z^2)^\kappa}+\frac{\varepsilon}{3}.
\end{aligned} \tag{80}
$$

Hence,

$$\|\lambda\|_2\sup_{z\in[z_0,\infty)}\frac{\left(Y_l\left(1+t^2\right)\right)(z)}{(1+z^2)^{1+\kappa}} \le \frac{\|\lambda\|_2}{\left(1+z_0^2\right)^\kappa}+\frac{\varepsilon}{3},\forall l \ge L_1. \tag{81}$$

Thus,

$$I_2+I_3 < \frac{2\|\lambda\|_2}{\left(1+z_0^2\right)^\kappa}+\frac{\varepsilon}{3},\forall l \ge L_1. \tag{82}$$

Next, for sufficiently large z_0 such that $\|\lambda\|_2/(1+z_0^2)^\kappa < \varepsilon/6$, then $I_2+I_3 < 2\varepsilon/3, \forall l \ge L_1$. Applying Lemma 5, there exists $L_2 \in \mathbb{N}$, such that for all $l > L_2$,

$$\|(Y_l\lambda(t))(z)-\lambda\|_{(0,z_0)} < \frac{\varepsilon}{3}. \tag{83}$$

Let $L = \max\{L_1, L_2\}$. Combining (80) (82), and (83), we have

$$\sup_{z\in\mathbb{R}_+}\frac{|(Y_l\lambda(t))(z)-\lambda(z)|}{(1+z^2)^{1+\kappa}} < \varepsilon,\forall l \ge L. \tag{84}$$

Hence, the proof of Theorem 17 is completed.

Theorem 18. *If $\lambda \in C_2^0(\mathbb{R}_+)$, then we have*

$$\lim_{l\longrightarrow\infty}\|(Y_l\lambda(t))(z)-\lambda\|_2 = 0. \tag{85}$$

Proof. Applying the Korovkin theorem [27], it is sufficient to show the following three conditions:

$$\lim_{l\longrightarrow\infty}\|(Y_l t^p)(z)-z^p\|_2 = 0, p = 0, 1, 2. \tag{86}$$

Since $(Y_l 1)(z) = 1$, the condition (86) holds for $p = 0$. From Lemma (11), we have

$$\|(Y_l t)(z)-z\|_2 = \sup_{z\in\mathbb{R}_+}\frac{1}{1+z^2}\left|\frac{l+3}{l}z-z\right| \le \frac{3}{l}. \tag{87}$$

Thus, $\lim_{l\longrightarrow\infty}\|(Y_l t)(z)-z\|_2 = 0$. Finally, we have

$$\left\|\left(Y_l t^2\right)(z)-z^2\right\|_2 = \sup_{z\in\mathbb{R}_+}\frac{1}{1+z^2}\left|\frac{(l+2)(l+9)}{l(l-1)}z^2-z^2\right| \le \frac{12l+19}{l(l-1)}, \tag{88}$$

which implies that $\lim_{l\longrightarrow\infty}\|(Y_l t^2)(z)-z^2\|_2 = 0$.

6. Some Voronovskaya-Type Approximation Theorem

As is known, if $\lambda \in C(\mathbb{R}_+)$ is not uniform, the limit $\lim_{\delta \longrightarrow 0^+} \omega(\lambda;\delta) = 0$ may be not true. In [28], Yüksel and Ispir defined the following weighted modulus of continuity:

$$\Omega(\lambda;\delta) = \sup_{z\in\mathbb{R}_+, 0<h\le\delta} \frac{|\lambda(z+h)-\lambda(z)|}{(1+z^2)(1+h^2)} \text{ for } \lambda \in C_2^0(\mathbb{R}_+) \quad (89)$$

and proved the properties of monotone increasing about $\Omega(\lambda;\delta)$ as $\delta > 0$, $\lim_{\delta\longrightarrow 0^+} \Omega(\lambda;\delta) = 0$, and the inequality

$$\Omega(\lambda;\tau\delta) \le 2(1+\tau)(1+\delta^2)\Omega(\lambda;\delta), \tau > 0. \quad (90)$$

For any $\lambda \in C_2^0(\mathbb{R}_+)$, it follows from (89) and (90) that

$$\begin{aligned} |\lambda(t)-\lambda(z)| &\le (1+(t-z)^2)(1+z^2)\Omega(\lambda;|t-z|) \\ &\le 2\left(1+\frac{|t-z|}{\delta}\right)(1+\delta^2)\Omega(\lambda;\delta)(1+(t-z)^2)(1+z^2). \end{aligned} \quad (91)$$

In the next theorem, we obtain the degree of approximation of λ by the operators (2) in the weighted space of continuous functions $C_2^0(\mathbb{R}_+)$ in terms of the weighted modulus of smoothness $\Omega(\lambda;\delta)$, $\delta > 0$.

6.1. Quantitative Voronovskaya-Type Theorem

Theorem 19. *If $\lambda \in C_2^0(\mathbb{R}_+)$ satisfies $\lambda', \lambda'' \in C_2^0(\mathbb{R}_+)$, then for sufficiently large l and any $z \in \mathbb{R}_+$,*

$$l\left|(Y_l\lambda(t))(z)-\lambda(z)-\lambda'(z)A_l(z)-\frac{\lambda''(z)}{2!}B_l(z)\right| \le O(1)\Omega\left(\lambda'';\frac{1}{\sqrt{l}}\right). \quad (92)$$

Proof. By Taylors' expansion formula for λ, we have

$$\begin{aligned} \lambda(t) &= \lambda(z)+\lambda'(z)(t-z)+\frac{\lambda''(y)}{2!}(t-z)^2 \\ &= \lambda(z)+\lambda'(z)(t-z)+\frac{\lambda''(z)}{2!}(t-z)^2+R_1(t,z), \end{aligned} \quad (93)$$

where $|y-z| \le |t-z|$ and hence

$$R_1(t,z) = \frac{\lambda''(y)-\lambda''(z)}{2!}(t-z)^2. \quad (94)$$

Applying the inequality (91) of the weighted modulus of continuity, we have

$$\begin{aligned} |\lambda''(y)-\lambda''(z)| &\le (1+(y-z)^2)(1+z^2)\Omega(\lambda'';|y-z|) \\ &\le (1+(t-z)^2)(1+z^2)\Omega(\lambda'';|t-z|) \\ &\le 2\left(1+\frac{|t-z|}{\delta}\right)(1+\delta^2)\Omega(\lambda'';\delta) \\ &\quad \cdot (1+(t-z)^2)(1+z^2) \\ &\le \begin{cases} 4(1+\delta^2)^2(1+z^2)\Omega(\lambda'';\delta), & |t-z|\le\delta, \\ 4(1+\delta^2)^2(1+z^2)\Omega(\lambda'';\delta)\dfrac{(t-z)^4}{\delta^4}, & |t-z|>\delta, \end{cases} \\ &\le 4(1+\delta^2)^2(1+z^2)\Omega(\lambda'';\delta)\left(1+\frac{(t-z)^4}{\delta^4}\right). \end{aligned} \quad (95)$$

Combining (94) and (95) and choosing $\delta \in (0,1)$, we have

$$|R_1(t,z)| \le 2(1+\delta^2)^2(1+z^2)\Omega(\lambda'';\delta)\left(1+\frac{(t-z)^4}{\delta^4}\right)(t-z)^2. \quad (96)$$

Using the operator (2) and Lemma 4 on both sides of (94), we have

$$\begin{aligned} &\left|(Y_l\lambda(t))(z)-\lambda(z)-\lambda'(z)A_l(z)-\frac{\lambda''(z)}{2!}B_l(z)\right| \\ &\quad \le (Y_l|R_1(t,z)|)(z). \end{aligned} \quad (97)$$

Applying (16), (18), and (96), we have

$$\begin{aligned} (Y_l|R_1(t,z)|)(z) &\le 2(1+\delta^2)^2(1+z^2)\Omega(\lambda'';\delta) \\ &\quad \cdot \left(Y_l\left((t-z)^2+\frac{(t-z)^6}{\delta^4}\right)\right)(z) \\ &\le 2(1+\delta^2)^2(1+z^2)\Omega(\lambda'';\delta) \\ &\quad \cdot \left(B_l(z)+\frac{1}{\delta^4}(Y_l(t-z)^6)(z)\right) \\ &\le 2(1+\delta^2)^2(1+z^2)\Omega(\lambda'';\delta) \\ &\quad \cdot \left(O\left(\frac{1}{l}\right)+\frac{1}{\delta^4}O\left(\frac{1}{l^3}\right)\right). \end{aligned} \quad (98)$$

Choosing $\delta = 1/\sqrt{l}$, we have

$$l(Y_l|R_1(t,z)|)(z) \le O(1)\Omega\left(\lambda'';\frac{1}{\sqrt{l}}\right). \quad (99)$$

Combining (97)-(99), we complete the proof of Theorem 19.

6.2. Grüss Voronovskaya-Type Theorem

Theorem 20. *If $\lambda, \mu \in C_2^0(\mathbb{R}_+)$ satisfy $\lambda\mu, \lambda', \mu', (\lambda\mu)', \lambda'', \mu''$ and $(\lambda\mu)'' \in C_2^0(\mathbb{R}_+)$. Then, for any $z \in \mathbb{R}_+$,*

$$\lim_{l\longrightarrow\infty} l((Y_l(\lambda\cdot\mu)(t))(z) - (Y_l\lambda(t))(z)\cdot(Y_l\mu(t))(z)) = 6\lambda'(z)\mu'(z)z^2. \tag{100}$$

Proof. Using the equalities

$$\begin{aligned}(\lambda\cdot\mu)(z) &= \lambda(z)\cdot\mu(z), (\lambda\cdot\mu)'(z) = \lambda'(z)\cdot\mu(z) + \lambda(z)\cdot\mu'(z),\\ (\lambda\cdot\mu)''(z) &= \lambda''(z)\cdot\mu(z) + 2\lambda'(z)\cdot\mu'(z) + \lambda(z)\cdot\mu''(z),\end{aligned} \tag{101}$$

by simple computations, for any $z \in \mathbb{R}_+$, we have

$$\begin{aligned}&(Y_l(\lambda\cdot\mu)(t))(z) - (Y_l\lambda(t))(z)\cdot(Y_l\mu(t))(z)\\ &\quad= \left\{(Y_l(\lambda\cdot\mu)(t))(z) - (\lambda\cdot\mu)(z) - (\lambda\cdot\mu)'(z)A_l(z) - \frac{(\lambda\cdot\mu)''(z)}{2!}B_l(z)\right\}\\ &\quad- \mu(z)\left\{(Y_l\lambda(t))(z) - \lambda(z) - \lambda'(z)A_l(z) - \frac{\lambda''(z)}{2!}B_l(z)\right\}\\ &\quad- (Y_l\lambda(t))(z)\left\{(Y_l\mu(t))(z) - \mu(z) - \mu'(z)A_l(z) - \frac{\mu''(z)}{2!}B_l(z)\right\}\\ &\quad+ \frac{1}{2!}B_l(z)\left\{\lambda(z)\cdot\mu''(z) + 2\lambda'(z)\cdot\mu'(z) - \mu''(z)\cdot(Y_l\lambda(t))(z)\right\}\\ &\quad+ A_l(z)\left\{\lambda(z)\cdot\mu'(z) - \mu'(z)\cdot(Y_l\lambda(t))(z)\right\}.\end{aligned} \tag{102}$$

By using (16), Lemma 5, and Theorem 19, we have

$$\lim_{l\longrightarrow\infty} l((Y_l(\lambda\cdot\mu)(t))(z) - (Y_l\lambda(t))(z)\cdot(Y_l\mu(t))(z)) = 6\lambda'(z)\mu'(z)z^2, \tag{103}$$

which proves our theorem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Approximation by One and Two Variables of the Bernstein-Schurer-Type Operators and Associated GBS Operators on Symmetrical Mobile Interval

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In this article, we purpose to study some approximation properties of the one and two variables of the Bernstein-Schurer-type operators and associated GBS (Generalized Boolean Sum) operators on a symmetrical mobile interval. Firstly, we define the univariate Bernstein-Schurer-type operators and obtain some preliminary results such as moments, central moments, in connection with a modulus of continuity, the degree of convergence, and Korovkin-type approximation theorem. Also, we derive the Voronovskaya-type asymptotic theorem. Further, we construct the bivariate of this newly defined operator, discuss the order of convergence with regard to Peetre's K-functional, and obtain the Voronovskaya-type asymptotic theorem. In addition, we consider the associated GBS-type operators and estimate the order of approximation with the aid of mixed modulus of smoothness. Finally, with the help of the Maple software, we present the comparisons of the convergence of the bivariate Bernstein-Schurer-type and associated GBS operators to certain functions with some graphical illustrations and error estimation tables.

1. Introduction

In [1], Bernstein suggested his polynomials that still inspire many studies today as follows:

$$B_r(\mu\,;x)=\sum_{j=0}^{r}\binom{r}{j}x^j(1-x)^{r-j}\mu\left(\frac{j}{r}\right),\quad x\in[0,1],\tag{1}$$

for any $r\in\mathbb{N}$ and any $\mu\in C[0,1]$.

In 1962, the operators $S_r(\mu\,;x)$: $C[0,p+1]\longrightarrow C[0,1]$, which are called Bernstein-Schurer, are proposed by Schurer [2] as follows:

$$S_r(\mu\,;x)=\sum_{j=0}^{r+p}\binom{r+p}{j}x^j(1-x)^{r+p-j}\mu\left(\frac{j}{r}\right),\quad x\in[0,1],\tag{2}$$

for any $r\in\mathbb{N}$ and $\mu\in C[0,p+1]$.

Very recently, many modifications and generalizations of the Bernstein or Bernstein-Schurer operators for univariate and bivariate cases are discussed by many authors. For instance, Acar et al. [3] established local and global approximation results in terms of modulus of continuity for a new type of the Bernstein-Durrmeyer operators on mobile interval. Izgi [4] presented a new type of the Bernstein polynomials and studied several approximation results of the univariate and bivariate of these operators. For the parameter $\alpha\in\mathbb{R}$, Chen et al. [5] defined a new generalization of the Bernstein operator and derived the order of convergence and Voronovskaya-type asymptotic relation for the $\alpha-$ Bernstein operator. Kajla and Acar [6] constructed a new kind of the $\alpha-$ Bernstein operator and studied a uniform convergence estimate, some direct results involving the asymptotic theorems for these operators. Acar et al. [7] introduced the Kantorovich modifications of the $(p,q)-$ Bernstein operators for bivariate functions using a new $(p,q)-$ integral and obtained the uniform convergence and rate of approximation in terms of modulus of continuity for these operators. Further, for $\lambda \in [-1, 1]$, Cai [8] introduced the Bézier version of the Kantorovich-type $\lambda-$Bernstein polynomials and gained the global and direct approximation theorems. Acar and Kajla [9] introduced an extension of the bivariate generalized Bernstein operators with nonnegative real parameters and studied the degree of approximation with regard to Peetre's K-functional and Lipschitz-type functions. Bărbosu [10] demonstrated the uniform convergence and estimated the degree of approximation of the bivariate of the Bernstein-Schurer operators. Căbulea [11] considered the generalizations of the Kantorovich and Durrmeyer type of the Bernstein-Schurer operators and evaluated in connection with the modulus of continuity the order of approximation of these operators. Also, for some recent works, we can refer the readers to ([12–23]).

By the motivation of the all the above-mentioned works, we define the univariate Bernstein-Schurer-type operators on a symmetrical mobile interval. Let the intervals be $D_r = [-r/(r+1)-p, r/(r+1)+p]$, $I = [-1, 1]$, and $C(D_r)$ be the set of all continuous and bounded functions on D_r. For a function $\mu \in C(D_r)$ and $x \in D_r$, the univariate Bernstein-Schurer operators $F_r : C(D_r) \longrightarrow C(I)$ are defined as

$$F_r(\mu;x) = \left(\frac{r+1}{2r}\right)^{r+p} \sum_{j=0}^{r+p} \varphi_{r,j}^{p}(x)\mu\left(\frac{2j-r}{r+1}\right), \tag{3}$$

where $\in \mathbb{N}$, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\varphi_{r,j}^{p}(x) = \binom{r+p}{j} (r/(r+1)+x)^j (r/(r+1)-x)^{r+p-j}$.

The goal of the present work is to obtain some approximation features of the operators given by (3). We show the uniform convergence, estimate the degree of convergence with the help of modulus of continuity, and prove the Voronovskaya-type asymptotic theorem for the (3) operators. Next, we define the bivariate of (3) operators, compute the order of convergence by using Peetre's K-functional, and derive the Voronovskaya-type asymptotic theorem for the bivariate case. Further, we construct the associated GBS type of bivariate operators and estimate their degree of convergence in terms of mixed modulus of smoothness. Finally, by the help of the Maple software, we give comparisons of the convergence of bivariate of (3) operators and related GBS operators to the certain functions with some graphics and error estimation tables.

2. Main Results

Lemma 1. *Let the operators $F_r(\mu;x)$ be defined by (3). Then, for all $x \in D_r$, the following moments verify*

$$F_r(1;x) = 1, \tag{4}$$

$$F_r(t;x) = x + \frac{p((r+1)x+r)}{r(r+1)}, \tag{5}$$

$$F_r(t^2;x) = x^2 + \frac{(p^2+2pr-p-r)}{r^2}x^2 + \frac{2p(r+p)}{r(r+1)}x + \frac{p^2+p+r}{(r+1)^2}, \tag{6}$$

$$\begin{aligned} F_r(t^3;x) &= x^3 + \frac{(p^3+3pr^2+3p^2r-3p^2-3r^2-3pr+2p+2r)}{r^3}x^3 \\ &+ \frac{3p(r+p)(r+p-1)}{r^2(r+1)}x^2 + \frac{(r+p)(3p^2+3p+3r-2)}{r(r+1)^2}x \\ &+ \frac{p(p^2+3p+3r)}{(r+1)^3}, \end{aligned} \tag{7}$$

$$\begin{aligned} F_r(t^4;x) &= x^4 + \left(\frac{p^4+4p^3r+6p^2r^2+4pr^3-6p^3-6r^3-18pr^2-18p^2r}{r^4}\right. \\ &+ \left.\frac{11p^2+22pr+11r^2-6p-6r}{r^4}x^4\right) \\ &+ \frac{4(p-1)(r+p)(r+p-1)(r+p-2)}{r^3(r+1)}x^3 \\ &+ \frac{2(r+p)(r+p-1)(3p^2-3p-3r+8)}{r^2(r+1)^2}x^2 \\ &+ \frac{4(r+p)(p^3-3pr-3r^2+7p+9r-6)}{r(r+1)^3}x \\ &+ \frac{p^4+2p^3-6p^2r-12pr^2-4r^3+15p^2+30pr+15r^2-10p-10r}{(r+1)^4}. \end{aligned} \tag{8}$$

Proof. From (3), it becomes

$$\begin{aligned} F_r(1;x) &= \left(\frac{r+1}{2r}\right)^{r+p} \sum_{j=0}^{r+p} \binom{r+p}{j} \left(\frac{r}{r+1}+x\right)^j \left(\frac{r}{r+1}-x\right)^{r+p-j} \\ &= \left(\frac{r+1}{2r}\right)^{r+p} \left(\frac{2r}{r+1}\right)^{r+p} = 1, \end{aligned}$$

$$\begin{aligned} F_r(t;x) &= \left(\frac{r+1}{2r}\right)^{r+p} \sum_{j=0}^{r+p} \binom{r+p}{j} \left(\frac{r}{r+1}+x\right)^j \\ &\cdot \left(\frac{r}{r+1}-x\right)^{r+p-j} \left(\frac{2j-r}{r+1}\right) \\ &= \left(\frac{r+1}{2r}\right)^{r+p} \frac{2(r+p)}{r+1} \sum_{j=0}^{r+p-1} \binom{r+p-1}{j} \\ &\cdot \left(\frac{r}{r+1}+x\right)^{j+1} \left(\frac{r}{r+1}-x\right)^{r+p-j-1} - \frac{r}{r+1} \\ &\cdot \left(\frac{r+1}{2r}\right)^{r+p} \sum_{j=0}^{r+p} \binom{r+p}{j} \left(\frac{r}{r+1}+x\right)^j \left(\frac{r}{r+1}-x\right)^{r+p-j} \\ &= \frac{r+1}{2r}\frac{2(r+p)}{r+1}\left(\frac{r}{r+1}+x\right) - \frac{r}{r+1} = x + \frac{p((r+1)x+r)}{r(r+1)}, \end{aligned}$$

$$\begin{aligned}
F_r(t^2;x) &= \left(\frac{r+1}{2r}\right)^{r+p} \sum_{j=0}^{r+p} \binom{r+p}{j} \left(\frac{r}{r+1}+x\right)^{j} \\
&\quad \cdot \left(\frac{r}{r+1}-x\right)^{r+p-j} \left(\frac{2j-r}{r+1}\right)^{2} \\
&= \left(\frac{r+1}{2r}\right)^{r+p} \frac{4(r+p)(r+p-1)}{(r+1)^2} \sum_{j=0}^{r+p-2} \binom{r+p-2}{j} \\
&\quad \cdot \left(\frac{r}{r+1}+x\right)^{j+2} \left(\frac{r}{r+1}-x\right)^{r+p-j-2} \\
&\quad + \left(\frac{r+1}{2r}\right)^{r+p} \frac{4(r+p)}{(r+1)^2} \sum_{j=0}^{r+p-1} \binom{r+p-1}{j} \\
&\quad \cdot \left(\frac{r}{r+1}+x\right)^{j+1} \left(\frac{r}{r+1}-x\right)^{r+p-j-1} - \left(\frac{r+1}{2r}\right)^{r+p} \\
&\quad \cdot \frac{4r(r+p)}{(r+1)^2} \sum_{j=0}^{r+p-1} \binom{r+p-1}{j} \left(\frac{r}{r+1}+x\right)^{j+1} \\
&\quad \cdot \left(\frac{r}{r+1}-x\right)^{r+p-j-1} + \left(\frac{r+1}{2r}\right)^{r+p} \left(\frac{r}{r+1}\right)^{2} \sum_{j=0}^{r+p} \binom{r+p}{j} \\
&\quad \cdot \left(\frac{r}{r+1}+x\right)^{j} \left(\frac{r}{r+1}-x\right)^{r+p-j} \\
&= \left(\frac{r+1}{2r}\right)^{2} \frac{4(r+p)(r+p-1)}{(r+1)^2} \left(\frac{r}{r+1}+x\right)^{2} \\
&\quad + \left(\frac{r+1}{2r}\right) \frac{4(r+p)}{(r+1)^2} \left(\frac{r}{r+1}+x\right) - \left(\frac{r+1}{2r}\right) \frac{4r(r+p)}{(r+1)^2} \\
&\quad \cdot \left(\frac{r}{r+1}+x\right) + \left(\frac{r}{r+1}\right)^{2} \\
&= x^2 + \frac{(p^2+2pr-p-r)}{r^2} x^2 + \frac{2p(r+p)}{r(r+1)} x + \frac{p^2+p+r}{(r+1)^2}.
\end{aligned} \tag{9}$$

The last two identities can be obtained by applying similar methods; hence, we have omitted the details.

Lemma 2. *For all $x \in D_r$, we obtain the following central moments:*

$$\begin{aligned}
F_r(t-x;x) &= \frac{p((r+1)x+r)}{r(r+1)}, \\
F_r((t-x)^2;x) &= \frac{(p^2-p-r)}{r^2} x^2 + \frac{2p^2}{r(r+1)} x + \frac{p^2+p+r}{(r+1)^2},
\end{aligned}$$

$$\begin{aligned}
&F_r((t-x)^4;x) \\
&= \frac{(p^4-6p^3-6p^2r-12pr^2+11p^2+14pr+3r^2-6p-6r)}{r^4} x^4 \\
&\quad + \frac{4(p^4-4p^3-6p^2r-3pr^2+5p^2+8pr+3r^2-2p-2r)}{r^3(r+1)} x^3 \\
&\quad + \frac{2(3p^4-6p^3-18p^2-18pr^2+11p^2+26pr+15r^2-8p-8r)}{r^2(r+1)^2} x^2 \\
&\quad + \frac{4(p^4-6p^2r-9pr^2+7p^2+16pr+9r^2-6p-6r)}{r(r+1)^3} x \\
&\quad + \frac{p^4+2p^3-6p^2r-12pr^2+15p^2+30pr+15r^2-10p-10r}{(r+1)^4}.
\end{aligned} \tag{10}$$

Proof. The proof of this lemma can be directly obtained by using the linearity of (3) operators and as a consequence of Lemma 1.

Corollary 3. *For all $x \in D_r$, the following identities hold:*

$$\begin{aligned}
\lim_{r\to\infty} r(F_r(t-x;x)) &= p(x+1), \\
\lim_{r\to\infty} r(F_r((t-x)^2;x)) &= 1-x^2, \\
\lim_{r\to\infty} r^2(F_r((t-x)^4;x)) &= 3(1-3x^2)(x^2+1).
\end{aligned} \tag{11}$$

In the next theorem we show the uniform convergence of (3) operators. As it is known, the space $C(D_r)$ denotes the real-valued continuous functions on D_r, and it is equipped with the norm for a function μ as follows:

$$\|\mu\|_{C(D_r)} = \sup_{x\in D_r} |\mu(x)|. \tag{12}$$

Theorem 4. *Let the operators $F_r(\mu;x)$ be given by (3). Then, for all $x \in D_r$, $F_r(\mu;x)$ converges to μ uniformly on D_r.*

Proof. From (4), it is obvious that

$$\lim_{r\to\infty} \|F_r(1;.)-1\|_{C(D_r)} = 0. \tag{13}$$

By (5), we arrive

$$\begin{aligned}
\lim_{r\to\infty} \|F_r(t;.)-x\|_{C(D_r)} &= \lim_{r\to\infty} \max_{x\in D} \left| px\frac{(2r+1)}{r(r+1)} \right| \\
&< \lim_{r\to\infty} \left| \frac{p(pr+p+2r)}{r(r+1)} \right| = 0.
\end{aligned} \tag{14}$$

Similarly, using (6), then

$$\begin{aligned}
&\lim_{r\to\infty} \left\| F_r(t^2;.)-x^2 \right\|_{C(D_r)} \\
&= \lim_{r\to\infty} \max_{x\in D_r} \left| \frac{(p^2-p-r)}{r^2} x^2 + \frac{2p^2}{r(r+1)} x + \frac{p^2+p+r}{(r+1)^2} \right| \\
&< \lim_{r\to\infty} \left| \frac{p^2-3p+r+4}{(r+1)^2} \right| = 0.
\end{aligned} \tag{15}$$

Hence, according to the Korovkin theorem [24], the (3) operators converge uniformly to μ on D_r.

Further, we will obtain the degree of approximation of (3) operators. Let the modulus of continuity for a function $\mu \in C[a,b]$ be given by

$$\omega(\mu,\delta) = \sup_{\substack{t,y\in[a,b] \\ |t-y|\le\delta}} |\mu(t)-\mu(y)|. \tag{16}$$

Since $\delta > 0$, $\omega(\mu,\delta)$ has some useful properties which can be found in [25].

Theorem 5. *Let $\mu \in C(D_r)$. Then, for every $x \in D_r$, the following inequality is verified:*

$$|F_r(\mu ; x) - \mu(x)| \leq 2\omega(\mu ; \gamma_r(x)), \tag{17}$$

where $\gamma_r(x) := \sqrt{F_r((t-x)^2 ; x)}$.

Proof. Taking into account the following common property of the modulus of continuity:

$$|\mu(t) - \mu(x)| \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega(\mu ; \delta), \tag{18}$$

by the linearity of the operator (3), then

$$|F_r(\mu ; x) - \mu(x)| \leq \left(1 + \frac{1}{\delta} F_r(|t-x| ; x)\right)\omega(\mu ; \delta). \tag{19}$$

Utilizing the Cauchy-Schwarz inequality yields

$$\begin{aligned} |F_r(\mu ; x) - \mu(x)| &\leq \left(1 + \frac{1}{\delta}\sqrt{F_r\left((t-x)^2 ; x\right)}\right)\omega(\mu ; \delta) \\ &\leq \left(1 + \frac{\gamma_r(x)}{\delta}\right)\omega(\mu ; \delta). \end{aligned} \tag{20}$$

If we take $\delta = \gamma_r(x) := \sqrt{F_r((t-x)^2 ; x)}$, thus Theorem 5 is proven.

Theorem 6. *Let the operators $F_r(\mu ; x)$ be given by (3). Then, for any $\mu \in C(D_r)$ such that μ', $\mu'' \in C(D_r)$, the following identity holds:*

$$\lim_{r \longrightarrow \infty} r(F_r(\mu ; x) - \mu(x)) = p(x+1)\mu'(x) + \frac{1}{2}\left(1 - x^2\right)\mu''(x), \tag{21}$$

uniformly on D_r.

Proof. Suppose that $\mu, \mu', \mu'' \in C(D_r)$ and $x \in D_r$ are fixed. By the Taylor formula, hence

$$\mu(t) = \mu(x) + \mu'(x)(t-x) + \frac{1}{2}\mu''(x)(t-x)^2 + \xi(t ; x)(t-x)^2, \tag{22}$$

where $\xi(t ; x)$ is a form of Peano of the rest term, and since $\xi(.;x) \in C(D_r)$, $\lim_{t \longrightarrow x} \xi(t ; x) = 0$.

Operating $F_r(.;x)$ to (22), then

$$\begin{aligned} F_r(\mu ; x) - \mu(x) &= \mu'(x) F_r(t-x ; x) + \frac{1}{2}\mu''(x) F_r\left((t-x)^2 ; x\right) \\ &\quad + F_r\left(\xi(t ; x)(t-x)^2 ; x\right). \end{aligned} \tag{23}$$

From Lemma 2, it becomes

$$\begin{aligned} \lim_{r \longrightarrow \infty} r(F_r(\mu ; x) - \mu(x)) &= p(x+1)\mu'(x) + \frac{1}{2}\left(1 - x^2\right)\mu''(x) \\ &\quad + \lim_{r \to \infty} r\left(F_r\left(\xi(t ; x)(t-x)^2 ; x\right)\right). \end{aligned} \tag{24}$$

Applying the Cauchy-Schwarz inequality, one has

$$r\left(F_r\left(\xi(t ; x)(t-x)^2 ; x\right)\right) \leq \sqrt{F_r\left(\xi^2(t ; x) ; x\right)}\sqrt{r^2 F_r\left((t-x)^4 ; x\right)}. \tag{25}$$

Owing to $\xi(t ; x) \in C(D_r)$, $\lim_{t \longrightarrow x} \xi(t ; x) = 0$, thus

$$\lim_{r \longrightarrow \infty} F_r\left(\xi^2(t ; x) ; x\right) = 0, \tag{26}$$

uniformly on D_r with Theorem 4.

Combining (25) and (26) and by Lemma 2, we get

$$\lim_{r \longrightarrow \infty} r\left(F_r\left(\xi(t ; x)(t-x)^2 ; x\right)\right) = 0. \tag{27}$$

Hence,

$$\lim_{r \longrightarrow \infty} r(F_r(\mu ; x) - \mu(x)) = p(x+1)\mu'(x) + \frac{1}{2}\left(1 - x^2\right)\mu''(x), \tag{28}$$

which gives the proof.

3. Construction of the Bivariate Bernstein-Schurer-Type Operators

Let the intervals be $D_{r_1,r_2} = [(-r_1/r_1+1) - p_1, (r_1/r_1+1) + p_1] \times [(-r_2/r_2+1) - p_2, (r_2/r_2+1) + p_2]$, $I^2 = I \times I = [-1, 1] \times [-1, 1]$, and by $C(D_{r_1,r_2})$, we denote the set of all real-valued continuous functions on D_{r_1,r_2}, and it is equipped with the norm $\|\mu\|_{C(D_{r_1,r_2})} = \sup_{(x,y) \in D_{r_1,r_2}} |\mu(x, y)|$.

We define the bivariate $F_{r_1,r_2} : C(D_{r_1,r_2}) \longrightarrow C(I^2)$ of operators given by (3) as follows:

$$\begin{aligned} F_{r_1,r_2}(\mu ; x, y) = \left(\frac{r_1+1}{2r_1}\right)^{r_1+p_1} \left(\frac{r_2+1}{2r_2}\right)^{r_2+p_2} \sum_{j_1=0}^{r_1+p_1} \sum_{j_2=0}^{r_2+p_2} \varphi^{p_1,p_2}_{r_1,r_2,j_1,j_2} \\ \cdot (x, y)\mu\left(\frac{2j_1 - r_1}{r_1+1}, \frac{2j_2 - r_2}{r_2+1}\right), \end{aligned} \tag{29}$$

where $\varphi^{p_1,p_2}_{r_1,r_2,j_1,j_2}(x, y) = \binom{r_1+p_1}{j_1}\binom{r_2+p_2}{j_2}(r_1/(r_1+1) + x)^{j_1}\,(r_2/(r_2+1) + y)^{j_2}\,(r_1/(r_1+1) - x)^{r_1+p_1-j_1}\,(r_2/(r_2+1) - y)^{r_2+p_2-j_2}$, $(r_1, r_2) \in \mathbb{N} \times \mathbb{N}$, $(p_1, p_2) \in \mathbb{N}_0 \times \mathbb{N}_0$, and $(x, y) \in D_{r_1,r_2}$.

It can be seen that the operators given by (29) are positive and linear.

Lemma 7. *Let $e_{a_1,a_2}(x,y)=x^{a_1}y^{a_2}$, $(x,y)\in D_{r_1,r_2}$, $(a_1,a_2)\in\mathbb{N}_0\times\mathbb{N}_0$ with $a_1+a_2\leq 4$, be the bivariate test functions. Then, one has*

$$F_{r_1,r_2}(e_{0,0};x,y)=1,$$

$$F_{r_1,r_2}(e_{1,0};x,y)=x+\frac{p_1((r_1+1)x+r_1)}{r_1(r_1+1)},$$

$$F_{r_1,r_2}(e_{0,1};x,y)=y+\frac{p_2((r_2+1)y+r_2)}{r_2(r_2+1)},$$

$$\begin{aligned}&F_{r_1,r_2}(e_{2,0};x,y)\\&\quad=x^2+\frac{(p_1^2+2p_1r_1-p_1-r_1)}{r_1^2}x^2+\frac{2p_1(r_1+p_1)}{r_1(r_1+1)}x+\frac{p_1^2+p_1+r_1}{(r_1+1)^2},\end{aligned}$$

$$\begin{aligned}&F_{r_1,r_2}(e_{0,2};x,y)\\&\quad=y^2+\frac{(p_2^2+2p_2r_2-p_2-r_2)}{r_2^2}y^2+\frac{2p_2(r_2+p_2)}{r_2(r_2+1)}y+\frac{p_2^2+p_2+r_2}{(r_2+1)^2},\end{aligned}$$

$$\begin{aligned}&F_{r_1,r_2}(e_{3,0};x,y)\\&\quad=x^3+\frac{(p_1^3+3p_1r_1^2+3p_1^2r_1-3p_1^2-3r_1^2-3p_1r_1+2p_1+2r_1)}{r_1^3}x^3\\&\qquad+\frac{3p_1(r_1+p_1)(r_1+p_1-1)}{r_1^2(r_1+1)}x^2+\frac{(r_1+p_1)(3p_1^2+3p_1+3r_1-2)}{r_1(r_1+1)^2}x\\&\qquad+\frac{p_1(p_1^2+3p_1+3r_1)}{(r_1+1)^3},\end{aligned}$$

$$\begin{aligned}&F_{r_1,r_2}(e_{0,3};x,y)\\&\quad=y^3+\frac{(p_2^3+3p_2r_2^2+3p_2^2r_2-3p_2^2-3r_2^2-3p_2r_2+2p_2+2r_2)}{r_2^3}y^3\\&\qquad+\frac{3p_2(r_2+p_2)(r_2+p_2-1)}{r_2^2(r_2+1)}y^2+\frac{(r_2+p_2)(3p_2^2+3p_2+3r_2-2)}{r_2(r_2+1)^2}y\\&\qquad+\frac{p_2(p_2^2+3p_2+3r_2)}{(r_2+1)^3},\end{aligned}$$

$$\begin{aligned}&F_{r_1,r_2}(e_{4,0};x,y)\\&\quad=x^4+\left(\frac{p_1^4+4p_1^3r_1+6p_1^2r_1^2+4p_1r_1^3-6p_1^3-6r_1^3-18p_1r_1^2-18p_1^2r_1}{r_1^4}\right.\\&\qquad\left.+\frac{11p_1^2+22p_1r_1+11r_1^2-6p_1-6r_1}{r_1^4}\right)x^4\\&\qquad+\frac{4(p_1-1)(r_1+p_1)(r_1+p_1-1)(r_1+p_1-2)}{r_1^3(r_1+1)}x^3\\&\qquad+\frac{2(r_1+p_1)(r_1+p_1-1)(3p_1^2-3p_1-3r_1+8)}{r_1^2(r_1+1)^2}x^2\\&\qquad+\frac{4(r_1+p_1)(p_1^3-3p_1r_1-3r_1^2+7p_1+9r_1-6)}{r_1(r_1+1)^3}x\\&\qquad+\frac{p_1^4+2p_1^3-6p_1^2r_1-12p_1r_1^2+-4r_1^3+15p_1^2+30p_1r_1+15r_1^2-10p_1-10r_1}{(r_1+1)^4},\end{aligned}$$

$$\begin{aligned}&F_{r_1,r_2}(e_{0,4};x,y)\\&\quad=y^4+\left(\frac{p_2^4+4p_2^3r_2+6p_2^2r_2^2+4p_2r_2^3-6p_2^3-6r_2^3-18p_2r_2^2-18p_2^2r_2}{r_2^4}\right.\\&\qquad\left.+\frac{11p_2^2+22p_2r_2+11r_2^2-6p_2-6r_2}{r_2^4}\right)y^4\\&\qquad+\frac{4(p_2-1)(r_2+p_2)(r_2+p_2-1)(r_2+p_2-2)}{r_2^3(r_2+1)}y^3\\&\qquad+\frac{2(r_2+p_2)(r_2+p_2-1)(3p_2^2-3p_2-3r_2+8)}{r_2^2(r_2+1)^2}y^2\\&\qquad+\frac{4(r_2+p_2)(p_2^3-3p_2r_2-3r_2^2+7p_2+9r_2-6)}{r_2(r_2+1)^3}y\\&\qquad+\frac{p_2^4+2p_2^3-6p_2^2r_2-12p_2r_2^2+-4r_2^3+15p_2^2+30p_2r_2+15r_2^2-10p_2-10r_2}{(r_2+1)^4}.\end{aligned}\tag{30}$$

Proof. The proof of the above equalities can be reached easily as a consequence of Lemma 1 and by (29); hence, we have omitted the details.

Corollary 8. *In view of Lemma 7, the following relations hold true:*

$$\begin{aligned}&F_{r_1,r_2}(t_0-x;x,y)=\frac{p_1((r_1+1)x+r_1)}{r_1(r_1+1)},\\&F_{r_1,r_2}(s_0-y;x,y)=\frac{p_2((r_2+1)y+r_2)}{r_2(r_2+1)},\\&F_{r_1,r_2}((t_0-x)^2;x,y)=\frac{(p_1^2-p_1-r_1)}{r_1^2}x^2+\frac{2p_1^2}{r_1(r_1+1)}x+\frac{p_1^2+p_1+r_1}{(r_1+1)^2},\\&F_{r_1,r_2}((s_0-y)^2;x,y)=\frac{(p_2^2-p_2-r_2)}{r_2^2}y^2+\frac{2p_2^2}{r_2(r_2+1)}y+\frac{p_2^2+p_2+r_2}{(r_2+1)^2},\\&F_{r_1,r_2}((t_0-x)^4;x,y)\\&\quad=\frac{(p_1^4-6p_1^3-6p_1^2r_1-12p_1r_1^2+11p_1^2+14p_1r_1+3r_1^2-6p_1-6r_1)}{r_1^4}x^4\\&\qquad+\frac{4(p_1^4-4p_1^3-6p_1^2r_1-3p_1r_1^2+5p_1^2+8p_1r_1+3r_1^2-2p_1-2r_1)}{r_1^3(r_1+1)}x^3\\&\qquad+\frac{2(3p_1^4-6p_1^3-18p_1^2-18p_1r_1^2+11p_1^2+26p_1r_1+15r_1^2-8p_1-8r_1)}{r_1^2(r_1+1)^2}x^2\\&\qquad+\frac{4(p_1^4-6p_1^2r_1-9p_1r_1^2+7p_1^2+16p_1r_1+9r_1^2-6p_1-6r_1)}{r_1(r_1+1)^3}x\\&\qquad+\frac{p_1^4+2p_1^3-6p_1^2r_1-12p_1r_1^2+15p_1^2+30p_1r_1+15r_1^2-10p_1-10r_1}{(r_1+1)^4},\\&F_{r_1,r_2}((s_0-y)^4;x,y)\\&\quad=\frac{(p_2^4-6p_2^3-6p_2^2r_2-12p_2r_2^2+11p_2^2+14p_2r_2+3r_2^2-6p_2-6r_2)}{r_2^4}y^4\\&\qquad+\frac{4(p_2^4-4p_2^3-6p_2^2r_2-3p_2r_2^2+5p_2^2+8p_2r_2+3r_2^2-2p_2-2r_2)}{r_2^3(r_2+1)}y^3\\&\qquad+\frac{2(3p_2^4-6p_2^3-18p_2^2-18p_2r_2^2+11p_2^2+26p_2r_2+15r_2^2-8p_2-8r_2)}{r_2^2(r_2+1)^2}y^2\\&\qquad+\frac{4(p_2^4-6p_2^2r_2-9p_2r_2^2+7p_2^2+16p_2r_2+9r_2^2-6p_2-6r_2)}{r_2(r_2+1)^3}y\\&\qquad+\frac{p_2^4+2p_2^3-6p_2^2r_2-12p_2r_2^2+15p_2^2+30p_2r_2+15r_2^2-10p_2-10r_2}{(r_2+1)^4}.\end{aligned}\tag{31}$$

Theorem 9. *Let the operators $F_{r_1,r_2}(\mu;x,y)$ be given by (29). Then for any $\mu\in C(D_{r_1,r_2})$, we arrive at*

$$\lim_{r_1,r_2\longrightarrow\infty}\left\|F_{r_1,r_2}(\mu)-\mu\right\|=0. \tag{32}$$

Proof. It is seen from the following that

$$\begin{aligned}&\left\|F_{r_1,r_2}(e_{0,0})-e_{0,0}\right\|\longrightarrow 0,\left\|F_{r_1,r_2}(e_{1,0})-e_{1,0}\right\|\longrightarrow 0,\\&\left\|F_{r_1,r_2}(e_{0,1})-e_{0,1}\right\|\longrightarrow 0,\left\|F_{r_1,r_2}(e_{2,0}+e_{0,2})-(e_{2,0}+e_{0,2})\right\|\longrightarrow 0,\end{aligned} \tag{33}$$

as $r_1,r_2\longrightarrow\infty$. Thus, these results complete the proof, as required by the Volkov theorem [26].

Moreover, for the operators given by (29), we want to derive the Voronovskaya-type asymptotic theorem and estimate the degree of convergence with the help of Peetre's K-functional.

Suppose that $C^2(D_{r_1,r_2})$ represents the space of all functions of $\mu\in C(D_{r_1,r_2})$ such that $\partial_i\mu/\partial x_i, \partial_i\mu/\partial y_i\in C(D_{r_1,r_2})$ (for $i=1,2$) . The norm on $C^2(D_{r_1,r_2})$ and Peetre's K-functional of $\mu\in C(D_{r_1,r_2})$ are given as follows, respectively.

$$\begin{aligned}&\|\mu\|_{C^2(D_{r_1,r_2})}=\|\mu\|_{C(D_{r_1,r_2})}+\sum_{i=1}^{2}\left(\left\|\frac{\partial_i\mu}{\partial x_i}\right\|_{C(D_{r_1,r_2})}+\left\|\frac{\partial_i\mu}{\partial y_i}\right\|_{C(D_{r_1,r_2})}\right),\\&K(\mu,\zeta)=\inf\left\{\|\mu-\varrho\|_{C(D_{r_1,r_2})}+\zeta\|\varrho\|_{C^2(D_{r_1,r_2})}:\varrho\in C^2\left(D_{r_1,r_2}\right)\right\},\end{aligned} \tag{34}$$

where $\zeta>0$.

For a constant $C>0$, the following inequality

$$K(\mu,\zeta)\le C\overset{*}{\omega}_2\left(\mu,\sqrt{\zeta}\right) \tag{35}$$

holds, where $\omega_2{}^*(\mu,\sqrt{\zeta})$ denotes the second order of the modulus of continuity of $\mu\in C(D_{r_1,r_2})$. Also, for $\mu\in C(D_{r_1,r_2})$, the ordinary modulus of continuity is defined as

$$\begin{aligned}\omega(\mu,\delta)=\sup&\left\{|\mu(u_1,u_2)-\mu(v_1,v_2)|:(u_1,u_2),(v_1,v_2)\right.\\&\left.\in C\left(D_{r_1,r_2}\right),\sqrt{(u_1-v_1)^2+(u_2-v_2)^2}\le\delta\right\}.\end{aligned} \tag{36}$$

Theorem 10. *Suppose that $\mu\in C^2(D_{r_1,r_2})$. Then, the following relation holds*

$$\begin{aligned}&\lim_{r_1\longrightarrow\infty}r_1\left[F_{r_1,r_1}(\mu;x,y)-\mu;x,y\right]\\&\quad=p_1(x+1)\mu_x'(x,y)+p_2(y+1)\mu_y'(x,y)+\frac{1}{2}\\&\qquad\times\left\{\left(1-x^2\right)\mu_{xx}''(x,y)+\left(1-y^2\right)\mu_{yy}''(x,y)\right\},\end{aligned} \tag{37}$$

uniformly on D_{r_1,r_2}.

Proof. For the arbitrary $(x,y)\in D_{r_1,r_2}$, using the Taylor formula, it becomes

$$\begin{aligned}\mu(t_0,s_0)&=\mu(x,y)+\mu_x'(x,y)(t_0-x)+\mu_y'(x,y)(s_0-y)+\frac{1}{2}\\&\cdot\left\{\mu_{xx}''(x,y)(t_0-x)^2+\mu_{yy}''(x,y)(s_0-y)^2+2\mu_{xy}''\right.\\&\left.\cdot(x,y)(t_0-x)(s_0-y)\right\}+\chi(t_0,s_0;x,y)\\&\cdot\sqrt{(t_0-x)^4+(s_0-y)^4},\end{aligned} \tag{38}$$

for $(t_0,s_0)\in D_{r_1,r_2}$, where $\chi(.,.;x,y)\in C(D_{r_1,r_2})$ and $\chi(t_0,s_0;x,y)\longrightarrow 0$, as $(t_0,s_0)\longrightarrow(x,y)$.

Operating $F_{r_1,r_1}(.;x,y)$ on (38) yields

$$\begin{aligned}&F_{r_1,r_1}(\mu(t_0,s_0);x,y)\\&\quad=\mu(x,y)+\mu_x'(x,y)F_{r_1,r_1}(t_0-x;x,y)+\mu_y'(x,y)F_{r_1,r_1}\\&\qquad\times(s_0-y;x,y)+\frac{1}{2}\left\{\mu_{xx}''(x,y)F_{r_1,r_1}\left((t_0-x)^2;x,y\right)\right.\\&\qquad+\mu_{yy}''(x,y)F_{r_1,r_1}\left((s_0-y)^2;x,y\right)+2\mu_{xy}''(x,y)F_{r_1,r_1}\\&\qquad\left.\times((t_0-x)(s_0-y);x,y)\right\}+F_{r_1,r_1}\\&\qquad\times\left(\chi(t_0,s_0;x,y)\sqrt{(t_0-x)^4+(s_0-y)^4};x,y\right).\end{aligned} \tag{39}$$

If we use the Cauchy-Schwarz inequality to the last part of (39), one has

$$\begin{aligned}&\left|F_{r_1,r_1}\left(\chi(t_0,s_0;x,y)\sqrt{(t_0-x)^4+(s_0-y)^4};x,y\right)\right|\\&\quad\le\sqrt{F_{r_1,r_1}\left(\chi^2(t_0,s_0;x,y);x,y\right)}\\&\qquad\times\sqrt{F_{r_1,r_1}\left((t_0-x)^4;x,y\right)+F_{r_1,r_1}\left((s_0-y)^4;x,y\right)}.\end{aligned} \tag{40}$$

Considering Theorem 9 and because of $\chi(.,.;x,y)\in C(D_{r_1,r_2})$, $\chi(t_0,s_0;x,y)\longrightarrow 0$, as $(t_0,s_0)\longrightarrow(x,y)$, then

$$\lim_{r_1\longrightarrow\infty}F_{r_1,r_1}\left(\chi^2(t_0,s_0;x,y);x,y\right)=0, \tag{41}$$

uniformly on D_{r_1,r_2}. Also, from Corollary 8, it is clear to see

$$\lim_{r_1\longrightarrow\infty}r_1\left[F_{r_1,r_1}((t_0-x)(s_0-y);x,y)\right]=0. \tag{42}$$

Further, by Corollary 3, it follows that

$$\begin{aligned}&\lim_{r_1\longrightarrow\infty}r_1^2\left[F_{r_1,r_1}\left((t_0-x)^4;x,y\right)\right]=3\left(1-3x^2\right)\left(x^2+1\right),\\&\lim_{r_1\longrightarrow\infty}r_1^2\left[F_{r_1,r_1}\left((s_0-y)^4;x,y\right)\right]=3\left(1-3y^2\right)\left(y^2+1\right).\end{aligned} \tag{43}$$

Thus,

$$\lim_{r_1\longrightarrow\infty} r_1\left[F_{r_1,r_1}\left(\chi(t_0,s_0;x,y)\sqrt{(t_0-x)^4+(s_0-y)^4};x,y\right)\right]=0. \tag{44}$$

Hence, the desired sequel is arrived as follows:

$$\begin{aligned}\lim_{r_1\longrightarrow\infty} r_1\left[F_{r_1,r_1}(\mu;x,y)-\mu;x,y\right] &= p_1(x+1)\mu'_x(x,y)+p_2(y+1)\mu'_y(x,y)+\frac{1}{2}\Big\{\left(1-x^2\right)\mu''_{xx}\\ &\quad\times(x,y)+\left(1-y^2\right)\mu''_{yy}(x,y)\Big\},\end{aligned} \tag{45}$$

uniformly on D_{r_1,r_2}.

Theorem 11. *Suppose that $\mu\in C(D_{r_1,r_2})$. Then, the following inequality is verified:*

$$\begin{aligned}&\left|F_{r_1,r_2}(\mu;x,y)-\mu(x,y)\right|\\ &\quad\le C_1\left\{\overset{*}{\omega}_2\left(\mu;\sqrt{S_{r_1,r_2}(x,y)}\right)+\min\left\{1,S_{r_1,r_2}(x,y)\|\mu\|_{C^2(D_{r_1,r_2})}\right\}\right.\\ &\qquad\left.+\,\omega\left(\mu;\xi_{r_1,r_2}(x,y)\right)\right),\end{aligned} \tag{46}$$

where $C_1>0$ is a constant and independent of μ and $S_{r_1,r_2}(x,y)$ and $\gamma^2_{r_1}(x)=F_{r_1,r_2}((t_0-x)^2;x,y)$, $\gamma^2_{r_2}(y)=F_{r_1,r_2}((s_0-y)^2;x,y)$, $\xi_{r_1,r_2}(x,y)=\sqrt{(p_1((r_1+1)x+r_1)/r_1(r_1+1))^2+(p_2((r_2+1)y+r_2)/r_2(r_2+1))^2}$, $S_{r_1,r_2}(x,y)=\gamma^2_{r_1}(x)+\gamma^2_{r_2}(y)+\xi^2_{r_1,r_2}$.

Proof. Let us define the following auxiliary operators:

$$\begin{aligned}F^{-}_{r_1,r_2}(\mu;x,y)&=F_{r_1,r_2}(\mu;x,y)+\mu(x,y)-\mu\\ &\quad\cdot\left(x+\frac{p_1((r_1+1)x+r_1)}{r_1(r_1+1)},y+\frac{p_2((r_2+1)y+r_2)}{r_2(r_2+1)}\right).\end{aligned} \tag{47}$$

By Lemma 7, we obtain

$$\begin{aligned}F^{-}_{r_1,r_2}(t_0-x;x,y)&=0,\\ F^{-}_{r_1,r_2}(s_0-y;x,y)&=0.\end{aligned} \tag{48}$$

Suppose that $\mu\in C^2(D_{r_1,r_2})$ and $(t_0,s_0)\in D_{r_1,r_2}$; by the Taylor formula, we may write

$$\begin{aligned}\mu(t_0,s_0)-\mu(x,y)&=\frac{\partial\mu(x,y)}{\partial x}(t_0-x)+\int_x^{t_0}(t_0-u)\frac{\partial^2\mu(u,y)}{\partial^2u}\,du\\ &\quad+\frac{\partial\mu(x,y)}{\partial y}(s_0-y)+\int_y^{s_0}(s_0-v)\frac{\partial^2\mu(x,v)}{\partial^2 v}\,dv.\end{aligned} \tag{49}$$

Now, operating $F^{-}_{r_1,r_2}$ on (49), we get

$$\begin{aligned}&F^{-}_{r_1,r_2}(\mu;x,y)-\mu(x,y)\\ &\quad=F^{-}_{r_1,r_2}\left(\int_x^{t_0}(t_0-u)\frac{\partial^2\mu(u,y)}{\partial^2u}du;x,y\right)\\ &\qquad+F^{-}_{r_1,r_2}\left(\int_y^{s_0}(s_0-v)\frac{\partial^2\mu(x,v)}{\partial^2v}dv;x,y\right)\\ &\quad=F_{r_1,r_2}\left(\int_x^{t_0}(t_0-u)\frac{\partial^2\mu(u,y)}{\partial^2u}du;x,y\right)\\ &\qquad-\int_x^{x+(p_1((r_1+1)x+r_1)/r_1(r_1+1))}\left(x+\frac{p_1((r_1+1)x+r_1)}{r_1(r_1+1)}-u\right)\\ &\qquad\times\frac{\partial^2\mu(u,y)}{\partial^2u}du+F_{r_1,r_2}\left(\int_y^{s_0}(s_0-v)\frac{\partial^2\mu(x,v)}{\partial^2v}dv;x,y\right)\\ &\qquad-\int_y^{y+(p_2((r_2+1)y+r_2)/r_2(r_2+1))}\left(y+\frac{p_2((r_2+1)y+r_2)}{r_2(r_2+1)}-v\right)\\ &\qquad\times\frac{\partial^2\mu(x,v)}{\partial^2v}dv.\end{aligned} \tag{50}$$

Hence,

$$\begin{aligned}&\left|F_{r_1,r_2}(\mu;x,y)-\mu(x,y)\right|\\ &\quad\le F_{r_1,r_2}\left(\left|\int_x^{t_0}|t_0-u|\left|\frac{\partial^2\mu(u,y)}{\partial^2u}\right|du\right|;x,y\right)\\ &\qquad+\left|\int_x^{x+(p_1((r_1+1)x+r_1)/r_1(r_1+1))}\left|x+\frac{p_1((r_1+1)x+r_1)}{r_1(r_1+1)}-u\right|\right.\\ &\qquad\left.\times\left|\frac{\partial^2\mu(u,y)}{\partial^2u}\right|du\right|+F_{r_1,r_2}\left(\left|\int_y^{s_0}|s_0-v|\left|\frac{\partial^2\mu(x,v)}{\partial^2v}\right|dv\right|;x,y\right)\\ &\qquad+\left|\int_y^{y+(p_2((r_2+1)y+r_2)/r_2(r_2+1))}\left|y+\frac{p_2((r_2+1)y+r_2)}{r_2(r_2+1)}-v\right|\right.\\ &\qquad\left.\times\left|\frac{\partial^2\mu(x,v)}{\partial^2v}\right|dv\right|\\ &\quad\le\left\{F_{r_1,r_2}\left((t_0-x)^2;x,y\right)+\left(x+\frac{p_1((r_1+1)x+r_1)}{r_1(r_1+1)}-x\right)^2\right.\\ &\qquad\left.+F_{r_1,r_2}\left((s_0-y)^2;x,y\right)+\left(y+\frac{p_2((r_2+1)y+r_2)}{r_2(r_2+1)}-y\right)^2\right\}\\ &\qquad\times\|\mu\|_{C^2(D_{r_1,r_2})}.\end{aligned} \tag{51}$$

If we choose $\xi_{r_1,r_2}=\sqrt{(x+p_1((r_1+1)x+r_1)/r_1(r_1+1))^2+(p_2((r_2+1)y+r_2)/r_2(r_2+1))^2}$, $\gamma^2_{r_1}(x)=F_{r_1,r_2}((t_0-x)^2;x,y)$, $\gamma^2_{r_2}(y)=F_{r_1,r_2}((s_0-y)^2;x,y)$,

$S_{r_1,r_2}(x,y)=\gamma_{r_1}^2(x)+\gamma_{r_2}^2(y)+\xi_{r_1,r_2}^2$, then

$$\left|F_{r_1,r_2}(\mu;x,y)-\mu(x,y)\right|\le S_{r_1,r_2}(x,y)\|\mu\|_{C^2\left(D_{r_1,r_2}\right)}. \tag{52}$$

Also, by Lemma 7 and (52), we arrive at

$$\begin{aligned}\left|F^{-}_{r_1,r_2}(\mu;x,y)\right|&\le\left|F_{r_1,r_2}(\mu;x,y)\right|+|\mu(x,y)|\\&+\left|\mu\left(x+\frac{p_1((r_1+1)x+r_1)}{r_1(r_1+1)},y\right.\right.\\&\left.\left.+\frac{p_2((r_2+1)y+r_2)}{r_2(r_2+1)}\right)\right|\\&\le 3\|\mu\|_{C\left(D_{r_1,r_2}\right)}.\end{aligned} \tag{53}$$

Next, from (53)

$$\begin{aligned}&\left|F_{r_1,r_2}(\mu;x,y)-\mu(x,y)\right|\\&\le\left|F^{-}_{r_1,r_2}(\mu-\kappa;x,y)\right|+\left|F^{-}_{r_1,r_2}(\kappa;x,y)-\kappa(x,y)\right|\\&+\left|\mu\left(x+\frac{p_1((r_1+1)x+r_1)}{r_1(r_1+1)},y+\frac{p_2((r_2+1)y+r_2)}{r_2(r_2+1)}\right)\right.\\&\left.-\mu(x,y)\right|+|\kappa(x,y)-\mu(x,y|\\&\le\left(4\|\mu-\kappa\|_{C\left(D_{r_1,r_2}\right)}+S_{r_1,r_2}(x,y)\|\kappa\|_{C^2\left(D_{r_1,r_2}\right)}\right)\\&+\omega\left(\mu;\xi_{r_1,r_2}(x,y)\right).\end{aligned} \tag{54}$$

Consequently, on (54), utilizing the infimum on the right-hand side over all $\mu\in C^2(D_{r_1,r_2})$ and by (35), it becomes

$$\begin{aligned}&\left|F_{r_1,r_2}(\mu;x,y)-\mu(x,y)\right|\\&\le C_1\left\{\overset{*}{\omega}_2\left(\mu;\sqrt{S_{r_1,r_2}(x,y)}\right)+\min\left\{1,S_{r_1,r_2}(x,y)\|\mu\|_{C^2\left(D_{r_1,r_2}\right)}\right\}\right.\\&+\omega\left(\mu;\xi_{r_1,r_2}(x,y)\right),\end{aligned} \tag{55}$$

which ends the proof.

4. Construction of the GBS Type of $F_{r_1,r_2}(\mu;x,y)$

The notion of the B-continuous and B-differentiable functions was firstly used by Bögel [27, 28]. Dobrescu and Matei [29] considered the GBS (Generalized Boolean Sum) kind of the bivariate of the Bernstein polynomials. Next, using the B-continuous functions by the GBS operators, which is related to quantitative version of the Korovkin-type convergence theorem, firstly has been improved by Badea et al. [30, 31]. Pop and Fărcas [32] obtained some approximation of the B-continuous and B-differentiable functions by GBS type of the Bernstein bivariate operators. İspir [33] established quantitative estimates for the GBS of the Chlodowsky-Szász-kind operators. Recently, some authors introduced the GBS operators of various operators (we refer the readers to [34–42]).

Let a function $\mu:U\times V\longrightarrow\mathbb{R}$, where U,V are compact real intervals of $\mathbb{R}$. With $(x,y),(t_0,s_0)\in U\times V$, the mixed difference of the function μ is given as

$$\phi_{(x,y)}\mu[t_0,s_0;x,y]=\mu(x,y)-\mu(x,s_0)-\mu(t_0,y)+\mu(t_0,s_0). \tag{56}$$

A function $\mu:U\times V\longrightarrow\mathbb{R}$ is named Bögel-continuous (B-continuous) at $(t_0,s_0)\in U\times V$, if

$$\lim_{(x,y)\longrightarrow(t_0,s_0)}\phi_{(x,y)}\mu[t_0,s_0;x,y]=0. \tag{57}$$

A function $\mu:U\times V\longrightarrow\mathbb{R}$ is named Bögel-differentiable (B-differentiable) at $(t_0,s_0)\in U\times V$, if the below result which denoted by $D_Bf(x,y)$ exists and finite

$$\lim_{(x,y)\longrightarrow(t_0,s_0)}\frac{\phi_{(x,y)}\mu[t_0,s_0;x,y]}{(x-t_0)(y-s_0)}=D_Bf(x,y). \tag{58}$$

Note that, by $C_b(U\times V)$ and $D_b(U\times V)$, we represent the sets of all B-differentiable and B-continuous functions on $U\times V$, respectively.

A function $\mu:Y\subset U\times V\longrightarrow\mathbb{R}$ is named Bögel-bounded ($B-$bounded) on Y, if there consists $W>0$ such that $|\phi_{(x,y)}\mu[t_0,s_0;x,y]|\le W$ for every $(t_0,s_0),(x,y)\in Y$.

Also, if Y is a compact subset of $\mathbb{R}^2$, hence all B-continuous functions are $B-$bounded on $Y\longrightarrow\mathbb{R}$.

Further, by $B_b(Y)$, we represent the set of all B-bounded functions on Y and equipped with the norm $\|\mu\|_B=\sup_{(x,y),(t_0,s_0)\in Y}|\phi_{(x,y)}\mu[t_0,s_0;x,y])|$. Considering the definition of B-continuous, then $C(Y)\subset C_b(Y)$ (see details by [43]).

The mixed modulus of smoothness for $\mu\in C_b(D_{r_1,r_2})$ is given by

$$\begin{aligned}&\omega_{\text{mixed}}(\mu;\delta_1,\delta_2)\\&\coloneqq\sup\left\{\left|\phi_{(x,y)}\mu[t_0,s_0;x,y]\right|:|t_0-x|<\delta_1,|s_0-y|<\delta_2\right\},\end{aligned} \tag{59}$$

where $(x,y),(t_0,s_0)\in D_{r_1,r_2}$, and $\delta_1,\delta_2\in\mathbb{R}^+$. Also, for all κ_1, $\kappa_2\ge 0$, the following inequality satisfy

$$\omega_{\text{mixed}}(\mu;\kappa_1\delta_1,\kappa_2\delta_2)\le(1+\kappa_1)(1+\kappa_2)\omega_{\text{mixed}}(\mu;\delta_1,\delta_2). \tag{60}$$

More details about the mixed modulus of smoothness can be found in [30, 31].

Now, for all $(x,y)\in D_{r_1,r_2}$, for any $\mu\in C_b(D_{r_1,r_2})$ and $(r_1,r_2)\in\mathbb{N}\times\mathbb{N}$, $(p_1,p_2)\in\mathbb{N}_0\times\mathbb{N}_0$, we construct the GBS-type operators given by (29) operators as follows:

$$G_{r_1,r_2}(\mu;x,y)=F_{r_1,r_2}(\mu(x,s_0)+\mu(t_0,y)-\mu(t_0,s_0);x,y). \tag{61}$$

Exactly, for any $\mu \in C_b(D_{r_1,r_2})$ and for all $(x,y) \in D_{r_1,r_2}$, the GBS type of F_{r_1,r_2} operators is given as

$$G_{r_1,r_2}(\mu;x,y) = \left(\frac{r_1+1}{2r_1}\right)^{r_1+p_1}\left(\frac{r_2+1}{2r_2}\right)^{r_2+p_2}\sum_{j_1=0}^{r_1+p_1}\sum_{j_2=0}^{r_2+p_2}\varphi_{r_1,r_2,j_1,j_2}^{p_1,p_2}(x,y) \times \left(\mu\left(x,\frac{2j_2-r_2}{r_2+1}\right)+\mu\left(\frac{2j_1-r_1}{r_1+1},y\right) - \mu\left(\frac{2j_1-r_1}{r_1+1},\frac{2j_2-r_2}{r_2+1}\right)\right), \quad (62)$$

where $\varphi_{r_1,r_2,j_1,j_2}^{p_1,p_2}(x,y)$ is defined as in (29).

It is clear that the operators given by (62) are linear and positive. In the following theorem, with regard to the mixed modulus of smoothness, we estimate the order of approximation of the (62) operators.

Theorem 12. *For all $\mu \in C_b(D_{r_1,r_2})$ and for each $(x,y) \in D_{r_1,r_2}$, the operators given by (62) satisfy the following inequality:*

$$\left|G_{r_1,r_2}(\mu;x,y)-\mu(x,y)\right| \le 4\omega_{mixed}\left(\mu;\sqrt{(\alpha_{r_1,p_1})},\sqrt{(\alpha_{r_2,p_2})}\right), \quad (63)$$

where $\alpha_{r_1,p_1} = (p_1^2-3p_1+r_1+4)/(r_1+1)^2$ *and* $\alpha_{r_2,p_2} = (p_2^2-3p_2+r_2+4)/(r_2+1)^2$.

Proof. In view of (60), it gives

$$\left|\phi_{(x,y)}\mu[t_0,s_0;x,y]\right| \le \omega_{\text{mixed}}(\mu;|t_0-x|,|s_0-y|) \le \left(1+\frac{|t_0-x|}{\delta_1}\right)\left(1+\frac{|s_0-y|}{\delta_2}\right)\omega_{\text{mixed}} \cdot (\mu;\delta_1,\delta_2), \quad (64)$$

for all $(x,y),(t_0,s_0) \in D_{r_1,r_2}$, and for any $\delta_1,\delta_2 \in \mathbb{R}^+$. By (56), we have

$$\mu(x,s_0)+\mu(t_0,y)-\mu(t_0,s_0) = \mu(x,y)-\phi_{(x,y)}\mu[t_0,s_0;x,y]. \quad (65)$$

Operating F_{r_1,r_2} and by the definition of G_{r_1,r_2}, then

$$G_{r_1,r_2}(\mu;x,y) = \mu(x,y)F_{r_1,r_2}(e_{0,0};x,y) - F_{r_1,r_2} \cdot \left(\phi_{(x,y)}\,\mu[t_0,s_0;x,y];x,y\right). \quad (66)$$

It follows that

$$\left|G_{r_1,r_2}(\mu;x,y)-\mu(x,y)\right| \le F_{r_1,r_2}\left(\left|\phi_{(x,y)}\mu[t_0,s_0;x,y]\right|;x,y\right). \quad (67)$$

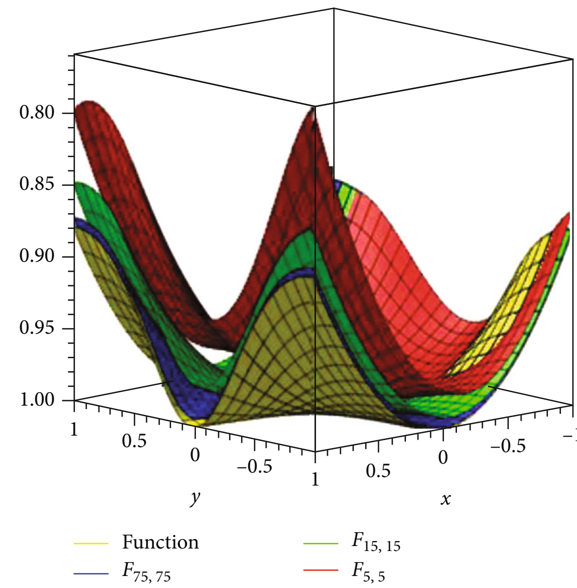

Figure 1: The convergence of $F_{r_1,r_2}(\mu;x,y)$ operators to $\mu(x,y) = \cos(xy/(1+y^2))$ (yellow) for $r_1=r_2=5$ (red), $r_1=r_2=15$ (green), $r_1=r_2=75$ (blue), and $p_1=p_2=1$.

From (64) and by utilizing the Cauchy-Schwarz inequality, one has

$$\left|G_{r_1,r_2}(\mu;x,y)-\mu(x,y)\right| \le \left(1+\frac{\sqrt{F_{r_1,r_2}\left((t_0-x)^2;x,y\right)}}{\delta_1}+\frac{\sqrt{F_{r_1,r_2}\left((s_0-y)^2;x,y\right)}}{\delta_2} + \frac{\sqrt{F_{r_1,r_2}\left((t_0-x)^2;x,y\right)F_{r_1,r_2}\left((s_0-y)^2;x,y\right)}}{\delta_1\delta_2}\right) \times \omega_{\text{mixed}}(\mu;\delta_1,\delta_2). \quad (68)$$

Taking into account Corollary 8, it becomes

$$F_{r_1,r_2}\left((t_0-x)^2;x,y\right) \le \left(\frac{p_1}{r_1}x\right)^2+\frac{2p_1^2}{r_1(r_1+1)}x+\frac{p_1^2+p_1+r_1}{(r_1+1)^2} \le \frac{p_1^2-3p_1+r_1+4}{(r_1+1)^2} = \alpha_{r_1,p_1},$$

$$F_{r_1,r_2}\left((s_0-y)^2;x,y\right) \le \left(\frac{p_2}{r_2}y\right)^2+\frac{2p_2^2}{r_2(r_2+1)}y+\frac{p_2^2+p_2+r_2}{(r_2+1)^2} \le \frac{p_2^2-3p_2+r_2+4}{(r_2+1)^2} = \alpha_{r_2,p_2}. \quad (69)$$

Table 1: Error of approximation $F_{r_1,r_2}(\mu; x, y)$ operators to $\mu(x,y) = \cos(xy/(1+y^2))$ for $r_1 = r_2 = 5$, $r_1 = r_2 = 50$, $r_1 = r_2 = 500$, and $p_1 = p_2 = 1$.

(x,y)	$\lvert F_{5,5}(\mu; x,y) - \mu(x,y)\rvert$	$\lvert F_{50,50}(\mu; x,y) - \mu(x,y)\rvert$	$\lvert F_{500,500}(\mu; x,y) - \mu(x,y)\rvert$
(0.6, 0.6)	0.0589662082	0.0057059967	0.0005642954
(0.8, −0.1)	0.0532558351	0.0049072492	0.0004809529
(0.5, −0.5)	0.0271614376	0.0024426683	0.0002455107
(0.4, −0.3)	0.0203534842	0.0014131511	0.0001327585
(0.3, −0.7)	0.0170542963	0.0010060800	0.0000896256
(−0.2, 0.8)	0.0148469930	0.0015555907	0.0001550674
(−0.6, 0.4)	0.0127417986	0.0019020417	0.0001971948
(−0.1, 0.1)	0.0091694597	0.0003325468	0.0000204350
(−0.3, −0.7)	0.0081301508	0.0007964789	0.0000801801
(−0.9, 0.9)	0.0020362863	0.0003014692	0.0000358039

Hence,

$$\begin{aligned}
&\left|G_{r_1,r_2}(\mu; x,y) - \mu(x,y)\right| \\
&\le \left(1 + \frac{\sqrt{(p_1^2 - 3p_1 + r_1 + 4)/(r_1+1)^2}}{\delta_1} + \frac{\sqrt{(p_2^2 - 3p_2 + r_2 + 4)/(r_2+1)^2}}{\delta_2}\right. \\
&\quad \left. + \frac{\sqrt{\left((p_1^2 - 3p_1 + r_1 + 4)/(r_1+1)^2\right)\left((p_2^2 - 3p_2 + r_2 + 4)/(r_2+1)^2\right)}}{\delta_1\delta_2}\right) \\
&\quad \times \omega_{\text{mixed}}(\mu; \delta_1, \delta_2).
\end{aligned} \tag{70}$$

If we choose $\delta_1 = \sqrt{\alpha_{r_1,p_1}}$ and $\delta_2 = \sqrt{\alpha_{r_2,p_2}}$, then the desired result is arrived.

5. Graphics and Error Estimation Tables

In this section, with the help of the Maple software, we give some plots and error estimation tables for the comparison of the convergence behavior of (29) and (62) operators to the certain functions.

Example 1. Let $\mu(x,y) = \cos(xy/(1+y^2))$. In Figure 1, for $r_1 = r_2 = 5$ (red), $r_1 = r_2 = 15$ (green), $r_1 = r_2 = 75$ (blue), and by choosing $p_1 = p_2 = 1$, we illustrate the convergence of (29) operators to $\mu(x,y) = \cos(xy/(1+y^2))$ (yellow). Also, in Table 1, we determine the error of approximation (29) operators to $\mu(x,y) = \cos(xy/(1+y^2))$ for the certain values of $-1 \le x, y \le 1$ and $r_1 = r_2 = 5{,}50{,}500$, respectively. It is clear from Table 1 that as the values of r_1 and r_2 increase then the error of approximation (29) operators to $\mu(x,y) = \cos(xy/(1+y^2))$ decreases.

Example 2. Let $\mu(x,y) = \sin((5/2)xy)e^{-y/3}$. In Figure 2, for $r_1 = r_2 = 15$, $p_1 = p_2 = 2$, we compare the convergence of (29) (red) and the associated GBS operators (62) (blue) to $\mu(x,y) = \sin((5/2)xy)e^{-y/3}$ (yellow). Also, in Table 2, we evaluate the error of approximation (29) and (62) operators to $\mu(x,y) = \sin((5/2)xy)e^{-y/3}$ for $r_1 = r_2 = 300$, $p_1 = p_2 = 2$, and the certain points of $-1 \le x, y \le 1$. It is obvious that, for $r_1 = r_2 = 300$, $p_1 = p_2 = 2$, the absolute difference between

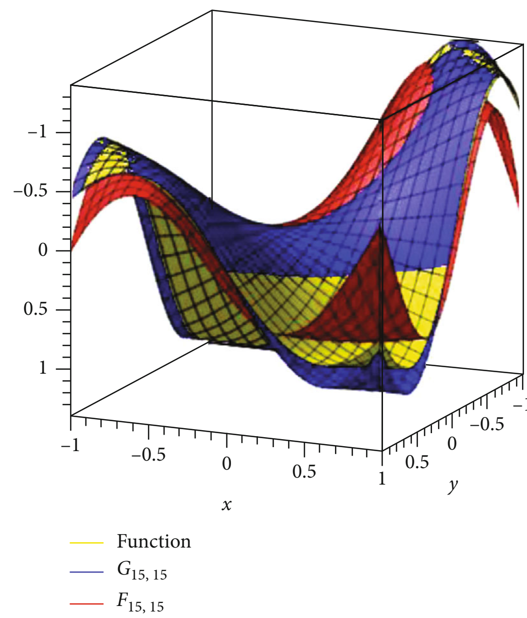

Figure 2: The convergence of $F_{r_1,r_2}(\mu; x,y)$ (red) and $G_{r_1,r_2}(\mu; x,y)$ (blue) operators to $\mu(x,y) = \sin((5/2)xy)e^{-y/3}$ (yellow) for $r_1 = r_2 = 15$ and $p_1 = p_2 = 2$.

Table 2: Error of approximation $F_{r_1,r_2}(\mu; x,y)$ and $G_{r_1,r_2}(\mu; x,y)$ operators to $\mu(x,y) = \sin((5/2)xy)e^{-y/3}$ for $r_1 = r_2 = 300$ and $p_1 = p_2 = 2$.

(x,y)	$\lvert F_{300,300}(\mu; x,y) - \mu(x,y)\rvert$	$\lvert G_{300,300}(\mu; x,y) - \mu(x,y)\rvert$
(-0.9, 0.9)	0.0136597503	0.0000249663
(0.5, 0.5)	0.0127076638	0.0000753517
(-0.8, -0.1)	0.0118523844	0.0000165295
(-0.3, -0.6)	0.0117422817	0.0000287716
(-0.6, -0.4)	0.0102228713	0.0000248461
(0.4, 0.8)	0.0101988560	0.0001669195
(0.9, 0.7)	0.0081455565	0.0005454158
(0.1, -0.4)	0.0070195207	0.0000801529
(-0.7, 0.3)	0.0056112239	0.0000346506
(-0.2, 0.2)	0.0003416940	0.0000879462

(29) operators to $\mu(x, y)$ is greater than that of (62) operators to $\mu(x, y)$. Namely, the (62) operators has better approximation than (29) operators.

Data Availability

All data required for this paper are included within this paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All the authors contributed equally and significantly in writing this article.

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Research Article

On Controlled Rectangular Metric Spaces and an Application

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In this paper, we introduce the notion of controlled rectangular metric spaces as a generalization of rectangular metric spaces and rectangular b-metric spaces. Further, we establish some related fixed point results. Our main results extend many existing ones in the literature. The obtained results are also illustrated with the help of an example. In the last section, we apply our results to a common real-life problem in a general form by getting a solution for the Fredholm integral equation in the setting of controlled rectangular metric spaces.

1. Introduction

The fixed point theory is a growing and exciting field of mathematics with a variety of variant applications in mathematical sciences, proposing newer applications in discrete dynamics and super fractals. The fixed point theory is a fundamental tool to various theoretical and applied fields, such as variational and linear inequalities, the approximation theory, nonlinear analysis, integral and differential equations and inclusions, the dynamic systems theory, mathematics of fractals, mathematical economics (game theory, equilibrium problems, and optimization problems), and mathematical modeling; see [1–3]. In particular, fixed point techniques have been applied in such diverse fields; see [4, 5]. There are particular real-life problems, whose statements are fairly easy to understand, which can be argued using some versions of fixed point theorems; see [6, 7].

The notion of extended b-metric spaces was introduced by Kamran et al. [9] as a generalization of metric spaces and b-metric spaces [10, 11]. This metric type space has been generalized in several directions (for instance, controlled metric spaces [12], double controlled metric spaces [13], and others [14–19]). In a different perception, Branciari [20] proposed rectangular metric spaces. In the same order, Asim et al. [21] included a control function to initiate the concept of extended rectangular b-metric spaces as a generalization of rectangular b-metric spaces [22]. In [23], Mlaiki et al. introduced controlled rectangular b-metric spaces, which generalize rectangular metric spaces and rectangular b-metric spaces.

In this paper, our goal is to introduce the notion of controlled rectangular metric spaces, which is different from controlled rectangular b-metric spaces, and generalize rectangular metric spaces as well as rectangular b-metric spaces. Further, we prove some fixed point results on such spaces as a generalization of many preexisting results in the literature. Also, we give examples for the justification of our results. In the last, as an application, we give an existence theorem for the Fredholm integral equation in the setting of controlled rectangular metric spaces.

2. Preliminaries

In this section, we collect some basic concepts related to our main results.

Definition 1 [22]. A mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ on a nonempty set $\mathcal{M}$ is called a rectangular b-metric space, if there exists a constant $s \geq 1$ such that for all $m_1, m_2 \in \mathcal{M}$ and all distinct $\mu_1, \mu_2 \in \mathcal{M}$ different from m_1 *and* m_2, the following axioms are satisfied:

(i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$

(ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$

(iii) $d_\zeta(m_1, m_2) \leq s[d_\zeta(m_1, \mu_1) + d_\zeta(\mu_1, \mu_2) + d_\zeta(\mu_2, m_2)]$

In this case, the pair $(\mathcal{M}, d_\zeta)$ is called a rectangular b -metric space.

Definition 2 [9]. Let $\mathcal{M}$ be a nonempty set and $\zeta : \mathcal{M} \times \mathcal{M} \rightarrow [1,\infty)$ be a mapping. Then, a mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ is called an extended b-metric, if for all $m_1, m_2, m_3 \in \mathcal{M}$, it satisfies the following axioms:

(i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$

(ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$

(iii) $d_\zeta(m_1, m_3) \leq \zeta(m_1, m_3)[d_\zeta(m_1, m_2) + d_\zeta(m_2, m_3)]$

The pair $(\mathcal{M}, d_\zeta)$ is called an extended b-metric space.

Definition 3 [21]. A mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ on a nonempty set $\mathcal{M}$ is called an extended rectangular b -metric space, if for all $m_1, m_2 \in \mathcal{M}$ and all distinct $\mu_1, \mu_2 \in \mathcal{M}$ different from m_1 and m_2, the following axioms are satisfied:

(i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$

(ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$

(iii) $d_\zeta(m_1, m_2) \leq \zeta(m_1, m_2)[d_\zeta(m_1, \mu_1) + d_\zeta(\mu_1, \mu_2) + d_\zeta(\mu_2, m_2)]$

where $\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [1,\infty)$ is a mapping. In this case, the pair $(\mathcal{M}, d_\zeta)$ is called an extended rectangular b-metric space.

Note that the topology of rectangular metric spaces need not be Hausdorff. For more examples, see the papers of Sarma et al. [24] and Samet [25]. The topological structure of rectangular metric spaces is not compatible with the topology of classic metric spaces; see Example 7 in the paper of Suzuki [26]. In the same direction, extended rectangular b-metric spaces cannot be Hausdorff.

Definition 4 [23]. A mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ on a nonempty set $\mathcal{M}$ is called a controlled rectangular b-metric space, if for all distinct $m_1, m_2, \mu_1, \mu_2 \in \mathcal{M}$, the following axioms are satisfied:

(i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$

(ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$

(iii) $d_\zeta(m_1, m_2) \leq \zeta(m_1, m_2, \mu_1, \mu_2)[d_\zeta(m_1, \mu_1) + d_\zeta(\mu_1, \mu_2) + d_\zeta(\mu_2, m_2)]$

where $\zeta : \mathcal{M}^4 \longrightarrow [1,\infty)$ is a mapping. In this case, the pair $(\mathcal{M}, d_\zeta)$ is called a controlled rectangular b-metric space.

As a generalization of metric spaces, Mlaiki et al. in [12] introduced the concept of controlled metric spaces as follows.

Definition 5 [12]. Let $\mathcal{M}$ be a nonempty set and $\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [1,\infty)$. Then, a mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ is called a controlled metric, if for all $m_1, m_2, m_3 \in \mathcal{M}$, it satisfies the following axioms:

(i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$

(ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$

(iii) $d_\zeta(m_1, m_3) \leq \zeta(m_1, m_2) d_\zeta(m_1, m_2) + \zeta(m_2, m_3) d_\zeta(m_2, m_3)$

The pair $(\mathcal{M}, d_\zeta)$ is called a controlled metric space.

Note that Definition 5 generalizes b-metric spaces and is different from Definition 2.

Example 1 [12]. Let $\mathcal{M} = \{1, 2, \cdots\}$. Define $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ as

$$d_\zeta(x, y) = \begin{cases} 0, & \text{if } m_1 = m_2, \\ \dfrac{1}{m_1}, & \text{if } m_1 \text{ is even and } m_2 \text{ is odd}, \\ \dfrac{1}{m_2}, & \text{if } m_1 \text{ is odd and } m_2 \text{ is even}, \\ 1, & \text{otherwise}. \end{cases} \quad (1)$$

Hence, $(\mathcal{M}, d_\zeta)$ is a controlled metric space, where ζ

$: \mathscr{M} \times \mathscr{M} \longrightarrow [1,\infty)$ is defined as

$$\zeta(m_1, m_2) = \begin{cases} m_1, & \text{if } m_1 \text{ is even and } m_2 \text{ is odd,} \\ m_2, & \text{if } m_1 \text{ is odd and } m_2 \text{ is even,} \\ 1, & \text{otherwise.} \end{cases} \tag{2}$$

3. Main Results

In this section, we introduce the notion of controlled rectangular metric spaces. Also, we establish some fixed point results.

Definition 6. A mapping $d_\zeta : \mathscr{M} \times \mathscr{M} \longrightarrow [0,\infty)$ on a non-empty set $\mathscr{M}$ is called a controlled rectangular metric space, if for all $m_1, m_2 \in \mathscr{M}$ and all distinct $\mu_1, \mu_2 \in \mathscr{M}$ different from m_1 and m_2, the following axioms are satisfied:

(i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$

(ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$

(iii) $d_\zeta(m_1, m_2) \leq \zeta(m_1, \mu_1) d_\zeta(m_1, \mu_1) + \zeta(\mu_1, \mu_2) d_\zeta(\mu_1, \mu_2) + \zeta(\mu_2, m_2) d_\zeta(\mu_2, m_2)$

where $\zeta : \mathscr{M} \times \mathscr{M} \longrightarrow [1,\infty)$ is a mapping, In this case, the pair $(\mathscr{M}, d_\zeta)$ is called a controlled rectangular metric space.

Remark 7.

(i) Every rectangular metric space and rectangular b-metric is a controlled rectangular metric space

(ii) Clearly, Definition 6 is different from Definition 4

(iii) Every controlled metric space is a controlled rectangular metric space, but its converse is not true in general. See the following example

Example 2. Let $\mathscr{M} = [0,\infty)$. Define a mapping $d_\zeta : \mathscr{M} \times \mathscr{M} \longrightarrow [0,\infty)$ by

$$d_\zeta(m_1, m_2) = \begin{cases} 0, & \text{if } m_1 = m_2, \\ \dfrac{1}{m_1}, & \text{if } m_1 \geq 1 \text{ and } m_2 \in [0, 1), \\ \dfrac{1}{m_2}, & \text{if } m_2 \geq 1 \text{ and } m_1 \in [0, 1), \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

Then, d_ζ is a controlled rectangular metric space, where $\zeta : \mathscr{M} \times \mathscr{M} \longrightarrow [1,\infty)$ is a mapping defined as

$$\zeta(m_1, m_2) = \begin{cases} m_1, & \text{if } m_1, m_2 \geq 1, \\ 1, & \text{otherwise.} \end{cases} \tag{4}$$

Clearly, $(\mathscr{M}, d_\zeta)$ is not a controlled metric space if we take $m_1, m_3 \geq 1$ and $m_2 \in [0, 1)$. Then, $d_\zeta(m_1, m_3) = 1$, $d_\zeta(m_1, m_2) = 1/m_1$, $d_\zeta(m_2, m_3) = 1/m_3$, $\zeta(m_1, m_2) = 1$, and $\zeta(m_2, m_3) = 1$. Here, the triangle inequality is not satisfied:

$$d_\zeta(m_1, m_3) = 1 > \zeta(m_1, m_2) d_\zeta(m_1, m_2) + \zeta(m_2, m_3) d_\zeta(m_2, m_3) = \frac{1}{m_1} + \frac{1}{m_3}. \tag{5}$$

Example 3. Let $\mathscr{M} = \{1, 2, 3, 4\}$. Define $d_\zeta : \mathscr{M} \times \mathscr{M} \longrightarrow [o,\infty)$ as

$$\begin{aligned} &d_\zeta(1, 1) = d_\zeta(2, 2) = d_\zeta(3, 3) = d_\zeta(4, 4) = 0, \\ &d_\zeta(1, 2) = d_\zeta(2, 1) = d_\zeta(2, 3) = d_\zeta(3, 2) = d_\zeta(3, 4) = d_\zeta(4, 3) \\ &\quad = d_\zeta(1, 3) = d_\zeta(3, 1) = 80, \\ &d_\zeta(1, 4) = d_\zeta(4, 1) = 1000, \\ &d_\zeta(2, 4) = d_\zeta(4, 2) = 450. \end{aligned} \tag{6}$$

Then, $(\mathscr{M}, d_\zeta)$ is a controlled rectangular metric space with $\zeta : \mathscr{M} \times \mathscr{M} \longrightarrow [1,\infty)$ defined as $\zeta(m_1, m_2) = \max\{m_1, m_2\} + 2$, for all $m_1, m_2 \in \mathscr{M}$. However, $(\mathscr{M}, d_\zeta)$ is not a rectangular metric space; for instance, notice

$$d_\zeta(1, 4) = 1000 > d_\zeta(1, 2) + d_\zeta(2, 3) + d_\zeta(3, 4) = 240. \tag{7}$$

The concepts of convergence, Cauchyness, and completeness can simply be generalized in terms of controlled rectangular metric spaces.

Definition 8. Let $(\mathscr{M}, d_\zeta)$ be a controlled rectangular metric space. Then,

(i) A sequence $\{m_n\}$ in $(\mathscr{M}, d_\zeta)$ is said to be convergent to $m \in \mathscr{M}$, if $\lim_{n\longrightarrow\infty} d_\zeta(m_n, m) = 0$

(ii) A sequence $\{m_n\}$ in $(\mathscr{M}, d_\zeta)$ is called a Cauchy sequence, if $\lim_{n,r\longrightarrow\infty} d_\zeta(m_n, m_r) = 0$

(iii) $(\mathscr{M}, d_\zeta)$ is called a complete controlled rectangular metric space, if every Cauchy sequence in $\mathscr{M}$ is convergent to some point of $\mathscr{M}$

Definition 9. Let $(\mathscr{M}, d_\zeta)$ be a controlled rectangular metric space. Let $m \in \mathscr{M}$ and $\tau > 0$. Then,

(i) The open ball $B(m, \tau)$ is define as

$$B(m, \tau) = \{m_1 \in \mathscr{M}, d_\zeta(m, m_1) < \tau\}. \tag{8}$$

(ii) The mapping $\mathbf{f} : \mathscr{M} \longrightarrow \mathscr{M}$ is called continuous at $m \in \mathscr{M}$, if for $\upsilon > 0$, there is $\nu > 0$ such that $\mathbf{f}(B(m,$

$\nu)) \subseteq B(\mathbf{f}(m), \upsilon)$. Thus, if $\mathbf{f}$ is continuous at m, then for any sequence $\{m_n\}$ converging to m, we have $\lim_{n\longrightarrow\infty}\mathbf{f}m_n = \mathbf{f}m$

Note that a rectangular b-metric space is not continuous in general, and it is the same for controlled rectangular metric spaces.

Lemma 10. *Let $(\mathcal{M}, d_\zeta)$ be a controlled rectangular metric space and $\{m_n\}$ be a Cauchy sequence in $\mathcal{M}$ such that $m_n \neq m_r$, whenever $n \neq r$. If $\lim_{n,r\longrightarrow\infty}\zeta(m_n, m_r) < \infty$ for all $m_n, m_r \in \mathcal{M}$, then $\{m_n\}$ has a unique limit.*

Proof. Suppose that a sequence $\{m_n\}$ in $\mathcal{M}$ has two limit points $\mu, \nu \in \mathcal{M}$, that is, $\lim_{n\longrightarrow\infty}m_n = \mu$ and $\lim_{n\longrightarrow\infty}m_n = \nu$. $\{m_n\}$ is a Cauchy sequence for $m_n \neq m_r$, whenever $n \neq r$. Hence, from condition (iii) of Definition 6, we have

$$\begin{aligned} d_\zeta(\mu, \nu) \leq \zeta(\mu, m_n)d_\zeta(\mu, m_n) + \zeta(m_n, m_r)d_\zeta(m_n, m_r) \\ + \zeta(m_r, \nu)d_\zeta(m_r, \nu) \longrightarrow 0 \quad \text{as } n, r \longrightarrow \infty. \end{aligned} \tag{9}$$

This implies that

$$d_\zeta(\mu, \nu) = 0. \tag{10}$$

Hence, $\{m_n\}$ has a unique limit point in $\mathcal{M}$.

Definition 11. Let $(\mathcal{M}, d_\zeta)$ be a controlled rectangular metric space. Then,

(i) For a mapping $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$, we define

$$\begin{aligned} \mathcal{O}(m, n) &= \{m, \mathbf{f}m, \cdots, \mathbf{f}^n m\}, \\ \mathcal{O}(m,\infty) &= \{m, \mathbf{f}m, \cdots, \mathbf{f}^n m, \cdots\}, \end{aligned} \tag{11}$$

where $m \in \mathcal{M}$ and $n \in \mathbb{N}$. The $\mathcal{O}(m, \infty)$ is called an orbit of $\mathbf{f}$.

(ii) A mapping $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ is called $\mathbf{f}$-orbitally continuous, if $\lim_{k\longrightarrow\infty}\mathbf{f}_k^n m = m$ implies $\lim_{k\longrightarrow\infty}\mathbf{f}(\mathbf{f}_k^n m) = \mathbf{f}m$, where $m \in \mathcal{M}$

(iii) A mapping $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ is called $\mathbf{f}$-orbitally complete, if every Cauchy sequence in $\mathcal{O}(m, \infty)$ is convergent in $\mathcal{M}$

Our main result is similar to the Banach contraction principle in the setting of controlled rectangular metric spaces. Throughout this section, for a mapping $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ and $m \in \mathcal{M}$, we consider an orbit $\mathcal{O}(m, \infty)$.

Theorem 12. *Let $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ be a mapping on a controlled rectangular metric space $(\mathcal{M}, d_\zeta)$. Suppose that the following axioms hold:*

(i) For all $m_1, m_2 \in \mathcal{M}$, we have

$$d_\zeta(\mathbf{f}m_1, \mathbf{f}m_2) \leq \lambda d_\zeta(m_1, m_2), \tag{12}$$

where $\lambda \in [0, 1)$.

(ii) $\sup_{q\geq 1}\lim_{i\to\infty}\zeta(m_i, m_q)(\zeta(m_{i+1}, m_{i+2})/\zeta(m_{i-1}, m_i))\lambda < 1$, for any $m_n \in \mathcal{M}$

(iii) $(\mathcal{M}, d_\zeta)$ is $\mathbf{f}$-orbitally complete

(iv) $\mathbf{f}$ is orbitally continuous

(v) For each $m \in \mathcal{M}$, $\lim_{n\longrightarrow\infty}\varsigma(m_n, \mu)$ and $\lim_{n\longrightarrow\infty}\varsigma(\mu, m_n)$ exist and are finite

Then, $\mathbf{f}$ has a unique fixed point in $\mathcal{M}$.

Proof. Consider an arbitrary point $m_0 \in \mathcal{M}$, and define an iterative sequence $\{m_n\}$ over m_0 as follows:

$$m_1 = \mathbf{f}m_0, m_2 = \mathbf{f}m_1 = \mathbf{f}(\mathbf{f}m_0) = \mathbf{f}^2 m_0, \cdots, m_n = \mathbf{f}^n m_0, \cdots. \tag{13}$$

From equation (12), we have

$$d_\zeta(m_1, m_2) = d_\zeta\left(\mathbf{f}m_0, \mathbf{f}^2 m_0\right) \leq \lambda d_\zeta(m_0, \mathbf{f}m_0) = d_\zeta(m_0, m_1). \tag{14}$$

Recursively, we have

$$\begin{aligned} d_\zeta(m_n, m_{n+1}) = d_\zeta\left(\mathbf{f}^n m_0, \mathbf{f}^{n+1} m_0\right) &\leq \lambda d_\zeta\left(\mathbf{f}^{n-1} m_0, \mathbf{f}^n m_0\right) \\ &\leq \lambda^2 d_\zeta\left(\mathbf{f}^{n-2} m_0, \mathbf{f}^{n-1} m_0\right) \cdots \\ &\leq \lambda^n d_\zeta(m_0, m_1). \end{aligned} \tag{15}$$

That is,

$$d_\zeta(m_n, m_{n+1}) \leq \lambda^n d_\zeta(m_0, m_1). \tag{16}$$

By taking limit $n \longrightarrow \infty$, we get

$$\lim_{n\longrightarrow\infty} d_\zeta(m_n, m_{n+1}) = 0, \tag{17}$$

$$\lim_{n\longrightarrow\infty} d_\zeta(m_{n+1}, m_{n+2}) = 0. \tag{18}$$

Next, we show that $\{m_n\}$ is a Cauchy sequence. For this, we will take the following two cases.

Case 1. Let ρ be odd, that is, $\rho = 2r+1$, where $r \geq 1$. Then, from condition (iii) of Definition 6 and equation (16) for $n+\rho > n$, we have

$$\begin{aligned}
d_\zeta(m_n, m_{n+2r+1}) &\leq \zeta(m_n, m_{n+1}) d_\zeta(m_n, m_{n+1}) + \zeta \\
&\cdot (m_{n+1}, m_{n+2}) d_\zeta(m_{n+1}, m_{n+2}) + \zeta(m_{n+2}, m_{n+2r+1}) \\
&\times d_\zeta(m_{n+2}, m_{n+2r+1}) \leq [\zeta(m_n, m_{n+1})\lambda^n + \zeta \\
&\cdot (m_{n+1}, m_{n+2})\lambda^{n+1}] d_\zeta(m_0, m_1) + \zeta(m_{n+2}, m_{n+2r+1}) d_\zeta \\
&\cdot (m_{n+2}, m_{n+2r+1}) \leq [\zeta(m_n, m_{n+1})\lambda^n + \zeta \\
&\cdot (m_{n+1}, m_{n+2})\lambda^{n+1}] d_\zeta(m_0, m_1) + \zeta(m_{n+2}, m_{n+2r+1})\zeta \\
&\cdot (m_{n+2}, m_{n+3}) \times d_\zeta(m_{n+2}, m_{n+3}) + \zeta(m_{n+2}, m_{n+2r+1})\zeta \\
&\cdot (m_{n+3}, m_{n+4}) d_\zeta(m_{n+3}, m_{n+4}) + \zeta(m_{n+2}, m_{n+2r+1}) \times \zeta \\
&\cdot (m_{n+4}, m_{n+2r+1}) d_\zeta(m_{n+4}, m_{n+2r+1}) \leq d_\zeta \\
&\cdot (m_0, m_1)\left[\zeta(m_n, m_{n+1})\lambda^n + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}\right] \\
&+ d_\zeta(m_0, m_1) \times \left[\zeta(m_{n+2}, m_{n+2r+1})\zeta(m_{n+2}, m_{n+3})\lambda^{n+2}\right. \\
&\left.+ \zeta(m_{n+2}, m_{n+2r+1})\zeta(m_{n+3}, m_{n+4})\lambda^{n+3}\right] + \cdots + d_\zeta \\
&\cdot (m_0, m_1)\zeta(m_{n+2}, m_{n+2r+1}) \cdots \zeta(m_{n+2r-2}, m_{n+2n-1})\lambda^{n+2r-2} \\
&+ \zeta(m_{n+2}, m_{n+2r+1}) \cdots \zeta(m_{n+2r-1}, m_{n+2r})\lambda^{n+2r-1} \\
&+ \zeta(m_{n+2}, m_{n+2r+1}) \cdots \zeta(m_{n+2r}, m_{n+2r+1})\lambda^{n+2r} d_\zeta(m_0, m_1) \\
&\leq d_\zeta(m_0, m_1)\left[\zeta(m_n, m_{n+1})\lambda^n + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}\right] \\
&+ \sum_{i=1}^{r-1}\prod_{j=1}^{i} \zeta(m_{n+2j}, m_{n+2r+1}) \times \zeta(m_{n+2i}, m_{n+2i+1})\lambda^{n+2i} d_\zeta \\
&\cdot (m_0, m_1) + \sum_{i=1}^{r-1}\prod_{j=1}^{i} \zeta(m_{n+2j+1}, m_{n+2r+1})\zeta(m_{n+2i+1}, m_{n+2i+2}) \\
&\times \lambda^{n+2i+1} d_\zeta(m_0, m_1) + \prod_{i=1}^{r-1} \zeta(m_{n+2j}, m_{n+2r+1})\lambda^{n+2r} d_\zeta(m_0, m_1).
\end{aligned} \tag{19}$$

As

$$\begin{aligned}
&\sum_{i=1}^{r-1}\prod_{j=1}^{i} \zeta(m_{n+2j}, m_{n+2r+1})\zeta(m_{n+2i}, m_{n+2i+1})\lambda^{n+2i} \\
&\quad \leq \sum_{i=1}^{r-1}\prod_{j=1}^{i} \zeta(m_{2j}, m_{n+2r+1})\zeta(m_{2i}, m_{n+2i+1})\lambda^{2i},
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\sum_{i=1}^{r-1}\prod_{j=1}^{i} \zeta(m_{n+2j+1}, m_{n+2r+1})\zeta(m_{n+2i+1}, m_{n+2i+2})\lambda^{n+2i+1} \\
&\quad \leq \sum_{i=1}^{r-1}\prod_{j=1}^{i} \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{n+2i+2})\lambda^{2i+1},
\end{aligned} \tag{21}$$

therefore, we obtain

$$\begin{aligned}
d_\zeta(m_n, m_{n+2r+1}) &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n \\
&+ \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] + \sum_{i=1}^{r-1}\prod_{j=1}^{i} \zeta(m_{2j}, m_{n+2r+1}) \\
&\times \zeta(m_{2i}, m_{2i+1})\lambda^{2i} d_\zeta(m_0, m_1) \\
&+ \sum_{i=1}^{r-1}\prod_{j=1}^{i} \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2}) \\
&\times \lambda^{2i+1} d_\zeta(m_0, m_1) + \prod_{i=1}^{r-1} \zeta(m_{n+2j}, m_{n+2r+1})\lambda^{n+2r} d_\zeta(m_0, m_1).
\end{aligned} \tag{22}$$

Since $\sup_{q\geq 1}\lim_{i\longrightarrow\infty}\zeta(m_i, m_q)(\zeta(m_{i+1}, m_{i+2})/\zeta(m_{i-1}, m_i))\lambda < 1$, the series

$$\begin{aligned}
&\sum_{i=1}^{\infty}\prod_{j=1}^{i} \zeta(m_{2j}, m_{n+2r+1})\zeta(m_{2i}, m_{2i+1})\lambda^{2i}, \\
&\sum_{i=1}^{\infty}\prod_{j=1}^{i} \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1},
\end{aligned} \tag{23}$$

converge by the ratio test. Let

$$\begin{aligned}
\mathcal{S} &= \sum_{i=1}^{\infty}\prod_{j=1}^{i} \zeta(m_{2j}, m_{n+2r+1})\zeta(m_{2i}, m_{2i+1})\lambda^{2i}, \\
\mathcal{S}_n &= \sum_{i=1}^{n}\prod_{j=1}^{i} \zeta(m_{2j}, m_{n+2r+1})\zeta(m_{2i}, m_{2i+1})\lambda^{2i},
\end{aligned} \tag{24}$$

$$\begin{aligned}
\mathcal{S}' &= \sum_{i=1}^{\infty}\prod_{j=1}^{i} \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1}, \\
\mathcal{S}'_n &= \sum_{i=1}^{n}\prod_{j=1}^{i} \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1}.
\end{aligned} \tag{25}$$

Then, equation (22) takes the following form:

$$\begin{aligned}
d_\zeta(m_n, m_{n+2r+1}) &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n \\
&+ \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] + d_\zeta(m_0, m_1)[\mathcal{S}_{r-1} - \mathcal{S}_{n+1}] d_\zeta \\
&\cdot (m_0, m_1) + d_\zeta(m_0, m_1)\left[\mathcal{S}'_{r-1} - \mathcal{S}'_{n+1}\right] d_\zeta(m_0, m_1) \\
&+ \prod_{i=1}^{r-1} \zeta(m_{n+2j}, m_{n+2r+1})\lambda^{n+2r} d_\zeta(m_0, m_1).
\end{aligned} \tag{26}$$

By taking limit $n \longrightarrow \infty$ in equation (26), we get

$$\lim_{n\longrightarrow\infty} d_\zeta(m_n, m_{n+2r+1}) = 0. \tag{27}$$

Case 2. Let ρ be even, that is, $\rho = 2r$, where $r \geq 1$. Then, from condition (iii) of Definition 6 and equation (16) for $n+\rho > n$, we have

$$\begin{aligned}
d_\zeta(m_n, m_{n+2r}) &\leq \zeta(m_n, m_{n+1}) d_\zeta(m_n, m_{n+1}) \\
&\quad + \zeta(m_{n+1}, m_{n+2}) d_\zeta(m_{n+1}, m_{n+2}) + \zeta(m_{n+2}, m_{n+2r}) \\
&\quad \times d_\zeta(m_{n+2}, m_{n+2r}) \leq [\zeta(m_n, m_{n+1})\lambda^n \\
&\quad + \zeta(m_{n+1}, m_{n+2})\lambda^{n+1}] d_\zeta(m_0, m_1) \\
&\quad + \zeta(m_{n+2}, m_{n+2r}) d_\zeta(m_{n+2}, m_{n+2r}) \leq [\zeta(m_n, m_{n+1})\lambda^n \\
&\quad + \zeta(m_{n+1}, m_{n+2})\lambda^{n+1}] d_\zeta(m_0, m_1) \\
&\quad + \zeta(m_{n+2}, m_{n+2r})\zeta(m_{n+2}, m_{n+3}) \times d_\zeta(m_{n+2}, m_{n+3}) \\
&\quad + \zeta(m_{n+2}, m_{n+2r})\zeta(m_{n+3}, m_{n+4}) d_\zeta(m_{n+3}, m_{n+4}) \\
&\quad + \zeta(m_{n+2}, m_{n+2r}) \times \zeta(m_{n+4}, m_{n+2r}) d_\zeta(m_{n+4}, m_{n+2r}) \\
&\quad \leq d_\zeta(m_0, m_1)\left[\zeta(m_n, m_{n+1})\lambda^n + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}\right] \\
&\quad + d_\zeta(m_0, m_1) \times \left[\zeta(m_{n+2}, m_{n+2r})\zeta(m_{n+2}, m_{n+3})\lambda^{n+2}\right. \\
&\quad \left. + \zeta(m_{n+2}, m_{n+2r})\zeta(m_{n+3}, m_{n+4})\lambda^{n+3}\right] \\
&\quad + \cdots + d_\zeta(m_0, m_1)\zeta(m_{n+2}, m_{n+2r}) \cdots \zeta \\
&\quad \cdot (m_{n+2r-4}, m_{n+2n-3})\lambda^{n+2r-4} + \zeta(m_{n+2}, m_{n+2r}) \cdots \zeta \\
&\quad \cdot (m_{n+2r-3}, m_{n+2r-2})\lambda^{n+2r-3} + \zeta(m_{n+2}, m_{n+2r}) \cdots \zeta \\
&\quad \cdot (m_{n+2r-2}, m_{n+2r})\lambda^{n+2r-2} d_\zeta(m_0, m_2) \leq d_\zeta \\
&\quad \cdot (m_0, m_1)\left[\zeta(m_n, m_{n+1})\lambda^n + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}\right] \\
&\quad + \sum_{i=1}^{r-2}\prod_{j=1}^{i}\zeta(m_{n+2j}, m_{n+2r}) \times \zeta(m_{n+2i}, m_{n+2i+1})\lambda^{n+2i} d_\zeta \\
&\quad \cdot (m_0, m_1) + \sum_{i=1}^{r-2}\prod_{j=1}^{i}\zeta(m_{n+2j+1}, m_{n+2r})\zeta \\
&\quad \cdot (m_{n+2i+1}, m_{n+2i+2}) \times \lambda^{n+2i+1} d_\zeta(m_0, m_1) \\
&\quad + \prod_{i=1}^{r-1}\zeta(m_{n+2j}, m_{n+2r})\lambda^{n+2r-2} d_\zeta(m_0, m_2).
\end{aligned} \tag{28}$$

As

$$\begin{aligned}
&\sum_{i=1}^{r-2}\prod_{j=1}^{i}\zeta(m_{n+2j}, m_{n+2r})\zeta(m_{n+2i}, m_{n+2i+1})\lambda^{n+2i} \\
&\quad \leq \sum_{i=1}^{r-2}\prod_{j=1}^{i}\zeta(m_{2j}, m_{n+2r})\zeta(m_{2i}, m_{2i+1})\lambda^{2i},
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\sum_{i=1}^{r-2}\prod_{j=1}^{i}\zeta(m_{n+2j+1}, m_{n+2r+1})\zeta(m_{n+2i+1}, m_{n+2i+2})\lambda^{n+2i+1} \\
&\quad \leq \sum_{i=1}^{r-2}\prod_{j=1}^{i}\zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1},
\end{aligned} \tag{30}$$

therefore, we obtain

$$\begin{aligned}
d_\zeta(m_n, m_{n+2r}) &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n \\
&\quad + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] + \sum_{i=1}^{r-2}\prod_{j=1}^{i}\zeta(m_{2j}, m_{n+2r}) \\
&\quad \times \zeta(m_{2i}, m_{2i+1})\lambda^{2i} d_\zeta(m_0, m_1) \\
&\quad + \sum_{i=1}^{r-2}\prod_{j=1}^{i}\zeta(m_{2j+1}, m_{n+2r})\zeta(m_{2i+1}, m_{2i+2}) \\
&\quad \times \lambda^{2i+1} d_\zeta(m_0, m_1) + \prod_{i=1}^{r-1}\zeta \\
&\quad \cdot (m_{n+2j}, m_{n+2r})\lambda^{n+2r-2} d_\zeta(m_0, m_2).
\end{aligned} \tag{31}$$

Since $\sup_{q\geq 1}\lim_{i\longrightarrow\infty}\zeta(m_i, m_q)(\zeta(m_{i+1}, m_{i+2})/\zeta(m_{i-1}, m_i))\lambda < 1$, the series

$$\begin{aligned}
&\sum_{i=1}^{\infty}\prod_{j=1}^{i}\zeta(m_{2j}, m_{n+2r})\zeta(m_{2i}, m_{2i+1})\lambda^{2i}, \\
&\sum_{i=1}^{\infty}\prod_{j=1}^{i}\zeta(m_{2j+1}, m_{n+2r})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1},
\end{aligned} \tag{32}$$

converge by the ratio test. Let

$$\begin{aligned}
\mathcal{S} &= \sum_{i=1}^{\infty}\prod_{j=1}^{i}\zeta(m_{2j}, m_{n+2r})\zeta(m_{2i}, m_{2i+1})\lambda^{2i}, \\
\mathcal{S}_n &= \sum_{i=1}^{n}\prod_{j=1}^{i}\zeta(m_{2j}, m_{n+2r})\zeta(m_{2i}, m_{2i+1})\lambda^{2i},
\end{aligned} \tag{33}$$

$$\begin{aligned}
\mathcal{S}' &= \sum_{i=1}^{\infty}\prod_{j=1}^{i}\zeta(m_{2j+1}, m_{n+2r})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1}, \\
\mathcal{S}'_n &= \sum_{i=1}^{n}\prod_{j=1}^{i}\zeta(m_{2j+1}, m_{n+2r})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1}.
\end{aligned} \tag{34}$$

Then, equation (31) takes the following form:

$$\begin{aligned}
d_\zeta(m_n, m_{n+2r}) &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n \\
&\quad + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] + d_\zeta(m_0, m_1)[\mathcal{S}_{r-1} - \mathcal{S}_{n+1}] \\
&\quad + d_\zeta(m_0, m_1)\left[\mathcal{S}'_{r-1} - \mathcal{S}'_{n+1}\right] \\
&\quad + \prod_{i=1}^{r-1}\zeta(m_{n+2j}, m_{n+2r})\lambda^{n+2r-2} d_\zeta(m_0, m_1).
\end{aligned} \tag{35}$$

By taking limit $n \longrightarrow \infty$ in equation (35), we get

$$\lim_{n\longrightarrow\infty} d_\zeta(m_n, m_{n+2r}) = 0. \tag{36}$$

Hence, in both cases, $\lim_{n\longrightarrow\infty} d_\zeta(m_n, m_{n+\rho}) = 0$, which shows that $\{m_n\}$ is a Cauchy sequence. As $\mathcal{M}$ is $\mathbf{f}$-orbitally complete, so there exists $\mu \in \mathcal{M}$ such that $\lim_{n\longrightarrow\infty} m_n = \mu$. Next, we show that μ is a fixed point of $\mathbf{f}$. As $\mathbf{f}$ is orbitally continuous, so we have

$$d_\zeta(\mu, \mathbf{f}\mu) \le \zeta(\mu, m_n) d_\zeta(\mu, m_n) + \zeta(m_n, m_{n+1}) d_\zeta(m_n, m_{n+1}) + \zeta(m_{n+1}, \mathbf{f}\mu) d_\zeta(m_{n+1}, \mathbf{f}\mu). \tag{37}$$

Since for each $m \in \mathcal{M}$, $\lim_{n\longrightarrow\infty}\zeta(m_n, m)$ and $\lim_{n\longrightarrow\infty}\zeta(m, m_n)$ exist and are finite, so by taking limit $n \longrightarrow \infty$ and using equation (17), we get

$$\lim_{n\longrightarrow\infty} d_\zeta(\mu, \mathbf{f}\mu) = 0. \tag{38}$$

Therefore, $\mathbf{f}\mu = \mu$. Hence, μ is a fixed point of $\mathbf{f}$. In view of Lemma 10, m is the unique fixed point of $\mathbf{f}$.

Example 4. Let $X = [1, 2]$. Define $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ by $d_\zeta(m_1, m_2) = (m_1 - m_2)^2$. Then, $(\mathcal{M}, d_\zeta)$ is a complete controlled rectangular metric space with $\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [1, \infty)$ defined as $\zeta(m_1, m_2) = 3m_1 + 2m_2 + 5$. Define a mapping $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$\mathbf{f}m = \frac{m}{4}. \tag{39}$$

Clearly, all the axioms of Theorem 12 are satisfied, and hence, $m = 0$ is a fixed point of $\mathbf{f}$.

Corollary 13. *Let $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ be a mapping on a complete controlled rectangular metric space $(\mathcal{M}, d_\zeta)$. Suppose that the following axioms hold:*

(i) For all $m_1, m_2 \in \mathcal{M}$, we have

$$d_\zeta(\mathbf{f}m_1, \mathbf{f}m_2) \le \lambda d_\zeta(m_1, m_2), \quad \lambda \in [0, 1). \tag{40}$$

(ii) $\sup_{q\ge 1}\lim_{i\longrightarrow\infty}\zeta(m_i, m_q)(\zeta(m_{i+1}, m_{i+2})/\zeta(m_{i-1}, m_i))\lambda < 1$, for any $m_n \in \mathcal{M}$

(iii) $\mathbf{f}$ is continuous

Then, $\mathbf{f}$ has a unique fixed point.

Remark 14.

(i) By putting $\zeta(m_1, m_2) = s$, for all $m_1, m_2 \in \mathcal{M}$ in Theorem 12, we get Theorem 2.1 of George et al. [22]

(ii) By putting $\zeta(m_1, m_2) = 1$, for all $m_1, m_2 \in \mathcal{M}$ in Theorem 12, we get the following corollary in view of Das and Dey [27]

Corollary 15. *Let $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ be a mapping on a rectangular metric space $(\mathcal{M}, d_\zeta)$. Suppose that the following axioms hold:*

(i) For all $m_1, m_2 \in \mathcal{M}$, $d_\zeta(\mathbf{f}m_1, \mathbf{f}m_2) \le \lambda d_\zeta(m_1, m_2)$, where $\lambda \in [0, 1)$

(ii) $(\mathcal{M}, d_\zeta)$ is $\mathbf{f}$-orbitally complete

(iii) $\mathbf{f}$ is orbitally continuous

Then, $\mathbf{f}$ has a unique fixed point.

Theorem 16. *Let $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ be a mapping on complete controlled rectangular metric space $(\mathcal{M}, d_\zeta)$, which satisfies the following axioms:*

(i) For all $m_1, m_2 \in \mathcal{M}$, we have

$$d_\zeta(\mathbf{f}m_1, \mathbf{f}m_2) \le \lambda\left[d_\zeta(m_1, \mathbf{f}m_1) + d_\zeta(m_2, \mathbf{f}m_2)\right], \tag{41}$$

where $\lambda \in [0, 1/2)$.

(ii) $\sup_{q\ge 1}\lim_{i\longrightarrow\infty}\zeta(m_i, m_q)(\zeta(m_{i+1}, m_{i+2})/\zeta(m_{i-1}, m_i))\lambda < 1$, for any $m_n \in \mathcal{M}$, where $\lambda \ne 1/\zeta(m_1, m_2)$ for each $m_1, m_2 \in \mathcal{M}$

(iii) For each $m \in \mathcal{M}$, $\lim_{n\longrightarrow\infty}\zeta(m_n, m_{n+1}) \le 1$, $\lim_{n\longrightarrow\infty}\zeta(m_n, \mu)$, and $\lim_{n\longrightarrow\infty}\zeta(\mu, m_n)$ exist and are finite

Then, $\mathbf{f}$ has a unique fixed point in $\mathcal{M}$.

Proof. Let us take an arbitrary element $m_0 \in \mathcal{M}$ and choose $m_1 = \mathbf{f}m_0$ and $m_2 = \mathbf{f}m_1$. Then, from equation (41), we obtain

$$d_\zeta(m_1, m_2) = d_\zeta(\mathbf{f}m_0, \mathbf{f}m_1) \le \lambda\left[d_\zeta(m_0, \mathbf{f}m_0) + d_\zeta(m_1, \mathbf{f}m_1)\right] = \lambda\left[d_\zeta(m_0, m_1) + d_\zeta(m_1, m_2)\right]. \tag{42}$$

This implies that

$$d_\zeta(m_1, m_2) \le \frac{\lambda}{1 - \lambda} d_\zeta(m_0, m_1), \tag{43}$$

where $\omega = \lambda/(1 - \lambda) < 1$, as $\lambda < 1/2$. By recursively applying equation (41), we obtain

$$d_\zeta(m_n, m_{n+1}) \le \omega^n d_\zeta(m_0, m_1). \tag{44}$$

Thus, by taking the limit in equation (44), we have

$$\lim_{n\to\infty} d_\zeta(m_n, m_{n+1}) = 0. \tag{45}$$

Again from equation (41), we have

$$d_\zeta(m_n, m_{n+2}) = d_\zeta(\mathbf{f}m_{n-1}, \mathbf{f}m_{1n+1}) \leq \lambda\left[d_\zeta(m_{n-1}, \mathbf{f}m_{n-1}) + d_\zeta(m_{n+1}, \mathbf{f}m_{n+1})\right] = \lambda\left[d_\zeta(m_{n-1}, m_n) + d_\zeta(m_{n+1}, m_{n+2})\right]. \tag{46}$$

By using equation (45), we obtain

$$\lim_{n\to\infty} d_\zeta(m_n, m_{n+2}) = 0. \tag{47}$$

Now, we will show that $\{m_n\}$ is a Cauchy sequence. By following the same procedure as in the proof of Theorem 12 and using equations (45) and (47), we conclude that $\{m_n\}$ is a Cauchy sequence. As $\mathcal{M}$ is complete, so there exists $\mu \in \mathcal{M}$ such that

$$\lim_{n\to\infty} d_\zeta(m_n, \mu) = 0. \tag{48}$$

Next, we show that μ is a fixed point of $\mathbf{f}$. From condition (iii) of Definition 6, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d_\zeta(\mu, \mathbf{f}\mu) &\leq \zeta(\mu, m_n)d_\zeta(\mu, m_n) + \zeta(m_n, m_{n+1})d_\zeta(m_n, m_{n+1}) \\ &+ \zeta(m_{n+1}, \mathbf{f}\mu)d_\zeta(m_{n+1}, \mathbf{f}\mu) \leq \zeta(\mu, m_n)d_\zeta(\mu, m_n) \\ &+ \zeta(m_n, m_{n+1})d_\zeta(m_n, m_{n+1}) + \zeta(m_{n+1}, \mathbf{f}\mu)d_\zeta \\ &\cdot (\mathbf{f}m_n, \mathbf{f}\mu) \leq \zeta(\mu, m_n)d_\zeta(\mu, m_n) \\ &+ \zeta(m_n, m_{n+1})d_\zeta(m_n, m_{n+1}) \\ &+ \zeta(m_{n+1}, \mathbf{f}\mu)\lambda\left[d_\zeta(m_n, \mathbf{f}m_n) + d_\zeta(\mu, \mathbf{f}\mu)\right]. \end{aligned} \tag{49}$$

$\lambda \neq 1/\zeta(m_1, m_2)$ for each $m_1, m_2 \in \mathcal{M}$. This implies that

$$d_\zeta(\mu, \mathbf{f}\mu) \leq \frac{\zeta(\mu, m_n)}{(1 - \lambda\zeta(m_{n+1}, \mathbf{f}\mu))} d_\zeta(\mu, m_n) + \frac{\zeta(m_n, m_{n+1}) + \lambda\zeta(m_{n+1}, \mathbf{f}\mu)}{(1 - \lambda\zeta(m_{n+1}, \mathbf{f}\mu))} d_\zeta(m_n, m_{n+1}). \tag{50}$$

For each $m \in \mathcal{M}$, $\lim_{n\longrightarrow\infty}\zeta(m_n, m_{n+1}) \leq 1$, $\lim_{n\longrightarrow\infty}\zeta(m_n, m)$, and $\lim_{n\longrightarrow\infty}\zeta(m, m_n)$ exist and are finite. Therefore, by taking limit $n \longrightarrow \infty$ in equation (50) and using equations (47) and (48), we obtain

$$d_\zeta(\mu, \mathbf{f}\mu) = 0, \tag{51}$$

which implies that $\mathbf{f}\mu = \mu$. For the uniqueness, let ν be another fixed point of $\mathbf{f}$ and $\mu \neq \nu$. From equation (41), we obtain

$$\begin{aligned} d_\zeta(\mu, \nu) = d_\zeta(\mathbf{f}\mu, \mathbf{f}\nu) &\leq \lambda\left[d_\zeta(\mu, \mathbf{f}\mu) + d_\zeta(\nu, \mathbf{f}\nu)\right] \\ &= \lambda\left[d_\zeta(\mu, \mu) + d_\zeta(\nu, \nu)\right], \end{aligned} \tag{52}$$

where $d_\zeta(\mu, \mu) = 0$ and $d_\zeta(\nu, \nu) = 0$. Hence, from the above inequality, we obtain $d_\zeta(\mu, \nu) = 0$, that is, $\mu = \nu$, and μ is a unique fixed point of $\mathbf{f}$.

4. Application

In this section, we will apply Corollary 13 to prove the existence and uniqueness of a solution for the following Fredholm integral equation:

$$m(t) = \int_a^b \tau(t, r, m(r))\mathrm{d}r + \nu(t), \quad \text{for } t, r \in [a, b], \tag{53}$$

where $\nu : [a, b] \longrightarrow \mathbb{R}$ and $\tau : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ both are continuous functions. Let $\mathcal{M} = \mathbb{C}([a, b])$ be the space of all continuous real-valued functions defined on the closed interval $[a, b]$. Consider

$$d_\zeta(m_1, m_2) = \sup_{t\in[a,b]} |m_1(t) - m_2(t)|^2. \tag{54}$$

Clearly, $(\mathcal{M}, d_\zeta)$ is a complete controlled rectangular metric space with $\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [1,\infty)$ defined as $\zeta(m_1, m_2) = 3|m_1(t)| + 2|m_2(t)| + 5$. Next, we will prove our result as follows.

Theorem 17. *For all $m_1, m_2 \in \mathcal{M}$ and $t, r \in [a, b]$, the following condition holds:*

$$|\tau(t, r, m_1(r)) - \tau(t, r, m_2(r))| \leq \frac{1}{2(b - a)} |m_1(r) - m_2(r)|. \tag{55}$$

Then, the integral equation (53) has a unique solution.

Proof. Define $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$\mathbf{f}m(t) = \int_a^b \tau(t, r, m(r))\mathrm{d}r + \nu(t), \quad \text{for } t, r \in [a, b], \tag{56}$$

where $\nu : [a, b] \longrightarrow \mathbb{R}$ and $\tau : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ both are continuous functions. Clearly, γ is a fixed point of $\mathbf{f}$, if and only if γ is a solution of the integral equation (53). For all $m_1, m_2 \in \mathcal{M}$, we have

$$\begin{aligned} |\mathbf{f}m_1(t) - \mathbf{f}m_2(t)|^2 &= \left|\int_a^b [\tau(t, r, m_1(r)) - \tau(t, r, m_2(r))\, \mathrm{d}r\right|^2 \\ &\leq \frac{1}{4(b - a)^2} \sup_{s\in[a,b]} |m_1(s) - m_2(s)|^2 \left(\int_a^b \mathrm{d}s\right)^2 \\ &\leq \frac{1}{4} d_\zeta(m_1, m_2). \end{aligned} \tag{57}$$

It implies that

$$d_\zeta(\mathbf{f}m_1, \mathbf{f}m_2) \le \frac{1}{4} d_\zeta(m_1, m_2), \tag{58}$$

where $\lambda = 1/4 \in (0, 1)$. Thus, all the conditions of Corollary 13 are satisfied. Hence, $\mathbf{f}$ has a unique fixed point; that is, the Fredholm integral equation (53) has a solution.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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Research Article

Some Novel Sixth-Order Iteration Schemes for Computing Zeros of Nonlinear Scalar Equations and Their Applications in Engineering

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In this paper, we propose two novel iteration schemes for computing zeros of nonlinear equations in one dimension. We develop these iteration schemes with the help of Taylor's series expansion, generalized Newton-Raphson's method, and interpolation technique. The convergence analysis of the proposed iteration schemes is discussed. It is established that the newly developed iteration schemes have sixth order of convergence. Several numerical examples have been solved to illustrate the applicability and validity of the suggested schemes. These problems also include some real-life applications associated with the chemical and civil engineering such as adiabatic flame temperature equation, conversion of nitrogen-hydrogen feed to ammonia, the van der Wall's equation, and the open channel flow problem whose numerical results prove the better efficiency of these methods as compared to other well-known existing iterative methods of the same kind.

