# Dynamics of Delay Differential Equations with Their Applications 

Guest Editors: Chuanqxia Huang, Zhiming Guo, Zhichun Yang, and Yuming Chen


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## Abstract and Applied Analysis

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## Editorial

# Dynamics of Delay Differential Equations with Their Applications 

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Delay differential equations have attracted a rapidly growing attention in the field of nonlinear dynamics and have become a powerful tool for investigating the complexities of the real-world problems such as infectious diseases, biotic population, neuronal networks, and even economics and finance. When employing delay differential equations to solve practical problems, it is very crucial to be able to completely characterize the dynamical properties of the delay differential equations. In spite of the amount of published results recently focused on such systems, there remain many challenging open questions. The basic purpose of this special issue is to extend the applications of the relatively new approaches and theories for delay differential equations and to see the latest developments. The authors were invited to submit original research articles as well as review articles that stimulated the continuing efforts in delay differential equations and related theories. The topics included in this special issue are invariant sets and attractor; boundedness analysis; stability and bifurcation analysis; asymptotic analysis and synchronization; the existence and uniqueness or nonexistence of equilibrium point, periodic solutions, and almost periodic solutions; impulsive and stochastic control; and modeling and simulation analysis.

The response to this special issue on dynamics of delay differential equations with their Applications was beyond our expectation. We received 49 papers in the interdisciplinary research fields. This special issue includes twenty-six highquality peer-reviewed articles. These articles contain several
new, novel, and innovative techniques and ideas that may stimulate further research in every branch of pure and applied sciences.

## Acknowledgments

The authors would like to express their deepest gratitude to the reviewers, whose professional comments and valuable suggestions guaranteed the high quality of these selected papers. The editors would like to express their gratitude to the authors for their interesting and novel contributions. The interested readers are advised to explore these interesting and fascinating fields further. The authors hope that problems discussed and investigated in this special issue may inspire and motivate discovering new, innovative, and novel applications in all areas of delay differential equations.

Chuangxia Huang
Zhiming Guo
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## Research Article

# Positive Stability Analysis and Bio-Circuit Design for Nonlinear Biochemical Networks 

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#### Abstract

This paper is concerned with positive stability analysis and bio-circuits design for nonlinear biochemical networks. A fuzzy interpolation approach is employed to approximate nonlinear biochemical networks. Based on the Lyapunov stability theory, sufficient conditions are developed to guarantee the equilibrium points of nonlinear biochemical networks to be positive and asymptotically stable. In addition, a constrained bio-circuits design with positive control input is also considered. It is shown that the conditions can be formulated as a solution to a convex optimization problem, which can be easily facilitated by using the Matlab LMI control toolbox. Finally, a real biochemical network model is provided to illustrate the effectiveness and validity of the obtained results.


## 1. Introduction

In the past decades, biochemical networks, such as metabolic networks [1] and genetic networks [2], have received considerable attention and become a hot research topic [3-5]. A great number of results have been obtained, such as gene expression data modeling [6-9] and dynamic analysis of biochemical networks [ 10,11 ].

It is not surprising that dynamical system theory plays a central role in understanding biological and physiological processes $[4,12]$ since it provides a powerful tool to quantitatively analyze these biochemical networks from a systematic viewpoint. In addition, dynamical system theory is important and useful for the development of synthetic and systems biology, which has a great potential in gene therapy and drug design [13].

It is worth noting that dynamical models of many biological and physiological processes, such as metabolic systems and endocrine systems [1], biochemical reactions [12], are derived from mass and energy balance considerations that take nonnegative chemical concentrations as dynamic states. Hence, state trajectories of such biochemical systems remain in the nonnegative orthant of the state space for arbitrary nonnegative initial conditions. Such systems are commonly referred to as nonnegative systems or positive systems in the literature [14-17]. In this paper, we call them positive systems
for convenience. A subclass of positive dynamical systems are compartmental systems [18], which involve dynamical models that are characterized by conservation laws (e.g., mass and energy) capturing exchange of materials between coupled macroscopic subsystems known as compartments. Each compartment is assumed to be kinetically homogeneous. That is, any material entering the compartment is instantaneously mixed with materials of the compartment. There have been some studies on this kind of systems with application to biochemical networks [4, 19].

Recently, there have been some results on stability analysis of gene networks with some special regulation functions [20-25]. However, due to nonlinearity and complexity of biochemical networks, there does not exist a systematic approach to stability analysis of such nonlinear biochemical networks. It is noted that the fuzzy interpolation approach proposed in [26] can be seen as a promising way of dealing with nonlinear complex systems. In [27], the authors firstly used the fuzzy approximation method to investigate the robust stability of stochastic biochemical regulatory networks, where, however, the positive constraint of network states was ignored.

On the other hand, there have been some other results on bio-circuits design for biochemical networks. In [28], an external optimal control input was applied to stabilize a gene regulatory system. In [29], a simple robust circuit has been designed for the S-system model without considering
stochastic noises. In [30], a robust engineering principle was proposed for stochastic biochemical regulatory networks with parameter uncertainties and disturbances. However, most of the aforementioned results paid little attention to the positive constraint of the states of biochemical networks, not to mention constrained bio-circuits with positive control input. In many cases, the states of those studied systems would become negative in simulation, which could be seen as the main drawbacks of these results.

In this paper, the T-S fuzzy system is employed to approximate nonlinear biochemical networks by interpolating several local linear systems. A positive stability condition and a bio-circuits design procedure will be developed for the nonlinear biochemical networks. In addition, a constrained bio-circuits design with positive control input will also be considered. Finally, a real biochemical network is given to illustrate the effectiveness of the obtained results.

The rest of the paper is organized as follows. In Section 2, some useful definitions and lemmas for positive systems are introduced. In Section 3, by using the fuzzy approximation approach, a sufficient condition for positive stability will be derived for nonlinear biochemical networks. In Section 4, bio-circuits design will be developed from a systematic point of view. In addition, a constrained bio-circuits design with positive control input will also be considered. In Section 5, a real biochemical network is given to illustrate the effectiveness of the obtained results. Finally, the paper will be closed with a conclusion.

## 2. Notation and Preliminaries

Notation. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^{n}$ stands for the vector space of all $n$-tuples of real numbers, and $\mathbb{R}^{n \times m}$ is the space of $n \times m$ matrices with real entries. For $x$ in $\mathbb{R}^{n}, x_{i}$ denotes the $i$ th component of $x . \mathbb{R}_{+}^{n \times m}$ denotes the sets of all $n \times m$ real matrices with nonnegative entries and $\mathbb{R}_{+}^{n} \triangleq\{x \in$ $\left.\mathbb{R}^{n}: x \succeq 0\right\}$. For a real matrix $A, A \succeq 0(\succ 0)$ means that all its entries are nonnegative (positive). $A^{\mathrm{T}}$ is the transpose of $A$, and $A^{-1}$ is the inverse of $A$. The notation $P>0$ means that $P$ is symmetric and positive definite. The following notations of matrices are used throughout this paper: $A_{i}=\left[a_{k j}^{i}\right], B_{i}=$ $\left[b_{1}^{i} ; b_{2}^{i} ; \ldots ; b_{n}^{i}\right]$.

Consider a general nonlinear system

$$
\begin{gather*}
\frac{d x}{d t}=f(x(t), u(t))  \tag{1}\\
x(0)=x_{0}
\end{gather*}
$$

where $x(t) \in \mathbb{R}^{n}$ is the state vector and $u(t) \in \mathbb{R}^{m}$ is the control input. The nonlinear function $f(x(t), u(t))$ satisfies $f(0,0)=0$ and $f \in \mathscr{C}^{2}$; that is, $f$ has the second-order continuous derivative with respect to $x$ and $u$.

Firstly, some definitions and useful lemmas for positive nonlinear systems are given as follows.

Definition 1. Given any positive initial condition $x(0)=x_{0} \in$ $\mathbb{R}_{+}^{n}$, the unforced nonlinear system (1) is said to be positive if the corresponding trajectory $x(t) \in \mathbb{R}_{+}^{n}$ for all $t \geq 0$.

Definition 2. Let $f=\left[f_{1}, f_{2}, \ldots, f_{n}\right]^{\mathrm{T}}: \mathscr{D} \subseteq \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}^{n}$ the nonlinear function f is positive (or essentially nonnegative in [19]) if $f_{i}(x(t)) \geq 0$, for all $i=1,2, \ldots, n$ and $x(t) \in \mathscr{D} \subseteq \mathbb{R}_{+}^{n}$.

Lemma 3 (see [19]). Consider the unforced nonlinear system (1). If $f$ is positive and continuously differentiable in $\mathscr{D} \subseteq \mathbb{R}_{+}^{n}$ and $f(0)=0$, then $A \triangleq \partial f /\left.\partial x\right|_{x=0}$ is positive (essentially nonnegative).

Lemma 3 implies that if a nonlinear system is positive, then its linearization is also positive.

Theorem 4. For the unforced system (1), if the nonlinear function $f$ is positive, and if there exists a Lyapunov function $V(x(t))>0$ and $V(0)=0$ satisfying the following inequality:

$$
\begin{equation*}
\left(\frac{\partial V(x(t))}{\partial x}\right)^{\mathrm{T}} f(x(t))<0 \tag{2}
\end{equation*}
$$

for all nonzero $x(t) \in \mathbb{R}_{+}^{n}$, then the equilibrium point $x(t)=0$ of the nonlinear system (1) is asymptotically stable.

As a special case of the nonlinear system (1), the following linear system is considered

$$
\begin{gather*}
\frac{d x}{d t}=A x(t)+B u(t)  \tag{3}\\
x(0)=x_{0}
\end{gather*}
$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Some useful results from [14] are presented as follows.

Definition 5. A real matrix $M$ is called a Metzler matrix if its off-diagonal elements are nonnegative, that is

$$
\begin{equation*}
M_{i j} \geq 0, \quad i \neq j \tag{4}
\end{equation*}
$$

Lemma 6. The unforced linear system (3) is positive if and only if $A$ is a Metzler matrix.

Lemma 7. The unforced positive linear system (3) is asymptotically stable if and only if there exists a positive definite diagonal matrix $P$ such that

$$
\begin{equation*}
P A+A^{\mathrm{T}} P<0 \tag{5}
\end{equation*}
$$

Remark 8. It follows from the physical consideration that all the states of biochemical networks should stay in the positive orthant. Thus, positive systems are suitable for quantitatively studying biochemical networks, such as gene microarray data modeling [6-8] and bio-circuits design for biochemical networks [3, 4].

## 3. Positive Stability Analysis of Nonlinear Biochemical Networks

As pointed out in Introduction, many applications in biochemical processes give rise to nonlinear dynamical systems, such as genetic networks, metabolic pathways and membrane transports, to cite just a few examples. Consider the nonlinear
system (1) for representation of biochemical networks, which describes complex interactions between molecules. It is noted that the states of the system denote the concentrations of the molecules, the nonlinear function $f$ is the regulation function and the control input $u(t)$ can be seen as the external sources, such as drugs, proteins, or other chemical complexes.

For a general nonlinear biochemical network (1), it is very difficult if not impossible to find a suitable Lyapunov function $V$ such that the condition (2) is satisfied, especially when the positive constraint of the states should be also maintained. However, the T-S fuzzy interpolation approach provides a way to approximate the nonlinear biochemical network and, thus, potentially provides a simplified method for positive stability analysis and bio-circuits design.

Consider a nonlinear biochemical network described by a T-S fuzzy system

$$
\begin{align*}
& R^{i}: \text { IF } z_{1}(t) \text { is } M_{1}^{i} \text { and } \cdots z_{r}(t) \text { is } M_{r}^{i} \text {, THEN } \\
& \qquad \begin{array}{c}
x(t)=A_{i} x(t)+B_{i} u(t), \\
y(t)=C_{i} x(t),
\end{array} \tag{6}
\end{align*}
$$

where $i=1,2, \ldots, L$ and $L$ is the number of fuzzy rules; $z_{1}(t), z_{2}(t), \ldots, z_{r}(t)$ are the premise variables and $M_{l}^{i}(i=$ $1,2, \ldots, L, l=1,2, \ldots, r)$ are the fuzzy sets; $u(t)$ and $y(t)$ are the control input and output, respectively; $A_{i}, B_{i}$, and $C_{i}$ are the known matrices of appropriate dimensions.

By using a center average defuzzifier, product inference, and a singleton fuzzifier, the global dynamics of the T-S fuzzy system (6) cab be described by

$$
\begin{gather*}
\dot{x}(t)=\sum_{i=1}^{L} \alpha_{i}(z(t))\left(A_{i} x(t)+B_{i} u(t)\right), \\
y(t)=\sum_{i=1}^{L} \alpha_{i}(z(t)) C_{i} x(t), \tag{7}
\end{gather*}
$$

where $\alpha_{i}(z(t))$ 's are the so-called normalized activation functions in relation to the $i$ th submodel such that

$$
\begin{equation*}
\alpha_{i}(z(t))=\frac{\prod_{l=1}^{r} M_{l}^{i}\left(z_{l}(t)\right)}{\sum_{i=1}^{L} \prod_{l=1}^{r} M_{l}^{i}\left(z_{l}(t)\right)}, \quad 0 \leq \alpha_{i}(z(t)) \leq 1 . \tag{8}
\end{equation*}
$$

Now, we are in the position to develop the global positive stability results for the unforced biochemical network (7); that is, $u=0$.

Theorem 9. If there exists a diagonal matrix $0<P \in \mathbb{R}^{n \times n}$ such that the following LMI conditions

$$
\begin{gather*}
A_{i}^{\mathrm{T}} P+P A_{i}<0,  \tag{9}\\
a_{k j}^{i} p_{j j} \geq 0, \quad k, j=1,2, \ldots, n, k \neq j \tag{10}
\end{gather*}
$$

hold for $i=1,2, \ldots, L$, then the unforced biochemical network (7) is positive and asymptotically stable.

Proof. Construct the following quadratic Lyapunov function candidate for the unforced biochemical network (7)

$$
\begin{equation*}
V(x(t))=x^{\mathrm{T}}(t) P x(t) \tag{11}
\end{equation*}
$$

where the diagonal matrix $P>0$ is to be determined.

Taking the derivative along the trajectory of (7), one can readily get

$$
\begin{align*}
\frac{d V(x(t))}{d t}= & \dot{x}^{\mathrm{T}}(t) P x(t)+x^{\mathrm{T}}(t) P \dot{x}(t) \\
= & {\left[\sum_{i=1}^{L} \alpha_{i}(z(t)) A_{i} x(t)\right]^{\mathrm{T}} P x(t) } \\
& +x^{\mathrm{T}}(t) P\left[\sum_{i=1}^{L} \alpha_{i}(z(t)) A_{i} x(t)\right]  \tag{12}\\
= & \sum_{i=1}^{L} \alpha_{i}(z(t)) x^{\mathrm{T}}(t)\left[A_{i}^{\mathrm{T}} P+P A_{i}\right] x(t)
\end{align*}
$$

Then, it follows immediately from condition (9) that

$$
\begin{equation*}
\frac{d V(x(t))}{d t}<0 \tag{13}
\end{equation*}
$$

for all nonzero $x(t)$, and $d V(x(t)) / d t=0$ if and only if $x(t)=0$. Hence, the unforced biochemical network (7) is asymptotically stable.

Furthermore, since $P=\operatorname{diag}\left(p_{11}, \ldots, p_{n n}\right)$ is a positive definite diagonal matrix, that is, $p_{i i}>0, i=1,2, \ldots, n$, it can be deduced from the LMI condition (10) that matrix $A_{i}$ is a Metzler matrix for every local linear model of (7). Moreover, it follows from condition (8) that the membership function satisfies

$$
\begin{equation*}
\sum_{i=1}^{L} \alpha_{i}(z(t))=1, \quad 0 \leq \alpha_{i} \leq 1 \tag{14}
\end{equation*}
$$

which together with the fact that $A_{i}$ is Metzler can guarantee the unforced biochemical network (7) to be positive. Hence, together with condition (13), it can be concluded that the equilibrium point of the unforced biochemical network (7) is positive and asymptotically stable. The proof is thus completed.

Remark 10. It is noted that the obtained sufficient condition for positive stability is in the form of linear matrix inequalities (LMIs), which could be efficiently solved by using the Matlab LMI control toolbox. Compared with the results developed in [20-25], the regulation function is no longer needed to satisfy a sector condition; thus, the regulation function considered here is more general.

## 4. Bio-Circuits Design for Nonlinear Biochemical Networks

If the equilibrium point of the unforced nonlinear biochemical network (7) is not stable, bio-circuits design would become necessary for these complex biological systems to work properly, which would be useful for drug design and gene therapy. In this paper, the following smooth controller is employed to stabilize the biochemical network (7):

$$
\begin{gather*}
R^{i}: \text { IF } z_{1}(t) \text { is } M_{1}^{i} \text { and } \cdots, z_{r}(t) \text { is } M_{r}^{i}, \text { THEN }  \tag{15}\\
\qquad u(t)=K_{i} x(t), \quad i \in\{1,2, \ldots, L\},
\end{gather*}
$$

which can be rewritten as

$$
\begin{equation*}
u(t)=\sum_{i=1}^{L} \alpha_{i}(z(t)) K_{i} x(t) \tag{16}
\end{equation*}
$$

where $K_{i}, i=1,2, \ldots, L$ are the feedback gains of the biocircuits to be determined.

The controlled biochemical network (7) can be described as follows:

$$
\begin{gather*}
\dot{x}(t)=\sum_{j=1}^{L} \sum_{i=1}^{L} \alpha_{j} \alpha_{i}\left(A_{i}+B_{i} K_{j}\right) x(t),  \tag{17}\\
y(t)=\sum_{i=1}^{L} \alpha_{i}(z(t)) C_{i} x(t)
\end{gather*}
$$

Then, the following bio-circuits design results would guarantee the positivity and asymptotic stability of the nonlinear biochemical network (17).

Theorem 11. There exists a smooth control scheme $u(t)=$ $\sum_{i=1}^{L} \alpha_{i}(z(t)) K_{i} x(t)$ such that the biochemical network (17) is positive and asymptotically stable, if there exist a diagonal matrix $0<P \in \mathbb{R}^{n \times n}$ and matrices $Q_{i}, i=1,2, \ldots, L$ such that the following LMI conditions

$$
\begin{gather*}
P A_{i}^{\mathrm{T}}+Q_{j} B_{i}^{\mathrm{T}}+A_{i} P+B_{i} Q_{j}<0,  \tag{18}\\
a_{k s}^{i} p_{s s}+\sum_{t=1}^{m} b_{k t}^{i} q_{t s}^{j} \geq 0, \quad k, s=1,2, \ldots, n, k \neq s, \tag{19}
\end{gather*}
$$

hold for $i, j=1,2, \ldots, L$.
Moreover, the feedback gains can be computed as

$$
\begin{equation*}
K_{j}=Q_{j} P^{-1}, \quad j=1,2, \ldots, L \tag{20}
\end{equation*}
$$

Proof. Construct the following quadratic Lyapunov function for the biochemical network (17)

$$
\begin{equation*}
V(x(t))=x^{\mathrm{T}}(t) P^{-1} x(t) \tag{21}
\end{equation*}
$$

where the diagonal matrix $P>0$ is to be determined.

Taking the derivative along the trajectory of system (17), one has

$$
\frac{d V(x(t))}{d t}=\dot{x}^{\mathrm{T}}(t) P^{-1} x(t)+x^{\mathrm{T}}(t) P^{-1} \dot{x}(t)
$$

$$
=\left[\sum_{j=1}^{L} \sum_{i=1}^{L} \alpha_{j} \alpha_{i}\left(A_{i}+B_{i} K_{j}\right) x(t)\right]^{\mathrm{T}} P^{-1} x(t)
$$

$$
+x^{\mathrm{T}}(t) P^{-1}\left[\sum_{j=1}^{L} \sum_{i=1}^{L} \alpha_{j} \alpha_{i}\left(A_{i}+B_{i} K_{j}\right) x(t)\right]
$$

$$
=\sum_{j=1}^{L} \sum_{i=1}^{L} \alpha_{j} \alpha_{i} x^{\mathrm{T}}(t)
$$

$$
\times\left[\left(A_{i}+B_{i} K_{j}\right)^{\mathrm{T}} P^{-1}+P^{-1}\left(A_{i}+B_{i} K_{j}\right)\right] x(t)
$$

$$
\begin{equation*}
i, j=1,2, \ldots, L \tag{22}
\end{equation*}
$$

It is noted that

$$
\begin{equation*}
\left(A_{i}+B_{i} K_{j}\right)^{\mathrm{T}} P^{-1}+P^{-1}\left(A_{i}+B_{i} K_{j}\right)<0, \quad i, j=1,2, \ldots, L \tag{23}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
P\left(A_{i}+B_{i} K_{j}\right)^{\mathrm{T}}+\left(A_{i}+B_{i} K_{j}\right) P<0, \quad i, j=1,2, \ldots, L \tag{24}
\end{equation*}
$$

Therefore, it follows immediately from conditions (18) and (20) that

$$
\begin{equation*}
\frac{d V(x(t))}{d t}<0 \tag{25}
\end{equation*}
$$

for all the nonzero $x(t)$, and $d V(x(t)) / d t=0$ if and only if $x(t)=0$. Hence, the controlled biochemical network (17) is asymptotically stable.

On the other hand, since $P$ is a positive definite diagonal matrix, that is, $p_{i i}>0, i=1,2, \ldots, n$, it follows from conditions (19) and (20) that the off-diagonal elements of matrix $\left(A_{i}+B_{i} K_{j}\right)$ are nonnegative; that is,

$$
\begin{align*}
a_{k s}^{i}+\frac{\sum_{t=1}^{m} b_{k t}^{i} q_{t s}^{j}}{p_{s s}} & =a_{k s}^{i}+\sum_{t=1}^{m} b_{k t}^{i} k_{t s}^{j} \geq 0 \\
k, s & =1,2, \ldots, n, \quad k \neq s, \quad i, j=1,2, \ldots, L \tag{26}
\end{align*}
$$

This implies that for any $i, j=1,2, \ldots, L, A_{i}+B_{i} K_{j}$ is a Metzler matrix for every linear local model. Similar to the proof of Theorem 9, from condition (14), it can be concluded that the controlled biochemical network (17) is positive. Therefore, together with condition (25), one can conclude that the controlled biochemical network (17) is positive and asymptotically stable. The proof is, thus, completed.

It is worth pointing out that in many practical applications, drugs or chemical complexes delivered to the human body are often taken as control inputs to biochemical systems, and in this case, control signals have to be nonnegative. Therefore, it is necessary to consider the positive constraints of control inputs when designing bio-circuits. We have the following results.

Theorem 12. There exists a positive smooth control scheme $u(t)=\sum_{i=1}^{L} \alpha_{i}(z(t)) K_{i} x(t)$, that is, $0 \preceq u(t)$, such that the biochemical network (17) is positive and asymptotically stable, if there exist a diagonal matrix $0<P \in \mathbb{R}^{n \times n}$ and matrices $Q_{i}, i=1,2, \ldots, L$ such that the following LMI conditions

$$
\begin{align*}
& \qquad P A_{i}^{\mathrm{T}}+Q_{j} B_{i}^{\mathrm{T}}+A_{i} P+B_{i} Q_{j}<0, \\
& a_{k s}^{i} p_{s s}+\sum_{t=1}^{m} b_{k t}^{i} q_{t s}^{j} \geq 0, \quad k, s=1,2, \ldots, n, k \neq s,  \tag{27}\\
& q_{t s}^{j} \geq 0, \quad t=1,2, \ldots, m, s=1,2, \ldots, n, \\
& \text { hold for } i, j=1,2, \ldots, L .
\end{align*}
$$

Moreover, the feedback gains can be computed as

$$
\begin{equation*}
K_{j}=Q_{j} P^{-1}, \quad j=1,2, \ldots, L \tag{28}
\end{equation*}
$$

Remark 13. It is noted that the positive feedback control law is only available in some special cases. When $A_{i}$ is a Metzler matrix and $B_{i} \succeq 0$, it is impossible to design a positive feedback bio-circuit.

Remark 14. In many practical applications, the concentrations of some materials should be rigorously kept below or above a certain level, otherwise they may have side effects on blood or other chemical complexes. In other words the states of biochemical networks should be subject to some kinds of constraints. How to deal with such scenarios is one of our future research interests.

## 5. Design Example In Silico for the Proposed Method

Consider a two-compartment model [31], which describes the kinetics of a drug in the human body. The drug is injected into the blood where it exchanges linearly with the tissues; the drug is irreversibly removed with a nonlinear saturative characteristic from the blood and with a linear one from the tissue. The model can be expressed by the following nonlinear differential equations:

$$
\begin{gather*}
\dot{x}_{1}=-\left(a_{11}+\frac{V_{M}}{V_{m}+x_{1}}\right) x_{1}+a_{12} x_{2}+b_{1} u, \\
\dot{x}_{2}=a_{21} x_{1}-\left(a_{02}+a_{22}\right) x_{2},  \tag{29}\\
y=c_{1} x_{1}
\end{gather*}
$$

where $x_{1}, x_{2}$ are the drug masses in blood and tissues, respectively; $u$ is the drug input $y$ is the measured drug
output in the blood; $a_{11}, a_{12}, a_{21}, a_{22}$, and $a_{02}$ are the constant rate parameters; $V_{M}$ and $V_{m}$ are the Michaelis-Menten parameters; $b_{1}$ and $c_{1}$ are the input and output parameters, respectively.

Let the premise variable $z_{1}(t)=x_{1}(t)$, then the membership functions can be chosen as

$$
\begin{equation*}
M_{1}^{1}\left(z_{1}(t)\right)=\frac{V_{m}}{V_{m}+x_{1}(t)}, \quad M_{1}^{2}\left(z_{1}(t)\right)=\frac{x_{1}(t)}{V_{m}+x_{1}(t)} . \tag{30}
\end{equation*}
$$

By using $M_{1}^{1}$ and $M_{1}^{2}$, the biochemical network (29) can be expressed by the following T-S fuzzy model

Plant Rule 1: IF $z_{1}(t)$ is $M_{1}^{1}$, THEN

$$
\begin{gather*}
\dot{x}(t)=A_{1} x(t)+B_{1} u(t), \\
y(t)=C_{1} x(t), \tag{31}
\end{gather*}
$$

Plant Rule 2: IF $z_{1}(t)$ is $M_{1}^{2}$, THEN

$$
\begin{gathered}
\dot{x}(t)=A_{2} x(t)+B_{2} u(t), \\
y(t)=C_{2} x(t),
\end{gathered}
$$

where $x(t)=\left[x_{1}(t), x_{2}(t)\right]^{\mathrm{T}}$ and

$$
\begin{gather*}
A_{1}=\left[\begin{array}{cc}
-\left(a_{11}+V_{M} / V_{m}\right) & a_{12} \\
a_{21} & -\left(a_{02}+a_{22}\right)
\end{array}\right] \\
B_{1}=\left[\begin{array}{c}
b_{1} \\
0
\end{array}\right], \quad C_{1}=\left[\begin{array}{ll}
c_{1} & 0
\end{array}\right]  \tag{32}\\
A_{2}=\left[\begin{array}{cc}
-a_{11} & a_{12} \\
a_{21} & -\left(a_{02}+a_{22}\right)
\end{array}\right] \\
B_{2}=\left[\begin{array}{c}
b_{1} \\
0
\end{array}\right], \quad C_{2}=\left[\begin{array}{ll}
c_{1} & 0
\end{array}\right]
\end{gather*}
$$

Let $a_{11}=0.1, a_{21}=2, a_{12}=0.5, a_{22}=0.1, a_{02}=$ $0.6, V_{M}=1, V_{m}=0.5$, and $b_{1}=-1$; Figure 1 shows that the state trajectories of biochemical network (29) are divergent under $u=0$ with the initial condition $x(0)=[0.1,4]^{\mathrm{T}}$. For convenience, we use the same initial conditions in the following simulations.
5.1. Bio-Circuits Design without Constraints. In this case study, we will consider the bio-circuits design for the biochemical network (31) without constraints. Solving the LMI conditions (18)-(19) leads to the feasible solutions as follows:

$$
\begin{gather*}
P=\left[\begin{array}{cc}
8.5971 & 0 \\
0 & 39.6566
\end{array}\right],  \tag{33}\\
Q_{1}=Q_{2}=\left[\begin{array}{lll}
34.1204 & -10.3198
\end{array}\right] .
\end{gather*}
$$

Then the feedback gains can be calculated as $K_{1}=K_{2}=$ [3.9688, -0.2602], which could guarantee the controlled biochemical network (31) to be positive and asymptotically stable. The dynamic response of the controlled biochemical network can be seen in Figure 2, and the evolution of the control input can be seen in Figure 3. It can be observed that the control input is negative in some stage.


Figure 1: Time response of the biochemical network under $u=0$.


Figure 2: Time response of the controlled biochemical network.
5.2. Bio-Circuits Design with Positive Control Constraints. In most situations, the control input $u(t)$ is the drug or the mixture of some biochemical complexes, which should be constrained to be positive. From Theorem 12, the elements of nonsymmetric matrixes $Q_{1}, Q_{2}$ should satisfy conditions (27). By using the constrained LMI algorithm, the following feasible solutions are obtained:

$$
\begin{gather*}
P=\left[\begin{array}{cc}
1.1517 & 0 \\
0 & 9.6618
\end{array}\right],  \tag{34}\\
Q_{1}=Q_{2}=\left[\begin{array}{ll}
5.3937 & 3.7805
\end{array}\right] .
\end{gather*}
$$



Figure 3: Evolution of the control input without constraints.


Figure 4: Time response of the biochemical network under positive control.

Then, the feedback gains can be calculated as $K_{1}=K_{2}=$ [4.6831, 0.3913]. The state response of the biochemical network under this constrained control can be seen in Figure 4, and the evolution of the positive control input $u(t)$ can be seen in Figure 5, where the control signal remains to be positive all the time.

Remark 15. By using the fuzzy interpolation approach, biocircuits can be easily implemented for the nonlinear biochemical network (29), which could guarantee it to be positive and asymptotically stable. Although the stability conditions for this biochemical network can also be derived by the method proposed in [29], the positivity of the states cannot be guaranteed. This would greatly reduce the significance of those results in real application such as drug delivery.


Figure 5: Evolution of the control input with constraints.

## 6. Discussion

In this paper, the fuzzy interpolation approach has been employed to approximate nonlinear biochemical networks for positive stability analysis and bio-circuits design. A few results on sufficient conditions for positivity and asymptotic stability of the network have been obtained in terms of a number of linear matrix inequalities. In addition, the positive constraint on control input is also considered for bio-circuits design. Finally, a real biochemical network model was provided to illustrate the effectiveness and validity of the obtained results.

