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Means and Their Inequalities

Guest Editors: Mowaffaq Hajja, Peter S. Bullen, Janusz Matkowski, Edward Neuman, and Slavko Simic

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Editorial

Means and Their Inequalities

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The theory of means has its roots in the work of the Pythagoreans who introduced the harmonic, geometric, and arithmetic means with reference to their theories of music and arithmetic. Later, Pappus introduced seven other means and gave the well-known elegant geometric proof of the celebrated inequalities among the harmonic, geometric, and arithmetic means.

Nowadays, the families and types of means that are being investigated by researchers and the variety of questions that are being asked about them are beyond the scope of any single survey, with the voluminous book Handbook of Means and Their Inequalities by P. S. Bullen being the best such reference in this direction. The theory of means has grown to occupy a prominent place in mathematics with hundreds of papers on the subject appearing every year.

The strong relations and interactions of the theory of means with the theories of inequalities, functional equations, and probability and statistics add greatly to its importance.

Continuous versions of some means and inequalities among them tie it with real analysis and the theory of integration. The fact that centers of triangles and simplices can be viewed as means of points in the Euclidean spaces makes the subject of interest to geometers.

Positivity and copositivity tests in the theory of forms naturally give rise to questions on internality tests of means arising from forms, making this aspect of the subject of interest to algebraists as well. Extensions of Gauss's outstanding discoveries that relate the evaluation of certain elliptic integrals to iterations of the arithmetic and geometric means that led to the beautiful arithmeticogeometric mean resulted in so many interesting results and lines of research. A quick look at the table of contents of the book Pi and the AGM by J. M. Borwein and P. B. Borwein shows how extensive this line of research is and also shows that the subject is related to almost everything.

The theory of means has applications in so many other diverse fields. Quoting from the preface of the aforementioned book of P. S. Bullen, these include electrostatics, heat conduction, chemistry, and even medicine.

This issue contains several papers that pertain to some of the the aforementioned subjects.

One of the papers is an exposition of certain elementary aspects of the subject, together with several open problems that are within the comprehension of a graduate student. It is hoped that such questions will lead to contributions from experts and amateurs alike.

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Research Article

Some Elementary Aspects of Means

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We raise several elementary questions pertaining to various aspects of means. These questions refer to both known and newly introduced families of means, and include questions of characterizations of certain families, relations among certain families, comparability among the members of certain families, and concordance of certain sequences of means. They also include questions about internality tests for certain mean-looking functions and about certain triangle centers viewed as means of the vertices. The questions are accessible to people with no background in means, and it is also expected that these people can seriously investigate, and contribute to the solutions of, these problems. The solutions are expected to require no more than simple tools from analysis, algebra, functional equations, and geometry.

1. Definitions and Terminology

In all that follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{J}$ denotes an interval in $\mathbb{R}$.

By a data set (or a list) in a set S, we mean a finite subset of S in which repetition is allowed. Although the order in which the elements of a data set are written is not significant, we sometimes find it convenient to represent a data set in S of size n by a point in S^n, the cartesian product of n copies of S.

We will call a data set $A = (a_1, \ldots, a_n)$ in $\mathbb{R}$ *ordered* if $a_1 \leq \cdots \leq a_n$. Clearly, every data set in $\mathbb{R}$ may be assumed ordered.

A *mean* of k variables (or a *k-dimensional mean*) on $\mathbb{J}$ is defined to be any function $\mathscr{M} : \mathbb{J}^k \to \mathbb{J}$ that has the *internality* property

$$\min \{a_1, \ldots, a_k\} \leq \mathscr{M}(a_1, \ldots, a_k) \leq \max \{a_1, \ldots, a_k\} \quad (1)$$

for all a_j in $\mathbb{J}$. It follows that a mean $\mathscr{M}$ must have the property $\mathscr{M}(a, \ldots, a) = a$ for all a in $\mathbb{J}$.

Most means that we encounter in the literature, and all means considered below, are also symmetric in the sense that

$$\mathscr{M}(a_1, \ldots, a_k) = \mathscr{M}(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \quad (2)$$

for all permutations σ on $\{1, \ldots, n\}$, and *1-homogeneous* in the sense that

$$\mathscr{M}(\lambda a_1, \ldots, \lambda a_k) = \lambda \mathscr{M}(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \quad (3)$$

for all permissible $\lambda \in \mathbb{R}$.

If $\mathscr{M}$ and $\mathscr{N}$ are two k-dimensional means on $\mathbb{J}$, then we say that $\mathscr{M} \leq \mathscr{N}$ if $\mathscr{M}(a_1, \ldots, a_k) \leq \mathscr{N}(a_1, \ldots, a_k)$ for all $a_j \in \mathbb{J}$. We say that $\mathscr{M} < \mathscr{N}$ if $\mathscr{M}(a_1, \ldots, a_k) < \mathscr{N}(a_1, \ldots, a_k)$ for all $a_j \in \mathbb{J}$ for which $a_1, \ldots, a_k$ are not all equal. This exception is natural since $\mathscr{M}(a, \ldots, a)$ and $\mathscr{N}(a, \ldots, a)$ must be equal, with each being equal to a. We say that $\mathscr{M}$ and $\mathscr{N}$ are *comparable* if $\mathscr{M} \leq \mathscr{N}$ or $\mathscr{N} \leq \mathscr{M}$.

A *distance* (or *a distance function*) on a set S is defined to be any function $d : S \times S \to [0, \infty)$ that is *symmetric* and *positive definite*, that is,

$$d(a,b) = d(b,a), \quad \forall a, b \in S,$$
$$d(a,b) = 0 \Longleftrightarrow a = b. \quad (4)$$

Thus a *metric* is a distance that satisfies the triangle inequality

$$d(a,b) + d(b,c) \geq d(a,c), \quad \forall a, b, c \in S, \quad (5)$$

a condition that we find too restrictive for our purposes.

2. Examples of Means

The *arithmetic*, *geometric*, and *harmonic* means of two positive numbers were known to the ancient Greeks; see [1, pp. 84–90]. They are usually denoted by $\mathscr{A}$, $\mathscr{G}$, and $\mathscr{H}$, respectively, and are defined, for $a, b > 0$, by

$$\mathscr{A}(a,b) = \frac{a+b}{2},$$
$$\mathscr{G}(a,b) = \sqrt{ab},$$
$$\mathscr{H}(a,b) = \frac{2}{1/a + 1/b} = \frac{2ab}{a+b}. \tag{6}$$

The celebrated inequalities

$$\mathscr{H}(a,b) < \mathscr{G}(a,b) < \mathscr{A}(a,b) \quad \forall a, b > 0 \tag{7}$$

were also known to the Greeks and can be depicted in the well-known figure that is usually attributed to Pappus and that appears in [2, p. 364]. Several other less well known means were also known to the ancient Greeks; see [1, pp. 84–90].

The three means above, and their natural extensions to any number n of variables, are members of a large two-parameter family of means, known now as the *Gini* means and defined by

$$G_{r,s}(x_1, \ldots, x_n) = \left(\frac{N_r(x_1, \ldots, x_n)}{N_s(x_1, \ldots, x_n)}\right)^{1/(r-s)}, \tag{8}$$

where N_j are the Newton polynomials defined by

$$N_j(x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k^j. \tag{9}$$

Means of the type $G_{r,r-1}$ are known as *Lehmer*'s means, and those of the type $G_{r,0}$ are known as *Hölder* or *power* means. Other means that have been studied extensively are the *elementary symmetric polynomial* and *elementary symmetric polynomial ratio* means defined by

$$\left(\frac{\sigma_r}{C_r^n}\right)^{1/r}, \qquad \frac{\sigma_r / C_r^n}{\sigma_{r-1} / C_{r-1^n}}, \tag{10}$$

where σ_r is the rth elementary symmetric polynomial in n variables, and where

$$C_r^n = \binom{n}{r}. \tag{11}$$

These are discussed in full detail in the encyclopedic work [3, Chapters III and V].

It is obvious that the power means $\mathscr{P}_r$ defined by

$$\mathscr{P}_r(a_1, \ldots, a_n) = G_{r,0}(a_1, \ldots, a_n) = \left(\frac{a_1^r + \cdots + a_n^r}{n}\right)^{1/r} \tag{12}$$

that correspond to the values $r = -1$ and $r = 1$ are nothing but the harmonic and arithmetic means $\mathscr{H}$ and $\mathscr{A}$, respectively. It is also natural to set

$$\mathscr{P}_0(a_1, \ldots, a_n) = \mathscr{G}(a_1, \ldots, a_n) = (a_1 \ldots a_n)^{1/n}, \tag{13}$$

since

$$\lim_{r \to 0} \left(\frac{a_1^r + \cdots + a_n^r}{n}\right)^{1/r} = (a_1 \ldots a_n)^{1/n} \tag{14}$$

for all $a_1, \ldots, a_n > 0$.

The inequalities (7) can be written as $\mathscr{P}_{-1} < \mathscr{P}_0 < \mathscr{P}_1$. These inequalities hold for any number of variables and they follow from the more general fact that $\mathscr{P}_r(a_1, \ldots, a_n)$, for fixed $a_1, \ldots, a_n > 0$, is strictly increasing with r. Power means are studied thoroughly in [3, Chapter III].

3. Mean-Producing Distances and Distance Means

It is natural to think of the mean of any list of points in any set to be the point that is *closest* to that list. It is also natural to think of a point as closest to a list of points if the sum of its distances from these points is minimal. This mode of thinking associates means to distances.

If d is a distance on S, and if $A = (a_1, \ldots, a_n)$ is a data set in S, then a d*-mean* of A is defined to be any element of S at which the function

$$f(x) = \sum_{i=1}^{n} d(x, a_i) \tag{15}$$

attains its minimum. It is conceivable that (15) attains its minimum at many points, or nowhere at all. However, we shall be mainly interested in distances d on $\mathbb{J}$ for which (15) attains its minimum at a unique point x_A that, furthermore, has the property

$$\min\{a : a \in A\} \le x_A \le \max\{a : a \in A\} \tag{16}$$

for every data set A. Such a distance is called a *mean-producing* or a *mean-defining* distance, and the point x_A is called the d*-mean* of A or the *mean of A arising from the distance d* and will be denoted by $\mu_d(A)$. A mean $\mathscr{M}$ is called a *distance mean* if it is of the form μ_d for some distance d.

Problem Set 1. (1-a) Characterize those distances on $\mathbb{J}$ that are mean-producing.

(1-b) Characterize those pairs of mean producing distances on $\mathbb{J}$ that produce the same mean.

(1-c) Characterize distance means.

4. Examples of Mean-Producing Distances

If d_0 is the discrete metric defined on $\mathbb{R}$ by

$$d_0(a,b) = \begin{cases} 1 & \text{if } a \ne b, \\ 0 & \text{if } a = b, \end{cases} \tag{17}$$

then the function $f(x)$ in (15) is nothing but the number of elements in the given data set A that are different from x, and therefore every element having maximum frequency in A minimizes (15) and is hence a d_0-mean of A. Thus the discrete metric gives rise to what is referred to in statistics as "the" *mode* of A. Due to the nonuniqueness of the mode, the discrete metric is not a mean-producing distance.

Similarly, the usual metric $d = d_1$ defined on $\mathbb{R}$ by

$$d_1(a, b) = |a - b| \tag{18}$$

is not a mean-producing distance. In fact, it is not very difficult to see that if $A = (a_1, \ldots, a_n)$ is an ordered data set of even size $n = 2m$, then any number in the closed interval $[a_m, a_{m+1}]$ minimizes

$$\sum_{j=1}^{n} \left| x - a_j \right| \tag{19}$$

and is therefore a d_1-mean of A. Similarly, one can show that if A is of an odd size $n = 2m - 1$, then a_m is the unique d_1-mean of A. Thus the usual metric on $\mathbb{R}$ gives rise to what is referred to in statistics as "the" *median* of A.

On the other hand, the distance d_2 defined on $\mathbb{R}$ by

$$d_2(a, b) = (a - b)^2 \tag{20}$$

is a mean-producing distance, although it is not a metric. In fact, it follows from simple derivative considerations that the function

$$\sum_{j=1}^{n} \left(x - a_j \right)^2 \tag{21}$$

attains its minimum at the unique point

$$x = \frac{1}{n} \left(\sum_{j=1}^{n} a_j \right). \tag{22}$$

Thus d_2 is a mean-producing distance, and the corresponding mean is nothing but the arithmetic mean.

It is noteworthy that the three distances that come to mind most naturally give rise to the three most commonly used "means" in statistics. In this respect, it is also worth mentioning that a fourth mean of statistics, the so-called *midrange*, will be encountered below as a very natural *limiting* distance mean.

The distances d_1 and d_2 (and in a sense, d_0 also) are members of the family d_p of distances defined by

$$d_p(a, b) = |a - b|^p. \tag{23}$$

It is not difficult to see that if $p > 1$, then d_p is a mean-producing distance. In fact, if $A = (a_1, \ldots, a_n)$ is a given data set, and if

$$f(x) = \sum_{j=1}^{n} \left| x - a_j \right|^p, \tag{24}$$

then

$$f''(x) = p(p - 1) \sum_{j=1}^{n} \left| x - a_j \right|^{p-2} \geq 0, \tag{25}$$

with equality if and only if $a_1 = \cdots = a_n = x$. Thus f is convex and cannot attain its minimum at more than one point. That it attains its minimum follows from the continuity of $f(x)$, the compactness of $[a_1, a_n]$, and the obvious fact that $f(x)$ is increasing on $[a_n, \infty)$ and is decreasing on $(-\infty, a_1]$. If we denote the mean that d_p defines by μ_p, then $\mu_p(A)$ is the unique zero of

$$\sum_{j=1}^{n} \operatorname{sign}\left(x - a_j \right) \left| x - a_j \right|^{p-1}, \tag{26}$$

where $\operatorname{sign}(t)$ is defined to be 1 if t is nonnegative and -1 otherwise.

Note that no matter what $p > 1$ is, the two-dimensional mean μ_p arising from d_p is the arithmetic mean. Thus when studying μ_p, we confine our attention to the case when the number k of variables is greater than two. For such k, it is impossible in general to compute $\mu_p(A)$ in closed form.

Problem 2. It would be interesting to investigate comparability among $\{\mu_p : p > 1\}$.

It is highly likely that no two means μ_p are comparable.

5. Deviation and Sparseness

If d is a mean-producing distance on S, and if μ_d is the associated mean, then it is natural to define the d-deviation $\mathcal{D}_d(A)$ of a data set $A = (a_1, \ldots, a_n)$ by an expression like

$$\mathcal{D}_d(A) = \mu_d \left\{ d\left(\mu_d(A), a_i \right) : 1 \leq i \leq n \right\}. \tag{27}$$

Thus if d is defined by

$$d(x, y) = (x - y)^2, \tag{28}$$

then μ_d is nothing but the arithmetic mean or ordinary *average* μ defined by

$$\mu = \mu(a_1, \ldots, a_n) = \frac{a_1 + \cdots + a_n}{n}, \tag{29}$$

and $\mathcal{D}_d$ is the (squared) standard deviation $\sigma^{(2)}$ given by

$$\sigma^{(2)}(a_1, \ldots, a_n) = \frac{\left| a_1 - \mu \right|^2 + \cdots + \left| a_n - \mu \right|^2}{n}. \tag{30}$$

In a sense, this provides an answer to those who are puzzled and mystified by the choice of the exponent 2 (and not any other exponent) in the standard definition of the standard deviation given in the right-hand side of (30). In fact, distance means were devised by the author in an attempt to remove that mystery. Somehow, we are saying that the ordinary average μ and the standard deviation $\sigma^{(2)}$ must be taken or discarded together, being both associated with the same distance d given in (28). Since few people question the sensibility of the definition of μ given in (29), accepting the standard definition of the standard deviation given in (30) as is becomes a *must*.

It is worth mentioning that choosing an exponent other than 2 in (30) would result in an *essentially* different notion of deviations. More precisely, if one defines $\sigma^{(k)}$ by

$$\sigma^{(k)}(a_1,\ldots,a_n) = \frac{|a_1-\mu|^k+\cdots+|a_n-\mu|^k}{n}, \tag{31}$$

then $\sigma^{(k)}$ and $\sigma^{(2)}$ would of course be unequal, but more importantly, they would not be *monotone* with respect to each other, in the sense that there would exist data sets A and B with $\sigma^{(2)}(A) > \sigma^{(k)}(B)$ and $\sigma^{(2)}(A) < \sigma^{(k)}(B)$. Thus the choice of the exponent k in defining deviations is not as arbitrary as some may feel. On the other hand, it is (27) and not (31) that is the natural generalization of (30). This raises the following, expectedly hard, problem.

Problem 3. Let d be the distance defined by $d(x,y) = |x-y|^k$, and let the associated deviation $\mathscr{D}_d$ defined in (27) be denoted by $\mathscr{D}_k$. Is $\mathscr{D}_k$ monotone with respect to $\mathscr{D}_2$ for any $k \neq 2$, in the sense that

$$\mathscr{D}_k(A) > \mathscr{D}_k(B) \Longrightarrow \mathscr{D}_2(A) > \mathscr{D}_2(B)? \tag{32}$$

We end this section by introducing the notion of *sparseness* and by observing its relation with deviation. If d is a mean-producing distance on $\mathbb{J}$, and if μ_d is the associated mean, then the *d-sparseness* $\mathscr{S}_d(A)$ of a data set $A = (a_1,\ldots,a_n)$ in $\mathbb{J}$ can be defined by

$$\mathscr{S}_d(A) = \mu_d\left\{d\left(a_i,a_j\right) : 1 \le i < j \le n\right\}. \tag{33}$$

It is interesting that when d is defined by (28), the standard deviation coincides, up to a constant multiple, with the sparsenss. One wonders whether this pleasant property characterizes this distance d.

Problem Set 4. (4-a) Characterize those mean-producing distances whose associated mean is the arithmetic mean.

(4-b) If d is as defined in (28), and if d' is another mean-producing distance whose associated mean is the arithmetic mean, does it follow that $\mathscr{D}_{d'}$ and $\mathscr{D}_d$ are monotone with respect to each other?

(4-c) Characterize those mean-producing distances δ for which the deviation $\mathscr{D}_\delta(A)$ is determined by the sparseness $\mathscr{S}_\delta(A)$ for every data set A, and vice versa.

6. Best Approximation Means

It is quite transparent that the discussion in the previous section regarding the distance mean μ_p, $p > 1$, can be written in terms of best approximation in ℓ_p^n, the vector space $\mathbb{R}^n$ endowed with the p-norm $\|\cdots\|_p$ defined by

$$\|(a_1,\ldots,a_n)\|_p = \left(\sum_{j=1}^n |a_j|^p\right)^{1/p}. \tag{34}$$

If we denote by $\Delta = \Delta_n$ the line in $\mathbb{R}^n$ consisting of the points $(x_1,\ldots,x_n)$ with $x_1 = \cdots = x_n$, then to say that $a = \mu_p(a_1,\ldots,a_n)$ is just another way of saying that the point $(a,\ldots,a)$ is a best approximant in Δ_n of the point $(a_1,\ldots,a_n)$ with respect to the p-norm given in (34). Here, a point s_t in a subset S of a metric (or distance) space (T,D) is said to be a *best approximant* in S of $t \in T$ if $D(t,s_t) = \min\{D(t,s) : s \in S\}$. Also, a subset S of (T,D) is said to be *Chebyshev* if every t in T has exactly one best approximant in S; see [4, p. 21].

The discussion above motivates the following definition.

Definition 1. Let $\mathbb{J}$ be an interval in $\mathbb{R}$ and let D be a distance on $\mathbb{J}^n$. If the diagonal $\Delta(\mathbb{J}^n)$ of $\mathbb{J}^n$ defined by

$$\Delta(\mathbb{J}^n) = \{(a_1,\ldots,a_n) \in \mathbb{J}^n : a_1 = \cdots = a_n\} \tag{35}$$

is Chebyshev (with respect to D), then the n-dimensional mean M_D on $\mathbb{J}$ defined by declaring $M_D(a_1,\ldots,a_n) = a$ if and only if $(a,\ldots,a)$ is the best approximant of $(a_1,\ldots,a_n)$ in $\Delta(\mathbb{J}^n)$ is called the *Chebyshev* or *best approximation D-mean* or the *best approximation mean arising from D*.

In particular, if one denotes by M_p the best approximation n-dimensional mean on $\mathbb{R}$ arising from (the distance on $\mathbb{R}^n$ induced by) the norm $\|\cdots\|_p$, then the discussion above says that M_p exists for all $p > 1$ and that it is equal to μ_p defined in Section 4.

In view of this, one may also define M_∞ to be the best approximation mean arising from the ∞-norm of ℓ_∞^n, that is, the norm $\|\cdots\|_\infty$ defined on $\mathbb{R}^n$ by

$$\|(a_1,\ldots,a_n)\|_\infty = \max\left\{|a_j| : 1 \le j \le n\right\}. \tag{36}$$

It is not very difficult to see that $\mu_\infty(A)$ is nothing but what is referred to in statistics as the *mid-range* of A. Thus if $A = (a_1,\ldots,a_n)$ is an ordered data set, then

$$M_\infty(A) = \frac{a_1+a_n}{2}. \tag{37}$$

In view of the fact that d_∞ cannot be defined by anything like (23) and μ_∞ is thus meaningless, natural question arises as to whether

$$M_\infty(A) = \lim_{p\to\infty}\mu_p(A) \quad \left(\text{or equivalently} = \lim_{p\to\infty} M_p(A)\right) \tag{38}$$

for every A. An affirmative answer is established in [5, Theorem 1]. In that theorem, it is also established that

$$\lim_{p\to q}\mu_p(A) \quad \left(\text{or equivalently } \lim_{p\to q} M_p(A)\right) = M_q(A) \tag{39}$$

for all q and all A. All of this can be expressed by saying that μ_p is continuous in p for $p \in (1,\infty]$ for all A.

We remark that there is no *obvious* reason why (38) should immediately follow from the well known fact that

$$\lim_{p\to\infty}\|A\|_p = \|A\|_\infty \tag{40}$$

for all points A in $\mathbb{R}^n$.

Problem Set 5. Suppose that δ_p is a sequence of distances on a set S that converges to a distance δ_∞ (in the sense that $\lim_{p\to\infty}\delta_p(a,b) = \delta_\infty(a,b)$ for all a, b in S). Let $T \subseteq S$.

(5-a) If T is Chebyshev with respect to each δ_p, is it necessarily true that T is Chebyshev with respect to δ_∞?

(5-b) If T is Chebyshev with respect to each δ_p and with respect to δ_∞ and if x_p is the best approximant in T of x with respect to δ_p and x_∞ is the best approximant in T of x with respect to δ_∞, does it follow that x_p converges to x_∞?

We end this section by remarking that if $M = M_d$ is the n-dimensional best approximation mean arising from a distance d on $\mathbb{J}^n$, then d is significant only up to its values of the type $d(u,v)$, where $u \in \Delta(\mathbb{J}^n)$ and $v \notin \Delta(\mathbb{J}^n)$. Other values of d are not significant. This, together with the fact that

$$\text{every mean is a best approximation mean arising from a metric,} \tag{41}$$

makes the study of best approximation means less interesting. Fact (41) was proved in an unduly complicated manner in [6], and in a trivial way based on a few-line set-theoretic argument in [7].

Problem 6. Given a mean $\mathscr{M}$ on $\mathbb{J}$, a metric D on $\mathbb{J}$ is constructed in [6] so that $\mathscr{M}$ is the best approximation mean arising from D. Since the construction is extremely complicated in comparison with the construction in [7], it is desirable to examine the construction of D in [6] and see what other nice properties (such as continuity with respect to the usual metric) D has. This would restore merit to the construction in [6] and to the proofs therein and provide raison d'être for the so-called *generalized* means introduced there.

7. Towards a Unique Median

As mentioned earlier, the distance d_1 on $\mathbb{R}$ defined by (23) does not give rise to a (distance) mean. Equivalently, the 1-norm $\|\cdots\|_1$ on $\mathbb{R}^n$ defined by (34) does not give rise to a (best approximation) mean. These give rise, instead, to the many-valued function known as *the median*. Thus, following the statistician's mode of thinking, one may set

$$\begin{aligned}\mu_1(A) = M_1(A) &= \text{ the median interval of } A\\ &= \text{ the set of all medians of } A.\end{aligned} \tag{42}$$

From a mathematician's point of view, however, this leaves a lot to be desired, to say the least. The feasibility and naturality of defining μ_∞ as the limit of μ_p as p approaches ∞ gives us a clue on how the median μ_1 may be defined. It is a pleasant fact, proved in [5, Theorem 4], that the limit of $\mu_p(A)$ (equivalently of $M_p(A)$) as p decreases to 1 exists for every $A \in \mathbb{R}^n$ and equals one of the medians described in (42). This limit can certainly be used as *the* definition of *the* median.

Problem Set 7. Let μ_p be as defined in Section 4, and let μ^* be the limit of μ_p as p decreases to 1.

(7-a) Explore how the value of $\mu^*(A)$ compares with the common practice of taking the median of A to be the midpoint of the median interval (defined in (42) for various values of A.

(7-b) Is μ^* continuous on $\mathbb{R}^n$? If not, what are its points of discontinuity?

(7-c) Given $A \in \mathbb{R}^n$, is the convergence of $\mu_p(A)$ (as p decreases to 1) to $\mu^*(A)$ monotone?