1. Introduction

The solution of nonlinear scalar equations plays a vital role in many fields of applied sciences such as Engineering, Physics, and Mathematics. Analytical methods do not help us to solve such equations, and therefore, we need iterative methods for approximate the solution. In an iterative process, the first step is to choose an initial guess x_0 which is improved step by step by means of iterations till the approximate solution is achieved with the required accuracy. Some basic iterative methods are given in literature [1–8] and the references therein. In the last few years, a lot of researchers worked on iterative methods with their applications and proposed some new iterative schemes which possesses either a high convergence rate or have less number of functional evaluations per iteration, see [9–21] and the references therein. The convergence rate of an iterative method can be increased by involving predictor and corrector steps which results multi-step iterative methods whereas the number of functional evaluations can be reduced by removing second and higher derivatives in the considered iterative method using different mathematical techniques. When we try to raise the convergence rate of an iterative scheme, we have to use more functional evaluations per iteration, and similarly, less number of functional evaluations per iterations causes low order of convergence which is the main drawback. It is much difficult to manage both terms, i.e., the convergence rate and functional evaluations per iterations as it seems that there exists an inverse relation between them. In twenty-first century, many mathematicians try to modify the existing methods with less number of functional evaluations per iterations and higher convergence order by applying different techniques such as predictor-corrector technique, finite difference scheme, interpolation technique, Taylor's series, and quadrature formula etc. In 2007, Noor et al. [22] introduced a two-step Halley's method with sextic convergence and then approximated its second derivative by the utilization of finite difference scheme and suggested a novel second-derivative free iterative algorithm which have fifth convergence order. In 2012, Hafiz and Al-Goria [23] suggested two new algorithms with order seven and nine, respectively, which were based on the weight combination of midpoint with Simpson quadrature formulas and using the predictor-corrector technique. Nazeer et al. [24] in 2016 proposed a new second derivative free generalized Newton-Raphson's method with convergence of order five by means of finite difference scheme. In 2017, Kumar et al. [25] suggested a sixth-order parameter-based family of algorithms for solving nonlinear equations. In the same year, Salimi et al. [26] proposed an optimal class of eighth-order methods by using weight functions and Newton interpolation technique. Very recently, Naseem et al. [27] presented some new sixth-order algorithms for finding zeros of nonlinear equations and then investigated their dynamics by means of polynomiography and presented some novel mathematical art through the execution of the presented algorithms.

In this paper, we suggested two novel iteration schemes in the form of predictor-corrector type numerical methods, namely, Algorithms 1 and 2, by taking Newton's iteration method as a predictor step. The derivation of the first iteration scheme is purely based on the Taylor's series expansion and generalized Newton-Raphson's method whereas in second one, we use interpolation technique for removing its second derivative which results the higher efficiency index. We examined the convergence criteria of the suggested schemes and proved that these iteration schemes bearing sextic convergence and superior to the other well-known methods of the similar nature. The efficiency indices of the presented schemes have been compared with the other similar existing two-step iteration schemes. The proposed iteration schemes have been applied to solve some real life problems along with the arbitrary transcendental and algebraic equations in order to assess its applicability, validity, and accuracy.

2. Main Results

Consider the nonlinear algebraic equation

$$f(x)=0. \tag{1}$$

We assume that α is a simple zero of (1) and x_0 is an initial guess sufficiently close to α. Using the Taylor's series around x_0 for (1), we have

$$f(x_0)+(x-x_0)f'(x_0)+\frac{1}{2!}(x-x_0)^2 f''(x_0)+\cdots=0. \tag{2}$$

If $f'(x_0)\neq 0$, we can evaluate the above expression as follows:

$$f(x_0)+(x-x_0)f'(x_0)=0. \tag{3}$$

If we choose x_{j+1} the root of equation, then we have

$$x_{j+1}=x_j-\frac{f(x_j)}{f'(x_j)}. \tag{4}$$

This is quadratically convergent Newton's method [2–4] for root-finding of nonlinear functions and needs two computations for its execution. From (2), one can evaluate

$$x=x_0-\frac{f'(x_0)-\sqrt{f'2(x_0)-2f(x_0)f''(x_0)}}{f''(x_0)}. \tag{5}$$

In iterative form:

$$x_{j+1}=x_j-\frac{f'(x_j)-\sqrt{f'2(x_j)-2f(x_j)f''(x_j)}}{f''(x_j)}, \tag{6}$$

which is cubically convergent generalize Newton-Raphson's method [28] and requires three functional evaluations per iteration for the execution. After simplification of (2), one can obtain:

$$x=x_0-\frac{f(x_0)}{f'(x_0)}-\frac{(x-x_0)^2 f''(x_0)}{2f'(x_0)}. \tag{7}$$

Now from generalized Newton-Raphson's method in (5)

$$x-x_0=-\frac{f'(x_0)-\sqrt{f'2(x_0)-2f(x_0)f''(x_0)}}{f''(x_0)}. \tag{8}$$

Using (8) in (7), we obtain

$$x=x_0-\frac{f(x_0)}{f'(x_0)}-\frac{\left[f'(x_0)-\sqrt{f'2(x_0)-2f(x_0)f''(x_0)}\right]^2}{2f'(x_0)f''(x_0)}. \tag{9}$$

After rewriting the above obtained equality in the general form with the insertion of Newton's iteration method as a predictor, we arrive at a new algorithm of the form:

Algorithm 1. For a given x_0, compute the approximate solution x_{j+1} by the following iterative schemes:

$$y_j=x_j-\frac{f(x_j)}{f'(x_j)},\, j=0,1,2,\cdots,$$

$$x_{j+1} = y_j - \frac{f(y_j)}{f'(y_j)} - \frac{\left[f'(y_j) - \sqrt{f'2(y_j) - 2f(y_j)f''(y_j)}\right]^2}{2f'(y_j)f''(y_j)}, \quad (10)$$

which is the modification of the generalized Newton-Raphson's method for determining the approximate roots of the nonlinear algebraic equations. To find the approximate root of the given nonlinear equation by means of the above described algorithm, one has to find the first as well as the second derivative of the given function $f(x)$. But in several cases, we have to deal with such functions in which second derivative does not exists and our proposed algorithm fails to find approximate root in that situation. To resolve this issue, we apply interpolation technique for the approximation of the second derivative as follows:

Consider the function

$$\rho(u) = a_1 + a_2(u - y_j) + a_3(u - y_j)^2 + a_4(u - y_j)^3, \quad (11)$$

where the values of the unknowns a_1, a_2, a_3, and a_4 can be found by applying the following interpolation conditions:

$$\begin{aligned} &f(x_j) = \rho(x_j), f(w_j) = \rho(w_j), f'(w_j) = \rho'(w_j), \\ &f'(x_j) = \rho'(x_j), f''(w_j) = \rho''(w_j). \end{aligned} \quad (12)$$

From the above conditions, we gain a system containing four linear equations with four variables, the solution of which gives the following equality:

$$\begin{aligned} f''(y_j) &= \frac{6\left[f(x_j) - f(y_j)\right] - 2\left[x_j - y_j\right]\left[2f'(y_j) + f'(y_j)\right]}{(x_j - y_j)^2} \\ &= \rho(x_j, y_j). \end{aligned} \quad (13)$$

After putting the value of $f''(y_j)$ from the above equality in Algorithm 1, we gain novel second-derivative free algorithm as follows:

Algorithm 2. For a given x_0, compute the approximate solution x_{j+1} by the following iterative schemes:

$$y_j = x_j - \frac{f(x_j)}{f'(x_j)}, j = 0, 1, 2, \cdots,$$

$$x_{j+1} = y_j - \frac{f(y_j)}{f'(y_j)} - \frac{\left[f'(y_j) - \sqrt{f'2(y_j) - 2f(y_j)\rho(x_j, y_j)}\right]^2}{2f'(y_j)\rho(x_j, y_j)}, \quad (14)$$

which is a novel second-derivative free iterative algorithm for computing the approximate solutions of the nonlinear algebraic equations. One of the main features of the suggested algorithm is that it can be applied to all those nonlinear functions in which second derivative does not exist. The removal of second derivative causes less number of functional evaluations per iteration which yields the best efficiency index as compared to those methods which require second derivative. The results of the given test examples certified its best performance in comparison with the other similar existing methods in literature.

3. Convergence Analysis

This section includes the discussion regarding the convergence criteria of the suggested iteration schemes.

Theorem 3. *Assuming α as a simple zero of the given equation $f(x) = 0$, where $f(x)$ is sufficiently smooth in the neighborhood of α, then the convergence orders of Algorithms 1 and 2 are at least six.*

Proof. To prove the convergence of Algorithms 1 and 2, we assume that α is the simple root of the equation $f(x) = 0$ and e_j be the error at nth iteration; then, $e_j = x_j - \alpha$ and by using Taylor series about $x = \alpha$, we have

$$\begin{aligned} f(x_j) = &f'(\alpha)e_j + \frac{1}{2!}f''(\alpha)e_j^2 + \frac{1}{3!}f'''(\alpha)e_j^3 + \frac{1}{4!}f^{(iv)}(\alpha)e_j^4 \\ &+ \frac{1}{5!}f^{(v)}(\alpha)e_j^5 + \frac{1}{6!}f^{(vi)}(\alpha)e_j^6 + O(e_j^7), \end{aligned} \quad (15)$$

$$f(x_j) = f'(\alpha)\left[e_j + c_2e_j^2 + c_3e_j^3 + c_4e_j^4 + c_5e_j^5 + c_6e_j^6 + O(e_j^7)\right], \quad (16)$$

$$\begin{aligned} f'(x_j) = &f'(\alpha)\left[1 + 2c_2e_j + 3c_3e_j^2 + 4c_4e_j^3 + c_5e_j^4 + 6c_6e_j^5 \right. \\ &\left. + 7c_7e_j^6 + O(e_j^7)\right], \end{aligned} \quad (17)$$

where

$$c_n = \frac{1}{n!}\frac{f^{(n)}(\alpha)}{f'(\alpha)}. \quad (18)$$

With the help of equations (16) and (17), we get

$$\begin{aligned} y_j = f'(\alpha)\Big[&\alpha + c_2 e_j^2 + \left(2c_3 - 2c_2^2\right)e_j^3 + 3c_4 - 7c_2c_3 + 4c_2^3\big)e_j^4 \\ &+ \left(-6c_3^2 + 20c_3c_2^2 - 10c_2c_4 + 4c_5 - 8c_2^4\right)e_j^5 \\ &+ \big(-17c_4c_3 + 28c_4c_2^2 - 13c_2c_5 + 5c_6 \\ &+ 33c_2c_3^2 - 52c_3c_2^3 + 16c_2^5\big)e_j^6 + O\left(e_j^7\right)\Big], \end{aligned} \tag{19}$$

$$\begin{aligned} f\left(y_j\right) = f'(\alpha)\Big[&c_2 e_j^2 + \left(2c_3 - 2c_2^2\right)e_j^3 + \left(5c_2^3 - 7c_2c_3 + 3c_4\right)e_j^4 \\ &+ \left(24c_3c_2^2 - 12c_2^4 - 10c_2c_4 + 4c_5 - 6c_3^2\right)e_j^5 \\ &+ \big(-73c_3c_2^3 + 34c_4c_2^2 + 28c_2^5 + 37c_2c_3^2 - 17c_4c_3 \\ &- 13c_2c_5 + 5c_6\big)e_j^6 + O\left(e_j^7\right)\Big], \end{aligned} \tag{20}$$

$$\begin{aligned} f'\left(y_j\right) = f'(\alpha)\Big[&1 + 2c_2^2 e_j^2 + \left(4c_2c_3 - 4c_2^3\right)e_j^3 \\ &+ \left(6c_2c_4 - 11c_3c_2^2 + 8c_2^4\right)e_j^4 + \big(28c_3c_2^3 - 20c_4c_2^2 \\ &+ 8c_2c_5 - 16c_2^5\big)e_j^5 + \big(-16c_4c_2c_3 - 68c_3c_2^4 + 12c_3^3 \\ &+ 60c_4c_2^3 - 26c_5c_2^2 + 10c_2c_6 + 32c_2^6\big)e_j^6 + O\left(e_j^7\right)\Big], \end{aligned} \tag{21}$$

$$\begin{aligned} f''\left(y_j\right) = f'(\alpha)\Big[&2c_2 + 6c_2c_3e_j^2 + \left(12c_3^2 - 12c_3c_2^2\right)e_j^3 \\ &+ \left(-42c_2c_3^2 + 18c_4c_3 + 24c_3c_2^3 + 12c_4c_2^2\right)e_j^4 \\ &+ \big(-12c_2c_4c_3 + 24c_5c_3 - 36c_3^3 + 120c_3^2c_2^2 \\ &- 48c_3c_2^4 - 48c_4c_2^3\big)e_j^5 + \big(-78c_3c_2c_5 + 30c_3c_6 \\ &- 54c_4c_3^2 - 96c_3c_4c_2^2 + 198c_2c_3^3 - 312c_3^2c_2^3 + 96c_3c_2^5 \\ &+ 72c_2c_4^2 + 144c_4c_2^4 + 20c_5c_2^3\big)e_j^6 + O\left(e_j^8\right)\Big]. \end{aligned} \tag{22}$$

With the help of equations (16)–(21), we have

$$\begin{aligned} \rho\left(x_j, y_j\right) = f'(\alpha)\Big[&2c_2 + (6c_2c_3 - 2c_4)e_j^2 + \big(12c_3^2 - 12c_3c_2^2 \\ &+ 4c_2c_4 - 4c_5\big)e_j^3 + \big(2c_2c_5 + 26c_3c_4 - 42c_2c_3^2 \\ &+ 24c_3c_2^3 + 2c_4c_2^2 - 6c_6\big)e_j^4 + \big(-48c_4c_2c_3 + 12c_4^2 \\ &- 24c_4c_2^3 + 28c_5c_3 + 4c_5c_2^2 + 120c_3^2c_2^2 - 48c_3c_2^4 \\ &- 8c_7 - 36c_3^3\big)e_j^5 + \big(-60c_5c_2c_3 + 28c_4c_3c_2^2 - 2c_2c_7 \\ &+ 22c_5c_4 - 10c_5c_2^3 + 30c_6c_3 + 6c_6c_2^2 + 20c_2c_4^2 \\ &- 86c_4c_3^2 + 88c_4c_2^4 + 198c_2c_3^3 - 312c_3^2c_2^3 \\ &+ 96c_3c_2^5 - 10c_8\big)e_j^6 + O\left(e_j^7\right)\Big]. \end{aligned} \tag{23}$$

Using equations (19)–(23) in Algorithms 1 and 2, we get the following equalities

$$\begin{aligned} x_{j+1} &= \alpha + \left(-c_3c_2^3\right)e_j^6 + O\left(e^7\right), \\ x_{j+1} &= \alpha + \left(-c_3c_2^3 + c_4c_2^2\right)e_j^6 + O\left(e^7\right), \end{aligned} \tag{24}$$

which imply that

$$e_{n+1} = \left(-c_3c_2^3\right)e_j^6 + O\left(e^7\right), \tag{25}$$

$$e_{n+1} = \left(-c_3c_2^3 + c_4c_2^2\right)e_j^6 + O\left(e^7\right). \tag{26}$$

Equations (25) and (26) show that the orders of convergence of Algorithms 1 and 2 are atleast six.

4. Comparison of Efficiency Index

In numerical analysis, the efficiency index of an algorithm provides us the information about the speed and performance of the algorithm which is being under the consideration. It is actually a numerical quantity that relates to the number of computational resources needed to execute the considered algorithm. The efficiency of an algorithm can be thought of as analogous to the engineering productivity for a process that includes iterations. The term efficiency index is used to analyze the numeric behavior of different algorithms. In iterative algorithms, this quantity totally depends upon the two factors. The first one is the convergence order of the algorithm whereas the second factor is the number of computations per iteration, i.e., the number of functional and derivatives evaluations, required to execute the algorithm for the purpose of root-finding of the nonlinear functions. If the convergence order is represented by P and the number of computations per iteration by n_f, then the efficiency index can be written mathematically as:

$$\text{Efficiency Index} = P^{1/n_f}. \tag{27}$$

Since Noor's method one [11] has quadratic convergence and requires three computations per iteration for execution, so its efficiency index will be $2^{1/3} \approx 1.2599$. In the same way, the cubically convergent Noor's method two [11] requires three computations per iteration and has $3^{1/3} \approx 1.4422$ as an efficiency index. Similarly, the efficiency index of the Traub's methods [6] is $4^{1/4} \approx 1.4142$ because it possesses the convergence of order four with four computations for execution. Since the modified Halley's method [22] has fifth convergence order with four computations per iteration, so its efficiency index will be $5^{1/4} \approx 1.4953$. Now, we calculate the efficiency indices of the suggested algorithms. Both algorithms bearing the convergence of order six. The number of computations per iteration for the execution of the first algorithm is five whereas the second proposed algorithm requires only four evaluations per iteration. So, their efficiency indices will be $6^{1/5} \approx 1.4310$ and $6^{1/4} \approx 1.5651$, respectively. The efficiency indices of the different iterative methods, we have discussed above, are summarized in the following Table 1.

Table 1: Comparison of efficiency indices of different iterative methods.

Method	Convergence order	No. of required computations	Efficiency index
Noor's method one	2	3	1.2599
Noor's method two	3	3	1.4422
Traub's method	4	4	1.4142
Modified Halley's method	5	4	1.4953
Algorithm 1	6	5	1.4310
Algorithm 2	6	4	1.5651

Table 1 clearly shows that the presented method, namely, Algorithm 2, has better efficiency index among the other compared methods.

5. Numerical Comparisons and Applications

In this section, we include four real-life engineering problems and seven arbitrary problems in the form of transcendental and algebraic equations to illustrate the applicability and efficiency of our newly developed iterative methods. We compare these methods with the following similar existing two-step iteration schemes:

5.1. Noor's Method One (NM1). For a provided initial guess x_0, determine the approximate root x_{j+1} with the iteration schemes given below:

$$\begin{aligned} x_{j+1} &= x_j - \frac{f(x_j)}{f'(x_j)}, j = 0, 1, 2, 3, \cdots, \\ x_{j+1} &= x_j - \frac{f(x_j)}{f'(x_j)} + \left[\frac{f(x_j)}{f'(x_j)}\right]\frac{f'(y_j)}{f'(x_j)}, \end{aligned} \tag{28}$$

which is quadratically convergent Noor's method one [11] for root-finding of nonlinear equations.

5.2. Noor's Method Two (NM2). For a provided initial guess x_0, determine the approximate root x_{j+1} with the iteration schemes given below:

$$\begin{aligned} y_j &= x_j - \frac{f(x_j)}{f'(x_j)}, j = 0, 1, 2, 3, \cdots, \\ x_{j+1} &= x_j - \frac{2f(x_j)}{f'(x_j) + f'(y_j)}, \end{aligned} \tag{29}$$

which is cubically convergent Noor's method two [11] for root-finding of nonlinear equations.

5.3. Traub's Method (TM). For a provided initial guess x_0, determine the approximate root x_{j+1} with the iteration schemes given below:

$$\begin{aligned} y_j &= x_j - \frac{f(x_j)}{f'(x_j)}, j = 0, 1, 2, 3, \cdots, \\ x_{j+1} &= y_j - \frac{f(y_j)}{f'(y_j)}, \end{aligned} \tag{30}$$

which is two-step fourth order Traub's method [6] for root-finding of nonlinear equations which bearing the convergence of order four.

5.4. Modified Halley's Method (MHM). For a provided initial guess x_0, determine the approximate root x_{j+1} with the iteration schemes given below:

$$\begin{aligned} y_j &= x_j - \frac{f(x_j)}{f'(x_j)}, j = 0, 1, 2, 3, \cdots, \\ x_{j+1} &= y_j - \frac{2f(x_j)f(y_j)f'(y_j)}{2f(x_j)f'2(y_j) - f'2(x_j)f(y_j) + f'(x_j)f'(y_j)f(y_j)}, \end{aligned} \tag{31}$$

which is two-step Halley' method [22] for root-finding of nonlinear equations which has the convergence of fifth order. In order to make the numerical comparison of the above defined methods with the presented algorithms, we consider the following test Examples 1–5.

The general algorithm for finding the approximate solution of the given nonlinear functions is given as:

In Algorithm 3, we take the accuracy $\varepsilon = 10^{-15}$ in the stopping criteria $|x_{j+1} - x_j| < \varepsilon$. We did all the calculations of the numerical examples with the aid of the computer program Maple 13, and their numerical results can be seen in the following presented Tables 2–6.

Example 1. Adiabatic flame temperature equation. The adiabatic flame temperature equation is represented by the following relation:

$$f_1(x) = \Delta H + a_1(x - 298) + \frac{a_2}{2}\left(x^2 - 298^2\right) + \frac{a_3}{3}\left(x^3 - 298^3\right), \tag{32}$$

where $\Delta H = -57798$, $a_1 = 7.256$, $a_2 = 0.002298$, and $a_3 = 0.00000283$. For further details, see [29, 30] and the references therein. The above function is actually a polynomial of degree three, and by the fundamental theorem of Algebra, it must have exactly three roots. Among these roots, $\alpha = 4305.3099136661$ is a simple one which we approximated

Input: $f \in \mathbb{R}$ — non-linear function, k — maximum number of iterations, I — iteration method, ε — accuracy.
Output: Approximated root of the given non-linear function.
for $x_0 \in A$ **do**
  $i = 0$.
  while $i \leq k$ **do**
    $x_{j+1} = I(x_j)$
    if $|x_{j+1} - x_j| < \varepsilon$ **then**
      break
    $i = i + 1$
  x_{j+1} is the required solution.

Algorithm 3: General root's finding algorithm.

Table 2: Numerical comparison among different algorithms for the engineering problem f_1.

Method	N	$\|f(x_{j+1})\|$	x_{j+1}	$\sigma = \|x_{j+1} - x_j\|$	COC
$f_1(x)$, $x_0 = 2050.0$					
NR1	9	$3.688522e - 28$	4305.30991366612556300000	$3.947209e - 13$	2
NR2	4	$2.919985e - 37$	4305.30991366612556300000	$9.938805e - 11$	3
TM	3	$6.063382e - 31$	4305.30991366612556300000	$1.002459e - 05$	4
MHM	3	$1.311971e - 69$	4305.30991366612556300000	$1.526816e - 11$	5
Algorithm 1	2	$3.738643e - 18$	4305.30991366612556300000	$1.795691e - 00$	6
Algorithm 2	2	$3.738643e - 18$	4305.30991366612556300000	$1.795691e - 00$	6

Table 3: Numerical comparison among different algorithms for the engineering problem f_2.

Method	N	$\|f(x_{j+1})\|$	x_{j+1}	$\sigma = \|x_{j+1} - x_j\|$	COC
$f_2(x)$, $x_0 = 0.1$					
NR1	7	$9.675391e - 26$	0.27775954284172065910	$1.053628e - 13$	2
NR2	3	$1.203488e - 18$	0.27775954284172065910	$6.173552e - 07$	3
TM	3	$3.260304e - 39$	0.27775954284172065910	$1.412011e - 10$	4
MHM	2	$2.057683e - 15$	0.27775954284172065910	$1.970771e - 04$	5
Algorithm 1	2	$6.942638e - 22$	0.27775954284172065910	$2.202374e - 04$	6
Algorithm 2	2	$1.726207e - 21$	0.27775954284172065910	$2.502938e - 04$	6

Table 4: Numerical comparison among different algorithms for the engineering problem f_3.

Method	N	$\|f(x_{j+1})\|$	x_{j+1}	$\sigma = \|x_{j+1} - x_j\|$	COC
$f_3(x)$, $x_0 = 2.0$					
NR1	4	$1.319023e - 19$	1.92984624284786221696	$5.000588e - 10$	2
NR2	3	$3.958485e - 15$	1.92984624284786221696	$1.021691e - 05$	3
TM	3	$8.395139e - 34$	1.92984624284786221696	$2.556739e - 09$	4
MHM	2	$8.089146e - 19$	1.92984624284786221696	$1.079121e - 04$	5
Algorithm 1	2	$4.275791e - 23$	1.92984624284786221696	$7.584886e - 05$	6
Algorithm 2	2	$4.275791e - 23$	1.92984624284786221696	$7.584886e - 05$	6

through the proposed methods by choosing the initial guess $x_0 = 2050$, and the numerical results have been shown in Table2.

Example 2. Fraction conversion of nitrogen-hydrogen to ammonia. We take this example from [31], which describe the fraction conversion of nitrogen-hydrogen feed to

Table 5: Numerical comparison among different algorithms for the engineering problem f_4.

Method	N	$\lvert f(x_{j+1})\rvert$	x_{j+1}	$\sigma = \lvert x_{j+1} - x_j\rvert$	COC
$f_4(x)$, $x_0 = 0.4$					
NR1	6	$2.230770e-24$	1.46509122029582464238	$1.280653e-12$	2
NR2	3	$5.751429e-27$	1.46509122029582464238	$4.220179e-09$	3
TM	3	$7.765624e-44$	1.46509122029582464238	$4.884637e-11$	4
MHM	3	$4.020841e-64$	1.46509122029582464238	$5.749371e-13$	5
Algorithm 1	2	$2.394555e-20$	1.46509122029582464238	$1.957777e-03$	6
Algorithm 2	2	$1.709800e-15$	1.46509122029582464238	$9.937512e-03$	6

ammonia, usually known as fractional conversion. In this problem, the values of temperature and pressure have been taken as 500°C and 250 atm, respectively. This problem has the following nonlinear form:

$$f_2(x) = -0.186 - \frac{8x^2(x-4)^2}{9(x-2)^3}, \tag{33}$$

which can be easily reduced to the following polynomial:

$$f_2(x) = x^4 - 7.79075x^3 + 14.7445x^2 + 2.511x - 1.674. \tag{34}$$

Since the degree of the above polynomial is four, so, it must have exactly four roots. By definition, the fraction conversion lies in $(0,1)$ interval, so only one real root exists in this interval which is 0.2777595428. The other three roots have no physical meanings. We started the iteration process by the initial guess $x_0 = 0.1$. The numerical results through different methods have been shown in Table 3.

Example 3. Finding volume from van der Waal's equation. In Chemical Engineering, the van der Waal's equation has been used for interpreting real and ideal gas behavior [32], having the following form:

$$\left(P + \frac{A_1 n^2}{V^2}\right)(V - nA_2) = nRT. \tag{35}$$

By taking the specific values of the parameters of the above equation, we can easily convert it to the following nonlinear function:

$$f_3(x) = 0.986x^3 - 5.181x^2 + 9.067x - 5.289, \tag{36}$$

where s represents the volume that can easily be found by solving the function f_3. Since the degree of the polynomial is three, so it must possess three roots. Among these roots, there is only one positive real root 1.9298462428 which is feasible because the volume of the gas can never be negative. We start the iteration process with the initial guess $x_0 = 2.0$, and their results can be seen in Table 4.

Example 4. Open channel flow problem. The water flow in an open channel with uniform flow condition is given by Manning's equation [33], having the following standard form:

$$\text{Water Flow} = F = \frac{\sqrt{s}ar^{2/3}}{n}, \tag{37}$$

where s, a, and r represent the slope, area, and hydraulic radius of the corresponding channel, respectively, and n denotes Manning's roughness coefficient. For a rectangular-shaped channel, having width b and depth of water in channel x, then we may write:

$$\begin{aligned} a &= bx, \\ r &= \frac{bx}{b+2x}. \end{aligned} \tag{38}$$

Using these values in (37), we obtain:

$$F = \frac{\sqrt{s}bx}{n}\left(\frac{bx}{b+2x}\right)^{2/3}. \tag{39}$$

To find the depth of water in the channel for a given quantity of water, the above equation may written in the form of nonlinear function as:

$$f_4(x) = \frac{\sqrt{s}bx}{n}\left(\frac{bx}{b+2x}\right)^{2/3} - F. \tag{40}$$

We take the values of different parameters as $F = 14.15$ m^3/s, $b = 4.572$ m, $s = 0.017$, and $n = 0.0015$. We choose the initial guess $x_0 = 0.4$ to start the iteration process, and the corresponding results through different iteration schemes are given in Table 5.

Table 6: Numerical comparison among different algorithms for transcendental and algebraic problems $f_5 - f_{11}$.

Method	N	$\lvert f(x_{j+1}) \rvert$	x_{j+1}	$\sigma = \lvert x_{j+1} - x_j \rvert$	COC
$f_5(x)$, $x_0 = -2$					
NR1	9	$1.485377e-16$	−0.52248077281054548914	$1.517821e-08$	2
NR2	7	$4.247308e-18$	−0.52248077281054548914	$1.202925e-06$	3
TM	25	$1.821056e-17$	−0.52248077281054548914	$1.970771e-04$	4
MHM	3	$8.137892e-19$	−0.52248077281054548914	$3.574246e-04$	5
Algorithm 1	2	$6.584167e-72$	−0.52248077281054548914	$2.829449e-12$	6
Algorithm 2	2	$3.852650e-57$	−0.52248077281054548914	$8.860340e-10$	6
$f_6(x)$, $x_0 = 2$					
NR1	5	$3.163807e-17$	0.40999201798913713162	$5.629386e-09$	2
NR2	45	$3.382231e-18$	0.40999201798913713162	$1.512669e-06$	3
TM	4	$6.063382e-56$	0.40999201798913713162	$1.586231e-14$	4
MHM	3	$4.499472e-42$	0.40999201798913713162	$1.880726e-08$	5
Algorithm 1	2	$5.856963e-25$	0.40999201798913713162	$2.788223e-04$	6
Algorithm 2	2	$6.737587e-25$	0.40999201798913713162	$2.852107e-04$	6
$f_7(x)$, $x_0 = 1.2$					
NR1	7	$3.278748e-27$	0.56714329040978387300	$4.592634e-14$	2
NR2	4	$1.127879e-31$	0.56714329040978387300	$3.980665e-11$	3
TM	3	$3.135655e-36$	0.56714329040978387300	$1.588919e-09$	4
MHM	3	$7.204101e-59$	0.56714329040978387300	$2.420306e-12$	5
Algorithm 1	2	$2.894187e-17$	0.56714329040978387300	$2.115592e-03$	6
Algorithm 2	2	$7.355480e-18$	0.56714329040978387300	$1.597483e-03$	6
$f_8(x)$, $x_0 = 1.5$					
NR1	141	$3.196546e-22$	2.15443469003188372180	$7.032555e-12$	2
NR2	4	$7.066989e-31$	2.15443469003188372180	$5.866631e-11$	3
TM	3	$7.857615e-27$	2.15443469003188372180	$2.740790e-07$	4
MHM	3	$2.248861e-57$	2.15443469003188372180	$5.868237e-12$	5
Algorithm 1	2	$2.929220e-17$	2.15443469003188372180	$2.575011e-03$	6
Algorithm 2	2	$2.929220e-17$	2.15443469003188372180	$2.575011e-03$	6
$f_9(x)$, $x_0 = 0.6$					
NR1	78	$2.720561e-26$	1.00000000000000000000	$8.247062e-14$	2
NR2	4	$3.726794e-29$	1.00000000000000000000	$3.726794e-29$	3
TM	3	$1.853180e-25$	1.00000000000000000000	$5.187039e-07$	4
MHM	3	$2.707444e-54$	1.00000000000000000000	$1.950103e-11$	5
Algorithm 1	2	$8.487620e-17$	1.00000000000000000000	$2.340774e-03$	6
Algorithm 2	2	$8.487620e-17$	1.00000000000000000000	$2.340774e-03$	2
$f_{10}(x)$, $x_0 = -3.0$					
NR1	6	$7.104675e-20$	−1.40449164821534122600	$1.911132e-10$	2
NR2	3	$2.634964e-20$	−1.40449164821534122600	$2.526930e-07$	3
TM	3	$2.820318e-22$	−1.40449164821534122600	$3.920088e-06$	4
MHM	3	$5.741591e-44$	−1.40449164821534122600	$3.100243e-09$	5
Algorithm 1	2	$7.567005e-19$	−1.40449164821534122600	$1.391725e-03$	6
Algorithm 2	2	$5.794073e-16$	−1.40449164821534122600	$3.524255e-03$	2

Table 6: Continued.

Method	N	$\lvert f(x_{j+1})\rvert$	x_{j+1}	$\sigma = \lvert x_{j+1} - x_j\rvert$	COC
$f_{11}(x)$, $x_0 = 2.0$					
NR1	4	$2.455690e-16$	0.00000000000000000000	$9.031605e-06$	2
NR2	4	$3.424681e-40$	0.00000000000000000000	$1.271327e-13$	3
TM	3	$7.907266e-122$	0.00000000000000000000	$5.282400e-14$	4
MHM	3	$1.556811e-66$	0.00000000000000000000	$6.000887e-10$	5
Algorithm 1	2	$2.328631e-21$	0.00000000000000000000	$8.308383e-03$	6
Algorithm 2	2	$3.198129e-15$	0.00000000000000000000	$3.593833e-02$	2

Table 7: Comparison of the iterations consumed by different algorithms for the accuracy $\varepsilon = 10^{-100}$.

Function with initial guess	Method					
	NM1	NM2	TM	MHM	Algorithm 1	Algorithm 2
$f_1(x)$, $x_0 = 2050.0$	09	05	04	04	03	03
$f_2(x)$, $x_0 = 0.1$	09	06	06	06	06	05
$f_3(x)$, $x_0 = 2.0$	07	05	04	04	03	03
$f_4(x)$, $x_0 = 0.4$	08	05	04	05	03	04
$f_5(x)$, $x_0 = -2.0$	12	09	27	05	04	04
$f_6(x)$, $x_0 = 2.0$	08	06	05	04	03	03
$f_7(x)$, $x_0 = 1.2$	09	06	04	04	03	03
$f_8(x)$, $x_0 = 1.5$	144	06	04	04	03	03
$f_9(x)$, $x_0 = 0.6$	80	06	04	04	03	03
$f_{10}(x)$, $x_0 = -3.0$	09	05	05	04	03	04
$f_{11}(x)$, $x_0 = 2.0$	06	05	03	04	03	03

Example 5. Transcendental and algebraic problems. To numerically analyze the suggested algorithms, we consider the following seven transcendental and algebraic equations:

$$\begin{aligned}
f_5(x) &= e^x + \cos(\pi x) + x, x_0 = -2.0,\\
f_6(x) &= x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}, x_0 = 2.0,\\
f_7(x) &= \ln x + x, x_0 = 1.2,\\
f_8(x) &= x^3 - 10, x_0 = 1.5,\\
f_9(x) &= x^3 + x^2 - 2, x_0 = 0.6,\\
f_{10}(x) &= \sin^2 x - x^2 + 1, x_0 = -3.0,\\
f_{11}(x) &= \tan^{-1}(x) + x, x_0 = 2.0,
\end{aligned} \tag{41}$$

and their numerical results can be seen in Table 6.

Tables 2–6 exhibit the numerical comparison of the suggested algorithms with other similar-nature existing algorithms. In the columns of the above presented tables, N represents the iterations consumed by different algorithms, $\lvert f(x)\rvert$ denotes the absolute value of $f(x)$ at final approximation, x_{j+1} shows the final approximated root, $\lvert x_{j+1} - x_j\rvert$ represents the absolute distance between the two consecutive approximations, and (COC) denotes the computational order of convergence having the following approximated formula:

$$\mathrm{COC} \approx \frac{\ln\left(\left|x_{j+1} - \alpha\right| / \left|x_j - \alpha\right|\right)}{\ln\left(\left|x_j - \alpha\right| / \left|x_{j-1} - \alpha\right|\right)}. \tag{42}$$

The above approximation was firstly suggested in 2000 by Weerakoon and Fernando [34]. When we look at the numerical results of Tables 2–6, we come to know that the presented methods are showing best performance as compared to the other ones. For example, in second, fourth, fifth, tenth, and eleventh test examples, Algorithm 1 is the best as it took less number of iterations among the all other compared methods with great precision. In the seventh test example, Algorithm 2 showing the best performance than the other ones whereas in first, third, sixth, eighth, and ninth test examples, both proposed algorithms behave alike and looks better than all the other ones. In short, we can say that the proposed algorithms are superior in terms of accuracy, speed, number of iterations, and computational order of convergence to the other well-known existing iteration schemes.

Table 7 exhibits the comparison of the iterations consumed by different algorithms with the newly proposed methods for the root-finding of nonlinear algebraic functions with the accuracy $\varepsilon = 10^{-100}$. Here, the columns of the table denote the iterations' number for various test functions together with the initial guess x_0. The numerical results as shown in Table 7 again certified the fast and best performance of the presented algorithms in terms of number of iterations for the above defined stopping criteria with the given accuracy. In all test examples, the proposed algorithms consumed less number of iterations in comparison with the other iterative algorithms. We did all the calculations with the aid of the computer program Maple 13.

6. Concluding Remarks

In this work, two novel iteration schemes for computing the zeros of nonlinear functions have been established which possess the sextic convergence. The first iteration scheme is derived using the Taylor's series expansion and generalized Newton-Raphson's method whereas in second one, we apply the basic idea of interpolation technique for approximating second derivative which results higher efficiency index. A comparison table regarding the efficiency indices of different methods of the similar nature has been presented which shows that the presented method has higher efficiency index among the other compared methods. By solving some engineering and arbitrary test problems with the aid of computer program, the validity and applicability of the suggested iteration schemes have been analyzed. The numerical results of the Tables 1–7 certified the superiority of the suggested iteration schemes to the other existing two-step iteration schemes of the similar nature. Using the basic idea of interpolation technique, one can derive a broad range of new iteration schemes for computing zeros of one-dimensional nonlinear equations.

Data Availability

All data required for this paper is including within this paper.

Conflicts of Interest

The authors do not have any conflicts.

Authors' Contributions

All authors contribute equally in this paper.

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Research Article

On the Porous-Elastic System with Thermoelasticity of Type III and Distributed Delay: Well-Posedness and Stability

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The paper deals with a one-dimensional porous-elastic system with thermoelasticity of type III and distributed delay term. This model is dealing with dynamics of engineering structures and nonclassical problems of mathematical physics. We establish the well posedness of the system, and by the energy method combined with Lyapunov functions, we discuss the stability of system for both cases of equal and nonequal speeds of wave propagation.

1. Introduction

Let $\mathscr{H} = (0,1) \times (\tau_1, \tau_2) \times (0,\infty)$, $\tau_1, \tau_2 > 0$. For $(x, s, t) \in \mathscr{H}$, we consider the following porous-elastic system:

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\phi_x, \\ \rho_2 \phi_{tt} = \delta \phi_{xx} - b u_x - \xi \phi - \beta \theta_x - \mu_1 \phi_t - \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| \phi_t(x, t-s) ds, \\ \rho_3 \theta_{tt} = l\theta_{xx} - \gamma \phi_{ttx} + k\theta_{txx}, \end{cases} \tag{1}$$

with the initial data

$$\begin{aligned} & u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ & \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), \phi_t(x,-t) = f_0(x,t), \\ & \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), x \in (0,1), \quad t > 0 \end{aligned} \tag{2}$$

and boundary conditions

$$u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \quad t \geq 0. \tag{3}$$

Here, ϕ is the volume fraction of the solid elastic material, u is the longitudinal displacement, and θ is the difference in temperatures. The parameters $\rho_1, \rho_2, \rho_3, \mu, b, \delta, \xi, l, \gamma, \beta, k$ are positive constants with $\mu\xi > b^2$. The integral represents the distributed delay term withτ_1, τ_2which are time delays, μ_1 is positive constant, and μ_2 is an L^∞ function such that

(Hyp1) $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds < \mu_1. \tag{4}$$

This type of problem was mainly based on the following equations for one-dimensional theories of porous materials with temperature

$$\begin{cases} \rho_1 u_{tt} - T_x = 0, \\ \rho_2 \phi_{tt} - H_x - G = 0, \\ \rho_3 \theta_t + q_x + \gamma \phi_{tx} = 0, \end{cases} \quad (5)$$

where $(x, t) \in (0, L) \times (0, \infty)$.

According to Green and Naghdis theory, the constitutive equations of system (5) are given by

$$T = \mu u_x + b\phi, \quad (6)$$

$$G = -bu_x - \xi\phi - \mu_1 \phi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \phi_t(x, t - s)\, ds, \quad (7)$$

$$H = \delta \phi_x - \beta\theta, \quad (8)$$

$$q = -l\Phi_x - k\Phi_{tx}, \quad (9)$$

where $l, k > 0$ are the thermal conductivity and Φ is the thermal displacement whose time derivative is the empirical temperature θ, that is $\Phi_t = \theta$.

We substitute (9) in (5) with the condition $b \neq 0$, which results in

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\phi_x, \\ \rho_2 \phi_{tt} = \delta \phi_{xx} - bu_x - \xi\phi - \mu_1 \phi_t - \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| \phi_t(x, t - s)\, ds - \beta\theta_x, \\ \rho_3 \theta_t = l\Phi_{xx} - \gamma \phi_{tx} + k\Phi_{txx}. \end{cases} \quad (10)$$

By using $\Phi_t = \theta$ in the system (10), we find directly our system (1).

By using the multiplier techniques, the exponential decay results have been established. Next, in [1–3], the authors considered three types of thermoelastic theories based on an entropy equality instead of the usual entropy inequality (see [1–21] for more details).

According to the distributed delay, we mention, as a matter of course, the work by Nicaise and Pignotti in [16], where the authors studied the following system with distributed delay:

$$\begin{cases} u_{tt} - \Delta u = 0, \\ u = 0, \\ \dfrac{du}{dv}(t) + \displaystyle\int_{\tau_1}^{\tau_2} \mu(s) u_t(t - s)\, ds + \mu_0 u_t = 0, \\ u(.,0) = u_0, u_t(.,0) = u_1, u_t(x, -t) = f_0(x, t), \end{cases} \quad (11)$$

and proved the exponential stability result with condition

$$\int_{\tau_1}^{\tau_2} \mu(s)\, ds < \mu_0. \quad (12)$$

See for example [8, 22, 23]. Hao and Wei [24] considered the following problem:

$$\begin{cases} \rho_1 \phi_{tt} - K(\phi_x + \psi_x)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\phi_x + \psi) + \beta\theta_{tx} + \mu_1 \psi_t + \mu_2 \psi_t(t - s) + f(\psi_t) = 0, \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \gamma \varphi_{ttx} - k\theta_{txx} = 0, \end{cases} \quad (13)$$

and obtained the well-posedness and stability of system.

There are many other works done by the authors in this context; our work differs from all of them, since we took the delay in the second equation to make the distributed delay in the rotation angle of the filament, which makes the contributions clear and important. In addition, we established the well-posedness of the system, and we obtain the exponential decay rate when $\delta/\rho_2 = \mu/\rho_1$ and the energy takes the algebraic rate for the case $\delta/\rho_2 \neq \mu/\rho_1$; these results are mainly stated in Theorem 8.

In order to show the dissipativity of systems (1)–(3), we introduce the new variables $\varphi = u_t$ and $\psi = \phi_t$. So, problems (1)–(3) take the form

$$\begin{cases} \rho_1 \varphi_{tt} = \mu \varphi_{xx} + b\psi_x, \\ \rho_2 \psi_{tt} = \delta \psi_{xx} - b\varphi_x - \xi\psi - \mu_1 \psi_t - \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| \psi_t(x, t - s)\, ds - \beta\theta_{tx}, \\ \rho_3 \theta_{tt} = l\theta_{xx} - \gamma \psi_{tx} + k\theta_{txx}, \end{cases} \quad (14)$$

with the initial data

$$\begin{aligned} &\varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \psi(x, 0) = \psi_0(x), \\ &\psi_t(x, 0) = \psi_1(x), \theta(x, 0) = \theta_0(x), \theta_t(x, 0) = \theta_1(x), \\ &\psi_t(x, -t) = -f_0(x, t), \quad x \in (0, 1) \end{aligned} \quad (15)$$

and boundary conditions

$$\varphi_x(0, t) = \varphi_x(1, t) = \psi(0, t) = \psi(1, t) = \theta_x(0, t) = \theta_x(1, t) = 0, \quad t \geq 0. \quad (16)$$

First, as in [16], taking the following new variable:

$$z(x, \rho, s, t) = \psi_t(x, t - s\rho), \quad (17)$$

then we obtain

$$\begin{cases} s z_t(x, \rho, s, t) + z_\rho(x, \rho, s, t) = 0, \\ z(x, 0, s, t) = \psi_t(x, t). \end{cases} \quad (18)$$

Consequently, the problem was rewritten as

$$\begin{cases} \rho_1\varphi_{tt} = \mu\varphi_{xx} + b\psi_x, \\ \rho_2\psi_{tt} = \delta\psi_{xx} - b\varphi_x - \xi\psi - \mu_1\psi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t)\,ds - \beta\theta_{tx}, \\ \rho_3\theta_{tt} = l\theta_{xx} - \gamma\psi_{tx} + k\theta_{txx}, \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \end{cases} \tag{19}$$

where

$$(x,\rho,s,t) \in (0,1) \times \mathscr{H}, \tag{20}$$

with the boundary and the initial conditions

$$\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \quad t \geq 0. \tag{21}$$

$$\varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x), \tag{22}$$

$$\psi_t(x,0) = \psi_1(x), \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), \quad x \in (0,1), \tag{23}$$

$$z(x,\rho,s,0) = -f_0(x,\rho s) = h_0(x,\rho s), \quad x \in (0.1), \rho \in (0.1), s \in (0,\tau_2). \tag{24}$$

Meanwhile, from (19) and (24), it follows that

$$\frac{d^2}{dt^2}\int_0^1 \varphi(x,t)\,dx = 0. \tag{25}$$

So, by solving (25) and using (24), we get

$$\int_0^1 \varphi(x,t)\,dx = t\int_0^1 \varphi_1(x)\,dx + \int_0^1 \varphi_0(x)\,dx. \tag{26}$$

Consequently, if we let

$$\bar{\varphi}(x,t) = \varphi(x,t) - t\int_0^1 \varphi_1(x)\,dx - \int_0^1 \varphi_0(x)\,dx, \tag{27}$$

we get

$$\int_0^1 \bar{\varphi}(x,t)\,dx = 0, \quad \forall t \geq 0, \tag{28}$$

and from (19), we have

$$\frac{d^2}{dt^2}\int_0^1 \theta(x,t)\,dx = 0. \tag{29}$$

So, by solving (29) and using (24), we get

$$\int_0^1 \theta(x,t)\,dx = t\int_0^1 \theta_1(x)\,dx + \int_0^1 \theta_0(x)\,dx. \tag{30}$$

Consequently, if we let

$$\bar{\theta}(x,t) = \theta(x,t) - t\int_0^1 \theta_1(x)\,dx - \int_0^1 \theta_0(x)\,dx, \tag{31}$$

we get

$$\int_0^1 \bar{\theta}(x,t)\,dx = 0, \quad \forall t \geq 0. \tag{32}$$

Then, the Poincaré's inequality was used for $\bar{\varphi}$ and θ^- which are justified. A simple substitution shows that $(\bar{\varphi}, \psi, \bar{\theta})$ satisfies system (19) with initial data for $\bar{\varphi}$ and $\bar{\theta}$ given as

$$\begin{aligned} \bar{\varphi}_0(x) &= \varphi_0(x) - \int_0^1 \varphi_0(x)\,dx, \\ \bar{\varphi}_1(x) &= \varphi_1(x) - \int_0^1 \varphi_1(x)\,dx, \\ \bar{\theta}_0(x) &= \theta_0(x) - \int_0^1 \theta_0(x)\,dx, \\ \bar{\theta}_1(x) &= \theta_1(x) - \int_0^1 \theta_1(x)\,dx. \end{aligned} \tag{33}$$

Now, we use $\bar{\varphi}, \bar{\theta}$ instead of φ, θ and writing φ, θ for simplicity.

2. Well-Posedness

In this section, we give the existence and uniqueness result of the system (19)–(24) using the semigroup theory.

First, we introduce the vector function

$$U = (\varphi, \varphi_t, \psi, \psi_t, \theta, \theta_t, z)^T, \tag{34}$$

and the new dependent variables $u = \varphi_t$, $v = \psi_t$, $w = \theta_t$; then the system (19) can be written as follows:

$$\begin{cases} U_t = \mathscr{A}U, \\ U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1, h_0)^T, \end{cases} \tag{35}$$

where $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathscr{H} : \longrightarrow \mathscr{H}$ is the linear operator defined by

$$\mathscr{A}U = \begin{pmatrix} u \\ \frac{1}{\rho_1}[\mu\varphi_{xx} + b\psi_x] \\ v \\ \frac{1}{\rho_2}\left[\delta\psi_{xx} - b\varphi_x - \xi\psi - \beta w_x - \mu_1\psi_t - -\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z(x,1,s,t)\,ds\right] \\ w \\ \frac{1}{\rho_3}[l\theta_{xx} - \gamma v_x + k w_{xx}] \\ -\frac{1}{s} z_\rho \end{pmatrix}, \tag{36}$$

and $\mathscr{H}$ is the energy space given by

$$\mathscr{H} = H_*^1 \times L_*^2(0,1) \times H_0^1 \times L^2(0,1) \times H_*^1 \times L^2(0,1) \times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \quad (37)$$

where

$$L_*^2(0,1) = \left\{ \frac{\phi \in L^2(0,1)}{\int_0^1 \phi(x)dx} = 0 \right\},$$
$$H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1), \quad (38)$$
$$H_*^2(0,1) = \left\{ \frac{\phi \in H^2(0,1)}{\phi_x(1) = \phi_x(0)} = 0 \right\}.$$

For every

$$U = (\varphi, u, \psi, v, \theta, w, z)^T \in \mathscr{H},$$
$$\widehat{U} = (\varphi\wedge, u\wedge, \psi\wedge, v\wedge, \theta\wedge, w\wedge, z\wedge)^T \in \mathscr{H}, \quad (39)$$

we equip $\mathscr{H}$ with the inner product defined by

$$\begin{aligned} <U, \widehat{U}>_{\mathscr{H}} &= \gamma\rho_1\int_0^1 u\widehat{u}dx + \gamma\rho_2\int_0^1 v\widehat{v}dx + \gamma\xi\int_0^1 \psi\widehat{\psi}dx \\ &+ \beta\rho_3\int_0^1 w\widehat{w}dx + \gamma\mu\int_0^1 \varphi_x\widehat{\varphi}_x dx + \gamma\delta\int_0^1 \psi_x\widehat{\psi}_x dx \\ &+ \gamma b\int_0^1 \left(\varphi_x\widehat{\psi} + \psi\widehat{\varphi}\right)dx + l\beta\int_0^1 \theta_x\widehat{\theta}_x dx \\ &+ \gamma\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z\widehat{z}dsd\rho dx. \end{aligned} \quad (40)$$

The domain of $\mathscr{A}$ is given by

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{l} U \in \mathscr{H}/\varphi, \theta \in H_*^2(0,1) \cap H_*^1(0,1), \psi \in H^2(0,1) \cap H_0^1(0,1) \\ u, w \in H_*^1(0,1), v \in H_0^1(0,1), z(x,0,s,t) = v \\ z, z_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \end{array} \right\} \quad (41)$$

Clearly, $\mathscr{D}(\mathscr{A})$ is dense in $\mathscr{H}$. Now, we can give the following existence result.

Theorem 1. *Let $U_0 \in \mathscr{H}$ and assume that (4) holds. Then, there exists a unique solution $U \in \mathscr{C}(\mathbb{R}_+, \mathscr{H})$ of problem (19).*

Moreover, if $U_0 \in \mathscr{D}(\mathscr{A})$, then

$$U \in \mathscr{C}(\mathbb{R}_+, \mathscr{D}(\mathscr{A})) \cap \mathscr{C}^1(\mathbb{R}_+, \mathscr{H}). \quad (42)$$

Proof. First, we prove that the operator $\mathscr{A}$ is dissipative. For any $U_0 \in \mathscr{D}(\mathscr{A})$ and by using (40), we have

$$\begin{aligned} <\mathscr{A}U, U>_{\mathscr{H}} &= -\gamma\mu_1\int_0^1 v^2 dx - \gamma\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|vz(x,1,s,t)dsdx \\ &- \gamma\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|z_\rho z dsd\rho dx - \beta k\int_0^1 w_x^2 dx. \end{aligned} \quad (43)$$

For the third term of the right-hand side of (43), we have

$$\begin{aligned} -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|z_\rho z dsd\rho dx &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 |\mu_2(s)|\frac{d}{d\rho}z^2 d\rho dsdx \\ &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,s,t)dsdx \\ &+ \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,0,s,t)dsdx. \end{aligned} \quad (44)$$

By using Young's inequality, we get

$$\begin{aligned} -\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|vz(x,1,s,t)dsdx &\le \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\right)\int_0^1 v^2 dx \\ &+ \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,s,t)dsdx. \end{aligned} \quad (45)$$

Substituting (44) and (45) into (43), using the fact that $z(x,0,s,t) = v(x,t)$ and (4), we obtained

$$<\mathscr{A}U, U>_{\mathscr{H}} \le -\gamma\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\right)\int_0^1 v^2 dx - \beta k\int_0^1 w_x^2 dx \le 0. \quad (46)$$

Hence, the operator $\mathscr{A}$ is dissipative.

Next, we prove the operator $\mathscr{A}$ is maximal. It is sufficient to show that the operator $(Id - \mathscr{A})$ is surjective.

Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathscr{H}$, we prove that there exists a unique $V = (\varphi, u, \psi, v, \theta, w, z) \in \mathscr{D}(\mathscr{A})$ such that

$$(Id - \mathscr{A})V = F. \quad (47)$$

That is

$$\begin{cases} \varphi - u = f_1, \\ \rho_1 u - \mu\varphi_{xx} - b\psi_x = \rho_1 f_2, \\ \psi - v = f_3, \\ \rho_2 v - \delta\psi_{xx} + b\varphi_x + \xi\psi + \beta w_x + \mu_1 v + \int_{\tau_1}^{\tau_2} |\mu_2(s)|z(x,1,s,t)ds = \rho_2 f_4, \\ \theta - w = f_5, \\ \rho_3 w - l\theta_{xx} + \gamma v_x - kw_{xx} = \rho_3 f_6, \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = sf_7, \end{cases} \quad (48)$$

We note that the last equation in (48) with $z(x,0,s,t) = v(x,t)$ has a unique solution given by

$$z(x,\rho,s,t) = e^{-\rho s}v + se^{s\rho}\int_0^{\rho} e^{s\sigma} f_7(x,\sigma,s,t)d\sigma, \quad (49)$$

then

$$z(x,1,s,t) = e^{-s}v + se^{s}\int_0^{1} e^{s\sigma} f_7(x,\sigma,s,t)d\sigma, \quad (50)$$

we have

$$u = \varphi - f_1, v = \psi - f_3, w = \theta - f_5. \quad (51)$$

Inserting (50) and (51) into (48), (48), and (48), we get

$$\begin{cases} \rho_1\varphi - \mu\varphi_{xx} - b\psi_x = h_1, \\ \mu_4\psi - \delta\psi_{xx} + b\varphi_x + \beta\theta_x = h_2, \\ rho_3\theta - (l+k)\theta_{xx} + \gamma\psi_x = h_3, \end{cases} \quad (52)$$

where

$$\begin{cases} \mu_4 = \rho_2 + \xi + \mu_1 + \frac{4}{3}\gamma + \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} ds, \\ h_1 = \rho_1(f_1 + f_2), \\ h_2 = \rho_2(f_3 + f_4) + \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} ds) f_3 ds - \int_{\tau_1}^{\tau_2} s|\mu_2(s)| e^{s} \int_0^1 e^{s\sigma} f_7(x,\sigma,s,t) d\sigma ds + \beta f_{5x} \\ h_3 = \rho_3(f_5 + f_6) + \gamma f_{3x} - k f_{5xx}. \end{cases} \quad (53)$$

We multiply (52) by $\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}$, respectively, and integrate their sum over $(0,1)$ to get the following variational formulation:

$$B\Big((\varphi,\psi,\theta), \Big(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\Big)\Big) = \Gamma\Big(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\Big), \quad (54)$$

where

$$B : \left(H_*^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1)\right)^2 \longrightarrow \mathbb{R} \quad (55)$$

is the bilinear form defined by

$$\begin{aligned} B\Big((\varphi,\psi,\theta), \Big(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\Big)\Big) = {} & \gamma\rho_1\int_0^1 \varphi\widehat{\varphi}dx + \gamma\mu\int_0^1 \varphi_x\widehat{\varphi}_x dx \\ & + \gamma b\int_0^1 \left(\psi\widehat{\varphi}_x + \varphi\widehat{\psi}_x\right)dx \\ & + \gamma\mu_4\int_0^1 \psi\widehat{\psi}dx + \gamma\delta\int_0^1 \psi_x\widehat{\psi}_x dx \\ & + \gamma\beta\int_0^1 \theta_x\widehat{\psi}dx + \beta\gamma\int_0^1 \psi_x\widehat{\theta}dx \\ & + \beta\rho_3\int_0^1 \theta\widehat{\theta}dx + \beta(l+k)^2\int_0^1 \theta_x\widehat{\theta}_x dx, \end{aligned}$$

$$\Gamma : \left(H_*^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1)\right) \longrightarrow \mathbb{R} \quad (56)$$

is the linear functional given by

$$\Gamma\Big(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta}\Big) = \int_0^1 h_1\widehat{\varphi}dx + \int_0^1 h_2\widehat{\psi}dx + \int_0^1 h_3\widehat{\theta}dx. \quad (57)$$

Now, for $V = H_*^1(0,L) \times H_0^1(0,L) \times H_*^1(0,L)$, equipped with the norm

$$\|(\varphi,\psi,\theta)\|_V^2 = \|\varphi\|_2^2 + \|\varphi_x\|_2^2 + \|\psi\|_2^2 + \|\psi_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2, \quad (58)$$

then, we have

$$\begin{aligned} B((\varphi,\psi,\theta),(\varphi,\psi,\theta)) = {} & \gamma\rho_1\int_0^1 \varphi^2 dx + \gamma\mu\int_0^1 \varphi_x^2 dx \\ & + \gamma\mu_4\int_0^1 \psi^2 dx + \gamma\delta\int_0^1 \psi_x^2 dx \\ & + \rho_3\beta\int_0^1 \theta^2 dx + \beta(l+k)\int_0^1 \theta_x^2 dx \\ & + 2\gamma b\int_0^1 \varphi_x\psi dx, \end{aligned} \quad (59)$$

we have

$$\begin{aligned} \mu\varphi_x^2 + \mu_4\psi^2 + 2b\varphi_x\psi = {} & \frac{1}{2}\left[\mu\left(\varphi_x + \frac{b}{\mu}\psi\right)^2 + \mu_4\left(\psi + \frac{b}{\mu_4}\varphi_x\right)^2 \right. \\ & \left. + \left(\mu - \frac{b^2}{\mu_4}\right)\varphi_x^2 + \left(\mu_4 - \frac{b^2}{\mu}\right)\psi^2\right] \\ > {} & \frac{1}{2}\left[\left(\mu - \frac{b^2}{\mu_4}\right)\varphi_x^2 + \left(\mu_4 - \frac{b^2}{\mu}\right)\psi^2\right], \end{aligned} \quad (60)$$

by assuming $\mu\xi - b^2 > 0$, we get

$$\mu - \frac{b^2}{\mu_4} > 0, \mu_4 - \frac{b^2}{\mu} > 0, \quad (61)$$

then, for some $M_0 > 0$,

$$B((\varphi, \psi, \theta), (\varphi, \psi, \theta)) \geq M_0 \|(\varphi, \psi, \theta)\|_V^2. \quad (62)$$

Thus, B is coercive. Consequently, using the Lax-Milgram theorem, we conclude that the existence of a unique solution $((\varphi, \psi, \theta))$ in V satisfies

$$\begin{aligned} u &= \varphi - f_1 \in H_*^1(0,1), \\ v &= \psi - f_3 \in H_0^1(0,1), \\ w &= \theta - f_5 \in H_*^1(0,1). \end{aligned} \quad (63)$$

Substituting φ, ψ, θ into (50) and (51), respectively, we have

$$\begin{aligned} &u, \theta \in H_*^1(0,1), \\ &\psi \in H_0^1(0,1), \\ &z, z_\rho \in L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)). \end{aligned} \quad (64)$$

Let $\widehat{\varphi} \in H_0^1(0,1)$ and denote

$$\hat{\hat{\hat{\varphi}}} = \widehat{\varphi}(x) - \int_0^1 \widehat{\varphi}(\xi) d\xi, \quad (65)$$

which gives us $\hat{\hat{\hat{\varphi}}} \in H_*^1(0,1)$. Now, we replace $(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta})$ by $(\hat{\hat{\hat{\varphi}}}, 0, 0)$ in (54) to obtain

$$\gamma\rho_1 \int_0^1 \varphi \hat{\hat{\hat{\varphi}}} dx + \gamma\mu \int_0^1 \varphi_x \hat{\hat{\hat{\varphi}}}_x dx + \gamma b \int_0^1 \psi_x \hat{\hat{\hat{\varphi}}} dx = \int_0^1 h_1 \hat{\hat{\hat{\varphi}}} dx. \quad (66)$$

We get

$$\gamma\mu \int_0^1 \varphi_x \hat{\hat{\hat{\varphi}}}_x dx = \int_0^1 (h_1 - \gamma\rho_1\varphi - \gamma b \psi_x) \hat{\hat{\hat{\varphi}}} dx, \quad (67)$$

which yields

$$\gamma\mu\varphi_{xx} = \gamma\rho_1\varphi - \gamma b\psi_x - h_1 \in L^2(0,1). \quad (68)$$

Thus,

$$\varphi \in H^2(0,1). \quad (69)$$

Moreover, (52) also holds for any every $\widehat{\varphi} \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$\gamma\mu \int_0^1 \varphi_x \widehat{\varphi}_x dx = \int_0^1 (h_1 - \gamma\rho_1\varphi - \gamma b\psi_x) \widehat{\varphi} dx. \quad (70)$$

Then, we get for any $\widehat{\varphi} \in C^1([0,1])$

$$\varphi_x(1)\widehat{\varphi}(1) - \varphi_x(0)\widehat{\varphi}(0) = 0. \quad (71)$$

Since $\widehat{\varphi}$ is arbitrary, we get that $\varphi_x(0) = \varphi_x(1) = 0$. Hence, $\varphi \in H_*^2(0,1)$. Using similar arguments as above, we can obtain

$$\begin{gathered} \psi \in H^2(0,1) \cap H_0^1(0,1), \\ \theta \in H_*^2(0,1). \end{gathered} \quad (72)$$

Finally, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in \mathscr{D}(\mathscr{A})$ such that (47) is satisfied.

Consequently, we conclude that $\mathscr{A}$ is a maximal dissipative operator. Hence, by Lumer-Philips theorem (see [25, 26]), we have the well-posedness result. This completes the proof.

3. Stability Results

We prepare the next lemmas (Lemmas 2–7) which will be useful to introduce the Lyapunov function in (104).

Lemma 2. *The energy functional E associated with our problem defined by*

$$\begin{aligned} E(t) = &\frac{\gamma}{2} \left\{ \int_0^1 \left[\rho_1\varphi_t^2 + \mu\varphi_x^2 + \rho_2\psi_t^2 + \delta\psi_x^2 + \xi\psi^2 + 2b\varphi_x\psi \right] dx \right\} \\ &+ \frac{\beta}{2} \left\{ \int_0^1 \left[l\theta_x^2 + \rho_3\theta_t^2 \right] dx \right\} \\ &+ \frac{\gamma}{2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx \end{aligned} \quad (73)$$

satisfies

$$E'(t) \leq -k\beta \int_0^1 \theta_{tx}^2 dx - \gamma\eta_0 \int_0^1 \psi_t^2 dx \leq 0, \quad (74)$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \geq 0.$

Proof. Multiplying (19) by $\gamma\varphi_t$, (19) by $\gamma\psi_t$, and (19) by $\beta\theta_t$ then integration by parts over $(0,1)$, we get

$$\begin{aligned}\frac{\gamma}{2}\frac{d}{dt}\int_0^1\left[\rho_1\varphi_t^2+\mu\varphi_x^2+\rho_2\psi_t^2+\delta\psi_x^2+\xi\psi^2+2b\varphi_x\psi\right]dx\\+\gamma\mu_1\int_0^1\psi_t^2dx+\frac{\beta}{2}\frac{d}{dt}\int_0^1\left[l\theta_x^2+\rho_3\theta_t^2\right]dx\\+\gamma\int_0^1\psi_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx=0.\end{aligned}\tag{75}$$

Now, multiplying (19) by $z\,|\,\mu_2(s)\,|$ and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}&\frac{d}{dt}\frac{\gamma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx\\&=-\gamma\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|zz_\rho(x,\rho,s,t)dsd\rho dx\\&=-\frac{\gamma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\frac{d}{d\rho}z^2(x,\rho,s,t)dsd\rho dx\\&=\frac{\gamma}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\left(z^2(x,0,s,t)-z^2(x,1,s,t)\right)dsdx\\&=\frac{\gamma}{2}\left(\int_{\tau_1}^{\tau_2}|\,\mu_2(s)\,|\,ds\right)\int_0^1\psi_t^2dx-\frac{\gamma}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx.\end{aligned}\tag{76}$$

From (75) and (76), we get (73) and (74).

Now, using Young's inequality, (74) can be written as

$$E'(t)\le-k\beta\int_0^1\theta_{tx}^2dx-\gamma\left(\mu_1-\int_{\tau_1}^{\tau_2}|\,\mu_2(s)\,|\,ds\right)\int_0^1\psi_t^2dx.\tag{77}$$

Then, by (4), there exists a positive constant η_0 such that

$$E'(t)\le-k\beta\int_0^1\theta_{tx}^2dx-\gamma\eta_0\int_0^1\psi_t^2dx.\tag{78}$$

Thus, the functional E is nonincreasing.

Lemma 3. *The function*

$$F_1(t):=\rho_2\int_0^1\psi_t\psi dx+\frac{b\rho_1}{\mu}\int_0^1\psi\int_0^x\varphi_t(y)dydx+\frac{\mu_1}{2}\int_0^1\psi^2dx\tag{79}$$

satisfies

$$\begin{aligned}F_1'(t)\le-\frac{\delta}{2}\int_0^1\psi_x^2dx-\mu_3\int_0^1\psi^2dx+\varepsilon_1\int_0^1\varphi_t^2dx\\+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\psi_t^2dx+c\int_0^1\theta_{tx}^2dx\\+c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx,\end{aligned}\tag{80}$$

where $\mu_3=\xi-(b^2/\mu)>0$.

Proof. Direct computation, using integration by parts and Young's inequality, for $\varepsilon_1>0$, yields

$$\begin{aligned}F_1'(t)&=-\delta\int_0^1\psi_x^2dx-\left(\xi-\frac{b^2}{\mu}\right)\int_0^1\psi^2dx+\rho_2\int_0^1\psi_t^2dx\\&+\frac{b\rho_1}{\mu}\int_0^1\psi_t\int_0^x\varphi_t(y)dydx-\beta\int_0^1\psi\theta_{tx}dx\\&-\int_0^1\psi\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx\le-\delta\int_0^1\psi_x^2dx\\&-\left(\xi-\frac{b^2}{\mu}\right)\int_0^1\psi^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\psi_t^2dx\\&+\varepsilon_1\int_0^1\left(\int_0^x\varphi_t(y)dy\right)^2dx-\beta\int_0^1\psi\theta_{tx}dx\\&-\int_0^1\psi\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx.\end{aligned}\tag{81}$$

By Cauchy-Schwartz's inequality, it is clear that

$$\int_0^1\left(\int_0^x\varphi_t(y)dy\right)^2dx\le\int_0^1\left(\int_0^1\varphi_tdx\right)^2dx\le\int_0^1{\varphi_t}^2dx.\tag{82}$$

So, estimate (81) becomes

$$\begin{aligned}F_1'(t)\le-\delta\int_0^1\psi_x^2dx-\left(\xi-\frac{b^2}{\mu}\right)\int_0^1\psi^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\psi_t^2dx\\+\varepsilon_1\int_0^1{\varphi_t}^2dx-\beta\int_0^1\psi\theta_{tx}dx-\int_0^1\psi\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx,\end{aligned}\tag{83}$$

where the Cauchy-Schwartz, Young, and Poincaré's inequalities have been used, for $\varepsilon_1>0$.

By the fact that $\mu\xi>b^2$, we get the desired result (80).

Lemma 4. *Assume that ((4)) holds. Then, the function*

$$F_2(t):=\int_0^1\psi_x\varphi_tdx+\int_0^1\psi_t\varphi_xdx\tag{84}$$

satisfies

$$F_2'(t) \le -\frac{b}{2\rho_2}\int_0^1 \varphi_x^2 dx + c\int_0^1 \psi_x^2 dx + c\int_0^1 \psi_t^2 + c\int_0^1 \theta_{tx}^2 + c\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dx + \left(\frac{\delta}{\rho_2} - \frac{\mu}{\rho_1}\right)\int_0^1 \varphi_x\psi_{xx}dx. \quad (85)$$

Proof. By differentiating F_2, then using (19), integration by parts gives

$$F_2'(t) = -\frac{b}{\rho_2}\int_0^1 \varphi_x^2 dx + \left(\frac{\delta}{\rho_2} - \frac{\mu}{\rho_1}\right)\int_0^1 \varphi_x\psi_{xx}dx + \frac{b}{\rho_1}\int_0^1 \psi_x^2 dx - \frac{\xi}{\rho_2}\int_0^1 \varphi_x\psi dx - \frac{\mu_1}{\rho_2}\int_0^1 \psi_t\varphi_x dx - \frac{\beta}{\rho_2}\int_0^1 \theta_{tx}\varphi_x dx - \frac{1}{\rho_2}\int_0^1 \varphi_x\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dsdx. \quad (86)$$

Thanks to Young, Cauchy-Schwartz, and Poincaré's inequalities to estimate terms in RHS of (86). For $\delta_1, \delta_2, \delta_3, \delta_4 > 0$, we have

$$-\frac{\xi}{\rho_2}\int_0^1 \varphi_x\psi dx \le \delta_1\int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_1}\int_0^1 \psi^2 dx, \quad (87)$$

$$-\frac{\mu_1}{\rho_2}\int_0^1 \psi_t\varphi_x dx \le \delta_2\int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_2}\int_0^1 \psi_t^2 dx, \quad (88)$$

$$-\frac{\beta}{\rho_2}\int_0^1 \theta_{tx}\varphi_x dx \le \delta_3\int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_3}\int_0^1 \theta_{tx}^2 dx, \quad (89)$$

$$-\frac{1}{\rho_2}\int_0^1 \varphi_x\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t)dsdx \le \delta_4\int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_4}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)ds. \quad (90)$$

The replacement of (87)–(90) into (86) and setting$\delta_1 = \delta_2 = \delta_3 = \delta_4 = b/8\rho_2$helps to obtain (85).

Lemma 5. *The function*

$$F_3(t) \coloneqq -\rho_1\int_0^1 \varphi_t\varphi dx \quad (91)$$

satisfies

$$F_3'(t) \le -\rho_1\int_0^1 \varphi_t^2 dx + \frac{3\mu}{2}\int_0^1 \varphi_x^2 dx + c\int_0^1 \psi_x^2 dx. \quad (92)$$

Proof. Direct computations give

$$F_3'(t) = -\rho_1\int_0^1 \varphi_t^2 dx + \mu\int_0^1 \varphi_x^2 dx + b\int_0^1 \varphi_x\psi dx. \quad (93)$$

Estimate (92) easily follows by using Young's and Poincaré's inequalities

$$F_3'(t) \le -\rho_1\int_0^1 \varphi_t^2 dx + \mu\int_0^1 \varphi_x^2 dx + \delta_5\int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_5}\int_0^1 \psi_x^2 dx, \quad (94)$$

setting $\delta_5 = \mu/2$ to obtain (92).

Lemma 6. *The function*

$$F_4(t) \coloneqq -\rho_3\int_0^1 \theta_t\theta dx \quad (95)$$

satisfies

$$F_4'(t) \le -\frac{l}{2}\int_0^1 \theta_x^2 dx + c\int_0^1 \psi_t^2 dx + c\int_0^1 \theta_{tx}^2 dx. \quad (96)$$

Proof. Direct computations give

$$F_4'(t) = -l\int_0^1 \theta_x^2 dx + \gamma\int_0^1 \theta_x\psi_t dx - k\int_0^1 \theta_x\theta_{tx}dx + \rho_3\int_0^1 \theta_t^2 dx. \quad (97)$$

By using Young and Poincaré's inequalities, we get (96).

Lemma 7. *The function*

$$F_5(t) \coloneqq \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)| z^2(x,\rho,s,t)dsd\rho dx \quad (98)$$

satisfies

$$F_5'(t) \le -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)dsd\rho dx + \mu_1\int_0^1 \psi_t^2 dx - \eta_1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dsdx, \quad (99)$$

where η_1 is a given positive constant.

Proof. By differentiating F_5 with respect to t and using the last equation in $(Hyp1)$, we have

$$F_5'(t) = -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\mu_2(s)| zz_\rho(x,\rho,s,t)dsd\rho dx = -\frac{d}{d\rho}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)| z^2(x,\rho,s,t)dsd\rho dx - \int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\left[e^{-s}z^2(x,1,s,t) - z^2(x,0,s,t)\right]dsdx. \quad (100)$$

Using the fact that $z(x,0,s,t)=\psi_t(x,t-s)$ and $e^{-s}\le e^{-s\rho}\le 1$, for all $0<\rho<1$, we obtain

$$\begin{aligned}F_5'(t)=&-\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,ds\,d\rho\,dx\\&-\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,s,t)\,ds\,dx\\&+\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_0^1\psi_t^2dx.\end{aligned} \tag{101}$$

We have $-e^{-s}\le -e^{-\tau_2}\forall s\in[\tau_1,\tau_2]$. Set $\eta_1=e^{-\tau_2}$, and by (4), we get (99).

We state and prove the decay result in Theorem 8.

Theorem 8. *Let ((4)) hold. Then, there exist positive constants λ_1 and λ_2 such that the function ((73)) satisfies, for any $t>0$*

$$E(t)\le\lambda_2 e^{-\lambda_1 t},\quad \text{if } \frac{\delta}{\rho_2}=\frac{\mu}{\rho_1}, \tag{102}$$

$$E(t)\le C(E_1(0)+E_2(0))t^{-1},\quad \text{if } \frac{\delta}{\rho_2}\ne\frac{\mu}{\rho_1}. \tag{103}$$

Proof. We define a class of an appropriate Lyapunov function as

$$\mathscr{L}(t):=NE(t)+N_1F_1(t)+N_2F_2(t)+F_3(t)+F_4(t)+N_5F_5(t), \tag{104}$$

where N, N_1, N_2, and N_5 are positive constants to be selected later.

Differentiating (104) and by (74), (80), (85), (92), (96), and (99), we have

$$\begin{aligned}\mathscr{L}'(t)\le&-\left[\frac{\delta N_1}{2}-cN_2-c\right]\int_0^1\psi_x^2dx-[\rho_1-N_1\varepsilon_1]\int_0^1\varphi_t^2dx\\&-\left[\gamma\eta_0N-cN_1\left(1+\frac{1}{\varepsilon_1}\right)-N_2c-\mu_1N_5-c\right]\int_0^1\psi_t^2dx\\&-\left[\frac{bN_2}{2\rho_2}-\frac{3\mu}{2}\right]\int_0^1\varphi_x^2dx-N_1\mu_3\int_0^1\psi^2dx\\&-[N_5\eta_1-cN_1-cN_2]\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)\,ds\,dx\\&-\frac{l}{2}\int_0^1\theta_x^2dx-N_5\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)\,ds\,d\rho\,dx\\&-[Nk\beta-cN_1-cN_2-c]\int_0^1\theta_{tx}^2dx+N_2\left(\frac{\delta}{\rho_2}-\frac{\mu}{\rho_1}\right)\int_0^1\varphi_x\psi_{xx}dx.\end{aligned} \tag{105}$$

By setting $\varepsilon_1=\rho_1/2N_1$, we obtain

$$\begin{aligned}\mathscr{L}'(t)\le&-\left[\frac{\delta N_1}{2}-cN_2-c\right]\int_0^1\psi_x^2dx-\frac{\rho_1}{2}\int_0^1\varphi_t^2dx\\&-\left[\frac{bN_2}{2\rho_2}-\frac{3\mu}{2}\right]\int_0^1\varphi_x^2dx\\&-[\gamma\eta_0N-cN_1(1+N_1)-cN_2-\mu_1N_5-c]\int_0^1\psi_t^2dx\\&-[N_5\eta_1-cN_1-cN_2]\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)\,ds\,dx\\&-N_1\mu_3\int_0^1\psi^2dx-[Nk\beta-cN_1-cN_2-c]\int_0^1\theta_{tx}^2dx\\&-\frac{l}{2}\int_0^1\theta_x^2dx-N_5\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)\,ds\,d\rho\,dx\\&+N_2\left(\frac{\delta}{\rho_2}-\frac{\mu}{\rho_1}\right)\int_0^1\varphi_x\psi_{xx}dx.\end{aligned} \tag{106}$$

Next, we carefully choose the constants, starting by N_2 to be large enough such that

$$\alpha_1=\frac{bN_2}{2J}-\frac{3\mu}{2}>0, \tag{107}$$

and N_1 so that

$$\alpha_2=\frac{\delta N_1}{2}-cN_2-c>0, \tag{108}$$

and N_5 large enough such that

$$\alpha_3=N_5\eta_1-cN_1-cN_2>0. \tag{109}$$

We arrive at

$$\begin{aligned}\mathscr{L}'(t)\le&-\alpha_2\int_0^1\psi_x^2dx-\alpha_0\int_0^1\psi^2dx-\frac{\rho}{2}\int_0^1\varphi_t^2dx-\alpha_1\int_0^1\varphi_x^2dx\\&-[\gamma\eta_0N-c]\int_0^1\psi_t^2dx-[k\beta N-c]\int_0^1\theta_{tx}^2dx-\frac{l}{2}\int_0^1\theta_x^2dx\end{aligned} \tag{110}$$

$$\begin{aligned}&-\alpha_3\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)\,ds\,dx+\alpha_5\int_0^1\varphi_x\psi_{xx}dx\\&-\alpha_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)\,ds\,d\rho\,dx,\end{aligned} \tag{111}$$

where $\alpha_0=\mu_3N_1=(\xi-(b^2/\mu))N_1$, $\alpha_4=N_5\eta_1$, $\alpha_5=N_2k_0=N_2((\delta/\rho_2)-(\mu/\rho_1))$.

Now, let us define the related function

$$\mathfrak{L}(t)=N_1F_1(t)+N_2F_2(t)+F_3(t)+F_4(t)+N_5F_5(t), \tag{112}$$

then

$$
\begin{aligned}
|\mathfrak{L}(t)| \le{}& JN_1 \int_0^1 |\psi\psi_t| dx + \frac{b\rho_1 N_1}{\mu} \int_0^1 \left|\psi \int_0^x \varphi_t(y) dy\right| dx \\
&+ \frac{\mu_1 N_1}{2} \int_0^1 \psi^2 dx + N_2 \int_0^1 |\psi_x \varphi_t + \varphi_x \psi_t| dx \\
&+ \rho_1 \int_0^1 |\varphi_t \varphi| dx + \rho_3 \int_0^1 |\theta_t \theta| dx \\
&+ N_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx.
\end{aligned} \tag{113}
$$

Thanks to Young, Cauchy-Schwartz, and Poincaré's inequalities, we get

$$
\begin{aligned}
|\mathfrak{L}(t)| \le{}& c \int_0^1 \left(\varphi_t^2 + \psi_t^2 + \psi_x^2 + \varphi_x^2 + \psi^2 + \theta_t^2 + \theta_x^2\right) dx \\
&+ c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho \le cE(t).
\end{aligned} \tag{114}
$$

Then,

$$|\mathfrak{L}(t)| = |\mathscr{L}(t) - NE(t)| \le cE(t). \tag{115}$$

Thus,

$$(N - c)E(t) \le \mathscr{L}(t) \le (N + c)E(t). \tag{116}$$

One can now$N$large enough such that

$$N - c > 0, k\beta N - c > 0, N\gamma\eta_0 - c > 0. \tag{117}$$

We get

$$c_2 E(t) \le \mathscr{L}(t) \le c_3 E(t), \quad \forall t \ge 0, \tag{118}$$

and using (73), (110), and (116), and the fact that

$$\int_0^1 \theta_t^2 dx \le \int_0^1 \theta_{tx}^2 dx, \tag{119}$$

which gives

$$\mathscr{L}'(t) \le -k_1 E(t) + \alpha_5 \int_0^1 \varphi_x \psi_{xx} dx, \quad \forall t \ge 0. \tag{120}$$

for some $k_1, c_2, c_3 > 0$.

Case 1. If $k_0 = (\delta/\rho_2) - (\mu/\rho_1) = 0$, in this case, ((120)) takes the form

$$\mathscr{L}'(t) \le -k_1 E(t), \quad \forall t \ge 0. \tag{121}$$

The combination of (118) and (121) gives

$$\mathscr{L}'(t) \le -\lambda_1 \mathscr{L}(t), \quad \forall t \ge 0, \lambda_1 = \frac{k_1}{c_2}. \tag{122}$$

Finally, by integrating (122) and recalling (118), we obtain the first result of (103).

Case 2. If $k_0 = (\delta/\rho_2) - (\mu/\rho_1) \ne 0$, then

$$
\begin{cases}
k_0 < \dfrac{k_1 \mu^2 \gamma \delta}{2N_2(\rho_1 + b)}, & \text{if } k_0 > 0, \\[2ex]
|k_0| < \dfrac{k_1 \mu^2 \gamma}{2N_2 \rho_1}, & \text{if } k_0 < 0.
\end{cases} \tag{123}
$$

Let

$$E(t) = E(\varphi, \psi, \theta, z) = E_1(t), \tag{124}$$

be denoted by

$$E_2(t) = E(\varphi_t, \psi_t, \theta_t, z_t). \tag{125}$$

Then, we have

$$E_2'(t) \le -k\beta \int_0^1 \theta_{ttx}^2 dx - \gamma\eta_0 \int_0^1 \psi_{tt}^2 dx. \tag{126}$$

The last term in (120), by using (19), and Young's inequality, and by setting $K = -\rho_1 \alpha_5/\mu$, we have

$$
\begin{aligned}
\alpha_5 \int_0^1 \varphi_x \psi_{xx} dx ={}& -\frac{\alpha_5 \rho_1}{\mu} \int_0^1 \psi_x \varphi_{tt} dx + \frac{b\alpha_5}{\mu} \int_0^1 \psi_x^2 dx \\
={}& -K\left(\frac{d}{dt}\left[\int_0^1 \psi \varphi_{xt} dx - \int_0^1 \psi_t \varphi_x dx\right]\right) \\
&- K \int_0^1 \varphi_x \psi_{tt}^2 dx + \frac{b\alpha_5}{\mu} \int_0^1 \psi_x^2 dx \\
\le{}& -K\left(\frac{d}{dt}\left[\int_0^1 \psi \varphi_{xt} dx - \int_0^1 \psi_t \varphi_x dx\right]\right) \\
&+ \frac{b\alpha_5}{\mu} \int_0^1 \psi_x^2 dx + \frac{|K|}{4} \int_0^1 \psi_{tt}^2 dx + |K| \int_0^1 \varphi_x^2 dx.
\end{aligned} \tag{127}
$$

Let

$$\mathscr{N}(t) = \int_0^1 \psi \varphi_{xt} dx - \int_0^1 \psi_t \varphi_x dx, \tag{128}$$

then (120)

$$\mathscr{L}'(t) + K\mathscr{N}'(t) \le -k_1 E_1'(t) + \frac{b\alpha_5}{\mu}\int_0^1 \psi_x^2 dx + \frac{|K|}{4}\int_0^1 \psi_{tt}^2 dx + |K|\int_0^1 \varphi_x^2 dx \le -k_2 E_1'(t) + \frac{|K|}{4}\int_0^1 \psi_{tt}^2 dx, \tag{129}$$

where

$$k_2 = k_1 - \frac{2}{\mu\gamma}\left(|K| + \frac{b\alpha_5}{\delta}\right). \tag{130}$$

Let

$$G(t) = \mathscr{L}(t) + K\mathscr{N}(t) + N_3(E_1(t) + E_2(t)). \tag{131}$$

If $N_3 > \max\{C_0 \mid K| - c_1, |K|, |K|/4C\}$, indeed,

$$|\mathscr{N}(t)| = |\int_0^1 \psi\varphi_{xt} dx| + |\int_0^1 \psi_t \varphi_x dx| \le \frac{1}{2}\int_0^1 \varphi_{tx}^2 dx + \frac{1}{2}\int_0^1 \psi_t^2 dx + \frac{1}{2}\int_0^1 \psi^2 dx + \frac{1}{2}\int_0^1 \varphi_x^2 dx \le E_2(t) + C_0 E_1(t), \tag{132}$$

where $C_0 = \max\{2/\gamma\xi, 2/\gamma\mu, 2/\gamma\rho_2\}$. By (118), we obtain

$$\begin{aligned} G(t) &\le c_1 E_1(t) - |K|(E_2(t) + C_0 E_1(t)) + N_3(E_1(t) + E_2(t)) \\ &\le (N_3 + c_1 - C_0 \mid K \mid) E_1(t) + (N_3 - |K \mid) E_2(t). \end{aligned} \tag{133}$$

It is not hard to prove

$$m_1(E_1(t) + E_2(t)) \le G(t) \le m_2(E_1(t) + E_2(t)), \tag{134}$$

where $m_1, m_2 > 0$. By using (129) and (128), we obtain

$$\begin{aligned} G'(t) &= \mathscr{L}'(t) + K\mathscr{N}'(t) + N_3\left(E_1'(t) + E_2'(t)\right) \\ &\le -k_2 E_1(t) + \left(-CN_3 + \frac{|K|}{4}\right)\int_0^1 \psi_{tt}^2 dx. \end{aligned} \tag{135}$$

Choosing N_3 such that

$$CN_3 - \frac{|K|}{4} > 0, \tag{136}$$

we have

$$G'(t) \le -k_2 E_1(t). \tag{137}$$

Integrating (137), we get

$$\int_0^t E_1(y)dy \le \frac{1}{k_2}(G(0) - G(1)) \le \frac{1}{k_2}G(0) \le \frac{m_2}{k_2}(E_1(0) + E_2(0)), \tag{138}$$

using the fact that

$$(tE_1(t))' = tE_1'(t) + E_1(t) \le E_1(t). \tag{139}$$

We get that

$$tE_1(t) \le \frac{m_2}{C_2}(E_1(0) + E_2(0)), \tag{140}$$

which is desired to be the second result of (103). This completes the proof.

4. Conclusion

This paper studied the asymptotic behavior of a one-dimensional thermoelastic system with distributed time delay; namely, an integral damping term on a time interval $[t-\tau_2, t-\tau_1]$ is taken into account. Beside the distributed delay term, a standard undelayed damping is included in the model $(-\mu_1\phi_t)$. We established the well-posedness of the system, and we proved stability estimates by means of appropriate Lyapunov functions. Exponential decay estimates are proved by nonclassical condition between the delay damping coefficient and the coefficient of the undelayed one which is satisfied. Several papers have been proposed for models including both undelayed and delayed damping of the same form, and exponential stability results have been obtained if the coefficient of the delay is smaller than the one of the undelayed term. This analysis has been extended to the case of a distributed delay in [16]. Also in this case, there are now a few literature, dealing with different PDE models, including thermoelastic systems. Typically, under the assumption (4), the system keeps the same properties, the one without delay but only with a standard frictional damping $c\phi_t$, for some coefficient c. Then, this paper introduced a considerable novelties different from those of [15].

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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Research Article

Nonunique Coincidence Point Results via Admissible Mappings

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This paper is aimed at presenting some coincidence point results using admissible mapping in the framework of the partial b-metric spaces. Observed results of the article cover a number of existing works on the topic of "investigation of nonunique fixed points." We express an example to indicate the validity of the observed outcomes.

1. Introduction and Preliminaries

In 1974, Ćirić [1] published the first paper on nonunique fixed point theory. Despite Banach's theorem, Ćirić [1] focused only on the existence of a fixed point, but not the uniqueness. The motivation of Ćirić [1] was inspired by Banach's motivation. As it is known, Banach's fixed point theorem is abstracted from Picard's paper, in which Picard [2] analyzed both the existence and uniqueness of the solution of the certain differential equation (see [3–5]). On the other hand, not all differential or integral equations have a unique solution. In the differential/integral equations, nonunique solutions are also crucial, for example, periodic solutions. Consequently, Ćirić [1] investigated the corresponding fixed point theorems that would be a tool in finding periodic solutions of the differential/integral equations. In the last five decades, a number of nonunique fixed point results have been reported in two ways: either proposing a new contraction type or changing the structure. The first example for the changing the contraction inequality, in the standard set-up, was given by Achari [6] in 1976 and Pachpatte [7] in 1973. Fifteen years later, Ćirić and Jotić [8] proposed a new type of contraction inequalities in the context of complete metric space. This trend was followed by the attractive results [9–13]. On the other hand side, in [14–17], the authors observed several characterizations of the unique fixed point results in the setting of complete b-metric spaces. Indeed, among the several extensions of metric structure, the true extension is the b-metric space. For this reason, observed nonunique fixed theorems in the context of b-metric space is very interesting and important, see also [18–20]. In addition, in [21–23], the characterization of fixed point theorems in partial metric spaces is crucial due to the potential application in the domain theory of computer science. Regarding the applied mathematics, nonunique fixed point results in cone metric spaces have taken attention [24].

In this paper, we consider a nonunique fixed point theorem in the context of the very general frame, partial b-metric spaces. An illustrative example is a set-up to indicate the validity of the main theorem.

Let M be a nonempty set, a real number $s \geq 1$, and $\mathbb{N} = \{1, 2, 3, \cdots\}$.In this case, the triplet (M, p_b, s) forms a partial b *-metric space*, on short p_b-ms.Undoubtedly, b-metric spaces (and ordinary metric spaces) are closely related to partial b -metric spaces. Definitely, a b-metric space $(s \geq 1)$ is a partial b-metric space with zero self-distance and a partial metric space is a partial b-metric space with $s = 1$. Moreover, a partial b-metric can define a b-metric space. Indeed, for example, let p_b be a partial b-metric on M. Then, the functions b'_p, b_p, $b_{p,m} : M \longrightarrow M$, where

$$b_p(u,v) = \begin{cases} p_b(u,v), & \text{if } u \neq v, \\ 0, & \text{if } u = v, \end{cases} \tag{1}$$

$$b'_p(u,y) = 2p_b(u,y) - p_b(u,u) - p_b(y,y), \tag{2}$$

$$b_{p,m}(u,y) = p_b(u,y) - \min\{p_b(u,u), p_b(y,y)\} \tag{3}$$

are b-metrics on M.

Definition 1. A function $p_b : M \times M \longrightarrow [0,\infty)$ is a partial b-metric on M if for all $u, y, w \in M$, it satisfies the following conditions:

$(p_b)_1 u = y \Longleftrightarrow p_b(u,u) = p_b(u,y) = p_b(y,y)$
$(p_b)_2 p_b(u,u) \leq p_b b(u,y)$
$(p_b)_3 p_b(u,y) \leq p(y,u)$
$(p_b)_4 p_b(u,y) \leq s[p_b(u,w) + p_b(w,u)] - p_b(w,w)$

Example 1. (see [25]). Let p_b be a partial metric on the set M. Then, the functions $p_b : M \times M \longrightarrow [0,\infty)$ are given for all $u, y \in M$ by

(1) $p_b(u,y) = p(u,y) + b(u,y)$ is a partial b-metric on M (where b is a b-metric $(s > 1)$ on M)

(2) $p_b(u,y) = [p(u,y)]^r$ for $r \geq 1$, define a partial b-metrics on M with coefficient $s = 2^{r-1}$

Remark 2. From $(pb)_1$ and $(pb)_2$, it follows that if $u, y \in M$ are such that $p_b(u,y) = 0$, then $u = y$.

Definition 3. (see [26, 27]). Let $\{u_n\}$ be a sequence on the p_b-*ms*$(M, p_b, s \geq 1)$

(1) $\{u_n\}$ is p_b-convergent to $u \in M$ if $\lim_{n\to\infty} p_b(u, u_n) = p_b(u,u)$

(2) $\{u_n\}$ is p_b-Cauchy if $\lim_{n,q\to\infty} p_b(u_n, u_q)$ exists and is finite

(3) $\{u_n\}$ is 0-p_b-Cauchy if $\lim_{n,q\to\infty} p_b(u_n, u_q) = 0$

(4) $(M, p_b, s \geq 1)$ is p_b-complete if every p_b-Cauchy sequence in M is p_b-convergent

$$\lim_{n,q\to\infty} p_b(u_n, u_q) = \lim_{n\to\infty} p_b(u_n, u) = p_b(u,u) \tag{4}$$

(5) $(M, p_b, s \geq 1)$ is 0-p_b-complete if every 0-p_b-Cauchy sequence we can find $u \in M$ such that

$$\lim_{n,q\to\infty} p_b(u_n, u_q) = \lim_{n\to\infty} p_b(u_n, u) = p_b(u,u) = 0 \tag{5}$$

Moreover, in [26], the following interesting results were proved.

Lemma 4. *(see [26]). Every p_b-complete p_b-ms $(M, p_b, s \geq 1)$ is 0 - p_b -complete.*

Lemma 5. *(see [26]). The p_b-ms $(M, p_b, s \geq 1)$ is 0 - p_b -complete if and only if the b -metric space $(M, b_p, s \geq 1)$ is complete, where the b -metric b_p was defined in (3).*

They also showed that the converse affirmation does not hold.

Let R, S to self-mappings on the set M. We say that

(i) S commutes with R on M if $RSu = SRu$ for all $u \in M$

(ii) a point $z \in M$ is a point of coincidence of R and S if we can find $u^* \in M$ such that $z = Ru^* = Su^*$

(iii) a point $u^* \in M$ is a common fixed point of R and S if $Ru^* = u^* = Su^*$

We will use the following notations:

$$C_c(R,S)_M = \{u \in M \mid Ru = Su\} M^* = M \setminus C_c(R,S)_M. \tag{6}$$

In [28], the notion of R-β-admissible mapping was introduced as follows:

(i) Let the function $\beta : M \times M \longrightarrow [0,\infty)$ and $R, S : M \longrightarrow M$. The mapping S is said to be R-β-admissible if

$$\beta(Ru, Ry) \geq 1 \text{ implies } \beta(Su, Sy) \geq 1, \tag{7}$$

for all $u, y \in M$.

In case that $R = I_M$, the mapping S is said to be β -admissible.

Let $(M, p_b, s \geq 1)$ be a p_b-ms and $\beta ; M \times M \longrightarrow [0,+\infty)$. The space M is β-regular if for every sequence $\{z_n\}$ in M such that $z_n \longrightarrow z$ and $\beta(z_n, z_{n+1}) \geq 1$, there exists a subsequence $\{z_{n_l}\}$ of $\{z_n\}$ such that

$$\beta(z_{n_l}, z_*) \geq 1, \tag{8}$$

for all $l \in \mathbb{N}$.

Lemma 6. *Let $R, S : M \longrightarrow M$ such that S is a R-β-admissible. If there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \geq 1$, then*

$$\beta(Ru_n, Ru_{n+1}) \geq 1, \tag{9}$$

where the sequence $\{u_n\}$ in M is defined by $Su_n = Ru_{n+1}$, for each $n \in \mathbb{N} \cup \{0\}$.

Proof. By the assumption $\beta(Ru_0, Su_0) \geq 1$, since the mapping S is R-β-admissible, we get

$$\beta(Ru_0, Ru_1) = \beta(Ru_0, Su_0) \geq 1 \text{ implies } \beta(Ru_1, Ru_2) = \beta(Su_0, Su_1) \geq 1, \tag{10}$$

and by induction, it follows that

$$\beta(Ru_n, Ru_{n+1}) \geq 1, \tag{11}$$

for $n \in \mathbb{N} \cup \{0\}$.

2. Main Results

Following the idea in [29], we state the following results useful in the sequel.

Lemma 7. *Let $(M, p_b, s \geq 1)$ be a p_b-ms. If $\{u_n\}$ is a sequence in M such that there exists $\{z_n\}$ in M, satisfying the inequality*

$$p_b(u_n, u_{n+1}) \leq c p_b(u_{n-1}, u_n), \tag{12}$$

for any $n \in \mathbb{N}$, then the sequence is $\{u_n\}$ and is 0-p_b-Cauchy.

Proof. First of all, by (12), we get

$$p_b(u_n, u_{n+1}) \leq c^n p_b(u_0, u_1), \tag{13}$$

for all $n \in \mathbb{N}$. On the other hand, by using $(pb)_4$, we can derive that

$$\begin{aligned} p_b(u_n, u_{n+q}) &\leq s(p_b(u_n, u_{n+1}) \\ &\quad + p_b(u_{n+1}, u_{n+q})) - p_b(u_{n+1}, u_{n+1}) \\ &\leq s p_b(u_n, u_{n+1}) \\ &\quad + s^2(p_b(u_{n+1}, u_{n+2}) + p_b(u_{n+2}, u_{n+2}, u_{n+q}) \\ &\quad -- p_b(u_{n+1}, u_{n+1}) - p_b(u_{n+2}, u_{n+2}) \cdots \\ &\leq s p_b(u_n, u_{n+1}) + s^2 p_b(u_{n+1}, u_{n+2}) + \cdots \\ &\quad + s^q p_b(u_{n+q-1}, u_{n+q}) -- \sum_{l=1}^{q-1} p_b(u_{n+l}, u_{n+l}) \\ &\leq s^q [p_b(u_n, u_{n+1}) + p_b(u_{n+1}, u_{n+2}) + \cdots \\ &\quad + p_b(u_{n+q-1}, u_{n+q})] -- \sum_{l=1}^{q-1} p_b(u_{n+l}, u_{n+l}). \end{aligned} \tag{14}$$

(1) If $c \in [0, 1/s)$, by (13) and (14), we get

$$\begin{aligned} p_b(u_n, u_{n+q}) &\leq \sum_{l=0}^{q-1} s^{l+1} c^{n+l} p_b(u_0, u_1) - \sum_{l=1}^{q-1} p_b(u_{n+l}, u_{n+l}) \\ &\leq s c^n \sum_{l=0}^{q-1} (sc)^l p_b(u_0, u_1) \\ &= s c^n \frac{1 - (sc)^q}{1 - sc} \longrightarrow 0 \text{ as } n, q \longrightarrow \infty. \end{aligned} \tag{15}$$

Therefore, $\{u_n\}$ is a 0-p_b-Cauchy sequence.

(2) If $c \in [1/s, 1)$, thus $c^n \longrightarrow 0$ (as $n \longrightarrow \infty$). Moreover, there exits $l \in \mathbb{N}$ such that $c^l < 1/s$. This means $l > -\log s/\log c$. Again, by (13) together with (14), we have

$$\begin{aligned} p_b(u_{nl}, u_{(n+1)l}) &\leq s^l \left[p_b(unl, u_{nl+1}) + \cdots + p_b(u_{nl+l-1}, u_{(n+1)l})\right] \\ &\quad -- \sum_{j=1}^{l-1} p_b(u_{nl+j}, u_{nl+j}) \\ &\leq s^l \sum_{j=0}^{l-1} c^{nl+j} p_b(u_0, u_1) - \sum_{j=1}^{l-1} p_b(u_{nl+j}, u_{nl+j}) \\ &\leq s^l c^{nl} \sum_{j=0}^{l-1} p_b(u_0, u_1) \\ &\leq c^{nl} \frac{s^l \cdot p_b(u_0, u_1)}{1 - c} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{16}$$

Thereby, letting $\lambda = c^l < 1/s$ by Case (i), we get that the sequence $\{u_{nl}\}$ is 0-p_b-Cauchy sequence, which means that

$$\lim_{n,q \to \infty} p_b(u_{nl}, u_{ql}) = 0. \tag{17}$$

On the other hand,

$$\begin{aligned} p_b(u_{l[n/l]}, u_n) &\leq s\Big(p_b(u_{l[n/l]}, u_{l[n/l]+1}) \\ &\quad + p_b(u_{l[n/l]+1}, u_n)\Big) - p_b(u_{l[n/l]+1}, u_{l[n/l]+1}) \\ &\leq s^l \left[p_b(u_{l[n/l]}, u_{l[n/l]+1}) + \cdots + p_b(u_{n-1}, u_n)\right] \\ &\quad -- \left(p_b(u_{l[n/l]+1}, u_{l[n/l]+1}) + \cdots + p_b(u_{n-1}, u_{n-1})\right), \end{aligned} \tag{18}$$

and using (13), we have

$$\begin{aligned} p_b(u_{l[n/l]}, u_n) &\leq s^l \left[c^{l[n/l]} + \cdots + c^{n-1}\right] p_b(u_0, u_1) \\ &\leq s^l c^{l[n/l]} \frac{p_b(u_0, u_1)}{1 - c} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{19}$$

Finally, combining relations (19) and (17) and keeping in mind $(pb)_4$, we have

$$\begin{aligned} p_b(u_n, u_q) &\leq s\left[p_b(u_n, u_{l[n/l]}) \right. \\ &\quad \left. + p_b(u_{l[n/l]}, u_q)\right] - p_b(u_{l[n/l]}, u_{l[n/l]}) \\ &\leq s p_b(u_n, u_{l[n/l]}) \\ &\quad + s^2 p_b(u_{l[n/l]}, u_{l[q/l]}) + s^2 p_b(u_{l[q/l]}, u_q) \\ &\quad -- p_b(u_{l[n/l]}, u_{l[n/l]}) - p_b(u_{l[q/l]}, u_{l[q/l]}) \\ &\leq s p_b(u_n, u_{l[n/l]}) \\ &\quad + s^2 p_b(u_{l[n/l]}, u_{l[q/l]}) + s^2 p_b(u_{l[q/l]}, u_q) \longrightarrow 0 \text{ as } n, q \longrightarrow \infty. \end{aligned} \tag{20}$$

Thereupon, the sequence $\{u_n\}$ is 0-p_b-Cauchy.

Theorem 8. *Let $(M, p_b, s \geq 1)$ be a complete p_b-ms and two mappings $R, S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0, 1)$ such that*

$$\begin{aligned} &\beta(Ru, Ry) \min \{p_b(Su, Sy), p_b(Sy, Ry)\} \\ &\quad -- \min \{b_p(Su, Ry), b_p(Sy, Ru)\} \\ &\leq \kappa \max \{p_b(Ru, Ry), p_b(Su, Ru)\}, \end{aligned} \tag{21}$$

for all $u, y \in M$, such that $u \neq y$ when $u, y \in C_c(R, S)_M$. Suppose also that

(a) $S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms

(b) S is R-β-admissible, and there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \geq 1$

(c) M is β-regular

Then, the mappings S and R have a point of coincidence.

Proof. Let u_0 be an arbitrary point in M, such that $\beta(Ru_0, S u_0) \geq 1$. Thus, since $S(M) \subset R(M)$, there exists $u_1 \in M$ such that $Su_0 = Ru_1$. Thereupon, $Su_1 \in S(M) \subset R(M)$ and we can find $u_2 \in M$ such that $Su_1 = Ru_2$. In this way, we can build a sequence $\{u_n\} \subseteq M$ as follows:

$$\text{having defined } u_n \in M, \text{ we let } u_{n+1} \in M \text{ such that } Su_n = Ru_{n+1}, \tag{22}$$

for all $n \in \mathbb{N} \cup \{0\}$. Letting $u = u_n$ and $y = u_{n+1}$ in (ref1T1) and taking into account Lemma 6, we have

$$\begin{aligned} &\min \{p_b(Su_n, Su_{n+1}), p_b(Su_{n+1}, Ru_{n+1})\} \\ &\quad -- \min \{b_{p,m}(Su_n, Ru_{n+1}), b_{p,m}(Su_{n+1}, Ru_n)\} \\ &\leq \beta(Ru_n, Ru_{n+1}) \min \{p_b(Su_n, Su_{n+1}), p_b(Su_{n+1}, Ru_{n+1})\} \\ &\quad -- \min \{b_p(Su_{n+1}, Ru_n)\} \\ &\leq \kappa \max \{p_b(Ru_n, Ru_{n+1}), p_b(Su_n, Ru_n)\}. \end{aligned} \tag{23}$$

Keeping in mind (22), we get

$$\begin{aligned} &\min \{p_b(Ru_{n+1}, Ru_{n+2}), p_b(Ru_{n+1}, Ru_n), p_b(Ru_{n+2}, Ru_{n+1})\} \\ &\quad -- \min \{b_p(Ru_{n+1}, Ru_{n+1}), b_p(Ru_{n+2}, Ru_n)\} \\ &\leq \kappa \max \{p_b(Ru_n, Ru_{n+1}), p_b(Ru_{n+1}, R(u_n))\} \\ &= \kappa p_b(Ru_n, Ru_{n+1}), \end{aligned} \tag{24}$$

which is equivalent with

$$\begin{aligned} &\min \{p_b(Ru_{n+1}, Ru_{n+2}), p_b(Ru_{n+1}, Ru_n)\} \\ &\quad -- \min \{b_p(Ru_{n+1}, Ru_{n+1}), b_p(Ru_{n+2}, Ru_n)\} \\ &\leq \kappa p_b(Ru_n, Ru_{n+1}). \end{aligned} \tag{25}$$

Therefore, we get

$$p_b(Ru_{n+1}, Ru_{n+2}) \leq \kappa p_b(Ru_n, Ru_{n+1}), \tag{26}$$

for any $n \in \mathbb{N} \cup \{0\}$.Let now $\{z_n\}$ be a sequence in M, with $z_n = Ru_{n+1} = Su_n$, $n \in \mathbb{N} \cup \{0\}$. First of all, we mention that $z_n \neq z_{n+1}$ for every $n \in \mathbb{N}$. Indeed, if we suppose that there exists $m_0 \in \mathbb{N} \cup \{0\}$ such that $z_{m_0} = z_{m_0+1}$, thus by (22), we have

$$Ru_{m_0+1} = Su_{m_0} = z_{m_0} = z_{m_0+1} = Su_{m_0+1}, \tag{27}$$

so that z_{m_0+1} is a point of coincidence. Thus, $z_n \neq z_{n+1}$ for every $\mathbb{N} \cup \{0\}$ and (28) can be rewritten as

$$p_b(z_n, z_{n+1}) \leq \kappa p_b(z_{n-1}, z_n). \tag{28}$$

Therefore, according to Lemma 7, the sequence $\{z_n\}$ is 0 -p_b-Cauchy. Since the space is 0-p_b-complete, it follows that there is $z \in M$ such that

$$\lim_{n,q\to\infty} p_b(z_n, z_q) = \lim_{n\to\infty} p_b(z_n, z) = \mathrm{p}_b(z, z) = 0. \tag{29}$$

But, on the other hand, since $z_n = Ru_{n+1}$ and the space $(R(M), p_b, s)$ is 0-p_b-complete, we can find $u_* \in M$, with $z = Ru_*$. Thus,

$$\lim_{n\to\infty} p_b(Su_n, Ru_*) = \lim_{n\to\infty} p_b(Ru_n, Ru_*) = p_b(Ru_*, Ru_*) = 0. \tag{30}$$

Supposing that $Ru_* \neq Su_*$ for $u = u_{n_l}$ and $y = u_*$ and taking into account the β-regularity of the space M, we have

$$\begin{aligned} &\min \{p_b(Su_{n_l}, Su_*), p_b(Su_*, Ru_*)\} \\ &\quad - \min \{b_p(Su_{n_l}, Ru_*), b_p(Su_*, Ru_{n_l})\} \\ &\leq \leq \beta(z_{n_l}, z) \min \{p_b(Su_{n_l}, Su_*), p_b(Su_*, Ru_*)\} \\ &\quad - \min \{b_p(Su_{n_l}, Ru_*), b_p(Su_*, Ru_{n_l})\} \\ &= \beta(Ru_{n_l}, Ru_*) \min \{p_b(Su_{n_l}, Su_*), p_b(Su_*, Ru_*)\} \\ &\quad - \min \{b_p(Su_{n_l}, Ru_*), b_p(Su_*, Ru_{n_l})\} \\ &\leq \kappa \max \{p_b(Ru_{n_l}, Ru_*), p_b(Su_{n_l}, Ru_{n_l})\}. \end{aligned} \tag{31}$$

If $\min \{p_b(Su_{n_l}, Su_*), p_b(Su_*, Ru_*)\} = p_b(Su_*, Ru_*)$, the above inequality becomes

$$\begin{aligned} &p_b(Su_*, Ru_*) - \min \{p_b(Su_{n_l}, Su_*), p_b(Su_*, Ru_*)\} \\ &\quad - \min \{b_p(Su_{n_l}, Ru_*), b_p(Su_*, Ru_{n_l})\} \\ &\leq \kappa \max \{p_b(Ru_{n_l}, Ru_*), p_b(Su_{n_l}, Ru_{n_l})\}. \end{aligned} \tag{32}$$

Letting $l \longrightarrow \infty$ and taking into account (28) and (30), we get

$$p_b(Su_*, Ru_*) = 0, \tag{33}$$

and by $(pb)_1$, $(pb)_1$, we have $Su_* = Ru_*$.If $\min \{p_b(Su_{n_l}, Su_*), p_b(Su_*, Ru_*)\} = p_b(Su_{n_l}, Su_*)$, we find that $\lim_{l\to\infty} p_b(Su_{n_l}, Su_*) = 0$. On the other hand, by $(pb)_4$,

$$p_b(Su_*, Ru_*) \le s[p_b(Su_*, Su_{n_l}) + p_b(S(u_{n_l}, Ru_*)] - p_b(S(u_{n_l}, Su_{n_l}), \tag{34}$$

and then, $p_b(Su_*, Ru_*) = 0$, as $l \longrightarrow \infty$. This proves that $z = Su_* = Ru_*$, that is, z is a point of coincidence for S and R.

Example 2. Let $M = [0,\infty)$ and $p_b : M \times M \longrightarrow [0,\infty)$ be a partial b-metric, where $p_b(u, y) = (\max \{u, y\})^2$. Let the mappings $S, R : M \longrightarrow M$,

$$\begin{aligned} Su &= \begin{cases} \dfrac{u+1}{2}, & \text{if } u \in [0,1], \\ 3, & \text{if } u > 1, \end{cases} \\ Ru &= \begin{cases} \dfrac{u+2}{4}, & \text{if } u \in [0,1], \\ \dfrac{u+5}{10}, & \text{if } u > 1, \end{cases} \end{aligned} \tag{35}$$

and the function $\beta : M \times M \longrightarrow [0,\infty)$,

$$\beta(x, v) = \begin{cases} 2, & \text{for } x = v = \dfrac{1}{2}, \\ 3, & \text{for } x = v = 3, \\ 1, & \text{for } x, v \ge 4, \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

Obviously, since $x = Ru \ge 4$ for $u \ge 35$ we have

(i) For $u, y \ge 35$

$$\begin{aligned} &\beta(Ru, Ry) = 1 \Longrightarrow \beta(Su, Sv) = \beta(3, 3) = 3 > 1, \\ &\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \beta(R(0), R(0)) = 2 \Longrightarrow \beta(S(0), S(0)) = \beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2, \\ &\beta(3, 3) = \beta(R(25), R(25)) = 3 \Longrightarrow \beta(S(25), S(25)) = \beta(3, 3) = 3. \end{aligned} \tag{37}$$

Moreover,

$$\begin{aligned} &\beta(Ru, Ry) \min \{p_b(Su, Sy), p_b b(Sy, Ry)\} \\ &\quad -- \min \{b_p(Su, Ry), b_p(Sy, Ru)\} \\ &\le \min \{p_b(3, 3), p_b(3, Ry)\} \\ &= 9, \le \kappa \cdot 16 \le \kappa \cdot \max \{p_b(Ru, Ry), p_b(Su, Ru)\}, \end{aligned} \tag{38}$$

for any $9/16 < \kappa < 1$.

(ii) All other cases are uninteresting due to the way the function β was defined

Consequently, by Theorem 8, the mappings S, R have points of coincidence. These are $1/2 = S(0) = R(0)$, respectively, $3 = S(25) = R(25)$.

Corollary 9. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and two mappings $R, S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0, 1)$ such that*

$$\begin{aligned} &\min \{p_b(Su, Sy), p_b(Sy, Ry)\} \\ &\quad -- \min \{b_p(Su, Ry), b_p(Sy, Ru)\} \\ &\le \kappa \max \{p_b(Ru, Rv), p_b(Su, Ru)\}, \end{aligned} \tag{39}$$

for every $u, y \in M$, such that $u \ne y$ when $u, y \in C_c(R, S)_M$. If $S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms, then the mappings S and R have a point of coincidence.

Proof. It is enough to choose $\beta(u, y) = 1$ in Theorem 8.

Theorem 10. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and a mapping $S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0, 1)$ such that*

$$\begin{aligned} &\beta(u, y) \min \{p_b(Su, Sy), p_b(Sy, y)\} \\ &\quad -- \min \{b_p(Su, y), b_p(Sy, u)\} \\ &\le \kappa \max \{p_b(u, y), p_b(Su, u)\}, \end{aligned} \tag{40}$$

for every $u, y \in M$, such that $u \ne y$. Suppose also that

(a) S is β-admissible, and there exists $u_0 \in M$ such that $\beta(u_0, Su_0) \ge 1$

(b) M is β-regular

Then, the mapping S has a fixed point.

Proof. Put $R = I_M$ in Theorem 8.

Corollary 11. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and a mapping $S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0, 1)$ such that*

$$\begin{aligned} &\min \{p_b(Su, Sy), p_b(Sy, y)\} \\ &\quad -- \min \{b_p(Su, y), b_p(Sy, u)\} \\ &\le \kappa \max \{p_b(u, y), p_b(Su, u)\}, \end{aligned} \tag{41}$$

for every $u, y \in M$,$u \ne y$. Then, the mapping S has a fixed point.

Proof. It is enough to choose $\beta(u, y) = 1$ in Theorem 10.

Theorem 12. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and two mappings $R, S : M \longrightarrow M$. Suppose that there exist $\kappa \in (0, 1)$*

and $a > 0$ such that

$$\beta(Ru, Ry)M^1_S(u, y) - a \cdot N^1_{S,R}(u, y) \leq \kappa p_b(Su, Ru)p_b(Sy, Ry), \tag{42}$$

where

$$M^1_{S,R}(u, y) = \min\left\{[p_b(Su, Sy)]^2, [p_b(Sy, Ry)]^2\right\},$$
$$N^1_{S,R}(u, y) = \min\left\{b_p(Su, Ry)b_p(Sy, Ru), p_b(Su, Ry)p_b(Su, Sy), p_b(Sy, Ru)p_b(Ru, Ry)\right\}, \tag{43}$$

for every $u, y \in M$, such that $u \neq y$ when $u, y \in C_c(R, S)_M$. Suppose also that:

(a) $S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms

(b) S is R-β-admissible, and there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \geq 1$

(c) M is β-regular

Then, the mappings S and R have a point of coincidence.

Proof. Starting with a point $u_0 \in M$ such that $\beta(Ru_0, Su_0) \geq 1$, we build the sequences $\{u_n\}$, $\{z_n\}$ as in Theorem 8,

$$z_n = Ru_{n+1} = Su_n, \text{ for all } n \in \mathbb{N}. \tag{44}$$

Using the same arguments, we can assume that $z_n \neq z_{n+1}$, also, for all $n \in \mathbb{N}$. Thus, for $u = u_n$, $y = u_{n+1}$,

$$\begin{aligned} M^1_{S,R}(u_n, u_{n+1}) &= \min\left\{[p_b(Su_n, Su_{n+1})]^2, [p_b(Su_{n+1}, Ru_{n+1})]^2\right\} \\ &= \min\left\{p_b(z_n, z_{n+1})]^2, [p_b(z_{n+1}, z_n)]^2\right\} \\ &= [p_b(z_{n+1}, z_n)]^2, \end{aligned} \tag{45}$$

$$\begin{aligned} N^1_{S,R}(u_n, u_{n+1}) &= \min\left\{\begin{array}{c} b_p(Su_n, Ru_{n+1})b_p(Su_{n+1}, Ru_n), p_b(Su_n, Ru_{n+1})p_b(Su_n, Su_{n+1}), \\ p_b(Su_{n+1}, Ru_n)p_b(Ru_n, Ru_{n+1}) \end{array}\right\} \\ &= \min\left\{\begin{array}{c} b_p(z_n, z_n)b_p(z_{n+1}, z_{n-1}), p_b(z_n, z_{n+1})p_b(z_n, z_{n+1}), \\ p_b(z_{n+1}, u_{n-1})p_b(u_{n-1}, z_n) \end{array}\right\} = 0, \end{aligned} \tag{46}$$

and taking into account Lemma 6, (42) becomes

$$\begin{aligned} M^1_{S,R}(u_n, u_{n+1}) &\leq \beta(Ru_n, Ru_{n+1})M^1_{S,R}(u_n, u_{n+1}) - a \cdot N^1_{S,R}(u_n, u_{n+1}) \\ &\leq \kappa p_b(Su_n, Ru_n) \cdot p_b(Su_{n+1}, Ru_{n+1}). \end{aligned} \tag{47}$$

Taking into account (46), the above inequality turns into

$$[p_b(z_n, z_{n+1})]^2 \leq \kappa p_b(z_n, z_{n-1})p_b(z_{n+1}, z_n), \tag{48}$$

or equivalent (since $z_n \neq z_{n+1}$)

$$p_b(z_n, z_{n+1}) \leq \kappa p_b(z_n, z_{n-1}). \tag{49}$$

Accordingly, from Lemma 7, it follows that the sequence $\{z_n\}$ is 0-p_b-Cauchy and due to the completeness of the space, there exists $z \in M$ such that $\lim n \longrightarrow \infty p_b(z_n), z) = p_b(z, z) = 0$. Following the corresponding lines in Theorem 8, we can find $u_* \in M$ such that $Ru_* = z$. Supposing that $Ru_* \neq Su^*$ for $u = u_{n_l}$ and $y = u_*$ and taking into account the assumption (c),

$$\begin{aligned} M^1_{S,R}(u_{n_l}, u_*) &\leq \beta(Ru_{n_l}, Ru_*)M^1_{S,R}(u_{n_l}, u_*) - a \cdot N^1_{S,R}(u_{n_l}, u_*) \\ &\leq \kappa p_b(Su_{n_l}, Ru_{n_l}) \cdot p_b(Su_*, Ru_*), \end{aligned} \tag{50}$$

where

$$\begin{aligned} M^1_{S,R}(u_{n_l}, u_*) &= \min\left\{[p_b(Su_{n_l}, Su_*)]^2, [p_b(Su_*, Ru_*)]^2\right\}, \\ N^1_{S,R}(u_{n_l}, u_*) &= \min\left\{b_p(Su_{n_l}, Ru_*)b_p(Su_*, Ru_{n_l}), p_b(Su_{n_l}, Ru_*)p_b(Su_*, Su_{n_l}), p_b(Su_*, Ru_{n_l})p_b(Ru_*, Ru_{n_l})\right\}. \end{aligned} \tag{51}$$

Since $\lim_{l\longrightarrow\infty} N^1_{S,R}(u_{n_l}, u_*) = 0$ and $\lim_{l\longrightarrow\infty} = p_b(Su_{n_l}, Ru_{n_l}) \cdot p_b(Su_*, Ru_*) = 0$ (by) letting $l \longrightarrow \infty$ in (50), we have

$$\text{either } [p_b(Su_*, Ru_*)]^2 = 0 \text{ or} \lim_{l\longrightarrow\infty} \left[p_b\left(Su_{n_l}, Su_*\right)\right]^2 = 0. \quad (52)$$

(1) If $[p_b(Su_*, Ru_*)]^2 = 0$, it follows that $Su_* = Ru_*$.

(2) If $\lim_{l\longrightarrow\infty}[p_b(Su_{n_l}, Su_*)]^2 = 0$, by $(pb)_4$

$$\begin{aligned} p_b(Ru_*, Su_*) &\le s\left[p_b\left(Ru_*, Su_{n_l}\right) + p_b\left(Su_{n_l}, Su_*\right)\right] - p_b\left(Su_{n_l}, Su_{n_l}\right) \\ &\le s\left[p_b\left(Ru_*, Su_{n_l}\right) + p_b\left(Su_{n_l}, Su_*\right)\right] \longrightarrow 0 \text{ as } l \longrightarrow \infty, \end{aligned} \quad (53)$$

so $p_b(Ru_*, Su_*) = 0$.

Thereupon, $Ru_* = Su_* = z$ and z is a point of coincidence of R and S.

Example 3. Let $M = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and the partial b-metric $p_b : M \times M \longrightarrow [0,+\infty)$ defined as follows (Table 1).

Let the function $\beta : M \times M \longrightarrow [0,+\infty)$, with

$$\beta(u, y) = \begin{cases} 1, & \text{for } (u, y) \in \{(\alpha_5, \alpha_3), (\alpha_3, \alpha_2)\}, \\ 2, & \text{for } (u, y) = (\alpha_2, \alpha_2), \\ 0, & \text{otherwise}, \end{cases} \quad (54)$$

and two mappings $S, R : M \longrightarrow M$ (Table 2).
First of all, we remark that

$$\begin{aligned} \beta(\alpha_5, \alpha_3) &= \beta(R\alpha_2, R\alpha_5) = 1 \Longrightarrow \beta(S\alpha_2, S\alpha_5) = \beta(\alpha_3, \alpha_2) = 1, \\ \beta(\alpha_3, \alpha_2) &= \beta(R\alpha_5, R\alpha_4) = 1 \Longrightarrow \beta(S\alpha_5, S\alpha_4) = \beta(\alpha_2, \alpha_2) = 2, \\ \beta(\alpha_2, \alpha_2) &= \beta(R\alpha_4, R\alpha_4) = 2 \Longrightarrow \beta(S\alpha_4, S\alpha_4) = \beta(\alpha_2, \alpha_2) = 2, \end{aligned} \quad (55)$$

which shows as that (b) holds. Also, it is easy to see that (a) and (c) are satisfied, so it remains to be verified (42). We distinguish two cases as follows:

(1) $(u, y) = (\alpha_2, \alpha_5)$

$$\begin{aligned} M^1_{S,R}(\alpha_2, \alpha_5) &= \min\left\{[p_b(S\alpha_2, S\alpha_5)]^2, [p_b(S\alpha_5, R\alpha_5)]^2\right\} = \min\left\{[p_b(\alpha_3, \alpha_2)]^2, [p_b b(\alpha_2, \alpha_3)]^2\right\} = 9, \\ N^1_{S,R}(\alpha_2, \alpha_5) &= \min\left\{b_p(S\alpha_2, R\alpha_5) b_p(S\alpha_5, R\alpha_2), \cdots\right\} = \min\left\{b_p(\alpha_3, \alpha_3) b_p(\alpha_2, \alpha_5), \cdots\right\} = 0, \\ p_b(S\alpha_2, R\alpha_2) p_b(S\alpha_5, R\alpha_5) &= p_b(\alpha_3, \alpha_5) p_b(\alpha_2, \alpha_3) = 22 \cdot 3 = 66. \end{aligned} \quad (56)$$

(2) $(u, y) = (\alpha_5, \alpha_4)$

$$\begin{aligned} M^1_{S,R}(\alpha_5, \alpha_4) &= \min\left\{[p_b(S\alpha_5, S\alpha_4)]^2, [p_b(S\alpha_4, R\alpha_4)]^2\right\} = \min\left\{[p_b(\alpha_2, \alpha_2)]^2, [p_b(\alpha_2, \alpha_2)]^2\right\} = 1, \\ N^1_{S,R}(\alpha_5, \alpha_4) &= \min\left\{b_p(S\alpha_5, R\alpha_4) b_p(S\alpha_4, R\alpha_5), \cdots\right\} = \min\left\{b_p(\alpha_2, \alpha_2) b_p(\alpha_2, \alpha_3), \cdots\right\} = 0, \\ p_b(S\alpha_5, R\alpha_5) p_b(S\alpha_4, R\alpha_4) &= p_b(\alpha_2, \alpha_3) p_b(\alpha_2, \alpha_2) = 3 \cdot 1 = 3. \end{aligned} \quad (57)$$

So, for any $\kappa \in (0, 1)$, the inequality (42) holds. Therefore, the mappings S, R have a point of coincidence, which is $z = \alpha_2$.

Corollary 13. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and two mappings $R, S : M \to M$. Suppose that there exist $\kappa \in (0, 1)$ and $a > 0$ such that*

$$M^1_{S,R}(u, y) - a \cdot N^1_{S,R}(u, y) \le \kappa p_b(Su, Ru) p_b(Sy, Ry), \quad (58)$$

where

$$\begin{aligned} M^1_{S,R}(u, y) &= \min\left\{[p_b(Su, Sy)]^2, [p_b(Sy, Ry)]^{22}\right\}, \\ N^1_{S,R}(u, y) &= \min\left\{b_p(Su, Ry) b_p(Sy, Ru), p_b(Su, Ry) p_b(Su, Sy), p_b(Sy, Ru) p_b(Ru, Ry)\right\}, \end{aligned} \quad (59)$$

Table 1

$p_b(u,y)$	α_1	α_2	α_3	α_4	α_5
α_1	0	2	6	30	42
α_2	2	1	3	21	31
α_3	6	3	2	14	22
α_4	30	21	14	5	7
α_5	42	31	22	7	6

Table 2

	α_1	α_2	α_3	α_4	α_5
S	α_5	α_3	α_2	α_2	α_2
R	α_1	α_5	α_1	α_2	α_3

for every $u, y \in M$, such that $u \neq y$ when $u, y \in C_c(R,S)_M$. Then, the mappings S and R have a point of coincidence providing that $S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms.

Proof. Put $\beta(u,y)=1$ in Theorem 12.

Theorem 14. *Let $(M, p_b, s \geq 1)$ be a complete p_b-ms a mapping $S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0,1)$ and $a > 0$ such that*

$$\beta(u,y) M_S^1(u,y) - a \cdot N_S^1(u,y) \leq \kappa p_b(Su,u) p_b(Sy,y), \quad (60)$$

where

$$M_S^1(u,y) = \min\left\{[p_b(Su,Sy)]^2, [p_b(Sy,y)]^2\right\},$$
$$N_S^1(u,y) = \min\left\{b_p(Su,y) b_p(Sy,u), p_b(Su,y) p_b(Su,Sy), p_b(Sy,u) p_b(u,y)\right\}, \quad (61)$$

for every $u, y \in M, u \neq y$. Suppose also that

(a) *S is β-admissible, and there exists $u_0 \in M$ such that $\beta(u_0, Su_0) \geq 1$*

(b) *M is β-regular*

Then, the mapping S possesses a fixed point.