Due to the transcription, translation, diffusion, and translocation processes of genes, time delays are inevitable in describing the dynamics of biochemical networks [7]. In addition, biochemical networks are often subject to intrinsic and extrinsic perturbations such as gene expression noises, mutation and disturbance from uncertain environment, and the fractal and chaotic features of systems [32]. Therefore, how to design constrained robust bio-circuits for such biochemical networks will be an interesting and challenging task.

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## Research Article

# Stability of a Functional Differential System with a Finite Number of Delays 

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#### Abstract

The paper is devoted to the study of asymptotic properties of a real two-dimensional differential system with unbounded nonconstant delays. The sufficient conditions for the stability and asymptotic stability of solutions are given. Used methods are based on the transformation of the considered real system to one equation with complex-valued coefficients. Asymptotic properties are studied by means of Lyapunov-Krasovskii functional. The results generalize some previous ones, where the asymptotic properties for two-dimensional systems with one or more constant delays or one nonconstant delay were studied.


## 1. Introduction

There are a lot of papers dealing with the stability and asymptotic behaviour of $n$-dimensional real vector equations with delay. Among others we should mention the recent results [1-13]. Since the plane has special topological properties different from those of $n$-dimensional space, where $n \geq 3$ or $n=1$, it is interesting to study asymptotic behaviour of two-dimensional systems by using tools which are typical and effective for two-dimensional systems. The convenient tool is the combination of the method of complexification and the method of Lyapunov-Krasovskii functional. Using these techniques we obtain new and easy applicable results on stability, asymptotic stability, or boundedness of solutions of real two-dimensional differential system

$$
\begin{align*}
x^{\prime}(t)= & \mathrm{A}(t) x(t) \\
& +\sum_{k=1}^{m} \mathrm{~B}_{k}(t) x\left(\theta_{k}(t)\right)  \tag{1}\\
& +h\left(t, x(t), x\left(\theta_{1}(t)\right), \ldots, x\left(\theta_{m}(t)\right)\right),
\end{align*}
$$

where $\theta_{k}(t)$ are real functions, $\mathrm{A}(t)=\left(a_{i j}(t)\right), \mathrm{B}_{k}(t)=$ $\left(b_{i j k}(t)\right)(i, j=1,2 ; k=1, \ldots, m)$ are real square matrices, and
$h(t, x, y)=\left(h_{1}\left(t, x, y_{1}, \ldots, y_{m}\right), h_{2}\left(t, x, y_{1}, \ldots, y_{m}\right)\right)$ is a real vector function, $x=\left(x_{1}, x_{2}\right), y_{k}=\left(y_{1 k}, y_{2 k}\right)$. It is supposed that the functions $\theta_{k}, a_{i j}$ are locally absolutely continuous on $\left[t_{0}, \infty\right), b_{i j k}$ are locally Lebesgue integrable on $\left[t_{0}, \infty\right)$, and the function $h$ satisfies Carathéodory conditions on $\left[t_{0}, \infty\right) \times$ $\mathbb{R}^{2(m+1)}$.

Delayed differential equations recently gain more importance in applications in science and real world. They can be found in applications in medicine (control of drug therapies and neurological, physiological, and epidemiological models), biology (predator-prey models and blowflies lifecycle), chemistry (chemical kinetics), physics (private communication and signal masking), and engineering (machining operation on a lathe). Equation (1) represents a generalization of many of these models. Particularly, (1) in this general form has an application in modeling of multiple regenerative effect in tool chatter. Obtained results on stability give the possibility to find the best spindle speeds and depth-of-cut for the machines for chatter-free high-productivity operation. For more details, see [14].

The main idea of the investigation, the combination of the method of complexification and the method of LyapunovKrasovskii functional, was introduced for ordinary differential equations in the paper by Ráb and Kalas [15].

The principle was transferred to differential equations with delay by Kalas and Baráková [16]. The results for several constant delays can be found in papers by Rebenda [17, 18]. Differential equations with one nonconstant delay are studied by Kalas [19] and Rebenda [20].

We extend such type of results to differential equations with a finite number of nonconstant delays. We introduce the transformation of the considered real system to one equation with complex-valued coefficients. We present sufficient conditions for stability and asymptotic stability of a solution and the conditions under which all solutions tend to zero. The applicability of the results is demonstrated with an example.

At the end of this introduction we append an overview of notations used in the paper and the transformation of the real system to one equation with complex-valued coefficients.

Consider the following:
$\mathbb{R}$ : set of all real numbers,
$\mathbb{R}_{+}$: set of all positive real numbers,
$\mathbb{R}_{+}^{0}$ : set of all nonnegative real numbers,
$\mathbb{R}_{-}$: set of all negative real numbers,
$\mathbb{R}_{-}^{0}$ : set of all nonpositive real numbers,
$\mathbb{C}$ : set of all complex numbers,
$\mathscr{C}:$ class of all continuous functions $[-r, 0] \rightarrow \mathbb{C}$,
$A C_{\text {loc }}(I, M)$ : class of all locally absolutely continuous functions $I \rightarrow M$,
$L_{\text {loc }}(I, M)$ : class of all locally Lebesgue integrable functions $I \rightarrow M$,
$K(I \times \Omega, M)$ : class of all functions $I \times \Omega \rightarrow M$ satisfying Carathéodory conditions on $I \times \Omega$,
$\operatorname{Re} z$ : real part of $z$,
$\operatorname{Im} z$ : imaginary part of $z$,
$\bar{z}$ : complex conjugate of $z$.
Introducing complex variables $z=x_{1}+i x_{2}, w_{1}=y_{11}+$ $i y_{12}, \ldots, w_{m}=y_{m 1}+i y_{m 2}$, we can rewrite system (1) into an equivalent equation with complex-valued coefficients:

$$
\begin{align*}
z^{\prime}(t)= & a(t) z(t)+b(t) \bar{z}(t) \\
& +\sum_{k=1}^{m}\left[A_{k}(t) z\left(\theta_{k}(t)\right)+B_{k}(t) \bar{z}\left(\theta_{k}(t)\right)\right]  \tag{2}\\
& +g\left(t, z(t), z\left(\theta_{1}(t)\right), \ldots, z\left(\theta_{m}(t)\right)\right),
\end{align*}
$$

where $\theta_{k} \in A C_{\mathrm{loc}}(J, \mathbb{R})$ for $k=1, \ldots, m, A_{k}, B_{k} \in L_{\mathrm{loc}}(J, \mathbb{C})$, $a, b \in A C_{\mathrm{loc}}(J, \mathbb{C}), g \in K\left(J \times \mathbb{C}^{m+1}, \mathbb{C}\right), J=\left[t_{0}, \infty\right)$.

The relations between the functions are as follows:

$$
\begin{align*}
& a(t)=\frac{1}{2}\left(a_{11}(t)+a_{22}(t)\right)+\frac{i}{2}\left(a_{21}(t)-a_{12}(t)\right), \\
& b(t)=\frac{1}{2}\left(a_{11}(t)-a_{22}(t)\right)+\frac{i}{2}\left(a_{21}(t)+a_{12}(t)\right), \\
& A_{k}(t)=\frac{1}{2}\left(b_{11 k}(t)+b_{22 k}(t)\right)+\frac{i}{2}\left(b_{21 k}(t)-b_{12 k}(t)\right), \\
& B_{k}(t)= \frac{1}{2}\left(b_{11 k}(t)-b_{22 k}(t)\right)+\frac{i}{2}\left(b_{21 k}(t)+b_{12 k}(t)\right), \\
& g\left(t, z, w_{1}, \ldots, w_{m}\right)  \tag{3}\\
&= h_{1}\left(t, \frac{1}{2}(z+\bar{z}), \frac{1}{2 i}(z-\bar{z})\right. \\
&\left.\frac{1}{2}\left(w_{1}+\bar{w}_{1}\right), \ldots, \frac{1}{2 i}\left(w_{m}-\bar{w}_{m}\right)\right) \\
&+i h_{2}\left(t, \frac{1}{2}(z+\bar{z}), \frac{1}{2 i}(z-\bar{z}), \frac{1}{2}\left(w_{1}+\bar{w}_{1}\right)\right. \\
&\left.\frac{1}{2 i}\left(w_{1}-\bar{w}_{1}\right), \ldots, \frac{1}{2 i}\left(w_{m}-\bar{w}_{m}\right)\right)
\end{align*}
$$

Conversely, putting

$$
\begin{gather*}
a_{11}(t)=\operatorname{Re}[a(t)+b(t)], \\
a_{12}(t)=\operatorname{Im}[b(t)-a(t)], \\
a_{21}(t)=\operatorname{Im}[a(t)+b(t)], \\
a_{22}(t)=\operatorname{Re}[a(t)-b(t)], \\
b_{11 k}(t)=\operatorname{Re}\left[A_{k}(t)+B_{k}(t)\right], \\
b_{12 k}(t)=\operatorname{Im}\left[B_{k}(t)-A_{k}(t)\right],  \tag{4}\\
b_{21 k}(t)=\operatorname{Im}\left[A_{k}(t)+B_{k}(t)\right], \\
b_{22 k}(t)=\operatorname{Re}\left[A_{k}(t)-B_{k}(t)\right], \\
h_{1}\left(t, x, y_{1}, \ldots, y_{m}\right) \\
=\operatorname{Re} g\left(t, x_{1}+i x_{2}, y_{11}+i y_{12}, \ldots, y_{m 1}+i y_{m 2}\right), \\
h_{2}\left(t, x, y_{1}, \ldots, y_{m}\right) \\
=\operatorname{Im} g\left(t, x_{1}+i x_{2}, y_{11}+i y_{12}, \ldots, y_{m 1}+i y_{m 2}\right),
\end{gather*}
$$

equation (2) can be written in real form (1) as well.

## 2. Preliminaries

We consider (2) in the case when

$$
\begin{equation*}
\liminf _{t \rightarrow \infty}(|\operatorname{Im} a(t)|-|b(t)|)>0 \tag{5}
\end{equation*}
$$

and study the behavior of solutions of (2) under this assumption. This situation corresponds to the case when the equilibrium 0 of the autonomous homogeneous system

$$
\begin{equation*}
x^{\prime}=\mathbf{A} x \tag{6}
\end{equation*}
$$

where $\mathbf{A}$ is supposed to be regular constant matrix, is a centre or a focus.

This case is included in the case $\liminf _{t \rightarrow \infty}(|a(t)|-$ $|b(t)|)>0$ considered in [21], but in this special case, we are able to derive more useful results as we will see later in an example. The idea is based on the well-known result that the condition $|a|>|b|$ in an autonomous equation $z^{\prime}=a z+b \bar{z}$ ensures that zero is a focus, a centre, or a node while under the condition $|\operatorname{Im} a|>|b|$ zero can be just a focus or a centre. Details are found in [15].

A simple example shows that, in some cases, the results of this paper can be applied more suitably than those given in [21].

Regarding (5) and since the delay functions $\theta_{k}$ satisfy $\lim _{t \rightarrow \infty} \theta_{k}(t)=\infty$, there are numbers $T_{1} \geq t_{0}, T \geq T_{1}$, and $\mu>0$ such that

$$
\begin{gather*}
|\operatorname{Im} a(t)|>|b(t)|+\mu \quad \text { for } t \geq T_{1}  \tag{7}\\
t \geq \theta_{k}(t) \geq T_{1} \quad \text { for } t \geq T \quad(k=1, \ldots, m) .
\end{gather*}
$$

Denote

$$
\begin{gather*}
\widetilde{\gamma}(t)=\operatorname{Im} a(t)+\sqrt{(\operatorname{Im} a(t))^{2}-|b(t)|^{2}} \operatorname{sgn}(\operatorname{Im} a(t))  \tag{8}\\
\widetilde{c}(t)=-i b(t)
\end{gather*}
$$

Notice that, unlike the function $\gamma$ introduced in [21], the previously defined function $\widetilde{\gamma}$ need not be positive.

Since $|\widetilde{\gamma}(t)|>|\operatorname{Im} a(t)|$ and $|\widetilde{c}(t)|=|b(t)|$, the inequality

$$
\begin{equation*}
|\widetilde{\gamma}(t)|>|\widetilde{c}(t)|+\mu \tag{9}
\end{equation*}
$$

is valid for $t \geq T_{1}$. It can be easily verified that $\widetilde{\gamma}, \widetilde{c} \in$ $A C_{\text {loc }}\left(\left[T_{1}, \infty\right), \mathbb{C}\right)$.

For the rest of this section, denote that
$\widetilde{\mathcal{V}}(t)$

$$
\begin{equation*}
=\frac{\operatorname{Re}\left(\widetilde{\gamma}(t) \widetilde{\gamma}^{\prime}(t)-\overline{\widetilde{c}}(t) \widetilde{c}^{\prime}(t)\right)-\left|\widetilde{\gamma}(t) \widetilde{c}^{\prime}(t)-\widetilde{\gamma}^{\prime}(t) \widetilde{c}(t)\right|}{\widetilde{\gamma}^{2}(t)-|\widetilde{c}(t)|^{2}} . \tag{10}
\end{equation*}
$$

The stability and asymptotic stability are studied under the following assumptions.
(i) The numbers $T_{1} \geq t_{0}, T \geq T_{1}$, and $\mu>0$ are such that (7) holds.
(ii) There exist functions $\tilde{\mathcal{\chi}}, \tilde{\kappa}_{k}, \widetilde{\varrho}:[T, \infty) \rightarrow \mathbb{R}$ such that

$$
\begin{align*}
& \mid \widetilde{\gamma}(t) g\left(t, z, w_{1}, \ldots, w_{m}\right) \\
& \quad+\widetilde{c}(t) \bar{g}\left(t, z, w_{1}, \ldots, w_{m}\right) \mid \\
& \quad \leq \tilde{\varkappa}(t)|\widetilde{\gamma}(t) z+\widetilde{c}(t) \bar{z}|  \tag{11}\\
& \quad+\sum_{k=1}^{m} \widetilde{\kappa}_{k}(t) \mid \widetilde{\gamma}\left(\theta_{k}(t)\right) w_{k} \\
& \quad+\widetilde{c}\left(\theta_{k}(t)\right) \bar{w}_{k} \mid+\widetilde{\varrho}(t)
\end{align*}
$$

## 3. Main Results

The aim is to generalize the results for ordinary differential equations published in [15] as well as the results contained in [16] (one constant delay), [18] (a finite number of constant delays), and [20] (one nonconstant delay). In the proof of the crucial theorem, we use the following auxiliary result.

Lemma 1. Let $a_{1}, a_{2}, b_{1}, b_{2} \in \mathbb{C}$ and $\left|a_{2}\right|>\left|b_{2}\right|$. Then

$$
\begin{equation*}
\operatorname{Re} \frac{a_{1} z+b_{1} \bar{z}}{a_{2} z+b_{2} \bar{z}} \leq \frac{\operatorname{Re}\left(a_{1} \bar{a}_{2}-b_{1} \bar{b}_{2}\right)+\left|a_{1} b_{2}-a_{2} b_{1}\right|}{\left|a_{2}\right|^{2}-\left|b_{2}\right|^{2}} \tag{17}
\end{equation*}
$$

for $z \in \mathbb{C}, z \neq 0$.
The proof of Lemma 1 can be found, for example, in [15, page 131] or [17, page 101].

Theorem 2. Let the conditions (i), (ii), (iii), and (iv) hold and $\widetilde{\varrho}(t) \equiv 0$.
(a) If

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int^{t} \tilde{\Lambda}(s) d s<\infty \tag{18}
\end{equation*}
$$

then the trivial solution of (2) is stable on $[T, \infty)$.
(b) If

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \int^{t} \widetilde{\Lambda}(s) d s=-\infty \tag{19}
\end{equation*}
$$

then the trivial solution of (2) is asymptotically stable on $[T, \infty)$.

Proof. Choose arbitrary $t_{1} \geq T$. Let $z(t)$ be any solution of (2) satisfying the condition $z(t)=z_{0}(t)$ for $t \in\left[T_{1}, t_{1}\right]$, where $z_{0}(t)$ is a continuous complex-valued initial function defined on $t \in\left[T_{1}, t_{1}\right]$. Consider the Lyapunov functional

$$
\begin{equation*}
V(t)=U(t)+\widetilde{\beta}(t) \sum_{k=1}^{m} \int_{\theta_{k}(t)}^{t} U(s) d s \tag{20}
\end{equation*}
$$

where

$$
\begin{equation*}
U(t)=|\widetilde{\gamma}(t) z(t)+\widetilde{c}(t) \bar{z}(t)| . \tag{21}
\end{equation*}
$$

To simplify the computations, denote that $w_{k}(t)=$ $z\left(\theta_{k}(t)\right)$ and write the functions of variable $t$ without brackets, for example, $z$ instead of $z(t)$.

From (20) we get

$$
\begin{align*}
V^{\prime}= & U^{\prime}+\widetilde{\beta}^{\prime} \sum_{k=1}^{m} \int_{\theta_{k}(t)}^{t} U(s) d s+m \widetilde{\beta}|\widetilde{\gamma} z+\widetilde{c} \bar{z}| \\
& -\sum_{k=1}^{m} \theta_{k}^{\prime} \widetilde{\beta}\left|\widetilde{\gamma}\left(\theta_{k}\right) w_{k}+\widetilde{c}\left(\theta_{k}\right) \bar{w}_{k}\right| \tag{22}
\end{align*}
$$

for almost all $t \geq t_{1}$ for which $z(t)$ is defined and $U^{\prime}(t)$ exists.
Denote that $\mathscr{K}=\left\{t \geq t_{1}: z(t)\right.$ exists, $\left.U(t) \neq 0\right\}$ and $\mathscr{M}=$ $\left\{t \geq t_{1}: z(t)\right.$ exists, $\left.U(t)=0\right\}$. It is clear that the derivative $U^{\prime}(t)$ exists for almost all $t \in \mathscr{K}$; hence, we focus on the set $\boldsymbol{M}$.

In view of (9) we have $z(t)=0$ for $t \in \mathscr{M}$. For almost all $t \in \mathscr{M}$, we compute

$$
\begin{align*}
U_{ \pm}^{\prime}(t) & =\lim _{\tau \rightarrow t \pm} \frac{U(\tau)-U(t)}{\tau-t} \\
& =\lim _{\tau \rightarrow t \pm} \frac{U(\tau)}{\tau-t} \\
& =\lim _{\tau \rightarrow t \pm} \frac{|\widetilde{\gamma}(\tau)[z(\tau)-z(t)]-\widetilde{c}(\tau)[\bar{z}(\tau)-\bar{z}(t)]|}{\tau-t} \\
& = \pm\left|\widetilde{\gamma}(t) z^{\prime}(t)+\widetilde{c}(t) \bar{z}^{\prime}(t)\right| \\
& = \pm\left|\widetilde{\gamma}(t) g^{*}(t)+\widetilde{c}(t) \bar{g}^{*}(t)\right|, \tag{23}
\end{align*}
$$

where

$$
\begin{align*}
g^{*}(t)= & \sum_{k=1}^{m}\left(A_{k}(t) w_{k}(t)+B_{k}(t) \bar{w}_{k}(t)\right)  \tag{24}\\
& +g\left(t, 0, w_{1}(t), \ldots, w_{m}(t)\right) .
\end{align*}
$$

Hence, $U$ has one-sided derivatives a.e. in $\mathscr{M}$. According to [22, Chapter IX., Theorem (1.1)] or [23], the set of all $t$ such that $U_{+}^{\prime}(t) \neq U_{-}^{\prime}(t)$ can be at most countable; thus, the derivative $U^{\prime}$ exists for almost all $t \in \mathscr{M}$, and for these $t$, $U^{\prime}(t)=0$.

In particular, the derivative $U^{\prime}$ exists for almost all $t \geq t_{1}$ for which $z(t)$ is defined; thus, (22) holds for almost all $t \geq t_{1}$ for which $z(t)$ is defined.

Now return the attention to the set $\mathscr{K}$. For almost all $t \in$ $\mathscr{K}$, it holds that $U U^{\prime}=U(\sqrt{(\tilde{\gamma} z+\widetilde{c} \bar{z})(\overline{\tilde{\gamma}} \bar{z}+\overline{\tilde{c}} z)})^{\prime}=\operatorname{Re}[(\tilde{\gamma} \bar{z}+$ $\left.\overline{\tilde{c}} z)\left(\tilde{\gamma}^{\prime} z+\tilde{\gamma} z^{\prime}+\tilde{c}^{\prime} \bar{z}+\widetilde{c} \bar{z}^{\prime}\right)\right]$. As $z(t)$ is a solution of (2), we have

$$
\begin{aligned}
& U U^{\prime}=\operatorname{Re}\{(\tilde{\gamma} \bar{z}+\overline{\tilde{c}} z) \\
& \times\left[\tilde{\gamma}^{\prime} z+\tilde{c}^{\prime} \bar{z}\right. \\
& +\tilde{\gamma}\left(a z+b \bar{z}+\sum_{k=1}^{m}\left(A_{k} w_{k}+B_{k} \bar{w}_{k}\right)+g\right) \\
& \left.\left.+\widetilde{c}\left(\bar{a} \bar{z}+\bar{b} z+\sum_{k=1}^{m}\left(\bar{A}_{k} \bar{w}_{k}+\bar{B}_{k} w_{k}\right)+\bar{g}\right)\right]\right\} \\
& =\operatorname{Re}\{(\tilde{\gamma} z+\overline{\tilde{c}} z) \\
& \times\left[\tilde{\gamma}^{\prime} z+\tilde{c}^{\prime} \bar{z}+(\tilde{\gamma} a+\bar{c} \bar{b}) z\right. \\
& +(\widetilde{\gamma} b+\widetilde{c} a) \bar{z}
\end{aligned}
$$

$$
\begin{align*}
& +\widetilde{\gamma}\left(\sum_{k=1}^{m}\left(A_{k} w_{k}+B_{k} \bar{w}_{k}\right)+g\right) \\
& \left.\left.+\tilde{c}\left(\sum_{k=1}^{m}\left(\bar{A}_{k} \bar{w}_{k}+\bar{B}_{k} w_{k}\right)+\bar{g}\right)\right]\right\} \tag{25}
\end{align*}
$$

for almost all $t \in \mathscr{K}$.
Short computation gives $(\widetilde{\gamma} a+\widetilde{c} \bar{b}) \widetilde{c}=(\widetilde{\gamma} b+\widetilde{c} \bar{a}) \widetilde{\gamma}$, and from this we get

$$
\begin{aligned}
U U^{\prime} \leq & \operatorname{Re}\left\{(\widetilde{\gamma} \bar{z}+\overline{\tilde{c}} z)(\widetilde{\gamma} a+\tilde{c} \bar{b})\left(z+\frac{\widetilde{c}}{\widetilde{\gamma}} \bar{z}\right)\right\} \\
& +\operatorname{Re}\left\{( \widetilde { \gamma } \overline { z } + \overline { \tilde { c } } z ) \left[\widetilde{\gamma} \sum_{k=1}^{m}\left(A_{k} w_{k}+B_{k} \bar{w}_{k}\right)\right.\right. \\
& \left.\left.+\widetilde{c} \sum_{k=1}^{m}\left(\bar{A}_{k} \bar{w}_{k}+\bar{B}_{k} w_{k}\right)\right]\right\} \\
& +\operatorname{Re}[(\tilde{\gamma} \bar{z}+\overline{\widetilde{c}} z)(\widetilde{\gamma} g+\widetilde{c} \bar{g})] \\
& +\operatorname{Re}\left[(\widetilde{\gamma} \bar{z}+\overline{\tilde{c}} z)\left(\widetilde{\gamma}^{\prime} z+\tilde{c}^{\prime} \bar{z}\right)\right] \\
\leq & U^{2} \operatorname{Re}\left(a+\frac{\widetilde{c}}{\widetilde{\gamma}} \bar{b}\right) \\
& +U(|\tilde{\gamma}|+|\widetilde{c}|)\left(\sum_{k=1}^{m}\left|A_{k} w_{k}+B_{k} \bar{w}_{k}\right|\right) \\
& +U|\widetilde{\gamma} g+\widetilde{c} \bar{g}|+U^{2} \operatorname{Re} \frac{\widetilde{\gamma}^{\prime} z+\widetilde{c}^{\prime} \bar{z}}{\widetilde{\gamma} z+\widetilde{c} z}
\end{aligned}
$$

for almost all $t \in \mathscr{K}$.
Applying Lemma 1 to the last term, we obtain

$$
\begin{equation*}
\operatorname{Re} \frac{\tilde{\gamma}^{\prime} z+\widetilde{c}^{\prime} \bar{z}}{\widetilde{\gamma} z+\widetilde{c} z} \leq \widetilde{\vartheta} \tag{27}
\end{equation*}
$$

Using this inequality together with (13), assumption (ii), and the relation $\operatorname{Re}(a+(\widetilde{c} / \widetilde{\gamma}) \bar{b})=\operatorname{Re} a$, we obtain

$$
\begin{aligned}
U U^{\prime} \leq & U^{2}(\operatorname{Re} a+\widetilde{\vartheta}+\tilde{\varkappa}) \\
& +U \sum_{k=1}^{m}\left(\widetilde{\kappa}_{k}\left|\widetilde{\gamma}\left(\theta_{k}\right) w_{k}+\widetilde{c}\left(\theta_{k}\right) \bar{w}_{k}\right|\right) \\
& +U(|\widetilde{\gamma}|+|\widetilde{c}|) \\
& \times\left(\sum_{k=1}^{m} \frac{\left|A_{k}\right|\left|w_{k}\right|+\left|B_{k}\right|\left|\bar{w}_{k}\right|}{\left|\widetilde{\gamma}\left(\theta_{k}\right)\right|-\left|\widetilde{c}\left(\theta_{k}\right)\right|}\left(\left|\widetilde{\gamma}\left(\theta_{k}\right)\right|-\left|\widetilde{c}\left(\theta_{k}\right)\right|\right)\right) \\
\leq & U^{2}(\operatorname{Re} a+\widetilde{\vartheta}+\widetilde{\varkappa})
\end{aligned}
$$

$$
\begin{align*}
& \quad+U\left\{\sum_{k=1}^{m}\left[\tilde{\kappa}_{k}+\left(\left|A_{k}\right|+\left|B_{k}\right|\right) \frac{|\widetilde{\gamma}|+|\widetilde{c}|}{\left|\widetilde{\gamma}\left(\theta_{k}\right)\right|-\left|\widetilde{c}\left(\theta_{k}\right)\right|}\right]\right. \\
& \left.\quad \times\left|\widetilde{\gamma}\left(\theta_{k}\right) w_{k}+\widetilde{c}\left(\theta_{k}\right) \bar{w}_{k}\right|\right\} \\
& \leq U^{2}(\operatorname{Re} a+\widetilde{\vartheta}+\tilde{\varkappa}) \\
& \quad+U \sum_{k=1}^{m} \widetilde{\lambda}_{k}\left|\widetilde{\gamma}\left(\theta_{k}\right) w_{k}+\widetilde{c}\left(\theta_{k}\right) \bar{w}_{k}\right| \tag{28}
\end{align*}
$$

for almost all $t \in \mathscr{K}$.
Consequently,

$$
\begin{equation*}
U^{\prime} \leq U(\operatorname{Re} a+\tilde{\vartheta}+\tilde{\varkappa})+\sum_{k=1}^{m} \tilde{\lambda}_{k}\left|\widetilde{\gamma}\left(\theta_{k}\right) w_{k}+\widetilde{c}\left(\theta_{k}\right) \bar{w}_{k}\right| \tag{29}
\end{equation*}
$$

for almost all $t \in \mathscr{K}$.
Recalling that $U^{\prime}(t)=0$ for almost all $t \in \mathscr{M}$, we can see that inequality (29) is valid for almost all $t \geq t_{1}$ for which $z(t)$ is defined.

From (29) we have

$$
\begin{align*}
V^{\prime} \leq & U(\operatorname{Re} a+\widetilde{\vartheta}+\tilde{\varkappa}+m \tilde{\beta}) \\
& +\sum_{k=1}^{m}\left(\widetilde{\lambda}_{k}-\theta_{k}^{\prime} \widetilde{\beta}\right)\left|\widetilde{\gamma}\left(\theta_{k}\right) w_{k}+\widetilde{c}\left(\theta_{k}\right) \bar{w}_{k}\right|  \tag{30}\\
& +\widetilde{\beta}^{\prime} \sum_{k=1}^{m} \int_{\theta_{k}(t)}^{t}|\widetilde{\gamma}(s) z(s)+\widetilde{c}(s) \bar{z}(s)| d s .
\end{align*}
$$

As $\widetilde{\beta}(t)$ fulfills condition (12), we obtain

$$
\begin{align*}
V^{\prime}(t) \leq & U(t) \widetilde{\Theta}(t) \\
& +\widetilde{\beta}^{\prime}(t) \sum_{k=1}^{m} \int_{\theta_{k}(t)}^{t}|\widetilde{\gamma}(s) z(s)+\widetilde{c}(s) \bar{z}(s)| d s \tag{31}
\end{align*}
$$

Hence,

$$
\begin{equation*}
V^{\prime}(t)-\widetilde{\Lambda}(t) V(t) \leq 0 \tag{32}
\end{equation*}
$$

for almost all $t \geq t_{1}$ for which the solution $z(t)$ exists.
Notice that, with respect to (9),

$$
\begin{equation*}
V(t) \geq(|\widetilde{\gamma}(t)|-|\widetilde{c}(t)|)|z(t)| \geq \mu|z(t)| \tag{33}
\end{equation*}
$$

for all $t \geq t_{1}$ for which $z(t)$ is defined.
Suppose that condition (18) holds, and choose arbitrary $\varepsilon>0$. Put

$$
\begin{gather*}
\Delta=\max _{s \in\left[T_{1}, t_{1}\right]}(|\widetilde{\gamma}(s)|+|\widetilde{c}(s)|), \quad L=\sup _{T \leq t<\infty} \int_{T}^{t} \widetilde{\Lambda}(s) d s \\
\delta=\mu \varepsilon \Delta^{-1}\left(1+m \widetilde{\beta}\left(t_{1}\right)\left(t_{1}-T_{1}\right)\right)^{-1} \exp \left\{\int_{T}^{t_{1}} \widetilde{\Lambda}(s) d s-L\right\}, \tag{34}
\end{gather*}
$$

where $\mu>0, T_{1} \geq t_{0}$, and $T \geq T_{1}$ are the numbers from condition (i).