The convergence of $\mu_p(A)$ (as p decreases to 1) to $\mu^*(A)$ is described in [5, Theorem 4], where it is proved that the convergence is ultimately monotone. It is also proved in [5, Theorem 5] that when $n = 3$, then the convergence is monotone.

It is of course legitimate to question the usefulness of defining the median to be μ^*, but that can be left to statisticians and workers in relevant disciplines to decide. It is also legitimate to question the path that we have taken the limit along. In other words, it is conceivable that there exists, in addition to d_p, a sequence d'_p of distances on $\mathbb{R}$ that converges to d_1 such that the limit μ^{**}, as p decreases to 1, of their associated distance means μ'_p is not the same as the limit μ^* of μ_p. In this case, μ^{**} would have as valid a claim as μ^* to being *the* median. However, the naturality of d_p may help accepting μ^* as a most legitimate median.

Problem Set 8. Suppose that δ_p and δ'_p, $p \in \mathbb{N}$, are sequences of distances on a set S that converge to the distances δ_∞ and δ'_∞, respectively (in the sense that $\lim_{p\to\infty}\delta_p(a,b) = \delta_\infty(a,b)$ for all a, b in S, etc.).

(8-a) If each δ_p, $p \in \mathbb{N}$, is mean producing with corresponding mean m_p, does it follow that δ_∞ is mean producing? If so, and if the mean produced by δ_∞ is m_∞, is it necessarily true that m_p converges to m_∞?

(8-b) If δ_p and δ'_p, $p \in \mathbb{N} \cup \{\infty\}$, are mean producing distances with corresponding means m_p and m'_p, and if $m_p = m'_p$ for all $p \in \mathbb{N}$, does it follow that $m_\infty = m'_\infty$?

8. Examples of Distance Means

It is clear that the arithmetic mean is the distance mean arising from the the distance d_2 given by $d_2(a,b) = (a-b)^2$. Similarly, the geometric mean on the set of positive numbers is the distance mean arising from the distance $d_{\mathscr{G}}$ given by

$$d_{\mathscr{G}}(a,b) = (\ln a - \ln b)^2. \tag{43}$$

In fact, this should not be amazing since the arithmetic mean $\mathscr{A}$ on $\mathbb{R}$ and the geometric mean $\mathscr{G}$ on $(0,\infty)$ are *equivalent* in the sense that there is a bijection $g : (0,\infty) \to \mathbb{R}$, namely $g(x) = \ln x$, for which $\mathscr{G}(a,b) = g^{-1}\mathscr{A}(g(a), g(b))$ for all a, b. Similarly, the harmonic and arithmetic means on $(0,\infty)$ are equivalent via the bijection $h(x) = 1/x$, and therefore the harmonic mean is the distance mean arising from the distance $d_{\mathscr{H}}$ given by

$$d_{\mathscr{H}}(a,b) = \left(\frac{1}{a} - \frac{1}{b}\right)^2. \tag{44}$$

The analogous question pertaining to the logarithmic mean $\mathscr{L}$ defined by

$$\mathscr{L}(a,b) = \frac{a-b}{\ln a - \ln b}, \quad a,b > 0, \tag{45}$$

remains open.

Problem 9. Decide whether the mean $\mathscr{L}$ (defined in (45)) is a distance mean.

9. Quasi-Arithmetic Means

A k-dimensional mean $\mathscr{M}$ on $\mathbb{J}$ is called a *quasi-arithmetic* mean if there is a continuous strictly monotone function g from $\mathbb{J}$ to an interval $\mathbb{I}$ in $\mathbb{R}$ such that

$$\mathscr{M}(a_1,\ldots,a_k) = g^{-1}(\mathscr{A}(g(a_1),\ldots,g(a_k))) \tag{46}$$

for all a_j in $\mathbb{J}$. We have seen that the geometric and harmonic means are quasi-arithmetic and concluded that they are distance means. To see that $\mathscr{L}$ is not quasi-arithmetic, we observe that the (two-dimensional) arithmetic mean, and hence any quasi-arithmetic mean $\mathscr{M}$, satisfies the elegant functional equation

$$\mathscr{M}(\mathscr{M}(\mathscr{M}(a,b),b),\mathscr{M}(\mathscr{M}(a,b),a)) = \mathscr{M}(a,b) \tag{47}$$

for all $a,b > 0$. However, a quick experimentation with a random pair (a,b) shows that (47) is not satisfied by $\mathscr{L}$.

This shows that $\mathscr{L}$ is not quasi-arithmetic, but does not tell us whether $\mathscr{L}$ is a distance mean, and hence does not answer Problem 9.

The functional equation (47) is a weaker form of the functional equation

$$\mathscr{M}(\mathscr{M}(a,b),\mathscr{M}(c,d)) = \mathscr{M}(\mathscr{M}(a,c),\mathscr{M}(b,d)) \tag{48}$$

for all $a,b,c,d > 0$. This condition, together with the assumption that $\mathscr{M}$ is strictly increasing in each variable, characterizes two-dimensional quasi-arithmetic means; see [8, Theorem 1, pp. 287–291]. A thorough discussion of quasi-arithmetic means can be found in [3, 8].

Problem 10. Decide whether a mean $\mathscr{M}$ that satisfies the functional equation (47) (together with any necessary smoothness conditions) is necessarily a quasi-arithmetic mean.

10. Deviation Means

Deviation means were introduced in [9] and were further investigated in [10]. They are defined as follows.

A real-valued function $E = E(x,t)$ on $\mathbb{R}^2$ is called a *deviation* if $E(x,x) = 0$ for all x and if $E(x,t)$ is a strictly decreasing continuous function of t for every x. If E is a deviation, and if $x_1,\ldots,x_n$ are given, then the *E-deviation mean* of $x_1,\ldots,x_n$ is defined to be the unique zero of

$$E(x_1,t) + \cdots + E(x_n,t). \tag{49}$$

It is direct to see that (49) has a unique zero and that this zero does indeed define a mean.

Problem 11. Characterize deviation means and explore their exact relationship with distance means.

If E is a deviation, then (following [11]), one may define d_E by

$$d_E(x,t) = \int_x^t E(x,s)\,ds. \tag{50}$$

Then $d_E(x,t) \geq 0$ and $d_E(x,t)$ is a strictly convex function in t for every x. The E-deviation mean of $x_1,\ldots,x_n$ is nothing but the unique value of t at which $d_E(x_1,t)+\cdots+d_E(x_n,t)$ attains its minimum. Thus if d_E happens to be symmetric, then d_E would be a distance and the E-deviation mean would be the distance mean arising from the distance d_E.

11. Other Ways of Generating New Means

If f and g are differentiable on an open interval $\mathbb{J}$, and if $a < b$ are points in $\mathbb{J}$ such that $f(b) \neq f(a)$, then there exists, by Cauchy's mean value theorem, a point c in (a,b), such that

$$\frac{f'(c)}{g'(c)} = \frac{g(b)-g(a)}{f(b)-f(a)}. \tag{51}$$

If f and g are such that c is unique for every a, b, then we call c the *Cauchy* mean of a and b corresponding to the functions f and g, and we denote it by $\mathscr{C}_{f,g}(a,b)$.

Another natural way of defining means is to take a continuous function F that is strictly monotone on $\mathbb{J}$, and to define the mean of $a,b \in \mathbb{J}$, $a \neq b$, to be the unique point c in (a,b) such that

$$F(c) = \frac{1}{b-a}\int_a^b F(x)\,dx. \tag{52}$$

We call c the *mean value* (mean) of a and b corresponding to F, and we denote it by $\mathscr{V}(a,b)$.

Clearly, if H is an antiderivative of F, then (53) can be written as

$$H'(c) = \frac{H(b)-H(a)}{b-a}. \tag{53}$$

Thus $\mathscr{V}_F(a,b) = \mathscr{C}_{H,E}(a,b)$, where E is the identity function.

For more on the these two families of means, the reader is referred to [12] and [13], and to the references therein.

In contrast to the attitude of thinking of the mean as the number that minimizes a certain function, there is what one may call the Chisini attitude that we now describe. A function f on $\mathbb{J}^n$ may be called a *Chisini function* if and only if the equation

$$f(a_1,\ldots,a_n) = f(x,\ldots,x) \tag{54}$$

has a unique solution $x = a \in [a_1, a_n]$ for every ordered data set $(a_1, \ldots, a_n)$ in $\mathbb{J}$. This unique solution is called the *Chisini mean* associated to f. In Chisini's own words, *x is said to be the mean of n numbers $x_1, \ldots, x_n$ with respect to a problem, in which a function of them $f(x_1, \ldots, x_n)$ is of interest, if the function assumes the same value when all the x_h are replaced by the mean value x: $f(x_1, \ldots, x_n) = f(x, \ldots, x)$*; see [14, page 256] and [1]. Examples of such Chisini means that arise in geometric configurations can be found in [15].

Problem 12. Investigate how the families of distance, deviation, Cauchy, mean value, and Chisini means are related.

12. Internality Tests

According to the definition of a mean, all that is required of a function $\mathscr{M} : \mathbb{J}^n \to \mathbb{J}$ to be a mean is to satisfy the internality property

$$\min\{a_1, \ldots, a_k\} \le \mathscr{M}(a_1, \ldots, a_k) \le \max\{a_1, \ldots, a_k\} \quad (55)$$

for all $a_j \in \mathbb{J}$. However, one may ask whether it is sufficient, for certain types of functions $\mathscr{M}$, to verify (55) for a finite, preferably small, number of well-chosen n-tuples. This question is inspired by certain elegant theorems in the theory of copositive forms that we summarize below.

12.1. Copositivity Tests for Quadratic and Cubic Forms. By a (real) *form* in n variables, we shall always mean a homogeneous polynomial $F = F(x_1, \ldots, x_n)$ in the indeterminates $x_1, \ldots, x_n$ having coefficients in $\mathbb{R}$. When the degree t of a form F is to be emphasized, we call F a t-form. Forms of degrees 1, 2, 3, 4, and 5 are referred to as *linear*, *quadratic*, *cubic*, *quartic*, and *quintic* forms, respectively.

The set of all t-forms in n variables is a vector space (over $\mathbb{R}$) that we shall denote by $\mathbb{F}_t^{(n)}$. It may turn out to be an interesting exercise to prove that the set

$$\left\{\prod_{j=1}^{d} N_j^{e_j} : \sum_{j=1}^{d} j e_j = d\right\} \quad (56)$$

is a basis, where N_j is the Newton polynomial defined by

$$N_j = \sum_{k=1}^{n} x_k^j. \quad (57)$$

The statement above is quite easy to prove in the special case $d \le 3$, and this is the case we are interested in in this paper. We also discard the trivial case $n = 1$ and assume always that $n \ge 2$.

Linear forms can be written as aN_1, and they are not worth much investigation. Quadratic forms can be written as

$$Q = aN_1^2 + bN_2 = a\left(\sum_{k=1}^{n} x_k\right)^2 + b\left(\sum_{k=1}^{n} x_k^2\right). \quad (58)$$

Cubic and quartic forms can be written, respectively, as

$$\begin{gathered} aN_1^3 + bN_1N_2 + cN_3, \\ aN_1^4 + bN_1^2N_2 + cN_1N_3 + dN_2^2. \end{gathered} \quad (59)$$

A form $F = F(x_1, \ldots, x_n)$ is said to be *copositive* if $f(a_1, \ldots, a_n) \ge 0$ for all $x_i \ge 0$. Copositive forms arise in the theory of inequalities and are studied in [14] (and in references therein). One of the interesting questions that one may ask about forms pertains to algorithms for deciding whether a given form is copositive. This problem, in full generality, is still open. However, for quadratic and cubic forms, we have the following satisfactory answers.

Theorem 2. *Let $F = F(x_1, \ldots, x_n)$ be a real symmetric form in any number $n \ge 2$ of variables. Let $\mathbf{v}_m^{(n)}$, $1 \le m \le n$, be the n-tuple whose first m coordinates are 1's and whose remaining coordinates are $0'$s.*

(i) *If F is quadratic, then F is copositive if and only if $F \ge 0$ at the two test n-tuples*

$$\mathbf{v}_1^{(n)} = (1, 0, \ldots, 0), \qquad \mathbf{v}_n^{(n)} = (1, 1, \ldots, 1). \quad (60)$$

(ii) *If F is cubic, then F is copositive if and only if $F \ge 0$ at the n test n-tuples*

$$\mathbf{v}_m^{(n)} = \Bigl(\overbrace{1, \ldots, 1}^{m}, \overbrace{0, \ldots, 0}^{n-m}\Bigr), \quad 1 \le m \le n. \quad (61)$$

Part (i) is a restatement of Theorem 1(a) in [16]. Theorem 1(b) there is related and can be restated as

$$\begin{gathered} F(a_1, \ldots, a_n) \ge 0, \qquad \forall a_i \in \mathbb{R}, \\ \Longleftrightarrow F \ge 0 \text{ at the 3 } n\text{-tuples} \\ (1, 0, \ldots, 0), (1, 1, \ldots, 1), (1, -1, 0, \ldots, 0). \end{gathered} \quad (62)$$

Part (ii) was proved in [17] for $n \le 3$ and in [18] for all n. Two very short and elementary inductive proofs are given in [19].

It is worth mentioning that the n test n-tuples in (61) do not suffice for establishing the copositivity of a quartic form even when $n = 3$. An example illustrating this that uses methods from [20] can be found in [19]. However, an algorithm for deciding whether a symmetric quartic form f in n variables is copositive that consists in testing f at n-tuples of the type

$$\begin{gathered} \Bigl(\overbrace{a, \ldots, a}^{m}, \overbrace{1, \ldots, 1}^{r}, \overbrace{0, \ldots, 0}^{n-m-r}\Bigr), \\ 0 \le m, \ r \le n, \ m + r \le n \end{gathered} \quad (63)$$

is established in [21]. It is also proved there that if $n = 3$, then the same algorithm works for quintics but does not work for forms of higher degrees.

12.2. Internality Tests for Means Arising from Symmetric Forms. Let $\mathbb{F}_t^{(n)}$ be the vector space of all real t-forms in n variables, and let N_j, $1 \le j \le d$, be the Newton polynomials defined in (57). Means of the type

$$\mathscr{M} = \left(\frac{F_r}{F_s}\right)^{1/(r-s)}, \quad (64)$$

where F_j is a symmetric form of degree j, are clearly symmetric and 1-homogeneous, and they abound in the literature. These include the family of Gini means $G_{r,s}$ defined in (8) (and hence the Lehmer and Hölder means). They also include the elementary symmetric polynomial and elementary symmetric polynomial ratio means defined earlier in (10).

In view of Theorem 2 of the previous section, it is tempting to ask whether the internality of a function $\mathscr{M}$ of the type described in (64) can be established by testing it at a finite set of test n-tuples. Positive answers for some special cases of (64), and for other related types, are given in the following theorem.

Theorem 3. *Let L, Q, and C be real symmetric forms of degrees 1, 2, and 3, respectively, in any number $n \geq 2$ of nonnegative variables. Let $\mathbf{v}_k^{(n)}$, $1 \leq k \leq n$, be as defined in Theorem 2.*

(i) *$\sqrt{Q}$ is internal if and only if it is internal at the two test n-tuples: $\mathbf{v}_n^{(n)} = (1,1,\ldots,1)$ and $v_{n-1}^{(n)} = (1,1,\ldots,1,0)$.*

(ii) *Q/L is internal if and only if it is internal at the two test n-tuples: $\mathbf{v}_n^{(n)} = (1,1,\ldots,1)$ and $v_1^{(n)} = (1,0,\ldots,0)$.*

(iii) *If $n \leq 4$, then $\sqrt[3]{C}$ is internal if and only if it is internal at the n test n-tuples*

$$\mathbf{v}_m^{(n)} = \left(\overbrace{1,\ldots,1}^{m},\overbrace{0,\ldots,0}^{n-m}\right), \quad 1 \leq m \leq n. \tag{65}$$

Parts (i) and (ii) are restatements of Theorems 3 and 5 in [16]. Part (iii) is proved in [22] in a manner that leaves a lot to be desired. Besides being rather clumsy, the proof works for $n \leq 4$ only. The problem for $n \geq 5$, together with other open problems, is listed in the next problem set.

Problem Set 13. Let L, Q, and C be real symmetric cubic forms of degrees 1, 2, and 3, respectively, in n non-negative variables.

(13-a) Prove or disprove that $\sqrt[3]{C}$ is internal if and only if it is internal at the n test n-tuples

$$\mathbf{v}_m^{(n)} = \left(\overbrace{1,\ldots,1}^{m},\overbrace{0,\ldots,0}^{n-m}\right), \quad 1 \leq m \leq n. \tag{66}$$

(13-b) Find, or prove the nonexistence of, a finite set T of test n-tuples such that the internality of C/Q at the n-tuples in T gurantees its internality at all nonnegative n-tuples.

(13-c) Find, or prove the nonexistence of, a finite set T of test n-tuples such that the internality of $L \pm \sqrt{Q}$ at the n-tuples in T guarantees its internality at all nonnegative n-tuples.

Problem (13-b) is open even for $n = 2$. In Section 6 of [15], it is shown that the two pairs $(1,0)$ and $(1,1)$ do not suffice as test pairs.

As for Problem (13-c), we refer the reader to [23], where means of the type $L \pm \sqrt{Q}$ were considered. It is proved in Theorem 2 there that when Q has the special form $a\prod_{1\leq i<j\leq n}(x_i - x_j)^2$, then $L \pm \sqrt{Q}$ is internal if and only if it is internal at the two test n-tuples $\mathbf{v}_n^{(n)} = (1,1,\ldots,1)$ and $\mathbf{v}_{n-1}^{(n)} = (1,1,\ldots,1,0)$. In the general case, sufficient and necessary conditions for internality of $L \pm \sqrt{Q}$, in terms of the coefficients of L and Q, are found in [23, Theorem 3]. However, it is not obvious whether these conditions can be rewritten in terms of test n-tuples in the manner done in Theorem 3.

13. Extension of Means, Concordance of Means

The two-dimensional arithmetic mean $\mathscr{A}^{(2)}$ defined by

$$\mathscr{A}^{(2)}(a_1, a_2) = \frac{a_1 + a_2}{2} \tag{67}$$

can be extended to any dimension k by setting

$$\mathscr{A}^{(k)}(a_1, \ldots, a_k) = \frac{a_1 + \cdots + a_k}{k}. \tag{68}$$

Although very few people would disagree on this, nobody can possibly give a mathematically sound justification of the feeling that the definition in (68) is the only (or even the best) definition that makes the sequence $A^{(k)}$ of means *harmonious* or *concordant*. This does not seem to be an acceptable definition of the notion of concordance.

In a private communication several years ago, Professor Zsolt Páles told me that Kolmogorov suggested calling a sequence $\mathscr{M}^{(k)}$ of means on $\mathbb{J}$, where $\mathscr{M}^{(k)}$ is k-dimensional, concordant if for every m and n and every a_i, b_i in $\mathbb{J}$, we have

$$\begin{aligned}&\mathscr{M}^{(n+m)}(a_1, \ldots, a_n, b_1, \ldots, b_m)\\ &\quad = \mathscr{M}^{(2)}\left(\mathscr{M}^{(n)}(a_1, \ldots, a_n), \mathscr{M}_m(b_1, \ldots, b_m)\right).\end{aligned} \tag{69}$$

He also told me that such a definition is too restrictive and seems to confirm concordance in the case of the quasi-arithmetic means only.

Problem 14. Suggest a definition of concordance, and test it on sequences of means that you feel concordant. In particular, test it on the existing generalizations, to higher dimensions, of the logarithmic mean $\mathscr{L}$ defined in (45).

14. Distance Functions in Topology

Distance functions, which are not necessarily metrics, have appeared early in the literature on topology. Given a distance function d on any set X, one may define the *open ball* $B(a, r)$ in the usual manner, and then one may declare a subset $A \subseteq X$ *open* if it contains, for every $a \in A$, an open ball $B(a, r)$ with $r > 0$. If d has the triangle inequality, then one can proceed in the usual manner to create a topology. However, for a general distance d, this need not be the case, and distances that give rise to a coherent topology in the usual manner are called *semimetrics* and they are investigated and characterized in [24–29]. Clearly, these are the distances d for which the family $\{B(a, r) : r > 0\}$ of open balls centered at $a \in S$ forms a local base at a for every a in X.

15. Centers and Center-Producing Distances

A distance d may be defined on any set S whatsoever. In particular, if d is a distance on $\mathbb{R}^2$ and if the function $f(X)$ defined by

$$f(X) = \sum_{i=1}^{n} d(X, A_i) \tag{70}$$

attains its minimum at a unique point X_0 that lies in the convex hull of $\{A_1, \ldots, A_n\}$ for every choice of $A_1, \ldots, A_n$ in $\mathbb{R}^2$, then d will be called *a center-producing distance.*

The Euclidean metric d_1 on $\mathbb{R}^2$ produces the *Fermat-Torricelli* center. This is defined to be the point whose distances from the given points have a minimal sum. Its square, d_2, which is just a distance but not a metric, produces the *centroid*. This is the center of mass of equal masses placed at the given points. It would be interesting to explore the centers defined by d_p for other values of p.

Problem 15. Let d_p, $p > 1$, be the distance defined on $\mathbb{R}^2$ by $d_p(A, B) = \|A - B\|^p$, and let ABC be a triangle. Let $Z_p = Z_p(A, B, C)$ be the point that minimizes

$$\begin{aligned} d_p(Z, A) + d_p(Z, B) + d_p(Z, C) \\ = \|Z - A\|^p + \|Z - B\|^p + \|Z - C\|^p. \end{aligned} \tag{71}$$

Investigate how Z_p, $p \geq 1$, are related to the known triangle centers, and study the curve traced by them.

The papers [30, 31] may turn out to be relevant to this problem.

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Research Article

On Some Intermediate Mean Values

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We give a necessary and sufficient mean condition for the quotient of two Jensen functionals and define a new class $\Lambda_{f,g}(a,b)$ of mean values where f, g are continuously differentiable convex functions satisfying the relation $f''(t) = tg''(t)$, $t \in \mathbb{R}^+$. Then we asked for a characterization of f, g such that the inequalities $H(a,b) \le \Lambda_{f,g}(a,b) \le A(a,b)$ or $L(a,b) \le \Lambda_{f,g}(a,b) \le I(a,b)$ hold for each positive a, b, where H, A, L, I are the harmonic, arithmetic, logarithmic, and identric means, respectively. For a subclass of Λ with $g''(t) = t^s$, $s \in \mathbb{R}$, this problem is thoroughly solved.

1. Introduction

It is said that the mean P is intermediate relating to the means M and N, $M \le N$ if the relation

$$M(a,b) \le P(a,b) \le N(a,b) \tag{1}$$

holds for each two positive numbers a, b.

It is also well known that

$$\begin{aligned} \min\{a,b\} &\le H(a,b) \le G(a,b) \\ &\le L(a,b) \le I(a,b) \le A(a,b) \le S(a,b) \\ &\le \max\{a,b\}, \end{aligned} \tag{2}$$

where

$$H = H(a,b) := 2\left(\frac{1}{a}+\frac{1}{b}\right)^{-1};$$

$$G = G(a,b) := \sqrt{ab}; \qquad L = L(a,b) := \frac{b-a}{\log b - \log a};$$

$$I = I(a,b) := \frac{\left(b^b/a^a\right)^{1/(b-a)}}{e};$$

$$A = A(a,b) := \frac{a+b}{2}; \qquad S = S(a,b) := a^{a/(a+b)}b^{b/(a+b)} \tag{3}$$

are the harmonic, geometric, logarithmic, identric, arithmetic, and Gini mean, respectively.

An easy task is to construct intermediate means related to two given means M and N with $M \le N$. For instance, for an arbitrary mean P, we have that

$$M(a,b) \le P(M(a,b), N(a,b)) \le N(a,b). \tag{4}$$

The problem is more difficult if we have to decide whether the given mean is intermediate or not. For example, the relation

$$L(a,b) \le S_s(a,b) \le I(a,b) \tag{5}$$

holds for each positive a and b if and only if $0 \le s \le 1$, where the Stolarsky mean S_s is defined by (cf [1])

$$S_s(a,b) := \left(\frac{b^s - a^s}{s(b-a)}\right)^{1/(s-1)}. \tag{6}$$

Also,

$$G(a,b) \le A_s(a,b) \le A(a,b) \tag{7}$$

holds if and only if $0 \le s \le 1$, where the Hölder mean of order s is defined by

$$A_s(a,b) := \left(\frac{a^s + b^s}{2}\right)^{1/s}. \tag{8}$$

An inverse problem is to find best possible approximation of a given mean P by elements of an ordered class of means S. A good example for this topic is comparison between the logarithmic mean and the class A_s of Hölder means of order s. Namely, since $A_0 = \lim_{s\to 0} A_s = G$ and $A_1 = A$, it follows from (2) that

$$A_0 \le L \le A_1. \tag{9}$$

Since A_s is monotone increasing in s, an improving of the above is given by Carlson [2]:

$$A_0 \le L \le A_{1/2}. \tag{10}$$

Finally, Lin showed in [3] that

$$A_0 \le L \le A_{1/3} \tag{11}$$

is the best possible approximation of the logarithmic mean by the means from the class A_s.

Numerous similar results have been obtained recently. For example, an approximation of Seiffert's mean by the class A_s is given in [4, 5].