Proof. Choose $R = I_M$ in Theorem 12.

Corollary 15. *Let $(M, p_b, s \geq 1)$ be a complete p_b-ms a mapping $S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0,1)$ and $a > 0$ such that*

$$M_S^1(u,y) - a \cdot N_S^1(u,y) \leq \kappa p_b(Su,u) p_b(Sy,y), \quad (62)$$

where

$$M_S^1(u,y) = \min\left\{[p_b(Su,Sy)]^2, [p_b(Sy,y)]^2\right\},$$
$$N_S^1(u,y) = \min\left\{b_p(Su,y) b_p(Sy,u), p_b(Su,y) p_b(Su,Sy), p_b(Sy,u) p_b(u,y)\right\}, \quad (63)$$

for every $u, y \in M$, $u \neq y$. Then, the mapping S possesses a fixed point.

Proof. Put $\beta(u,y)=1$ in Theorem 14.

Theorem 16. *Let $(M, p_b, s \geq 1)$ be a complete p_b-ms and two mappings $R, S : M \longrightarrow M$. Suppose that there exist $\kappa \in (0,1)$ and $a > 0$ such that*

$$\beta(Ru, Ry) M_{S,R}^2(u,y) \leq \kappa \cdot N_{S,R}^2(u,y), \quad (64)$$

where

$$M_{S,R}^2(u,y) = p_b(Su,Sy) p_b(Sy,Ry) - a \cdot \min\left\{b_p(Su,Ry), b_p(Sy,Ru)\right\},$$
$$N_{S,R}^2(u,y) = p_b(Ru,Ry) \cdot \max\left\{p_b(Su,Ru), p_b(Sy,Ry), \frac{p_b(Su,Ry) + p_b(Sy,Ru)}{2s}\right\}, \quad (65)$$

for every $u, y \in M$, such that $u \neq y$ when $u, y \in C_c(R,S)_M$. Suppose also that

(a) *$S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms*

(b) *S is R-β-admissible and there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \geq 1$*

(c) *M is β-regular*

Then, the mappings S and R have a point of coincidence.

Proof. We will only sketch the proof, because, basically, we use the same technique that was used in the above theorems. Indeed, for $u = u_n$, $y = u_{n+1}$, where the sequences $\{z_n\}$, $\{u_n\}$ are defined in Theorem 8, we have

$$\begin{aligned} M_{S,R}^2(u_n, u_{n+1}) &= p_b(Su_n, Su_{n+1}) p_b(Su_{n+1}, Ru_{n+1}) \\ &\quad --a \cdot \min\left\{b_p(Su_n, Ru_{n+1}), b_p(Su_{n+1}, Ru_n)\right\} \\ &= p_b(z_n, z_{n+1}) p_b(z_{n+1}, z_n) \\ &\quad - a \cdot \min\left\{b_p(z_n, z_n), b_p(z_{n+1}, z_{n-1})\right\} \\ &= [p_b(z_n, z_{n+1})]^2, \end{aligned}$$

$$
\begin{aligned}
N_{S,R}^2(u_n, u_{n+1}) &= p_b(Ru_n, Ru_{n+1}) \cdot \max \left\{ \begin{array}{c} p_b(Su_n, Ru_n), p_b(Su_{n+1}, Ru_{n+1}), \\ \dfrac{p_b(Su_n, Ru_{n+1}) + p_b(Su_{n+1}, Ru_n)}{2s} \end{array} \right\} \\
&= \max \left\{ \begin{array}{c} p_b(z_{n-1}, z_n), p_b(z_n, z_{n+1}), \\ \dfrac{p_b(z_n, z_n) + p_b(z_{n+1}, z_{n-1})}{2s} \end{array} \right\} \\
&\le p_b(z_{n-1}, z_n) \cdot \mathrm{m} p_b(z_{n-1}, z_n) \cdot \max \\
&\quad \cdot \left\{ \begin{array}{c} p_b(z_{n-1}, z_n), p_b(z_n, z_{n+1}), \\ \dfrac{p_b(z_n, z_n) + s[p_b(z_{n+1}, z_n) + p_b(z_n, z_{n+1})] - p_b(z_n, z_n)}{2s} \end{array} \right\} \\
&= p_b(z_{n-1}, z_n) \cdot \max \\
&\quad \cdot \left\{ p_b(z_{n-1}, z_n), p_b(z_n, z_{n+1}), \frac{p_b(z_{n+1}, z_n) + p_b(z_n, z_{n+1})}{2} \right\} \\
&= p_b(z_{n-1}, z_n) \cdot \max \left\{ p_b(z_{n-1}, z_n), p_b(z_n, z_{n+1}) \right\}.
\end{aligned} \tag{66}
$$

Thus, the inequality (64) becomes

$$
[p_b(z_n, z_{n+1})]^2 \le \kappa p_b(z_{n-1}, z_n) \cdot \max \{p_b(z_{n-1}, z_n), p_b(z_n, z_{n+1})\}. \tag{67}
$$

Since for the case $\max \{p_b(z_{n-1}, z_n), p_b(z_n, z_{n+1})\} = p_b(z_n, z_{n+1})$ we get $[p_b(z_n, z_{n+1})]^2 \le \kappa p_b(z_{n-1}, z_n) \cdot p_b(z_n, z_{n+1})$, or $p_b(z_n, z_{n+1}) \le \kappa p_b(z_{n-1}, z_n) < p_b(z_{n-1}, z_n)$, which is a contradiction, we conclude that $\max \{p_b(z_{n-1}, z_n), p_b(z_n, z_{n+1})\} = p_b(z_{n-1}, z_n)$ and then (67) becomes

$$
p_b(z_n, z_{n+1}) \le \kappa p_b(z_{n-1}, z_n), \tag{68}
$$

for any $n \in \mathbb{N}$. Therefore, by Lemma L2A and using similar arguments as in Theorems 8 and 12, there exists $u_* \in M$ such that

$$
\lim_{n\to\infty} p_b(Su_n, Ru_*) = \lim_{n\to\infty} p_b(Ru_n, Ru_*) = p_b(Ru_*, Ru_*) = 0. \tag{69}
$$

Finally, we claim that $Su_* = Ru_*$. From the assumptions (c), there exists a subsequences $\{u_{n_l}\}$ of $\{u_n\}$ such that $\beta(u_{n_l}, u_*) \ge 1$. Thus, replacing u by u_{n_l} and y by u_*, we get (as $l \longrightarrow \infty$)

$$
\begin{aligned}
\lim_{n\to\infty} M_{S,R}^2(u_{n_l}, u_*) &= \lim_{n\to\infty} \left[p_b(Su_{n_l}, Su_*) p_b(Su_*, Ru_*) \right. \\
&\quad \left. --a \cdot \min \left\{ b_p(Su_{n_l}, Ru_*), b_p(Su_*, Ru_{n_l}) \right\} \right] \\
&= p_b(Su_*, Ru_*) \cdot \lim_{n\to\infty} \left[p_b(Su_{n_l}, Su_*) \right), \\
\lim_{n\to\infty} N_{S,R}^2(u_{n_l}, u_*) &= \lim_{n\to\infty} p_b(Ru_{n_l}, Ru_*) \\
&\quad \cdot \max \left\{ \begin{array}{c} p_b(Su_{n_l}, Ru_{n_l}), p_b(Su_*, Ru_*), \\ \dfrac{p_b(Su_{n_l}, Ru_*) + p_b(Su_*, Ru_{n_l})}{2s} \end{array} \right\} = 0.
\end{aligned} \tag{70}
$$

Consequently, (64) becomes $p_b(Su_*, Ru_*) \cdot \lim_{n\to\infty} [p_b(Su_{n_l}, Su_*) = 0$ and the rest is just a verbatim repetition of the lines in the previous proofs.

Corollary 17. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and two mappings $R, S : M \longrightarrow M$. Suppose that there exist $\kappa \in (0, 1)$ and $a > 0$ such that*

$$
M_{S,R}^2(u, y) \le \kappa \cdot N_{S,R}^2(u, y), \tag{71}
$$

where

$$
\begin{aligned}
M_{S,R}^2(u, y) &= p_b(Su, Sy) p_b(Sy, Ry) - a \cdot \min \left\{ b_p(Su, Ry), b_p(Sy, Ru) \right\}, \\
N_{S,R}^2(u, y) &= p_b(Ru, Ry) \cdot \max \left\{ p_b(Su, Ru), p_b(Sy, Ry), \frac{p_b(Su, Ry) + p_b(Sy, Ru)}{2s} \right\},
\end{aligned} \tag{72}
$$

for every $u, y \in M$, such that $u \ne y$ when $u, y \in C_c(R, S)_M$. If $S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms, then, the mappings S and R have a point of coincidence.

Proof. Let $\beta(u, y) = 1$ in Theorem 16.

Theorem 18. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and a mapping $S : M \longrightarrow M$. Suppose that there exist $\kappa \in (0, 1)$ and $a > 0$ such that*

$$
\beta(u, y) M_S^2(u, y) \le \kappa \cdot N_S^2(u, y), \tag{73}
$$

where

$$
\begin{aligned}
M_S^2(u, y) &= p_b(Su, Sy) p_b(Sy, y) - a \cdot \min \left\{ b_p(Su, y), b_p(Sy, u) \right\}, \\
N_S^2(u, y) &= p_b(u, y) \cdot \max \left\{ p_b(Su, u), p_b(Sy, y), \frac{p_b(Su, y) + p_b(Sy, u)}{2s} \right\},
\end{aligned} \tag{74}
$$

for every $u, y \in M$. Suppose also that

(i) S is β-admissible, and there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \ge 1$

(ii) M is β-regular

Then, the mapping S admits a fixed point.

Proof. Choose $R = I_M$.

Corollary 19. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and two mappings $R, S : M \longrightarrow M$. Suppose that there exist $\kappa \in (0, 1)$ and $a > 0$ such that*

$$
M_S^2(u, y) \le \kappa \cdot N_S^2(u, y), \tag{75}
$$

where

$$M_S^2(u,y)M_S^2(u,y) = p_b(Su,Sy)p_b(Sy,y) - a \cdot \min\left\{b_p(Su,y), b_p(Sy,u)\right\},$$
$$N_S^2(u,y) = p_b(u,y) \cdot \max\left\{p_b(Su,u), p_b(Sy,y), \frac{p_b(Su,y)+p_b(Sy,u)}{2s}\right\}, \tag{76}$$

for every $u, y \in M$. Then, the mapping S has a fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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Research Article

Differences of Positive Linear Operators on Simplices

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The aim of the paper is twofold: we introduce new positive linear operators acting on continuous functions defined on a simplex and then estimate differences involving them and/or other known operators. The estimates are given in terms of moduli of smoothness and K-functionals. Several applications and examples illustrate the general results.

1. Introduction

Differences of positive linear operators were intensively investigated in the last years; see [1–14] and the references therein. The operators involved in these studies act usually on continuous functions defined on real intervals, and the differences are estimated in terms of moduli of smoothness and K-functionals. In some papers, operators having equal central moments up to a certain order are considered. Other articles deal with operators constructed with the same fundamental functions and different functionals in front of them.

The study of differences of positive linear operators is important from a theoretical point of view, but also from a practical one. Let (U_n) and (V_n) be certain positive linear operators. If we know that $|U_n(f)-V_n(f)|$ is small, we can choose (U_n) or (V_n) taking into account other qualities of them like shape-preserving properties and smoothness/Lipschitz preserving properties.

This paper is concerned with differences of positive linear operators acting on continuous functions defined on simplices. For the sake of simplicity, we consider only the case of the canonical simplex in $\mathbb{R}^2$, where the notation is simpler, but the results can be easily translated to an arbitrary simplex in $\mathbb{R}^n$.

We consider the bivariate versions of some classical operators like Bernstein, Durrmeyer, Kantorovich, and genuine Bernstein-Durrmeyer operators. These bivariate versions were already studied in literature from other points of view. We introduce the bivariate versions of other operators: U_n^ρ (see [15, 16]) and the operators defined in [17]. All these operators are constructed with the fundamental Bernstein polynomials on the two-dimensional simplex. A different kind of operator is the bivariate version of the univariate Beta operator of Mühlbach and Lupas (see [18–20]); we introduce it and use it in composition with the Bernstein operator to get a useful representation of U_n^ρ.

We get estimates of differences of the abovementioned operators, in terms of suitable moduli of smoothness and K-functionals.

To resume, the aim of our paper is twofold: we introduce new operators on a simplex and then estimate differences involving them and other known operators.

The list of applications and examples can be enlarged. In particular, we will be interested for a future work in studying differences of bivariate versions of operators, which preserve exponential functions (see [21–23]). We also intend to deepen the study of the newly introduced Beta operators on the simplex and to consider the composition of it with other operators, leading to new applications and-why not-new theoretical aspects/problems. Given a Markov operator (i.e., a positive linear operator which preserves the constant functions), the study of its iterate is important not only in Approximation Theory but also in Ergodic Theory and other areas of research. We intend to investigate from this point of view the newly introduced operators, which are in fact Markov operators.

We end this Introduction by presenting some notation and a fundamental inequality expressed in Lemma 1. Section 2 contains the main theoretical results, while Section 3 is devoted to applications and examples.

Let $S := \{(x,y) \in \mathbb{R}^2 | x, y \geq 0, x+y \leq 1\}$ be the canonical simplex in $\mathbb{R}^2$ and $E(S)$ denote a space of real-valued continuous functions of two variables defined on S, containing the polynomials. Throughout the paper, we will denote by $\mathbf{1}$ the constant function, namely,

$$\mathbf{1} : S \longrightarrow \mathbb{R}, \mathbf{1}(x,y) = 1, (x,y) \in S, \tag{1}$$

and $pr_i : S \longrightarrow \mathbb{R}, i = 1, 2,$ will denote the ith coordinate functions restricted on S, which are given by

$$pr_1(x,y) = x \text{ and } pr_2(x,y) = y, (x,y) \in S. \tag{2}$$

Let $F : E(S) \longrightarrow \mathbb{R}$ be a positive linear functional such that $F(\mathbf{1}) = 1$. Set

$$\begin{gathered} b_1^F := F(pr_1), b_2^F := F(pr_2), \\ \mu_{i,j}^F := F\left(\left(pr_1 - b_1^F \mathbf{1}\right)^i \left(pr_2 - b_2^F \mathbf{1}\right)^j\right), i, j \in \mathbb{N}. \end{gathered} \tag{3}$$

Then, one has

$$\begin{gathered} \mu_{1,0}^F = 0, \mu_{2,0}^F = F\left(pr_1^2\right) - \left(b_1^F\right)^2 \geq 0, \\ \mu_{0,1}^F = 0, \mu_{0,2}^F = F\left(pr_2^2\right) - \left(b_2^F\right)^2 \geq 0. \end{gathered} \tag{4}$$

Let $C^2(S)$ be the space of all real-valued (continuous) functions, differentiable on int (S) and whose partial derivatives of order ≤ 2 can be continuously extended to S, having

$$\|f\| = \sup \{|f(x,y)| : (x,y) \in S\} < \infty. \tag{5}$$

Lemma 1. *If $f \in C^2(S)$, then*

$$\begin{gathered} \left|F(f) - f\left(b_1^F, b_2^F\right)\right| \leq M_f \{\mu_{2,0}^F + \mu_{0,2}^F\}, \\ \text{where } M_f := \max\left\{\|f_{xx}\|, \left\|f_{xy}\right\|, \left\|f_{yy}\right\|\right\}. \end{gathered} \tag{6}$$

Proof. Consider the line segment connecting (b_1^F, b_2^F) with $(t_1, t_2) \in S$. From Taylor's formula (see [24], p.245), there is a point (c_1, c_2) on this line segment, different from (b_1^F, b_2^F) and (t_1, t_2), such that

$$\begin{aligned} f(t_1,t_2) = {} & f\left(b_1^F, b_2^F\right) + f_x\left(b_1^F, b_2^F\right)\left(t_1 - b_1^F\right) \\ & + f_y\left(b_1^F, b_2^F\right)\left(t_2 - b_2^F\right) \\ & + \frac{1}{2}\left\{f_{xx}(c_1,c_2)\left(t_1 - b_1^F\right)^2\right. \\ & + 2f_{xy}(c_1,c_2)\left(t_1 - b_1^F\right)\left(t_2 - b_2^F\right) \\ & \left. \cdot + f_{yy}(c_1,c_2)\left(t_2 - b_2^F\right)^2\right\}. \end{aligned} \tag{7}$$

Therefore, we can write

$$\begin{aligned} & \left|f - f\left(b_1^F, b_2^F\right)\mathbf{1} - f_x\left(b_1^F, b_2^F\right)\left(pr_1 - b_1^F\mathbf{1}\right)\right. \\ & \left. - f_y\left(b_1^F, b_2^F\right)\left(pr_2 - b_2^F\mathbf{1}\right)\right| \\ & \leq \frac{1}{2}\left\{\|f_{xx}\|\left(pr_1 - b_1^F\mathbf{1}\right)^2 + 2\left\|f_{xy}\right\|\left|\left(pr_1 - b_1^F\mathbf{1}\right)\right.\right. \\ & \quad \left.\left. \cdot \left(pr_2 - b_2^F\mathbf{1}\right)\right| + \left\|f_{yy}\right\|\left(pr_2 - b_2^F\mathbf{1}\right)^2\right\} \\ & \leq \frac{1}{2}\left\{\left(\|f_{xx}\| + \left\|f_{xy}\right\|\right)\left(pr_1 - b_1^F\mathbf{1}\right)^2\right. \\ & \quad \left. + \left(\left\|f_{xy}\right\| + \left\|f_{yy}\right\|\right)\left(pr_2 - b_2^F\mathbf{1}\right)^2\right\} \\ & \leq M_f\left(\left(pr_1 - b_1^F\mathbf{1}\right)^2 + \left(pr_2 - b_2^F\mathbf{1}\right)^2\right), \end{aligned} \tag{8}$$

which gives the result.

2. Difference of Bivariate Positive Linear Operators

Denote by $C(S)$ the space of real-valued continuous functions on S with the norm $\|f\| = \max_{(x,y)\in S} |f(x,y)|, f \in C(S)$. Let K be a set of nonnegative integers and for $k, l \in K$ let $p_{k,l} \in C(S), p_{k,l} \geq 0$, satisfy $\sum_{k,l\in K} p_{k,l} = \mathbf{1}$. Let $F_{k,l} : E(S) \longrightarrow \mathbb{R}$ and $G_{k,l} : E(S) \longrightarrow \mathbb{R}, k, l \in K$, be positive linear functionals such that $F_{k,l}(\mathbf{1}) = 1$ and $G_{k,l}(\mathbf{1}) = 1$. Moreover, let $D(S)$ be the set of all $f \in E(S)$ for which

$$\begin{gathered} \sum_{k,l\in K} F_{k,l}(f) p_{k,l} \in C(S), \\ \sum_{k,l\in K} G_{k,l}(f) p_{k,l} \in C(S). \end{gathered} \tag{9}$$

Now, consider the bivariate positive linear operators V

and W acting from $D(S)$ into $C(S)$ defined, for $f \in D(S)$, by

$$V(f)(x,y) = \sum_{k,l\in K} F_{k,l}(f) p_{k,l}(x,y), \\ W(f)(x,y) = \sum_{k,l\in K} G_{k,l}(f) p_{k,l}(x,y), \tag{10}$$

respectively. For future correspondences, we denote

$$\sigma(x,y) := \sum_{k,l\in K} \left(\mu_{2,0}^{F_{k,l}} + \mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{F_{k,l}} + \mu_{0,2}^{G_{k,l}}\right) p_{k,l}(x,y), \tag{11}$$

$$\delta := \sup_{k,l\in K} \left| \left(b_1^{F_{k,l}}, b_2^{F_{k,l}}\right) - \left(b_1^{G_{k,l}}, b_2^{G_{k,l}}\right) \right|, \tag{12}$$

where $|\cdot|$ is the l_1-norm in $\mathbb{R}^2$.

In the following, we adopt the definitions of K-functional and modulus of smoothness from [25, 26]. Let

$$S(h) = \{x \in S | x + ht \in S \text{ for } 0 \le t \le 1\}, h \in \mathbb{R}^2. \tag{13}$$

For $r \in \mathbb{N}$, rth order differences on the subset $S(rh)$ are defined as

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x+kh). \tag{14}$$

The rth order modulus of smoothness of f is a function $\omega_r : C(S) \times (0,\infty) \longrightarrow [0,\infty)$ given by

$$\omega_r(f,\alpha) = \sup_{0<|h|\le\alpha} \|\Delta_h^r f\|, \alpha > 0. \tag{15}$$

Let $C^r(S)$ be the space of all real-valued (continuous) functions, differentiable on int (S) and whose partial derivatives of order $\le r$ can be continuously extended to S, with the seminorm

$$|g|_{C^r(S)} = \sum_{\gamma_1+\gamma_2=r} \left\| \frac{\partial^r g}{\partial x^{\gamma_1} \partial y^{\gamma_2}} \right\| < \infty, \gamma_i \ge 0, i = 1, 2, \gamma_1 + \gamma_2 = r. \tag{16}$$

For $f \in C(S)$, we shall use the following K-functional:

$$K_r(f,t) = \inf \left\{ \|f-g\| + t|g|_{C^r(S)} : g \in C^r(S) \right\}. \tag{17}$$

Then, there exist $c_1, c_2 > 0$ such that for any $t > 0$ (see [25, 26])

$$c_1 K_r(f,t^r) \le \omega_r(f,t) \le c_2 K_r(f,t^r). \tag{18}$$

Here, c_2 depends only on r (for the general definition on the $L_p, 1 \le p \le \infty$, spaces of functions on bounded domains, see [25] or, on unbounded domains see [27], p.341.

Theorem 2. *If $f \in D(S) \cap C^2(S)$, then*

$$|(V-W)(f)(x,y)| \le M_f \sigma(x,y) + \omega_1(f,\delta), \tag{19}$$

where M_f is defined in Lemma 1.

Proof. Let $(x,y) \in S$. From Lemma 1, we get

$$\begin{aligned} |(V-W)(f)(x,y)| &\le \sum_{k,l\in K} \left|F_{k,l}(f) - G_{k,l}(f)\right| p_{k,l}(x,y) \\ &\le \sum_{k,l\in K} p_{k,l}(x,y) \Big\{ \left| F_{k,l}(f) - f\left(b_1^{F_{k,l}}, b_2^{F_{k,l}}\right) \right| \\ &+ \left| G_{k,l}(f) - f\left(b_1^{G_{k,l}}, b_2^{G_{k,l}}\right) \right| + \Big| f\left(b_1^{F_{k,l}}, b_2^{F_{k,l}}\right) \\ &- f\left(b_1^{G_{k,l}}, b_2^{G_{k,l}}\right) \Big| \Big\} \le M_f \sum_{k,l\in K} p_{k,l}(x,y) \left[\mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}} + \mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}}\right] \\ &+ \omega_1\left(f ; \left| \left(b_1^{F_{k,l}}, b_2^{F_{k,l}}\right) - \left(b_1^{G_{k,l}}, b_2^{G_{k,l}}\right) \right| \right) \le M_f \sigma(x,y) + \omega_1(f,\delta). \end{aligned} \tag{20}$$

Theorem 3. *If $f \in C(S)$, then*

$$|(V-W)(f)(x,y)| \le \eta_1 \omega_1(f,\delta) + \eta_2 \omega_2\left(f, \sqrt{\sigma(x,y)}\right), \tag{21}$$

where $\eta_1, \eta_2 > 0$, and $\delta = \sup_{k,l\in K} \{|b_1^{F_{k,l}} - b_1^{G_{k,l}}| + |b_2^{F_{k,l}} - b_2^{G_{k,l}}|\}$.

Proof. Let $g \in C^2(S)$. From Theorem 2, we get

$$\begin{aligned} |(V-W)(f)(x,y)| &\le |V(f-g)(x,y)| + |W(g-f)(x,y)| \\ &+ |(V-W)(g)(x,y)| \le 2\|f-g\| + M_g \sigma(x,y) \\ &+ \left| g\left(b_1^{F_{k,l}}, b_2^{F_{k,l}}\right) - g\left(b_1^{G_{k,l}}, b_2^{G_{k,l}}\right) \right|, \end{aligned} \tag{22}$$

where M_g is the same notation as in Lemma 1 for g. Since partial derivatives of g exist and are continuous everywhere in S, it follows that g is differentiable at every point of the line segment connecting the points $(b_1^{F_{k,l}}, b_2^{F_{k,l}})$ and $(b_1^{G_{k,l}}, b_2^{G_{k,l}})$ in S, $k, l \in K$. By the mean value theorem (see, e.g., [24], p. 239), there is a point (a_1, a_2) on this line segment such that

$$\begin{aligned} g\left(b_1^{F_{k,l}}, b_2^{F_{k,l}}\right) - g\left(b_1^{G_{k,l}}, b_2^{G_{k,l}}\right) &= g_x(a_1,a_2)\left(b_1^{F_{k,l}} - b_1^{G_{k,l}}\right) \\ &+ g_y(a_1,a_2)\left(b_2^{F_{k,l}} - b_2^{G_{k,l}}\right). \end{aligned} \tag{23}$$

From (16), we get

$$\begin{aligned} \left| g\left(b_1^{F_{k,l}}, b_2^{F_{k,l}}\right) - g\left(b_1^{G_{k,l}}, b_2^{G_{k,l}}\right) \right| &\le \|g_x\| \left| b_1^{F_{k,l}} - b_1^{G_{k,l}} \right| \\ &+ \left\|g_y\right\| \left| b_2^{F_{k,l}} - b_2^{G_{k,l}} \right| \le |g|_{C^1(S)} \sup_{k,l\in K} \Big\{ \left| b_1^{F_{k,l}} - b_1^{G_{k,l}} \right| \\ &+ \left| b_2^{F_{k,l}} - b_2^{G_{k,l}} \right| \Big\} \le \delta |g|_{C^1(S)}, k, l \in K. \end{aligned} \tag{24}$$

Moreover, since $M_g \le |g|_{C^2(S)}$, (22) gives that

$$\begin{aligned} |(V-W)(f)(x,y)| &\le 2\|f-g\| + \delta |g|_{C^1(S)} \\ &+ \sigma(x,y)|g|_{C^2(S)} \le K_1(f,\delta) + K_2(f,\sigma(x,y)). \end{aligned} \tag{25}$$

Finally, from (18), we obtain

$$|(V-W)(f)(x,y)| \le \eta_1 \omega_1(f,\delta) + \eta_2 \omega_2\left(f, \sqrt{\sigma(x,y)}\right). \tag{26}$$

3. Applications

3.1. Difference of Bivariate Bernstein Operators and Their Durrmeyer Variants. For every $n \ge 1, f \in C(S)$, and $(x,y) \in S$, the nth bivariate Bernstein operator $B_n : C(S) \longrightarrow C(S)$ is defined by

$$B_n(f)(x,y) = \sum_{\substack{k,l=0,\cdots,n \\ k+l\le n}} f\left(\frac{k}{n}, \frac{l}{n}\right) p_{n,k,l}(x,y), \tag{27}$$

where

$$p_{n,k,l}(x,y) \coloneqq \frac{n!}{k!l!(n-k-l)!} x^k y^l (1-x-y)^{n-k-l}, \tag{28}$$

with $k, l = 0, \cdots, n, k+l \le n, (x,y) \in S$, (see, e.g., [28], p. 115).

For $f \in L^1(S)$, the bivariate Durrmeyer operators $M_n : L^1(S) \longrightarrow C(S)$ are defined by

$$\begin{aligned} M_n(f)(x,y) = \sum_{\substack{k,l=0,\cdots,n \\ k+l\le n}} &\left((n+1)(n+2) \int_0^1 \int_0^{1-s} p_{n,k,l}(s,t) \right. \\ &\left. \cdot f(s,t) dt\, ds \right) p_{n,k,l}(x,y), \end{aligned} \tag{29}$$

see, e.g., [29].

Now, denoting

$$\begin{aligned} F_{n,k,l}(f) &\coloneqq f\left(\frac{k}{n}, \frac{l}{n}\right), 0 \le k+l \le n, \\ G_{n,k,l}(f) &\coloneqq (n+1)(n+2) \int_0^1 \int_0^{1-s} p_{n,k,l}(s,t) f(s,t) dt\, ds, \end{aligned} \tag{30}$$

the bivariate Bernstein operators and bivariate Durrmeyer operators can be written as

$$\begin{aligned} B_n(f)(x,y) &= \sum_{\substack{k,l=0,\cdots,n \\ k+l\le n}} F_{n,k,l}(f) p_{n,k,l}(x,y), \\ M_n(f)(x,y) &= \sum_{\substack{k,l=0,\cdots,n \\ k+l\le n}} G_{n,k,l}(f) p_{n,k,l}(x,y), \end{aligned} \tag{31}$$

respectively.

Proposition 4. *For bivariate Bernstein operators and their Durrmeyer variants, the following properties hold:*

(i) If $f \in C^2(S)$, then

$$|(B_n - M_n)(f)(x,y)| \le M_f \sigma(x,y) + \omega_1\left(f, \frac{3}{n+3}\right), \tag{32}$$

where M_f is the same as in Lemma 1 and

$$\sigma(x,y) = \frac{\left(-x^2 - y^2 + x + y\right)n^2 + \left(x^2 + y^2 + 2\right)n + 4}{(n+3)^2(n+4)} \le \frac{1}{n+4}. \tag{33}$$

(ii) If $f \in C(S)$, then

$$|(B_n - M_n)(f)(x,y)| \le \eta_1 \omega_1\left(f, \frac{3}{n+3}\right) + \eta_2 \omega_2\left(f, \sqrt{\sigma(x,y)}\right). \tag{34}$$

Proof. We need to evaluate the terms in (11). So, we get the following results:

$$\begin{aligned} b_1^{F_{n,k,l}} &= \frac{k}{n}, b_2^{F_{n,k,l}} = \frac{l}{n}, \\ b_1^{G_{n,k,l}} &= \frac{k+1}{n+3}, b_2^{G_{n,k,l}} = \frac{l+1}{n+3}, \end{aligned} \tag{35}$$

$0 \le k+l \le n$. Therefore, we easily obtain that

$$\begin{aligned} \mu_{2,0}^{F_{n,k,l}} &= 0, \mu_{2,0}^{G_{n,k,l}} = \frac{(k+1)(n+2-k)}{(n+3)^2(n+4)}, \\ \mu_{0,2}^{F_{n,k,l}} &= 0, \mu_{0,2}^{G_{n,k,l}} = \frac{(l+1)(n+2-l)}{(n+3)^2(n+4)}. \end{aligned} \tag{36}$$

Using Maple, one obtains

$$\sigma(x,y)=\sum_{\substack{k,l=0,\cdots,n\\ k+l\le n}}\left[\frac{(k+1)(n+2-k)}{(n+3)^2(n+4)}+\frac{(l+1)(n+2-l)}{(n+3)^2(n+4)}\right]$$
$$p_{n,k,l}(x,y)=\frac{\left(-x^2-y^2+x+y\right)n^2+\left(x^2+y^2+2\right)n+4}{(n+3)^2(n+4)}. \tag{37}$$

It is easy to verify that $\sigma(x,y)\le 1/(n+4)$.
Now, for δ, we obtain

$$\begin{aligned}\delta&=\max_{0\le k+l\le n}\left\{\left|b_1^{F_{n,k,l}}-b_1^{G_{n,k,l}}\right|+\left|b_2^{F_{n,k,l}}-b_2^{G_{n,k,l}}\right|\right\}\\&=\max_{0\le k+l\le n}\left\{\left|\frac{n-3k}{n(n+3)}\right|+\left|\frac{n-3l}{n(n+3)}\right|\right\}=\frac{3}{n+3}.\end{aligned} \tag{38}$$

The rest of the proof follows from Theorems 2 and 3.

3.2. Difference of Bivariate Bernstein Operators and the Bivariate Operators A_n. Let Π_n be the space of polynomials over [0, 1] of degree at most n. In [17], Aldaz et al. introduced a Bernstein operator $A_n : C[0,1]\longrightarrow \Pi_n$ that fixes 1 and x^2. The operators A_n are given by

$$A_n(f)(x)=\sum_{k=0}^{n} f\left(\left(\frac{k(k-1)}{n(n-1)}\right)^{1/2}\right)\binom{n}{k}x^k(1-x)^{n-k}. \tag{39}$$

Here, for $f\in C(S)$ and $(x,y)\in S$, we introduce the bivariate form of the operators A_n as follows

$$A_n(f)(x,y)=\sum_{\substack{k,l=0,\cdots,n\\ k+l\le n}} f\left(\sqrt{\frac{k(k-1)}{n(n-1)}},\sqrt{\frac{l(l-1)}{n(n-1)}}\right)p_{n,k,l}(x,y). \tag{40}$$

Denoting

$$\begin{aligned}F_{n,k,l}(f)&=f\left(\frac{k}{n},\frac{l}{n}\right),G_{n,k,l}(f)\\&=f\left(\sqrt{\frac{k(k-1)}{n(n-1)}},\sqrt{\frac{l(l-1)}{n(n-1)}}\right),\end{aligned} \tag{41}$$

for $k,l=0,\cdots,n$, $k+l\le n$, we get

$$\begin{gathered}b_1^{F_{n,k,l}}=\frac{k}{n},\ b_2^{F_{n,k,l}}=\frac{l}{n},\\ b_1^{G_{n,k,l}}=\sqrt{\frac{k(k-1)}{n(n-1)}},\ b_2^{G_{n,k,l}}=\sqrt{\frac{l(l-1)}{n(n-1)}},\\ \mu_{2,0}^{F_{n,k,l}}=\mu_{0,2}^{F_{n,k,l}}=\mu_{2,0}^{G_{n,k,l}}=\mu_{0,2}^{G_{n,k,l}}=0,\\ \delta=\max_{\substack{k,l=0,\cdots,n\\ k+l\le n}}\left\{\left|b_1^{F_{n,k,l}}-b_1^{G_{n,k,l}}\right|+\left|b_2^{F_{n,k,l}}-b_2^{G_{n,k,l}}\right|\right\}=\frac{2}{n}.\end{gathered} \tag{42}$$

Proposition 5. *For bivariate Bernstein operators and bivariate operators A_n, the following properties hold:*

(i) If $f\in C^2(S)$, then

$$\left|(B_n-A_n)(f)(x,y)\right|\le\omega_1\left(f,\frac{2}{n}\right). \tag{43}$$

(ii) If $f\in C(S)$, then

$$\left|(B_n-A_n)(f)(x,y)\right|\le\eta_1\omega_1\left(f,\frac{2}{n}\right). \tag{44}$$

3.3. Difference of Bivariate Bernstein Operators and Bivariate Genuine Bernstein-Durrmeyer Operators. In 1987, Chen [30] and Goodman and Sharma [31] constructed the following positive linear operators

$$\begin{aligned}U_{n,1}(f)(x)=&f(0)p_{n,0}(x)+f(1)p_{n,n}(x)\\&+\sum_{k=1}^{n-1}p_{n,k}(x)(n-1)\\&\cdot\int_0^1 p_{n-2,k-1}(t)f(t)\,dt,\end{aligned} \tag{45}$$

where $n\in\mathbb{N}, f\in C[0,1]$, and

$$p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k},\ x\in[0,1],\ 0\le k\le n. \tag{46}$$

For the historical background of these operators, we refer to [32]. In 1991, Goodman and Sharma [33] constructed and studied the multivariate form of the operators $U_{n,1}$ on a simplex. In [34], Sauer deeply studied the multivariate genuine Bernstein-Durrmeyer operators. Here, for $f\in L^1(S)$, we consider the bivariate form given by

$$\begin{aligned} U_n(f)(x,y) &= f(0,0)(1-x-y)^n + f(1,0)x^n + f(0,1)y^n \\ &+ \sum_{l=1}^{n-1} p_{n,0,l}(x,y)(n-1)\int_0^1 p_{n-2,l-1}(t)f(0,t)dt \\ &+ \sum_{k=1}^{n-1} p_{n,k,0}(x,y)(n-1)\int_0^1 p_{n-2,k-1}(s)f(s,0)ds \\ &+ \sum_{k=1}^{n-1} p_{n,k,n-k}(x,y)(n-1) \\ &\cdot \int_0^1 p_{n-2,k-1}(t)f(t,1-t)dt \\ &+ \sum_{\substack{k+l\le n-1\\ k\ge 1, l\ge 1}} p_{n,k,l}(x,y)(n-1)(n-2) \\ &\cdot \int_0^1\int_0^{1-t} p_{n-3,k-1,l-1}(s,t)f(s,t)ds\,dt \end{aligned} \tag{47}$$

with the bivariate Bernstein's fundamental functions given by (28) (see [33], Formula 1.7). These operators satisfy $U_n(f)(x,y) = f(x,y)$ at the vertices of S.

Proposition 6. *For bivariate Bernstein operators and bivariate genuine Bernstein-Durrmeyer operators, the following properties hold:*

(i) If $f \in C^2(S)$, then

$$|(B_n - U_n)(f)(x,y)| \le M_f \sigma(x,y), \tag{48}$$

where M_f is the same as in Lemma 1 and

$$\sigma(x,y) = \frac{(x+y-x^2-y^2)(n-1)}{n(n+1)} \le \frac{1}{2(n+1)}. \tag{49}$$

(ii) If $f \in C(S)$, then

$$|(B_n - U_n)(f)(x,y)| \le \eta_2 \omega_2\left(f;\sqrt{\sigma(x,y)}\right). \tag{50}$$

Proof. If we denote

$$\begin{aligned} F_{n,k,l}(f) &:= f\left(\frac{k}{n},\frac{l}{n}\right), 0 \le k+l \le n, \\ G_{n,0,0}(f) &= f(0,0), k=l=0, \\ G_{n,n,0}(f) &= f(1,0), k=n, l=0, \\ G_{n,0,n}(f) &= f(0,1), k=0, l=n, \\ G_{n,k,0}(f) &= (n-1)\int_0^1 p_{n-2,k-1}(s)f(s,0)ds, 1\le k\le n-1, l=0, \\ G_{n,0,l}(f) &= (n-1)\int_0^1 p_{n-2,l-1}(t)f(0,t)dt, k=0, 1\le l\le n-1, \\ G_{n,k,n-k}(f) &= (n-1)\int_0^1 p_{n-2,k-1}(t)f(t,1-t) \\ &\cdot dt, 1\le k\le n-1, l=n-k, \\ G_{n,k,l}(f) &= (n-1)(n-2)\int_0^1\int_0^{1-t} p_{n-3,k-1,l-1}(s,t)f(s,t) \\ &\cdot ds\,dt, 1\le k+l\le n-1, \end{aligned} \tag{51}$$

then for the bivariate Bernstein operators, we have

$$B_n(f)(x,y) = \sum_{\substack{k,l=0,\cdots,n\\ k+l\le n}} F_{n,k,l}(f)p_{n,k,l}(x,y). \tag{52}$$

The bivariate genuine Bernstein-Durrmeyer operators are given by

$$U_n(f)(x,y) = \sum_{\substack{k,l=0,\cdots,n\\ k+l\le n}} G_{n,k,l}(f)p_{n,k,l}(x,y). \tag{53}$$

Now, for $0 \le k+l \le n$, we get

$$b_1^{F_{n,k,l}} = \frac{k}{n}, b_2^{F_{n,k,l}} = \frac{l}{n}, b_1^{G_{n,k,l}} = \frac{k}{n}, b_2^{G_{n,k,l}} = \frac{l}{n}. \tag{54}$$

Hence, we obtain

$$\begin{aligned} \mu_{2,0}^{F_{n,k,l}} &= 0, \mu_{2,0}^{G_{n,k,l}} = \frac{k(n-k)}{n^2(n+1)}, \\ \mu_{0,2}^{F_{n,k,l}} &= 0, \mu_{0,2}^{G_{n,k,l}} = \frac{l(n-l)}{n^2(n+1)}. \end{aligned} \tag{55}$$

Therefore, $\delta = 0$ and

$$\begin{aligned} \sigma(x,y) &= \sum_{\substack{k,l=0,\cdots,n\\ k+l\le n}} \left[\frac{k(n-k)}{n^2(n+1)} + \frac{l(n-l)}{n^2(n+1)}\right] p_{n,k,l}(x,y) \\ &= \frac{(x+y-x^2-y^2)(n-1)}{n(n+1)}, (x,y)\in S. \end{aligned} \tag{56}$$

The proof is concluded by using Theorems 2 and 3.

4. The Difference $U_n^\rho - U_n^r$

Let $\rho > 0$ and $n \in \mathbb{N}$. The operators $U_{n,1}^\rho : C[0,1] \longrightarrow \prod_n$ are introduced by Paltanea in [35] (see also [15, 16]). These operators are defined by

$$U_{n,1}^\rho(f;x) := \sum_{k=1}^{n-1}\left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{\beta(k\rho,(n-k)\rho)} f(t)\,dt\right) \cdot p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n, \tag{57}$$

where $f \in C[0,1]$, $x \in [0,1]$, and $\beta(\cdot,\cdot)$ are Euler's Beta function.

Here, for $f \in L^1(S)$, we consider the bivariate form of these operators, given by

$$\begin{aligned} U_n^\rho f(x,y) = {} & f(0,0)(1-x-y)^n + f(1,0)x^n + f(0,1)y^n \\ & + \sum_{l=1}^{n-1} F_{n,0,l}^\rho(f) p_{n,0,l}(x,y) \\ & + \sum_{k=1}^{n-1} F_{n,k,0}^\rho(f) p_{n,k,0}(x,y) \\ & + \sum_{k=1}^{n-1} F_{n,k,n-k}^\rho(f) p_{n,k,n-k}(x,y) \\ & + \sum_{\substack{k\ge1,l\ge1\\ k+l\le n-1}} F_{n,k,l}^\rho(f) p_{n,k,l}(x,y), \end{aligned} \tag{58}$$

where

$$\begin{aligned} F_{n,0,l}^\rho(f) &:= \frac{\int_0^1 t^{l\rho-1}(1-t)^{(n-l)\rho-1} f(0,t)\,dt}{B(l\rho,(n-l)\rho)}, \\ F_{n,k,0}^\rho(f) &:= \frac{\int_0^1 s^{k\rho-1}(1-s)^{(n-k)\rho-1} f(s,0)\,ds}{B(k\rho,(n-k)\rho)}, \\ F_{n,k,n-k}^\rho(f) &:= \frac{\int_0^1 t^{k\rho-1}(1-t)^{(n-k)\rho-1} f(t,1-t)\,dt}{B(k\rho,(n-k)\rho)}, \\ F_{n,k,l}^\rho(f) &:= \frac{\iint_S s^{k\rho-1}t^{l\rho-1}(1-s-t)^{(n-k-l)\rho-1} f(s,t)\,ds\,dt}{\iint_S s^{k\rho-1}t^{l\rho-1}(1-s-t)^{(n-k-l)\rho-1}\,ds\,dt}. \end{aligned} \tag{59}$$

It can be easily seen that, for $\rho = 1$, we obtain the genuine Bernstein-Durrmeyer operators U_n. On the other hand, these operators have the following limiting behavior.

Theorem 7. *For any $f \in C(S)$, one has $\lim_{\rho\longrightarrow\infty} U_n^\rho(f) = B_n(f)$, uniformly.*

Proof. Let $f = pr_1^j$, $j = 0, 1, \cdots$. Then,

$$\begin{aligned} F_{n,k,l}^\rho\left(pr_1^j\right) &= \frac{(k\rho+j-1)\cdots(k\rho)}{(n\rho+j-1)\cdots(n\rho)}, \quad \text{for} \quad k,l \ge 1, \\ \lim_{\rho\longrightarrow\infty} F_{n,k,l}^\rho\left(pr_1^j\right) &= \left(\frac{k}{n}\right)^j = pr_1^j\left(\frac{k}{n},\frac{l}{n}\right). \end{aligned} \tag{60}$$

Since

$$\begin{aligned} F_{n,0,l}^\rho\left(pr_1^j\right) = \begin{cases} 1, j = 0, \\ 0, j > 0, \end{cases} F_{n,k,0}^\rho\left(pr_1^j\right) &= F_{n,k,n-k}^\rho\left(pr_1^j\right) \\ &= \frac{(k\rho+j-1)\cdots(k\rho)}{(n\rho+j-1)\cdots(n\rho)}, \end{aligned} \tag{61}$$

we get

$$\begin{aligned} \lim_{\rho\longrightarrow\infty} F_{n,0,l}^\rho\left(pr_1^j\right) = pr_1^j\left(0,\frac{l}{n}\right), \lim_{\rho\longrightarrow\infty} F_{n,k,0}^\rho\left(pr_1^j\right) = \left(\frac{k}{n}\right)^j = pr_1^j\left(\frac{k}{n},0\right), \\ \lim_{\rho\longrightarrow\infty} F_{n,k,n-k}^\rho\left(pr_1^j\right) = \left(\frac{k}{n}\right)^j = pr_1^j\left(\frac{k}{n},\frac{\text{n}-k}{n}\right). \end{aligned} \tag{62}$$

Similar results can be obtained for pr_2^j, $j = 0, 1, \cdots$.

Using Korovkin's theorem (see [36], p. 534, C.4.3.3), it follows $\lim_{\rho\longrightarrow\infty} F_{n,\mu,\nu}^\rho(f) = f(\mu/n,\nu/n)$. Therefore,

$$\lim_{\rho\longrightarrow\infty} U_n^\rho(f) = B_n(f), f \in C(S). \tag{63}$$

Proposition 8. *For the bivariate operators U_n^ρ, the following properties hold:*

(i) If $f \in C^2(S)$, then

$$\left|\left(U_n^\rho - U_n^r\right)(f)(x,y)\right| \le M_f \sigma(x,y), \tag{64}$$

where M_f is the same as in Lemma 1 and

$$\sigma(x,y) = \frac{(nr+n\rho+2)\left(x+y-x^2-y^2\right)(n-1)}{n(n\rho+1)(nr+1)}. \tag{65}$$

(ii) If $f \in C(S)$, then

$$\left|\left(U_n^\rho - U_n^r\right)(f)(x,y)\right| \le \eta_2\omega_2\left(f;\sqrt{\sigma(x,y)}\right). \tag{66}$$

Proof. Since $b_1^{F_{n,k,l}^\rho} = k/n$, $b_1^{F_{n,k,l}^\rho} = l/n$, we get

$$\mu_{2,0}^{F_{n,k,l}^\rho} = \frac{k(n-k)}{n^2(n\rho+1)}, \mu_{0,2}^{F_{n,k,l}^\rho} = \frac{l(n-l)}{n^2(n\rho+1)}. \tag{67}$$

Therefore, $\delta = 0$ and

$$\sigma(x,y) = \frac{(nr + n\rho + 2)(x + y - x^2 - y^2)(n-1)}{n(n\rho + 1)(nr + 1)}. \tag{68}$$

5. Difference of Bivariate Bernstein Operators and Their Kantorovich Variants

In 2017, F. Altomare et al. [37] introduced Kantorovich operators on S as follows

$$\mathbb{C}_n(f)(x,y) = \sum_{\substack{k,l=0,\cdots,n\\ k+l\le n}} p_{n,k,l}(x,y) 2\iint_S f\left(\frac{k+as}{n+a}, \frac{l+at}{n+a}\right) ds dt, \tag{69}$$

where $p_{n,k,l}(x,y)$ is given by (28). It can be easily seen that, for $a=1$, we obtain Kantorovich operators K_n introduced in [38].

If we denote

$$G_{k,l}(f) := 2\iint_S f\left(\frac{k+as}{n+a}, \frac{l+at}{n+a}\right) ds dt, \tag{70}$$

the bivariate Kantorovich operators can be written as

$$\mathbb{C}_n(f)(x,y) = \sum_{\substack{k,l=0,\cdots,n\\ k+l\le n}} G_{k,l}(f) p_{n,k,l}(x,y). \tag{71}$$

Proposition 9. *For bivariate Bernstein operators and bivariate Bernstein-Kantorovich operators, the following properties hold:*

(i) If $f \in C^2(S)$, then

$$|(B_n - \mathbb{C}_n)(f)(x,y)| \le M_f \sigma(x,y) + \omega_1\left(f, \frac{4a}{3(n+a)}\right), \tag{72}$$

where M_f is the same as in Lemma 1 and

$$\sigma(x,y) = \frac{a^2}{9(n+a)^2}. \tag{73}$$

(ii) If $f \in C(S)$, then

$$|(B_n - \mathbb{C}_n)(f)(x,y)| \le \eta_1 \omega_1\left(f, \frac{4a}{3(n+a)}\right) + \eta_2 \omega_2\left(f, \sqrt{\sigma(x,y)}\right). \tag{74}$$

Proof. As in the previous examples, taking Bernstein operators as

$$B_n(f)(x,y) = \sum_{\substack{k,l=0,\cdots,n\\ k+l\le n}} F_{k,l}(f) p_{n,k,l}(x,y), \tag{75}$$

we get

$$\begin{aligned} b_1^{F_{k,l}} &= \frac{k}{n}, b_2^{F_{k,l}} = \frac{l}{n}, \\ b_1^{G_{k,l}} &= \frac{3k+a}{3(n+a)}, b_2^{G_{k,l}} = \frac{3l+a}{3(n+a)}. \end{aligned} \tag{76}$$

Therefore, we easily obtain that

$$\begin{aligned} \mu_{2,0}^{F_{k,l}} &= 0, \mu_{2,0}^{G_{k,l}} = \frac{a^2}{18(n+a)^2}, \\ \mu_{0,2}^{F_{k,l}} &= 0, \mu_{0,2}^{G_{k,l}} = \frac{a^2}{18(n+a)^2}. \end{aligned} \tag{77}$$

Then

$$\sigma(x,y) = \sum_{\substack{k,l=0,\cdots,n\\ k+l\le n}} \frac{a^2}{9(n+a)^2} p_{n,k,l}(x,y) = \frac{a^2}{9(n+a)^2}. \tag{78}$$

Moreover, we have

$$\begin{aligned} \delta &= \max_{0\le k+l\le n} \left\{ \left| b_1^{F_{k,l}} - b_1^{G_{k,l}} \right| + \left| b_2^{F_{k,l}} - b_2^{G_{k,l}} \right| \right\} \\ &= \max_{0\le k+l\le n} \left\{ \frac{a|3k-n|}{3n(n+a)} + \frac{a|3l-n|}{3n(n+a)} \right\} \le \frac{4a}{3(n+a)}. \end{aligned} \tag{79}$$

Then, the proof follows from Theorems 2 and 3.

6. A Beta Operator on $C(S)$

For $\rho \in (0,\infty)$, $f \in C(S)$, and $(x,y) \in S$, let us define

$$\mathscr{B}_\rho(f)(x,y) = \begin{cases} f(x,y), (x,y) \in \{(0,0),(1,0),(0,1)\}, \\ \dfrac{\int_0^1 s^{\rho x-1}(1-s)^{\rho(1-x)-1} f(s,0) ds}{B(\rho x, \rho(1-x))}, x \in (0,1), y = 0, \\ \dfrac{\int_0^1 t^{\rho y-1}(1-t)^{\rho(1-y)-1} f(0,t) dt}{B(\rho y, \rho(1-y))}, x = 0, y \in (0,1), \\ \dfrac{\int_0^1 u^{\rho x-1}(1-u)^{\rho(1-x)-1} f(u,1-u) du}{B(\rho x, \rho(1-x))}, y = 1-x, x \in (0,1), \\ \dfrac{\iint_S s^{\rho x-1} t^{\rho y-1}(1-s-t)^{\rho-\rho x-\rho y-1} f(s,t) ds dt}{\iint_S s^{\rho x-1} t^{\rho y-1}(1-s-t)^{\rho-\rho x-\rho y-1} ds dt}, (x,y) \in \operatorname{int}(S). \end{cases} \tag{80}$$

For $\rho = n \in \mathbb{N}$, this is the bivariate version of the operator $\bar{\mathbb{B}}_n$; see [39] and the references therein.

Theorem 10. *$\mathscr{B}_\rho$ is a positive linear operator acting between $C(S)$ and $C(S)$. Moreover,*

$$\mathscr{B}_\rho(1) = 1, \quad (81)$$

and if $\varphi_{i,j}(x,y) = pr_1^i pr_2^j(x,y)$, $(x,y) \in S$, $i \geq 0$, $j \geq 0$, integers, then

$$\mathscr{B}_\rho\left(\varphi_{i,j}\right)(x,y) = \frac{\rho x(\rho x+1)\cdots(\rho x+i-1)}{\rho(\rho+1)\cdots(\rho+i-1)} \cdot \frac{\rho y(\rho y+1)\cdots(\rho y+j-1)}{\rho(\rho+1)\cdots(\rho+j-1)}. \quad (82)$$

Proof. It is easy to prove (81) and (82) by direct calculation. It remains to prove that if $f \in C(S)$, then, $\mathscr{B}_\rho(f) \in C(S)$. To do this, it suffices to verify that $\mathscr{B}_\rho(f)$ is continuous at each point of the boundary of S. Let us prove that if $0 < a < 1$ then

$$\lim_{\substack{(x,y)\longrightarrow(a,0)\\(x,y)\in \operatorname{int}(S)}} \mathscr{B}_\rho(f)(x,y) = \mathscr{B}_\rho(f)(a,0). \quad (83)$$

Let $V_{(a,0)} : C(S) \longrightarrow \mathbb{R}$, $V_{(a,0)}(g) = \mathscr{B}_\rho(g)(a,0)$. For $(x,y) \in \operatorname{int}(S)$ define $U_{(x,y)} : C(S) \longrightarrow \mathbb{R}$, $U_{(x,y)}(g) = \mathscr{B}_\rho(g)(x,y)$, $g \in C(S)$. Then, $U_{(x,y)}$ and $V_{(a,0)}$ are positive linear functionals of norm 1.

Let $\varepsilon > 0$. Then, there exists a polynomial function p on S such that $\|f-p\|_\infty \leq \varepsilon/4$. Using (82), it is easy to verify that

$$\lim_{\substack{(x,y)\longrightarrow(a,0)\\(x,y)\in \operatorname{int}(S)}} U_{(x,y)}(p) = V_{(a,0)}(\mathrm{p}). \quad (84)$$

Consequently, there exists $\delta > 0$ with

$$\left|U_{(x,y)}(p) - V_{(a,0)}(p)\right| \leq \frac{\varepsilon}{2}, \|(x,y)-(a,0)\|_1 \leq \delta. \quad (85)$$

So, if $\|(x,y)-(a,0)\|_1 \leq \delta$, we have

$$\begin{aligned}\left|U_{(x,y)}(f) - V_{(a,0)}(f)\right| &\leq \left|U_{(x,y)}(f) - U_{(x,y)}(p)\right| \\ &+ \left|U_{(x,y)}(p) - V_{(a,0)}(p)\right| + \left|V_{(a,0)}(p) - V_{(a,0)}(f)\right| \\ &\leq \|f-p\|_\infty + \frac{\varepsilon}{2} + \|p-f\|_\infty \leq \varepsilon.\end{aligned} \quad (86)$$

This shows that

$$\lim_{\substack{(x,y)\longrightarrow(a,0)\\(x,y)\in \operatorname{int}(S)}} U_{(x,y)}(f) = V_{(a,0)}(f), f \in C(S), \quad (87)$$

and then (83) is proved.

The continuity of $\mathscr{B}_\rho(f)$ at the other boundary points can be proved similarly.

Proposition 11. *For each $f \in C(S)$, one has*

$$\lim_{\rho\longrightarrow\infty} \mathscr{B}_\rho(f) = f. \quad (88)$$

Proof. Using Theorem 10, it is easy to verify that (88) is valid for the functions $1, pr_1, pr_2, pr_1^2 + pr_2^2$. But these functions form a Korovkin test system (see [36], p. 534, C.4.3.3), so that (88) holds for each $f \in C(S)$.

In what follows, we formulate a

Conjecture 12. *If $f \in C(S)$ is convex and $(x,y) \in S$, then, the function $\rho \longrightarrow \mathscr{B}_\rho(f)(x,y)$ is decreasing on $(0,\infty)$.*

It is supported by the following facts.

(i) The unidimensional version of the conjecture is valid: see [14, 40].

(ii) $\mathscr{B}_\rho$ is a positive linear operator preserving the affine functions; this implies

$$\mathscr{B}_\rho(f) \geq f, f \in C(S) convex. \quad (89)$$

Now, (88) combined with (89) support the conjecture.

(iii) The conjecture is valid for the functions pr_1^k, pr_2^k, $(1-pr_1-pr_2)^k, k \in \mathbb{N}$

In the sequel, we present two results under the hypothesis that the conjecture is true. To this end, let us introduce some notation.

Let $f \in C^2(S)$ and

$$\begin{aligned}m_1(f) &:= \min\left\{f_{xx}(x,y) - \left|f_{xy}(x,y)\right| : (x,y) \in S\right\}, \\ m_2(f) &:= \min\left\{f_{yy}(x,y) - \left|f_{xy}(x,y)\right| : (x,y) \in S\right\}, \\ M_1(f) &:= \max\left\{f_{xx}(x,y) + \left|f_{xy}(x,y)\right| : (x,y) \in S\right\}, \\ M_2(f) &:= \max\left\{f_{yy}(x,y) + \left|f_{xy}(x,y)\right| : (x,y) \in S\right\}.\end{aligned} \quad (90)$$

Then, the functions $\varphi(x,y) := f(x,y) - 1/2 m_1(f)x^2 - 1/2 m_2(f)y^2$ and $\psi(x,y) := 1/2 M_1(f)x^2 + 1/2 M_2(f)y^2 - f(x,y)$ are convex on S; indeed, for each of them, the Hessian matrix is positive semidefinite.

Theorem 13. *If $f \in C^2(S)$ and $\sigma > \rho > 0$, then*

$$\begin{aligned}&\frac{\sigma-\rho}{2(\sigma+1)(\rho+1)}\left[m_1(f)x(1-x) + m_2(f)y(1-y)\right] \\ &\leq \mathscr{B}_\rho(f)(x,y) - \mathscr{B}_\sigma(f)(x,y) \\ &\leq \frac{\sigma-\rho}{2(\sigma+1)(\rho+1)}\left[M_1(f)x(1-x) + M_2(f)y(1-y)\right], (x,y) \in S.\end{aligned} \quad (91)$$

Proof. Let $\sigma > \rho > 0$. If the Conjecture is true, we have $\mathscr{B}_\rho(\varphi) \geq \mathscr{B}_\sigma(\varphi)$ and $\mathscr{B}_\rho(\psi) \geq \mathscr{B}_\sigma(\psi)$. Thus

$$0 \leq \mathscr{B}_\rho(\varphi)(x,y) - \mathscr{B}_\sigma(\varphi)(x,y) = \mathscr{B}_\rho(f)(x,y) - \mathscr{B}_\sigma(f)(x,y) + \frac{x}{2}m_1(f)\frac{(\rho-\sigma)(1-x)}{(\sigma+1)(\rho+1)} + \frac{y}{2}m_2(f)\frac{(\rho-\sigma)(1-y)}{(\sigma+1)(\rho+1)}. \quad (92)$$

Consequently,

$$\mathscr{B}_\rho(f)(x,y) - \mathscr{B}_\sigma(f)(x,y) \geq \frac{\sigma-\rho}{2(\sigma+1)(\rho+1)}[m_1(f)x(1-x) + m_2(f)y(1-y)]. \quad (93)$$

Moreover,

$$0 \leq \mathscr{B}_\rho(\psi)(x,y) - \mathscr{B}_\sigma(\psi)(x,y) = \frac{x}{2}M_1(f)\frac{(\sigma-\rho)(1-x)}{(\sigma+1)(\rho+1)} + \frac{y}{2}M_2(f)\frac{(\sigma-\rho)(1-y)}{(\sigma+1)(\rho+1)} - \mathscr{B}_\rho(f)(x,y) + \mathscr{B}_\sigma(f)(x,y). \quad (94)$$

Now,

$$\mathscr{B}_\rho(f)(x,y) - \mathscr{B}_\sigma(f)(x,y) \leq \frac{\sigma-\rho}{2(\sigma+1)(\rho+1)}[M_1(f)x(1-x) + M_2(f)y(1-y)]. \quad (95)$$

So, combining (93) and (95), we have proved the theorem.

Theorem 14. *If $f \in C^2(S)$ and $\sigma > \rho > 0$, then*

$$\frac{(n-1)(\sigma-\rho)}{2(n\sigma+1)(n\rho+1)}[m_1(f)x(1-x) + m_2(f)y(1-y)] \leq U_n^\rho(f)(x,y) - U_n^\sigma(f)(x,y) \leq \frac{(n-1)(\sigma-\rho)}{2(n\sigma+1)(n\rho+1)}[M_1(f)x(1-x) + M_2(f)y(1-y)],\ (x,y) \in S. \quad (96)$$

Proof. It is easy to verify that $U_n^\rho = B_n \circ \mathscr{B}_{n\rho}$ and, if $u := pr_1^2$, then

$$\mathscr{B}_{n\rho}(u)(x,y) = \frac{(n-1)\rho x^2 + (\rho+1)x}{n\rho+1}. \quad (97)$$

Using these facts and supposing that the conjecture is true, we have

$$U_n^\rho(\varphi) \geq U_n^\sigma(\varphi) \text{ and } U_n^\rho(\psi) \geq U_n^\sigma(\psi), \sigma > \rho > 0. \quad (98)$$

Now,

$$0 \leq U_n^\rho(\varphi)(x,y) - U_n^\sigma(\varphi)(x,y) = U_n^\rho(f)(x,y) - U_n^\sigma(f)(x,y) - \frac{(n-1)(\sigma-\rho)}{2(n\sigma+1)(n\rho+1)}x(1-x)m_1(f) - \frac{(n-1)(\sigma-\rho)}{2(n\sigma+1)(n\rho+1)}y(1-y)m_2(f). \quad (99)$$

Thus,

$$U_n^\rho(f)(x,y) - U_n^\sigma(f)(x,y) \geq \frac{(n-1)(\sigma-\rho)}{2(n\sigma+1)(n\rho+1)}[m_1(f)x(1-x) + m_2(f)y(1-y)]. \quad (100)$$

From $U_n^\rho(\psi) \geq U_n^\sigma(\psi)$, we get a similar upper bound for $U_n^\rho(f)(x,y) - U_n^\sigma(f)(x,y)$, which concludes the proof.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no competing financial interests.

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Research Article

Approximation of Functions by Dunkl-Type Generalization of Szász-Durrmeyer Operators Based on (p, q)-Integers

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In this article, our main purpose is to define the (p, q)-variant of Szász-Durrmeyer type operators with the help of Dunkl generalization generated by an exponential function. We estimate moments and establish some direct results of the aforementioned operators. Moreover, we establish some approximation results in weighted spaces.

1. Introduction and Preliminaries

The well-known Bernstein operators [1] and the q-Bernstein operators have become very important tools in the study of approximation theory and several branches of applied sciences and engineering [2, 3]. A good approach to introduce the (p, q)-analogues in approximation theory is given by Mursaleen et al. [4] by an idea of newly introduced integers known as(p, q)-integers and which is $[\alpha]_{p,q} = (p^\alpha - q^\alpha/p - q)$, $\alpha = 0, 1, 2, \cdots, q \in (0, p)$ and $p \in (q, 1]$. In (p, q)-calculus, there are generally two types of exponential functions which are defined as follows:

$$
\begin{aligned}
e_{p,q}(y) &= \sum_{\hbar=0}^{\infty} p^{(\hbar(\hbar-1)/2)} \frac{y^\hbar}{[\hbar]_{p,q}!}, \\
E_{p,q}(y) &= \sum_{\hbar=0}^{\infty} q^{(\hbar(\hbar-1)/2)} \frac{y^\hbar}{[\hbar]_{p,q}!}.
\end{aligned} \tag{1}
$$

In 1950, Szász [5] defined positive linear operators on $[0, \infty)$, and the Dunkl modification of these operators were given by Sucu [6] who was motivated by the work of Cheikh et al. [7]. The new generalization of these Szász operators [6] in quantum calculus (via q-analogue) was introduced in [8] by Içöz and Çekim. Very recent work on the quantum Dunkl analogue in postquantum calculus studied in [9] for a set of all continuous functions$f$defined on$[0, \infty)$denote it as$f \in C[0,\infty)$; for parameter$\lambda > -(1/2)$, they designed the following operators:

$$
D_{\alpha,p,q}(f ; y) = \frac{1}{e_{\lambda,p,q}\left([\alpha]_{p,q} y\right)} \sum_{\hbar=0}^{\infty} \frac{\left([\alpha]_{p,q} y\right)^\hbar}{\gamma_{\lambda,p,q}(\hbar)} p^{(\hbar(\hbar-1)/2)} f \cdot \left(\frac{p^{\hbar+2\lambda\theta_\hbar} - q^{\hbar+2\lambda\theta_\hbar}}{p^{\hbar-1}(p^\alpha - q^\alpha)}\right). \tag{2}
$$

Lemma 1. *For $f(t) = 1, t, t^2$, we have*

$$
\begin{aligned}
&D_{\alpha,p,q}(1 ; y) = 1, \\
&D_{\alpha,p,q}(t ; y) = y, \\
&y^2 + \frac{q^{2\lambda}}{[\alpha]_{p,q}} [1 - 2\lambda]_{p,q} \frac{e_{\lambda,p,q}\left((q/p)[\alpha]_{p,q} y\right)}{e_{\lambda,p,q}\left([\alpha]_{p,q} y\right)} y \leqq D_{\alpha,p,q}(t^2 ; y) \leqq y^2 + \frac{1}{[\alpha]_{p,q}} [1 + 2\lambda]_{p,q} y.
\end{aligned} \tag{3}
$$

Moreover, for every $\lambda > -(1/2)$ and $0 < q < p \leqq 1$, where exponential functions and recursion relations are given by

$$e_{\lambda,p,q}(y) = \sum_{\hbar=0}^{\infty} p^{(\hbar(\hbar-1)/2)} \frac{y^{\hbar}}{\gamma_{\lambda,p,q}(\hbar)}, \tag{4}$$

$$\gamma_{\lambda,p,q}(\hbar) = \frac{\prod_{i=0}^{[(\hbar+1)/2]-1} p^{2\lambda(-1)^{i+1}+1}\left(\left(p^2\right)^i p^{2\lambda+1} - \left(q^2\right)^i q^{2\lambda+1}\right)\prod_{j=0}^{[\hbar/2]-1} p^{2\lambda(-1)^{j}+1}\left(\left(p^2\right)^j p^2 - \left(q^2\right)^j q^2\right)}{(p-q)^{\hbar}}, \tag{5}$$

$$\gamma_{\lambda,p,q}(\hbar+1) = \frac{p^{2\lambda(-1)^{\ell+1}+1}\left(p^{2\lambda\theta_{\hbar+1}+\hbar+1} - q^{2\lambda\theta_{\hbar+1}+\hbar+1}\right)}{(p-q)}\gamma_{\lambda,p,q}(\hbar), \tag{6}$$

$$\theta_{\hbar} = \begin{cases} 0, & \text{for } \hbar = 0, 2, 4, \cdots, \\ 1, & \text{for } \hbar = ,1, 3, 5, \cdots. \end{cases} \tag{7}$$

For $\hbar = 0, 1, 2, \cdots n$, the number $[\hbar/2]$ denotes the greatest integer functions.

The q-analogues of Szász operators on the Dunkl type have been studied by several authors in [10–12] and for post-quantum calculus in [9, 13–15]. We also refer some useful research articles on these topic (see [16–33]). Some convergence properties of operators through summability techniques can be examined in [34–39].

2. New Operators and Estimations of Moments

Here, with the motivational work of [9, 28], we design a different version of the(p,q)-Szász-Durrmeyer operators compared to the previous one, and we define it by ((9)). To obtain a generalized version of the approximation in Dunkl form generally, we take positive sequences $p = p_\alpha$ and $q = q_\alpha$ for every $0 < q_\alpha < 1$ and $q_\alpha < p_\alpha \leqq 1$, and also satisfy the following results:

$$\begin{aligned} &\lim_{\alpha\longrightarrow\infty} p_\alpha \longrightarrow 1, \\ &\lim_{\alpha\longrightarrow\infty} q_\alpha \longrightarrow 1, \\ &\lim_{\alpha\longrightarrow\infty} p_\alpha^\alpha \longrightarrow m, \\ &\lim_{\alpha\longrightarrow\infty} q_\alpha^\alpha \longrightarrow n, \end{aligned} \tag{8}$$

where the numbers m and n belong to $(0,1]$.

Definition 2. Let $0 < q < p \leqq 1, \lambda > -(1/2)$ and $\theta_\hbar$ be defined by (7). Then, for every $f \in C[0,\infty)$ and $y \in [0,\infty)$, we have

$$\mathcal{S}_{\alpha,p,q}^{\lambda}(f;y) = \sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{1}{[\hbar+2\lambda\theta_\hbar]_{p,q}!}\int_0^\infty \mathfrak{S}_{\alpha,p,q}(t) f\left(\frac{p^{\hbar+2\lambda\theta_\hbar}t}{p^{\hbar-1}}\right)\mathrm{d}_{p,q}t, \tag{9}$$

where

$$\mathfrak{R}_{\alpha,p,q}(y) = \frac{[\alpha]_{p,q}}{e_{\lambda,p,q}\left([\alpha]_{p,q}y\right)}\frac{\left([\alpha]_{p,q}y\right)^{\hbar}}{\gamma_{\lambda,p,q}(\hbar)}p^{(\hbar(\hbar-1)/2)},$$

$$\mathfrak{S}_{\alpha,p,q}(t) = p^{((\hbar+2\lambda\theta_\hbar)(\hbar+2\lambda\theta_\hbar-1)/2)}\left([\alpha]_{p,q}t\right)^{\hbar+2\lambda\theta_\hbar}E_{p,q}\left(-q[\alpha]_{p,q}t\right). \tag{10}$$

$\alpha \in \mathbb{N}$ and $\zeta > \alpha$. Moreover, for all $\alpha \in \mathbb{N}$, the gamma functions in the postquantum calculus are defined as follows:

$$\Gamma_{p,q}(\alpha) = \int_0^\infty p^{((\alpha-1)(\alpha-2)/2)}t^{\alpha-1}E_{p,q}(-qt)d_{p,q}t, \tag{11}$$

and

$$\Gamma_{p,q}(\alpha+1) = \frac{(p\ominus q)_{p,q}^{\alpha}}{(p-q)^{\alpha}} = [\alpha]_{p,q}[\alpha-1]_{p,q}\Gamma_{p,q}(\alpha-1) = [\alpha]_{p,q}!. \tag{12}$$

Note that

$$[\hbar+1+2\lambda\theta_\hbar]_{p,q} = q[\hbar+2\lambda\theta_\hbar]_{p,q} + p^{\hbar+2\lambda\theta_\hbar}, \tag{13}$$

$$[\hbar+2+2\lambda\theta_\hbar]_{p,q} = q^2[\hbar+2\lambda\theta_\hbar]_{p,q} + (p+q)p^{\hbar+2\lambda\theta_\hbar}. \tag{14}$$

For more detailed properties of the (p,q)-analogue of the beta and gamma functions, see [40, 41].

Lemma 3. *For the operators in (9), we have $\mathcal{S}_{\alpha,p,q}^{\lambda}(1;y) = 1$:*

$$\mathcal{S}_{\alpha,p,q}^{\lambda}(f;y) \leqq \begin{cases} y + \dfrac{1}{[\alpha]_{p,q}}, & \text{for } f(t) = t, \\ y^2 + \dfrac{1}{[\alpha]_{p,q}}\left(1 + [2]_{p,q} + [1+2\lambda]_{p,q}\right)y + \dfrac{[2]_{p,q}}{[\alpha]_{p,q}^2}, & \text{for } f(t) = t^2, \end{cases} \tag{15}$$

and

$$\mathcal{S}^{\lambda}_{\alpha,p,q}(f\,;y) \geqq \begin{cases} qy + \dfrac{p^{1+2\lambda}}{[\alpha]_{p,q}}, & \text{for } f(t)=t, \\ \dfrac{q}{[\alpha]_{p,q}[\alpha-2]_{p,q}}\left(q^{2+2\lambda}[1-2\lambda]_{p,q}\dfrac{e_{\lambda,p,q}\left((q/p)[\alpha]_{p,q}y\right)}{e_{\lambda,p,q}\left([\alpha]_{p,q}y\right)} + \left(q+[2]_{p,q}\right)p^{2\lambda}\right)y + q^3y^2, & \text{for } f(t)=t^2, \end{cases} \tag{16}$$

Proof. We prove this Lemma by using the results obtained in (11), (12), (13), and (14). Therefore, for $f(t)=1$, we easily see that

$$\begin{aligned} \mathcal{S}^{\lambda}_{\alpha,p,q}(1\,;y) &= \sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{1}{[\hbar+2\lambda\theta_\hbar]_{p,q}!}\int_0^\infty \mathfrak{S}_{\alpha,p,q}(t)d_{p,q}t \\ &= \sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{1}{[\alpha]_{p,q}[\hbar+2\lambda\theta_\hbar]_{p,q}!}\Gamma_{p,q}(\hbar+2\lambda\theta_\hbar+1) \\ &= 1. \end{aligned} \tag{17}$$

Take $f(t)=t$. Then, we have

$$\begin{aligned} \mathcal{S}^{\lambda}_{\alpha,p,q}(t\,;y) &= \sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{p^{\hbar+2\lambda\theta_\hbar}}{p^{\hbar-1}[\alpha]_{p,q}[\hbar+2\lambda\theta_\hbar]_{p,q}!} \\ &\quad\times\int_0^\infty p^{((\hbar+2\lambda\theta_\hbar)(\hbar+2\lambda\theta_\hbar-1)/2)}\left([\alpha]_{p,q}t\right)^{\hbar+2\lambda\theta_\hbar+1}E_{p,q}\left(-q[\alpha]_{p,q}t\right)d_{p,q}t \\ &= \frac{1}{[\alpha]^2_{p,q}}\sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{1}{p^{\hbar-1}[\hbar+2\lambda\theta_\hbar]_{p,q}!} \\ &\quad\times\int_0^\infty p^{((\hbar+2\lambda\theta_\hbar)(\hbar+2\lambda\theta_\hbar+1)/2)}t^{\hbar+2\lambda\theta_\hbar+1}E_{p,q}(-qt)d_{p,q}t \\ &= \frac{1}{[\alpha]^2_{p,q}}\sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{1}{p^{\hbar-1}[\hbar+2\lambda\theta_\hbar]_{p,q}!}\Gamma_{p,q}(\hbar+2\lambda\theta_\hbar+2) \\ &= \frac{1}{[\alpha]^2_{p,q}}\sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{1}{p^{\hbar-1}}[\hbar+2\lambda\theta_\hbar+1]_{p,q} \\ &= \frac{1}{[\alpha]^2_{p,q}}\sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\left(q[\hbar+2\lambda\theta_\hbar]_{p,q}+p^{\hbar+2\lambda\theta_\hbar}\right)p^{1-\hbar} \\ &= \frac{q}{e_{\lambda,p,q}\left([\alpha]_{p,q}y\right)}\sum_{\hbar=0}^{\infty}\frac{\left([\alpha]_{p,q}y\right)^\hbar}{\gamma_{\lambda,p,q}(\hbar)}p^{(\hbar(\hbar-1)/2)}\left(\frac{p^{\hbar+2\lambda\theta_\hbar}-q^{\hbar+2\lambda\theta_\hbar}}{p^{\hbar-1}(p^\alpha-q^\alpha)}\right) \\ &\quad+\frac{1}{[\alpha]_{p,q}e_{\lambda,p,q}\left([\alpha]_{p,q}y\right)}\sum_{\hbar=0}^{\infty}\frac{\left([\alpha]_{p,q}y\right)^\hbar}{\gamma_{\lambda,p,q}(\hbar)}p^{(\hbar(\hbar-1)/2)}p^{2\lambda\theta_\hbar+1} \\ &= qD_{\alpha,p,q}(t\,;y)+\frac{p}{[\alpha]_{p,q}e_{\lambda,p,q}\left([\alpha]_{p,q}y\right)}\sum_{\hbar=0}^{\infty}\frac{\left([\alpha]_{p,q}y\right)^{2\hbar}}{\gamma_{\lambda,p,q}(2\hbar)}p^{(2\hbar(2\hbar-1)/2)}p^{2\lambda\theta_{2\hbar}} \\ &\quad+\frac{p}{[\alpha]_{p,q}e_{\lambda,p,q}\left([\alpha]_{p,q}y\right)}\sum_{\hbar=0}^{\infty}\frac{\left([\alpha]_{p,q}y\right)^{2\hbar+1}}{\gamma_{\lambda,p,q}(2\hbar+1)}p^{(2\hbar(2\hbar+1)/2)}p^{2\lambda\theta_{2\hbar+1}} \\ &\geqq qy+\frac{p^{2\lambda+1}}{[\alpha]_{p,q}e_{\lambda,p,q}\left([\alpha]_{p,q}y\right)}\sum_{\hbar=0}^{\infty}\frac{\left([\alpha]_{p,q}y\right)^\hbar}{\gamma_{\lambda,p,q}(\hbar)}p^{(\hbar(\hbar-1)/2)} \\ &= qy+\frac{p^{2\lambda+1}}{[\alpha]_{p,q}}. \end{aligned} \tag{18}$$

Similarly,

$$\mathcal{S}^{\lambda}_{\alpha,p,q}(t\,;y) \leqq y+\frac{1}{[\alpha]_{p,q}}. \tag{19}$$

If $f(t)=t^2$, then

$$\begin{aligned} \mathcal{S}^{\lambda}_{\alpha,p,q}(t^2\,;y) &= \sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{p^{\hbar+2\lambda\theta_\hbar}}{p^{\hbar-1}[\alpha]^2_{p,q}[\hbar+2\lambda\theta_\hbar]_{p,q}!} \\ &\quad\times\int_0^\infty p^{((\hbar+2\lambda\theta_\hbar)(\hbar+2\lambda\theta_\hbar-1)/2)}\left([\alpha]_{p,q}t\right)^{\hbar+2\lambda\theta_\hbar+2}E_{p,q}\left(-q[\alpha]_{p,q}t\right)d_{p,q}t \\ &= \frac{1}{p[\alpha]^3_{p,q}}\sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{1}{p^{2\hbar-2}[\hbar+2\lambda\theta_\hbar]_{p,q}!}\Gamma_{p,q}(\hbar+2\lambda\theta_\hbar+3) \\ &= \frac{1}{p[\alpha]^3_{p,q}}\sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)\frac{1}{p^{2\hbar-2}}[1+\hbar+2\lambda\theta_\hbar]_{p,q}[2+\hbar+2\lambda\theta_\hbar]_{p,q} \\ &= \frac{q^3}{p[\alpha]^3_{p,q}}\sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)p^{2-2\hbar}[\hbar+2\lambda\theta_\hbar]^2_{p,q} \\ &\quad+\frac{q(p+2q)}{p[\alpha]^3_{p,q}}\sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)p^{2-\hbar+2\lambda\theta_\hbar}[\hbar+2\lambda\theta_\hbar]_{p,q} \\ &\quad+\frac{(p+q)}{p[\alpha]^3_{p,q}}\sum_{\hbar=0}^{\infty} \mathfrak{R}_{\alpha,p,q}(y)p^{2+4\lambda\theta_\hbar} \\ &= \frac{q^3}{pe_{\lambda,p,q}\left([\alpha]_{p,q}y\right)}\sum_{\hbar=0}^{\infty}\frac{\left([\alpha]_{p,q}y\right)^\hbar}{\gamma_{\lambda,p,q}(\hbar)}p^{(\hbar(\hbar-1)/2)} \\ &\quad\cdot\left(\frac{p^{\hbar+2\lambda\theta_\hbar}-q^{\hbar+2\lambda\theta_\hbar}}{p^{\hbar-1}(p^\alpha-q^\alpha)}\right)^2+\frac{q(p+2q)}{p[\alpha]_{p,q}}\sum_{\hbar=0}^{\infty}\frac{\left([\alpha]_{p,q}y\right)^\hbar}{\gamma_{\lambda,p,q}(\hbar)}p^{(\hbar(\hbar-1)/2)}p^{1+2\lambda\theta_\hbar} \\ &\quad\cdot\left(\frac{p^{\hbar+2\lambda\theta_\hbar}-q^{\hbar+2\lambda\theta_\hbar}}{p^{\hbar-1}(p^\alpha-q^\alpha)}\right)+\frac{(p+q)}{p[\alpha]^2_{p,q}}\sum_{\hbar=0}^{\infty}\frac{\left([\alpha]_{p,q}y\right)^\hbar}{\gamma_{\lambda,p,q}(\hbar)}p^{(\hbar(\hbar-1)/2)}p^{2+2\lambda\theta_\hbar}. \end{aligned} \tag{20}$$

We apply the results $\theta_\hbar$ defined by (7) and separate it into even and odd terms, i.e., take $\hbar=2m$ and $\hbar=2m+1$ for all $m=0,1,2,\cdots$, and applying (2) and Lemma 1, we easily see that

$$\begin{aligned} \mathcal{S}^{\lambda}_{\alpha,p,q}\left(t^2\,;y\right) &\geqq q^3D_{\alpha,p,q}\left(t^2\,;y\right) \\ &\quad+\frac{q(p+2q)p^{2\lambda}}{[\alpha]_{p,q}}D_{\alpha,p,q}(t\,;y) \\ &\quad+\frac{(p+q)p^{2\lambda+1}}{[\alpha]^2_{p,q}}D_{\alpha,p,q}(1\,;y), \end{aligned} \tag{21}$$

and

$$\mathcal{S}^{\lambda}_{\alpha,p,q}(t^2;y) \leqq D_{\alpha,p,q}(t^2;y) + \frac{\left(1+[2]_{p,q}\right)}{[\alpha]_{p,q}} D_{\alpha,p,q}(t;y) + \frac{[2]_{p,q}}{[\alpha]^2_{p,q}} D_{\alpha,p,q}(1;y). \tag{22}$$

These conclusions complete the proof of Lemma 3.

Lemma 4. *Let $\Delta_l = (t-y)^l$, for $l = 1, 2$; then, we have*

$$\mathcal{S}^{\lambda}_{\alpha,p,q}(\Delta_l;y) \leqq \begin{cases} \dfrac{1}{[\alpha]_{p,q}}, & \text{for } l = 1, \\ \dfrac{1}{[\alpha]_{p,q}}\left([2]_{p,q} + [1+2\lambda]_{p,q} - 1\right)y + \dfrac{[2]_{p,q}}{[\alpha]^2_{p,q}}, & \text{for } l = 2. \end{cases} \tag{23}$$

3. Approximation in Weighted Spaces

To obtain the approximation in weighted Korovkin spaces, we take the weight function $\sigma(y) = 1 + y^2$ and on $[0,\infty)$ consider $B_{\sigma(y)}$, $C_{\sigma(y)}$, and $C^k_{\sigma(y)}$ such that

$$B_{\sigma(y)} = \left\{f : |f(y)| \leqq \mathcal{M}_f\left(1+y^2\right)\right\}, \tag{24}$$

where $\mathcal{M}_f$ depends on f,

$$C_{\sigma(y)} = \left\{f : f \in C[0,\infty) \cap B_{\sigma(y)}\right\}, \quad C^k_{\sigma(y)} = \left\{f : f \in C_{\sigma(y)} \text{ and } \lim_{y\to\infty}\frac{f(y)}{1+y^2} = k\right\}, \tag{25}$$

where k is a constant, $C[0,\infty)$ is the set of continuous functions on $[0,\infty)$, $C_B[0,\infty)$ is the set of all bounded and continuous functions on $[0,\infty)$ equipped with the norm $\|f\|_{C_B} = \sup_{y\in 0,\infty)}|f(y)|$, and on $\sigma(y)$, a norm is given by $\|f\|_{\sigma(y)} = \sup_{y\in[0,\infty)}(|f(y)|/1+y^2)$.

Theorem 5. *Let $q_\alpha \in (0,1)$, $p_\alpha \in (q_\alpha, 1]$, and $\Omega(y) = \{f(y)$: $y \in [0,\infty)$ and $([0,\infty)/1+y^2)$ is convergent as $y \longrightarrow \infty\}$. Then, for each $f \in \Omega(y) \cap C[0,\infty)$, the sequence $\{\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}\}_{\alpha\geqq 1}$ converges uniformly to f on each compact subset of $[0,\infty)$ if and only if $\lim_{\alpha\longrightarrow\infty} q_\alpha = 1$ and $\lim_{\alpha\longrightarrow\infty} p_\alpha = 1$.*

Proof. Since the operators $\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}$ defined by (9) are positive and linear on $[0,\infty)$, if $\lim_{\alpha\to\infty} q_\alpha \longrightarrow 1$ and $\lim_{\alpha\to\infty} p_\alpha \longrightarrow 1$, then $(1/[\alpha]_{p_\alpha,q_\alpha}) \longrightarrow 0$. Therefore, from Korovkin's theorem for every $f \in C[0,\infty) \cap \Omega(y)$, the operators $\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(f;y)$ converge uniformly to $f(y)$ as $\alpha \longrightarrow \infty$ if and only if

$$\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(t^i;y) \longrightarrow y^i, i = 1, 2. \tag{26}$$

In another way for all $f \in C[0,\infty) \cap \Omega(y)$, if we assume $\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(f;y)$ converges uniformly to $f(y)$, when α approaches to ∞, then clearly $q_\alpha \longrightarrow 1$ and $p_\alpha \longrightarrow 1$. Suppose in the case where the sequences (q_α) and (p_α) do not converge to 1 and $[\alpha]_{p_\alpha,q_\alpha} \longrightarrow (1/p-q)$ as $\alpha \longrightarrow \infty$. Thus, from Lemma 3, we have

$$\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(t;y) \longrightarrow y + (p-q) \nrightarrow y, \tag{27}$$

and

$$\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(t^2;y) \longrightarrow y^2 + (p-q)\left([2]_{p,q} + [2\lambda+1]_{p,q} + 1\right)y + (p-q)^2[2]_{p,q} \nrightarrow y^2, \tag{28}$$

which leads to contradiction, and hence, $q_\alpha \longrightarrow 1$ and $p_\alpha \longrightarrow 1$ as $\alpha \longrightarrow \infty$.

Theorem 6. *Let the sequences of positive numbers $0 < q_\alpha < 1$ and $q_\alpha < p_\alpha \leqq 1$ satisfy $q_\alpha \longrightarrow 1$ and $p_\alpha \longrightarrow 1$ as α approaches to ∞. Then, for every $f \in C^k_{\sigma(y)}$ on $[0,\infty)$, we have*

$$\lim_{\alpha\to\infty}\left\|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(f;y) - f\right\|_{\sigma(y)} = 0. \tag{29}$$

Proof. Take $f(t) = t^i$ for $i = 0, 1, 2$. Since by Theorem 5, $\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(t^i;y)$ converges to y^i uniformly for $i = 0, 1, 2$, from Lemma 3, we conclude that

$$\lim_{\alpha\to\infty}\left\|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(1;y) - 1\right\|_{\sigma(y)} = 0. \tag{30}$$

If $f(t) = t$,

$$\left\|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(t;y) - y\right\|_{\sigma(y)} = \sup_{y\geqq 0}\frac{\left|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(t;y) - y\right|}{1+y^2} \leqq \frac{1}{[\alpha]_{p_\alpha,q_\alpha}}\sup_{y\geqq 0}\frac{1}{1+y}. \tag{31}$$

Then, we have

$$\lim_{\alpha\to\infty}\left\|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(t;y) - y\right\|_{\sigma(y)} = 0. \tag{32}$$

Similarly, if we take $i = 2$, we have

$$\begin{aligned}\left\|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(t^2;y) - y^2\right\|_{\sigma(y)} &= \sup_{y\geqq 0}\frac{\left|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(t^2;y) - y^2\right|}{1+y^2} \\ &\leqq \frac{1}{[\alpha]_{p_\alpha,q_\alpha}}\left([2]_{p_\alpha,q_\alpha} + [2\lambda+1]_{p_\alpha,q_\alpha} + 1\right)\sup_{y\geqq 0}\frac{y}{1+y^2} \\ &\quad + \frac{[2]_{p_\alpha,q_\alpha}}{[\alpha]^2_{p_\alpha,q_\alpha}}\sup_{y\geqq 0}\frac{1}{1+y^2},\end{aligned} \tag{33}$$

which implies that

$$\lim_{\alpha\to\infty}\left\|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}\left(t^2;y\right)-y^2\right\|_{\sigma(y)}=0. \tag{34}$$

These explanations complete the proof of Theorem 6.

Let the modulus of the continuity of f for any $\delta>0$ and $\rho>0$ be defined as follows:

$$\omega_\rho(f;\delta)=\sup_{|t-y|\delta y,t\in[0,\rho]}\ \sup|f(t)-f(y)|, \tag{35}$$

where it is obvious that $\lim_{\delta\to0+}\omega_\rho(f;\delta)=0$ and for $f\in C[0,\infty)$

$$|f(t)-f(y)|\leqq\left(1+\frac{|t-y|}{\delta}\right)\omega_\rho(f;\delta). \tag{36}$$

Theorem 7. *Take the numbers $q=q_\alpha$, $p=p_\alpha$ with the positive sequences $q_\alpha\in(0,1)$, $p_\alpha\in(q_\alpha,1]$ satisfying $q_\alpha\longrightarrow1$ and $p_\alpha\longrightarrow1$ as $\alpha\longrightarrow\infty$. Let $\omega_\rho(f;\delta)$ be defined on the interval $[0,\rho+1]\subset[0,\infty)$ for $\rho>0$. Then, for every $f\in C^k_{\sigma(y)}$ on $[0,\infty)$, we have*

$$\left|\mathcal{S}^{\lambda}_{\alpha,p,q}(f;y)-f(y)\right|\leqq6\mathcal{M}_f\left(1+\rho^2\right)\delta_\alpha(y)+2\omega_{\rho+1}\left(f;\sqrt{\delta_\alpha(y)}\right), \tag{37}$$

where $\mathcal{M}_f$ is a constant depending only on f.

Proof. Let $y\in[0,\rho]$ and $t\leqq\rho+1$ for $\rho>0$. Then, clearly one has

$$|f(t)-f(y)|\leqq6\mathcal{C}_f\left(1+\rho^2\right)(t-y)^2. \tag{38}$$

Also, when $y\in[0,\rho]$ and $t>\rho+1$ for $\rho>0$, then for a given $\delta>0$

$$|f(t)-f(y)|\leqq\omega_{\rho+1}(f;|t-y|)\leqq\left(\frac{|t-y|}{\delta}+1\right)\omega_{\rho+1}(f;\delta). \tag{39}$$

From (38) and (39), we easily see that

$$|f(t)-f(y)|\leqq6\mathcal{C}_f\left(1+\rho^2\right)(t-y)^2+\left(1+\frac{|t-y|}{\delta}\right)\omega_{\rho+1}(f;\delta), \tag{40}$$

which implies that

$$\begin{aligned}&\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(|f(t)-f(y)|;y)\\&\quad\leqq6\mathcal{C}_f\left(1+\rho^2\right)\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(\Delta_2;y)\\&\qquad+\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}\left(1+\frac{|t-y|}{\delta};y\right)\omega_{\rho+1}(f;\delta).\end{aligned} \tag{41}$$

The Cauchy-Schwartz inequality gives us

$$\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(|t-y|;y)\leqq\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}\left((t-y)^2;y\right)^{(1/2)}. \tag{42}$$

From an easy calculation, this leads us to

$$\left|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(f;y)-f(y)\right|\leqq\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(|f(t)-f(y)|;y). \tag{43}$$

Therefore, in view of (41)–(43), clearly we get

$$\begin{aligned}&\left|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(f;y)-f(y)\right|\\&\quad\leqq6\mathcal{C}_f\left(1+\rho^2\right)\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(\Delta_2;y)\\&\qquad+\left(1+\frac{1}{\delta}\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(\Delta_2;y)^{(1/2)}\right)\omega_{\rho+1}(f;\delta).\end{aligned} \tag{44}$$

Finally, if we take $\delta=(\delta_\alpha(y))^{(1/2)}=\sqrt{\mathcal{S}^{\lambda}_{\alpha,p,q}(\Delta_2;y)}$, then we use a denumerable to get the result.

4. Pointwise Approximation

In an approximation process for measuring the smoothness of a continuous function, we need Peetre's K-functional [42] defined as follows.

Definition 8. Let$f\in C[0,\infty)$, and for a given$\delta>0$of the$K$ -functional, we have

$$K_2(f;\delta)=\inf_{y\in[0,\infty)}\left\{\left(\delta\left\|\psi''\right\|_{C_B[0,\infty)}+\|f-\psi\|_{C_B[0,\infty)}\right):\psi\in C^2_B[0,\infty)\right\}. \tag{45}$$

Now, from [43], there exists a positive constant $\mathcal{C}$ such that

$$K_2(f;\delta)\leqq\mathcal{C}\left\{\min(1,\delta)\|f\|_{C_B[0,\infty)}+\omega_2\left(f;\sqrt{\delta}\right)\right\}, \tag{46}$$

where the modulus of continuity of order two is given by

$$\omega_2(f;\delta)=\sup_{0<h<\delta}\ \sup_{y\in[0,\infty)}|f(y+2h)-2f(y+h)+f(y)|. \tag{47}$$

Moreover, the classical modulus of continuity is given by

$$\omega(f;\delta)=\sup_{0<h<\delta}\ \sup_{y\in[0,\infty)}|f(y+h)-f(y)|. \tag{48}$$

Theorem 9. *Suppose q_α and p_α are the sequences of positive numbers satisfying $q_\alpha\in(0,1)$, $p_\alpha\in(q_\alpha,1]$ such that $q_\alpha\longrightarrow1$, $p_\alpha\longrightarrow1$ as $\alpha\longrightarrow\infty$. Let us define an auxiliary operator such that $\mathcal{T}^{\lambda}_{\alpha,p,q}(f;y)=\mathcal{S}^{\lambda}_{\alpha,p,q}(f;y)+f(y)-f(([\alpha]_{p,q}y+1)/[\alpha]_{p,q})$. Then, for every $\psi\in C^2_B[0,\infty)$, we have*

$$\left|\mathcal{T}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(\psi;y)-\psi(y)\right|\leqq\Theta_\alpha(y)\left\|\psi''\right\|, \tag{49}$$

where $\Theta_\alpha(y)=\delta_\alpha(y)+(([\alpha]_{p_\alpha,q_\alpha}y+1)/[\alpha]_{p_\alpha,q_\alpha}-y)^2$, *and* $\delta_\alpha(y)$ *is defined in Theorem 7.*

Proof. Let $\psi\in C_B^2[0,\infty)$. We easily get $\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}(1;y)=1$ and

$$\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}(t;y)=\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}(t;y)+y-\frac{[\alpha]_{p_\alpha,q_\alpha}y+1}{[\alpha]_{p_\alpha,q_\alpha}}=y, \quad (50)$$

where we easily know that

$$\left\|\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}(f;y)\right\|\leqq\|f\|. \quad (51)$$

Therefore,

$$\left|\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}(f;y)\right|\leqq\left|\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}(f;y)+f(y)-f\left(\frac{[\alpha]_{p_\alpha,q_\alpha}y+1}{[\alpha]_{p_\alpha,q_\alpha}}\right)\right|\leqq 3\|f\|. \quad (52)$$

In view of the Taylor series expansion, we have

$$\psi(t)=\psi(y)+(t-y)\psi'(y)+\int_y^t(t-x)\psi''(x)dx. \quad (53)$$

On operating $\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}$, we conclude that

$$\begin{aligned}\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}(\psi;y)-\psi(y)&=\psi'(y)\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}(t-y;y)\\&\quad+\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}\left(\int_y^t(t-x)\psi''(x)dx;y\right)\\&=\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}\left(\int_y^t(t-x)\psi''(x)dx;y\right)\\&=\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}\left(\int_y^t(t-x)\psi''(x)dx;y\right)-\int_y^{(([\alpha]_{p_\alpha,q_\alpha}y+1)/[\alpha]_{p_\alpha,q_\alpha})}\\&\quad\cdot\left(\frac{[\alpha]_{p_\alpha,q_\alpha}y+1}{[\alpha]_{p_\alpha,q_\alpha}}-\alpha\right)\psi''(x)dx\left|\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}(\psi;y)-\psi(y)\right|\\&\leqq\left|\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}\left(\int_y^t(t-x)\psi''(x)dx;y\right)\right|\\&\quad+\left|\int_y^{(([\alpha]_{p_\alpha,q_\alpha}y+1)/[\alpha]_{p_\alpha,q_\alpha})}\left(\frac{[\alpha]_{p_\alpha,q_\alpha}y+1}{[\alpha]_{p_\alpha,q_\alpha}}-x\right)\psi''(x)dx\right|.\end{aligned} \quad (54)$$

Since we know that

$$\begin{gathered}\left|\int_y^t(t-x)\psi''(x)dx\right|\leqq(t-y)^2\|\psi''\|,\\ \left|\int_y^{(([\alpha]_{p_\alpha,q_\alpha}y+1)/[\alpha]_{p_\alpha,q_\alpha})}\left(\frac{[\alpha]_{p_\alpha,q_\alpha}y+1}{[\alpha]_{p_\alpha,q_\alpha}}-x\right)\psi''(x)dx\right|\leqq\left(\frac{[\alpha]_{p_\alpha,q_\alpha}y+1}{[\alpha]_{p_\alpha,q_\alpha}}-y\right)^2\|\psi''\|,\end{gathered} \quad (55)$$

we get

$$\left|\mathcal{T}^{h,\lambda}_{\alpha,p_\alpha,q_\alpha}(\psi;y)-\psi(y)\right|\leqq\left\{\mathcal{S}^{h,\lambda}_{\alpha,p_\alpha,q_\alpha}((t-y)^2;y)+\left(\frac{[\alpha]_{p_\alpha,q_\alpha}y+1}{[\alpha]_{p_\alpha,q_\alpha}}-y\right)^2\right\}\|\psi''\|. \quad (56)$$

Hence, the above discussion completes the proof.

Theorem 10. *Let $\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}$ be defined by (9); then, for every $\psi\in C_B^2[0,\infty)$, there exists an absolute constant $\mathcal{C}>0$ such that*

$$\begin{aligned}\left|\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}(\psi;y)-f(y)\right|&\leqq\mathcal{C}\left\{\omega_2\left(f;\frac{\sqrt{\Theta_\alpha(y)}}{2}\right)\right.\\&\quad\left.+\min\left(1,\frac{\Theta_\alpha(y)}{4}\right)\|f\|\right\}+\omega\left(f;\frac{1}{[\alpha]_{p_\alpha,q_\alpha}}\right),\end{aligned} \quad (57)$$

where $\Theta_\alpha(y)$ is defined by Theorem 9.

Proof. In the view of the result asserted by Theorem 9, we prove this theorem. For all $f\in C_B[0,\infty)$ and $\psi\in C_B^2[0,\infty)$, we have

$$\begin{aligned}\left|\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}(\psi;y)-f(y)\right|&=\left|\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}(\psi;y)-f(y)+f\left(\frac{[\alpha]_{p_\alpha,q_\alpha}y+1}{[\alpha]_{p,q}}\right)-f(y)\right|\\&\leqq\left|\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}(f-\psi;y)\right|+\left|\mathcal{T}^\lambda_{\alpha,p_\alpha,q_\alpha}(\psi;y)-\psi(y)\right|\\&\quad+|\psi(y)-f(y)|+\left|f\left(\frac{[\alpha]_{p_\alpha,q_\alpha}y+1}{[\alpha]_{p_\alpha,q_\alpha}}\right)-f(y)\right|\\&\leqq 4\|f-\psi\|+\Theta_\alpha(y)\|\psi''\|\\&\quad+\omega\left(f;\left|\left(\frac{[\alpha]_{p_\alpha,q_\alpha}}{[\alpha]_{p_\alpha,q_\alpha}}-1\right)y+\frac{1}{[\alpha]_{p_\alpha,q_\alpha}}\right|\right).\end{aligned} \quad (58)$$

By taking the infimum over all $\psi\in C_B^2[0,\infty)$ and using (45), we get

$$\begin{aligned}\left|\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}(\psi;y)-f(y)\right|&\leqq 4K_2\left(f;\frac{\Theta_\alpha(y)}{4}\right)+\omega\left(f;\left|\left(\frac{[\alpha]_{p_\alpha,q_\alpha}}{[\alpha]_{p_\alpha,q_\alpha}}-1\right)y+\frac{1}{[\alpha]_{p_\alpha,q_\alpha}}\right|\right)\\&\leqq\mathcal{C}\left\{\omega_2\left(f;\frac{\sqrt{\Theta_\alpha(y)}}{2}\right)+\min\left(1;\frac{\Theta_\alpha(y)}{4}\right)\|f\|\right\}\\&\quad+\omega\left(f;\frac{1}{[\alpha]_{p_\alpha,q_\alpha}}\right).\end{aligned} \quad (59)$$

We consider the Lipschitz-type maximal function by [44] and obtain the local approximation such as for $f\in C[0,\infty]$, $0<\kappa\leqq 1$, and $t,y\in[0,\infty)$. We recall that

$$\omega_\kappa(f;y)=\sup_{t\neq y,t\in[0,\infty)}\frac{|f(t)-f(y)|}{|t-y|^\kappa}. \quad (60)$$

Theorem 11. *For all $f\in C_B[0,\infty)$ and $\kappa\in(0,1]$, we have*

$$\left|\mathcal{S}^\lambda_{\alpha,p_\alpha,q_\alpha}(f;y)-f(y)\right|\leqq\omega_\kappa(f;y)(\delta_\alpha(y))^{(\kappa/2)}, \quad (61)$$

where the $\omega_\kappa(f;y)$ *Lipschitz maximal function is defined by (60), and* $\delta_\alpha(y)$ *is defined by Theorem 7.*

Proof. We prove Theorem11 by applying (60) and the well-known Hölder inequality:

$$\begin{aligned}\left|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(f;y)-f(y)\right| &\leqq \mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(|f(t)-f(y)|;y)\\ &\leqq \omega_\kappa(f;y)|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(|t-y|^\kappa;y)\\ &\leqq \omega_\kappa(f;y)\left(\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(1;y)\right)^{(2-\kappa/2)}\\ &\quad\cdot\left(\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(|t-y|^2;y)\right)^{(\kappa/2)}\\ &= \omega_\kappa(f;y)\left(\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(\Delta_2;y)\right)^{(\kappa/2)}.\end{aligned} \tag{62}$$

The desired results are proven.

We next denote

$$\begin{aligned}C_B^2[0,\infty) &= \left\{\psi:\psi\in C_B[0,\infty)\ \text{and}\ \psi',\psi''\in C_B[0,\infty)\right\},\\ \|\psi\|_{C_B^2(\mathbb{R}^+)} &= \|\psi\|_{C_B[0,\infty)}+\left\|\psi'\right\|_{C_B[0,\infty)}+\left\|\psi''\right\|_{C_B[0,\infty)},\\ \|\psi\|_{C_B[0,\infty)} &= \sup_{y\in[0,\infty)}|\psi(y)|.\end{aligned} \tag{63}$$

Theorem 12. *Let* $\psi\in C_B^2[0,\infty)$. *Then,* $\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}$ *defined by (9) satisfies*

$$\left|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(\psi;y)-\psi(y)\right|\leqq\left(\sqrt{\delta_\alpha(y)}+\frac{\delta_\alpha(y)}{2}\right)\|\psi\|_{C_B^2[0,\infty)}. \tag{64}$$

Proof. From the Taylor series expansion of order two, we have

$$\psi(t)=\psi(y)+\psi'(y)(t-y)+\psi''(\varphi)\frac{(t-y)^2}{2}, \tag{65}$$

for $\varphi\in(y,t)$. Let

$$\begin{aligned}\mathcal{S} &= \sup_{y[0,\infty)}\left|\psi'(y)\right| = \left\|\psi'\right\|_{C_B[0,\infty)}\leqq\|\psi\|_{C_B^2[0,\infty)},\\ \mathcal{T} &= \sup_{y[0,\infty)}\left|\psi''(y)\right| = \left\|\psi''\right\|_{C_B[0,\infty)}\leqq\|\psi\|_{C_B^2[0,\infty)}.\end{aligned} \tag{66}$$

Then, we have

$$|\psi(t)-\psi(y)|\leqq\mathcal{S}|t-y|+\frac{1}{2}\mathcal{T}(t-y)^2\leqq\left(|t-y|+\frac{1}{2}(t-y)^2\right)\|\psi\|_{C_B^2[0,\infty)}. \tag{67}$$

Therefore, we have

$$\begin{aligned}\left|\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(\psi;y)-\psi(y)\right| &\leqq\left(\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(|t-y|;y)+\frac{1}{2}\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}((t-y)^2;y)\right)\|\psi\|_{C_B^2[0,\infty)}\\ &\leqq\left(\left(\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(\Delta_2;y)\right)^{(1/2)}+\frac{1}{2}\mathcal{S}^{\lambda}_{\alpha,p_\alpha,q_\alpha}(\Delta_2;y)\right)\|\psi\|_{C_B^2[0,\infty)}.\end{aligned} \tag{68}$$

This completes the proof of Theorem 12.

Data Availability

Not applicable.

Conflicts of Interest

The author declares there are no conflicts of interest.

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Research Article

An Extension of the Picard Theorem to Fractional Differential Equations with a Caputo-Fabrizio Derivative

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In this paper, we consider fractional differential equations with the new fractional derivative involving a nonsingular kernel, namely, the Caputo-Fabrizio fractional derivative. Using a successive approximation method, we prove an extension of the Picard-Lindelöf existence and uniqueness theorem for fractional differential equations with this derivative, which gives a set of conditions, under which a fractional initial value problem has a unique solution.

1. Introduction

Due to the demonstrated applications of fractional operators in various and widespread fields of many sciences, such as mathematics, physics, chemistry, engineering, and statistics [1–4], various operators of a fractional calculus have been found to be remarkably popular for modelling of numerous varied problems in these sciences. We mention here some of these definitions, such as Riemann-Liouville, Hadamard, Grünwald-Letnikov, Weyl, Riesz, Erdélyi-Kober, and Caputo. Compared with an integer order, a significant feature of a fractional order differential operator appeared in its hereditary property. In other words, when we describe a process by a fractional operator, we predict the future state by its current as well as its past states. Therefore, the memory and hereditary properties of materials and systems can be intervened in the modeling of a process by making use of differential equations of an arbitrary order. So, in recent years, fractional differential equations have been paid a great interest and also have appeared in new areas for applications of initial and boundary value problems of such equations. The Riemann-Liouville definition for the fractional derivative is one of the most widely used definitions and has many applications. But this definition had its drawbacks, such as the fact that the derivative of a constant function is not zero, and in practical examples, we need the value of fractional derivatives as initial values. The Caputo fractional derivative does not have the above weaknesses and is believed to be one of the most efficient definitions of fractional derivative applied in many areas of science and engineering.

However, the new definition suggested by Caputo and Fabrizio [5], which has all the characteristics of the old definitions, assumes two different representations for the temporal and spatial variables. In fact, they claimed that the classical definition given by Caputo appears to be particularly convenient for mechanical phenomena, related with plasticity, fatigue, damage, and with electromagnetic hysteresis. When these effects are not present, it seems more appropriate to use the new Caputo-Fabrizio operator.

The main advantage of the Caputo-Fabrizio approach is that the boundary conditions of the fractional differential equations with Caputo-Fabrizio derivatives admit the same form as for the integer-order differential equations. On the other hand, the Caputo-Fabrizio fractional derivative has many significant properties, such as its ability in describing matter heterogeneities and configurations with different scales [6–8]. Therefore, there are some certain phenomena that cannot be well-modeled using the Riemann-Liouville, Caputo, or other standard fractional operators [5, 9–13]. For an example, in issues related to material heterogeneities, we encounter some problems that are not well described by the above fractional operators. Also, later, some other definitions with a nonsingular kernel, such as the Atangana-Baleanu [6] fractional derivative, were defined.

Many researchers have shared their contributions to obtain properties of many models with new and old definitions of fractional derivatives. In [14], we have the analytic solutions of a viscous fluid with the Caputo and Caputo-Fabrizio fractional derivatives. In [15], the authors used the fractional derivative with a nonsingular kernel to model a Maxwell fluid and found semianalytical solutions. In [16], we found a comparison approach of two latest fractional derivatives models, namely, Atangana-Baleanu and Caputo-Fabrizio, for a generalized Casson fluid and obtained exact solutions. In [17–19], the authors also used the Caputo-Fabrizio fractional derivative to model some important examples.

Due to the abovementioned applications, the existence of solutions for nonlinear differential equations is an attractive research topic and has been studied using different techniques of nonlinear analysis [20–23]. One of the most important theorems in ordinary differential equations is Picard's existence and uniqueness theorem. This theorem, which is applied on first-order ordinary differential equations, can be generalized to establish existence and uniqueness results for both higher-order ordinary differential equations and for systems of differential equations. This theorem is a good introduction to the broad class of existence and uniqueness theorems that are based on fixed-point techniques [24–30].

In this paper, we obtain an extension of Picard's theorem for differential equations with the Caputo-Fabrizio fractional derivative. This theorem provides conditions for which a fractional initial value problem involving the Caputo-Fabrizio derivative has a unique solution. On the other hand, the proof of this extension of Picard's theorem provides a way of constructing successive approximations to the solution.

2. Preliminaries

In this section, we recall some notations and definitions which are needed throughout this paper. Further, some lemmas and theorems are stated as preparations for the main results. First, in the following, we provide some basic concepts and definitions in connection with the new Caputo-Fabrizio derivative.

The well-known left-sided Caputo fractional derivative ${}^{C}D^{\alpha}_{0^+,t}$ of a function $f(x) \in H^1(0,b)$ with $0<\alpha<1$, is defined by

$$ {}^{C}D^{\alpha}_{0^+,t}g(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t f'(s)(t-s)^{-\alpha}ds. \quad (1)$$

In [5], Caputo and Fabrizio proposed the new operator by replacing the singular kernel $(x-t)^{-\alpha}$ with $e^{-\alpha(x-t)/(1-\alpha)}$ and $1/\Gamma(1-\alpha)$ with $N(\alpha)/(1-\alpha)$ in the Caputo definition to obtain the following definition.

Definition 1. Let g be a given function in $H^1(a,b)$. The Caputo-Fabrizio derivative of fractional order $\alpha \in [0,1]$ is defined as

$$ {}^{CF}_{a}D^{\alpha}_{t}(g(t)) = \left(\frac{N(\alpha)}{1-\alpha}\right)\int_a^t g'(x)\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx, \quad (2)$$

where $N(\alpha)$ is a normalization function [5]. Also, if a certain function g does not satisfy in the restriction $g \in H^1(a,b)$, then its fractional derivative is redefined as

$$ {}^{CF}_{a}D^{\alpha}_{t}(g(t)) = \frac{\alpha N(\alpha)}{1-\alpha}\int_a^t (g(t)-g(x))\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx. \quad (3)$$

Clearly, as mentioned in [5], if one sets $\sigma = (1-\alpha)/\alpha \in [0,\infty]$ and $\alpha = 1/(1+\sigma) \in [0,1]$, then the Caputo-Fabrizio definition becomes

$$ {}^{CF}_{a}D^{\alpha}_{t}(g(t)) = \frac{N(\sigma)}{\sigma}\int_a^t g'(x)\exp\left[-\frac{t-x}{\sigma}\right]dx, \quad (4)$$

where $N(0) = N(\infty) = 1$, and

$$ \lim_{\sigma\to 0}\exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t). \quad (5)$$

Also, the fractional derivative of order $(n+\alpha)$ when $n \ge 1$ and $\alpha \in [0,1]$ is defined by the following

$$ {}^{CF}_{a}D^{(\alpha+n)}_{t}(g(t)) = {}^{CF}_{a}D^{(\alpha)}_{t}\left(D^{(n)}_{t}g(t)\right). \quad (6)$$

Definition 2. Let $g \in H^1(a,b)$, then its fractional integral of an arbitrary order is defined as follows:

$$ {}^{CF}_{a}I^{\alpha}_{t}(g(t)) = \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}g(t) + \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_a^t g(s)ds, \quad t \ge 0. \quad (7)$$

It is clear, in view of the above definition, that the αth Caputo-Fabrizio derivative of a function g is average between g and its first-order integral. Therefore,

$$ \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} + \frac{2\alpha}{(2-\alpha)N(\alpha)} = 1. \quad (8)$$

So, we arrive at the following

$$ N(\alpha) = \frac{2}{2-\alpha}, \quad 0 \le \alpha \le 1. \quad (9)$$

The Laplace transform of the Caputo-Fabrizio derivative is

$$L\{{}^{CF}D_t^{\alpha}(g(t))\} = \frac{sL\{g(t)\} - g(0)}{(1-\alpha)s + \alpha}. \quad (10)$$

Theorem 3 (Picard theorem [31]). *Let D be an open set in (t, x)-space. Let $(t_0, x^0) \in D$ and a and b be positive constants such that the set*

$$R = \{(t, x) \,|\, |t - t_0| \le a, |x - x_0| \le b\}, \quad (11)$$

is contained in D. Suppose that the function g is defined, continuous on D, and satisfies a Lipschitz condition with respect to x in R. Let

$$\begin{aligned} M &= \max_{(t,x)\in R} |g(t, x)|, \\ A &= \min\left\{a, \frac{b}{M}\right\}. \end{aligned} \quad (12)$$

Then, the following initial value problem

$$x' = g(t, x), x(t_0) = x^0, \quad (13)$$

has a unique solution, $x(t)$, on the interval $(t_0 - A, t_0 + A)$. For this solution in the domain $(t_0 - A, t_0 + A)$, we have

$$|x(t) - x^0| \le MA. \quad (14)$$

Note that by the mean-value theorem, the Lipschitz condition will be satisfied if we have $|(\partial/\partial x)g(t, x)| \le K$.

3. Extension of Picard Theorem

Picard's Theorem 3 guarantees the existence and uniqueness of the solution of the following initial value problem of first-order differential equations:

$$\frac{dy}{dt} = f(t, y(t))\; t \ge t_0, \quad (15)$$

$$y(t_0) = y_0. \quad (16)$$

In proving this theorem, the solution is obtained by the well-known successive approximations method (Picard-Lindelöf method) [31]. In this method, the approximate solution for solving (15) is defined by

$$y_{k+1} = y_0 + \int_{t_0}^{t} f(s, y_k(s))ds, \quad k \in N. \quad (17)$$

By continuing this process, when $k \to \infty$, the exact solution is obtained. In practice, the exact solution is approximated for a sufficient large k by y_k.

In this section, we consider the following differential equation

$${}_0^{CF}D_t^{\alpha}u(t) = g(t, u), \quad (18)$$

such that $t \in J = [0, 1]$, with the initial condition $u(0) = u_0$, where ${}_0^{CF}D_t^{\alpha}$ denotes the fractional Caputo-Fabrizio derivative. We extend Picard's theorem to this problem, and by the successive approximation method, an iterative process is provided to obtain the solution. We state the following generalized Picard existence and uniqueness theorem.

Theorem 4. *Suppose that the function g is defined, continuous on an open set Ω in (t, u)-space, and satisfies*

$$|g(t, u) - g(t, v)| \le k|u - v|, \quad 0 < k < 1. \quad (19)$$

Let $M = \max_{t\in J} |g(t, u)|$. Then, the fractional differential equation (18) has a unique solution such that $u(0) = u_0$.

To prove the theorem, first, we need to establish the following lemma.

Lemma 5. *The function $u(t)$ is the solution of (18) under the initial condition $u(0) = u_0$ if and only if it satisfies the following integral equation:*

$$\begin{aligned} u(t) = u_0 &+ \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u(t)) \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^t g(s, u(s))ds. \end{aligned} \quad (20)$$

Proof. If $u(t)$ is a solution of (18), then taking the fractional integral of order α, we obtain (20). The second part of the theorem comes from differentiating equation (20).

In the reminder of the proof, using the successive approximation method, we show that the sequence defined by

$$\begin{aligned} u_0(t) &= u_0, \\ u_1(t) &= u_0 + \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u_0) \\ &\quad + \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^t g(s, u_0)ds, \\ &\vdots \\ u_m(t) &= u_0 + \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u_{m-1}(t)) \\ &\quad + \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^t g(s, u_{m-1}(s))ds, \end{aligned} \quad (21)$$

converges to a function, which is a solution of (20), and then we show that this solution is unique.

Lemma 6. *For each m, the function $u_m(t)$ is defined, continuous on J and satisfies*

$$|u_m(t) - u_0| \le M. \quad (22)$$

Proof. We prove the lemma by induction. Since

$$\begin{aligned}|u_1(t)-u_0(t)| &\leq \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}|g(t,u_0(t))| \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t |g(s,u_0(s))|ds \\ &\leq \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}M+\frac{2\alpha}{(2-\alpha)N(\alpha)}M\int_0^t \\ &\quad\cdot ds \leq \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}M \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}M=M,\end{aligned} \tag{23}$$

the result is obviously true for $m=0$. Let us suppose that for $t\in J$,

$$|u_m(t)-u_0|\leq M. \tag{24}$$

This yields that $f(t,u_m(t))$ is defined on J, and since $f(t, u_m(t))$ is continuous at t, one asserts that

$$\begin{aligned}u_{m+1}(t)&=u_0+\frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}g(t,u_m(t)) \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t g(s,u_m(s))ds,\end{aligned} \tag{25}$$

is defined and continuous. Indeed, we have

$$\begin{aligned}|u_{m+1}(t)-u_0| &\leq \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}|g(t,u_m(t))| \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t |g(s,u_m(s))|ds\leq \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}M \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}M\int_0^t ds=M.\end{aligned} \tag{26}$$

Lemma 7. *The sequence $\{u_m(t)\}$ converges uniformly on J to a continuous function $u(t)$.*

Proof. It is obvious that the convergence of the series

$$u_0(t)+\sum_{n=0}^{\infty}[u_{n+1}(t)-u_n(t)], \tag{27}$$

yields convergence of the sequence $\{u_m(t)\}$. For $t\in J$, let us denote

$$\begin{aligned}d_n(t)&=|u_{n+1}(t)-u_n(t)|, \\ F_n(t)&=g(t,u_{n+1}(t))-g(t,u_n(t)).\end{aligned} \tag{28}$$

Then, for each n, one has

$$\begin{aligned}d_n(t) &\leq \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}|F_n(t)| \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t |F_n(s)|ds \\ &\leq \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}k|u_n(t)-u_{n-1}(t)| \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}k\int_0^t |u_n(s)-u_{n-1}(s)|ds \\ &=\frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}kd_{n-1}(t)+\frac{2\alpha}{(2-\alpha)N(\alpha)}k\int_{t_0}^t \\ &\quad\cdot d_{n-1}(s)ds=k_0^{CF}I_t^{\alpha}d_{n-1}(t),\end{aligned} \tag{29}$$

where k is the Lipschitz constant of g and $0<k<1$. Now, we show that for each n, we have

$$d_n(t)\leq Mk^n. \tag{30}$$

From Lemma 6, we have

$$d_0(t)=|u_1(t)-u_0(t)|\leq M. \tag{31}$$

By induction, let $d_n(t)\leq Mk^n$. Then, from (29) and (8), one writes

$$\begin{aligned}d_{n+1}(t) &\leq k_0^{CF}I_t^{\alpha}d_n(t)=\frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}kd_n(t) \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}k\int_{t_0}^t d_n(s)ds\leq \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}kMk^n \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}kMk^n\int_0^t ds\leq \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}Mk^{n+1} \\ &\quad+\frac{2\alpha}{(2-\alpha)N(\alpha)}Mk^{n+1}=Mk^{n+1}.\end{aligned} \tag{32}$$

Therefore,

$$\sum_{n=0}^{\infty}d_n(t)\leq M\sum_{n=0}^{\infty}k^n. \tag{33}$$

Since $0<k<1$, the uniform convergence of (27) follows from the Weierstrass test or by a simple comparison test.

Lemma 8. *The function $u(t)$ is satisfied in (18), and we have $u(0)=u_0$.*

Proof. First, let us show that $|u(t)-u_0|$ is bounded. That is,

$$|u(t)-u_0|<B, \tag{34}$$

for some constant B. We can deduce that $g(t,u(t))$ is defined for $t\in J$. For $t\in J$ and $\varepsilon>0$ and for a sufficiently large m, one has

$$|u(t) - u_0| \le |u(t) - u_m(t)| + |u_m(t) - u_0| \le \varepsilon + M < B. \quad (35)$$

Then, by the Lipschitz condition of g, we have

$$\begin{aligned} \left|\int_0^t g(s, u(s)) - g(s, u_m(s))ds\right| &\le \int_0^t |g(s, u(s)) \\ &- g(s, u_m(s))|ds \le k\int_0^t |u(s) - u_m(s)|ds \le k\varepsilon. \end{aligned} \quad (36)$$

Therefore, $\lim_{m\to\infty}\int_0^t g(s, u_m(s)) = \int_0^t g(s, u(s))ds$. Now, by taking the limit with respect to m on both sides of the following equation

$$\begin{aligned} u_m(t) = u_0 &+ \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u_{m-1}(t)) \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^t g(s, u_{m-1}(s))ds, \end{aligned} \quad (37)$$

we obtain

$$\begin{aligned} u(t) = u_0 &+ \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u(t)) \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^t g(s, u(s))ds. \end{aligned} \quad (38)$$

Now, we prove the uniqueness of the solution.

Lemma 9. *The solution $u(t)$ of the integral equation (7) satisfying the condition $u(t_0) = u_0$, is the unique solution of (18) with this initial condition.*

Proof. Suppose that there exist two solutions $u_1(t)$ and $u_2(t)$ of the integral equation (7) on J subject to the condition $u_1(t_0) = u_2(t_0) = u_0$. First, since $u_1(t)$ and $u_2(t)$ are continuous functions, there exists a constant $B > 0$ such that in the closed interval J, we have

$$|u_1(t) - u_2(t)| < B. \quad (39)$$

Let us suppose that for each positive integer m,

$$|u_1(t) - u_2(t)| < k^m B. \quad (40)$$

Then, from (7), we have $|u_1(t) - u_2(t)| < k^{m+1}B$. Therefore, by induction, $|u_1(t) - u_2(t)|$ is less than each term of the convergent geometric series of $B/(1-k)$. This yields that for each ε, $|u_1(t) - u_2(t)| < \varepsilon$, and therefore, we have $u_1(t) = u_2(t)$.

By proving the above lemma, the proof of Theorem 3 is completed. Note that the iterative process (21) provides a constructive approach to obtain the solution. We describe the following simple example where the hypotheses of Theorem (4) hold:

$$\begin{aligned} {}_0^{CF}D_t^{\alpha} u(t) &= \frac{1}{u+1}, \\ u(0) &= 0. \end{aligned} \quad (41)$$

By assuming $C = 2(1-\alpha)/(2-\alpha)N(\alpha)$ and $D = 2\alpha/(2-\alpha)N(\alpha)$, the results of using (21) are as follows:

$$\begin{aligned} u_0(t) &= 0, \\ u_1(t) &= C + Dt, \\ u_2(t) &= \frac{C}{Dt + C + 1} + \ln(Dt + C + 1), \\ u_3(t) &= \frac{C}{C/(Dt + C + 1) + \ln(Dt + C + 1) + 1} \\ &\quad + D\int_0^t \frac{1}{C/(Ds + C + 1) + \ln(Ds + C + 1) + 1}. \\ &\vdots \end{aligned} \quad (42)$$

To ensure the results, let us choose $\alpha = 1$. In this case, it is easy to show that the obtained sequence $0, t, \ln(t+1), \cdots$ converges to the exact solution $u(t) = \sqrt{2t+1} - 1$.

4. Conclusion

By Picard's theorem, we can study the existence and uniqueness of a solution of first-order differential equations. Also, this theorem can be applied to ensure the existence of a unique solution of higher-order ordinary differential equations and for systems of differential equations. On the other hand, this theorem is an essential tool in fixed-point theory. Therefore, a generalization of this theorem for fractional differential equations would be interesting. In this paper, we proved an extension of this theorem to the initial value problems of fractional ordinary differential equations with the Caputo-Fabrizio derivative, and by the successive approximation method, an iterative process was provided to obtain the solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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Research Article

Soft Fixed Point Theorems for the Soft Comparable Contractions

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In this article, we introduce the notions of a soft inf-comparable contraction and soft comparable Meir-Keeler contraction in a soft metric space. Furthermore, we prove two soft fixed point theorems which assure the existence of soft fixed points for these two types of comparable contractions. The obtained results not only generalize but also unify many recent fixed point results in the literature.

1. Introduction and Preliminaries

It is the main feature of mathematical study to produce different methods and tools to perceive the behavior of systems that we have difficulty understanding with known methods. In particular, it may be necessary to deal with systems that contain uncertainties and to use inaccurate data in different situations. With this motivation, one of the mathematical tools used to deal with the necessities of systems established with uncertainty and to analyze the models created by the uncertainties and uncertainties already existing in the data is the Fuzzy Set Theory. Fuzzy sets were introduced by Zadeh [1] for dealing with the uncertainties on its own limits. Another mathematical tool to deal with the uncertainties is the soft set that was introduced by Molodtsov [2]. In this paper, we shall focus on the soft set theory. The topology based on the soft sets was defined by Cagman et al. [3]. They also considered the basic topological notions over soft sets. On the other hand, a soft real set and soft real number were proposed successfully by Das and Samanta [4]. Furthermore, the same authors in considered the notions of a soft metric and its topology, properly. After then, Abbas et al. [5] proved a fixed point theorem by introducing the notion of soft contraction mapping over the soft metric space. Application potential of the soft sets in various distinct research topics is very rich and wide, for example, the smoothness of functions, game theory, operation research, probability theory, and measurement theory. For more details on soft sets and application, we can refer to, e.g., [3, 4, 6–12].

As usual, $\mathbb{R}$ denotes real numbers and $\mathbb{R}^+ := [0,\infty)$. Furthermore, the letters $\mathbb{Z}, \mathbb{N}$ denote integers and natural numbers, respectively. The symbol $B(\mathbb{R})$ denotes the collection of all nonempty bounded subsets of $\mathbb{R}$.