If the initial function $z_{0}(t)$ of the solution $z(t)$ satisfies $\max _{s \in\left[T_{1}, t_{1}\right]}\left|z_{0}(s)\right|<\delta$, then the multiplication of (32) by $\exp \left\{-\int_{t_{1}}^{t} \widetilde{\Lambda}(s) d s\right\}$ and the integration over $\left[t_{1}, t\right]$ yield

$$
\begin{equation*}
V(t) \exp \left\{-\int_{t_{1}}^{t} \widetilde{\Lambda}(s) d s\right\}-V\left(t_{1}\right) \leq 0 \tag{35}
\end{equation*}
$$

for all $t \geq t_{1}$ for which $z(t)$ is defined. From (33) and (35) we get

$$
\begin{align*}
\mu|z(t)| \leq & V(t) \leq V\left(t_{1}\right) \exp \left\{\int_{t_{1}}^{t} \widetilde{\Lambda}(s) d s\right\} \\
\leq & {\left[\left(\left|\widetilde{\gamma}\left(t_{1}\right)\right|+\left|\widetilde{c}\left(t_{1}\right)\right|\right)\left|z\left(t_{1}\right)\right|\right.} \\
& +\widetilde{\beta}\left(t_{1}\right) \max _{s \in\left[T_{1}, t_{1}\right]}|z(s)| \\
& \left.\times\left(\sum_{k=1}^{m} \int_{\theta_{k}\left(t_{1}\right)}^{t_{1}}(|\widetilde{\gamma}(s)|+|\widetilde{c}(s)|) d s\right)\right] \\
& \times \exp \left\{\int_{t_{1}}^{t} \widetilde{\Lambda}(s) d s\right\}  \tag{36}\\
\leq & {\left[\Delta \max _{s \in\left[T_{1}, t_{1}\right]}\left|z_{0}(s)\right|\right.} \\
& +\widetilde{\beta}\left(t_{1}\right) \max _{s \in\left[T_{1}, t_{1}\right]}\left|z_{0}(s)\right| \Delta \\
& \left.\times \sum_{k=1}^{m}\left(t_{1}-\theta_{k}\left(t_{1}\right)\right)\right] \\
& \times \exp \left\{\int_{t_{1}}^{t} \widetilde{\Lambda}(s) d s\right\}
\end{align*}
$$

that is,

$$
\begin{align*}
\mu|z(t)| \leq & \Delta \max _{s \in\left[T_{1}, t_{1}\right]}\left|z_{0}(s)\right|\left(1+m \widetilde{\beta}\left(t_{1}\right)\left(t_{1}-T_{1}\right)\right) \\
& \times \exp \left\{L-\int_{T}^{t_{1}} \widetilde{\Lambda}(s) d s\right\}<\mu \varepsilon . \tag{37}
\end{align*}
$$

Thus, we have $|z(t)|<\varepsilon$ for all $t \geq t_{1}$, and we conclude that the trivial solution of (2) is stable.

Now suppose that condition (19) is valid. Then, in view of the first part of Theorem 2, for $K>0$, there is a $\rho>0$ such that $\max _{s \in\left[T_{1}, t_{1}\right]}\left|z_{0}(s)\right|<\rho$ implies that the solution $z(t)$ of (2) exists for all $t \geq t_{1}$ and satisfies $|z(t)|<K$, where $K$ is arbitrary real constant. Hence, from this and (33), we have

$$
\begin{equation*}
|z(t)| \leq \mu^{-1} V(t) \leq \mu^{-1} V\left(t_{1}\right) \exp \left\{\int_{t_{1}}^{t} \widetilde{\Lambda}(s) d s\right\}, \tag{38}
\end{equation*}
$$

for all $t \geq t_{1}$. This inequality, with condition (19), gives

$$
\begin{equation*}
\lim _{t \rightarrow \infty} z(t)=0, \tag{39}
\end{equation*}
$$

which completes the proof.

Remark 3. Theorem 2 represents a generalization of previous results.

If we take $A_{1}(t)=A(t), A_{k} \equiv 0$, for $k=2, \ldots, m, B_{1}(t)=$ $B(t), B_{k} \equiv 0$, for $k=2, \ldots, m$, and $\theta_{1}(t)=t-r$, where $r>0$, we get Theorem 4 from [16].

If we take $\theta_{k}(t)=t-r_{k}$, where $r_{k}>0, k=1, \ldots, m$, we obtain Theorem 1 from [18].

If we take $A_{1}(t)=A(t), A_{k} \equiv 0$, for $k=2, \ldots, m$, $B_{1}(t)=B(t), B_{k} \equiv 0$, for $k=2, \ldots, m$, and $\theta_{1}(t)=\theta(t)$, we get Theorem 2.2 from [20].

The next theorem involves the function $\widetilde{\varrho}$ in (ii); thus, it is more general than Theorem 2. A part of the proof of Theorem 2 is utilized in the proof of Theorem 4.

Theorem 4. Let the assumptions (i), (ii), (iii), and (iv) hold and

$$
\begin{align*}
V(t)= & |\widetilde{\gamma}(t) z(t)+\widetilde{c}(t) \bar{z}(t)| \\
& +\widetilde{\beta}(t) \sum_{k=1}^{m} \int_{\theta_{k}(t)}^{t}|\widetilde{\gamma}(s) z(s)+\widetilde{c}(s) \bar{z}(s)| d s, \tag{40}
\end{align*}
$$

where $z(t)$ is any solution of (2) defined for $t \rightarrow \infty$. Then

$$
\begin{align*}
\mu|z(t)| \leq & V(s) \exp \left(\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right) \\
& +\int_{s}^{t} \widetilde{\varrho}(\xi) \exp \left(\int_{\xi}^{t} \widetilde{\Lambda}(\sigma) d \sigma\right) d \xi \tag{41}
\end{align*}
$$

for $t \geq s \geq t_{1}$, where $t_{1} \geq T$.
Proof. Following the proof of Theorem 2, we have

$$
\begin{align*}
V^{\prime}(t) \leq & |\widetilde{\gamma}(t) z(t)+\widetilde{c}(t) \bar{z}(t)| \Theta(t) \\
& +\widetilde{\beta}^{\prime}(t) \sum_{k=1}^{m} \int_{\theta_{k}(t)}^{t}|\widetilde{\gamma}(s) z(s)+\widetilde{c}(s) \bar{z}(s)| d s  \tag{42}\\
& +\widetilde{\varrho}(t) \\
\leq & \widetilde{\Lambda}(t) V(t)+\widetilde{\varrho}(t),
\end{align*}
$$

a.e. on $\left[t_{1}, \infty\right)$. Using this inequality, we get

$$
\begin{equation*}
V^{\prime}(t)-\widetilde{\Lambda}(t) V(t) \leq \widetilde{\varrho}(t), \tag{43}
\end{equation*}
$$

a.e. on $\left[t_{1}, \infty\right)$. Multiplying (43) by $\exp \left(-\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right)$ gives

$$
\begin{equation*}
\left[V(t) \exp \left(-\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right)\right] \leq \widetilde{\varrho}(t) \exp \left(-\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right) \tag{44}
\end{equation*}
$$

a.e. on $\left[t_{1}, \infty\right)$. Integration over $[s, t]$ yields

$$
\begin{align*}
& V(t) \exp \left(-\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right)-V(s) \\
& \quad \leq \int_{s}^{t} \widetilde{\varrho}(\xi) \exp \left(-\int_{s}^{\xi} \widetilde{\Lambda}(\sigma) d \sigma\right) d \xi \tag{45}
\end{align*}
$$

and multiplying (45) by $\exp \left(\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right)$, we obtain

$$
\begin{align*}
V(t) \leq & V(s) \exp \left(\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right) \\
& +\int_{s}^{t} \widetilde{\varrho}(\xi) \exp \left(\int_{\xi}^{t} \widetilde{\Lambda}(\sigma) d \sigma\right) d \xi \tag{46}
\end{align*}
$$

The statement now follows from (33).
Remark 5. Theorem 4 generalizes theorems contained in previous papers.

If we take $A_{1}(t)=A(t), A_{k} \equiv 0$, for $k=2, \ldots, m, B_{1}(t)=$ $B(t), B_{k} \equiv 0$, for $k=2, \ldots, m$, and $\theta_{1}(t)=t-r$, where $r>0$, we get Theorem 2 from [16].

If we take $\theta_{k}(t)=t-r_{k}$, where $r_{k}>0, k=1, \ldots, m$, we obtain Theorem 2 from [18].

If we take $A_{1}(t)=A(t), A_{k} \equiv 0$, for $k=2, \ldots, m$, $B_{1}(t)=B(t), B_{k} \equiv 0$, for $k=2, \ldots, m$, and $\theta_{1}(t)=\theta(t)$, we get Theorem 2.7 from [20].

The last of the main propositions gives the conditions under which all solutions of (2) tend to zero.

Theorem 6. Let the assumptions (i), (ii), (iii), and (iv) be satisfied. Let $\widetilde{\Lambda}(t) \leq 0$ a.e. on $\left[T^{*}, \infty\right)$, where $T^{*} \in[T, \infty)$. If

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \int^{t} \widetilde{\Lambda}(s) d s=-\infty, \quad \widetilde{\varrho}(t)=o(\widetilde{\Lambda}(t)) \tag{47}
\end{equation*}
$$

then any solution $z(t)$ of (2) existing for $t \rightarrow \infty$ satisfies

$$
\begin{equation*}
\lim _{t \rightarrow \infty} z(t)=0 \tag{48}
\end{equation*}
$$

Proof. Choose arbitrary $\varepsilon>0$. According to (47), there is $s \geq$ $T^{*}$ such that $\widetilde{\varrho}(t) \leq(1 / 2) \mu \varepsilon|\widetilde{\Lambda}(t)|$ for $t \geq s$ and

$$
\begin{align*}
\int_{s}^{t} \widetilde{\varrho} & (\tau) \exp \left(\int_{\tau}^{t} \widetilde{\Lambda}(\sigma) d \sigma\right) d \tau \\
& \leq \frac{\mu \varepsilon}{2} \int_{s}^{t}[-\widetilde{\Lambda}(\tau)] \exp \left(\int_{\tau}^{t} \widetilde{\Lambda}(\sigma) d \sigma\right) d \tau \\
& =\frac{\mu \varepsilon}{2} \int_{s}^{t}\left(\frac{d}{d \tau}\left[\exp \left(\int_{\tau}^{t} \widetilde{\Lambda}(\sigma) d \sigma\right)\right]\right) d \tau  \tag{49}\\
& =\frac{\mu \varepsilon}{2}\left[\exp \left(\int_{\tau}^{t} \widetilde{\Lambda}(\sigma) d \sigma\right)\right]_{s}^{t} \\
& =\frac{\mu \varepsilon}{2}\left[1-\exp \left(\int_{s}^{t} \widetilde{\Lambda}(\tau) d \tau\right)\right]<\frac{\mu \varepsilon}{2}
\end{align*}
$$

for $t \geq s$. From (47) we have $\exp \left(\int_{s}^{t} \widetilde{\Lambda}(\tau) d \tau\right) \rightarrow 0$ as $t \rightarrow \infty$; hence, there is $S \geq s$ such that $\exp \left(\int_{s}^{t} \widetilde{\Lambda}(\tau) d \tau\right)<\mu \varepsilon(2 V(s))^{-1}$ for $t \geq S$. Considering this fact and (41), we get

$$
\begin{equation*}
\mu|z(t)|<V(s) \frac{\mu \varepsilon}{2 V(s)}+\frac{\mu \varepsilon}{2}=\mu \varepsilon \tag{50}
\end{equation*}
$$

for $t \geq S$. This completes the proof.

Remark 7. Theorem 6 is a generalization of results published in the papers $[16,18,20]$.

If we take $A_{1}(t)=A(t), A_{k} \equiv 0$, for $k=2, \ldots, m, B_{1}(t)=$ $B(t), B_{k} \equiv 0$, for $k=2, \ldots, m$, and $\theta_{1}(t)=t-r$, where $r>0$, we get Theorem 3 from [16].

If we take $\theta_{k}(t)=t-r_{k}$, where $r_{k}>0, k=1, \ldots, m$, we obtain Theorem 3 from [18].

If we take $A_{1}(t)=A(t), A_{k} \equiv 0$, for $k=2, \ldots, m$, $B_{1}(t)=B(t), B_{k} \equiv 0$, for $k=2, \ldots, m$, and $\theta_{1}(t)=\theta(t)$, we get Theorem 2.14 from [20].

## 4. Corollaries and Examples

From Theorem 2 we easily obtain several corollaries. We give an example which shows that it is worth to consider the case (5).

Corollary 8. Let $a(t) \equiv a \in \mathbb{C}, b(t) \equiv b \in \mathbb{C},|\operatorname{Im} a|>|b|$. Suppose that $\lim _{t \rightarrow \infty} \theta_{k}(t)=\infty, \theta_{k}(t) \leq t$, for $t \geq T_{1}$, where $T_{1} \geq t_{0}$. Let $\rho_{0}, \rho_{1}, \ldots, \rho_{m}:\left[T_{1}, \infty\right) \rightarrow \mathbb{R}$ be such that

$$
\begin{equation*}
\left|g\left(t, z, w_{1}, \ldots, w_{m}\right)\right| \leq \rho_{0}(t)|z|+\sum_{k=1}^{m} \rho_{k}(t)\left|w_{k}\right| \tag{51}
\end{equation*}
$$

for $t \geq T_{1},|z|<R,\left|w_{k}\right|<R, k=1, \ldots, m, R>0$ and $\rho_{0} \in$ $L_{\text {loc }}\left(\left[T_{1}, \infty\right), \mathbb{R}\right)$.

Let $\widetilde{\beta} \in A C_{\mathrm{loc}}\left(\left[T_{1}, \infty\right), \mathbb{R}_{+}\right)$satisfy

$$
\begin{align*}
& \theta_{k}^{\prime}(t) \widetilde{\beta}(t) \\
& \quad \geq\left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{1 / 2}\left(\rho_{k}(t)+\left|A_{k}(t)\right|+\left|B_{k}(t)\right|\right) \tag{52}
\end{align*}
$$

a.e. on $\left[T_{1}, \infty\right)$ for $k=1, \ldots, m$. If

$$
\begin{gather*}
\limsup _{t \rightarrow \infty} \int^{t} \max \left(\operatorname{Re} a+\left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{1 / 2} \rho_{0}(s)\right. \\
\left.+m \tilde{\beta}(s), \frac{\widetilde{\beta}^{\prime}(s)}{\widetilde{\beta}(s)}\right) d s<\infty, \tag{53}
\end{gather*}
$$

then the trivial solution of $(2)$ is stable. If

$$
\begin{array}{r}
\lim _{t \rightarrow \infty} \int^{t} \max \left(\operatorname{Re} a+\left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{1 / 2} \rho_{0}(s)\right. \\
\left.+m \widetilde{\beta}(s), \frac{\widetilde{\beta}^{\prime}(s)}{\widetilde{\beta}(s)}\right) d s=-\infty \tag{54}
\end{array}
$$

then the trivial solution of (2) is asymptotically stable.
Proof. Choose $T \geq T_{1}$ such that $\theta_{k}(t) \geq T_{1}$ for $t \geq T, k=$ $1, \ldots, m$. Denote that $z=z(t)$ and $w_{k}=z\left(\theta_{k}(t)\right)$ again. Since
$a, b \in \mathbb{C}$ are constants, then also $\tilde{\gamma}$ and $\tilde{c}$ are constants, and we have $\widetilde{\mathcal{V}}(t) \equiv 0$. Using condition (51) we get

$$
\begin{align*}
& \left|\widetilde{\gamma} g\left(t, z, w_{1}, \ldots, w_{m}\right)+\widetilde{c} \bar{g}\left(t, z, w_{1}, \ldots, w_{m}\right)\right| \\
& \quad \leq(|\widetilde{\gamma}|+|\widetilde{c}|)\left(\rho_{0}(t)|z|+\sum_{k=1}^{m} \rho_{k}(t)\left|w_{k}\right|\right) \\
& \quad=\frac{|\widetilde{\gamma}|+|\widetilde{c}|}{|\widetilde{\gamma}|-|\widetilde{c}|}(|\widetilde{\gamma}|-|\widetilde{c}|)\left(\rho_{0}(t)|z|+\sum_{k=1}^{m} \rho_{k}(t)\left|w_{k}\right|\right) \\
& \quad \leq \frac{|\widetilde{\gamma}|+|\widetilde{c}|}{|\widetilde{\gamma}|-|\widetilde{c}|}\left(\rho_{0}(t)|\widetilde{\gamma} z+\widetilde{c} \bar{z}|+\sum_{k=1}^{m} \rho_{k}(t)\left|\widetilde{\gamma} w_{k}+\widetilde{c} \bar{w}_{k}\right|\right), \tag{55}
\end{align*}
$$

and it follows that condition (ii) holds with

$$
\begin{equation*}
\widetilde{\varkappa}(t)=\frac{|\widetilde{\gamma}|+|\widetilde{c}|}{|\widetilde{\gamma}|-|\widetilde{c}|} \rho_{0}(t), \quad \widetilde{\kappa}_{k}(t)=\frac{|\widetilde{\gamma}|+|\widetilde{c}|}{|\widetilde{\gamma}|-|\widetilde{c}|} \rho_{k}(t) \tag{56}
\end{equation*}
$$

and $\widetilde{\varrho}(t) \equiv 0$.
Condition (53) implies that Re $a \leq 0$. Since

$$
\begin{align*}
\frac{|\widetilde{\gamma}|+|\widetilde{c}|}{|\widetilde{\gamma}|-|\widetilde{c}|} & =\frac{|\operatorname{Im} a|+\sqrt{|\operatorname{Im} a|^{2}-|b|^{2}}+|b|}{|\operatorname{Im} a|+\sqrt{|\operatorname{Im} a|^{2}-|b|^{2}}-|b|}  \tag{57}\\
& =\left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{1 / 2}
\end{align*}
$$

in view of (14) we obtain

$$
\begin{align*}
\tilde{\lambda}_{k}(t)= & \left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{1 / 2} \\
& \times\left\{\rho_{k}(t)+\left|A_{k}(t)\right|+\left|B_{k}(t)\right|\right\}, \\
\widetilde{\Theta}(t)= & \operatorname{Re} a+\frac{|\widetilde{\gamma}|+|\widetilde{c}|}{|\widetilde{\gamma}|-|\widetilde{c}|} \rho_{0}(t)+m \widetilde{\beta}(t)  \tag{58}\\
= & \operatorname{Re} a+\left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{1 / 2} \rho_{0}(t)+m \widetilde{\beta}(t),
\end{align*}
$$

and the assertion follows from (16) and Theorem 2.

Now we show an example that, under certain circumstances, Corollary 8 is more useful than Corollary 1 from [21].

Example 9. Consider (2), where $a(t) \equiv-\sqrt{5}+2 i, b(t) \equiv 1$, $A_{k}(t) \equiv 0, B_{k}(t) \equiv 0$, for $k=1, \ldots, m$, and
$g\left(t, z, w_{1}, \ldots, w_{m}\right)=\frac{2}{\sqrt{3}} e^{i t} z+\sum_{k=1}^{m} \frac{k}{2 m t}(\sqrt{15}-\sqrt{14}) e^{-t} w_{k}$.

Assume that $t_{0}=m$ and $R=\infty, \theta_{k}(t)=k \ln t$. Put $T=$ $e^{t_{0}}=e^{m}$. Then, $\rho_{0}(t) \equiv 2 / \sqrt{3}, \rho_{k}(t)=(k / 2 m t)(\sqrt{15}-\sqrt{14}) e^{-t}$. We have

$$
\begin{align*}
& \max \left(\frac{|a|-|b|}{|a|} \operatorname{Re} a+\left(\frac{|a|+|b|}{|a|-|b|}\right)^{1 / 2} \rho_{0}(t)+m \beta(t), \frac{\beta^{\prime}(t)}{\beta(t)}\right) \\
&=\max \left(-\frac{2}{3} \sqrt{5}+\sqrt{2} \frac{2}{\sqrt{3}}+m \beta(t), \frac{\beta^{\prime}(t)}{\beta(t)}\right) \\
& \quad \geq \frac{2}{3}(\sqrt{6}-\sqrt{5})>0 \tag{60}
\end{align*}
$$

for

$$
\begin{align*}
\theta_{k}^{\prime}(t) \beta(t)= & \frac{k}{t} \beta(t) \geq\left(\frac{|a|+|b|}{|a|-|b|}\right)^{1 / 2} \\
& \times\left\{\rho_{k}(t)+\left|A_{k}(t)\right|+\left|B_{k}(t)\right|\right\}  \tag{61}\\
= & \frac{k}{m t \sqrt{2}}(\sqrt{15}-\sqrt{14}) e^{-t},
\end{align*}
$$

where $k \in\{1, \ldots, m\}$; hence, we cannot apply Corollary 1 from [21].

On the other hand, if we use

$$
\begin{align*}
\theta_{k}^{\prime}(t) \widetilde{\beta}(t) & =\frac{k}{t} \widetilde{\beta}(t)=\frac{k \sqrt{3}}{2 m t}(\sqrt{15}-\sqrt{14}) e^{-t} \\
& \geq\left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{1 / 2}\left\{\rho_{k}(t)+\left|A_{k}(t)\right|+\left|B_{k}(t)\right|\right\}, \tag{62}
\end{align*}
$$

where $k \in\{1, \ldots, m\}$, we have

$$
\begin{align*}
& \max \left(\operatorname{Re} a+\left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{1 / 2} \rho_{0}(t)+m \tilde{\beta}(t), \frac{\widetilde{\beta}^{\prime}(t)}{\widetilde{\beta}(t)}\right) \\
& \quad=\max \left(-\sqrt{5}+2+m \frac{\sqrt{3}}{2 m}(\sqrt{15}-\sqrt{14}) e^{-t},-1\right) \\
& \quad \leq-\sqrt{5}+2+\frac{\sqrt{3}}{2}(\sqrt{15}-\sqrt{14})<-\frac{12}{100}<0 \tag{63}
\end{align*}
$$

thus, Corollary 8 guarantees the stability and also asymptotic stability of the trivial solution of the considered equation.

The following corollary gives sufficient conditions for stability of the trivial solution of (2).

Corollary 10. Assume that the conditions (i), (ii), and (iii) are valid with $\widetilde{\varrho}(t) \equiv 0$. If $\widetilde{\beta}(t)$ is monotone and bounded on $[T, \infty)$ and if

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int^{t}[\widetilde{\Theta}(s)]_{+} d s<\infty \tag{64}
\end{equation*}
$$

where $[\widetilde{\Theta}(t)]_{+}=\max \{\widetilde{\Theta}(t), 0\}$, then the trivial solution of (2) is stable.

Proof. Suppose firstly that $\widetilde{\beta}$ is nonincreasing on $[T, \infty)$. Then, $\widetilde{\beta}^{\prime} \leq 0$ a.e. on $[T, \infty)$.

If $\widetilde{\beta}\left(T_{2}\right)=0$ for some $T_{2} \geq T$, then $\widetilde{\beta}(t) \equiv 0$ on $\left[T_{2}, \infty\right)$. Consequently, $\widetilde{\Lambda}$ has to satisfy only the inequality $\widetilde{\Theta}(t) \leq \widetilde{\Lambda}(t)$ a.e. on $\left[T_{2}, \infty\right)$, so we may choose $\widetilde{\Lambda}(t)=\widetilde{\Theta}(t)$ on $\left[T_{2}, \infty\right)$. It follows that $\widetilde{\Lambda}(t)=\widetilde{\Theta}(t) \leq \max \{\widetilde{\Theta}(t), 0\}=[\widetilde{\Theta}(t)]_{+}$.

On the other hand, if $\widetilde{\beta}(t)>0$ on $[T, \infty)$, we may put $\widetilde{\Lambda}(t)=\max \left\{\widetilde{\Theta}(t), \widetilde{\beta}^{\prime}(t) / \widetilde{\beta}(t)\right\}$. Then,

$$
\begin{equation*}
\widetilde{\Lambda}(t)=\max \left\{\widetilde{\Theta}(t), \frac{\widetilde{\beta}^{\prime}(t)}{\widetilde{\beta}(t)}\right\} \leq \max \{\widetilde{\Theta}(t), 0\}=[\widetilde{\Theta}(t)]_{+} \tag{65}
\end{equation*}
$$

In both cases, $\widetilde{\Lambda}$ satisfies condition (iv) and the inequality $\widetilde{\Lambda}(t) \leq[\widetilde{\Theta}(t)]_{+}$on $\left[T_{2}, \infty\right)$; hence,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int^{t} \widetilde{\Lambda}(s) d s \leq \limsup _{t \rightarrow \infty} \int^{t}[\widetilde{\Theta}(s)]_{+} d s<\infty . \tag{66}
\end{equation*}
$$

Now assume that $\widetilde{\beta}$ is nondecreasing on $[T, \infty)$. Then, $\widetilde{\beta}^{\prime} \geq 0$ a.e. on $[T, \infty)$.

If $\widetilde{\beta}(t) \equiv 0$ on $[T, \infty)$, we may treat it as previously mentioned.

Otherwise, there is some $T_{3} \geq T$ such that $\widetilde{\beta}(t)>0$ on $\left[T_{3}, \infty\right)$, and we may choose $\widetilde{\Lambda}(t)=\max \left\{\widetilde{\Theta}(t), \widetilde{\beta}^{\prime}(t) / \widetilde{\beta}(t)\right\}$ on $\left[T_{3}, \infty\right)$. Clearly $\widetilde{\Lambda}$ satisfies condition (iv) on $\left[T_{3}, \infty\right)$. Since $\widetilde{\beta}^{\prime} \geq 0$ a.e. on $[T, \infty)$, it follows that $\widetilde{\beta}^{\prime} / \widetilde{\beta} \geq 0$ a.e. on $\left[T_{3}, \infty\right)$. Hence,

$$
\begin{align*}
\widetilde{\Lambda}(t) & =\max \left\{\widetilde{\Theta}(t), \frac{\widetilde{\beta}^{\prime}(t)}{\widetilde{\beta}(t)}\right\} \\
& \leq \max \left\{[\widetilde{\Theta}(t)]_{+}, \frac{\widetilde{\beta}^{\prime}(t)}{\widetilde{\beta}(t)}\right\}  \tag{67}\\
& \leq[\widetilde{\Theta}(t)]_{+}+\frac{\widetilde{\beta}^{\prime}(t)}{\widetilde{\beta}(t)}
\end{align*}
$$

and then

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \int^{t} \widetilde{\Lambda}(s) d s \\
& \quad \leq \limsup _{t \rightarrow \infty} \int^{t}[\widetilde{\Theta}(s)]_{+} d s+\limsup _{t \rightarrow \infty} \int^{t} \frac{\widetilde{\beta}^{\prime}(t)}{\widetilde{\beta}(t)} d s  \tag{68}\\
& \quad \leq \limsup _{t \rightarrow \infty} \int^{t}[\widetilde{\Theta}(s)]_{+} d s+\limsup _{t \rightarrow \infty}(\ln (\widetilde{\beta}(t))) \\
& \quad-\ln \left(\widetilde{\beta}\left(T_{3}\right)\right)<\infty,
\end{align*}
$$

since $\widetilde{\beta}$ is bounded on $[T, \infty)$.
The statement follows from Theorem 2.

We can derive several consequences from Theorem 4.

Corollary 11. Let the conditions (i), (ii), (iii), and (iv) be fulfilled and

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{s}^{t} \widetilde{\varrho}(\xi) \exp \left(-\int_{s}^{\xi} \widetilde{\Lambda}(\sigma) d \sigma\right) d \xi<\infty \tag{69}
\end{equation*}
$$

for some $s \geq T$.
If $z(t)$ is any solution of (2) existing for $t \rightarrow \infty$, then

$$
\begin{equation*}
z(t)=O\left[\exp \left(\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right)\right] \tag{70}
\end{equation*}
$$

Proof. From the assumptions and (45) we can see that there are $K>0$ and $S \geq s$ such that for $t \geq S$ we have

$$
\begin{aligned}
& V(t) \exp \left(-\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right)-V(s) \\
& \quad \leq \int_{s}^{t} \widetilde{\varrho}(\xi) \exp \left(-\int_{s}^{\xi} \widetilde{\Lambda}(\sigma) d \sigma\right) d \xi \\
& \quad \leq K<\infty
\end{aligned}
$$

Then,

$$
\begin{equation*}
\mu|z(t)| \leq V(t) \leq(K+V(s)) \exp \left(\int_{s}^{t} \widetilde{\Lambda}(\xi) d \xi\right) \tag{72}
\end{equation*}
$$

Corollary 12. Let the assumptions (i), (ii), (iii), and (iv) hold, and let

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \widetilde{\Lambda}(t)<\infty, \quad \widetilde{\varrho}(t)=O\left(e^{\eta t}\right) \tag{73}
\end{equation*}
$$

where $\eta>\lim \sup _{t \rightarrow \infty} \tilde{\Lambda}(t)$. If $z(t)$ is any solution of (2) existing for $t \rightarrow \infty$, then $z(t)=O\left(e^{\eta t}\right)$.

Proof. In view of (73), there are $L>0, \eta^{*}<\eta$, and $s>T$ such that $\eta^{*}>\widetilde{\Lambda}(t)$ for $t \geq s$ and $\widetilde{\varrho}(t) e^{-\eta t}<L$ for $t \geq s$. From (41) we get

$$
\begin{align*}
\mu|z(t)| & \leq V(s) e^{\eta^{*}(t-s)}+L \int_{s}^{t} e^{\eta \tau} e^{\eta^{*}(t-\tau)} d \tau \\
& \leq V(s) e^{\eta^{*}(t-s)}+L e^{\eta^{*} t} \frac{e^{\left(\eta-\eta^{*}\right) t}-e^{\left(\eta-\eta^{*}\right) s}}{\eta-\eta^{*}}  \tag{74}\\
& \leq V(s) e^{\eta^{*}(t-s)}+\frac{L}{\eta-\eta^{*}} e^{\eta t}=O\left(e^{\eta t}\right) .
\end{align*}
$$

Remark 13. If $\widetilde{\rho}(t) \equiv 0$, we can take $L=0$ in the proof of Corollary 12, and taking inequalities (74) into account we obtain the following statement: there is an $\eta^{*}<\eta_{0}<\eta$ such that $z(t)=o\left(e^{\eta_{0} t}\right)$ holds for the solution $z(t)$ of (2).

## 5. Conclusion

We studied asymptotic behavior of real two-dimensional differential system with a finite number of nonconstant delays. We considered the case corresponding to the situation
when the equilibrium point 0 of autonomous system (6) is a stable focus or a stable centre. We utilized the method of complexification and the method of Lyapunov-Krasovskii functional. Criteria for stability and asymptotic stability of the solutions as well as conditions ensuring that all solutions of (2) tend to zero are derived. At the end we supplied several corollaries and an example which shows that in some cases the criteria obtained in this paper are more applicable than the criteria presented in [21].