In this paper we will give best possible approximations for a whole variety of elementary means (2) by the class λ_s defined below (see Theorem 5).

Let f, g be twice continuously differentiable (strictly) convex functions on $\mathbb{R}^+$. By definition (cf [6], page 5),

$$\begin{gathered}\overline{f}(a,b) := f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) > 0, \quad a \ne b,\\ \overline{f}(a,b) = 0,\end{gathered} \tag{12}$$

if and only if $a = b$.

It turns out that the expression

$$\Lambda_{f,g}(a,b) := \frac{\overline{f}(a,b)}{\overline{g}(a,b)} = \frac{f(a) + f(b) - 2f((a+b)/2)}{g(a) + g(b) - 2g((a+b)/2)} \tag{13}$$

represents a mean of two positive numbers a, b; that is, the relation

$$\min\{a,b\} \le \Lambda_{f,g}(a,b) \le \max\{a,b\} \tag{14}$$

holds for each $a, b \in \mathbb{R}^+$, if and only if the relation

$$f''(t) = tg''(t) \tag{15}$$

holds for each $t \in \mathbb{R}^+$.

Let $f, g \in C^\infty(0,\infty)$ and denote by Λ the set $\{(f,g)\}$ of convex functions satisfying the relation (15). There is a natural question how to improve the bounds in (14); in this sense we come upon the following intermediate mean problem.

Open Question. Under what additional conditions on $f, g \in \Lambda$, the inequalities

$$H(a,b) \le \Lambda_{f,g}(a,b) \le A(a,b), \tag{16}$$

or, more tightly,

$$L(a,b) \le \Lambda_{f,g}(a,b) \le I(a,b), \tag{17}$$

hold for each $a, b \in \mathbb{R}^+$?

As an illustration, consider the function $f_s(t)$ defined to be

$$f_s(t) = \begin{cases} \dfrac{t^s - st + s - 1}{s(s-1)}, & s(s-1) \ne 0;\\ t - \log t - 1, & s = 0;\\ t\log t - t + 1, & s = 1.\end{cases} \tag{18}$$

Since

$$\begin{gathered} f_s'(t) = \begin{cases} \dfrac{t^{s-1} - 1}{s-1}, & s(s-1) \ne 0;\\ 1 - \dfrac{1}{t}, & s = 0;\\ \log t, & s = 1,\end{cases}\\ f_s''(t) = t^{s-2}, \quad s \in \mathbb{R},\ t > 0,\end{gathered} \tag{19}$$

it follows that $f_s(t)$ is a twice continuously differentiable convex function for $s \in \mathbb{R}$, $t \in \mathbb{R}^+$.

Moreover, it is evident that $(f_{s+1}, f_s) \in \Lambda$.

We will give in the sequel a complete answer to the above question concerning the means

$$\frac{\overline{f}_{s+1}(a,b)}{\overline{f}_s(a,b)} := \lambda_s(a,b) \tag{20}$$

defined by

$$\lambda_s(a,b) = \begin{cases} \dfrac{s-1}{s+1}\,\dfrac{a^{s+1} + b^{s+1} - 2((a+b)/2)^{s+1}}{a^s + b^s - 2((a+b)/2)^s}, & s \in \mathbb{R}/\{-1,0,1\};\\ \dfrac{2\log((a+b)/2) - \log a - \log b}{1/2a + 1/2b - 2/(a+b)}, & s = -1;\\ \dfrac{a\log a + b\log b - (a+b)\log((a+b)/2)}{2\log((a+b)/2) - \log a - \log b}, & s = 0;\\ \dfrac{(b-a)^2}{4(a\log a + b\log b - (a+b)\log((a+b)/2))}, & s = 1.\end{cases} \tag{21}$$

Those means are obviously symmetric and homogeneous of order one.

As a consequence we obtain some new intermediate mean values; for instance, we show that the inequalities

$$\begin{gathered} H(a,b) \le \lambda_{-1}(a,b) \le G(a,b) \le \lambda_0(a,b) \le L(a,b)\\ \le \lambda_1(a,b) \le I(a,b)\end{gathered} \tag{22}$$

hold for arbitrary $a, b \in \mathbb{R}^+$. Note that

$$\begin{gathered}\lambda_{-1} = \frac{2G^2\log(A/G)}{A-H}; \qquad \lambda_0 = A\frac{\log(S/A)}{\log(A/G)};\\ \lambda_1 = \frac{1}{2}\frac{A-H}{\log(S/A)}.\end{gathered} \tag{23}$$

2. Results

We prove firstly the following

Theorem 1. *Let $f, g \in C^2(I)$ with $g'' > 0$. The expression $\Lambda_{f,g}(a,b)$ represents a mean of arbitrary numbers $a, b \in I$ if and only if the relation* (15) *holds for $t \in I$.*

Remark 2. In the same way, for arbitrary $p, q > 0$, $p+q=1$, it can be deduced that the quotient

$$\Lambda_{f,g}(p,q;a,b) := \frac{pf(a)+qf(b)-f(pa+qb)}{pg(a)+qg(b)-g(pa+qb)} \tag{24}$$

represents a mean value of numbers a, b if and only if (15) holds.

A generalization of the above assertion is the next.

Theorem 3. *Let $f, g : I \to \mathbb{R}$ be twice continuously differentiable functions with $g'' > 0$ on I and let $p = \{p_i\}$, $i = 1, 2, \ldots, \sum p_i = 1$ be an arbitrary positive weight sequence. Then the quotient of two Jensen functionals*

$$\Lambda_{f,g}(p,x) := \frac{\sum_1^n p_i f(x_i) - f\left(\sum_1^n p_i x_i\right)}{\sum_1^n p_i g(x_i) - g\left(\sum_1^n p_i x_i\right)}, \quad n \geq 2, \tag{25}$$

represents a mean of an arbitrary set of real numbers $x_1, x_2, \ldots, x_n \in I$ if and only if the relation

$$f''(t) = tg''(t) \tag{26}$$

holds for each $t \in I$.

Remark 4. It should be noted that the relation $f''(t) = tg''(t)$ determines f in terms of g in an easy way. Precisely,

$$f(t) = tg(t) - 2G(t) + ct + d, \tag{27}$$

where $G(t) := \int_1^t g(u)du$ and c and d are constants.

Our results concerning the means $\lambda_s(a,b)$, $s \in \mathbb{R}$ are included in the following.

Theorem 5. *For the class of means $\lambda_s(a,b)$ defined above, the following assertions hold for each $a, b \in \mathbb{R}^+$.*

(1) *The means $\lambda_s(a,b)$ are monotone increasing in s;*
(2) *$\lambda_s(a,b) \leq H(a,b)$ for each $s \leq -4$;*
(3) *$H(a,b) \leq \lambda_s(a,b) \leq G(a,b)$ for $-3 \leq s \leq -1$;*
(4) *$G(a,b) \leq \lambda_s(a,b) \leq L(a,b)$ for $-1/2 \leq s \leq 0$;*
(5) *there is a number $s_0 \in (1/12, 1/11)$ such that $L(a,b) \leq \lambda_s(a,b) \leq I(a,b)$ for $s_0 \leq s \leq 1$;*
(6) *there is a number $s_1 \in (1.03, 1.04)$ such that $I(a,b) \leq \lambda_s(a,b) \leq A(a,b)$ for $s_1 \leq s \leq 2$;*
(7) *$A(a,b) \leq \lambda_s(a,b) \leq S(a,b)$ for each $2 \leq s \leq 5$;*
(8) *there is no finite s such that the inequality $S(a,b) \leq \lambda_s(a,b)$ holds for each $a, b \in \mathbb{R}^+$.*

The above estimations are best possible.

3. Proofs

3.1. Proof of Theorem 1. We prove firstly the necessity of the condition (15).

Since $\Lambda_{f,g}(a,b)$ is a mean value for arbitrary $a, b \in I$; $a \neq b$, we have

$$\min\{a,b\} \leq \Lambda_{f,g}(a,b) \leq \max\{a,b\}. \tag{28}$$

Hence

$$\lim_{b\to a} \Lambda_{f,g}(a,b) = a. \tag{29}$$

From the other hand, due to l'Hospital's rule we obtain

$$\begin{aligned}\lim_{b\to a} \Lambda_{f,g}(a,b) &= \lim_{b\to a}\left(\frac{f'(b)-f'((a+b)/2)}{g'(b)-g'((a+b)/2)}\right) \\ &= \lim_{b\to a}\left(\frac{2f''(b)-f''((a+b)/2)}{2g''(b)-g''((a+b)/2)}\right) \\ &= \frac{f''(a)}{g''(a)}.\end{aligned} \tag{30}$$

Comparing (29) and (30) the desired result follows.

Suppose now that (15) holds and let $a < b$. Since $g''(t) > 0$ $t \in [a,b]$ by the Cauchy mean value theorem there exists $\xi \in ((a+t)/2, t)$ such that

$$\frac{f'(t)-f'((a+t)/2)}{g'(t)-g'((a+t)/2)} = \frac{f''(\xi)}{g''(\xi)} = \xi. \tag{31}$$

But,

$$a \leq \frac{a+t}{2} < \xi < t \leq b, \tag{32}$$

and, since g' is strictly increasing, $g'(t) - g'((a+t)/2) > 0$, $t \in [a,b]$.

Therefore, by (31) we get

$$\begin{aligned}a\left(g'(t)-g'\left(\frac{a+t}{2}\right)\right) &\leq f'(t)-f'\left(\frac{a+t}{2}\right) \\ &\leq b\left(g'(t)-g'\left(\frac{a+t}{2}\right)\right).\end{aligned} \tag{33}$$

Finally, integrating (33) over $t \in [a,b]$ we obtain the assertion from Theorem 1.

3.2. Proof of Theorem 3. We will give a proof of this assertion by induction on n.

By Remark 2, it holds for $n = 2$.

Next, it is not difficult to check the identity

$$\begin{aligned}&\sum_1^n p_i f(x_i) - f\left(\sum_1^n p_i x_i\right) \\ &= (1-p_n)\left(\sum_1^{n-1} p_i' f(x_i) - f\left(\sum_1^{n-1} p_i' x_i\right)\right) \\ &\quad + \left[(1-p_n) f(T) + p_n f(x_n) - f((1-p_n)T + p_n x_n)\right],\end{aligned} \tag{34}$$

where

$$T := \sum_1^{n-1} p_i' x_i; \quad p_i' := \frac{p_i}{(1-p_n)}, \quad i = 1, 2, \dots, n-1;$$
$$\sum_1^{n-1} p_i' = 1. \tag{35}$$

Therefore, by induction hypothesis and Remark 2, we get

$$\begin{aligned}
&\sum_1^n p_i f(x_i) - f\left(\sum_1^n p_i x_i\right) \\
&\quad \le \max\{x_1, x_2, \dots x_{n-1}\}(1-p_n) \\
&\quad \times \left(\sum_1^{n-1} p_i' g(x_i) - g\left(\sum_1^{n-1} p_i' x_i\right)\right) \\
&\quad + \max\{T, x_n\}\left[(1-p_n) g(T) + p_n g(x_n)\right. \\
&\qquad \left. - g((1-p_n)T + p_n x_n)\right] \\
&\quad \le \max\{x_1, x_2, \dots, x_n\} \\
&\quad \times \left((1-p_n)\left(\sum_1^{n-1} p_i' g(x_i) - g\left(\sum_1^{n-1} p_i' x_i\right)\right)\right. \\
&\quad \left. + \left[(1-p_n) g(T) + p_n g(x_n) - g((1-p_n)T + p_n x_n)\right]\right) \\
&\quad = \max\{x_1, x_2, \dots, x_n\}\left(\sum_1^n p_i g(x_i) - g\left(\sum_1^n p_i x_i\right)\right).
\end{aligned} \tag{36}$$

The inequality

$$\min\{x_1, x_2, \dots, x_n\} \le \Lambda_{f,g}(p, x) \tag{37}$$

can be proved analogously.

For the proof of necessity, put $x_2 = x_3 = \cdots = x_n$ and proceed as in Theorem 1.

Remark 6. It is evident from (15) that if $I \subseteq \mathbb{R}^+$ then f has to be also convex on I. Otherwise, it shouldn't be the case. For example, the conditions of Theorem 3 are satisfied with $f(t) = t^3/3$, $g(t) = t^2$, $t \in \mathbb{R}$. Hence, for an arbitrary sequence $\{x_i\}_1^n$ of real numbers, we obtain

$$\begin{aligned}
\min\{x_1, x_2, \dots, x_n\} &\le \frac{\sum_1^n p_i x_i^3 - \left(\sum_1^n p_i x_i\right)^3}{3\left(\sum_1^n p_i x_i^2 - \left(\sum_1^n p_i x_i\right)^2\right)} \\
&\le \max\{x_1, x_2, \dots, x_n\}.
\end{aligned} \tag{38}$$

Because the above inequality does not depend on n, a probabilistic interpretation of the above result is contained in the following.

Theorem 7. *For an arbitrary probability law F of random variable X with support on $(-\infty, +\infty)$, one has*

$$(EX)^3 + 3(\min X)\,\sigma_X^2 \le EX^3 \le (EX)^3 + 3(\max X)\,\sigma_X^2. \tag{39}$$

3.3. Proof of Theorem 5, Part (1). We will prove a general assertion of this type. Namely, for an arbitrary positive sequence $\mathbf{x} = \{x_i\}$ and an associated weight sequence $\mathbf{p} = \{p_i\}$, $i = 1, 2, \dots$, denote

$$\chi_s(\mathbf{p}, \mathbf{x}) := \begin{cases} \dfrac{\sum p_i x_i^s - \left(\sum p_i x_i\right)^s}{s(s-1)}, & s \in \mathbb{R}/\{0,1\}; \\ \log\left(\sum p_i x_i\right) - \sum p_i \log x_i, & s = 0; \\ \sum p_i x_i \log x_i - \left(\sum p_i x_i\right)\log\left(\sum p_i x_i\right), & s = 1. \end{cases} \tag{40}$$

For $s \in \mathbb{R}$, $r > 0$ we have

$$\chi_s(\mathbf{p}, \mathbf{x})\,\chi_{s+r+1}(\mathbf{p}, \mathbf{x}) \ge \chi_{s+1}(\mathbf{p}, \mathbf{x})\,\chi_{s+r}(\mathbf{p}, \mathbf{x}), \tag{41}$$

which is equivalent to

Theorem 8. *The sequence $\{\chi_{s+1}(\mathbf{p}, \mathbf{x})/\chi_s(\mathbf{p}, \mathbf{x})\}$ is monotone increasing in s, $s \in \mathbb{R}$.*

This assertion follows applying the result from [7, Theorem 2] which states the following.

Lemma 9. *For $-\infty < a < b < c < +\infty$, the inequality*

$$(\chi_b(\mathbf{p}, \mathbf{x}))^{c-a} \le (\chi_a(\mathbf{p}, \mathbf{x}))^{c-b}(\chi_c(\mathbf{p}, \mathbf{x}))^{b-a} \tag{42}$$

holds for arbitrary sequences $\mathbf{p}$, $\mathbf{x}$.

Putting there $a = s$, $b = s + 1$, $c = s + r + 1$ and $a = s$, $b = s + r$, $c = s + r + 1$, we successively obtain

$$\begin{aligned}
(\chi_{s+1}(\mathbf{p}, \mathbf{x}))^{r+1} &\le (\chi_s(\mathbf{p}, \mathbf{x}))^r \chi_{s+r+1}(\mathbf{p}, \mathbf{x}), \\
(\chi_{s+r}(\mathbf{p}, \mathbf{x}))^{r+1} &\le \chi_s(\mathbf{p}, \mathbf{x})(\chi_{s+r+1}(\mathbf{p}, \mathbf{x}))^r.
\end{aligned} \tag{43}$$

Since $r > 0$, multiplying those inequalities we get the relation (41), that is, the proof of Theorem 8.

The part (1) of Theorem 5 follows for $p_1 = p_2 = 1/2$.

A general way to prove the rest of Theorem 5 is to use an easy-checkable identity

$$\frac{\lambda_s(a, b)}{A(a, b)} = \lambda_s(1 + t, 1 - t), \tag{44}$$

with $t := (b - a)/(b + a)$.

Since $0 < a < b$, we get $0 < t < 1$. Also,

$$\frac{H(a,b)}{A(a,b)} = 1 - t^2; \qquad \frac{G(a,b)}{A(a,b)} = \sqrt{1-t^2};$$

$$\frac{L(a,b)}{A(a,b)} = \frac{2t}{\log(1+t) - \log(1-t)};$$

$$\frac{I(a,b)}{A(a,b)} = \exp\left(\frac{(1+t)\log(1+t) - (1-t)\log(1-t)}{2t} - 1\right);$$

$$\frac{S(a,b)}{A(a,b)} = \exp\left(\frac{1}{2}\left((1+t)\log(1+t) + (1-t)\log(1-t)\right)\right). \tag{45}$$

Therefore, we have to compare some one-variable inequalities and to check their validness for each $t \in (0,1)$.

For example, we will prove that the inequality

$$\lambda_s(a,b) \le L(a,b) \tag{46}$$

holds for each positive a, b if and only if $s \le 0$.

Since $\lambda_s(a,b)$ is monotone increasing in s, it is enough to prove that

$$\frac{\lambda_0(a,b)}{L(a,b)} \le 1. \tag{47}$$

By the above formulae, this is equivalent to the assertion that the inequality

$$\phi(t) \le 0 \tag{48}$$

holds for each $t \in (0,1)$, with

$$\begin{aligned}\phi(t) := {} & \frac{\log(1+t) - \log(1-t)}{2t} \\ & \times \left((1+t)\log(1+t) + (1-t)\log(1-t)\right) \\ & + \log(1+t) + \log(1-t).\end{aligned} \tag{49}$$

We will prove that the power series expansion of $\phi(t)$ have non-positive coefficients. Thus the relation (48) will be proved.

Since

$$\begin{aligned}\frac{\log(1+t) - \log(1-t)}{2t} &= \sum_0^\infty \frac{t^{2k}}{2k+1}; \\ \log(1+t) + \log(1-t) &= -t^2 \sum_0^\infty \frac{t^{2k}}{k+1}; \\ (1+t)\log(1+t) + (1-t)\log(1-t) &= t^2 \sum_0^\infty \frac{t^{2k}}{(k+1)(2k+1)},\end{aligned} \tag{50}$$

we get

$$\begin{aligned}\frac{\phi(t)}{t^2} &= \sum_{n=0}^\infty \left(-\frac{1}{n+1} + \sum_{k=0}^n \frac{1}{(2n-2k+1)(k+1)(2k+1)}\right) t^{2n} \\ &= \sum_0^\infty c_n t^{2n}.\end{aligned} \tag{51}$$

Hence,

$$c_0 = c_1 = 0; \qquad c_2 = -\frac{1}{90}, \tag{52}$$

and, after some calculation, we get

$$c_n = \frac{2}{(n+1)(2n+3)}\left((n+2)\sum_1^n \frac{1}{2k+1} - (n+1)\sum_1^n \frac{1}{2k}\right), \quad n > 1. \tag{53}$$

Now, one can easily prove (by induction, e.g.) that

$$d_n := (n+2)\sum_1^n \frac{1}{2k+1} - (n+1)\sum_1^n \frac{1}{2k} \tag{54}$$

is a negative real number for $n \ge 2$. Therefore $c_n \le 0$, and the proof of the first part is done. For $0 < s < 1$ we have

$$\begin{aligned}\frac{\lambda_s(a,b)}{L(a,b)} - 1 &= \frac{(1-s)\left((1+t)^{s+1} + (1-t)^{s+1} - 2\right)\log((1+t)/(1-t))}{2t(1+s)\left(2 - (1+t)^s - (1-t)^s\right)} - 1 \\ &= \frac{1}{6}st^2 + O\left(t^4\right) \quad (t \longrightarrow 0).\end{aligned} \tag{55}$$

Therefore, $\lambda_s(a,b) > L(a,b)$ for $s > 0$ and sufficiently small $t := (b-a)/(b+a)$.

Similarly, we will prove that the inequality

$$\lambda_s(a,b) \le I(a,b) \tag{56}$$

holds for each a, b; $0 < a < b$ if and only if $s \le 1$.

As before, it is enough to consider the expression

$$\frac{I(a,b)}{\lambda_1(a,b)} = e^{\mu(t)}\nu(t) := \psi(t), \tag{57}$$

with

$$\begin{aligned}\mu(t) &= \frac{(1+t)\log(1+t) - (1-t)\log(1-t)}{2t} - 1; \\ \nu(t) &= \frac{(1+t)\log(1+t) + (1-t)\log(1-t)}{t^2}.\end{aligned} \tag{58}$$

It is not difficult to check the identity

$$\psi'(t) = -\frac{e^{\mu(t)}\phi(t)}{t^3}. \tag{59}$$

Hence by (48), we get $\psi'(t) > 0$, that is, $\psi(t)$ is monotone increasing for $t \in (0,1)$.

Therefore

$$\frac{I(a,b)}{\lambda_1(a,b)} \geq \lim_{t\to 0^+}\psi(t) = 1. \tag{60}$$

By monotonicity it follows that $\lambda_s(a,b) \leq I(a,b)$ for $s \leq 1$.

For $s > 1$, $(b-a)/(b+a) = t$, we have

$$\lambda_s(a,b) - I(a,b) = \left(\frac{1}{6}(s-1)t^2 + O\left(t^4\right)\right)A(a,b) \quad \left(t \longrightarrow 0^+\right). \tag{61}$$

Hence, $\lambda_s(a,b) > I(a,b)$ for $s > 1$ and t sufficiently small.

From the other hand,

$$\lim_{t\to 1^-}\left[\frac{\lambda_s(a,b)}{I(a,b)} - 1\right] = \frac{e(s-1)\left(2^{s+1}-2\right)}{2(s+1)(2^s-2)} - 1 := \tau(s). \tag{62}$$

Examining the function $\tau(s)$, we find out that it has the only real zero at $s_0 \approx 1.0376$ and is negative for $s \in (1, s_0)$.

Remark 10. Since $\psi(t)$ is monotone increasing, we also get

$$\frac{I(a,b)}{\lambda_1(a,b)} \leq \lim_{t\to 1^-}\psi(t) = \frac{4\log 2}{e}. \tag{63}$$

Hence

$$1 \leq \frac{I(a,b)}{\lambda_1(a,b)} \leq \frac{4\log 2}{e}. \tag{64}$$

A calculation gives $4\log 2/e \approx 1.0200$.

Note also that

$$\lambda_2(a,b) \equiv A(a,b). \tag{65}$$

Therefore, applying the assertion from the part 1, we get

$$\begin{aligned} \lambda_s(a,b) &\leq A(a,b), \quad s \leq 2; \\ \lambda_s(a,b) &\geq A(a,b), \quad s \geq 2. \end{aligned} \tag{66}$$

Finally, we give a detailed proof of the part 7.

We have to prove that $\lambda_s(a,b) \leq S(a,b)$ for $s \leq 5$. Since $\lambda_s(a,b)$ is monotone increasing in s, it is sufficient to prove that the inequality

$$\lambda_5(a,b) \leq S(a,b) \tag{67}$$

holds for each $a, b \in \mathbb{R}^+$.

Therefore, by the transformation given above, we get

$$\begin{aligned} \log\frac{\lambda_5}{A} &= \log\left[\frac{2}{3}\frac{(1+t)^6 + (1-t)^6 - 2}{(1+t)^5 + (1-t)^5 - 2}\right] \\ &= \log\left[\frac{2}{15}\frac{15 + 15t^2 + t^4}{2 + t^2}\right] \\ &\leq \log\left[\frac{1 + t^2 + t^4/4}{1 + t^2/2}\right] = \log\left(1 + \frac{t^2}{2}\right) \\ &= \frac{t^2}{2} - \frac{t^4}{8} + \frac{t^6}{24} - \cdots \\ &\leq \frac{t^2}{2} + \frac{t^4}{12} + \frac{t^6}{30} + \cdots \\ &= \frac{1}{2}\left((1+t)\log(1+t) + (1-t)\log(1-t)\right) \\ &= \log\frac{S}{A}, \end{aligned} \tag{68}$$

and the proof is done.

Further, we have to show that $\lambda_s(a,b) > S(a,b)$ for some positive a, b whenever $s > 5$.

Indeed, since

$$(1+t)^s + (1-t)^s - 2 = \binom{s}{2}t^2 + \binom{s}{4}t^4 + O\left(t^6\right), \tag{69}$$

for $s > 5$ and sufficiently small t, we get

$$\begin{aligned} \frac{\lambda_s}{A} &= \frac{s-1}{s+1}\frac{\binom{s+1}{2}t^2 + \binom{s+1}{4}t^4 + O\left(t^6\right)}{\binom{s}{2}t^2 + \binom{s}{4}t^4 + O\left(t^6\right)} \\ &= \frac{1 + (s-1)(s-2)t^2/12 + O\left(t^4\right)}{1 + (s-2)(s-3)t^2/12 + O\left(t^4\right)} \\ &= 1 + \left(\frac{s}{6} - \frac{1}{3}\right)t^2 + O\left(t^4\right). \end{aligned} \tag{70}$$

Similarly,

$$\begin{aligned} \frac{S}{A} &= \exp\left(\frac{1}{2}\left((1+t)\log(1+t) + (1-t)\log(1-t)\right)\right) \\ &= \exp\left(\frac{t^2}{2} + O\left(t^4\right)\right) = 1 + \frac{t^2}{2} + O\left(t^4\right). \end{aligned} \tag{71}$$

Hence,

$$\frac{1}{A}\left(\lambda_s - S\right) = \frac{1}{6}(s-5)t^2 + O\left(t^4\right), \tag{72}$$

and this expression is positive for $s > 5$ and t sufficiently small, that is, a sufficiently close to b.