We shall denote an initial universe Ω. We set $\mathscr{P}$ as a set of parameters. As usual, 2^Ω denotes the collection of all subsets of Ω. For a nonempty subset S of $\mathscr{P}$, we consider a set-valued mapping $T : S \rightarrow 2^\Omega$ for all $\tau \notin A$ with $T(\tau) = \phi$. We define a pair (T, A) on Ω as

$$(T, S) = \{(T(\tau), \tau) \colon \tau \in \mathscr{P}\}. \tag{1}$$

Here, (T, S) is called a soft set [2]. The symbol $\mathcal{S}(\Omega)$ represents the collection of all soft sets on Ω.

A soft set (T, S) on Ω is called null soft [11] (respectively, absolute soft set [11]) represented by, respectively, $\tilde{S}$if$T(\tau) =$ (respectively,$T(\tau) = S$) for all$\tau \in S$. We presume that (T_1, S_1) and (T_2, S_2) are two soft sets on Ω. We define the intersection [11] of the mentioned two sets above as a soft set (T_3, S_3), denoted by $(T_1,S_1) \cap^{\sim} (T_2, S_2) = (T_3, S_3)$, where $S_3 = S_1 \cap S_2$, and for each $\tau \in S_3$, $T_3(\tau) = T_1(\tau) \cap T_2(\tau)$. As expected, we define the union of (T_1, S_1) and (T_2, S_2) [11] as a soft set

(T_3, S_3), denoted by $(T_1,A)\tilde{\cup}(T_2,B)=(T_3,C)$ where $S_3 = S_1 \cup S_2$ and for each $\tau \in S_3$,

$$T_3(\tau)=\begin{cases} T_1(\tau), & \text{if } \tau \in S_1 \setminus S_2, \\ T_2(\tau), & \text{if } \tau \in S_2 \setminus S_1, \\ T_1(\tau)\cup T_2(\tau), & \text{if } \tau \in S_1 \cap S_2. \end{cases} \quad (2)$$

We use the notation $(T^c, \mathscr{P})$ to indicate the complement [11] of soft set $(T, \mathscr{P})$ on Ω where $T^c : X \to 2^{\Omega}$ is a mapping given by $T^c(\tau)=\Omega \setminus T(\tau)$ for all $\tau \in \mathscr{P}$.

A mapping $T : \mathscr{P} \to B(\mathbb{R})$ is called a soft real set [13]. The symbol $\mathbb{R}^+(\mathscr{P})$ is used to denote the set of all nonnegative soft real numbers. If $(T, \mathscr{P})$ is a singleton soft set, then it is called a soft real number. Regarding the corresponding soft set, soft real numbers will be denoted as $\tilde{\gamma}, \tilde{\eta}, \tilde{\xi}$, etc. In particular, $\bar{0}$ and $\tilde{1}$ are the soft real numbers where $\bar{0}(\tau)=0$, $\tilde{1}(\tau)=1$ for all $\tau \in \mathscr{P}$.

For two soft real numbers, for all $\tau \in \mathscr{P}$, we have the following inequalities [13]:

(1) $\gamma^{\sim}\tilde{\le}\tilde{\eta}$ if $\gamma^{\sim}(\tau)\tilde{\le}\tilde{\eta}(\tau)$

(2) $\gamma^{\sim}\tilde{\ge}\tilde{\eta}$ if $\gamma^{\sim}(\tau)\tilde{\ge}\tilde{\eta}(\tau)$

(3) $\gamma^{\sim}\tilde{<}\tilde{\eta}$ if $\gamma^{\sim}(\tau)\tilde{<}\tilde{\eta}(\tau)$

(4) $\gamma^{\sim}\tilde{>}\tilde{\eta}$ if $\gamma^{\sim}(\tau)\tilde{>}\tilde{\eta}(\tau)$

Definition 1.

(1) The mapping $\phi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$ is called soft increasing, if

$$r^{\sim}\tilde{<}\tilde{t} \Rightarrow \phi(r^{\sim})\tilde{<}\phi(\tilde{t}). \quad (3)$$

(2) The mapping $\phi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$ is called soft continuous at $a^{\sim}\tilde{\in}\mathbb{R}^+(\mathscr{P})$, if for every $\gamma^{\sim}\tilde{>}\bar{0}$, there exists $\delta^{\sim}\tilde{>}\bar{0}$ such that $\bar{0}\tilde{<}\tilde{x}-a^{\sim}\tilde{<}\tilde{\delta}$ implies

$$\phi(\tilde{x})-\phi(a^{\sim})\tilde{<}\tilde{\gamma}. \quad (4)$$

Moreover, $\phi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$ is called soft continuous at every point $\tilde{a}$ of $\mathbb{R}^+(\mathscr{P})$, then we call ϕ as a continuous mapping.

A soft set $(T, \mathscr{P})$ on Ω is called a soft point [4, 14], denoted by $\widetilde{x_\tau}$, if there is a unique $\tau \in \mathscr{P}$ such that $T(\tau)=\{x\}$ for some $\tau \in \mathscr{P}$ and $T(\omega)=\phi$ for all $\omega \in \mathscr{P}\setminus\{\tau\}$.

Definition 2 (see). Let $\tilde{X}=(\mathrm{T}, \mathscr{P})$ be an absolute soft set, and let $\mathscr{SP}(\tilde{X})$ be the collection of all soft points of $\tilde{X}$. A mapping $\tilde{d} : \mathscr{SP}(\tilde{X})\times\mathscr{SP}(\tilde{X}) \to \mathbb{R}^+(\mathscr{P})$ is called a soft metric on $\tilde{\mathrm{X}}$ if $\tilde{d}$ satisfies the following conditions for all $\widetilde{x_{\tau_1}}, \widetilde{x_{\tau_2}}, \widetilde{x_{\tau_3}} \in \tilde{X}$:

$(M1)\tilde{d}(\widetilde{x_{\tau_1}}, x_{\tau_2}{}^{\sim})\tilde{\ge}\bar{0}$,

$(M2)\tilde{d}(\widetilde{x_{\tau_1}}, \widetilde{x_{\tau_2}})=\bar{0}$ if and only if $\widetilde{x_{\tau_1}}=\widetilde{x_{\tau_2}}$,

$(M3)\tilde{d}(\widetilde{x_{\tau_1}}, \widetilde{x_{\tau_2}})=\tilde{d}(\widetilde{x_{\tau_2}}, \widetilde{x_{\tau_1}})$,

$(M4)\tilde{d}(\widetilde{x_{\tau_1}}, x^{\sim}{}_{\tau_3})\tilde{\le}\tilde{d}(\widetilde{x_{\tau_1}}, \widetilde{x_{\tau_2}})+\tilde{d}(\widetilde{x_{\tau_2}}, \widetilde{x_{\tau_3}})$.

The triple $(\tilde{X}, \tilde{d}, \mathscr{P})$ is called a soft metric space, in short, s.m.s.

For the sake of simplicity, we set $\mathscr{M} := (\tilde{X}, \tilde{d}, \mathscr{P})$.

Suppose $\mathscr{M}$ is a s.m.s. and $\tilde{\gamma}$ is a nonnegative soft real number. A soft open ball with the center $\widetilde{x_e}$ and radius $\tilde{\gamma}$ is defined by $B(\widetilde{x_e}, \tilde{\gamma})=\{y_{e'}{}^{\sim}\tilde{\in}\tilde{X} : \tilde{d}(\widetilde{x_e}, y_{e'}{}^{\sim})\tilde{<}\tilde{\gamma}\}$. Analogously, a soft closed ball with center $\widetilde{x_e}$ and radius $\tilde{\gamma}$ is $B[\widetilde{x_e}, \tilde{\gamma}]=\{y_{e'}{}^{\sim}\tilde{\in}\tilde{X} : \tilde{d}(\widetilde{x_e}, y_{e'}{}^{\sim})\tilde{\le}\tilde{\gamma}\}$. We set that a soft set$(F, \mathscr{P})$is soft open in$\tilde{X}$with respect to$\tilde{d}$ if and only if all soft points of (F, E) are interior points of $(F, \mathscr{P})$.

In a soft metric space $\mathscr{M}$, a sequence of soft points $\{\widetilde{x_{\lambda,n}}\}_n$ is called convergent in $\mathscr{M}$ if there is a soft point $y_v{}^{\sim}\tilde{\in}\tilde{X}$ such that

$$\lim_{n\to\infty}\tilde{d}(\widetilde{x_{\lambda,n}}, \widetilde{y_v})=\bar{0}. \quad (5)$$

Furthermore, a sequence $\{\widetilde{x_{\lambda,n}}\}_n$ is said to be a Cauchy in $\mathscr{M}$ if

$$\lim_{i,j\to\infty}\tilde{d}(\widetilde{x_{\lambda,i}}, \widetilde{x_{\lambda,j}})=\bar{0}. \quad (6)$$

Moreover, if each Cauchy sequence in $\tilde{X}$ converges to some point of $\tilde{X}$, then $\mathscr{M}$ is called complete soft metric space.

Let $\mathscr{N}=(\tilde{Y}, \tilde{\sigma}, \mathscr{P}')$ be another soft metric space. A soft mapping (f, φ): $\mathscr{M} \to \mathscr{N}$ is soft continuous at a point $x_\lambda{}^{\sim}\tilde{\in}\mathscr{SP}(\tilde{X})$, if for each $B((f,\varphi)(\tilde{x}_\lambda), \tilde{\gamma})$ of $\mathscr{N}$, there exists $B(\widetilde{x_\lambda}), \tilde{\delta})$ such that

$$f\left(B(\widetilde{x_\lambda}), \tilde{\delta})\right)\tilde{\subset}B((f,\varphi)(\widetilde{x_\lambda}), \tilde{\gamma}). \quad (7)$$

In other words, for every $\gamma^{\sim}\tilde{>}\bar{0}$, there exists $\delta^{\sim}\tilde{>}\bar{0}$ such that $\tilde{d}(\widetilde{x_\lambda}, y_\mu{}^{\sim})\tilde{<}\tilde{\delta}$ implies that $\tilde{\sigma}((f,\varphi)(\widetilde{x_\lambda}), (f,\varphi)(x_\lambda{}^{\sim}))\tilde{<}\tilde{\gamma}$. Moreover, if (f, φ) is soft continuous for each point of $\mathscr{SP}(\tilde{X})$, then it is called soft continuous mapping.

2. Soft Fixed Points for the Soft Inf-Comparable Contraction

In this section, we first introduce the notion of soft inf -comparable mapping $\psi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$.

Definition 3 (see [15]). Let $\mathscr{P}$ be a parameter set and $\psi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$. We call ψ a soft inf-comparable mapping if it satisfies the following two axioms:

$(\psi_1)\psi(\tau^{\sim})\tilde{<}\tilde{\tau}$ for all $\tilde{\tau} \in \mathbb{R}^+(\mathscr{P})\setminus\{\bar{0}\}$ and $\psi(\bar{0})=\bar{0}$,

$(\psi_2)\liminf_{\widetilde{\tau_n}\to\tilde{\tau}}\psi(\tau_n{}^{\sim})\tilde{<}\tilde{\tau}$ for all $\tau^{\sim}\tilde{>}\bar{0}$.

Lemma 4. *Let $\psi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$ be a soft* inf*-comparable mapping. Then, $\lim_{n\to\infty}\psi^n(\tilde{\tau}) = \bar{0}$ for all $\tilde{\tau} > \bar{0}$, where ψ^n denotes the n-th iteration of ψ.*

Proof. Let $\tau\tilde{} >\tilde{} \bar{0}$ be fixed. If $\psi_0^n(\tilde{\tau}) = \bar{0}$ for some $n_0 \in \mathbb{N}$, then we have

$$\psi^{n_0+1}(\tilde{\tau}) = \psi(\psi^{n_0}(\tilde{\tau})) = \psi(\bar{0}) = \bar{0}, \tag{8}$$

which implies that

$$\psi^{n_0+r}(\tilde{\tau}) = \bar{0}, \quad \text{for all } r \in \mathbb{N}. \tag{9}$$

Thus, we conclude that

$$\lim_{n\to\infty}\psi^n(\tilde{\tau}) = \bar{0}. \tag{10}$$

If $\psi^n(\tau\tilde{})>\tilde{}\bar{0}$ for each $n \in \mathbb{N}$, then we take $\widetilde{\sigma_n} = \psi^n(\tilde{\tau})$, and

$$\widetilde{\sigma_{n+1}} = \psi^{n+1}(\tilde{\tau}) = \psi(\psi^n(\tilde{\tau})) = \psi(\widetilde{\sigma_n}), \tag{11}$$

for all $n \in \mathbb{N}$. By the condition (ψ_1) of the soft inf-comparable mapping ψ, we have that for all $n \in \mathbb{N}$,

$$\widetilde{\sigma_{n+1}} = \psi(\sigma_n\tilde{})<\tilde{}\widetilde{\sigma_n}. \tag{12}$$

Keeping (ψ_2) in mind and considering that the soft sequence$\{\sigma_{n\in\mathbb{N}}\}$is bounded from below and also that the soft sequence is strictly decreasing, one can find an $\nu\tilde{}\geq\tilde{}\bar{0}$ such that

$$\lim_{n\to\infty}\widetilde{\sigma_n} = \tilde{\nu}. \tag{13}$$

We assert that $\tilde{\nu} = \bar{0}$. If not, suppose that $\nu\tilde{}>\tilde{}\bar{0}$, then we find

$$\tilde{\nu} = \lim_{n\to\infty}\widetilde{\sigma_{n+1}} = \lim_{n\to\infty} \inf \psi(\widetilde{\sigma_n}) = \lim_{\widetilde{\sigma_n}\to\tilde{\nu}} \inf \psi(\sigma_n\tilde{})<\tilde{}\tilde{\nu}, \tag{14}$$

a contradiction. So we obtain that $\lim_{n\to\infty}\psi^n(\tilde{\tau}) = \bar{0}$.

We introduce the notion of soft inf-comparable contraction, as follows:

Definition 5. Let $\mathscr{M}$ be a soft metric space and let $\psi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$ be a soft inf-comparable mapping. A mapping (f,φ): $\mathscr{M} \to \mathscr{M}$ is called a soft inf-comparable contraction if for each soft points $\widetilde{x_p}, \widetilde{y_\tau} \in \mathcal{SP}(\tilde{X})$,

$$\begin{aligned}&\tilde{d}\left((f,\varphi)(\widetilde{x_p}),(f,\varphi)(y_\tau\tilde{})\right)\\ &\quad\leq\tilde{}\psi\left(\max\left\{\tilde{d}(\widetilde{x_p},\widetilde{y_\tau}),\tilde{d}(\widetilde{x_p},(f,\varphi)(\widetilde{x_p})),\tilde{d}(\widetilde{y_\tau},(f,\varphi)(\widetilde{y_\tau}))\right\}\right).\end{aligned} \tag{15}$$

Example 6. Set $\mathscr{R} = (\tilde{\mathbb{R}}, \tilde{d}, \mathscr{P})$ where the soft metric is expressed as

$$\begin{aligned}&d_\varphi(p,\tau) = \max\{p,\tau\}, d(x,y) = |x-y|,\\ &\tilde{d}(\widetilde{x_p},\widetilde{y_\tau}) = \frac{3}{5}d_\varphi(p,\tau) + d(x,y),\end{aligned} \tag{16}$$

with $\mathscr{P} = [0,\infty)$, $\varphi(t) = (2/3)t$ for $t \in [0,\infty)$.

Let $\psi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$ be denoted by

$$\psi(\tilde{\omega}) = \frac{5}{6}\tilde{\omega}. \tag{17}$$

and let $f(x) = (2/5)x$. Consequently, we find

$$\begin{aligned}&\tilde{d}\left((f,\varphi)(\widetilde{x_p}),(f,\varphi)(\widetilde{y_\tau})\right) = \tilde{d}\left(\widetilde{\frac{2}{5}x_{(2/3)p}},\widetilde{\frac{2}{5}y_{(2/3)\tau}}\right) = \frac{2}{5}\max\{p,\tau\} + \frac{2}{5}|x-y|,\\ &\tilde{d}(\widetilde{x_p},\widetilde{y_\tau}) = \frac{3}{5}\max\{p,\tau\} + |x-y|,\\ &\tilde{d}\left(\widetilde{x_p},\widetilde{\frac{1}{2}x_{(1/3)p}}\right) = \frac{3}{5}\max\left\{p,\frac{1}{3}p\right\} + \left|x-\frac{1}{2}x\right| = \frac{3}{5}p + \frac{1}{2}|x|,\\ &\tilde{d}\left(\widetilde{y_\tau},\widetilde{\frac{1}{2}y_{(1/3)\tau}}\right) = \frac{3}{5}\max\left\{\tau,\frac{1}{3}\tau\right\} + \left|y-\frac{1}{2}y\right| = \frac{3}{5}\tau + \frac{1}{2}|y|.\end{aligned} \tag{18}$$

As a result, (f,φ) forms a soft inf-comparable contraction on $\mathscr{R}$.

We say that a soft point $\widetilde{x_\tau} \in \mathcal{SP}(\tilde{X})$ is a soft fixed point of a self-soft-mapping (f,φ) if $(f,\varphi)(\widetilde{x_\tau}) = \widetilde{x_\tau}$.

Theorem 7. *Let $\psi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$ be a soft* inf*-comparable mapping. Let (f,φ): $\mathscr{M} \to \mathscr{M}$ be a soft* inf*-comparable contraction on a complete soft metric space $\mathscr{M}$. Then, a soft mapping (f,φ) possesses a soft fixed point.*

Proof. Let $\widetilde{x_{\tau_0}^0} \in \mathcal{SP}(\tilde{X})$ be given. For each $n \in \mathbb{N}\cup\{0\}$, we put

$$\widetilde{x_{\tau_{n+1}}^{n+1}} = \left((f,\varphi)\left(\widetilde{x_{\tau_n}^n}\right)\right) = \left(f^{n+1}\left(\widetilde{x_{\tau_0}^0}\right)\right)_{\varphi^{n+1}(\tau_0)}. \tag{19}$$

Then, we have for each $n \in \mathbb{N}\cup\{0\}$

$$\begin{aligned}\tilde{d}\left(\widetilde{x_{\tau_n}^n},\widetilde{x_{\tau_{n+1}}^{n+1}}\right) &= \tilde{d}\left((f,\varphi)\left(\widetilde{x_{\tau_{n-1}}^{n-1}}\right),(f,\varphi)\left(x_{\tau_n}^n\tilde{}\right)\right)\\ &\leq\tilde{}\psi\left(\max\left\{\tilde{d}\left(\widetilde{x_{\tau_{n-1}}^{n-1}},\widetilde{x_{\tau_n}^n}\right),\tilde{d}\left(\widetilde{x_{\tau_{n-1}}^{n-1}},(f,\varphi)\left(\widetilde{x_{\tau_{n-1}}^{n-1}}\right)\right),\tilde{d}\right.\right.\\ &\quad\left.\left.\cdot\left(\widetilde{x_{\tau_n}^n},(f,\varphi)\left(\widetilde{x_{\tau_n}^n}\right)\right)\right\}\right) = \psi\left(\max\left\{\tilde{d}\left(\widetilde{x_{\tau_{n-1}}^{n-1}},\widetilde{x_{\tau_n}^n}\right),\tilde{d}\right.\right.\\ &\quad\left.\left.\cdot\left(\widetilde{x_{\tau_{n-1}}^{n-1}},\widetilde{x_{\tau_n}^n}\right),\tilde{d}\left(\widetilde{x_{\tau_n}^n},\widetilde{x_{\tau_{n+1}}^{n+1}}\right)\right\}\right).\end{aligned} \tag{20}$$

Since $\psi : \mathbb{R}^+(\mathscr{P}) \to \mathbb{R}^+(\mathscr{P})$ is a soft inf-comparable mapping, we can conclude that for each $n \in \mathbb{N}\cup\{0\}$,

$$\tilde{d}\left(\widetilde{x_{\tau_n}^n},x_{\tau_{n+1}}^{n+1}\tilde{}\right)\leq\tilde{}\psi\left(\tilde{d}\left(\widetilde{x_{\tau_{n-1}}^{n-1}},\widetilde{x_{\tau_n}^n}\right)\right). \tag{21}$$

By induction, we obtain that

$$\tilde{d}\left(\widetilde{x_{\tau_n}^{n}}, x_{\tau_{n+1}}^{n+1\sim}\right) \tilde{\leq} \psi\left(\tilde{d}\left(\widetilde{x_{\tau_{n-1}}^{n-1}}, x_{\tau_n}^{n\sim}\right)\right) \tilde{\leq} \psi^{2}\left(\tilde{d}\left(\widetilde{x_{\tau_{n-2}}^{n-2}}, x_{\tau_{n-1}}^{n-1\sim}\right)\right) \tilde{\leq} \cdots \tilde{\leq} \psi^{n}\left(\tilde{d}\left(\widetilde{x_{\tau_0}^{0}}, \widetilde{x_{\tau_1}^{1}}\right)\right). \tag{22}$$

By Lemma 4, we obtained that

$$\lim_{n\to\infty} \tilde{d}\left(\widetilde{x_{\tau_n}^{n}}, \widetilde{x_{\tau_{n+1}}^{n+1}}\right) = \bar{0}. \tag{23}$$

In what follows, we check whether the sequence $\{\widetilde{x_{\tau_n}^{n}}\}$ is Cauchy: for each $\tilde{\varepsilon}$, there is $n_0 \in \mathbb{N}$ such that if $n, k \geq n_0$, then

$$\tilde{d}\left(\widetilde{x_{\tau_{k_r}}^{k_r}}, x_{\tau_{n_r}}^{n_r\sim}\right) \tilde{<} \tilde{\varepsilon}. \tag{24}$$

Suppose, on the contrary, that the statement $(*)$ is false. Then, there exists $\varepsilon^{\sim} \tilde{>} \bar{0}$ such that, for any $r \in \mathbb{N}$, there are $n_r, k_r \in \mathbb{N}$ with $n_r > k_r \geq r$ satisfying that

(1) n_r is even and k_r is odd

(2) $\tilde{d}(\widetilde{x_{\tau_{k_r}}^{k_r}}, x_{\tau_{n_r}}^{n_r\sim}) \tilde{\geq} \tilde{\varepsilon}$

(3) n_r is the smallest even number such that condition (2) holds

By (1) and (2), we conclude that

$$\varepsilon^{\sim} \tilde{\leq} \tilde{d}\left(\widetilde{x_{\tau_{k_r}}^{k_r}}, x_{\tau_{n_r}}^{n_r\sim}\right) \tilde{\leq} \tilde{d}\left(\widetilde{x_{\tau_{k_r}}^{k_r}}, \widetilde{x_{\tau_{n_r-2}}^{n_r-2}}\right) + \tilde{d}\left(\widetilde{x_{\tau_{n_r-2}}^{n_r-2}}, \widetilde{x_{\tau_{n_r-1}}^{n_r-1}}\right) + \tilde{d}\left(\widetilde{x_{\tau_{n_r-1}}^{n_r-1}}, x_{\tau_{n_r}}^{n_r\sim}\right) \tilde{\leq} \tilde{\varepsilon} + \tilde{d}\left(\widetilde{x_{\tau_{n_r-2}}^{n_r-2}}, \widetilde{x_{\tau_{n_r-1}}^{n_r-1}}\right) + \tilde{d}\left(\widetilde{x_{\tau_{n_r-1}}^{n_r-1}}, \widetilde{x_{\tau_{n_r}}^{n_r}}\right). \tag{25}$$

Letting $r \to \infty$, we obtain that

$$\lim_{r\to\infty} \tilde{d}\left(\widetilde{x_{\tau_{k_r}}^{k_r}}, \widetilde{x_{\tau_{n_r}}^{n_r}}\right) = \tilde{\varepsilon}. \tag{26}$$

On the other hand,

$$\varepsilon^{\sim} \tilde{\leq} \tilde{d}\left(\widetilde{x_{\tau_{k_r-1}}^{k_r-1}}, x_{\tau_{n_r-1}}^{n_r-1\sim}\right) \tilde{\leq} \tilde{d}\left(\widetilde{x_{\tau_{k_r-1}}^{k_r-1}}, \widetilde{x_{\tau_{n_r-3}}^{n_r-3}}\right) + \tilde{d}\left(\widetilde{x_{\tau_{n_r-3}}^{n_r-3}}, \widetilde{x_{\tau_{n_r-2}}^{n_r-2}}\right) + \tilde{d}\left(\widetilde{x_{\tau_{n_r-2}}^{n_r-2}}, x_{\tau_{n_r-1}}^{n_r-1\sim}\right) \tilde{\leq} \tilde{\varepsilon} + \tilde{d}\left(\widetilde{x_{\tau_{n_r-3}}^{n_r-3}}, \widetilde{x_{\tau_{n_r-2}}^{n_r-2}}\right) + \tilde{d}\left(\widetilde{x_{\tau_{n_r-2}}^{n_r-2}}, \widetilde{x_{\tau_{n_r-1}}^{n_r-1}}\right). \tag{27}$$

Letting $r \to \infty$, we obtain that

$$\lim_{r\to\infty} \tilde{d}\left(\widetilde{x_{\tau_{k_r-1}}^{k_r-1}}, \widetilde{x_{\tau_{n_r-1}}^{n_r-1}}\right) = \tilde{\varepsilon}. \tag{28}$$

By the above arguments, we obtain that

$$\begin{aligned}\tilde{d}\left(\widetilde{x_{\tau_{k_r}}^{k_r}}, \widetilde{x_{\tau_{n_r}}^{n_r}}\right) &= \tilde{d}\left((f,\varphi)(f,\varphi), (f,\varphi)\left(x_{\tau_{n_r-1}}^{n_r-1\sim}\right)\right) \\ &\tilde{\leq} \psi\left(\max\left\{\tilde{d}\left(\widetilde{x_{\tau_{k_r-1}}^{k_r-1}}, \widetilde{x_{\tau_{n_r-1}}^{n_r-1}}\right), d \cdot \left(\widetilde{x_{\tau_{k_r-1}}^{k_r-1}}, (f,\varphi)\left(\widetilde{x_{\tau_{kr-1}}^{k_r-1}}\right)\right), d \cdot \left(\widetilde{x_{\tau_{n_r-1}}^{n_r-1}}, (f,\varphi)\left(x_{\tau_{n_r-1}}^{n_r-1\sim}\right)\right)\right\}\right) \\ &\tilde{\leq} \psi\left(\max\left\{\tilde{d}\left(\widetilde{x_{\tau_{k_r-1}}^{k_r-1}}, \widetilde{x_{\tau_{n_r-1}}^{n_r-1}}\right), \tilde{d}\left(\widetilde{x_{\tau_{k_r-1}}^{k_r-1}}, \widetilde{x_{\tau_{kr}}^{k_r}}\right), d \cdot \left(\widetilde{x_{\tau_{n_r-1}}^{n_r-1}}, \widetilde{x_{\tau_{n_r}}^{n_r}}\right)\right\}\right).\end{aligned} \tag{29}$$

Taking $\lim_{r\to\infty}\inf$, we get $\varepsilon^{\sim} \tilde{<} \tilde{\varepsilon}$. This implies a contradiction. So the sequence $\{\widetilde{x_{\tau_n}^{n}}\}$ is Cauchy.

Since $\mathscr{M}$ is complete, there exists $\widetilde{x_\tau^*} \in \tilde{X}$ such that

$$\widetilde{x_{\tau_n}^{n}} \longrightarrow \widetilde{x_\tau^*} \text{ as } n \longrightarrow \infty, \tag{30}$$

that is,

$$\tilde{d}\left(\widetilde{x_{\tau_n}^{n}}, \widetilde{x_\tau^*}\right) \longrightarrow \bar{0} \text{ as } n \longrightarrow \infty. \tag{31}$$

Notice also that

$$\begin{aligned}\tilde{d}\left((f,\varphi)\left(\widetilde{x_\tau^*}\right), x_\tau^{*\sim}\right) &\tilde{\leq} \tilde{d}\left((f,\varphi)\left(\widetilde{x_{\tau_n}^{n}}\right), (f,\varphi)\left(\widetilde{x_\tau^*}\right)\right) + \tilde{d}\left((f,\varphi)\left(\widetilde{x_{\tau_n}^{n}}\right), x_\tau^{*\sim}\right) \\ &\tilde{<} \psi\left(\max\left\{\tilde{d}\left(\widetilde{x_{\tau_n}^{n}}, \widetilde{x_\tau^*}\right), \tilde{d}\left(\widetilde{x_{\tau_n}^{n}}, (f,\varphi)\left(\widetilde{x_{\tau_n}^{n}}\right)\right), \tilde{d}\left(\widetilde{x_\tau^*}, (f,\varphi) \cdot \left(\widetilde{x_\tau^*}\right)\right)\right\}\right) + \tilde{d}\left(\widetilde{x_{\tau_{n+1}}^{n+1}}, x_\tau^{*\sim}\right) \\ &\tilde{<} \psi\left(\max\left\{\tilde{d}\left(\widetilde{x_{\tau_n}^{n}}, \widetilde{x_\tau^*}\right), \tilde{d}\left(\widetilde{x_{\tau_n}^{n}}, \widetilde{x_{\tau_{n+1}}^{n+1}}\right), \tilde{d}\left(\widetilde{x_\tau^*}, (f,\varphi)\left(\widetilde{x_\tau^*}\right)\right)\right\}\right) + \tilde{d}\left(\widetilde{x_{\tau_{n+1}}^{n+1}}, \widetilde{x_\tau^*}\right).\end{aligned} \tag{32}$$

Taking $n \to \infty$, we get that

$$\tilde{d}\left((f,\varphi)\left(\widetilde{x_\tau^*}\right), x_\tau^{*\sim}\right) \tilde{\leq} \psi\left(\max\left\{\bar{0}, \bar{0}, ,\tilde{d}\left(\widetilde{x_\tau^*}, (f,\varphi)\left(\widetilde{x_\tau^*}\right)\right)\right\}\right) + \bar{0} \tilde{<} \tilde{d}\left((f,\varphi)\left(\widetilde{x_\tau^*}\right), \widetilde{x_\tau^*}\right), \tag{33}$$

and this is a contradiction unless $\tilde{d}((f,\varphi)(\widetilde{x_\tau^*}), \widetilde{x_\tau^*}) = \bar{0}$. Thus, $(f,\varphi)(\tilde{x}_\tau^*) = \tilde{x}_\lambda^*$ completes the proof.

Example 8. Consider Example 6. All hypotheses of Theorem 7 are fulfilled. Thus, we can conclude that $\bar{0}_0$ is a fixed soft point of the soft inf-comparable Meir-Keeler contraction (f,φ).

3. Observation on the Soft Comparable Meir-Keeler Contractions

We start this section by recalling the Meir-Keeler contraction in the standard setting.

Definition 9. (see). A self-mapping g on a metric space (X,d) is called a Meir-Keeler contraction if the following is fulfilled: for any $\eta>0$, there is $\gamma>0$ such that

$$\eta \le d(x,y) < \eta+\gamma \Rightarrow d(gx,gy) < \eta \quad \text{for all } x,y\in X. \tag{34}$$

The mapping $\phi:\mathbb{R}^+(\mathscr{P})\times\mathbb{R}^+(\mathscr{P})\times\mathbb{R}^+(\mathscr{P})\to\mathbb{R}^+(\mathscr{P})$ is said to be soft comparable, if the following two axioms are fulfilled:

(ϕ_1) ϕ is a soft increasing and soft continuous function in each coordinate

(ϕ_2) for $\tilde{\omega}\in\mathbb{R}^+(\mathscr{P})\setminus\{\bar{0}\}$, $\phi(\tilde{\omega},\tilde{\omega},\omega\tilde{\ })\tilde{<}\tilde{\omega}$, and $\phi(\widetilde{\omega_1},\widetilde{\omega_2},\widetilde{\omega_3})=\bar{0}$ if and only if $\widetilde{\omega_1}=\widetilde{\omega_2}=\widetilde{\omega_3}=\bar{0}$

Now, we introduce the notion of soft comparable Meir-Keeler contraction.

Definition 10. Let $\phi:\mathbb{R}^+(\mathscr{P})\times\mathbb{R}^+(\mathscr{P})\times\mathbb{R}^+(\mathscr{P})\to\mathbb{R}^+(\mathscr{P})$ be soft comparable. A self-soft-mapping (f,φ) on a soft metric space $\mathscr{M}$ is called a soft comparable Meir-Keeler contraction if for each soft real number $\eta\tilde{\ }\tilde{>}\bar{0}$, there is $\gamma\tilde{\ }\tilde{>}\bar{0}$ such that

$$\begin{aligned}&\eta\tilde{\ }\tilde{\le}\phi\left(\tilde{d}\left(\widetilde{x_p},\widetilde{y_\tau}\right),\tilde{d}\left(\widetilde{x_p},(f,\varphi)\left(\widetilde{x_p}\right)\right),\tilde{d}\left(\widetilde{y_\tau},(f,\varphi)\left(y_\tau\tilde{\ }\right)\right)\right)\\&\quad\tilde{<}\tilde{\eta}+\tilde{\gamma}\Rightarrow\tilde{d}\left((f,\varphi)\left(\widetilde{x_p}\right),(f,\varphi)\left(y_\tau\tilde{\ }\right)\right)\tilde{<}\tilde{\eta},\end{aligned}\tag{35}$$

for each soft points $\widetilde{x_\lambda},\widetilde{y_\mu}\in\mathcal{SP}(\tilde{X})$.

Example 11. Set $\mathscr{R}=(\tilde{\mathbb{R}},\tilde{d},\mathscr{P})$ where the soft metric is expressed as

$$\begin{aligned}&d_\varphi(p,\tau)=\max\{p,\tau\},\ d(x,y)=|x-y|,\\&\tilde{d}\left(\widetilde{x_p},\widetilde{y_\tau}\right)=d_\varphi(p,\tau)+d(x,y),\end{aligned}\tag{36}$$

with $\mathscr{P}=[0,\infty)$, $\varphi(t)=(1/3)t$ for $t\in[0,\infty)$.

Let $\phi:\mathbb{R}^+(\mathscr{P})\times\mathbb{R}^+(\mathscr{P})\times\mathbb{R}^+(\mathscr{P})\to\mathbb{R}^+(\mathscr{P})$ be denoted by

$$\phi(\widetilde{\omega_1},\widetilde{\omega_2},\widetilde{\omega_3})=\frac{3}{4}\cdot\max\{\widetilde{\omega_1},\widetilde{\omega_2},\widetilde{\omega_3}\},\tag{37}$$

where

$$\begin{aligned}&\widetilde{\omega_1}=\tilde{d}\left(\widetilde{x_p},\widetilde{y_\tau}\right),\\&\widetilde{\omega_2}=\tilde{d}\left(\widetilde{x_p},(f,\varphi)\left(\widetilde{x_p}\right)\right),\\&\widetilde{\omega_3}=\tilde{d}\left(\widetilde{y_\tau},(f,\varphi)\left(\widetilde{y_\tau}\right)\right).\end{aligned}\tag{38}$$

Let $f(x)=(1/2)x$. Then,

$$\begin{aligned}\tilde{d}\left((f,\varphi)\left(\widetilde{x_p}\right),(f,\varphi)\left(\widetilde{y_\tau}\right)\right)&=\tilde{d}\left(\widetilde{\frac{1}{2}x_{(1/3)p}},\widetilde{\frac{1}{2}y_{(1/3)\tau}}\right)\\&=\frac{1}{3}\max\{p,\tau\}+\frac{1}{2}|x-y|,\end{aligned}$$

$$\begin{aligned}&\tilde{\omega}_1=\tilde{d}\left(\widetilde{x_p},\widetilde{y_\tau}\right)=\max\{p,\tau\}+|x-y|,\\&\widetilde{\omega_2}=\tilde{d}\left(\widetilde{x_p},\widetilde{\frac{1}{2}x_{(1/3)p}}\right)=\max\left\{p,\frac{1}{3}p\right\}+\left|x-\frac{1}{2}x\right|=p+\frac{1}{2}|x|,\\&\widetilde{\omega_3}=\tilde{d}\left(\widetilde{y_\tau},\widetilde{\frac{1}{2}y_{(1/3)\tau}}\right)=\max\left\{\tau,\frac{1}{3}\tau\right\}+\left|y-\frac{1}{2}y\right|=\tau+\frac{1}{2}|y|.\end{aligned}\tag{39}$$

So we can conclude that

$$\begin{aligned}&\phi\left(\widetilde{\omega_1},\widetilde{\omega_2},\widetilde{\omega_3},\tilde{t}_4\right)\\&\quad=\frac{3}{4}\cdot\max\left\{\widetilde{\omega_1},\widetilde{\omega_2},\omega_3\tilde{\ }\right\}\tilde{\ge}\frac{3}{4}\widetilde{\omega_1}=\frac{3}{4}\left(\max\{p,\tau\}+|x-y|\right).\end{aligned}\tag{40}$$

Consequently, a soft mapping (f,φ) forms a soft comparable Meir-Keeler contraction on $\mathscr{R}$.

We establish the following fixed point results for the soft comparable Meir-Keeler contraction.

Theorem 12. *Let $\mathscr{M}$ be a complete soft metric space, and let $\phi:\mathbb{R}^+(\mathscr{P})\times\mathbb{R}^+(\mathscr{P})\times\mathbb{R}^+(\mathscr{P})\to\mathbb{R}^+(\mathscr{P})$ be a soft comparable. Let (f,φ): $\mathscr{M}\to\mathscr{M}$ be a soft comparable Meir-Keeler contraction on $\mathscr{M}$. Then, (f,φ) possesses a soft fixed point.*

Proof. Let $\widetilde{x^0_{\tau_0}}\in\mathcal{SP}(\tilde{X})$ be given. For each $n\in\mathbb{N}\cup\{0\}$, we put

$$\widetilde{x^{n+1}_{\tau_{n+1}}}=\left((f,\varphi)\left(\widetilde{x^n_{\tau_n}}\right)\right)=\left(f^{n+1}\left(\widetilde{x^0_{\tau_0}}\right)\right)_{\varphi^{n+1}(\tau_0)}.\tag{41}$$

So, for each $n\in\mathbb{N}\cup\{0\}$ we have

$$\begin{aligned}\tilde{d}\left(\widetilde{x^n_{\tau_n}},\widetilde{x^{n+1}_{\tau n+1}}\right)&=\tilde{d}\left((f,\varphi)\left(\widetilde{x^{n-1}_{\tau_{n-1}}}\right),(f,\varphi)\left(x^n_{\tau_n}\tilde{\ }\right)\right)\\&\tilde{\le}\phi\left(\tilde{d}\left(\widetilde{x^{n-1}_{\tau_{n-1}}},\widetilde{x^n_{\tau_n}}\right),\tilde{d}\left(\widetilde{x^{n-1}_{\tau_{n-1}}},(f,\varphi)\left(\widetilde{x^{n-1}_{\tau_{n-1}}}\right)\right),\tilde{d}\left(\widetilde{x^n_{\tau_n}},(f,\varphi)\left(\widetilde{x^n_{\tau_n}}\right)\right)\right)\\&=\phi\left(\tilde{d}\left(\widetilde{x^{n-1}_{\tau_{n-1}}},\widetilde{x^n_{\tau_n}}\right),\tilde{d}\left(\widetilde{x^{n-1}_{\tau_{n-1}}},\widetilde{x^n_{\tau_n}}\right),\tilde{d}\left(\widetilde{x^n_{\tau_n}},\widetilde{x^{n+1}_{\tau_{n+1}}}\right)\right).\end{aligned}\tag{42}$$

If $\tilde{d}(\widetilde{x^{k-1}_{\tau_{k-1}}}, \widetilde{x^{k}_{\tau_k}}) \tilde{<} \tilde{d}(\widetilde{x^{k}_{\tau_k}}, \widetilde{x^{k+1}_{\tau_{k+1}}})$ for some $k \in \mathbb{N}$, then by the above inequality and the conditions of the function ϕ, we have

$$\tilde{d}\left(\widetilde{x^{k}_{\tau_n}}, \widetilde{x^{k+1}_{\tau_{k+1}}}\right) \tilde{\le} \phi\left(\tilde{d}\left(\widetilde{x^{k-1}_{\tau_{k-1}}}, \widetilde{x^{k}_{\tau_k}}\right), \tilde{d}\left(\widetilde{x^{k-1}_{\tau_{k-1}}}, \widetilde{x^{k}_{\tau_k}}\right), \tilde{d}\left(\tilde{k}^{n}_{\tau_k}, \widetilde{x^{k+1}_{\tau_{k+1}}}\right)\right) \tilde{<} \tilde{d}\left(\widetilde{x^{k}_{\tau_k}}, \widetilde{x^{k+1}_{\tau_{k+1}}}\right), \tag{43}$$

which implies a contradiction. Hence, for each $n \in \mathbb{N}$, we find

$$\tilde{d}\left(\widetilde{x^{n}_{\tau_n}}, \widetilde{x^{n+1}_{\tau_{n+1}}}\right) \tilde{<} \tilde{d}\left(\widetilde{x^{k-1}_{\tau_{k-1}}}, \widetilde{x^{k}_{\tau_k}}\right). \tag{44}$$

Thus, the sequence $\{\tilde{d}(\widetilde{x^{n}_{\tau_n}}, \widetilde{x^{n+1}_{\tau_{n+1}}})\}$ is decreasing and converges to a soft real number, say $\gamma \tilde{\ge} \bar{0}$. In other words, $\tilde{d}(\widetilde{x^{n}_{\tau_n}}, \widetilde{x^{n+1}_{\tau_{n+1}}}) \to \tilde{\gamma}$, as $n \to \infty$.

Notice that $\tilde{\gamma} = \inf \{\tilde{d}(\widetilde{x^{n}_{\tau_n}}, \widetilde{x^{n+1}_{\tau_{n+1}}}) : n \in \mathbb{N} \cup \{0\}\}$. We claim that $\tilde{\gamma} = \bar{0}$. Suppose, on the contrary, that $\gamma \tilde{>} \bar{0}$. Since (f, φ) is a soft comparable Meir-Keeler contraction, corresponding to $\tilde{\gamma}$, there exists $\eta \tilde{>} \bar{0}$ and $k \in \mathbb{N}$ such that

$$\begin{aligned} &\gamma \tilde{\le} \phi\left(\tilde{d}\left(\widetilde{x^{k}_{\tau_k}}, \widetilde{x^{k+1}_{\tau_{k+1}}}\right), \tilde{d}\left(\widetilde{x^{k}_{\tau_k}}, \widetilde{x^{k+1}_{\tau_{k+1}}}\right), \tilde{d}\left(\widetilde{x^{k+1}_{\tau_{k+1}}}, \widetilde{x^{k+2}_{\tau_{k+2}}}\right)\right) \tilde{<} \tilde{\gamma} + \tilde{\eta} \\ &\Rightarrow \tilde{d}\left(\widetilde{x^{k+1}_{\tau_{k+1}}}, \widetilde{x^{k+2}_{\tau_{k+2}}}\right) = \tilde{d}\left((f, \varphi)\left(\widetilde{x^{k}_{\tau_k}}\right), (f, \varphi)\left(\widetilde{x^{k+1}_{\tau_{k+1}}}\right)\right) \tilde{<} \tilde{\gamma}. \end{aligned} \tag{45}$$

This is a contradiction since $\tilde{\gamma} = \inf \{\tilde{d}(\widetilde{x^{n}_{\tau_n}}, \widetilde{x^{n+1}_{\tau_{n+1}}}) : n \in \mathbb{N} \cup \{0\}\}$. Thus, we obtain that $\tilde{d}(\widetilde{x^{n}_{\tau_n}}, \widetilde{x^{n+1}_{\tau_{n+1}}}) \to \bar{0}$, as $n \to \infty$.

As a next step, we check whether the sequence $\{\widetilde{x^{n}_{\tau_n}}\}$ is Cauchy in $\mathscr{M}$. Suppose, on the contrary, it is not. Thus, there exists a soft real number $\varepsilon \tilde{>} \bar{0}$ such that for any $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $n_k > m_k \ge k$ satisfying

$$\tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right) \tilde{\ge} \tilde{\varepsilon}. \tag{46}$$

Further, corresponding to $m_k \ge k$, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k \ge k$ and $\tilde{d}(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}) \tilde{\ge} \tilde{\varepsilon}$. Therefore,

$$\tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k-2}_{\tau_{n_k-2}}}\right) \tilde{<} \tilde{\varepsilon}. \tag{47}$$

So, we derive that

$$\begin{aligned} \varepsilon \tilde{\le} \tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right) &\tilde{\le} \tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k-2}_{\tau_{n_k-2}}}\right) + \tilde{d}\left(\widetilde{x^{n_k-2}_{\tau_{n_k-2}}}, \widetilde{x^{n_k-1}_{\tau_{n_k-1}}}\right) \\ &+ \tilde{d}\left(\widetilde{x^{n_k-1}_{\tau_{n_k-1}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right) \tilde{<} \tilde{\varepsilon} + \tilde{d}\left(\widetilde{x^{n_k-2}_{\tau_{n_k-2}}}, \widetilde{x^{n_k-1}_{\tau_{n_k-1}}}\right) + \tilde{d}\left(\widetilde{x^{n_k-1}_{\tau_{n_k-1}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right), \end{aligned} \tag{48}$$

for all $k \in \mathbb{N}$. As $k \to \infty$, the inequality above yields that

$$\lim_{k\to\infty} \tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right) = \tilde{\varepsilon}. \tag{49}$$

On the other hand, we have

$$\begin{aligned} \varepsilon \tilde{\le} \tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right) &\tilde{\le} \tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{m_k+1}_{\tau_{m_k+1}}}\right) + \tilde{d}\left(\widetilde{x^{m_k+1}_{\tau_{m_k+1}}}, \widetilde{x^{n_k+1}_{\tau_{n_k+1}}}\right) \\ &+ \tilde{d}\left(\widetilde{x^{n_k+1}_{\tau_{n_k+1}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right) \tilde{\le} \tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{m_k+1}_{\tau_{m_k+1}}}\right) + \tilde{d}\left(\widetilde{x^{m_k+1}_{\tau_{m_k+1}}}, \widetilde{x^{m_k}_{\tau_{m_k}}}\right) \\ &+ \tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right) + \tilde{d}\left(\widetilde{x^{n_k}_{\tau_{n_k}}}, \widetilde{x^{n_k+1}_{\tau_{n_k+1}}}\right) + \tilde{d}\left(\widetilde{x^{n_k+1}_{\tau_{n_k+1}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right). \end{aligned} \tag{50}$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} \tilde{d}\left(\widetilde{x^{m_k+1}_{\lambda_{m_k+1}}}, \widetilde{x^{n_k+1}_{\lambda_{n_k+1}}}\right) = \tilde{\varepsilon}. \tag{51}$$

Since (f, φ) is a soft comparable Meir-Keeler contraction, we have

$$\begin{aligned} \tilde{d}\left(\widetilde{x^{m_k+1}_{\tau_{m_k+1}}}, \widetilde{x^{n_k+1}_{\tau_{n_k+1}}}\right) &= \tilde{d}\left((f, \varphi)\left(\widetilde{x^{m_k}_{\tau_{m_k}}}\right), (f, \varphi)\left(\widetilde{x^{n_k}_{\tau_{n_k}}}\right)\right) \\ &\tilde{<} \phi\left(\tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k}_{\tau_{n_k}}}\right), \tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{m_k+1}_{\tau_{m_k+1}}}\right), \tilde{d}\left(\widetilde{x^{n_k}_{\tau_{n_k}}}, \widetilde{x^{n_k+1}_{\tau_{n_k+1}}}\right)\right). \end{aligned} \tag{52}$$

Moreover, since

$$\begin{aligned} \tilde{d}\left(\widetilde{x^{m_k}_{\tau_{m_k}}}, \widetilde{x^{n_k+1}_{\tau_{n_k+1}}}\right) &\tilde{\le} \tilde{d}\left(\tilde{x}^{m_k}_{\tau_{m_k}}, \widetilde{x^{m_k+1}_{\tau_{m_k+1}}}\right) + \tilde{d}\left(\widetilde{x^{m_k+1}_{\tau_{m_k+1}}}, \widetilde{x^{n_k+1}_{\tau_{n_k+1}}}\right), \tilde{d}\left(\widetilde{x^{n_k}_{\tau_{n_k}}}, \widetilde{x^{m_k+1}_{\tau_{m_k+1}}}\right) \\ &\tilde{\le} \tilde{d}\left(\widetilde{x^{n_k}_{\tau_{n_k}}}, \widetilde{x^{n_k+1}_{\tau_{n_k+1}}}\right) + \tilde{d}\left(\widetilde{x^{n_k+1}_{\tau_{n_k+1}}}, \widetilde{x^{m_k+1}_{\tau_{m_k+1}}}\right). \end{aligned} \tag{53}$$

Taking $k \to \infty$ in the above inequalities, we get that

$$\varepsilon \tilde{\le} \phi(\tilde{\varepsilon}, \tilde{\varepsilon}, \varepsilon) \tilde{<} \tilde{\varepsilon}, \tag{54}$$

and this is a contradiction. Thus, the sequence $\{\widetilde{x^{n}_{\lambda_n}}\}$ is Cauchy.

Keeping the completeness of $\mathscr{M}$ in mind, one can find $x^*_\tau \in \tilde{X}$ such that

$$\widetilde{x^{n}_{\tau_n}} \to x^*_\tau \text{ as } n \to \infty, \tag{55}$$

that is,

$$\tilde{d}\left(\widetilde{x^{n}_{\tau_n}}, x^*_\tau\right) \to \bar{0} \text{ as } n \to \infty. \tag{56}$$

And, we also have

$$\begin{aligned}\tilde{d}\left((f,\varphi)\left(\widetilde{x_\tau^*}\right),x_\tau^{*\sim}\right)&\tilde{\leq}\tilde{d}\left((f,\varphi)\left(\widetilde{x_{\tau_n}^n}\right),(f,\varphi)\left(\widetilde{x_\tau^*}\right)\right)\\&+\tilde{d}\left((f,\varphi)\left(\widetilde{x_{\tau_n}^n}\right),x_\tau^{*\sim}\right)\tilde{<}\phi\left(\tilde{d}\left(\widetilde{x_{\tau_n}^n},\widetilde{x_\tau^*}\right),d\right.\\&\cdot\left(\widetilde{x_{\tau_n}^n},(f,\varphi)\left(\widetilde{x_{\tau_n}^n}\right)\right),\left.\tilde{d}\left(\widetilde{x_\tau^*},(f,\varphi)\left(\widetilde{x_\tau^*}\right)\right)\right)\\&+\tilde{d}\left(\widetilde{x_{\tau_{n+1}}^{n+1}},x_\tau^{*\sim}\right)\tilde{<}\phi\left(\tilde{d}\left(\widetilde{x_{\tau_n}^n},\widetilde{x_\tau^*}\right),\tilde{d}\left(\widetilde{x_{\tau_n}^n},\widetilde{x_{\tau_{n+1}}^{n+1}}\right),d\right.\\&\cdot\left.\left(\widetilde{x_\tau^*},(f,\varphi)\left(\widetilde{x_\tau^*}\right)\right)\right)+\tilde{d}\left(\widetilde{x_{\tau_{n+1}}^{n+1}},\widetilde{x_\tau^*}\right).\end{aligned}\tag{57}$$

Taking $k\to\infty$ in the inequality above,

$$\begin{aligned}\tilde{d}\left((f,\varphi)\left(\widetilde{x_\tau^*}\right),x_\tau^{*\sim}\right)&\tilde{\leq}\phi\left(\bar{0},\bar{0},\tilde{d}(f,\varphi)\left(\widetilde{x_\tau^*}\right)\right)+\bar{0}\\&\tilde{\leq}\phi\left(\tilde{d}\left((f,\varphi)\left(\widetilde{x_\tau^*}\right),\widetilde{x_\tau^*}\right),\tilde{d}\left((f,\varphi)\left(\widetilde{x_\tau^*}\right),\widetilde{x_\tau^*}\right),d\right.\\&\cdot\left.\left((f,\varphi)\left(\widetilde{x_\tau^*}\right),x_\tau^{*\sim}\right)\right)\tilde{<}\tilde{d}\left((f,\varphi)\left(\widetilde{x_\tau^*}\right),\widetilde{x_\tau^*}\right),\end{aligned}\tag{58}$$

a contradiction unless $\tilde{d}((f,\varphi)(\widetilde{x_\tau^*}),\widetilde{x_\tau^*})=\bar{0}$. Thus, $(f,\varphi)(\tilde{x}_\tau^*)=\tilde{x}_\tau^*$ which completes the proof.

Example 13. Consider Example 11. One can easily check all hypotheses of Theorem 12. Consequently, we conclude that $\bar{0}_0$ is a fixed soft point of the soft comparable Meir-Keeler contraction (f,φ).

We next introduce the notion of soft generalized Meir-Keeler contraction, as follows:

Definition 14. Let $(\tilde{X},\tilde{d},\mathscr{P})$ be a soft metric space. A mapping (f,φ): $(\tilde{X},\tilde{d},\mathscr{P})\to(\tilde{X},\tilde{d},\mathscr{P})$ is called a soft generalized Meir-Keeler contraction if for any soft real number $\eta\tilde{>}\bar{0}$, there exists $\gamma\tilde{>}\bar{0}$ such that for each soft point $\widetilde{x_p},\widetilde{y_\tau}\in\mathcal{SP}(\tilde{X})$,

$$\begin{aligned}\eta\tilde{\leq}\max&\left\{\tilde{d}(\widetilde{x_p},(f,\varphi)(\widetilde{x_p})),\tilde{d}(\widetilde{y_\tau},(f,\varphi)(\widetilde{y_\tau})),\tilde{d}\left(\widetilde{x_p},(f,\varphi)\left(y_\tau{}^\sim\right)\right)\right\}\\&\tilde{<}\tilde{\eta}+\tilde{\gamma}\Rightarrow\tilde{d}((f,\varphi)(\widetilde{x_p}),(f,\varphi)(y_\tau{}^\sim))\tilde{<}\tilde{\eta}.\end{aligned}\tag{59}$$

It is clear that the soft generalized Meir-Keeler contraction is a comparable soft Meir-Keeler contraction; we can easily conclude the following corollary.

Corollary 15. *A soft generalized Meir-Keeler contraction on(f,φ)which is a complete soft metric space$\mathscr{M}$possesses a fixed soft point.*

Data Availability

No data is used!

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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Research Article

Some Fixed-Point Results via Mix-Type Contractive Condition

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We consider a fixed-point problem for mappings involving a rational type and almost type contraction on complete metric spaces. To do this, we are using F-contraction and (H, φ)-contraction. We also present an example to illustrate our result.

1. Introduction

The beginning of metrical fixed point theory is related to Banach's Contraction Principle, presented in 1922 [1], which says that any contraction self-map on M has a unique fixed point whenever (M, d) is complete. Afterwards, the crucial role of the principle in existence and uniqueness problems arising in mathematics has been realized which fact directed the researchers to extend and generalize the principle in many ways (see [2–7]).

In the studies of generalizations and modifications of contractions, an interesting generalization was given by Wardowski [8] using a new concept F-contraction. Then, many authors gave some results using this concept in different type metric spaces. One of them is given by Jleli et al. [9] by introducing a family $\mathscr{H}$ of functions $H : [0,\infty)^3 \to 0,\infty)$ with the certain assumption. Also, you can find this type generalizations in [10–12].

In this paper, we consider a fixed-point problem for mappings involving a rational type contraction and almost contraction. Firstly, we recall some basic on the notions of F-contraction and (H, ϕ)-contraction.

2. Preliminaries

Let $\mathscr{F}$ be the family of all functions $F : \mathbb{R}^+ = [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

$(F1)$ F is nondecreasing;

$(F2)$ for every sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to+\infty}\alpha_n = 0$ if and only if $\lim_{n\to+\infty}F(\alpha_n) = -\infty$;

$(F3)$ there exists $k \in]0, 1[$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha) = 0$. ([8])

Definition 1. (see [8]). Let (M, d) be a metric space and $Y : M \to M$ be a mapping. Given $F \in \mathscr{F}$, we say that Y is F-contraction, if there exists $\tau > 0$ such that

$$\mu, \gamma \in M, d(Y\mu, Y\gamma) > 0 \Rightarrow \tau + F(d(Y\mu, Y\gamma)) \le F(d(\mu, \gamma)). \tag{1}$$

Taking in (1) different functions $F \in \mathscr{F}$, one gets a variety of F-contractions, and some of them being already known in the literature. You can see this contractions in [8]. In addition, Wardowski concluded that every F-contraction Y is a contractive mapping, i.e.,

$$d(Y\mu, Y\gamma) < d(\mu, \gamma), \text{forall}\mu, \gamma \in M, Y\mu \ne Y\gamma. \tag{2}$$

Thus, every F-contraction is a continuous mapping.

Theorem 2. (see [8]). *Let (M, d) be a complete metric space (C.M.S) and let $Y : M \to M$ be an F-contraction. Then, Y has a unique fixed point in M.*

In [9], Jleli et al. introduced a family $\mathscr{H}$ of functions $H : [0,+\infty^3 \to 0,+\infty$ satisfying the following conditions:

(H1) $\max\{\alpha, \beta\} \le H(\alpha, \beta, \gamma)$ *for all* $\alpha, \beta, \gamma \in 0, +\infty$;

(H2) $H(0, 0, 0) = 0$;

(H3) H is continuous.

Some examples of functions belonging to $\mathscr{H}$ are given as follows:

(i) $H(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$ *for all* $\alpha, \beta, \gamma \in 0, +\infty$

(ii) $H(\alpha, \beta, \gamma) = \max\{\alpha, \beta\} + \gamma$ *for all* $\alpha, \beta, \gamma \in 0, +\infty$

(iii) $H(\alpha, \beta, \gamma) = \alpha + \beta + \alpha\beta + \gamma$ *for all* $\alpha, \beta, \gamma \in 0, +\infty$

Using a function $H \in \mathscr{H}$*, the authors of [9] introduced the following notion of* (H, ϕ)*-contraction.*

Definition 3. (see [9]). Let (M, d) be a metric space, $\phi : M \to 0, +\infty$ be a given function, and $H \in \mathscr{H}$. Then, $Y : M \to M$ is called a (H, ϕ)-contraction with respect to the metric d if and only if

$$H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)) \leq kH(d(\mu, \gamma), \phi(\mu), \phi(\gamma)) \text{ for all } \mu, \gamma \in M, \tag{3}$$

for some constant $k \in]0, 1[$.

Now, we set

$$Z_\phi \coloneqq \{\mu \in M : \phi(\mu) = 0\}, \quad F_Y \coloneqq \{\mu \in M : Y\mu = \mu\}. \tag{4}$$

Furthermore, we say that Y is a ϕ-Picard operator if and only if the following condition holds

$$F_Y \cap Z_\phi = \{\varsigma\} \text{ and } Y^n\mu \to \varsigma, \text{ as } n \to +\infty, \text{for each } \mu \in M. \tag{5}$$

Theorem 4. (see [9]). *Let* (M, d) *be a C.M.S,* $\phi : M \to 0, +\infty$ *be a given function and* $H \in \mathscr{H}$*. Suppose that the following conditions hold*

(A1) ϕ *is lower semicontinuous (l.s.c.);*

(A2) $Y : M \to M$ *is a* (H, ϕ)*-contraction with respect to the metric* d*.*

Then,

$$F_Y \subset Z_\phi; \tag{6}$$

(i) Y *is a* ϕ*-Picard operator*

(ii) *For all* $\mu \in M$ *and for all* $n \in \mathbb{N}$*, we have*

$$d(Y^n\mu, \varsigma) \leq \frac{k^n}{1-k} H(d(Y\mu, \mu), \phi(Y\mu), \phi(\mu)), \tag{7}$$

where $\{\varsigma\} = F_Y \cap Z_\phi = F_Y$*.*

Recently, Vetro ([13]) generalized Theorem 4 by using F -H-contraction.

Definition 5. (see [13]). Let (M, d) be a metric space and let $Y : M \to M$ be a mapping. The mapping Y is called an F -H-contraction if there exists $F \in \mathscr{F}$, $H \in \mathscr{H}$, a real number, $\tau > 0$ and $\phi : M \to 0, +\infty)$ s.t.

$$\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma))) \leq F(H(d(\mu, \gamma), \phi(\mu), \phi(\gamma))), \tag{8}$$

for all $\mu, \gamma \in M$ with $H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)) > 0$.

We remark that every F-contraction is an F-H-contraction such that $H \in \mathscr{H}$ defined by $H(x, y, z) = x + y + z$ for all $x, y, z \in 0, +\infty$ and $\phi : M \to 0, +\infty$ defined by $\phi(\mu) = 0$ for all $\mu \in M$.

Lemma 6. (see [13]). *Let* (M, d) *be a metric space and let* $Y : M \to M$ *be an F - H-contraction with respect to the functions* $F \in \mathscr{F}$*,* $H \in \mathscr{H}$*,* $\phi : M \to 0, +\infty$*, and the real number* $\tau > 0$*. If* $\{\mu_n\}$ *is a sequence of Picard starting at* $\mu_0 \in M$*, then*

$$\lim_{n\to+\infty} H(d(\mu_{n-1}, \mu_n), \phi(\mu_{n-1}), \phi(\mu_n)) = 0, \tag{9}$$

and hence

$$\lim_{n\to+\infty} d(\mu_{n-1}, \mu_n) = 0 \text{ and } \lim_{n\to+\infty} \phi(\mu_n) = 0. \tag{10}$$

Theorem 7. (see [13]). *Let* (M, d) *be a C.M.S and* $Y : M \to M$ *be an F - H-contraction with respect to the functions* $F \in \mathscr{F}$*,* $H \in \mathscr{H}$*, the real number* $\tau > 0$*, and a l.s.c. function* $\phi : M \to 0, +\infty$ *such that (8) holds; that is,*

$$\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma))) \leq F(H(d(\mu, \gamma), \phi(\mu), \phi(\gamma))), \tag{11}$$

for all $\mu, \gamma \in M$ *with* $H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)) > 0$*. Then,* Y *has a unique fixed point* ς *such that* $\phi(\varsigma) = 0$*.*

Theorem 8. (see [13]). *Let* (M, d) *be a C.M.S and let* $Y : M \to M$ *be a mapping. Assume that there exists a continuous function F that satisfies the conditions* (F_1) *and* (F_2)*, a function* $H \in \mathscr{H}$*, a real number* $\tau > 0$*, and a l.s.c. function* $\phi : M \to 0, +\infty$ *such that (8) holds; that is,*

$$\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma))) \leq F(H(d(\mu, \gamma), \phi(\mu), \phi(\gamma))), \tag{12}$$

for all $\mu, \gamma \in M$ *with* $H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)) > 0$*. Then,* Y *has a unique fixed point* ς *such that* $\phi(\varsigma) = 0$*.*

3. Main Results

We first introduce the rational type F-H-contraction.

Definition 9. Let (M, d) be a metric space and $Y : M \to M$ be a mapping. Y is called a rational type F - H-contraction if there exists $F \in \mathscr{F}$, $H \in \mathscr{H}$, a real number $\tau > 0$, and $\phi : M \to 0, +\infty$ *s.t.*

$$\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma))) \leq F(H(M(\mu, \gamma), \phi(\mu), \phi(\gamma))), \tag{13}$$

for all $\mu, \gamma \in M$ with $H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)) > 0$ where

$$M(\mu,\gamma) = \max\left\{d(\mu,\gamma), \frac{d(\mu, Y\mu)[1+d(\gamma, Y\gamma))]}{1+d(Y\mu, Y\gamma)}\right\}. \quad (14)$$

Lemma 10. *Let (M, d) be a metric space and $Y : M \to M$ be a rational type F - H-contraction with respect to the functions $F \in \mathscr{F}$, $H \in \mathscr{H}$, $\phi : M \to 0, +\infty$, and the real number $\tau > 0$. If $\{\mu_n\}$ is a sequence of Picard starting at $\mu_0 \in M$, then*

$$\lim_{n\to+\infty} H(d(\mu_{n-1}, \mu_n), \phi(\mu_{n-1}), \phi(\mu_n)) = 0, \quad (15)$$

and hence

$$\lim_{n\to+\infty} d(\mu_{n-1}, \mu_n) = 0 \text{ and } \lim_{n\to+\infty} \phi(\mu_n) = 0. \quad (16)$$

Proof. By replacing the contradiction in [[13], (29)] with contradiction (13) and following the proof of [[13], Lemma 1], we immediately have the desired result.

Theorem 11. *Let (M, d) be a C.M.S and let $Y : M \to M$ be an rational type F - H-contraction with respect to the functions $F \in \mathscr{F}$, $H \in \mathscr{H}$, the real number $\tau > 0$, and a l.s.c. function $\phi : M \to 0, +\infty$ such that (13) holds for all $\mu, \gamma \in M$ with $H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)) > 0$. Then, Y has a unique fixed point ς such that $\phi(\varsigma) = 0$.*

Proof. First, we shall proof the uniqueness. Arguing by contradiction, we assume that there exist $\varsigma, w \in M$ such that $\varsigma = Y\varsigma$, $w = Yw$, and $\varsigma \neq w$. The hypothesis $\varsigma \neq w$ ensures, by the property (H_1) of the function H, that

$$H(d(Y\varsigma, Yw), \phi(Y\varsigma), \phi(Yw)) \geq d(Y\varsigma, Yw) = d(\varsigma, w) > 0. \quad (17)$$

Using (13) with $\mu = \varsigma$ and $\gamma = w$, we obtain

$$\begin{aligned}
&\tau + F(H(d(Y\varsigma, Yw), \phi(Y\varsigma), \phi(Yw)))\\
&\quad = \tau + F(H(d(\varsigma, w), \phi(\varsigma), \phi(w)))\\
&\quad \leq F(H(M(\varsigma, w), \phi(\varsigma), \phi(w)))\\
&\quad \leq F\left(H\left(\max\left\{d(\varsigma, w), \frac{d(\varsigma, Y\varsigma)[1+d(w, Yw))]}{1+d(Y\varsigma, Yw)}\right\}, \phi(\varsigma), \phi(w)\right)\right)\\
&\quad \leq F\left(H\left(\max\left\{d(\varsigma, w), \frac{d(\varsigma, \varsigma)[1+d(w, w))]}{1+d(\varsigma, w)}\right\}, \phi(\varsigma), \phi(w)\right)\right)\\
&\quad \leq F(H(d(\varsigma, w), \phi(\varsigma), \phi(w))),
\end{aligned} \quad (18)$$

which is a contradiction. So, we have $w = \varsigma$, and the fixed point is unique.

Now, we can show the existence of a fixed point. Take a point $\mu_0 \in M$ and create the $\{\mu_n\}$ sequence starting at μ_0. We emphasize that if $\mu_{k-1} = \mu_k$ for some $k \in \mathbb{N}$, then $\varsigma = \mu_{k-1} = \mu_k = Y\mu_{k-1} = Y\varsigma$; that is, ς is a fixed point of Y such that $\phi(\varsigma) = 0$. In fact, by Lemma 10, $H(d(\mu_{k-1}, \mu_k), \phi(\mu_{k-1}), \phi(\mu_k)) = 0$ and by the property (H_1) of the function H, we have $\phi(\varsigma) = 0$. So, we can suppose that $\mu_{n-1} \neq \mu_n$ for every $n \in \mathbb{N}$.

In this step, we show that $\{\mu_n\}$ is a Cauchy. By Lemma 10, we say that

$$0 < h_{n-1} = H(d(\mu_{n-1}, \mu_n), \phi(\mu_{n-1}), \phi(\mu_n)) \to 0 \text{ as } n \to +\infty. \quad (19)$$

There exists $k \in]0, 1[$ such that $h_n^k F(h_n) \to 0$ as $n \to +\infty$ by he property (F_3) of F. Using (13) with $\mu = \mu_{n-1}$ and $\gamma = \mu_n$, we get

$$\begin{aligned}
&F(H(d(\mu_n, \mu_{n+1}), \phi(\mu_n), \phi(\mu_{n+1})))\\
&\quad \leq F(H(M(\mu_{n-1}, \mu_n), \phi(\mu_{n-1}), \phi(\mu_n))) - \tau\\
&\quad \leq F(H(\max\{d(\mu_{n-1}, \mu_n),\\
&\quad\quad \frac{d(\mu_{n-1}, Y\mu_{n-1})[1+d(\mu_n, Y\mu_n))]}{1+d(Y\mu_{n-1}, Y\mu_n)}\right\}, \phi(\mu_{n-1}), \phi(\mu_n))) - \tau\\
&\quad \leq F(H(d(\mu_{n-1}, \mu_n), \phi(\mu_{n-1}), \phi(\mu_n))) - \tau\\
&\quad \leq F(H(d(\mu_0, \mu_1), \phi(\mu_0), \phi(\mu_1))) - n\tau,
\end{aligned} \quad (20)$$

for all $n \in \mathbb{N}$; that is,

$$F(h_n) \leq F(h_{n-1}) - \tau \leq \cdots \leq F(h_0) - n\tau \text{ for all } n \in \mathbb{N}. \quad (21)$$

From

$$0 = \lim_{n\to+\infty} h_n^k F(h_n) \leq \lim_{n\to+\infty} h_n^k (F(h_0) - n\tau) \leq 0, \quad (22)$$

we deduce that

$$\lim_{n\to+\infty} h_n^k n = 0. \quad (23)$$

This provides that $\sum_{n=1}^{+\infty} h_n$ is convergent. By the property (H_1) of the function H, also, the series $\sum_{n=1}^{+\infty} d(\mu_n, \mu_{n+1})$ is convergent and hence $\{\mu_n\}$ is a Cauchy sequence. Now, since (M, d) is complete, there exists $\varsigma \in M$ such that

$$\lim_{n\to+\infty} \mu_n = \varsigma. \quad (24)$$

By (13), taking into account that ϕ is a l.s.c. function, we have

$$0 \leq \phi(\varsigma) \leq \liminf_{n\to+\infty} \phi(\mu_n) = 0\,; \quad (25)$$

that is, $\phi(\varsigma) = 0$. Now, show that ς is a fixed point. If there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\mu_{n_k} = \varsigma$ or $Y\mu_{n_k} = Y\varsigma$, for all $k \in \mathbb{N}$, then ς is a fixed point. Otherwise, we can assume that $\mu_n \neq \varsigma$ and $Y\mu_n \neq Y\varsigma$ for all $n \in \mathbb{N}$. So, using (13) with $\mu = \mu_n$ and $\gamma = \varsigma$, we deduce that

$$\begin{aligned}
&\tau + F(H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma)))\\
&\quad \leq F(H(M(\mu_n, \varsigma), \phi(\mu_n), \phi(\varsigma))).
\end{aligned} \quad (26)$$

Since $\tau > 0$, we obtain

$$\begin{aligned} &H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma)) \\ &\quad < H(M(\mu_n, \varsigma), \phi(\mu_n), \phi(\varsigma)) \text{ for } all\ n \in \mathbb{N}, \end{aligned} \tag{27}$$

and so

$$\begin{aligned} d(\varsigma, Y\varsigma) &\le d(\varsigma, \mu_{n+1}) + d(Y\mu_n, Y\varsigma) \\ &\le d(\varsigma, \mu_{n+1}) + H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma)) \\ &< d(\varsigma, \mu_{n+1}) + H(M(\mu_n, \varsigma), \phi(\mu_n), \phi(\varsigma)) \\ &< d(\varsigma, \mu_{n+1}) + H\Big(\max\Big\{d(\mu_n, \varsigma), \\ &\quad \cdot \frac{d(\mu_n, Y\mu_n)[1 + d(\varsigma, Y\varsigma)]}{1 + d(Y\mu_n, Y\varsigma)}\Big\}, \phi(\mu_n), \phi(\varsigma)\Big) \\ &\le d(\varsigma, \mu_{n+1}) + H\Big(\max\Big\{d(\mu_n, \varsigma), \\ &\quad \cdot \frac{d(\mu_n, \mu_{n+1})[1 + d(\varsigma, Y\varsigma)]}{1 + d(\mu_{n+1}, Y\varsigma)}\Big\}, \phi(\mu_n), \phi(\varsigma)\Big), \end{aligned} \tag{28}$$

for all $n \in \mathbb{N}$.

Finally, letting $n \to +\infty$ in the above calculations and using that H is continuous in $(0, 0, 0)$, we deduce that $d(\varsigma, Y\varsigma) \le H(0, 0, 0) = 0$; that is, $\varsigma = Y\varsigma$.

Imposing that F is a continuous function and relaxing the hypothesis (F_3), we can give t Theorem 12.

Theorem 12. *Let (M, d) be a C.M.S and $Y : M \to M$ be a mapping. Assume that there exists a continuous function F that satisfies the conditions (F_1) and (F_2), a function $H \in \mathscr{H}$, a real number $\tau > 0$, and a l.s.c. function $\phi : M \to 0, +\infty$ s.t.*

$$\begin{aligned} &\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma))) \\ &\quad \le F(H(M(\mu, \gamma), \phi(\mu), \phi(\gamma))), \end{aligned} \tag{29}$$

for all $\mu, \gamma \in M$ with $H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)) > 0$. Then, Y has a unique fixed point ς such that $\phi(\varsigma) = 0$.

Proof. Following the similar arguments as in the proof of Theorem 11, we obtain easily the uniqueness of the fixed point. The existence of a fixed point, we take a point $\mu_0 \in M$ and create the $\{\mu_n\}$ sequence starting at μ_0. Clearly, if $\mu_{k-1} = \mu_k$ for some $k \in \mathbb{N}$, then $\varsigma = \mu_{k-1} = \mu_k = Y\mu_{k-1} = Y\varsigma$; that is, ς is a fixed point of Y such that $\phi(\varsigma) = 0$ (see the proof of Theorem 11), and so we have already done.

So, we can suppose that $\mu_{n-1} \neq \mu_n$ for every $n \in \mathbb{N}$. Now, showing that $\{\mu_n\}$ is a Cauchy. Let us admit the opposite. Then, there exists a positive real number ε and two sequences $\{m_k\}$ and $\{n_k\}$ such that

$$n_k > m_k \ge k \text{ and } d\left(\mu_{m_k}, \mu_{n_k}\right) \ge \varepsilon > d\left(\mu_{m_k}, \mu_{n_k - 1}\right) \text{ for all } k \in \mathbb{N}. \tag{30}$$

By Lemma 10, we say that $d(\mu_{n-1}, \mu_n) \to 0$, $\phi(\mu_n) \to 0$, as $n \to +\infty$. This implies

$$\lim_{k \to +\infty} d\left(\mu_{m_k}, \mu_{n_k}\right) = \lim_{k \to +\infty} d\left(\mu_{m_k - 1}, \mu_{n_k - 1}\right) = \varepsilon. \tag{31}$$

Now, the hypothesis that $d(\mu_{m_k}, \mu_{n_k}) > \varepsilon$ ensures that

$$H\left(d\left(\mu_{m_k}, \mu_{n_k}\right), \phi\left(\mu_{m_k}\right), \phi\left(\mu_{n_k}\right)\right) > 0 \text{ for all } k \in \mathbb{N}. \tag{32}$$

Using the continuity of H, we have

$$\begin{aligned} &\lim_{k \to +\infty} H\left(d\left(\mu_{m_k - 1}, \mu_{n_k - 1}\right), \phi\left(\mu_{m_k - 1}\right), \phi\left(\mu_{n_k - 1}\right)\right) \\ &\quad = \lim_{k \to +\infty} H\left(d\left(\mu_{m_k}, \mu_{n_k}\right), \phi\left(\mu_{m_k}\right), \phi\left(\mu_{n_k}\right)\right) \\ &\quad = H(\varepsilon, 0, 0) > 0. \end{aligned} \tag{33}$$

Using again (29), with $\mu = \mu_{m_k - 1}$ and $\gamma = \mu_{n_k - 1}$, we get

$$\begin{aligned} &\tau + F\left(H\left(d\left(\mu_{m_k}, \mu_{n_k}\right), \phi\left(\mu_{m_k}\right), \phi\left(\mu_{n_k}\right)\right)\right) \\ &\quad \le F\left(H\left(M\left(\mu_{m_k-1}, \mu_{n_k-1}\right), \phi\left(\mu_{m_k-1}\right), \phi\left(\mu_{n_k-1}\right)\right)\right) \\ &\quad \le F\left(H\left(\max\left\{\begin{matrix} d\left(\mu_{m_k-1}, \mu_{n_k-1}\right), \\ \dfrac{d\left(\mu_{m_k-1}, Y\mu_{m_k-1}\right)\left[1 + d\left(\mu_{n_k-1}, Y\mu_{n_k-1}\right)\right)\right]}{1 + d\left(Y\mu_{m_k-1}, Y\mu_{n_k-1}\right)} \end{matrix}\right\}, \phi\left(\mu_{m_k-1}\right), \phi\left(\mu_{n_k-1}\right)\right)\right) \\ &\quad \le F\left(H\left(\max\left\{\begin{matrix} d\left(\mu_{m_k-1}, \mu_{n_k-1}\right), \\ \dfrac{d\left(\mu_{m_k-1}, \mu_{m_k}\right)\left[1 + d\left(\mu_{n_k-1}, \mu_{n_k}\right)\right)\right]}{1 + d\left(\mu_{m_k}, \mu_{n_k}\right)} \end{matrix}\right\}, \phi\left(\mu_{m_k-1}\right), \phi\left(\mu_{n_k-1}\right)\right)\right), \end{aligned} \tag{34}$$

for all $k \in \mathbb{N}$. Letting $k \to +\infty$ in the previous inequality, since the function F is continuous, we get

$$\tau + F(H(\varepsilon, 0, 0))) \le F(H(\varepsilon, 0, 0))), \tag{35}$$

which leads to contradiction. It follows that $\{\mu_n\}$ is a Cauchy sequence.

Now, since (M, d) is complete, there exists some $\varsigma \in M$ such that

$$\lim_{n \to +\infty} \mu_n = \varsigma. \tag{36}$$

By (29), using lower semicontinuity of ϕ, we get

$$0 \le \phi(\varsigma) \le \liminf_{n \to +\infty} \phi(\mu_n) = 0\,; \tag{37}$$

that is, $\phi(\varsigma) = 0$. Now, show that ς is a fixed point of Y. Clearly, ς is a fixed point of Y if there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\mu_{n_k} = \varsigma$ or $Y\mu_{n_k} = Y\varsigma$, for all $k \in \mathbb{N}$. Otherwise, we can assume that $\mu_n \neq \varsigma$ and $Y\mu_n \neq Y\varsigma$ for all $n \in \mathbb{N}$. Then, the property (H_1) of the function H ensures that $H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma)) > 0$ for all $n \in \mathbb{N}$. So, using (29) with $\mu = \mu_n$ and $\gamma = \varsigma$, we deduce that

$$\begin{aligned}
&\tau + F(H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma))) \\
&\quad \leq F(H(M(\mu_n, \varsigma), \phi(\mu_n), \phi(\varsigma))) \\
&\quad \leq F\Big(H\Big(\max\Big\{d(\mu_n, \varsigma), \\
&\qquad \cdot \frac{d(\mu_n, Y\mu_n)[1 + d(\varsigma, Y\varsigma)]}{1 + d(Y\mu_n, Y\varsigma)}\Big\}, \phi(\mu_n), \phi(\varsigma)\Big)\Big) \text{ for all } n \in \mathbb{N}.
\end{aligned} \tag{38}$$

Since $\tau > 0$, we conclude that

$$\begin{aligned}
&H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma)) \\
&\quad < F\Big(H\Big(\max\Big\{d(\mu_n, \varsigma), \\
&\qquad \cdot \frac{d(\mu_n, Y\mu_n)[1 + d(\varsigma, Y\varsigma)]}{1 + d(Y\mu_n, Y\varsigma)}\Big\}, \phi(\mu_n), \phi(\varsigma)\Big)\Big) \text{ for all } n \in \mathbb{N},
\end{aligned} \tag{39}$$

and so

$$\begin{aligned}
d(\varsigma, Y\varsigma) &\leq d(\varsigma, \mu_{n+1}) + d(Y\mu_n, Y\varsigma) \\
&\leq d(\varsigma, \mu_{n+1}) + H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma)) \\
&< d(\varsigma, \mu_{n+1}) + H\Big(\max\Big\{d(\mu_n, \varsigma), \\
&\qquad \cdot \frac{d(\mu_n, Y\mu_n)[1 + d(\varsigma, Y\varsigma)]}{1 + d(Y\mu_n, Y\varsigma)}\Big\}, \phi(\mu_n), \phi(\varsigma)\Big) \\
&= d(\varsigma, \mu_{n+1}) + H\Big(\max\Big\{d(\mu_n, \varsigma), \\
&\qquad \cdot \frac{d(\mu_n, \mu_{n+1})[1 + d(\varsigma, Y\varsigma)]}{1 + d(\mu_{n+1}, Y\varsigma)}\Big\}, \phi(\mu_n), \phi(\varsigma)\Big),
\end{aligned} \tag{40}$$

for all $n \in \mathbb{N}$. Finally, letting $n \to +\infty$ and using that H is continuous in $(0,0,0)$, we deduce that $d(\varsigma, Y\varsigma) \leq H(0,0,0) = 0$; that is, $\varsigma = Y\varsigma$.

Definition 13. Let (M, d) be a metric space and let $Y : M \to M$ be a mapping. The mapping Y is called almost F - H -contraction if there exists a function $F \in \mathcal{F}$, $H \in \mathcal{H}$, a real number $\tau > 0$, and $L \geq 0$ and a *l.s.c.* function $\phi : M \to 0, +\infty$ such that

$$\begin{aligned}
&\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma))) \\
&\quad \leq F(H(d(\mu, \gamma) + Ld(\gamma, Y\mu), \phi(\mu), \phi(\gamma))),
\end{aligned} \tag{41}$$

for all $\mu, \gamma \in M$ with $H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)) > 0$.

Theorem 14. *Let (M, d) be a C.M.S and let $Y : M \to M$ be an almost F - H-contraction with respect to the functions $F \in \mathcal{F}$, $H \in \mathcal{H}$, the real number $\tau > 0$, and $L \geq 0$ and a l.s.c. function $\phi : M \to 0, +\infty$ s.t.*

$$\begin{aligned}
&\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma))) \\
&\quad \leq F(H(d(\mu, \gamma) + Ld(\gamma, Y\mu), \phi(\mu), \phi(\gamma))),
\end{aligned} \tag{42}$$

for all $\mu, \gamma \in M$ with $H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)) > 0$. Then, Y has a fixed point ς such that $\phi(\varsigma) = 0$.

Proof. The existence of a fixed point we take a point $\mu_0 \in M$ and create the $\{\mu_n\}$ sequence starting at μ_0. We stress that if $\mu_{k-1} = \mu_k$ for some $k \in \mathbb{N}$, then $\varsigma = \mu_{k-1} = \mu_k = Y\mu_{k-1} = Y\varsigma$; that is, ς is a fixed point of Y such that $\phi(\varsigma) = 0$. In fact, by Lemma 10, $H(d(\mu_{k-1}, \mu_k), \phi(\mu_{k-1}), \phi(\mu_k)) = 0$ and by the property (H_1) of the function H, we have $\phi(\varsigma) = 0$. So, we can suppose that $\mu_{n-1} \neq \mu_n$ for every $n \in \mathbb{N}$.

Now, showing that $\{\mu_n\}$ is a Cauchy. By Lemma 10, we say that

$$0 < h_{n-1} = H(d(\mu_{n-1}, \mu_n), \phi(\mu_{n-1}), \phi(\mu_n)) \to 0 \textit{ as } n \to +\infty. \tag{43}$$

The property (F_3) of the function F ensures that there exists $k \in]0,1[$ such that $h_n^k F(h_n) \to 0$ as $n \to +\infty$. Using (42), with $\mu = \mu_{n-1}$ and $\gamma = \mu_n$, we get

$$\begin{aligned}
&F(H(d(\mu_n, \mu_{n+1}), \phi(\mu_n), \phi(\mu_{n+1}))) \\
&\quad \leq F(H(d(\mu_{n-1}, \mu_n) + Ld(\mu_n, Y\mu_{n-1}), \phi(\mu_{n-1}), \phi(\mu_n))) - \tau \\
&\quad \leq F(H(d(\mu_{n-1}, \mu_n), \phi(\mu_0), \phi(\mu_1))) - \tau \\
&\quad \leq F(H(d(\mu_0, \mu_1), \phi(\mu_0), \phi(\mu_1))) - n\tau,
\end{aligned} \tag{44}$$

for $\forall n \in \mathbb{N}$; that is,

$$F(h_n) \leq F(h_{n-1}) - \tau \leq \cdots \leq F(h_0) - n\tau \text{ for } \textit{all } n \in \mathbb{N}. \tag{45}$$

From

$$0 = \lim_{n \to +\infty} h_n^k F(h_n) \leq \lim_{n \to +\infty} h_n^k (F(h_0) - n\tau) \leq 0, \tag{46}$$

we deduce that

$$\lim_{n \to +\infty} h_n^k n = 0. \tag{47}$$

This ensures that the series $\sum_{n=1}^{+\infty} h_n$ is convergent. By the property (H_1) of the function H, also, the series $\sum_{n=1}^{+\infty} d(\mu_n, \mu_{n+1})$ is convergent, and hence $\{\mu_n\}$ is a Cauchy sequence. Now, since (M, d) is complete, there exists some $\varsigma \in M$ such that

$$\lim_{n \to +\infty} \mu_n = \varsigma. \tag{48}$$

By (42), using lower semicontinuity of ϕ, we get

$$0 \leq \phi(\varsigma) \leq \liminf_{n \to +\infty} \phi(\mu_n) = 0\,; \tag{49}$$

that is, $\phi(\varsigma) = 0$. We assert that ς is a fixed point of Y. Clearly, ς is a fixed point of Y if there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\mu_{n_k} = \varsigma$ or $Y\mu_{n_k} = Y\varsigma$, for all $k \in \mathbb{N}$. Otherwise, we can assume that $\mu_n \neq \varsigma$ and $Y\mu_n \neq Y\varsigma$ for all $n \in \mathbb{N}$. So, using (42) with $\mu = \mu_n$ and $\gamma = \varsigma$, we deduce that

$$\begin{aligned}&\tau + F(H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma)))\\&\quad \leq F(H(d(\mu_n, \varsigma) + Ld(\varsigma, Y\mu_n), \phi(\mu_n), \phi(\varsigma))).\end{aligned} \tag{50}$$

Since $\tau > 0$, this inequality leads to

$$\begin{aligned}&H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma))\\&\quad < H(d(\mu_n, \varsigma) + Ld(\varsigma, Y\mu_n), \phi(\mu_n), \phi(\varsigma)) \text{ for all } n \in \mathbb{N},\end{aligned} \tag{51}$$

and so

$$\begin{aligned}d(\varsigma, Y\varsigma) &\leq d(\varsigma, \mu_{n+1}) + d(Y\mu_n, Y\varsigma)\\&\leq d(\varsigma, \mu_{n+1}) + H(d(Y\mu_n, Y\varsigma), \phi(Y\mu_n), \phi(Y\varsigma))\\&< d(\varsigma, \mu_{n+1}) + H(d(\mu_n, \varsigma) + Ld(\varsigma, Y\mu_n), \phi(\mu_n), \phi(\varsigma)),\end{aligned} \tag{52}$$

for all $n \in \mathbb{N}$.

Finally, letting $n \to +\infty$ in the above calculations and using that H is continuous in $(0, 0, 0)$, we deduce that $d(\varsigma, Y\varsigma) \leq H(0, 0, 0) = 0$; that is, $\varsigma = Y\varsigma$.

Example 15. Let $M = [0, 1]$ endowed with the standart metric $d(\mu, \gamma) = |\mu - \gamma|$ for all $\mu, \gamma \in M$. Consider the mapping $Y : M \to M$ defined by

$$Y\mu = \begin{cases} \mu/2\,; & \mu \in 0, 1) \\ 1\,; & \mu = 1 \end{cases}. \tag{53}$$

Clearly, Y is not a $F-$contraction but Y is an almost F-H-contraction with respect to the functions $F \in \mathscr{F}$ defined by $F(\alpha) = \ln \alpha$ for all $\alpha > 0, H \in \mathscr{H}$ defined by $H(a, b, c) = \max \{a, b\} + c$ for all $a, b, c \in 0, +\infty$, the real number $\tau = \ln 2$ and $L = 4$, and a l.s.c. function $\phi : M \to 0, +\infty, \phi(t) = t$ for all $t \in M$, indeed.

Case 1. $\mu = 0, \gamma = 1$, we have

$$\begin{aligned}&\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)))\\&\quad = \tau + F(H(d(Y0, Y1), \phi(Y0), \phi(Y1)))\\&\quad = \tau + F(H(d(0, 1), \phi(0), \phi(1)))\\&\quad = \tau + F(H(1, 0, 1)) = \ln 4\\&\quad \leq \ln 6 = F(H(d(0, 1) + 4d(1, Y0), \phi(0), \phi(1)))\\&\quad = F(H(d(\mu, \gamma) + Ld(\gamma, Y\mu), \phi(\mu), \phi(\gamma))).\end{aligned} \tag{54}$$

Case 2. $\mu = 1, \gamma = 0$, we have

$$\begin{aligned}&\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)))\\&\quad = \tau + F(H(d(Y1, Y0), \phi(Y1), \phi(Y0)))\\&\quad = \tau + F(H(d(1, 0), \phi(1), \phi(0)))\\&\quad = \tau + F(H(1, 1, 0)) = \ln 2 \leq \ln 5\\&\quad = F(H(d(1, 0) + 4d(0, Y1), \phi(1), \phi(0)))\\&\quad = F(H(d(\mu, \gamma) + Ld(\gamma, Y\mu), \phi(\mu), \phi(\gamma))).\end{aligned} \tag{55}$$

Case 3. $\mu, \gamma \in (0, 1)$ with $\mu > \gamma$, we have

$$\begin{aligned}&\tau + F(H(d(Y\mu, Y\gamma), \phi(Y\mu), \phi(Y\gamma)))\\&\quad = \tau + F\left(H\left(d\left(\frac{\mu}{2}, \frac{\gamma}{2}\right), \phi\left(\frac{\mu}{2}\right), \phi\left(\frac{\gamma}{2}\right)\right)\right)\\&\quad = \tau + F\left(H\left(\frac{\mu - \gamma}{2}, \frac{\mu}{2}, \frac{\gamma}{2}\right)\right)\\&\quad = \tau + F\left(\frac{\mu + \gamma}{2}\right) = \ln(\mu + \gamma)\\&\quad \leq \max\{\ln(\mu + \gamma), \ln(4\gamma - \mu)\}\\&\quad = F(H(d(\mu, \gamma) + Ld(\gamma, Y\mu), \phi(\mu), \phi(\gamma))).\end{aligned} \tag{56}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Global Well-Posedness for Coupled System of mKdV Equations in Analytic Spaces

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The main result in this paper is to prove, in Bourgain type spaces, the existence of unique local solution to system of initial value problem described by integrable equations of modified Korteweg-de Vries (mKdV) by using linear and trilinear estimates, together with contraction mapping principle. Moreover, owing to the approximate conservation law, we prove the existence of global solution.

1. Introduction and Main Results

For an effective approach to solving problems arising in modern science and technology, one cannot do without researching nonlinear problems of mathematical physics. The rapid development of new technology and the emergence of its high speed allow researchers to build and consider increasingly complex multidimensional models describing various phenomena, which are modeled, as a rule, using nonlinear partial differential equations (systems). However, now it has become clear that without the development of analytical methods, it is impossible to get a complete idea of the essence of the phenomenon. Analytical methods provide not only a reliable tool for debugging and comparing various numerical methods but also sometimes anticipate some scientific discoveries, make it possible to study the properties of models, to detect the presence of certain effects as a result of the existence or nonexistence of objects (solutions) with the required properties. Therefore, at present, fundamental research is being intensively carried out aimed at proving theorems of existence, uniqueness, and regularity of solutions of nonlinear partial differential equations.

In the present paper, a coupled system of modified Korteweg-de Vries equations is considered as follows:

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x\left(uv^2\right) = 0, \\ \partial_t v + \beta\partial_x^3 v + \partial_x\left(u^2 v\right) = 0, (x, t) \in \mathbb{R}^2, 0 < \beta < 1, \\ u(x, 0) = u_0(x), \\ v(x, 0) = v_0(x). \end{cases} \tag{1}$$

The dynamics of solutions in the Korteweg-de Vries equations (KdV) and the modified Korteweg-de Vries equations (mKdV) are well studied due to the complete integrability of these equations (see [1–6]). For KdV equations, the studies date back to the 1970s, although some results have been obtained very recently (please see [7]). We extend the results in [7] and consider a coupled system of mKdV-type equations on the line in Equation (1).

For mKdV equations, many problems have been studied. It is proved that the mKdV equation is locally [8] and globally [9] well-posed in $H^s(\mathbb{T})$ for $s \geq 1/2$. Global well-posedness in $L^2(\mathbb{T})$ is shown in [10].

For $0 < \beta < 1$, the author in [7] proved that the IVP (Equation (1)) is locally well-posed for the given data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -1/2$. Oh in [11] used the Fourier transform restriction norm method and proved that the next IVP

$$\begin{aligned} &\partial_t u + \partial_x^3 u + \partial_x\left(v^2\right) = 0, \qquad u(x,0) = u_0(x), \\ &\partial_t v + \beta\partial_x^3 v + \partial_x(uv) = 0, \quad v(x,0) = v_0(x), 0 < \beta < 1, \end{aligned} \tag{2}$$

is locally well-posed for data with regularity $s \geq 0$.

For $\beta = 1$, the system (Equation (1)) reduces to a special case of a broad class of nonlinear evolution equations considered by Ablowitz et al. [12] in the inverse scattering context. In this case, the well-posedness issues along with existence and stability of solitary waves for this system are widely studied in the literature, using the technique developed by Kenig et al. in [13, 14].

Well-posedness for the nonperiodic gKdV equation in spaces of analytic functions has been proved by Grujic and Kalisch [15].

A class of suitable analytic functions for our analysis is the analytic Gevrey class $G^{\delta,s}(\mathbb{R}) = G^{\delta,s}$ introduced by Foias and Temam [16], defined as follows:

$$G^{\delta,s} = \left\{ f \in L^2 \, ; \|f\|_{G^{\delta,s}} = \int_{\mathbb{R}} e^{2\delta|\zeta|} \langle\zeta\rangle^{2s} |f\wedge(\zeta)|^2 d\zeta < \infty \right\}, \tag{3}$$

for $s \in \mathbb{R}$ and $\delta > 0$ with $\langle\cdot\rangle = (1 + |\cdot|)$. For $\delta = 0$, the space $G^{\delta,s}$ coincides with the standard Sobolev space H^s. For all $0 < \delta' < \delta$ and $s, s' \in \mathbb{R}$, we have

$$G^{\delta,s} \subset G^{\delta',s'}, \text{ i.e.,} \|f\|_{G^{\delta',s'}} \leq c_{s,s}{}'^{,\delta,\delta'} \|f\|_{G^{\delta,s}}, \tag{4}$$

which is the embedding property of the Gevrey spaces.

New minimal conditions are used to show the local well-posedness of solution by using linear and trilinear estimates, together with contraction mapping principle. By imposing a more appropriate conditions with the help of the approximate conservation law, we obtain an unusual global existence result in Gevery spaces.

Proposition 1 (Paley-Wiener Theorem) [17]. *Let $\delta > 0$, $s \in \mathbb{R}$. Then, $f \in G^{\delta,s}$ if and only if it is the restriction to the real line of a function F which is holomorphic in the strip $\{x + iy : x, y \in \mathbb{R}, |y| < \delta\}$ and satisfies*

$$\sup_{|y|<\delta} \|F(x+iy)\|_{H_x^s} < \infty. \tag{5}$$

Remark 2. In the view of the Paley-Wiener Theorem, it is natural to take initial data in $G^{\delta,s}$, to obtain the best behavior of solution and may be extended to be globally in time. It means that given $(u_0, v_0) \in G^{\delta,s} \times G^{\delta,s}$ for some initial radius $\delta > 0$, we then estimate the behavior of the radius of analyticity $\delta(T)$ over time.

The first main result on local well-posedness of Equation (1) in analytic spaces reads as follows.

Theorem 3. *Let $\delta > 0$ and $s > -1/2$. Then for any $(u_0, v_0) \in G^{\delta,s} \times G^{\delta,s}$, there exists $T = T(\|(u_0, v_0)\|_{G^{\delta,s}\times G^{\delta,s}})$ and unique solution (u, v) of Equation (1) on $[0, T]$ such that*

$$(u, v) \in C\left([0, T], G^{\delta,s}\right) \times C\left([0, T], G^{\delta,s}\right). \tag{6}$$

Moreover, the solution depends on (u_0, v_0), where

$$T = \frac{1}{\left(16C^3 + 16C^3\|(u_0, v_0)\|^2_{G^{\delta,s}\times G^{\delta,s}}\right)^{1/\varepsilon}}. \tag{7}$$

Furthermore, the solution satisfies the following:

$$\|(u, v)\|_{X_{\delta,s,b}\times X^{\beta}_{\delta,s,b}} \leq 2C\|(u_0, v_0)\|_{G^{\delta,s}\times G^{\delta,s}}, \quad b = \frac{1}{2} + \varepsilon, \tag{8}$$

with constant $C > 0$ depending only on s and b.

An effective method for studying lower bounds on the radius of analyticity, including this type of problem, was introduced in [18] for 1D Dirac-Klein-Gordon equations. It was applied in [19] to the modified Kawahara equation and in [20] to the nonperiodic KdV equation (for more details, please see [20–23]).

The second result for the problem (Equation(1)) is given in the next theorem.

Theorem 4. *Let $s > -1/2$, $0 < \beta < 1$, and $\delta_0 > 0$. Assume that $(u_0, v_0) \in G^{\delta,s} \times G^{\delta,s}$, then the solution in Theorem 3 can be extended to be global in time and for any $T' > 0$, we have the following:*

$$(u, v) \in C\left(\left[0, T'\right], G^{\delta(T'),s}\right) \times C\left(\left[0, T'\right], G^{\delta(T'),s}\right), \tag{9}$$

with

$$\delta\left(T'\right) = \min\left\{\delta_0, C_1 T'^{-(2+\sigma_0)}\right\}, \tag{10}$$

where $\sigma_0 > 0$ can be taken arbitrarily small and $C_1 > 0$ is a constant depending on w_0, δ_0, s, and σ_0.

The third result is Gevrey's temporal regularity of the unique solution obtained in the Theorem 3. A nonperiodic function $f(x)$ is the Gevrey class of order r, i.e., $f(x) \in G^r$, if there exists a constant $C > 0$ such that

$$|\partial_x^l f(x)| \leq C^{l+1} (l!)^r \, l = 0, 1, 2, \cdots, \tag{11}$$

if $r = 1 f(x)$ is analytic.

Here, we will show that for $x \in \mathbb{R}$, for every $t \in [0, T]$ and $j, l \in \{0, 1, 2, \cdots\}$, there exist $C > 0$ such that

$$\begin{aligned} \left|\partial_t^j \partial_x^l u(x,t)\right| &\le C^{j+l+1}(j!)^3(l!), \\ \left|\partial_t^j \partial_x^l v(x,t)\right| &\le C^{j+l+1}(j!)^3(l!), \end{aligned} \tag{12}$$

i.e., $(u(\cdot,t), v(\cdot,t)) \in G^1(\mathbb{R}) \times G^1(\mathbb{R})$ in spacial variable and $(u(x,\cdot), v(x,\cdot)) \in G^3([0,T]) \times G^3([0,T])$ in time variable. Also,

$$\left|\partial_t^j \partial_x^l u(x,t)\right| \le C^{j+l+1}(j!)^d(l!), \tag{13}$$

$$\left|\partial_t^j \partial_x^l v(x,t)\right| \le C^{j+l+1}(j!)^d(l!), \tag{14}$$

where Equations (13) and (14) do not hold for$1 \le d < 3$.

Theorem 5. *Let $s > -1/2$, $0 < \beta < 1$, and $\delta > 0$. If $(u_0, v_0) \in G^{\delta,s} \times G^{\delta,s}$, then the solution $(u, v) \in C([0,T], G^{\delta(T),s}) \times C([0,T], G^{\delta(T),s})$ given by Theorem 4 belongs to the Gevrey class $G^3([0,T]) \times G^3([0,T])$ in time variable. Furthermore, it is not belong to $G^d([0,T]) \times G^d([0,T])$, $1 \le d < 3$ in t.*

The proof of Theorem 5 is similar to that in [1].

The paper is organized as follows. In Section 2, we define the function spaces and linear and trilinear estimates. In Section 3, we prove Theorem 3, using the linear and trilinear estimates, together with contraction mapping principle. In Section 4, we prove the existence of fundamental approximate conservation law. In the last section, Theorem 4 will be proved using the approximate conservation law.

2. Preliminary Tools and Analytic Function Spaces

2.1. Function Spaces. We define the analytic Bourgain spaces related to the modified Korteweg-de Vries type equations. The completion of the Schwartz class $S(\mathbb{R}^2)$ is given by $X^{\beta}_{\delta,s,b}(\mathbb{R}^2) = X^{\beta}_{\delta,s,b}$, for $s, b \in \mathbb{R}$, $\delta > 0$, subjected to the norm:

$$\|w\|_{X^{\beta}_{\delta,s,b}} = \left(\int_{\mathbb{R}^2} e^{2\delta|\zeta|}\langle\zeta\rangle^{2s}\left\langle\eta - \beta\zeta^3\right\rangle^{2b}|w\wedge(\zeta,\eta)|^2 d\zeta d\eta\right)^{1/2}. \tag{15}$$

We often use without mention, the definition $X^1_{\delta,s,b} = X_{\delta,s,b}$, where

$$\|w\|_{X_{\delta,s,b}} = \left(\int_{\mathbb{R}^2} e^{2\delta|\zeta|}\langle\zeta\rangle^{2s}\left\langle\eta - \zeta^3\right\rangle^{2b}|w\wedge(\zeta,\eta)|^2 d\zeta d\eta\right)^{1/2}. \tag{16}$$

For any interval I, we define the localized spaces $X^{\beta}_{\delta,s,b}(\mathbb{R} \times I) = X^{\beta,I}_{\delta,s,b}$ with norm:

$$\|w\|_{X^{\beta,I}_{\delta,s,b}} = \inf\left\{\|W\|_{X^{\beta}_{\delta,s,b}};\, W\Big|_{\mathbb{R}\times I} = w\right\}. \tag{17}$$

2.2. Linear Estimates. We have the trilinear estimate (Equations (15) and (16)) defined in the analytic Bourgain spaces. Since the spaces $X^{\beta}_{\delta,s,b}$ is continuously embedded in $C([0,T], G^{\delta,s})$, provided $b > 1/2$.

Lemma 6. *Let $b > 1/2$, $s \in \mathbb{R}$, and $\delta > 0$. Then, for all $T > 0$, we have the following:*

$$X_{\delta,s,b} \hookrightarrow C\left([0,T], G^{\delta,s}\right). \tag{18}$$

Proof. First, we note that the operator A defined by

$$Aw\wedge^x(\zeta,t) = e^{\delta|\zeta|}w\wedge^x(\zeta,t), \tag{19}$$

satisfies

$$\|w\|_{X^{\beta}_{\delta,s,b}} = \|Aw\|_{X^{\beta}_{s,b}},\ \|w\|_{G^{\delta,s}} = \|Aw\|_{H^s}, \tag{20}$$

where $X^{\beta}_{s,b}$ is introduced in [7]. We observe that Aw belongs to $C(\mathbb{R}, H^s)$ and for some $C > 0$, we have the following:

$$\|Aw\|_{C(\mathbb{R},H^s)} \le C\|Aw\|_{X^{\beta}_{s,b}}. \tag{21}$$

Thus, it follows that $w \in C([0,T], G^{\delta,s})$ and

$$\|w\|_{C\left([0,T],G^{\delta,s}\right)} \le C\|w\|_{X^{\beta}_{\delta,s,b}}. \tag{22}$$

Taking the Fourier transform with respect to x of the Cauchy problems (Equation (1)), after an ordinary calculation, we localize in t by using a cut-off function, satisfying $\psi \in C_0^{\infty}$, with $\psi = 1$ in $[-1, 1]$, supp$\psi \subset [-2, 2]$, and $\psi_T(t) = \psi(t/T)$. We consider the operator Λ, Γ given by the following:

$$\begin{cases} \Lambda[u,v](t) = \psi(t)S(t)u_0 - \psi_T(t)\displaystyle\int_0^t S(t-\nu)\partial_x F_1(\nu)d\nu, \\ \Gamma[u,v](t) = \psi(t)S_{\beta}(t)v_0 - \psi_T(t)\displaystyle\int_0^t S_{\beta}(t-\nu)\partial_x F_2(\nu)d\nu, \end{cases} \tag{23}$$

where $S(t) = e^{-t\partial_x^3}$ and $S_{\beta}(t) = e^{-t\beta\partial_x^3}$ are the unitary groups associated with the linear problems.

The nonlinear terms defined by $F_1 = (uv^2)$ and $F_2 = (u^2 v)$ will be treated in the next lemmas.

Lemma 7. *Let $s, b \in \mathbb{R}$ and $\delta > 0$. For some constant $C > 0$, we have the following:*

$$\|\psi(t)S(t)u_0\|_{X_{\delta,s,b}} \le C\|u_0\|_{G^{\delta,s}},$$
$$\left\|\psi(t)S_\beta(t)v_0\right\|_{X^\beta_{\delta,s,b}} \le C\,\|v_0\|_{G^{\delta,s}}, \tag{24}$$

for all $u_0, v_0 \in G^{\delta,s}$.

Proof. By definition, we have the following:

$$\begin{aligned}\psi(t)S_\beta(t)u_0 &= C\psi(t)\int_{\mathbb{R}} e^{i\left(x\zeta+t\beta\zeta^3\right)}\widehat{u_0}(\zeta)d\zeta\\ &= C\int_{\mathbb{R}^2} e^{i(x\zeta+t\eta)}\,\widehat{\psi}\left(\eta-\beta\zeta^3\right)\Big)\widehat{u_0}(\zeta)d\zeta d\eta.\end{aligned} \tag{25}$$

It follows that

$$\begin{aligned}\|\psi(t)S(t)u_0\|^2_{X^\beta_{\delta,s,b}} &= C\int_{\mathbb{R}^2} e^{2\delta|\zeta|}(1+|\zeta\,|)^{2s}\left(1+\left|\eta-\beta\zeta^3\right|\right)^{2b}\\ &\quad\cdot\left|\psi\wedge\left(\eta-\beta\zeta^3\right)\right|^2|u_0\wedge(\zeta)|^2 d\zeta d\eta\\ &= C\int_{\mathbb{R}} e^{2\delta|\zeta|}(1+|\zeta|)^{2s}|u_0\wedge(\zeta)|^2\\ &\quad\cdot\left(\int_{\mathbb{R}}\left|\psi\wedge\left(\eta-\beta\zeta^3\right)\right|^2\left(1+\left|\eta-\beta\zeta^3\right|\right)^{2b}d\eta\right)d\zeta.\end{aligned} \tag{26}$$

Since $b > 1/2$, we have the following:

$$\begin{aligned}&\int_{\mathbb{R}}\left|\psi\wedge\left(\eta-\beta\zeta^3\right)\right|^2\left(1+|\eta-\beta\zeta^3\,|\right)^{2b}d\eta\\ &\le C\int_{\mathbb{R}}\left|\psi\wedge\left(\eta-\beta\zeta^3\right)\right|^2 d\eta + C\int_{\mathbb{R}}\left|\psi\wedge\left(\eta-\beta\zeta^3\right)\right|^2\\ &\quad\times\left(1+|\eta-\beta\zeta^3\,|\right)^{2b}d\eta \le C.\end{aligned} \tag{27}$$

Lemma 8. *Let $s \in \mathbb{R}$, $-1/2 < b' \le 0 \le b < b' + 1$, $0 \le T \le 1$, and $\delta > 0$, then for some constant $C > 0$, we have the following:*

$$\left\|\psi_T(t)\int_0^t S(t-\nu)\partial_x F_1(x,\nu)d\nu\right\|_{X_{\delta,s,b}} \le CT^{1-b+b'}\|\partial_x F_1\|_{X_{\delta,s,b}{}'},$$
$$\left\|\psi_T(t)\int_0^t S_\beta(t-\nu)\partial_x F_2(x,\nu)d\nu\right\|_{X^\beta_{\delta,s,b}} \le CT^{1-b+b'}\|\partial_x F_2\|_{X^\beta_{\delta,s,b'}}. \tag{28}$$

Proof. Define

$$W = \psi_T(t)\int_0^t S_\beta(t-\nu)\partial_x F_2(x,\nu)d\nu. \tag{29}$$

We have, by Equation (19), the following:

$$\begin{aligned}AW\wedge^x(\zeta,t) &= \psi_T(t)\int_0^t\left(e^{-i(t-\nu)\beta\zeta^3}\right)e^{\delta|\zeta|}\partial_x F_2\wedge^x(\zeta,\nu)d\nu\\ &= \psi_T(t)\int_0^t\left[S_\beta(t-\nu)(\partial_x AF_2)\right]\wedge^x(\zeta,\nu)d\nu.\end{aligned} \tag{30}$$

Thus,

$$\|W\|_{X^\beta_{\delta,s,b}} = \|AW\|_{X^\beta_{s,b}} = \left\|\psi_T(t)\int_0^t S_\beta(t-\nu)\partial_x AF_2(x,\nu)d\nu\right\|_{X^\beta_{s,b}}. \tag{31}$$

Owing to Lemma 6 in [7], we get the following:

$$\begin{aligned}&\left\|\psi_T(t)\int_0^t S_\beta(t-\nu)\partial_x AF_2(x,\nu)d\nu\right\|_{X^\beta_{s,b}} \le CT^{1-b+b'}\|\partial_x AF_2\|_{X^\beta_{s,b'}}\\ &= CT^{1-b+b'}\|\partial_x F_2\|_{X^\beta_{\delta,s,b'}}.\end{aligned} \tag{32}$$

This completes the proof.

Lemma 9. *Let $\Theta \in S(\mathbb{R})$ be a Schwartz function in time, $s \in \mathbb{R}$, and $\delta \ge 0$. If $-1/2 < b \le b' < 1/2$, then for any $T > 0$, we have the following:*

$$\|\Theta_T(t)w\|_{X_{\delta,s,b}} \le CT^{b'-b}\|w\|_{X_{\delta,s,b}{}'},$$
$$\|\Theta_T(t)w\|_{X^\beta_{\delta,s,b}} \le CT^{b'-b}\|w\|_{X^\beta_{\delta,s,b'}}, \tag{33}$$

where C depends only on b and b'.

Proof. The proof of Lemma 9 for $\delta = 0$ can be found in Lemma 13 of [14], for $\delta > 0$ as one merely has to replace w by Aw, where the operator is defined in Equation (19).

Lemma 10 [20]. *Let $s \in \mathbb{R}$, $\delta \ge 0$, $-1/2 < b < 1/2$, and $T > 0$. Then, for any time interval $I \subset [0, T]$, we have the following:*

$$\|\chi_I(t)w\|_{X_{\delta,s,b}} \le C\|w\|_{X^T_{\delta,s,b}},$$
$$\|\chi_I(t)w\|_{X^\beta_{\delta,s,b}} \le C\|w\|_{X^{\beta,T}_{\delta,s,b}}, \tag{34}$$

where $\chi_I(t)$ is the characteristic function of I and C depends only on b.

2.3. Trilinear Estimates. We have the trilinear estimate in the following lemmas.

Lemma 11. *Let $s > -1/2$, $\delta > 0$, $b > 1/2$, and b' be as in Lemma 8. Then,*

$$\begin{aligned}\left\|\partial_x\left(uv^2\right)\right\|_{X_{\delta,s,b'}} &\le C\|u\|_{X_{\delta,s,b}}\|v\|^2_{X^{\beta}_{\delta,s,b}},\\ \left\|\partial_x\left(u^2v\right)\right\|_{X^{\beta}_{\delta,s,b'}} &\le C\|u\|^2_{X_{\delta,s,b}}\|v\|_{X^{\beta}_{\delta,s,b}}.\end{aligned} \tag{35}$$

Proof. We observe, by considering the operator A in (19), that

$$\begin{aligned}e^{\delta|\zeta|}\widehat{uuv} &= (2\pi)^{-2}e^{\delta|\zeta|}\widehat{u} * \widehat{u} * \widehat{v} \le (2\pi)^{-2}\int_{\mathbb{R}^4} e^{\delta|\zeta-\zeta_1|}\widehat{u}(\zeta-\zeta_1,\eta\\ &\quad -\eta_1)e^{\delta|\zeta_1-\zeta_2|}\widehat{u}(\zeta_1-\zeta_2,\eta_1-\eta_2)e^{\delta|\zeta_2|}\widehat{v}(\zeta_2,\eta_2)d\zeta_1 d\zeta_2 d\eta_1 d\eta_2\\ &= \widehat{AuAuAv},\end{aligned} \tag{36}$$

since $\delta|\zeta| \le \delta|\zeta-\zeta_1| + \delta|\zeta_1-\zeta_2| + \delta|\zeta_2|$.

Then,

$$\begin{aligned}\left\|\partial_x\left(u^2v\right)\right\|_{X^{\beta}_{\delta,s,b'}} &= \left\|e^{\delta|\zeta|}\langle\zeta\rangle^s\left\langle\eta-\beta\zeta^3\right\rangle^b\widehat{\partial_x(uuv)}(\zeta,\eta)\right\|_{L^2_{\zeta,\eta}}\\ &\le \|\partial_x(AuAuAv)\|_{X^{\beta}_{s,b'}}.\end{aligned} \tag{37}$$

Thanks to Proposition 2.3 of [7], for some $C > 0$, we have the following:

$$\|\partial_x(AuAuAv)\|_{X^{\beta}_{s,b'}} \le C\|Au\|^2_{X_{s,b}}\|Av\|_{X^{\beta}_{s,b}} = C\|u\|^2_{X_{\delta,s,b}}\|v\|_{X^{\beta}_{\delta,s,b}}. \tag{38}$$

This completes the proof.

3. Proof of Theorem 3

3.1. Existence of Solution. We estimate terms in Equation (23). For this end, we define $B_{\delta,s,b} = X_{\delta,s,b} \times X^{\beta}_{\delta,s,b}$ and $N^{\delta,s} = G^{\delta,s} \times G^{\delta,s}$, with norms $\|(u,v)\|_{B_{\delta,s,b}} = \max\{\|u\|_{X_{\delta,s,b}}, \|v\|_{X^{\beta}_{\delta,s,b}}\}$ and similar for $N^{\delta,s}$.

Lemma 12. *Let $s > -1/2$, $\delta > 0$, and $b > 1/2$. Then, for all $(u_0, v_0) \in N^{\delta,s}$ and $0 < T < 1$, with some constant $C > 0$, we have the following:*

$$\|(\Lambda[u,v],\Gamma[u,v])\|_{B_{\delta,s,b}} \le C\left(\|(u_0,v_0)\|_{N^{\delta,s}} + T^{\varepsilon}\|(u,v)\|^3_{B_{\delta,s,b}}\right), \tag{39}$$

$$\begin{aligned}&\|\Lambda[u,v]-\Lambda[u^*,v^*],\Gamma[u,v]-\Gamma[u^*,v^*]\|_{B_{\delta,s,b}}\\ &\quad\le CT^{\varepsilon}\|(u-u^*,v-v^*)\|_{B_{\delta,s,b}}\\ &\qquad\times\left(\|(u,v)\|^2_{B_{\delta,s,b}} + \|(u,v)\|_{B_{\delta,s,b}}\|(u^*,v^*)\|_{B_{\delta,s,b}} + \|(v^*,v^*)\|^2_{B_{\delta,s,b}}\right),\end{aligned} \tag{40}$$

for all $(u,v), (u^*,v^*) \in B_{\delta,s,b}$.

Proof. To prove estimate of Equation (39), we have the following:

$$\begin{aligned}\|\Lambda[u,v]\|_{X_{\delta,s,b}} &\le C\|u_0\|_{G^{\delta,s}} + CT^{\varepsilon}\|u\|_{X_{\delta,s,b}}\|v\|^2_{X^{\beta}_{\delta,s,b}}\\ &\le C\|(u_0,v_0)\|_{N^{\delta,s}} + CT^{\varepsilon}\|(u,v)\|^3_{B_{\delta,s,b}},\end{aligned} \tag{41}$$

$$\begin{aligned}\|\Gamma[u,v]\|_{X^{\beta}_{\delta,s,b}} &\le C\|v_0\|_{G^{\delta,s}} + CT^{\varepsilon}\|u\|^2_{X_{\delta,s,b}}\|v\|_{X^{\beta}_{\delta,s,b}}\\ &\le C\|(u_0,v_0)\|_{N^{\delta,s}} + CT^{\varepsilon}\|(u,v)\|^3_{B_{\delta,s,b}}.\end{aligned} \tag{42}$$

Therefore, from Equations (41) and (42), we obtain the following:

$$\|(\Lambda[u,v],\Gamma[u,v])\|_{B_{\delta,s,b}} \le C\left(\|(u_0,v_0)\|_{N^{\delta,s}} + T^{\varepsilon}\|(u,v)\|^3_{B_{\delta,s,b}}\right). \tag{43}$$

For the estimate of Equation (40), we observe that

$$\begin{aligned}\Lambda[u,v]-\Lambda[u^*,v^*] &= \psi_T(t)\int_0^t S(t-\nu)\partial_x\left(uv^2-u^*v^{*2}\right)(x,\nu)d\nu,\\ \Gamma[u,v]-\Gamma[u^*,v^*] &= \psi_T(t)\int_0^t S_\beta(t-\nu)\partial_x\left(u^2v-u^{*2}v^*\right)(x,\nu)d\nu,\end{aligned} \tag{44}$$

where

$$\begin{aligned}\omega &= \partial_x\left(u^2v-u^{*2}v^*\right) = \partial_x\left[v(u+u^*)(u-u^*)+u^{*2}(v-v^*)\right],\\ \omega' &= \partial_x\left(uv^2-u^*v^{*2}\right) = \partial_x\left[u(v+v^*)(v-v^*)+v^{*2}(u-u^*)\right].\end{aligned} \tag{45}$$

We will show that $\Lambda \times \Gamma$ is a contraction on the ball $\mathbb{B}(0,R)$ to $\mathbb{B}(0,R)$, where $\mathbb{B}(0,R)$ is given in Equation (46).

Lemma 13. *Let $s \ge -1/4$, $\delta > 0$, and $b > 1/2$. Then, for all $(u_0, v_0) \in N^{\delta,s}$, such that the map $\Lambda \times \Gamma : \mathbb{B}(0,R) \longrightarrow \mathbb{B}(0,R)$ is a contraction, where $\mathbb{B}(0,R)$ is given by the following:*

$$\mathbb{B}(0,R) = \left\{(u,v) \in B_{\delta,s,b} \,;\, \|u,v\|_{B_{\delta,s,b}} \le R\right\}, \tag{46}$$

with $R = 2C\|(u_0,v_0)\|_{N^{\delta,s}}$.

Proof. From Lemma 12, for all $(u,v) \in \mathbb{B}(0,R)$, we have the following:

$$\begin{aligned}\|(\Lambda[u,v],\Gamma[u,v])\|_{B_{\delta,s,b}} &\le C\|(u_0,v_0)\|_{N^{\delta,s}} + CT^{\varepsilon}\|(u,v)\|^3_{B_{\delta,s,b}}\\ &\le \frac{R}{2} + CT^{\varepsilon}R^3.\end{aligned} \tag{47}$$

We choose T sufficiently small such that $T^{\varepsilon} \le 1/4CR^2$; hence,

$$\|(\Lambda[u,v],\Gamma[u,v])\|_{B_{\delta,s,b}} \le R,\ \forall (u,v) \in \mathbb{B}(0,R). \tag{48}$$

Thus, $\Lambda \times \Gamma$ maps $\mathbb{B}(0,R)$ into $\mathbb{B}(0,R)$, which is a contraction, since

$$\begin{aligned}
&\|(\Lambda[u,v]-\Lambda[u^*,v^*],\Gamma[u,v]-\Gamma[u^*,v^*]\|_{B_{\delta,s,b}}\\
&\quad\le CT^{\varepsilon}\|(u-u^*,v-v^*)\|_{B_{\delta,s,b}}\\
&\qquad\times\left(\|(u,v)\|^2_{B_{\delta,s,b}}+\|(u,v)\|_{B_{\delta,s,b}}\|(u^*,v^*)\|_{B_{\delta,s,b}}+\|(v^*,v^*)\|^2_{B_{\delta,s,b}}\right),\\
&\quad\le 3CT^{\varepsilon}R^2\|(u-u^*,v-v^*)\|_{B_{\sigma,\delta,s,b}}\le\frac{3}{4}\|(u-u^*,v-v^*)\|_{B_{\sigma,\delta,s,b}},
\end{aligned}\tag{49}$$

for all $(u,v)\in\mathbb{B}(0,R)$. Hence, (Λ,Γ): $\mathbb{B}(0,R)\longrightarrow\mathbb{B}(0,R)$ is a contraction.

3.2. The Uniqueness. Uniqueness of the solution in $C([0,T],G^{\delta,s})\times C([0,T],G^{\delta,s})$ can be proved by the following standard argument.

Suppose that $(u,v),(u^*,v^*)\in C([0,T],G^{\delta,s})\times C([0,T],G^{\delta,s})$ are solutions to Equation (1) with $(v(\cdot,0),u(\cdot,0))=(v^*(\cdot,0),u^*(\cdot,0))$ in $G^{\delta,s}\times G^{\delta,s}$. Setting $\vartheta=u-u^*$ and $\omega=v-v^*$, we see that ϑ,ω solves the Cauchy problem:

$$\partial_t\vartheta+\partial_x^3\vartheta+\partial_x\left(uv^2-u^*v^2\right)=0,\quad \vartheta(0)=0,\tag{50}$$

$$\partial_t\omega+\partial_x^3\omega+\partial_x\left(u^2v-u^2v^*\right)=0,\quad \omega(0)=0.\tag{51}$$

Thus, by Equation (50), we have the following:

$$\begin{aligned}
\frac{1}{2}\partial_t\|\vartheta(t,\cdot)\|^2_{L^2}&=\frac{1}{2}\partial_t\int_{\mathbb{R}}\vartheta^2(t,x)dx=\int_{\mathbb{R}}\vartheta(t,x)\partial_t\vartheta(t,x)dx\\
&=-\int_{\mathbb{R}}\vartheta(t,x)\partial_x\left(uv^2-u^*v^2\right)dx=0,
\end{aligned}\tag{52}$$

since we have the following:

$$\int_{\mathbb{R}}\vartheta(t,x)\partial_x^3\vartheta(t,x)dx=0.\tag{53}$$

Thanks to Equation (53), we have the following:

$$\partial_t\|\vartheta(t,\cdot)\|^2_{L^2}=-2\int_{\mathbb{R}}\vartheta(t,x)\partial_x\left[v^2\vartheta(t,x)\right]dx.\tag{54}$$

Integrating by parts of the last integral, we obtain the following:

$$\partial_t\|\vartheta(t,\cdot)\|^2_{L^2}=-\int_{\mathbb{R}}\partial_x v^2(t,x)\vartheta^2(t,x)dx,\tag{55}$$

from which we deduce the inequality as follows:

$$\left|\partial_t\|\vartheta(t,\cdot)\|^2_{L^2}\right|\le\left\|\partial_x v^2\right\|_{L^\infty}\|\vartheta(t)\|^2_{L^2}.\tag{56}$$

Since $u,u^*\in C([0,T],G^{\delta,s})$, we have that u and u^* are continuous in t on the compact set $[0,T]$ and are $G^{\delta,s}$ in x. Thus, we can conclude that

$$\left\|\partial_x v^2\right\|_{L^\infty}\le c<\infty.\tag{57}$$

Therefore, from Equations (56) and (57), we obtain the differential inequality:

$$\left|\partial_t\|\vartheta(t,\cdot)\|^2_{L^2}\right|\le c\|\vartheta(t)\|^2_{L^2},\quad 0\le t\le T.\tag{58}$$

Solving it gives the following:

$$\|\vartheta(t)\|^2_{L^2}\le e^c\|\vartheta(0)\|^2_{L^2},\quad 0\le t\le T.\tag{59}$$

Since $\|\vartheta(0)\|^2_{L^2}=0$, from Equation (59), we obtain that $\vartheta(t)=0$, $0\le t\le T$, or $u=u^*$.

Now by Equation (51), we have the following:

$$\left|\partial_t\|\omega(t,\cdot)\|^2_{L^2}\right|\le c\|\omega(t)\|^2_{L^2},\quad 0\le t\le T.\tag{60}$$

Solving it gives the following:

$$\|\omega(t)\|^2_{L^2}\le e^c\|\omega(0)\|^2_{L^2},\quad 0\le t\le T.\tag{61}$$

Since $\|\omega(0)\|^2_{L^2}=0$, from Equation (61), we obtain that $\omega(t)=0$, $0\le t\le T$, or $v=v^*$.

3.3. Continuous Dependence of the Initial Data. To prove continuous dependence of the initial data, we will prove the following.

Lemma 14. *Let $s>-1/2$, $\delta>0$, and $b>1/2$. Then, for all $(u_0,v_0),(u_0^*,v_0^*)\in N^{\delta,s}$, if (u,v) and (u^*,v^*) are two solutions to Equation (1) corresponding to initial data (u_0,v_0) and (u_0^*,v_0^*), we have the following:*

$$\|(u-u^*,v-v^*)\|_{C\left([0,T],G^{\delta,s}\right)^2}\le 4C_0C\|(u_0-u_0^*,v_0-v_0^*)\|_{N^{\delta,s}}.\tag{62}$$

Proof. If (u,v) and (u^*,v^*) are two solutions to Equation (1), corresponding to initial data (u_0,v_0) and (u_0^*,v_0^*), we have from Lemma 6 as follows:

$$\begin{aligned}
\|u-u^*\|_{C\left([0,T],G^{\delta,s}\right)}&\le C_0\|u-u^*\|_{X_{\delta,s,b}},\\
\|v-v^*\|_{C\left([0,T],G^{\delta,s}\right)}&\le C_0\|v-v^*\|_{X^{\beta}_{\delta,s,b}}.
\end{aligned}\tag{63}$$

By taking $(u,v),(u^*,v^*)\in\mathbb{B}(0,R)$ and $T^{\varepsilon}\le 1/4CR$, we have the following:

$$\begin{aligned}
\|u-u^*\|_{X_{\delta,s,b}}&\le C\|(u_0-u_0^*,v_0-v_0^*)\|_{N^{\delta,s}}+\frac{3}{4}\|(u-u^*,v-v^*)\|_{B_{\delta,s,b}},\\
\|v-v^*\|_{X^{\beta}_{\delta,s,b}}&\le C\|(u_0-u_0^*,v_0-v_0^*)\|_{N^{\delta,s}}+\frac{3}{4}\|(u-u^*,v-v^*)\|_{B_{\delta,s,b}}.
\end{aligned}\tag{64}$$

Thus,

$$\|(u-u^*, v-v^*)\|_{B_{\delta,s,b}} \leq 4C\|(u_0-u_0^*, v_0-v_0^*)\|_{N^{\delta,s}}. \quad (65)$$

Then,

$$\|(u-u^*, v-v^*)\|_{C\left([0,T],G^{\delta,s}\right)^2} \leq 4C_0C\|(u_0-u_0^*, v_0-v_0^*)\|_{N^{\delta,s}}. \quad (66)$$

This completes the proof of Theorem 3.