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## Research Article

# Multiple Nonlinear Oscillations in a $\mathbb{D}_{3} \times \mathbb{D}_{3}$-Symmetrical Coupled System of Identical Cells with Delays 

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#### Abstract

A coupled system of nine identical cells with delays and $\mathbb{D}_{3} \times \mathbb{D}_{3}$-symmetry is considered. The individual cells are modelled by a scalar delay differential equation which includes linear decay and nonlinear delayed feedback. By analyzing the corresponding characteristic equations, the linear stability of the equilibrium is given. We also investigate the simultaneous occurrence of multiple periodic solutions and spatiotemporal patterns of the bifurcating periodic oscillations by using the equivariant bifurcation theory of delay differential equations combined with representation theory of Lie groups. Numerical simulations are carried out to illustrate our theoretical results.


## 1. Introduction

Over the past decades, symmetry has become a topic of considerable attention in the research of nonlinear dynamical systems [1-11]. In general, the symmetry reflects a certain spatial invariant of the dynamical systems. The work of Golubitsky et al. [1] shows that systems with symmetry can lead to multiple patterns of oscillation, which are predictable based on the theory of equivariant bifurcations. It is well known that the introduction of time delays into some systems is more reasonable and realistic [12]. Wu and coworkers $[3,13,14]$ extended the theory of equivariant Hopf bifurcation to delay differential equations.

An artificial neural network is an information processing device that is inspired by the way biological nervous systems, such as the brain, process information simultaneously. It has many applications in different areas including pattern recognition, associative memory, signal processing, and combinatorial optimization. There has been an increasing interest in the investigation of neural networks (see, e.g., [4-6, 9-11, 15]) since Hopfield [16] constructed a simplified neural network model. Ring networks have been found in a variety of neural structures such as cerebellum [17] and even in chemistry and electrical engineering. In the field of neural networks, rings are studied to gain insight into the mechanisms underlying the behavior of recurrent networks [15, 18]. The dynamical
behavior of ring networks has been investigated in more detail. For example, in order to understand which patterns occur in a particular system, Huang and Wu [4] studied the following ring neural network of three identical neurons with delayed feedback:

$$
\begin{align*}
\dot{x}_{i}(t)= & -x_{i}(t)+f\left(x_{i}(t-\tau)\right) \\
& -\left[g\left(x_{i-1}(t-\tau)\right)+g\left(x_{i+1}(t-\tau)\right)\right], \quad i(\bmod 3), \tag{1}
\end{align*}
$$

where $\dot{x}=\mathrm{d} x / \mathrm{d} t, x_{i}(t)$ represents the state of the $i$ th neuron at time $t, f$ represents the nonlinear self-feedback function, $g$ is the connection function between neurons, and $\tau \geq 0$ is the time delay. Afterward, some researchers have been studying many ring networks with $\mathbb{D}_{n}$-symmetry (see [5-8]). However, previous work just has considered the individual network but not investigated the interactions between multiple networks.

In fact, a wide variety of natural and artificial systems possess a hierarchic structure or functioning and can naturally be modeled by coupled subnetwork. For example, the brain may be conceived as a dynamic network of coupled neurons. In order to describe the complicated interaction between billions of neurons in large neural network, the neurons are often lumped into highly connected subnetworks [19]. Coupled networks of nonlinear dynamical systems can


Figure 1: Architecture of model (2).
exhibit rich dynamics, such as synchronization, symmetric bifurcation, and chaos. The spatio-temporal dynamics of systems of several coupled nonlinear oscillators is presently receiving great attention and a significant body of research has been carried out [ $9-11,20,21$ ]. It must be pointed out that the hierarchical network of neuronal oscillators with $\mathbb{D}_{3} \times \mathbb{D}_{3}$ symmetry investigated in $[20,21]$ is described by a system of ordinary differential equations (ODEs), and the effect of time delays is not considered.

Motivated by the above ideas, in this paper, we consider the two-level hierarchical system which is composed of three coupled modules of interacting nonlinear neuron oscillators with time delays, modeled by the following system of delay differential equations (DDEs):

$$
\begin{align*}
\dot{x}_{i, j}(t)= & -x_{i, j}(t)+f\left(x_{i, j}(t-\tau)\right)+g\left(x_{i, j-1}(t-\tau)\right) \\
& +g\left(x_{i, j+1}(t-\tau)\right)+h\left(x_{i-1, j}(t-\tau)\right)  \tag{2}\\
& +h\left(x_{i+1, j}(t-\tau)\right), \quad i, j=0,1,2(\bmod 3)
\end{align*}
$$

where $f, g, h \in C^{1}(\mathbb{R}, \mathbb{R}), f(0)=g(0)=h(0)=0$, and $h$ represents the connection function between different modules. The individual elements are represented by a scalar equation, consisting of a linear decay term and a nonlinear time-delayed self-feedback. The architecture of the model is given in Figure 1.

It is easy to see that all cells are identical, all couplings within each group are identical, and all groups are identically coupled to each other in this model. Therefore, these lead to a $\mathbb{D}_{3} \times \mathbb{D}_{3}$-symmetry of the associated system. the model is a natural extension of system (1) and is a particularly simple example of a symmetric system exhibiting a hierarchical structure with two levels: a "macro" level concerning the interactions between the groups and a "micro" level concerning the interactions within the groups. On the other hand, system (2) can be regarded as a special example of the general Hopfield neural networks with delays [16].

Although model (2) is a little simple, it would be of great significance for applications to have a detailed analysis and then to understand possible mechanisms behind the observed behaviour. In this paper, our main purpose is to reveal how the time delay can affect the stability of system (2),
the simultaneous occurrence of multiple periodic solutions, and spatio-temporal patterns of the bifurcating periodic oscillations depending on the $\mathbb{D}_{3} \times \mathbb{D}_{3}$-symmetry.

The rest of the paper is organized as follows. In Section 2, the associated characteristic equation is analyzed and the linear stability of the equilibrium is given. In Section 3, we discuss the existence of multiple branches of periodic oscillations and their spatio-temporal patterns with the help of symmetric bifurcation theory of delay differential equations coupled with representation theory of Lie groups. An example and numerical simulations are presented to illustrate the results in Section 4. In Section 5, a brief discussion is drawn to conclude this paper.

## 2. Distribution of Characteristic Roots and Linear Stability

It is clear that (2) admits the trivial solution, $\widehat{x}=0$. The linearization of (2) at this equilibrium point is given by

$$
\begin{align*}
\dot{x}_{i, j}(t)= & -x_{i, j}(t)+a x_{i, j}(t-\tau)+b x_{i, j-1}(t-\tau) \\
& +b x_{i, j+1}(t-\tau)+c x_{i-1, j}(t-\tau)  \tag{3}\\
& +c x_{i+1, j}(t-\tau), \quad i, j=0,1,2(\bmod 3),
\end{align*}
$$

where $a=f^{\prime}(0), b=g^{\prime}(0)$, and $c=h^{\prime}(0)$. Regarding $\tau$ as the parameter, let $A(\tau)$ denote the infinitesimal generator of the semigroup generated by linear system (3). We first determine when $A(\tau)$ has a pair of purely imaginary eigenvalues.

The characteristic matrix of (3) is

$$
\begin{equation*}
\Delta(\tau, \lambda)=(\lambda+1) \mathrm{Id}_{9}-M \mathrm{e}^{-\lambda \tau}, \quad \lambda \in \mathbb{C} \tag{4}
\end{equation*}
$$

where $\mathrm{Id}_{n}$ denotes the identity matrix of order $n, M=$ $\operatorname{circ}\left(M_{1}, c \operatorname{Id}_{3}, c \operatorname{Id}_{3}\right)$ is a circle block matrix, and $M_{1}=$ $\operatorname{circ}(a, b, b)$ is a circulant matrix of order 3. Then we have the following lemma.

Lemma 1. The associated characteristic equation of (3) is

$$
\begin{equation*}
\operatorname{det} \Delta(\tau, \lambda)=\prod_{q=0}^{2} \prod_{p=0}^{2} \Delta_{p q}=0 \tag{5}
\end{equation*}
$$

where $\Delta_{p q}=\lambda+1-(a+2 b \cos (2 q \pi / 3)+2 c \cos (2 p \pi / 3)) e^{-\lambda \tau}$.
Proof. Let $\chi=\mathrm{e}^{\mathrm{i}(2 \pi / 3)}, v_{q}=\left(1, \chi^{q}, \chi^{2 q}\right)^{\mathrm{T}}$, and $v_{p q}=$ $\left(v_{q}, \chi^{p} v_{q}, \chi^{2 p} v_{q}\right)^{\mathrm{T}}$. Then

$$
\begin{equation*}
M v_{p q}=\left(a+2 b \cos \frac{2 q \pi}{3}+2 c \cos \frac{2 p \pi}{3}\right) v_{p q} \tag{6}
\end{equation*}
$$

Hence

$$
\begin{align*}
\Delta & (\tau, \lambda) v_{p q} \\
& =\left[\lambda+1-\left(a+c \chi^{p}+c \chi^{-p}+b \chi^{q}+b \chi^{-q}\right) \mathrm{e}^{-\lambda \tau}\right] v_{p q} \\
& =\left[\lambda+1-\left(a+2 b \cos \frac{2 q \pi}{3}+2 c \cos \frac{2 p \pi}{3}\right) \mathrm{e}^{-\lambda \tau}\right] v_{p q} \tag{7}
\end{align*}
$$

The conclusion is obtained.

## Note that

$$
\begin{align*}
& \Delta_{00}=\lambda+1-(a+2 b+2 c) \mathrm{e}^{-\lambda \tau} \\
& \Delta_{01}=\Delta_{02}=\lambda+1-(a+2 c-b) \mathrm{e}^{-\lambda \tau}  \tag{8}\\
& \Delta_{10}=\Delta_{20}=\lambda+1-(a+2 b-c) \mathrm{e}^{-\lambda \tau} \\
& \Delta_{11}=\Delta_{12}=\Delta_{21}=\Delta_{22}=\lambda+1-(a-b-c) \mathrm{e}^{-\lambda \tau},
\end{align*}
$$

we have

$$
\begin{equation*}
\operatorname{det} \Delta(\tau, \lambda)=\Delta_{00}\left(\Delta_{01}\right)^{2}\left(\Delta_{10}\right)^{2}\left(\Delta_{11}\right)^{4} \tag{9}
\end{equation*}
$$

In this paper, for the sake of simplicity, we consider only the case where characteristic equation (5) may have a pair of purely imaginary roots of multiplicity 4 , that is, focus on the distribution of zeros of the factor $\Delta_{11}$. The other cases are simpler and can be handled in a similar way, and we omit them. Therefore, throughout this paper we suppose

$$
\begin{aligned}
& \text { (H) }|a+2 b+2 c|<1,|a+2 c-b|<1,|a+2 b-c|<1 \text {, } \\
& a-b-c<-1 \text {. }
\end{aligned}
$$

The following result about the distribution of the characteristic roots is obtained.

Lemma 2. Assume that (H) holds. Define

$$
\begin{gather*}
\tau_{s}=\frac{\arccos (1 /(a-b-c))+2 s \pi}{\sqrt{(a-b-c)^{2}-1}}, \quad s=0,1,2, \ldots,  \tag{10}\\
\beta=\sqrt{(a-b-c)^{2}-1} \tag{11}
\end{gather*}
$$

Then
(i) for all $\tau \geqslant 0$, all zeros of the factors $\Delta_{00}, \Delta_{10}$, and $\Delta_{01}$ have negative real parts,
(ii) when $\tau \in\left[0, \tau_{0}\right)$, all roots of the characteristic equation (5) have negative real parts; when $\tau \in\left(\tau_{s}, \tau_{s+1}\right)$, the characteristic equation (5) has exactly $2 s+2$ roots with positive real parts; the other roots have negative real parts; at (and only at) $\tau=\tau_{s}, A(\tau)$ has a pair of purely imaginary eigenvalues $\pm i \beta$ of multiplicity 4 , and all other eigenvalues of $A(\tau)$ are not integer multiples of $\pm i \beta$,
(iii) for each fixed $s \in \mathbb{N}$, there exist $\delta>0$ and $C^{1}$-smooth mapping $\lambda:\left(\tau_{s}-\delta, \tau_{s}+\delta\right) \rightarrow \mathbb{C}$ such that $\lambda\left(\tau_{s}\right)=i \beta$ and $\lambda(\tau)+1-(a-b-c) e^{-\tau \lambda(\tau)}=0$ for all $\tau \in\left(\tau_{s}-\right.$ $\left.\delta, \tau_{s}+\delta\right)$. Moreover, $\left.(d / d \tau) \operatorname{Re} \lambda(\tau)\right|_{\tau=\tau_{s}}>0$,
(iv) the generalized eigenspace $U_{ \pm i \beta}\left(A\left(\tau_{s}\right)\right)$ consists of eigenvector of $A\left(\tau_{s}\right)$ associated with $\pm i \beta$. Moreover,

$$
\begin{align*}
& U_{ \pm i \beta}\left(A\left(\tau_{s}\right)\right) \\
& =\left\{\sum_{q=1}^{2} \sum_{p=1}^{2}\left(y_{p q} \epsilon^{p q}+z_{p q} \varsigma^{p q}\right), y_{p q}, z_{p q} \in \mathbb{R}, p, q=1,2\right\} \tag{12}
\end{align*}
$$

where

$$
\begin{align*}
\epsilon^{p q}(\theta) & =\operatorname{Re}\left\{e^{i \beta \theta} v_{p q}\right\} \\
& =\cos (\beta \theta) \operatorname{Re}\left\{v_{p q}\right\}-\sin (\beta \theta) \operatorname{Im}\left\{v_{p q}\right\} \\
\varsigma^{p q}(\theta) & =\operatorname{Im}\left\{e^{i \beta \theta} v_{p q}\right\}  \tag{13}\\
& =\sin (\beta \theta) \operatorname{Re}\left\{v_{p q}\right\}+\cos (\beta \theta) \operatorname{Im}\left\{v_{p q}\right\}
\end{align*}
$$

$$
\text { for } \theta \in[-\tau, 0]
$$

Proof. For $\gamma \in \mathbb{R}$, let $q_{\gamma}(\lambda)=\lambda+1-\gamma \mathrm{e}^{-\lambda \tau}$. If $\lambda=\alpha+\mathrm{i} \beta$ is a zero of $q_{\gamma}(\lambda)$, then $1+\alpha+\mathrm{i} \beta=\gamma \mathrm{e}^{-(\alpha+\mathrm{i} \beta) \tau}$, from which it follows that

$$
\begin{equation*}
(1+\alpha)^{2}+\beta^{2}=\gamma^{2} \mathrm{e}^{-2 \alpha \tau} \tag{14}
\end{equation*}
$$

Thus, we claim that all the zeros of $q_{\gamma}(\lambda)$ have negative real parts provided that $|\gamma|<1$. Noticing that (H) holds and applying the above discussions to the factors $\Delta_{00}, \Delta_{10}$, and $\Delta_{01}$, we obtain conclusion (i).

In what follows, we consider the distribution of zeros of the factor $\Delta_{11}$. For $\tau=0, \Delta_{11}=0$ becomes $\lambda=a-b-c-1<$ -2 . Let $\mathrm{i} \beta(\beta>0)$ be a root of $\Delta_{11}=0$, then

$$
\begin{align*}
& \mathrm{i} \beta+1-(b+c-a) \mathrm{e}^{\mathrm{i}(\pi-\tau \beta)} \\
& =\sqrt{1+\beta^{2}} \mathrm{e}^{\mathrm{i} \operatorname{arc} \cos \left(1 / \sqrt{1+\beta^{2}}\right)}-(b+c-a) \mathrm{e}^{\mathrm{i}(\pi-\tau \beta)}=0 . \tag{15}
\end{align*}
$$

Thus,

$$
\begin{gather*}
\sqrt{1+\beta^{2}}=b+c-a \\
\arccos \frac{1}{\sqrt{1+\beta^{2}}}=\pi-\tau \beta+2 s \pi, \quad s \in \mathbb{Z} \tag{16}
\end{gather*}
$$

Therefore, $\Delta_{11}=0$ has a pair of purely imaginary roots $\pm \mathrm{i} \beta$ if and only if

$$
\begin{gather*}
\beta=\sqrt{(a-b-c)^{2}-1} \\
\tau=\frac{\operatorname{arc} \cos (1 /(a-b-c))+2 s \pi}{\sqrt{(a-b-c)^{2}-1}}  \tag{17}\\
:=\tau_{s}, \quad \text { for some } s \in \mathbb{N} .
\end{gather*}
$$

Clearly, for each fixed $s, \tau_{s} \geqslant 0$. Since

$$
\begin{align*}
\left.\frac{\mathrm{d}}{\mathrm{~d} \lambda} \Delta_{11}(\lambda)\right|_{\lambda=\mathrm{i} \beta, \tau=\tau_{s}} & =1+\left.(a-b-c) \tau \mathrm{e}^{-\lambda \tau}\right|_{\lambda=\mathrm{i} \beta, \tau=\tau_{s}} \\
& =1+\tau_{s}(\mathrm{i} \beta+1)  \tag{18}\\
& \neq 0
\end{align*}
$$

there exist $\delta>0$ and $C^{1}$-smooth mapping $\lambda:\left(\tau_{s}-\delta, \tau_{s}+\delta\right) \rightarrow$ $\mathbb{C}$ such that $\lambda\left(\tau_{s}\right)=\mathrm{i} \beta$ and $\lambda(\tau)+1-(a-b-c) \mathrm{e}^{-\tau \lambda(\tau)}=0$ for
all $\tau \in\left(\tau_{s}-\delta, \tau_{s}+\delta\right)$. Differentiating $\Delta_{11}=0$ with respect to $\tau$, we have

$$
\begin{align*}
\lambda^{\prime}\left(\tau_{s}\right) & =\frac{-(a-b-c) \lambda\left(\tau_{s}\right) \mathrm{e}^{-\lambda\left(\tau_{s}\right) \tau_{s}}}{1+(a-b-c) \tau_{s} \mathrm{e}^{-\lambda\left(\tau_{s}\right) \tau_{s}}}  \tag{19}\\
& =\frac{-\lambda\left(\tau_{s}\right)\left[\lambda\left(\tau_{s}\right)+1\right]}{1+\tau_{s}\left[\lambda\left(\tau_{s}\right)+1\right]} .
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\operatorname{Re} \lambda^{\prime}\left(\tau_{s}\right)=\frac{\beta^{2}}{\left(1+\tau_{s}\right)^{2}+\tau_{s}^{2} \beta^{2}}>0 \tag{20}
\end{equation*}
$$

So far, the proof of (ii) and (iii) is complete (for more details see XI. 2 in [22]). It remains to verify (iv).

Note that $v_{11}=\overline{v_{22}}, v_{12}=\overline{v_{21}}$; it follows from the proof of Lemma 1 and the above discussions that the eigenspace of $A\left(\tau_{s}\right)$ associated with $\pm \mathrm{i} \beta$ is spanned by $\mathrm{e}^{\mathrm{i} \beta \theta} v_{11}, \mathrm{e}^{\mathrm{i} \beta \theta} v_{12}, \mathrm{e}^{\mathrm{i} \beta \theta} v_{21}, \mathrm{e}^{\mathrm{i} \beta \theta} v_{22}, \mathrm{e}^{-\mathrm{i} \beta \theta} v_{11}, \mathrm{e}^{-\mathrm{i} \beta \theta} v_{12}, \mathrm{e}^{-\mathrm{i} \beta \theta} v_{21}$, and $\mathrm{e}^{-\mathrm{i} \beta \theta} v_{22}$. Hence, the space has the real basis $\left\{\epsilon^{11}, \varsigma^{11}\right.$, $\left.\epsilon^{12}, \varsigma^{12}, \epsilon^{21}, \varsigma^{21}, \epsilon^{22}, \varsigma^{22}\right\}$. On the other hand, the eigenspace $A\left(\tau_{s}\right)$ associated with $\mathrm{i} \beta_{p q}$ is of dimension 4, and the multiplicity of the characteristic root $\lambda=\mathrm{i} \beta_{p q}$ is also 4 . According to the folk theorem in functional differential equations (see [23]), $U_{ \pm \mathrm{i} \beta}\left(A\left(\tau_{s}\right)\right)$ must coincide with the eigenspace of $A\left(\tau_{s}\right)$ associated with $\pm i \beta$. Therefore, conclusion (iv) is correct. This completes the proof.

From Lemma 2 (ii), we can draw the following conclusion about the linear stability of system (2).

Theorem 3. If the assumption $(\mathbf{H})$ is satisfied, then the equilibrium $\widehat{x}=0$ of system (2) is asymptotically stable for $\tau \in\left[0, \tau_{0}\right)$; the equilibrium $\widehat{x}=0$ is unstable for $\tau>\tau_{0}$.

## 3. Multiple Patterns of Oscillation

It follows from Lemma 2 that $A(\tau)$ has a pair of purely imaginary eigenvalues $\pm \mathrm{i} \beta$ of multiplicity 4 and all other eigenvalues of $A(\tau)$ are not integer multiples of $\pm \mathrm{i} \beta$ at $\tau=$ $\tau_{s}(s=0,1, \ldots)$. Thus, the Hopf bifurcation may provide some asynchronous periodic solutions at each $\tau_{s}$.

The symmetry of a system is important in determining the patterns of oscillation. In order to explore the symmetry of system and analyze the spatio-temporal patterns of the bifurcated periodic solutions, we need the following definition.

Let $\mathbb{D}_{3}$ denote the dihedral group of order 6 , which is generated by cyclic group $\mathbb{Z}_{3}$ together with the flip $\kappa$ of order 2. Define the action of $\Gamma:=\mathbb{D}_{3} \times \mathbb{D}_{3}$ on $\mathbb{R}^{9}$ by

$$
\begin{gather*}
((\rho, 1) \cdot x)_{i, j}=x_{i+1, j}, \quad((1, \rho) \cdot x)_{i, j}=x_{i, j+1} \\
((\kappa, 1) \cdot x)_{i, j}=x_{3-i, j}, \quad((1, \kappa) \cdot x)_{i, j}=x_{i, 3-j}, \quad i, j \bmod 3 \tag{21}
\end{gather*}
$$

Lemma 4. System (2) is $\Gamma$-equivalent.
Proof. Let mapping $\mathscr{F}: C\left([-\tau, 0], \mathbb{R}^{9}\right) \rightarrow \mathbb{R}^{9}$ be

$$
\begin{align*}
(\mathscr{F}(\phi))_{i, j}= & -(\phi)_{i, j}(0)+f\left((\phi)_{i, j}(-\tau)\right) \\
& +g\left((\phi)_{i, j-1}(-\tau)\right)+g\left((\phi)_{i, j+1}(-\tau)\right)  \tag{22}\\
& +h\left((\phi)_{i-1, j}(-\tau)\right) \\
& +h\left((\phi)_{i+1, j}(-\tau)\right), \quad i, j(\bmod 3),
\end{align*}
$$

where

$$
\begin{gather*}
\phi=\left(\phi_{00}, \phi_{01}, \phi_{02}, \phi_{10}, \phi_{11}, \phi_{12}, \phi_{20}, \phi_{21}, \phi_{22}\right)^{\mathrm{T}} \\
\in C\left([-\tau, 0], \mathbb{R}^{9}\right), \\
\mathscr{F}(\phi)=\left((\mathscr{F}(\phi))_{00},(\mathscr{F}(\phi))_{01},(\mathscr{F}(\phi))_{02},\right.  \tag{23}\\
\\
(\mathscr{F}(\phi))_{10},(\mathscr{F}(\phi))_{11},(\mathscr{F}(\phi))_{12}, \\
\\
\left.(\mathscr{F}(\phi))_{20},(\mathscr{F}(\phi))_{21},(\mathscr{F}(\phi))_{22}\right)^{\mathrm{T}} \in \mathbb{R}^{9} .
\end{gather*}
$$

Then

$$
\begin{aligned}
(\mathscr{F} & ((\rho, 1) \phi))_{i, j} \\
= & -((\rho, 1) \phi)_{i, j}(0)+f\left(((\rho, 1) \phi)_{i, j}(-\tau)\right) \\
& +g\left(((\rho, 1) \phi)_{i, j-1}(-\tau)\right)+g\left(((\rho, 1) \phi)_{i, j+1}(-\tau)\right) \\
& +h\left(((\rho, 1) \phi)_{i-1, j}(-\tau)\right)+h\left(((\rho, 1) \phi)_{i+1, j}(-\tau)\right) \\
= & -(\phi)_{i+1, j}(0)+f\left((\phi)_{i+1, j}(-\tau)\right) \\
& +g\left((\phi)_{i+1, j-1}(-\tau)\right) \\
& +g\left((\phi)_{i+1, j+1}(-\tau)\right)+h\left((\phi)_{i, j}(-\tau)\right) \\
& +h\left((\phi)_{i+2, j}(-\tau)\right) \\
= & ((\rho, 1) \mathscr{F}(\phi))_{i, j} \\
(\mathscr{F} & ((\kappa, 1) \phi)_{i, j} \\
= & -((\kappa, 1) \phi)_{i, j}(0)+f\left(((\kappa, 1) \phi)_{i, j}(-\tau)\right) \\
& +g\left(((\kappa, 1) \phi)_{i, j-1}(-\tau)\right)+g\left(((\kappa, 1) \phi)_{i, j+1}(-\tau)\right) \\
& +h\left(((\kappa, 1) \phi)_{i-1, j}(-\tau)\right)+h\left(((\kappa, 1) \phi)_{i+1, j}(-\tau)\right) \\
= & -(\phi)_{3-i, j}(0)+f\left((\phi)_{3-i, j}(-\tau)\right) \\
& +g\left((\phi)_{3-i, j-1}(-\tau)\right)+g\left((\phi)_{3-i, j+1}(-\tau)\right) \\
& +h\left((\phi)_{4-i, j}(-\tau)\right)+h\left((\phi)_{2-i, j}(-\tau)\right)
\end{aligned}
$$

$$
\begin{equation*}
=((\kappa, 1) \mathscr{F}(\phi))_{i, j} \tag{24}
\end{equation*}
$$

Similarly, we can prove that

$$
\begin{align*}
& (\mathscr{F}((1, \rho) \phi))_{i, j}=((1, \rho) \mathscr{F}(\phi))_{i, j}  \tag{25}\\
& (\mathscr{F}((1, \kappa) \phi))_{i, j}=((1, \kappa) \mathscr{F}(\phi))_{i, j} .
\end{align*}
$$

Therefore, $\mathscr{F}$ is $\mathbb{D}_{3} \times \mathbb{D}_{3}$-equivalent. This completes the proof.

Lemma 5. Let $\Gamma$ act on $\mathbb{R}^{4}$ by

$$
\begin{align*}
(\rho, 1) \cdot\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right)=\left(\begin{array}{cccc}
-\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\
0 & -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 \\
0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2}
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right), \\
(1, \rho) \cdot\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right)=\left(\begin{array}{ccc}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\
0 \\
\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\
0 & 0 & -\frac{1}{2} \\
-\frac{\sqrt{3}}{2} \\
0 & 0 & \frac{\sqrt{3}}{2} \\
\hline
\end{array}\right)\left(\begin{array}{l}
-\frac{1}{2}
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right), \\
(\kappa, 1) \cdot\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right)=\left(\begin{array}{c}
x_{1} \\
x_{2} \\
-x_{3} \\
-x_{4}
\end{array}\right), \\
(1, \kappa) \cdot\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right)=\left(\begin{array}{c}
x_{1} \\
-x_{2} \\
x_{3} \\
-x_{4}
\end{array}\right) \tag{26}
\end{align*}
$$

Then $\mathbb{R}^{4}$ is an absolutely irreducible representation of $\Gamma$, and the restricted action of $\Gamma$ on $\operatorname{Ker} \Delta(\tau, i \beta)$ is isomorphic to the action of $\Gamma$ on $\mathbb{R}^{4} \oplus \mathbb{R}^{4}$.

Proof. It is straightforward to verify the absolute irreducibility of the representation of $\Gamma$ on $\mathbb{R}^{4}$ by the definition (see [1]). Note that

$$
\begin{align*}
& \text { Ker } \Delta(\tau, \mathrm{i} \beta) \\
& =\left\{\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}, a_{p q}, b_{p q} \in \mathbb{R}, p, q=1,2\right\} . \tag{27}
\end{align*}
$$

Define $J: \operatorname{Ker} \Delta(\tau, \mathrm{i} \beta) \rightarrow \mathbb{R}^{4} \oplus \mathbb{R}^{4}$ by

$$
\begin{align*}
\sum_{p=1}^{2} & \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}  \tag{28}\\
& \longmapsto B\left(a_{11}, b_{11}, a_{12}, b_{12}, a_{21}, b_{21}, a_{22}, b_{22}\right)^{\mathrm{T}}
\end{align*}
$$

where the matrix

$$
B=\left(\begin{array}{cccccccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0  \tag{29}\\
0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\
0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\
-1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 & 0 & -1
\end{array}\right)
$$

The nonsingularity of the matrix $B$ implies that $J$ : $\operatorname{Ker} \Delta(\tau, \mathrm{i} \beta) \cong \mathbb{R}^{4}$ is a linear isomorphism. It is easy to see that

$$
\begin{align*}
& (\rho, 1) \cdot\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right) \\
& =\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) \mathrm{e}^{\mathrm{i}\left((-1)^{p+1} 2 \pi / 3\right)} v_{p q}, \\
& (1, \rho) \cdot\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right) \\
& = \\
& \sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) \mathrm{e}^{\mathrm{i}\left((-1)^{q+1} 2 \pi / 3\right)} v_{p q}  \tag{30}\\
& (\kappa, 1) \cdot \\
& =\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right) \\
& = \\
& \quad+\left(a_{21}+\mathrm{i} b_{21}\right) v_{11}+\left(a_{21}+\mathrm{i} b_{11}\right) v_{21}+\left(a_{12}+\mathrm{i} b_{12}\right) v_{12} \\
& (1, \kappa) \cdot \\
& \\
& \begin{aligned}
& \left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right) \\
= & \left(a_{12}+\mathrm{i} b_{12}\right) v_{11}+\left(a_{11}+\mathrm{i} b_{11}\right) v_{12} \\
& +\left(a_{22}+\mathrm{i} b_{22}\right) v_{21}+\left(a_{21}+\mathrm{i} b_{21}\right) v_{22} .
\end{aligned}
\end{align*}
$$

Therefore, a straightforward calculation shows that

$$
\begin{aligned}
& J\left((\rho, 1) \cdot\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right)\right) \\
& \quad=(\rho, 1) \cdot J\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right), \\
& J\left((1, \rho) \cdot\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right)\right) \\
& \quad=(1, \rho) \cdot J\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right),
\end{aligned}
$$

$$
\begin{align*}
& J\left((\kappa, 1) \cdot\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right)\right) \\
& \quad=(\kappa, 1) \cdot J\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right), \\
& J\left((1, \kappa) \cdot\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right)\right) \\
& \quad=(1, \kappa) \cdot J\left(\sum_{p=1}^{2} \sum_{q=1}^{2}\left(a_{p q}+\mathrm{i} b_{p q}\right) v_{p q}\right) . \tag{31}
\end{align*}
$$

This concludes the proof.