As for the part 8, applying the above transformation we obtain

$$\frac{\lambda_s(a,b)}{S(a,b)} = \frac{s-1}{s+1}\frac{(1+t)^{s+1}+(1-t)^{s+1}-2}{(1+t)^{s}+(1-t)^{s}-2} \times \exp\left(-\frac{1}{2}\left((1+t)\log(1+t)+(1-t)\log(1-t)\right)\right), \tag{73}$$

where $0 < a < b$, $t = (b-a)/(b+a)$.

Since for $s > 5$,

$$\lim_{t\to 1^-}\frac{\lambda_s}{S} = \frac{s-1}{s+1}\frac{2^s-1}{2^s-2}, \tag{74}$$

and the last expression is less than one, it follows that the inequality $S(a,b) < \lambda_s(a,b)$ cannot hold whenever b/a is sufficiently large.

The rest of the proof is straightforward.

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Research Article

The Monotonicity Results for the Ratio of Certain Mixed Means and Their Applications

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We continue to adopt notations and methods used in the papers illustrated by Yang (2009, 2010) to investigate the monotonicity properties of the ratio of mixed two-parameter homogeneous means. As consequences of our results, the monotonicity properties of four ratios of mixed Stolarsky means are presented, which generalize certain known results, and some known and new inequalities of ratios of means are established.

1. Introduction

Since the Ky Fan [1] inequality was presented, inequalities of ratio of means have attracted attentions of many scholars. Some known results can be found in [2–14]. Research for the properties of ratio of bivariate means was also a hotspot at one time.

In this paper, we continue to adopt notations and methods used in the paper [13, 14] to investigate the monotonicity properties of the functions Q_{if} ($i = 1, 2, 3, 4$) defined by

$$\begin{aligned} Q_{1f}(p) &:= \frac{g_{1f}(p;a,b)}{g_{1f}(p;c,d)}, \\ Q_{2f}(p) &:= \frac{g_{2f}(p;a,b)}{g_{2f}(p;c,d)}, \\ Q_{3f}(p) &:= \frac{g_{3f}(p;a,b)}{g_{3f}(p;c,d)}, \\ Q_{4f}(p) &:= \frac{g_{4f}(p;a,b)}{g_{4f}(p;c,d)}, \end{aligned} \tag{1.1}$$

where

$$g_{1f}(p) = g_{1f}(p; a, b) := \sqrt{\mathcal{H}_f(p, q)\mathcal{H}_f(2k - p, q)}, \tag{1.2}$$

$$g_{2f}(p) = g_{2f}(p; a, b) := \sqrt{\mathcal{H}_f(p, p + m)\mathcal{H}_f(2k - p, 2k - p + m)}, \tag{1.3}$$

$$g_{3f}(p) = g_{3f}(p; a, b) := \sqrt{\mathcal{H}_f(p, 2m - p)\mathcal{H}_f(2k - p, 2m - 2k + p)}, \tag{1.4}$$

$$g_{4f}(p) = g_{4f}(p; a, b) := \sqrt{\mathcal{H}_f(pr, ps)\mathcal{H}_f((2k - p)r, (2k - p)s)}, \tag{1.5}$$

the $q, r, s, k, m \in \mathbb{R}$, $a, b, c, d \in \mathbb{R}_+$ with $b/a > d/c \geq 1$, $\mathcal{H}_f(p, q)$ is the so-called two-parameter homogeneous functions defined by [15, 16]. For conveniences, we record it as follows.

Definition 1.1. Let $f\colon \mathbb{R}_+^2 \setminus \{(x, x), x \in \mathbb{R}_+\} \to \mathbb{R}_+$ be a first-order homogeneous continuous function which has first partial derivatives. Then, $\mathcal{H}_f : \mathbb{R}^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a homogeneous function generated by f with parameters p and q if $\mathcal{H}_f$ is defined by for $a \neq b$

$$\begin{aligned} \mathcal{H}_f(p, q; a, b) &= \left(\frac{f(a^p, b^p)}{f(a^q, b^q)}\right)^{1(p-q)}, \quad \text{if } pq(p - q) \neq 0, \\ \mathcal{H}_f(p, p; a, b) &= \exp\left(\frac{a^p f_x(a^p, b^p)\ln a + b^p f_y(a^p, b^p)\ln b}{f(a^p, b^p)}\right), \quad \text{if } p = q \neq 0, \end{aligned} \tag{1.6}$$

where $f_x(x, y)$ and $f_y(x, y)$ denote first-order partial derivatives with respect to first and second component of $f(x, y)$, respectively.

If $\lim_{y \to x} f(x, y)$ exits and is positive for all $x \in \mathbb{R}_+$, then further define

$$\begin{aligned} \mathcal{H}_f(p, 0; a, b) &= \left(\frac{f(a^p, b^p)}{f(1, 1)}\right)^{1/p}, \quad \text{if } p \neq 0,\ q = 0, \\ \mathcal{H}_f(0, q; a, b) &= \left(\frac{f(a^q, b^q)}{f(1, 1)}\right)^{1/q}, \quad \text{if } p = 0,\ q \neq 0, \\ \mathcal{H}_f(0, 0; a, b) &= a^{f_x(1,1)/f(1,1)} b^{f_y(1,1)/f(1,1)}, \quad \text{if } p = q = 0, \end{aligned} \tag{1.7}$$

and $\mathcal{H}_f(p, q; a, a) = a$.

Remark 1.2. Witkowski [17] proved that if the function $(x, y) \to f(x, y)$ is a symmetric and first-order homogeneous function, then for all p, q $\mathcal{H}_f(p, q; a, b)$ is a mean of positive numbers a and b if and only if f is increasing in both variables on $\mathbb{R}_+$. In fact, it is easy to see that the condition "$f(x, y)$ is symmetric" can be removed.

If $\mathcal{H}_f(p, q; a, b)$ is a mean of positive numbers a and b, then it is called two-parameter homogeneous mean generated by f.

For simpleness, $\mathcal{H}_f(p, q; a, b)$ is also denoted by $\mathcal{H}_f(p, q)$ or $\mathcal{H}_f(a, b)$.

The two-parameter homogeneous function $\mathcal{H}_f(p, q; a, b)$ generated by f is very important because it can generates many well-known means. For example, substituting

$L = L(x, y) = (x - y)/(\ln x - \ln y)$ if $x, y > 0$ with $x \neq y$ and $L(x, x) = x$ for f yields Stolarsky means $\mathscr{H}_L(p, q; a, b) = S_{p,q}(a, b)$ defined by

$$S_{p,q}(a,b) = \begin{cases} \left(\dfrac{q}{p}\dfrac{a^p - b^p}{a^q - b^q}\right)^{1/(p-q)}, & \text{if } pq(p-q) \neq 0, \\ L^{1/p}(a^p, b^p), & \text{if } p \neq 0,\ q = 0, \\ L^{1/q}(a^q, b^q), & \text{if } q \neq 0, p = 0, \\ I^{1/p}(a^p, b^p), & \text{if } p = q \neq 0, \\ \sqrt{ab}, & \text{if } p = q = 0, \end{cases} \tag{1.8}$$

where $I(x, y) = e^{-1}(x^x/y^y)^{1/(x-y)}$ if $x, y > 0$, with $x \neq y$, and $I(x, x) = x$ is the identric (exponential) mean (see [18]). Substituting $A = A(x, y) = (x + y)/2$ for f yields Gini means $\mathscr{H}_A(p, q; a, b) = G_{p,q}(a, b)$ defined by

$$G_{p,q}(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{a^q + b^q}\right)^{1/(p-q)}, & \text{if } p \neq q, \\ Z^{1/p}(a^p, b^p), & \text{if } p = q, \end{cases} \tag{1.9}$$

where $Z(a, b) = a^{a/(a+b)} b^{b/(a+b)}$ (see [19]).

As consequences of our results, the monotonicity properties of four ratios of mixed Stolarsky means are presented, which generalize certain known results, and some known and new inequalities of ratios of means are established.

2. Main Results and Proofs

In [15, 16, 20], two decision functions play an important role, that are,

$$\begin{aligned} \mathcal{I} = \mathcal{I}(x,y) &= \frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = (\ln f(x,y))_{xy} = (\ln f)_{xy}, \\ \mathcal{J} = \mathcal{J}(x,y) &= (x-y)\frac{\partial(x\mathcal{I})}{\partial x} = (x-y)(x\mathcal{I})_x. \end{aligned} \tag{2.1}$$

In [14], it is important to another key decision function defined by

$$\mathcal{T}_3(x,y) := -xy(x\mathcal{I})_x \ln^3\left(\frac{x}{y}\right), \quad \text{where } \mathcal{I} = (\ln f)_{xy},\ x = a^t,\ y = b^t. \tag{2.2}$$

Note that the function T defined by

$$T(t) := \ln f(a^t, b^t), \quad t \neq 0 \tag{2.3}$$

has well properties (see [15, 16]). And it has shown in [14, (3.4)], [16, Lemma 4] the relation among $T'''(t)$, $\mathcal{J}(x,y)$ and $\mathcal{T}_3(x,y)$:

$$T'''(t) = t^{-3}\mathcal{T}_3(x,y), \quad \text{where } x = a^t,\ y = b^t, \tag{2.4}$$

$$T'''(t) = -Ct^{-3}\mathcal{J}(x,y), \quad \text{where } C = xy(x-y)^{-1}(\ln x - \ln y)^3 > 0. \tag{2.5}$$

Moreover, it has revealed in [14, (3.5)] that

$$\mathcal{T}_3(x,y) = \mathcal{T}_3\left(\frac{x}{y},1\right) = \mathcal{T}_3\left(1,\frac{y}{x}\right). \tag{2.6}$$

Now, we observe the monotonicities of ratio of certain mixed means defined by (1.1).

Theorem 2.1. *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a symmetric, first-order homogenous, and three-time differentiable function, and $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$. Then, for any $a,b,c,d > 0$ with $b/a > d/c \geq 1$ and fixed $q \geq 0$, $k \geq 0$, but q,k are not equal to zero at the same time, Q_{1f} is strictly increasing (decreasing) in p on (k,∞) and decreasing (increasing) on $(-\infty,k)$.*

The monotonicity of Q_{1f} is converse if $q \leq 0$, $k \leq 0$, but q,k are not equal to zero at the same time.

Proof. Since $f(x,y) > 0$ for $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$, so $T'(t)$ is continuous on $[p,q]$ or $[q,p]$ for $p,q \in \mathbb{R}$, then (2.13) in [13] holds. Thus we have

$$\ln g_{1f}(p) = \frac{1}{2}\ln \mathcal{H}_f(p,q) + \frac{1}{2}\ln \mathcal{H}_f(2k-p,q) = \frac{1}{2}\int_0^1 T'(t_{11})dt + \frac{1}{2}\int_0^1 T'(t_{12})dt, \tag{2.7}$$

where

$$t_{12} = tp + (1-t)q, \qquad t_{11} = t(2k-p) + (1-t)q. \tag{2.8}$$

Partial derivative leads to

$$\begin{aligned}(\ln g_{1f}(p))' &= \frac{1}{2}\int_0^1 tT''(t_{12})dt - \frac{1}{2}\int_0^1 tT''(t_{11})dt \\ &= \frac{1}{2}\int_0^1 tT''(|t_{12}|)dt - \frac{1}{2}\int_0^1 tT''(|t_{11}|)dt \quad (\text{by}[13], (2.7)) \\ &= \frac{1}{2}\int_0^1 t\int_{|t_{11}|}^{|t_{12}|} T'''(v)dv\, dt,\end{aligned} \tag{2.9}$$

and then

$$
\begin{aligned}
(\ln Q_{1f}(p))' &= (\ln g_{1f}(p;a,b))' - (\ln g_{1f}(p;c,d))' \\
&= \frac{1}{2}\int_0^1 t\int_{|t_{11}|}^{|t_{12}|} T'''(v)dv\,dt - \frac{1}{2}\int_0^1 t\int_{|t_{11}|}^{|t_{12}|} T'''(v;c,d)dv\,dt \\
&= \int_0^1 t(|t_{12}|-|t_{11}|)\frac{\int_{|t_{11}|}^{|t_{12}|}(T'''(v;a,b)-T'''(v;c,d))dv}{|t_{12}|-|t_{11}|}dt \\
&:= \int_0^1 t(|t_{12}|-|t_{11}|)h(|t_{11}|,|t_{12}|)dt,
\end{aligned}
\tag{2.10}
$$

where

$$
h(x,y) := \begin{cases} \dfrac{\int_x^y (T'''(v;a,b)-T'''(v;c,d))dv}{y-x}, & \text{if } x \neq y, \\ T'''(x;a,b)-T'''(x;c,d), & \text{if } x = y. \end{cases}
\tag{2.11}
$$

Since $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u>1$ and decrease (increase) with $0<u<1$, (2.4) and (2.6) together with $b/a > d/c \geq 1$ yield

$$
\begin{aligned}
T'''(v;a,b) - T'''(v;c,d) &= v^{-3}(\mathcal{T}_3(a^v,b^v) - \mathcal{T}_3(c^v,d^v)) \\
&= v^{-3}\left(\mathcal{T}_3\left(1,\left(\frac{b}{a}\right)^v\right) - \mathcal{T}_3\left(1,\left(\frac{d}{c}\right)^v\right)\right) > (<)0, \quad \text{for } v>0,
\end{aligned}
\tag{2.12}
$$

and therefore $h(x,y) > (<)0$ for $x,y>0$. Thus, in order to prove desired result, it suffices to determine the sign of $(|t_{12}|-|t_{11}|)$. In fact, if $q \geq 0$, $k \geq 0$, then for $t \in [0,1]$

$$
|t_{12}|-|t_{11}| = \frac{t_{12}^2 - t_{11}^2}{|t_{12}|+|t_{11}|} = 4t\frac{q(1-t)+kt}{t_{12}+t_{11}}(p-k) = \begin{cases} >0, & \text{if } p>k, \\ <0, & \text{if } p<k. \end{cases}
\tag{2.13}
$$

It follows that

$$
(\ln Q_{1f}(p))' = \begin{cases} > (<)0, & \text{if } p>k, \\ < (>)0, & \text{if } p<k. \end{cases}
\tag{2.14}
$$

Clearly, the monotonicity of Q_{1f} is converse if $q \leq 0$, $k \leq 0$.

This completes the proof. □

Theorem 2.2. *The conditions are the same as those of Theorem 2.1. Then, for any $a,b,c,d>0$ with $b/a > d/c \geq 1$ and fixed m,k with $k \geq 0$, $k+m \geq 0$, but m,k are not equal to zero at the same time, Q_{2f} is strictly increasing (decreasing) in p on (k,∞) and decreasing (increasing) on $(-\infty,k)$.*

The monotonicity of Q_{2f} is converse if $k \le 0$ and $k+m \le 0$, but m, k are not equal to zero at the same time.

Proof. By (2.13) in [13] we have

$$\begin{aligned} \ln g_{2f}(p) &= \frac{1}{2}\ln \mathscr{H}_f(p, p+m) + \frac{1}{2}\ln \mathscr{H}_f(2k-p, 2k-p+m) \\ &= \frac{1}{2}\int_0^1 T'(t_{22})dt + \frac{1}{2}\int_0^1 T'(t_{21})dt, \end{aligned} \tag{2.15}$$

where

$$t_{22} = tp + (1-t)(p+m), \qquad t_{21} = t(2k-p) + (1-t)(2k-p+m). \tag{2.16}$$

Direct calculation leads to

$$(\ln g_{2f}(p))' = \frac{1}{2}\int_0^1 T''(t_{22})dt - \frac{1}{2}\int_0^1 T''(t_{21})dt = \frac{1}{2}\int_0^1 \int_{|t_{21}|}^{|t_{22}|} T'''(v)dv\,dt, \tag{2.17}$$

and then

$$\begin{aligned} (\ln Q_{2f}(p))' &= (\ln g_{2f}(p; a, b))' - (\ln g_{2f}(p; c, d))' \\ &= \frac{1}{2}\int_0^1 \int_{|t_{21}|}^{|t_{22}|} T'''(v; a, b)dv\,dt - \frac{1}{2}\int_0^1 \int_{|t_{21}|}^{|t_{22}|} T'''(v; c, d)dv\,dt \\ &= \frac{1}{2}\int_0^1 (|t_{22}| - |t_{21}|)h(|t_{21}|, |t_{22}|)dt, \end{aligned} \tag{2.18}$$

where $h(x, y)$ is defined by (2.11). As shown previously, $h(x, y) > (<)0$ for $x, y > 0$ if $\mathcal{T}_3(1, u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$; it remains to determine the sign of $(|t_{22}| - |t_{21}|)$. It is easy to verify that if $k \ge 0$ and $k+m \ge 0$, then

$$|t_{22}| - |t_{21}| = \frac{t_{22}^2 - t_{21}^2}{|t_{22}| + |t_{21}|} = 4\frac{k + m(1-t)}{|t_{22}| + |t_{21}|}(p-k) = \begin{cases} > 0, & \text{if } p > k, \\ < 0, & \text{if } p < k. \end{cases} \tag{2.19}$$

Thus, we have

$$(\ln Q_{2f}(p))' = \begin{cases} > (<)0, & \text{if } p > k, \\ < (>)0, & \text{if } p < k. \end{cases} \tag{2.20}$$

Clearly, the monotonicity of Q_{2f} is converse if $k \le 0$ and $k+m \le 0$.

The proof ends. □

Theorem 2.3. *The conditions are the same as those of Theorem 2.1. Then, for any $a, b, c, d > 0$ with $b/a > d/c \geq 1$ and fixed $m > 0$, $0 \leq k \leq 2m$, Q_{3f} is strictly increasing (decreasing) in p on (k, ∞) and decreasing (increasing) on $(-\infty, k)$.*

The monotonicity of Q_{2f} is converse if $m < 0$, $2m \leq k \leq 0$.

Proof. From (2.13) in [13], it is derived that

$$\begin{aligned} \ln g_{3f}(p) &= \frac{1}{2}\ln \mathscr{H}_f(p, 2m-p) + \frac{1}{2}\ln \mathscr{H}_f(2k-p, 2m-2k+p) \\ &= \frac{1}{2}\int_0^1 T'(t_{32})dt + \frac{1}{2}\int_0^1 T'(t_{31})dt, \end{aligned} \tag{2.21}$$

where

$$t_{32} = \left(tp + (1-t)(2m-p)\right), \qquad t_{31} = \left(t(2k-p) + (1-t)(2m-2k+p)\right). \tag{2.22}$$

Simple calculation yields

$$\left(\ln g_{3f}(p)\right)' = \frac{1}{2}\int_0^1 (2t-1)\left(T''(t_{32}) - T''(t_{31})\right)dt = \frac{1}{2}\int_0^1 (2t-1)\int_{|t_{31}|}^{|t_{32}|} T'''(v; a, b)dv\, dt. \tag{2.23}$$

Hence,

$$\begin{aligned} \left(\ln Q_{3f}(p)\right)' &= \left(\ln g_{3f}(p; a, b)\right)' - \left(\ln g_{3f}(p; c, d)\right)' \\ &= \frac{1}{2}\int_0^1 (2t-1)\int_{|t_{31}|}^{|t_{32}|} \left(T'''(v; a, b) - T'''(v; c, d)\right)dv\, dt \\ &= \frac{1}{2}\int_0^1 (2t-1)(|t_{32}| - |t_{31}|)h(|t_{31}|, |t_{32}|)dt, \end{aligned} \tag{2.24}$$

where $h(x, y)$ is defined by (2.11). It has shown that $h(x, y) > (<)0$ for $x, y > 0$ if $\mathcal{T}_3(1, u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$, and we have also to check the sign of $(2t-1)(|t_{32}| - |t_{31}|)$. Easy calculation reveals that if $m > 0$, $0 \leq k \leq 2m$, then

$$\begin{aligned} (2t-1)(|t_{32}| - |t_{31}|) &= (2t-1)\frac{\left(t_{32}^2 - t_{31}^2\right)}{|t_{32}| + |t_{31}|} \\ &= 4(2t-1)^2\frac{tk + (1-t)(2m-k)}{|t_{32}| + |t_{31}|}(p-k) \\ &= \begin{cases} > 0, & \text{if } p > k, \\ < 0, & \text{if } p < k, \end{cases} \end{aligned} \tag{2.25}$$

which yields

$$(\ln Q_{3f}(p))' = \begin{cases} > (<)0, & \text{if } p > k, \\ < (>)0, & \text{if } p < k. \end{cases} \tag{2.26}$$

It is evident that the monotonicity of Q_{3f} is converse if $m < 0, 2m \le k \le 0$.

Thus the proof is complete. □

Theorem 2.4. *The conditions are the same as those of Theorem 2.1. Then, for any* $a, b, c, d > 0$ *with* $b/a > d/c \ge 1$ *and fixed* $k, r, s \in \mathbb{R}$ *with* $r + s \ne 0$, Q_{4f} *is strictly increasing (decreasing) in* p *on* (k, ∞) *and decreasing (increasing) on* $(-\infty, k)$ *if* $k(r+s) > 0$.

The monotonicity of Q_{4f} *is converse if* $k(r+s) < 0$.

Proof. By (2.13) in [13], $\ln \mathscr{H}_f(pr, ps)$ can be expressed in integral form

$$\ln \mathscr{H}_f(pr, ps) = \begin{cases} \dfrac{1}{r-s} \int_s^r T'(pt)\,dt, & \text{if } r \ne s, \\ T'(pr), & \text{if } r = s. \end{cases} \tag{2.27}$$

The case $r = s \ne 0$ has no interest since it can come down to the case of $m = 0$ in Theorem 2.2. Therefore, we may assume that $r \ne s$. We have

$$\begin{aligned} \ln g_{4f}(p) &= \ln \sqrt{\mathscr{H}_f(pr, ps)\mathscr{H}_f((2k-p)r, (2k-p)s)} \\ &= \frac{1}{2}\frac{1}{r-s}\int_s^r T'(pt)\,dt + \frac{1}{2}\frac{1}{r-s}\int_s^r T'((2k-p)t)\,dt, \end{aligned} \tag{2.28}$$

and then

$$\begin{aligned} (\ln g_{4f}(p))' &= \frac{1}{2}\frac{1}{r-s}\int_s^r tT''(pt)\,dt - \frac{1}{2}\frac{1}{r-s}\int_s^r tT''((2k-p)t)\,dt \\ &= \frac{1}{2}\frac{1}{r-s}\int_s^r t(T''(pt) - T''((2k-p)t)). \end{aligned} \tag{2.29}$$

Note that $T''(t)$ is even (see [13, (2.7)]) and so $t(T''(pt) - T''((2k-p)t))$ is odd, then make use of Lemma 3.3 in [13], $(\ln g_{4f}(p))'$ can be expressed as

$$\begin{aligned} (\ln g_{4f}(p))' &= \frac{1}{2}\frac{r+s}{|r|-|s|}\int_{|s|}^{|r|} t(T''(|pt|) - T''(|(2k-p)t|))\,dt \\ &= \frac{1}{2}\frac{r+s}{|r|-|s|}\int_{|s|}^{|r|} t\int_{|t_{41}|}^{|t_{42}|} T'''(v)\,dv\,dt, \end{aligned} \tag{2.30}$$

where

$$t_{42} = pt, \qquad t_{41} = (2k - p)t. \tag{2.31}$$

Hence,

$$\begin{aligned}
(\ln Q_{4f}(p))' &= (\ln g_{4f}(p; a, b))' - (\ln g_{4f}(p; c, d))' \\
&= \frac{1}{2}\frac{r+s}{|r|-|s|}\int_{|s|}^{|r|} t \int_{|t_{41}|}^{|t_{42}|} (T'''(v; a, b) - T'''(v; c, d))\,dv\,dt \\
&= \frac{1}{2}\frac{r+s}{|r|-|s|}\int_{|s|}^{|r|} t(|t_{42}| - |t_{41}|)h(|t_{41}|, |t_{42}|)dt,
\end{aligned} \tag{2.32}$$

where $h(x, y)$ is defined by (2.11). We have shown that $h(x, y) > (<)0$ for $x, y > 0$ if $\mathcal{T}_3(1, u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$, and we also have

$$\operatorname{sgn}(|t_{42}| - |t_{41}|) = \operatorname{sgn}\left(t_{42}^2 - t_{41}^2\right) = \operatorname{sgn}(k)\operatorname{sgn}(p - k). \tag{2.33}$$

It follows that

$$\begin{aligned}
\operatorname{sgn} Q'_{4f}(p) &= \operatorname{sgn}(r + s)\operatorname{sgn}(k)\operatorname{sgn}(p - k)\operatorname{sgn} h(|t_{41}|, |t_{42}|) \\
&= \begin{cases} > (<)0, & \text{if } k(r+s) > 0,\ p > k, \\ < (>)0, & \text{if } k(r+s) > 0,\ p < k, \\ < (>)0, & \text{if } k(r+s) < 0,\ p > k, \\ > (<)0, & \text{if } k(r+s) < 0,\ p < k. \end{cases}
\end{aligned} \tag{2.34}$$

This proof is accomplished. □

3. Applications

As shown previously, $S_{p,q}(a, b) = \mathcal{H}_L(p, q; a, b)$, where $L = L(x, y)$ is the logarithmic mean. Also, it has been proven in [14] that $\mathcal{T}'_3(1, u) < 0$ if $u > 1$ and $\mathcal{T}'_3(1, u) > 0$ if $0 < u < 1$. From the applications of Theorems 2.1–2.4, we have the following.