4. Approximate Conservation Law

We have the following:

$$\|(u,v)\|_{L^2} = \int_{\mathbb{R}} \left(u^2+v^2\right) dx, \quad (67)$$

which is conserved for a solution (u,v) of Equation (1). We are going to show an approximate conservation law for a solution to Equation (1) based on the conservation of the $L^2(\mathbb{R})$ norm of solution.

Theorem 15. *Let $\kappa \in [0, 1/2)$ and $0 < T_1 < T_0 < 1$, T_0 be as in Theorem 3 with $s=0$; there exist $b = 1/2+\varepsilon$ and $C>0$, such that for any $\delta > 0$ and any solution $(u,v) \in B_{\delta,0,b}^{T_0}$ to the Cauchy problem (Equation (1)) on the time interval $[0, T_1]$, we have the estimate:*

$$\sup_{t\in[0,T_1]} \|(u(t),v(t))\|_{N^{\delta,0}}^2 \leq \|(u(0),v(0))\|_{N^{\delta,0}}^2 + C\delta^\kappa \|(u,v)\|_{B_{\delta,0,b}}^4. \quad (68)$$

Moreover, we have the following:

$$\sup_{t\in[0,T_1]} \|u(t),v(t)\|_{N^{\delta,0}}^2 \leq \|(u(0),v(0))\|_{N^{\delta,0}}^2 + C\delta^\kappa \|(u(0),v(0)\|_{N^{\delta,0}}^4. \quad (69)$$

We need the following estimate.

Lemma 16. *Given $\kappa \in [0,-1/2)$, there exist $b = 1/2+\varepsilon$, $C>0$, and $(u,v) \in B_{\delta,0,b}$; we have the following:*

$$\|(G_1,G_2)\|_{B_{0,b-1}} \leq C\delta^\kappa \|(u,v)\|_{B_{\delta,0,b}}^3, \quad (70)$$

where $G_1 = \partial_x[(AuAvAv) - A(uv^2)]$, $G_2 = \partial_x[(AuAuAv) - A(u^2v)]$, and the operator A is given by Equation (19).

Proof. Let $L_1 = (AuAvAv) - A(uv^2)$. Then,

$$\|G_1\|_{X_{0,b-1}} = \left\| \frac{\zeta}{\left\langle \eta-\zeta^3 \right\rangle^{1-b}} \widehat{L_1}(\zeta,\eta) \right\|_{L^2_{\zeta,\eta}} = \left(\int_{\mathbb{R}^2} \frac{|\zeta|^2}{\left\langle \eta-\zeta^3 \right\rangle^{2(1-b)}} |L_1\wedge(\zeta,\eta)|^2 d\zeta d\eta \right). \quad (71)$$

We shall calculate the Fourier transform of L_1 as follows:

$$\begin{aligned}
\left|\widehat{L}(\zeta,\eta)\right| &= \left|(\widehat{AuAvAv}) - A(\widehat{uv^2})\right| \\
&= C\left|\left(e^{\delta|\zeta|}\widehat{u} * e^{\delta|\zeta|}\widehat{v} * e^{\delta|\zeta|}\widehat{v} *\right)(\zeta,\eta) - e^{\delta|\zeta|}(\widehat{u} * \widehat{v} * \widehat{v} *)(\zeta,\eta)\right| \\
&= C\left|\int_{\mathbb{R}^4} \left(e^{\delta|\zeta_1|}\widehat{u}(\zeta_1,\eta_1)e^{\delta|\zeta_2|}\widehat{v}(\zeta_2,\eta_2)e^{\delta|\zeta-\zeta_1-\zeta_2|}\widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)\right.\right. \\
&\quad \left.\left. - e^{\delta|\zeta|}\widehat{u}(\zeta_1,\eta_1)\widehat{v}(\zeta_2,\eta_2)\widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)\right) d\zeta_1 d\zeta_2 d\eta_1 d\eta_2\right| \\
&\leq C\int_{\mathbb{R}^4} \left(e^{\delta|\zeta_1|}e^{\delta|\zeta_2|}e^{\delta|\zeta-\zeta_1-\zeta_2|} - e^{\delta|\zeta|}\right) \\
&\quad \times |\widehat{u}(\zeta_1,\eta_1)\widehat{v}(\zeta_2,\eta_2)\widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)| d\zeta_1 d\zeta_2 d\eta_1 d\eta_2.
\end{aligned} \quad (72)$$

Now using Corollary 7.3 in [21], let $\theta \in [0,1]$; we have the following:

$$\begin{aligned}
&e^{\delta|\zeta_1|}e^{\delta|\zeta_2|}e^{\delta|\zeta-\zeta_1-\zeta_2|} - e^{\delta|\zeta|} \\
&\quad \leq \left[4\delta \frac{\langle \zeta-\zeta_1-\zeta_2\rangle\langle\zeta_1\rangle\langle\zeta_2\rangle}{\langle\zeta\rangle}\right]^\theta e^{\delta|\zeta_1|}e^{\delta|\zeta_2|}e^{\delta|\zeta-\zeta_1-\zeta_2|}.
\end{aligned} \quad (73)$$

For $\kappa \in [0, 1/2) \subset [0,1]$, one can see that

$$\begin{aligned}
\|G_1\|_{X_{0,b-1}}^2 &= \left\| \frac{\zeta}{\left\langle \eta-\zeta^3 \right\rangle^{1-b}} \widehat{L_1}(\zeta,\eta) \right\|_{L^2_{\zeta,\eta}}^2 \leq (C4\delta)^{2\kappa} \int_{\mathbb{R}^2} \frac{|\zeta|^2}{\left\langle \eta-\zeta^3 \right\rangle^{2(1-b)}} \\
&\quad \cdot \left[\int_{\mathbb{R}^4} \left(\frac{\langle \zeta-\zeta_1-\zeta_2\rangle\langle\zeta_1\rangle\langle\zeta_2\rangle}{\langle\zeta\rangle}\right)^\kappa \times e^{\delta|\zeta_1|}e^{\delta|\zeta_2|}e^{\delta|\zeta-\zeta_1-\zeta_2|} \times |u\wedge(\zeta_1,\eta_1)v\wedge(\zeta_2,\eta_2)v\wedge(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)| d\zeta_1 d\zeta_2 d\eta_1 d\eta_2\right]^2 d\zeta d\eta \\
&= C(4\delta)^{2\kappa} \left\| \frac{\zeta\langle\zeta\rangle^{-\kappa}}{\left\langle \eta-\zeta^3 \right\rangle^{1-b}} \int_{\mathbb{R}^4} e^{\delta|\zeta_1|}\langle\zeta_1\rangle^\kappa \widehat{u}(\zeta_1,\eta_1)e^{\delta|\zeta_2|}\langle\zeta_2\rangle^\kappa \widehat{v}(\zeta_2,\eta_2) \times e^{\delta|\zeta-\zeta_1-\zeta_2|}\langle\zeta-\zeta_1-\zeta_2\rangle^\kappa \widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2) d\zeta_1 d\zeta_2 d\eta_1 d\eta_2 \right\|_{L^2_{\zeta,\eta}}^2.
\end{aligned} \quad (74)$$

Now by taking $s=-\kappa\in(-1/2,0]$, we obtain the following:

$$\begin{aligned}\|G_1\|_{X_{0,b-1}} &\le C(4\delta)^\kappa\left\|\frac{\zeta\langle\zeta\rangle^s}{\left\langle\eta-\zeta^3\right\rangle^{1-b}}\int_{\mathbb{R}^4}\frac{e^{\delta|\zeta_1|}\widehat{u}(\zeta_1,\eta_1)}{\langle\zeta_1\rangle^s}\frac{e^{\delta|\zeta_2|}\widehat{v}(\zeta_2,\eta_2)}{\langle\zeta_2\rangle^s}\cdot\frac{e^{\delta|\zeta-\zeta_1-\zeta_2|}\widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)}{\langle\zeta-\zeta_1-\zeta_2\rangle^s}d\zeta_1 d\zeta_2 d\eta_1 d\eta_2\right\|_{L^2_{\zeta,\eta}}\\ &\le C(4\delta)^\kappa\left\|\frac{\zeta\langle\zeta\rangle^s}{\left\langle\eta-\zeta^3\right\rangle^{1-b}}\int_{\mathbb{R}^4}\frac{e^{\delta|\zeta_1|}\left\langle\eta_1-\zeta_1^3\right\rangle^b\widehat{u}(\zeta_1,\eta_1)}{\langle\zeta_1\rangle^s\left\langle\eta_1-\zeta_1^3\right\rangle^b}\frac{e^{\delta|\zeta_2|}\left\langle\eta_2-\beta\zeta_2^3\right\rangle^b\widehat{v}(\zeta_2,\eta_2)}{\langle\zeta_2\rangle^s\left\langle\eta_2-\beta\zeta_2^3\right\rangle^b}\times\frac{e^{\delta|\zeta-\zeta_1-\zeta_2|}\left\langle\eta-\eta_1-\eta_2-\beta(\zeta-\zeta_1-\zeta_2)^3\right\rangle^b\widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)}{\langle\zeta-\zeta_1-\zeta_2\rangle^s\left\langle\eta-\eta_1-\eta_2-\beta(\zeta-\zeta_1-\zeta_2)^3\right\rangle^b}\times d\zeta_1 d\zeta_2 d\eta_1 d\eta_2\right\|_{L^2_{\zeta,\eta}}.\end{aligned}\tag{75}$$

Then,

$$\begin{aligned}\|G_1\|_{X_{0,b-1}} &\le C\delta^\kappa\|Au\|_{X_{0,b}}\|Av\|^2_{X^\beta_{0,b}}=C\delta^\kappa\|u\|_{X_{\delta,0,b}}\|v\|^2_{X^\beta_{\delta,0,b}}\\ &\le C\delta^\kappa\|(u,v)\|^3_{B_{\delta,0,b}}.\end{aligned}\tag{76}$$

Now let $L_2=(AuAuAv)-A(u^2v)$. Then,

$$\begin{aligned}\|G_2\|_{X_{0,b-1}} &\le C\delta^\kappa\|Au\|^2_{X_{0,b}}\|Av\|_{X^\beta_{0,b}}=C\delta^\kappa\|u\|^2_{X_{\delta,0,b}}\|v\|_{X^\beta_{\delta,0,b}}\\ &\le C\delta^\kappa\|(u,v)\|^3_{B_{\delta,0,b}}.\end{aligned}\tag{77}$$

By Equations (76) and (77), we have the following:

$$\|(G_1,G_2)\|_{B_{0,b-1}}\le C\delta^\kappa\|(u,v)\|^3_{B_{\delta,0,b}}.\tag{78}$$

Proof (Theorem 15). Let $U(t,x)=Au(t,x)$, $V(t,x)=Av(t,x)$ which are real-valued since the multiplier A is even and u,v are real-valued. Applying A to Equation (1), we obtain the following:

$$\partial_t U+\partial_x^3 U+\partial_x\left(UV^2\right)=G_1,\tag{79}$$

$$\partial_t U+\partial_x^3 U+\partial_x\left(U^2V\right)=G_2,\tag{80}$$

where $G_1=\partial_x[(AuAvAv)-A(uv^2)]$ and $G_2=\partial_x[(AuAuAv)-A(u^2v)]$.

By multiplying both sides of Equation (79) by U and Equation (80) by V and integrating with respect to space variable, we get the following:

$$\begin{aligned}&\int_{\mathbb{R}}U\partial_t U dx+\int_{\mathbb{R}}U\partial_x^3 U dx+\int_{\mathbb{R}}U\partial_x\left(UV^2\right)dx=\int_{\mathbb{R}}UG_1 dx,\\ &\int_{\mathbb{R}}V\partial_t V dx+\int_{\mathbb{R}}V\partial_x^3 V dx+\int_{\mathbb{R}}V\partial_x\left(U^2V\right)dx=\int_{\mathbb{R}}VG_2 dx.\end{aligned}\tag{81}$$

Then,

$$\begin{aligned}&\int_{\mathbb{R}}(U\partial_t U+V\partial_t V)dx+\int_{\mathbb{R}}\left(U\partial_x^3 U+V\partial_x^3 V\right)dx+\int_{\mathbb{R}}\left[U\partial_x\left(UV^2\right)\right.\\ &\quad\left.+V\partial_x\left(U^2V\right)\right]dx=\int_{\mathbb{R}}(UG_1+VG_2)dx,\\ &\int_{\mathbb{R}}(U\partial_t U+V\partial_t V)dx+\int_{\mathbb{R}}\partial_x(\partial_x U\partial_x U+\partial_x V\partial_x V)dx\\ &\quad+\int_{\mathbb{R}}\partial_x\left(U^2V^2\right)dx=\int_{\mathbb{R}}(UG_1+VG_2)dx.\end{aligned}\tag{82}$$

Noting that $\partial_x^j U(x,t)\longrightarrow 0$ as $|x|\longrightarrow\infty$ (see [20]), we use integration by parts to obtain the following:

$$\frac{1}{2}\partial_t\int_{\mathbb{R}}\left(U^2+V^2\right)dx=\int_{\mathbb{R}}(UG_1+VG_2)dx.\tag{83}$$

Integrating the last equality with respect to $t\in[0,T_1]$, we obtain the following:

$$\begin{aligned}\int_{\mathbb{R}}\left(U^2(T_1,x)+V^2(T_1,x)\right)dx&=\int_{\mathbb{R}}\left(U^2(0,x)+V^2(0,x)\right)dx\\ &\quad+2\int_{\mathbb{R}^2}\chi_{[0,T_1]}(t)(UG_1+VG_2)dxdt.\end{aligned}\tag{84}$$

Thus,

$$\begin{aligned}\|u(T_1)\|^2_{G^{\delta,0}}+\|v(T_1)\|^2_{G^{\delta,0}}&=\|u(0)\|^2_{G^{\delta,0}}+\|v(0)\|^2_{G^{\delta,0}}\\ &\quad+2\left|\int_{\mathbb{R}^2}\chi_{[0,T_1]}(t)(UG_1+VG_2)dxdt\right|.\end{aligned}\tag{85}$$

By using Holder's inequality, Lemma 10, Lemma 9, and the fact that

$$\frac{1}{2}<1-b<\frac{1}{2},\ \frac{1}{2}<b-1<\frac{1}{2}.\tag{86}$$

Since $b > 1/2 + \varepsilon$, we obtain the following:

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\chi_{[0,T_1]}(t)(UG_1+VG_2)dxdt\right| &\le \left\|\chi_{[0,T_1]}(t)U\right\|_{X_{0,1-b}} \\ &\times \left\|\chi_{[0,T_1]}(t)G_1\right\|_{X_{0,b-1}} + \left\|\chi_{[0,T_1]}(t)V\right\|_{X^{\beta}_{0,1-b}}\left\|\chi_{[0,T_1]}(t)G_2\right\|_{X^{\beta}_{0,b-1}} \\ &\le C\|U\|_{X^{T_1}_{0,1-b}}\|G_1\|_{X^{T_1}_{0,b-1}} + C\|V\|_{X^{\beta,T_1}_{0,1-b}}\|G_2\|_{X^{\beta,T_1}_{0,b-1}} \\ &\le C\|\Theta_{T_1}U\|_{X_{0,1-b}}\|\Theta_{T_1}G_1\|_{X_{0,b-1}} + C\|\Theta_{T_1}V\|_{X^{\beta}_{0,1-b}}\|\Theta_{T_1}G_2\|_{X^{\beta}_{0,b-1}} \\ &\le C\|U\|_{X_{0,1-b}}\|G_1\|_{X_{0,b-1}} + C\|V\|_{X^{\beta}_{0,1-b}}\|G_2\|_{X^{\beta}_{0,b-1}},\end{aligned} \tag{87}$$

where $\Theta_{T_1} = 1$ for $t \in [0, T_1]$; we can conclude from Lemma 16:

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\chi_{[0,T_1]}(t)(UG_1+VG_2)dxdt\right| &\le C\|U\|_{X_{0,1-b}}\|G_1\|_{X_{0,b-1}} \\ &+ C\|V\|_{X^{\beta}_{0,1-b}}\|G_2\|_{X^{\beta}_{0,b-1}} \le C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}}\|v\|^2_{X^{\beta}_{\delta,0,b}} \\ &+ C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}}\|v\|^2_{X^{\beta}_{\delta,0,b}} = 2C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}}\|v\|^2_{X^{\beta}_{\delta,0,b}} \\ &\le 2C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}}.\end{aligned} \tag{88}$$

Therefore,

$$\begin{aligned}&\|u(T_1)\|^2_{G^{\delta,0}} + \|v(T_1)\|^2_{G^{\delta,0}} \le \|u(0)\|^2_{G^{\delta,0}} + \|v(0)\|^2_{G^{\delta,0}} + 2C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}}, \\ &2\|(u(T_1),v(T_1))\|^2_{N^{\delta,0}} \le 2\|(u(0),v(0))\|^2_{N^{\delta,0}} + 2C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}}, \\ &\sup_{t\in[0,T_1]}\|(u(t),v(t))\|^2_{N^{\delta,0}} \le \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}}.\end{aligned} \tag{89}$$

Finally, by using Equation (8), we conclude that

$$\sup_{t\in[0,T_1]}\|(u(t),v(t))\|^2_{N^{\delta,0}} \le \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^{\kappa}\|(u(0),v(0))\|^4_{N^{\delta,0}}. \tag{90}$$

5. Proof of Theorem 4

Let $\delta_0 > 0$, $s > -1/2$, and $\kappa \in (0, 1/2)$ be fixed, and $(u_0, v_0) \in N^{\delta_0,s}$. Then, we have to prove that the solution (u, v) of Equation (1) satisfies the following:

$$(u,v) \in C\left(\left[0,T'\right], G^{\delta\left(T'\right),s}\right) \times C\left(\left[0,T'\right], G^{\delta\left(T'\right),s}\right), \tag{91}$$

where

$$\delta\left(T'\right) = \min\left\{\delta_0, C_1 {T'}^{-1/\kappa}\right\}, \quad \text{for all } T' > 0, \tag{92}$$

and $C_1 > 0$ is a constant depending on u_0, v_0, δ_0, s, and κ. By Theorem 3, there is a maximal time $T^* = T^*(u_0, v_0, \delta_0, s) \in (0,\infty]$, such that

$$(u,v) \in C\left([0,T^*], G^{\delta_0,s}\right) \times C\left([0,T^*], G^{\delta_0,s}\right). \tag{93}$$

If $T^* = \infty$, it is done.

If $T^* < \infty$, as we assume henceforth, it remains to prove the following:

$$(u,v) \in C\left(\left[0,T'\right], G^{C_1T'^{-1/\kappa},s}\right) \times C\left(\left[0,T'\right], G^{C_1T'^{-1/\kappa},s}\right), \quad \text{for all } T' \ge T^*. \tag{94}$$

5.1. The Case S = 0. Fixed $T' \ge T^*$; we will show that, for $\delta > 0$, sufficiently small

$$\sup_{t\in\left[0,T'\right]\|(u(0),v(0))\|^2_{N^{\delta,0}} \le 2\|(u(0),v(0))\|^2_{N^{\delta_0,0}}.} \tag{95}$$

In this case, by Theorem 3 and Theorem 15 with

$$T_0 = \frac{1}{\left(16C^3 + 32C^3\|(u(0),v(0))\|^2_{N^{\delta_0,0}}\right)^{1/\varepsilon}}, \tag{96}$$

the smallness conditions on δ will be

$$\delta < \delta_0, \frac{2T'}{T_0}C\delta^{\kappa}2^2\|(u(0),v(0))\|^2_{N^{\delta_0,0}} \le 1, \quad C > 0. \tag{97}$$

Here, C is the constant in Theorems 15.

By induction, we check that

$$\sup_{t\in[0,nT_0]}\|(u(t),v(t))\|^2_{N^{\delta,0}} \le \|(u(0),v(0))\|^2_{N^{\delta,0}} + nC\delta^{\kappa}2^2\|(u(0),v(0))\|^4_{N^{\delta_0,0}}, \tag{98}$$

$$\sup_{t\in[0,nT_0]}\|(u(t),v(t))\|^2_{N^{\delta,0}} \le 2\|(u(0),v(0))\|^2_{N^{\delta_0,0}}, \tag{99}$$

for $n \in \{1, \cdots, m+1\}$, where $m \in \mathbb{N}$ is chosen so that $T' \in [mT_0, (m+1)T_0)$. This m does exist; by Theorem 3 and the definition of T^*, we have the following:

$$T_0 < \frac{1}{\left(16C^3 + 16C^3\|(u(0),v(0))\|^2_{N^{\delta_0,0}}\right)^{1/\varepsilon}} < T^*, \text{hence } T_0 < T'. \tag{100}$$

In the first step, we cover the interval $[0, T_0]$, and by Theorem 15, we have the following:

$$\begin{aligned}\sup_{t\in[0,T_0]}\|(u(t),v(t))\|^2_{N^{\delta,0}} &\le \|(u(0),v(0))\|^2_{N^{\delta,0}} \\ &+ C\delta^{\kappa}\|(u(0),v(0))\|^4_{N^{\delta,0}} \le \|(u(0),v(0))\|^2_{N^{\delta,0}} \\ &+ C\delta^{\kappa}\|(u(0),v(0))\|^4_{N^{\delta_0,0}},\end{aligned} \tag{101}$$

since $\delta \leq \delta_0$; we used the following:

$$\|(u(0), v(0))\|_{N^{\delta,0}} \leq \|(u(0), v(0))\|_{N^{\delta_0,0}}. \tag{102}$$

This satisfies Equation (98) for $n = 1$, and Equation (99) is following and using again $\|(u(0), v(0))\|_{N^{\delta,0}} \leq \|(u(0), v(0))\|_{N^{\delta_0,0}}$ as well as the following:

$$C\delta^{\kappa}\|(u(0), v(0))\|^2_{N^{\delta_0,0}} \leq 1. \tag{103}$$

Suppose now that Equations (98) and (99) hold for some $n \in \{1, \cdots, m\}$ and we prove that it holds for $n + 1$, we estimate the following:

$$\begin{aligned}
&\sup_{t\in[nT_0,(n+1)T_0]} \|(u(t), v(t))\|^2_{N^{\delta,0}} \leq \|w(nT_0)\|^2_{N^{\delta,0}} \\
&\quad + C\delta^{\kappa}\|(u(nT_0), v(nT_0))\|^4_{N^{\delta,0}} \leq \|(u(nT_0nT_0), v(nT_0))\|^2_{N^{\delta,0}} \\
&\quad + C\delta^{\kappa}2^2\|(u(0), v(0))\|^4_{N^{\delta_0,0}} \leq \|(u(0), v(0))\|^2_{N^{\delta,0}} \\
&\quad + nC\delta^{\kappa}2^2\|(u(0), v(0))\|^4_{N^{\delta_0,0}} + C\delta^{\kappa}2^2\|(u(0), v(0))\|^4_{N^{\delta_0,0}},
\end{aligned} \tag{104}$$

satisfying Equation (98) with n replaced by $n + 1$. To get Equation (99) with n replaced by $n + 1$, it is then enough to have the following:

$$(n + 1)C\delta^{\kappa}2^2\|(u(0), v(0))\|^2_{N^{\delta_0,0}} \leq 1, \tag{105}$$

but this holds by Equation (97), since $n + 1 \leq m + 1 \leq T'/T_0 + 1 < 2T'/T_0$.

Finally, Equation (97) is satisfied for $\delta \in (0, \delta_0)$ such that

$$\frac{2T'}{T_0} C\delta^{\kappa}2^2\|(u(0), v(0))\|^2_{N^{\delta_0,0}} = 1. \tag{106}$$

Thus, $\delta = C_1 T'^{-1/\kappa}$, where

$$C_1 = \left(\frac{1}{C2^3\|(u(0), v(0))\|^2_{N^{\delta_0,0}}\left(16C^3 + 32C^3\|(u(0), v(0))\|^2_{N^{\delta_0,0}}\right)^{1/\varepsilon}}\right)^{1/\kappa}. \tag{107}$$

5.2. The General Case. For all s, by Equation (4), we have $u_0, v_0 \in G^{\delta_0,s} \subset G^{\delta_0/2,0}$.

For case $s = 0$, it is proved that there is a $T_2 > 0$, such that

$$\begin{aligned}
&(u, v) \in C\left([0, T_2), G^{\delta_0/2,0}\right) \times C\left([0, T_2), G^{\delta_0/2,0}\right), \\
&(u, v) \in C\left(\left[0, T'\right], G^{2\sigma T'^{-1/\kappa},0}\right) \times C\left(\left[0, T'\right], G^{2\sigma T'^{-1/\kappa},0}\right), \quad \text{for } T' \geq T_2,
\end{aligned} \tag{108}$$

where $\sigma > 0$ depends on u_0, v_0, δ_0, and κ.

Applying again the embedding Equation (4), we now conclude that

$$\begin{aligned}
&(u, v) \in C\left([0, T_2), G^{\delta_0/4,s}\right) \times C\left([0, T_2), G^{\delta_0/4,s}\right), \\
&(u, v) \in C\left(\left[0, T'\right], G^{\sigma T'^{-1/\kappa},s}\right) \times C\left(\left[0, T'\right], G^{\sigma T'^{-1/\kappa},s}\right), \quad \text{for } T' \geq T_2,
\end{aligned} \tag{109}$$

which imply Equation (94). The proof of Theorem 4 is now completed.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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Research Article

Phillips-Type q-Bernstein Operators on Triangles

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The purpose of the paper is to introduce a new analogue of Phillips-type Bernstein operators $(\mathscr{B}^u_{m,q}f)(u,v)$ and $(\mathscr{B}^v_{n,q}f)(u,v)$, their products $(\mathscr{P}_{mn,q}f)(u,v)$ and $(\mathscr{Q}_{nm,q}f)(u,v)$, their Boolean sums $(\mathscr{S}_{mn,q}f)(u,v)$ and $(\mathscr{T}_{nm,q}f)(u,v)$ on triangle $\mathscr{T}_h$, which interpolate a given function on the edges, respectively, at the vertices of triangle using quantum analogue. Based on Peano's theorem and using modulus of continuity, the remainders of the approximation formula of corresponding operators are evaluated. Graphical representations are added to demonstrate consistency to theoretical findings. It has been shown that parameter q provides flexibility for approximation and reduces to its classical case for $q=1$.

1. Introduction and Essential Preliminaries

In 1912, Bernstein constructed polynomials to provide a constructive proof of the Weierstrass approximation theorem [1, 2] using probabilistic interpolation, which is now known as Bernstein polynomials in approximation theory. In computer-aided geometric design (CAGD), the basis of Bernstein polynomials plays a significant role to preserve the shape of the curves and surfaces.

Further, with the development of q-calculus (quantum analogue), the first q-analogue of Bernstein operators (rational) was constructed by Lupas in [3]. In 1997, Phillips [4] initiated another generalization of Bernstein polynomials based on the q-integers (quantum analogue) called q-Bernstein polynomials. The q-Bernstein polynomials attracted a lot of attention and were studied broadly by several researchers. One can find a survey of the obtained results and references on the subject in [5].

Computer-aided geometric design (CAGD) is a discipline which deals with computational aspects of geometric objects. It emphasizes on the mathematical development of curves and surfaces such that it becomes compatible with computers. Popular programs, like Adobe's Illustrator and Flash, and font imaging systems, such as Postscript, utilize Bernstein polynomials to form what are known as Bézier curves [6–9].

The approximating operators on triangles and their basis have important applications in finite element analysis and computer-aided geometric design [10] etc. Starting with the paper [11] of Barnhill et al., the blending interpolation operators were considered in the papers [12–14].

In this paper, we construct new operators based on quantum analogue of Phillips. Bernstein-type operators also interpolate the value of a given function on the boundary of the triangle. Also, we will discuss some particular cases. Using modulus of continuity and Peano's theorem, the remainders of the corresponding approximation formulas are evaluated. The accuracy of the approximation is also illustrated by graphics of given functions with suitable Bernstein-type approximation. For more information regarding such operators, their properties and their remainders one can refer to [15–28].

In this paper, we would like to draw attention to the Phillips q-analogue of the Bernstein operators and obtain new results using q-analogue on triangles. To present results by Phillips, we recall the following definitions. For other relevant works, one can see [29].

Let $q>0$. For any $m=0,1,2,\cdots$, the q-integer $[m]_q$ is defined by

$$[m]_q := 1+q+\cdots+q^{m-1}, \quad m=1,2,\cdots, [0]_q := 0, \tag{1}$$

and the q-factorial $[m]_q!$ by

$$[m]_q! := [1]_q [2]_q \cdots [m]_q, \quad m = 1, 2, \cdots, [0]_q! = 1. \tag{2}$$

For integers $0 \le i \le m$, the q-binomial or the Gaussian coefficient is defined by

$$\begin{bmatrix} m \\ i \end{bmatrix}_q := \frac{[m]_q!}{[i]_q! [m-i]_q!}. \tag{3}$$

Clearly, for $q = 1$,

$$[m]_1 = m, [m]_1! = m!, \begin{bmatrix} m \\ i \end{bmatrix}_1 = \begin{pmatrix} m \\ i \end{pmatrix}. \tag{4}$$

The q-binomial coefficients are involved in Cauchy's q-binomial theorem (cf. [30], Chapter 10, Section 10.2). The first one is a q-analogue as an extension to Newton's binomial formula:

$$(au + bv)_q^m := \sum_{i=0}^{m} q^{(i(i-1))/2} \begin{bmatrix} m \\ i \end{bmatrix}_q a^{m-i} b^i u^{m-i} v^i, \tag{5}$$

$$(1+u)(1+qu) \cdot \cdots \cdot \left(1 + q^{m-1} u\right) = \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q q^{i(i-1)/2} u^i. \tag{6}$$

Following Phillips, we denote

$$b_{m,i}(u, v) = \begin{bmatrix} m \\ i \end{bmatrix}_q \prod_{s=0}^{m-i-1} (1 - q^s u). \tag{7}$$

It follows from (6) that

$$\sum_{i=0}^{m} b_{m,i}(q; u) = 1, \quad u \in [0, 1], \tag{8}$$

for integers $k \ge i \ge 0$. These recurrence relations are satisfied by q-binomial coefficients

$$\begin{aligned} \begin{bmatrix} k+1 \\ i \end{bmatrix}_q &= q^{k-i+1} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q + \begin{bmatrix} k \\ i \end{bmatrix}_q, \\ \begin{bmatrix} k+1 \\ i \end{bmatrix}_q &= \begin{bmatrix} k \\ i-1 \end{bmatrix}_q + q^i \begin{bmatrix} k \\ i \end{bmatrix}_q, \end{aligned} \tag{9}$$

when $q = 1$, both the relations reduce to the Pascal identity. In the next section, we construct quantum analogue of operators studied in [31] on triangles.

2. Construction of New Univariate Operators on Triangle

In [31], the authors considered only the standard triangle sufficient due to affine invariance as

$$\mathcal{T}_h = \left\{(u, v) \in \mathbb{R}^2 \mid u \ge 0, v \ge 0, u + v \le h\right\}, \quad \text{for } h > 0. \tag{10}$$

Let $\Delta_m^u = \{i((h-v)/m), i = \overline{0, m}\}$ and $\Delta_n^v = \{j((h-u)/n), j = \overline{0, n}\}$ be uniform partitions of the intervals $[0, h-v]$ and $[0, h-u]$, respectively.

In 2009, they [31] constructed some univariant Bernstein-type operators on triangle $\mathcal{T}_h$ as follows:

$$\begin{aligned} (\mathcal{B}_m^u f)(u, v) &= \sum_{i=0}^{m} p_{m,i}(u, v) f\left(\frac{i}{m}(h-v), v\right), \\ (\mathcal{B}_n^v f)(u, v) &= \sum_{j=0}^{n} q_{n,j}(u, v) f\left(u, \frac{j}{n}(h-u)\right), \end{aligned} \tag{11}$$

where

$$p_{m,i}(u, v) = \frac{\begin{pmatrix} m \\ i \end{pmatrix} u^i (h-u-v)^{m-i}}{(h-v)^m}, \quad 0 \le u + v \le h, \tag{12}$$

$$q_{n,j}(u, v) = \frac{\begin{pmatrix} n \\ j \end{pmatrix} v^j (h-u-v)^{n-j}}{(h-u)^n}, \quad 0 \le u + v \le h, \tag{13}$$

respectively.

Consider a real-valued function f defined on $\mathcal{T}_h$ as done in [31]. Through the point $(u, v) \in \mathcal{T}_h$, one considers the parallel lines to the coordinate axes which intersect the edges Γ_i, $i = 1, 2, 3$, of the triangle at the points $(0, v)$ and $(h-v, v)$, respectively $(u, 0)$ and $(u, h-u)$ ([31], Figure 1).

Let $\Delta_m^u = \{i((h-v)/m), i = \overline{0, m}\}$ and $\Delta_n^v = \{j((h-u)/n), j = \overline{0, n}\}$ be uniform partitions of the intervals $[0, h-v]$ and $[0, h-u]$, respectively.

We define the new Phillips-type Bernstein operators $\mathcal{B}_{m,q}^u$ and $\mathcal{B}_{n,q}^v$ on triangle by using quantum calculus as follows:

$$\begin{aligned} \left(\mathcal{B}_{m,q}^u f\right)(u, v) &= \begin{cases} \sum_{i=0}^{m} \tilde{p}_{m,i}(u, v) f\left(\frac{[i]_q}{[m]_q}(h-v), v\right), & (u, v) \in \mathcal{T}_h \setminus (0, h), \\ f(0, h), & (0, h) \in \mathcal{T}_h, \end{cases} \\ \left(\mathcal{B}_{n,q}^v f\right)(u, v) &= \begin{cases} \sum_{j=0}^{n} \tilde{q}_{n,j}(u, v) f\left(u, \frac{[j]_q}{[n]_q}(h-u)\right), & (u, v) \in \mathcal{T}_h \setminus (h, 0), \\ f(h, 0), & (h, 0) \in \mathcal{T}_h, \end{cases} \end{aligned} \tag{14}$$

where

$$\tilde{p}_{m,i}(u,v) = \frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i \prod_{s=0}^{m-i-1} (h-v-q^s u)}{(h-v)^m}, \quad 0 \le u+v \le h \text{ (except the point} (0,h)), \tag{15}$$

$$\tilde{q}_{n,j}(u,v) = \frac{\begin{bmatrix} n \\ j \end{bmatrix}_q v^j \prod_{t=0}^{n-j-1} (h-u-q^t v)}{(h-u)^n}, \quad 0 \le u+v \le h \text{ (except the point} (0,h)), \tag{16}$$

respectively. These operators reduce to Phillips-type operator on $[0,1]$. One can note that the bases (15) and (16) of the operators constructed using quantum calculus are different from the bases (12) and (13) of the operators constructed by Blaga and Coman [31]. In case $q=1$, corresponding operators reduce to its classical case on triangles. Now, we generalize various results of [31] in quantum calculus frame.

For the sake of convenience, we use the following notation onwards:

$$\begin{aligned} (h-v)^m &:= \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q u^i \prod_{s=0}^{m-i-1} (h-v-q^s u), \\ (h-u)^n &:= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q v^i \prod_{s=0}^{n-i-1} (h-u-q^s v). \end{aligned} \tag{17}$$

Theorem 1. *If f is a real-valued function defined on $\mathcal{T}_h$, then*

(i) $\mathcal{B}^u_{m,q} f = f$ *on* $\Gamma_2 \cup \Gamma_3$

(ii) $(\mathcal{B}^u_{m,q} e_{i0})(u,v) = u^i$, $i = 0, 1 (dex(\mathcal{B}^u_{m,q}) = 1)$

(iii) $(\mathcal{B}^u_{m,q} e_{20})(u,v) = u^2 + ((u(h-u-v))/[m]_q)$

$$\left(\mathcal{B}^u_{m,q} e_{ij}\right)(u,v) = \begin{cases} v^j u^i, & i = 0, 1, j \in N, \\ v^j \left(u^2 + \dfrac{u(h-u-v)}{[m]_q} \right), & i = 2, j \in \mathbb{N}, \end{cases} \tag{18}$$

where $e_{ij}(u,v) = u^i v^j$ and dex $(\mathcal{B}^u_{m,q})$ is the degree of exactness of the operator $\mathcal{B}^u_{m,q}$.

Proof. By definition, $(B^u_{m,q} f)(0,h) = f(0,h)$. So we will calculate the moments only on $\mathcal{T}_h \setminus (0,h)$. The interpolation property (i) follows from the relations

$$\begin{aligned} \tilde{p}_{m,i}(0,v) &= \begin{cases} 1, & \text{if } i = 0, \\ 0, & i \neq 0, \end{cases} \\ \tilde{p}_{m,i}(h-v,v) &= \begin{cases} 1, & \text{if } i = m, \\ 0, & i \neq m. \end{cases} \end{aligned} \tag{19}$$

Regarding the property (ii), we have

$$\left(\mathcal{B}^u_{m,q} e_{00}\right)(u,v) = \sum_{i=0}^{m} \frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i \prod_{s=0}^{m-i-1} (h-v-q^s u)}{(h-v)^m} = \frac{(h-v)^m}{(h-v)^m} = 1,$$

$$\begin{aligned} \left(\mathcal{B}^u_{m,q} e_{10}\right)(u,v) &= \sum_{i=0}^{m} \frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i \prod_{s=0}^{m-i-1} (h-v-q^s u)}{(h-v)^m} \frac{[i]_q}{[m]_q} (h-v) \\ &= \sum_{i=0}^{m} \frac{\left([i]_q/[m]_q\right) \begin{bmatrix} m \\ i \end{bmatrix}_q u^i \prod_{s=0}^{m-i-1} (h-v-q^s u)}{(h-v)^{m-1}} \\ &= \sum_{i=0}^{m-1} \frac{\begin{bmatrix} m-1 \\ i \end{bmatrix}_q u^{i+1} \prod_{s=0}^{m-i-2} (h-v-q^s u)}{(h-v)^{m-1}} \\ &= u \sum_{i=0}^{m-1} \frac{\begin{bmatrix} m-1 \\ i \end{bmatrix}_q u^i \prod_{s=0}^{(m-1)-i-1} (h-v-q^s u)}{(h-v)^{m-1}} = u, \end{aligned}$$

$$\begin{aligned} \left(\mathcal{B}^u_{m,q} e_{20}\right)(u,v) &= \sum_{i=0}^{m} \frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i \prod_{s=0}^{m-i-1} (h-v-q^s u)}{(h-v)^m} \frac{[i]_q^2}{[m]_q^2} (h-v)^2 \\ &= (h-v)^2 \sum_{i=0}^{m-1} \frac{([i+1]q/[m]q) \begin{bmatrix} m \\ i \end{bmatrix}_q u^{i+1} \prod_{s=0}^{m-i-2} (h-v-q^s u)}{(h-v)^m} \\ &= (h-v)^2 u \sum_{i=0}^{m-1} \frac{((1+q[i]q)/[m]q) \begin{bmatrix} m \\ i \end{bmatrix}_q u^i \prod_{s=0}^{m-i-2} (h-v-q^s u)}{(h-v)^m} \\ &= (h-v) \frac{u}{[m]_q} \sum_{i=0}^{m-1} \frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i \prod_{s=0}^{(m-1)-i-1} (h-v-q^s u)}{(h-v)^{m-1}} + (h-v)^2 u \\ &\quad \cdot \sum_{i=0}^{m-1} \frac{\left(q[m-1]_q/[m]_q\right)\left([i]_q/[m-1]_q\right) \begin{bmatrix} m-1 \\ i \end{bmatrix}_q u^i \prod_{s=0}^{m-i-2} (h-v-q^s u)}{(h-v)^m} \\ &= (h-v) \frac{u}{[m]_q} + \frac{q[m-1]_q u^2}{[m]_q} \sum_{i=0}^{m-2} \frac{\begin{bmatrix} m-2 \\ i \end{bmatrix}_q u^i \prod_{s=0}^{(m-2)-i-1} (h-v-q^s u)}{(h-v)^{m-2}}, \end{aligned}$$

$$\left(\mathcal{B}^u_{m,q} e_{20}\right)(u,v) = (h-v) \frac{u}{[m]_q} + \frac{q[m-1]_q u^2}{[m]_q}, \tag{20}$$

or equivalently,

$$\left(\mathcal{B}^u_{m,q} e_{20}\right)(u,v) = (h-v) \frac{u}{[m]_q} + u^2 \left(1 - \frac{1}{[m]_q}\right) = u^2 + \frac{u(h-u-v)}{[m]_q}. \tag{21}$$

Remark 2. In the same way, it can be proved that if f is a real-valued function defined on $\mathcal{T}_h$, then

(i) $\mathcal{B}^v_{n,q}f = f$ on $\Gamma_1 \cup \Gamma_3$

(ii) $(\mathcal{B}^v_{n,q}e_{0j})(u,v) = v^j$, $j = 0, 1 (\mathrm{dex}(\mathcal{B}^v_{n,q}) = 1)$

(iii) $(\mathcal{B}^v_{n,q}e_{02})(u,v) = v^2 + ((v(h-u-v))/[n]_q)$

$$\left(\mathcal{B}^v_{n,q}e_{ij}\right)(u,v) = \begin{cases} u^i v^j, & j = 0, 1, i \in N, \\ u^i\left(v^2 + \dfrac{v(h-u-v)}{[n]_q}\right), & j = 2, i \in \mathbb{N}. \end{cases} \tag{22}$$

Based on the following approximation formula

$$f = \mathcal{B}^u_{m,q}f + \mathcal{R}^u_{m,q}f, \tag{23}$$

we present the following results.

Theorem 3. *If $f(.,v) \in C[0,h-v]$, then*

$$\left|\left(\mathcal{R}^u_{m,q}f\right)(u,v)\right| \le \left(1+\frac{h}{2\delta\sqrt{[m]_q}}\right) w(f(.,v);\delta), v \in [0,h], \tag{24}$$

where modulus of continuity of the function f with respect to the variable u is denoted by $w(f(.,v);\delta)$.

Further, if $\delta = 1/\sqrt{[m]_q}$, then

$$\left|\left(\mathcal{R}^u_{m,q}f\right)(u,v)\right| \le \left(1+\frac{h}{2}\right) w\left(f(.,v);\frac{1}{\sqrt{[m]_q}}\right), v \in [0,h]. \tag{25}$$

Proof. Since by definition, $(B^u_{m,q}f)(0,h) = f(0,h)$ and hence remainder will be zero at $(0,h)$ due to interpolation. We have

$$\left|\left(\mathcal{R}^u_{m,q}f\right)(u,v)\right| \le \sum_{i=0}^{m} \tilde{p}_{m,i}(u,v)\left|f(u,v) - f\left(\frac{[i]_q(h-v)}{[m]_q}, v\right)\right|. \tag{26}$$

Since

$$|f(u,v) - f\left(\frac{[i]_q(h-v)}{[m]_q}, v\right)| \le \left(\frac{1}{\delta}\,|\,u - \frac{[i]_q(h-v)}{[m]_q}| + 1\right) w(f(.,v);\delta), \tag{27}$$

one obtains

$$\begin{aligned} |\left(\mathcal{R}^u_{m,q}f\right)(u,v)| &\le \sum_{i=0}^{m} \tilde{p}_{m,i}(u,v)\left(\frac{1}{\delta}\,|\,u - \frac{[i]_q(h-v)}{[m]_q}| + 1\right) w(f(.,v);\delta) \\ &\le \left[1 + \frac{1}{\delta}\left(\sum_{i=0}^{m} \tilde{p}_{m,i}(u,v)\left(u - \frac{[i]_q(h-v)}{[m]_q}\right)^2\right)^{1/2}\right] w(f(.,v);\delta) \\ &= \left[1 + \frac{1}{\delta}\sqrt{\frac{u(h-u-v)}{[m]_q}}\right] w(f(.,v);\delta). \end{aligned} \tag{28}$$

As

$$\max_{\mathcal{T}_h}[u(h-u-v)] = \frac{h^2}{4}, \tag{29}$$

it follows that

$$|\left(\mathcal{R}^u_{m,q}f\right)(u,v)| \le \left(1+\frac{h}{2\delta\sqrt{[m]_q}}\right) w(f(.,v);\delta). \tag{30}$$

For $\delta = 1/\sqrt{[m]_q}$, we obtain

$$|\left(\mathcal{R}^u_{m,q}f\right)(u,v)| \le \left(1+\frac{h}{2}\right) w\left(f(.,v);\frac{1}{\sqrt{[m]_q}}\right). \tag{31}$$

Theorem 4. *If $f(.,v) \in C^2[0,h]$, then*

$$\left(\mathcal{R}^u_{m,q}f\right)(u,v) = -\frac{u(h-u-v)}{2[m]_q} f^{(2,0)}(\xi, y), \xi \in [0, h-v], \tag{32}$$

$$\left|\left(\mathcal{R}^u_{m,q}f\right)(u,v)\right| \le \frac{h^2}{8[m]_q}\mathcal{M}_{20}f, (u,v) \in \mathcal{T}_h, \tag{33}$$

where

$$\mathcal{M}_{ij}f = \max_{\mathcal{T}_h}\left|f^{(i,j)}(u,v)\right|. \tag{34}$$

Proof. As $\mathrm{dex}(\mathcal{B}^u_{m,q}) = 1$, by Peano's theorem, one obtains

$$\left(\mathcal{R}^u_{m,q}f\right)(u,v) = \int_0^{h-v} \mathcal{K}_{20}(u,v;t) f^{(2,0)}(t,v)\,dt, \tag{35}$$

where the kernel

$$\begin{aligned}\mathcal{K}_{20}(u,v;t) &:= \mathcal{R}^u_{m,q}\left[(u-t)_+\right]\\ &= (u-t)_+ - \sum_{i=0}^{m}\tilde{p}_{m,i}(u,v)\left([i]_q\frac{h-v}{[m]_q}-t\right)_+\end{aligned} \quad (36)$$

does not change the sign$(\mathcal{K}_{20}(u,v;t)\le 0, u\in[0,h-v])$. By the Mean Value Theorem, it follows that

$$\left(\mathcal{R}^u_{m,q}f\right)(u,v) = f^{(2,0)}(\xi,v)\int_0^{h-v}\mathcal{K}_{20}(u,v;t)dt, \quad \xi\in[0,h-v]. \quad (37)$$

After an easy calculation, we get

$$\left(\mathcal{R}^u_{m,q}f\right)(u,v) = -\frac{u(h-u-v)}{2[m]_q}f^{(2,0)}(\xi,v), \quad (38)$$

where $\xi\in[0,h-v]$.

By using it in Equation (32), we get

$$\left|\left(\mathcal{R}^u_{m,q}f\right)(u,v)\right| \le \frac{h^2}{8[m]_q}\mathcal{M}_{20}f, \quad (u,v)\in\mathcal{T}_h. \quad (39)$$

Remark 5. From (32), it follows that

(i) if $f(.,v)$ is a concave function, then $(\mathcal{R}^u_{m,q}f)(u,v)\ge 0$, i.e.,

$$\left(\mathcal{B}^u_{m,q}f\right)(u,v)\le f(u,v), \quad (40)$$

(ii) if $f(.,v)$ is a convex function, then $(\mathcal{R}^u_{m,q}f)(u,v)\le 0$, i.e.,

$$\left(\mathcal{B}^u_{m,q}f\right)(u,v)\ge f(u,v), \quad (41)$$

for $u\in[0,h-v]$ and $v\in[0,h]$.

Remark 6. For the remainder $\mathcal{R}^v_{n,q}f$ of the approximation formula

$$f = \mathcal{B}^v_{n,q}f + \mathcal{R}^v_{n,q}f. \quad (42)$$

We also have the following:

(A) If $f(u,.)\in C[0,h-u]$, then

$$\left|\left(\mathcal{R}^v_{n,q}f\right)(u,v)\right| \le \left(1+\frac{h}{2\delta\sqrt{[n]_q}}\right)w(f(u,.);\delta), \quad u\in[0,h]. \quad (43)$$

And for $\delta = 1/\sqrt{[n]_q}$,

$$\left|\left(\mathcal{R}^v_{n,q}f\right)(u,v)\right| \le \left(1+\frac{h}{2}\right)w\left(f(u,.);\frac{1}{\sqrt{[n]_q}}\right), \quad u\in[0,h]. \quad (44)$$

(B) If $f(u,.)\in C^2[0,h]$, then

$$\begin{aligned}&\left(\mathcal{R}^v_{n,q}f\right)(u,v) = -\frac{v(h-u-v)}{2[n]_q}f^{(0,2)}(u,\eta), \quad \eta\in[0,h-u],\\ &\left|\left(\mathcal{R}^v_{n,q}f\right)(u,v)\right| \le \frac{h^2}{8[n]_q}\mathcal{M}_{02}f, \quad (u,v)\in\mathcal{T}_h,\end{aligned} \quad (45)$$

where

$$\mathcal{M}_{ij}f = \max_{\mathcal{T}_h}\left|f^{(i,j)}(u,v)\right|. \quad (46)$$

3. Product Operators

Let $\mathcal{P}_{mn,q} = \mathcal{B}^u_{m,q}\mathcal{B}^v_{n,q}$ and $\mathcal{Q}_{mn,q} = \mathcal{B}^v_{n,q}\mathcal{B}^u_{m,q}$ be the products of operators $\mathcal{B}^u_{m,q}$ and $\mathcal{B}^v_{n,q}$.

We have

$$\left(\mathcal{P}_{mn,q}f\right)(u,v) = \begin{cases}\displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q},v\right)f\left([i]_q\frac{(h-v)}{[m]_q},[j]_q\frac{\left([m]_q-[i]_q\right)h+[i]_qv}{[m]_q[n]_q}\right), & (u,v)\in\mathcal{T}_h\setminus\{(0,h),(h,0)\},\\ f(0,h), \quad (0,h)\in\mathcal{T}_h,\\ f(h,0), \quad (h,0)\in\mathcal{T}_h.\end{cases} \quad (47)$$

Remark 7. The nodes of the operator $\mathscr{P}_{mn,q}$ are the q-analogue of the nodes, which are given in [31], Figure 2, for $i=\overline{0,m}\,; j=\overline{0,n}$, and $v\in[0,h]$.

Theorem 8. *The product operator $\mathscr{P}_{mn,q}$ satisfies the following relations:*

(i) $(\mathscr{P}_{mn,q}f)(u,0)=(\mathscr{B}^{u}_{m,q}f)(u,0)$

(ii) $(\mathscr{P}_{mn,q}f)(0,v)=(\mathscr{B}^{v}_{n,q}f)(0,v)$

(iii) $(\mathscr{P}_{mn,q}f)(u,h-u)=f(u,h-u),\ u,v\in[0,h]$

The above proofs follow from some simple computation.

The property (i) or (ii) implies that $(\mathscr{P}_{mn,q}f)(0,0)=f(0,0)$.

Remark 9. The product operator $\mathscr{P}_{mn,q}$ interpolates the function f at the vertex $(0,0)$ and on the hypotenuse $u+v=h$ of the triangle $\mathscr{T}_h$.

The product operator $\mathscr{Q}_{mn,q}$, given by

$$(\mathscr{Q}_{nm,q}f)(u,v)=\begin{cases}\displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}\left(u,[j]_q\frac{(h-u)}{[n]_q}\right)\tilde{q}_{n,j}(u,v)f\left([i]_q\frac{\left([n]_q-[j]_q\right)h+[j]_q u}{[m]_q[n]_q},[j]_q\frac{(h-u)}{[n]_q}\right), & (u,v)\in\mathscr{T}_h\setminus\{(0,h),(h,0)\},\\ f(0,h), \quad (0,h)\in\mathscr{T}_h, & \\ f(h,0), \quad (h,0)\in\mathscr{T}_h, & \end{cases} \tag{48}$$

has the nodes, which are q-analogue of nodes given in [31], Figure 3, for $i=\overline{0,m}$, $j=\overline{0,n}$, $u\in[0,h]$, and the properties:

(i) $(\mathscr{Q}_{nm,q}f)(u,0)=(\mathscr{B}^{u}_{m,q}f)(u,0)$

(ii) $(\mathscr{Q}_{nm,q}f)(0,v)=(\mathscr{B}^{v}_{n,q}f)(0,v)$

(iii) $(\mathscr{Q}_{nm,q}f)(h-v,v)=f(h-v,v),\ u,v\in[0,h]$

Let us consider the approximation formula

$$f=\mathscr{P}_{mn,q}f+\mathscr{R}^{\mathscr{P}}_{mn,q}f. \tag{49}$$

Theorem 10. *If $f\in C(\mathscr{T}_h)$ and $0<q\le 1$, then*

$$\left|\left(\mathscr{R}^{\mathscr{P}}_{mn,q}f\right)(u,v)\right|\le(1+h)w\left(f;\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right),\quad (u,v)\in\mathscr{T}_h. \tag{50}$$

Proof. We have

$$\begin{aligned}&\left|\left(\mathscr{R}^{\mathscr{P}}_{mn,q}f\right)(u,v)\right|\\ &\le\left[\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q},v\right)\left|u-[i]_q\frac{(h-v)}{[m]_q}\right|\right.\\ &\quad+\frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q},v\right)\left|v-[j]_q\frac{\left([m]_q-[i]_q\right)h+[i]_q v}{[m]_q[n]_q}\right|\\ &\quad\left.+\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q},v\right)\right]w(f;\delta_1,\delta_2).\end{aligned} \tag{51}$$

After some transformations, one obtains

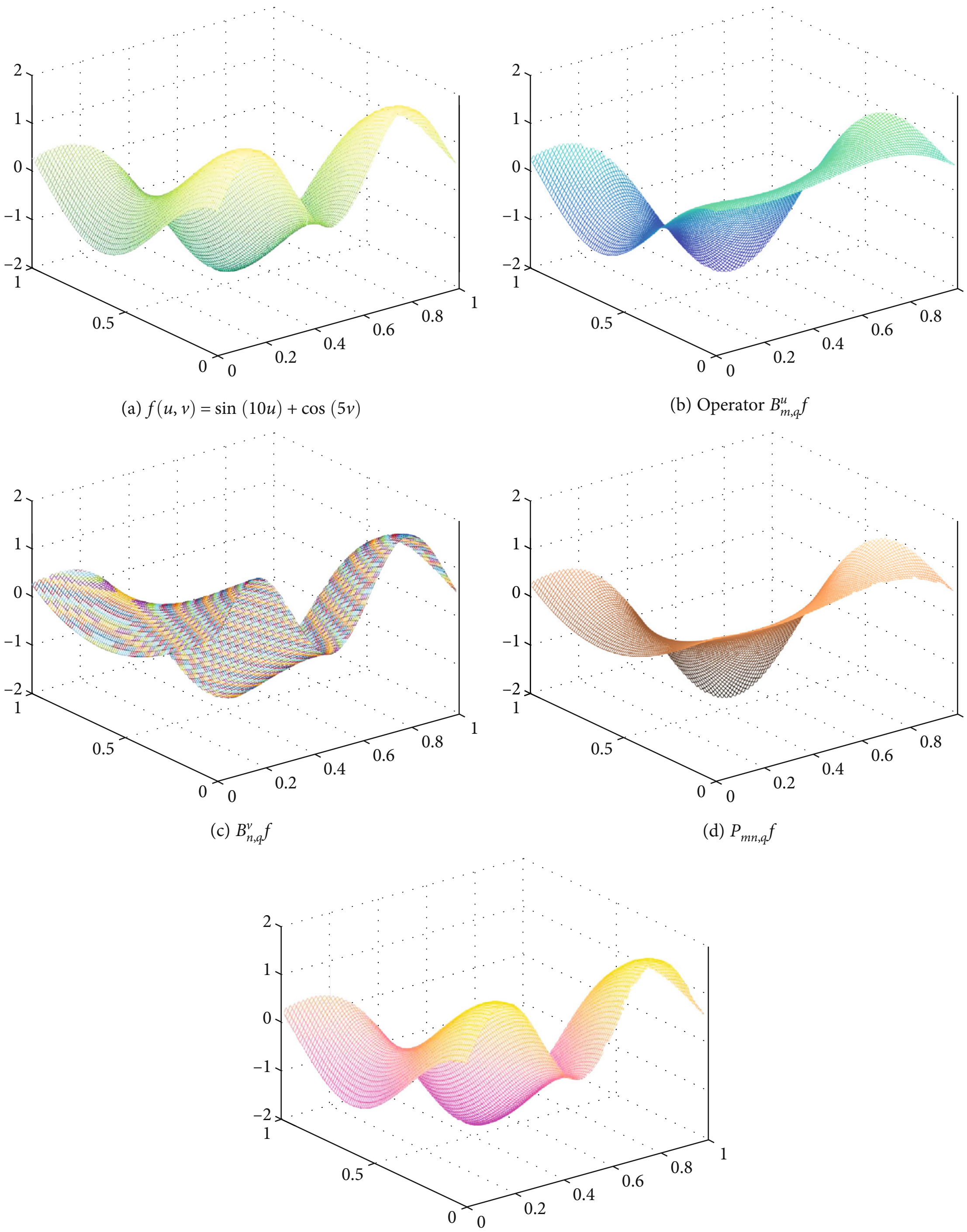
$$\begin{aligned}&\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q},v\right)\left|u-[i]_q\frac{(h-v)}{[m]_q}\right|\le\sqrt{\frac{u(h-u-v)}{[m]_q}},\\ &\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q},v\right)\left|v-[j]_q\frac{\left([m]_q-[i]_q\right)h+[i]_q v}{[m]_q[n]_q}\right|\le\sqrt{\frac{v(h-u-v)}{[n]_q}},\end{aligned} \tag{52}$$

while

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q},v\right)=1. \tag{53}$$

It follows

$$\left|\left(\mathscr{R}^{\mathscr{P}}_{mn,q}f\right)(u,v)\right|\le\left(\frac{1}{\delta_1}\sqrt{\frac{u(h-u-v)}{[m]_q}}+\frac{1}{\delta_2}\sqrt{\frac{v(h-u-v)}{[n]_q}}+1\right)w(f;\delta_1,\delta_2). \tag{54}$$



(a) $f(u,v) = \sin(10u) + \cos(5v)$

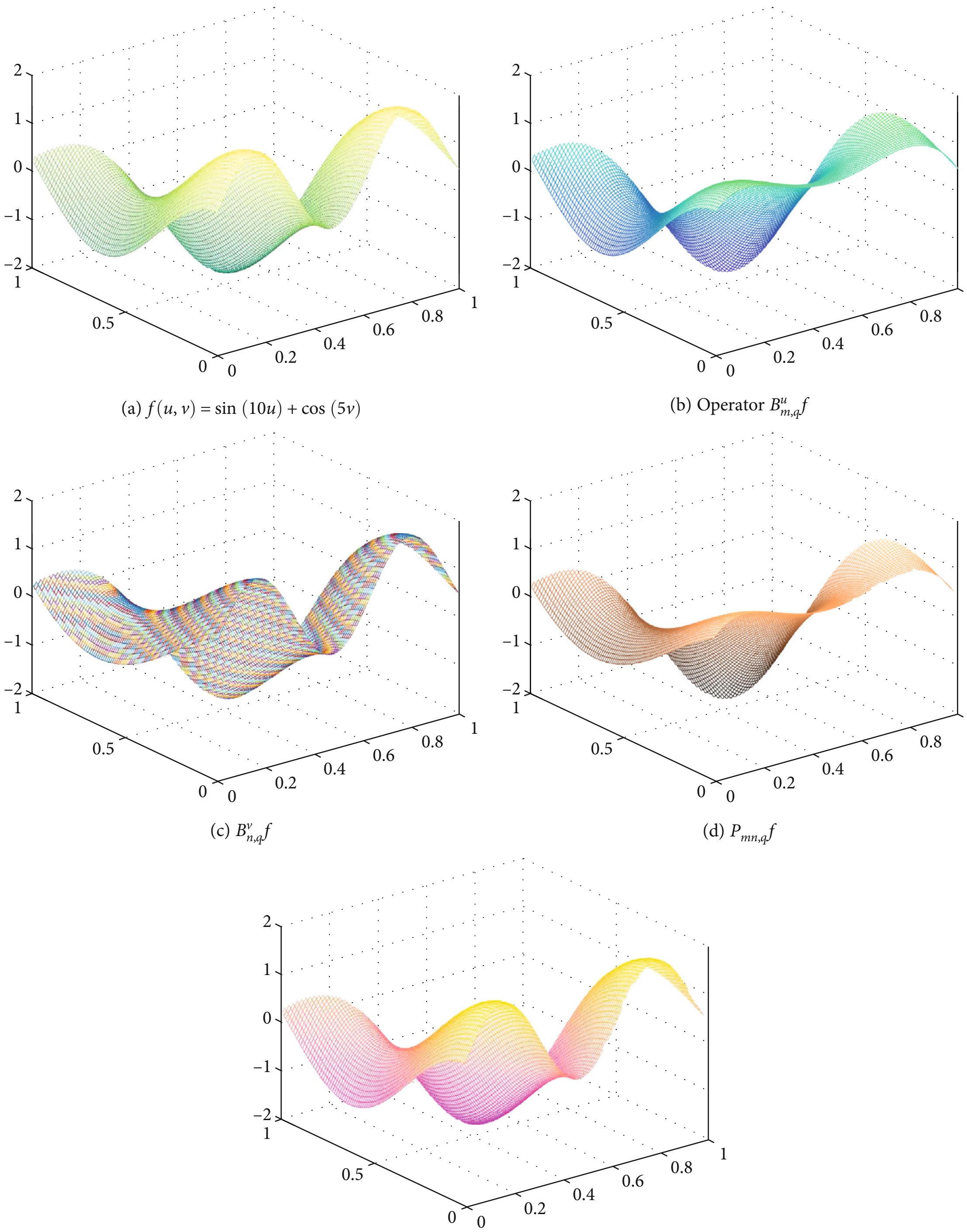
(b) Operator $B^{u}_{m,q}f$

(c) $B^{v}_{n,q}f$

(d) $P_{mn,q}f$

(e) $S_{mn,q}f$

Figure 1: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h = 1$, $m = 6$, $n = 6$, and $q = 0.70$.



(a) $f(u,v) = \sin(10u) + \cos(5v)$

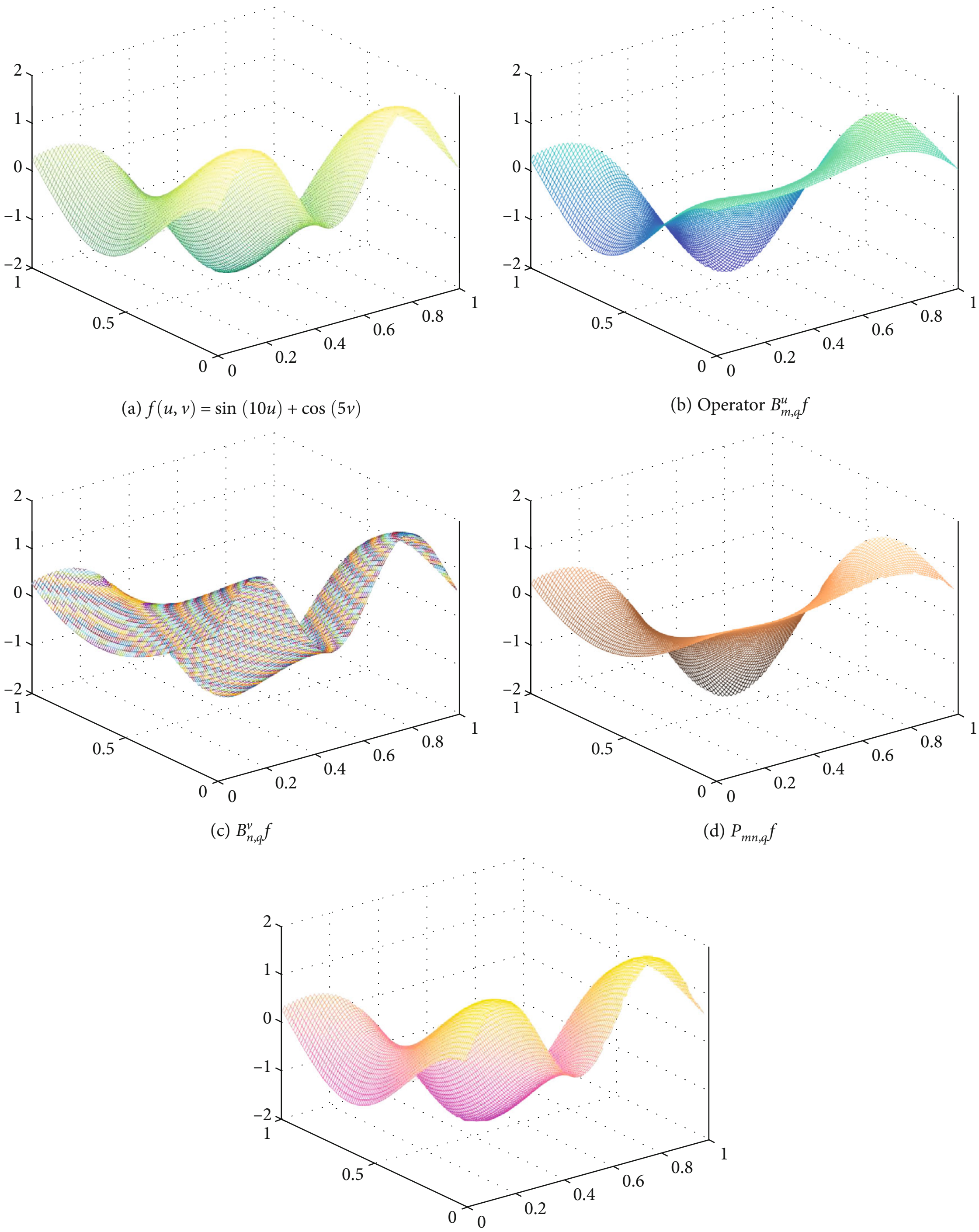
(b) Operator $B^{u}_{m,q}f$

(c) $B^{v}_{n,q}f$

(d) $P_{mn,q}f$

(e) $S_{mn,q}f$

FIGURE 2: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h = 1$, $m = 6$, $n = 6$, and $q = 0.99$.



(a) $f(u, v) = \sin(10u) + \cos(5v)$

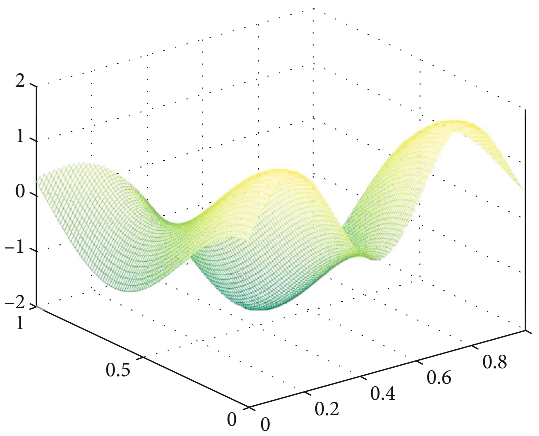
(b) Operator $B^{u}_{m,q}f$

(c) $B^{v}_{n,q}f$

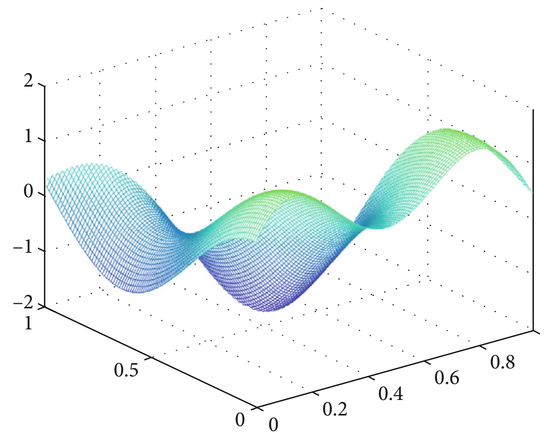
(d) $P_{mn,q}f$

(e) $S_{mn,q}f$

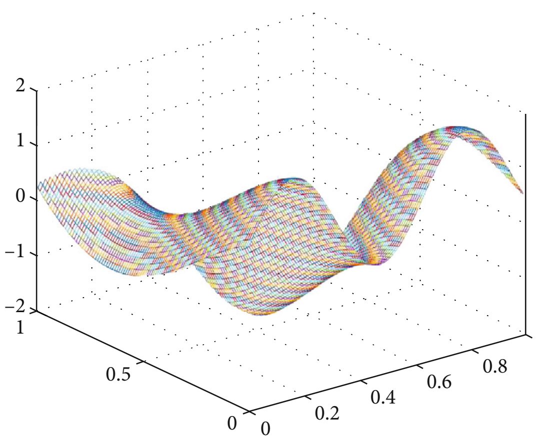
Figure 3: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h = 1$, $m = 15$, $n = 15$, and $q = 0.70$.



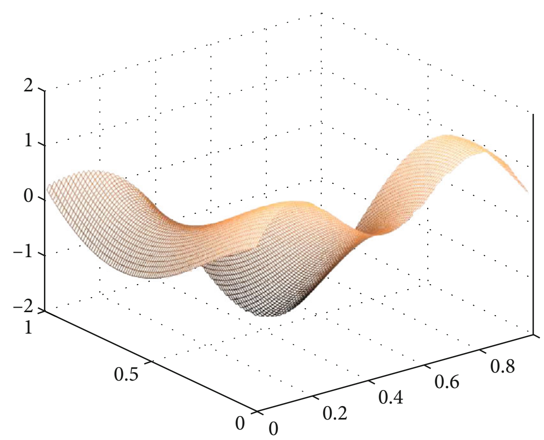
(a) $f(u,v) = \sin(10u) + \cos(5v)$



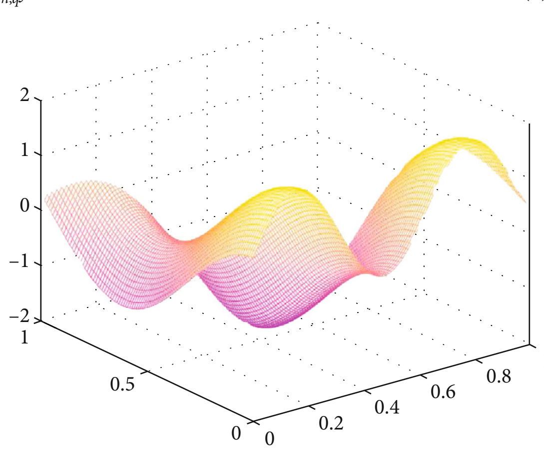
(b) Operator $B^{u}_{m,q}f$



(c) $B^{v}_{n,q}f$

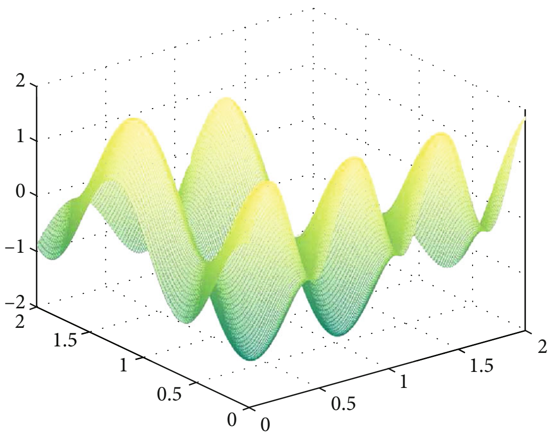


(d) $P_{mn,q}f$

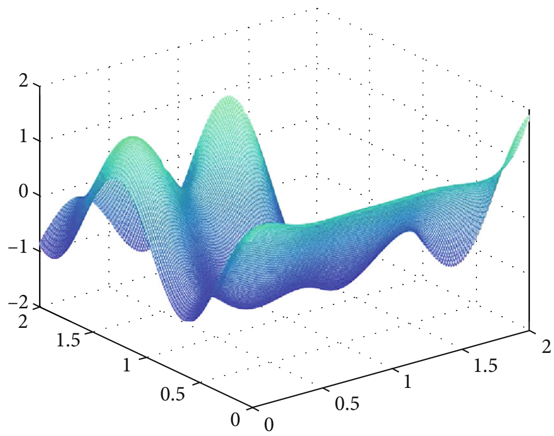


(e) $S_{mn,q}f$

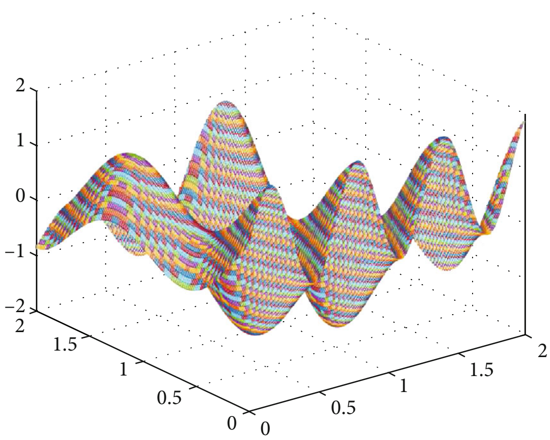
FIGURE 4: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h = 1$, $m = 15$, $n = 10$, and $q = 0.99$.



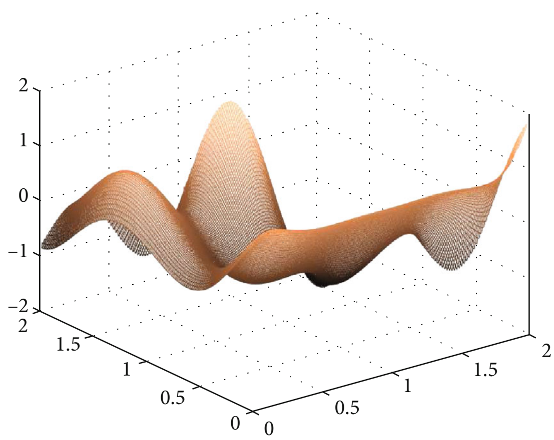
(a) $f(u,v)=\sin(10u)+\cos(5v)$



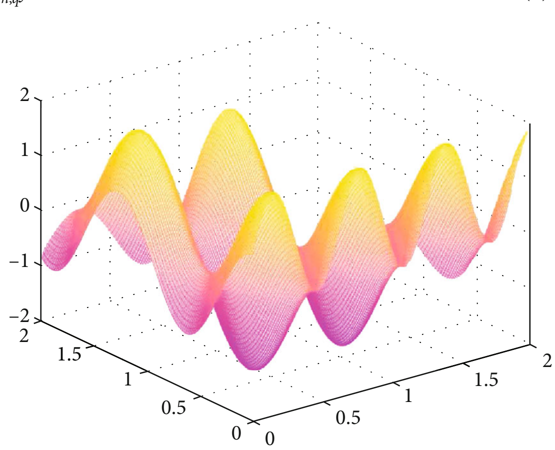
(b) Operator $B^{u}_{m,q}f$



(c) $B^{v}_{n,q}f$



(d) $P_{mn,q}f$



(e) $S_{mn,q}f$

FIGURE 5: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h=2$, $m=10$, $n=10$, and $q=0.99$.

Since

$$\frac{u(h-u-v)}{[m]_q} \le \frac{h^2}{4[m]_q}, \quad \frac{v(h-u-v)}{[n]_q} \le \frac{h^2}{4[n]_q}, \quad \text{for all } (u,v) \in \mathcal{T}_h. \tag{55}$$

We have

$$\left|\left(\mathcal{R}^{\mathcal{P}}_{mn,q}f\right)(u,v)\right| \le \left(\frac{h}{2\delta_1\sqrt{[m]_q}} + \frac{h}{2\delta_2\sqrt{[n]_q}} + 1\right) w(f;\delta_1,\delta_2)| \cdot \left(\mathcal{R}^{\mathcal{P}}_{mn,q}f\right)(u,v)| \le (1+h)w\left(f;\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right). \tag{56}$$

4. Boolean Sum Operators

Let

$$\mathcal{S}_{mn,q} := \mathcal{B}^u_{m,q} \bigoplus \mathcal{B}^v_{n,q} = \mathcal{B}^u_{m,q} + \mathcal{B}^v_{n,q} - \mathcal{B}^u_{m,q}\mathcal{B}^v_{n,q}, \quad \mathcal{T}_{nm,q} := \mathcal{B}^v_{n,q} \bigoplus \mathcal{B}^u_{m,q} = \mathcal{B}^v_{n,q} + \mathcal{B}^u_{m,q} - \mathcal{B}^v_{n,q}\mathcal{B}^u_{m,q}, \tag{57}$$

be the Boolean sums of the Phillips-type Bernstein operators $\mathcal{B}^u_{m,q}$ and $\mathcal{B}^v_{n,q}$.

Theorem 11. *For the real-valued function f defined on $\mathcal{T}_h$, we have*

$$\mathcal{S}_{mn,q}f\big|_{\partial\mathcal{T}_h} = f\big|_{\partial\mathcal{T}_h}. \tag{58}$$

Proof. We have

$$\mathcal{S}_{mn,q}f = \left(\mathcal{B}^u_{m,q} + \mathcal{B}^v_{n,q} - \mathcal{B}^u_{m,q}\mathcal{B}^v_{n,q}\right)f. \tag{59}$$

The interpolation properties of $\mathcal{B}^u_{m,q}$,$\mathcal{B}^v_{n,q}$ together with properties (i)–(iii) of the operator $\mathcal{P}_{mn,q}$ imply that

$$\begin{aligned}(\mathcal{S}_{mn,q}f)(u,0) &= \left(\mathcal{B}^u_{m,q}f\right)(u,0) + f(u,0) - \left(\mathcal{B}^u_{m,q}f\right)(u,0) = f(u,0),\\ (\mathcal{S}_{mn,q}f)(0,v) &= f(0,v) - \left(\mathcal{B}^v_{n,q}f\right)(0,v) + \left(\mathcal{B}^v_{n,q}f\right)(0,v) = f(0,v),\\ (\mathcal{S}_{mn,q}f)(u,h-u) &= f(u,h-u) + f(u,h-u) - f(u,h-u) = f(u,h-u),\end{aligned} \tag{60}$$

for all $u,v \in [0,h]$.

Let $\mathcal{R}^{\mathcal{S}}_{mn,q}f$ be the remainder of the Boolean sum approximation formula

$$f = \mathcal{S}_{mn,q}f + \mathcal{R}^{\mathcal{S}}_{mn,q}f. \tag{61}$$

Theorem 12. *If $f \in C(\mathcal{T}_h)$, then*

$$\begin{aligned}\left|\left(\mathcal{R}^{\mathcal{S}}_{mn,q}f\right)(u,v)\right| \le{}& \left(1+\frac{h}{2}\right)w\left(f(.,v);\frac{1}{\sqrt{[m]_q}}\right) + \left(1+\frac{h}{2}\right)w\left(f(u,.);\frac{1}{\sqrt{[n]_q}}\right) + (1+h)w\\ &\cdot\left(f;\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right),\end{aligned} \tag{62}$$

for all $(u,v) \in \mathcal{T}_h$.

Proof. From the equality

$$f - \mathcal{S}_{mn,q}f = f - \mathcal{B}^u_{m,q}f + f - \mathcal{B}^v_{n,q}f - \left(f - \mathcal{P}_{mn,q}f\right), \tag{63}$$

we get

$$\left|\left(\mathcal{R}^{\mathcal{S}}_{mn,q}f\right)(u,v)\right| \le \left|\left(\mathcal{R}^u_{m,q}f\right)(u,v)\right| + \left|\left(\mathcal{R}^v_{n,q}f\right)(u,v)\right| + \left|\left(\mathcal{R}^{\mathcal{P}}_{mn,q}f\right)(u,v)\right|. \tag{64}$$

Now, from (25), (44), and (50), we follow the proof (62).

Remark 13. Analogous relations can be obtained for the remainders of the product approximation formula

$$f = \mathcal{Q}_{nm,q}f + \mathcal{R}^{\mathcal{Q}}_{nm,q}f = \mathcal{B}^v_{n,q}\mathcal{B}^u_{m,q}f + \mathcal{R}^{\mathcal{Q}}_{nm,q}f, \tag{65}$$

and for the Boolean sum formula

$$f = \mathcal{T}_{nm,q}f + \mathcal{R}^{\mathcal{T}}_{nm,q}f = \left(\mathcal{B}^v_{n,q} \oplus \mathcal{B}^u_{m,q}\right)f + \mathcal{R}^{\mathcal{T}}_{nm,q}f. \tag{66}$$

5. Graphical Analysis

Let us consider a function for graphical analysis. In Figure 1(a), we have presented the graph of function $f(u,v) = \sin(10u) + \cos(5v)$ on triangular domain. The graph of Phillips Bernstein operator $\mathcal{B}^u_{m,q}f$ based on quantum analogue on triangular domain is shown in Figure 1(b). Similarly, other operators $\mathcal{B}^v_{n,q}f$, $\mathcal{P}_{mn,q}f$, and $\mathcal{S}_{mn,q}f$ approximating function are shown in Figures 1(c)–1(e) for various values of q, m, n, and h. One can observe from Figures 1–5 that operators are approximating function better as q approaches to 1 for fixed value of m and n.

Also from these figures, one can observe that operator is approximating function better with increasing values of m and n and by fixing q on triangular domain.

Thus, we have constructed Phillips-type q-Bernstein operators over triangular domain which hold the end point interpolation property on some edges and vertices of triangle.

Hence, it can be concluded that after introducing one extra parameter q in Lupas Bernstein operators, we have more modeling flexibility for approximation on triangular domain.

Data Availability

No data are available.

Conflicts of Interest

The authors declare that they have no competing interests.

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Research Article

A Novel Value for the Parameter in the Dai-Liao-Type Conjugate Gradient Method

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A new rule for calculating the parameter t involved in each iteration of the MHSDL (Dai-Liao) conjugate gradient (CG) method is presented. The new value of the parameter initiates a more efficient and robust variant of the Dai-Liao algorithm. Under proper conditions, theoretical analysis reveals that the proposed method in conjunction with backtracking line search is of global convergence. Numerical experiments are also presented, which confirm the influence of the new value of the parameter t on the behavior of the underlying CG optimization method. Numerical comparisons and the analysis of obtained results considering Dolan and Moré's performance profile show better performances of the novel method with respect to all three analyzed characteristics: number of iterative steps, number of function evaluations, and CPU time.

1. Introduction and Background Results

The topic of our research is solving the unconstrained nonlinear optimization problem

$$\min f(x), \quad x \in \mathbb{R}^n, \tag{1}$$

where the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuously differentiable and bounded below. Following the standard notation, $g_k = \nabla f(x_k)$ denotes the gradient, $s_{k-1} = x_k - x_{k-1}$ and $y_{k-1} = g_k - g_{k-1}$. Using an extended conjugacy condition

$$d_k^{\mathrm{T}} y_{k-1} = -t g_k^{\mathrm{T}} s_{k-1}, \quad t > 0, \tag{2}$$

Dai and Liao in [1] proposed the conjugate gradient (CG) method

$$x_{k+1} = x_k + \alpha_k d_k, \tag{3}$$

where the step size α_k is a positive parameter, x_k is an already generated point, x_{k+1} is a new iterative point, and d_k is a suitable search direction. The search directions d_k are generated by the conceptual formula

$$d_k = \begin{cases} -g_0, & k = 0, \\ -g_k + \beta_k^{\mathrm{DL}} d_{k-1}, & k \geq 1, \end{cases} \tag{4}$$

where the conjugate gradient coefficient β_k^{DL} is defined by

$$\beta_k^{\mathrm{DL}} = Y(t) := \frac{g_k^{\mathrm{T}} y_{k-1}}{d_{k-1}^{\mathrm{T}} y_{k-1}} - t \frac{g_k^{\mathrm{T}} s_{k-1}}{d_{k-1}^{\mathrm{T}} y_{k-1}}, \quad t > 0, \tag{5}$$

wherein $t > 0$ is a scalar.

Some well-known formulas for defining β_k have been created by modifying the conjugate gradient parameter β_k^{DL} [2–9]. One of them is denoted as β_k^{MHSDL} and defined

The backtracking line search.
Require: Nonlinear objective function $f(x)$, search direction d_k, previous point x_k, and real quantities $0 < \omega < 0.5$ and $\varphi \in (0, 1)$.
1: $\alpha = 1$.
2: While $f(x_k + \alpha d_k) > f(x_k) + \omega\alpha g_k^T d_k$, do $\alpha \coloneqq \alpha\varphi$.
3: Return $\alpha_k = \alpha$.

Algorithm 1:

in [7] by

$$\beta_k^{\mathrm{MHSDL}} = Y_1(t) \coloneqq \frac{g_k^T \widehat{y_{k-1}}}{d_{k-1}^T y_{k-1}} - t\frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}, \tag{6}$$

where $t > 0$ is a scalar as in (5) and $\widehat{y_{k-1}} = g_k - (\|g_k\|/\|g_{k-1}\|)g_{k-1}$.

The family of CG methods for nonlinear optimization has reached great popularity lately, thanks to the various benefits and advantages it possesses. The most important property is based on computationally efficient iterations arising from a simple CG rule. This property initiates the high efficiency of CG methods with respect to analogous methods for nonlinear optimization. Moreover, global convergence is ensured under suitable conditions. Finally, the application of various CG methods in solving image restoration problems has become an important research topic [10, 11].

Since the parameter t is important for the numerical behavior of Dai-Liao (DL) CG methods [12], one of the most important problems in the implementation of the DL class CG method is to determine a proper value $t > 0$ which will give desirable results. Many scientists have invested a lot of time and effort in the previous period to determine the best definition of the nonnegative parameter t in the DL class CG methods. So far, the research in finding the appropriate value of t has evolved in two directions. One group of methods is aimed at finding an appropriate fixed value for t [1, 2, 6–8], while methods from another group promote appropriate rules for computing values of t in each iteration, which ensure a satisfactory decrease of the objective. In our research, we will pay attention to the second research stream: find the parameter t whose values change through iterations so that the faster convergence is achieved. The value of the parameter t defined in the kth iteration will be denoted by $t(k) \coloneqq t_k$.

In order to complete the presentation, we will restate the main principles proposed so far for computing t_k. Hager and Zhang in [13, 14] proposed the DL CG method (5), known as CG-DESCENT, where $t(k) \equiv t_{k1}$ is defined by

$$t(k) \equiv t_{k1} \coloneqq 2\frac{\|y_{k-1}\|^2}{y_{k-1}^T s_{k-1}}. \tag{7}$$

Dai and Kou [15] suggested the conjugate gradient coefficient β_k^{DK} of the form

$$\beta_k^{\mathrm{DK}} = Y\left(\tau_k + \frac{\|y_{k-1}\|^2}{y_{k-1}^T s_{k-1}} - \frac{y_{k-1}^T s_{k-1}}{\|s_{k-1}\|^2}\right) = \frac{g_k^T y_{k-1}}{y_{k-1}^T d_{k-1}} - \left(\tau_k + \frac{\|y_{k-1}\|^2}{y_{k-1}^T s_{k-1}} - \frac{y_{k-1}^T s_{k-1}}{\|s_{k-1}\|^2}\right)\frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}, \tag{8}$$

where τ_k is the scaling parameter arising from the self-scaling memoryless BFGS method. Clearly, the Dai and Kou (DK) method is a member of the DL class CG methods, which is determined by

$$t(k) \equiv t_{k2} \coloneqq \tau_k + \frac{\|y_{k-1}\|^2}{y_{k-1}^T s_{k-1}} - \frac{y_{k-1}^T s_{k-1}}{\|s_{k-1}\|^2}. \tag{9}$$

The results given in [15] confirm that the DK iterations outperform many existing CG methods. Following the development of DL methods, Babaie-Kafaki and Ghanbari [16] defined two new ways to calculate the value of the parameter t in (5), as in the following two formulas:

$$\begin{aligned} t(k) \equiv t_{k3} &\coloneqq \frac{s_{k-1}^T y_{k-1}}{\|s_{k-1}\|^2} + \frac{\|y_{k-1}\|}{\|s_{k-1}\|}, \\ t(k) \equiv t_{k4} &\coloneqq \frac{\|y_{k-1}\|}{\|s_{k-1}\|}. \end{aligned} \tag{10}$$

Andrei in [17] proposed the new rule for calculating t in order to define $Y(t)$ in (5) and defined a new variant of the DL class CG methods, denoted by DLE, with

$$t(k) \equiv t_{k5} \coloneqq \frac{s_{k-1}^T y_{k-1}}{\|s_{k-1}\|^2}. \tag{11}$$

Lotfi and Hosseini in [18] proposed the following rule for determining the parameter $t(k)$, using the expression

$$t(k) \equiv t_{k6} \coloneqq \max\left\{t_{k6}^*, \upsilon\frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}}\right\}, \tag{12}$$

Effective Dai-Liao (EDL) CG method.
Require: An initial point x_0 and quantities $0 < \varepsilon < 1$, $0 < \delta < 1$.
1: Assign $k = 0$ and $d_0 = -g_0$.
2: If
$$\|g_k\| \le \varepsilon \text{ and } ((|f(x_{k+1}) - f(x_k)|)/(1 + |f(x_k)|)) \le \delta,$$
STOP;
else go to Step 3.
3: Calculate $\alpha_k \in (0,1)$ using Algorithm 1 (backtracking line search).
4: Compute $x_{k+1} = x_k + \alpha_k d_k$.
5: Calculate g_{k+1}, $y_k = g_{k+1} - g_k$, $s_k = x_{k+1} - x_k$.
6: Compute t_k^* by (16).
7: Calculate $\beta_{k+1}^{\mathrm{EDL}}$ by (18).
8: Compute $d_{k+1} = -g_{k+1} + \beta_{k+1}^{\mathrm{EDL}} d_k$.
9: Let $k := k + 1$, and go to Step 2.

Algorithm 2:

where

$$t_{k6}^* := \frac{(1 - h_k\|g_{k-1^r}\|)s_{k-1}^T g_k + (g_k^T y_{k-1}/y_{k-1}^T s_{k-1}) h_k \|g_{k-1}\|^r \|s_{k-1}\|^2}{g_k^T s_{k-1} + (g_k^T s_{k-1}/s_{k-1}^T y_{k-1}) h_k \|g_{k-1}\|^r \|s_{k-1}\|^2},$$
$$h_k = C + \max\left\{-\frac{s_{k-1}^T y_{k-1}}{\|s_{k-1}\|^2}, 0\right\} \|g_{k-1}\|^{-r}, \tag{13}$$

and $\upsilon > 1/4$, C, and r are three positive constants.

On the basis of the above overview of the main CG methods and motivated by the strong theoretical properties and computational efficiency of modified Dai-Liao CG methods proposed by many researchers, we suggest a new way of calculating the value of the parameter $t(k)$. As a consequence, the corresponding CG method of DL type, termed as the *Effective Dai-Liao* (EDL) method, is proposed and its convergence is proven. Numerical testing and comparison with other known DL variants are presented in order to show the effectiveness of the introduced method. Analysis of generated numerical results exhibits that the proposed EDL method is efficient compared with other DL-type methods.

The global organization of sections is described as follows. Introduction, motivation, and a brief overview of the preliminary results are given in Section 1. A new rule for calculating the variable parameter $t(k)$ is proposed in Section 2. An effective algorithm and global convergence of the EDL method initiated by $t(k)$ are given in the same section. The new EDL method is tested in Section 3 on some unlimited optimization test problems and compared against some known variants of the DL class methods. Finally, concluding remarks are presented in the last concluding section.

2. A Modified Dai-Liao Method and Its Convergence

Popularity in defining new rules for calculating $t(k)$ is a guarantee that such an approach is effective and still insufficiently explored. The idea for defining a new parameter t_k^* comes from previously described rules for computing $t(k)$, particularly from the paper Li and Ruan [19] and from the idea which can be found in the paper Yuan et al. [11]. Further, analyzing the results from [1, 2, 6–8], we conclude that the scalar t was defined by a fixed value of 0.1 in related numerical experiments. Also, numerical experience related to the fixed valued $t = 1$ was reported in [1]. According to this experience, our intention is to define variable values $t(k)$ inside the interval $(0, 1)$.

To successfully define $t(k)$ with values belonging to the interval $(0, 1)$, let us start from the definition of the quantity L_k which was used in defining the direction d_k in [19]. The parameter L_k was defined by $L_k = s_{k-1}^T s_{k-1}/s_{k-1}^T y_{k-1}^* \in (0,1)$, $k \ge 0$, where

$$y_{k-1}^* = y_{k-1} + \left(\max\left\{0, -\frac{s_{k-1}^T y_{k-1}}{\|s_{k-1}\|^2}\right\} + 1\right) s_{k-1}. \tag{14}$$

By putting y_{k-1}^* into L_k, the following can be obtained:

$$L_k = \frac{s_{k-1}^T s_{k-1}}{s_{k-1}^T \left(y_{k-1} + \left(\max\left\{0, -\left(s_{k-1}^T y_{k-1}/\|s_{k-1}\|^2\right)\right\} + 1\right) s_{k-1}\right)}$$
$$= \frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1} + \left(\max\left\{0, -\left(s_{k-1}^T y_{k-1}/\|s_{k-1}\|^2\right)\right\} + 1\right)\|s_{k-1}\|^2}. \tag{15}$$

Further, with certain modifications and substitutions in the equation defining L_k, as well as using the function max, which chooses the maximum between the value of the expression $d_{k-1}^T g_k$ and 1, we come to a new definition of the parameter $t(k)$. As described in advance imposed desired restrictions, the novel parameter t_k^* is defined by

$$t_k^* = \frac{\|g_k\|^2}{\max\left\{1, d_{k-1}^T g_k\right\} + \left(\max\left\{0, \left(d_{k-1}^T g_k/\|g_k\|^2\right)\right\} + 1\right)\|g_k\|^2}. \tag{16}$$

Table 1: Summary results of EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods with respect to NI.

Test function	MHSDL3	MHSDL4	MHSDL5	EDL	MHSDL6
Extended penalty	1466	2243	2231	1259	1371
Perturbed quadratic	1203710	754291	746557	305622	423037
Raydan 1	159055	110587	106586	55477	75154
Raydan 2	1636	441	441	70	209
Diagonal 1	116788	78844	73512	30978	20332
Diagonal 2	176983	270434	271595	515000	271295
Diagonal 3	150328	98647	104417	47155	37711
Hager	8666	5219	5157	3234	3625
Generalized tridiagonal 1	1862	1471	1485	639	877
Extended TET	1357	5954	5915	4030	2664
Diagonal 4	30693	19589	19332	8040	12012
Diagonal 5	1721	25120	25120	60	216
Extended Himmelblau	1777	8023	7946	1376	3682
Perturbed quadratic diagonal	2940970	2115659	2027128	1136414	1352704
Quadratic QF1	1270802	799192	786032	309509	325415
Extended quadratic penalty QP1	770	594	575	560	543
Extended quadratic penalty QP2	399671	240530	245254	96620	137799
Extended quadratic exponential EP1	462	606	606	513	526
Extended tridiagonal 2	3119	2176	2177	1132	1455
ARWHEAD (CUTE)	88824	69868	67413	40713	48669
ENGVAL1 (CUTE)	2323	1407	1415	552	820
INDEF (CUTE)	20	31	1080	23	36240
QUARTC (CUTE)	173913	262291	262291	524299	262181
Diagonal 6	1824	508	508	70	227
Generalized quartic	1208	1403	2846	1265	1154
Diagonal 7	3217	655	655	653	580
Diagonal 8	511	698	698	686	596
Full Hessian FH3	1456	5353	5350	2523	3176

It is easy to verify that t_k^* defined by (16) satisfies

$$0 < t_k^* \leq \frac{\|g_k\|^2}{1+(0+1)\|g_k\|^2} = \frac{\|g_k\|^2}{1+\|g_k\|^2} < 1. \tag{17}$$

Accordingly, $t_k^* \in (0,1)$, which was our initial intention. Clearly, greater values of $\|g_k\|$ lead to values $t_k^* \nearrow 1$. Further, since the trend $\|g_k\| \longrightarrow 0$ is expectable, we can expect smaller values $t_k^* \searrow 0$ in late iterations. Therefore, t_k^* is suitable for defining corresponding conjugate gradient coefficient $Y(t)$ or $Y_1(t)$ and further DL CG iterations (4).

Considering $t = t_k^*$ in (6), it is reasonable to propose a novel variant of the Dai-Liao CG parameter β_k^{EDL} which is subject to the following rule during the iterative process:

$$\beta_k^{\mathrm{EDL}} = Y_1(t_k^*) := \frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right|}{d_{k-1}^{\mathrm{T}} y_{k-1}} - t_k^* \frac{g_k^{\mathrm{T}} s_{k-1}}{d_{k-1}^{\mathrm{T}} y_{k-1}}. \tag{18}$$

Before the main algorithm, it is necessary to define the backtracking line search as one of the most popular and practical methods for computing the step length α_k in (3). The procedure for the backtracking line search proposed in [20] starts from the initial value $\alpha = 1$ and generates output values which ensure that the goal function decreases in each iteration. Consequently, it is appropriate to use Algorithm 1, restated from [21], in order to determine the primary step size α_k.

Algorithm 2 describes a computational framework for the EDL method.

It is necessary to examine the properties of the EDL method and prove its convergence.

Assumption 1.

(1) The level set $\mathscr{M} = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$, defined upon the initial point x_0 of the iterative method (3), is bounded.

(2) The goal function f is continuous and differentiable in a neighborhood $\mathscr{P}$ of M with the Lipschitz

Table 2: Summary results of EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods with respect to NFE.

Test function	MHSDL3	MHSDL4	MHSDL5	EDL	MHSDL6
Extended penalty	54876	73764	73429	46820	49791
Perturbed quadratic	56691737	34287604	33885701	13168688	18486375
Raydan 1	5066739	3364983	3236335	1551846	2170553
Raydan 2	6554	1162	1162	159	428
Diagonal 1	5004640	3256274	3022015	1200086	744278
Diagonal 2	353976	540878	543200	1030010	542600
Diagonal 3	6339146	3998904	4229565	1798032	1400076
Hager	192474	107413	106534	59187	69735
Generalized tridiagonal 1	37429	27860	28138	10760	15177
Extended TET	19546	77422	76925	40340	29334
Diagonal 4	713120	425023	418666	155027	242443
Diagonal 5	6874	50460	50460	140	442
Extended Himmelblau	45972	192362	190524	26104	80854
Perturbed quadratic diagonal	135901222	94177165	90238441	48147512	57702654
Quadratic QF1	55972697	33836473	33243711	12316721	12853424
Extended quadratic penalty QP1	17016	12882	12565	11116	10544
Extended quadratic penalty QP2	13015888	7454686	7584960	2743358	4030601
Extended quadratic exponential EP1	14914	18463	18463	14132	15133
Extended tridiagonal 2	36450	22564	22379	9687	12920
ARWHEAD (CUTE)	4296028	3305257	3182138	1846606	2230650
ENGVAL1 (CUTE)	40462	22432	22898	8209	12858
INDEF (CUTE)	1808	2182	5995	2060	104962
QUARTC (CUTE)	347926	524662	524662	1048648	524422
Diagonal 6	7394	1416	1408	159	468
Generalized quartic	14364	21842	48770	16695	14103
Diagonal 7	6454	6838	6838	3891	4521
Diagonal 8	6098	6938	6938	4161	5494
Full Hessian FH3	60792	212799	212701	89890	114962

continuous gradient g. This assumption implies the existence of a positive constant $L > 0$ satisfying

$$\|g(u) - g(v)\| \leq L\|u - v\|, \quad \forall u, v \in \mathcal{P}. \tag{19}$$

Assumption 1 initiates the existence of positive constants D and γ satisfying

$$\begin{aligned} \|u - v\| &\leq D, \quad \forall u, v \in \mathcal{P}, \\ \|g(u)\| &\leq \gamma, \quad \forall u \in \mathcal{P}. \end{aligned} \tag{20}$$

The conditions from Assumption 1 are assumed. In view of the uniform convexity of f, there is a constant $\theta > 0$ that satisfies

$$(g(u) - g(v))^{\mathrm{T}}(u - v) \geq \theta\|u - v\|^2, \quad \text{for all } u, v \in \mathcal{M}, \tag{21}$$

or equivalently,

$$f(u) \geq f(v) + g(v)^{\mathrm{T}}(u - v) + \frac{\theta}{2}\|u - v\|^2, \quad \text{for all } u, v \in \mathcal{M}. \tag{22}$$

It follows from (21) and (22) that

$$s_{k-1}^{\mathrm{T}} y_{k-1} \geq \theta\|s_{k-1}\|^2, \tag{23}$$

$$f(x_{k-1}) - f(x_k) \geq -g(x_k)^{\mathrm{T}} s_{k-1} + \frac{\theta}{2}\|s_{k-1}\|^2. \tag{24}$$

By (19) and (23), one concludes

$$\theta\|s_{k-1}\|^2 \leq s_{k-1}^{\mathrm{T}} y_{k-1} \leq L\|s_{k-1}\|^2, \tag{25}$$

where the inequality implies $\theta \leq L$.

The inequality (25) initiates

$$s_{k-1}^{\mathrm{T}} y_{k-1} = \alpha_{k-1} d_{k-1}^{\mathrm{T}} y_{k-1} > 0. \tag{26}$$

Table 3: Summary results of EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods with respect to CPU time (sec).

Test function	MHSDL3	MHSDL4	MHSDL5	EDL	MHSDL6
Extended penalty	29.75	34.11	31.42	18.30	24.27
Perturbed quadratic	40532.66	24358.20	24947.84	8335.80	13225.80
Raydan 1	3054.67	1904.48	1692.06	690.91	1184.86
Raydan 2	6.77	1.58	1.66	0.31	0.77
Diagonal 1	7834.03	5106.41	4592.28	1476.89	486.09
Diagonal 2	885.13	1428.05	1447.02	2352.11	1513.50
Diagonal 3	13614.27	8416.77	9064.30	3132.02	1916.30
Hager	586.63	325.75	333.41	142.06	198.13
Generalized tridiagonal 1	66.14	35.59	34.42	15.19	21.63
Extended TET	20.50	78.34	82.94	41.23	31.45
Diagonal 4	134.53	77.86	87.88	30.41	55.34
Diagonal 5	18.06	134.73	121.09	0.56	1.84
Extended Himmelblau	11.13	44.47	44.36	6.19	18.30
Perturbed quadratic diagonal	91655.55	58226.16	60920.06	32179.38	36383.83
Quadratic QF1	62610.50	31552.48	28679.91	8832.11	8465.34
Extended quadratic penalty QP1	7.56	7.25	6.98	4.98	4.94
Extended quadratic penalty QP2	3814.16	2128.86	2288.55	671.52	1204.72
Extended quadratic exponential EP1	9.11	10.23	8.55	8.00	8.02
Extended tridiagonal 2	11.13	8.83	6.95	4.08	5.25
ARWHEAD (CUTE)	2709.42	2336.92	2369.28	1266.80	1689.80
ENGVAL1 (CUTE)	19.47	11.33	11.81	4.03	6.70
INDEF (CUTE)	2.44	2.89	10.70	1.92	774.34
QUARTC (CUTE)	3106.56	4818.58	4808.70	7138.72	4735.39
Diagonal 6	6.75	1.92	2.03	0.38	1.34
Generalized quartic	7.16	11.53	21.05	7.53	9.78
Diagonal 7	5.98	8.20	8.28	4.56	6.25
Diagonal 8	6.17	8.20	8.08	4.72	7.69
Full Hessian FH3	30.08	66.45	79.48	35.77	43.42

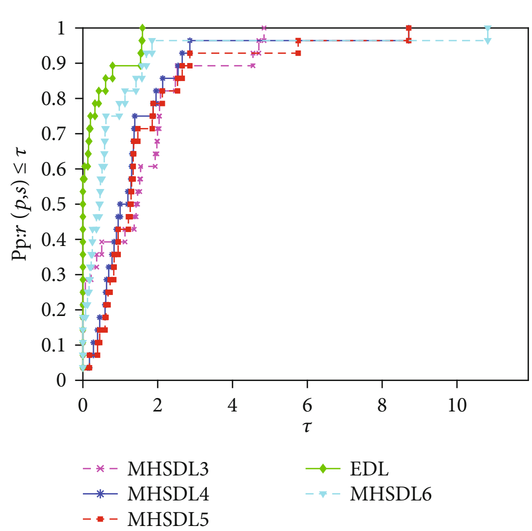

Figure 1: NI performance profile for EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods.

Taking into account $\alpha_{k-1} > 0$ and the last inequality, we conclude

$$d_{k-1}^{\mathrm{T}} y_{k-1} > 0. \tag{27}$$

Lemma 2. *[22, 23]. Let Assumption 1 be accomplished and the points $\{x_k\}$ be generated by the method (3)–(4). Then, it holds*

$$\sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} < +\infty. \tag{28}$$

Lemma 3. *Consider the proposed Dai-Liao CG method, including (3), (4), and (18). If the search procedure guarantees (27), for all $k \geq 0$, then the next inequality holds*

$$g_k^T d_k \leq -c\|g_k\|^2, \tag{29}$$

for some $0 \leq c \leq 1$.

Proof. The inequality (29) will be verified by induction. In the initial situation $k = 0$, one obtains $g_0^{\mathrm{T}} d_0 = -\|g_0\|^2$. Since $c \leq 1$,

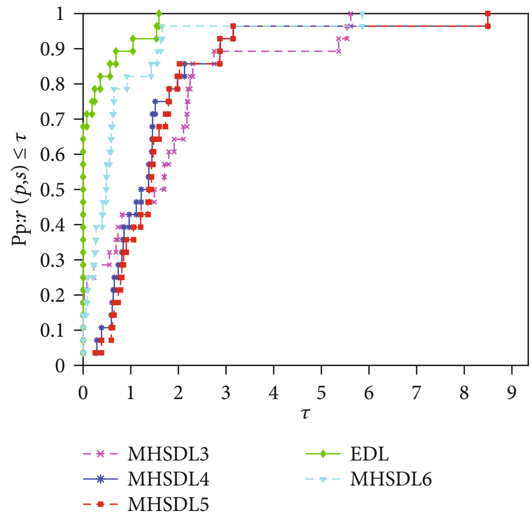

FIGURE 2: NFE performance profile for EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods.

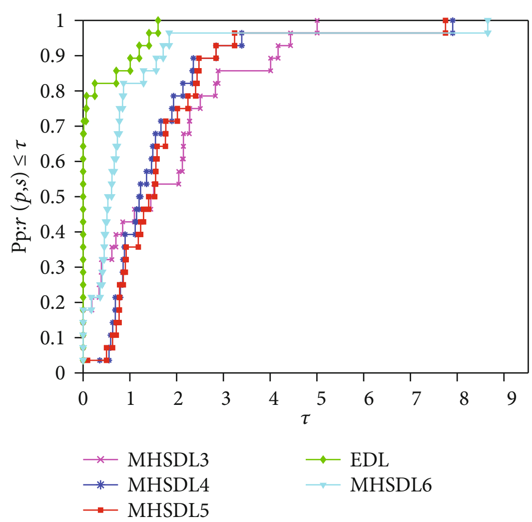

FIGURE 3: CPU performance profile for EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods.

obviously (29) is satisfied in the basic case. Suppose that (29) is valid for some $k \geq 1$. Taking the inner product of both the left- and right-hand sides in (4) with the vector g_k^{T}, the following can be obtained:

$$\begin{aligned} g_k^{\mathrm{T}} d_k &= -\|g_k\|^2 + \beta_k^{\mathrm{EDL}} g_k^{\mathrm{T}} d_{k-1} \\ &= -\|g_k\|^2 + \left(\frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right|}{d_{k-1}^{\mathrm{T}} y_{k-1}} - t_k^* \frac{g_k^{\mathrm{T}} s_{k-1}}{d_{k-1}^{\mathrm{T}} y_{k-1}} \right) g_k^{\mathrm{T}} d_{k-1} \\ &= -\|g_k\|^2 + \frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right|}{d_{k-1}^{\mathrm{T}} y_{k-1}} g_k^{\mathrm{T}} d_{k-1} - t_k^* \frac{g_k^{\mathrm{T}} s_{k-1}}{d_{k-1}^{\mathrm{T}} y_{k-1}} g_k^{\mathrm{T}} d_{k-1} \\ &= -\|g_k\|^2 + \frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right|}{d_{k-1}^{\mathrm{T}} y_{k-1}} g_k^{\mathrm{T}} d_{k-1} - t_k^* \frac{\alpha_{k-1}\left(g_k^{\mathrm{T}} d_{k-1}\right)^2}{d_{k-1}^{\mathrm{T}} y_{k-1}}. \end{aligned} \tag{30}$$

Using (17) in common with (27) and $\alpha_{k-1} > 0$, we conclude

$$t_k^* \frac{\alpha_{k-1}\left(g_k^{\mathrm{T}} d_{k-1}\right)^2}{d_{k-1}^{\mathrm{T}} y_{k-1}} > 0. \tag{31}$$

Now from (30), (31), and

$$0 \leq \beta_k^{\mathrm{MHS}} = \frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right|}{d_{k-1}^{\mathrm{T}} y_{k-1}} \leq \frac{\|g_k\|^2}{\lambda \left| g_k^{\mathrm{T}} d_{k-1} \right|}, \quad \lambda \geq 1, \tag{32}$$

it follows that

$$\begin{aligned} g_k^{\mathrm{T}} d_k &\leq -\|g_k\|^2 + \frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right|}{d_{k-1}^{\mathrm{T}} y_{k-1}} g_k^{\mathrm{T}} d_{k-1} \\ &\leq -\|g_k\|^2 + \frac{\|g_k\|^2}{\lambda\left|g_k^{\mathrm{T}} d_{k-1}\right|}\left|g_k^{\mathrm{T}} d_{k-1}\right| = -\left(1 - \frac{1}{\lambda}\right)\|g_k\|^2. \end{aligned} \tag{33}$$

In view of $\lambda \geq 1$, the inequality (29) is satisfied for $c = (1 - (1/\lambda))$ in (33) and arbitrary $k \geq 0$.

The global convergence of the proposed EDL method is confirmed by Theorem 4.

Theorem 4. *Let Assumption 1 be true and f be uniformly convex. Then, the sequence $\{x_k\}$ generated by (3), (4), and (18) fulfills*

$$\liminf_{k \to \infty} \|g_k\| = 0. \tag{34}$$

Proof. Suppose the opposite, i.e., (34) is not true. This implies the existence of a constant $c_1 > 0$ such that

$$\|g_k\| \geq c_1, \quad \text{for all } k. \tag{35}$$

Squaring both sides of (4) implies

$$\|d_k\|^2 = \|g_k\|^2 - 2\beta_k^{\mathrm{EDL}} g_k^{\mathrm{T}} d_{k-1} + \left(\beta_k^{\mathrm{EDL}}\right)^2 \|d_{k-1}\|^2. \tag{36}$$

Taking into account (18), we can get

$$\begin{aligned} -2\beta_k^{\mathrm{EDL}} g_k^{\mathrm{T}} d_{k-1} &= -2\left(\frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right|}{d_{k-1}^{\mathrm{T}} y_{k-1}} - t_k^* \frac{g_k^{\mathrm{T}} s_{k-1}}{d_{k-1}^{\mathrm{T}} y_{k-1}} \right) g_k^{\mathrm{T}} d_{k-1} \\ &= -2\left(\frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right|}{d_{k-1}^{\mathrm{T}} y_{k-1}} g_k^{\mathrm{T}} d_{k-1} - t_k^* \frac{\alpha_{k-1}\left(g_k^{\mathrm{T}} d_{k-1}\right)^2}{d_{k-1}^{\mathrm{T}} y_{k-1}} \right). \end{aligned} \tag{37}$$

Now from (31) and (32), it follows that

$$-2\beta_k^{\mathrm{EDL}} g_k^{\mathrm{T}} d_{k-1} \le 2\left|\frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right|}{d_{k-1}^{\mathrm{T}} y_{k-1}}\right|\left|g_k^{\mathrm{T}} d_{k-1}\right| \le 2\frac{\|g_k\|^2}{\lambda\left|g_k^{\mathrm{T}} d_{k-1}\right|}\left|g_k^{\mathrm{T}} d_{k-1}\right| = 2\frac{\|g_k\|^2}{\lambda}. \tag{38}$$

Now, an application of (18) initiates

$$\begin{aligned}
\beta_k^{\mathrm{EDL}} &= \frac{\|g_k\|^2 - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right| - t_k^* g_k^{\mathrm{T}} s_{k-1}}{d_{k-1}^{\mathrm{T}} y_{k-1}} \\
&\le \left|\frac{g_k^{\mathrm{T}} g_k - (\|g_k\|/\|g_{k-1}\|)\left|g_k^{\mathrm{T}} g_{k-1}\right| - t_k^* g_k^{\mathrm{T}} s_{k-1}}{d_{k-1}^{\mathrm{T}} y_{k-1}}\right| \\
&\le \frac{\left|g_k^{\mathrm{T}}\left(g_k - (\|g_k\|/\|g_{k-1}\|) g_{k-1} - t_k^* s_{k-1}\right)\right|}{\theta\alpha_{k-1}\|d_{k-1}\|^2} \\
&= \frac{\left|g_k^{\mathrm{T}}\left(g_k - g_{k-1} + g_{k-1} - (\|g_k\|/\|g_{k-1}\|) g_{k-1} - t_k^* s_{k-1}\right)\right|}{\theta\alpha_{k-1}\|d_{k-1}\|^2} \\
&\le \frac{\|g_k\|\left(\|g_k - g_{k-1}\| + \|g_{k-1}(1 - (\|g_k\|/\|g_{k-1}\|))\| + t_k^*\|s_{k-1}\|\right)}{\theta\alpha_{k-1}\|d_{k-1}\|^2} \\
&= \frac{\|g_k\|\left(\|g_k - g_{k-1}\| + |1 - (\|g_k\|/\|g_{k-1}\|)|\|g_{k-1}\| + t_k^*\|s_{k-1}\|\right)}{\theta\alpha_{k-1}\|d_{k-1}\|^2} \\
&= \frac{\|g_k\|\left(\|g_k - \mathrm{g}_{k-1}\| + |\|g_{k-1}\| - \|g_k\|| + t_k^*\|s_{k-1}\|\right)}{\theta\alpha_{k-1}\|d_{k-1}\|^2} \\
&\le \frac{\|g_k\|\left(\|g_k - g_{k-1}\| + \|g_{k-1} - g_k\| + t_k^*\|s_{k-1}\|\right)}{\theta\alpha_{k-1}\|d_{k-1}\|^2} \\
&= \frac{\|g_k\|\left(2\|g_k - g_{k-1}\| + t_k^*\|s_{k-1}\|\right)}{\theta\alpha_{k-1}\|d_{k-1}\|^2} \le \frac{\|g_k\|\left(2L\|s_{k-1}\| + t_k^*\|s_{k-1}\|\right)}{\theta\alpha_{k-1}\|d_{k-1}\|^2} \\
&= \frac{(2L + t_k^*)\|g_k\|\|s_{k-1}\|}{\theta\alpha_{k-1}\|d_{k-1}\|^2} = \frac{(2L + t_k^*)\|g_k\|\alpha_{k-1}\|d_{k-1}\|}{\theta\alpha_{k-1}\|d_{k-1}\|^2} \\
&= \frac{(2L + t_k^*)\|g_k\|}{\theta\|d_{k-1}\|}.
\end{aligned} \tag{39}$$

Using $t_k^* \in (0, 1)$ and (38) and (39) in (36), we obtain

$$\begin{aligned}
\|d_k\|^2 &\le \|g_k\|^2 + 2\frac{\|g_k\|^2}{\lambda} + \frac{(2L + t_k^*)^2\|g_k\|^2}{\theta^2\|d_{k-1}\|^2}\|d_{k-1}\|^2 \\
&= \|g_k\|^2 + 2\frac{\|g_k\|^2}{\lambda} + \frac{(2L + t_k^*)^2}{\theta^2}\|g_k\|^2 \\
&= \left(1 + \frac{2}{\lambda} + \frac{(2L + t_k^*)^2}{\theta^2}\right)\|g_k\|^2 = \left(\frac{\lambda + 2}{\lambda} + \frac{(2L + t_k^*)^2}{\theta^2}\right)\|g_k\|^2 \\
&= \frac{(\lambda + 2)\theta^2 + \lambda(2L + t_k^*)^2}{\lambda\theta^2}\|g_k\|^2.
\end{aligned} \tag{40}$$

Next, dividing both sides of (40) by $\|g_k\|^4$ and using (35), it can be concluded that

$$\begin{aligned}
\frac{\|d_k\|^2}{\|g_k\|^4} &\le \frac{(\lambda + 2)\theta^2 + \lambda(2L + t_k^*)^2}{\lambda\theta^2} \cdot \frac{1}{c_1^2}, \\
\frac{\|g_k\|^4}{\|d_k\|^2} &\ge \frac{\lambda\theta^2 \cdot c_1^2}{(\lambda + 2)\theta^2 + \lambda(2L + t_k^*)^2}.
\end{aligned} \tag{41}$$

The inequalities in (41) imply

$$\sum_{k=0}^{\infty}\frac{\|g_k\|^4}{\|d_k\|^2} \ge \sum_{k=0}^{\infty}\frac{\lambda\theta^2 \cdot c_1^2}{(\lambda + 2)\theta^2 + \lambda(2L + t_k^*)^2} = \infty. \tag{42}$$

Therefore, $\|g_k\| \ge c_1$ causes a contradiction with Lemma 2.

3. Numerical Experiments

The implementation of the EDL method is based on Algorithm 2. This section is intended to analyze and compare the numerical results obtained by the EDL method and four variants of the MHSDL class methods (6). These variants are defined by $t \equiv t_{k3}$, $t \equiv t_{k4}$, $t \equiv t_{k5}$, and $t \equiv t_{k6}$ and denoted, respectively, as MHSDL3, MHSDL4, MHSDL5, and MHSDL6. The obtained results are not compared with the values t_{k1} and t_{k2}, because in [16], the authors have already shown that t_{k3} and t_{k4} initiate better numerical performances compared to t_{k1} and t_{k2}.

The codes used in the testing experiments for the above methods are written in MATLAB R2017a and executed on the Intel Core i3 2.0 GHz workstation with the Windows 10 operating system. Three important criteria are analyzed in each individual test case: number of iterations (NI), number of function evaluations (NFE), and processor time (CPU).

The numerical experiment is performed using 28 test functions presented in [24], where much of the problems are taken over from the CUTEr collection [25]. All methods used in the testing of an arbitrary objective function start from the same initialization x_0. Each function is tested 10 times with gradually increasing dimensions $n = 100$, 500, 1000, 3000, 5000, 7000, 8000, 10000, 15000, and 20000.

The uniform terminating criteria for each of the five considered algorithms (EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6) are

$$\begin{aligned}
\|g_k\| &\le \varepsilon, \\
\frac{|f(x_{k+1}) - f(x_k)|}{1 + |f(x_k)|} &\le \delta,
\end{aligned} \tag{43}$$

where $\varepsilon = 10^{-6}$ and $\delta = 10^{-16}$. The backtracking line search is based on the parameters $\omega = 0.0001$ and $\varphi = 0.8$ for all five algorithms. Specific parameters used only in the MHSDL6 method are defined as $C = 1$, $\upsilon = 0.26$, and $r = r_k = \upsilon\|g_{k-1}\|$.

Summary numerical results for EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods, executed on 28 test functions, are arranged in Tables 1–3. Tables 1–3 show the numerical outcomes corresponding to all three criteria (NI, NFE, and CPU) for the EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods.

We utilized the performance profile given in [26] to compare numerical results for three criteria (NI, NFE, and CPU) generated by five methods (EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6). The upper curve of the selected performance profile corresponds to the method that shows the best performance.

Figures 1–3 plot the performance profiles for the numerical values included in Tables 1–3, respectively. Figure 1 presents the performance profiles of the NI criterion generated by the EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods. In this figure, it is noticeable that EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods solved all tested functions, wherein the EDL method shows the best performances in 57.14% of test functions compared with MHSDL3 (25.00%), MHSDL4 (0.00%), MHSDL5 (0.00%), and MHSDL6 (17.86%). From Figure 1, it is observable that the graph of the EDL method comes first to the top, which means that the EDL outperforms other considered methods with respect to the NI.

Figure 2 presents the performance profiles of the NFE of the EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods. It is observable that EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 generated solutions to all tested cases, and the EDL method is the best in 67.86% of the functions compared with MHSDL3 (17.86%), MHSDL4 (0.00%), MHSDL5 (0.00%), and MHSDL6 (14.28%). From Figure 2, it is observed that the EDL graph first comes to the top, which confirms that the EDL is the winner with respect to the NFE.

Figure 3 contains graphs of the performance profiles corresponding to the CPU time of the EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 methods. It is obvious that EDL, MHSDL3, MHSDL4, MHSDL5, and MHSDL6 solved all tested functions. Further analysis gives that the EDL method is the winner in 67.86% of the test cases compared with MHSDL3 (17.86%), MHSDL4 (0.00%), MHSDL5 (0.00%), and MHSDL6 (14.28%). Figure 3 demonstrates that the graph of the EDL method first comes to level 1, which indicates its superiority with respect to the CPU time.

From the previous analysis of the results shown in Tables 1–3 and Figures 1–3, it can be concluded that the EDL method produces superlative results in terms of all three basic metrics: NI, NFE, and CPU.

4. Conclusion

A novel rule which determines the value $t(k)$ of the parameter t in each iteration of the Dai-Liao-type CG method is presented. The proposed expression for defining $t(k)$ is denoted by t_k^*. Considering $t = t_k^*$ in (6), a novel variant of the Dai-Liao CG parameter β_k^{EDL} is defined and a novel Effective Dai-Liao (EDL) conjugate gradient method is proposed. The convergence of the EDL method is investigated, and the global convergence on a class of uniformly convex functions is established. By numerical testing, we have shown that there is a significant influence of the scalar size of t_k^* on the convergence speed of the EDL method. Numerical comparisons on large-scale unconstrained optimization test functions of different structures and complexities confirm the computational efficiency of the algorithm EDL and its superiority over the previously known DL CG variants, such as MHSDL3, MHSDL4, MHSDL5, and MHSDL6. During the testing, we tracked the number of iterations (NI), number of function evaluations (NFE), and spanned processor time (CPU) performances for each function and each method. Analysis of the obtained performance profiles introduced by Dolan and Moré revealed that the EDL method is the most efficient.

We are convinced that the obtained results will be a motivation for further research in defining new values of the parameter t_k in the Dai-Liao CG methods. Future research would include research in finding some more efficient rules to calculate the parameter t_k during the iterative process. We hope that our proposal of the new expression for defining the parameter t will initiate further research in that direction. It is evident that finding novel approaches in defining different values of t and the conjugate gradient parameter β_k is an inexhaustible topic for scientific research, and our approach is only one possible direction in this research.

Data Availability

Data will be provided on request to the first author.

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

A Note on the Górnicki-Proinov Type Contraction

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In this paper, we propose a notion of the Górnicki-Proinov type contraction. Then, we prove the uniqueness and existence of the fixed point for such mappings in the framework of the complete metric spaces. Some illustrative examples are also expressed to strengthen the observed results.

1. Introduction and Preliminaries

The history of the fixed point theory goes back about a century. Banach's result initiated the metric fixed point theory in 1922 [1]. The first outstanding extension of this initial theorem was given by Kannan [2] in 1968. In this first generalization, Kannan [2] removed the necessity of the continuity of the contraction mapping. Recently, Górnicki [3] expressed an extension of Kannan type of contraction but the continuity condition was assumed. After then, Bisht [4] refined the result of Górnicki [3] by replacing the continuity condition for the considered mapping with orbitally continuity or p-continuity. Very recently, Górnicki [5] improved these two mentioned results by introducing new contractions, "Geraghty-Kannan type" and "ϕ-Kannan type." He proved the existence of a fixed point for such mappings. On the other hand, Proinov [6] discussed some existing results and noted that these results are particular cases of Skof [7]. He also proposed a very general fixed point theorem that also contains the result of Skof [7].

We first recall the pioneer theorem of Banach [1] and Kannan [2]. On a complete metric space (X, d), a mapping $T : X \to X$ admits a unique fixed point if there exists $0 \leq \mathscr{K} < 1$ such that

$$d(Tu, Tv) \leq k \cdot d(u, v), \tag{1}$$

and

$$d(Tu, Tv) \leq K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{2}$$

for all $u, v \in X$. The inequality (1) belongs to Banach [1] and (2) belongs to Kannan [2]. By using the "asymptotic regularity" concept, Górnicki [3] proved an extension of Kannan Theorem 1.2. Before giving this interesting result, we recollect the interesting concepts:

Let T be a self-mapping on a metric space (X, d) and $\{T^n u\}$ be the Picard iterative sequence, for an initial point $u \in X$.

(o) The set $O(T, u) = \{T^n u : n = 0, 1, 2, \cdots\}$ is called *the orbit of the mapping* T at u.

The mapping T is said to be [3, 5]:

(o-c) *orbitally continuous at a point* $w \in X$ if for any sequence $\{u_n\}$ in $O(T, u)$ for some $u \in X$, $\lim_{n\to\infty} d(u_n, w) = 0$ implies $\lim_{n\to\infty} d(Tu_n, Tw) = 0$.