Let $\omega=2 \pi / \beta$ and let $P_{\omega}$ be the Banach space of all continuous $\omega$-periodic functions $x: \mathbb{R} \rightarrow \mathbb{R}^{9}$. Then for the circle group $S^{1}, \Gamma \times S^{1}$ acts on $P_{\omega}$ by

$$
\left.\begin{array}{r}
\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right) \cdot x(t)=\gamma \cdot x\left(t+\frac{\omega}{2 \pi} \theta\right)  \tag{32}\\
\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right)
\end{array}\right) \in \Gamma \times S^{1}, x \in P_{\omega} .
$$

Denote by $S P_{\omega}$ the subspace of $P_{\omega}$ consisting of all $\omega$-periodic solutions of system (3) with $\tau=\tau_{s}$. Then

$$
\begin{align*}
& S P_{\omega} \\
& =\left\{\sum_{q=1}^{2} \sum_{p=1}^{2}\left(y_{p q} \epsilon^{p q}+z_{p q} s^{p q}\right), y_{p q}, z_{p q} \in \mathbb{R}, p, q=1,2\right\}, \tag{33}
\end{align*}
$$

where

$$
\begin{align*}
\epsilon^{p q}(t) & =\operatorname{Re}\left\{\mathrm{e}^{\mathrm{i} \beta t} v_{p q}\right\} \\
& =\cos (\beta t) \operatorname{Re}\left\{v_{p q}\right\}-\sin (\beta t) \operatorname{Im}\left\{v_{p q}\right\}, \\
\varsigma^{p q}(t) & =\operatorname{Im}\left\{\mathrm{e}^{\mathrm{i} \beta t} v_{p q}\right\}  \tag{34}\\
& =\sin (\beta t) \operatorname{Re}\left\{v_{p q}\right\}+\cos (\beta t) \operatorname{Im}\left\{v_{p q}\right\} .
\end{align*}
$$

Therefore, for $\epsilon^{p q}(t), \varsigma^{p q}(t)(p, q=1,2)$, we have the following properties.

## Lemma 6.

$$
\begin{align*}
& (\rho, 1) \cdot \epsilon^{p q}=\epsilon^{p q} \cos \frac{2 p \pi}{3}-\varsigma^{p q} \sin \frac{2 p \pi}{3}, \\
& (\rho, 1) \cdot \varsigma^{p q}=\varsigma^{p q} \cos \frac{2 p \pi}{3}+\epsilon^{p q} \sin \frac{2 p \pi}{3}, \\
& (1, \rho) \cdot \epsilon^{p q}=\epsilon^{p q} \cos \frac{2 q \pi}{3}-\varsigma^{p q} \sin \frac{2 q \pi}{3},  \tag{35}\\
& (1, \rho) \cdot \varsigma^{p q}=\varsigma^{p q} \cos \frac{2 q \pi}{3}+\epsilon^{p q} \sin \frac{2 q \pi}{3}, \\
& (\kappa, 1) \cdot \epsilon^{p q}=\epsilon^{3-p, q}, \quad(\kappa, 1) \cdot \varsigma^{p q}=\varsigma^{3-p, q}, \\
& (1, \kappa) \cdot \epsilon^{p q}=\epsilon^{p, 3-q}, \quad(1, \kappa) \cdot \varsigma^{p q}=\varsigma^{p, 3-q} .
\end{align*}
$$

Proof. For $i, j(\bmod 3)$ and $t \in \mathbb{R}$, note that

$$
\begin{align*}
& \epsilon_{i, j}^{p q}(t)=\cos \left(\beta t+\frac{2 i p \pi}{3}+\frac{2 j q \pi}{3}\right), \\
& \varsigma_{i, j}^{p q}(t)=\sin \left(\beta t+\frac{2 i p \pi}{3}+\frac{2 j q \pi}{3}\right), \tag{36}
\end{align*}
$$

we have

$$
\begin{aligned}
& \left((\rho, 1) \cdot \epsilon^{p q}(t)\right)_{i, j} \\
& \quad=\epsilon_{i+1, j}^{p q}(t)=\cos \left(\beta t+\frac{2(i+1) p \pi}{3}+\frac{2 j q \pi}{3}\right) \\
& \quad=\cos \left(\beta t+\frac{2 i p \pi}{3}+\frac{2 j q \pi}{3}+\frac{2 p \pi}{3}\right) \\
& \quad=\epsilon_{i, j}^{p q} \cos \frac{2 p \pi}{3}-\epsilon_{i, j}^{p q} \sin \frac{2 p \pi}{3},
\end{aligned}
$$

$$
\left((\rho, 1) \cdot \varsigma^{p q}(t)\right)_{i, j}
$$

$$
=\epsilon_{i+1, j}^{p q}(t)=\sin \left(\beta t+\frac{2(i+1) p \pi}{3}+\frac{2 j q \pi}{3}\right)
$$

$$
=\sin \left(\beta t+\frac{2 i p \pi}{3}+\frac{2 j q \pi}{3}+\frac{2 p \pi}{3}\right)
$$

$$
=c_{i, j}^{p q} \cos \frac{2 p \pi}{3}+\epsilon_{i, j}^{p q} \sin \frac{2 p \pi}{3}
$$

$$
\left((\kappa, 1) \cdot \epsilon^{p q}(t)\right)_{i, j}
$$

$$
\begin{aligned}
& =\epsilon_{3-i, j}^{p q}(t)=\cos \left(\beta t+\frac{2(3-i) p \pi}{3}+\frac{2 j q \pi}{3}\right) \\
& =\cos \left(\beta t+\frac{2 i(3-p) \pi}{3}+\frac{2 j q \pi}{3}\right)=\epsilon_{i, j}^{3-p, q}
\end{aligned}
$$

$$
\left((\kappa, 1) \cdot \zeta^{p q}(t)\right)_{i, j}
$$

$$
=\epsilon_{3-i, j}^{p q}(t)=\sin \left(\beta t+\frac{2(3-i) p \pi}{3}+\frac{2 j q \pi}{3}\right)
$$

$$
\begin{equation*}
=\sin \left(\beta t+\frac{2 i(3-p) \pi}{3}+\frac{2 j q \pi}{3}\right)=\varsigma_{i, j}^{3-p, q} . \tag{37}
\end{equation*}
$$

TABLE 1: The maximal isotropy subgroups of $\Gamma \times S^{1}$ and associated fixed point subspaces.

| No. | $\Sigma$ | $\operatorname{Fix}\left(\Sigma, S P_{\omega}\right)$ |
| :--- | :---: | :---: |
| 1 | $\langle(\kappa, 1,1),(1, \kappa, 1)\rangle$ | $(y, z, y, z, y, z, y, z)$ |
| 2 | $\langle(\kappa, 1,1),(1, \kappa,-1)\rangle$ | $(y, z,-y,-z, y, z,-y,-z)$ |
| 3 | $\langle(\kappa, 1,-1),(1, \kappa, 1)\rangle$ | $(y, z, y, z,-y,-z,-y,-z)$ |
| 4 | $\langle(\kappa, 1,-1),(1, \kappa,-1)\rangle$ | $(y, z,-y,-z,-y,-z, y, z)$ |
| 5 | $\left\langle\left(\rho, 1, e^{-2 \pi i / 3}\right),\left(1, \rho, e^{-2 \pi i / 3}\right)\right\rangle$ | $(y, z, 0,0,0,0,0,0)$ |
| 6 | $\left\langle\left(\rho, 1, e^{-2 \pi i / 3}\right),\left(1, \rho, e^{2 \pi i / 3}\right)\right\rangle$ | $(0,0, y, z, 0,0,0,0)$ |
| 7 | $\left\langle\left(\rho, 1, e^{2 \pi i / 3}\right),\left(1, \rho, e^{-2 \pi i / 3}\right)\right\rangle$ | $(0,0,0,0, y, z, 0,0)$ |
| 8 | $\left\langle\left(\rho, 1, e^{2 \pi i / 3}\right),\left(1, \rho, e^{2 \pi i / 3}\right)\right\rangle$ | $(0,0,0,0,0,0, y, z)$ |
| 9 | $\left\langle\left(\rho, 1, e^{-2 \pi i / 3}\right),(1, \kappa, 1)\right\rangle$ | $(y, z, y, z, 0,0,0,0)$ |
| 10 | $\left\langle\left(\rho, 1, e^{-2 \pi i / 3}\right),(1, \kappa,-1)\right\rangle$ | $(y, z,-y,-z, 0,0,0,0)$ |
| 11 | $\left\langle\left(\rho, 1, e^{2 \pi i / 3}\right),(1, \kappa, 1)\right\rangle$ | $(0,0,0,0, y, z, y, z)$ |
| 12 | $\left\langle\left(\rho, 1, e^{2 \pi i / 3}\right),(1, \kappa,-1)\right\rangle$ | $(0,0,0,0, y, z,-y,-z)$ |
| 13 | $\left\langle\left(1, \rho, e^{-2 \pi i / 3}\right),(\kappa, 1,1)\right\rangle$ | $(y, z, 0,0, y, z, 0,0)$ |
| 14 | $\left\langle\left(1, \rho, e^{-2 \pi i / 3}\right),(\kappa, 1,-1)\right\rangle$ | $(y, z, 0,0,-y,-z, 0,0)$ |
| 15 | $\left\langle\left(1, \rho, e^{2 \pi i / 3}\right),(\kappa, 1,1)\right\rangle$ | $(0,0, y, z, 0,0, y, z)$ |
| 16 | $\left\langle\left(1, \rho, e^{2 \pi i / 3}\right),(\kappa, 1,-1)\right\rangle$ | $(0,0, y, z, 0,0,-y,-z)$ |
| 17 | $\langle(\rho, \rho, 1),(\kappa, \kappa, 1)\rangle$ | $(0,0, y, z, y, z, 0,0)$ |
| 18 | $\langle(\rho, \rho, 1),(\kappa, \kappa,-1)\rangle$ | $(0,0, y, z,-y,-z, 0,0)$ |
| 19 | $\left\langle\left(\rho^{2}, \rho, 1\right),(\kappa, \kappa, 1)\right\rangle$ | $(y, z, 0,0,0,0, y, z)$ |
| 20 | $\left\langle\left(\rho^{2}, \rho, 1\right),(\kappa, \kappa,-1)\right\rangle$ | $(y, z, 0,0,0,0,-y,-z)$ |

Therefore

$$
\begin{gather*}
(\rho, 1) \cdot \epsilon^{p q}=\epsilon^{p q} \cos \frac{2 p \pi}{3}-\varsigma^{p q} \sin \frac{2 p \pi}{3} \\
(\rho, 1) \cdot \varsigma^{p q}=\varsigma^{p q} \cos \frac{2 p \pi}{3}+\epsilon^{p q} \sin \frac{2 p \pi}{3}  \tag{38}\\
(\kappa, 1) \cdot \epsilon^{p q}=\epsilon^{3-p, q}, \quad(\kappa, 1) \cdot \varsigma^{p q}=\varsigma^{3-p, q} .
\end{gather*}
$$

Similarly, we can prove that

$$
\begin{gather*}
(1, \rho) \cdot \epsilon^{p q}=\epsilon^{p q} \cos \frac{2 q \pi}{3}-\varsigma^{p q} \sin \frac{2 q \pi}{3} \\
(1, \rho) \cdot \varsigma^{p q}=\varsigma^{p q} \cos \frac{2 q \pi}{3}+\epsilon^{p q} \sin \frac{2 q \pi}{3}  \tag{39}\\
(1, \kappa) \cdot \epsilon^{p q}=\epsilon^{p, 3-q}, \quad(1, \kappa) \cdot \varsigma^{p q}=\varsigma^{p, 3-q} .
\end{gather*}
$$

It is clear that if $x$ is a periodic solution of system (2), then so is $\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right) x$ for every $\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right) \in \Gamma \times S^{1}$. The spatialtemporal symmetry of a bifurcation of periodic solutions $x(t)$ can be completely characterized by the isotropy group $\Sigma_{x}=$ $\left\{\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right) \in \Gamma \times S^{1} \mid\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right) x=x\right\} \leq \Gamma \times S^{1}$, and it is easy to verify that the isotropy group of $\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right) x$ is $\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right) \Sigma_{x}\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right)^{-1}$, which is conjugate to $\Sigma_{x}$. The maximal isotropy subgroups $\Sigma^{m}(m=1,2, \ldots, 20)$ of $\Gamma \times S^{1}$ are listed in Table 1. For each subgroup $\Sigma^{m}$, the $\Sigma^{m}$-fixed-point set

$$
\begin{align*}
\operatorname{Fix} & \left(\Sigma^{m}, S P_{\omega}\right) \\
& =\left\{x \in S P_{\omega} \mid\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right) x=x, \forall\left(\gamma, \mathrm{e}^{\mathrm{i} \theta}\right) \in \Sigma^{m}\right\} \tag{40}
\end{align*}
$$

is a subspace of $S P_{\omega}$. According to Lemma 6, it is easy to verify that dim $\operatorname{Fix}\left(\Sigma^{m}, S P_{\omega}\right)=2, m=1,2, \ldots, 20$.

Together with Lemma 2, Lemmas 4-6 allow us to apply the equivariant Hopf bifurcation theorem for delay differential equations due to Wu [3] to obtain the following result on the spatio-temporal patterns of the bifurcated periodic solutions.

Theorem 7. Assume that (H) is satisfied. Then for system (2), near each $\tau_{s}(s=0,1, \ldots)$ there exist 100 distinct branches of periodic solutions bifurcated from the equilibrium $\widehat{x}=0$. More precisely, for each isotropy group $\Sigma^{m}(m=1,2, \ldots, 20)$ and a chosen basis $\left\{\delta_{1}, \delta_{2}\right\}$ of $\operatorname{Fix}\left(\Sigma^{m}, S P_{\omega}\right)$, there exist $\alpha_{0}>0, \tau_{0}^{*}>$ $0, \sigma_{0}>0$, and a $C^{1}$-smooth mapping $\left(\tau^{*}, \omega^{*}, x^{*}\right): \mathbb{R}_{\alpha_{0}}^{2} \rightarrow$ $\mathbb{R} \times \mathbb{R}^{+} \times C\left(\mathbb{R}, \mathbb{R}^{9}\right)$, where $\mathbb{R}_{\alpha_{0}}^{2}=\left\{\alpha \in \mathbb{R}^{2}| | \alpha \mid<\alpha_{0}\right\}$, such that for each $\alpha \in \mathbb{R}_{\alpha_{0}}^{2}, x^{*}=x^{*}(t ; \alpha)$ is an $\omega^{*}(\alpha)$-periodic solution of system (2) with $\tau=\tau_{s}+\tau^{*}(\alpha)$, and

$$
\begin{gather*}
\gamma \cdot x^{*}(t)=x^{*}\left(t-\frac{\omega^{*}(\alpha)}{2 \pi} \theta\right), \quad\left(\gamma, e^{i \theta}\right) \in \Sigma^{m} \\
\omega^{*}(0)=\frac{2 \pi}{\beta}, \quad \tau^{*}(0)=0  \tag{41}\\
x^{*}(t ; \alpha)=\left(\delta_{1}, \delta_{2}\right) \alpha+o(|\alpha|), \quad \text { as }|\alpha| \longrightarrow 0
\end{gather*}
$$

Furthermore, for $\left|\tau-\tau_{s}\right|<\tau_{0}^{*}$, $|\widetilde{\omega}-2 \pi / \beta|<\sigma_{0}$, every $\widetilde{\omega}$ periodic solution $x(t)$ of system (2) with $\left\|x_{t}\right\|<\sigma_{0}, \gamma \cdot x(t)=$ $x(t-(\widetilde{\omega} / 2 \pi) \theta)$ for $\left(\gamma, e^{i \theta}\right) \in \Sigma^{m}$ and $t \in \mathbb{R}$ must be of the above type.


Figure 2: Trajectories $x_{i, j}(i, j=0,1,2)$ of system (42) when $\tau=2.6$.


Figure 3: System (42) has multiple periodic solutions when $\tau=3$.

## 4. An Example and Numerical Simulations

As a simple example, we consider the following specific Hopfield model with $\mathbb{D}_{3} \times \mathbb{D}_{3}$-symmetry:

$$
\begin{aligned}
\dot{x}_{i, j}(t)= & -x_{i, j}(t)-0.6 \tanh \left(x_{i, j}(t-\tau)\right) \\
& +0.3 \tanh \left(x_{i, j-1}(t-\tau)\right)
\end{aligned}
$$

$$
\begin{align*}
& +0.3 \tanh \left(x_{i, j+1}(t-\tau)\right) \\
& +0.4 \tanh \left(x_{i-1, j}(t-\tau)\right)  \tag{42}\\
& +0.4 \tanh \left(x_{i+1, j}(t-\tau)\right),
\end{align*}
$$

where $i, j=0,1,2(\bmod 3)$.
Clearly, the origin is an equilibrium of system (42). It is easy to compute $a=-0.6, b=0.3$, and $c=0.4$ and verify that the hypothesis (H) holds. According to (10), we obtain

$$
\begin{equation*}
\tau_{s}=2.9476+7.5641 s, \quad s=0,1, \ldots \tag{43}
\end{equation*}
$$

From Theorem 3, the origin is asymptotically stable if $\tau<$ $\tau_{0}=2.9476$. It follows from Theorem 7 that the equivariant Hopf bifurcation occurs at $\tau_{0}=2.9476$ and there exist 100 distinct branches of periodic solutions bifurcated from the origin. To illustrate the analytical results found, we give some numerical simulations. Figure 2 shows that the origin of system (42) is stable when $\tau=2.6$. Figure 3 shows that four periodic orbits occur simultaneously when $\tau=3$. Unfortunately, we cannot verify the existence of all other bifurcated periodic orbits since they may be unstable.

## 5. Discussion

In this paper, we have studied a coupled system of nine identical cells with delays and $\mathbb{D}_{3} \times \mathbb{D}_{3}$-symmetry. By choosing the time delay $\tau$ as a bifurcation parameter and analyzing the corresponding characteristic equation, we have shown that under some assumption, the equilibrium of the model loses its stability and periodic solutions via Hopf bifurcation occur when $\tau$ passes through a critical value. This implies that the time delay can be regarded as a source of instability and oscillatory response of the networks and is able to alter the dynamics of system (2) significantly. Moreover, the spatio-temporal patterns of bifurcating periodic solutions are explored clearly by employing the symmetric bifurcation theory of delay differential equations combined with representation theory of Lie groups. From Theorem 7, we have obtained the conclusion that the small-scale network with a special structure may have a large number of periodic oscillations. Therefore, it is natural that the large-scale network possesses complicated dynamics generally.

Further investigations such as the stability, direction and global existence of the periodic solutions bifurcating from the local Hopf bifurcations are essential in order to fully understand the periodic phenomenon of the system. We can compute the normal forms directly by using the method due to Faria and Magalháes [24, 25]. However, this is a complex and prolix task. In addition, we would like to point out that codimension two mode interactions may take place if the assumption (H) is not satisfied. For example, if $a+2 b+2 c=1$, $|a+2 c-b|<1,|a+2 b-c|<1$, and $a-b-c<-1$, then system (2) undergoes a fold-Hopf bifurcation when $\tau$ passes through a critical value. Therefore, it is possible to study secondary bifurcations and more complex behaviours in this coupled network. We leave them for our future work.

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# Research Article 

# Global Exponential Stability Criteria for Bidirectional Associative Memory Neural Networks with Time-Varying Delays 

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#### Abstract

The global exponential stability for bidirectional associative memory neural networks with time-varying delays is studied. In our study, the lower and upper bounds of the activation functions are allowed to be either positive, negative, or zero. By constructing new and improved Lyapunov-Krasovskii functional and introducing free-weighting matrices, a new and improved delay-dependent exponential stability for BAM neural networks with time-varying delays is derived in the form of linear matrix inequality (LMI). Numerical examples are given to demonstrate that the derived condition is less conservative than some existing results given in the literature.


## 1. Introduction

A class of neural networks related to bidirectional associative memory (BAM) has been introduced by Kosko [1]. This model generalized the single-layer autoassociative Hebbian correlator to a two-layer pattern-matched heteroassociative circuit. It is an important model with the ability of information memory and information association, which is crucial for various applications such as pattern recognition, solving optimization problems, and automatic control engineering [2-10]. In $[1,11]$, Kosko investigates the global stability of BAM models and obtains a severe constraint of having a symmetric connection weight matrix. Since it is impossible to maintain an absolutely symmetric connection weight matrix, asymmetric connection has been a focus of this field. Some of these applications require that there should be a welldefined computable solution for all possible initial states. From a mathematical point of view, this means that the equilibrium point of the designed cellular neural networks (CNNs) is globally asymptotically stable (GAS) or globally exponentially stable (GES). Moreover, in biological and artificial neural networks, time delays arise in the process of information transmission; for example, in the electronic implementation of analogue neural networks, time delays
occur in the communication and response of neurons owing to the finite switching speed of amplifiers. It is known that they can create an oscillatory or an unstable phenomenon. Therefore, the study of the stability and convergent dynamics of BAM neural networks with delays has raised considerable interest in recent years; see for examples [5, 7, 9, 10, 12-23] and the references cited therein. In $[14,15,18,20-22,24-$ 27], several sufficient conditions on the global exponential stability of BAM neural networks with time-varying delays have been derived. It is worth pointing out that the given criteria in [14, 15, 18, 20-22, 24-27] required the following hypothesis: the time-varying delays are continuously differentiable, the derivative of time-varying delays is smaller than one, and activation functions are bounded and monotonically nondecreasing. The common approach for studying stability of BAM neural networks is Lyapunov stability theory. With a properly designed Lyapunov-Krasovskii functional as well as introducing free-weighting matrices, one may derive stability criteria in term of linear matrix inequality (LMI) which is easily solved by several available algorithms.

Based on the above discussion, we propose to study the problem of global exponential stability of BAM neural networks with time-varying delays and generalized activation functions. The main contributions of our works are
that the system consists of both memoryless and delayed activation functions, and the lower and upper bounds of the activation functions are allowed to be either positive, negative, or zero which is more general than systems considered in $[14,15,18,21,22,24-27]$. By constructing a new and improved Lyapunov-Krasovskii functional which contains some integral terms of the activation functions, less conservative results are obtained by introducing appropriate free-weighting matrices and by using some improved integral inequality. Finally, two numerical examples are presented to show that our result is less conservative than some existing ones.

Notations. Throughout the paper, $\mathbb{R}$ denotes the set of all real numbers. $*$ denotes the elements below the main diagonal of a symmetric block matrix. $\operatorname{diag}\{\cdots\}$ denotes the diagonal matrix. For symmetric matrices $X$ and $Y$, the notation $X>Y$ (resp., $X \geq Y$ ) means that the matrix $X-Y$ is positive definite (resp., nonnegative). $\lambda_{m}(\cdot)$ and $\lambda_{M}(\cdot)$ denote the smallest and largest eigenvalue of given square matrix, respectively.

## 2. Model Description and Preliminaries

Consider the following BAM neural network with timevarying delays of the form

$$
\begin{align*}
\dot{u}_{i}(t)= & -c_{i} u_{i}(t)+\sum_{j=1}^{m} a_{i j}^{(1)} \widetilde{g}_{j}\left(v_{j}(t)\right) \\
& +\sum_{j=1}^{m} a_{i j}^{(2)} \widetilde{g}_{j}\left(v_{j}(t-h(t))\right)+I_{i}, \\
\dot{v}_{j}(t)= & -d_{j} v_{j}(t)+\sum_{i=1}^{n} b_{j i}^{(1)} \widetilde{f}_{i}\left(u_{i}(t)\right)  \tag{1}\\
& +\sum_{i=1}^{n} b_{j i}^{(2)} \widetilde{f}_{i}\left(u_{i}(t-d(t))\right)+I_{j}, \\
& i=1,2, \ldots, n, j=1,2, \ldots, m
\end{align*}
$$

where $u_{i}(t)$ and $v_{j}(t)$ are the state of the $i$ th neurons from the neural field $F_{u}$ and the $j$ th neurons from the neural field $F_{v}$, at time $t$, respectively; $c_{i}$ and $d_{j}$ denote the neuron charging time constants and passive delay rates, respectively; $a_{i j}^{(1)}$ and $b_{j i}^{(1)}$ are the synaptic connection strengths; $a_{i j}^{(2)}$ and $b_{j i}^{(2)}$ are delayed synaptic connection strengths; $\widetilde{f}_{i}(\cdot)$ and $\widetilde{g}_{j}(\cdot)$ denote the activation functions of the $i$ th neurons from the neural field $F_{u}$ and the $j$ th neurons from the neural field $F_{v}$, respectively; $I_{i}$ and $I_{j}$ denote the external inputs; and $d(t)$ and $h(t)$ represent the time-varying differentiable functions which satisfy

$$
\begin{array}{ll}
\text { (i) } 0 \leq d(t) \leq d, & \dot{d}(t) \leq \tau<1 \\
\text { (ii) } 0 \leq h(t) \leq h, & \dot{h}(t) \leq \mu<1 \tag{2}
\end{array}
$$

where $d, h, \mu$, and $\tau$ are positive scalars.

The initial conditions associated with (1) are assumed to be

$$
\begin{array}{r}
u_{i}(s)=\widetilde{\phi}_{i}(s), \quad v_{j}(s)=\widetilde{\varphi}_{j}(s), \quad s \in[-\max \{d, h\}, 0], \\
i=1,2, \ldots, n, j=1,2, \ldots, m \tag{3}
\end{array}
$$

Throughout this paper, we make the following assumption on the activation function $\widetilde{f}_{i}(\cdot), \widetilde{g}_{j}(\cdot)$.
(A1) $\tilde{f}_{i}(\cdot)$ and $\widetilde{g}_{j}(\cdot)$ are bounded on $\mathbb{R}$.
(A2) For any $\alpha, \beta \in \mathbb{R}, \alpha \neq \beta$, there exist four constant matrices $E=\operatorname{diag}\left(E_{1}, E_{2}, \ldots, E_{n}\right), F=$ $\operatorname{diag}\left(F_{1}, F_{2}, \ldots, F_{n}\right), M=\operatorname{diag}\left(M_{1}, M_{2}, \ldots, M_{m}\right)$, and $N=\operatorname{diag}\left(N_{1}, N_{2}, \ldots, N_{m}\right)$ satisfying

$$
\begin{gather*}
E_{i} \leq \frac{\tilde{f}_{i}(\alpha)-\tilde{f}_{i}(\beta)}{\alpha-\beta} \leq F_{i}, \quad i=1,2, \ldots, n \\
N_{j} \leq \frac{\widetilde{g}_{j}(\alpha)-\widetilde{g}_{j}(\beta)}{\alpha-\beta} \leq M_{j}, \quad j=1,2, \ldots, m \tag{4}
\end{gather*}
$$

It is clear that under (A1) and (A2), the system (1) has at least one equilibrium; see [20]. In order to simplify our proof, we shift the equilibrium point $u^{*}=\left(u_{1}^{*}, u_{2}^{*}, \ldots, u_{n}^{*}\right)^{T}, v^{*}=$ $\left(v_{1}^{*}, v_{2}^{*}, \ldots, v_{m}^{*}\right)^{T}$ of system (1) to the origin. Let $x_{i}(t)=u_{i}(t)-$ $u_{i}^{*}, y_{j}(t)=v_{j}(t)-v_{j}^{*}, f_{i}\left(x_{i}(t)\right)=\widetilde{f}_{i}\left(x_{i}(t)+u_{i}^{*}\right)-\widetilde{f}_{i}\left(u_{i}^{*}\right)$, $g_{j}\left(y_{j}(t)\right)=\widetilde{g}_{j}\left(y_{j}(t)+v_{j}^{*}\right)-\widetilde{g}_{j}\left(v_{j}^{*}\right), f_{i}\left(x_{i}(t-d(t))\right)=\widetilde{f}_{i}\left(x_{i}(t-\right.$ $\left.d(t))+u_{i}^{*}\right)-\widetilde{f}_{i}\left(u_{i}^{*}\right), g_{j}\left(y_{j}(t-h(t))\right)=\widetilde{g}_{j}\left(y_{j}(t-h(t))+v_{j}^{*}\right)-$ $\widetilde{g}_{j}\left(v_{j}^{*}\right), i=1,2, \ldots, n$ and $j=1,2, \ldots, m$. Then the system (1) can be transformed to

$$
\begin{align*}
\dot{x}(t)= & -c_{i} x_{i}(t)+\sum_{j=1}^{m} a_{i j}^{(1)} g_{j}\left(y_{j}(t)\right) \\
& +\sum_{j=1}^{m} a_{i j}^{(2)} g_{j}\left(y_{j}(t-h(t))\right),  \tag{5}\\
\dot{y}(t)= & -d_{j} y_{j}(t)+\sum_{i=1}^{n} b_{j i}^{(1)} f_{i}\left(x_{i}(t)\right) \\
& +\sum_{i=1}^{n} b_{j i}^{(2)} f_{i}\left(x_{i}(t-d(t))\right) .
\end{align*}
$$

The activation functions $f_{i}(\cdot)$ and $g_{j}(\cdot)$ satisfy the following properties.
(H1) $f_{i}$ and $g_{j}$ are bounded on $\mathbb{R}$.
(H2) For any $\alpha, \beta \in \mathbb{R}, \alpha \neq \beta$, there exist constant matrices $E=\operatorname{diag}\left(E_{1}, E_{2}, \ldots, E_{n}\right), F=\operatorname{diag}\left(F_{1}, F_{2}, \ldots, F_{n}\right)$, $M=\operatorname{diag}\left(M_{1}, M_{2}, \ldots, M_{m}\right)$, and $N=\operatorname{diag}\left(N_{1}\right.$, $N_{2}, \ldots, N_{m}$ ) satisfying

$$
E_{i} \leq \frac{f_{i}(\alpha)-f_{i}(\beta)}{\alpha-\beta} \leq F_{i}, \quad i=1,2, \ldots, n
$$

$$
\begin{equation*}
N_{j} \leq \frac{g_{j}(\alpha)-g_{j}(\beta)}{\alpha-\beta} \leq M_{j}, \quad j=1,2, \ldots, m \tag{6}
\end{equation*}
$$

(H3) $f_{i}(0)=0$ and $g_{j}(0)=0, i=1,2, \ldots, n, j=1,2, \ldots, m$.
Rewrite the system (7) into the vector form

$$
\begin{gather*}
\dot{x}(t)=-C x(t)+A_{1} g(y(t))+A_{2} g(y(t-h(t))), \\
\dot{y}(t)=-D y(t)+B_{1} f(x(t))+B_{2} f(x(t-d(t))) . \tag{7}
\end{gather*}
$$

The initial conditions associated with (7) are assumed to be

$$
\begin{array}{r}
x_{i}(s)=\phi_{i}(s), \quad y_{j}(s)= \\
\varphi_{j}(s), \quad s \in[-\max \{d, h\}, 0]  \tag{8}\\
\\
i=1,2, \ldots, n, j=1,2, \ldots, m
\end{array}
$$

where $x(t)=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T}, y(t)=\left(y_{1}(t)\right.$, $\left.y_{2}(t), \ldots, y_{m}(t)\right)^{T}, C=\operatorname{diag}\left(c_{1}, c_{2}, \ldots, c_{n}\right), D=$ $\operatorname{diag}\left(d_{1}, d_{2}, \ldots, d_{n}\right), A_{1}=\left(a_{i j}^{(1)}\right)_{m \times n}, A_{2}=\left(a_{i j}^{(2)}\right)_{m \times n}, B_{1}=$ $\left(b_{i j}^{(1)}\right)_{n \times m}, B_{2}=\left(b_{i j}^{(2)}\right)_{n \times m}, f(\cdot)=\left(f_{1}(\cdot), f_{2}(\cdot), \ldots, f_{n}(\cdot)\right)^{T}$, $g(\cdot)=\left(g_{1}(\cdot), g_{2}(\cdot), \ldots, g_{m}(\cdot)\right)^{T}$.