Corollary 3.1. *Let $a, b, c, d > 0$ with $b/a > d/c \geq 1$. Then, the following four functions are all strictly decreasing (increasing) on (k, ∞) and increasing (decreasing) on $(-\infty, k)$:*

(i) *Q_{1L} is defined by*

$$Q_{1L}(p) = \frac{\sqrt{S_{p,q}(a, b)S_{2k-p,q}(a, b)}}{\sqrt{S_{p,q}(c, d)S_{2k-p,q}(c, d)}}, \tag{3.1}$$

for fixed $q \geq (\leq)0$, $k \geq (\leq)0$, but q, k are not equal to zero at the same time,

(ii) Q_{2L} *is defined by*

$$Q_{2L}(p) = \frac{\sqrt{S_{p,p+m}(a,b)S_{2k-p,2k-p+m}(a,b)}}{\sqrt{S_{p,p+m}(c,d)S_{2k-p,2k-p+m}(c,d)}}, \tag{3.2}$$

for fixed m, k *with* $k \geq (\leq) 0$ *and* $k + m \geq (\leq) 0$, *but* m, k *are not equal to zero at the same time,*

(iii) Q_{3L} *is defined by*

$$Q_{3L}(p) = \frac{\sqrt{S_{p,2m-p}(a,b)S_{2k-p,2m-2k+p}(a,b)}}{\sqrt{S_{p,2m-p}(c,d)S_{2k-p,2m-2k+p}(c,d)}}, \tag{3.3}$$

for fixed $m > (<) 0$, $k \in [0, 2m]$ $([2m, 0])$.

(iv) Q_{4L} *is defined by*

$$Q_{4L}(p) = \frac{\sqrt{S_{pr,ps}(a,b)S_{(2k-p)r,(2k-p)s}(a,b)}}{\sqrt{S_{pr,ps}(c,d)S_{(2k-p)r,(2k-p)s}(c,d)}}, \tag{3.4}$$

for fixed $k, r, s \in \mathbb{R}$ *with* $k(r+s) > (<) 0$.

Remark 3.2. Letting in the first result of Corollary 3.1, $q = k$ yields Theorem 3.4 in [13] since $\sqrt{S_{p,k}S_{2k-p,k}} = S_{p,2k-p}$. Letting $q = 1$, $k = 0$ yields

$$\frac{G(a,b)}{G(c,d)} = Q_{1L}(\infty) < \frac{\sqrt{S_{p,1}(a,b)S_{-p,1}(a,b)}}{\sqrt{S_{p,1}(c,d)S_{-p,1}(c,d)}} < Q_{1L}(0) = \frac{L(a,b)}{L(c,d)}. \tag{3.5}$$

Inequalities (3.5) in the case of $d = c$ were proved by Alzer in [21]. By letting $q = 1$, $k = 1/2$ from $Q_{1L}(1/2) > Q_{1L}(1) > Q_{1L}(2)$, we have

$$\frac{A(a,b)+G(a,b)}{A(c,d)+G(c,d)} > \frac{\sqrt{L(a,b)I(a,b)}}{\sqrt{L(c,d)I(c,d)}} > \frac{\sqrt{A(a,b)G(a,b)}}{\sqrt{A(c,d)G(c,d)}}. \tag{3.6}$$

Inequalities (3.6) in the case of $d = c$ are due to Alzer [22].

Remark 3.3. Letting in the second result of Corollary 3.1, $m = 1$, $k = 0$ yields Cheung and Qi's result (see [23, Theorem 2]). And we have

$$\frac{G(a,b)}{G(c,d)} = Q_{2L}(\infty) < \frac{\sqrt{S_{p,p+1}(a,b)S_{-p,-p+1}(a,b)}}{\sqrt{S_{p,p+1}(c,d)S_{-p,-p+1}(c,d)}} < Q_{2L}(0) = \frac{L(a,b)}{L(c,d)}. \tag{3.7}$$

When $d = c$, inequalities (3.7) are changed as Alzer's ones given in [24].

Remark 3.4. In the third result of Corollary 3.1, letting $k = m$ also leads to Theorem 3.4 in [13]. Put $m = 1/2$, $k = 1/4$. Then from $Q_{3L}(1/4) > Q_{3L}(1/2)$, we obtain a new inequality

$$\frac{He_{1/2}(a,b)}{He_{1/2}(c,d)} > \frac{\sqrt{L(a,b)I_{1/2}(a,b)}}{\sqrt{L(c,d)I_{1/2}(c,d)}}. \tag{3.8}$$

Putting $m = 1/2$, $k = 1/3$ leads to another new inequality

$$\frac{A_{1/3}(a,b)}{A_{1/3}(c,d)} > \frac{\sqrt{S_{1/6,5/6}(a,b)I_{1/2}(a,b)}}{\sqrt{S_{1/6,5/6}(c,d)I_{1/2}(c,d)}}. \tag{3.9}$$

Remark 3.5. Letting in the third result of Corollary 3.1, $k = 1/2$ and $(r,s) = (1,0), (1,1), (2,1)$, and we deduce that all the following three functions

$$p \longrightarrow \frac{\sqrt{L_p(a,b)L_{1-p}(a,b)}}{\sqrt{L_p(c,d)L_{1-p}(c,d)}}, \qquad p \longrightarrow \frac{\sqrt{I_p(a,b)I_{1-p}(a,b)}}{\sqrt{I_p(c,d)I_{1-p}(c,d)}}, \qquad p \longrightarrow \frac{\sqrt{A_p(a,b)A_{1-p}(a,b)}}{\sqrt{A_p(c,d)A_{1-p}(c,d)}}, \tag{3.10}$$

are strictly decreasing on $(1/2,\infty)$ and increasing on $(-\infty,1/2)$, where $L_p = L^{1/p}(a^p,b^p)$, $I_p = I^{1/p}(a^p,b^p)$, and $A_p = A^{1/p}(a^p,b^p)$ are the p-order logarithmic, identric (exponential), and power mean, respectively, particularly, so are the functions $\sqrt{L_pL_{1-p}}$, $\sqrt{I_pI_{1-p}}$, $\sqrt{A_pA_{1-p}}$.

4. Other Results

Let $d = c$ in Theorems 2.1–2.4. Then, $\mathcal{H}_f(p,q;c,d) = c$ and $T'''(t;c,c) = 0$. From the their proofs, it is seen that the condition "$\mathfrak{T}_3(1,u)$ strictly increases (decreases) with $u > 1$ and decreases (increases) with $0 < u < 1$" can be reduce to "$T'''(v) > (<)0$ for $v > 0$", which is equivalent with $\mathcal{Q} = (x-y)(x\mathcal{I})_x < (>)0$, where $\mathcal{I} = (\ln f)_{xy}$, by (2.4). Thus, we obtain critical theorems for the monotonicities of g_{if}, $i = 1-4$, defined as (1.2)–(1.5).

Theorem 4.1. *Suppose that $f: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a symmetric, first-order homogenous, and three-time differentiable function and $\mathcal{Q} = (x-y)(x\mathcal{I})_x < (>)0$, where $\mathcal{I} = (\ln f)_{xy}$. Then, for $a,b > 0$ with $a \neq b$, the following four functions are strictly increasing (decreasing) in p on (k,∞) and decreasing (increasing) on $(-\infty,k)$:*

(i) *g_{1f} is defined by (1.2), for fixed $q,k \geq 0$, but q,k are not equal to zero at the same time;*

(ii) *g_{2f} is defined by (1.3), for fixed m,k with $k \geq 0$ and $k+m \geq 0$, but m,k are not equal to zero at the same time;*

(iii) *g_{3f} is defined by (1.4), for fixed $m > 0$ and $0 \leq k \leq 2m$;*

(iv) *g_{4f} is defined by (1.5), for fixed $k,r,s \in \mathbb{R}$ with $k(r+s) > 0$.*

If f is defined on $\mathbb{R}_+^2 \setminus \{(x,x), x \in \mathbb{R}_+\}$, then $T'(t)$ may be not continuous at $t = 0$, and (2.13) in [13] may not hold for $p,q \in \mathbb{R}$ but must be hold for $p,q \in \mathbb{R}_+$. And then, we easily derive the following from the proofs of Theorems 2.1–2.4.

Theorem 4.2. *Suppose that $f: \mathbb{R}_+^2 \setminus \{(x,x), x \in \mathbb{R}_+\} \to \mathbb{R}_+$ is a symmetric, first-order homogenous and three-time differentiable function and $\mathcal{Q} = (x-y)(x\mathcal{I})_x < (>)0$, where $\mathcal{I} = (\ln f)_{xy}$. Then for $a, b > 0$ with $a \neq b$ the following four functions are strictly increasing (decreasing) in p on $(k, 2k)$ and decreasing (increasing) on $(0, k)$:*

(i) g_{1f} *is defined by* (1.2), *for fixed* $q, k > 0$;

(ii) g_{2f} *is defined by* (1.3), *for fixed* m, k *with* $k > 0$ *and* $k + m > 0$;

(iii) g_{3f} *is defined by* (1.4), *for fixed* $m > 0$ *and* $0 \leq k \leq 2m$;

(iv) g_{4f} *is defined by* (1.5), *for fixed* $k, r, s > 0$.

If we substitute L, A, and I for f, where L, A, and I denote the logarithmic, arithmetic, and identric (exponential) mean, respectively, then from Theorem 4.1, we will deduce some known and new inequalities for means. Similarly, letting in Theorem 4.2 $f(x,y) = D(x,y) = |x-y|$, $K(x,y) = (x+y)|\ln(x/y)|$, where $x, y > 0$ with $x \neq y$, we will obtain certain companion ones of those known and new ones. Here no longer list them.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

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Research Article

Refinements of Inequalities among Difference of Means

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In this paper, for the difference of famous means discussed by Taneja in 2005, we study the Schur-geometric convexity in $(0, \infty) \times (0, \infty)$ of the difference between them. Moreover some inequalities related to the difference of those means are obtained.

1. Introduction

In 2005, Taneja [1] proved the following chain of inequalities for the binary means for $(a, b) \in R_+^2 = (0, \infty) \times (0, \infty)$:

$$H(a,b) \le G(a,b) \le N_1(a,b) \le N_3(a,b) \le N_2(a,b) \le A(a,b) \le S(a,b), \tag{1.1}$$

where

$$\begin{gathered} A(a,b) = \frac{a+b}{2}, \\ G(a,b) = \sqrt{ab}, \\ H(a,b) = \frac{2ab}{a+b}, \\ N_1(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 = \frac{A(a,b)+G(a,b)}{2}, \end{gathered} \tag{1.2}$$

$$N_3(a,b) = \frac{a+\sqrt{ab}+b}{3} = \frac{2A(a,b)+G(a,b)}{3},$$
$$N_2(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right), \tag{1.3}$$
$$S(a,b) = \sqrt{\frac{a^2+b^2}{2}}.$$

The means A, G, H, S, N_1 and N_3 are called, respectively, the arithmetic mean, the geometric mean, the harmonic mean, the root-square mean, the square-root mean, and Heron's mean. The N_2 one can be found in Taneja [2, 3].

Furthermore Taneja considered the following difference of means:

$$\begin{aligned}
M_{SA}(a,b) &= S(a,b) - A(a,b),\\
M_{SN_2}(a,b) &= S(a,b) - N_2(a,b),\\
M_{SN_3}(a,b) &= S(a,b) - N_3(a,b),\\
M_{SN_1}(a,b) &= S(a,b) - N_1(a,b),\\
M_{SG}(a,b) &= S(a,b) - G(a,b),\\
M_{SH}(a,b) &= S(a,b) - H(a,b),\\
M_{AN_2}(a,b) &= A(a,b) - N_2(a,b),\\
M_{AG}(a,b) &= A(a,b) - G(a,b),\\
M_{AH}(a,b) &= A(a,b) - H(a,b),\\
M_{N_2N_1}(a,b) &= N_2(a,b) - N_1(a,b),\\
M_{N_2G}(a,b) &= N_2(a,b) - G(a,b)
\end{aligned} \tag{1.4}$$

and established the following.

Theorem A. *The difference of means given by* (1.4) *is nonnegative and convex in* $R_+^2 = (0,\infty) \times (0,\infty)$.

Further, using Theorem A, Taneja proved several chains of inequalities; they are refinements of inequalities in (1.1).

Theorem B. *The following inequalities among the mean differences hold:*

$$M_{SA}(a,b) \le \frac{1}{3}M_{SH}(a,b) \le \frac{1}{2}M_{AH}(a,b) \le \frac{1}{2}M_{SG}(a,b) \le M_{AG}(a,b), \tag{1.5}$$

$$\frac{1}{8}M_{AH}(a,b) \le M_{N_2N_1}(a,b) \le \frac{1}{3}M_{N_2G}(a,b) \le \frac{1}{4}M_{AG}(a,b) \le M_{AN_2}(a,b), \tag{1.6}$$

$$M_{SA}(a,b) \le \frac{4}{5}M_{SN_2}(a,b) \le 4M_{AN_2}(a,b), \tag{1.7}$$

$$M_{SH}(a,b) \le 2M_{SN_1}(a,b) \le \frac{3}{2}M_{SG}(a,b), \tag{1.8}$$

$$M_{SA}(a,b) \le \frac{3}{4}M_{SN_3}(a,b) \le \frac{2}{3}M_{SN_1}(a,b). \tag{1.9}$$

For the difference of means given by (1.4), we study the Schur-geometric convexity of difference between these differences in order to further improve the inequalities in (1.1). The main result of this paper reads as follows.

Theorem I. *The following differences are Schur-geometrically convex in* $R_+^2 = (0,\infty) \times (0,\infty)$*:*

$$\begin{aligned}
D_{SH-SA}(a,b) &= \frac{1}{3}M_{SH}(a,b) - M_{SA}(a,b),\\
D_{AH-SH}(a,b) &= \frac{1}{2}M_{AH}(a,b) - \frac{1}{3}M_{SH}(a,b),\\
D_{SG-AH}(a,b) &= M_{SG}(a,b) - M_{AH}(a,b),\\
D_{AG-SG}(a,b) &= M_{AG}(a,b) - \frac{1}{2}M_{SG}(a,b),\\
D_{N_2N_1-AH}(a,b) &= M_{N_2N_1}(a,b) - \frac{1}{8}M_{AH}(a,b),\\
D_{N_2G-N_2N_1}(a,b) &= \frac{1}{3}M_{N_2G}(a,b) - M_{N_2N_1}(a,b),\\
D_{AG-N_2G}(a,b) &= \frac{1}{4}M_{AG}(a,b) - \frac{1}{3}M_{N_2G}(a,b),\\
D_{AN_2-AG}(a,b) &= M_{AN_2}(a,b) - \frac{1}{4}M_{AG}(a,b),\\
D_{SN_2-SA}(a,b) &= \frac{4}{5}M_{SN_2}(a,b) - M_{SA}(a,b),\\
D_{AN_2-SN_2}(a,b) &= 4M_{AN_2}(a,b) - \frac{4}{5}M_{SN_2}(a,b),\\
D_{SN_1-SH}(a,b) &= 2M_{SN_1}(a,b) - M_{SH}(a,b),\\
D_{SG-SN_1}(a,b) &= \frac{3}{2}M_{SG}(a,b) - 2M_{SN_1}(a,b),
\end{aligned} \tag{1.10}$$

$$D_{SN_3-SA}(a,b) = \frac{3}{4}M_{SN_3}(a,b) - M_{SA}(a,b),$$
$$D_{SN_1-SN_3}(a,b) = \frac{2}{3}M_{SN_1}(a,b) - \frac{3}{4}M_{SN_3}(a,b). \tag{1.11}$$

The proof of this theorem will be given in Section 3. Applying this result, in Section 4, we prove some inequalities related to the considered differences of means. Obtained inequalities are refinements of inequalities (1.5)–(1.9).

2. Definitions and Auxiliary Lemmas

The Schur-convex function was introduced by Schur in 1923, and it has many important applications in analytic inequalities, linear regression, graphs and matrices, combinatorial optimization, information-theoretic topics, Gamma functions, stochastic orderings, reliability, and other related fields (cf. [4–14]).

In 2003, Zhang first proposed concepts of "Schur-geometrically convex function" which is extension of "Schur-convex function" and established corresponding decision theorem [15]. Since then, Schur-geometric convexity has evoked the interest of many researchers and numerous applications and extensions have appeared in the literature (cf. [16–19]).

In order to prove the main result of this paper we need the following definitions and auxiliary lemmas.

Definition 2.1 (see [4, 20]). Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} \prec \mathbf{y}$) if $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(ii) $\Omega \subseteq \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$ for every $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iii) Let $\Omega \subseteq \mathbb{R}^n$. The function φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex.

Definition 2.2 (see [15]). Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$.

(i) $\Omega \subseteq \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^{\alpha} y_1^{\beta}, \ldots, x_n^{\alpha} y_n^{\beta}) \in \Omega$ for all $\mathbf{x}$,$\mathbf{y} \in \Omega$ and α,$\beta \in [0, 1]$ such that $\alpha + \beta = 1$.

(ii) Let $\Omega \subseteq \mathbb{R}^n_+$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. The function φ is said to be a Schur-geometrically concave on Ω if and only if $-\varphi$ is Schur-geometrically convex.

Definition 2.3 (see [4, 20]). (i) The set $\Omega \subseteq \mathbb{R}^n$ is called symmetric set, if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.

(ii) The function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if, for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

Lemma 2.4 (see [15]). *Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric and geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial\varphi}{\partial x_1} - x_2 \frac{\partial\varphi}{\partial x_2} \right) \ge 0 \quad (\le 0) \tag{2.1}$$

holds for any $\mathbf{x} = (x_1, \dots, x_n) \in \Omega^0$, then φ is a Schur-geometrically convex (Schur-geometrically concave) function.

Lemma 2.5. *For $(a,b) \in R_+^2 = (0,\infty) \times (0,\infty)$ one has*

$$1 \ge \frac{a+b}{\sqrt{2(a^2+b^2)}} \ge \frac{1}{2} + \frac{2ab}{(a+b)^2}, \tag{2.2}$$

$$\frac{a+b}{\sqrt{2(a^2+b^2)}} - \frac{ab}{(a+b)^2} \le \frac{3}{4}, \tag{2.3}$$

$$\frac{3}{2} \ge \frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}} \ge \frac{5}{4} + \frac{ab}{(a+b)^2}. \tag{2.4}$$

Proof. It is easy to see that the left-hand inequality in (2.2) is equivalent to $(a-b)^2 \ge 0$, and the right-hand inequality in (2.2) is equivalent to

$$\frac{\sqrt{2(a^2+b^2)} - (a+b)}{\sqrt{2(a^2+b^2)}} \le \frac{(a+b)^2 - 4ab}{2(a+b)^2}, \tag{2.5}$$

that is,

$$\frac{(a-b)^2}{2(a^2+b^2) + \sqrt{2(a^2+b^2)}(a+b)} \le \frac{(a-b)^2}{2(a+b)^2}. \tag{2.6}$$

Indeed, from the left-hand inequality in (2.2) we have

$$2\left(a^2+b^2\right) + \sqrt{2(a^2+b^2)}(a+b) \ge 2\left(a^2+b^2\right) + (a+b)^2 \ge 2(a+b)^2, \tag{2.7}$$

so the right-hand inequality in (2.2) holds.

The inequality in (2.3) is equivalent to

$$\frac{\sqrt{2(a^2+b^2)} - (a+b)}{\sqrt{2(a^2+b^2)}} \ge \frac{(a-b)^2}{4(a+b)^2}. \tag{2.8}$$

Since

$$\frac{\sqrt{2(a^2+b^2)}-(a+b)}{\sqrt{2(a^2+b^2)}} = \frac{2(a^2+b^2)-(a+b)^2}{\sqrt{2(a^2+b^2)}\left(\sqrt{2(a^2+b^2)}+(a+b)\right)} = \frac{(a-b)^2}{2(a^2+b^2)+(a+b)\sqrt{2(a^2+b^2)}}, \tag{2.9}$$

so it is sufficient prove that

$$2\left(a^2+b^2\right)+(a+b)\sqrt{2(a^2+b^2)} \le 4(a+b)^2, \tag{2.10}$$

that is,

$$(a+b)\sqrt{2(a^2+b^2)} \le 2\left(a^2+b^2+4ab\right), \tag{2.11}$$

and, from the left-hand inequalities in (2.2), we have

$$(a+b)\sqrt{2(a^2+b^2)} \le 2\left(a^2+b^2\right) \le 2\left(a^2+b^2+4ab\right), \tag{2.12}$$

so the inequality in (2.3) holds.

Notice that the functions in the inequalities (2.4) are homogeneous. So, without loss of generality, we may assume $\sqrt{a}+\sqrt{b}=1$, and set $t=\sqrt{ab}$. Then $0<t\le 1/4$ and (2.4) reduces to

$$\frac{3}{2} \ge \frac{\sqrt{1-2t}}{\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{1-2t}} \ge \frac{5}{4} + \frac{t^2}{(1-2t)^2}. \tag{2.13}$$

Squaring every side in the above inequalities yields

$$\frac{9}{4} \ge \frac{1-2t}{2} + \frac{1}{2-4t} + 1 \ge \frac{25}{16} + \frac{t^4}{(1-2t)^4} + \frac{5t^2}{2(1-2t)^2}. \tag{2.14}$$

Reducing to common denominator and rearranging, the right-hand inequality in (2.14) reduces to

$$\frac{(1-2t)\left(16t^2(2t-1)^2+(1/8)(16t-7)^2+(7/8)\right)}{16(2t-1)^4} \ge 0, \tag{2.15}$$

and the left-hand inequality in (2.14) reduces to

$$\frac{2(1-2t)^2+2-5(1-2t)}{2(1-2t)}=-\frac{1+2t}{2}\le 0, \tag{2.16}$$

so two inequalities in (2.4) hold. □

Lemma 2.6 (see [16]). *Let $a\le b$, $u(t)=ta+(1-t)b$, $v(t)=tb+(1-t)a$. If $1/2\le t_2\le t_1\le 1$ or $0\le t_1\le t_2\le 1/2$, then*

$$\left(\frac{a+b}{2},\frac{a+b}{2}\right)\prec (u(t_2),v(t_2))\prec (u(t_1),v(t_1))\prec (a,b). \tag{2.17}$$

3. Proof of Main Result

Proof of Theorem I. Let $(a,b)\in R_+^2$.

(1) For

$$D_{SH-SA}(a,b)=\frac{1}{3}M_{SH}(a,b)-M_{SA}(a,b)=\frac{a+b}{2}-\frac{2ab}{3(a+b)}-\frac{2}{3}\sqrt{\frac{a^2+b^2}{2}}, \tag{3.1}$$

we have

$$\begin{aligned}\frac{\partial D_{SH-SA}(a,b)}{\partial a}&=\frac{1}{2}-\frac{2b^2}{3(a+b)^2}-\frac{2}{3}\frac{a}{\sqrt{2(a^2+b^2)}},\\ \frac{\partial D_{SH-SA}(a,b)}{\partial b}&=\frac{1}{2}-\frac{2a^2}{3(a+b)^2}-\frac{2}{3}\frac{b}{\sqrt{2(a^2+b^2)}},\end{aligned} \tag{3.2}$$

whence

$$\begin{aligned}\Lambda&:=(\ln a-\ln b)\left(a\frac{\partial D_{SH-SA}(a,b)}{\partial a}-b\frac{\partial D_{SH-SA}(a,b)}{\partial b}\right)\\ &=(a-b)(\ln a-\ln b)\left(\frac{1}{2}+\frac{2ab}{3(a+b)^2}-\frac{2}{3}\frac{a+b}{\sqrt{2(a^2+b^2)}}\right).\end{aligned} \tag{3.3}$$

From (2.3) we have

$$\frac{1}{2}+\frac{2ab}{3(a+b)^2}-\frac{2}{3}\frac{a+b}{\sqrt{2(a^2+b^2)}}\ge 0, \tag{3.4}$$

which implies $\Lambda\ge 0$ and, by Lemma 2.4, it follows that D_{SH-SA} is Schur-geometrically convex in R_+^2.

(2) For

$$D_{AH-SH}(a,b)=\frac{1}{2}M_{AH}(a,b)-\frac{1}{3}M_{SH}(a,b)=\frac{a+b}{4}-\frac{ab}{3(a+b)}-\frac{1}{3}\sqrt{\frac{a^2+b^2}{2}}. \tag{3.5}$$

To prove that the function D_{AH-SH} is Schur-geometrically convex in R_+^2 it is enough to notice that $D_{AH-SH}(a,b) = (1/2)D_{SH-SA}(a,b)$.