(p-c) p *-continuous at a point* $w \in X$ $(p = 1, 2, 3, \cdots)$ if for any sequence $\{u_n\}$ in $X \lim_{n\to\infty} d(T^{p-1} u_n, w) = 0$ implies $\lim_{n\to\infty} d(T^p u_n, Tw) = 0$.

(a-r) *asymptotically regular at a point* $u \in X$ if $\lim_{n\to\infty} d(T^n u, T^{n+1} u) = 0$. If T is asymptotically regular at each point of X, we say that it is asymptotically regular.

Remark 1. In [8], it is shown that p-continuity of T and the continuity of Tp are independent conditions for the case $p > 1$.

Theorem 2 (see [3, 5]). *On a complete metric space (X, d), a continuous asymptotically regular mapping $T : X \to X$ admits a unique fixed point if there exist $0 \le k < 1$ and $0 \le K < +\infty$ such that*

$$d(Tu, Tv) \le k \cdot d(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{3}$$

for all $u, v \in X$.

Later, the assumption of continuity of the mapping T was replaced with weaker notions of continuity.

Theorem 3 (see [4]). *On a complete metric space (X, d) and a mapping $T : X \to X$. Suppose that there exists $0 \le K < 1$ such that*

$$d(Tu, Tv) \le k \cdot d(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{4}$$

for all $u, v \in X$. Then, T admits a unique fixed point if either T is (o-c) or (p-c) for $p \ge 1$.

In [5], some generalizations of Theorems 2 and 3 are considered, by replacing the constant k with some real-valued functions.

Theorem 4 (see [5]). *Let (X, d) be a complete metric space and $T : X \to X$ be an (a-r) mapping such that there exist $\psi : [0,\infty) \to [0,\infty)$ and $0 \le K < \infty$ such that*

$$d(Tu, Tv) \le \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{5}$$

for all $u, v \in X$. Suppose that:

(i) $\phi(\theta) < \theta$ for all $\theta > 0$ and ϕ is upper semicontinuous

(ii) either T is (o-c) or T is (p-c) for some $p \ge 1$

Then, T has a unique fixed point $u_* \in X$ and for each $u \in X$, $T^n u \to u_*$ as $n \to \infty$.

Theorem 5 (see [5]). *Let (X, d) be a complete metric space and $T : X \to X$ be an (a-r) mapping such that there exist $\varsigma : [0,\infty) \to [0, 1)$ and $0 \le K < \infty$ such that*

$$d(Tu, Tv) \le \varsigma(d(u, v)) \cdot d(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{6}$$

for all $u, v \in X$. Suppose that:

(1) $\varsigma(\theta_n) \to 1 \Rightarrow \theta_n \to 0$;

(2) either T is (o-c) or T is (p-c) for some $p \ge 1$

Then, T has a unique fixed point $u_* \in X$ and for each $u \in X$, $T^n u \to u_*$ as $n \to \infty$.

On the other hand, very recently, Proinov announced some results which unify many known results [6].

Theorem 6 (see [6]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$\psi(d(Tu, Tv)) \le \phi(d(u, v)), \tag{7}$$

for all $u, v \in X$ with $d(Tu, Tv) > 0$, where the functions ψ, $\phi : (0,\infty) \to \mathbb{R}$ are such that the following conditions are satisfied:

(p_1) $\phi(\theta) < \psi(\theta)$ for any $\theta > 0$;

(p_2) ψ is nondecreasing;

(p_3) $\limsup_{\theta\to e+} \phi(\theta) < \psi(e+)$ for any $e > 0$.

Then, T admits a unique fixed point.

Theorem 7 (see [6]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$\psi(d(Tu, Tv)) \le \phi(d(u, v)), \tag{8}$$

for all $u, v \in X$ with $d(Tu, Tv) > 0$, where $\psi, \phi : (0,\infty) \to \mathbb{R}$ are two functions such that the following conditions are satisfied:

(p_1) $\phi(\theta) < \psi(\theta)$ for any $\theta > 0$;

(p_4) $\inf_{\theta > e} \psi(\theta) > -\infty$ for any e;

(p_5) $\limsup_{\theta\to e+} \phi(\theta) < \liminf_{\theta\to e} \psi(\theta)$ or $\limsup_{\theta\to e} \phi(\theta) < \liminf_{\theta\to e+} \psi(\theta)$ for any $e > 0$;

(p_6) $\limsup_{\theta\to 0+} \phi(\theta) < \liminf_{\theta\to e} \psi(\theta)$ for any $e > 0$;

(p_7) if the sequences $(\psi(\theta_n))$ and $(\phi(\theta_n))$ are convergent with the same limit and $(\psi(\theta_n))$ is strictly decreasing, then $\theta_n \to 0$ as $n \to \infty$.

Then, T admits a unique fixed point.

Lemma 8 (see [6]). *Let (u_n) be a sequence in a metric space (X, d) such that $d(u_n, u_{n+1}) \to 0$ as $n \to \infty$. If the sequence (u_n) is not Cauchy, then there exist $e > 0$ and two subsequences $\{s_k\}$, $\{r_k\}$ of positive integers such that*

$$\lim_{k\to\infty} d\left(u_{s_k+1}, u_{r_k+1}\right) = e+,$$
$$\lim_{k\to\infty} d\left(u_{s_k}, u_{r_k}\right) = \lim_{k\to\infty} d\left(u_{s_k+1}, u_{r_k}\right) = \lim_{k\to\infty} d\left(u_{s_k}, u_{r_k+1}\right) = e \tag{9}$$

Lemma 9 (see [6]). *Let (u_n) be a sequence in a metric space (X, d) such that $d(u_n, u_{n+1}) \to 0$ as $n \to \infty$. If the sequence (u_n) is not Cauchy, then there exist $e > 0$ and two subsequences $\{s_k\}$, $\{r_k\}$ of positive integers such that*

$$\lim_{k\to\infty} d\left(u_{s_k}, u_{r_k}\right) = e+,$$
$$\lim_{k\to\infty} d\left(u_{s_k+1}, u_{r_k+1}\right) = \lim_{k\to\infty} d\left(u_{s_k+1}, u_{r_k}\right) = \lim_{k\to\infty} d\left(u_{s_k}, u_{r_k+1}\right) = e \tag{10}$$

In the end of this section, we recall the notions of α-orbital admissible and triangular α-orbital admissible mappings [9] with mention that these notions were extended in many directions, see, e.g., [10] and it could be potentially extended also to several approaches of recent developments in fixed point theory. See, for instance, [11–21].

On a metric space (X, d), a self-mapping T is called

(i) α-orbital admissible if

$$\alpha(u, Tu) \geq 1 \Rightarrow \alpha\left(Tu, T^2u\right) \geq 1, \tag{11}$$

for any $u, v \in X$, where $\alpha : X \times X \to [0,\infty)$

(ii) triangular α-orbital admissible if it is α-orbital admissible and the following condition is satisfied

$$\alpha(u, v) \geq 1 \text{ and } \alpha(v, Tv) \geq 1 \Rightarrow \alpha(u, Tv) \geq 1, \tag{12}$$

for any $u, v, w \in X$

Lemma 10. *If for an triangular α-orbital admissible mapping $T : X \to X$ there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$, then*

$$\alpha\left(u_n, u_p\right) \geq 1, \text{for all } n, p \in \mathbb{N}, \tag{13}$$

where the sequence $\{u_n\}$ is defined as $u_{n+1} = Tu_n$.

Let (X, d) be a metric space and the function $\alpha : X \times X \to [0,\infty)$. The following conditions will be used further:

$\mathscr{R}$ If for a sequence $\{u_n\}$ in X such that $u_n \to u$ and $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{u_{p_k}\}$ of $\{u_n\}$ such that $\alpha(u_{p_k}, u) \geq 1$.

$(\mathscr{U})$ For all $u, v \in Fix_X T = \{z \in X : Tz = z\}$, we have $\alpha(u, v) \geq 1$.

2. Main Results

Let Λ be the set of all functions $\phi : (0,\infty) \to \mathbb{R}$. For $\phi, \psi \in \Lambda$, we are considering the following conditions:

(a$_1$) $\phi(\theta) < \psi(\theta)$ for $\theta > 0$

(a$_2$) $\limsup_{\theta\to e+}\phi(\theta) < \liminf_{\theta\to e}\psi(\theta)$, for any $e > 0$

(a$_3$) $\limsup_{\theta\to e}\phi(\theta) < \liminf_{\theta\to e+}\psi(\theta)$, for any $e > 0$

(a$_4$) $\limsup_{\theta\to e+}\phi(\theta) < \psi(e+)$, for any $e > 0$

Definition 11. Let (X, d) be a metric space, the functions ψ, $\phi \in \Lambda$ and $\alpha : X \times X \to [0,\infty)$. An *(a-r)* mapping $T : X \to X$ is said to be (α, ψ, ϕ)-contraction if there exists $0 \leq K < \infty$ such that

$$\alpha(u, v)\psi(d(u, v)) \leq \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{14}$$

for each $u, v \in X$ with $d(Tu, Tv) > 0$.

Theorem 12. *On a complete metric space (X, d) an (α, ψ, ϕ)-contraction $T : X \to X$ has a fixed point provided that*

(1) the functions $\psi, \phi \in \Lambda$ satisfy (a_1) and either (a_2) or (a_3)

(2) T is triangular α-orbital admissible and there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$

(3) either T is (o-c) or T is (p-c), for some $p \geq 1$

Moreover, if property (U) is satisfied, then the fixed point of T is unique.

Proof. Let u be any point (but fixed) in X and we build the sequence $\{u_n\}$, where $u_0 = u$ and $u_n = T^n u$ for any $n \in \mathbb{N}$. If there exists $m_0 \in \mathbb{N}$ such that $T^{m_0}u = T^{m_0+1}u = T(T^{m_0}u)$, then $T^{m_0}u$ is a fixed point of T. For this reason, we can suppose that $T^n u \neq T^{n+1}u$, for every $n \in \mathbb{N} \cup \{0\}$ and we claim that $\{u_n\}$ is Cauchy sequence. Assuming the contrary, that the sequence $\{u_n\}$ is not Cauchy, from Lemma 1, it follows that we can find e and two subsequences $\{s_k\}$ and $\{r_k\}$ of positive integers such that (9) holds. Letting $u = u_{s_k}$ and $v = u_{r_k}$ in (14), we have $\alpha(u_{s_k}, u_{r_k}) \geq 1$ (taking into account (1.8)), and then,

$$\begin{aligned}\psi\left(d\left(u_{s_k+1}, u_{r_k+1}\right)\right) &\leq \alpha\left(u_{s_k}, u_{r_k}\right)\psi\left(d\left(u_{s_k+1}, u_{r_k+1}\right)\right) \\ &= \alpha\left(u_{s_k}, u_{r_k}\right)\psi\left(d\left(Tu_{s_k}, Tu_{r_k}\right)\right) \\ &\leq \phi\left(d\left(u_{s_k}, u_{r_k}\right)\right) + K \\ &\quad \cdot \left\{d\left(u_{s_k}, u_{s_k+1}\right) + d\left(u_{r_k}, u_{r_k+1}\right)\right\},\end{aligned} \tag{15}$$

or denoting $\xi_k = d(u_{s_k+1}, u_{r_k+1})$ and $\zeta_k = d(u_{s_k}, u_{r_k})$

$$\psi(\xi_k) \leq \phi(\zeta_k) + K \cdot \left\{d\left(u_{s_k}, u_{s_k+1}\right) + d\left(u_{r_k}, u_{r_k+1}\right)\right\}. \tag{16}$$

Taking into account the asymptotically regularity of T, from (9), it follows that

$$\xi_k \to e + \text{and } \zeta_k \to e. \tag{17}$$

Thus, letting the limit in (16), we have

$$\liminf_{\theta\to e+}\psi(\theta) \leq \liminf_{k\to\infty}\psi(\xi_k) \leq \limsup_{k\to\infty}\phi(\zeta_k) \leq \limsup_{\theta\to e}\phi(\theta). \tag{18}$$

This contradicts the assumption (a$_2$).

Similarly, if we consider that the functions ψ, ϕ satisfy (a$_3$), the conclusion follows in the same way, but taking into account Lemma 2.

Therefore, $\{u_n\}$ is a Cauchy sequence, and because the space (X, d) is complete, there exists u_* such that

$$\lim_{n\to\infty} u_n = u_*. \tag{19}$$

We claim that u_* is a fixed point of T.

If T is orbitally continuous, then since $\{u_n\} \in O(T, u)$ and $u_n \to u_*$, we have $u_{n+1} = Tu_n \to Tu_*$ as $n \to \infty$. The uniqueness of the limit gives $Tu_* = u_*$.

If T is p-continuous, for some $p \geq 1$, by (19), we have $\lim_{n\to\infty} T^{p-1}u_n = u^*$ which implies $\lim_{n\to\infty} T^p u_n = Tu^*$ (because T is p-continuous). Therefore, by uniqueness of the limit, we have $Tu_* = u_*$.

Now, supposing that there exists $v_* \in X$ such that $Tv_* = v_* \neq u_* = Tu_*$, from (14) and taking into account the property (U), we have

$$\begin{aligned}\psi(d(u_*, v_*)) &\leq \alpha(u_*, v_*)\psi(d(Tu_*, Tv_*)) \\ &\leq \phi(d(u_*, v_*)) + K \cdot \{(d(u_*, Tu_*) + (d(v_*, Tv_*))\} \\ &= \phi(d(u_*, v_*)) < \psi((d(u_*, v_*)),\end{aligned} \tag{20}$$

which is a contradiction. Therefore, $u_* = v_*$.

Letting $\alpha(u, v) = 1$ in Theorem 12, we get the following:

Corollary 13. *Let (X, d) be a complete metric space and an (a-r) mapping $T : X \to X$. Suppose that there exists $0 \leq K < \infty$ such that*

$$\psi(d(u, v)) \leq \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{21}$$

for each $u, v \in X$ with $d(Tu, Tv) > 0$, where $\psi, \phi \in \Lambda$. Suppose also that:

(1) the functions $\psi, \phi \in \Lambda$ satisfy (a_1) and either (a_2) or (a_3)

(2) either T is (o-r) or T is (p-o), for some $p \geq 1$

Then, T has a unique fixed point.

Corollary 14. *Let (T, d) be a complete metric space and $T : X \to X$ be an (a-r) mapping such that*

$$d(Tu, Tv) \leq \varsigma(d(u, v))d(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{22}$$

for each $u, v \in X$, where $0 \leq K < \infty$ and the function $\varsigma : (0, \infty) \to (0, 1)$ is such that $\limsup_{\theta\to e+}\varsigma(\theta) < 1$ for any $e > 0$. If T is either (o-c) or (p-c) for some $p \geq 1$, then T has a unique fixed point.

Proof. Let $\psi(\theta) = \theta$ in Corollary 1.

Taking $\psi(\theta) = \theta$ and $\phi(\theta) = k \cdot \theta$, with $k \in [0, 1)$ Corollary 1 becomes:

Corollary 15. *Let (T, d) be a complete metric space and $T : X \to X$ be an (a-r) mapping. If there exist $k \in [0, 1)$ and $0 \leq K < \infty$ such that*

$$d(Tu, Tv) \leq kd(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\} \tag{23}$$

for each $u, v \in X$, then T admits a unique fixed point provided that T is (o-c) or (p-c) for some $p \geq 1$.

Theorem 16. *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$, $\psi, \varphi \in \Lambda$ such that (a_1) and (a_2) are satisfied. Let $T : X \to X$ be an (a-r) mapping. Suppose that there exists $0 \leq K < \infty$ such that*

$$\psi(\alpha(u, v)d(Tu, Tv)) \leq \varphi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{24}$$

for each $u, v \in X$ with $d(Tu, Tv) > 0$. Suppose also that

(i) ψ is nondecreasing and $\limsup_{\theta\to e+} < \psi(e+)$ for any $e > 0$

(ii) T is triangular α-orbital admissible and there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$

(iii) the mapping T is either (o-c) or (p-c)

Then, the mapping T possesses a fixed point. Moreover, the fixed point is unique, provided that property (U) is satisfied.

Proof. Let $\{u_n\}$ be the sequence defined as in the previous theorem, as $u_n = T^n u$, where $u \in X$ is arbitrary but fixed. Letting $u = u_{s_k}$ and $v = u_{r_k}$ in (2.7), we have

$$\begin{aligned}&\psi\left(\alpha\left(u_{s_k}, u_{r_k}\right)d\left(u_{s_k+1}, u_{r_k+1}\right)\right) \\ &\quad \leq \phi\left(d\left(u_{s_k}, u_{r_k}\right)\right) + K \cdot \left\{d\left(u_{s_k}, u_{s_k+1}\right) + d\left(u_{s_k}, u_{s_k+1}\right)\right\},\end{aligned} \tag{25}$$

and taking into account the assumptions (i), (ii), and Lemma 3, we get

$$\begin{aligned}\psi\left(d\left(u_{s_k+1}, u_{r_k+1}\right)\right) \leq \phi\left(d\left(u_{s_k}, u_{r_k}\right)\right) + K \\ \cdot \left\{d\left(u_{s_k}, u_{s_k+1}\right) + d\left(u_{r_k}, u_{r_k+1}\right)\right\},\end{aligned} \tag{26}$$

Setting $\xi_k = d(u_{s_k+1}, u_{r_k+1})$ and $\zeta_k = d(u_{s_k}, u_{r_k})$ and since $\phi(\theta) < \psi(\theta)$, we get

$$\begin{aligned}\psi(\xi_k) &\leq \phi(\zeta_k) + K \cdot \left\{d\left(u_{s_k}, u_{s_k+1}\right) + d\left(u_{r_k}, u_{r_k+1}\right)\right\} \\ &< \psi(\zeta_k) + K \cdot \left\{d\left(u_{s_k}, u_{s_k+1}\right) + d\left(u_{r_k}, u_{r_k+1}\right)\right\}.\end{aligned} \tag{27}$$

On the other hand, from 1.5 that $\xi_k \to e+$, $\zeta_k \to e+$ and then, letting the limit as $k \to \infty$ in the above inequality, since T is an (a-r) mapping and taking into account the second part of the assumption (i), we have

$$\psi(e+) = \lim_{k\to\infty}\psi(\xi_k) \le \limsup_{k\to\infty}\phi(\zeta_k) \le \limsup_{\theta\to e+}\phi(\theta) < \psi(e+), \tag{28}$$

which is a contradiction. Thus, the sequence $\{u_n\}$ is Cauchy on a metric space, so there exists u_* such that $u_n \to u_*$ as $n \to \infty$ and following the lines of the previous proof, we get that u_* is the unique fixed point of T.

Again, letting $\alpha(u,v) = 1$ for any $u, v \in X$ we get the following:

Theorem 17. *Let (X,d) be a complete metric space, and two functions $\psi, \varphi \in \Lambda$ such that (a_1) is satisfied. Let $T : X \to X$ be an (a-r) mapping. Suppose that there exists $0 \le K < \infty$ such that*

$$\psi(d(Tu,Tv)) \le \phi(d(u,v)) + K\cdot\{d(u,Tu) + d(v,Tv)\}, \tag{29}$$

for each $u, v \in X$ with $d(Tu,Tv) > 0$, where $\psi, \phi \in \Lambda$. Suppose also that

(i) ψ is nondecreasing and $\limsup_{\theta\to e+} < \psi(e+)$ *for any $e > 0$*

(ii) the mapping T is either (o-c) or (p-c)

Then, the mapping T possesses a unique fixed point.

Theorem 18. *Let (X,d) be a complete metric space, and two functions $\psi, \varphi \in \Lambda$ such that (a_1) is satisfied. Let $T : X \to X$ be an (a-r) mapping. Suppose that there exists $0 \le K < \infty$ such that*

$$\psi(d(Tu,Tv)) \le \varsigma(d(u,v))\psi(d(u,v)) + K\cdot\{d(u,Tu) + d(v,Tv)\}, \tag{30}$$

for each $u, v \in X$ with $d(Tu,Tv) > 0$, where $\psi \in \Lambda$ and $\varsigma : (0, \infty) \to (0,1)$. Suppose also that

(i) ψ is nondecreasing and $\limsup_{\theta\to e+}\varsigma(\theta) < 1$ *for any $e > 0$*

(ii) the mapping T is either (o-c) or (p-c)

Then, the mapping T possesses a unique fixed point.

Proof. Take $\phi(\theta) = \alpha(\theta)\psi(\theta)$, for $\theta > 0$ in Theorem 17.

Next, we consider mappings that satisfy a similar condition as (14), but for which the asymptotic regularity condition is not necessary.

Definition 19. Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ and $\psi, \varphi \in \Lambda$. A mapping $T : X \to X$ is called (α,ψ,ϕ)-contraction of type 2 if there exists $0 \le K < \infty$ such that

$$\alpha(u,v)\psi(d(Tu,Tv)) \le \phi(d(u,v)) + K\cdot\{d(u,Tu) + d(v,Tv)\} \cdot d(u,Tv)d(v,Tu), \tag{31}$$

for each $u, v \in X$ with $d(Tu,Tv) > 0$.

Theorem 20. *On a complete metric space (X,d), an (α,ψ,ϕ) -contraction of type 2, $T : X \to X$ has a fixed point provided that property (R) and the following conditions hold:*

(A) T is triangular α-orbital admissible and there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \ge 1$

(B) ψ, ϕ satisfy the assumptions (a_1) and (a_4)

(C) ψ is nondecreasing

(D) $\limsup_{\theta\to 0+}\phi(\theta) < \liminf_{\theta\to e}\psi(\theta)$, *for any $e > 0$*

Moreover, if the property (U) holds, the fixed point of T is unique.

Proof. Let $\{u\}$ be a sequence in X defined as

$$u_n = T^n u_0, foreveryn \in \mathbb{N}, \tag{32}$$

where u_0 is an arbitrary but fixed point in X. Replacing in (31) and taking into account (11), we have

$$\begin{aligned}\psi(d(u_n,u_{n+1})) &\le \alpha(u_{n-1},u_n)\psi(d(Tu_{n-1},Tu_n)) \\ &\le \phi(d(u_{n-1},u_n)) + +K\cdot\{d(u_{n-1},Tu_{n-1}) + d(u_n,Tu_n)\} \\ &\quad\cdot d(u_{n-1},Tu_n)d(u_n,Tu_{n-1}) \\ &= \phi(d(u_{n-1},u_n)) + K\cdot\{d(u_{n-1},u_n) + d(u_n,u_{n+1})\} \\ &\quad\cdot d(u_{n-1},u_{n+1})d(u_n,u_n) = \phi(d(u_{n-1},u_n)),\end{aligned} \tag{33}$$

or setting $x_n = d(u_{n-1},u_n)$ (we can suppose that $x_n > 0$) and taking into account the condition (a_1) for any $\theta > 0$, we get

$$\psi(x_n) \le \phi(x_{n-1}) < \psi(x_{n-1}). \tag{34}$$

If the condition (C) holds, from the above inequality, we get $x_n < x_{n-1}$, for every $n \in \mathbb{N}$. Consequently, being positive and strictly decreasing, the sequence $\{x_n\}$ is convergent and there is $x \ge 0 \in X$ such that $x_n \to x$. If we assume that $x > 0$, then from the above inequality, we have

$$\phi(x+) = \lim_{n\to\infty}\psi(x_n) \le \limsup_{n\to\infty}\phi(x_n) \le \limsup_{\theta\to x+}\phi(\theta) < \phi(x+), \tag{35}$$

which is a contradiction. Thus,

$$\lim_{n\to\infty} x_n = x = 0. \tag{36}$$

The aim for the next step is to prove that the sequence $\{u_n\}$ is Cauchy. Supposing by contradiction, the sequence

$\{u_n\}$ is not Cauchy, by (36), and taking into account Lemma 1, we can find $e > 0$ and two subsequences $\{u_{s_k}\}$ and $\{u_{r_k}\}$ of $\{u_n\}$ such that (9) holds. Taking $u = u_{s_k}$ and $v = u_{r_k}$ in (14) and keeping in mind (1.7), we have

$$\begin{aligned}\psi(d(u_{s_k+1}, u_{r_k+1})) &\le \alpha(u_{s_k}, u_{r_k})\psi(d(Tu_{s_k}, Tu_{r_k})) \\ &\le \phi(d(u_{s_k}, u_{r_k})) + +K \cdot \{d(u_{s_k}, u_{s_k+1}) + d(u_{r_k}, u_{r_k+1})\} \\ &\quad \cdot d(u_{s_k}, u_{r_k+1}) d(u_{r_k}, u_{s_k+1}).\end{aligned} \tag{37}$$

Letting the limit as $k \to \infty$ in the previous inequality (since $d(u_{s_k+1}, u_{r_k+1}) \to e+$ and $d(u_{s_k}, u_{r_k}) \to e$ and using (2.11), we get

$$\begin{aligned}\liminf_{\theta \to e+} \psi(\theta) &\le \liminf_{k\to\infty} \psi(d(u_{s_k+1}, u_{r_k+1})) \\ &\le \limsup_{k\to\infty} \phi(d(u_{s_k}, u_{r_k})) \le \limsup_{\theta \to e} \phi(\theta).\end{aligned} \tag{38}$$

This is a contradiction to (a_4). Thus, $\{u_n\}$ is a Cauchy sequence on a complete metric space, so it is convergent. Let $u_* = \lim_{n\to\infty} u_n$ and we claim that u_* is a fixed point of T. From (31) and (R), for $u = u_n$ and $v = u_*$, we have

$$\begin{aligned}\psi(d(u_{n+1}, Tu_*)) &\le \alpha(u_n, u_*)\psi(d(Tu_n, Tu_*)) \\ &\le \phi(d(u_n, u_*)) + +K \cdot \{d(u_n, u_{n+1}) + d(u_*, Tu_*)\} \\ &\quad \cdot d(u_n, Tu_*) d(u_*, u_{n+1}).\end{aligned} \tag{39}$$

Since $\lim_{n\to\infty} d(u_{n+1}, Tu_*) = d(u_*, Tu_*)$ and $\lim_{n\to\infty} d(u_n, u_*) = 0$ if we suppose that $d(u_*, Tu_*) > 0$, the above inequality yields

$$\begin{aligned}\liminf_{\theta \to d(u_*, Tu_*)} \psi(\theta) &\le \liminf_{n\to\infty} \psi(d(u_{n+1}, Tu_*)) \\ &\le \limsup_{n\to\infty} \phi(d(u_n, u_*)) \le \limsup_{\theta\to 0} \phi(\theta),\end{aligned} \tag{40}$$

which is a contradiction to (D). Therefore, $d(u_*, Tu_*) = 0$, that is u_* is a fixed point of T. As in the Theorem 12, adding the condition (U) to the statement of Theorem 20, we are able to prove that the fixed point is unique. Indeed, if we suppose that $v_* \in X$ is such that $Tv_* = v_* \ne u_* = Tu_*$, from (2.10), we have

$$\begin{aligned}\psi(d(u_*, v_*)) &\le \alpha(u_*, v_*)\psi(d(Tu_*, Tv_*)) \\ &\le \phi(d(u_*, v_*)) + +K \cdot \{d(u_*, Tu_*) + d(v_*, Tv_*)\} \\ &\quad \cdot (d(u_*, Tv_*) + d(v_*, Tu_*)).\end{aligned} \tag{41}$$

Letting $n \to \infty$ in the above inequality and keeping in mind (a_1), we have

$$\psi(d(u_*, v_*)) \le \phi(d(u_*, v_*)) < \psi(d(u_*, v_*)), \tag{42}$$

which is a contradiction.

Example 21. Let the set $X = \{A_1, A_2, A_3, A_4, A_5\}$ endowed with the distance $d : X \times X \to [0,\infty)$, where $d(u, u) = 0$, $d(u, v) = d(v, u)$ for any $u, v \in X$ and

$$\begin{gathered} d(A_1, A_2) = d(A_2, A_3) = d(A_3, A_4) = 1, \\ d(A_1, A_3) = d(A_2, A_4) = d(A_4, A_5) = 2, \\ d(A_1, A_4) = d(A_2, A_5) = d(A_3, A_5) = 3, d(A_1, A_5) = 4. \end{gathered} \tag{43}$$

Let the mapping $T : X \to X$ defined by

$$TA_1 = A_1, TA_2 = A_3, TA_3 = A_5, TA_4 = A_2, TA_5 = A_2. \tag{44}$$

Let also the function $\alpha : X \times X \to [0,\infty)$, with

$$\alpha(u, v) = \begin{cases} 2, & \text{if}\,(u, v) = (A_i, A_1), \text{for}\, i = 1, 2, 3, 4, 5 \\ 1, & \text{if}\,(u, v) \in \{(A_3, A_4), (A_4, A_3)\} \\ 0, & \text{otherwise} \end{cases} \tag{45}$$

Then, T does not satisfy Banach, neither Kannan type condition. Indeed, letting for example $u = A_1, v = A_3$,

$$\begin{aligned} d(TA_1, TA_3) = d(A_1, A_5) = 4 &> 2k \\ &= kd(A_1, A_3) for any 0 \le k < 1, \end{aligned}$$

$$\begin{aligned} d(TA_1, TA_3) = d(A_1, A_5) = 4 &> 3K \\ &= K \cdot \{d(A_1, A_1) + d(A_3, A_5)\} \\ &= K \cdot \{d(A_1, TA_1) + d(A_3, TA_3)\}, \\ &\quad for any 0 \le K < \frac{1}{2}. \end{aligned} \tag{46}$$

On the other hand, T is not (a-r), so Theorem 3 cannot be applied. Let the functions $\psi, \phi \in \Lambda$, $\phi(\theta) = \theta$, $\psi(\theta) = \theta/2$, for $\theta > 0$ and $K = 8$. For an easier reading, we will set

$$\begin{aligned} A(u, v) &= \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\} \\ &\quad \cdot d(u, Tv) d(v, Tu) = \frac{d(u, v)}{2} + 8 \\ &\quad \cdot \{d(u, Tu) + d(v, Tv)\} d(u, Tv) d(v, Tu). \end{aligned} \tag{47}$$

Let us check that the mapping T is an (α, ψ, ϕ)-contraction of type 2. For this purpose, we must consider the following cases:

(i) $u = A_1, v = A_2$,

$$\begin{aligned} \alpha(A_1, A_2)\psi(d(TA_1, TA_2)) = 2d(A_1, A_3) = 4 &< \frac{33}{2} \\ &= A(A_1, A_2) \end{aligned} \tag{48}$$

(ii) $u = A_1, v = A_3$,

$$\alpha(A_1, A_3)\psi(d(TA_1, TA_3)) = 2d(A_1, A_5) = 8 < 193 = A(A_1, A_3) \tag{49}$$

(iii) $u = A_1, v = A_4$,

$$\alpha(A_1, A_4)\psi(d(TA_1, TA_4)) = 2d(A_1, A_2) = 2 < \frac{99}{2} = A(A_1, A_4) \tag{50}$$

(iv) $u = A_1, v = A_5$,

$$\alpha(A_1, A_5)\psi(d(TA_1, TA_5)) = 2d(A_1, A_2) = 2 < 98 = A(A_1, A_5) \tag{51}$$

(v) $u = A_3, v = A_4$,

$$\alpha(A_3, A_4)\psi(d(TA_3, TA_4)) = d(A_5, A_2) = 3 < \frac{97}{2} = A(A_3, A_4) \tag{52}$$

Moreover, it is easy to see that all the assumptions of Theorem 20 are satisfied, so that T has a unique fixed point.

Example 22. Let the set $X = [0,\infty)$ be endowed with the usual distance d on $\mathbb{R}$. Consider the mapping $T : X \to X$ defined by

$$Tu = \begin{cases} 1 - u, & \text{if} 0 \le u \le 1 \\ \ln(1 + e^u), & if u > 1 \end{cases}. \tag{53}$$

Then,$T$is neither continuous, a contraction, nor (a-r). Define the function $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(u, av) = \begin{cases} 2, & \text{if } u \in \left\{\frac{1}{4}, \frac{1}{2}, 1\right\},\ v = \frac{1}{2} \\ 1, & \text{if } u = 2, v = 1 \\ 0, & \text{otherwise} \end{cases}. \tag{54}$$

Consider also, the functions $\psi, \phi \in \Lambda$, where $\psi(\theta) = e^{\theta}$ and $\phi(\theta) = \theta + 1$, for $\theta > 0$. Let, for example, $K = 64$. Using the same notation as in Example 1, taking into account the definition of the function α, we have the following:

(i) $u = 1/4$, $v = 1/2$

$$\alpha\left(\frac{1}{4}, \frac{1}{2}\right)\psi\left(d\left(T\frac{1}{4}, T\frac{1}{2}\right)\right) = 2e^{\frac{1}{4}} \le \frac{17}{4} = \phi\left(d\left(\frac{1}{4}, \frac{1}{2}\right)\right) + A\left(\frac{1}{4}, \frac{1}{2}\right) \tag{55}$$

(ii) $u = 1$, $v = 1/2$

$$\begin{aligned} \alpha\left(1, \frac{1}{2}\right)\psi\left(d\left(T1, T\frac{1}{2}\right)\right) &= 2e^{\frac{1}{2}} \le \frac{35}{2} \\ &= \phi\left(d\left(1, \frac{1}{2}\right)\right) + A\left(1, \frac{1}{2}\right) \end{aligned} \tag{56}$$

(iii) $u = 2$, $v = 1$

$$\alpha(2, 1)\psi(d(T2, T1)) = e^{\ln(1+e^2)} = 1 + e^2 \le \phi(d(2, 1)) + A(2, 1). \tag{57}$$

Since it easy to check that all the assumptions of Theorem 20 are verified, we can conclude that T has a unique fixed point.

Corollary 23. *Let (X, d) be a complete metric space and a mapping $T : X \to X$ such that for all $u, v \in X$ with $d(Tu, Tv) > 0$,*

$$\begin{aligned} \psi(d(Tu, Tv)) \le \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\} \\ \cdot d(u, Tv)d(v, Tu), \end{aligned} \tag{58}$$

where $0 \le K < 1$ and the functions $\psi, \phi \in \Lambda$ are such that

(a) ψ, ϕ satisfy (a_1) and (a_4)

(b) ψ is not decreasing

Then, T admits a unique fixed point.

Corollary 24 (Theorem 6). *Let (X, d) be a complete metric space and a mapping $T : X \to X$ such that for all $u, v \in X$ with $d(Tu, Tv) > 0$,*

$$\psi(d(Tu, Tv)) \le \phi(d(u, v)), \tag{59}$$

where the functions $\psi, \phi \in \Lambda$ are such that

(a) ψ, ϕ satisfy the assumptions (a_1) and (a_4)

(b) ψ is not decreasing

Then, T admits a unique fixed point.

Proof. Let $\alpha(u, v) = 0$ and $K = 0$ in Theorem 20.

Corollary 25. *Let (T, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$\begin{aligned} \psi(d(Tu, Tv)) \le \varsigma(d(u, v))\psi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\} \\ \cdot d(u, Tv)d(v, Tu), \end{aligned} \tag{60}$$

for each $u, v \in X$ *with* $d(Tu, Tv) > 0$*, where* $0 \le K < \infty$ *and the functions* $\varsigma : (0,\infty) \to (0, 1)$*,* $\psi : (0,\infty) \to (0, 1)$ *are such that*

(i) $\limsup_{\theta \to e+} \varsigma(\theta) < 1$ *for any* $e > 0$

(ii) ψ *is nondecreasing*

Then, T has a unique fixed point.

Proof. Let $\phi(\theta) = \varsigma(d(u, v))\psi(d(u, v))$ in Corollary 4.

Corollary 26. *Let* (X, d) *be a complete metric space and a mapping* $T : X \to X$*. Suppose that there exist* $0 \le k < 1$ *and* $0 \le K < \infty$ *such that for all* $u, v \in X$*,*

$$d(Tu, Tv) \le kd(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\}d(u, Tv)d(v, Tv). \tag{61}$$

Then, T admits a unique fixed point.

Proof. Let $\alpha(u, v) = 0$, $\psi(\theta) = \theta$ and $\phi(\theta) = k \cdot \theta$, with $0 \le k < 1$ in Theorem 20.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Quantum Integral Inequalities with Respect to Raina's Function via Coordinated Generalized Ψ-Convex Functions with Applications

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In accordance with the quantum calculus, we introduced the two variable forms of Hermite-Hadamard- ($\mathscr{H}\mathscr{H}$-) type inequality over finite rectangles for generalized Ψ-convex functions. This novel framework is the convolution of quantum calculus, convexity, and special functions. Taking into account the $\widehat{q}_1\widehat{q}_2$-integral identity, we demonstrate the novel generalizations of the $\mathscr{H}\mathscr{H}$-type inequality for $\widehat{q}_1\widehat{q}_2$-differentiable function by acquainting Raina's functions. Additionally, we present a different approach that can be used to characterize $\mathscr{H}\mathscr{H}$-type variants with respect to Raina's function of coordinated generalized Ψ-convex functions within the quantum techniques. This new study has the ability to generate certain novel bounds and some well-known consequences in the relative literature. As application viewpoint, the proposed study for changing parametric values associated with Raina's functions exhibits interesting results in order to show the applicability and supremacy of the obtained results. It is expected that this method which is very useful, accurate, and versatile will open a new venue for the real-world phenomena of special relativity and quantum theory.

1. Introduction

Recently, a nonrestricted analysis is recognized as quantum calculus (in short, $\widehat{q}$-calculus) and has initiated numerous $\widehat{q}$-mathematical formulation as $\widehat{q} \mapsto 1^-$. In 1707–1783, Euler proposed $\widehat{q}$-calculus theory. Accordingly, Jackson [1] explored the investigation of $\widehat{q}$-integrals efficiently. The previously mentioned outcomes prompted an escalated presentation on quantum theory in the 20$^{\text{th}}$ Century. As an application perspective, the concept of $\widehat{q}$-calculus has been potentially utilized in quantum mechanics, special relativity theory, anomalous diffusion equations, orthogonal polynomials, fractional calculus, and henceforth. In [2, 3], authors contemplated the $\widehat{q}$-derivatives on finite intervals of real line and amplified several new generalizations of classical convexity, $\widehat{q}$-version of Grüss, $\widehat{q}$-Cebyšev's, and $\widehat{q}$-Pólya-Szegö type inequalities. Over the most recent couple of years, the subject of $\widehat{q}$-theory has become a fascinating theme for several researchers, and new developments have been investigated in the relative literature (see [4–6]).

Within the framework of $\widehat{q}$-calculus, mechanothermodynamics, translimiting states, analysis, and generalization of experimental data, several special approaches are being developed to assess the quantum calculus in terms of a generalized energy states (see [7, 8]).

Convex functions have potential applications in many intriguing and captivating fields of research and furthermore played a remarkable role in numerous areas, such as coding theory, optimization, physics, information theory, engineering, and inequality theory. Several new classes of classical convexity have been proposed in the literature (see [9–14]). Mathematical inequalities are viewed as the prominent framework for assembling the qualitative and quantitative characterization in the area of applied analysis. A persistent development of intrigue has emerged to address the prerequisites of issue for rich utilization of these variants. Numerous generalizations were investigated by several scientists who thus utilized different procedures for introducing and proposing these bounds [15–17]. Additionally, many authors demonstrated various forms of inequalities such as Ostrowski, Lyenger, Opial, Hardy, and Olsen, and the most distinguished one is the Hermite-Hadamard inequality. Here, we intend to find the novel version of $\mathscr{H}\mathscr{H}$-type inequality in the frame of $\widehat{q}_1\widehat{q}_2$-integral on coordinated generalized Ψ-convex functions that correlates with Raina's function. Also, we shall represent the application of our findings in the Mittag-Leffler and hypergeometric functions which show the applicability of the suggested scheme.

Let $\mathscr{G}:\mathscr{I}\subseteq\mathbb{R}\mapsto\mathbb{R}$ be a convex function such that $\varphi_1 < \varphi_2$. Then,

$$\mathscr{G}\left(\frac{\varphi_1+\varphi_2}{2}\right) \le \frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathscr{G}(z)dz \le \frac{\mathscr{G}(\varphi_1)+\mathscr{G}(\varphi_2)}{2}. \quad (1)$$

The inequality (1) is a well-known paramount in related literature and plays a pivotal role in optimization, coding, and fractional calculus theory [18, 19].

In [20], Dragomir proposed the two-variable version of the $\mathscr{H}\mathscr{H}$-type inequality for convex functions as follows:

Theorem 1. (see [20]). *Let $\mathscr{G}:\Delta\mapsto\mathbb{R}$ be the coordinated convex on Δ. Then, the following inequalities hold:*

$$\begin{aligned}
\mathscr{G}\left(\frac{\varphi_1+\varphi_2}{2},\frac{\phi_1+\phi_2}{2}\right) &\le \frac{1}{2}\left[\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathscr{G}\left(\mu,\frac{\phi_1+\phi_2}{2}\right)d\mu\right.\\
&\quad\left.+\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\mathscr{G}\left(\frac{\varphi_1+\varphi_2}{2},\nu\right)d\nu\right]\\
&\le \frac{1}{(\varphi_2-\varphi_1)(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_2}\int_{\phi_1}^{\phi_2}\mathscr{G}(\mu,\nu)d\mu d\nu\\
&\le \frac{1}{4}\left[\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathscr{G}(\mu,\phi_1)d\mu+\frac{1}{\varphi_2-\varphi_1}\right.\\
&\quad\cdot\int_{\varphi_1}^{\varphi_2}\mathscr{G}(\mu,\phi_2)d\mu+\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\mathscr{G}(\varphi_1,\nu)d\nu\\
&\quad\left.+\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\mathscr{G}(\varphi_2,\nu)d\nu\right]\\
&\le \frac{\mathscr{G}(\varphi_1,\phi_1)+\mathscr{G}(\varphi_1,\phi_2)+\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{4}.
\end{aligned} \quad (2)$$

In [21], Kunt et al. established the $\widehat{q}$-$\mathscr{H}\mathscr{H}$-type inequality for functions of two variables utilizing convexity on rectangle from the plane $\mathbb{R}^2$.

Theorem 2. *Let $\mathscr{G}:\Delta=[\varphi_1,\varphi_2]\times[\phi_1,\phi_2]\subseteq\mathbb{R}^2\mapsto\mathbb{R}$ be convex on the coordinates on Δ with $0<\widehat{q}_1,\widehat{q}_2<1$ and $\varphi_1<\varphi_2$, $\phi_1<\phi_2$. Then, one has the following inequalities:*

$$\begin{aligned}
\mathscr{G}\left(\frac{\widehat{q}_1\varphi_1+\varphi_2}{\widehat{q}_1+1},\frac{\widehat{q}_2\phi_1+\phi_2}{\widehat{q}_2+1}\right) &\le \frac{1}{2}\left[\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathscr{G}\left(\mu,\frac{\widehat{q}_2\phi_1+\phi_2}{\widehat{q}_2+1}\right)_{\varphi_1}d_{\widehat{q}_1}\mu+\frac{1}{\phi_2-\phi_1}\right.\\
&\quad\left.\cdot\int_{\phi_1}^{\phi_2}\mathscr{G}\left(\frac{\widehat{q}_1\varphi_1+\varphi_2}{\widehat{q}_1+1},\nu\right)_{\phi_1}d_{\widehat{q}_2}\nu\right]\\
&\le \frac{1}{(\varphi_2-\varphi_1)(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_2}\int_{\phi_1}^{\phi_2}\mathscr{G}(\mu,\nu)_{\varphi_1}d_{\widehat{q}_1}\mu_{\phi_1}d_{\widehat{q}_2}\nu\\
&\le \frac{1}{2}\left[\frac{\widehat{q}_2}{(1+q2)(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_2}\mathscr{G}(\mu,\phi_1)_{\varphi_1}d_{\widehat{q}_1}\mu\right.\\
&\quad+\frac{\widehat{q}_2}{(1+\widehat{q}_2)(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_2}\mathscr{G}(\mu,\phi_2)_{\varphi_1}d_{\widehat{q}_1}\mu\\
&\quad+\frac{\widehat{q}_1}{(1+\widehat{q}_1)(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_2}\mathscr{G}(\varphi_1,\nu)_{\phi_1}d_{\widehat{q}_2}\nu\\
&\quad\left.+\frac{\widehat{q}_1}{(1+\widehat{q}_1)(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_2}\mathscr{G}(\varphi_2,\nu)_{\phi_1}d_{\widehat{q}_2}\nu\right]\\
&\le \frac{\widehat{q}_1\widehat{q}_2\mathscr{G}(\varphi_1,\phi_1)+\widehat{q}_1\mathscr{G}(\varphi_1,\phi_2)+\widehat{q}_2\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{(1+\widehat{q}_1)(1+\widehat{q}_2)}.
\end{aligned} \quad (3)$$

For many useful consequences on the coordinates on rectangle from the plane $\mathbb{R}^2$ with the various sorts of variants for mappings that hold numerous types of convex mappings, see [22–24] and the references cited therein.

Owing to the above-mentioned work, this research is aimed at exploring the novel generalizations of $\mathscr{H}\mathscr{H}$-type inequalities on the coordinates by the use of generalized Ψ-convex functions which are elaborated. An auxiliary identity is derived with respect to the $\widehat{q}_1\widehat{q}_2$-derivative by the correlation of Raina's function. Considering this new approach, we derive certain novel quantum bounds of $\mathscr{H}\mathscr{H}$-type variants for coordinated generalized Ψ-convex mappings. Meanwhile, we recapture remarkable cases in the relative literature. For the change of parameter in Raina's function, we generate numerous new outcomes depending on hypergeometric and Mittag-Leffler functions. This new study may stimulate further investigation in this dynamic field of inequality theory.

2. Prelude

This segment evokes certain earlier ideas and necessary details related to the notion of a coordinated generalized Ψ-convex set and coordinated generalized Ψ-convex function by considering Raina's function.

Assume that a finite interval of real numbers $\mathscr{I}$, and we say that a mapping $\mathscr{G}:\mathscr{I}\mapsto\mathbb{R}$ is known to be convex if

$$\mathscr{G}(\zeta x+(1-\zeta)y)\le\zeta\mathscr{G}(x)+(1-\zeta)\mathscr{G}(y), x,y\in\mathscr{I},\zeta\in[0,1]. \quad (4)$$

In [20], Dragomir introduced a new term in convexity theory, which is known as the coordinated convex function described as follows:

Definition 3. Let a mapping $\mathscr{G}:\nabla^{\sim}\to\mathbb{R}$ be said to be convex on the coordinates, for all $\zeta,\theta\in[0,1]$ with $(x,y),(u,v)\in\tilde{\nabla}$, if the partial functions

$$\begin{aligned}&\mathscr{G}(\zeta x+(1-\zeta)u,\theta y+(1-\theta)v)\\&\quad\leq\zeta\theta\mathscr{G}(x,y)+\zeta(1-\theta)\mathscr{G}(x,v)\\&\qquad+(1-\zeta)\theta\mathscr{G}(u,y)+(1-\zeta)(1-\theta)\mathscr{G}(u,v),\end{aligned}\tag{5}$$

holds for all $\zeta,\theta\in 0,1]$ and $(x,y),(u,v)\in\tilde{\nabla}$.

In [25], Raina contemplated the subsequent class of function

$$\mathscr{F}^{\lambda}_{\gamma,\rho}(t)=\mathscr{F}^{\lambda(0),\lambda(1),\cdots}_{\gamma,\rho}(t)=\sum_{p=0}^{\infty}\frac{\lambda(p)}{\Gamma(\gamma p+\rho)}t^{p},\tag{6}$$

where $\gamma,\rho>0,|t|<\mathbb{R}$ and

$$\lambda=(\lambda(0),\lambda(1),\cdots\lambda(p),\cdots),\tag{7}$$

is a bounded sequence of $\mathbb{R}^{+}$. Also, setting $\gamma=1,\rho=0$ in (6) and

$$\lambda(p)=\frac{(\vartheta_1)_p(\vartheta_2)_p}{(\vartheta_3)_p}\quad\text{for } p=0,1,2,3\cdots,\tag{8}$$

where the parameters $\vartheta_i,(i=1,2,3)$ are assumed to be real or complex (provided that $\vartheta_3=0,-1,-2,\cdots$) and the symbol $(z)_p$ mentions the value

$$(z)_p=\frac{\Gamma(z+p)}{\Gamma(z)}=z(z+1)\cdots(z+p-1),\quad p=0,1,2,\cdots,\tag{9}$$

and its domain is restricted as $|t|\leq 1$ (with $t\in\mathbb{C}$), then we attain the subsequent hypergeometric function,

$$\mathscr{F}^{\lambda}_{\gamma,\rho}(t)=F(\vartheta_1;\vartheta_2;\vartheta_3;t)=\sum_{p=0}^{\infty}\frac{(\vartheta_1)_p(\vartheta_2)_p}{p!(\vartheta_3)_p}t^{p}.\tag{10}$$

Furthermore, if $\lambda=(1,1,\cdots)$ with $\gamma=\vartheta_1,(\Re(\vartheta_1)>0)$, $\lambda=1$ and its domain is restricted as $t\in\mathbb{C}$ in (6), then we attain the subsequent Mittag-Leffler function

$$E_{\vartheta_1}(t)=\sum_{p=0}^{\infty}\frac{1}{\Gamma(1+\vartheta_1 p)}t^{p}.\tag{11}$$

Next, we mention a novel concept that reunites the coordinated convex function and Raina's function as mentioned above.

Definition 4. For $\gamma,\lambda>0$ and $\lambda=(\lambda(0),\lambda(1),\cdots\lambda(p),\cdots)$ is assumed to be a bounded sequence of $\mathbb{R}^{+}$. A nonempty set $\tilde{\Delta}$ is known to be a coordinated generalized Ψ -convex set

$$\mathscr{G}\left(z+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(x-z),w+\theta\mathscr{F}^{\lambda}_{\sigma,\rho}(y-w)\right)\in\tilde{\Delta},\tag{12}$$

holds for all $\zeta\theta\in 0,1]$, $(x,y),(z,w)\in\tilde{\Delta}$, and $\mathscr{F}^{\lambda}_{\gamma,\rho}(.)$ denotes Raina's function.

Definition 5. For $\gamma,\lambda>0$ and $\lambda=(\lambda(0),\lambda(1),\cdots\lambda(p),\cdots)$ is assumed to be a bounded sequence of $\mathbb{R}^{+}$. A mapping $\mathscr{G}:\tilde{\Delta}\to\mathbb{R}$ is said to be a coordinated generalized Ψ -convex, if

$$\begin{aligned}&\mathscr{G}\left(z+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(x-z),w+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(y-w)\right)\\&\quad\leq\zeta\theta\mathscr{G}(x,y)+\zeta(1-\theta)\mathscr{G}(x,w)\\&\qquad+(1-\zeta)\theta\mathscr{G}(z,y)+(1-\zeta)(1-\theta)\mathscr{G}(z,w),\end{aligned}\tag{13}$$

holds for all $\zeta,\theta\in 0,1]$ and $(x,y),(z,w)\in\tilde{\Delta}$.

Remark 6. Setting $\mathscr{F}^{\lambda}_{\gamma,\rho}(x-\varphi_1)=x-\varphi_1>0$ and $\mathscr{F}^{\lambda}_{\gamma,\rho}(y-\phi_1)=y-\phi_1>0$ in Definition 5, we get Definition 3.

Furthermore, we demonstrate some essential ideas and preliminaries in $\hat{q}$-analog for a single and two-variable senses.

Let $\mathscr{J}=[\varrho_1,\varrho_2]\subseteq\mathbb{R}$, and let $\mathscr{U}=[\varrho_1,\varrho_2]\times\varrho_3,\varrho_4]\subseteq\mathbb{R}^2$ with constants $\hat{q},\hat{q}_k\in(0,1)$, $k=1,2$.

Tariboon and Ntouyas [2, 3] studied the concept of $\hat{q}$ -derivative, $\hat{q}$-integral, and characteristics for finite interval, which has been shown as

Definition 7. Assume that a continuous mapping $\mathscr{G}:\mathscr{J}\to\mathbb{R}$ and $t\in\mathscr{J}$. Then, one has $\hat{q}$ -derivative of $\mathscr{G}$ on $\mathscr{J}$ at t which is stated as

$${}_{\varrho_1}\mathscr{D}_{\hat{q}}\mathscr{G}(t)=\frac{\mathscr{G}(t)-\mathscr{G}(qt+(1-q)\varrho_1)}{(1-q)(t-\varrho_1)},\quad t\neq\varrho_1.\tag{14}$$

Clearly, we see that

$$\lim_{t\to\varrho_1}{}_{\varrho_1}\mathscr{D}_{\hat{q}}\mathscr{G}(t)={}_{\varrho_1}\mathscr{D}_{\hat{q}}\mathscr{G}(\varrho_1).\tag{15}$$

We say that the mapping $\mathscr{G}$ is $\hat{q}$-differentiable over $\mathscr{J}$, also ${}_{\rho_1}\mathscr{D}_{\hat{q}}\mathscr{G}(t)$ exists $\forall t\in\mathscr{J}$.

Observe that if $\varrho_1=0$ in (14), then ${}_0\mathscr{D}_{\hat{q}}\mathscr{G}=\mathscr{D}_{\hat{q}}\mathscr{G}$, where $\mathscr{D}_{\hat{q}}\mathscr{G}$ is a well-defined $\hat{q}$-derivative of $\mathscr{G}(t)$, i.e, it is mentioned as

$$\mathscr{D}_{\hat{q}}\mathscr{G}(t)=\frac{\mathscr{G}(t)-\mathscr{G}(qt)}{(1-q)(t)}.\tag{16}$$

Definition 8. Assume that a continuous mapping $\mathscr{G}:\mathscr{J}\to\mathbb{R}$ is symbolized as ${}_{\varrho_1}\mathscr{D}^2_{\hat{q}}\mathscr{G}$, given that ${}_{\varrho_1}\mathscr{D}^2_{\hat{q}}\mathscr{G}$ is $\hat{q}$ -differentiable from $\mathscr{J}\to\mathbb{R}$ defined by

$${}_{\varrho_1}\mathscr{D}^2_{\hat{q}}\mathscr{G}={}_{\varrho_1}\mathscr{D}_{\hat{q}}\left({}_{\varrho_1}\mathscr{D}_{\hat{q}}\mathscr{G}\right).\tag{17}$$

Therefore, the higher order $\hat{q}$-differentiable is defined as ${}_{\varrho_1}\mathscr{D}^n_{\hat{q}}\mathscr{G}:\mathscr{J}\to\mathbb{R}$.

Definition 9. Assume that a continuous mapping $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ and the $\widehat{q}$ -integral on $\mathscr{J}$ is stated as

$$\int_{\varrho_1}^{t} \mathscr{G}(z)_{\varrho_1} d_{\widehat{q}} z = (1-\widehat{q})(t-\varrho_1) \sum_{n=0}^{\infty} q\wedge^n \mathscr{G}(q\wedge^n t + (1-q\wedge^n)\varrho_1), \quad \forall t \in \mathscr{J}. \tag{18}$$

Next, if $\varrho_1 = 0$ in (18), then we have a new formulation of $\widehat{q}$-integral, which is pointed out as

$$\int_{0}^{t} \mathscr{G}(z)_{0} d_{\widehat{q}} z = (1-\widehat{q}) t \sum_{n=0}^{\infty} q\wedge^n \mathscr{G}(q\wedge^n t). \tag{19}$$

Theorem 10. *Assuming that a continuous mapping $\mathscr{G} : \mathscr{J} \to \mathbb{R}$, the following assumptions hold:*

$$\begin{gathered} {}_{\varrho_1}D_{\widehat{q}} \int_{\varrho_1}^{t} G(z)_{\varrho_1} d_{\widehat{q}} z = G(t), \\ \int_{\varrho_1}^{t} {}_{\varrho_1}\mathscr{D}_{\widehat{q}} \mathscr{G}(z)_{\varrho_1} d_{\widehat{q}} z = \mathscr{G}(t), \\ \int_{\varrho_2}^{t} {}_{\varrho_1}\mathscr{D}_{\widehat{q}} \mathscr{G}(z)_{\varrho_1} d_{\widehat{q}} z = \mathscr{G}(t) - \mathscr{G}(\varrho_2), \quad \varrho_2 \in (\varrho_1, t). \end{gathered} \tag{20}$$

Theorem 11. *Assuming that a continuous mapping $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ and $a \in \mathbb{R}$, then the following assumptions hold:*

$$\begin{gathered} \int_{\varrho_1}^{t} [\mathscr{G}_1(z) + \mathscr{G}_2(z)]_{\varrho_1} d_{\widehat{q}} z = \int_{\varrho_1}^{t} \mathscr{G}_1(z)_{\varrho_1} d_{\widehat{q}} z + \int_{\varrho_1}^{t} \mathscr{G}_2(z)_{\varrho_1} d_{\widehat{q}} z, \\ \int_{\varrho_1}^{t} (a\mathscr{G}_1(z))_{\varrho_1} d_{\widehat{q}} z = a \int_{\varrho_1}^{t} \mathscr{G}_1(z)_{\varrho_1} d_{\widehat{q}} z. \end{gathered} \tag{21}$$

In [26], Kalsoom et al. introduced the quantum integral identities in a two-variable sense as follows:

Definition 12. Consider a continuous mapping in two-variable sense$\mathscr{G} : \mathscr{U} \to \mathbb{R}$, then the partial $\widehat{q}_1$ -derivative, $\widehat{q}_2$ -derivative, and $\widehat{q}_1\widehat{q}_2$ -derivative at $(z, w) \in \varrho_1, \varrho_2] \times \varrho_3, \varrho_4]$ are, respectively, stated as

$$\begin{aligned} \frac{{}_{\varrho_1}\partial_{\widehat{q}_1} \mathscr{G}(z,w)}{{}_{\varrho_1}\partial_{\widehat{q}_1} z} &= \frac{\mathscr{G}(z,w) - \mathscr{G}(\widehat{q}_1 z + (1-\widehat{q}_1)\varrho_1, w)}{(1-\widehat{q}_1)(z-\varrho_1)}, \quad z \neq \varrho_1, \\ \frac{{}_{\varrho_3}\partial_{\widehat{q}_2} \mathscr{G}(z,w)}{{}_{\varrho_3}\partial_{\widehat{q}_2} w} &= \frac{\mathscr{G}(z,w) - \mathscr{G}(z, \widehat{q}_2 w + (1-\widehat{q}_2)\varrho_3)}{(1-\widehat{q}_2)(w-\varrho_3)}, \quad w \neq \varrho_3, \\ \frac{{}_{\varrho_1,\varrho_3}\partial^2_{\widehat{q}_1,\widehat{q}_2} \mathscr{G}(z,w)}{{}_{\varrho_1}\partial_{\widehat{q}_1} z_{\varrho_3}\partial_{\widehat{q}_2} w} &= \frac{1}{(1-\widehat{q}_1)(1-\widehat{q}_2)(z-\varrho_1)(w-\varrho_3)} \\ &\quad \times [\mathscr{G}(\widehat{q}_1 z + (1-\widehat{q}_1)\varrho_1, \widehat{q}_2 w + (1-\widehat{q}_2)\varrho_3) \\ &\quad - \mathscr{G}(\widehat{q}_1 z + (1-\widehat{q}_1)\varrho_1, w) \\ &\quad - \mathscr{G}(z, \widehat{q}_2 w + (1-\widehat{q}_2)\varrho_3) + \mathscr{G}(z,w)], \quad z \neq \varrho_1, w \neq \varrho_3. \end{aligned} \tag{22}$$

Definition 13. Consider a continuous mapping in two-variable sense$\mathscr{G} : \mathscr{U} \to \mathbb{R}$, then the definite $\widehat{q}_1\widehat{q}_2$ -integral on $[\varrho_1, \varrho_2] \times \varrho_3, \varrho_4]$ is stated as

$$\int_{\varrho_3}^{t} \int_{\varrho_1}^{t} \mathscr{G}(z,w)_{\varrho_1} d_{\widehat{q}_1} z_{\varrho_3} d_{\widehat{q}_2} w = (1-\widehat{q}_1)(1-\widehat{q}_2)(t-\varrho_1)(t_1-\varrho_3) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \widehat{q}_1^n \widehat{q}_2^m \mathscr{G}(\widehat{q}_1^n t + (1-\widehat{q}_1^n)\varrho_1, \widehat{q}_2^m t_1 + (1-\widehat{q}_2^m)\varrho_3), \tag{23}$$

for $(t, t_1) \in \varrho_1, \varrho_2] \times \varrho_3, \varrho_4]$.

Theorem 14. *Consider a continuous mapping in two-variable sense$\mathscr{G} : \mathscr{B} \to \mathbb{R}$, then the following assumptions hold:*

$$\begin{aligned} &\frac{{}_{\varrho_1,\varrho_3}\partial^2_{\widehat{q}_1,\widehat{q}_2}}{{}_{\varrho_1}\partial_{\widehat{q}_1} t_{\varrho_3}\partial_{\widehat{q}_2} t_1} \int_{\varrho_4}^{t_1} \int_{\varrho_1}^{t} \mathscr{G}(z,w)_{\varrho_1} d_{\widehat{q}_1} z_{\varrho_3} d_{\widehat{q}_2} w = \mathscr{G}(t, t_1), \\ &\int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} \frac{{}_{\varrho_1,\varrho_3}\partial^2_{\widehat{q}_1,\widehat{q}_2} \mathscr{G}(z,w)}{{}_{\varrho_1}\partial_{\widehat{q}_1} z_{\varrho_3}\partial_{\widehat{q}_2} w} \varrho_1 d_{\widehat{q}_1} z_{\varrho_3} d_{\widehat{q}_2} w = \mathscr{G}(t, t_1), \\ &\int_{t_2}^{t_1} \int_{y_1}^{t} \frac{{}_{\varrho_1,\varrho_3}\partial^2_{\widehat{q}_1,\widehat{q}_2} \mathscr{G}(z,w)}{{}_{\varrho_1}\partial_{\widehat{q}_1} z_{\varrho_3}\partial_{\widehat{q}_2} w} \varrho_1 d_{\widehat{q}_1} z_{\varrho_3} d_{\widehat{q}_2} w; \\ &\quad = \mathscr{G}(t, t_1) - \mathscr{G}(t, t_2) - \mathscr{G}(y_1, t_1) \\ &\quad + \mathscr{G}(y_1, t_2), (y_1, t_2) \in (\varrho_1, t) \times (\varrho_4, t_1). \end{aligned} \tag{24}$$

Theorem 15. *Suppose that $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{U} \to \mathbb{R}$ are continuous mappings of two variables. Then, the following properties hold for $(t, t_1) \in \varrho_1, \varrho_2] \times \varrho_3, \varrho_4]$,*

$$\begin{aligned} &\int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} [\mathscr{G}_1(z,w) + \mathscr{G}_2(z,w)]_{\varrho_1} d_{\widehat{q}_1} z_{\varrho_4} d_{\widehat{q}_2} w \\ &\quad = \int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} \mathscr{G}_1(z,w)_{\varrho_1} d_{\widehat{q}_1} z_{\varrho_3} d_{\widehat{q}_2} w \\ &\quad + \int_{\varrho_3}^{t} \int_{\varrho_1}^{t} \mathscr{G}_2(z,w)_{\varrho_1} d_{\widehat{q}_1} z_{\varrho_3} d_{\widehat{q}_2} w, \\ &\int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} a\mathscr{G}(z,w)_{\varrho_1} d_{\widehat{q}_1} z_{\varrho_3} d_{\widehat{q}_2} w \\ &\quad = a \int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} \mathscr{G}(z,w)_{\varrho_1} d_{\widehat{q}_1} z_{\varrho_3} d_{\widehat{q}_2} w. \end{aligned} \tag{25}$$

3. Quantum $\mathscr{H}\mathscr{H}$-Type Inequality for Generalized Ψ-Convex on the Coordinates

This section addresses the $\hat{q}_1$-$\mathscr{H}\mathscr{H}$-type inequality on the coordinates via generalized Ψ-convex functions.

Theorem 16. *For $\gamma, \rho > 0$ with $\lambda = (\lambda(0), \cdots, \lambda(p))$ as the bounded sequence of positive real numbers and let $\mathscr{G} : \Delta \mapsto \mathbb{R}$ be the coordinated generalized Ψ -convex and partially differentiable function on Δ° with $0 < \hat{q}_1, \hat{q}_2 < 1$, then the following inequalities hold:*

$$\begin{aligned}
&\mathscr{G}\left(\frac{(\hat{q}_1+1)\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\hat{q}_1}, \frac{(\hat{q}_2+1)\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{1+\hat{q}_2}\right)\\
&\le \frac{1}{2\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\\
&\quad\cdot\left(\mu, \frac{(\hat{q}_2+1)\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{1+\hat{q}_2}\right)_{\varphi_1} d_{\hat{q}_1}\mu\\
&\quad+\frac{1}{2\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\\
&\quad\cdot\left(\frac{(\hat{q}_1+1)\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\hat{q}_1}, \nu\right)_{\phi_1} d_{\hat{q}_2}\nu\\
&\le \frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\\
&\quad\cdot\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)_{\phi_1} d_{\hat{q}_2}\nu_{\varphi_1} d_{\hat{q}_1}\mu\\
&\le \frac{\hat{q}_2}{2(1+\hat{q}_2)}\left(\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)_{\varphi_1} d_{\hat{q}_1}\mu\right)\\
&\quad+\frac{\hat{q}_2}{2(1+\hat{q}_2)}\left(\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)_{\varphi_1} d_{\hat{q}_1}\mu\right)\\
&\quad+\frac{\hat{q}_1}{2(1+\hat{q}_1)}\left(\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)_{\phi_1} d_{\hat{q}_2}\mu\right)\\
&\quad+\frac{\hat{q}_1}{2(1+\hat{q}_1)}\left(\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)_{\phi_1} d_{\hat{q}_2}\nu\right)\\
&\le \frac{\hat{q}_1\hat{q}_2\mathscr{G}(\varphi_1,\phi_1)+\hat{q}_1\mathscr{G}(\varphi_1,\phi_2)+\hat{q}_2\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{(1+\hat{q}_1)(1+\hat{q}_2)}.
\end{aligned}\tag{26}$$

Proof. Since $\mathscr{G}$ is the coordinated generalized Ψ-convex on Δ and partially differentiable mappings on Δ°, clearly, we see that the mapping $\mathscr{G}_\mu : [\phi_1, \phi_1 + \mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)] \mapsto \mathbb{R}$, $\mathscr{G}_\mu(\nu) := \mathscr{G}(\mu,\nu)$ is a generalized Ψ-convex on $[\phi_1, \phi_1 + \mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)]$ and a differentiable function on $(\phi_1, \phi_1 + \mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1))$ for all $\mu \in [\varphi_1, \varphi_1 + \mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)]$. Then, by using the $\hat{q}_1$-$\mathscr{H}\mathscr{H}$-type inequality, we obtain

$$\begin{aligned}
&\mathscr{G}_\mu\left(\frac{(\hat{q}_2+1)\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{1+\hat{q}_2}\right)\\
&\le \frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}_\mu(\nu)_\mu d_{\hat{q}_2}\nu\\
&\le \frac{\hat{q}_2\mathscr{G}_\mu(\phi_1)+\mathscr{G}_{\phi_1}(\phi_2)}{1+\hat{q}_2}, \left(\mu \in \left[\varphi_1, \varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\right]\right),
\end{aligned}\tag{27}$$

which can be written as

$$\begin{aligned}
&\mathscr{G}\left(\mu, \frac{(\hat{q}_2+1)\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{1+\hat{q}_2}\right)\\
&\le \frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)_{\phi_1} d_{\hat{q}_2}\nu\\
&\le \frac{\hat{q}_2\mathscr{G}(\mu,\phi_1)+\mathscr{G}(\mu,\phi_2)}{1+\hat{q}_2}, \left(\mu \in \left[\varphi_1, \varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\right]\right).
\end{aligned}\tag{28}$$

Applying $\hat{q}_1$-inegration on the above inequalities over $[\varphi_1, \varphi_1 + \mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)]$, we have

$$\begin{aligned}
&\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu, \frac{(\hat{q}_2+1)\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{1+\hat{q}_2}\right)_{\varphi_1} d_{\hat{q}_1}\mu\\
&\le \frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\\
&\quad\cdot\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)_{\phi_1} d_{\hat{q}_2}\nu_{\varphi_1} d_{\hat{q}_1}\mu\\
&\le \frac{1}{1+\hat{q}_2}\left[\frac{\hat{q}_2}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)_{\varphi_1} d_{\hat{q}_1}\mu\right.\\
&\quad\left.+\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)_{\varphi_1} d_{\hat{q}_1}\mu\right].
\end{aligned}\tag{29}$$

Adopting the same procedure for the mapping $\mathscr{G}_\nu : [\varphi_1, \varphi_1 + \mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)] \mapsto \mathbb{R}, \mathscr{G}_\nu(\mu) := \mathscr{G}(\mu,\nu)$, we have

$$\begin{aligned}
&\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\\
&\quad\cdot\left(\frac{(\hat{q}_1+1)\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\hat{q}_1}, \nu\right)_{\phi_1} d_{\hat{q}_1}\mu\\
&\le \frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\\
&\quad\cdot\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)_{\phi_1} d_{\hat{q}_2}\nu_{\varphi_1} d_{\hat{q}_1}\mu\\
&\le \frac{1}{1+\hat{q}_1}\left[\frac{\hat{q}_1}{\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)_{\phi_1} d_{\hat{q}_2}\nu\right.\\
&\quad\left.+\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)_{\phi_1} d_{\hat{q}_2}\nu\right].
\end{aligned}\tag{30}$$

Adding (29) and (30), yields

$$\frac{1}{2\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\cdot\left(\mu,\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu+\frac{1}{2\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\cdot\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\nu\right)\,{}_{\phi_1}d_{\widehat{q}_2}\nu+$$
$$\le\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\cdot\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\nu\,{}_{\varphi_1}d_{\widehat{q}_1}\mu$$
$$\le\left[\frac{\widehat{q}_2}{2(1+\widehat{q}_2)\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu+\frac{1}{2(1+\widehat{q}_2)\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu+\frac{\widehat{q}_1}{2(1+\widehat{q}_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\nu+\frac{1}{2(1+\widehat{q}_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\nu\right].\tag{31}$$

Also, by considering the $\widehat{q}$-$\mathscr{H}\mathscr{H}$-type inequality, we have

$$\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\le\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu,\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu,\tag{32}$$

$$\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\le\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\nu\right)\,{}_{\phi_1}d_{\widehat{q}_2}\nu.\tag{33}$$

Adding the inequalities (32) and (33), we have the following inequality:

$$\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\le\frac{1}{2\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu,\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu+\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\nu\right)\,{}_{\phi_1}d_{\widehat{q}_2}\nu.\tag{34}$$

Consequently, we have

$$\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu\right)\le\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{\widehat{q}_1\mathscr{G}(\varphi_1,\phi_1)+\mathscr{G}(\varphi_2,\phi_1)}{1+\widehat{q}_1}\right),\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\cdot\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu\right)\le\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{\widehat{q}_1\mathscr{G}(\varphi_1,\phi_2)+\mathscr{G}(\varphi_2,\phi_2)}{1+\widehat{q}_1}\right),\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\mu\right)\le\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{\widehat{q}_2\mathscr{G}(\varphi_1,\phi_1)+\mathscr{G}(\varphi_1,\phi_2)}{1+\widehat{q}_2}\right),\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\cdot\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\nu\right)\le\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{\widehat{q}_2\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{1+\widehat{q}_2}\right).\tag{35}$$

Adding the above inequalities yields

$$\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu\right)+\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu\right)+\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\mu\right)+\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\nu\right)\le\frac{\widehat{q}_1\widehat{q}_2\mathscr{G}(\varphi_1,\phi_1)+\widehat{q}_1\mathscr{G}(\varphi_1,\phi_2)+\widehat{q}_2\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{(1+\widehat{q}_1)(1+\widehat{q}_2)}.\tag{36}$$

A combination of (31), (34), and (36) gives (36). This completes the proof.

Corollary 17. *In Theorem 16, if we choose $\widehat{q}_1, \widehat{q}_2 \mapsto 1^-$, we have the following new double inequality:*

$$
\begin{aligned}
&\mathscr{G}\left(\frac{2\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{2}, \frac{2\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)\\
&\quad\le \frac{1}{2\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu, \frac{2\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)d\mu\\
&\qquad+\frac{1}{2\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \nu\right)d\nu\\
&\quad\le \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)d\nu d\mu\\
&\quad\le \frac{1}{4}\left\{\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu\right)\right.\\
&\qquad+\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)\,{}_{\varphi_1}d_{\widehat{q}_1}\mu\right)\\
&\qquad+\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\mu\right)\\
&\qquad\left.+\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\nu\right)\right\}\\
&\quad\le \frac{\mathscr{G}(\varphi_1,\phi_1)+\mathscr{G}(\varphi_1,\phi_2)+\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{4}.
\end{aligned}
\tag{37}
$$

Remark 18. In Theorem 16,

(i) letting $\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$ along with $\widehat{q}_1, \widehat{q}_2 \mapsto 1^-$, then we attain Theorem 1 in [20]

(ii) letting $\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$, then we attain Theorem 4 in [21]

4. Quantum Integral Identity for Coordinated Generalized Ψ-Convex Functions

The following identity plays a significant role in inaugurating the main consequences of this paper. The identification is expressed as follows.

Lemma 19. *For $\gamma, \rho > 0$ with $\lambda = (\lambda(0), \cdots, \lambda(p))$ as the bounded sequence of positive real numbers and let a twice partially $\widehat{q}_1\widehat{q}_2$ -differentiable mapping $\mathscr{G}: \Delta \mapsto \mathbb{R}$ be defined on Δ° (the interior of Δ). If the second-order partial $\widehat{q}_1\widehat{q}_2$ -derivatives are continuous and integrable over Δ with $0 < \widehat{q}_1, \widehat{q}_2 < 1$, then the following equality holds:*

$$
\begin{aligned}
&Y_{\widehat{q}_1,\widehat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G})\\
&\quad:= \mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\\
&\qquad-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu, \frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right){}_{\varphi_1}d_{\widehat{q}_1}\mu\\
&\qquad-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \nu\right){}_{\phi_1}d_{\widehat{q}_2}\nu\\
&\qquad+\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)\,{}_{\phi_1}d_{\widehat{q}_2}\nu\,{}_{\varphi_1}d_{\widehat{q}_1}\mu,
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
&Y_{\widehat{q}_1,\widehat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G})\\
&:= \widehat{q}_1\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\mathscr{A}(\zeta,\theta)\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1), \phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta,
\end{aligned}
$$

$$
\mathscr{A}(\zeta,\theta)=\begin{cases}
\zeta\theta, & (\zeta,\theta)\in\left[0,\dfrac{1}{1+\widehat{q}_1}\right]\times\left[0,\dfrac{1}{1+\widehat{q}_2}\right],\\
\zeta\left(\theta-\dfrac{1}{\widehat{q}_2}\right), & (\zeta,\theta)\in\left[0,\dfrac{1}{1+\widehat{q}_1}\right]\times\left(\dfrac{1}{1+\widehat{q}_2},1\right],\\
\theta\left(\zeta-\dfrac{1}{\widehat{q}_1}\right), & (\zeta,\theta)\in\left(\dfrac{1}{1+\widehat{q}_1},1\right]\times\left[0,\dfrac{1}{1+\widehat{q}_2}\right],\\
\left(\zeta-\dfrac{1}{\widehat{q}_1}\right)\left(\theta-\dfrac{1}{\widehat{q}_2}\right), & (\zeta,\theta)\in\left(\dfrac{1}{1+\widehat{q}_1},1\right]\times\left(\dfrac{1}{1+\widehat{q}_2},1\right].
\end{cases}
\tag{39}
$$

Proof. Consider

$$
\begin{aligned}
&\widehat{q}_1\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\mathscr{A}(\zeta,\theta)\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1), \phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta\\
&=\widehat{q}_1\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\times\left\{\int_0^{1/(1+\widehat{q}_1)}\int_0^{1/(1+\widehat{q}_2)}\zeta\theta\right.\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1), \phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta\\
&\quad+\int_{\frac{1}{1+\widehat{q}_1}}^1\int_0^{1/(1+\widehat{q}_2)}\theta\left(\zeta-\frac{1}{\widehat{q}_1}\right)\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1), \phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta\\
&\quad+\int_0^{1/(1+\widehat{q}_1)}\int_{1/(1+\widehat{q}_2)}^1\zeta\left(\theta-\frac{1}{\widehat{q}_2}\right)\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1), \phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta\\
&\quad+\int_{1/(1+\widehat{q}_1)}^1\int_{1/(1+\widehat{q}_2)}^1\left(\zeta-\frac{1}{\widehat{q}_1}\right)\left(\theta-\frac{1}{\widehat{q}_2}\right)\\
&\quad\left.\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1), \phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \hat{q}_1\hat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\Bigg\{\int_0^{1/(1+\hat{q}_1)}\int_0^{1/(1+\hat{q}_2)}\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad+\int_{1/(1+\hat{q}_1)}^1\int_0^{1/(1+\hat{q}_2)}\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad+\int_0^{1/(1+\hat{q}_1)}\int_{1/(1+\hat{q}_2)}^1\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad+\int_{1/(1+\hat{q}_1)}^1\int_{1/(1+\hat{q}_2)}^1\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad-\frac{1}{\hat{q}_1}\int_0^{1/(1+\hat{q}_1)}\int_0^{1/(1+\hat{q}_2)}\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad-\frac{1}{\hat{q}_2}\int_{1/(1+\hat{q}_1)}^1\int_0^{1/(1+\hat{q}_2)}\zeta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad-\frac{1}{\hat{q}_2}\int_0^{1/(1+\hat{q}_1)}\int_{1/(1+\hat{q}_2)}^1\zeta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad\cdot\frac{1}{\hat{q}_1\hat{q}_2}\int_{1/(1+\hat{q}_1)}^1\int_{1/(1+\hat{q}_2)1/(1+\hat{q}_2)}^1 \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{e}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta\Bigg\} \\
&= \Bigg\{\hat{q}_1\hat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad-\hat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad-\hat{q}_1\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\zeta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad+\hat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\hat{q}_1)}\int_0^1\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad+\hat{q}_1\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\hat{q}_1)}\zeta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad+\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1 \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad-\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\hat{q}_1)}\int_0^1 \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad-\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\hat{q}_1)} \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&\quad+\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\hat{q}_1)}\int_0^{\frac{1}{1+\hat{q}_2}} \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta\Bigg\}.
\end{aligned}
\tag{40}
$$

In view of Definition 12 and Definition 13, we conclude the following identities with the aid of the last nine integrals appearing in the aforementioned identities as follows:

$$
\begin{aligned}
&\hat{q}_1\hat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1} \\
&\quad= -\mathscr{G}(\varphi_2,\phi_2)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)\,{}_0d_{\hat{q}_1}\mu \\
&\quad-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)\,{}_0d_{\hat{q}_2}\nu \\
&\quad+\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)} \\
&\quad\cdot\mathscr{G}(\mu,\nu)\,{}_{\phi_1}d_{\hat{q}_2}\nu\,{}_{\varphi_1}d_{\hat{q}_1}\mu,
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&\hat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\hat{q}_1}\zeta\,\phi_1\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta \\
&= -\mathscr{G}(\varphi_2,\phi_2)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)\,{}_0d_{\hat{q}_2}\nu,
\end{aligned}
\tag{42}
$$

$$\widehat{q}_1\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\zeta \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}(\varphi_2,\phi_2)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)\,{}_0d_{\widehat{q}_1}\nu, \tag{43}$$

$$\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\widehat{q}_1)}\int_0^1\theta \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\phi_2\right)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)} \cdot \int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\nu\right)\,{}_0d_{\widehat{q}_1}\nu, \tag{44}$$

$$\widehat{q}_1\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\widehat{q}_1)}\zeta \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\varphi_2,\frac{(\widehat{q}_1+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_1}\right)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)} \cdot \int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu,\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\,{}_0d_{\widehat{q}_1}\mu, \tag{45}$$

$$\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\widehat{q}_1)} \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}(\varphi_2,\phi_2), \tag{46}$$

$$\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\widehat{q}_1)}\int_0^1 \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\phi_2\right), \tag{47}$$

$$\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\widehat{q}_1)} \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\varphi_2,\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right), \tag{48}$$

$$\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\widehat{q}_1)}\int_0^{1/(1+\widehat{q}_2)} \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right). \tag{49}$$

Combining (42), (43), (44), (45), (46), (47), (48), and (49), we have the identity (38). This is the proof of Lemma 19.

Corollary 20. *In Lemma 19, if we choose $\widehat{q}_1,\widehat{q}_2\mapsto 1^-$, we have the following new identity:*

$$\Lambda(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G}) \coloneqq \mathscr{G}\left(\frac{2\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\frac{2\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right) - \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G} \cdot \left(\mu,\frac{2\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)d\mu - \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)} \cdot \int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{2\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\nu\right)d\nu + \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)} \cdot \int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)\,d\nu\,d\mu, \tag{50}$$

where

$$\Lambda(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G}) \coloneqq \left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\mathscr{A}(\zeta,\theta) \cdot \frac{\partial^2\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\partial\zeta\partial\theta}\,d\theta\,d\zeta,$$

$$A(\zeta,\theta)\begin{cases}\zeta\theta, & (\zeta-\theta)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right],\\ \zeta(\theta-1), & (\zeta-\theta)\in\left[0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right],\\ \theta(\zeta-1), & (\zeta-\theta)\in\left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right],\\ (\zeta-1)(\theta-1), & (\zeta-\theta)\in\left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right].\end{cases}\quad(51)$$

5. Certain New $\widehat{q}_1\widehat{q}_2$-Integral Estimates for Generalized Ψ-Convex Functions

The following results exhibit some practice related to Lemma 19 on quantum calculus for generalized Ψ-convex on coordinates.

Theorem 21. *For $\gamma,\rho>0$ with $\lambda=(\lambda(0),\cdots,\lambda(p))$ as the bounded sequence of positive real numbers and let a mapping $\mathcal{G}:\Delta\mapsto\mathbb{R}$ be a twice partially $\widehat{q}_1\widehat{q}_2$ -differentiable on Δ° such that continuous partial $\widehat{q}_1\widehat{q}_2$ -derivatives ${}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathcal{G}/{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta$ is integrable on Δ with $0<\widehat{q}_1,\widehat{q}_2<1$. If $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ is a generalized Ψ -convex on the coordinates on Δ for $\sigma\geq 1$, where $\sigma^{-1}+\beta^{-1}=1$. Then, the following inequality holds:*

$$\begin{aligned}\left|Y_{\widehat{q}_1,\widehat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathcal{G})\right| &\leq \widehat{q}_1\widehat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\cdot(\mathbb{B}_1(q\wedge_1,q\wedge_2))^{1-(1/\sigma)}\times\left[\mathbb{B}_2(q\wedge_1,q\wedge_2)\right.\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_3(q\wedge_1,q\wedge_2)\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_4(q\wedge_1,q\wedge_2)\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_5(q\wedge_1,q\wedge_2)\\ &\cdot\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right]^{1/\sigma},\end{aligned}\quad(52)$$

where

$$\mathbb{B}_1(\widehat{q}_1,\widehat{q}_2)\coloneqq\frac{4}{(1+q\wedge_1)^3(1+q\wedge_2)^3},\quad(53)$$

$$\mathbb{B}_2(\widehat{q}_1,\widehat{q}_2)\coloneqq\frac{9}{(1+q\wedge_1)^3(1+q\wedge_2)^3\left(1+\widehat{q}_1+\widehat{q}_1^2\right)\left(1+\widehat{q}_2+\widehat{q}_2^2\right)},\quad(54)$$

$$\mathbb{B}_3(\widehat{q}_1,\widehat{q}_2)\coloneqq\frac{1+\widehat{q}_1+\widehat{q}_2+\widehat{q}_2^2-3\widehat{q}_1\widehat{q}_2-\widehat{q}_1\widehat{q}_2^3}{\widehat{q}_1\widehat{q}_2(1+q\wedge_1)^3(1+q\wedge_2)^3\left(1+\widehat{q}_1+\widehat{q}_1^2\right)\left(1+\widehat{q}_2+\widehat{q}_2^2\right)},\quad(55)$$

$$\mathbb{B}_4(\widehat{q}_1,\widehat{q}_2)\coloneqq\frac{1+\widehat{q}_1+\widehat{q}_2+\widehat{q}_1^2+\widehat{q}_2^2-3\widehat{q}_1\widehat{q}_2-\widehat{q}_2\widehat{q}_1^3-\widehat{q}_1^3\widehat{q}_2^2+6\widehat{q}_1\widehat{q}_2^2-\widehat{q}_1^2\widehat{q}_2^3-\widehat{q}_1^3\widehat{q}_2^3+5\widehat{q}_1\widehat{q}_2^3}{\widehat{q}_1\widehat{q}_2(1+q\wedge_1)^3(1+q\wedge_2)^3\left(1+\widehat{q}_1+\widehat{q}_1^2\right)\left(1+\widehat{q}_2+\widehat{q}_2^2\right)}\quad(56)$$

$$\begin{aligned}\mathbb{B}_5(\widehat{q}_1,\widehat{q}_2)\coloneqq&\left(-2\widehat{q}_1^5-6\widehat{q}_1^4+2\widehat{q}_1^4\widehat{q}_2^3+2\widehat{q}_1^3\widehat{q}_2^4-4\widehat{q}_1^4\widehat{q}_2^2-4\widehat{q}_1^4\widehat{q}_2-2\widehat{q}_1\widehat{q}_2^5\right.\\ &-2\widehat{q}_1^5\widehat{q}_2-2\widehat{q}_1^5\widehat{q}_2^3+16\widehat{q}_1^3\widehat{q}_2^3-4\widehat{q}_1^2+2\widehat{q}_1^3\widehat{q}_2-2\widehat{q}_1^2\widehat{q}_2^5-4\widehat{q}_1^2\widehat{q}_2^4\\ &+4\widehat{q}_1^2\widehat{q}_2+6\widehat{q}_1^3+8\widehat{q}_1^2\widehat{q}_2^2+10\widehat{q}_1^3\widehat{q}_2^2+10\widehat{q}_1^2\widehat{q}_2^3-6\widehat{q}_2^4-6\widehat{q}_2^3\\ &\left.-4\widehat{q}_2^2-4\widehat{q}_1\widehat{q}_2^4+2\widehat{q}_1\widehat{q}_2^3+4\widehat{q}_1\widehat{q}_2^2+9\widehat{q}_1\widehat{q}_2-2\widehat{q}_2^5\right)\\ &/\widehat{q}_1\widehat{q}_2(1+q\wedge_1)^3(1+q\wedge_2)^3\left(1+\widehat{q}_1+\widehat{q}_1^2\right)\left(1+\widehat{q}_2+\widehat{q}_2^2\right)\end{aligned}\quad(57)$$

Proof. Taking into consideration the $\widehat{q}_1\widehat{q}_2$-integral power mean inequality, the generalized Ψ-convexity of $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ on the coordinates on Δ with the aid of Lemma 19, we have

$$\begin{aligned}&\left|Y_{\widehat{q}_1,\widehat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathcal{G})\right|\\ &\leq\widehat{q}_1\widehat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\times\left\{\int_0^1\int_0^1|\mathcal{A}(\zeta,\theta)|\right.\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathcal{G}\left(\varphi_1+\zeta\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta}\right|_0\\ &\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathcal{G}\left(\varphi_1+\zeta\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta}\right|\cdot d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta\right\}\\ &\leq\widehat{q}_1\widehat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\cdot\left(\int_0^1\int_0^1|\mathcal{A}(\zeta,\theta)|_0 d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\right)^{1-(1/\sigma)}\\ &\times\left(\int_0^1\int_0^1|\mathcal{A}(\zeta,\theta)|\right.\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}\left(\varphi_1+\zeta\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma_0\\ &\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}\left(\varphi_1+\zeta\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma_0\cdot d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\right)^{1/\sigma}\\ &=\widehat{q}_1\widehat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\cdot\left[\left[\int_0^{1/(1+\widehat{q}_1)}\int_0^{1/(1+\widehat{q}_1)}\zeta\theta_0 d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta+\int_0^{1/(1+\widehat{q}_1)}\int_{1/(1+\widehat{q}_2)}^1\zeta\right.\right.\\ &\qquad\cdot\left(\frac{1}{\widehat{q}_2}-\theta\right)_0 d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta\end{aligned}$$

$$+\int_{1/(1+\widehat{q}_1)}^{1}\int_{0}^{1/(1+\widehat{q}_2)}\theta\left(\frac{1}{\widehat{q}_1}-\zeta\right)_0 d_{\widehat{q}_2}\theta\, {}_0d_{\widehat{q}_1}\zeta$$

$$\left.+\int_{1/(1+\widehat{q}_1)}^{1}\int_{1/(1+\widehat{q}_2)}^{1}\left(\frac{1}{\widehat{q}_1}-\zeta\right)\left(\frac{1}{\widehat{q}_2}-\theta\right)_0 d_{\widehat{q}_2}\theta\, {}_0d_{\widehat{q}_1}\zeta\right]$$

$$\times\left[\int_0^{1/(1+q\wedge_1)}\int_0^{1/(1+q\wedge_1)}\zeta\theta\left[\begin{array}{c}\zeta\theta\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \theta(1-\zeta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \zeta(1-\theta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ (1-\zeta)(1-\theta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\end{array}\right]_0\right.$$

$$\cdot d_{q\wedge_2}\theta\,{}_0d_{q\wedge_1}\zeta \quad +\int_0^{1/(1+q\wedge_1)}\int_{1/(1+q\wedge_2)}^{1}\zeta\left(\frac{1}{q\wedge_2}-\theta\right)$$

$$\cdot\left[\begin{array}{c}\zeta\theta\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \theta(1-\zeta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \zeta(1-\theta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ (1-\zeta)(1-\theta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\end{array}\right]_0$$

$$\cdot d_{q\wedge_2}\theta\,{}_0d_{q\wedge_1}\zeta \quad +\int_{1/(1+q\wedge_1)}^{1}\int_0^{1/(1+q\wedge_2)}\theta\left(\frac{1}{q\wedge_1}-\zeta\right)$$

$$\cdot\left[\begin{array}{c}\zeta\theta\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \theta(1-\zeta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \zeta(1-\theta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ (1-\zeta)(1-\theta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\end{array}\right]_0$$

$$\cdot d_{q\wedge_2}\theta\,{}_0d_{q\wedge_1}\zeta \quad +\int_{1/(1+q\wedge_1)}^{1}\int_{1/(1+q\wedge_2)}^{1}\left(\frac{1}{q\wedge_1}-\zeta\right)\left(\frac{1}{q\wedge_2}-\theta\right)$$

$$\left.\times\left[\begin{array}{c}\zeta\theta\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \theta(1-\zeta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \zeta(1-\theta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ (1-\zeta)(1-\theta)\left|\dfrac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\end{array}\right]_0 d_{q\wedge_2}\theta\,{}_0d_{q\wedge_1}\zeta\right]^{1/\sigma}$$

$$\begin{aligned}&=\widehat{q}_1\widehat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\left(\mathbb{B}_1(q\wedge_1,q\wedge_2)\right)^{1-1/\sigma}\\&\times\left[\mathbb{B}_2(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_3(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right.\\&\left.+\mathbb{B}_4(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_5(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta\,{}_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right]^{1/\sigma}.\end{aligned}\tag{58}$$

This completes the proof of Theorem 21.

Corollary 22. *In Theorem 21, if we choose $\widehat{q}_1,\widehat{q}_2\mapsto 1^-$, we have the following new inequality:*

$$\begin{aligned}&\left|\mathcal{G}\left(\frac{2\varphi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\frac{2\phi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)\right.\\&-\frac{1}{\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}\left(\mu,\frac{2\phi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)d\mu\\&-\frac{1}{\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}\left(\frac{2\varphi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\nu\right)d\nu\\&\left.-\frac{1}{\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\mu,\nu)d\nu d\mu\right|\\&\leq\frac{\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{16}\\&\cdot\left\{\frac{\left|\partial^2\mathcal{G}(\varphi_2,\phi_2)/\partial\zeta\partial\theta\right|^\sigma+\left|\partial^2\mathcal{G}(\varphi_1,\phi_2)/\partial\zeta\partial\theta\right|^\sigma+\left|\partial^2\mathcal{G}(\varphi_2,\phi_1)/\partial\zeta\partial\theta\right|^\sigma+\left|\partial^2\mathcal{G}(\varphi_1,\phi_1)/\partial\zeta\partial\theta\right|^\sigma}{4}\right\}.\end{aligned}\tag{59}$$

Remark 23. In Theorem 21,

(i) letting $\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$, then we attain Theorem 5 in [21]

(ii) letting $\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$ along with $\widehat{q}_1,\widehat{q}_2\mapsto 1^-$, then we attain Corollary 1 in [21] and Theorem 4 in [27], respectively

Theorem 24. *For $\gamma,\rho>0$ with $\lambda=(\lambda(0),\cdots,\lambda(1))$ as the bounded sequence of positive real numbers and let a mapping $\mathcal{G}:\Delta\mapsto\mathbb{R}$ be a twice partially $\widehat{q}_1\widehat{q}_2$ -differentiable on Δ°(the* *interior of Δ) such that continuous partial $\hat{q}_1\hat{q}_2$ -derivatives ${}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathcal{G}/{}_{\varphi_1}\partial_{\hat{q}_1}\zeta_{\phi_1}\partial_{\hat{q}_2}\theta$ is integrable on Δ with $0<\hat{q}_1,\hat{q}_2<1$. If $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ is a generalized Ψ -convex on the coordinates on Δ for $\sigma>1$ where $\sigma^{-1}+\beta^{-1}=1$. Then, the following inequality holds*

$$
\begin{aligned}
&\left|Y_{\hat{q}_1,\hat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathcal{G})\right| \le \hat{q}_1\hat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\
&\cdot\left(\int_0^1\int_0^1|\mathcal{A}(\zeta,\theta)|^\beta{}_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\beta}\\
&\times\left[\left(\left|{}_{\varphi_1,\varphi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)/{}_{\phi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right.\right.\\
&+q\wedge_1\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\
&+q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\
&\left.+q\wedge_1q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right)\\
&\left./((1+q\wedge_1)(1+q\wedge_2))\right]^{1/\sigma},
\end{aligned}
\tag{60}
$$

where $\mathcal{A}(\zeta,\theta)$ is defined as in (38).

Proof. Taking into consideration the $\hat{q}_1\hat{q}_2$-Hölder integral inequality, the generalized Ψ-convexity of $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ on the coordinates on Δ with the aid of Lemma 19, we have

$$
\begin{aligned}
&\left|Y_{\hat{q}_1,\hat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathcal{G})\right| \le \hat{q}_1\hat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\
&\times\left\{\int_0^1\int_0^1|\mathcal{A}(\zeta,\theta)|\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathcal{G}\left(\varphi_1+\zeta\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\sigma_1}\partial_{\hat{q}_1}\zeta_{\phi_1}\partial_{\hat{q}_2}\theta}\right|_0\right.\\
&\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathcal{G}\left(\varphi_1+\zeta\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\hat{q}_1}\zeta_{\phi_1}\partial_{\hat{q}_2}\theta}\right|\cdot d_{\hat{q}_2}\theta_0d_{\hat{q}_1}\zeta\right\}\\
&\le\hat{q}_1\hat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\
&\times\left[\left(\int_0^1\int_0^1|\mathcal{A}(\zeta,\theta)|^\beta{}_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\beta}\right.\\
&\times\left(\int_0^1\int_0^1|\mathcal{A}(\zeta,\theta)|\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}\left(\varphi_1+\zeta\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma_0\right.\\
&\left.\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}\left(\varphi_1+\zeta\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma_0\cdot d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\sigma}\right]\\
&=\hat{q}_1\hat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\left(\int_0^1\int_0^1|\mathcal{A}(\zeta,\theta)|^\beta{}_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\beta}\\
&\times\left[\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\int_0^1\int_0^1\zeta\theta_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right.\\
&+\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\int_0^1\int_0^1\theta(1-\zeta)_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\\
&+\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\int_0^1\int_0^1\zeta(1-\theta)_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\\
&\left.+\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\int_0^1\int_0^1(1-\zeta)(1-\theta)_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right]^{1/\sigma}\\
&=\hat{q}_1\hat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\left(\int_0^1\int_0^1|\mathcal{A}(\zeta,\theta)|^\beta{}_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\beta}\\
&\times\left[\left(\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma+q\wedge_1\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right.\right.\\
&+q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\
&\left.+q\wedge_1q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right)\\
&\left./((1+q\wedge_1)(1+q\wedge_2))\right]^{1/\sigma}.
\end{aligned}
\tag{61}
$$

This completes the proof of Theorem 21.

Corollary 25. *In Theorem 21, if we choose $\hat{q}_1,\hat{q}_2\mapsto 1^-$, we have the following new inequality:*

$$
\begin{aligned}
&\left|\mathcal{G}\left(\frac{2\varphi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\frac{2\phi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)\right.\\
&-\frac{1}{\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}\left(\mu,\frac{2\phi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)d\mu\\
&-\frac{1}{\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}\left(\frac{2\varphi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\nu\right)d\nu\\
&-\frac{1}{\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\\
&\left.\cdot\int_{\phi_1}^{\phi_1+\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\mu,\nu)d\nu d\mu\right|\le\frac{\hat{q}_1\hat{q}_2\left(\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{4(\beta+1)^{2/\beta}}\\
&\times\left[\left(\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right.\right.\\
&+\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\
&+\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\
&\left.\left.+\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right)/(4)\right]^{1/\sigma}.
\end{aligned}
\tag{62}
$$

Remark 26. In Theorem 21,

(i) letting $\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$, then we attain Theorem 6 in [21]

(ii) letting $\mathcal{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathcal{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$ along with $\hat{q}_1\hat{q}_2\mapsto 1^-$, then we attain Theorem 3 in [27]

6. Applications

This section contains some useful utilities of our findings derived in the previous sections. For appropriate and suitable selections of parameters γ, ρ, and λ in the special functions stated in (6), (10), and (11). Taking into account Raina's function (6), we shall derive outcomes for the hypergeometric function and Mittag-Leffler function as particular cases.

6.1. Hypergeometric Function. Letting $\gamma = 1$ and $\rho = 0$, and

$$\lambda(p) = \frac{(\vartheta_1)_p(\vartheta_2)_p}{(\vartheta_3)_p}, \quad \text{for } p = 0, 1, 2, \cdots, \tag{63}$$

then for Theorem 16, Lemma 19, and Theorems 21–24, the following results hold.

Theorem 27. *Suppose $\lambda = (\lambda(0), \cdots, \lambda(p))$ is the bounded sequence of positive real numbers and let $\mathscr{G} : O = [\varphi_1, \varphi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ is the coordinated generalized Ψ -convex and partially differentiable function on O° with $0 < \widehat{q}_1, \widehat{q}_2 < 1$, then the following inequalities hold:*

$$\begin{aligned}
&\mathscr{G}\left(\frac{(\widehat{q}_1 + 1)\varphi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)}{1 + \widehat{q}_1}, \frac{(\widehat{q}_2 + 1)\phi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)}{1 + \widehat{q}_2}\right) \\
&\le \frac{1}{2\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)} \int_{\varphi_1}^{\varphi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \mathscr{G} \\
&\quad \cdot \left(\mu, \frac{(\widehat{q}_2 + 1)\phi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)}{1 + \widehat{q}_2}\right)_{\varphi_1} d_{\widehat{q}_1}\mu \\
&\quad + \frac{1}{2\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\phi_1}^{\phi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathscr{G} \\
&\quad \cdot \left(\frac{(\widehat{q}_1 + 1)\varphi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)}{1 + \widehat{q}_1}, \nu\right)_{\phi_1} d_{\widehat{q}_2}\nu \\
&\le \frac{1}{\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\varphi_1}^{\varphi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \\
&\quad \cdot \int_{\phi_1}^{\phi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathscr{G}(\mu, \nu)_{\phi_1} d_{\widehat{q}_2}\nu_{\varphi_1} d_{\widehat{q}_1} \\
&\le \frac{\widehat{q}_2}{2(1 + \widehat{q}_2)} \left(\frac{1}{\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)} \int_{\varphi_1}^{\varphi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \mathscr{G}(\mu, \phi_1)_{\varphi_1} d_{\widehat{q}_1}\mu\right) \\
&\quad + \frac{\widehat{q}_2}{2(1 + \widehat{q}_2)} \left(\frac{1}{\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)} \int_{\varphi_1}^{\varphi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \mathscr{G}(\mu, \phi_2)_{\varphi_1} d_{\widehat{q}_1}\mu\right) \\
&\quad + \frac{\widehat{q}_1}{2(1 + \widehat{q}_1)} \left(\frac{1}{\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\phi_1}^{\phi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathscr{G}(\varphi_1, \nu)_{\phi_1} d_{\widehat{q}_2}\mu\right) \\
&\quad + \frac{\widehat{q}_1}{2(1 + \widehat{q}_1)} \left(\frac{1}{\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\phi_1}^{\phi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathscr{G}(\varphi_2, \nu)_{\phi_1} d_{\widehat{q}_2}\nu\right) \\
&\le \frac{\widehat{q}_1\widehat{q}_2\mathscr{G}(\varphi_1, \phi_1) + \widehat{q}_1\mathscr{G}(\varphi_1, \phi_2) + \widehat{q}_2\mathscr{G}(\varphi_2, \phi_1) + \mathscr{G}(\varphi_2, \phi_2)}{(1 + \widehat{q}_1)(1 + \widehat{q}_2)}.
\end{aligned} \tag{64}$$

Lemma 28. *Suppose $\lambda = (\lambda(0), \cdots, \lambda(p))$ be the bounded sequence of positive real numbers and let a twice partially $\widehat{q}_1$ $\widehat{q}_2$ -differentiable mapping $\mathscr{G} : O = [\varphi_1, \varphi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ defined on O° (the interior of O). If the second-order partial $\widehat{q}_1\widehat{q}_2$ -derivatives are continuous and integrable over O with $0 < \widehat{q}_1, \widehat{q}_2 < 1$, then the following equality holds:*

$$\begin{aligned}
&\tilde{Y}_{\widehat{q}_1,\widehat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathscr{G}) \\
&\coloneqq \mathscr{G}\left(\frac{(\widehat{q}_1 + 1)\varphi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)}{1 + \widehat{q}_1}, \frac{(\widehat{q}_2 + 1)\phi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 -_c 1)}{1 + \widehat{q}_2}\right) \\
&\quad - \frac{1}{\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)} \int_{\varphi_1}^{\varphi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \mathscr{G} \\
&\quad \cdot \left(\mu, \frac{(\widehat{q}_2 + 1)\phi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)}{1 + \widehat{q}_2}\right)_{\varphi_1} d_{\widehat{q}_1}\mu \quad - \frac{1}{\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \\
&\quad \cdot \int_{\phi_1}^{\phi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathscr{G}\left(\frac{(\widehat{q}_1 + 1)\varphi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)}{1 + \widehat{q}_1}, \nu\right)_{\phi_1} d_{\widehat{q}_2}\nu \\
&\quad + \frac{1}{\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\varphi_1}^{\varphi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \\
&\quad \cdot \int_{\phi_1}^{\phi_1 + \mathscr{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathscr{G}(\mu, \nu)_{\phi_1} d_{\widehat{q}_2}\nu_{\varphi_1} d_{\widehat{q}_1}\mu,
\end{aligned} \tag{65}$$

where

$$\begin{aligned}
&\tilde{Y}_{\widehat{q}_1,\widehat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathscr{G}) \coloneqq \widehat{q}_1\widehat{q}_2(\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1) \\
&\quad \cdot \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)) \times \int_0^1\int_0^1 \mathscr{A}(\zeta, \theta) \\
&\quad \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}(\varphi_1 + \zeta\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1), \phi_1 + \theta\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1))}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\, {}_{\phi_1}\partial_{\widehat{q}_2}\theta}{}_0 \\
&\quad \cdot d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta f\nu,
\end{aligned} \tag{66}$$

and $\mathscr{A}(\zeta, \theta)$ given in (38).

Theorem 29. *Suppose $\lambda = (\lambda(0), \cdots, \lambda(p))$ is the bounded sequence of positive real numbers and let a mapping $\mathscr{G}$ $: O = [\varphi_1, \varphi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ;$ $\vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ be a twice partially $\widehat{q}_1\widehat{q}_2$ -differentiable on O° such that continuous partial $\widehat{q}_1\widehat{q}_2$ -derivatives ${}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}$ $\mathscr{G}/{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta$ is integrable on O with $0 < \widehat{q}_1, \widehat{q}_2 < 1$. If $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ is a generalized Ψ -convex on the coordinates on O for $\sigma \ge 1$ where $\sigma^{-1} + \beta^{-1} = 1$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|\tilde{Y}_{\widehat{q}_1,\widehat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathscr{G})\right| \le \widehat{q}_1\widehat{q}_2(\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1) \\
&\quad \cdot \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1))(\mathbb{B}_1(q\wedge_1, q\wedge_2))^{1-1/\sigma} \\
&\quad \times \left[\mathbb{B}_2(q\wedge_1, q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2, \phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma + \mathbb{B}_3(q\wedge_1, q\wedge_2)\right. \\
&\quad \cdot \left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1, \phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma + \mathbb{B}_4(q\wedge_1, q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2, \phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma \\
&\quad \left. + \mathbb{B}_5(q\wedge_1, q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1, \phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right]^{1/\sigma},
\end{aligned} \tag{67}$$

where $\mathbb{B}_1(\widehat{q}_1, \widehat{q}_2), \mathbb{B}_2(\widehat{q}_1, \widehat{q}_2), \mathbb{B}_3(\widehat{q}_1, \widehat{q}_2), \mathbb{B}_4(\widehat{q}_1, \widehat{q}_2)$, and $\mathbb{B}_5(\widehat{q}_1, \widehat{q}_2)$ are given in (53), (54), (55), (56), and (57), respectively.

Theorem 30. *Suppose $\lambda = (\lambda(0), \cdots, \lambda(p))$ is the bounded sequence of positive real numbers and let a mapping $\mathscr{G} : O$*

$= [\varphi_1, \varphi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ *be a twice partially* $\widehat{q}_1\widehat{q}_2$ *-differentiable on* O° *such that continuous partial* $\widehat{q}_1\widehat{q}_2$ *-derivatives* ${}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}/{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta$ *is integrable on* O *with* $0 < \widehat{q}_1, \widehat{q}_2 < 1$. *If* $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ *is a generalized* Ψ *-convex on the coordinates on O for* $\sigma > 1$ *where* $\sigma^{-1} + \beta^{-1} = 1$. *Then, the following inequality holds:*

$$\begin{aligned}
&\left|Y_{\widehat{q}_1,\widehat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathscr{G})\right| \le \widehat{q}_1\widehat{q}_2(\mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1) \\
&\cdot \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1))\left(\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|^\beta {}_0d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\right)^{1/\beta} \\
&\times\left[\left(\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right.\right. \\
&+ q\wedge_1\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma \\
&+ q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma \\
&\left.+ q\wedge_1 q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right)/ \\
&\left.((1+q\wedge_1)(1+q\wedge_2))\right]^{1/\sigma},
\end{aligned} \tag{68}$$

where $\mathscr{A}(\zeta,\theta)$ *is defined as in (38).*

6.2. Mittag-Leffler Function. Setting $\nu = (1, 1, \cdots)$ having $\gamma = \vartheta_1, \Re(\vartheta_1) > 0$ and $\rho = 1$, then from Theorem 16, Lemma 19, and Theorems 21–24, the following results hold.

Theorem 31. *Let* $\mathscr{G} : \mathscr{S} = [\varphi_1, \varphi_1 + E_{\vartheta_1}(\varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + E_{\vartheta_1}(\phi_2 - \phi_1)] \mapsto \mathbb{R}$ *be the coordinated generalized* Ψ*-convex and partially differentiable function on* $\mathscr{S}^\circ$ *with* $0 < \widehat{q}_1, \widehat{q}_2 < 1$, *then the following inequalities hold:*

$$\begin{aligned}
&\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1 + E_{\vartheta_1}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \frac{(\widehat{q}_2+1)\phi_1 + E_{\vartheta_1}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right) \\
&\le \frac{1}{2E_{\vartheta_1}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\mathscr{G} \\
&\cdot\left(\mu, \frac{(\widehat{q}_2+1)\phi_1 + E_{\vartheta_1}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)_{\varphi_1} d_{\widehat{q}_1}\mu + \frac{1}{2E_{\vartheta_1}(\phi_2-\phi_1)} \\
&\cdot\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1 + E_{\vartheta_1}(\varphi_2-\phi_1)}{1+\widehat{q}_1}, \nu\right)_{\phi_1} d_{\widehat{q}_2}\nu \\
&\le \frac{1}{E_{\vartheta_1}(\varphi_2-\varphi_1)E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathscr{G} \\
&\cdot(\mu,\nu)_{\phi_1} d_{\widehat{q}_2}\nu_{\varphi_1} d_{\widehat{q}_1}\mu \le \frac{\widehat{q}_2}{2(1+\widehat{q}_2)} \\
&\cdot\left(\frac{1}{E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)_{\varphi_1} d_{\widehat{q}_1}\mu\right) \\
&+ \frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{E_{\vartheta_1}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)_{\varphi_1} d_{\widehat{q}_1}\mu\right) \\
&+ \frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)_{\phi_1} d_{\widehat{q}_2}\mu\right) \\
&+ \frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)_{\phi_1} d_{\widehat{q}_2}\nu\right) \\
&\le \frac{\widehat{q}_1\widehat{q}_2\mathscr{G}(\varphi_1,\phi_1) + \widehat{q}_1\mathscr{G}(\varphi_1,\phi_2) + \widehat{q}_2\mathscr{G}(\varphi_2,\phi_1) + \mathscr{G}(\varphi_2,\phi_2)}{(1+\widehat{q}_1)(1+\widehat{q}_2)}.
\end{aligned} \tag{69}$$

Lemma 32. *Let a twice partially* $\widehat{q}_1\widehat{q}_2$ *-differentiable mapping* $\mathscr{G} : \mathscr{S} = [\varphi_1, \varphi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathscr{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ *defined on* $\mathscr{S}^\circ$*(the interior of* $\mathscr{S}$*). If the second-order partial* $\widehat{q}_1\widehat{q}_2$*-derivatives are continuous and integrable over* $\mathscr{S}$ *with* $0 < \widehat{q}_1, \widehat{q}_2 < 1$, *then the following equality holds:*

$$\begin{aligned}
&\widetilde{\widetilde{Y}}_{\widehat{q}_1,\widehat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathscr{G}) \\
&\coloneqq \mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1 + E_{\vartheta_1}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \frac{(\widehat{q}_2+1)\phi_1 + E_{\vartheta_1}(\phi_2 -_c 1)}{1+\widehat{q}_2}\right) \\
&- \frac{1}{E_{\vartheta_1}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu, \frac{(\widehat{q}_2+1)\phi_1 + E_{\vartheta_1}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)_{\varphi_1} \\
&\cdot d_{\widehat{q}_1}\mu - \frac{1}{E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1 + E_{\vartheta_1}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \nu\right)_{\phi_1} \\
&\cdot d_{\widehat{q}_2}\nu + \frac{1}{E_{\vartheta_1}(\varphi_2-\varphi_1)E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)} \\
&\cdot\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)_{\phi_1} d_{\widehat{q}_2}\nu_{\varphi_1} d_{\widehat{q}_1}\mu,
\end{aligned} \tag{70}$$

where

$$\begin{aligned}
&\widetilde{\widetilde{Y}}_{\widehat{q}_1,\widehat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathscr{G}) \\
&\coloneqq \widehat{q}_1\widehat{q}_2\left(E_{\vartheta_1}(\varphi_2-\varphi_1)E_{\vartheta_1}(\phi_2-\phi_1)\right) \times \int_0^1\int_0^1 \mathscr{A}(\zeta,\theta) \\
&\cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1 + \zeta E_{\vartheta_1}(\varphi_2-\varphi_1), \phi_1 + \theta E_{\vartheta_1}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0 \\
&\cdot d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta,
\end{aligned} \tag{71}$$

and $\mathscr{A}(\zeta,\theta)$ *given in (38).*

Theorem 33. *Let a mapping* $\mathscr{G} : \mathscr{S} = [\varphi_1, \varphi_1 + E_{\vartheta_1}(\varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + E_{\vartheta_1}(\phi_2 - \phi_1)] \mapsto \mathbb{R}$ *be a twice partially* $\widehat{q}_1\widehat{q}_2$ *-differentiable on* $\mathscr{S}^\circ$ *such that continuous partial* $\widehat{q}_1\widehat{q}_2$*-derivatives* ${}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}/{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta$ *is integrable on* $\mathscr{S}$ *with* $0 < \widehat{q}_1, \widehat{q}_2 < 1$. *If* $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ *is a generalized* Ψ *-convex on the coordinates on* $\mathscr{S}$ *for* $\sigma \ge 1$ *where* $\sigma^{-1} + \beta^{-1} = 1$. *Then, the following inequality holds:*

$$\begin{aligned}
&\left|\widetilde{\widetilde{Y}}_{\widehat{q}_1,\widehat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathscr{G})\right| \\
&\le \widehat{q}_1\widehat{q}_2\left(E_{\vartheta_1}(\varphi_2-\varphi_1)E_{\vartheta_1}(\phi_2-\phi_1)\right)(\mathbb{B}_1(q\wedge_1, q\wedge_2))^{1-1/\sigma} \\
&\times\left[\mathbb{B}_2(q\wedge_1, q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma + \mathbb{B}_3(q\wedge_1, q\wedge_2)\right. \\
&\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma + \mathbb{B}_4(q\wedge_1, q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma \\
&\left.+ \mathbb{B}_5(q\wedge_1, q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right]^{1/\sigma},
\end{aligned} \tag{72}$$

where $\mathbb{B}_1(\hat{q}_1,\hat{q}_2), \mathbb{B}_2(\hat{q}_1,\hat{q}_2), \mathbb{B}_3(\hat{q}_1,\hat{q}_2), \mathbb{B}_4(\hat{q}_1,\hat{q}_2)$, *and* $\mathbb{B}_5(\hat{q}_1,\hat{q}_2)$ *are given in (53), (54), (55), (56), and (57), respectively.*

Theorem 34. *Let a mapping* $\mathcal{G} : \mathcal{S} = [\varphi_1, \varphi_1 + E_{\vartheta_1}(\varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + E_{\vartheta_1}(\phi_2 - \phi_1)] \mapsto \mathbb{R}$ *be a twice partially* $\hat{q}_1\hat{q}_2$ *-differ-*
where $\mathcal{A}(\zeta,\theta)$ *is defined as in (38).*

7. Conclusion

The main objective of this paper will be a motivation source for future studies. An auxiliary result in $\hat{q}_1\hat{q}_2$-integrals has been derived. We established some new generalizations for the $\mathcal{H}\mathcal{H}$-type inequality pertaining to $\hat{q}_1\hat{q}_2$-differentiable mappings for generalized Ψ-convex functions on coordinates in the special Raina's function sense that correlates with the $\hat{q}_1\hat{q}_2$-identity. Some useful applications of our findings have been illustrated with the association of the well-known special functions (hypergeometric and Mittag-Leffler function). Moreover, our findings are essentially applicable for obtaining the solution of integral equations that interact with$n$ bodies subject to mixed boundary conditions (see [7, 8]). For further potential investigation, we left the details for futuristic research. Every aspect of the suggested scheme is versatile and simple to execute. We apprehended noteworthy special cases for varying the parametric values in the involvement of special functions. This new study is explicit and viable and can be effectively utilized in inequality theory, special relativity theory, and quantum mechanics.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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entiable on $\mathcal{S}^\circ$ *such that continuous partial* $\hat{q}_1\hat{q}_2$ *-derivatives* ${}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathcal{G}/{}_{\varphi_1}\partial_{\hat{q}_1}\zeta_{\phi_1}\partial_{\hat{q}_2}\theta$ *is integrable on* $\mathcal{S}$ *with* $0 < \hat{q}_1, \hat{q}_2 < 1$. *If* $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ *is a generalized* Ψ *-convex on the coordinates on* $\mathcal{S}$ *for* $\sigma > 1$ *where* $\sigma^{-1} + \beta^{-1} = 1$. *Then, the following inequality holds:*

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Research Article

New Generalizations of Set Valued Interpolative Hardy-Rogers Type Contractions in b-Metric Spaces

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Debnath and De La Sen introduced the notion of set valued interpolative Hardy-Rogers type contraction mappings on b-metric spaces and proved that on a complete b-metric space, whose all closed and bounded subsets are compact, the set valued interpolative Hardy-Rogers type contraction mapping has a fixed point. This article presents generalizations of above results by omitting the assumption that all closed and bounded subsets are compact.

1. Introduction

There are numerous studies on interpolation inequalities in literature. In 1999, Chua [1] gave some weighted Sobolev interpolation inequalities on product spaces. Badr and Russ [2] proved some Littlewood-Paley inequalities and interpolation results for Sobolev spaces. Interpolation is considered as one of the central concepts in pure logic. Various interpolation properties find their applications in computer science and have many deep purely logical consequences (see [3, 4]). Gogatishvili and Koskela [5] presented variant interpolation properties of Besov spaces defined on metric spaces. Going in the same direction in the setting of metric spaces via contraction mappings, Karapinar [6] presented the concept of an interpolative Kannan contraction mapping and proved that this mapping admits a fixed point on complete metric spaces. Later on, this notation has been extended into several directions (see [7–18]).

In [6], Karapinar presented the interpolative Kannan contraction as follows: a mapping $K : (W, d_W) \to (W, d_W)$ is an interpolative Kannan contraction if

$$d_W\left(Kw^a, Kw^b\right)\Big] \leq \delta[d_W(w^a, Kw^a)]^{\iota_1}\left[d_W\left(w^b, Kw^b\right)\right]^{1-\iota_1} \tag{1}$$

for all $w^a, w^b \in W$ with $w^a \neq Kw^a$, where $\delta \in [0, 1)$ and $\iota_1 \in (0, 1)$. This inequality was further refined by Karapinar et al. [7] by

$$d_W\left(Kw^a, Kw^b\right)\Big] \leq \delta[d_W(w^a, Kw^a)]^{\iota_1}\left[d_W\left(w^b, Kw^b\right)\right]^{1-\iota_1} \tag{2}$$

for all $w^a, w^b \in W \setminus \text{fix}(K)$, where $\delta \in [0,1)$, $\iota_1 \in (0,1)$, and $\text{fix}(K) = \{w^a \in W : Kw^a = w^a\}$.

Gaba and Karapinar [9] further modified the interpolative Kannan contraction concept in the following way: a mapping $K : (W, d_W) \to (W, d_W)$ is a $(\delta, \iota_1, \iota_2)$-interpolative Kannan contraction, if

$$d_W\left(Kw^a, Kw^b\right)\Big] \leq \delta[d_W(w^a, Kw^a)]^{\iota_1}\left[d_W\left(w^b, Kw^b\right)\right]^{\iota_2} \tag{3}$$

for all $w^a, w^b \in W \setminus \text{fix}(K)$, where $\delta \in [0,1)$, $\iota_1, \iota_2 \in (0,1)$ with $\iota_1 + \iota_2 < 1$. Karapinar et al. [10] gave the interpolative Hardy-Rogers type contraction as follows: a mapping $K : (W, d_W) \to (W, d_W)$ is called an interpolative Hardy-Rogers type contraction if

$$\begin{aligned} d_W&\left(Kw^a, Kw^b\right) \\ &\leq \delta\left[\left[d_W\left(w^a, w^b\right)\right]^{\iota_1}[d_W(w^a, Kw^a)]^{\iota_2}\left[d_W\left(w^b, Kw^b\right)\right]^{\iota_3}\right. \\ &\quad \left.\times\left[\frac{1}{2\rho}\left(d_W\left(w^a, Kw^b\right) + d_W\left(Kw^a, w^b\right)\right)\right]^{1-\iota_1-\iota_2-\iota_3}\right] \end{aligned} \tag{4}$$

for each $w^a, w^b \in W \setminus \text{fix}(K)$, where $\delta \in [0,1)$ and $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1 + \iota_2 + \iota_3 < 1$.

Later on, Debnath and De La Sen [12] extended the above definition to set valued interpolative Hardy-Rogers type contraction mappings on b-metric spaces and proved that on complete b-metric spaces, whose all closed and bounded subsets are compact, the set valued interpolative Hardy-Rogers type contraction mapping has a fixed point.

On the other hand, Bakhtin [19] and Czerwik [20] introduced the notion of b-metric spaces.

Definition 1 (see [19, 20]). Let W be a nonempty set and $d_W : W \times W \to [0,\infty)$ be a function so that for all $i, j, \ell \in X$ and some $\rho \geq 1$,

$$\begin{aligned} &d_W(i,j) = 0 \Leftrightarrow i = j, \\ &d_W(i,j) = d_W(j,i), \\ &d_W(i,j) \leq \rho[d_W(i,\ell) + d_W(\ell,j)]. \end{aligned} \tag{5}$$

Then, d_W is a b-metric on W, and (W, d_W, ρ) is called a b-metric space with a coefficient $\rho \geq 1$.

For related works in this setting, see [21–23]. From now on, (W, d_W, ρ) is a b-metric space with a coefficient $\rho \geq 1$. In the whole paper, $\rho \geq 1$ is the coefficient of the b-metric space.

Definition 2 (see [20]). We have the following:

(a) A sequence $\{\eta_n\}$ in W is said to be Cauchy if $\lim_{n,m\to\infty} d_W(\eta_n, \eta_m) = 0$

(b) A sequence $\{\eta_n\}$ in W is said to be convergent to η if $\lim_{n,m\to\infty} d_W(\eta_n, \eta) = 0$

(c) (W, d_W, ρ) is said to be complete if every Cauchy sequence $\{\eta_n\}$ in W is convergent

Denote by $CB(W)$ the set of nonempty closed bounded subsets of W. For $A, B \in CB(X)$, consider

$$\Delta_W(A,B) = \sup\{d_W(\omega, B)\,;\, \omega \in A\}, \tag{6}$$

where $d_W(\omega, B) = \inf\{d_W(\omega,\mu), \mu \in B\}$. The functional $H_W : CB(W) \times CB(W) \to [0,\infty)$ defined by

$$H_W(A,B) = \max\{\Delta_W(A,B), \Delta_W(B,A)\} \tag{7}$$

is known as the Pompieu-Hausdorff b-metric on $CB(W)$. We state the following known lemma.

Lemma 3 (see [24]). *Let (W, d_W, ρ) be a b-metric space ($\rho \geq 1$). Let $A, B \in CB(W)$ and $a \in A$. We have the two following statements:*

(i) For each $\varepsilon > 0$, there is $b \in B$ so that

$$d_W(a,b) \leq H_W(A,B) + \varepsilon \tag{8}$$

(ii) For each $h > 1$, there is $\nu \in B$ so that

$$d_W(a,\nu) \leq hH_W(A,B) \tag{9}$$

This article presents two new generalizations of set valued interpolative Hardy-Rogers type contraction mappings. Namely, we ensure the existence of fixed points of such maps on a complete b-metric space without considering the assumption that all closed and bounded subsets must be compact. Two examples are also presented.

2. Main Results

First, we define the notion of ξ-interpolative Hardy-Rogers type contractions.

Definition 4. Consider a b-metric space (W, d_W, ρ). Also, consider maps $K : W \to CB(W)$ and $\xi : W \times W \to \mathbb{R} \setminus \{0\}$. Such a map K is called an ξ-interpolative Hardy-Rogers type contraction if

$$\begin{aligned} &\left[H_W\left(Kw^a, Kw^b\right)\right]^{\xi(w^a,w^b)} \\ &\quad\leq \delta\left[\left[d_W\left(w^a, w^b\right)\right]^{\iota_1}[d_W(w^a, Kw^a)]^{\iota_2}\left[d_W\left(w^b, Kw^b\right)\right]^{\iota_3}\right. \\ &\qquad \left.\times\left[\frac{1}{2\rho}\left(d_W\left(w^a, Kw^b\right) + d_W\left(Kw^a, w^b\right)\right)\right]^{1-\iota_1-\iota_2-\iota_3}\right] \end{aligned} \tag{10}$$

for each $w^a, w^b \in W$ with

$$\min\left\{d_W\left(w^a, w^b\right), d_W(w^a, Kw^a), d_W\left(w^b, Kw^b\right)\right\} > 0, \tag{11}$$

where $\delta \in [0, 1/\rho^2)$ and $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1 + \iota_2 + \iota_3 < 1$.

The following result ensures the existence of a fixed point of ξ-interpolative Hardy-Rogers type contractions.

Theorem 5. *Consider a complete b-metric space (W, d_W, ρ) and consider an ξ-interpolative Hardy-Rogers type contraction map K. Also, consider the given assertions.*

(I) There must exist $w_0^a \in W$ and $w_1^a \in Kw_0^a$ such that $\xi(w_0^a, w_1^a) = 1$

(II) For each $w^a, w^b \in W$ with $\xi(w^a, w^b) = 1$, we have $\xi(w^c, w^d) = 1 \forall w^c \in Kw^a, w^d \in Kw^b$

(III) For each $\{w_m^a\}$ in W with $w_m^a \to w$ and $\xi(w_m^a, w_{m+1}^a) = 1 \forall m \in \mathbb{N}$, we have $\xi(w_m^a, w) = 1 \forall m \in \mathbb{N}$

Then, K must have a fixed point in W.