Definition 1 (see [14]). The trivial solution of system (7) is said to be globally exponentially stable if there exist constants $k>$ 0 and $\rho \geq 1$ such that

$$
\begin{equation*}
\|x(t)\|^{2}+\|y(t)\|^{2} \leq \rho e^{-2 k t}\left(\|\phi\|^{2}+\|\varphi\|^{2}\right), \quad \forall t \geq 0 \tag{9}
\end{equation*}
$$

where one denotes

$$
\begin{align*}
\|\phi\|^{2}+\|\varphi\|^{2}= & \sup _{-\max \{d, h\} \leq s \leq 0}\|\phi(s)\|^{2} \\
& +\sup _{-\max \{d, h\} \leq s \leq 0}\|\varphi(s)\|^{2} . \tag{10}
\end{align*}
$$

Lemma 2 (see [28]). If there exist symmetric positive-definite matrix $X_{33}>0$ and arbitrary matrices $X_{11}, X_{12}, X_{13}, X_{22}$, and $X_{23}$ such that

$$
X=\left[\begin{array}{ccc}
X_{11} & X_{12} & X_{13}  \tag{11}\\
X_{12}^{T} & X_{22} & X_{23} \\
X_{13}^{T} & X_{23}^{T} & X_{33}
\end{array}\right] \geq 0
$$

then,

$$
\begin{align*}
& -\int_{t-h(t)}^{t} \dot{x}(s) X_{33} \dot{x}(s) d s \\
& \quad \leq \int_{t-h(t)}^{t}\left[\begin{array}{lll}
x^{T}(t) & x^{T}(t-h(t)) & \dot{x}^{T}(s)
\end{array}\right]  \tag{12}\\
& \quad \times\left[\begin{array}{ccc}
X_{11} & X_{12} & X_{13} \\
X_{12}^{T} & X_{22} & X_{23} \\
X_{13}^{T} & X_{23}^{T} & 0
\end{array}\right]\left[\begin{array}{c}
x(t) \\
x(t-h(t)) \\
\dot{x}(s)
\end{array}\right] d s
\end{align*}
$$

Lemma 3 (see [25]). For any real vectors $a, b$ and any matrix $Q>0$ with appropriate dimensions, it follows that

$$
\begin{equation*}
2 a^{T} b \leq a^{T} Q a+b^{T} Q^{-1} b \tag{13}
\end{equation*}
$$

Lemma 4 (see [25]). Suppose that (H2) holds; then

$$
\begin{array}{r}
\int_{v}^{u}\left[f_{i}(s)-f_{i}(v)\right] d s \leq[u-v]\left[f_{i}(u)-f_{i}(v)\right] \\
i=1,2, \ldots, n \\
\int_{v}^{u}\left[g_{j}(s)-g_{j}(v)\right] d s \leq[u-v]\left[g_{j}(u)-g_{j}(v)\right]  \tag{14}\\
j=1,2, \ldots, m
\end{array}
$$

## 3. Main Result

In this section, we present a theorem which states the conditions that guarantee the global exponential stability of the system (7) employing the Lyapunov stability theory and linear matrix inequality approach.

Theorem 5. Under the assumptions (H1)-(H3), for given four diagonal matrices $E=\operatorname{diag}\left(E_{1}, E_{2}, \ldots, E_{n}\right), F=$ $\operatorname{diag}\left(F_{1}, F_{2}, \ldots, F_{n}\right), M=\operatorname{diag}\left(M_{1}, M_{2}, \ldots, M_{m}\right)$, and $N=$ $\operatorname{diag}\left(N_{1}, N_{2}, \ldots, N_{m}\right)$ and positive constants $d, h, \tau, \mu$, and $k$, the system (7) is globally exponentially stable with the convergent rate $k$, if there exist positive matrices $P_{i}, W_{i}, Z_{i}$, $i=1,2, Q_{j}, j=1,2,3,4$, positive diagonal matrices $K=$ $\operatorname{diag}\left(k_{1}, k_{2}, \ldots, k_{n}\right), R=\operatorname{diag}\left(r_{1}, r_{2}, \ldots, r_{m}\right)$, and positivedefinite matrices

$$
\begin{align*}
& S=\left[\begin{array}{lll}
S_{11} & S_{12} & S_{13} \\
S_{12}^{T} & S_{22} & S_{23} \\
S_{13}^{T} & S_{23}^{T} & Z_{1}
\end{array}\right],  \tag{15}\\
& T=\left[\begin{array}{lll}
T_{11} & T_{12} & T_{13} \\
T_{12}^{T} & T_{22} & T_{23} \\
T_{13}^{T} & T_{23}^{T} & Z_{2}
\end{array}\right],
\end{align*}
$$

such that the following LMI holds:

$$
\Xi=\left[\begin{array}{cccccccc}
\Sigma_{1}^{1} & \Sigma_{2}^{1} & \Sigma_{3}^{1} & 0 & 0 & 0 & \Sigma_{7}^{1} & \Sigma_{8}^{1}  \tag{16}\\
* & \Sigma_{2}^{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Sigma_{3}^{3} & \Sigma_{4}^{3} & \Sigma_{5}^{3} & 0 & \Sigma_{7}^{3} & \Sigma_{8}^{3} \\
* & * & * & \Sigma_{4}^{4} & \Sigma_{5}^{4} & 0 & \Sigma_{7}^{4} & 0 \\
* & * & * & * & \Sigma_{5}^{5} & \Sigma_{6}^{5} & \Sigma_{7}^{5} & 0 \\
* & * & * & * & * & \Sigma_{6}^{6} & 0 & 0 \\
* & * & * & * & * & * & \Sigma_{7}^{7} & \Sigma_{8}^{7} \\
* & * & * & * & * & * & * & \Sigma_{8}^{8}
\end{array}\right]<0
$$

where

$$
\begin{aligned}
& \Sigma_{1}^{1}= 2 k P_{1}-P_{1} C-C^{T} P_{1}-2 k E^{T} K-2 k K^{T} E \\
&+E^{T} K C+C^{T} K E+Q_{1}+d C^{T} Z_{1} C \\
&+e^{-2 k d}\left(d S_{11}+S_{13}^{T}+S_{13}\right)+W_{1}, \\
& \Sigma_{2}^{1}= e^{-2 k d}\left(d S_{12}+S_{23}^{T}-S_{13}\right), \\
& \Sigma_{3}^{1}= 2 k K, \\
& \Sigma_{7}^{1}= P_{1} A_{1}-E^{T} K A_{1}-d C^{T} Z_{1} A_{1}, \\
& \Sigma_{8}^{1}= P_{2} A_{2}-E^{T} K A_{2}-d C^{T} Z_{1} A_{2}, \\
& \Sigma_{2}^{2}= e^{-2 k d}\left(d S_{22}-S_{23}^{T}-S_{23}\right)-e^{-2 k d} Q_{1}, \\
& \Sigma_{3}^{3}= h B_{1}^{T} Z_{2} B_{1}+Q_{3}-2 K C F^{-1}, \\
& \Sigma_{4}^{3}= h B_{1}^{T} Z_{2} B_{2}, \\
& \Sigma_{5}^{3}= P_{2} B_{1}-N^{T} R B_{1}-h B_{1}^{T} Z_{2} D, \\
& \Sigma_{7}^{3}= K A_{1}+R B_{1}, \\
& \Sigma_{8}^{3}= K A_{2}, \\
& \Sigma_{4}^{4}= h B_{2}^{T} Z_{2} B_{2}-(1-\tau) e^{-2 k d} Q_{3} \\
&-(1-\tau) e^{-2 k d} F^{-1} W_{1} F^{-1}, \\
& \Sigma_{8}^{7}= d A_{1}^{T} Z_{1}^{T} A_{1}+Z_{1} A_{2}, \\
& \Sigma_{8}^{8}= d A_{2}^{T} Z_{1} A_{2}-(1-\mu) e^{-2 k h} M^{-1} W_{2} M^{-1} . \\
& \Sigma_{5}^{4}= P_{2} B_{2}-N^{T} R B_{2}-h B_{2}^{T} Z_{2} D, \\
& \Sigma_{7}^{4}= R B_{2}, \\
& \Sigma_{5}^{5}= 2 k P_{2}-P_{2} D-D^{T} P_{2}-2 k N^{T} R-2 k R^{T} N \\
&+N^{T} R D+D^{T} R N+Q_{2}+h D^{T} Z_{2} D \\
&+e^{-2 k h}\left(h T_{11}+T_{13}^{T}+T_{13}\right)+W_{2}, \\
& \Sigma_{6}^{5}= e^{-2 k h}\left(h T_{12}+T_{23}^{T}-T_{13}\right), \\
& 2 k R, \\
& \\
& \\
& \hline
\end{aligned}
$$

Proof. Choose the Lyapunov-Krasovskii function candidate for the system (7) to be

$$
\begin{equation*}
V(t)=\sum_{l=1}^{5} V_{l}(t), \tag{18}
\end{equation*}
$$

where

$$
\begin{align*}
V_{1}(t)= & e^{2 k t} x^{T}(t) P_{1} x(t) \\
& +2 \sum_{i=1}^{n} k_{i} e^{2 k t} \int_{0}^{x_{i}(t)}\left[f_{i}(s)-E_{i} s\right] d s \\
& +e^{2 k t} y^{T}(t) P_{2} y(t) \\
& +2 \sum_{j=1}^{m} r_{j} e^{2 k t} \int_{0}^{y_{j}(t)}\left[g_{j}(s)-N_{j} s\right] d s \\
V_{2}(t)= & \int_{t-d}^{t} e^{2 k s} x^{T}(s) Q_{1} x(s) d s \\
& +\int_{t-h}^{t} e^{2 k s} y^{T}(s) Q_{2} y(s) d s \\
V_{3}(t)= & \int_{t-d(t)}^{t} e^{2 k s} f^{T}(x(s)) Q_{3} f(x(s)) d s  \tag{19}\\
& +\int_{t-h(t)}^{t} e^{2 k s} g^{T}(y(s)) Q_{4} g(y(s)) d s, \\
V_{4}(t)= & \int_{t-d(t)}^{t} e^{2 k s} x^{T}(s) W_{1} x(s) d s \\
& +\int_{t-h(t)}^{t} e^{2 k s} y^{T}(s) W_{2} y(s) d s \\
V_{5}(t)= & \int_{t-d}^{t} \int_{s}^{t} e^{2 k \theta} \dot{x}^{T}(\theta) Z_{1} \dot{x}(\theta) d \theta d s \\
& +\int_{t-h}^{t} \int_{s}^{t} e^{2 k \theta} \dot{y}^{T}(\theta) Z_{2} \dot{y}(\theta) d \theta d s \\
&
\end{align*}
$$

The derivative of $V(t)$ along the trajectories of system (7) is given by

$$
\begin{aligned}
\dot{V}_{1}(t)= & 2 k e^{2 k t} x^{T}(t) P_{1} x(t)+2 e^{2 k t} x^{T}(t) P_{1} \dot{x}(t) \\
& +4 k \sum_{i=1}^{n} k_{i} e^{2 k t} \int_{0}^{x_{i}(t)}\left[f_{i}(s)-E_{i} s\right] d s \\
& +2 \sum_{i=1}^{n} k_{i} e^{2 k t} \dot{x}_{i}(t)\left[f_{i}\left(x_{i}(t)\right)-E_{i} x_{i}(t)\right] \\
& +2 k e^{2 k t} y^{T}(t) P_{2} y(t)+2 e^{2 k t} y^{T}(t) P_{2} \dot{y}(t) \\
& +4 k \sum_{j=1}^{m} r_{j} e^{2 k t} \int_{0}^{y_{j}(t)}\left[g_{j}(s)-N_{j} s\right] d s \\
& +2 \sum_{j=1}^{m} r_{j} e^{2 k t} \dot{y}_{j}(t)\left[g_{j}\left(y_{j}(t)\right)-N_{j} y_{j}(t)\right]
\end{aligned}
$$

$$
\begin{align*}
& \leq e^{2 k t}\left\{2 k x^{T}(t) P_{1} x(t)+2 x^{T}(t)\right. \\
& \times P_{1}\left[-C x(t)+A_{1} g(y(t))\right. \\
&\left.+A_{2} g(y(t-h(t)))\right] \\
&+ 4 k[f(x(t))-E x(t)]^{T} K x(t) \\
&+ 2[f(x(t))-E x(t)]^{T} \\
& \times K\left[-C x(t)+A_{1} g(y(t))\right. \\
&\left.+A_{2} g(y(t-h(t)))\right] \\
&+ 2 k y^{T}(t) P_{2} y(t) \\
&+ 2 y^{T}(t) P_{2}\left[-D y(t)+B_{1} f(x(t))\right. \\
&\left.\quad+B_{2} f(x(t-d(t)))\right] \\
&+ 4 k[g(y(t))-N y(t)]^{T} R y(t) \\
&+ 2[g(y(t))-N y(t)]^{T} R \\
& \times {\left[-D y(t)+B_{1} f(x(t))\right.} \\
&\left.\left.+B_{2} f(x(t-d(t)))\right]\right\} . \tag{20}
\end{align*}
$$

By (H2), we have

$$
\begin{aligned}
& -2 f^{T}(x(t)) K C x(t) \leq-2 f^{T}(x(t)) K C F^{-1} f(x(t)) \\
& -2 g^{T}(y(t)) R D y(t) \leq-2 g^{T}(y(t)) R D M^{-1} g(y(t))
\end{aligned}
$$

Substituting (21) into (20), we obtain

$$
\begin{aligned}
& \dot{V}_{1}(t) \leq e^{2 k t} \\
& \times\left\{2 k x^{T}(t) P_{1} x(t)-2 x^{T}(t) P_{1} C x(t)\right. \\
&+2 x^{T}(t) P_{1} A_{1} g(y(t))+2 x^{T}(t) P_{1} A_{2} \\
& \times g(y(t-h(t)))+4 k f^{T}(x(t)) K x(t) \\
&-4 k x^{T}(t) E^{T} K x(t)-2 f^{T}(x(t)) K C \\
& \times F^{-1} f(x(t))+2 f^{T}(x(t)) K A_{1} g(y(t)) \\
&+2 f^{T}(x(t)) K A_{2} g(y(t-h(t))) \\
&+2 x^{T}(t) E^{T} K C x(t) \\
&-2 x^{T}(t) E^{T} K A_{1} g(y(t)) \\
&-2 x^{T}(t) E^{T} K A_{2} g(y(t-h(t))) \\
&+2 k y^{T}(t) P_{2} y(t)-2 y^{T}(t) P_{2} D y(t) \\
&+2 y^{T}(t) P_{2} B_{1} f(x(t))
\end{aligned}
$$

$$
\dot{V}_{4}(t)=e^{2 k t}\left\{x^{T}(t) W_{1} x(t)\right.
$$

$$
-(1-\dot{d}(t)) e^{-2 k d(t)} x^{T}(t-d(t))
$$

$$
\times W_{1} x(t-d(t))
$$

$$
+y^{T}(t) W_{2} y(t)-(1-\dot{h}(t)) e^{-2 k h(t)}
$$

$$
\left.\times y^{T}(t-h(t))(s) W_{2} y^{T}(t-h(t))\right\}
$$

$$
\begin{aligned}
& +2 y^{T}(t) P_{2} B_{2} f(x(t-d(t))) \\
& +4 k g^{T}(y(t)) R y(t)-4 k y^{T}(t) N^{T} R y(t) \\
& -2 g^{T}(y(t)) R D M^{-1} g(y(t)) \\
& +2 g^{T}(y(t)) R B_{1} f(x(t)) \\
& +2 g^{T}(y(t)) R B_{2} f(x(t-d(t))) \\
& +2 y^{T}(t) N^{T} R D y(t) \\
& -2 y^{T}(t) N^{T} R B_{1} f(x(t)) \\
& \left.-2 y^{T}(t) N^{T} R B_{2} f(x(t-d(t)))\right\}, \\
& \dot{V}_{2}(t)=e^{2 k t}\left\{x^{T}(t) Q_{1} x(t)\right. \\
& -e^{-2 k d} x^{T}(t-d) Q_{1} x(t-d) \\
& +y^{T}(t) Q_{2} y(t) \\
& \left.-e^{-2 k h} y^{T}(t-h) Q_{2} y(t-h)\right\}, \\
& \dot{V}_{3}(t)=e^{2 k t} \\
& \times\left\{f^{T}(x(t)) Q_{3} f(x(t))\right. \\
& -(1-\dot{d}(t)) e^{-2 k d(t)} f^{T}(x(t-d(t))) \\
& \times Q_{3} f(x(t-d(t))) \\
& +g^{T}(y(t)) Q_{4} g(y(t)) \\
& -(1-\dot{h}(t)) e^{-2 k h(t)} \\
& \left.\times g^{T}(y(t-h(t))) Q_{4} g(y(t-h(t)))\right\} \\
& \leq e^{2 k t}\left\{f^{T}(x(t)) Q_{3} f(x(t))\right. \\
& -(1-\tau) e^{-2 k d} f^{T}(x(t-d(t))) \\
& \times Q_{3} f(x(t-d(t))) \\
& +g^{T}(y(t)) Q_{4} g(y(t)) \\
& -(1-\mu) e^{-2 k h} g^{T}(y(t-h(t))) \\
& \left.\times Q_{4} g(y(t-h(t)))\right\},
\end{aligned}
$$

$$
\begin{aligned}
\leq e^{2 k t}\{ & x^{T}(t) W_{1} x(t)-(1-\tau) e^{-2 k d} \\
& \times x^{T}(t-d(t)) W_{1} x(t-d(t)) \\
& +y^{T}(t) W_{2} y(t)-(1-\mu) e^{-2 k h} \\
& \left.\times y^{T}(t-h(t))(s) W_{2} y^{T}(t-h(t))\right\}
\end{aligned}
$$

By (H2), we have

$$
\begin{aligned}
&-e^{-2 k d} x^{T}(t-d(t)) W_{1} x(t-d(t)) \\
& \leq-e^{-2 k d} f^{T}(x(t-d(t))) F^{-1} W_{1} \\
& \times F^{-1} f(x(t-d(t))) \\
&-e^{-2 k h} y^{T}(t-h(t))(s) W_{2} y^{T}(t-h(t)) \\
& \leq-e^{-2 k h} g^{T}(y(t-h(t))) M^{-1} W_{2} \\
& \times M^{-1} g(y(t-h(t)))
\end{aligned}
$$

By (24), we conclude that

$$
\begin{aligned}
& \dot{V}_{4}(t) \leq e^{2 k t}\left\{x^{T}(t) W_{1} x(t)-(1-\tau) e^{-2 k d}\right. \\
& \times f^{T}(x(t-d(t))) F^{-1} W_{1} F^{-1} \\
& \times f(x(t-d(t))) \\
&+y^{T}(t) W_{2} y(t)-(1-\mu) e^{-2 k h} \\
& \times g^{T}(y(t-h(t))) M^{-1} W_{2} M^{-1} \\
&\times g(y(t-h(t)))\} \\
& \dot{V}_{5}(t)=d e^{2 k t} \dot{x}^{T}(t) Z_{1} \dot{x}(t) \\
&-\int_{t-d}^{t} e^{2 k s} \dot{x}^{T}(s) Z_{1} \dot{x}(s) d s \\
&+h e^{2 k t} \dot{y}^{T}(t) Z_{2} \dot{y}(t) \\
&-\int_{t-h}^{t} e^{2 k s} \dot{y}^{T}(s) Z_{2} \dot{y}(s) d s \\
& \leq e^{2 k t}\left\{d \dot{x}^{T}(t) Z_{1} \dot{x}(t)\right. \\
&-e^{-2 k d} \int_{t-d}^{t} \dot{x}^{T}(s) Z_{1} \dot{x}(s) d s
\end{aligned}
$$

$$
\begin{align*}
& \times h \dot{y}^{T}(t) Z_{2} \dot{y}(t) \\
& \left.-e^{-2 k h} \int_{t-h}^{t} \dot{y}^{T}(s) Z_{2} \dot{y}(s) d s\right\} \tag{26}
\end{align*}
$$

By Lemma 2, we obtain

$$
\begin{align*}
& -\int_{t-h}^{t} \dot{y}^{T}(s) Z_{2} \dot{y}(s) d s \\
& \leq \int_{t-h}^{t}\left[\begin{array}{llll}
y^{T}(t) & y^{T}(t-h) & \dot{y}^{T}(s)
\end{array}\right] \\
& \times\left[\begin{array}{ccc}
T_{11} & T_{12} & T_{13} \\
T_{12}^{T} & T_{22} & T_{23} \\
T_{13}^{T} & T_{23}^{T} & 0
\end{array}\right] \\
& \times\left[\begin{array}{c}
y(t) \\
y(t-h) \\
\dot{y}(s)
\end{array}\right] d s \\
& =\int_{t-h}^{t}\left(y^{T}(t) T_{11}+y^{T}(t-h) T_{12}^{T}+\dot{y}^{T}(s) T_{13}^{T}\right) y(t) d s \\
& +\int_{t-h}^{t}\left(y^{T}(t) T_{12}+y^{T}(t-h) T_{22}+\dot{y}^{T}(s) T_{23}^{T}\right) \\
& \times y(t-h) d s \\
& +\int_{t-h}^{t}\left(y^{T}(t) T_{13}+y^{T}(t-h) T_{23}\right) \dot{y}(s) d s \\
& =y^{T}(t) h T_{11} y(t)+y^{T}(t-h) h T_{12}^{T} y(t) \\
& +\int_{t-h}^{t} \dot{y}^{T}(s) d s T_{13}^{T} y(t) \\
& +y^{T}(t) h T_{12} y(t-h)+y^{T}(t-h) h T_{22} y(t-h) \\
& +\int_{t-h}^{t} \dot{y}^{T}(s) d s T_{23}^{T} y(t-h) \\
& +y^{T}(t) T_{13} \int_{t-h}^{t} \dot{y}(s) d s \\
& +y^{T}(t-h) T_{23} \int_{t-h}^{t} \dot{y}(s) d s \\
& =y^{T}(t) h T_{11} y(t)+y^{T}(t-h) h T_{12}^{T} y(t) \\
& +y^{T}(t) h T_{12} y(t-h) \\
& +y^{T}(t-h) h T_{22} y(t-h)+[y(t)-y(t-h)]^{T} \\
& \times\left[T_{13}^{T} y(t)+T_{23}^{T} y(t-h)\right] \\
& +\left[y^{T}(t) T_{13}+y^{T}(t-h) T_{23}\right] \\
& \times[y(t)-y(t-h)] . \tag{27}
\end{align*}
$$

Substituting (27) into (26), we have

$$
\begin{aligned}
\dot{V}_{5}(t) \leq e^{2 k t}\{ & d \dot{x}^{T}(t) Z_{1} \dot{x}(t) \\
& +e^{-2 k d}\left[d x^{T}(t) S_{11} x(t)\right.
\end{aligned}
$$

$$
\begin{align*}
&+ 2 d x^{T}(t) S_{12}^{T} x(t-d) \\
&+ 2[x(t)-x(t-d)]^{T} \\
& \times {\left[S_{13}^{T} x(t)+S_{23}^{T} x(t-d)\right] } \\
&+\left.d x^{T}(t-d) S_{22} x(t-d)\right] \\
&+h \dot{y}^{T}(t) Z_{2} \dot{y}(t) \\
&+e^{-2 k h}\left[h y^{T}(t) T_{11} y(t)\right. \\
&+ 2 h y^{T}(t) T_{12} y(t-h) \\
&+ 2[y(t)-y(t-h)]^{T} \\
& \times {\left[T_{13}^{T} y(t)+T_{23}^{T} y(t-h)\right] } \\
&+\left.\left.h y^{T}(t-h) T_{22} y(t-h)\right]\right\} \tag{28}
\end{align*}
$$

From (22), (25), and (28) we obtain

$$
\begin{equation*}
\dot{V}(t) \leq e^{2 k t} \xi^{T}(t) \Xi \xi(t) \tag{29}
\end{equation*}
$$

where $\Xi$ is defined as in (16), and $\xi(t)=[x(t), x(t-$ d), $f(x(t)), f(x(t-d(t))), y(t), y(t-h), g(y(t)), g(y(t-h(t)))]$. Since the matrix $\Xi$ given in Theorem 5 is the negative definite matrix, we have $\dot{V}(t) \leq 0$, for all $t \geq 0$ which implies that $V(t) \leq V(0)$. From the definition of $V(t)$ in (20), we obtain

$$
\begin{aligned}
V(0)= & x^{T}(0) P_{1} x(0) \\
& +2 \sum_{i=1}^{n} k_{i} \int_{0}^{x_{i}(0)}\left[f_{i}(s)-E_{i} s\right] d s \\
& +\int_{-d}^{0} e^{2 k s} x^{T}(s) Q_{1} x(s) d s \\
& +\int_{-d(0)}^{0} e^{2 k s} f^{T}(x(s)) Q_{3} f(x(s)) d s \\
& +\int_{-d(0)}^{0} e^{2 k s} x^{T}(s) W_{1} x(s) d s \\
& +y^{T}(0) P_{2} y(0)
\end{aligned}
$$

$$
+2 \sum_{j=1}^{m} r_{j} \int_{0}^{y_{j}(0)}\left[g_{j}(s)-N_{j} s\right] d s
$$

$$
+\int_{-h}^{0} e^{2 k s} y^{T}(s) Q_{2} y(s) d s
$$

$$
\begin{aligned}
& +\int_{-h(0)}^{0} e^{2 k s} g^{T}(y(s)) Q_{4} g(y(s)) d s \\
& +\int_{-h(0)}^{0} y^{T}(s) W_{2} y(s) d s \\
& +\int_{-d}^{0} \int_{s}^{0} e^{2 k \theta} \dot{x}^{T}(\theta) Z_{1} \dot{x}(\theta) d \theta d s \\
& +\int_{-h}^{0} \int_{s}^{0} e^{2 k \theta} \dot{y}^{T}(\theta) Z_{2} \dot{y}(\theta) d \theta d s \\
\leq & \lambda_{M}\left(P_{1}\right)\|\phi\|^{2}+2 K_{M}(F-E)\|\phi\|^{2} \\
& +\left[\lambda_{M}\left(Q_{1}\right)+\lambda_{M}\left(Q_{3}\right)(F-E)+\lambda_{M}\left(W_{1}\right)\right] \\
& \times \int_{-d}^{0} e^{2 k s} x^{T}(s) x(s) d s+\lambda_{M}\left(P_{2}\right)\|\varphi\|^{2} \\
& +2 R_{M}(M-N)\|\varphi\|^{2} \\
& +\left[\lambda_{M}\left(Q_{2}\right)+\lambda_{M}\left(Q_{4}\right)(M-N)+\lambda_{M}\left(W_{2}\right)\right] \\
& \times \int_{-h}^{0} e^{2 k s} y^{T}(s) y(s) d s \\
& +\lambda_{M}\left(Z_{1}\right) \int_{-d}^{0} \int_{s}^{0} \dot{x}^{T}(\theta) \dot{x}(\theta) d \theta d s \\
& +\lambda_{M}\left(Z_{2}\right) \int_{-h}^{0} \int_{s}^{0} \dot{y}^{T}(\theta) \dot{y}(\theta) d \theta d s \\
&
\end{aligned}
$$

$$
\begin{align*}
& \leq 3 x^{T}(\theta) C^{T} C x(\theta) \\
&+3 g^{T}(y(\theta)) A_{1}^{T} A_{1} g(y(\theta)) \\
&+3 g^{T}(y(\theta-h(\theta))) A_{2}^{T} \\
& \times A_{2} g(y(\theta-h(\theta))), \\
& \dot{y}^{T}(\theta) \dot{y}(\theta) \\
&= {\left[-D y(\theta)+B_{1} f(x(\theta))+B_{2} f(x(\theta-d(\theta)))\right]^{T} } \\
& \times\left[-D y(\theta)+B_{1} f(x(\theta))+B_{2} f(x(\theta-d(\theta)))\right] \\
&= y^{T}(\theta) D^{T} D y(\theta)+f^{T}(x(\theta)) B_{1}^{T} B_{1} f(x(\theta)) \\
&+f^{T}(x(\theta-d(\theta))) \\
& \times B_{2}^{T} B_{2} f(x(\theta-d(\theta))) \\
&-2 y^{T}(\theta) D^{T} B_{1} f(x(\theta))-2 y^{T}(\theta) D^{T} B_{2} \\
& \times f(x(\theta-d(\theta))) \\
&+2 f^{T}(x(\theta)) B_{1}^{T} B_{2} f(x(\theta-d(\theta))) \\
& \leq 3 y^{T}(\theta) D^{T} D y(\theta)+3 f^{T}(x(\theta)) B_{1}^{T} B_{1} f(x(\theta)) \\
&+3 f^{T}(x(\theta-d(\theta))) B_{2}^{T} \times B_{2} g(y(\theta-h(\theta))) \tag{31}
\end{align*}
$$

Substituting (31) into (30), we obtain the bound of $V(0)$ as follows:

$$
\begin{aligned}
V(0) \leq & \lambda_{M}\left(P_{1}\right)\|\phi\|^{2}+2 K_{M}(F-E)\|\phi\|^{2} \\
& +\left(\frac{1-e^{-2 k d}}{2 k}\right) \\
& \times\left[\lambda_{M}\left(Q_{1}\right)+\lambda_{M}\left(Q_{3}\right) \times(F-E)+\lambda_{M}\left(W_{1}\right)\right] \\
& +\lambda_{M}\left(P_{2}\right)\|\varphi\|^{2}+2 R_{M}(M-N)\|\varphi\|^{2} \\
& +\left(\frac{1-e^{-2 k h}}{2 k}\right) \\
& \times\left[\lambda_{M}\left(Q_{2}\right)+\lambda_{M}\left(Q_{4}\right)(M-N)+\lambda_{M}\left(W_{2}\right)\right] \\
& +\frac{d^{2}}{2} \lambda_{M}\left(Z_{1}\right)\left[3 \lambda_{M}\left(C^{T} C\right)\|\phi\|^{2}\right. \\
& +3 \lambda_{M}\left(A_{1}^{T} A_{1}\right)\|\varphi\|^{2} \\
& \left.+3 \lambda_{M}\left(A_{2}^{T} A_{2}\right)\|\varphi\|^{2}\right]
\end{aligned}
$$

$$
\begin{align*}
+\frac{h^{2}}{2} \lambda_{M}\left(Z_{2}\right) & {\left[3 \lambda_{M}\left(D^{T} D\right)\|\varphi\|^{2}\right.} \\
& +3 \lambda_{M}\left(B_{1}^{T} B_{1}\right)\|\phi\|^{2} \\
& \left.+3 \lambda_{M}\left(B_{2}^{T} B_{2}\right)\|\phi\|^{2}\right] \tag{32}
\end{align*}
$$

Thus,

$$
\begin{equation*}
V(0) \leq \chi_{1}\|\phi\|^{2}+\chi_{2}\|\varphi\|^{2} \tag{33}
\end{equation*}
$$

where

$$
\begin{align*}
\chi_{1}= & \lambda_{M}\left(P_{1}\right)+2 K_{M}(F-E)+\left(\frac{1-e^{-2 k d}}{2 k}\right) \\
& \times\left[\lambda_{M}\left(Q_{1}\right)+\lambda_{M}\left(Q_{3}\right)(F-E)+\lambda_{M}\left(W_{1}\right)\right] \\
& +\frac{3 d^{2}}{2} \lambda_{M}\left(Z_{1}\right) \lambda_{M}\left(C^{T} C\right) \\
& +\frac{h^{2}}{2} \lambda_{M}\left(Z_{2}\right)\left[3 \lambda_{M}\left(B_{1}^{T} B_{1}\right)+3 \lambda_{M}\left(B_{2}^{T} B_{2}\right)\right]  \tag{34}\\
\chi_{2}= & \lambda_{M}\left(P_{2}\right)+2 R_{M}(M-N)+\left(\frac{1-e^{-2 k h}}{2 k}\right) \\
& \times\left[\lambda_{M}\left(Q_{2}\right)+\lambda_{M}\left(Q_{4}\right)(M-N)+\lambda_{M}\left(W_{2}\right)\right] \\
& +\frac{3 h^{2}}{2} \lambda_{M}\left(Z_{2}\right) \lambda_{M}\left(D^{T} D\right) \\
& +\frac{d^{2}}{2} \lambda_{M}\left(Z_{1}\right)\left[3 \lambda_{M}\left(A_{1}^{T} A_{1}\right)+3 \lambda_{M}\left(A_{2}^{T} A_{2}\right)\right] .
\end{align*}
$$

On the other hand, we have

$$
\begin{equation*}
V(t) \geq e^{2 k t}\left\{\lambda_{m}\left(P_{1}\right)\|x(t)\|^{2}+\lambda_{m}\left(P_{2}\right)\|y(t)\|^{2}\right\} . \tag{35}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\|x(t)\|^{2}+\|y(t)\|^{2} \leq \rho e^{-2 k t}\left\{\|\phi\|^{2}+\|\varphi\|^{2}\right\} \tag{36}
\end{equation*}
$$

where $\rho=\max \left(\chi_{1}, \chi_{2}\right) / \min \left(\lambda_{m}\left(P_{1}\right), \lambda_{m}\left(P_{2}\right)\right) \geq 1$. Therefore, the system (7) is global exponentially stable with the convergent rate $k>0$. This completes the proof.