(3) For

$$D_{SG-AH}(a,b) = M_{SG}(a,b) - M_{AH}(a,b) = \sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab} - \frac{a+b}{2} + \frac{2ab}{a+b}, \tag{3.6}$$

we have

$$\begin{aligned} \frac{\partial D_{SG-AH}(a,b)}{\partial a} &= \frac{a}{\sqrt{2(a^2+b^2)}} - \frac{b}{2\sqrt{ab}} - \frac{1}{2} + \frac{2b^2}{(a+b)^2}, \\ \frac{\partial D_{SG-AH}(a,b)}{\partial b} &= \frac{b}{\sqrt{2(a^2+b^2)}} - \frac{a}{2\sqrt{ab}} - \frac{1}{2} + \frac{2a^2}{(a+b)^2}, \end{aligned} \tag{3.7}$$

and then

$$\begin{aligned} \Lambda &:= (\ln a - \ln b)\left(a\frac{\partial D_{SH-SA}(a,b)}{\partial a} - b\frac{\partial D_{SH-SA}(a,b)}{\partial b}\right) \\ &= (a-b)(\ln a - \ln b)\left(\frac{a+b}{\sqrt{2(a^2+b^2)}} - \frac{1}{2} - \frac{2ab}{(a+b)^2}\right). \end{aligned} \tag{3.8}$$

From (2.2) we have $\Lambda \geq 0$, so by Lemma 2.4, it follows that D_{SH-SA} is Schur-geometrically convex in R_+^2.

(4) For

$$D_{AG-SG}(a,b) = M_{AG}(a,b) - \frac{1}{2}M_{SG}(a,b) = \frac{1}{2}\left(a+b-\sqrt{ab}-\sqrt{\frac{a^2+b^2}{2}}\right), \tag{3.9}$$

we have

$$\begin{aligned} \frac{\partial D_{AG-SG}(a,b)}{\partial a} &= \frac{1}{2}\left(1 - \frac{b}{2\sqrt{ab}} - \frac{a}{\sqrt{2(a^2+b^2)}}\right), \\ \frac{\partial D_{AG-SG}(a,b)}{\partial b} &= \frac{1}{2}\left(1 - \frac{a}{2\sqrt{ab}} - \frac{b}{\sqrt{2(a^2+b^2)}}\right), \end{aligned} \tag{3.10}$$

and then

$$\begin{aligned} \Lambda &:= (\ln a - \ln b)\left(a\frac{\partial D_{SH-SA}(a,b)}{\partial a} - b\frac{\partial D_{SH-SA}(a,b)}{\partial b}\right) \\ &= (a-b)(\ln a - \ln b)\left(1 - \frac{a+b}{\sqrt{2(a^2+b^2)}}\right). \end{aligned} \tag{3.11}$$

By (2.2) we infer that

$$1-\frac{a+b}{\sqrt{2(a^2+b^2)}}\geq 0, \tag{3.12}$$

so $\Lambda \geq 0$. By Lemma 2.4, we get that D_{AG-SG} is Schur-geometrically convex in R_+^2.

(5) For

$$\begin{aligned} D_{N_2N_1-AH}(a,b) &= M_{N_2N_1}(a,b)-\frac{1}{8}M_{AH}(a,b)\\ &= \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right)-\frac{1}{4}(a+b)-\frac{1}{2}\sqrt{ab}-\frac{1}{8}\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right), \end{aligned} \tag{3.13}$$

we have

$$\begin{aligned} \frac{\partial D_{N_2N_1-AH}(a,b)}{\partial a} &= \frac{1}{4\sqrt{a}}\sqrt{\frac{a+b}{2}}+\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}\\ &\quad -\frac{1}{4}-\frac{b}{4\sqrt{ab}}-\frac{1}{8}\left(\frac{1}{2}-\frac{2b^2}{(a+b)^2}\right),\\ \frac{\partial D_{N_2N_1-AH}(a,b)}{\partial b} &= \frac{1}{4\sqrt{b}}\sqrt{\frac{a+b}{2}}+\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}\\ &\quad -\frac{1}{4}-\frac{a}{4\sqrt{ab}}-\frac{1}{8}\left(\frac{1}{2}-\frac{2a^2}{(a+b)^2}\right), \end{aligned} \tag{3.14}$$

and then

$$\begin{aligned} \Lambda &= (\ln a-\ln b)\left(a\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial a}-b\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial b}\right)\\ &= \frac{1}{4}(a-b)(\ln a-\ln b)\left(\frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}}-\frac{5}{4}-\frac{ab}{(a+b)^2}\right). \end{aligned} \tag{3.15}$$

From (2.4) we have

$$\frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}}-\frac{5}{4}-\frac{ab}{(a+b)^2}\geq 0, \tag{3.16}$$

so $\Lambda \geq 0$; it follows that $D_{N_2N_1-AH}$ is Schur-geometrically convex in R_+^2.

(6) For

$$D_{N_2G-N_2N_1}(a,b) = \frac{1}{3}M_{N_2G}(a,b) - M_{N_2N_1}(a,b)$$
$$= \frac{a+b}{4} + \frac{\sqrt{ab}}{6} - \frac{2}{3}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right), \tag{3.17}$$

we have

$$\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial a} = \frac{1}{4} + \frac{b}{12\sqrt{ab}} - \frac{1}{6\sqrt{a}}\sqrt{\frac{a+b}{2}} - \frac{1}{6}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2},$$
$$\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial b} = \frac{1}{4} + \frac{a}{12\sqrt{ab}} - \frac{1}{6\sqrt{b}}\sqrt{\frac{a+b}{2}} - \frac{1}{6}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}, \tag{3.18}$$

and then

$$\Lambda = (\ln a - \ln b)\left(a\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial a} - b\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial b}\right)$$
$$= (\ln a - \ln b)\left(\frac{1}{4}(a-b) - \frac{\sqrt{a}-\sqrt{b}}{6}\sqrt{\frac{a+b}{2}} - \frac{(a-b)\left(\sqrt{a}+\sqrt{b}\right)}{12}\left(\frac{a+b}{2}\right)^{-1/2}\right) \tag{3.19}$$
$$= \frac{1}{6}(a-b)(\ln a - \ln b)\left(\frac{3}{2} - \frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)} - \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}}\right).$$

By (2.4) we infer that $\Lambda \geq 0$, which proves that $D_{N_2G-N_2N_1}$ is Schur-geometrically convex in R_+^2.

(7) For

$$D_{AG-N_2G}(a,b) = \frac{1}{4}M_{AG}(a,b) - \frac{1}{3}M_{N_2G}(a,b)$$
$$= \frac{a+b}{8} + \frac{1}{12}\sqrt{ab} - \frac{1}{3}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right), \tag{3.20}$$

we have

$$\frac{\partial D_{AG-N_2G}(a,b)}{\partial a} = \frac{1}{8} + \frac{b}{24\sqrt{ab}} - \frac{\sqrt{a+b}}{12\sqrt{2a}} - \frac{\sqrt{a}+\sqrt{b}}{12\sqrt{2(a+b)}},$$
$$\frac{\partial D_{AG-N_2G}(a,b)}{\partial b} = \frac{1}{8} + \frac{a}{24\sqrt{ab}} - \frac{\sqrt{a+b}}{12\sqrt{2b}} - \frac{\sqrt{a}+\sqrt{b}}{12\sqrt{2(a+b)}}, \tag{3.21}$$

and then

$$\begin{aligned}
\Lambda &= (\ln a - \ln b)\left(a\frac{\partial D_{AG-N_2G}(a,b)}{\partial a} - b\frac{\partial D_{AG-N_2G}(a,b)}{\partial b}\right) \\
&= (\ln a - \ln b)\left(\frac{a-b}{8} - \frac{\sqrt{a+b}\left(\sqrt{a}-\sqrt{b}\right)}{12\sqrt{2}} - \frac{(a-b)\left(\sqrt{a}+\sqrt{b}\right)}{12\sqrt{2(a+b)}}\right) \\
&= \frac{(a-b)(\ln a - \ln b)}{8}\left(1 - \frac{2}{3}\left(\frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}}\right)\right).
\end{aligned} \tag{3.22}$$

From (2.4) we have $\Lambda \geq 0$, and, consequently, by Lemma 2.4, we obtain that D_{AG-N_2G} is Schur-geometrically convex in R_+^2.

(8) In order to prove that the function $D_{AN_2-AG}(a,b)$ is Schur-geometrically convex in R_+^2 it is enough to notice that

$$D_{AN_2-AG}(a,b) = M_{AN_2}(a,b) - \frac{1}{4}M_{AG}(a,b) = 3D_{AG-N_2G}(a,b). \tag{3.23}$$

(9) For

$$\begin{aligned}
D_{SN_2-SA}(a,b) &= \frac{4}{5}M_{SN_2}(a,b) - M_{SA}(a,b) \\
&= \frac{a+b}{2} - \frac{1}{5}\sqrt{\frac{a^2+b^2}{2}} - \frac{1}{5}\left(\sqrt{a}+\sqrt{b}\right)\sqrt{2(a+b)},
\end{aligned} \tag{3.24}$$

we have

$$\begin{aligned}
\frac{\partial D_{SN_2-SA}(a,b)}{\partial a} &= \frac{1}{2} - \frac{a}{5\sqrt{2(a^2+b^2)}} - \frac{1}{5}\sqrt{\frac{a+b}{2a}} - \frac{\sqrt{a}+\sqrt{b}}{5\sqrt{2(a+b)}}, \\
\frac{\partial D_{SN_2-SA}(a,b)}{\partial b} &= \frac{1}{2} - \frac{b}{5\sqrt{2(a^2+b^2)}} - \frac{1}{5}\sqrt{\frac{a+b}{2b}} - \frac{\sqrt{a}+\sqrt{b}}{5\sqrt{2(a+b)}},
\end{aligned} \tag{3.25}$$

and then

$$\begin{aligned}
\Lambda &= (\ln a - \ln b)\left(\frac{\partial D_{SN_2-SA}(a,b)}{\partial a} - \frac{\partial D_{SN_2-SA}(a,b)}{\partial b}\right) \\
&= (\ln a - \ln b)\left(\frac{a-b}{2} - \frac{a^2-b^2}{5\sqrt{2(a^2+b^2)}} - \frac{1}{5}\left(\sqrt{\frac{a(a+b)}{2}} - \sqrt{\frac{b(a+b)}{2}}\right) - \frac{\left(\sqrt{a}+\sqrt{b}\right)(a-b)}{5\sqrt{2(a+b)}}\right) \\
&= \frac{(a-b)(\ln a - \ln b)}{5\sqrt{2}}\left(\frac{5}{\sqrt{2}} - \frac{a+b}{\sqrt{a^2+b^2}} - \frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}\right).
\end{aligned} \tag{3.26}$$

From (2.2) and (2.4) we obtain that

$$\frac{5}{\sqrt{2}} - \frac{a+b}{\sqrt{a^2+b^2}} - \frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}} \geq \frac{5}{\sqrt{2}} - \sqrt{2} - \frac{3}{\sqrt{2}} = 0, \tag{3.27}$$

so $\Lambda \geq 0$, which proves that the function $D_{SN_2-SA}(a,b)$ is Schur-geometrically convex in R_+^2.

(10) One can easily check that

$$D_{AN_AN_2-SN_2}(a,b) = 4D_{SN_2-SA}(a,b), \tag{3.28}$$

and, consequently, the function $D_{AN_2-SN_2}$ is Schur-geometrically convex in R_+^2.

(11) To prove that the function

$$D_{SN_1-SH}(a,b) = 2M_{SN_1}(a,b) - M_{SH}(a,b) = \sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2} - \sqrt{ab} + \frac{2ab}{a+b} \tag{3.29}$$

is Schur-geometrically convex in R_+^2 it is enough to notice that

$$D_{SN_1-SH}(a,b) = D_{SG-AH}(a,b). \tag{3.30}$$

(12) For

$$\begin{aligned}
D_{SG-SN_1}(a,b) &= \frac{3}{2}M_{SG}(a,b) - 2M_{SN_1}(a,b) \\
&= \frac{1}{2}\left(a+b-\sqrt{ab}-\sqrt{\frac{a^2+b^2}{2}}\right),
\end{aligned} \tag{3.31}$$

we have

$$\begin{aligned}\frac{\partial D_{SG-SN_1}(a,b)}{\partial a} &= \frac{1}{2}\left(1-\frac{b}{2\sqrt{ab}}-\frac{a}{\sqrt{2(a^2+b^2)}}\right),\\ \frac{\partial D_{SG-SN_1}(a,b)}{\partial b} &= \frac{1}{2}\left(1-\frac{a}{2\sqrt{ab}}-\frac{b}{\sqrt{2(a^2+b^2)}}\right),\end{aligned} \tag{3.32}$$

and then

$$\begin{aligned}\Lambda &= (\ln a-\ln b)\left(a\frac{\partial D_{SG-SN_1}(a,b)}{\partial a}-b\frac{\partial D_{SG-SN_1}(a,b)}{\partial b}\right)\\ &= \frac{(a-b)(\ln a-\ln b)}{2}\left(1-\frac{a+b}{\sqrt{2(a^2+b^2)}}\right).\end{aligned} \tag{3.33}$$

By the inequality (2.2) we get that $\Lambda \geq 0$, which proves that D_{SG-SN_1} is Schur-geometrically convex in R_+^2.

(13) It is easy to check that

$$D_{SN_3-SA}(a,b) = \frac{1}{2}D_{AG-SG}(a,b), \tag{3.34}$$

which means that the function D_{SN_3-SA} is Schur-geometrically convex in R_+^2.

(14) To prove that the function $D_{SN_1-SN_3}$ is Schur-geometrically convex in R_+^2 it is enough to notice that

$$D_{SN_1-SN_3}(a,b) = \frac{1}{6}D_{AG-SG}(a,b). \tag{3.35}$$

The proof of Theorem I is complete. □

4. Applications

Applying Theorem I, Lemma 2.6, and Definition 2.2 one can easily prove the following.

Theorem II. *Let* $0 < a \leq b$. $1/2 \leq t \leq 1$ *or* $0 \leq t \leq 1/2$, $u = a^t b^{1-t}$ *and* $v = b^t a^{1-t}$. *Then*

$$\begin{aligned}
M_{SA}(a,b) &\leq \frac{1}{3}M_{SH}(a,b) - \left(\frac{1}{3}M_{SH}(u,v) - M_{SA}(u,v)\right) \leq \frac{1}{3}M_{SH}(a,b) \\
&\leq \frac{1}{2}M_{AH}(a,b) - \left(\frac{1}{2}M_{AH}(u,v) - \frac{1}{3}M_{SH}(u,v)\right) \leq \frac{1}{2}M_{AH}(a,b) \\
&\leq \frac{1}{2}M_{SG}(a,b) - \left(\frac{1}{2}M_{SG}(u,v) - \frac{1}{2}M_{AH}(u,v)\right) \leq \frac{1}{2}M_{SG}(a,b) \\
&\leq M_{AG}(a,b) - \left(M_{AG}(u,v) - \frac{1}{2}M_{SG}(u,v)\right) \leq M_{AG}(a,b),
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\frac{1}{8}M_{AH}(a,b) &\leq M_{N_2N_1}(a,b) - \left(M_{N_2N_1}(u,v) - \frac{1}{8}M_{AH}(u,v)\right) \leq M_{N_2N_1}(a,b) \\
&\leq \frac{1}{3}M_{N_2G}(a,b) - \left(\frac{1}{3}M_{N_2G}(u,v) - M_{N_2N_1}(u,v)\right) \leq \frac{1}{3}M_{N_2G}(a,b) \\
&\leq \frac{1}{4}M_{AG}(a,b) - \left(\frac{1}{4}M_{AG}(u,v) - \frac{1}{3}M_{N_2G}(u,v)\right) \leq \frac{1}{4}M_{AG}(a,b) \\
&\leq M_{AN_2}(a,b) - \left(M_{AN_2}(u,v) - \frac{1}{4}M_{AG}(u,v)\right) \leq M_{AN_2}(a,b),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
M_{SA}(a,b) &\leq \frac{4}{5}M_{SN_2}(a,b) - \left(\frac{4}{5}M_{SN_2}(u,v) - \frac{4}{5}M_{SN_2}(u,v)\right) \leq \frac{4}{5}M_{SN_2}(a,b) \\
&\leq 4M_{AN_2}(a,b) - \left(4M_{AN_2}(u,v) - \frac{4}{5}M_{SN_2}(u,v)\right) \leq 4M_{AN_2}(a,b),
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
M_{SH}(a,b) &\leq 2M_{SN_1}(a,b) - (2M_{SN_1}(u,v) - M_{SH}(u,v)) \leq 2M_{SN_1}(a,b) \\
&\leq \frac{3}{2}M_{SG}(a,b) - \left(\frac{3}{2}M_{SG}(u,v) - \frac{3}{2}M_{SG}(u,v)\right) \leq \frac{3}{2}M_{SG}(a,b),
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
M_{SA}(a,b) &\leq \frac{3}{4}M_{SN_3}(a,b) - \left(\frac{3}{4}M_{SN_3}(u,v) - M_{SA}(u,v)\right) \leq \frac{3}{4}M_{SN_3}(a,b) \\
&\leq \frac{2}{3}M_{SN_1}(a,b) - \left(\frac{2}{3}M_{SN_1}(u,v) - \frac{3}{4}M_{SN_3}(u,v)\right) \leq \frac{2}{3}M_{SN_1}(a,b).
\end{aligned} \tag{4.5}$$

Remark 4.1. Equation (4.1), (4.2), (4.3), (4.4), and (4.5) are a refinement of (1.5), (1.6), (1.7), (1.8), and (1.9), respectively.

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Research Article

Complete Moment Convergence of Weighted Sums for Arrays of Rowwise φ-Mixing Random Variables

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The complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables is investigated. By using moment inequality and truncation method, the sufficient conditions for complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables are obtained. The results of Ahmed et al. (2002) are complemented. As an application, the complete moment convergence of moving average processes based on a φ-mixing random sequence is obtained, which improves the result of Kim et al. (2008).

1. Introduction

Hsu and Robbins [1] introduced the concept of complete convergence of $\{X_n\}$. A sequence $\{X_n, n = 1, 2, \ldots\}$ is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \epsilon) < \infty, \quad \forall \epsilon > 0. \tag{1.1}$$

Moreover, they proved that the sequence of arithmetic means of independent identically distributed (i.i.d.) random variables converge completely to the expected value if the variance of the summands is finite. The converse theorem was proved by Erdös [2]. This result has been generalized and extended in several directions, see Baum and Katz [3], Chow [4], Gut [5], Taylor et al. [6], and Cai and Xu [7]. In particular, Ahmed et al. [8] obtained the following result in Banach space.

Theorem A. *Let $\{X_{ni}; i \geq 1, n \geq 1\}$ be an array of rowwise independent random elements in a separable real Banach space $(B, \|\cdot\|)$. Let $P(\|X_{ni}\| > x) \leq CP(|X| > x)$ for some random variable X, constant C and all n, i and $x > 0$. Suppose that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants such that*

$$\begin{gathered}\sup_{i\geq 1}|a_{ni}| = O(n^{-r}), \quad \text{for some } r > 0, \\ \sum_{i=1}^{\infty}|a_{ni}| = O(n^{\alpha}), \quad \text{for some } \alpha \in [0, r).\end{gathered} \tag{1.2}$$

Let β be such that $\alpha + \beta \neq -1$ and fix $\delta > 0$ such that $1 + \alpha/r < \delta \leq 2$. Denote $s = \max(1 + (\alpha + \beta + 1)/r,\ \delta)$. If $E|X|^s < \infty$ and $S_n = \sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then $\sum_{n=1}^{\infty} n^{\beta} P(\|S_n\| > \epsilon) < \infty$ for all $\epsilon > 0$.

Chow [4] established the following refinement which is a complete moment convergence result for sums of (i.i.d.) random variables.

Theorem B. *Let $EX_1 = 0$, $1 \leq p < 2$ and $r \geq p$. Suppose that $E[|X_1|^r + |X_1|\log(1 + |X_1|)] < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2-(1/p)} E\left(\left|\sum_{i=1}^{n} X_i\right| - \epsilon n^{1/p}\right)^{+} < \infty, \quad \forall \epsilon > 0. \tag{1.3}$$

The main purpose of this paper is to discuss again the above results for arrays of rowwise φ-mixing random variables. The author takes the inspiration in [8] and discusses the complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables by applying truncation methods. The results of Ahmed et al. [8] are extended to φ-mixing case. As an application, the corresponding results of moving average processes based on a φ-mixing random sequence are obtained, which extend and improve the result of Kim and Ko [9].

For the proof of the main results, we need to restate a few definitions and lemmas for easy reference. Throughout this paper, C will represent positive constants, the value of which may change from one place to another. The symbol $I(A)$ denotes the indicator function of A; $[x]$ indicates the maximum integer not larger than x. For a finite set B, the symbol $\sharp B$ denotes the number of elements in the set B.

Definition 1.1. A sequence of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be a sequence of φ-mixing random variables, if

$$\varphi(m) = \sup_{k\geq 1}\left\{|P(B \mid A) - P(B)|\ ;\, A \in \Im_1^k, B \in \Im_{k+m}^{\infty},\ P(A) > 0\right\} \longrightarrow 0, \quad \text{as } m \longrightarrow \infty, \tag{1.4}$$

where $\Im_j^k = \sigma\{X_i; j \leq i \leq k\}$, $1 \leq j \leq k \leq \infty$.

Definition 1.2. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X (write $\{X_i\} \prec X$) if there exists a constant C, such that $P\{|X_n| > x\} \leq CP\{|X| > x\}$ for all $x \geq 0$ and $n \geq 1$.

The following lemma is a well-known result.

Lemma 1.3. *Let the sequence* $\{X_n, n \geq 1\}$ *of random variables be stochastically dominated by a random variable* X*. Then for any* $p > 0, x > 0$

$$E|X_n|^p I(|X_n| \leq x) \leq C\left[E|X|^p I(|X| \leq x) + x^p P\{|X| > x\}\right], \tag{1.5}$$

$$E|X_n|^p I(|X_n| > x) \leq CE|X|^p I(|X| > x). \tag{1.6}$$

Definition 1.4. A real-valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if $\lim_{x\to\infty} l(x\lambda)/l(x) = 1$ for each $\lambda > 0$.

By the properties of slowly varying function, we can easily prove the following lemma. Here we omit the details of the proof.

Lemma 1.5. *Let* $l(x) > 0$ *be a slowly varying function as* $x \to \infty$*, then there exists* C *(depends only on* r*) such that*

(i) $Ck^{r+1}l(k) \leq \sum_{n=1}^{k} n^r l(n) \leq Ck^{r+1}l(k)$ *for any* $r > -1$ *and positive integer* k*,*

(ii) $Ck^{r+1}l(k) \leq \sum_{n=k}^{\infty} n^r l(n) \leq Ck^{r+1}l(k)$ *for any* $r < -1$ *and positive integer* k*.*

The following lemma will play an important role in the proof of our main results. The proof is due to Shao [10].

Lemma 1.6. *Let* $\{X_i, 1 \leq i \leq n\}$ *be a sequence of* φ*-mixing random variables with mean zero. Suppose that there exists a sequence* $\{C_n\}$ *of positive numbers such that* $E(\sum_{i=k+1}^{k+m} X_i)^2 \leq C_n$ *for any* $k \geq 0, n \geq 1, m \leq n$*. Then for any* $q \geq 2$*, there exists* $C = C(q, \varphi(\cdot))$ *such that*

$$E\max_{1\leq j\leq n}\left|\sum_{i=k+1}^{k+j} X_i\right|^q \leq C\left[C_n^{q/2} + E\max_{k+1\leq i\leq k+n}|X_i|^q\right]. \tag{1.7}$$

Lemma 1.7. *Let* $\{X_i, 1 \leq i \leq n\}$ *be a sequence of* φ*-mixing random variables with* $\sum_{i=1}^{\infty} \varphi^{1/2}(i) < \infty$*, then there exists* C *such that for any* $k \geq 0$ *and* $n \geq 1$

$$E\left(\sum_{i=k+1}^{k+n} X_i\right)^2 \leq C\sum_{i=k+1}^{k+n} EX_i^2. \tag{1.8}$$

Proof. By Lemma 5.4.4 in [11] and Hölder's inequality, we have

$$
\begin{aligned}
E\left(\sum_{i=k+1}^{k+n} X_i\right)^2 &= \sum_{i=k+1}^{k+n} EX_i^2 + 2\sum_{k+1\le i<j\le k+n} EX_iX_j \\
&\le \sum_{i=k+1}^{k+n} EX_i^2 + 4\sum_{k+1\le i<j\le k+n} \varphi^{1/2}(j-i)\left(EX_i^2\right)^{1/2}\left(EX_j^2\right)^{1/2} \\
&\le \sum_{i=k+1}^{k+n} EX_i^2 + 2\sum_{i=k+1}^{k+n-1}\sum_{j=i+1}^{k+n} \varphi^{1/2}(j-i)\left(EX_i^2 + EX_j^2\right) \\
&\le \left(1 + 4\sum_{i=1}^{n}\varphi^{1/2}(i)\right)\sum_{i=k+1}^{k+n} EX_i^2.
\end{aligned}
\tag{1.9}
$$

Therefore, (1.8) holds. □

2. Main Results

Now we state our main results. The proofs will be given in Section 3.