Proof. By assertion (I) there are $w_0^a \in W$ and $w_1^a \in Kw_0^a$ with $\xi(w_0^a, w_1^a) = 1$. If

$$\min\{d_W(w_0^a, w_1^a), d_W(w_0^a, Kw_0^a), d_W(w_1^a, Kw_1^a)\} = 0, \tag{12}$$

then K has a fixed point. Suppose that

$$\min\{d_W(w_0^a, w_1^a), d_W(w_0^a, Kw_0^a), d_W(w_1^a, Kw_1^a)\} > 0. \tag{13}$$

By (10), we obtain

$$\begin{aligned} &H_W(Kw_0^a, Kw_1^a) \\ &\quad = [H_W(Kw_0^a, Kw_1^a)]^{\xi(w_0^a, w_1^a)} \\ &\quad \le \delta\left[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, Kw_0^a)]^{\iota_2}[d_W(w_1^a, Kw_1^a)]^{\iota_3}\right. \\ &\qquad \left.\times\left[\frac{1}{2\rho}(d_W(w_0^a, Kw_1^a) + d_W(Kw_0^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{14}$$

This leads to

$$\begin{aligned} &\frac{1}{\sqrt{\delta}} d_W(w_1^a, Kw_1^a) \\ &\quad \le \frac{1}{\sqrt{\delta}} H_W(Kw_0^a, Kw_1^a) \\ &\quad \le \sqrt{\delta}\left[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, Kw_0^a)]^{\iota_2}[d_W(w_1^a, Kw_1^a)]^{\iota_3}\right. \\ &\qquad \left.\times\left[\frac{1}{2\rho}(d_W(w_0^a, Kw_1^a) + d_W(Kw_0^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{15}$$

Since $1/\sqrt{\delta} > 1$, there is $w_2^a \in Kw_1^a$ such that

$$d_W(w_1^a, w_2^a) \le \frac{1}{\sqrt{\delta}} d_W(w_1^a, Kw_1^a). \tag{16}$$

Thus, by (15),

$$\begin{aligned} d_W(w_1^a, w_2^a) \le \sqrt{\delta}&\left[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, w_1^a)]^{\iota_2}[d_W(w_1^a, w_2^a)]^{\iota_3}\right. \\ &\left.\times\left[\frac{1}{2\rho}(d_W(w_0^a, w_2^a) + d_W(w_1^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{17}$$

Note that $d_W(w_0^a, w_2^a) \le \rho[d_W(w_0^a, w_1^a) + d_W(w_1^a, w_2^a)] \le 2\rho \max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\}$. Hence, by (17), we get

$$\begin{aligned} d_W(w_1^a, w_2^a) \le \sqrt{\delta}&\left[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, w_1^a)]^{\iota_2}[d_W(w_1^a, w_2^a)]^{\iota_3}\right. \\ &\left.\times[\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\}]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{18}$$

Now, we consider $\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\} = d_W(w_0^a, w_1^a)$. Then, by (18), we get

$$\begin{aligned} d_W(w_1^a, w_2^a) \le \sqrt{\delta}&\left[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, w_1^a)]^{\iota_2}[d_W(w_0^a, w_1^a)]^{\iota_3}\right. \\ &\left.\times[d_W(w_0^a, w_1^a)]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{19}$$

This implies

$$d_W(w_1^a, w_2^a) \le \sqrt{\delta} d_W(w_0^a, w_1^a). \tag{20}$$

Note that when we take $\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\} = d_W(w_1^a, w_2^a)$ in (18), then we get $d_W(w_1^a, w_2^a) = 0$, that is, $w_1^a \in Kw_1^a$; hence, this choice is not possible. As $\xi(w_0^a, w_1^a) = 1$ and $w_1^a \in Kw_0^a$ and $w_2^a \in Kw_1^a$, then by assertion (II), we get $\xi(w_1^a, w_2^a) = 1$. Again, we consider

$$\min\{d_W(w_1^a, w_2^a), d_W(w_1^a, Kw_1^a), d_W(w_2^a, Kw_2^a)\} > 0, \tag{21}$$

then by (10), we get

$$\begin{aligned} &\frac{1}{\sqrt{\delta}} d_W(w_2^a, Kw_2^a) \\ &\quad \le \frac{1}{\sqrt{\delta}} H_W(Kw_1^a, Kw_2^a) \\ &\quad = \frac{1}{\sqrt{\delta}}[H_W(Kw_1^a, Kw_2^a)]^{\xi(w_1^a, w_2^a)} \\ &\quad \le \sqrt{\delta}\left[[d_W(w_1^a, w_2^a)]^{\iota_1}[d_W(w_1^a, Kw_1^a)]^{\iota_2}[d_W(w_2^a, Kw_2^a)]^{\iota_3}\right. \\ &\qquad \left.\times\left[\frac{1}{2\rho}(d_W(w_1^a, Kw_2^a) + d_W(Kw_1^a, w_2^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{22}$$

Since $1/\sqrt{\delta} > 1$, there is $w_3^a \in Kw_2^a$ such that

$$d_W(w_2^a, w_3^a) \leq \frac{1}{\sqrt{\delta}} d_W(w_2^a, Kw_2^a). \tag{23}$$

Thus, by (22), we conclude

$$d_W(w_2^a, w_3^a) \leq \sqrt{\delta}\left[[d_W(w_1^a, w_2^a)]^{\iota_1}[d_W(w_1^a, w_2^a)]^{\iota_2}[d_W(w_2^a, w_3^a)]^{\iota_3} \times \left[\frac{1}{2\rho}(d_W(w_1^a, w_3^a) + d_W(w_2^a, w_2^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]. \tag{24}$$

Note that $d_W(w_1^a, w_3^a) \leq \rho[d_W(w_1^a, w_2^a) + d_W(w_2^a, w_3^a)] \leq 2\rho \max\{d_W(w_1^a, w_2^a), d_W(w_2^a, w_3^a)\}$. Hence, by (24), we get

$$d_W(w_2^a, w_3^a) \leq \sqrt{\delta}\left[[d_W(w_1^a, w_2^a)]^{\iota_1}[d_W(w_1^a, w_2^a)]^{\iota_2}[d_W(w_2^a, w_3^a)]^{\iota_3} \times [\max\{d_W(w_1^a, w_2^a), d_W(w_2^a, w_3^a)\}]^{1-\iota_1-\iota_2-\iota_3}\right]. \tag{25}$$

Now, we consider $\max\{d_W(w_1^a, w_2^a), d_W(w_2^a, w_3^a)\} = d_W(w_1^a, w_2^a)$. Then, by (18), we get

$$d_W(w_2^a, w_3^a) \leq \sqrt{\delta}\left[[d_W(w_1^a, w_2^a)]^{\iota_1}[d_W(w_1^a, w_2^a)]^{\iota_2}[d_W(w_1^a, w_2^a)]^{\iota_3} \times [d_W(w_1^a, w_2^a)]^{1-\iota_1-\iota_2-\iota_3}\right]. \tag{26}$$

This yields that

$$d_W(w_2^a, w_3^a) \leq \sqrt{\delta} d_W(w_1^a, w_2^a). \tag{27}$$

Note that if we take $\max\{d_W(w_1^a, w_2^a), d_W(w_2^a, w_3^a)\} = d_W(w_2^a, w_3^a)$ in (25), then $d_W(w_2^a, w_3^a) = 0$, that is, $w_2^a \in Kw_2^a$, which is not possible. From (27) and (20), we get

$$d_W(w_2^a, w_3^a) \leq \left(\sqrt{\delta}\right)^2 d_W(w_0^a, w_1^a). \tag{28}$$

Proceeding in this way, we can obtain a sequence $\{w_m^a\}$ in W with $w_{m+1}^a \in Kw_m^a$, $\xi(w_m^a, w_{m+1}^a) = 1$ for all $m \in \mathbb{W}$ and

$$d_W(w_m^a, w_{m+1}^a) \leq \left(\sqrt{\delta}\right)^m d_W(w_0^a, w_1^a) \forall m \in \mathbb{N}. \tag{29}$$

Also, by the construction of $\{w_m^a\}$, we get

$$\min\{d_W(w_m^a, w_{m+1}^a), d_W(w_m^a, Kw_m^a), d_W(w_{m+1}^a, Kw_{m+1}^a)\} > 0 \forall m \in \mathbb{N}. \tag{30}$$

By a triangular inequality, we have for $n > m$,

$$d_W(w_n^a, w_m^a) \leq \sum_{j=m}^{n-1} \rho^j d_W\left(w_j^a, w_{j+1}^a\right) \leq \sum_{j=m}^{n-1} \rho^j \left(\sqrt{\delta}\right)^j d_W(w_0^a, w_1^a). \tag{31}$$

Since the above series is convergent, $\{w_m^a\}$ is a Cauchy sequence in W. Completeness of W gives w_*^a in W such that $w_m^a \to w_*^a$. By considering assertion (III), we get $\xi(w_m^a, w_*^a) = 1 \forall m \in \mathbb{N}$. Here, we claim $w_*^a \in Kw_*^a$. If the claim is wrong, then $\min\{d_W(w_m^a, w_*^a), d_W(w_m^a, Kw_m^a), d_W(w_*^a, Kw_*^a)\} > 0$ for all $m > m_0$, for some $m_0 \in \mathbb{N}$. From (10), we get

$$\begin{aligned} d_W(w_{m+1}^a, Kw_*^a) &\leq H_W(Kw_m^a, Kw_*^a) \\ &= [H_W(Kw_m^a, Kw_*^a)]^{\xi(w_m^a, w_*^a)} \\ &\leq \delta\left[[d_W(w_m^a, w_*^a)]^{\iota_1}[d_W(w_m^a, Kw_m^a)]^{\iota_2}[d_W(w_*^a, Kw_*^a)]^{\iota_3} \times \left[\frac{1}{2\rho}(d_W(w_m^a, Kw_*^a) + d_W(Kw_m^a, w_*^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right] \\ &\leq \delta\left[[d_W(w_m^a, w_*^a)]^{\iota_1}[d_W(w_m^a, Kw_m^a)]^{\iota_2}[d_W(w_*^a, Kw_*^a)]^{\iota_3} \times \left[\frac{1}{2\rho}(d_W(w_m^a, Kw_*^a) + d_W(w_{m+1}^a, w_*^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right] \forall m > m_0. \end{aligned} \tag{32}$$

From the above, we get $\lim_{m\to\infty} d_W(w_{m+1}^a, Kw_*^a) = 0$. By the triangular inequality, we have

$$d_W(w_*^a, Kw_*^a) \leq \rho[d_W(w_*^a, w_{m+1}^a) + d_W(w_{m+1}^a, Kw_*^a)] \forall m \in \mathbb{N}. \tag{33}$$

By taking the limit $m \to \infty$, we get $d_W(w_*^a, Kw_*^a) = 0$, that is, $w_*^a \in Kw_*^a$. Therefore, our claim is valid.

Example 1. Consider $W = \mathbb{Z}$ with $d_W(w_n, w_m) = (w_n - w_m)^2$ for all $w_n, w_m \in W$. Define $K : W \to CB(W)$ by

$$K(w_n) = \begin{cases} \{0\}, & w_n \in \{0, 1, 2, 3, \cdots\} \\ \{-(w_n - 2)^2\}, & w_n \in \{-1, -2, -3, \cdots\} \end{cases} \tag{34}$$

and $\xi : W \times W \to \mathbb{R} \setminus \{0\}$ by

$$\xi(w_n, w_m) = \begin{cases} 1, & w_n, w_m \in \{0, 1, 2, 3, \cdots\} \\ -[|w_n| + |w_m| + 4], & \text{otherwise}. \end{cases} \tag{35}$$

Note that

Case 1. If $w_n, w_m \geq 0$ with $w_n \neq w_m$, we get $H_W(Kw_n, Kw_m)^{\xi(w_n, w_m)} = 0$.

Case 2. If $w_n, w_m < 0$ with $w_n \neq w_m$, we get

$$H_W(Kw_n, Kw_m)^{\xi(w_n,w_m)} = \frac{1}{\left[\left(-(w_n-2)^2+(w_m-2)^2\right)^2\right]^{|w_n|+|w_m|+4}}. \quad (36)$$

Case 3. If $w_n < 0$ and $w_m \geq 0$, we get $H_W(Kw_n, Kw_m)^{\xi(w_n,w_m)} = 1/[(-(w_n-2)^2)^2]^{|w_n|+|w_m|+4}$.

After calculating the values, it is easy to see that

For Case1: if $w_n, w_m > 0$ with $w_n \neq w_m$, we get

$$\begin{aligned}[H_W(Kw_n, Kw_m)]^{\xi(w_n,w_m)} &= 0 < \frac{1}{5}\left[1\cdot 1\cdot 1\cdot \frac{1}{4}\right] \\ &\leq \frac{1}{5}\left[[d_W(w_n,w_m)]^{\iota_1}[d_W(w_n,Kw_n)]^{\iota_2}[d_W(w_m,Kw_m)]^{\iota_3}\right. \\ &\quad \left.\times\left[\frac{1}{2\rho}(d_W(w_n,Kw_m)+d_W(Kw_n,w_m))\right]^{1-\iota_1-\iota_2-\iota_3}\right]\end{aligned} \quad (37)$$

for each $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1+\iota_2+\iota_3 < 1$.

For Case2: if $w_n, w_m < 0$ with $w_n \neq w_m$, we get

$$\begin{aligned}[H_W(Kw_n, Kw_m)]^{\xi(w_n,w_m)} &= \frac{1}{\left[\left(-(w_n-2)^2+(w_m-2)^2\right)^2\right]^{|w_n|+|w_m|+4}} \\ &\leq \frac{1}{(49)^7} < \frac{1}{5}\left[1\cdot 1\cdot 1\cdot \frac{1}{4}\right] \\ &\leq \frac{1}{5}\left[[d_W(w_n,w_m)]^{\iota_1}[d_W(w_n,Kw_n)]^{\iota_2}[d_W(w_m,Kw_m)]^{\iota_3}\right. \\ &\quad \left.\times\left[\frac{1}{2\rho}(d_W(w_n,Kw_m)+d_W(Kw_n,w_m))\right]^{1-\iota_1-\iota_2-\iota_3}\right]\end{aligned} \quad (38)$$

for each $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1+\iota_2+\iota_3 < 1$.

For Case3: if $w_n < 0$ and $w_m > 0$, we get

$$\begin{aligned}[H_W(Kw_n, Kw_m)]^{\xi(w_n,w_m)} &= \frac{1}{\left[\left(-(w_n-2)^2\right)^2\right]^{|w_n|+|w_m|+4}} \leq \frac{1}{(81)^6} < \frac{1}{5}\left[1\cdot 1\cdot 1\cdot \frac{1}{4}\right] \\ &\leq \frac{1}{5}\left[[d_W(w_n,w_m)]^{\iota_1}[d_W(w_n,Kw_n)]^{\iota_2}[d_W(w_m,Kw_m)]^{\iota_3}\right. \\ &\quad \left.\times\left[\frac{1}{2\rho}(d_W(w_n,Kw_m)+d_W(Kw_n,w_m))\right]^{1-\iota_1-\iota_2-\iota_3}\right]\end{aligned} \quad (39)$$

for each $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1+\iota_2+\iota_3 < 1$. By keeping these calculations in mind, one can check that all the hypotheses of Theorem 5 are valid. Hence, K must have a fixed point.

The following definition presents a multiplicative ξ-interpolative Hardy-Rogers type contraction.

Definition 6. Consider a b-metric space (W, d_W, ρ). Also, consider the maps $K: W \to CB(W)$ and $\xi: W\times W \to [0, \infty)$. Such K is called a multiplicative ξ-interpolative Hardy-Rogers type contraction if

$$\begin{aligned}\xi\left(w^a,w^b\right)H_W\left(Kw^a,Kw^b\right) &\leq \delta\left[\left[d_W\left(w^a,w^b\right)\right]^{\iota_1}[d_W(w^a,Kw^a)]^{\iota_2}\left[d_W\left(w^b,Kw^b\right)\right]^{\iota_3}\right. \\ &\quad \left.\times\left[\frac{1}{2\rho}\left(d_W\left(w^a,Kw^b\right)+d_W\left(Kw^a,w^b\right)\right)\right]^{1-\iota_1-\iota_2-\iota_3}\right]\end{aligned} \quad (40)$$

for each $w^a, w^b \in W$ with

$$\min\left\{d_W\left(w^a,w^b\right), d_W(w^a,Kw^a), d_W\left(w^b,Kw^b\right)\right\} > 0, \quad (41)$$

where $\delta \in (0, 1/\rho^2)$ and $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1+\iota_2+\iota_3 < 1$.

The following result concerns the existence of fixed points for the above-defined mapping.

Theorem 7. *Consider a complete b-metric space (W, d_W, ρ) and consider a multiplicative ξ-interpolative Hardy-Rogers type contraction map K. Also, consider the given assertions:*

(i) There must exist $w_0^a \in W$ and $w_1^a \in Kw_0^a$ such that $\xi(w_0^a, w_1^a) \geq 1$

(ii) For each $w^a, w^b \in W$ with $\xi(w^a,w^b) \geq 1$, we have $\xi(w^c,w^d) \geq 1 \forall w^c \in Kw^a, w^d \in Kw^b$

(iii) For each $\{w_m^a\}$ in W with $w_m^a \to w$ and $\xi(w_m^a, w_{m+1}^a) \geq 1 \forall m \in \mathbb{N}$, we have $\xi(w_m^a, w) \geq 1 \forall m \in \mathbb{N}$

Then K possesses a fixed point in W.

Proof. Assertion (i) implies the existence of $w_0^a \in W$ and $w_1^a \in Kw_0^a$ with $\xi(w_0^a, w_1^a) \geq 1$. We consider

$$\min\{d_W(w_0^a,w_1^a), d_W(w_0^a,Kw_0^a), d_W(w_1^a,Kw_1^a)\} > 0. \quad (42)$$

Otherwise, K has a fixed point. Then, by (40), we obtain

$$\begin{aligned} &H_W(Kw_0^a, Kw_1^a) \\ &\quad \le \xi(w_0^a, w_1^a) H_W(Kw_0^a, Kw_1^a) \\ &\quad \le \delta\left[[d_W(w_0^a, w_1^a)]^{\iota_1} [d_W(w_0^a, Kw_0^a)]^{\iota_2} [d_W(w_1^a, Kw_1^a)]^{\iota_3} \right. \\ &\qquad \left. \times \left[\frac{1}{2\rho}(d_W(w_0^a, Kw_1^a) + d_W(Kw_0^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{43}$$

This yields that

$$\begin{aligned} &\frac{1}{\sqrt{\delta}} d_W(w_1^a, Kw_1^a) \\ &\quad \le \frac{1}{\sqrt{\delta}} H_W(Kw_0^a, Kw_1^a) \\ &\quad \le \sqrt{\delta}\left[[d_W(w_0^a, w_1^a)]^{\iota_1} [d_W(w_0^a, Kw_0^a)]^{\iota_2} [d_W(w_1^a, Kw_1^a)]^{\iota_3} \right. \\ &\qquad \left. \times \left[\frac{1}{2\rho}(d_W(w_0^a, Kw_1^a) + d_W(Kw_0^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{44}$$

Since $1/\sqrt{\delta} > 1$, there is $w_2^a \in Kw_1^a$ satisfying

$$d_W(w_1^a, w_2^a) \le \frac{1}{\sqrt{\delta}} d_W(w_1^a, Kw_1^a). \tag{45}$$

Thus, by (44), we get

$$\begin{aligned} d_W(w_1^a, w_2^a) \le \sqrt{\delta}&\left[[d_W(w_0^a, w_1^a)]^{\iota_1} [d_W(w_0^a, w_1^a)]^{\iota_2} [d_W(w_1^a, w_2^a)]^{\iota_3} \right. \\ &\left. \times \left[\frac{1}{2\rho}(d_W(w_0^a, w_2^a) + d_W(w_1^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{46}$$

Since $d_W(w_0^a, w_2^a) \le \rho[d_W(w_0^a, w_1^a) + d_W(w_1^a, w_2^a)] \le 2\rho \max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\}$, we get using (46),

$$\begin{aligned} d_W(w_1^a, w_2^a) \le \sqrt{\delta}&\left[[d_W(w_0^a, w_1^a)]^{\iota_1} [d_W(w_0^a, w_1^a)]^{\iota_2} [d_W(w_1^a, w_2^a)]^{\iota_3} \right. \\ &\left. \times [\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\}]^{1-\iota_1-\iota_2-\iota_3}\right]. \end{aligned} \tag{47}$$

Consider $\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\} = d_W(w_0^a, w_1^a)$. Then, by (47), we get

$$\begin{aligned} d_W(w_1^a, w_2^a) \le \sqrt{\delta}&[d_W(w_0^a, w_1^a)]^{\iota_1} [d_W(w_0^a, w_1^a)]^{\iota_2} [d_W(w_0^a, w_1^a)]^{\iota_3} \\ &\times [d_W(w_0^a, w_1^a)]^{1-\iota_1-\iota_2-\iota_3}]. \end{aligned} \tag{48}$$

This implies that

$$d_W(w_1^a, w_2^a) \le \sqrt{\delta} d_W(w_0^a, w_1^a). \tag{49}$$

If we take $\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\} = d_W(w_1^a, w_2^a)$ in (47), then we get $d_W(w_1^a, w_2^a) = 0$, that is, $w_1^a \in Kw_1^a$, which is not possible. Since $\xi(w_0^a, w_1^a) \ge 1$, $w_1^a \in Kw_0^a$, and $w_2^a \in Kw_1^a$, by assertion (ii), we get $\xi(w_1^a, w_2^a) \ge 1$. Applying (40) and again assertion (ii), we can obtain a sequence $\{w_m^a\}$ in W with $w_{m+1}^a \in Kw_m^a$, $\xi(w_m^a, w_{m+1}^a) \ge 1$ for all $m \in \mathbb{W}$ and

$$d_W(w_m^a, w_{m+1}^a) \le \left(\sqrt{\delta}\right)^m d_W(w_0^a, w_1^a) \forall m \in \mathbb{N}. \tag{50}$$

Also, by construction of $\{w_m^a\}$, we know that

$$\begin{aligned} &\min\{d_W(w_m^a, w_{m+1}^a), d_W(w_m^a, Kw_m^a), d_W(w_{m+1}^a, Kw_{m+1}^a)\} \\ &\quad > 0 \forall m \in \mathbb{N}. \end{aligned} \tag{51}$$

By a triangular inequality, we have for $n > m$,

$$d_W(w_n^a, w_m^a) \le \sum_{j=m}^{n-1} \rho^j d_W\left(w_j^a, w_{j+1}^a\right) \le \sum_{j=m}^{n-1} \rho^j \left(\sqrt{\delta}\right)^j d_W(w_0^a, w_1^a). \tag{52}$$

This implies that $\{w_m^a\}$ is a Cauchy sequence in W. Since W is complete, $w_m^a \to w_*^a \in W$. By assertion (iii), we get $\xi(w_m^a, w_*^a) \ge 1$ for all $m \in \mathbb{N}$. Now, we claim that $w_*^a \in Kw_*^a$. Assume the claim is wrong, then $\min\{d_W(w_m^a, w_*^a), d_W(w_m^a, Kw_m^a), d_W(w_*^a, Kw_*^a)\} > 0$ for all $m > m_0$ for some $m_0 \in \mathbb{N}$. Then by (40), we get

$$\begin{aligned} &d_W(w_{m+1}^a, Kw_*^a) \\ &\quad \le H_W(Kw_m^a, Kw_*^a) \\ &\quad = \xi(w_m^a, w_*^a) H_W(Kw_m^a, Kw_*^a) \\ &\quad \le \delta\left[[d_W(w_m^a, w_*^a)]^{\iota_1} [d_W(w_m^a, Kw_m^a)]^{\iota_2} [d_W(w_*^a, Kw_*^a)]^{\iota_3} \right. \\ &\qquad \left. \times \left[\frac{1}{2\rho}(d_W(w_m^a, Kw_*^a) + d_W(Kw_m^a, w_*^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right] \\ &\quad \le \delta\left[[d_W(w_m^a, w_*^a)]^{\iota_1} [d_W(w_m^a, Kw_m^a)]^{\iota_2} [d_W(w_*^a, Kw_*^a)]^{\iota_3} \right. \\ &\qquad \left. \times \left[\frac{1}{2\rho}(d_W(w_m^a, Kw_*^a) + d_W(w_{m+1}^a, w_*^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right] \forall m > m_0. \end{aligned} \tag{53}$$

From the above inequality, we get $\lim_{m\to\infty} d_W(w_{m+1}^a, Kw_*^a) = 0$. By a triangular inequality, we have

$$d_W(w_*^a, Kw_*^a) \le \rho[d_W(w_*^a, w_{m+1}^a) + d_W(w_{m+1}^a, Kw_*^a)] \forall m \in \mathbb{N}. \tag{54}$$

Hence, by taking the limit $m \longrightarrow \infty$, we get $d_W(w_*^a, K w_*^a) = 0$, that is, $w_*^a \in K w_*^a$.

Example 2. Consider $W = \mathbb{Z}$ with $d_W(w_n, w_m) = |w_n - w_m|$ for all $w_n, w_m \in W$. Define $K : W \longrightarrow CB(W)$ by

$$K(w_n) = \begin{cases} \{0\}, & w_n \in \{0, 1, 2, 3, \cdots\} \\ \{0, 2w_n\}, & w_n \in \{-1, -2, -3, \cdots\} \end{cases} \tag{55}$$

and $\xi : W \times W \to \mathbb{R} - \{0\}$ by

$$\xi(w_n, w_m) = \begin{cases} 1, & w_n, w_m \in \{0, 1, 2, 3, \cdots\} \\ 0, & \text{otherwise.} \end{cases} \tag{56}$$

One can see that all the hypotheses of Theorem 7 are valid. Hence, K must have a fixed point.

Remark 8. Note that ([12], Theorem 2) is not applicable in Example 2. It suffices to take $x = -1$ and $y = -2$, then $Kx = \{0, -2\}$ and $Ky = \{0, -4\}$. Thus, we have $H(Kx, Ky) = 2$, $d(x, y) = 1$, $d(x, Kx) = 1$, $d(y, Ky) = 2$, $d(y, Kx) = 0$, and $d(x, Ky) = 1$. One then writes

$$H(Kx, Ky) = 2 > \delta\left[(1^{\iota_1})(1^{\iota_2})(2^{\iota_3})\left(\left(\frac{1}{2}\right)^{1-\iota_1-\iota_2-\iota_3}\right)\right] \tag{57}$$

for all $\delta, \iota_1, \iota_2, \iota_3 \in (0, 1)$. Thus, our main results generalize and improve the result given in [12]. Moreover, when considering the single valued case in Theorem 5 and Theorem 7, that is, for a self-mapping $K : W \longrightarrow W$, we get generalizations of the main results in [9].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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Research Article

On Stancu-Type Generalization of Modified (p, q)-Szász-Mirakjan-Kantorovich Operators

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In the present article, we construct (p, q)-Szász-Mirakjan-Kantorovich-Stancu operators with three parameters λ, α, β. First, the moments and central moments are estimated. Then, local approximation properties of these operators are established via K-functionals and Steklov mean in means of modulus of continuity. Also, a Voronovskaja-type theorem is presented. Finally, the pointwise estimates, rate of convergence, and weighted approximation of these operators are studied.

1. Introduction

During this decades, the applications of (p, q)-calculus transpired as a new area in the field of operator approximation theory. Many researchers constructed and discussed many positive linear operators based on (p, q)-integers, (p, q)-exponential functions, (p, q)-Gamma functions [1], (p, q)-Beta functions, and so on. Since Mursaleen et al. first constructed (p, q)-Bernstein operators [2] and (p, q)-Bernstein-Stancu operators [3], several generalizations of well-known positive linear operators based on (p, q)-calculus have been introduced and studied (see [4–11]). In [12], Acar first proposed (p, q)-Szász-Mirakjan operators defined on $[0, \infty)$. In [13], Kara et al. constructed a modified (p, q)-Szász-Mirakjan as follows:

$$S_n^{p,q}(f ; t) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(t) f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right), t \in [0,\infty), \tag{1}$$

where $0 < q < p \le 1, f \in C[0,\infty)$ and $s_{n,k}^{p,q}(t) = (p^{k(k-n)}/q^{k(k-1)/2})([n]_{p,q}^k t^k/[k]_{p,q}!) e_{p,q}(-[n]_{p,q} p^{k-n+1} q^{-k} t)$. Certain basic notations of (p, q)-calculus are mentioned below (for details see [14]): For each real number λ, (p, q)-analogue of λ named $[\lambda]_{p,q}$ is defined by

$$[\lambda]_{p,q} = \frac{p^\lambda - q^\lambda}{p - q}, p \neq q. \tag{2}$$

And for each nonnegative integer n, the (p, q)-integer $[n]_{p,q}$ and (p, q)-factorial $[n]_{p,q}!$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \dfrac{p^n - q^n}{n - n}, & p \neq q ; \\ np^{n-1}, & p = q ; \\ [n]_q, & p = 1 ; \\ n, & p = q = 1, \end{cases}$$

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q} \cdots [n]_{p,q}, & n \ge 1 ; \\ 1, & n = 0. \end{cases} \tag{3}$$

The (p, q)-analogue of the exponential function is defined by

$$e_{p,q}(t) = \sum_{n=0}^{\infty} \frac{p^{n(n-1)/2} t^n}{[n]_{p,q}!}. \tag{4}$$

Let f be an arbitrary function and $a \in \mathbb{R}$. The (p,q)-Jackson integral [15] was defined by

$$\int_0^a f(u)\mathrm{d}_{p,q}u = (p-q)a\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}f\left(\frac{q^i}{p^{i+1}}\right), 0<q<p\le 1. \quad (5)$$

And the (p,q)-Jackson integral over an interval $[a,b]$ $(a<b)$ can be defined by

$$\int_a^b f(u)\mathrm{d}_{p,q}u = \int_0^b f(u)\mathrm{d}_{p,q}u - \int_0^a f(u)\mathrm{d}_{p,q}u. \quad (6)$$

We easily know that (p,q)-Jackson integral (6) is not positive unless it is assumed that f is a nondecreasing function. To solve this problem, Acar et al. [16] defined the (p,q)-integral of the arbitrary function f on interval $[a,b](a<b)$ as follows:

$$\int_a^b f(u)\mathrm{d}_{p,q}u = (p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(a+(b-a)\frac{q^n}{p^{n+1}}\right), 0<q<p\le 1. \quad (7)$$

It is obvious that integral (6) and integral (7) of f on $[0,1]$ are equivalence.

The Kantorovich modification of positive linear operators on $[0,\infty)$ is a method to approximate the Riemann integrable functions. The idea behind the Kantorovich modifications mainly depends on replacing the sample value $f(k/n)$ by $n\int_{k+1/n}^{k/n} f(u)\mathrm{d}u$ (see [17, 18]). By definite integral substitution, we have $n\int_{k+1/n}^{k/n} f(u)\mathrm{d}u = \int_0^1 f(k+u/n)\mathrm{d}u$. However, two Kantorovich modifications may be not equivalence or cannot use definite integral substitution in q-calculus and (p,q)-calculus. For the researches about (p,q)-Szász-Mirakjan-Kantorovich-operators, we can see [19–21]. Meantime, the idea behind the Stancu modifications mainly depends on replacing the sample value $f(k/n)$ by $f(k+\alpha/n+\beta)$ with two parameters $0\le\alpha\le\beta$ (see [22]). For the researches about the Stancu modification of (p,q)-operators, we can see [23, 24]. All these achievements motivate us to construct the Stancu and Kantorovich generalizations of (p,q)-Szász-Mirakjan (1) with three parameters λ,α,β as follows:

Definition 1. For $n\in\mathbb{N}$, $0<q<p\le 1$, $\lambda>0$, $0\le\alpha\le\beta$ and $f\in C[0,\infty)$, the (p, q)-Szász-Mirakjan-Kantorovich-Stancu operators can be defined by

$$S_{n,\alpha,\beta}^{p,q,\lambda}(f;t) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(t)\int_0^1 f\left(\frac{p^{n-k}[k]_{p,q}+u^\lambda+\alpha}{[n]_{p,q}+\beta}\right)\mathrm{d}_{p,q}u, t\in[0,\infty). \quad (8)$$

2. Auxiliary Results

In order to obtain the approximation properties of the operators $S_{n,\alpha,\beta}^{p,q,\lambda}(f;t)$, we need the following lemmas and corollaries.

Lemma 2. *For $t\in[0,\infty)$, $0<q<p\le 1$, $\lambda>0$, we have $\int_0^1 t^\lambda d_{p,q}t = 1/[\lambda+1]_{p,q}$.*

Proof. Using (7),

$$\int_0^1 t^\lambda \mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\lambda+1} = \frac{p-q}{p^{\lambda+1}}\sum_{n=0}^{\infty}\left(\frac{q^{\lambda+1}}{p^{\lambda+1}}\right)^n = \frac{1}{[\lambda+1]_{p,q}}. \quad (9)$$

Lemma 3. *([13], Lemma 4) For $0<q<p\le 1$, $n\in\mathbb{N}$, and $t\in[0,\infty)$, we have*

$$S_n^{p,q}(1;t)=1, S_n^{p,q}(u;t)=t, S_n^{p,q}(u^2;t)=t^2+\frac{p^{n-1}}{[n]_{p,q}}t,$$
$$S_n^{p,q}(u^3;t)=t^3+\frac{(2p+q)p^{n-2}}{[n]_{p,q}}t^2+\frac{p^{2n-2}}{[n]_{p,q}^2}t,$$
$$S_n^{p,q}(u^4;t)=t^4+\frac{(3p^2+2qp+q^2)p^{n-3}}{[n]_{p,q}}t^3 + \frac{(3p^2+3qp+q^2)p^{2n-4}}{[n]_{p,q}^2}t^2+\frac{p^{3n-3}}{[n]_{p,q}^3}t. \quad (10)$$

The following lemma will tell us the relation between the moment of the operators $S_n^{p,q}$ and the moment of the operators $S_{n,\alpha,\beta}^{p,q,\lambda}$:

Lemma 4. *For $t\in[0,\infty)$, $n,m\in\mathbb{N}$, $0<q<p\le 1$, $\lambda>0$, $0\le\alpha\le\beta$, we have the following recursive relation:*

$$S_{n,\alpha,\beta}^{p,q,\lambda}(u^m;t) = \frac{1}{\left([n]_{p,q}+\beta\right)^m}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\frac{m!}{i!j!(m-i-j)!}[n]_{p,q}^i S_n^{p,q}(u^i;t)\frac{\alpha^{m-i-j}}{[\lambda j+1]_{p,q}}. \quad (11)$$

Proof. By direct computation, we have

$$\begin{aligned} S_{n,\alpha,\beta}^{p,q,\lambda}(u^m;t) &= \sum_{k=0}^{\infty}s_{n,k}^{p,q}(t)\int_0^1\left(\frac{p^{n-k}[k]_{p,q}+u^\lambda+\alpha}{[n]_{p,q}+\beta}\right)^m \mathrm{d}_{p,q}u \\ &= \frac{1}{\left([n]_{p,q}+\beta\right)^m}\sum_{k=0}^{\infty}\int_0^1\sum_{i=0}^{m}\sum_{j=0}^{m-i}\frac{m!}{i!j!(m-i-j)!}\left(p^{n-k}[k]_{p,q}\right)^i u^{\lambda j}\alpha^{m-i-j}\mathrm{d}_{p,q}u \\ &= \frac{1}{\left([n]_{p,q}+\beta\right)^m}\sum_{i=0}^{m}\sum_{j=0}^{m-i}[n]_{p,q}^i\left(\sum_{k=0}^{\infty}s_{n,k}^{p,q}(t)\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)^i\right)\frac{\alpha^{m-i-j}}{[\lambda j+1]_{p,q}} \\ &= \frac{1}{\left([n]_{p,q}+\beta\right)^m}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\frac{m!}{i!j!(m-i-j)!}[n]_{p,q}^i S_n^{p,q}(u^i;t)\frac{\alpha^{m-i-j}}{[\lambda j+1]_{p,q}}. \end{aligned} \quad (12)$$

Hence, the proof of Lemma 4 is completed.

Then, the following lemma can be obtain immediately:

Lemma 5. *For $t \in [0,\infty)$, $0 < q < p \le 1$, $\lambda > 0$, $0 \le \alpha \le \beta$, we have*

$$S_{n,\alpha,\beta}^{p,q,\lambda}(1;t)=1,\ S_{n,\alpha,\beta}^{p,q,\lambda}(u;t)=\frac{[n]_{p,q}}{[n]_{p,q}+\beta}t+\frac{1}{[n]_{p,q}+\beta}\left(\frac{1}{[\lambda+1]_{p,q}}+\alpha\right),$$

$$\begin{aligned}S_{n,\alpha,\beta}^{p,q,\lambda}(u^2;t)&=\frac{[n]_{p,q}^2}{\left([n]_{p,q}+\beta\right)^2}\left(t^2+\frac{p^{n-1}}{[n]_{p,q}}t\right)+\frac{2[n]_{p,q}}{\left([n]_{p,q}+\beta\right)^2}\\&\cdot\left(\frac{1}{[\lambda+1]_{p,q}}+\alpha\right)t+\frac{1}{\left([n]_{p,q}+\beta\right)^2}\\&\cdot\left(\frac{1}{[2\lambda+1]_{p,q}}+\frac{2\alpha}{[\lambda+1]_{p,q}}+\alpha^2\right).\end{aligned} \tag{13}$$

Lemma 6. *Under the condition of Lemma 5, we can easily obtain the following formulas for the first and second central moments:*

$$A_{n,\alpha,\beta}^{p,q,\lambda}(t)\coloneqq S_{n,\alpha,\beta}^{p,q,\lambda}(u-t;t)=\frac{1}{[n]_{p,q}+\beta}\left(\frac{1}{[\lambda+1]_{p,q}}+\alpha-\beta t\right),$$

$$\begin{aligned}B_{n,\alpha,\beta}^{p,q,\lambda}(t)&\coloneqq S_{n,\alpha,\beta}^{p,q,\lambda}\left((u-t)^2;t\right)=\frac{p^{n-1}[n]_{p,q}t}{\left([n]_{p,q}+\beta\right)^2}+\frac{1}{\left([n]_{p,q}+\beta\right)^2}\\&\cdot\left(\left(\beta t-\alpha-\frac{1}{[\lambda+1]_{p,q}}\right)^2+\frac{1}{[2\lambda+1]_{p,q}}-\frac{1}{[\lambda+1]_{p,q}^2}\right).\end{aligned} \tag{14}$$

Lemma 7. *The sequences (p_n), (q_n) satisfy $0 < q_n < p_n \le 1$, such that $q_n \longrightarrow 1$, $p_n^n \longrightarrow \eta \in [0,1]$, $[n]_{p_n,q_n} \longrightarrow \infty$ as $n \longrightarrow \infty$; then for any $t \in [0,\infty)$, $0 < q < p \le 1$, $\lambda > 0$, $0 \le \alpha \le \beta$, we have*

$$\lim_{n\to\infty}[n]_{p_n,q_n}A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)=-\beta t+\alpha+\frac{1}{\lambda+1}, \tag{15}$$

$$\lim_{n\to\infty}[n]_{p_n,q_n}B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)=\eta t, \tag{16}$$

$$\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^4;t\right)=0. \tag{17}$$

Proof. By $[\lambda+1]_{p_n,q_n}=(\lambda+1)\xi_n^\lambda$, $\xi_n\in(q_n,p_n)$, we have $\lim_{n\to\infty}[\lambda+1]_{p_n,q_n}=\lambda+1$. Thus, we easily obtain (15) and (16). As $n\longrightarrow\infty$, we can rewrite

$$\begin{aligned}S_n^{p_n,q_n}\left(u^3;t\right)&=t^3+\frac{\left(2+q_np_n^{-1}\right)p_n^{n-1}}{[n]_{p_n,q_n}}t^2+o\left(\frac{1}{[n]_{p_n,q_n}}\right),\\S_n^{p_n,q_n}\left(u^4;t\right)&=t^4+\frac{\left(3+2q_np_n^{-1}+q_n^2p_n^{-2}\right)p_n^{n-1}}{[n]_{p_n,q_n}}t^3+o\left(\frac{1}{[n]_{p_n,q_n}}\right).\end{aligned} \tag{18}$$

Set $A(n)=(1/[\lambda+1]_{p_n,q_n})+\alpha$. Applying Lemma 4 and $([n]_{p_n,q_n}/[n]_{p_n,q_n}+\beta)^i=1-(i\beta/[n]_{p_n,q_n}+\beta)+o(1/[n]_{p_n,q_n})$, $i=1,2,3,4$, we can also rewrite

$$\begin{aligned}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u;t)&=\frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}t+\frac{A(n)}{[n]_{p_n,q_n}+\beta}=\left(1-\frac{\beta}{[n]_{p_n,q_n}+\beta}\right)t\\&+\frac{A(n)}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right),\\S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^2;t)&=\frac{[n]_{p_n,q_n}^2}{\left([n]_{p_n,q_n}+\beta\right)^2}\left(t^2+\frac{p^{n-1}}{[n]_{p_n,q_n}}t\right)+\frac{2[n]_{p_n,q_n}A(n)}{\left([n]_{p_n,q_n}+\beta\right)^2}t\\&=\left(1-\frac{2\beta}{[n]_{p_n,q_n}+\beta}\right)t^2+\frac{p^{n-1}}{[n]_{p_n,q_n}}t\\&+\frac{2A(n)}{[n]_{p_n,q_n}+\beta}t+o\left(\frac{1}{[n]_{p_n,q_n}}\right),\\S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^3;t)&=\frac{[n]_{p_n,q_n}^3}{\left([n]_{p_n,q_n}+\beta\right)^3}\left(t^3+\frac{\left(2+q_np_n^{-1}\right)p_n^{n-1}}{[n]_{p_n,q_n}}t^2\right)\\&+\frac{3[n]_{p_n,q_n}^2A(n)}{\left([n]_{p_n,q_n}+\beta\right)^3}t^2+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\\&=\left(1-\frac{3\beta}{[n]_{p_n,q_n}+\beta}\right)t^3+\frac{\left(2+q_np_n^{-1}\right)p^{n-1}}{[n]_{p_n,q_n}}t^2\\&+\frac{3A(n)}{[n]_{p_n,q_n}+\beta}t^2+o\left(\frac{1}{[n]_{p_n,q_n}}\right),\\S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^4;t)&=\frac{[n]_{p_n,q_n}^4}{\left([n]_{p_n,q_n}+\beta\right)^4}\left(t^4+\frac{\left(3+2q_np_n^{-1}+q_n^2p_n^{-2}\right)p_n^{n-1}}{[n]_{p_n,q_n}}t^3\right)\\&+\frac{4[n]_{p_n,q_n}^3A(n)}{\left([n]_{p_n,q_n}+\beta\right)^4}t^3+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\\&=\left(1-\frac{4\beta}{[n]_{p_n,q_n}+\beta}\right)t^4+\frac{\left(3+2q_np_n^{-1}+q_n^2p_n^{-2}\right)p^{n-1}}{[n]_{p_n,q_n}}t^3\\&+\frac{4\mathrm{A}(n)}{[n]_{p_n,q_n}+\beta}t^3+o\left(\frac{1}{[n]_{p_n,q_n}}\right).\end{aligned} \tag{19}$$

Combining $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}((u-t)^4;t)=\sum_{m=0}^4\binom{4}{m}(-1)^mS_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^{4-m};t)t^m$, we can obtain

$$\begin{aligned}[n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^4;t\right)&=\left(1-2q_np_n^{-1}+q_n^2p_n^{-2}\right)p_n^{n-1}t^3\\&+o(1)\longrightarrow0,\ \text{as}\ n\longrightarrow\infty,\end{aligned} \tag{20}$$

we obtain the required result.

Lemma 8. *Let $C_B[0,\infty)$ be the set of real-valued continuous bounded functions defined on $[0,\infty)$ endowed with the norm*

$\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. *Under the condition of Lemma 5, for any* $f \in C_B[0,\infty)$, *we have*

$$\left\| S_{n,\alpha,\beta}^{p,q,\lambda}(f\,;t) \right\| \le \|f\|. \tag{21}$$

Proof. In view of (8) and Lemma 5, the proof of this lemma can be obtained easily.

3. Local Approximation

In this section, we will establish local approximation theorem for the operators. For any $f \in C_B[0,\infty)$, we consider the following $\mathscr{K}$-functional:

$$\mathscr{K}(f\,;\delta) = \inf_{h\in \mathbf{W}^2} \left\{ \|f-h\| + \delta\|h''\| \right\}, \tag{22}$$

where $\delta \in (0,\infty)$ and $\mathbf{w}^2 = \{h \in C_B[0,\infty)\colon h', h'' \in C_B[0,\infty)\}$. The usual modulus of continuity and the second-order modulus of smoothness of f can be defined as

$$\begin{aligned} \omega(f\,;\delta) &= \sup_{0<|u|<\delta} \sup_{x\in[0,\infty)} |f(t+u)-f(t)|, \\ \omega_2(f\,;\delta) &= \sup_{0<|u|<\delta} \sup_{x\in[0,\infty)} |f(t+2u)-f(t+u)+f(t)|. \end{aligned} \tag{23}$$

By ([25], p.177, Theorem 2.4), there exists an absolute positive constant C such that

$$\mathscr{K}(f\,;\delta) \le C\omega_2\left(f\,;\sqrt{\delta}\right), \delta > 0. \tag{24}$$

In the meantime, for $f \in C_B[0,\infty)$ and $h > 0$, the Steklov mean is defined as

$$f_h(t) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2} [2f(t+u+v) - f(t+2(u+v))]\mathrm{d}u\mathrm{d}v. \tag{25}$$

Thus, $f_h \in C_B[0,\infty)$, and we can write

$$\begin{aligned} f_h(t) - f(t) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2} &[2f(t+u+v) \\ &- f(t+2(u+v)) - f(t)]\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{26}$$

It is obvious that $|f_h(t)-f(t)| \le \omega_2(f\,;h)$ and $\|f_h - f\| \le \omega_2(f\,;h)$. If f is continuous, then $f_h', f_h'' \in C_B[0,\infty)$ and

$$\begin{aligned} f_h'(t) = \frac{4}{h^2}&\left[2\int_0^{h/2}\left(f\left(t+u+\frac{h}{2} - f(t+u)\right)\right)\mathrm{d}u \right.\\ &- \left.\frac{1}{2}\int_0^{h/2} (f(t+h+2u) - f(t+2u))\mathrm{d}u\right]. \end{aligned} \tag{27}$$

Thus, we have $\|f_h'\| \le (5/h)\omega(f\,;h)$. Similarly, $\|f_h''\| \le (9/h^2)\omega_2(f\,;h)$.

Theorem 9. *Under the condition of Lemma 7, then for all* $f \in C_B[0,\infty)$ *and* $t \in [0,\infty)$, *we have*

$$\left| S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t) - f(t) \right| \le 2\omega\left(f\,;\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right). \tag{28}$$

Proof. For any $\delta > 0$, we have $|f(u)-f(t)| \le \omega(f;|u-t|) \le (1 + (|u-t|/\delta))\omega(f\,;\delta)$. Applying $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ to both ends and using Lemma 5, we can obtain

$$\begin{aligned} \left| S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t) - f(t) \right| &\le S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f(u)-f(t)|\,;t) \\ &\le \left(1 + \frac{1}{\delta} S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|;t)\right)\omega(f\,;\delta). \end{aligned} \tag{29}$$

By using the Chauchy-Schwarz inequality and taking $\delta = \sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}$, we have

$$\begin{aligned} \left| S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t) - f(t) \right| &\le \left(1 + \frac{1}{\delta}\sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}((u-t)^2\,;t)}\right)\omega(f\,;\delta) \\ &\le 2\omega\left(f\,;\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right). \end{aligned} \tag{30}$$

Theorem 9 is proved.

Theorem 10. *Under the condition of Lemma 7, then for all* $f \in C_B[0,\infty)$ *and* $t \in [0,\infty)$, *there exists an absolute positive constant* $C_1 = 4C$ *such that*

$$\begin{aligned} \left| S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t) - f(t) \right| &\le C_1\omega_2\left(f\,;\sqrt{\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^2 + B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right) \\ &\quad + \omega\left(f\,;\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\right). \end{aligned} \tag{31}$$

Proof. First, we define the following new positive linear operators as follows:

$$\begin{aligned} T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t) = S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t) &- f\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t) + t\right) \\ &+ f(t), t \in [0,\infty). \end{aligned} \tag{32}$$

It is apparent from Lemma 5, Lemma 6, and Lemma 8 that

$$T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(1\,;t) = 1\,; T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u-t\,;t) = 0, \tag{33}$$

$$\left\| T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t) \right\| \le 3\|f\|. \tag{34}$$

Now for any given function $h \in \mathbf{W}^2$ and $u, t \in [0,\infty)$, we write Taylor's expansion formula as follows:

$$h(u) = h(t) + h'(t)(u-t) + \int_t^u h''(v)(u-v)\mathrm{d}v. \tag{35}$$

By applying $T_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ operators to both sides of the above equality, we can obtain

$$\begin{aligned} T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(h;t) &= T_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(h(t) + h'(t)(u-t) + \int_t^u h''(v)(u-v)\mathrm{d}v; t\right) \\ &= h(t) + T_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(h'(t)(u-t); t\right) \\ &\quad + T_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\int_t^u h''(v)(u-v)\mathrm{d}v; t\right). \end{aligned} \tag{36}$$

Using (32), (33), and the following inequality,

$$\begin{aligned} \left|\int_t^u h''(v)(u-v)\mathrm{d}v\right| &\le \left|\int_t^u |h''(v)||u-v|\mathrm{d}v\right| \\ &\le \|h''\|\left|\int_t^u |u-v|\mathrm{d}v\right| \\ &\le (u-t)^2\|h''\|, \end{aligned} \tag{37}$$

we can get

$$\begin{aligned} \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(h;t) - h(t)\right| &= \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\int_t^u h''(v)(u-v)\mathrm{d}v; t\right)\right| \\ &\le S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\left|\int_t^u h''(v)\,|u-v|\,\mathrm{d}v\right|; t\right) \\ &\quad + \left|\int_t^{A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)+t} h''(v)\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t) + t - v\right)\mathrm{d}v\right| \\ &\le S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2; t\right)\|h''\| + \left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^2\|h''\| \\ &= \left(\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^2 + B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)\|h''\|. \end{aligned} \tag{38}$$

By using (32) and (34), we have

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(h;t) - h(t)\right| &= \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) + f\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t) + t\right) - 2f(t)\right| \\ &\le \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f-h;t) - (f-g)(t)\right| \\ &\quad + \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(h;t) - h(t)\right| \\ &\quad + \left|f\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t) + t\right) - f(t)\right| \\ &\le 4\|f-h\| + \left(\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^2 + B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)\|h''\| \\ &\quad + \omega\left(f; \left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\right). \end{aligned} \tag{39}$$

Taking the infimum on the right-hand side over all $h \in \mathbf{W}^2$ and using (24), we complete the proof of Theorem 10.

Theorem 11. *Under the condition of Lemma 7, then for all $f' \in C_B[0,\infty)$ and $t \in [0,\infty)$, we have*

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right| &\le \left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\left|f'(t)\right| + 2\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\,\omega \\ &\quad \cdot \left(f'; \sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right). \end{aligned} \tag{40}$$

Proof. Applying $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ to both sides of the equality $f(u) = f(t) + f'(t)(u-t) + f(u) - f(t) - f'(t)(u-t)$, using mean value theorem and the Chauchy-Schwarz inequality and taking $\delta = \sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}$, we can obtain

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right| &\le \left|f'(t)\right|\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u-t;t)\right| + S_{n,\alpha,\beta}^{p_n,q_n,\lambda} \\ &\quad \cdot \left(\left|f(u) - f(t) - f'(t)(u-t)\right|; t\right) \\ &\le |f'(t)|\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u-t;t)\right| + S_{n,\alpha,\beta}^{p_n,q_n,\lambda} \\ &\quad \cdot \left(|u-t|\left(1 + \frac{|u-t|}{\delta}\right)\omega\left(f';\delta\right); t\right) \\ &\le |f'(t)|\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right| + \omega\left(f';\delta\right) \\ &\quad \cdot \left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|;t) + \frac{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)}{\delta}\right) \\ &\le |f'(t)|\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right| + \omega\left(f';\delta\right) \\ &\quad \cdot \sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)} \\ &\quad \cdot \left(1 + \frac{\sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)}}{\delta}\right) \\ &\le \left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\left|f'(t)\right| + 2\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\,\omega \\ &\quad \cdot \left(f'; \sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right). \end{aligned} \tag{41}$$

Theorem 12. *Under the condition of Lemma 7, if $f \in C_B[0,\infty)$, then*

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right| &\le 5\sqrt{[n]_{p_n,q_n}}\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\omega\left(f; \frac{1}{\sqrt{[n]_{p_n,q_n}}}\right) \\ &\quad + \left(\frac{9}{2}[n]_{p_n,q_n} B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t) + 2\right)\omega_2 \\ &\quad \cdot \left(f; \frac{1}{\sqrt{[n]_{p_n,q_n}}}\right). \end{aligned} \tag{42}$$

Proof. For $t \in [0,\infty)$, using the Steklov mean function f_h, we can write

$$\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right| \le S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f-f_h|;t)+\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f_h-f_h(t);t)\right| + |f_h(t)-f(t)|. \tag{43}$$

By Lemma 8 and properties of the Steklov mean, we can obtain

$$S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f-f_h|;t) \le \|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f-f_h|)\| \le \|f-f_h\| \le \omega_2(f;h). \tag{44}$$

By Taylor's expansion formula, we have

$$\begin{aligned}\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f_h-f_h(t);t)\right| &\le |f_{h'}(t)|\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|+\frac{1}{2}\|f_h{''}\|B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\\ &\le \frac{5}{h}\omega(f;h)\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\\ &\quad+\frac{9}{2h^2}\omega_2(f;h)B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t).\end{aligned} \tag{45}$$

Hence,

$$\begin{aligned}\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right| &\le \frac{5}{h}\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\omega(f;h)\\ &\quad+\left(\frac{9}{2h^2}B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)+2\right)\omega_2(f;h).\end{aligned} \tag{46}$$

Setting $h=1/\sqrt{[n]_{p_n,q_n}}$, we can get the desired result.

By the classic Korovkin theorem, we easily get the following corollary:

Corollary 13. *Under the condition of Lemma 7, then for all $f \in C_B[0,\infty)$ and any $A>0$, the the sequence $\{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)\}$ converges to f uniformly on $[0,A]$.*

4. Voronovskaja-Type Theorem for $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$

In this section, we show a Voronovskaja-type asymptotic formula for the operators $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ by means of the first, second and fourth central moments.

Theorem 14. *Under the condition of Lemma 7, then for all $f \in C_B[0,\infty)$ satisfying $f''(t)$ that exists at a point $t\in[0,\infty)$, we can obtain*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right)=\left(\alpha-\beta t+\frac{1}{\lambda+1}\right)f'(t)+\frac{\eta}{2}f''(t)t. \tag{47}$$

Proof. By Taylor's expansion formula for f, we have

$$f(u)=f(t)+f'(t)(u-t)+\frac{1}{2}f''(t)(u-t)^2+\phi(u;t)(u-t)^2, \tag{48}$$

where

$$\phi(u;t)=\begin{cases}\dfrac{f(u)-f(t)-f'(t)(u-t)-1/2f''(t)(u-t)^2}{(u-t)^2}, & u\ne t;\\ 0, & u=t.\end{cases} \tag{49}$$

Applying L'Hospital's Rule,

$$\lim_{u\to t}\phi(u;t)=\frac{1}{2}\lim_{u\to t}\frac{f'(u)-f'(t)}{u-t}-\frac{1}{2}f''(t)=0. \tag{50}$$

Thus, $\phi(.;t)\in C_B[0,\infty)$. Consequently, we can write

$$\begin{aligned}[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(x)\right) &= [n]_{p_n,q_n}A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)+\frac{1}{2}[n]_{p_n,q_n}B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\\ &\quad+[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi(u;t)(u-t)^2;t\right)\right).\end{aligned} \tag{51}$$

By Schwarz's inequality, we have

$$\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi(u;t)(u-t)^2;t\right)\le\sqrt{\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi^2(u;t);t\right)\right.}\cdot\sqrt{\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^4;t\right)\right.}. \tag{52}$$

We observe that $\phi^2(t;t)=0$ and $\phi^2(.;t)\in C_B[0,\infty)$. Then, it follows in Corollary 13 that

$$\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi^2(u;t);t\right)=\phi^2(t;t)=0. \tag{53}$$

Hence, from (17), we can obtain

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi(u;t)(u-t)^2;t\right)\right)=0. \tag{54}$$

Combining, we complete the proof of Theorem 14.

Corollary 15. *Under the condition of Lemma 7, then for all $f', f''\in C_B[0,\infty)$, we have*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right)=\left(\alpha-\beta t+\frac{1}{\lambda+1}\right)f'(t)+\frac{\eta}{2}f''(t)t, \tag{55}$$

uniformly with respect to any finite interval $I\subset[0,\infty)$.

5. Pointwise Estimates

In this section, we establish two pointwise estimates of the operators $S^{p,q,\lambda}_{n,\alpha,\beta}$. First, we compute the rate of convergence locally by using functions belonging to the Lipschitz class. We denote that $f \in C_B[0,\infty)$ is in $\mathrm{Lip}_M(\gamma, D)$, $\gamma \in (0,1]$, $D \subset [0,\infty)$ if it satisfies the following condition:

$$|f(u) - f(t)| \le M|u-t|^{\gamma}, u \in D, t \in [0,\infty), \tag{56}$$

where M is a positive constant depending only on γ and f.

Theorem 16. *The sequences (p_n), (q_n) satisfy $0 < q_n < p_n \le 1$, $\gamma \in (0,1]$ and D be any bounded subset on $[0,\infty)$. If $f \in C_B[0,\infty) \cap Lip_M(\gamma, D)$, then for any $t \in [0,\infty)$, we have*

$$\left| S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(f\,;t) - f(x) \right| \le M\left(\left(B^{p_n,q_n,\lambda}_{n,\alpha,\beta}(t) \right)^{\gamma/2} + 2d^{\gamma}(t\,;D) \right), \tag{57}$$

where $d(t\,;D) = \inf\{|u-t| : u \in D\}$ denotes the distance between t and D.

Proof. Let $\bar{D}$ be the closure of D. Using the properties of infimum, and there is at least a point $t_0 \in \bar{D}$ such that $\mathrm{d}(t\,;E) = |t - t_0|$. By the triangle inequality

$$|f(u) - f(t)| \le |f(u) - f(t_0)| + |f(t) - f(t_0)|. \tag{58}$$

By the monotonicity of $S^{p_n,q_n,\lambda}_{n,\alpha,\beta}$, we get

$$\begin{aligned} \left| S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(f\,;t) - f(x) \right| &\le S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(|f(u) - f(t_0)|;t) + S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(|f(t) - \mathrm{f}(t_0)|;t) \\ &\le M\left\{ S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(|u-t_0|^{\gamma}\,;t) + S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(|t-t_0|^{\gamma}\,;t) \right\} \\ &\le M\left\{ S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(|u-t|^{\gamma} + |t-t_0|^{\gamma}\,;t) + |t-t_0|^{\gamma} \right\} \\ &= M\left\{ S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(|u-t|^{\gamma}\,;t) + 2|t-t_0|^{\gamma} \right\}. \end{aligned} \tag{59}$$

Applying the well-known Hölder inequality with $a_1 = 2/\gamma$, $a_2 = 2/2-\gamma$, we obtain

$$\begin{aligned} \left| S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(f\,;t) - f(x) \right| &\le M\left\{ S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(|u-t|^{a_1\gamma}\,;t)^{1/a_1} + 2d^{\gamma}(t\,;D) \right\} \\ &\le M\left\{ S^{p_n,q_n,\lambda}_{n,\alpha,\beta}\left(|u-t|^{2}\,;t\right)^{1/a_1} + 2d^{\gamma}(t\,;D) \right\} \\ &= M\left\{ \left(B^{p_n,q_n,\lambda}_{n,\alpha,\beta}(t) \right)^{\gamma/2} + 2d^{\gamma}(t\,;D) \right\}. \end{aligned} \tag{60}$$

Second, we will give a local direct estimation of the operators $S^{p_n,q_n,\lambda}_{n,\alpha,\beta}$ by using the Lipschitz-type maximal function of the order γ introduced by Lenze [26] as

$$\tilde{\omega}_{\gamma}(f\,;t) = \sup_{u \ne t, u \in [0,\infty)} \frac{|f(u) - f(t)|}{|u-t|^{\gamma}}, t \in [0,\infty) \text{ and } \gamma \in (0,1]. \tag{61}$$

Theorem 17. *The sequences (p_n), (q_n) satisfy $0 < q_n < p_n \le 1$ and $\gamma \in (0,1]$. If $f \in C_B[0,\infty)$, then for any $t \in [0,\infty)$, we have*

$$\left| S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(f\,;t) - f(t) \right| \le \tilde{\omega}_{\gamma}(f\,;t)\left(B^{p_n,q_n,\lambda}_{n,\alpha,\beta}(t) \right)^{\gamma/2}. \tag{62}$$

Proof. Using the equality (61), we obtain

$$\left| S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(f\,;t) - f(t) \right| \le \tilde{\omega}_{\gamma}(f\,;t) S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(|u-t|^{\gamma}\,;t). \tag{63}$$

By the well-known Hölder inequality, we have

$$\begin{aligned} \left| S^{p_n,q_n,\lambda}_{n,\alpha,\beta}(f\,;t) - f(t) \right| &\le \tilde{\omega}_{\gamma}(f\,;t) S^{p_n,q_n,\lambda}_{n,\alpha,\beta}\left(|u-t|^{2}\,;t\right)^{\gamma/2} \\ &\le \tilde{\omega}_{\gamma}(f\,;t)\left(B^{p_n,q_n,\lambda}_{n,\alpha,\beta}(t) \right)^{\gamma/2}. \end{aligned} \tag{64}$$

Thus, the proof of Theorem 17 is completed.

6. Rate of Convergence

Let $B_2[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(t)| \le C_f(1+t^2)$ with an absolute constant $C_f > 0$ which may depend only on f. $C_2[0,\infty)$ denotes the subspace of all continuous functions $f \in B_2[0,\infty)$ with the norm $\|f\|_2 = \sup_{x \in [0,\infty)} |f(t)|/1+t^2$. By $C_2^0[0,\infty)$, and we denote the subspace of all functions $f \in C_2[0,\infty)$ for which $\lim_{x \to +\infty} |f(t)|/1+t^2$ is finite. Meantime, we denote the modulus of continuity of f on the interval $[0,a]$, $a > 0$ by

$$\omega_a(f\,;\delta) = \sup_{|u-t| \le \delta} \sup_{u,t \in [0,a]} |f(u) - f(t)|. \tag{65}$$

Theorem 18. *Let $f \in C_2[0,\infty)$, $0 < q < p \le 1$, and $a > 0$. Then, for all $t \in [0,a]$, we have*

$$\left| S^{p,q,\lambda}_{n,\alpha,\beta}(f\,;t) - f(t) \right| \le C_f\left(4 + 3a^2\right) B^{p,q,\lambda}_{n,\alpha,\beta}(t) + 2\omega_{a+1}\left(f\,; \sqrt{B^{p,q,\lambda}_{n,\alpha,\beta}(t)} \right). \tag{66}$$

Proof. For any $t \in [0,a]$ and $u > a+1$, we easily have $1 \le (u-a)^2 \le (u-t)^2$; thus

$$\begin{aligned} |f(u) - f(t)| &\le |f(u)| + |f(t)| \le C_f\left(2 + u^2 + t^2\right) \\ &= C_f\left(2 + t^2 + (u-t+t)^2\right) \le C_f\left(2 + 3t^2 + 2(u-t)^2\right) \\ &\le C_f\left(4 + 3t^2\right)(u-t)^2 \le M_f\left(4 + 3a^2\right)(u-t)^2, \end{aligned} \tag{67}$$

and for any $t \in [0, a]$, $u \in [0, a+1]$) and $\delta > 0$, we have

$$|f(u) - f(t)| \le \omega_{a+1}(|u-t|;t) \le \left(1 + \frac{|u-t|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{68}$$

For (67) and (68), we can get

$$|f(u) - f(t)| \le C_f\left(4 + 3a^2\right)(u-t)^2 + \left(1 + \frac{|u-t|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{69}$$

Applying the Cauchy-Schwarz inequality and choosing $\delta = \sqrt{B_{n,\alpha,\beta}^{p,q,\lambda}(t)}$, we have

$$\begin{aligned}\left|S_{n,\alpha,\beta}^{p,q,\lambda}(f;t) - f(t)\right| &\le S_{n,\alpha,\beta}^{p,q,\lambda}(|f(u) - f(t)|;t)\\ &\le C_f\left(4 + 3a^2\right)S_{n,\alpha,\beta}^{p,q,\lambda}\left((u-t)^2;t\right)\\ &\quad + S_{n,\alpha,\beta}^{p,q,\lambda}\left(\left(1 + \frac{|u-t|}{\delta}\right);t\right)\omega_{a+1}(f;\delta)\\ &\le C_f\left(4 + 3a^2\right)B_{n,\alpha,\beta}^{p,q,\lambda}(t) + \omega_{a+1}(f;\delta)\\ &\quad \cdot \left(1 + \frac{\sqrt{B_{n,\alpha,\beta}^{p,q,\lambda}(t)}}{\delta}\right) = C_f\left(4 + 3a^2\right)B_{n,\alpha,\beta}^{p,q,\lambda}(t)\\ &\quad + 2\omega_{a+1}\left(f;\sqrt{B_{n,\alpha,\beta}^{p,q,\lambda}(t)}\right).\end{aligned} \tag{70}$$

This completes the proof of Theorem 18.

7. Weighted Approximation

As is known, if $f \in C[0,\infty)$ is not uniform, the limit $\lim_{\delta\to0^+}\omega(f;\delta) = 0$ may be not true. In [27], Ispir defined the following weighted modulus of continuity:

$$\Omega(f;\delta) = \sup_{t\in[0,\infty),0<h\le\delta}\frac{|f(t+h) - f(t)|}{\left(1+t^2\right)\left(1+h^2\right)} \text{ for } f \in C_2^0[0,\infty), \tag{71}$$

and proved the properties of monotone increasing about $\Omega(f;\delta)$ as $\delta > 0$, $\lim_{\delta\to0^+}\Omega(f;\delta) = 0$ and the inequality

$$\Omega(f;\tau\delta) \le 2(1+\tau)\left(1+\delta^2\right)\Omega(f;\delta), \tau > 0. \tag{72}$$

Theorem 19. *Under the condition of Lemma 7, $f \in C_2^0[0,\infty)$, then for sufficiently large n, the inequality*

$$\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right| \le K\left(1+t^2\right)^{2+\theta}\Omega\left(f;\frac{1}{[n]_{p_n,q_n}}\right) \tag{73}$$

holds, where $\theta \ge 1/2$ and K is a positive constant depending only on f and n.

Proof. Applying (71) and (72), we can obtain

$$\begin{aligned}|f(u) - f(t)| &\le \left(1 + (u-t)^2\right)\left(1+t^2\right)\Omega(f;|u-t|)\\ &\le 2\left(1 + \frac{|u-t|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta)\left(1 + (u-t)^2\right)\left(1+t^2\right)\\ &\le \begin{cases} 4\left(1+\delta^2\right)^2\left(1+t^2\right)\Omega(f;\delta), & |u-t| \le \delta,\\ 4\left(1+\delta^2\right)\left(1+t^2\right)\Omega(f;\delta)\dfrac{|u-t|+|u-t|^3}{\delta}, & |u-t| > \delta.\end{cases}\end{aligned} \tag{74}$$

Thus, for any $\delta \in (0, 1/2)$ and $u, t \in [0,\infty)$, the above inequality can be rewritten

$$|f(u) - f(t)| \le 5\left(1+t^2\right)\Omega(f;\delta)\left(\frac{5}{4} + \frac{|u-t|+|u-t|^3}{\delta}\right). \tag{75}$$

Applying (16) and (17), there exists sufficiently large n such that

$$\begin{aligned}[n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right) &\le K_1^2\left(1+t^2\right),\\ [n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^4;t\right) &\le K_2^2\left(1+t^2\right)^2.\end{aligned} \tag{76}$$

By Schwarz's inequality, we can obtain

$$\begin{aligned}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|;t) &\le \sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)}\\ &\le \frac{K_1}{[n]_{p_n,q_n}}\sqrt{1+t^2},\\ S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(|u-t|^3;t\right) &\le \sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^4;t\right)}\\ &\le \frac{K_2}{[n]_{p_n,q_n}}\sqrt{\left(1+t^2\right)^3}.\end{aligned} \tag{77}$$

Using $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ as linear and positive and choosing $\delta = 1/[n]_{p_n,q_n}$, we can obtain

$$\begin{aligned}\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right| &\le 5\left(1+t^2\right)\Omega\left(f;\frac{1}{[n]_{p_n,q_n}}\right)\\ &\quad \cdot \left(\frac{5}{4} + K_1\left(1+t^2\right) + K_2\sqrt{\left(1+t^2\right)^3}\right)\\ &\le K\left(1+t^2\right)^{5/2}\Omega\left(f;\frac{1}{[n]_{p_n,q_n}}\right),\end{aligned} \tag{78}$$

for sufficiently large n and $t \in [0,\infty)$, where $K := 5\max\{5/4, K_1, K_2\}$.

Theorem 20. *Under the condition of Lemma 7, then for any $f \in C_2^0[0,\infty)$, we have*

$$\lim_{n\to\infty} \|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f\|_2 = 0. \tag{79}$$

Proof. Applying the Korovkin theorem [28], we only see that it is sufficient to prove the following three conditions:

$$\lim_{n\to\infty} \|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^m;t) - t^m\|_2 = 0, m = 0, 1, 2. \tag{80}$$

Since $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(1;t) = 1$, the condition holds for $m = 0$. By Lemma 6, we can obtain

$$\begin{aligned}\lim_{n\to\infty}\left\|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u;t)-t\right\|_2 &= \lim_{n\to\infty}\left\|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right\|_2 \\ &\le \frac{1}{[n]_{p_n,q_n}+\beta}\left(\left(\frac{1}{[\lambda+1]_{p_n,q_n}}+\alpha\right)\sup_{t\in[0,\infty)}\frac{1}{1+t^2} + \beta\sup_{t\in[0,\infty)}\frac{t}{1+t^2}\right) \longrightarrow 0, \text{as} n \longrightarrow \infty.\end{aligned} \tag{81}$$

Hence, (80) holds for $m = 1$. Similarly, by Lemma 5, we can write for $m = 2$,

$$\begin{aligned}\lim_{n\to\infty}\|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^2;t)-t^2\|_2 &\le \left|\frac{[n]_{p_n,q_n}^2}{\left([n]_{p_n,q_n}+\beta\right)^2}-1\right|\sup_{t\in[0,\infty)}\frac{t^2}{1+t^2}+\frac{[n]_{p_n,q_n}}{\left([n]_{p_n,q_n}+\beta\right)^2} \\ &\cdot\left(p_n^{n-1}+\frac{2}{[\lambda+1]_{p_n,q_n}}+2\alpha\right)\sup_{t\in[0,\infty)}\frac{t}{1+t^2}+\frac{1}{\left([n]_{p_n,q_n}+\beta\right)^2} \\ &\cdot\left(\frac{1}{[2\lambda+1]_{p_n,q_n}}+\frac{2\alpha}{[\lambda+1]_{p_n,q_n}}+\alpha^2\right)\sup_{t\in[0,\infty)}\frac{1}{1+t^2} \longrightarrow 0, \text{as} n \longrightarrow \infty.\end{aligned} \tag{82}$$

Thus, (80) holds for $m = 2$. Hence, the proof is completed.

Theorem 21. *Under the condition of Lemma 7, then for any $f \in C_2^0[0,\infty)$ and $\kappa > 0$, we have*

$$\lim_{n\to\infty}\sup_{t\in[0,\infty)}\frac{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)}{(1+t^2)^{1+\kappa}} = 0. \tag{83}$$

Proof. Let $t_0 \in (0,\infty)$ be arbitrary but fixed.

$$\begin{aligned}\sup_{t\in[0,\infty)}\frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right|}{(1+t^2)^{1+\kappa}} &\le \sup_{t\in[0,t_0)}\frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right|}{(1+t^2)^{1+\kappa}} + \sup_{t\in[t_0,\infty)}\frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right|}{(1+t^2)^{1+\kappa}} \\ &\le \|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f\|_{[0,t_0)}+\|f\|_2\sup_{t\in[t_0,\infty)}\frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(1+u^2;t)\right|}{(1+t^2)^{1+\kappa}} \\ &+ \sup_{t\in[t_0,\infty)}\frac{|f(t)|}{(1+t^2)^{1+\kappa}} := I_1+I_2+I_3.\end{aligned} \tag{84}$$

Applying $|f(t)| \le \|f\|_2(1+t^2)$, we have

$$I_3 = \sup_{t\in[t_0,\infty)}\frac{|f(t)|}{(1+t^2)^{1+\kappa}} \le \sup_{t\in[t_0,\infty)}\frac{\|f\|_2(1+t^2)}{(1+t^2)^{1+\kappa}} \le \frac{\|f\|_2}{(1+t_0^2)^\kappa}. \tag{85}$$

Let $\varepsilon > 0$. By Lemma 5, there exists $N_1 \in \mathbb{N}$, such that for all $n > N_1$:

$$\begin{aligned}\frac{\|f\|_2\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(1+u^2;t)\right|}{(1+t^2)^{1+\kappa}} &\le \frac{\|f\|_2}{(1+t^2)^{1+\kappa}}\left((1+t^2)+\frac{\varepsilon}{3\|f\|_2}\right) \\ &\le \frac{\|f\|_2}{(1+t^2)^\kappa}+\frac{\varepsilon}{3}.\end{aligned} \tag{86}$$

Hence

$$\|f\|_2 \sup_{t\in[t_0,\infty)} \frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(1+u^2;t\right)\right|}{\left(1+t^2\right)^{1+\kappa}} \le \frac{\|f\|_2}{\left(1+t_0^2\right)^{\kappa}} + \frac{\varepsilon}{3}, \forall n \ge N_1. \tag{87}$$

Thus

$$I_2 + I_3 < \frac{2\|f\|_2}{\left(1+t_0^2\right)^{\kappa}} + \frac{\varepsilon}{3}, \forall n \ge N_1. \tag{88}$$

Next, for sufficiently large t_0 such that $\|f\|_2/(1+t_0^2)^{\kappa} < \varepsilon/6$. Then, $I_2 + I_3 < 2\varepsilon/3, \forall n \ge N_1$. Applying Corollary 13, there exists $N_2 \in \mathbb{N}$, such that for all $n > N_2$,

$$\left\|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f\right\|_{[0,t_0)} < \frac{\varepsilon}{3}. \tag{89}$$

Let $N = \max\{N_1, N_2\}$. Combining (86), (88), and (89), we have

$$\sup_{t\in[0,\infty)} \frac{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)}{\left(1+t^2\right)^{1+\kappa}} < \varepsilon, \forall n \ge N. \tag{90}$$

Hence, the proof of Theorem 21 is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Unified Framework of Approximating and Interpolatory Subdivision Schemes for Construction of Class of Binary Subdivision Schemes

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In this paper, a generalized algorithm to develop a class of approximating binary subdivision schemes is presented. The proposed algorithm is based on three-point approximating binary and four-point interpolating binary subdivision schemes. It contains a parameter which classifies members of the new class of subdivision schemes. A set of efficient properties, for instance, polynomial generation and reproduction, support, continuity, and Hölder continuity, is discussed. Moreover, applications of the proposed subdivision schemes are given in order to demonstrate their variety, flexibility, and visual performance.

1. Introduction

Subdivision is a competent way of producing smooth curves or surfaces in geometric modeling and computer graphics. It repeatedly refines the initial polygonal shape. After each split average step, we get closer to the limit curve, which is the limit of an infinite series. A nice property of subdivision schemes is that they are simple and local, which means that local change in initial data will only have a local effect in the resulting object. Subdivision schemes have become celebrated because of their simplicity and efficiency. There are generally two main categories of subdivision schemes: interpolatory and approximating. For interpolating subdivision schemes, limit curve always passes through initial control points while for approximating subdivision schemes it may or may not. Subdivision schemes play an integral role in computer graphics due to their wide range of applications in many fields such as engineering, medical science, space science, graphic visualization, and image processing. Differential equations are used for mathematical modeling of many phenomena. Different techniques are being used to solve boundary value problems [1] and nonlinear problems [2]. Nowadays, subdivision schemes are also becoming a popular tool to numerically solve boundary value problems [3]. Subdivision algorithms are also a major field in many multiscale techniques applied in data compression. In some applications, the given data need not be reproduced at each step of the subdivision process, which needs the applications of subdivision schemes. Several researchers in the area of continuous geometry have been established classical subdivision schemes for various kinds of initial control data. In geometric modeling and engineering, practical applications of subdivision curves are restricted due to their shortcomings, and to overcome these shortcomings, a lot of work has been carried out [4–17].

Deslauriers and Dubuc [18] presented a family of interpolating binary subdivision schemes. They used Lagrange interpolating polynomial for construction of schemes. Horman and Sabin (*HS*) [19] proposed a family of binary subdivision schemes having cubic precision. Some members of the *HS* family are interpolating and some are approximating. Mustafa et al. [20] offered a family of binary subdivision schemes which has alternating primal and dual symbols. Ashraf et al. [21] discussed a family of binary subdivision schemes based on Lane-Riesenfeld algorithm. Members of the proposed family have quintic precision. Mustafa and Bari [22] developed a family of univariate subdivision schemes for curve generation and data fitting. Asghar and Mustafa [23] presented a unified framework of stationary and nonstationary subdivision schemes. Keeping in view this practice, we present a generalized algorithm to develop a new class of approximating binary subdivision schemes. Ghaffar et al. [24–29] constructed geometric continuity conditions for the construction of free-form generalized subdivision curves with single shape parameter. These free-form complex shape adjustable generalized curves can be obtained by using shape-adjustable generalized subdivision schemes. These newly proposed approaches not only take over the benefits of classical subdivision curve and surface schemes but also resolve the issue of shape adjustability of subdivision curves and surfaces with the help of tension control shape parameters. They modeled some complex curves and surfaces using higher continuity conditions. The proposed masks of the schemes provide an alternative approach to generate the complex curves using higher continuity conditions with simple and straightforward calculation for the proposed algorithm because they are blended with linear polynomials rather than trigonometric functions. In 2020, Ashraf et al. [17, 30, 31] proposed a new approach using the generalized hybrid subdivision curve with shape parameters to solve the problem in construction of some symmetric curves and surfaces. These curves are easily modified by the changing the values of shape parameters.

In this paper, we offer a Lane-Riesenfeld-like algorithm to derive a class of binary approximating subdivision schemes. Our algorithm is based on the well-known four-point interpolating binary subdivision scheme [18], which is C^1 continuous, and three-point approximating binary subdivision scheme [32], which has C^3 continuity. Considering ϕ smoothing stages as in the Lane-Riesenfeld algorithm, our proposed algorithm allows us to derive a class of univariate subdivision schemes. In fact, each member of the proposed class is enumerated by ϕ, and higher values of ϕ give schemes with wider masks and support, higher continuity, higher Holder regularity, and higher degree of polynomial generation. The first member of the proposed class (corresponding to $\phi = 0$) coincides with the three-point approximating binary subdivision scheme [32]. The proposed class of schemes generates schemes of higher continuities and visually more smooth limit curves as compared to existing families of schemes. The content of the paper is structured as follows. In Section 2, fundamental definitions and concepts are given. Section 3 presents a generalized algorithm for construction of new class of subdivision schemes. Section 4 is devoted for properties of proposed schemes, such as continuity, Hölder continuity, and support of basic limit function. Geometrical analysis and some beautiful examples of limit curve are given in Section 5. Section 6 presents a summary of the paper.

2. Preliminaries

Let the initial data be given by a set of control points $\mathbf{G}^0 = \{g_i^0 \in \mathbb{R}, i \in \mathbb{Z}\}$, and the set of control points at refinement level $h(h \geq 0, h \in \mathbb{N})$ is given by $\mathbf{G}^h = \{g_i^h \in \mathbb{R}, i \in \mathbb{Z}\}$. Define $\mathbf{G}^{h+1} = \{g_i^{h+1} \in \mathbb{R}, i \in \mathbb{Z}\}$ recursively by the following binary refinement rules:

$$g_i^{h+1} = \sum_{k \in \mathbb{Z}} b_{i-2k} g_i^h, \quad i \in \mathbb{Z}, \tag{1}$$

where the finite set $\mathbf{B} = \{b_i, i \in \mathbb{Z}\}$ is called mask. The recursive algorithm associated with the repeated application of (1) is called subdivision scheme and denoted by S. The Laurent polynomial or symbol of the scheme S is defined as

$$B(z) = \sum_{l \in \mathbb{Z}} b_l z^l. \tag{2}$$

Theorem 1 (see [33]). *If a binary scheme S is convergent, then the mask $\mathbf{B} = \{b_i, i \in \mathbb{Z}\}$ satisfies*

$$\sum_{l \in \mathbb{Z}} b_{2l} = \sum_{l \in \mathbb{Z}} b_{2l+1} = 1. \tag{3}$$

The symbol of a convergent scheme can be also be written as

$$B(z) = B_{even}\left(z^2\right) + zB_{odd}\left(z^2\right), \tag{4}$$

with $B_{even}(z) = \sum_{l \in \mathbb{Z}} b_{2l} z^l$ and $B_{odd}(z) = \sum_{l \in \mathbb{Z}} b_{2l+1} z^l$.

Theorem 2 (see [33]). *A binary scheme S associated with the symbol*

$$B(z) = \frac{(z+1)^{m+1}}{2^m} L(z) \tag{5}$$

is said to be C^m continuous if the subdivision scheme associated with the symbol $L(z)$ is contractive.

Proposition 3 (see [34]). *A binary scheme S generates polynomials of degree m if and only if*

$$B(1) = 2, B(-1) = 0 \text{ and } B^{(j)}(-1) = 0, j = 1, 2, \cdots, m. \tag{6}$$

Table 1: Mask of the R_φ schemes corresponding to different values of parameter φ.

φ	Scheme	Mask
0	3-point	$\frac{1}{16}[1, 5, 10,10,5, 1]$
1	5-point	$\frac{1}{256}[-1, 4, 44,124,170,124,44,4,-1]$
2	6-point	$\frac{1}{4096}[1,-13,-17,309,1338,2478,2478,1338,309,-17,-13, 1]$
3	8-point	$\frac{1}{65536}[-1, 22,-91,-580,1303,12362,31557,41928,31557,12362, 1303,-580,-91, 22,-1]$

Proposition 4 (see [34]). *A binary scheme S reproduces polynomials of degree n with respect to parametrization $\{e_j^h = (j+\tau)/2^h\}_{j\in\mathbb{Z}}$ with $\tau = B^{(j)}(1)/2$, if and only if it generates polynomials of degree n and*

$$B^{(j)}(1) = 2\prod_{i=0}^{j-1}(\tau - i), \quad j = 1, 2, \cdots, n. \tag{7}$$

3. Algorithm for Construction of Schemes

The well-known four-point interpolating binary subdivision scheme [18] is given by

$$\begin{cases} g_{2i}^{h+1} = g_i^h, \\ g_{2i+1}^{h+1} = -\frac{1}{16}g_{i-1}^h + \frac{9}{16}g_i^h + \frac{9}{16}g_{i+1}^h - \frac{1}{16}g_{i+2}^h. \end{cases} \tag{8}$$

By considering (4), the symbol of the even part of scheme (8) is as follows:

$$P_{\text{even}}(z) = \left(\frac{z+1}{2}\right)\left(\frac{-z^2+10z-1}{8}\right). \tag{9}$$

Now, consider the three-point approximating binary subdivision scheme [32]

$$\begin{cases} g_{2i}^{h+1} = \frac{1}{16}g_{i-1}^h + \frac{10}{16}g_i^h + \frac{5}{16}g_{i+1}^h, \\ g_{2i+1}^{h+1} = \frac{5}{16}g_{i-1}^h + \frac{10}{16}g_i^h + \frac{1}{16}g_{i+1}^h. \end{cases} \tag{10}$$

The symbol of scheme (10) is given by

$$T(z) = 2\left(\frac{z+1}{2}\right)^5. \tag{11}$$

Let us now present the class of subdivision schemes, namely, $\mathbf{R} = \{R_\varphi : \varphi \ge 0,\ \varphi \in \mathbb{N}\}$. The symbol of the scheme R_φ is obtained by applying symbol of the even part of scheme (8) φ-times on symbol of scheme (10) and given by

$$R_\varphi(z) = (P_{\text{even}}(z))^\varphi T(z). \tag{12}$$

So by (9), (11), and (12), we have

$$R_\varphi(z) = 2\left(\frac{z+1}{2}\right)^{\varphi+5}\left(\frac{-z^2+10z-1}{8}\right)^\varphi, \tag{13}$$

where $\{\varphi \ge 0,\ \varphi \in \mathbb{N}\}$. The members of the class R of subdivision schemes can be categorized by varying $\varphi = 0, 1, 2, \cdots$, in (13). By taking $\varphi = 0$ in (13), we get three-point approximating binary scheme [32]. Table 1 presents mask of some members of the proposed class.

4. Properties of the Proposed Schemes

In this section, we present some desirable properties of class $\mathbf{R}$ of subdivision schemes, comprising of polynomial generation and reproduction, support, continuity, and Hölder continuity.

4.1. Polynomial Generation and Reproduction. If a subdivision scheme generates polynomials of degree up to d_G, then the polynomial generation degree of the scheme is d_G. Also, if the initial data $\mathbf{G}^0 = \{g_i^0,\ i \in \mathbb{Z}\}$ is sampled from a polynomial $\widehat{\mathbf{P}}$ of degree d_R and the scheme yields precisely the same polynomial in the limit, then the reproduction degree d_R is the maximal degree of polynomials that can be reproduced by the scheme. Clearly, the reproduction degree is always less than or equal to the generation degree. Now, we establish few results about polynomial generation and polynomial reproduction of the proposed subdivision schemes.

Proposition 5. *R_φ -scheme generates space of polynomials up to degree $\varphi + 4$.*

Proof. Since symbol of R_φ -scheme satisfies the conditions

$$R_\varphi(1) = 2, R_\varphi(-1) = 0 \text{ and } R_\varphi^j(-1) = 0, j = 1, 2, \cdots, \varphi + 4, \tag{14}$$

so by Proposition 3, R_φ-scheme has $\varphi + 4$ polynomial generation degree.

In the view of Conti and Hormann [35], the standard parametrization $e_j^h = j/2^h$ at level $h \in \mathbb{N}$ is not appropriate to analyze a subdivision scheme to reproduce space of polynomials, and the relative shift $\tau_h = (e_0^h - e_0^{h+1})/2^{h+1}$

Table 2: Support, degree of polynomial generation (d_G), degree of polynomial reproduction (d_R), continuity (C), and HC of R_φ -scheme for $\varphi = 0, 1, 2,$ and 3.

φ	Support	d_G	d_R	C	Hölder continuity LB	UB
0	5	4	1	3	4	4
1	8	5	1	4	4.678	4.678
2	11	6	1	5	5.299	5.332
3	14	7	1	5	5.871	5.968

between the parameterizations at iteration level h and $h+1$ is important for polynomial reproduction of degree $d_R \geq 1$. By applying a more suitable parametrization $e_j^h = (j+\tau_\varphi)/2^h$ with shift parameter $\tau_\varphi = R_\varphi^{(1)}(1)/2 = (3\varphi+5)/2$, we have the following result.

Proposition 6. *R_φ-scheme reproduces linear polynomial with respect to parametrization $\{e_j^h = (j+\tau_\varphi)/2^h\}_{j\in\mathbb{Z}}$ with shift $\tau_\varphi = (3\varphi+5)/2$.*

Proof. It can be easily verified that

$$\begin{aligned} R_\varphi^{(1)}(1) &= 2\tau_\varphi = 3\varphi + 5, \\ R_\varphi^{(j)}(1) &= 2\prod_{i=0}^{j-1}\left(\tau_\varphi - j\right), \quad j = 0, 1. \end{aligned} \tag{15}$$

Thus, by Propositions 4 and 5, R_φ-scheme reproduces polynomial of degree one.

Table 2 presents the degree of polynomial generation and reproduction of some of the proposed R_φ-schemes. It is observed that the degree of polynomial generation is increasing linearly with the value of parameter φ.

4.2. Support. The support of a subdivision scheme quantifies how much one vertex brought change in its neighboring vertices, and its measure represents local support of the limit curve. Basic limit function (BLF) of a convergent subdivision scheme is a limit function of the initial data $\mathbf{G}^0 = \{g_i^0, i \in \mathbb{Z}\}$ which is of the form

$$g_i^0 = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases} \tag{16}$$

By following [36], we determine that support of BLF of R_φ-scheme is $3\varphi + 5$. BLF generated by the proposed R_0 and R_1 schemes are demonstrated in Figure 1.

4.3. Continuity Analysis. Continuity of a subdivision scheme is an essential parameter on which efficiency of a scheme depends. To investigate continuity of our proposed class, we follow the approach as given in [33] and use the symbol of R_φ-scheme.

Theorem 7. *The R_φ -scheme has $C^{\varphi+3-v_\varphi}$ continuity, where $\varphi = 1, 2, 3, \cdots$, and $v_\varphi = \lfloor(\varphi-1)/2\rfloor$ (floor function).*

Proof. The symbol of R_φ -scheme (13) can be simplified as

$$R_\varphi(z) = \frac{(z+1)^{\varphi+4-v_\varphi}}{2^{\varphi+3-v_\varphi}} r_\varphi(z), \tag{17}$$

with

$$r_\varphi(z) = \left(\frac{z+1}{2}\right)^{1+v_\varphi}\left(\frac{-z^2+10z-1}{8}\right)^{\varphi}, \quad \varphi = 1, 2, 3, \cdots, \tag{18}$$

where $v_\varphi = \lfloor(\varphi-1)/2\rfloor$.

Let S_{r_φ} be the subdivision scheme associated with the symbol $r_\varphi(z)$. The scheme S_{r_φ} is contractive provided that $\|S_{r_\varphi}\|_\infty = \max\{\sum_{l\in\mathbb{Z}}|r_{2l}|, \sum_{l\in\mathbb{Z}}|r_{2l+1}|\} < 1$. So, by Theorem 2, R_φ-scheme has $C^{\varphi+3-V_\varphi}$ continuity.

In Theorem 7, we discuss continuity of R_φ-scheme for $\varphi = 1, 2, 3, \cdots$. It is to be noted that R_0-scheme has C^3 continuity which is analyzed in [32].

Corollary 8. *The R_1-scheme has C^4 continuity.*

Proof. By letting $\varphi = 1$, the symbol of R_1-scheme from (17) and (18) is given by

$$R_1(z) = \frac{(z+1)^5}{2^4} r_1(z), \tag{19}$$

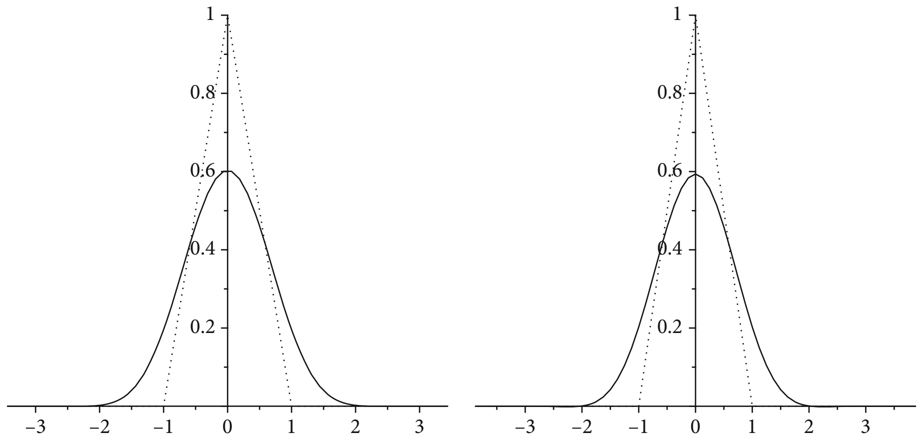
with

$$r_1(z) = \frac{1}{16}\left(-z^3 + 9z^2 + 9z - 1\right). \tag{20}$$

Let S_{r_1} be the scheme corresponding to the symbol $r_1(z)$. The scheme S_{r_1} is contractive, as $\|S_{r_1}\|_\infty = \max\{10/16, 10/16\} = (10/16) < 1$. So, by Theorem 7, R_1-scheme has C^4 continuity.

Similarly, for different values of parameter φ, continuity of R_φ-scheme can be easily computed by using Theorem 7.

4.4. Hölder Continuity Analysis. Continuity of a subdivision scheme is related to the existence of derivative of subdivision curve. For example, subdivision curve is said to be C^m continuous if the m^{th} derivative of the curve exists and is continuous everywhere in the given interval. On the other hand,



(a) (b)

FIGURE 1: (a, b) Basic limit functions generated by the proposed schemes R_0 and R_1, respectively.

Hölder continuity (HC) of a subdivision scheme tells how continuous the highest continuous derivative is. Therefore, it is also important to find HC of subdivision schemes along with continuity. Lower bound (LB) on HC of the proposed class is calculated by using an interesting property of symbol of R_φ-scheme, i.e., odd coefficients in $R_\varphi(z)$ are nonnegative and even coefficients are nonpositive.

Theorem 9. *LB on the HC of R_φ-scheme is $\varphi + 5 - \log_2((3/2)^\varphi + 1)$, where $\varphi = 0, 1, 2, \cdots$.*

Proof. By (13), symbol of R_φ -scheme can be expressed as

$$R_\varphi(z) = \left(\frac{z+1}{2}\right)^{\varphi+5} U_\varphi(z), \tag{21}$$

where $U_\varphi(z) = (a(z))^\varphi b(z)$, $a(z) = (-z^2 + 10z - 1)/8$, and $b(z) = 2$. So LB on HC of R_φ-scheme is given by $\varphi + 5 - \log_2\|U_\varphi\|$. As we know $\|U_\varphi\| = \max(u^\diamond, u_\diamond)$, where $u^\diamond$ is the sum of odd and $u_\diamond$ is the sum of even coefficients of $U_\varphi(z)$. We can write coefficients of $U_\varphi(z)$ in the following manner:

$$\begin{pmatrix} u^\diamond \\ u^\diamond \end{pmatrix} = \begin{pmatrix} a^\diamond & a_\diamond \\ a_\diamond & a^\diamond \end{pmatrix}^\varphi \begin{pmatrix} b^\diamond \\ b^\diamond \end{pmatrix}. \tag{22}$$

Thus, we have

$$\begin{pmatrix} u^\diamond \\ u^\diamond \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{5}{4} \end{pmatrix}^\varphi \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{23}$$

By eigenvalue decomposition, we have

$$\begin{pmatrix} u^\diamond \\ u^\diamond \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{pmatrix}^\varphi \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \tag{24}$$

which implies that

$$\begin{pmatrix} u^\diamond \\ u^\diamond \end{pmatrix} = \begin{pmatrix} -\left(\frac{3}{2}\right)^\varphi & +1 \\ \left(\frac{3}{2}\right)^\varphi & +1 \end{pmatrix}^\varphi. \tag{25}$$

Thus, we have

$$\|U_\varphi\| = \left(\frac{3}{2}\right)^\varphi + 1. \tag{26}$$

Consequently, LB on HC of R_φ-scheme is $\varphi + 5 - \log_2((3/2)^\varphi + 1)$, where $\varphi = 0, 1, \cdots$.

Upper bound (UB) on HC of R_φ-scheme is as follows.

Theorem 10. *UB on HC of R_φ-scheme is $\varphi + 5 - \log_2(\zeta_\varphi)$, where $\varphi = 0, 1, \cdots$, and ζ_φ be the joint spectral radius of the matrices Q_0 and Q_1 which are obtained by using symbol of R_φ -scheme.*

Proof. By (13), symbol of R_φ -scheme can be expressed as

$$R_\varphi(z) = \left(\frac{z+1}{2}\right)^{\varphi+5} Q_\varphi(z), \tag{27}$$

where $Q_\varphi(z) = 2((-z^2 + 10z - 1)/8)^\varphi$. Let $q_0, q_1, \cdots, q_d$ be the nonzero real coefficients of $Q_\varphi(z)$. Also, Q_0 and Q_1 are the matrices of order $d \times d$ defined by

$$(Q_0)_{mn} = q_{d+m-2n}, \text{ and } (Q_1)_{mn} = q_{d+m-2n+1}, \tag{28}$$

where $m, n = 1, 2, \cdots, d$.

Let us denote joint spectral radius of both matrices Q_0 and Q_1 by ζ_φ. Then, by Rioul [37] and Dyn [33], UB on HC of R_φ-scheme is given by $\varphi + 5 - \log_2(\zeta_\varphi)$.

For different values of parameter φ, upper and lower bounds on the HC of R_φ-schemes can be straightforwardly computed by using Theorems 9 and 10. Table 2 summarizes the continuity and HC of the proposed class of subdivision schemes. It clearly indicates that as we go up for higher values of parameter φ, continuity and HC of R_φ-schemes also increase. Moreover, newly generated R_φ-schemes have higher order of continuity and HC as compared to their parent subdivision schemes.

5. Geometrical Analysis of Proposed Schemes

The shape of an object is generally controlled by a control polygon. The purpose of applying a subdivision scheme on the control polygon is to generate visually smooth curves. Figure 2 presents the behavior of some of the proposed schemes. R_0, R_1, and R_2 schemes are applied on the same initial polygon, and limit curves are obtained after three iterations. It is evident that the proposed class offers more choices to meet different designing needs.

5.1. Subdivision Rules for Endpoints. For closed curves, the subdivision rules of R_0, R_1, R_2, and R_3 schemes can be defined by their corresponding Laurent polynomial from (13). The limit curves generated by these schemes are C^3, C^4, C^5, and C^5 continuous, respectively. In case of dealing with open polygons, these rules can be used to improve the interior of the curve, while it is quite troublesome to improve the first and last edges with the help of subdivision rules of the original proposed schemes. So to handle the endpoints of an open polygon, we need to supply additional points which are not usually required in case of a closed polygon. Let $g_0^h g_1^h$ be the first edge of the open polygon $\{G^h = g_k^h : k = 0, \cdots, 2^h m\}$. Now, we define an additional control point g_{-1}^h, as an extrapolatory rule in the nonrefined polygon G^h, and then we can compute the point g_1^{h+1} through the proposed schemes by applying subdivision to the subpolygon $\{g_{-1}^h, g_0^h, g_1^h, g_2^h\}$. We select the point as $g_{-1}^h = 2g_0^h - g_1^h$. The first edge of an open control polygon $\{G^h = g_k^h : k = 0, 1, \cdots, 2^h m\}$ can be refined by using the following rules.

(i) Refinement rules of the proposed three-point scheme R_0 are given by

$$\begin{aligned} g_0^{h+1} &= \frac{5}{4} g_0^h - \frac{1}{4} g_1^h, \\ g_1^{h+1} &= \frac{3}{4} g_0^h + \frac{1}{4} g_1^h. \end{aligned} \tag{29}$$

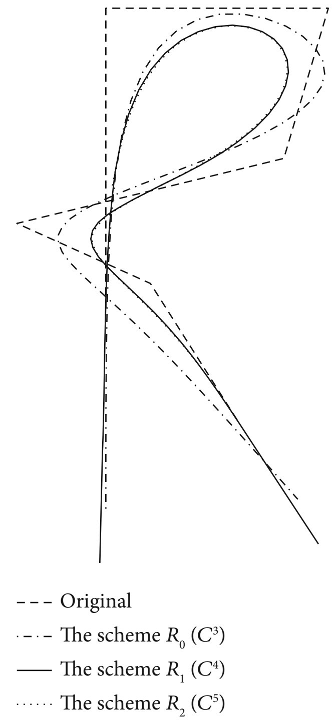


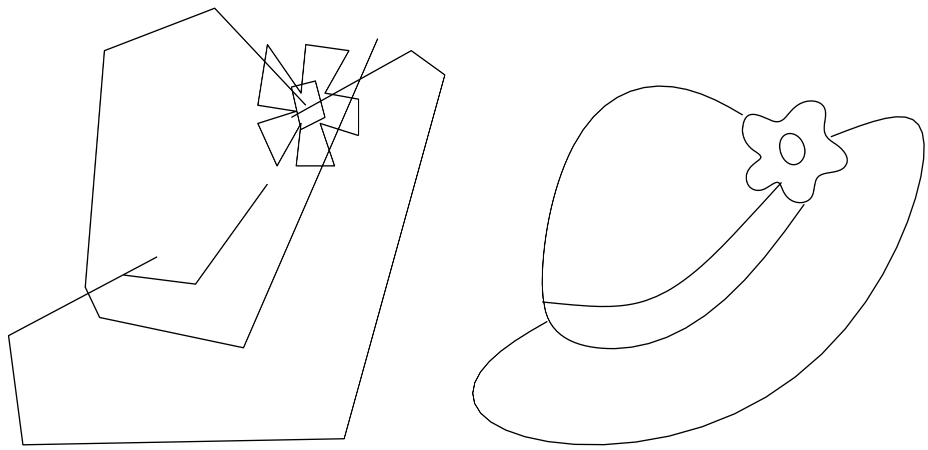
FIGURE 2: Behavior of the proposed R_0, R_1, and R_2 schemes after three iterations.

(ii) Refinement rules of the proposed five-point scheme R_1 are given by

$$\begin{aligned} g_0^{h+1} &= \frac{380}{256} g_0^h - \frac{120}{256} g_1^h - \frac{4}{256} g_2^h, \\ g_1^{h+1} &= \frac{255}{256} g_0^h + \frac{2}{256} g_1^h - \frac{1}{256} g_2^h, \\ g_2^{h+1} &= \frac{132}{256} g_0^h + \frac{120}{256} g_1^h + \frac{4}{256} g_2^h, \\ g_3^{h+1} &= \frac{42}{256} g_0^h + \frac{171}{256} g_1^h + \frac{44}{256} g_2^h - \frac{1}{256} g_3^h. \end{aligned} \tag{30}$$

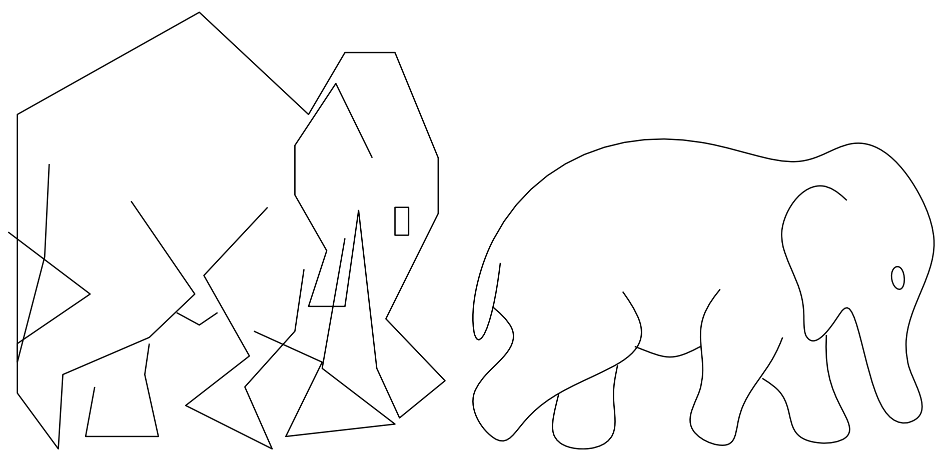
(iii) Refinement rules of the proposed six-point scheme R_2 are given by

$$\begin{aligned} g_0^{h+1} &= \frac{3070}{4096} g_0^h + \frac{1029}{4096} g_1^h - \frac{4}{4096} g_2^h + \frac{1}{4096} g_3^h, \\ g_1^{h+1} &= \frac{1306}{4096} g_0^h + \frac{2495}{4096} g_1^h + \frac{308}{4096} g_2^h - \frac{13}{4096} g_3^h, \\ g_2^{h+1} &= \frac{283}{4096} g_0^h + \frac{2491}{4096} g_1^h + \frac{1338}{4096} g_2^h - \frac{17}{4096} g_3^h + \frac{1}{4096} g_4^h, \\ g_3^{h+1} &= -\frac{15}{4096} g_0^h + \frac{1337}{4096} g_1^h + \frac{2478}{4096} g_2^h + \frac{309}{4096} g_3^h - \frac{13}{4096} g_4^h. \end{aligned} \tag{31}$$



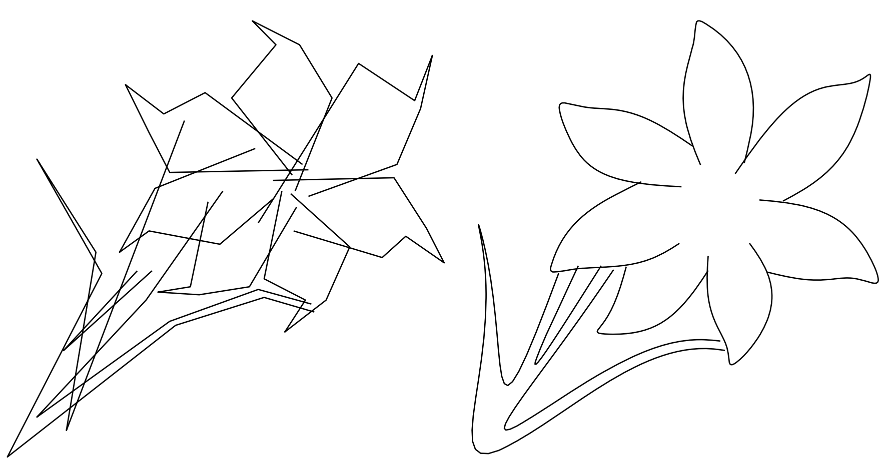
(a) (b)

Figure 3: Application of R_0-scheme: (a) initial polygon and (b) the limit curve generated by R_0-scheme at the third subdivision level.



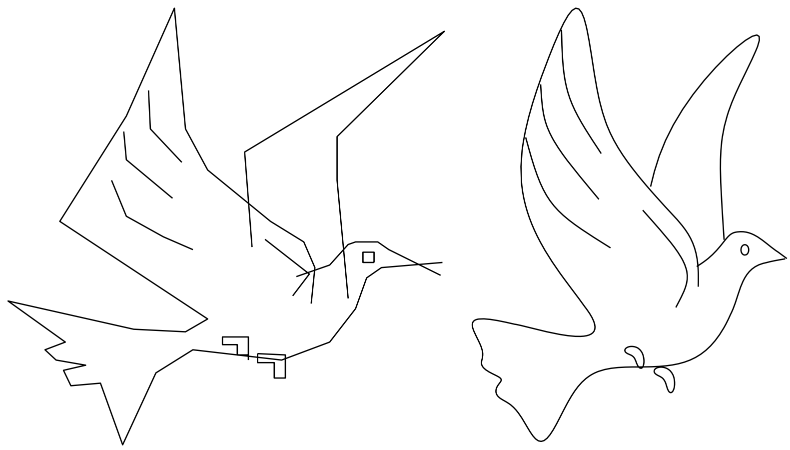
(a) (b)

Figure 4: Application of R_0 -scheme: (a) initial polygon and (b) the limit curve generated by R_0-scheme at the third subdivision level.



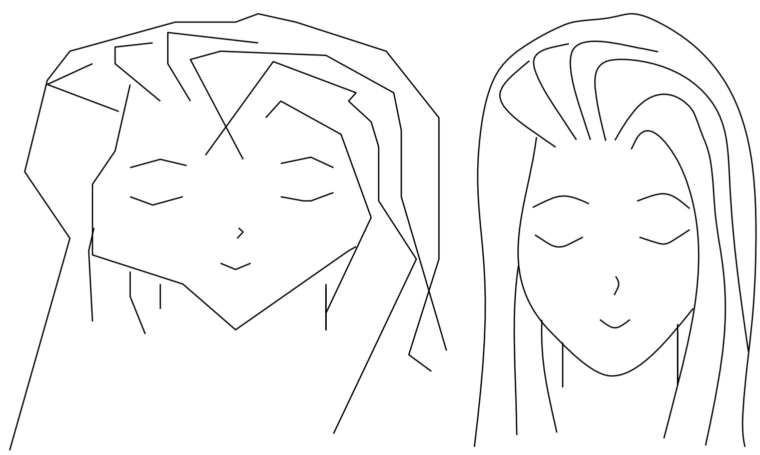
(a) (b)

Figure 5: Application of R_1-scheme: (a) initial polygon and (b) the limit curve generated by R_1-scheme at the third subdivision level.



(a) (b)

Figure 6: Application of R_1 -scheme: (a) initial polygon and (b) the limit curve generated by R_1 -scheme at the third subdivision level.



(a) (b)

Figure 7: Application of R_2-scheme: (a) initial polygon and (b) the limit curve generated by R_2-scheme at the third subdivision level.

(a) (b)

Figure 8: Application of R_3 -scheme: (a) initial polygon and (b) the limit curve generated by R_3 -scheme at the third subdivision level.

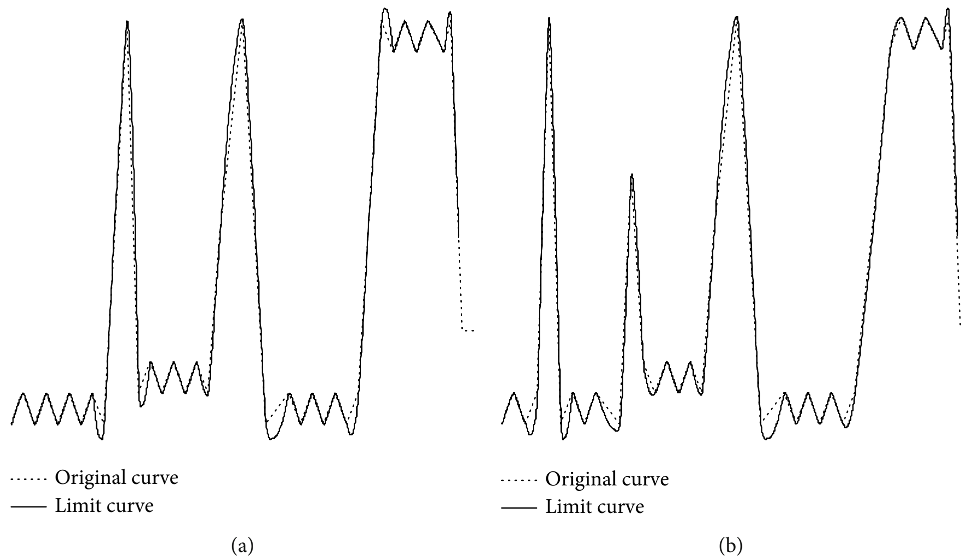


FIGURE 9: Application of R_3 -scheme: (a, b) the initial polygon along with sharp features of limit curve generated by R_3-scheme at the third subdivision level.

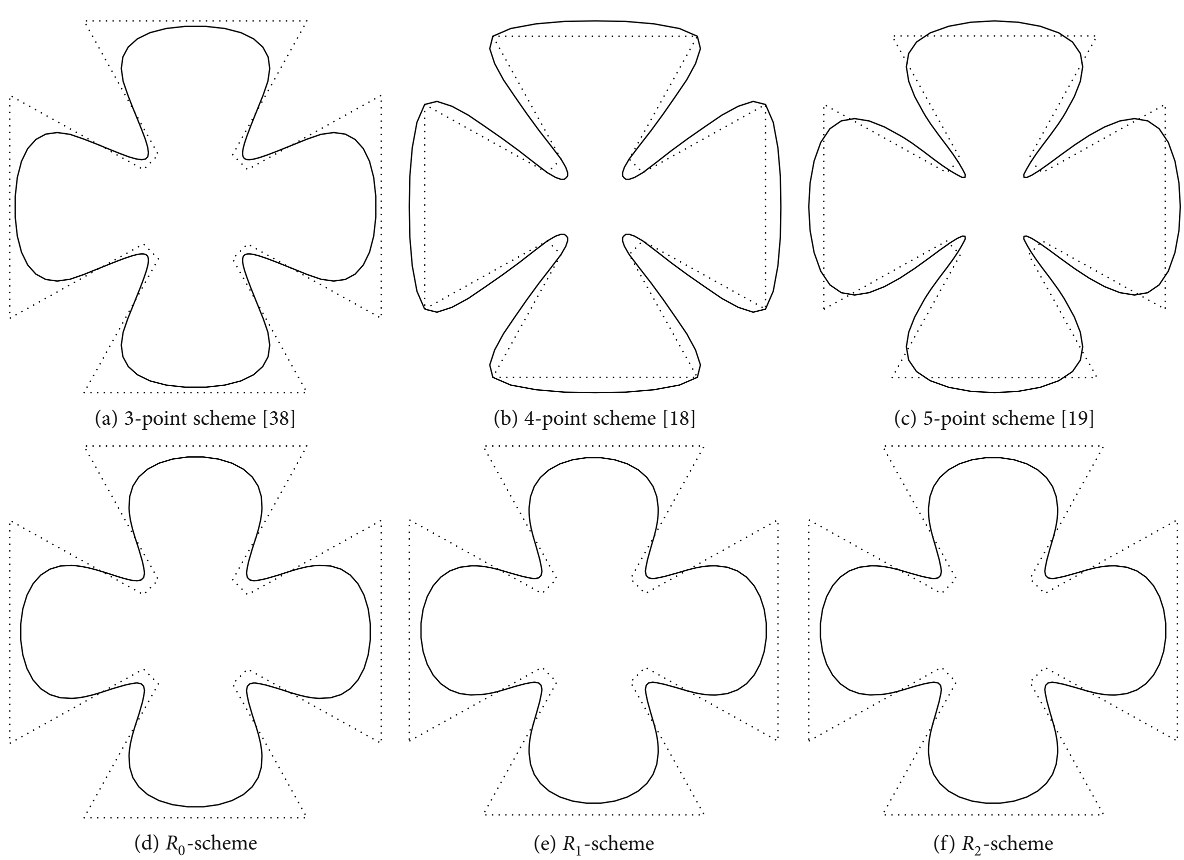
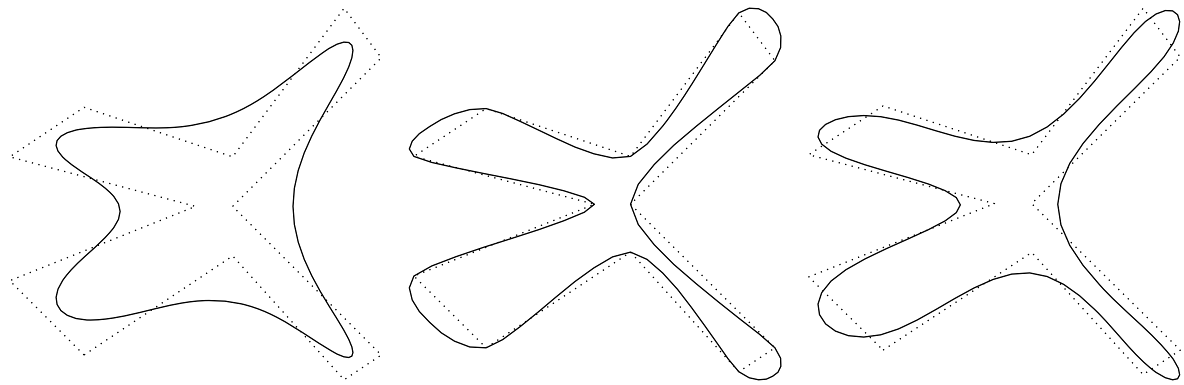
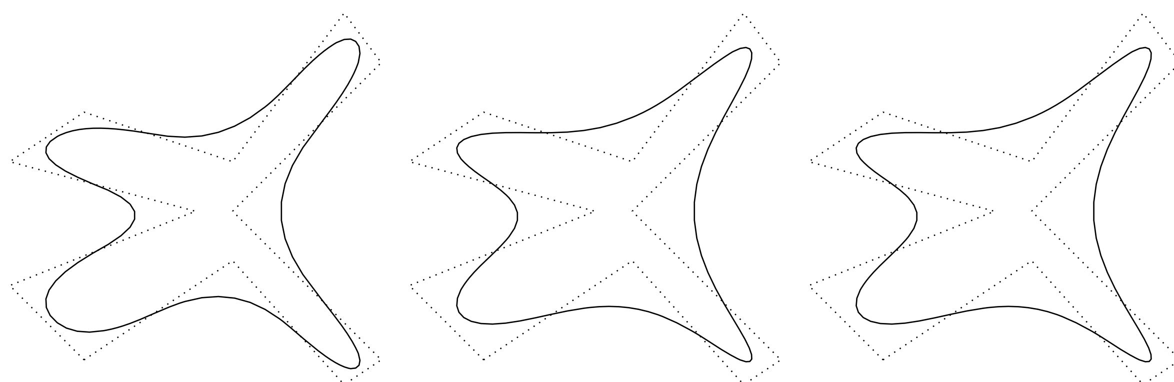


FIGURE 10: Comparison of the existing and proposed subdivision schemes at the third subdivision level.



(a) 3-point scheme [38] (b) 4-point scheme [18] (c) 5-point scheme [19]



(d) R_0-scheme (e) R_1-scheme (f) R_2-scheme

FIGURE 11: Comparison of the existing and proposed subdivision schemes at the third subdivision level.

(iv) Refinement rules of the proposed eight-point scheme R_3 are given by

$$\begin{aligned}
g_0^{h+1} &= \frac{65066}{65536} g_0^h + \frac{1028}{65536} g_1^h - \frac{580}{65536} g_2^h + \frac{22}{65536} g_3^h, \\
g_1^{h+1} &= \frac{33976}{65536} g_0^h + \frac{30254}{65536} g_1^h + \frac{1394}{65536} g_2^h - \frac{90}{65536} g_3^h \\
&\quad - \frac{1}{65536} g_4^h, \\
g_2^{h+1} &= \frac{11246}{65536} g_0^h - \frac{42508}{65536} g_1^h + \frac{12340}{65536} g_2^h - \frac{580}{65536} g_3^h \\
&\quad + \frac{22}{65536} g_4^h, \\
g_3^{h+1} &= \frac{1119}{65536} g_0^h + \frac{31648}{65536} g_1^h + \frac{31558}{65536} g_2^h + \frac{1303}{65536} g_3^h \\
&\quad - \frac{91}{65536} g_4^h - \frac{1}{65536} g_5^h, \\
g_4^{h+1} &= \frac{536}{65536} g_0^h - \frac{12340}{65536} g_1^h + \frac{41928}{65536} g_2^h + \frac{12362}{65536} g_3^h \\
&\quad - \frac{580}{65536} g_4^h + \frac{22}{65536} g_5^h, \\
g_5^{h+1} &= -\frac{93}{65536} g_0^h + \frac{1304}{65536} g_1^h + \frac{31557}{65536} g_2^h + \frac{31557}{65536} g_3^h \\
&\quad + \frac{1303}{65536} g_4^h - \frac{91}{65536} g_5^h - \frac{1}{65536} g_6^h.
\end{aligned} \tag{32}$$

Similarly, we can refine the final edges of the open polygon.

5.2. Applications and Comparison. Geometrical performance of R_0, R_1, R_2, and R_3 schemes is depicted through several examples. The proposed schemes have good continuity and present smooth limit curves. Figures 3(a) and 4(a) present initial control polygons of cap and elephant, respectively, while Figures 3(b) and 4(b) are the limit curves obtained by applying three iterations of R_0-scheme on these initial polygons. Figures 5(a) and 6(a) present initial control polygons of flower and bird, respectively, while Figures 5(b) and 6(b) are the limit curves obtained by applying three iterations of R_1-scheme on these initial polygons. Figures 7(a) and 8(a) present initial control polygons of face of girls, while Figures 7(b) and 8(b) are the limit curves obtained by applying three iterations of R_2-scheme on these initial polygons, respectively.

Figure 9 represents the initial polygon along with sharp features of limit curve generated by R_3-scheme at the third subdivision level. Figures 10 and 11 present comparison of some existing subdivision schemes (3-point scheme [38], 4-point scheme [18], and 5-point scheme [19]) with the proposed subdivision schemes (R_0, R_1, and R_2 schemes). We have chosen two different initial polygons, and limit curves are generated after three subdivision levels. It is clear from the figures that the proposed schemes generate smooth limit curves.

6. Conclusion

Subdivision is an efficient way of constructing smooth curves or surfaces in geometric modeling and computer graphics. In this paper, we have presented an elegant way of constructing a class of approximating binary subdivision schemes by using two well-known binary subdivision schemes. Several examples are provided to illustrate that the proposed schemes give wide choice to geometric designers for generation of smooth geometric models as per their own needs. Comparison with some existing schemes is also given. Moreover, several important properties like polynomial reproduction and generation, support of BLF, continuity, and HC of the proposed scheme are discussed. Geometrical analysis of the limit curve is also carried out.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally, and they read and approved the final manuscript for publication.

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Research Article

Some Identities Involving Derangement Polynomials and Numbers and Moments of Gamma Random Variables

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The problem of counting derangements was initiated by Pierre Rémond de Montmort in 1708. A derangement is a permutation that has no fixed points, and the derangement number D_n is the number of fixed point free permutations on an n element set. Furthermore, the derangement polynomials are natural extensions of the derangement numbers. In this paper, we study the derangement polynomials and numbers, their connections with cosine-derangement polynomials and sine-derangement polynomials, and their applications to moments of some variants of gamma random variables.

1. Introduction and Preliminaries

The problem of counting derangements was initiated by Pierre Rémond de Montmort in 1708 (see [1, 2]). A derangement is a permutation of the elements of a set, such that no element appears in its original position. In other words, a derangement is a permutation that has no fixed points. The derangement number D_n is the number of fixed point free permutations on an $n(n \geq 1)$ element set.

The aim of this paper is to study derangement polynomials and numbers, their connections with cosine-derangement polynomials and sine-derangement polynomials, and their applications to moments of some variants of gamma random variables. Here, the derangement polynomials $D_n(x)$ are natural extensions of the derangement numbers.

The outline of our main results is as follows. We show a recurrence relation for derangement polynomials. Then, we derive identities involving derangement polynomials, Bell polynomials, and Stirling numbers of both kinds. In addition, we also have an identity relating Bell polynomials, derangement polynomials, and Euler numbers. Next, we introduce the two variable polynomials, namely, cosine-derangement polynomials $D_n^{(c)}(x, y)$ and sine-derangement polynomials $D_n^{(s)}(x, y)$, in a natural manner by means of derangement polynomials. We obtain, among other things, their explicit expressions and recurrence relations. Lastly, in the final section, we show that if X is the gamma random variable with parameters $1, 1$, then $D_n(p), D_n^{(c)}(p, q), D_n^{(s)}(p, q)$ are given by the "moments" of some variants of X.

In the rest of this section, we recall the derangement numbers, especially their explicit expressions, generating function, and recurrence relations. Also, we give the derangement polynomials and give their explicit expressions. Then, we recall the gamma random variable with parameters α, λ along with their moments and the Bell polynomials. Finally, we give the definitions of the Stirling numbers of the first and second kinds.

As before, let D_n denote the derangement number for $n \geq 1$, and let $D_0 = 1$. Then, the first few derangement numbers D_n $(n \geq 0)$ are 1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, $\cdots$. For $n \geq 0$, the derangement numbers are given by [3–5]

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n \binom{n}{n} 0! \\ &= \sum_{k=0}^{n} \binom{n}{k}(n-k)!(-1)^k = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \end{aligned} \quad (1)$$

From (1), we note that [1–4, 6, 7]

$$\sum_{n=0}^{\infty} D_n \frac{t^n}{n!} = \frac{1}{1-t} e^{-t}. \tag{2}$$

By (2), we get

$$e^{-t} = (1-t) \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} = 1 + \sum_{n=1}^{\infty} (D_n - nD_{n-1}) \frac{t^n}{n!}. \tag{3}$$

From (3), we can easily derive the following recurrence relation [5, 8–11]:

$$(-1)^n = D_n - nD_{n-1}, (n \geq 1). \tag{4}$$

Now, we consider the derangement polynomials which are given by [10]

$$\frac{e^{-t}}{1-t} e^{xt} = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}. \tag{5}$$

From (5), we have

$$\sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} = \frac{1}{1-t} e^{-t} e^{xt} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} D_l x^{n-l} \right) \frac{t^n}{n!}. \tag{6}$$

By comparing the coefficients on both sides of (6), we get [10]

$$D_n(x) = \sum_{l=0}^{n} \binom{n}{l} D_l x^{n-l}, (n \geq 0). \tag{7}$$

On the other hand,

$$\begin{aligned} \frac{e^{-t}}{1-t} e^{xt} &= \frac{1}{1-t} e^{(x-1)t} = \sum_{l=0}^{\infty} t^l \sum_{m=0}^{\infty} (x-1)^m \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(n! \sum_{m=0}^{n} \frac{(x-1)^m}{m!} \right) \frac{t^n}{n!}. \end{aligned} \tag{8}$$

From (6), (7), and (8), we have

$$D_n(x) = n! \sum_{m=0}^{n} \frac{(x-1)^m}{m!} = \sum_{l=0}^{n} \binom{n}{l} D_l x^{n-l}, (n \geq 0). \tag{9}$$

A continuous random variable X whose density function is given by [12–14]

$$f(x) = \begin{cases} \lambda e^{-\lambda x} \dfrac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{10}$$

for some $\lambda > 0$ and $\alpha > 0$ is said to be the gamma random variable with parameter α, λ which is denoted by $X \sim \Gamma(\alpha, \lambda)$.

For $X \sim \Gamma(\alpha, \lambda)$, the n-th moment of X is given by

$$\begin{aligned} E[X^n] &= \frac{\lambda}{\Gamma(\alpha)} \int_0^{\infty} x^n e^{-\lambda x} (\lambda x)^{\alpha-1} dx \\ &= \frac{1}{\lambda^n \Gamma(\alpha)} \int_0^{\infty} t^{n+\alpha-1} e^{-t} dt \\ &= \frac{\Gamma(\alpha+n)}{\lambda^n \Gamma(\alpha)} = \frac{(\alpha+n) \cdots (\alpha+1)\alpha}{\lambda^n}. \end{aligned} \tag{11}$$

It is well known that the Bell polynomials are defined by [15]

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_n(x) \frac{t^n}{n!}. \tag{12}$$

When $x = 1$, $\mathrm{Bel}_n = \mathrm{Bel}_n(1) (n \geq 0)$ are called the Bell numbers.

The Stirling numbers of the first kind are defined as [16, 17]

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l, (n \geq 0), \tag{13}$$

where $(x)_0 = 1, (x)_n = x(x-1) \cdots (x-n+1) (n \geq 1)$.