Theorem 2.1. *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise φ-mixing random variables with $EX_{ni} = 0$, $\{X_{ni}\} \prec X$ and $\sum_{m=1}^{\infty}\varphi^{1/2}(m) < \infty$. Let $l(x) > 0$ be a slowing varying function, and $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of constants such that*

$$
\begin{aligned}
&\sup_{i\ge 1}|a_{ni}| = O\left(n^{-r}\right), \quad for\ some\ r > 0, \\
&\sum_{i=1}^{\infty}|a_{ni}| = O(n^{\alpha}), \quad for\ some\ \alpha \in [0, r).
\end{aligned}
\tag{2.1}
$$

(a) *If $\alpha + \beta + 1 > 0$ and there exists some $\delta > 0$ such that $(\alpha/r) + 1 < \delta \le 2$, and $s = \max(1 + ((\alpha+\beta+1)/r), \delta)$, then $E|X|^s l(|X|^{1/r}) < \infty$ implies*

$$
\sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\sup_{k\ge 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \epsilon\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.2}
$$

(b) *If $\beta = -1, \alpha > 0$, then $E|X|^{1+(\alpha/r)}(1 + l(|X|^{1/r})) < \infty$ implies*

$$
\sum_{n=1}^{\infty} n^{-1} l(n) E\left[\sup_{k\ge 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \epsilon\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.3}
$$

Remark 2.2. If $\alpha+\beta+1<0$, then $E|X|<\infty$ implies that (2.2) holds. In fact,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon\right]^{+} &\leq \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{\infty} |a_{ni}| E|X_{ni}| + \epsilon \sum_{n=1}^{\infty} n^{\beta} l(n) \\ &\leq C\sum_{n=1}^{\infty} n^{\beta+\alpha} l(n) E|X| + \epsilon \sum_{n=1}^{\infty} n^{\beta} l(n) < \infty.\end{aligned} \tag{2.4}$$

Remark 2.3. Note that

$$\begin{aligned}\infty > \sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon\right]^{+} &= \sum_{n=1}^{\infty} n^{\beta} l(n) \int_{0}^{\infty} P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon > x\right\} dx \\ &= \int_{0}^{\infty} \sum_{n=1}^{\infty} n^{\beta} l(n) P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| > x+\epsilon\right\} dx.\end{aligned} \tag{2.5}$$

Therefore, from (2.5), we obtain that the complete moment convergence implies the complete convergence, that is, under the conditions of Theorem 2.1, result (2.2) implies

$$\sum_{n=1}^{\infty} n^{\beta} l(n) P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| > \epsilon\right\} < \infty, \tag{2.6}$$

and (2.3) implies

$$\sum_{n=1}^{\infty} n^{-1} l(n) P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| > \epsilon\right\} < \infty. \tag{2.7}$$

Corollary 2.4. *Under the conditions of Theorem 2.1,*

(1) *if $\alpha+\beta+1>0$ and there exists some $\delta>0$ such that $(\alpha/r)+1<\delta\leq 2$, and $s=\max(1+((\alpha+\beta+1)/r),\delta)$, then $E|X|^{s} l(|X|^{1/r})<\infty$ implies*

$$\sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right|-\epsilon\right]^{+} < \infty, \quad \forall\epsilon>0, \tag{2.8}$$

(2) *if $\beta=-1, \alpha>0$, then $E|X|^{1+(\alpha/r)}(1+l(|X|^{1/r}))<\infty$ implies*

$$\sum_{n=1}^{\infty} n^{-1} l(n) E\left[\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right|-\epsilon\right]^{+} < \infty, \quad \forall\epsilon>0. \tag{2.9}$$

Corollary 2.5. *Let $\{X_{ni}, i\geq 1, n\geq 1\}$ be an array of rowwise φ-mixing random variables with $EX_{ni}=0, \{X_{ni}\} \prec X$ and $\sum_{m=1}^{\infty}\varphi^{1/2}(m)<\infty$. Suppose that $l(x)>0$ is a slowly varying function.*

(1) *Let* $p > 1$ *and* $1 \le t < 2$. *If* $E|X|^{pt} l(|X|^t) < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{p-2-(1/t)} l(n) E\left[\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}\right| - \epsilon n^{1/t}\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.10}$$

(2) *Let* $1 < t < 2$. *If* $E|X|^t[1 + l(|X|^t)] < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{-1-(1/t)} l(n) E\left[\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}\right| - \epsilon n^{1/t}\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.11}$$

Corollary 2.6. *Suppose that* $X_n = \sum_{i=-\infty}^{\infty} a_{i+n} Y_i$, $n \ge 1$, *where* $\{a_i, -\infty < i < \infty\}$ *is a sequence of real numbers with* $\sum_{-\infty}^{\infty} |a_i| < \infty$, *and* $\{Y_i, -\infty < i < \infty\}$ *is a sequence of* φ*-mixing random variables with* $EY_i = 0$, $\{Y_i\} \prec Y$ *and* $\sum_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$. *Let* $l(x)$ *be a slowly varying function.*

(1) *Let* $1 \le t < 2, r \ge 1 + (t/2)$. *If* $E|Y|^r l(|Y|^t) < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{(r/t)-2-(1/t)} l(n) E\left[\left|\sum_{i=1}^{n} X_i\right| - \epsilon n^{1/t}\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.12}$$

(2) *Let* $1 < t < 2$. *If* $E|Y|^t[1 + l(|Y|^t)] < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{-1-(1/t)} l(n) E\left[\left|\sum_{i=1}^{n} X_i\right| - \epsilon n^{1/t}\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.13}$$

Remark 2.7. Corollary 2.6 obtains the result about the complete moment convergence of moving average processes based on a φ-mixing random sequence with different distributions. We extend the results of Chen et al. [12] from the complete convergence to the complete moment convergence. The result of Kim and Ko [9] is a special case of Corollary 2.6 (1). Moreover, our result covers the case of $r = t$, which was not considered by Kim and Ko.

3. Proofs of the Main Results

Proof of Theorem 2.1. Without loss of generality, we can assume

$$\sup_{i\ge 1} |a_{ni}| \le n^{-r}, \qquad \sum_{i=1}^{\infty} |a_{ni}| \le n^{\alpha}. \tag{3.1}$$

Let $S_{nk}(x) = \sum_{i=1}^{k} a_{ni}X_{ni}I(|a_{ni}X_{ni}| \le n^{-r}x)$ for any $k \ge 1$, $n \ge 1$, and $x \ge 0$. First note that $E|X|^s l(|X|^{1/r}) < \infty$ implies $E|X|^t < \infty$ for any $0 < t < s$. Therefore, for $x > n^r$,

$$\begin{aligned}
x^{-1}n^r \sup_{k\ge1} E|S_{nk}(x)| &= x^{-1}n^r \sup_{k\ge1} E\left|\sum_{i=1}^{k} a_{ni}X_{ni}I\left(|a_{ni}X_{ni}| > n^{-r}x\right)\right| \qquad (EX_{ni}=0)\\
&\le \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|I\left(|a_{ni}X_{ni}| > n^{-r}x\right) \le \sum_{i=1}^{\infty} E|a_{ni}X|I\left(|a_{ni}X| > n^{-r}x\right)\\
&\le \sum_{i=1}^{\infty}|a_{ni}|E|X|I(|X|>x) \le n^{\alpha}E|X|I(|X|>x)\\
&\le x^{\alpha/r}E|X|I(|X|>x) \le E|X|^{1+(\alpha/r)}I(|X|>n^r) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.2}$$

Hence, for n large enough we have $\sup_{k\ge1}E|S_{nk}(x)| < (\epsilon/2)n^{-r}x$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta}l(n)E\left[\sup_{k\ge1}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right| - \epsilon\right]^{+}\\
&\quad= \sum_{n=1}^{\infty} n^{\beta}l(n)\int_{\epsilon}^{\infty} P\left\{\sup_{k\ge1}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right| \ge x\right\}dx\\
&\quad= \sum_{n=1}^{\infty} n^{\beta-r}l(n)\epsilon\int_{n^r}^{\infty} P\left\{\sup_{k\ge1}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right| \ge \epsilon n^{-r}x\right\}dx\\
&\quad\le C\sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^r}^{\infty} P\left\{\sup_{i}|a_{ni}X_{ni}| > n^{-r}x\right\}dx\\
&\qquad+ C\sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^r}^{\infty} P\left\{\sup_{k\ge1}|S_{nk}(x) - ES_{nk}(x)| \ge n^{-r}x\frac{\epsilon}{2}\right\}dx := I_1 + I_2.
\end{aligned} \tag{3.3}$$

Noting that $\alpha + \beta > -1$, by Lemma 1.5, Markov inequality, (1.6), and (3.1), we have

$$\begin{aligned}
I_1 &\le C\sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^r}^{\infty}\sum_{i=1}^{\infty}P\{|a_{ni}X_{ni}| > n^{-r}x\}dx\\
&\le C\sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^r}^{\infty} n^r x^{-1}\sum_{i=1}^{\infty}E|a_{ni}X_{ni}|I\left(|a_{ni}X_{ni}| > n^{-r}x\right)dx\\
&\le C\sum_{n=1}^{\infty} n^{\beta+\alpha}l(n)\int_{n^r}^{\infty} x^{-1}E|X|I(|X|>x)dx\\
&\le C\sum_{n=1}^{\infty} n^{\beta+\alpha}l(n)\sum_{k=n}^{\infty}\int_{k^r}^{k^{r+1}} x^{-1}E|X|I(|X|>x)dx
\end{aligned}$$

$$\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{\beta+\alpha}l(n)\sum_{k=n}^{\infty} k^{-1}E|X|I(|X|>k^r) \leq C\sum_{k=1}^{\infty} k^{-1}E|X|I(|X|>k^r)\sum_{n=1}^{k} n^{\beta+\alpha}l(n) \\
&\leq C\sum_{k=1}^{\infty} k^{\beta+\alpha}l(k)E|X|I(|X|>k^r) \leq CE|X|^{1+((1+\alpha+\beta)/r)}l\left(|X|^{1/r}\right) < \infty.
\end{aligned} \tag{3.4}$$

Now we estimate I_2, noting that $\sum_{m=1}^{\infty}\varphi^{1/2}(m) < \infty$, by Lemma 1.7, we have

$$\begin{aligned}
&\sup_{1\leq m<\infty} E\left(\sum_{i=1}^{m} a_{ni}X_{ni}I\left(|a_{ni}X_{ni}| \leq n^{-r}x\right) - E\sum_{i=1}^{m} a_{ni}X_{ni}I\left(|a_{ni}X_{ni}| \leq n^{-r}x\right)\right)^2 \\
&\quad \leq C\sum_{i=1}^{\infty} Ea_{ni}^2X_{ni}^2 I\left(|a_{ni}X_{ni}| \leq n^{-r}x\right).
\end{aligned} \tag{3.5}$$

By Lemma 1.6, Markov inequality, C_r inequality, and (1.5), for any $q \geq 2$, we have

$$\begin{aligned}
&P\left\{\sup_{k\geq 1}|S_{nk}(x) - ES_{nk}(x)| \geq n^{-r}x\frac{\epsilon}{2}\right\} \leq Cn^{rq}x^{-q}E\sup_{k\geq 1}|S_{nk}(x) - ES_{nk}(x)|^q \\
&\quad \leq Cn^{rq}x^{-q}\left[\left(\sum_{i=1}^{\infty} Ea_{ni}^2X_{ni}^2 I\left(|a_{ni}X_{ni}| \leq n^{-r}x\right)\right)^{q/2} + \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^q I\left(|a_{ni}X_{ni}| \leq n^{-r}x\right)\right] \\
&\quad \leq Cn^{rq}x^{-q}\left(\sum_{i=1}^{\infty} Ea_{ni}^2X^2 I\left(|a_{ni}X| \leq n^{-r}x\right)\right)^{q/2} + Cn^{rq}x^{-q}\sum_{i=1}^{\infty} E|a_{ni}X|^q I\left(|a_{ni}X| \leq n^{-r}x\right) \\
&\qquad + C\left(\sum_{i=1}^{\infty} P\left\{|a_{ni}X| > n^{-r}x\right\}\right)^{q/2} + C\sum_{i=1}^{\infty} P\left\{|a_{ni}X| > n^{-r}x\right\} \\
&\quad := J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.6}$$

So,

$$I_2 \leq \sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^r}^{\infty}(J_1 + J_2 + J_3 + J_4)dx. \tag{3.7}$$

From (3.4), we have $\sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^r}^{\infty} J_4 dx < \infty$.

For J_1, we consider the following two cases.

If $s > 2$, then $EX^2 < \infty$. Taking $q \geq 2$ such that $\beta + (q(\alpha - r)/2) < -1$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_1 dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \int_{n^r}^{\infty} x^{-q} \left(\sum_{i=1}^{\infty} a_{ni}^2 \right)^{q/2} dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) n^{q(\alpha-r)/2} n^{r(-q+1)} \leq C \sum_{n=1}^{\infty} n^{\beta+(q(\alpha-r)/2)} l(n) < \infty.
\end{aligned} \tag{3.8}$$

If $s \leq 2$, we choose s' such that $1 + (\alpha/r) < s' < s$. Taking $q \geq 2$ such that $\beta + (qr/2)(1 + (\alpha/r) - s') < -1$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_1 dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \int_{n^r}^{\infty} x^{-q} \left(\sum_{i=1}^{\infty} |a_{ni}| |a_{ni}|^{s'-1} E |a_{ni} X|^{2-s'} |X|^{s'} I\left(|a_{ni} X| \leq n^{-r} x\right) \right)^{q/2} dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) n^{q\alpha/2} n^{-(qr/2)(s'-1)} \int_{n^r}^{\infty} x^{-q} \left(n^{-r} x\right)^{(q/2)(2-s')} dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta+(qr/2)(1+(\alpha/r)-s')} l(n) < \infty.
\end{aligned} \tag{3.9}$$

So, $\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_1 dx < \infty$.

Now, we estimate J_2. Set $I_{nj} = \{i \geq 1 \mid (n(j+1))^{-r} < |a_{ni}| \leq (nj)^{-r}\}$, $j = 1, 2, \ldots$. Then $\cup_{j \geq 1} I_{nj} = N$, where N is the set of positive integers. Note also that for all $k \geq 1, n \geq 1$,

$$\begin{aligned}
n^{\alpha} &\geq \sum_{i=1}^{\infty} |a_{ni}| = \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}| \\
&\geq \sum_{j=1}^{\infty} \left(\sharp I_{nj}\right) \left(n(j+1)\right)^{-r} \geq n^{-r} \sum_{j=k}^{\infty} \left(\sharp I_{nj}\right) (j+1)^{-rq} (k+1)^{rq-r}.
\end{aligned} \tag{3.10}$$

Hence, we have

$$\sum_{j=k}^{\infty} \left(\sharp I_{nj}\right) j^{-rq} \leq C n^{\alpha+r} k^{r-rq}. \tag{3.11}$$

Note that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_2 dx \\
&= C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \int_{n^r}^{\infty} x^{-q} \sum_{j=1}^{\infty} \sum_{i\in I_{nj}} E|a_{ni}X|^q I\left(|a_{ni}X| \le n^{-r}x\right) dx \\
&= C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \sum_{j=1}^{\infty} \left(\sharp I_{nj}\right) (nj)^{-rq} \sum_{k=n}^{\infty} \int_{k^r}^{(k+1)^r} x^{-q} E|X|^q I\left(|X| \le x(j+1)^r\right) dx \\
&\le C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \sum_{j=1}^{\infty} \left(\sharp I_{nj}\right) (nj)^{-rq} \sum_{k=n}^{\infty} k^{r(-q+1)-1} E|X|^q I\left(|X| \le (k+1)^r (j+1)^r\right) \\
&= C \sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{j=1}^{\infty} \left(\sharp I_{nj}\right) j^{-rq} \sum_{i=0}^{(k+1)(j+1)-1} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&\le C \sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{j=1}^{\infty} \left(\sharp I_{nj}\right) j^{-rq} \sum_{i=0}^{2(k+1)-1} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&\quad + C \sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{j=1}^{\infty} \left(\sharp I_{nj}\right) j^{-rq} \sum_{i=2(k+1)}^{(k+1)(j+1)} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&:= J_2' + J_2''.
\end{aligned}
\tag{3.12}
$$

Taking $q \ge 2$ large enough such that $\beta + \alpha - rq + r < -1$, for J_2', by Lemma 1.6 and (3.11), we get

$$
\begin{aligned}
J_2' &\le C \sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} n^{\alpha+r} \sum_{i=0}^{2(k+1)-1} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&= C \sum_{k=1}^{\infty} k^{r(-q+1)-1} \sum_{i=0}^{2(k+1)-1} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \sum_{n=1}^{k} n^{\beta+\alpha} l(n) \\
&\le C \sum_{k=1}^{\infty} k^{\beta+\alpha-rq+r} l(k) \sum_{i=0}^{2(k+1)-1} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&\le C + C \sum_{i=3}^{\infty} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \sum_{k=[i/2]}^{\infty} k^{\beta+\alpha-rq+r} l(k) \\
&\le C + C \sum_{i=3}^{\infty} i^{\beta+\alpha-rq+r+1} l(i) E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \le C + CE|X|^{1+((\beta+\alpha+1)/r)} l\left(|X|^{1/r}\right) < \infty.
\end{aligned}
\tag{3.13}
$$

For J_2'', we obtain

$$
\begin{aligned}
J_2'' &\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rq} \sum_{i=2(k+1)}^{(j+1)(k+1)} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{i=2(k+1)}^{\infty} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \sum_{j=\left[i(k+1)^{-1}\right]-1}^{\infty} (\sharp I_{nj}) j^{-rq} \\
&\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{i=2(k+1)}^{\infty} n^{r+\alpha} i^{r(1-q)} k^{-r(1-q)} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&= C\sum_{k=1}^{\infty} k^{-1} \sum_{i=2(k+1)}^{\infty} i^{r(1-q)} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \sum_{n=1}^{k} n^{\beta+\alpha} l(n) \\
&\le C\sum_{k=1}^{\infty} k^{\beta+\alpha} l(k) \sum_{i=2(k+1)}^{\infty} i^{r(1-q)} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&\le C\sum_{i=4}^{\infty} i^{\beta+\alpha+1+r-rq} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \le CE|X|^{1+((\beta+\alpha+1)/r)} l\left(|X|^{1/r}\right) < \infty.
\end{aligned}
\tag{3.14}
$$

So $\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_2 dx < \infty$. Finally, we prove $\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_3 dx < \infty$. In fact, noting $1 + (a/r) < s' < s$ and $\beta + (qr/2)(1 + (\alpha/r) - s') < -1$, using Markov inequality and (3.1), we get

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_3 dx &\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} \left(\sum_{i=1}^{\infty} n^{rs'} x^{-s'} E|a_{ni}X|^{s'}\right)^{q/2} dx \\
&\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) n^{qrs'/2} n^{-r(s'-1)(q/2)} n^{\alpha(q/2)} \int_{n^r}^{\infty} x^{-s'(q/2)} dx \\
&\le C\sum_{n=1}^{\infty} n^{\beta-r+r(q/2)+\alpha(q/2)} l(n) n^{r(-s'(q/2)+1)} \le C\sum_{n=1}^{\infty} n^{\beta+(qr/2)(1+(\alpha/2)-s')} l(n) < \infty.
\end{aligned}
\tag{3.15}
$$

Thus, we complete the proof in (a). Next, we prove (b). Note that $E|X|^{1+\alpha/r} < \infty$ implies that (3.2) holds. Therefore, from the proof in (a), to complete the proof of (b), we only need to prove

$$
I_2 = C\sum_{n=1}^{\infty} n^{-1-r} l(n) \int_{n^r}^{\infty} P\left\{\sup_{k\ge 1} |S_{nk}(x) - ES_{nk}(x)| \ge n^{-r} x \frac{\epsilon}{2}\right\} dx < \infty. \tag{3.16}
$$

In fact, noting $\beta = -1$, $\alpha + \beta + 1 > 0$, $\alpha + \beta - r < -1$ and $E|X|^{1+\alpha/r} l(|X|^{1/r}) < \infty$. By taking $q = 2$ in the proof of (3.12), (3.13), and (3.14), we get

$$C \sum_{n=1}^{\infty} n^{-1+r} l(n) \int_{n^r}^{\infty} x^{-2} \sum_{i=1}^{\infty} E a_{ni}^2 X^2 I(|a_{ni} X| \le n^{-r} x) dx \le C + CE|X|^{1+(\alpha/r)} l\left(|X|^{1/r}\right) < \infty. \quad (3.17)$$

Then, by (3.17), we have

$$\begin{aligned}
I_2 &\le C \sum_{n=1}^{\infty} n^{-1-r} l(n) \int_{n^r}^{\infty} n^{2r} x^{-2} E|S_{xn} - ES_{xn}|^2 dx \\
&\le C \sum_{n=1}^{\infty} n^{-1+r} l(n) \int_{n^r}^{\infty} x^{-2} \sum_{i=1}^{\infty} E a_{ni}^2 X_{ni}^2 I(|a_{ni} X_{ni}| \le n^{-r} x) dx \\
&\le C \sum_{n=1}^{\infty} n^{-1+r} l(n) \int_{n^r}^{\infty} x^{-2} \sum_{i=1}^{\infty} E a_{ni}^2 X^2 I(|a_{ni} X| \le n^{-r} x) dx \\
&\quad + C \sum_{n=1}^{\infty} n^{-1-r} l(n) \int_{n^r}^{\infty} \sum_{i=1}^{\infty} P\{|a_{ni} X| > n^{-r} x\} dx \\
&\le C \sum_{n=1}^{\infty} n^{-1+r} l(n) \int_{n^r}^{\infty} x^{-2} \sum_{i=1}^{\infty} E a_{ni}^2 X^2 I(|a_{ni} X| \le n^{-r} x) dx + C < \infty.
\end{aligned} \quad (3.18)$$

The proof of Theorem 2.1 is completed. □

Proof of Corollary 2.4. Note that

$$\left[\left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| - \epsilon\right]^{+} \le \left[\sup_{k \ge 1} \left|\sum_{i=1}^{k} a_{ni} X_{ni}\right| - \epsilon\right]^{+}. \quad (3.19)$$

Therefore, (2.8) and (2.9) hold by Theorem 2.1. □

Proof of Corollary 2.5. By applying Theorem 2.1, taking $\beta = p - 2$, $a_{ni} = n^{-1/t}$ for $1 \le i \le n$, and $a_{ni} = 0$ for $i > n$, then we obtain (2.10). Similarly, taking $\beta = -1$, $a_{ni} = n^{-1/t}$ for $1 \le i \le n$, and $a_{ni} = 0$ for $i > n$, we obtain (2.11) by Theorem 2.1. □

Proof of Corollary 2.6. Let $X_{ni} = Y_i$ and $a_{ni} = n^{-1/t} \sum_{j=1}^{n} a_{i+j}$ for all $n \ge 1$, $-\infty < i < \infty$. Since $\sum_{-\infty}^{\infty} |a_i| < \infty$, we have $\sup_i |a_{ni}| = O(n^{-1/t})$ and $\sum_{i=-\infty}^{\infty} |a_{ni}| = O(n^{1-1/t})$. By applying Corollary 2.4, taking $\beta = (r/t) - 2$, $r = 1/t$, $\alpha = 1 - (1/t)$, we obtain

$$\sum_{n=1}^{\infty} n^{(r/t)-2-(1/t)} l(n) E\left[\left|\sum_{i=1}^{n} X_i\right| - \epsilon n^{1/t}\right]^{+} = \sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\left|\sum_{i=-\infty}^{\infty} a_{ni} X_{ni}\right| - \epsilon\right]^{+} < \infty, \quad \forall \epsilon > 0. \quad (3.20)$$

Therefore, (2.12) and (2.13) hold. □

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Research Article

On Huygens' Inequalities and the Theory of Means

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By using the theory of means, various refinements of Huygens' trigonometric and hyperbolic inequalities will be proved. New Huygens' type inequalities will be provided, too.

1. Introduction

The famous Huygens' trigonometric inequality (see e.g., [1–3]) states that for all $x \in (0, \pi/2)$ one has

$$2\sin x + \tan x > 3x. \tag{1.1}$$

The hyperbolic version of inequality (1.1) has been established recently by Neuman and Sándor [3]:

$$2\sinh x + \tanh x > 3x, \quad \text{for } x > 0. \tag{1.2}$$

Let $a, b > 0$ be two positive real numbers. The logarithmic and identric means of a and b are defined by

$$\begin{gathered} L = L(a,b) := \frac{b-a}{\ln b - \ln a} \quad (\text{for } a \neq b);\ L(a,a) = a, \\ I = I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \quad (\text{for } a \neq b);\ I(a,a) = a, \end{gathered} \tag{1.3}$$

respectively. Seiffert's mean P is defined by

$$P = P(a,b) := \frac{a-b}{2\arcsin((a-b)/(a+b))} \quad (\text{for } a \neq b), P(a,a) = a. \tag{1.4}$$

Let

$$\begin{gathered} A = A(a,b) := \frac{a+b}{2}, \qquad G = G(a,b) = \sqrt{ab}, \\ H = H(a,b) = 2\Big/\left(\frac{1}{a} + \frac{1}{b}\right) \end{gathered} \tag{1.5}$$

denote the arithmetic, geometric, and harmonic means of a and b, respectively. These means have been also in the focus of many research papers in the last decades. For a survey of results, see, for example, [4–6]. In what follows, we will assume $a \neq b$.

Now, by remarking that letting $a = 1 + \sin x$, $b = 1 - \sin x$, where $x \in (0, \pi/2)$, in $P, G,$ and A, we find that

$$P = \frac{\sin x}{x}, \qquad G = \cos x, \qquad A = 1, \tag{1.6}$$

so Huygens' inequality (1.1) may be written also as

$$P > \frac{3AG}{2G + A} = 3\Big/\left(\frac{2}{A} + \frac{1}{G}\right) = H(A, A, G). \tag{1.7}$$

Here $H(a,b,c)$ denotes the harmonic mean of the numbers a, b, c:

$$H(a,b,c) = 3\Big/\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \tag{1.8}$$

On the other hand, by letting $a = e^x$, $b = e^{-x}$ in $L, G,$ and A, we find that

$$L = \frac{\sinh x}{x}, \qquad G = 1, \qquad A = \cosh x, \tag{1.9}$$

so Huygens' hyperbolic inequality (1.2) may be written also as

$$L > \frac{3AG}{2A + G} = 3\Big/\left(\frac{2}{G} + \frac{1}{A}\right) = H(G, G, A). \tag{1.10}$$

2. First Improvements

Suppose $a, b > 0$, $a \neq b$.

Theorem 2.1. *One has*

$$P > H(L,A) > \frac{3AG}{2G+A} = H(A,A,G), \tag{2.1}$$

$$L > H(P,G) > \frac{3AG}{2A+G} = H(G,G,A). \tag{2.2}$$

Proof. The inequalities $P > H(L,A)$ and $L > H(P,G)$ have been proved in paper [7] (see Corollary 3.2). In fact, stronger relations are valid, as we will see in what follows.

Now, the interesting fact is that the second inequality of (2.1), that is, $2LA/(L+A) > 3AG/(2G+A)$ becomes, after elementary transformations, exactly inequality (1.10), while the second inequality of (2.2), that is, $2PG/(P+G) > 3AG/(2A+G)$ becomes inequality (1.7). □

Another improvements of (1.7), respectively, (1.10) are provided by

Theorem 2.2. *One has the inequalities:*

$$P > \sqrt[3]{A^2G} > \frac{3AG}{2G+A}, \tag{2.3}$$

$$L > \sqrt[3]{G^2A} > \frac{3AG}{2A+G}. \tag{2.4}$$

Proof. The first inequality of (2.3) is proved in [6], while the first inequality of (2.8) is a well-known inequality due to Leach and Sholander [8] (see [4] for many related references). The second inequalities of (2.3) and (2.4) are immediate consequences of the arithmetic-geometric inequality applied for A, A, G and A, G, G, respectively. □

Remark 2.3. By (2.3) and (1.6), we can deduce the following improvement of the Huygens' inequality (1.1):

$$\frac{\sin x}{x} > \sqrt[3]{\cos x} > \frac{3\cos x}{2\cos x + 1}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{2.5}$$

From (2.1) and (1.6), we get

$$\frac{\sin x}{x} > \frac{2L^*}{L^*+1} > \frac{3\cos x}{2\cos x + 1}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{2.5$'$}$$

Similarly, by (2.4) and (1.9), we get

$$\frac{\sinh x}{x} > \sqrt[3]{\cosh x} > \frac{3\cosh x}{2\cosh x + 1}, \quad x > 0. \tag{2.6}$$

From (2.2) and (1.9), we get

$$\frac{\sinh x}{x} > \frac{2P^*}{P^*+1} > \frac{3\cosh x}{2\cosh x + 1}, \quad x > 0. \tag{2.6$'$}$$

Here, $L^* = L(1+\sin x, 1-\sin x)$, $P^* = P(e^x, e^{-x})$.

We note that the first inequality of (2.5) has been discovered by Adamović and Mitrinović (see [3]), while the first inequality of (2.6) by Lazarević (see [3]).

Now, we will prove that inequalities (2.2) of Theorem 2.1 and (2.4) of Theorem 2.2 may be compared in the following way.

Theorem 2.4. *One has*

$$L > \sqrt[3]{G^2 A} > H(P, G) > \frac{3AG}{2A + G}. \tag{2.7}$$

Proof. We must prove the second inequality of (2.7). For this purpose, we will use the inequality (see [6]):

$$P < \frac{2A + G}{3}. \tag{2.8}$$

This implies $G/P > 3G/(G + 2A)$, so $(1/2)(1 + G/P) > (2G + A)/(G + 2A)$.

Now, we will prove that

$$\frac{2G + A}{G + 2A} > \sqrt[3]{\frac{G}{A}}. \tag{2.9}$$

By letting $x = G/A \in (0, 1)$, inequality (2.9) becomes

$$\frac{2x + 1}{x + 2} > \sqrt[3]{x}. \tag{2.10}$$

Put $x = a^3$, where $a \in (0, 1)$. After elementary transformations, inequality (2.10) becomes $(a + 1)(a - 1)^3 < 0$, which is true. □

Note. The Referee suggested the following alternative proof: since $P < (2A + G)/3$ and the harmonic mean increases in both variables, it suffices to prove stronger inequality $\sqrt[3]{A^2G} > H((2A + G)/3, G)$ which can be written as (2.9).

Remark 2.5. The following refinement of inequalities $(2.6')$ is true:

$$\frac{\sinh x}{x} > \sqrt[3]{\cosh x} > \frac{2P^*}{P^* + 1} > \frac{3\cosh x}{2\cosh x + 1}, \quad x > 0. \tag{2.11}$$

Unfortunately, a similar refinement to (2.7) for the mean P is not possible, as by numerical examples one can deduce that generally $H(L, A)$ and $\sqrt[3]{A^2G}$ are not comparable. However, in a particular case, the following result holds true.

Theorem 2.6. *Assume that $A/G \geq 4$. Then one has*

$$P > H(L, A) > \sqrt[3]{A^2G} > \frac{3AG}{2G + A}. \tag{2.12}$$

First, prove one the following auxiliary results.

Lemma 2.7. *For any $x \geq 4$, one has*

$$\sqrt[3]{(x+1)^2}\left(2\sqrt[3]{x}-1\right) > x\sqrt[3]{4}. \tag{2.13}$$

Proof. A computer computation shows that (2.13) is true for $x = 4$. Now put $x = a^3$ in (2.13). By taking logarithms, the inequality becomes

$$f(a) = 2\ln\left(\frac{a^3+1}{2}\right) - 9\ln a + 3\ln(2a-1) > 0. \tag{2.14}$$

An easy computation implies

$$a(2a-1)\left(a^3+1\right)f'(a) = 3(a-1)\left(a^2+a-3\right). \tag{2.15}$$

As $\sqrt[3]{4^2} + \sqrt[3]{4} - 3 = 2\sqrt[3]{2} + (\sqrt[3]{2})^2 - 3 = (\sqrt[3]{2}-1)(\sqrt[3]{2}+3) > 0$, we get that $f'(a) > 0$ for $a \geq \sqrt[3]{4}$. This means that $f(a) > f(\sqrt[3]{4}) > 0$, as the inequality is true for $a = \sqrt[3]{4}$. □

Proof of the theorem. We will apply the inequality:

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2}, \tag{2.16}$$

due to the author [9]. This implies

$$\frac{1}{2}\left(1+\frac{A}{L}\right) < \frac{1}{2}\left(1+\sqrt[3]{\frac{4A^3}{G(A+G)^2}}\right) = N. \tag{2.17}$$

By letting $x = A/G$ in (2.13), we can deduce

$$N < \sqrt[3]{\frac{A}{G}}. \tag{2.18}$$

So

$$\frac{1}{2}\left(1+\frac{A}{L}\right) < \sqrt[3]{\frac{A}{G}}. \tag{2.19}$$

This immediately gives $H(L, A) > \sqrt[3]{A^2G}$. □

Remark 2.8. If $\cos x \leq 1/4$, $x \in (0, \pi/2)$, then

$$\frac{\sin x}{x} > \frac{2L^*}{L^*+1} > \sqrt[3]{\cos x} > \frac{3\cos x}{2\cos x + 1}, \tag{2.20}$$

which is a refinement, in this case, of inequality $(2.5')$.

3. Further Improvements

Theorem 3.1. *One has*

$$P > \sqrt{LA} > \sqrt[3]{A^2G} > \frac{AG}{L} > \frac{3AG}{2G+A}, \tag{3.1}$$

$$L > \sqrt{GP} > \sqrt[3]{G^2A} > \frac{AG}{P} > \frac{3AG}{2A+G}. \tag{3.2}$$

Proof. The inequalities $P > \sqrt{LA}$ and $L > \sqrt{GP}$ are proved in [10]. We will see, that further refinements of these inequalities are true. Now, the second inequality of (3.1) follows by the first inequality of (2.3), while the second inequality of (3.2) follows by the first inequality of (2.4). The last inequality is in fact an inequality by Carlson [11]. For the inequalities on AG/P, we use (2.3) and (2.8). □

Remark 3.2. One has

$$\frac{\sin x}{x} > \sqrt{L^*} > \sqrt[3]{\cos x} > \frac{\cos x}{L^*} > \frac{3\cos x}{2\cos x + 1}, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{3.3}$$

$$\frac{\sinh x}{x} > \sqrt{P^*} > \sqrt[3]{\cosh x} > \frac{\cosh x}{P^*} > \frac{3\cosh x}{2\cosh x + 1}, \quad x > 0, \tag{3.4}$$

where L^* and P^* are the same as in $(2.6')$ and $(2.5')$.

Theorem 3.3. *One has*

$$P > \sqrt{LA} > H(A,L) > \frac{AL}{I} > \frac{AG}{L} > \frac{3AG}{2G+A}, \tag{3.5}$$

$$L > L \cdot \frac{I-G}{A-L} > \sqrt{IG} > \sqrt{PG} > \sqrt[3]{G^2A} > \frac{3AG}{2A+G}. \tag{3.6}$$

Proof. The first two inequalities of (3.5) one followed by the first inequality of (3.1) and the fact that $G(x,y) > H(x,y)$ with $x = L$, $y = A$.

Now, the inequality $H(A,L) > AL/I$ may be written also as

$$I > \frac{A+L}{2}, \tag{3.7}$$

which has been proved in [4] (see also [12]).

Further, by Alzer's inequality $L^2 > GI$ (see [13]) one has

$$\frac{L}{I} > \frac{G}{L} \tag{3.8}$$

and by Carlson's inequality $L < (2G+A)/3$ (see [11]), we get

$$\frac{AL}{I} > \frac{AG}{L} > \frac{3AG}{2G+A}, \tag{3.9}$$

so (3.5) is proved.

The first two inequalities of (3.6) have been proved by the author in [5]. Since $I > P$ (see [14]) and by (3.2), inequalities (3.6) are completely proved. □

Remark 3.4. One has the following inequalities:

$$\frac{\sin x}{x} > \sqrt{L^*} > \frac{2L^*}{L^*+1} > \frac{L^*}{I^*} > \frac{\cos x}{L^*} > \frac{3\cos x}{2\cos x+1}, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{3.10}$$

where $I^* = I(1+\sin x, 1-\sin x)$;

$$\frac{\sinh x}{x} > \frac{\sinh x}{x}\left(\frac{e^{x\coth x-1}-1}{\cosh x - \sinh x/x}\right) > e^{(x\coth x-1)/2} > \sqrt{P^*} > \sqrt[3]{\cosh x} > \frac{3\cosh x}{2\cosh x+1}. \tag{3.11}$$

Theorem 3.5. *One has*

$$P > \sqrt[3]{A\left(\frac{A+G}{2}\right)^2} > \sqrt{A\left(\frac{A+2G}{3}\right)} > \sqrt{AL} > H(A,L) > \frac{AL}{I} > \frac{3AG}{2G+A}, \tag{3.12}$$

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2} > \sqrt{IG} > \sqrt{G\left(\frac{2A+G}{3}\right)} > \sqrt{PG} > \sqrt[3]{G^2A} > \frac{3AG}{2A+G}. \tag{3.13}$$

Proof. In (3.12), we have to prove the first three inequalities, the rest are contained in (3.5).

The first inequality of (3.12) is proved in [6]. For the second inequality, put $A/G = t > 1$ By taking logarithms, we have to prove that

$$g(t) = 4\ln\left(\frac{t+1}{2}\right) - 3\ln\left(\frac{t+2}{3}\right) - \ln t > 0. \tag{3.14}$$

As $g'(t)t(t+1)(t+2) = 2(t-1) > 0$, $g(t)$ is strictly increasing, so

$$g(t) > g(1) = 0. \tag{3.15}$$

The third inequality of (3.12) follows by Carlson's relation $L < (2G+A)/3$ (see [11]).

The first inequality of (3.13) is proved in [9], while the second one in [15]. The third inequality follows by $I > (2A+G)/3$ (see [12]), while the fourth one by relation (2.9). The fifth one is followed by (2.3). □

Remark 3.6. The first three inequalities of (3.12) offer a strong improvement of the first inequality of (3.1); the same is true for (3.13) and (3.2).

4. New Huygens Type Inequalities

The main result of this section is contained in the following:

Theorem 4.1. *One has*

$$P > \sqrt[3]{A\left(\frac{A+G}{2}\right)^2} > \frac{3A(A+G)}{5A+G} > \frac{A(2G+A)}{2A+G} > \frac{3AG}{2G+A}, \tag{4.1}$$

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2} > \frac{3G(A+G)}{5G+A} > \frac{G(2A+G)}{2G+A} > \frac{3AG}{2A+G}. \tag{4.2}$$

Proof. The first inequalities of (4.1), respectively, (4.2) are the first ones in relations (3.12), respectively, (3.13).

Now, apply the geometric mean-harmonic mean inequality:

$$\sqrt[3]{xy^2} = \sqrt[3]{x \cdot y \cdot y} > \frac{3}{\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{y}\right)} = \frac{3}{\left(\dfrac{1}{x} + \dfrac{2}{y}\right)}, \tag{4.3}$$

for $x = A$, $y = (A+G)/2$ in order to deduce the second inequality of (4.1). The last two inequalities become, after certain transformation,

$$(A-G)^2 > 0. \tag{4.4}$$

□

The proof of (4.2) follows on the same lines, and we omit the details.

Theorem 4.2. *For all* $x \in \left(0, \dfrac{\pi}{2}\right)$, *one has*

$$\sin x + 4\tan\frac{x}{2} > 3x. \tag{4.5}$$

For all $x > 0$, *one has*

$$\sinh x + 4\tanh\frac{x}{2} > 3x. \tag{4.6}$$

Proof. Apply (1.6) for $P > (3A(A+G))/(5A+G)$ of (4.1).

As $\cos x + 1 = 2\cos^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$, we get inequality (4.5). A similar argument applied to (4.6), by an application of (4.2) and the formulae $\cosh x + 1 = 2\cosh^2(x/2)$ and $\sinh x = 2\sinh(x/2)\cosh(x/2)$. □

Remarks 4.3. By (4.1), inequality (4.5) is a refinement of the classical Huygens inequality (1.1):

$$2\sin x + \tan x > \sin x + 4\tan\frac{x}{2} > 3x. \tag{4.3$'$}$$

Similarly, (4.6) is a refinement of the hyperbolic Huygens inequality (1.2):

$$2\sinh x + \tanh x > \sinh x + 4\tanh\frac{x}{2} > 3x. \tag{4.4'}$$

We will call (4.5) as the second Huygens inequality, while (4.6) as the second hyperbolic Huygens inequality.

In fact, by (4.1) and (4.2) refinements of these inequalities may be stated, too.

The inequality $P > A(2G + A)/(2A + G)$ gives

$$\frac{\sin x}{x} > \frac{2\cos x + 1}{\cos x + 2}, \tag{4.7}$$

or written equivalently:

$$\frac{\sin x}{x} + \frac{3}{\cos x + 2} > 2, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{4.8}$$

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Research Article

A Nice Separation of Some Seiffert-Type Means by Power Means

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Seiffert has defined two well-known trigonometric means denoted by $\mathcal{P}$ and $\mathcal{T}$. In a similar way it was defined by Carlson the logarithmic mean $\mathcal{L}$ as a hyperbolic mean. Neuman and Sándor completed the list of such means by another hyperbolic mean $\mathcal{M}$. There are more known inequalities between the means $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{L}$ and some power means $\mathcal{A}_p$. We add to these inequalities two new results obtaining the following nice chain of inequalities $\mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/2} < \mathcal{P} < \mathcal{A}_1 < \mathcal{M} < \mathcal{A}_{3/2} < \mathcal{T} < \mathcal{A}_2$, where the power means are evenly spaced with respect to their order.

1. Means

A *mean* is a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, with the property

$$\min(a,b) \le M(a,b) \le \max(a,b), \quad \forall a,b > 0. \tag{1.1}$$

Each mean is *reflexive*; that is,

$$M(a,a) = a, \quad \forall a > 0. \tag{1.2}$$

This is also used as the definition of $M(a,a)$.

We will refer here to the following means:

(i) the power means $\mathcal{A}_p$, defined by

$$\mathcal{A}_p(a,b) = \left[\frac{a^p + b^p}{2}\right]^{1/p}, \quad p \ne 0; \tag{1.3}$$

(ii) the geometric mean $\mathcal{G}$, defined as $\mathcal{G}(a,b) = \sqrt{ab}$, but verifying also the property

$$\lim_{p \to 0} \mathcal{A}_p(a,b) = \mathcal{A}_0(a,b) = \mathcal{G}(a,b); \tag{1.4}$$

(iii) the first Seiffert mean $\mathcal{P}$, defined in [1] by

$$\mathcal{P}(a,b) = \frac{a-b}{2\sin^{-1}((a-b)/(a+b))}, \quad a \neq b; \tag{1.5}$$

(iv) the second Seiffert mean $\mathcal{T}$, defined in [2] by

$$\mathcal{T}(a,b) = \frac{a-b}{2\tan^{-1}((a-b)/(a+b))}, \quad a \neq b; \tag{1.6}$$

(v) the Neuman-Sándor mean $\mathcal{M}$, defined in [3] by

$$\mathcal{M}(a,b) = \frac{a-b}{2\sinh^{-1}((a-b)/(a+b))}, \quad a \neq b; \tag{1.7}$$

(vi) the Stolarsky means $\mathcal{S}_{p,q}$ defined in [4] as follows:

$$\mathcal{S}_{p,q}(a,b) = \begin{cases} \left[\dfrac{q(a^p - b^p)}{p(a^q - b^q)}\right]^{1/(p-q)}, & pq(p-q) \neq 0 \\ \dfrac{1}{e^p}\left(\dfrac{a^{a^p}}{b^{b^p}}\right)^{1/(a^p - b^p)}, & p = q \neq 0 \\ \left[\dfrac{a^p - b^p}{p(\ln a - \ln b)}\right]^{1/p}, & p \neq 0,\ q = 0 \\ \sqrt{ab}, & p = q = 0. \end{cases} \tag{1.8}$$

The mean $\mathcal{A}_1 = \mathcal{A}$ is the arithmetic mean and the mean $\mathcal{S}_{1,0} = \mathcal{L}$ is the logarithmic mean. As Carlson remarked in [5], the logarithmic mean can be represented also by

$$\mathcal{L}(a,b) = \frac{a-b}{2\tanh^{-1}((a-b)/(a+b))}; \tag{1.9}$$

thus the means $\mathcal{P}, \mathcal{T}, \mathcal{M}$, and $\mathcal{L}$ are very similar. In [3] it is also proven that these means can be defined using the nonsymmetric Schwab-Borchardt mean $\mathcal{SB}$ given by

$$\mathcal{SB}(a,b) = \begin{cases} \dfrac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)}, & \text{if } a < b \\[2ex] \dfrac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)}, & \text{if } a > b \end{cases} \tag{1.10}$$

(see [6, 7]). It has been established in [3] that

$$\mathcal{L} = \mathcal{SB}(\mathcal{A}, \mathcal{G}), \qquad \mathcal{P} = \mathcal{SB}(\mathcal{G}, \mathcal{A}), \qquad \mathcal{T} = \mathcal{SB}(\mathcal{A}, \mathcal{A}_2), \qquad \mathcal{M} = \mathcal{SB}(\mathcal{A}_2, \mathcal{A}). \tag{1.11}$$

2. Interlacing Property of Power Means

Given two means M and N, we will write $M < N$ if

$$M(a,b) < N(a,b), \quad \text{for } a \neq b. \tag{2.1}$$

It is known that the family of power means is an increasing family of means, thus

$$\mathcal{A}_p < \mathcal{A}_q, \quad \text{if } p < q. \tag{2.2}$$

Of course, it is more difficult to compare two Stolarsky means, each depending on two parameters. To present the comparison theorem given in [8, 9], we have to give the definitions of the following two auxiliary functions:

$$\begin{gathered} k(x,y) = \begin{cases} \dfrac{|x| - |y|}{x - y}, & x \neq y \\ \operatorname{sign}(x), & x = y, \end{cases} \\ l(x,y) = \begin{cases} \mathcal{L}(x,y), & x > 0,\ y > 0 \\ 0, & x \geq 0,\ y \geq 0,\ xy = 0. \end{cases} \end{gathered} \tag{2.3}$$

Theorem 2.1. *Let $p, q, r, s \in \mathbb{R}$. Then the comparison inequality*

$$\mathcal{S}_{p,q} \leq \mathcal{S}_{r,s} \tag{2.4}$$

holds true if and only if $p+q \leq r+s$, and (1) $l(p,q) \leq l(r,s)$ if $0 \leq \min(p,q,r,s)$, (2) $k(p,q) \leq k(r,s)$ if $\min(p,q,r,s) < 0 < \max(p,q,r,s)$, or (3) $-l(-p,-q) \leq -l(-r,-s)$ if $\max(p,q,r,s) \leq 0$.

We need also in what follows an important double-sided inequality proved in [3] for the Schwab-Borchardt mean:

$$\sqrt[3]{ab^2} < \mathcal{SB}(a,b) < \frac{a+2b}{3}, \quad a \neq b. \tag{2.5}$$

Being rather complicated, the Seiffert-type means were evaluated by simpler means, first of all by power means. The *evaluation* of a given mean M by power means assumes the determination of some real indices p and q such that $\mathcal{A}_p < M < \mathcal{A}_q$. The evaluation is *optimal* if p is the the greatest and q is the smallest index with this property. This means that M cannot be compared with $\mathcal{A}_r$ if $p < r < q$.

For the logarithmic mean in [10], it was determined the optimal evaluation

$$\mathcal{A}_0 < L < \mathcal{A}_{1/3}. \tag{2.6}$$

For the Seiffert means, there are known the evaluations

$$\mathcal{A}_{1/3} < P < \mathcal{A}_{2/3}, \tag{2.7}$$

proved in [11] and

$$\mathcal{A}_1 < T < \mathcal{A}_2, \tag{2.8}$$

given in [2]. It is also known that

$$\mathcal{A}_1 < M < \mathcal{T}, \tag{2.9}$$

as it was shown in [3]. Moreover in [12] it was determined the optimal evaluation

$$\mathcal{A}_{\ln 2/\ln \pi} < P < \mathcal{A}_{2/3}. \tag{2.10}$$

Using these results we deduce the following chain of inequalities:

$$\mathcal{A}_0 < L < \mathcal{A}_{1/2} < P < \mathcal{A}_1 < M < \mathcal{T} < \mathcal{A}_2. \tag{2.11}$$

To prove the full interlacing property of power means, our aim is to show that $\mathcal{A}_{3/2}$ can be put between $\mathcal{M}$ and $\mathcal{T}$. We thus obtain a nice separation of these Seiffert-type means by power means which are evenly spaced with respect to their order.

3. Main Results

We add to the inequalities (2.11) the next results.

Theorem 3.1. *The following inequalities*

$$\mathcal{M} < \mathcal{A}_{3/2} < T \tag{3.1}$$

are satisfied.

Proof. First of all, let us remark that $\mathcal{A}_{3/2} = \mathcal{S}_{3,3/2}$. So, for the first inequality in (3.1), it is sufficient to prove that the following chain of inequalities

$$\mathcal{M} < \frac{\mathcal{A}_2 + 2\mathcal{A}}{3} < \mathcal{S}_{3,1} < \mathcal{S}_{3,3/2} \tag{3.2}$$

is valid. The first inequality in (3.2) is a simple consequence of the property of the mean $\mathcal{M}$ given in (1.11) and the second inequality from (2.5). The second inequality can be proved by direct computation or by taking $a = 1 + t,\ b = 1 - t,\ (0 < t < 1)$ which gives

$$\frac{\sqrt{1+t^2}+2}{3} < \sqrt{\frac{3+t^2}{3}}, \tag{3.3}$$

which is easy to prove. The last inequality in (3.2) is given by the comparison theorem of the Stolarsky means. In a similar way, the second inequality in (3.1) is given by the relations

$$\mathcal{S}_{3,3/2} < \mathcal{S}_{4,1} = \sqrt[3]{\mathcal{A}\mathcal{A}_2^2} < T. \tag{3.4}$$

The first inequality is again given by the comparison theorem of the Stolarsky means. The equality in (3.4) is shown by elementary computations, and the last inequality is a simple consequence of the property of the mean $\mathcal{T}$ given in (1.11) and the first inequality from (2.5). □

Corollary 3.2. *The following two-sided inequality*

$$\frac{x}{\sinh^{-1}x} < \mathcal{A}_{3/2}(1-x, 1+x) < \frac{x}{\tan^{-1}x}, \tag{3.5}$$

is valid for all $0 < x < 1$.

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