As an inversion formula of (13), the Stirling numbers of the second kind are defined by [16–18]

$$x^n = \sum_{l=0}^{n} S_2(n,l) (x)_l (n \geq 0). \tag{14}$$

2. Derangement Polynomials and Numbers

From (5), we have

$$e^{(x-1)t} = \left(\sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} \right) (1-t) = 1 + \sum_{n=1}^{\infty} (D_n(x) - nD_{n-1}(x)) \frac{t^n}{n!}. \tag{15}$$

On the other hand,

$$e^{(x-1)t} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} t^n = 1 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n!} t^n. \tag{16}$$

Therefore, by (15) and (16), we obtain the following lemma.

Lemma 1. *For $n \geq 1$, we have*

$$D_n(x) - nD_{n-1}(x) = (x-1)^n. \tag{17}$$

Replacing t by $1-e^t$ in (5), we get

$$\begin{aligned} e^{(1-x)(e^t-1)} &= e^t \sum_{l=0}^{\infty} D_l(x) \frac{1}{l!}\left(1-e^t\right)^l \\ &= \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{l=0}^{\infty} (-1)^l D_l(x) \sum_{j=l}^{\infty} S_2(j,l) \frac{t^j}{j!} \\ &= \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j=0}^{\infty} \left(\sum_{l=0}^{j} (-1)^l D_l(x) S_2(j,l) \frac{t^j}{j!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} (-1)^l D_l(x) S_2(j,l) \right) \frac{t^n}{n!}. \end{aligned} \tag{18}$$

From (18), we have

$$\mathrm{Bel}_n(1-x) = \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} (-1)^l D_l(x) S_2(j,l), \ (n \geq 0). \tag{19}$$

It is easy to show that

$$\begin{aligned} \frac{1}{e^t} e^{(1-x)(e^t-1)} &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} t^l \sum_{m=0}^{\infty} \mathrm{Bel}_m(1-x) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \mathrm{Bel}_m(1-x)(-1)^{n-m} \right) \frac{t^n}{n!}. \end{aligned} \tag{20}$$

Replacing t by $\log(1-t)$ in (20), we get

$$\begin{aligned} \frac{1}{1-t} e^{-t} e^{xt} &= \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} \mathrm{Bel}_m(1-x)(-1)^{l-m} \frac{1}{l!} (\log(1-t))^l \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} \mathrm{Bel}_m(1-x)(-1)^{l-m} \sum_{n=l}^{\infty} (-1)^n S_1(n,l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} \mathrm{Bel}_m(1-x)(-1)^{n-l-m} S_1(n,l) \right) \frac{t^n}{n!}. \end{aligned} \tag{21}$$

From (5) and (21), we have

$$D_n(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} \mathrm{Bel}_m(1-x)(-1)^{n-m-l} S_1(n,l), \ (n \geq 0). \tag{22}$$

Therefore, by (19) and (22), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$\begin{aligned} Bel_n(1-x) &= \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} (-1)^l D_l(x) S_2(j,l), \\ D_n(x) &= \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} Bel_m(1-x)(-1)^{n-m-l} S_1(n,l). \end{aligned} \tag{23}$$

Corollary 3. *For $n \geq 0$, we have*

$$\begin{aligned} Bel_n &= \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} (-1)^l D_l S_2(j,l), \\ D_n &= \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} Bel_m (-1)^{n-m-l} S_1(n,l). \end{aligned} \tag{24}$$

Replacing t by $-e^t$ in (5), we get

$$\begin{aligned} \frac{1}{e^t+1} e^{(1-x)e^t} &= \sum_{m=0}^{\infty} D_m(x) \frac{(-1)^m}{m!} e^{mt} \\ &= \sum_{m=0}^{\infty} \frac{D_m(x)(-1)^m}{m!} \sum_{n=0}^{\infty} m^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{(-1)^m D_m(x)}{m!} m^n \right) \frac{t^n}{n!}. \end{aligned} \tag{25}$$

On the other hand, we have

$$\begin{aligned} \frac{1}{e^t+1} e^{(1-x)e^t} &= \frac{e^{1-x}}{2} \frac{2}{e^t+1} e^{(1-x)(e^t-1)} \\ &= \frac{e^{1-x}}{2} \sum_{l=0}^{\infty} E_l \frac{t^l}{l!} \sum_{m=0}^{\infty} \mathrm{Bel}_m(1-x) \frac{t^m}{m!} \\ &= \frac{e^{1-x}}{2} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \mathrm{Bel}_m(1-x) E_{n-m} \binom{n}{m} \right) \frac{t^n}{n!}, \end{aligned} \tag{26}$$

where E_n are the ordinary Euler numbers.

Therefore, by (25) and (26), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} Bel_m(1-x) E_{n-m} \binom{n}{m} = 2e^{x-1} \sum_{m=0}^{\infty} (-1)^m \frac{D_m(x)}{m!} m^n. \tag{27}$$

Now, we observe that

$$\begin{aligned}\left(\frac{1}{1-t}\right)^r &= \left(\frac{1}{1-t}\right)^r e^{-rt}e^{rt} = \left(\frac{1}{1-t}e^{-t}\right)^{r-1}\frac{e^{-t}}{1-t}e^{rt}\\ &= \sum_{k=0}^{\infty}\sum_{l_1+\cdots+l_{r-1}=k}\binom{k}{l_1,\cdots,l_{r-1}}D_{l_1}D_{l_2}\cdots D_{l_{r-1}}\\ &\quad\cdot\frac{t^k}{k!}\sum_{m=0}^{\infty}D_m(r)\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l_1+\cdots+l_{r-1}=k}\binom{k}{l_1,\cdots,l_{r-1}}\right.\\ &\quad\left.\cdot\binom{n}{k}D_{l_1}D_{l_2}\cdots D_{l_{r-1}}D_{n-k}(r)\right)\frac{t^n}{n!},\end{aligned} \tag{28}$$

where r is a positive integer.

On the other hand,

$$\left(\frac{1}{1-t}\right)^r = \sum_{n=0}^{\infty}\binom{-r}{n}(-1)^n t^n = \sum_{n=0}^{\infty}n!\binom{r+n-1}{n}\frac{t^n}{n!}. \tag{29}$$

Therefore, by (28) and (29), we obtain the following proposition.

Proposition 5. *For $r\in\mathbb{N}$, we have*

$$\binom{r+n-1}{n} = \frac{1}{n!}\sum_{k=0}^{n}\sum_{l_1+\cdots+l_{r-1}=k}\binom{k}{l_1,\cdots,l_{r-1}}\binom{n}{k}D_{l_1}\cdots D_{l_{r-1}}D_{n-k}(r). \tag{30}$$

It is well known that [16, 18, 19]

$$e^{ix} = \cos x + i\sin x, \quad i=\sqrt{-1}. \tag{31}$$

From (5), we note that

$$\frac{e^{-t}}{1-t}e^{(x+iy)t} = \sum_{n=0}^{\infty}D_n(x+iy)\frac{t^n}{n!},\ (x,y\in\mathbb{R}), \tag{32}$$

$$\frac{e^{-t}}{1-t}e^{(x-iy)t} = \sum_{n=0}^{\infty}D_n(x-iy)\frac{t^n}{n!}. \tag{33}$$

By (9), (32), and (33), we get

$$D_n(x+iy) = n!\sum_{m=0}^{n}\frac{(x-1+iy)^m}{m!}, \tag{34}$$

$$D_n(x-iy) = n!\sum_{m=0}^{n}\frac{(x-1-iy)^m}{m!},\ (n\geq 0). \tag{35}$$

From (34) and (35), we can derive the following equations:

$$\frac{e^{-t}}{1-t}e^{xt}\cos(yt) = \sum_{n=0}^{\infty}\left(\frac{D_n(x+iy)+D_n(x-iy)}{2}\right)\frac{t^n}{n!}, \tag{36}$$

$$\frac{e^{-t}}{1-t}e^{xt}\sin(yt) = \sum_{n=0}^{\infty}\left(\frac{D_n(x+iy)-D_n(x-iy)}{2i}\right)\frac{t^n}{n!}. \tag{37}$$

We define cosine-derangement polynomials and sine-derangement polynomials, respectively, by

$$\frac{e^{-t}}{1-t}e^{xt}\cos yt = \sum_{n=0}^{\infty}D_n^{(c)}(x,y)\frac{t^n}{n!}, \tag{38}$$

$$\frac{e^{-t}}{1-t}e^{xt}\sin yt = \sum_{n=0}^{\infty}D_n^{(s)}(x,y)\frac{t^n}{n!}. \tag{39}$$

Thus, we have

$$\begin{aligned}D_n^{(c)}(x,y) &= \frac{D_n(x+iy)+D_n(x-iy)}{2},\\ D_n^{(s)}(x,y) &= \frac{D_n(x+iy)-D_n(x-iy)}{2i},\ (n\geq 0).\end{aligned} \tag{40}$$

Therefore, we obtain the following theorem.

Theorem 6. *For $n\geq 0$, we have*

$$\begin{aligned}D_n^{(c)}(x,y) &= \frac{n!}{2}\sum_{m=0}^{n}\frac{1}{m!}\left((x-1+iy)^m+(x-1-iy)^m\right),\\ D_n^{(s)}(x,y) &= \frac{n!}{2i}\sum_{m=0}^{n}\frac{1}{m!}\left((x-1+iy)^m-(x-1-iy)^m\right).\end{aligned} \tag{41}$$

Before proceeding further, we recall that

$$\cos yt = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}y^{2n}t^{2n}. \tag{42}$$

From (38)and (42), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}D_n^{(c)}(x,y)\frac{t^n}{n!} &= \frac{e^{-t}}{1-t}e^{xt}\cos(yt)\\ &= \sum_{l=0}^{\infty}\frac{D_l}{l!}t^l\sum_{k=0}^{\infty}\sum_{m=0}^{[k/2]}\binom{k}{2m}(-1)^m y^{2m}x^{k-2m}\frac{t^k}{k!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}D_{n-k}\sum_{m=0}^{[k/2]}\binom{k}{2m}(-1)^m y^{2m}x^{k-2m}\right)\frac{t^n}{n!}.\end{aligned} \tag{43}$$

Therefore, by comparing the coefficients on both sides of (43), we obtain the following theorem.

Theorem 7. *For $n \geq 0$, we have*

$$D_n^{(c)}(x,y) = \sum_{m=0}^{[n/2]} \sum_{k=2m}^{n} \binom{n}{k}\binom{k}{2m} D_{n-k}(-1)^m y^{2m} x^{k-2m}. \tag{44}$$

Corollary 8. *For $n \geq 0$, we have*

$$\begin{aligned}\frac{n!}{2}\sum_{m=0}^{n} \frac{1}{m!}\left((x-1+iy)^m + (x-1-iy)^m\right) \\ = \sum_{m=0}^{[n/2]} \sum_{k=2m}^{n} \binom{n}{k}\binom{k}{2m} D_{n-k}(-1)^m y^{2m} x^{k-2m}.\end{aligned} \tag{45}$$

By (38), we get

$$\begin{aligned}e^{(x-1)t}\cos yt &= (1-t)\sum_{n=0}^{\infty} D_n^{(c)}(x,y)\frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty}\left(D_n^{(c)}(x,y) - nD_{n-1}^{(c)}(x,y)\right)\frac{t^n}{n!}.\end{aligned} \tag{46}$$

Thus, we have

$$\begin{aligned}\cos yt &= e^{(1-x)t} + e^{(1-x)t}\sum_{m=1}^{\infty}\left(D_m^{(c)}(x,y) - mD_{m-1}^{(c)}(x,y)\right)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}(1-x)^n\frac{t^n}{n!} + \sum_{l=0}^{\infty}(1-x)^l\frac{t^l}{l!}\sum_{m=1}^{\infty} \\ &\quad \cdot \left(D_m^{(c)}(x,y) - mD_{m-1}^{(c)}(x,y)\right)\frac{t^m}{m!} \\ &= 1 + \sum_{n=1}^{\infty}\left((1-x)^n + \sum_{m=1}^{n}\binom{n}{m}(1-x)^{n-m}\right. \\ &\quad \left.\cdot \left(D_m^{(c)}(x,y) - mD_{m-1}^{(c)}(x,y)\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{47}$$

Therefore, by (47) and (42), we obtain the following theorem.

Theorem 9. *For $k \in \mathbb{N}$, we have*

$$\begin{aligned}(1-x)^n + \sum_{m=1}^{n}\binom{n}{m}(1-x)^{n-m}\left(D_m^{(c)}(x,y) - mD_{m-1}^{(c)}(x,y)\right) \\ = \begin{cases}(-1)^k y^{2k}, & \text{if } n = 2k, \\ 0, & \text{if } n = 2k-1.\end{cases}\end{aligned} \tag{48}$$

By (38), we get

$$\begin{aligned}e^{(x-1)t}\cos yt &= \sum_{n=0}^{\infty} D_n^{(c)}(x,y)\frac{t^n}{n!}(1-t) \\ &= \sum_{n=1}^{\infty}\left(D_n^{(c)}(x,y) - nD_{n-1}^{(c)}(x,y)\right)\frac{t^n}{n!} + 1.\end{aligned} \tag{49}$$

On the other hand,

$$\begin{aligned}e^{(x-1)t}\cos yt &= \sum_{l=0}^{\infty}(x-1)^l\frac{t^l}{l!}\sum_{m=0}^{\infty} y^{2m}(-1)^m\frac{t^{2m}}{(2m)!} \\ &= 1 + \sum_{n=1}^{\infty}\left(\sum_{m=0}^{[n/2]}\binom{n}{2m}(-1)^m(x-1)^{n-2m}y^{2m}\right)\frac{t^n}{n!}.\end{aligned} \tag{50}$$

Therefore, by (49) and (50), we obtain the following theorem.

Theorem 10. *For $n \geq 1$, we have*

$$D_n^{(c)}(x,y) - nD_{n-1}^{(c)}(x,y) = \sum_{m=0}^{[n/2]}\binom{n}{2m}(-1)^m(x-1)^{n-2m}y^{2m}. \tag{51}$$

It is not difficult to show that

$$\sum_{n=0}^{\infty} D_n^{(c)}(x+r,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}D_l^{(c)}(x,y)r^{n-l}\right)\frac{t^n}{n!}, \tag{52}$$

where r is a positive integer.

By comparing the coefficients on both sides of (47), we get

$$D_n^{(c)}(x+r,y) = \sum_{l=0}^{n}\binom{n}{l}D_l^{(c)}(x,y)r^{n-l}. \tag{53}$$

Now, we observe that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial}{\partial x}D_n^{(c)}(x,y)\frac{t^n}{n!} &= \frac{\partial}{\partial x}\left(\frac{e^{-t}}{1-t}e^{xt}\cos yt\right) \\ &= t\frac{e^{-t}}{1-t}e^{xt}\cos yt = t\sum_{n=0}^{\infty} D_n^{(c)}(x,y)\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} nD_{n-1}^{(c)}(x,y)\frac{t^n}{n!}.\end{aligned} \tag{54}$$

Form (54), we note that

$$D_0^{(c)}(x,y)=1,\ \frac{\partial}{\partial x}D_n^{(c)}(x,y)=nD_{n-1}^{(c)}(x,y),\ (n\geq 1). \tag{55}$$

Therefore, we obtain the following theorem.

Theorem 11. *For $n\geq 0$, we have*

$$D_0^{(c)}(x,y)=1,\ \frac{\partial}{\partial x}D_n^{(c)}(x,y)=nD_{n-1}^{(c)}(x,y),\ (n\geq 1). \tag{56}$$

In particular,

$$\frac{d}{dx}D_n(x)=\frac{\partial}{\partial x}D_n^{(c)}(x,0)=nD_{n-1}^{(c)}(x,0)=nD_{n-1}^{(c)}(x),\ (n\geq 1). \tag{57}$$

Corollary 12. *$D_n^{(c)}(x,y)$ as a polynomial in x, for each fixed y, and $D_n(x)$ are Appell sequences.*

Before proceeding further, we recall that

$$\sin yt=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}y^{2n-1}t^{2n-1}. \tag{58}$$

From (39)and (58), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}D_n^{(s)}(x,y)\frac{t^n}{n!}&=\frac{1}{1-t}e^{-t}e^{xt}\sin yt\\&=\sum_{k=0}^{\infty}\frac{D_k}{k!}t^k\sum_{j=1}^{\infty}\sum_{m=0}^{[(j-1)/2]}\cdot\binom{j}{2m+1}x^{j-2m-1}y^{2m+1}\frac{t^j}{j!}\\&=\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\sum_{m=0}^{[(j-1)/2]}\binom{j}{2m+1}\cdot\binom{n}{j}x^{j-2m-1}y^{2m+1}D_{n-j}\right)\frac{t^n}{n!}.\end{aligned} \tag{59}$$

Therefore, by (59), we obtain the following theorem.

Theorem 13. *For $n\geq 0$, we have*

$$D_0^{(s)}(x,y)=0,\ D_n^{(s)}(x,y)=\sum_{j=1}^{n}\sum_{m=0}^{[(j-1)/2]}\binom{j}{2m+1}\binom{n}{j}x^{j-2m-1}y^{2m+1}D_{n-j}. \tag{60}$$

By (35) and (37) and Theorem 13, we obtain the following corollary.

Corollary 14. *For $n\geq 1$, we have*

$$\frac{D_n(x+iy)-D_n(x-iy)}{2i}=\sum_{j=1}^{n}\sum_{m=1}^{[(j-1)/2]}\binom{j}{2m+1}\cdot\binom{n}{j}x^{j-2m-1}y^{2m+1}D_{n-j}. \tag{61}$$

By (59), we see that

$$\begin{aligned}\sin yt&=e^{(1-x)t}\sum_{k=1}^{\infty}\left(D_k^{(s)}(x,y)-kD_{k-1}^{(s)}(x,y)\right)\frac{t^k}{k!}\\&=\sum_{m=0}^{\infty}(1-x)^m\frac{t^m}{m!}\sum_{k=1}^{\infty}\left(D_k^{(s)}(x,y)-kD_{k-1}^{(s)}(x,y)\right)\frac{t^k}{k!}\\&=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\binom{n}{k}\cdot\left(D_k^{(s)}(x,y)-kD_{k-1}^{(s)}(x,y)\right)(1-x)^{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{62}$$

Therefore, by (62) and (58), we obtain the following theorem.

Theorem 15. *For $m\in\mathbb{N}$, we have*

$$\sum_{k=1}^{n}\binom{n}{k}\left(D_k^{(s)}(x,y)-kD_{k-1}^{(s)}(x,y)\right)(1-x)^{n-k}=\begin{cases}(-1)^{m-1}y^{2m-1}, & \text{if } n=2m-1,\\ 0, & \text{if } n=2m.\end{cases} \tag{63}$$

It is easy to show that $(\partial/\partial x)D_n^{(s)}(x,y)=nD_{n-1}^{(s)}(x,y)$. However, $D_n^{(s)}(x,y)$ is not an Appell sequence, since $D_0^{(s)}(x,y)=0$.

We observe that

$$\begin{aligned}\sum_{n=0}^{\infty}D_n^{(s)}(x,y)\frac{t^n}{n!}&=\frac{e^{-t}}{1-t}e^{xt}\sin yt\\&=\sum_{l=0}^{\infty}D_l(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}(-1)^m y^{2m+1}\frac{t^{2m+1}}{(2m+1)!}\\&=\sum_{n=1}^{\infty}\left(\sum_{m=0}^{[(n-1)/2]}\binom{n}{2m+1}\cdot(-1)^m y^{2m+1}D_{n-2m-1}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{64}$$

Comparing the coefficients on both sides of (64), we have the following theorem.

Theorem 16. *For $n \geq 1$, we have*

$$D_n^{(s)}(x, y) = \sum_{m=0}^{[(n-1)/2]} \binom{n}{2m+1} (-1)^m y^{2m+1} D_{n-2m-1}(x). \tag{65}$$

For $r \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_n^{(s)}(x+r, y) &= \frac{e^{-t}}{1-t} e^{(x+r)t} \sin yt = \frac{e^{-t}}{1-t} e^{xt} \sin yte^{rt} \\ &= \sum_{l=0}^{\infty} D_l^{(s)}(x, y) \frac{t^l}{l!} \sum_{m=0}^{\infty} r^m \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} D_l^{(s)}(x, y) r^{n-l} \right) \frac{t^n}{n!}.\end{aligned} \tag{66}$$

Thus, we obtain

$$D_n^{(s)}(x+r, y) = \sum_{l=0}^{n} \binom{n}{l} D_l^{(s)}(x, y) r^{n-l}, \ (n \geq 0). \tag{67}$$

3. Further Remarks

As applications, we want to show that if X is the gamma random variable with parameters $1, 1$, then $D_n(p), D_n^{(c)}(\mathrm{p}, q)$, $D_n^{(s)}(p, q)$ are given by the "moments" of some variants of X . We let the reader refer to the papers [20–22] for some recent papers related to this section.

Let X be a gamma random variable with parameters $1, 1$ which is denoted by $X \sim \Gamma(1, 1)$. Then, we observe that

$$E\left[e^{(X-1+p)t}\right] = \int_0^{\infty} e^{(x-1+p)t} f(x) dx, \tag{68}$$

where $f(x)$ is the density function of X and $p \in \mathbb{R}$.
From (10) and (68), we can derive the following equation:

$$\begin{aligned}E\left[e^{(X-1+p)t}\right] &= \int_0^{\infty} e^{(x-1+p)t} e^{-x} dx = e^{-t+pt} \cdot \int_0^{\infty} e^{-x(1-t)} dx \\ &= \frac{e^{-t}}{1-t} e^{pt} = \sum_{n=0}^{\infty} D_n(p) \frac{t^n}{n!}.\end{aligned} \tag{69}$$

On the other hand, by Taylor expansion, we get

$$E\left[e^{(X-1+p)t}\right] = \sum_{n=0}^{\infty} E[(X-1+p)^n] \frac{t^n}{n!}. \tag{70}$$

Therefore, by (69) and (70), we obtain the following theorem.

Theorem 17. *For $n \geq 0$, $X \sim \Gamma(1, 1)$, the moment of $X - 1 + p$ is given by*

$$E[(X-1+p)^n] = D_n(p). \tag{71}$$

When $p = 0$, $D_n = D_n(0) = E[(X-1)^n]$, $(n \geq 0)$.
Thus, we note that

$$D_n = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} E\left[X^l\right]. \tag{72}$$

For $X \sim \Gamma(1, 1)$, we note that the moment of X is given by $E[X^n] = n!$, $(n \geq 0)$.
Therefore, by (72), we obtain the following corollary.

Corollary 18. *For $n \geq 0$, $X \sim \Gamma(1, 1)$, we have*

$$\begin{aligned}D_n &= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} l!, \\ D_n(p) &= \sum_{l=0}^{n} \binom{n}{l} (p-1)^{n-l} l!.\end{aligned} \tag{73}$$

For $X \sim \Gamma(1, 1)$, we have

$$E\left[e^{(X-1+p+iq)t}\right] = \frac{e^{-t}}{1-t} e^{(p+iq)t}, \tag{74}$$

where $p, q \in \mathbb{R}$.
From (74), we note that

$$E\left[e^{(X-1+p-iq)t}\right] = \frac{e^{-t}}{1-t} e^{(p-iq)t}. \tag{75}$$

By (74) and (75), we get

$$\begin{aligned}E\left[e^{(X-1+p+iq)t}\right] + E\left[e^{(X-1+p-iq)t}\right] &= \frac{2e^{-t}}{1-t} e^{pt} \cos qt \\ &= \sum_{n=0}^{\infty} 2D_n^{(c)}(p, q) \frac{t^n}{n!}.\end{aligned} \tag{76}$$

On the other hand, by Taylor expansion, we get

$$\begin{aligned}E\left[e^{(X-1+p+iq)t}\right] + E\left[e^{(X-1+p-iq)t}\right] &= \sum_{n=0}^{\infty} E[(X-1+p+iq)^n \\ &\quad + (X-1+p-iq)^n] \frac{t^n}{n!}.\end{aligned} \tag{77}$$

Therefore, by (76) and (77), we obtain the following theorem.

Theorem 19. *For $n \geq 0$, $X \sim \Gamma(1,1)$, we have*

$$E\left[\frac{(X-1+p+iq)^n + (X-1+p-iq)^n}{2}\right] = D_n^{(c)}(p,q). \quad (78)$$

It is easy to show that

$$E\left[e^{(X-1+p+iq)t}\right] - E\left[e^{(X-1+p-iq)t}\right] = 2i\frac{e^{-t}}{1-t}e^{pt}\sin qt = (2i)\sum_{n=1}^{\infty} D_n^{(s)}(p,q)\frac{t^n}{n!}, \quad (79)$$

where $X \sim \Gamma(1,1)$.

Thus, we have

$$E\left[\frac{(X-1+p+iq)^n - (X-1+p-iq)^n}{2i}\right] = D_n^{(s)}(p,q), (n \geq 0), \quad (80)$$

where $X \sim \Gamma(1,1)$.

4. Conclusion

The introduction of derangement numbers D_n goes back to as early as 1708 when Pierre Rémond de Montmort considered some counting problem on derangements. In this paper, we dealt with derangement polynomials $D_n(x)$ which are natural extensions of the derangement numbers. We showed a recurrence relation for derangement polynomials. We derived identities involving derangement polynomials, Bell polynomials, and Stirling numbers of both kinds. In addition, we also obtained an identity relating Bell polynomials, derangement polynomials, and Euler numbers. Next, we introduced the cosine-derangement polynomials $D_n^{(c)}(x,y)$ and sine-derangement polynomials $D_n^{(s)}(x,y)$, by means of derangement polynomials. Then, we derived, among other things, their explicit expressions and recurrence relations. Lastly, as applications, we showed that if X is the gamma random variable with parameters 1, 1, then $D_n(p)$, $D_n^{(c)}(p,q)$, $D_n^{(s)}(p,q)$ are given by the "moments" of some variants of X.

We have witnessed that the study of some special numbers and polynomials was done intensively by using several different means, which include generating functions, combinatorial methods, umbral calculus, p-adic analysis, probability theory, special functions, and differential equations. Moreover, the same has been done for various degenerate versions of quite a few special numbers and polynomials in recent years with their interests not only in combinatorial and arithmetical properties but also in their applications to symmetric identities, differential equations, and probability theories. It would have been nicer if we were able to find abundant applications in other disciplines.

It is one of our future projects to continue to investigate many ordinary and degenerate special numbers and polynomials by various means and find their applications in physics, science, engineering, and mathematics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Fixed Point Results via G-Function over the Complete Partial b-Metric Space

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In this paper, we consider an auxiliary function G to combine and unify several existing fixed point theorems in the setting of the complete partial b-metric space. We consider also some examples to support the observed main results.

1. Introduction and Preliminaries

The notion of the distance has been investigated and improved from the beginning of the mathematics sciences. The first formal definition was given by Hausdorff and Frechet under the name of metric spaces. The formal definition was extended, improved, and generalized in several ways. In this paper, we shall consider the combination of notions of partial metric space and b-metric space. Partial metric space, defined by Matthews [1, 2] is the most economical way to calculate the distance in computer science. So, it is important in the setting of theoretical computer science. On the other hand, b-metric is the most interesting and real generalization of metric spaces; in this case, the triangle inequality is replaced by a modified version of triangle inequality.For more details on the advances of fixed point theory in the setting of b-metric spaces, see e.g. [13]-[27].

In this paper, we shall propose a fixed point theorem by using an auxiliary function G to combine, generalize, and unify several fixed point results in the setting of the complete partial b-metric spaces.

In [3], the authors proposed a new fixed point theorem in the setting of metric spaces.

We consider the follow sets of functions:

(1) $\mathbb{G}$ be the set of the functions $\mathscr{G}:[0,\infty)^3 \longrightarrow [0,\infty)$ that satisfy the following conditions:

(f_1) $\mathscr{G}$ is continuous,
(f_2) $\mathscr{G}(0,0,0)=0$,
(f_3) $\max\{\tau,\upsilon\} \le \mathscr{G}(\tau,\upsilon,\omega)$, for all $\tau,\upsilon,\omega \in [0,\infty)$.
In [3], some examples of such a function were given.

(i) $\mathscr{G}(\tau,\upsilon,\omega)=\tau+\upsilon+\omega$

(ii) $\mathscr{G}(\tau,\upsilon,\omega)=\max\{\tau,\upsilon,\omega\}$

(iii) $\mathscr{G}(\tau,\upsilon,\omega)=(\tau+\upsilon)(1+\omega)$

(2) Φ be the set of functions $\psi:[0,\infty)\longrightarrow[0,\infty)$ that satisfy the following conditions:

(b_1) ψ is nondecreasing,
(b_2) $\sum_{i\ge1}\psi^i(u)<\infty$ for each $u>0$. (Here, by ψ^i, we denote the ith iterate ohψ.)

We mention that the functions $\psi\in\Phi$ are called (c)-comparison functions. Moreover, it is not difficult to check that $\phi(u)<u$ for every $u>0$.

(3) $\Gamma=\{\gamma : X\longrightarrow[0,\infty) \mid \gamma \text{ is lower semicontinuous}\}$

Theorem 1 (see [3]). *Let (X,d) be a complete metric space, a lower semicontinuous function $\gamma:X\longrightarrow[0,\infty)$, and a* *self-mapping $T : X \longrightarrow X$. If there exist $\psi \in \Phi$ and $\mathscr{G} \in \mathbb{G}$ such that*

$$\mathscr{G}(d(Tx,Ty),\gamma(Tx),\gamma(Ty)) \le \psi\left(\max\left\{\begin{array}{c}\mathscr{G}(d(x,y),\gamma(x),\gamma(y)),\\ \dfrac{\mathscr{G}(d(x,Tx),\gamma(Tx),\gamma(x))+\mathscr{G}(d(y,Ty),\gamma(Ty),\gamma(y))}{2}\end{array}\right\}\right), \tag{1}$$

for every $x, y \in X$, then T has a unique fixed point.

Let X be a nonempty set.

(i) A function $b : X \times X \longrightarrow [0,\infty)$ is a b-metric on X if for a given real number $s \ge 1$ and for all $x, y, z \in X$ the following conditions hold:

$$\begin{aligned}&(b_1)\, b(x,y)=0 \Leftrightarrow x=y,\\ &(b_2) b(x,y)=b(y,x),\\ &(b_3) b(x,y)\le s[b(x,z)+b(z,y)].\end{aligned} \tag{2}$$

The triplet $(X, b, s \ge 1)$ is called a b-metric space.

(ii) A function $\rho : X \times X \longrightarrow [0,\infty)$ is a partial metric on X if for all $x, y, z \in X$ the following conditions hold:

$$\begin{aligned}&(\rho_1)\, x=y \Leftrightarrow \rho(x,x)=\rho(y,y)=\rho(x,y) \Leftrightarrow x=y,\\ &(\rho_2)\, \rho(x,x)\le\rho(x,y),\\ &(\rho_3)\, \rho(x,y)=\rho(y,x)\\ &(\rho_4)\, \rho(x,y)\le\rho(x,z)+\rho(z,y)-\rho(z,z).\end{aligned} \tag{3}$$

The pair (X, ρ) is said to be a partial metric space.

Combining these two concepts, Shukla [4] introduced the notion of partial b-metric space as follows.

(iii) A function $\rho_b : X \times X \longrightarrow [0,\infty)$ is a partial b-metric on X if for all $x, y, z \in X$ the following conditions hold:

$$\begin{aligned}&\left(\rho_{b_1}\right) x=y \Leftrightarrow \rho_b(x,x)=\rho_b(y,y)=\rho_b(x,y),\\ &\left(\rho_{b_2}\right) \rho_b(x,x)\le\rho_b(x,y),\\ &\left(\rho_{b_3}\right) \rho_b(x,y)=\rho_b(y,x),\\ &\left(\rho_{b_4}\right) \rho_b(x,y)\le s[\rho_b(x,z)+\rho_b(z,y)]-\rho_b(z,z).\end{aligned} \tag{4}$$

The triplet $(X, \rho_b, s \ge 1)$ is said to be a partial b-metric space.

On a partial b-metric space $(X, \rho_b, s \ge 1)$ a sequence $\{x_n\}$ is said to be

(i) convergent to $x \in X$ if $\lim_{n\to\infty}\rho_b(x_n, x) = b(x, x)$ (the limit of a convergent sequence is not necessarily unique)

(ii) Cauchy if $\lim_{n,m\to\infty}\rho_b(x_n, x_p)$ exists and its finite

Moreover, the partial b-metric space is complete if for every Cauchy sequence $\{ax_n\}$ there exists $x \in X$ such that

$$\lim_{n,p\to\infty}\rho_b\left(x_n,x_p\right)=\lim_{n\to\infty}\rho_b(x_n,x)=\rho_b(x,x). \tag{5}$$

Let $(X, \rho_b, s \ge 1)$ be a partial b-metric space. We say that a self-mapping T on X is continuous if for every sequence $\{x_n\}$ in X which converges to a point $x \in X$ we have

$$\lim_{n\to\infty}\rho_b(Tx_n,Tx)=\lim_{n\to\infty}\rho_b\left(Tx_n,Tx_{n+j}\right)=\rho_b(Tx,Tx). \tag{6}$$

In [5], the authors introduced the following new notions.

(i) On a partial b-metric space, a sequence $\{x_n\}$ is a 0 -Cauchy sequence if $\lim_{n\to\infty}\rho_b(x_n, x_p) = 0$

(ii) The space $(X, \rho_b, s \ge 1)$ is said to be 0-complete if for each 0-Cauchy sequence $\{x_n\}$ in X, there exists a point $x \in X$ such that

$$\lim_{n,p\to\infty}\rho_b\left(x_n,x_p\right)=\lim_{n\to\infty}\rho_b(x_n,x)=\rho_b(x,x)=0. \tag{7}$$

Moreover, they proved that if the partial b-metric space $(X, \rho_b, s \ge 1)$ is complete, then it is 0-complete.

For a better understanding of the connections between these spaces (partial metric space, b-metric space, and partial b-metric space), we mention some papers that can be consulted [6–12].

Let Φ_b be the set of functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ that satisfy the following conditions:

$(\phi_1)\phi$ is nondecreasing,

$(\phi_2)\sum_{i\ge1} s^i\phi^i(u) < \infty$ for each $u > 0$. (Here, by ϕ^i, we denote the ith iterate oh ϕ.)

2. Main Results

The following is the main result of the paper.

Theorem 2. *Let $(X, \rho_b, s \ge 1)$ be a 0-complete partial b -metric space, a function $\gamma \in \Gamma$, $\mathscr{G} \in \mathbb{G}$, and a self-mapping $T : X \longrightarrow X$. If there exists $\phi \in \Phi_b$ such that*

$$\mathscr{G}(\rho_b(Tx,Ty),\gamma(Tx),\gamma(Ty)) \le \phi\left(\max\left\{\begin{array}{c}\mathscr{G}(\rho_b(x,y),\gamma(x),\gamma(y)),\\ \dfrac{\mathscr{G}(\rho_b(x,Tx),\gamma(Tx),\gamma(x))+\mathscr{G}(d(y,Ty),\gamma(Ty),\gamma(y))}{2s}\end{array}\right\}\right), \tag{8}$$

for every $x, y \in X$. If T is continuous or ρ_b is continuous, then T has a unique fixed point.

Proof. Starting with a point $x_0 \in X$, we consider the sequence $\{x\}$ defined by $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. Without losing the generality, we can assume that for any $n \in \mathbb{N}$, we have $b(x_n, x_{n+1}) > 0$. Indeed, on the contrary, if there exists a positive integer j_0 such that $x_{j_0} = x_{j_0+1}$, we get that x_{j_0} is a fixed point of T, because due to the way the sequence was $\{x\}$ defined, it follows that $x_{j_0} = Tx_{j_0}$. Moreover, using this remark, we can easily see that

$$\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) > 0, \quad \text{for every } n \in \mathbb{N}. \tag{9}$$

Again supposing that $\mathcal{G}(\rho_b(x_{j_0}, x_{j_0+1}), \gamma(x_{j_0}), \gamma(x_{j_0+1})) = 0$ for some j_0 from (f_3), we have

$$\begin{aligned} 0 < \rho_b\left(x_{j_0}, x_{j_0+1}\right) &\leq \max\left\{\rho_b\left(x_{j_0}, x_{j_0+1}\right), \gamma\left(x_{j_0}\right)\right\} \\ &\leq \mathcal{G}\left(\rho_b\left(x_{j_0}, x_{j_0+1}\right), \gamma\left(x_{j_0}\right), \gamma\left(x_{j_0+1}\right)\right), \end{aligned} \tag{10}$$

which is a contradiction. Taking $x = x_n$ and $y = x_{n+1}$ in (8) we get

$$\begin{aligned} &\mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})) \\ &\quad \leq \mathcal{G}(\rho_b(Tx_n, Tx_{n+1}), \gamma(Tx_n), \gamma(Tx_{n+1}) \\ &\quad \leq \phi\left(\max\left\{\begin{array}{c} \mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ \dfrac{\mathcal{G}(\rho_b(x_n, Tx_n), \gamma(x_n), \gamma(Tx_n)) + \mathcal{G}(\rho_b(x_{n+1}, Tx_{n+1}), \gamma(x_{n+1}), \gamma(Tx_{n+1}))}{2s} \end{array}\right\}\right) \\ &\quad \leq \phi\left(\max\left\{\begin{array}{c} \mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ \dfrac{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) + \mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2}))}{2s} \end{array}\right\}\right) \\ &\quad \leq \phi\left(\max\left\{\begin{array}{c} \mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ \mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})) \end{array}\right\}\right). \end{aligned} \tag{11}$$

There are two possibilities, namely,

$$\begin{aligned} &\max\{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ &\mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2}))\} \\ &\quad = \mathcal{G}(d(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})), \end{aligned} \tag{12}$$

which leads us (since $\phi(u) < u$ for any $u > 0$) to

$$\begin{aligned} &\mathcal{G}(d(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})) \\ &\quad \leq \phi(\mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2}))) \\ &\quad < \mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})). \end{aligned} \tag{13}$$

But, this is a contradiction, and then

$$\begin{aligned} &\max\{\mathcal{G}(d(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ &\mathcal{G}(d(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2}))\} \\ &\quad = \mathcal{G}(d(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})). \end{aligned} \tag{14}$$

Therefore, by (11) and taking into account (f_3), we have

$$\begin{aligned} \rho_b(x_{n+1}, x_{n+2}) &\leq \max\{\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1})\} \\ &\leq \mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})) \\ &\leq \phi(\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1}))), \\ &\qquad \text{for every } n \in \mathbb{N} \cup 0. \end{aligned} \tag{15}$$

Consequently, for every $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \rho_b(x_n, x_{n+1}) &\leq \phi(\mathcal{G}(\rho_b(x_{n-1}, x_n), \gamma(x_{n-1}), \gamma(x_n))) \\ &\leq \phi^n(\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1))). \end{aligned} \tag{16}$$

Let $p, m \in \mathbb{N}$ such that $p < m$. By applying the (triangle-type inequality) (ρ_{b4}), we have

$$\begin{aligned} \rho_b\left(x_p, x_m\right) &\leq s\left[\rho_b\left(x_p, x_{p+1}\right) + \rho_b\left(x_{p+1}, x_m\right)\right] - \rho_b\left(x_{p+1}, x_{p+1}\right) \\ &\leq s\left[\rho_b\left(x_p, x_{p+1}\right) + \rho_b\left(x_{p+1}, x_m\right)\right] \\ &\leq s\rho_b\left(x_p, x_{p+1}\right) + s^2\left[\rho_b\left(x_{p+1}, x_{p+2}\right)\right. \\ &\quad \left. + s\left[\rho_b\left(x_{p+2}, x_m\right)\right] - \rho_b\left(x_{p+2}, x_{p+2}\right)\right. \\ &\leq s\rho_b\left(x_p, x_{p+1}\right) + s^2\left[\rho_b\left(x_{p+1}, x_{p+2}\right)\right. \\ &\quad \left. + s\left[\rho_b\left(x_{p+2}, x_m\right)\right] \cdots \leq s\rho_b\left(x_p, x_{p+1}\right)\right. \\ &\quad + s^2\rho_b\left(x_{p+1}, x_{p+2}\right) + \cdots + s^{m-p-1}\rho_b\left(x_{m-1}, x_m\right), \end{aligned} \tag{17}$$

and (17) leads us to

$$\begin{aligned}\rho_b(x_p, x_m) &\leq \frac{1}{s^{p-1}} \sum_{i=p}^{m-1} s^i \phi^i (\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1))) \\ &< \frac{1}{s^{p-1}} \sum_{i=p}^{m-1} s^i \phi^i (\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1))) \\ &= \frac{1}{s^{p-1}} [S_{m-1} - S_{p-1}],\end{aligned} \tag{18}$$

where $S_n = \sum_{i=0}^{n} s^i \phi^i (\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1)))$. Keeping in mind (ϕ_2), we deduce that there exists $S_n \longrightarrow S$ as $n \longrightarrow \infty$, and from (18), we get

$$\lim_{p,m\to\infty} \rho_b(x_p, x_m) = 0. \tag{19}$$

Consequently, $\{x_n\}$ is a 0-Cauchy sequence in a 0-complete partial b-metric space, and then there exists $\varsigma \in X$ such that

$$\lim_{p,m\to\infty} \rho_b(x_p, x_m) = \lim_{p\to\infty} \rho_b(x_p, \varsigma) = \rho_b(\varsigma, \varsigma) = 0. \tag{20}$$

Moreover, by (f_3) together with (16), we have

$$\begin{aligned}\gamma(x_n) &\leq \max\{\rho_b(x_n, x_{n+1}), \gamma(x_n)\} \\ &\leq \mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) \\ &\leq \phi^n (\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1))\end{aligned} \tag{21}$$

and using (ϕ_2)

$$\lim_{n\to\infty} \gamma(x_n) = 0. \tag{22}$$

Plus, by (ϕ_1) and (20),

$$\gamma(\varsigma) = 0. \tag{23}$$

We claim that this point ς is in fact a fixed point of the mapping T. If the mapping T is continuous, then by (6), we have

$$\rho_b(\varsigma, \varsigma) = \lim_{p\to\infty} \rho_b(Tx_p, T\varsigma) = \lim_{p\to\infty} \rho_b(Tx_p, Tx_{p+i}) = 0. \tag{24}$$

Thus, applying the triangle inequality (ρ_4),

$$\rho_b(\varsigma, T\varsigma) \leq s[\rho_b(\varsigma, x_{n+1}) + \rho_b x_{n+1}, T\varsigma] - \rho_b(x_{n+1}, x_{n+1}), \tag{25}$$

and together with (20) and (24), letting $n \longrightarrow \infty$, we get $\rho_b(\varsigma, T\varsigma) = 0$, that is, ς is a fixed point of T.

Let assume now that ρ_b is continuous, that is, $\lim_{n\to\infty}\rho_b(x_n, T\varsigma) = \rho_b(\varsigma, T\varsigma)$ Replacing x by x_n and y by ς in (8), we have (for every $n \in \mathbb{N}$)

$$\begin{aligned}&\mathcal{G}(\rho_b(x_{n+1}, T\varsigma), \gamma(x_{n+1}), \gamma(T\varsigma)) = \mathcal{G}(\rho_b(Tx_n, T\varsigma), \gamma(Tx_n), \gamma(T\varsigma)) \\ &\quad\leq \phi\left(\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(x_n, \varsigma), \gamma(x_n), \gamma(\varsigma), \\ \dfrac{\mathcal{G}(\rho_b(x_n, Tx_n), \gamma(x_n), \gamma(Tx_n)) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma))}{2s}\end{array}\right\}\right) \\ &\quad\leq \phi\left(\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(x_n, \varsigma), \gamma(x_n), \gamma(\varsigma)), \\ \dfrac{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma))}{2s}\end{array}\right\}\right) \\ &\quad< \max\left\{\begin{array}{c}\mathcal{G}(\rho_b(x_n, \varsigma), \gamma(x_n), \gamma(\varsigma)), \\ \dfrac{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma))}{2s}\end{array}\right\}.\end{aligned} \tag{26}$$

Letting $n \longrightarrow \infty$ and taking into account (f_1), we have

$$\begin{aligned}&\mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma)) \\ &\quad= \lim_{n\to\infty} \mathcal{G}(\rho_b(x_{n+1}, T\varsigma), \gamma(x_{n+1}), \gamma(T\varsigma)) \\ &\quad< \lim_{n\to\infty} \max\left\{\begin{array}{c}\mathcal{G}(\rho_b(x_n, \varsigma), \gamma(x_n), \gamma(\varsigma)), \\ \dfrac{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma))}{2s}\end{array}\right\} \\ &\quad= \max\left\{\mathcal{G}(0,0,0), \frac{\mathcal{G}(0,0,0) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma))}{2}\right\} \\ &\quad= \frac{\mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma))}{2s}.\end{aligned} \tag{27}$$

Consequently, $\mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma)) = 0$. But, taking (f_3) into account, we get

$$0 \leq \max\{\rho_b(\varsigma, T\varsigma), 0\} \leq \mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma)) = 0, \tag{28}$$

which means $\rho_b(\varsigma, T\varsigma) = 0$. Thus, $T\varsigma = \varsigma$.

As a last step, we claim that ς is the unique fixed point of T. Supposing on the contrary, that there exists another point $\upsilon \in X$ such that $T\varsigma = \varsigma \neq \upsilon = T\upsilon$. First of all, applying (8) with $x = \upsilon = y$, we have

$$\mathcal{G}(0, \gamma(\upsilon), \gamma(\upsilon)) \leq \phi(\mathcal{G}(0, \gamma(\upsilon), \gamma(\upsilon))) < \mathcal{G}(0, \gamma(\upsilon), \gamma(\upsilon)), \tag{29}$$

which implies that $\gamma(\upsilon) = \gamma(T\upsilon) = 0$. Let now $x = \varsigma$ and $y = \upsilon$ in (8). We have

$$\begin{aligned}&\mathcal{G}(\rho_b(\varsigma, \upsilon), 0, 0) = \mathcal{G}(\rho_b(T\varsigma, T\upsilon), \gamma(T\varsigma), \gamma(T\upsilon)) \\ &\quad\leq \phi\left(\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(\varsigma, \upsilon), \gamma(\varsigma), \gamma(\upsilon)), \\ \dfrac{\mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma)) + \mathcal{G}(\rho_b(\upsilon, T\upsilon), \gamma(\upsilon), \gamma(T\upsilon))}{2s}\end{array}\right\}\right) \\ &\quad= \phi\left(\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(\varsigma, \upsilon), \gamma(\varsigma), \gamma(\upsilon)), \\ \dfrac{\mathcal{G}(0, \gamma(\varsigma), \gamma(T\varsigma)) + \mathcal{G}(0, \gamma(\upsilon), \gamma(T\upsilon))}{2s}\end{array}\right\}\right) \\ &\quad\leq \phi(\mathcal{G}(\rho_b(\varsigma, \upsilon), 0, 0)) < \mathcal{G}(\rho_b(\varsigma, \upsilon), 0, 0).\end{aligned} \tag{30}$$

This is a contradiction. Therefore, $\rho_b(\varsigma, \upsilon) = 0$, that is, T admits a unique fixed point.

In particular, letting $G(\tau, \upsilon, \omega) = \tau + \upsilon + \omega$, for $\tau, \upsilon, \omega \in [0, \infty)$, we can omit the continuity conditions of the mapping T or the partial b-metric ρ_b.

Theorem 3. *Let $(X, \rho_b, s \geq 1)$ be a 0-complete partial b-metric space, a function $\gamma \in \Gamma$, $G \in \mathcal{F}$, and a self-mapping $T : X \longrightarrow X$. If there exists $\phi \in \Phi_b$ such that*

$$\rho_b(Tx, Ty) + \gamma(Tx) + \gamma(Ty)) \leq \phi\left(\max\left\{\begin{array}{c}\rho_b(x, y) + \gamma(x) + \gamma(y)), \\ \dfrac{\rho_b(x, Tx) + \gamma(Tx) + \gamma(x) + \rho_b(y, Ty) + \gamma(Ty) + \gamma(y))}{2s}\end{array}\right\}\right), \tag{31}$$

for every $x, y \in X$, then T has a unique fixed point.

Proof. Of course, since the function $G(\tau, \upsilon, \omega) = \tau + \upsilon + \omega \in \mathcal{F}$, by Theorem 2, we have that the sequence $\{x_n\}$ defined as $x_n = Tx_{n-1}$ is convergent to a point $\varsigma \in X$, and moreover, (22) and (23) hold. We claim that this point ς is a fixed point of T. For this purpose, by (31), for $x = \varsigma$ and $y = \varsigma$, we get

$$\begin{aligned}
&\rho_b(\varsigma, T\varsigma) + \gamma(x_{n+1}) + \gamma(T\varsigma)) \\
&\quad \leq s[\rho_b(\varsigma, Tx_n) + \rho_b(Tx_n, T\varsigma)] \\
&\qquad - \rho_b(Tx_n, Tx_n) + \gamma(x_{n+1}) + \gamma(T\varsigma)) \\
&\quad \leq s\rho_b(\varsigma, x_{n+1}) + s[\rho_b(Tx_n, T\varsigma) + \gamma(Tx_n) + \gamma(T\varsigma)] \\
&\quad \leq s\rho_b(\varsigma, x_{n+1}) + s\phi\left(\max\left\{\rho_b(x_n, \varsigma) + \gamma(x_n) + \gamma(\varsigma), \frac{\rho_b(x_n, x_{n+1}) + \gamma(x_n) + \gamma(x_{n+1})) + \rho_b(\varsigma, T\varsigma) + \gamma(\varsigma) + \gamma(T\varsigma)}{2s}\right\}\right) \\
&\quad < s\rho_b(\varsigma, x_{n+1}) + s\max\left\{\rho_b(x_n, \varsigma) + \gamma(x_n) + \gamma(\varsigma), \frac{\rho_b(x_n, x_{n+1}) + \gamma(x_n) + \gamma(x_{n+1})) + \rho_b(\varsigma, T\varsigma) + \gamma(\varsigma) + \gamma(T\varsigma))}{2s}\right\}.
\end{aligned} \tag{32}$$

Letting $n \longrightarrow \infty$, in the above inequality and keeping in mind (19), (22), and (23), we get

$$\begin{aligned}
\rho_b(\varsigma, T\varsigma) + \gamma(T\varsigma) &\leq s\phi\left(\frac{\rho_b(\varsigma, T\varsigma) + \gamma(T\varsigma)}{2s}\right) \\
&< \frac{\rho_b(\varsigma, T\varsigma) + \gamma(T\varsigma)}{2},
\end{aligned} \tag{33}$$

which is a contradiction. Therefore, $\rho_b(\varsigma, T\varsigma) = 0$, that is, $T\varsigma = \varsigma$.

As in the previous theorem, supposing that there exists υ, another fixed point of T, by (31), we have

$$2\gamma(\upsilon) = \rho_b(T\upsilon, T\upsilon) + \gamma(\upsilon) + \gamma(T\upsilon) \leq \phi(2\gamma(\upsilon)) < 2\gamma(\upsilon), \tag{34}$$

which is a contradiction. Thus, $\gamma(\upsilon) = 0$ and taking $x = \varsigma$ and $y = \upsilon$ in (31), we have

$$\begin{aligned}
\rho_b(\varsigma, \upsilon) &= \rho_b(\varsigma, \upsilon) + \gamma(\varsigma) + \gamma(\upsilon) = \rho_b(\varsigma, \upsilon) + \gamma(\varsigma) + \gamma(\upsilon) \\
&\leq \phi\left(\max\left\{\rho_b(\varsigma, \upsilon) + \gamma(\varsigma) + \gamma(\upsilon), \frac{\rho_b(\varsigma, T\varsigma) + \gamma(\varsigma) + \gamma(T\varsigma) + \rho_b(\upsilon, \upsilon) + \gamma(\upsilon) + \gamma(T\upsilon)}{2s}\right\}\right) \\
&= \phi(\rho_b(\varsigma, \upsilon)) < \rho_b(\varsigma, \upsilon).
\end{aligned} \tag{35}$$

But, this is a contraction, so $\rho_b(\varsigma, \upsilon) = 0$ which proves the uniqueness of the fixed point.

Example 4. Let the set $X = [0, 1]$ and the function $\rho_b : X \times X \longrightarrow [0, \infty)$ be defined by $\rho_b(x, 1/2) = \rho_b(1/2, x) = 1$ for any $x \in X$ and $\rho_b(x, y) = (\max\{x, y\})^2$, otherwise. It easy to see that ρ_b is a partial b-metric space, with $s = 2$. Moreover, since $\lim_{n,m\to\infty}\rho_b(x_n, x_m) = \lim_{n,m\to\infty}(\max\{x_n, x_m\})^2 = 0$ implies $\lim_{n,m\to\infty}x_n = 0$, we have

$$\lim_{n\to\infty}\rho_b(x_n, 0) = \rho_b(0, 0) = 0, \tag{36}$$

which shows that (X, ρ_b, s) is 0-complete. On the other hand, taking, for example, the sequence $\{x_n\}$ in X, where $x_n = n/(2n+1)$, we have $\lim_{n,m\to\infty}\rho_b(x_n, x_m) = 1/4$, but $\lim_{n\to\infty}\rho_b(x_n, 1/2) = 1 = \rho_b(1/2, 1/2) = 1$. Thus, the space (X, ρ_b, s) is not complete.

Let the mapping $T : X \longrightarrow X$ be defined as

$$Tx = \begin{cases} \dfrac{x}{4}, & \text{if } x \in [0,1), \\ \dfrac{1}{2}, & \text{if } x = 1. \end{cases} \tag{37}$$

Choosing $\phi(u) = u/2$ and $\gamma(u) = u$, we have

(i) If $x = y = 1$, then

$$\begin{aligned} &\rho_b(T1, T1) + \gamma(T1) + \gamma(T1)) \\ &\quad = \rho_b\left(\frac{1}{2}, \frac{1}{2}\right) + 2\gamma\left(\frac{1}{2}\right) = \frac{5}{4} < \frac{3}{2} \\ &\quad = \phi\left(\max\left\{\begin{array}{c} \rho_b(1,1) + \gamma(1) + \gamma(1)), \\ \dfrac{\rho_b(1, T1) + \gamma(T1) + \gamma(1))}{2} \end{array}\right\}\right) \end{aligned} \tag{38}$$

(ii) If $x = 1$, $y \in [0, 1)$, then

$$\begin{aligned} &\rho_b(Tx, T1) + \gamma(Tx) + \gamma(T1)) \\ &\quad = \rho_b\left(\frac{x}{4}, \frac{1}{2}\right) + \gamma\left(\frac{x}{4}\right) + \gamma\left(\frac{1}{2}\right) = \frac{3+x}{4} < \frac{2+x}{2} \\ &\quad \le \phi\left(\max\left\{\begin{array}{c} \rho_b(x,1) + \gamma(x) + \gamma(1)), \\ \dfrac{\rho_b(x, Tx) + \gamma(Tx) + \gamma(x) + \rho_b(1, T1) + \gamma(T1) + \gamma(1))}{4} \end{array}\right\}\right) \end{aligned} \tag{39}$$

(iii) If $x, y \in [0, 1)$, then

$$\begin{aligned} &\rho_b(Tx, Ty) + \gamma(Tx) + \gamma(Ty)) \\ &\quad = \rho_b\left(\frac{x}{4}, \frac{y}{4}\right) + \gamma\left(\frac{x}{4}\right) + \gamma\left(\frac{y}{4}\right) \le \frac{x^2 + 4x + 4y}{16} \le \frac{x^2 + x + y}{2} \\ &\quad \le \phi\left(\max\left\{\begin{array}{c} \rho_b(x,y) + \gamma(x) + \gamma(y)), \\ \dfrac{\rho_b(x, Tx) + \gamma(Tx) + \gamma(x) + \rho_b(Ty, y) + \gamma(Ty) + \gamma(y))}{4} \end{array}\right\}\right) \end{aligned} \tag{40}$$

(We considered here $\max\{x, y\} = x$. The case $\max\{x, y\} = y$ is similar.)

Consequently, by Theorem 3, the mapping T admits a unique fixed point.

3. Conclusion

In this paper, we investigate the uniqueness and the existence of a fixed point for certain contraction via the G-function in one of the most general frames and complete the partial b-metric space. Regarding that the metric fixed point theory has a key role in the solution of not only differential equations and fractional differential equations but also integral equations, our results can be applied in these problems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

A Nonlinear Integral Equation Related to Infectious Diseases

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In this paper, a nonlinear integral equation related to infectious diseases is investigated. Namely, we first study the existence and uniqueness of solutions and provide numerical algorithms that converge to the unique solution. Next, we study the lower and upper subsolutions, as well as the data dependence of the solution.

1. Introduction

We consider the nonlinear integral equation

$$x(t)=\prod_{i=1}^{n}\left(f_i(t,x(t))+\int_{t-\tau_i}^{t} g_i(s,x(s))ds\right), t\in\mathbb{R}, \quad (1)$$

where $n\geq 2$ is an integer and $\tau_i>0$, $i=1,2,\cdots,n$. In the case $n=1$ and $f_1\equiv 0$, (1) reduces to

$$x(t)=\int_{t-\tau_1}^{t} g_1(s,x(s))ds\, t\in\mathbb{R}. \quad (2)$$

The integral equation (2) models the spread of certain infectious diseases with periodic contact rate that varies seasonally (see [1]). Several results related to certain mathematical aspects of (2) have been obtained by many authors (see, e.g., [1–9] and the references therein). In particular, in [3], using the Picard operator technique, the integral equation (2) was investigated regarding the existence and uniqueness of solutions and periodic solutions, lower and upper subsolutions, the data dependence, and the differentiability of solutions with respect to a parameter.

In this paper, we are concerned with the integral equation (1). We first investigate the existence and uniqueness of solutions and provide numerical algorithms that converge to the unique solution. Next, we study the lower and upper subsolutions, as well as the data dependence of the solution.

The next section is devoted to the main results of this paper. Namely, in Subsection 2.1, we fix some notations that will be used throughout this paper. In Subsection 2.2, we provide some lemmas that will be used in the proofs of our main results. In Subsection 2.3, the existence and uniqueness of solutions and periodic solutions are derived using the Banach contraction principle. Moreover, an iterative algorithm based on Picard iteration for approximating the unique solution is provided. In Subsection 2.4, a Prešic$'$-type iterative algorithm that converges to the unique solution is provided. Lower and upper subsolutions type results are obtained in Subsection 2.5. Finally, in Subsection 2.6, the data dependence of solutions is studied.

2. Results

We first fix some notations.

2.1. Notations. Let $I=[\alpha,\beta]$ and $J=[m,M]$, where $0<\alpha<\beta$ and $0<m<M$. Let

$$\begin{aligned}C(\mathbb{R}\times I,J)&=\{f:\mathbb{R}\times I\longrightarrow J, f \text{ is continuous}\},\\ X=C(\mathbb{R},I)&=\{f:\mathbb{R}\longrightarrow I, f \text{ is continuous}\}.\end{aligned} \quad (3)$$

The functional space X is equipped with the norm $\|\cdot\|_X$, where

$$\|x\|_X = \sup_{t\in\mathbb{R}}|x(t)|, x \in X. \quad (4)$$

Notice that $(X, \|\cdot\|_X)$ is a Banach space.

2.2. Preliminaries. The following lemma will be useful later. It can be easily proved by induction.

Lemma 1. *Let $\{a_n\}$ and $\{b_n\}$ be two real sequences. Then, for all $n \geq 2$,*

$$\begin{aligned} |a_1a_2 \cdots a_n - b_1b_2 \cdots b_n| &\leq |a_2a_3 \cdots a_n||a_1 - b_1| \\ &+ |b_1||a_3a_4 \cdots a_n||a_2 - b_2| \\ &+ |b_1b_2||a_4a_5 \cdots a_n||a_3 - b_3| + \cdots + |a_n||b_1b_2 \cdots b_{n-2}| \\ &\cdot |a_{n-1} - b_{n-1}| + |b_1b_2 \cdots b_{n-1}||a_n - b_n|. \end{aligned} \quad (5)$$

We recall the following result due to Prešic′ [10].

Lemma 2. *Let (X, d) be a complete metric space, k a positive integer and $\varphi : X^k \longrightarrow X$ a mapping satisfying the following condition:*

$$\begin{aligned} &d(\varphi(x_1, x_2, \cdots, x_k), \varphi(x_2, x_3, \cdots, x_{k+1})) \\ &\leq q_1d(x_1, x_2) + q_2d(x_2, x_3) + \cdots + q_kd(x_k, x_{k+1}), \end{aligned} \quad (6)$$

for all $x_1, \cdots, x_{k+1} \in X$, where $q_1, q_2, \cdots, q_k$ are nonnegative constants such that $q_1 + q_2 + \cdots + q_k < 1$. Then,

(i) There exists a unique $x^* \in X$ such that

$$x^* = \varphi(x^*, x^*, \cdots, x^*). \quad (7)$$

(ii) For all $x_1, x_2, \cdots, x_k \in X$, the sequence $\{x_p\} \subset X$ defined by

$$x_{p+k} = \varphi(x_p, x_{p+1}, \cdots, x_{p+k-1}), p \geq 1 \quad (8)$$

is convergent to x^*.

For more details about the above result, we refer to [11–15].

2.3. Existence and Uniqueness Result. Problem (1) is investigated under the following conditions:

(C1) $f_i, g_i \in C(\mathbb{R} \times I, J)$, $i = 1, 2, \cdots, n$.

(C2) For all $i = 1, 2, \cdots, n$, there exists a constant $L_{f_i} > 0$ such that for all $t \in \mathbb{R}$,

$$|f_i(t, u) - f_i(t, v)| \leq L_{f_i}|u - v|, u, v \in I. \quad (9)$$

(C3) For all $i = 1, 2, \cdots, n$, there exists a constant $L_{g_i} > 0$ such that for all $t \in \mathbb{R}$,

$$|g_i(t, u) - g_i(t, v)| \leq L_{g_i}|u - v|, u, v \in I. \quad (10)$$

(C4) $M^{n-1}(\prod_{i=1}^{n-1}(\tau_i + 1))\sum_{k=1}^{n}(L_{f_k} + L_{g_k}\tau_k) < 1$.

(C5) $\alpha/m^n \leq \prod_{i=1}^{n}(\tau_i + 1) \leq \beta/M^n$.

We have the following existence and uniqueness result.

Theorem 3. *Under conditions (C_1)–(C_5), problem (1) admits one and only one solution $x^* \in X$. Moreover, for all $x_0 \in X$, the sequence $\{x_p\} \subset X$ defined by*

$$x_{p+1}(t) = \prod_{i=1}^{n}\left(f_i(t, x_p(t)) + \int_{t-\tau_i}^{t} g_i(s, x_p(s))\,ds\right), t \in \mathbb{R} \quad (11)$$

converges uniformly to x^*.

Proof. Let us define the operator $T : X \longrightarrow C(\mathbb{R}, \mathbb{R})$ by

$$T(x)(t) = \prod_{i=1}^{n} T_i(x)(t), x \in X, t \in \mathbb{R}, \quad (12)$$

where

$$T_i(x)(t) = f_i(t, x(t)) + \int_{t-\tau_i}^{t} g_i(s, x(s))\,ds, i = 1, 2, \cdots, n. \quad (13)$$

By (C_1), for all $i = 1, 2, \cdots, n$ and $t \in \mathbb{R}$, one has

$$T_i(x)(t) \leq M + \int_{t-\tau_i}^{t} M\,ds = (\tau_i + 1)M, \quad (14)$$

which yields

$$T(x)(t) \leq M^n \prod_{i=1}^{n}(\tau_i + 1). \quad (15)$$

Then, using (C_5), one deduces that

$$T(x)(t) \leq \beta, t \in \mathbb{R}. \quad (16)$$

Similarly, by (C_1), one has

$$T_i(x)(t) \geq m + \int_{t-\tau_i}^{t} m\,ds = (\tau_i + 1)m, \quad (17)$$

which yields

$$T(x)(t) \geq m^n \prod_{i=1}^{n}(\tau_i + 1). \quad (18)$$

Hence, using (C_5), one obtains

$$T(x)(t) \geq \alpha, t \in \mathbb{R}. \quad (19)$$

Therefore, it follows from (16) and (19) that

$$TX \subset X. \tag{20}$$

Moreover, the set of solutions to the integral equation (1) coincides with the set of fixed points of the operator T. Next, by Lemma 1, for all $x, y \in X$ and $t \in \mathbb{R}$, one has

$$\begin{aligned} |T(x)(t) - T(y)(t)| &= \left| \prod_{i=1}^{n} T_i(x)(t) - \prod_{i=1}^{n} T_i(y)(t) \right| \\ &\le T_2(x)(t)T_3(x)(t) \cdots T_n(x)(t)|T_1(x)(t) - T_1(y)(t)| \\ &\quad + T_3(x)(t)T_4(x)(t) \cdots T_n(x)(t)T_1(y)(t)|T_2(x)(t) - T_2(y)(t)| \\ &\quad + T_4(x)(t)T_5(x)(t) \cdots T_n(x)(t)T_1(y)(t)T_2(y)(t) \\ &\quad \cdot |T_3(x)(t) - T_3(y)(t)| + \cdots + T_1(y)(t)T_2(y)(t) \cdots T_{n-1}(y)(t) \\ &\quad \cdot |T_n(x)(t) - T_n(y)(t)|. \end{aligned} \tag{21}$$

On the other hand, by (C_2) and (C_3), for all $i = 1, 2, \cdots, n$, one has

$$\begin{aligned} |T_i(x)(t) - T_i(y)(t)| &\le |f_i(t, x(t)) - f_i(t, y(t))| \\ &\quad + \int_{t-\tau_i}^{t} |g_i(s, x(s)) - g_i(s, y(s))| ds \le L_{f_i}|x(t) - y(t)| \\ &\quad + L_{g_i}\int_{t-\tau_i}^{t} |x(s) - y(s)| ds \le \left(L_{f_i} + L_{g_i}\tau_i\right)\|x - y\|_X. \end{aligned} \tag{22}$$

Therefore, using (14), (21), and (22), one obtains

$$|T(x)(t) - T(y)(t)| \le M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i + 1)\right)\sum_{k=1}^{n}\left(L_{f_k} + L_{g_k}\tau_k\right)\|x - y\|_X, \tag{23}$$

which yields

$$\|Tx - Ty\|_X \le M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i + 1)\right)\sum_{k=1}^{n}\left(L_{f_k} + L_{g_k}\tau_k\right)\|x - y\|_X, x, y \in X. \tag{24}$$

Finally, using (C_4), (20) and (24), the conclusion of the theorem follows from the Banach contraction principle.

Now, we consider problem (1) under the additional condition:

(C6) There exists $\omega > 0$ such that for all $i = 1, 2, \cdots, n$,

$$f_i(t + \omega, u) = f_i(t, u), g_i(t + \omega, u) = g_i(t, u), t \in \mathbb{R}, u \in I. \tag{25}$$

Theorem 4. *Under conditions (C_1)–(C_6), problem (1) admits one and only one ω-periodic solution $x^* \in X$. Moreover, for any ω-periodic function $x_0 \in X$, the sequence $\{x_p\}$ defined by (11) converges uniformly to x^*.*

Proof. Let $T : X \longrightarrow X$ be the operator defined by (12). Notice that from the proof of Theorem 3, we know that under conditions (C_1)–(C_5), one has $TX \subset X$. Let V be the closed subset of X (with respect to the norm $\|\cdot\|_X$) defined by

$$V = \{x \in X : x(t + \omega) = x(t), t \in \mathbb{R}\}. \tag{26}$$

For all $x \in V$ and $t \in \mathbb{R}$, using (C_6), one obtains

$$\begin{aligned} T(x)(t + \omega) &= \prod_{i=1}^{n}\left(f_i(t + \omega, x(t + \omega)) + \int_{t+\omega-\tau_i}^{t+\omega} g_i(s, x(s))\, ds\right) \\ &= \prod_{i=1}^{n}\left(f_i(t, x(t)) + \int_{t-\tau_i}^{t} g_i(\sigma + \omega, x(\sigma + \omega)d\sigma)\right) \\ &= \prod_{i=1}^{n}\left(f_i(t, x(t)) + \int_{t-\tau_i}^{t} g_i(\sigma, x(\sigma))d\sigma\right) = T(x)(t). \end{aligned} \tag{27}$$

Hence, one has $TV \subset V$. On the other hand, since $V \subset X$, it follows from (24) that

$$\|Tx - Ty\|_X \le M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i + 1)\right)\sum_{k=1}^{n}\left(L_{f_k} + L_{g_k}\tau_k\right)\|x - y\|_X, x, y \in V. \tag{28}$$

Then, the conclusion of the theorem follows from the Banach contraction principle.

2.4. Prešic′-Type Approximation of the Unique Solution. Let us consider the integral equation (1) under conditions (C_1)–(C_5). Notice that by Theorem 3, (1) admits one and only one solution $x^* \in X$.

Theorem 5. *Under conditions (C_1)–(C_5), for any $x_1, x_2, \cdots, x_n \in X$, the sequence $\{x_p\}$ defined by*

$$\begin{aligned} x_{p+n}(t) = &\left(f_1\left(t, x_p(t)\right) + \int_{t-\tau_1}^{t} g_1\left(s, x_p(s)\right)ds\right) \\ &\cdot \left(f_2\left(t, x_{p+1}(t)\right) + \int_{t-\tau_2}^{t} g_2\left(s, x_{p+1}(s)\right)ds\right)\cdots \\ &\cdot \left(f_n\left(t, x_{p+n-1}(t)\right) + \int_{t-\tau_n}^{t} g_2\left(s, x_{p+n-1}(s)\right)ds\right), \\ &p \ge 1, t \in \mathbb{R} \end{aligned} \tag{29}$$

converges uniformly to x^*.

Proof. Consider the function $\varphi : X^n \longrightarrow X$ defined by

$$\varphi(x_1, x_2, \cdots, x_n)(t) = \prod_{i=1}^{n}\left(f_i(t, x_i(t)) + \int_{t-\tau_i}^{t} g_1(s, x_i(s))ds\right), t \in \mathbb{R}, \tag{30}$$

that is,

$$\varphi(x_1, x_2,\cdots,x_n)(t) = \prod_{i=1}^{n} T_i(x_i)(t), \quad (31)$$

where for all $i = 1, 2, \cdots, n$, the operator T_i is defined by (13). Notice that from the considered assumptions, one has $\varphi(X^n) \subset X$, so φ is well-defined. On the other hand, using Lemma 1, for all $x_1, x_2, \cdots, x_n, x_{n+1} \in X$ and $t \in \mathbb{R}$, on has

$$\begin{aligned}
&|\varphi(x_1, x_2,\cdots,x_n)(t) - \varphi(x_2, x_3,\cdots,x_{n+1})(t)| \\
&= \left|\prod_{i=1}^{n} T_i(x_i)(t) - \prod_{i=1}^{n} T_i(x_{i+1})(t)\right| \\
&\le T_2(x_2)(t) \cdots T_n(x_n)(t)|T_1(x_1)(t) - T_1(x_2)(t)| \\
&\quad + T_1(x_2)(t)T_3(x_3)(t) \cdots T_n(x_n)(t)|T_2(x_2)(t) - T_2(x_3)(t)| \\
&\quad +\cdots+T_1(x_2)(t) \cdots T_{n-1}(x_n)(t)|T_n(x_n)(t) - T_n(x_{n+1}(t)|.
\end{aligned} \quad (32)$$

Next, using (14), it holds that

$$\begin{aligned}
&|\varphi(x_1, x_2,\cdots,x_n)(t) - \varphi(x_2, x_3,\cdots,x_{n+1})(t)| \\
&\le M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i + 1)\right)\sum_{k=1}^{n}|T_k(x_k)(t) - T_k(x_{k+1}(t)|.
\end{aligned} \quad (33)$$

On the other hand, under the considered assumptions, for all $k = 1, 2, \cdots, n$, one has

$$|T_k(x_k)(t) - T_k(x_{k+1}(t)| \le \left(L_{f_k} + \tau_k L_{g_k}\right)\|x_k - x_{k+1}\|_X. \quad (34)$$

Hence, one deduces that

$$\begin{aligned}
&\|\varphi(x_1, x_2,\cdots,x_n) - \varphi(x_2, x_3,\cdots,x_{n+1})\|_X \\
&\le M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i + 1)\right)\sum_{k=1}^{n}\left(L_{f_k} + \tau_k L_{g_k}\right)\|x_k - x_{k+1}\|_X.
\end{aligned} \quad (35)$$

Finally, using (C_4) and Lemma 2, the desired result follows.

2.5. Lower and Upper Subsolutions. We consider problem (1) under conditions (C_1)–(C_5). We recall that by Theorem 3, problem (1) admits one and only one solution $x^* \in X$. We suppose also that

For all $i = 1, 2, \cdots, n$ and $t \in \mathbb{R}$, the functions

$$f_i(t, \cdot): I \longrightarrow J \text{ and } g_i(t, \cdot): I \longrightarrow J \quad (36)$$

are nondecreasing.

Theorem 6. *Suppose that conditions (C_1)–(C_5) and (C_6') are satisfied. If $x \in C(\mathbb{R}, I)$ satisfies*

$$x(t) \le \prod_{i=1}^{n}\left(f_i(t, x(t)) + \int_{t-\tau_i}^{t} g_i(s, x(s))ds\right), t \in \mathbb{R}, \quad (37)$$

then

$$x(t) \le x^*(t), t \in \mathbb{R}. \quad (38)$$

Proof. Let $T : X \longrightarrow X$ be the operator defined by (12). Then, (39) is equivalent to

$$x(t) \le T(x)(t), t \in \mathbb{R}. \quad (39)$$

We shall prove that T is a nondecreasing operator, that is,

$$u, v \in X, u(t) \le v(t), t \in \mathbb{R} \Longrightarrow T(u)(t) \le T(v)(t), t \in \mathbb{R}. \quad (40)$$

Let $u, v \in X$ be such that

$$u(t) \le v(t), t \in \mathbb{R}. \quad (41)$$

By (C_6'), for all $i = 1, 2, \cdots, n$ and $t \in \mathbb{R}$, one obtains

$$0 \le f_i(t, u(t)) + \int_{t-\tau_i}^{t} g_i(s, u(s))ds \le f_i(t, v(t)) + \int_{t-\tau_i}^{t} g_i(s, v(s))ds, \quad (42)$$

which yields

$$\begin{aligned}
&\prod_{i=1}^{n}\left(f_i(t, u(t)) + \int_{t-\tau_i}^{t} g_i(s, u(s))ds\right) \\
&\le \prod_{i=1}^{n}\left(f_i(t, v(t)) + \int_{t-\tau_i}^{t} g_i(s, v(s))ds\right),
\end{aligned} \quad (43)$$

that is,

$$T(u)(t) \le T(v)(t). \quad (44)$$

This proves (40). Next, by (39), it holds that

$$x(t) \le T(x)(t) \le T^2(x)(t) \le \cdots \le T^p(x)(t), \quad (45)$$

for all nonnegative integer p and $t \in \mathbb{R}$, where

$$T^0(x)(t) = x(t) \text{ and } T^{p+1}(x)(t) = T(T^p(x))(t). \quad (46)$$

Hence, it holds that

$$x(t) \le x_p(t), t \in \mathbb{R}, \quad (47)$$

where $\{x_p\}$ is the sequence defined by (11) with $x_0 = x$.

On the other hand, by Theorem 3, one has

$$\lim_{p\to\infty} x_p(t) = x^*(t), t \in \mathbb{R}. \quad (48)$$

Therefore, passing to the limit as $p \longrightarrow \infty$ in (47), (38) follows.

(A1) For $j = 1, 2, 3$ and $i = 1, 2, \cdots, n$, let $f_i^{(j)}, g_i^{(j)} \in C(\mathbb{R} \times I, J)$. We suppose that

(A2) For all $i = 1, 2, \cdots, n$ and $j = 1, 2, 3$, there exists a constant $L_{f_i^{(j)}} > 0$ such that for all $t \in \mathbb{R}$,

$$\left|f_i^{(j)}(t,u) - f_i^{(j)}(t,v)\right| \le L_{f_i^{(j)}}|u-v|, u, v \in I. \quad (49)$$

For all $i = 1, 2, \cdots, n$ and $j = 1, 2, 3$, there exists a constant $L_{g_i^{(j)}} > 0$ such that for all $t \in \mathbb{R}$,

$$\left|g_i^{(j)}(t,u) - g_i^{(j)}(t,v)\right| \le L_{g_i^{(j)}}|u-v|, u, v \in I. \quad (50)$$

(A3) $M^{n-1}(\prod_{i=1}^{n-1}(\tau_i+1))\sum_{k=1}^{n}(L_{f_k^{(j)}} + L_{g_k^{(j)}}\tau_k) < 1,\ j = 1, 2, 3$.

(A4) $\alpha/m^n \le \prod_{i=1}^{n}(\tau_i+1) \le \beta/M^n$.

(A5) For all $i = 1, 2, \cdots, n$ and $t \in \mathbb{R}$, the functions

$$f_i^{(2)}(t,\cdot): I \longrightarrow J \text{ and } g_i^{(2)}(t,\cdot): I \longrightarrow J, \quad (51)$$

are nondecreasing.

(A6) For all $i = 1, 2, \cdots, n$, $t \in \mathbb{R}$ and $u \in I$,

$$f_i^{(1)}(t,u) \le f_i^{(2)}(t,u) \le f_i^{(3)}(t,u) \text{ and } g_i^{(1)}(t,u) \le g_i^{(2)}(t,u) \le g_i^{(3)}(t,u). \quad (52)$$

Notice that by (A_1)–(A_4), it follows from Theorem 3 that for all $j = 1, 2, 3$, the integral equation

$$x(t) = \prod_{i=1}^{n}\left(f_i^{(j)}(t,x(t)) + \int_{t-\tau_i}^{t} g_i^{(j)}(s,x(s))ds\right), t \in \mathbb{R}, \quad (53)$$

admits one and only one solution $x^{(j)} \in X$. Moreover, for all $j = 1, 2, 3$ and $x_0^{(j)} \in X$, the sequence $\{x_p^{(j)}\} \subset X$ defined by

$$x_{p+1}^{(j)}(t) = \prod_{i=1}^{n}\left(f_i^{(j)}\left(t, x_p^{(j)}(t)\right) + \int_{t-\tau_i}^{t} g_i^{(j)}\left(s, x_p^{(j)}(s)\right)ds\right), t \in \mathbb{R} \quad (54)$$

converges uniformly to $x^{(j)}$.

Theorem 7. *Under conditions (A_1)–(A_6), one has*

$$x^{(1)}(t) \le x^{(2)}(t) \le x^{(3)}(t), t \in \mathbb{R}. \quad (55)$$

Proof. For all $j = 1, 2, 3$, let $T^{(j)} : X \longrightarrow X$ be the operator defined by

$$T^{(j)}(x)(t) = \prod_{i=1}^{n}\left(f_i^{(j)}(t,x(t)) + \int_{t-\tau_i}^{t} g_i^{(j)}(s,x(s))ds\right), x \in X, t \in \mathbb{R}. \quad (56)$$

From condition (A_5), the operator $T^{(2)}$ is nondecreasing, that is,

$$u, v \in X, u(t) \le v(t), t \in \mathbb{R} \Longrightarrow T^{(2)}(u)(t) \le T^{(2)}(v)(t), t \in \mathbb{R}. \quad (57)$$

Moreover, by (A_6), one has

$$T^{(1)}(u)(t) \le T^{(2)}(u)(t) \le T^{(3)}(u)(t), u \in X, t \in \mathbb{R}. \quad (58)$$

Let $x_0^{(1)}, x_0^{(2)}, x_0^{(3)} \in X$ be such that

$$x_0^{(1)}(t) \le x_0^{(2)}(t) \le x_0^{(3)}(t), t \in \mathbb{R}. \quad (59)$$

Hence, by (57), one obtains

$$T^{(2)}\left(x_0^{(1)}\right)(t) \le T^{(2)}\left(x_0^{(2)}\right)(t) \le T^{(2)}\left(x_0^{(3)}\right)(t), t \in \mathbb{R}. \quad (60)$$

On the other hand, by (58), one has

$$T^{(1)}\left(x_0^{(1)}\right)(t) \le T^{(2)}\left(x_0^{(1)}\right)(t), t \in \mathbb{R}, \quad (61)$$

$$T^{(2)}\left(x_0^{(3)}\right)(t) \le T^{(3)}\left(x_0^{(3)}\right)(t), t \in \mathbb{R}. \quad (62)$$

Therefore, using (60), (61), and (62), one deduces that

$$T^{(1)}\left(x_0^{(1)}\right)(t) \le T^{(2)}\left(x_0^{(2)}\right)(t) \le T^{(3)}\left(x_0^{(3)}\right)(t), t \in \mathbb{R}, \quad (63)$$

that is,

$$x_1^{(1)}(t) \le x_1^{(2)}(t) \le x_1^{(3)}(t), t \in \mathbb{R}. \quad (64)$$

Repeating the same argument, by induction, one deduces that for all nonnegative integer p and $t \in \mathbb{R}$,

$$x_p^{(1)}(t) \le x_p^{(2)}(t) \le x_p^{(3)}(t), \quad (65)$$

where $\{x_p^{(j)}\}$, $j = 1, 2, 3$, is the sequence defined by (54). Finally, passing to the limit as $p \longrightarrow \infty$ in (65), the desired result follows.

2.6. Data Dependence of Solutions. Suppose that conditions (C_1)–(C_5) are satisfied. Then, by Theorem 3, the integral equation admits one and only one solution $x^* \in X$. Consider now the perturbed problem

$$y(t) = \prod_{i=1}^{n}\left(F_i(t,y(t)) + \int_{t-\tau_i}^{t} G_i(s,y(s))ds\right), t \in \mathbb{R}, \quad (66)$$

where $F_i, G_i \in C(\mathbb{R} \times I, J)$, $i = 1, 2, \cdots, n$. Suppose that $y^* \in X$ is a solution to the integral equation (66).

We have the following data dependence result.

Theorem 8. *Suppose that for all $i = 1, 2, \cdots, n$, there exist σ_i, $\eta_i > 0$ such that*

$$|f_i(t,u) - F_i(t,u)| \le \sigma_i, |g_i(t,u) - G_i(t,u)| \le \eta_i, t \in \mathbb{R}, u \in I. \tag{67}$$

Then,

$$\|x^* - y^*\|_X \le \frac{M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i+1)\right)\sum_{k=1}^n(\sigma_k + \tau_k\eta_k)}{\left[1 - M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i+1)\right)\sum_{k=1}^n\left(L_{f_k} + L_{g_k}\tau_k\right)\right]}. \tag{68}$$

Proof. Let

$$S(y)(t) = \prod_{i=1}^n S_i(y)(t), t \in \mathbb{R}, \tag{69}$$

where

$$S_i(y)(t) = F_i(t, y(t)) + \int_{t-\tau_i}^t G_i(s, y(s))\, ds, i = 1, 2, \cdots, n. \tag{70}$$

Then, for all $t \in \mathbb{R}$, one has

$$|x^*(t) - y^*(t)| = |T(x^*)(t) - S(y^*)(t)| = \left|\prod_{i=1}^n T_i(x^*)(t) - \prod_{i=1}^n S_i(y^*)(t)\right|, \tag{71}$$

where the operator T is defined by (12). Next, by Lemma 1 and (14), one obtains

$$\begin{aligned} |x^*(t) - y^*(t)| &\le T_2(x^*)(t) \cdots T_n(x^*)(t)|T_1(x^*)(t) - S_1(y^*)(t)| \\ &\quad + S_1(y^*)(t)T_3(x^*)(t) \cdots T_n(x^*)(t)|T_2(x^*)(t) - S_2(y^*)(t)| \\ &\quad + \cdots + S_1(y^*)(t) \cdots S_{n-1}(y^*)(t)|T_n(x^*)(t) - S_n(y^*)(t)| \\ &\le M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i+1)\right)\sum_{k=1}^n|T_k(x^*)(t) - S_k(y^*)(t)|. \end{aligned} \tag{72}$$

On the other hand, using (C_2) and (C_3) and (67), for all $k = 1, 2, \cdots, n$, one has

$$\begin{aligned} |T_k(x^*)(t) - S_k(y^*)(t)| &\le |f_k(t, x^*(t)) - F_k(t, y^*(t))| \\ &\quad + \int_{t-\tau_k}^t |g_k(s, x^*(s)) - G_k(s, y^*(s))| ds \\ &\le |f_k(t, x^*(t)) - f_k(t, y^*(t))| + |f_k(t, y^*(t)) - F_k(t, y^*(t))| \\ &\quad + \int_{t-\tau_k}^t |g_k(s, x^*(s)) - g_k(s, y^*(s))| ds \\ &\quad + \int_{t-\tau_k}^t |g_k(s, y^*(s)) - G_k(s, y^*(s))| ds \le L_{f_k}\|x^* - y^*\|_X \\ &\quad + \sigma_k + L_{g_k}\tau_k\|x^* - y^*\|_X + \tau_k\eta_k = \left(L_{f_k} + L_{g_k}\tau_k\right)\|x^* - y^*\|_X \\ &\quad + \sigma_k + \tau_k\eta_k. \end{aligned} \tag{73}$$

Hence, by (72), it holds that

$$\begin{aligned} \|x^* - y^*\|_X &\le M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i+1)\right) \\ &\quad \cdot \sum_{k=1}^n\left[\left(L_{f_k} + L_{g_k}\tau_k\right)\|x^* - y^*\|_X + \sigma_k + \tau_k\eta_k\right], \end{aligned} \tag{74}$$

which yields

$$\begin{aligned} &\left[1 - M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i+1)\right)\sum_{k=1}^n\left(L_{f_k} + L_{g_k}\tau_k\right)\right]\|x^* - y^*\|_X \\ &\le M^{n-1}\left(\prod_{i=1}^{n-1}(\tau_i+1)\right)\sum_{k=1}^n(\sigma_k + \tau_k\eta_k). \end{aligned} \tag{75}$$

Finally, by (C_4), the desired result follows.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

On $\mathfrak{R}$-Partial b-Metric Spaces and Related Fixed Point Results with Applications

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In this paper, we introduce the notion of $\mathfrak{R}$-partial b-metric spaces and prove some related fixed point results in the context of this notion. We also discuss an example to validate our result. Finally, as applications, we evince the importance of our work by discussing some fixed point results on graphical-partial b-metric spaces and on partially-ordered-partial b-metric spaces.

1. Introduction and Preliminaries

Due to the fact that fixed point theory plays a very crucial role for different mathematical models to obtain their solution existence and has a wide range of applications in different fields related to mathematics, this theory has intrigued many researchers.

By the inception of the Banach fixed point theorem [1], researchers are continuously trying to get the generalizations of this classical result through different methodologies. For instance, Czerwik [2] introduced the notion of b-metric spaces, with a triangle inequality weaker than that of metric spaces, in a view to generalize the Banach contraction principle. Moving on the same sequel, Matthews [3] introduced the notion of a partial metric space, which was a part of the study for denotational semantics of dataflow networks and gave a generalized version of the Banach contraction principle. The concept of partial metric spaces was further extended to partial b-metric spaces by Shukla in [4]. A number of researchers took keen interest in the generalized version of the metric spaces some work is available in [5–27].

Recently, Gordgi et al. [28] introduced the notion of orthogonal sets and gave a new extension for the classical Banach contraction principle. More details can be found in [29, 30].

After looking into the structure of orthogonal metric spaces, introduced by [29, 30], and the binary relation used with a metric, [31, 32], we introduce the notion of $\mathfrak{R}$-partial b-metric spaces. We are also improving and generalizing the concept of orthogonal contractions in the sense of $\mathfrak{R}$-partial b-metric spaces and establish some fixed point theorems for the proposed contractions.

Throughout this paper, we denote by $\mathbb{N}$,$\mathbb{R}$,$\mathbb{Z}$, and $\mathbb{R}^+$ the set of natural numbers, real numbers, integer numbers, and nonnegative real numbers, respectively.

Definition 1 (see [2]. *Let H be a nonempty set and $s \geq 1$. Suppose a mapping $d : H \times H \longrightarrow \mathbb{R}^+$ satisfies the following conditions for all $h,l,z \in H$:*

$(bM1) d(h, l) = 0$ if and only if $h = l$;
$(bM2) d(h, l) = d(l, h)$;
$(bM3) d(h, l) \le s[d(h, z) + d(z, l)]$.
Then d is called a b-metric on H, and (H, d) is called a b-metric space with coefficient s.

Definition 2 (see [3]. *Let H be a nonempty set. Let $p : H \times H \longrightarrow \mathbb{R}^+$ satisfy the following for all h,l,z $\in H$:*
$(pM1) h = l$ if and only if $p(h, h) = p(h, l) = p(l, l)$;
$(pM2) p(h, h) \le p(h, l)$;
$(pM3) p(h, l) = p(l, h)$;
$(pM4) p(h, l) \le p(h, z) + p(z, l) - p(z, z)$.
Then (H, p) is called a partial metric space.

Definition 3 [4]. *A partial b -metric on $H \ne \emptyset$ is a function $\sigma : H \times H \longrightarrow \mathbb{R}^+$ such that for all h,l,z $\in H$, and for some $s \ge 1$, we have*
$(\sigma 1) h = l$ if and only if $\sigma(h, h) = \sigma(h, l) = \sigma(l, l)$;
$(\sigma 2) \sigma(h, h) \le \sigma(h, l)$;
$(\sigma 3) \sigma(h, l) = \sigma(l, h)$;
$(\sigma 4) \sigma(h, l) \le s[\sigma(h, z) + \sigma(z, l)] - \sigma(z, z)$.
A partial b-metric space is denoted with (H, σ, s). The number s is called the coefficient of (H, σ, s).

Remark 4 (see [4]. *It is clear that every partial metric space is a partial b -metric space with coefficient $s = 1$ and every b -metric space is a partial b -metric space with the same coefficient and a zero self-distance. However, the converse of this fact need not hold.*

Example 1 [4]. *Let $H = \mathbb{R}^+$, $p > 1$ be a constant and $\sigma : H \times H \longrightarrow \mathbb{R}^+$ be defined by*

$$\sigma(h, l) = |h - l|^p + (\max\{h, l\})^p \quad \text{for all} \quad h, l \in H. \tag{1}$$

Then, (H, σ, s) is a partial b-metric space with coefficient $s = 2^p > 1$, but it is neither a b-metric nor a partial metric space.

Definition 5 [33]. *Let H be a nonempty set. A subset $\mathfrak{R}$ of H^2 is called a binary relation on H. Then, for any $h, l \in H$, we say that "h is $\mathfrak{R}$ -related to l", that is, $h\mathfrak{R}l$, or "h relates to l under $\mathfrak{R}$" if and only if $(h, l) \in \mathfrak{R}$. $(h, l) \notin \mathfrak{R}$ means that "h is not $\mathfrak{R}$ -related to l" or "h is not related to l under $\mathfrak{R}$".*

Definition 6 [33]. *A binary relation $\mathfrak{R}$ defined on a nonempty set H is called (a) reflexive if $(h, h) \in \mathfrak{R} \forall h \in H$;*
(b) irreflexive if $(h, h) \notin \mathfrak{R}$ for some $h \in H$;
(c) symmetric if $(h, l) \in \mathfrak{R}$ implies $(l, h) \in \mathfrak{R} \forall h, l \in H$;
(d) antisymmetric if $(h, l) \in \mathfrak{R}$ and $(l, h) \in \mathfrak{R}$ imply $h = l$ $\forall h,l \in H$;
(e) transitive if $(h, l) \in \mathfrak{R}$ and $(l, z) \in \mathfrak{R}$ imply $(h, z) \in \mathfrak{R}$ $\forall h,l, z \in H$;
(f) preorder if $\mathfrak{R}$ is reflexive and transitive;
(g) partial order if $\mathfrak{R}$ is reflexive, antisymmetric, and transitive.

Definition 7 [32]. *Let H be a nonempty set and let $\mathfrak{R}$ be a binary relation on H.*

(a) A sequence $\{h_n\}$ is called an $\mathfrak{R}$-sequence if

$$(\forall n \in \mathbb{N}, h_n \mathfrak{R} h_{n+1}). \tag{2}$$

(b) A map $T : H \longrightarrow H$ is $\mathfrak{R}$-preserving if

$$\forall h, l \in H, h\mathfrak{R}l \quad \text{implies} \quad Th\mathfrak{R}Tl. \tag{3}$$

Definition 8 [32]. *Let (H, d) be a metric space and $\mathfrak{R}$ be a binary relation on H. Then, $(H, d, \mathfrak{R})$ is called an $\mathfrak{R}$ -metric space.*

Definition 9 [31]. *A mapping $T : H \longrightarrow H$ is $\mathfrak{R}$ -continuous at $h_0 \in H$ if for each $\mathfrak{R}$ -sequence $\{h_n\}_{n\in\mathbb{N}}$ in H with $h_n \longrightarrow h_0$, we get $T(h_n) \longrightarrow T(h_0)$. Thus, T is $\mathfrak{R}$ -continuous on H if T is $\mathfrak{R}$ -continuous at each $h_0 \in H$.*

Definition 10 [31]. *A map $T : H \longrightarrow H$ is an $\mathfrak{R}$ -contraction, if*

$$d(Th, Tl) \le k d(h, l), \tag{4}$$

for all $h, l \in H$ with $h\mathfrak{R}l$, where $0 < k < 1$.

Khalehoghli et al. [31] extended the result of Banach in the following way.

Theorem 11 [31]. *If T is an $\mathfrak{R}$ -preserving and $\mathfrak{R}$ -continuous $\mathfrak{R}$ -contraction on an $\mathfrak{R}$ -complete $\mathfrak{R}$ -metric space with $h_0 \in H$ such that $h_0\mathfrak{R}l$ for each $l \in H$. Then, T has a unique fixed point.*

2. Main Results

Let us begin this section with the definition of $\mathfrak{R}$-partial b-metric spaces.

Definition 12. Let $H \ne \emptyset$ and $\mathfrak{R}$ be a reflexive binary relation on H, denoted as $(H, \mathfrak{R})$. A map $\sigma_{\mathfrak{R}} : H \times H \longrightarrow \mathbb{R}^+$ is called an $\mathfrak{R}$ -partial b -metric on the set H, if the following conditions are satisfied for all $h, l, z \in H$ with either ($h\mathfrak{R}l$ or $l\mathfrak{R}h$), either ($h\mathfrak{R}z$ or $z\mathfrak{R}h$) and either ($z\mathfrak{R}l$ or $l\mathfrak{R}z$):
$(\sigma_{\mathfrak{R}} 1) h = l$ if and only if $\sigma_{\mathfrak{R}}(h, h) = \sigma_{\mathfrak{R}}(h, l) = \sigma_{\mathfrak{R}}(l, l)$;
$(\sigma_{\mathfrak{R}} 2) \sigma_{\mathfrak{R}}(h, h) \le \sigma_{\mathfrak{R}}(h, l)$;
$(\sigma_{\mathfrak{R}} 3) \sigma_{\mathfrak{R}}(h, l) = \sigma_{\mathfrak{R}}(l, h)$;
$(\sigma_{\mathfrak{R}} 4) \sigma_{\mathfrak{R}}(h, l) \le s[\sigma_{\mathfrak{R}}(h, z) + \sigma_{\mathfrak{R}}(z, l)] - \sigma_{\mathfrak{R}}(z, z)$, where $s \ge 1$.
Then, $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ is called $\mathfrak{R}$-partial b-metric space with the coefficient $s \ge 1$.

Remark 13. In the above definition, a set H is endowed with a reflexive binary relation $\mathfrak{R}$ and $\sigma_{\mathfrak{R}} : H \times H \longrightarrow \mathbb{R}^+$ satisfies $(\sigma_{\mathfrak{R}} 1)$ - $(\sigma_{\mathfrak{R}} 4)$ only for those elements which are comparable under the reflexive binary relation $\mathfrak{R}$. Hence, the $\mathfrak{R}$ -partial b -metric may not be a partial b -metric, but the converse is true.

The following simplest example shows that the $\mathfrak{R}$-partial b-metric with $s \ge 1$ need not to be a partial b-metric with $s \ge 1$.

Example 2. Let $H = \{-1,-2, 1, 2\}$ and let the binary relation be defined by $h\mathfrak{R}l$ if and only if $h = l$ or $h, l > 0$. It is easy to prove that $\sigma_{\mathfrak{R}}(h, l) = \max\{|h|, |l|\}$ is an $\mathfrak{R}$ -partial b -metric on H with $s \geq 1$, but $\sigma_{\mathfrak{R}}$ is not a partial b -metric on H with $s \geq 1$. Indeed, for $h = -2$ and $l = 2$, we have $\sigma_{\mathfrak{R}}(h, h) = \sigma_{\mathfrak{R}}(h, l) = \sigma_{\mathfrak{R}}(l, l) = 2$.

In the coming definitions, let $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ be an $\mathfrak{R}$ -partial b-metric space with the coefficient $s \geq 1$.

Definition 14. Let $\{h_n\}$ be an $\mathfrak{R}$ -sequence in $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$, that is, $h_n\mathfrak{R}h_{n+1}$ or $h_{n+1}\mathfrak{R}h_n$ for each $n \in \mathbb{N}$. Then

(i) $\{h_n\}$ is a convergent sequence to some $h \in H$ if $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$ and $h_n\mathfrak{R}h$ for each $n \geq k$

(ii) $\{h_n\}$ is Cauchy if $\lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m)$ exists and is finite

Definition 15. $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ is said to be $\mathfrak{R}$ -complete if for every Cauchy $\mathfrak{R}$ -sequence in H , there is $h \in H$ with $\lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$ and $h_n \mathfrak{R}h$ for each $n \geq k$.

Definition 16. We say that $T : H \longrightarrow H$ is an R -property map, if for any iterative $\mathfrak{R}$ -sequence $\{h_n : h_n = T^n h, h \in H\}$ in $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ with $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$, $h_n\mathfrak{R}h$ for some $n \geq k$ and $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, Th) \leq \sigma_{\mathfrak{R}}(h, h)$, we have that $h\mathfrak{R}Th$ or $Th\mathfrak{R}h$.

Definition 17. We say that $T : H \longrightarrow H$ is $\mathfrak{R}$ -0-continuous at $h \in H$ if for each $\mathfrak{R}$ -sequence $\{h_n\}$ in $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ with $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h) = 0$, we have $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(Th_n, Th) = 0$. Also, T is $\mathfrak{R}$ -0-continuous on H if T is $\mathfrak{R}$ -0-continuous for each $h \in H$.

The following results help us to ensure the existence of fixed points for self maps. Throughout, we assume that $\mathfrak{R}$ is a preorder relation.

Theorem 18. *Let $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ be an $\mathfrak{R}$ -complete $\mathfrak{R}$ -partial b -metric space with the coefficient $s \geq 1$ and let $h_0 \in H$ be such that $h_0\mathfrak{R}l$ for each $l \in H$. Let $T : H \longrightarrow H$ be an $\mathfrak{R}$ -preserving and an R -property map satisfying the following*

$$\sigma_{\mathfrak{R}}(Th, Tl) \leq k\sigma_{\mathfrak{R}}(h, l) \quad \text{for all} \quad h, l \in H \quad \text{with} \quad h\mathfrak{R}l, \tag{5}$$

where $k \in [0, 1/s)$. Then, T has a fixed point $h^* \in H$ and $\sigma_{\mathfrak{R}}(h^*, h^*) = 0$.

Proof. As $h_0 \in H$ is such that $h_0\mathfrak{R}l$ for each $l \in H$, then by using the $\mathfrak{R}$-preserving nature of T, we construct an $\mathfrak{R}$ -sequence $\{h_n\}$ such that $h_n = Th_{n-1} = T^n h_0$ and $h_{n-1}\mathfrak{R}h_n$ for each $n \in \mathbb{N}$. We consider $h_n \neq h_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Thus, by (5), we get

$$\sigma_{\mathfrak{R}}(h_n, h_{n+1}) = \sigma_{\mathfrak{R}}(Th_{n-1}, Th_n) \leq k\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \tag{6}$$

for all $n \in \mathbb{N}$. This inequality yields

$$\sigma_{\mathfrak{R}}(h_n, h_{n+1}) \leq k^n\sigma_{\mathfrak{R}}(h_0, h_1), \tag{7}$$

for all $n \in \mathbb{N}$. To discuss the Cauchy criteria, we will consider an arbitrary integer $n \geq 1, m \geq 1$ with $m > n$ and use $\sigma_{\mathfrak{R}_4}$ along (7) in the following way.

$$\begin{aligned}
\sigma_{\mathfrak{R}}(h_n, h_m) &\leq s[\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + \sigma_{\mathfrak{R}}(h_{n+1}, h_m)] - \sigma_{\mathfrak{R}}(h_{n+1}, h_{n+1}) \\
&\leq s\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + s^2[\sigma_{\mathfrak{R}}(h_{n+1}, h_{n+2}) + \sigma_{\mathfrak{R}}(h_{n+2}, h_m)] \\
&\quad - \sigma_{\mathfrak{R}}(h_{n+2}, h_{n+2}) \leq s\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + s^2\sigma_{\mathfrak{R}}(h_{n+1}, h_{n+2}) \\
&\quad + s^3\sigma_{\mathfrak{R}}(h_{n+2}, h_{n+3}) + \cdots + s^{m-n}\sigma_{\mathfrak{R}}(h_{m-1}, h_m) \\
&\leq sk^n\sigma_{\mathfrak{R}}(h_0, h_1) + s^2k^{n+1}\sigma_{\mathfrak{R}}(h_0, h_1) \\
&\quad + s^3k^{n+2}\sigma_{\mathfrak{R}}(h_0, h_1) + \cdots + s^{m-n}k^{m-1}\sigma_{\mathfrak{R}}(h_0, h_1) \\
&\leq sk^n\left[1 + sk + (sk)^2 + \cdots\right]\sigma_{\mathfrak{R}}(h_0, h_1) \\
&= \frac{sk^n}{1 - sk}\sigma_{\mathfrak{R}}(h_0, h_1).
\end{aligned} \tag{8}$$

As $k \in [0, 1/s)$ and $s \geq 1$, it follows from the above inequality that

$$\lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = 0. \tag{9}$$

Therefore, $\{h_n\}$ is a Cauchy $\mathfrak{R}$-sequence. Since H is $\mathfrak{R}$ -complete, there exists $h^* \in H$ such that $\lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h^*) = \sigma_{\mathfrak{R}}(h^*, h^*)$ and $h_n\mathfrak{R}h^*$ for each $n \geq k$ (for some value of k). Thus, from above, we obtain $0 = \lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h^*) = \sigma_{\mathfrak{R}}(h^*, h^*)$ and $h_n\mathfrak{R}h^*$ for each $n \geq k$. As $h_n\mathfrak{R}h^*$ for each $n \geq k$, from (5), we get

$$\sigma_{\mathfrak{R}}(Th_n, Th^*) \leq k\sigma_{\mathfrak{R}}(h_n, h^*). \tag{10}$$

This inequality and the above findings imply

$$\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_{n+1}, Th^*) \leq \sigma_{\mathfrak{R}}(h^*, h^*) = 0. \tag{11}$$

As T is an $\mathfrak{R}$-property map, so we get $h^*\mathfrak{R}Th^*$ or $Th^*\mathfrak{R}h^*$. Without any loss of generality, we take $h^*\mathfrak{R}Th^*$. Thus, by using $\sigma_{\mathfrak{R}_4}$ with (5), we get the following for each $n \geq k$

$$\begin{aligned}
\sigma_{\mathfrak{R}}(h^*, Th^*) &\leq s\sigma_{\mathfrak{R}}(h^*, h_{n+1}) + s\sigma_{\mathfrak{R}}(h_{n+1}, Th^*) \\
&\quad - \sigma_{\mathfrak{R}}(h_{n+1}, h_{n+1}) \leq s\sigma_{\mathfrak{R}}(h^*, h_{n+1}) \\
&\quad + s\sigma_{\mathfrak{R}}(Th_n, Th^*) \leq s\sigma_{\mathfrak{R}}(h^*, h_{n+1}) \\
&\quad + sk\sigma_{\mathfrak{R}}(h_n, h^*).
\end{aligned} \tag{12}$$

When n tends to infinity, the above inequality yields $\sigma_{\mathfrak{R}}(h^*, Th^*) = 0$. Hence, we get $\sigma_{\mathfrak{R}}(h^*, Th^*) = 0$, $\sigma_{\mathfrak{R}}(h^*, h^*) =$

0 and $\sigma_{\mathfrak{R}}(Th^*, Th^*) = 0$. Therefore, $h^* = Th^*$, that is, h^* is a fixed point of T.

Remark 19. Note that the fixed point of T is unique if in the above theorem we add (I): for each fixed points h^* and l^* of T, we have $h^*\mathfrak{R}l^*$ or $l^*\mathfrak{R}h^*$.

Since h^* and l^* are fixed points of T such that $h^*\mathfrak{R}l^*$. Then, we have $T^n h^* = h^*, T^n l^* = l^*$ for all $n \in \mathbb{N}$. By the nature of h_0, we obtain

$$h_0\mathfrak{R}h^* \text{and} h_0\mathfrak{R}l^*. \tag{13}$$

Since T is $\mathfrak{R}$-preserving, we have

$$T^n h_0 \mathfrak{R} T^n h^* \text{and} T^n h_0 \mathfrak{R} T^n l^*, \tag{14}$$

for all $n \in \mathbb{N}$. Therefore, by the triangle inequality and (5), we get

$$\begin{aligned} \sigma_{\mathfrak{R}}(h^*, l^*) = \sigma_{\mathfrak{R}}(T^n h^*, T^n l^*) = s[\sigma_{\mathfrak{R}}(T^n h^*, T^n h_0) \\ + \sigma_{\mathfrak{R}}(T^n h_0, T^n l^*)] - \sigma_{\mathfrak{R}}(T^n h_0, T^n h_0) \\ \le sk^n \sigma_{\mathfrak{R}}(h^*, h_0) + sk^n \sigma_{\mathfrak{R}}(h_0, l^*). \end{aligned} \tag{15}$$

Taking limit as $n \longrightarrow \infty$ in the above inequality, we obtain

$$\sigma_{\mathfrak{R}}(h^*, l^*) = 0, \tag{16}$$

and so

$$h^* = l^*. \tag{17}$$

Remark 20. Note that the condition "let $h_0 \in H$ be such that $h_0\mathfrak{R}l$ for each $l \in H$" of Theorem18may be replaced with "let $h_0 \in H$ be such that $h_0\mathfrak{R}Th_0$."

Example 3. Let $H = \mathbb{R}$ and define $\sigma_{\mathfrak{R}} : H \times H \longrightarrow \mathbb{R}^+$ by

$$\sigma_{\mathfrak{R}}(h, l) = \begin{cases} |h - l|^2 & \text{if} \quad h, l \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

The relation on H is defined by $h\mathfrak{R}l$ if and only if $h = l$ or $h, l \ge 0$. Clearly, $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, 4)$ is an $\mathfrak{R}$-complete partial b-metric space. Define a map $T : H \longrightarrow H$ by

$$Th = \begin{cases} \dfrac{h}{4} & \text{if} \quad h \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Then, it is very simple to verify the following:

(1) If $h = l$, then $Th = Tl$. While if $h, l \ge 0$, then $Th, Tl \ge 0$. Thus, T is an $\mathfrak{R}$-preserving map

(2) Suppose that for any iterative $\mathfrak{R}$-sequence $\{h_n\}$ in H with $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$, $h_n\mathfrak{R}h$ for some $n \ge k$, and $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, Th) \le \sigma_{\mathfrak{R}}(h, h)$, then we get $h\mathfrak{R}Th$

(3) Consider $h_0 \ge 0$ any real number, then $Th_0 \ge 0$. Thus, we have $h_0, Th_0 \ge 0$, that is, $h_0\mathfrak{R}Th_0$

(4) For each $h, l \in H$ with $h\mathfrak{R}l$, we have

case (a) $h = l$:

$$\sigma_{\mathfrak{R}}(Th, Tl) = 0 = \frac{1}{16} \times 0 = \frac{1}{16}\sigma_{\mathfrak{R}}(h, l). \tag{20}$$

case (b) $h, l \ge 0$:

$$\sigma_{\mathfrak{R}}(Th, Tl) = \left|\frac{h}{4} - \frac{l}{4}\right|^2 = \frac{1}{16}|h - l|^2 = \frac{1}{16}\sigma_{\mathfrak{R}}(h, l). \tag{21}$$

Hence, by Theorem 18, T must has a fixed point.

Example 4. Let $H = \mathbb{R}$ and define $\sigma_{\mathfrak{R}} : H \times H \longrightarrow \mathbb{R}^+$ by

$$\sigma_{\mathfrak{R}}(h, l) = \begin{cases} |h - l|^2 + (\max\{h, l\})^2 & \text{if} \quad h, l \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

The relation on H is defined by $h\mathfrak{R}l$ if and only if $h = l$ or $h, l \ge 0$.

Clearly, $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, 4)$ is an $\mathfrak{R}$-complete partial b-metric space. Define a map $T : H \longrightarrow H$ by

$$Th = \begin{cases} \dfrac{h}{6} & \text{if} \quad h \ge 0, \\ -1 & \text{otherwise.} \end{cases} \tag{23}$$

Then, one can verify the following:

(1) If $h = l$, then $Th = Tl$. While if $h, l \ge 0$, then $Th, Tl \ge 0$. Thus, T is an $\mathfrak{R}$-preserving map

(2) Suppose that for any iterative $\mathfrak{R}$-sequence $\{h_n\}$ in H with $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$, $h_n\mathfrak{R}h$ for some $n \ge k$, and $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, Th) \le \sigma_{\mathfrak{R}}(h, h)$, then we get $h\mathfrak{R}Th$

(3) If $h_0 \ge 0$ be some real number, then $Th_0 \ge 0$. Thus, we get $h_0, Th_0 \ge 0$, that is, $h_0\mathfrak{R}Th_0$

(4) For each $h, l \in H$ with $h\mathfrak{R}l$, we have

Case (a) If $h = l \ge 0$, then $Th = Tl \ge 0$. Thus,

$$\begin{aligned} \sigma_{\mathfrak{R}}(Th, Tl) = 0 + \left(\max\left\{\frac{h}{6}, \frac{l}{6}\right\}\right)^2 = \frac{1}{36}(\max\{h, l\})^2 \\ = \frac{1}{36}\sigma_{\mathfrak{R}}(h, l). \end{aligned} \tag{24}$$

Case (b) If $h = l < 0$, then $Th = Tl = -1$. Thus,

$$\sigma_{\mathfrak{R}}(Th, Tl) = 0 = \sigma_{\mathfrak{R}}(h, l). \tag{25}$$

Case (c) If $h, l \geq 0$, then $Th, Tl \geq 0$. Thus,

$$\sigma_{\mathfrak{R}}(Th, Tl) = \left|\frac{h}{6} - \frac{l}{6}\right|^2 + \left(\max\left\{\frac{h}{6}, \frac{l}{6}\right\}\right)^2 = \frac{1}{36}\sigma_{\mathfrak{R}}(h, l). \tag{26}$$

Hence, by Theorem 18, T must has a fixed point.

Remark 21. Note that the function $\sigma_{\mathfrak{R}}$ defined in the above example is neither a metric nor a b -metric nor a partial b -metric on $\mathbb{R}$. Indeed, $\sigma_{\mathfrak{R}}(4, 1) = 25$, $\sigma_{\mathfrak{R}}(4,-1) = 0$, $\sigma_{\mathfrak{R}}(-1, 1) = 0$, $\sigma_{\mathfrak{R}}(-1,-1) = 0$, that is, $(\sigma 4)$ and $(bM3)$ do not exist.

Theorem 22. *Let $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ be an $\mathfrak{R}$ -complete $\mathfrak{R}$ -partial b -metric space with the coefficient $s \geq 1$ and let $h_0 \in H$ be such that $h_0\mathfrak{R}l$ for each $l \in H$. Let $T : H \longrightarrow H$ be an $\mathfrak{R}$ -preserving and $\mathfrak{R}$ -0-continuous map satisfying the following*

$$\sigma_{\mathfrak{R}}(Th, Tl) \leq k \max\{\sigma_{\mathfrak{R}}(h, l), \sigma_{\mathfrak{R}}(h, Th), \sigma_{\mathfrak{R}}(l, Tl)\}, \tag{27}$$

for all $h, l \in H$ with $h\mathfrak{R}l$, $h\mathfrak{R}Th$, and $l\mathfrak{R}Tl$, where $k \in [0, 1/s)$. Also, let for each $\mathfrak{R}$-sequence $\{h_n\}$ in H with $h_n\mathfrak{R}a$ and $h_n\mathfrak{R}b$, we have either $a\mathfrak{R}b$ or $b\mathfrak{R}a$. Then, T has a fixed point $h^* \in H$ and $\sigma_{\mathfrak{R}}(h^*, h^*) = 0$.

Proof. As $h_0 \in H$ is such that $h_0\mathfrak{R}l$ for each $l \in H$, then by using the $\mathfrak{R}$-preserving nature of T, we obtain an $\mathfrak{R}$ -sequence $\{h_n\}$ such that $h_n = Th_{n-1} = T^n h_0$ and $h_{n-1}\mathfrak{R}h_n$ for each $n \in \mathbb{N}$. We take $h_n \neq h_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Then by (27), for each $n \in \mathbb{N}$, we get

$$\begin{aligned}
\sigma_{\mathfrak{R}}(h_n, h_{n+1}) &= \sigma_{\mathfrak{R}}(Th_{n-1}, Th_n)\\
&\leq k \max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_{n-1}, Th_{n-1}), \sigma_{\mathfrak{R}}(h_n, Th_n)\}\\
&= k \max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_n, h_{n+1})\}\\
&= k \max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_n, h_{n+1})\}.
\end{aligned} \tag{28}$$

If $\max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_n, h_{n+1})\} = \sigma_{\mathfrak{R}}(h_n, h_{n+1})$, then from the above inequality, we obtain that $\sigma_{\mathfrak{R}}(h_n, h_{n+1}) \leq k \sigma_{\mathfrak{R}}(h_n, h_{n+1}) < \sigma_{\mathfrak{R}}(h_n, h_{n+1})$, which is a contradiction. Therefore, we must have $\max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_n, h_{n+1})\} = \sigma_{\mathfrak{R}}(h_{n-1}, h_n)$. Again, from the above inequality, we have

$$\sigma_{\mathfrak{R}}(h_n, h_{n+1}) \leq k\sigma_{\mathfrak{R}}(h_{n-1}, h_n) \forall n \in \mathbb{N}. \tag{29}$$

On repeating this process, we obtain

$$\sigma_{\mathfrak{R}}(h_n, h_{n+1}) \leq k^n \sigma_{\mathfrak{R}}(h_0, h_1) \forall n \in \mathbb{N}. \tag{30}$$

For $m,n \in \mathbb{N}$ with $m > n$, by $\sigma_{\mathfrak{R}}4$, we obtain

$$\begin{aligned}
\sigma_{\mathfrak{R}}(h_n, h_m) &\leq s[\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + \sigma_{\mathfrak{R}}(h_{n+1}, h_m)] - \sigma_{\mathfrak{R}}(h_{n+1}, h_{n+1})\\
&\leq s\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + s^2[\sigma_{\mathfrak{R}}(h_{n+1}, h_{n+2})\\
&\quad + \sigma_{\mathfrak{R}}(h_{n+2}, h_m)] - \sigma_{\mathfrak{R}}(h_{n+2}, h_{n+2})\\
&\leq s\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + s^2\sigma_{\mathfrak{R}}(h_{n+1}, h_{n+2})\\
&\quad + s^3\sigma_{\mathfrak{R}}(h_{n+2}, h_{n+3}) + \cdots + s^{m-n}\sigma_{\mathfrak{R}}(h_{m-1}, h_m).
\end{aligned} \tag{31}$$

Using (30) in the above inequality, we obtain

$$\begin{aligned}
\sigma_{\mathfrak{R}}(h_n, h_m) &\leq sk^n\sigma_{\mathfrak{R}}(h_0, h_1) + s^2k^{n+1}\sigma_{\mathfrak{R}}(h_0, h_1)\\
&\quad + s^3k^{n+2}\sigma_{\mathfrak{R}}(h_0, h_1) + \cdots + s^{m-n}k^{m-1}\sigma_{\mathfrak{R}}(h_0, h_1)\\
&\leq sk^n\left[1 + sk + (sk)^2 + \cdots\right]\sigma_{\mathfrak{R}}(h_0, h_1)\\
&= \frac{sk^n}{1 - sk}\sigma_{\mathfrak{R}}(h_0, h_1).
\end{aligned} \tag{32}$$

As $k \in [0, 1/s)$ and $s \geq 1$, it follows from the above inequality that

$$\lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = 0. \tag{33}$$

Therefore, $\{h_n\}$ is a Cauchy $\mathfrak{R}$-sequence. Since H is $\mathfrak{R}$-complete, there exists $h^* \in H$ such that $\lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h^*) = \sigma_{\mathfrak{R}}(h^*, h^*)$ and $h_n\mathfrak{R}h^*$ for each $n \geq k$. Thus, from above, we obtain $0 = \lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h^*) = \sigma_{\mathfrak{R}}(h^*, h^*)$ and $h_n\mathfrak{R}h^*$ for each $n \geq k$. Since T is $\mathfrak{R}$-0-continuous, one gets that $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h^*) = 0$, which leads to $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(Th_n, Th^*) = 0$. Obviously, we have $Th_n\mathfrak{R}Th^*$ for each $n \geq k$. Thus, $h_n\mathfrak{R}Th^*$ for each $n > k$. Since $h_n\mathfrak{R}h^*$ and $h_n\mathfrak{R}Th^*$ for each $n > k$, we have either $h^*\mathfrak{R}Th^*$ or $Th^*\mathfrak{R}h^*$. By using $\sigma_{\mathfrak{R}}4$, we get the following for each $n > k$:

$$\begin{aligned}
\sigma_{\mathfrak{R}}(h^*, Th^*) &\leq s\sigma_{\mathfrak{R}}(h^*, h_{n+1}) + s\sigma_{\mathfrak{R}}(h_{n+1}, Th^*)\\
&\quad - \sigma_{\mathfrak{R}}(h_{n+1}, h_{n+1}).
\end{aligned} \tag{34}$$

When n tends to infinity, the above inequality yields $\sigma_{\mathfrak{R}}(h^*, Th^*) = 0$. Hence, we get $\sigma_{\mathfrak{R}}(h^*, Th^*) = 0$, $\sigma_{\mathfrak{R}}(h^*, h^*) = 0$, and $\sigma_{\mathfrak{R}}(Th^*, Th^*) = 0$. Therefore, we say that $h^* = Th^*$, i.e., h^* is a fixed point of T.

Remark 23. Note that the fixed point of T is unique if in the above result, we add the condition: for each fixed points h^* and l^* of T , we have $h^*\mathfrak{R}l^*$ or $l^*\mathfrak{R}h^*$.

Since $h^* = Th^*$, we have $l^* = Tl^*$ and $h^* \mathfrak{R} l^*$. From (27), we get

$$\begin{aligned}\sigma_{\mathfrak{R}}(h^*, l^*) &= \sigma_{\mathfrak{R}}(Th^*, Tl^*) \\ &\le k \max\{\sigma_{\mathfrak{R}}(h^*, l^*), \sigma_{\mathfrak{R}}(h^*, Th^*), \sigma_{\mathfrak{R}}(l^*, Tl^*)\} \\ &= k \max\{\sigma_{\mathfrak{R}}(h^*, l^*), \sigma_{\mathfrak{R}}(h^*, h^*), \sigma_{\mathfrak{R}}(l^*, l^*)\} \\ &= k\sigma_{\mathfrak{R}}(h^*, l^*) < \sigma_{\mathfrak{R}}(h^*, l^*).\end{aligned} \tag{35}$$

It is a contradiction in the case $\sigma_{\mathfrak{R}}(h^*, l^*) \neq 0$. Therefore, we must have $\sigma_{\mathfrak{R}}(h^*, l^*) = 0$, that is, $h^* = l^*$.

3. Applications to Graphical Partial b-Metric Spaces and Partially-Ordered-Partial b-Metric Spaces

In this section, we define a directed graph G on H, denoted by $G = (V(H), E(H))$, with the vertex set $V(H) = H$ and the edge set $E(H)$ such that $E(H) \subset H \times H$ and $\{(h, h): h \in H\} \subset E(H)$. Also, $E(H)$ has no parallel edge. Note that hPl denotes the path between h and l, that is, there exists a finite sequence $\{k_i\}_{i=0}^{j}$, for some finite j, such that $k_0 = h$, $k_j = l$, and $(k_i, k_{i+1}) \in E(H)$ for $i \in \{0, 1, \cdots, j-1\}$.

Definition 24. Let $H \neq \emptyset$ be associated the above-defined G, denoted as (H, G). A map $\sigma_G : H \times H \to \mathbb{R}^+$ is called a G-partial b-metric on the set H, if the following conditions are satisfied for all $h, l, z \in H$ with hPl and $z \in hPl$:

$(\sigma_G 1)$ $h = l$ if and only if $\sigma_G(h, h) = \sigma_G(h, l) = \sigma_G(l, l)$;
$(\sigma_G 2)$ $\sigma_G(h, h) \le \sigma_G(h, l)$;
$(\sigma_G 3)$ $\sigma_G(h, l) = \sigma_G(l, h)$;
$(\sigma_G 4)$ $\sigma_G(h, l) \le s[\sigma_G(h, z) + \sigma_G(z, l)] - \sigma_G(z, z)$, where $s \ge 1$.

Then, (H, G, σ_G, s) is called a G-partial b-metric space with the coefficient $s \ge 1$.

Remark 25. If hPl and $z \in hPl$, then we get hPz and zPl. Also note if hPz and zPl, then we have hPl.

Thus, P is a preorder relation on H. Therefore, (H, G, σ_G, s) is also an $\mathfrak{R}$-partial b-metric space.

Definition 26. Let $\{h_n\}$ be a G-sequence in (H, G, σ_G, s), that is, $h_n P h_{n+1}$ or $h_{n+1} P h_n$ for each n. Then, we say that

(i) $\{h_n\}$ is a convergent sequence to $h \in H$ if $\lim_{n\to\infty}\sigma_G(h_n, h) = \sigma_G(h, h)$ and $h_n P h$ for each $n \ge k$

(ii) $\{h_n\}$ is Cauchy if $\lim_{n,m\to\infty}\sigma_G(h_n, h_m)$ exists and is finite

Definition 27. (H, G, σ_G, s) is said to be G-complete if for each Cauchy G-sequence in H there is $h \in H$ with $\lim_{n,m\to\infty}\sigma_G(h_n, h_m) = \lim_{n\to\infty}\sigma_G(h_n, h) = \sigma_G(h, h)$ and $h_n P h$ for each $n \ge k$.

Note that for a map $T : H \to H$, the G-0-continuity and G-property are defined in the same way as explained in the last section.

Theorem 28. *Let (H, G, σ_G, s) be a G-complete G-partial b-metric space with the coefficient $s \ge 1$ and let $h_0 \in H$ be such that $h_0 P l$ for each $l \in H$. Let $T : H \longrightarrow H$ be an edge preserving (if $(h, l) \in E(H)$, then $(Th, Tl) \in E(H)$) and a G-property map satisfying the following*

$$\sigma_G(Th, Tl) \le k\sigma_G(h, l) \quad \text{for all} \quad h, l \in H \quad \text{with} \quad hPl, \tag{36}$$

where $k \in [0, 1/s)$. Then, T has a fixed point $h^* \in H$ and $\sigma_G(h^*, h^*) = 0$.

By Remark 25, we know that P is a preorder relation on H and (H, G, σ_G, s) is an $\mathfrak{R}$-partial b-metric space. Also, an edge preserving map is path preserving. Thus, all the conditions of Theorem 18 hold. Hence, T has a fixed point.

In the following, we obtain partially-ordered-partial b-metric spaces from $\mathfrak{R}$-partial b-metric spaces, by considering $^\circ$ as a partial order on H.

Definition 29. Let $H \neq \emptyset$ be associated with a partial order $^\circ$, denoted as (H°). Given a map $\sigma_\circ : H \times H \to \mathbb{R}^+$. If the following conditions are satisfied for all $h, l, z \in H$ with $h^\circ l$ and $h^\circ z^\circ l$:

$(\sigma_\circ 1)$ $h = l$ if and only if $\sigma_\circ(h, h) = \sigma_\circ(h, l) = \sigma_\circ(l, l)$;
$(\sigma_\circ 2)$ $\sigma_\circ(h, h) \le \sigma_\circ(h, l)$;
$(\sigma_\circ 3)$ $\sigma_\circ(h, l) = \sigma_\circ(l, h)$;
$(\sigma_\circ 4)$ $\sigma_\circ(h, l) \le s[\sigma_\circ(h, z) + \sigma_\circ(z, l)] - \sigma_\circ(z, z)$, where $s \ge 1$,

then $(H, G, \sigma_\circ, s)$ is called a partially-ordered-partial b-metric space with the coefficient $s \ge 1$.

As we discussed in the above, we state the following result.

Theorem 30. *Let $(H, G, \sigma_\circ, s)$ be an $^\circ$-complete partially-ordered-partial b-metric space with the coefficient $s \ge 1$ and let $h_0 \in H$ be such that $h_0^\circ l$ for each $l \in H$. Let $T : H \longrightarrow H$ be order preserving (if $h^\circ l$ then $Th^\circ Tl$), and an $^\circ$-property map satisfying the following:*

$$\sigma_\circ(Th, Tl) \le k\sigma_\circ(h, l) \quad \text{for all} \quad h, l \in H \quad \text{with} \quad h^\circ l, \tag{37}$$

where $k \in [0, 1/s)$. Then, T has a fixed point $h^* \in H$ and $\sigma_\circ(h^*, h^*) = 0$.

Remark 31. $\preccurlyeq$– completeness is defined in the same way as G-completeness.

4. Conclusion

By combining the concepts of orthogonality and the binary relation, we introduced the notion of $\mathfrak{R}$-partial b-metric spaces. We presented some related fixed point results. Some illustrated examples and an application to graphical partial

b-metric spaces and partially-ordered-partial b-metric spaces have been provided. As perspectives, it would be interesting to consider in this setting more generalized contraction mappings involving simulation functions or more control functions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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