Remark 6. In hypothesis (H2), lower bounds $E_{i}, N_{j}$ and upper bounds $F_{i}, M_{j}, i=1,2, \ldots, n, j=1,2, \ldots, m$, of activation functions are allowed to be either positive, negative, or zero. Clearly, hypothesis (H2) in our paper is more general than those given in [14, 15, 18, 21, 22, 24-27]. Hence, our result is less conservative than some existing results given in the literature.

## 4. Numerical Examples

Example 1. Consider the BAM neural networks in (7) with $C=\operatorname{diag}(1,1,1), D=\operatorname{diag}(2,2,2), A_{1}=B_{1}=0$,

$$
\begin{align*}
& A_{2}=\left[\begin{array}{ccc}
0.05 & 0.25 & 0.05 \\
0.1 & 0.05 & 0.15 \\
0.15 & 0.15 & 0.05
\end{array}\right], \\
& B_{2}=\left[\begin{array}{ccc}
0.75 & 0.75 & 0.95 \\
0 & 0.50 & 0.15 \\
0.15 & 0.15 & 0.05
\end{array}\right] . \tag{37}
\end{align*}
$$

In this example, the activation function and time delay are given as follows: $f_{1}(x)=f_{2}(x)=f_{3}(x)=(1 / 2)(|x+1|-\mid x-$ $1 \mid), g_{1}(y)=g_{2}(y)=g_{3}(y)=(1 / 2)(|y+1|-|y-1|), d(t)=0.5$, $h(t)=1$. It follows that $d=0.5, \tau=0.3, h=1$, and $\mu=$ 0.3. The assumption (H2) is satisfied with $F=\operatorname{diag}(1,1,1)$, $M=\operatorname{diag}(1,1,1), E=N=0$. Let $k=0.1$. By using the LMI Toolbox in MATLAB, the LMI (16) of Theorem 5 is feasible with $k=0.1$ and a set of solutions of (16) is given by

$$
\begin{aligned}
& P_{1}=\left[\begin{array}{ccc}
28.6382 & 0.6843 & 0.7501 \\
0.6843 & 28.2061 & 1.0981 \\
0.7501 & 1.0981 & 27.7326
\end{array}\right], \\
& P_{2}=\left[\begin{array}{ccc}
11.7631 & 0.5671 & 0.4170 \\
0.5671 & 25.8296 & 0.7934 \\
0.4170 & 0.7934 & 26.2860
\end{array}\right], \\
& Q_{1}=\left[\begin{array}{ccc}
14.4157 & -0.0495 & -0.0610 \\
-0.0495 & 14.4217 & -0.0911 \\
-0.0610 & -0.0911 & 14.4462
\end{array}\right], \\
& Q_{2}=\left[\begin{array}{ccc}
12.8030 & -0.9294 & -0.9704 \\
-0.9294 & 16.9982 & -0.4287 \\
-0.9704 & -0.4287 & 17.2419
\end{array}\right], \\
& Q_{3}=\left[\begin{array}{ccc}
16.7725 & 0.4495 & 0.2528 \\
0.4495 & 15.0669 & 0.5969 \\
0.2528 & 0.5969 & 13.1708
\end{array}\right], \\
& Q_{4}=\left[\begin{array}{ccc}
11.9146 & -0.2010 & -0.3870 \\
-0.2010 & 15.1767 & -0.6757 \\
-0.3870 & -0.6757 & 15.3739
\end{array}\right], \\
& W_{1}=\left[\begin{array}{ccc}
14.6218 & 1.0858 & 0.9997 \\
1.0858 & 13.0260 & 1.3001 \\
0.9997 & 1.3001 & 11.6739
\end{array}\right], \\
& Z_{2}=\left[\begin{array}{ccc}
8.4065 & -0.0639 & -0.2523 \\
W_{2} & =\left[\begin{array}{ccc}
2.3511 & 0.7468 & 0.6736 \\
-0.0639 & 11.9959 & 0.1677 \\
-0.2523 & 0.1677 & 11.6957
\end{array}\right], \\
Z_{1}=\left[\begin{array}{ccc}
13.3993 & 0.2142 & 0.3899 \\
0.2142 & 13.5746 & 0.3941 \\
0.3899 & 0.3941 & 13.7729
\end{array}\right], \\
0.3500 & 11.7724
\end{array}\right],
\end{aligned}
$$

$$
\begin{align*}
& S_{11}=\left[\begin{array}{rrr}
12.4617 & -0.0242 & -0.0266 \\
-0.0242 & 12.4732 & -0.0399 \\
-0.0266 & -0.0399 & 12.4949
\end{array}\right] \text {, } \\
& S_{12}=\left[\begin{array}{lll}
-1.1535 & -0.0211 & -0.0257 \\
-0.0211 & -1.1444 & -0.0335 \\
-0.0257 & -0.0335 & -1.1320
\end{array}\right] \text {, } \\
& S_{13}=\left[\begin{array}{lll}
-0.9075 & -0.0415 & -0.0544 \\
-0.0415 & -0.9093 & -0.0805 \\
-0.0544 & -0.0805 & -0.9019
\end{array}\right] \text {, } \\
& S_{22}=\left[\begin{array}{rrr}
12.2314 & -0.0115 & -0.0101 \\
-0.0115 & 12.2463 & -0.0163 \\
-0.0101 & -0.0163 & 12.2626
\end{array}\right] \text {, } \\
& S_{23}=\left[\begin{array}{lll}
1.6306 & 0.0129 & 0.0208 \\
0.0129 & 1.6430 & 0.0270 \\
0.0208 & 0.0270 & 1.6490
\end{array}\right] \text {, } \\
& T_{11}=\left[\begin{array}{ccc}
8.6800 & -0.5650 & -0.6089 \\
-0.5650 & 12.0158 & -0.3052 \\
-0.6089 & -0.3052 & 12.2653
\end{array}\right] \text {, } \\
& T_{12}=\left[\begin{array}{lll}
-1.8596 & -0.2702 & -0.2795 \\
-0.2702 & -1.2902 & -0.1294 \\
-0.2795 & -0.1294 & -1.2317
\end{array}\right] \text {, } \\
& T_{13}=\left[\begin{array}{ccc}
-1.7089 & -0.3795 & -0.4069 \\
-0.3795 & -0.6642 & -0.2437 \\
-0.4069 & -0.2437 & -0.5078
\end{array}\right] \text {, } \\
& T_{22}=\left[\begin{array}{ccc}
8.2813 & -0.2103 & -0.2276 \\
-0.2103 & 10.2975 & -0.1027 \\
-0.2276 & -0.1027 & 10.4221
\end{array}\right] \text {, } \\
& T_{23}=\left[\begin{array}{lll}
1.9707 & 0.1871 & 0.1870 \\
0.1871 & 2.3062 & 0.0783 \\
0.1870 & 0.0783 & 2.3346
\end{array}\right] \text {, } \\
& K=\left[\begin{array}{ccc}
14.9427 & 0 & 0 \\
0 & 14.2734 & 0 \\
0 & 0 & 13.4467
\end{array}\right] \text {, } \\
& R=\left[\begin{array}{ccc}
5.8315 & 0 & 0 \\
0 & 8.1963 & 0 \\
0 & 0 & 8.3562
\end{array}\right] \text {. } \tag{38}
\end{align*}
$$

Thus, the system (7) is 0.1 -exponentially stable and the value $\rho=13.4606$. The solution of the closed-loop system satisfies

$$
\begin{align*}
& \|x(t)\|^{2}+\|y(t)\|^{2} \\
& \quad \leq 13.4606 e^{-2(0.1) t}\left\{\|\phi\|^{2}+\|\varphi\|^{2}\right\}, \quad \forall t \in \mathbb{R}^{+} . \tag{39}
\end{align*}
$$

By applying Theorem 5 and by solving the LMI (16) using MATLAB LMI Toolbox, we obtain the convergence rate $k$ which guarantees that the global exponential stability is 0.998 . In Table 1, we give comparison of maximum allowable convergence rate $k$ obtained by Theorem 5 and by other

Table 1: Maximum allowable convergence rate.

| $\tau=\mu$ | 0 | 0.3 | 0.5 | 0.7 | 0.9 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $[24]$ | - | 0.459 | 0.455 | 0.455 | 0.455 |
| $[29]$ | - | 0.445 | 0.424 | 0.408 | 0.407 |
| $[25]$ | - | 0.52 | 0.47 | 0.39 | 0.21 |
| This paper | 0.998 | 0.998 | 0.998 | 0.998 | 0.998 |

methods in some previous existing results. From Table 1, it is shown that the proposed global exponential stability criterion is less conservative than those obtained in [24, 25, 29].

Example 2. Consider the BAM neural networks in (7) with

$$
\begin{gather*}
C=\left[\begin{array}{lll}
5 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 5
\end{array}\right], \\
A_{1}=\left[\begin{array}{ccc}
-0.72 & -0.44 & -0.21 \\
-0.72 & -0.83 & -0.1 \\
-0.01 & 0.01 & -0.04
\end{array}\right], \\
A_{2}=\left[\begin{array}{ccc}
-0.01 & -0.12 & -0.24 \\
0.17 & -0.33 & -0.43 \\
-0.25 & 0.33 & -0.05
\end{array}\right],  \tag{40}\\
D=\left[\begin{array}{ccc}
5 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 5
\end{array}\right], \\
B_{1}=\left[\begin{array}{ccc}
-0.31 & -0.31 & 0.92 \\
0.34 & -0.33 & -0.78 \\
0.34 & 0.47 & 0.25
\end{array}\right], \\
B_{2}=\left[\begin{array}{ccc}
-0.83 & -0.12 & -0.52 \\
-0.65 & 0.5 & -0.14 \\
-0.05 & -0.14 & -0.65
\end{array}\right],
\end{gather*}
$$

In this example, the activation function and time-varying delay are given as follows: $f_{1}(x)=f_{2}(x)=f_{3}(x)=$ $(1 / 2)(|x+1|-|x-1|), g_{1}(y)=g_{2}(y)=g_{3}(y)=(1 / 2)(\mid y+$ $1|-|y-1|), d(t)=\sin ^{2}(0.5 t), h(t)=0.1 \cos ^{2}(t)$; the assumption (H2) is satisfied with $E=\operatorname{diag}(-0.2,-0.25,-0.2)$, $F=\operatorname{diag}(0.3,0.4,0.5), N=\operatorname{diag}(-0.2,-0.25,-0.2), M=$ $\operatorname{diag}(0.3,0.4,0.5)$. It follows that $d=0.5, \tau=0.3, h=0.1$, and $\mu=0.3$. Let $k=0.1$. By using the LMI Toolbox in MATLAB, the LMI (16) is feasible with $k=0.1$ and a set of solutions of (16) is given by

$$
\begin{gathered}
P_{1}=\left[\begin{array}{ccc}
110.3704 & -30.1135 & -5.4152 \\
-30.1135 & 90.3013 & 8.8948 \\
-5.4152 & 8.8948 & 115.0609
\end{array}\right], \\
P_{2}=\left[\begin{array}{ccc}
43.9625 & -2.3031 & 7.3341 \\
-2.3031 & 38.1671 & 4.9969 \\
7.3341 & 4.9969 & 62.3183
\end{array}\right],
\end{gathered}
$$

$$
\begin{aligned}
& Q_{1}=\left[\begin{array}{ccc}
125.4489 & -1.8802 & 0.9399 \\
-1.8802 & 127.5495 & -1.0063 \\
0.9399 & -1.0063 & 130.0399
\end{array}\right], \\
& Q_{2}=\left[\begin{array}{ccc}
100.3079 & -4.1254 & -0.7526 \\
-4.1254 & 101.8490 & 1.0897 \\
-0.7526 & 1.0897 & 114.9297
\end{array}\right], \\
& Q_{3}=\left[\begin{array}{ccc}
136.0956 & -4.2919 & 38.6712 \\
-4.2919 & 90.4808 & 19.2022 \\
38.6712 & 19.2022 & 113.1213
\end{array}\right], \\
& Q_{4}=\left[\begin{array}{ccc}
42.6986 & -34.8491 & -15.2241 \\
-34.8491 & 96.7728 & 2.0801 \\
-15.2241 & 2.0801 & 145.8684
\end{array}\right], \\
& W_{1}=\left[\begin{array}{ccc}
140.8059 & 25.3914 & -0.9357 \\
25.3914 & 146.4301 & -4.4760 \\
-0.9357 & -4.4760 & 11.6128
\end{array}\right], \\
& W_{2}=\left[\begin{array}{ccc}
105.4681 & -0.1405 & 5.1590 \\
-0.1405 & 107.4419 & 16.7637 \\
5.1590 & 16.7637 & 134.1991
\end{array}\right], \\
& Z_{1}=\left[\begin{array}{ccc}
69.1695 & -24.4444 & -4.5155 \\
-24.4444 & 48.7177 & 7.8545 \\
-4.5155 & 7.8545 & 65.4657
\end{array}\right], \\
& Z_{2}=\left[\begin{array}{ccc}
99.9490 & 1.3357 & 35.2470 \\
1.3357 & 80.2179 & 10.6682 \\
35.2470 & 10.6682 & 111.8243
\end{array}\right], \\
& S_{11}=\left[\begin{array}{ccc}
103.3199 & -2.9895 & 0.4043 \\
-2.9895 & 103.2019 & -0.1404 \\
0.4043 & -0.1404 & 106.3073
\end{array}\right] \text {, } \\
& S_{12}=\left[\begin{array}{ccc}
-2.9235 & -0.6314 & 0.1584 \\
-0.6314 & -2.4889 & -0.1672 \\
0.1584 & -0.1672 & -1.9572
\end{array}\right] \text {, } \\
& S_{13}=\left[\begin{array}{ccc}
1.0569 & -2.4649 & 0.0237 \\
-2.4649 & 0.4090 & 0.2113 \\
0.0237 & 0.2113 & 2.4210
\end{array}\right] \text {, } \\
& S_{22}=\left[\begin{array}{ccc}
97.0839 & -2.1772 & 0.0865 \\
-2.1772 & 96.3284 & 0.1980 \\
0.0865 & 0.1980 & 98.4407
\end{array}\right] \text {, } \\
& S_{23}=\left[\begin{array}{ccc}
8.9358 & -1.7429 & -0.3501 \\
-1.7429 & 7.0242 & 0.7119 \\
-0.3501 & 0.7119 & 8.3850
\end{array}\right] \text {, } \\
& T_{11}=\left[\begin{array}{ccc}
105.0332 & -0.4193 & 0.0331 \\
-0.4193 & 105.2266 & 0.3043 \\
0.0331 & 0.3043 & 107.5064
\end{array}\right] \text {, } \\
& T_{12}=\left[\begin{array}{ccc}
-1.5261 & -0.6096 & -0.1149 \\
-0.6096 & -1.2415 & 0.0894 \\
-0.1149 & 0.0894 & 0.3340
\end{array}\right] \text {, }
\end{aligned}
$$

Table 2: Maximum allowable convergence rate.

| $\tau=\mu$ | 0 | 0.3 | 0.5 | 0.7 |
| :--- | :---: | :---: | :---: | :---: |
| Convergence rate $k$ | 0.81 | 0.638 | 0.48 | 0.221 |



Figure 1: Time responses of state variables.

$$
\begin{align*}
T_{13} & =\left[\begin{array}{ccc}
-7.6562 & -3.2414 & -0.6934 \\
-3.2414 & -5.7650 & 0.5527 \\
-0.6934 & 0.5527 & 4.8321
\end{array}\right], \\
T_{22} & =\left[\begin{array}{ccc}
104.9648 & -0.5090 & -0.0471 \\
-0.5090 & 105.1940 & 0.0964 \\
-0.0471 & 0.0964 & 106.5444
\end{array}\right], \\
T_{23} & =\left[\begin{array}{ccc}
2.6072 & 0.1420 & 0.2077 \\
0.1420 & 2.6111 & -0.0404 \\
0.2077 & -0.0404 & 2.2053
\end{array}\right], \\
K & =\left[\begin{array}{ccc}
106.5957 & 0 & 0 \\
0 & 63.3921 & 0 \\
0 & 0 & 6.3380
\end{array}\right], \\
R & =\left[\begin{array}{ccc}
86.0059 & 0 & 0 \\
0 & 72.0231 & 0 \\
0 & 0 & 71.2919
\end{array}\right], \tag{41}
\end{align*}
$$

Thus, the system (7) is 0.1 -exponentially stable and the value $\rho=14.1811$. The solution of the closed-loop system satisfies

$$
\begin{align*}
& \|x(t)\|^{2}+\|y(t)\|^{2} \\
& \quad \leq 14.1811 e^{-2(0.1) t}\left\{\|\phi\|^{2}+\|\varphi\|^{2}\right\}, \quad \forall t \in \mathbb{R}^{+} . \tag{42}
\end{align*}
$$

The maximum allowable convergence rate $k$ for different values of $\tau=\mu$ is given in Table 2. The trajectory of solutions of BAM neural networks with time-varying delays is shown
in Figure 1, where the initial conditions are chosen as $\phi_{1}=$ $\cos (s), \phi_{2}=\sin (s), \phi_{3}=\sin (s)-1, \varphi_{1}=\cos (s)+1, \varphi_{2}=$ $\sin (s)-2, \varphi_{3}=\cos (s)+1, s \in[-0.5,0]$.

## 5. Conclusion

This paper has proposed a new sufficient condition guaranteeing the global exponential stability criteria for bidirectional associative memory neural networks with timevarying delays and generalized activation functions. The developed stability condition is in terms of LMI, which can be easily solved by some existing software packages. Furthermore, the proposed stability conditions are less conservative than some works in the literature.

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## Research Article

# Stochastic Delay Population Dynamics under Regime Switching: Permanence and Asymptotic Estimation 

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#### Abstract

This paper is concerned with a delay Lotka-Volterra model under regime switching diffusion in random environment. Permanence and asymptotic estimations of solutions are investigated by virtue of $V$-function technique, $M$-matrix method, and Chebyshev's inequality. Finally, an example is given to illustrate the main results.


## 1. Introduction

The delay differential equation

$$
\begin{equation*}
\frac{d x(t)}{d t}=x(t)(a-b x(t)+c x(t-\tau)) \tag{1}
\end{equation*}
$$

has been used to model the population growth of certain species and is known as the delay Lotka-Volterra model or the delay logistic equation. The delay Lotka-Volterra model for $n$ interacting species is described by the $n$-dimensional delay differential equation

$$
\begin{equation*}
\frac{d x(t)}{d t}=\operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right)(b+A x(t)+B x(t-\tau)) \tag{2}
\end{equation*}
$$

where $x=\left(x_{1}, \ldots, x_{n}\right)^{T} \in R^{n}, b=\left(b_{1}, \ldots, b_{n}\right)^{T} \in R_{+}^{n}, A=$ $\left(a_{i j}\right)_{n \times n} \in R^{n \times n}, B=\left(b_{i j}\right)_{n \times n} \in R^{n \times n}$. There is an extensive literature concerned with the dynamics of this delay model and have had lots of nice results and we here only mention the work of Ahmad and Rao [1], Bereketoglu and Győri [2], and Freedman and Ruan [3], and in particular, the books by Gopalsamy [4], Kolmanovskiĭ and Myshkis [5], and Kuang [6] among many others.

In the equations above, the state $x(t)$ denotes the population sizes of the species. Naturally, we focus on the positive solutions and also require the solutions not to explode at a finite time. To guarantee the positive solutions without
explosion (i.e., the global positive solutions), some conditions are in general needed to impose on the system parameters. For example, it is generally assumed that $a>0, b>0$, and $c<b$ for (1) while much more complicated conditions are required on matrices $A$ and $B$ for (2) [7] (and the references cited therein).

On the other hand, population systems are often subject to environmental noise, and the system will change significantly, which may change the dynamics behavior of solutions significantly [8, 9]. It is therefore necessary to reveal how the noise affects the dynamics of solutions for the delay population systems. In fact, many authors have discussed population systems subject to white noise [7-18]. Recall that the parameter $b_{i}$ in (2) represents the intrinsic growth rate of species $i$. In practice, we usually estimate it by an average value plus an error term. According to the well-known central limit theorem, the error term follows a normal distribution. In terms of mathematics, we can therefore replace the rate $b_{i}$ by $b_{i}+\sigma_{i} \dot{w}(t)$, where $\dot{w}(t)$ is a white noise (i.e., $w(t)$ is a Brownian motion) and $\sigma_{i} \geq 0$ represents the intensity of noise. As a result, (2) becomes a stochastic differential equation (SDE, in short)

$$
\begin{align*}
d x(t)= & \operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right)  \tag{3}\\
& \times[(b+A x(t)+B x(t-\tau)) d t+\sigma d w(t)]
\end{align*}
$$

where $\sigma=\left(\sigma_{1}, \ldots, \sigma_{n}\right)^{T}$. We refer to [7] for more details.

To our knowledge, much attention to environmental noise is focused on white noise. But another type of environmental noise, namely, color noise, say telegraph noise, has been studied by many authors (see [19-25] and the references cited therein). In this context, telegraph noise can be described as a random switching between two or more environmental regimes, which differ in terms of factors such as nutrition or rain falls [23, 24]. Usually, the switching between different environments is memoryless and the waiting time for the next switch has an exponential distribution. This indicates that we may model the random environments and other random factors in the system by a continuoustime Markov chain $r(t), t \geq 0$ with a finite state space $S=\{1,2, \ldots, N\}$. Therefore, stochastic delay population system (3) in random environments can be described by the following stochastic model with regime switching:

$$
\begin{align*}
d x(t)= & \operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right) \\
& \times[(b(r(t))+A(r(t)) x(t)+B(r(t)) x(t-\tau)) d t \\
& +\sigma(r(t)) d w(t)] . \tag{4}
\end{align*}
$$

The mechanism of ecosystem described by (4) can be explained as follows. Assume that, initially, the Markov chain $r(0)=\iota \in S$. Then the ecosystem (4) obeys the SDE

$$
\begin{align*}
& d x(t) \\
& =\operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right) \\
& \quad \times[(b(\iota)+A(\iota) x(t)+B(\iota) x(t-\tau)) d t+\sigma(\iota) d w(t)] \tag{5}
\end{align*}
$$

until the Markov chain $r(t)$ jumps to another state, say $\varsigma$. Therefore, the ecosystem (4) satisfies the SDE

$$
\begin{align*}
& d x(t) \\
& =\operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right) \\
& \quad \times[(b(\varsigma)+A(\varsigma) x(t)+B(\varsigma) x(t-\tau)) d t+\sigma(\varsigma) d w(t)], \tag{6}
\end{align*}
$$

for a random amount of time until the Markov chain $r(t)$ jumps to a new state again.

It should be pointed out that the stochastic population systems under regime switching have received much attention lately. For instance, the stochastic permanence and extinction of a logistic model under regime switching were considered in [20,24], asymptotic results of a competitive Lotka-Volterra model in random environment are obtained in [25], a new single-species model disturbed by both white noise and colored noise in a polluted environment was developed and analyzed in [26], and a general stochastic logistic system under regime switching was proposed and was treated in [27].

In [28], some results have been obtained for (4), such as existence of global positive solutions, stochastically ultimate boundedness, and extinction. In contrast to the existing results, our new contributions in this paper are as follows.
(i) The stochastic permanence of solutions is derived.
(ii) The asymptotic estimations of the solutions are obtained, which is related to the stationary probability distribution of the Markov chain.

The rest of the paper is arranged as follows. For convenience of the reader, we briefly recall the main result of [28] in Section 2. The main results of this paper are arranged in Sections 3 and 4. Section 3 is devoted to the stochastic permanence. The asymptotic estimations of the solutions are obtained in Section 4. Finally, an example is given to illustrate our main results.

## 2. Properties of the Solution

Throughout this paper, unless otherwise specified, let $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, P\right)$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_{0}$ contains all $P$-null sets). Let $w(t)$, $t \geq 0$, be a scalar standard Brownian motion defined on this probability space. We also denote by $R_{+}^{n}$ the positive cone in $R^{n}$, that is, $R_{+}^{n}=\left\{x \in R^{n}: x_{i}>0\right.$ for all $\left.1 \leq i \leq n\right\}$ and denote by $\bar{R}_{+}^{n}$ the nonnegative cone in $R^{n}$, that is, $\bar{R}_{+}^{n}=\{x \in$ $R^{n}: x_{i} \geq 0$ for all $\left.1 \leq i \leq n\right\}$. If $A$ is a vector or matrix, its transpose is denoted be $A^{T}$. If $A$ is a matrix, its trace norm is denoted by $|A|=\sqrt{\operatorname{trace}\left(A^{T} A\right)}$, whilst it operator norm is denoted by $\|A\|=\sup \{|A x|:|x|=1\}$. Moreover, let $\tau>0$ and denote by $C\left([-\tau, 0] ; R_{+}\right)$the family of continuous functions from $[-\tau, 0]$ to $R_{+}$.

In this paper we will use a lot of quadratic functions of the form $x^{T} A x$ for the state $x \in R_{+}^{n}$ only. Therefore, for a symmetric $n \times n$ matrix $A$, we naturally introduce the following definition:

$$
\begin{equation*}
\lambda_{\max }^{+}(A)=\sup _{x \in R_{+}^{n},|x|=1} x^{T} A x . \tag{7}
\end{equation*}
$$

For more properties of $\lambda_{\max }^{+}(A)$, please see the appendix in [7].

Let $r(t)$ be a right-continuous Markov chain on the probability space, taking values in a finite state space $S=$ $\{1,2, \ldots, N\}$, with the generator $\Gamma=\left(\gamma_{u v}\right)$ given by

$$
P\{r(t+\delta)=v \mid r(t)=u\}= \begin{cases}\gamma_{u v} \delta+o(\delta), & \text { if } u \neq v  \tag{8}\\ 1+\gamma_{u v} \delta+o(\delta), & \text { if } u=v\end{cases}
$$

where $\delta>0, \gamma_{u v}$ is the transition rate from $u$ to $v$ and $\gamma_{u v} \geq 0$ if $u \neq v$, while $\gamma_{u u}=-\sum_{v \neq u} \gamma_{u v}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a rightcontinuous step function with a finite number of jumps in any finite subinterval of $\bar{R}_{+}$. As a standing hypothesis, we assume in this paper that the Markov chain $r(t)$ is irreducible. This is a very reasonable assumption as it means that the system can switch from any regime to any other regime. This is equivalent to the condition that for, any $u, v \in S$, one can find finite numbers $i_{1}, i_{2}, \ldots, i_{k} \in S$ such that $\gamma_{u i_{1}} \gamma_{i_{1} i_{2}}, \ldots, \gamma_{i_{k} v}>0$.

Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi=\left(\pi_{1}, \pi_{2}, \ldots, \pi_{N}\right) \in R^{1 \times N}$ which can be determined by solving the following linear equation:

$$
\begin{equation*}
\pi \Gamma=0 \tag{9}
\end{equation*}
$$

subject to

$$
\begin{equation*}
\sum_{i=1}^{N} \pi_{i}=1, \quad \pi_{i}>0, \quad \forall i \in S \tag{10}
\end{equation*}
$$

For the fundamental theory of stochastic differential equations, one can refer to [12, 29].

For convenience and simplicity in the following discussion, for any constant sequence $f_{i}(k),(1 \leq i \leq n, k \in S)$, let

$$
\begin{array}{ll}
\check{f}=\max _{1 \leq i \leq n, k \in S} f_{i}(k), & \check{f}(k)=\max _{1 \leq i \leq n} f_{i}(k), \\
\widehat{f}=\min _{1 \leq i \leq n, k \in S} f_{i}(k), & \widehat{f}(k)=\min _{1 \leq i \leq n} f_{i}(k) . \tag{11}
\end{array}
$$

To proceed, we first state a result, whose proof can be found in [28].

Assumption 1. Assume that there exist positive numbers $c_{1}, \ldots, c_{n}$ such that

$$
\begin{equation*}
\max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right]\right\}+\max _{k \in S}\|\bar{C} B(k)\| \leq 0 \tag{12}
\end{equation*}
$$

where $\bar{C}=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$.
Assumption 2. Assume that there exist positive numbers $c_{1}, \ldots, c_{n}$ such that

$$
\begin{equation*}
\max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right]\right\}+\max _{k \in S}\|\bar{C} B(k)\|<0 \tag{13}
\end{equation*}
$$

where $\bar{C}=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$.
Assumption 3. Assume that there exist positive numbers $c_{1}, \ldots, c_{n}$ such that

$$
\begin{gather*}
|c|^{-1} \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right]\right\}  \tag{14}\\
+\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\| \leq 0
\end{gather*}
$$

where $\bar{C}=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$ and $\widehat{c}=\min _{1 \leq i \leq n} c_{i}$.
Theorem 1. (1) Under Assumption 1, for any given initial data $\{x(t):-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}^{n}\right)$, there is a unique solution $x(t)$ to (4) on $t \geq-\tau$ and the solution will remain in $R_{+}^{n}$ with probability 1 , namely, $x(t) \in R_{+}^{n}$ for all $t \geq-\tau$ almost surely.
(2) Under Assumption 2, for any given initial data $\{x(t)$ : $-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}^{n}\right)$ and any given positive constant
$p$, there are two positive constant $K_{1}(p)$ and $K_{2}(p)$, such that the solution $x(t)$ of (4) has the properties that

$$
\begin{gather*}
\limsup _{t \rightarrow \infty} E|x(t)|^{p} \leq K_{1}(p)  \tag{15}\\
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} E|x(s)|^{p+1} d s \leq K_{2}(p) \tag{16}
\end{gather*}
$$

(3) Solutions of (4) are stochastically ultimately bounded under Assumption 2; that is, for any $\varepsilon \in(0,1)$, there exists a positive constants $H=H(\varepsilon)$, such that the solutions of (4) with any positive initial value have the property that

$$
\begin{equation*}
\limsup _{t \rightarrow+\infty} P\{|x(t)|>H\}<\varepsilon \tag{17}
\end{equation*}
$$

(4) Under Assumption 3, for any given initial data $\{x(t)$ : $-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}\right)$, the solution $x(t)$ of (4) has the properties that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log |x(t)| \leq \sum_{i=1}^{n} \pi_{k} \beta(k) \quad \text { a.s., } \tag{18}
\end{equation*}
$$

where $\beta(k)=\check{b}(k)-(1 / 2) \widehat{\sigma}^{2}(k)$. Particularly, if $\sum_{k=1}^{N} \pi_{k} \beta(k)<$ 0 , then

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log |x(t)|<0 \quad \text { a.s. } \tag{19}
\end{equation*}
$$

That is, the population will become extinct exponentially with probability 1.

## 3. Stochastic Permanence

Definition 2. Equation (4) is said to be stochastically permanent if, for any $\varepsilon \in(0,1)$, there exist positive constants $H=H(\varepsilon), \delta=\delta(\varepsilon)$ such that

$$
\begin{align*}
& \liminf _{t \rightarrow+\infty} P\{|x(t)| \leq H\} \geq 1-\varepsilon \\
& \liminf _{t \rightarrow+\infty} P\{|x(t)| \geq \delta\} \geq 1-\varepsilon \tag{20}
\end{align*}
$$

where $x(t)$ is the solution of (4) with any positive initial value.
It is obvious that if a stochastic equation is stochastically permanent, its solutions must be stochastically ultimately bounded. For convenience, let

$$
\begin{equation*}
\alpha(k)=\widehat{b}(k)-\frac{1}{2} \check{\sigma}^{2}(k), \quad \beta(k)=\check{b}(k)-\frac{1}{2} \widehat{\sigma}^{2}(k), \tag{21}
\end{equation*}
$$

and we impose the following assumptions.
Assumption 4. For some $u \in S, \gamma_{i u}>0$ (for all $i \neq u$ ).
Assumption 5. $\sum_{k=1}^{N} \pi_{k} \alpha(k)>0$.
Assumption 6. For each $k \in S, \alpha(k)>0$.
Let $G$ be a vector or matrix. By $G \gg 0$, we mean all elements of $G$ are positive, and by $G \geq 0$, we mean all elements of $G$ are nonnegative. We also adopt here the traditional notation by letting

$$
\begin{equation*}
Z^{N \times N}=\left\{A=\left(a_{i j}\right)_{N \times N}: a_{i j} \leq 0, i \neq j\right\} \tag{22}
\end{equation*}
$$

Lemma 3 (see [29]). If $A=\left(a_{i j}\right)_{N \times N} \in Z^{N \times N}$ has all of its row sums positive, that is,

$$
\begin{equation*}
\sum_{j=1}^{N} a_{i j}>0, \quad \forall 1 \leq i \leq N, \tag{23}
\end{equation*}
$$

then $A>0$.
Lemma 4 (see [29]). If $A \in Z^{N \times N}$, then the following statements are equivalent:
(1) $A$ is a nonsingular $M$-matrix.
(2) All of the principal minors of A are positive; that is,

$$
\left|\begin{array}{ccc}
a_{11} & \cdots & a_{1 k}  \tag{24}\\
\vdots & \vdots & \vdots \\
a_{1 k} & \cdots & a_{k k}
\end{array}\right|>0 \quad \text { for every } k=1,2, \ldots, N .
$$

(3) A is semipositive; that is, there exists $x \gg 0$ in $R^{N}$ such that $A x \gg 0$.

Lemma 5 (see [23]). Assumptions 4 and 5 imply that there exists a constant $\theta>0$ such that the matrix

$$
\begin{equation*}
A(\theta)=\operatorname{diag}\left(\xi_{1}(\theta), \xi_{2}(\theta), \ldots, \xi_{N}(\theta)\right)-\Gamma \tag{25}
\end{equation*}
$$

is a nonsingular $M$-matrix, where

$$
\begin{equation*}
\xi_{k}(\theta)=\theta \alpha(k)-\frac{1}{2} \theta^{2} \check{\sigma}^{2}(k), \quad \forall k \in S \tag{26}
\end{equation*}
$$

Lemma 6 (see [23]). Assumption 6 implies that there exists a constant $\theta>0$ such that the matrix $A(\theta)$ is a nonsingular $M$ matrix.

Lemma 7. If there exists a constant $\theta>0$ such that $A(\theta)$ is a nonsingular $M$-matrix and $B(k) \geq 0(k=1,2, \ldots, N)$, then the global positive solution $x(t)$ of (4) has the property that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E\left(\frac{1}{|x(t)|^{\theta}}\right) \leq H, \tag{27}
\end{equation*}
$$

where $H$ is a fixed positive constant (defined by (42) in the proof).

Proof. Define $V(x)=\sum_{i=1}^{n} x_{i}$ on $x \in R_{+}^{n}$. Then

$$
\begin{align*}
d V(x(t))=x^{T}(t)[ & (b(r(t))+A(r(t)) x(t) \\
& +B(r(t)) x(t-\tau)) d t  \tag{28}\\
& +\sigma(r(t)) d w(t)] .
\end{align*}
$$

Define also

$$
\begin{equation*}
U(x)=\frac{1}{V(x)} \quad \text { on } x \in R_{+}^{n} . \tag{29}
\end{equation*}
$$

Let $y(t)=x(t-\tau)$. Applying the generalized Itô formula, we derive from (28) that

$$
\begin{align*}
d U= & -U^{2} d V \\
= & +U^{3}(d V)^{2} \\
& \quad x^{T}\{[b(r(t))+A(r(t)) x+B(r(t)) y] d t  \tag{30}\\
& +\sigma(r(t)) d w(t)\}+U^{3}\left|x^{T} \sigma(r(t))\right|^{2} d t \\
=\{ & -U^{2} x^{T}[b(r(t))+A(r(t)) x+B(r(t)) y] \\
& \left.+U^{3}\left|x^{T} \sigma(r(t))\right|^{2}\right\} d t-U^{2} x^{T} \sigma(r(t)) d w(t)
\end{align*}
$$

dropping $x(t)$ from $U(x(t)), V(x(t))$ and $t$ from $x(t), y(t)$, respectively. By Lemma 4 , for given $\theta$, there is a vector $\vec{q}=$ $\left(q_{1}, q_{2}, \ldots, q_{N}\right)^{T} \gg 0$ such that

$$
\begin{equation*}
\vec{\lambda}=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{N}\right)^{T}:=A(\theta) \vec{q} \gg 0, \tag{31}
\end{equation*}
$$

namely,

$$
\begin{equation*}
q_{k}\left(\theta \alpha(k)-\frac{1}{2} \theta^{2} \check{\sigma}^{2}(k)\right)-\sum_{l=1}^{N} \gamma_{k l} q_{l}>0 \quad \forall 1 \leq k \leq N . \tag{32}
\end{equation*}
$$

Define the function $\bar{V}: R_{+} \times S \rightarrow R_{+}$by $\bar{V}(U, k)=$ $q_{k}(1+U)^{\theta}$. It follows from the generalized Itô formula that

$$
\begin{align*}
E \bar{V}(U(t), r(t))= & \bar{V}(U(0), r(0)) \\
& +E \int_{0}^{t} L \bar{V}(U(s), x(s-\tau), r(s)) d s, \tag{33}
\end{align*}
$$

where

$$
\begin{aligned}
& L \bar{V}(U, x, y, k) \\
& \begin{aligned}
= & q_{k} \theta(1+U)^{\theta-1}\left\{-U^{2} x^{T}[b(k)+A(k) x+B(k) y]\right. \\
& \left.+U^{3}\left|x^{T} \sigma(k)\right|^{2}\right\}+q_{k} \frac{\theta(\theta-1)}{2} \\
\times & (1+U)^{\theta-2} U^{4}\left|x^{T} \sigma(k)\right|^{2}+\sum_{l=1}^{N} \gamma_{k l} q_{l}(1+U)^{\theta} \\
= & q_{k} \theta(1+U)^{\theta-2} \\
& \times\left\{-(1+U) U^{2} x^{T}[b(k)+A(k) x+B(k) y]\right. \\
& \left.\quad+(1+U) U^{3}\left|x^{T} \sigma(k)\right|^{2}+\frac{1}{2}(\theta-1) U^{4}\left|x^{T} \sigma(k)\right|^{2}\right\}
\end{aligned}
\end{aligned}
$$

$$
\begin{equation*}
+\sum_{l=1}^{N} \gamma_{k l} q_{l}(1+U)^{\theta} \tag{34}
\end{equation*}
$$

It is easy to see that, for all $x \in R_{+}^{n}$,

$$
\begin{gather*}
-\frac{x^{T} A(k) x}{V^{2}} \leq K \\
-\frac{x^{T} b(k)}{V}+\frac{\left|x^{T} \sigma(k)\right|^{2}-x^{T} A(k) x}{V^{2}} \leq K \tag{35}
\end{gather*}
$$

where $K$ is a positive constant, while

$$
\begin{align*}
\frac{x^{T} b(k)}{V}-\frac{1}{2}(\theta+1) \frac{\left|x^{T} \sigma(k)\right|^{2}}{V^{2}} & \geq \widehat{b}(k)-\frac{1}{2}(\theta+1) \check{\sigma}^{2}(k) \\
& =\widehat{\beta}(k)-\frac{1}{2} \theta \check{\sigma}^{2}(k) \tag{36}
\end{align*}
$$

Consequently,

$$
\begin{align*}
-(1+ & U) U^{2} x^{T}[b(k)+A(k) x+B(k) y] \\
& +(1+U) U^{3}\left|x^{T} \sigma(k)\right|^{2}+\frac{1}{2}(\theta-1) U^{4}\left|x^{T} \sigma(k)\right|^{2} \\
= & -U^{2} x^{T} b(k)-U^{3} x^{T} b(k)-U^{2} x^{T} A(k) x \\
& -U^{3} x^{T} A(k) x-U^{2} x^{T} B(k) y-U^{3} x^{T} B(k) y \\
& +U^{3}\left|x^{T} \sigma(k)\right|^{2}+\frac{1}{2}(\theta+1) U^{4}\left|x^{T} \sigma(k)\right|^{2} \\
\leq & -\left(\frac{x^{T} b(k)}{V}-\frac{1}{2}(\theta+1) \frac{\left|x^{T} \sigma(k)\right|^{2}}{V^{2}}\right) U^{2} \\
& +\left(-\frac{x^{T} b(k)}{V}+\frac{\left|x^{T} \sigma(k)\right|^{2}-x^{T} A(k) x}{V^{2}}\right) U \\
& -\frac{x^{T} A(k) x}{V^{2}} \\
\leq & -\left(\alpha(k)-\frac{1}{2} \theta \check{\sigma}^{2}(k)\right) U^{2}+K(1+U) . \tag{37}
\end{align*}
$$

Substituting (37) into (34) yields

$$
\begin{aligned}
& L \bar{V}(U, x, y, k) \\
& \leq q_{k} \theta(1+U)^{\theta-2}\left\{-\left(\alpha(k)-\frac{1}{2} \theta \check{\sigma}^{2}(k)\right) U^{2}+K(1+U)\right\} \\
& \quad+\sum_{l=1}^{N} \gamma_{k l} q_{l}(1+U)^{\theta} \\
& =(1+U)^{\theta-2}
\end{aligned}
$$

$$
\begin{aligned}
\times\{ & -\left[q_{k} \theta\left(\alpha(k)-\frac{1}{2} \theta \check{\sigma}^{2}(k)\right)-\sum_{l=1}^{N} \gamma_{k l} q_{l}\right] U^{2} \\
& +\left(q_{k} \theta K+2 \sum_{l=1}^{N} \gamma_{k l} q_{l}\right) U \\
& \left.+\left(q_{k} \theta K+2 \sum_{l=1}^{N} \gamma_{k l} q_{l}\right)\right\} .
\end{aligned}
$$

Now, choose a constant $\kappa>0$ sufficiently small such that it satisfies $\vec{\lambda}-\kappa \vec{q} \gg 0$, that is,

$$
\begin{equation*}
q_{k}\left(\theta \alpha(k)-\frac{1}{2} \theta^{2} \check{\sigma}^{2}(k)\right)-\sum_{l=1}^{N} \gamma_{k l} q_{l}-\kappa q_{k}>0 \quad \forall 1 \leq k \leq N \tag{39}
\end{equation*}
$$

Then, by the generalized Itô formula again,

$$
\begin{align*}
& E\left[e^{\kappa t} \bar{V}(U(t), r(t))\right] \\
& \quad=\bar{V}(U(0), r(0)) \\
& \quad+E \int_{0}^{t}\left[\kappa e^{\kappa t} \bar{V}(U(s), r(s))\right. \\
& \left.\quad+e^{\kappa t} L \bar{V}(U(s), x(s), x(s-\tau), r(s))\right] d s \tag{40}
\end{align*}
$$

It is computed that

$$
\begin{align*}
& \kappa e^{\kappa t} \bar{V}(U, i)+e^{\kappa t} L \bar{V}(U, x, y, i) \\
& \begin{aligned}
\leq e^{\kappa t}(1 & +U)^{\theta-2} \\
& \times\left\{\kappa q_{k}(1+U)^{2}\right. \\
& \quad-\left[q_{k} \theta\left(\alpha(k)-\frac{1}{2} \theta \check{\sigma}^{2}(k)\right)-\sum_{l=1}^{N} \gamma_{k l} q_{l}\right] U^{2} \\
& \left.+\left(q_{k} \theta K+2 \sum_{l=1}^{N} \gamma_{k l} q_{l}\right) U+\left(q_{k} \theta K+2 \sum_{l=1}^{N} \gamma_{k l} q_{l}\right)\right\} \\
\leq e^{\kappa t}(1 & +U)^{\theta-2} \\
\times\{ & -\left[q_{k} \theta\left(\alpha(k)-\frac{1}{2} \theta \check{\sigma}^{2}(k)\right)-\sum_{l=1}^{N} \gamma_{k l} q_{l}-\kappa q_{k}\right] U^{2} \\
& +\left(q_{k} \theta K+2 \sum_{l=1}^{N} \gamma_{k l} q_{l}+2 \kappa q_{k}\right) U \\
\leq n^{-\theta} & \\
& +\left(q_{k} \theta K H e^{\kappa t},\right.
\end{aligned}
\end{align*}
$$

where

$$
\begin{align*}
& H=\frac{1}{\widehat{q} \kappa} n^{\theta} \max _{i \in S}\left\{\sup _{U \in R_{+}}(1+U)^{\theta-2}\right. \\
& \times\left\{-\left[q_{k} \theta\left(\alpha(k)-\frac{1}{2} \theta \check{\sigma}^{2}(k)\right)\right.\right. \\
&\left.\quad-\sum_{l=1}^{N} \gamma_{k l} q_{l}-\kappa q_{k}\right] U^{2} \\
&+\left(q_{k} \theta K+2 \sum_{l=1}^{N} \gamma_{k l} q_{l}+2 \kappa q_{k}\right) U \\
&\left.\left.+\left(q_{k} \theta K+2 \sum_{l=1}^{N} \gamma_{k l} q_{l}+\kappa q_{k}\right)\right\} \vee 1\right\} \tag{42}
\end{align*}
$$

This implies

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E\left[U^{\theta}(x(t))\right] \leq \limsup _{t \rightarrow \infty} E\left[(1+U(x(t)))^{\theta}\right] \leq n^{-\theta} H \tag{43}
\end{equation*}
$$

For $x(t) \in R_{+}^{n}$, note that $\left(\sum_{i=1}^{n} x_{i}(t)\right)^{\theta} \leq\left(n \max _{1 \leq i \leq n} x_{i}(t)\right)^{\theta} \leq$ $n^{\theta}|x(t)|^{\theta}$. Consequently,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E\left(\frac{1}{|x(t)|^{\theta}}\right) \leq H \tag{44}
\end{equation*}
$$

The required assertion (27) is obtained.
Assumption 7. Assume that there exist positive numbers $c_{1}, \ldots, c_{n}$ such that

$$
\begin{equation*}
\max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right]\right\}+\max _{k \in S}\|\bar{C} B(k)\|<0 \tag{45}
\end{equation*}
$$

where $\bar{C}=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$. Moreover for each $k \in S, B(k) \geq 0$.
Theorem 8. Under Assumptions 4, 5, and 7, (4) is stochastically permanent.

The proof is a simple application of the Chebyshev's inequality, Lemmas 5 and 7, and Theorem 1(3). Similarly, we have the following result.

Theorem 9. Under Assumptions 6 and 7, (4) is stochastically permanent.

## 4. Asymptotic Properties

Lemma 10. Under Assumption 2, for any given initial data $\{x(t):-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}\right)$, the solution $x(t)$ of (4) with any positive initial value has the property

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log (x(t))}{\log t} \leq 1 \quad \text { a.s. } \tag{46}
\end{equation*}
$$

Proof. By Theorem 1 (1), the solution $x(t)$ will remain in $R_{+}^{n}$ for all $t \geq-\tau$ with probability 1 . Denote $V(x)=\sum_{i=1}^{n} x_{i}$, on $x \in R_{+}^{n}$. It is known that

$$
\begin{align*}
& d V(x(t)) \\
& =x^{T}(t)[(b(r(t))+A(r(t)) x(t)+B(r(t)) x(t-\tau)) d t \\
& \quad+\sigma(r(t)) d w(t)] . \tag{47}
\end{align*}
$$

We can also derive from this that

$$
\begin{align*}
& E\left(\sup _{t \leq u \leq t+1} V(x(u))\right) \\
& \quad \leq E V(x(t))+\max _{k \in S}|b(k)| \int_{t}^{t+1} E|x(s)| d s \\
& \quad+\max _{k \in S}|A(k)| \int_{t}^{t+1} E|x(s)|^{2} d s+\max _{k \in S}|B(k)|  \tag{48}\\
& \quad \times \int_{t}^{t+1} E(|x(s)||x(s-\tau)|) d s \\
& \quad+E\left(\sup _{t \leq u \leq t+1} \int_{t}^{u} x^{T}(s) \sigma(r(s)) d w(s)\right)
\end{align*}
$$

From (15), we know that $\lim _{\sup _{t \rightarrow \infty} E|x(t)| \leq K_{1}(1) \text { and }}$ $\lim \sup _{t \rightarrow \infty} E|x(t)|^{2} \leq K_{1}(2)$. By the well-known BDG's inequality [29] and the Hölder's inequality, we derive that

$$
\begin{align*}
& E\left(\sup _{t \leq u \leq t+1} \int_{t}^{u} x^{T}(s) \sigma(r(s)) d w(s)\right) \\
& \leq 3 \max _{k \in S}|b(k)| E\left(\int_{t}^{t+1}|x(s)|^{2} d s\right)^{1 / 2} \\
& \leq E\left(9 \check{\sigma} \int_{t}^{t+1} x^{2}(s) d s\right)^{1 / 2} \\
& \leq E\left(\sup _{t \leq u \leq t+1} x(u) \cdot 9 \check{\sigma} \int_{t}^{t+1} x(s) d s\right)^{1 / 2}  \tag{49}\\
& \quad \leq E\left[\left(\frac{1}{2} \sup _{t \leq u \leq t+1} x(u)\right)^{2}+9 \check{\sigma} \int_{t}^{t+1}(x(s) d s)^{2}\right]^{1 / 2} \\
& \quad \leq E\left(\frac{1}{2} \sup _{t \leq u \leq t+1} x(u)+9 \check{\sigma}^{2} \int_{t}^{t+1} x(s) d s\right) \\
& \quad \leq \frac{1}{2} E\left(\sup _{t \leq u \leq t+1} x(u)\right)+9 \check{\sigma}^{2} \int_{t}^{t+1} E(x(s)) d s
\end{align*}
$$

Combining the inequality above with

$$
\begin{align*}
& \int_{t}^{t+1} E(|x(s)||x(s-\tau)|) d s  \tag{50}\\
& \quad \leq \frac{1}{2} \int_{t}^{t+1} E|x(s)|^{2} d s+\frac{1}{2} \int_{t}^{t+1} E|x(s-\tau)|^{2} d s
\end{align*}
$$

we get that

$$
\begin{align*}
& E\left(\sup _{t \leq u \leq t+1} V(x(u))\right) \\
& \quad \leq E V x(t)+\max _{k \in S}|b(k)| \int_{t}^{t+1} E|x(s)| d s \\
&  \tag{51}\\
& \quad+\left(\max _{k \in S}|A(k)|+\frac{1}{2} \max _{k \in S}|B(k)|\right) \int_{t}^{t+1} E|x(s)|^{2} d s \\
& \\
& \quad+\frac{1}{2} \max _{k \in S}|B(k)| \int_{t}^{t+1} E|x(s-\tau)|^{2} d s \\
& \\
& \quad+3 \max _{k \in S}|b(k)| E\left(\int_{t}^{t+1}|x(s)|^{2} d s\right)^{1 / 2} .
\end{align*}
$$

Recalling the following inequality $|x| \leq \sum_{i=1}^{n} x_{i} \leq V(x)$ for any $x \in R_{+}^{n}$, we obtain

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} E\left(\sup _{t \leq u \leq t+1} x(u)\right) \\
& \quad \leq \max _{k \in S}|b(k)| K_{1}(1)  \tag{52}\\
& \quad+\left(\max _{k \in S}|A(k)|+\max _{k \in S}|B(k)|\right) K_{1}(2) \\
& \quad+3 \max _{k \in S}|b(k)|(K(2))^{1 / 2} .
\end{align*}
$$

It is following from (52) that there is a positive constant $M$ such that

$$
\begin{equation*}
E\left(\sup _{k \leq t \leq k+1}|x(t)|\right) \leq M, \quad k=1,2, \ldots \tag{53}
\end{equation*}
$$

Let $\varepsilon>0$ be arbitrary. Then, by Chebyshev's inequality, we have

$$
\begin{equation*}
P\left(\sup _{k \leq t \leq k+1}|x(u)|>k^{1+\varepsilon}\right) \leq \frac{M}{k^{1+\varepsilon}}, \quad k=1,2, \ldots \tag{54}
\end{equation*}
$$

Applying the well-known Borel-Cantelli lemma [24], we obtain that for almost all $\omega \in \Omega$

$$
\begin{equation*}
\sup _{k \leq t \leq k+1}|x(u)| \leq k^{1+\varepsilon} \tag{55}
\end{equation*}
$$

holds for all but finitely many $k$. Hence, there exists a $k_{0}(\omega)$, for almost all $\omega \in \Omega$, for which (55) holds whenever $k \geq k_{0}$. Consequently, for almost all $\omega \in \Omega$, if $k \geq k_{0}$ and $k \leq t \leq k+1$,

$$
\begin{equation*}
\frac{\log (|x(t)|)}{\log t} \leq \frac{(1+\varepsilon) \log k}{\log k}=1+\varepsilon . \tag{56}
\end{equation*}
$$

Therefore, $\lim \sup _{t \rightarrow \infty}(\log (|x(t)|) / \log t) \leq 1+\varepsilon$ a.s. Letting $\varepsilon \rightarrow 0$, we obtain the desired assertion (46).

Lemma 11. If there exists a constant $\theta>0$ such that $A(\theta)$ is a nonsingular $M$-matrix and for each $k \in S, B(k) \geq 0$, then the global positive solution $x(t)$ of SDE (4) has the property that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{\log (|x(t)|)}{\log t} \geq-\frac{1}{\theta} \quad \text { a.s. } \tag{57}
\end{equation*}
$$

Proof. Let $U: R_{+}^{n} \rightarrow R_{+}^{n}$ be the same as defined by (29); for convenience, we write $U(x(t))=U(t)$. Applying the generalized Itô formula, for the fixed constant $\theta>0$, we derive from (37) that

$$
\begin{align*}
d(1+U(t))^{\theta} \leq & \theta(1+U(t))^{\theta-2} \\
\times & \times\left[-U^{2}(t)\left(\alpha(r(t))-\frac{1}{2} \theta \check{\sigma}^{2}(r(t))\right)\right. \\
& \left.\quad+K_{1} U(t)+K_{1}+\check{b}\right] d t \\
& -\theta(1+U(t))^{\theta-1} U^{2}(t) \sigma(r(t)) d w(t) . \tag{58}
\end{align*}
$$

By (43), there exists a positive constant $M$ such that

$$
\begin{equation*}
E(1+U(t))^{\theta} \leq M \quad \text { on } t \geq 0 \tag{59}
\end{equation*}
$$

Let $\delta>0$ be sufficiently small such that

$$
\begin{equation*}
\theta\left[\left(|\widehat{\alpha}|+\frac{1}{2} \theta \check{\sigma}^{2}+K_{1}\right) \delta+3 \max _{k \in S}|\sigma(k)| \delta^{1 / 2}\right]<\frac{1}{2} \tag{60}
\end{equation*}
$$

Then (58) implies that

$$
\begin{align*}
& E\left[\sup _{(k-1) \delta \leq t \leq k \delta}(1+U(t))^{\theta}\right] \\
& \leq E\left[(1+U((k-1) \delta))^{\theta}\right] \\
& \quad+E\left\{\sup _{(k-1) \delta \leq t \leq k \delta} \mid \int_{(k-1) \delta}^{t} \theta(1+U(s))^{\theta-2}\right. \\
& \times\left[-U^{2}(s)(\widehat{\alpha}(r(s))\right. \\
& \left.-\frac{1}{2} \theta \check{\sigma}^{2}(r(s))\right)  \tag{61}\\
& \left.\quad+E\left\{K_{1}(U(s)+1)\right] d s \mid\right\} \\
& \quad \sup _{(k-1) \delta \leq t \leq k \delta} \mid \int_{(k-1) \delta}^{t} \theta(1+U(s))^{\theta-1} U^{2} \\
& \left.\times(s) x^{T}(s) \sigma(r(s)) d w(s) \mid\right\}
\end{align*}
$$

It is computed that

$$
\left.\begin{array}{l}
E\left\{\sup _{(k-1) \delta \leq t \leq k \delta} \mid \int_{(k-1) \delta}^{t} \theta(1+U(s))^{\theta-2}\right. \\
\times\left[-U^{2}(s)\left(\widehat{\alpha}(r(s))-\frac{1}{2} \theta \check{\sigma}^{2}(r(s))\right)\right. \\
\left.\left.\quad+K_{1}(U(s)+1)\right] d s \mid\right\} \\
\leq E\left\{\int_{(k-1) \delta}^{k \delta} \left\lvert\, \theta(1+U(s))^{\theta-2}\left[-U^{2}(s)\left(\widehat{\alpha}-\frac{1}{2} \theta \check{\sigma}^{2}\right)\right.\right.\right. \\
\left.\left.\quad+K_{1}(U(s)+1)\right] \mid d s\right\} \\
\leq \theta E\left\{\int_{(k-1) \delta}^{k \delta}\left(|\widehat{\alpha}|+\frac{1}{2} \theta \check{\sigma}^{2}+K_{1}\right)(1+U(s))^{\theta} d s\right\}
\end{array}\right\} \begin{aligned}
& \leq \theta\left(|\widehat{\alpha}|+\frac{1}{2} \theta \check{\sigma}^{2}+K_{1}\right) E\left[\int_{(k-1) \delta(k-1) \delta \leq s \leq k \delta}^{t} \sup ^{t}(1+U(s))^{\theta} d s\right] \\
& \leq \theta\left(|\widehat{\alpha}|+\frac{1}{2} \theta \check{\sigma}^{2}+K_{1}\right) \delta E\left[\sup _{(k-1) \delta \leq t \leq k \delta}(1+U(t))^{\theta}\right]
\end{aligned}
$$

On the other hand, by the BDG's inequality, we derive that

$$
\begin{align*}
& E\left\{\sup _{(k-1) \delta \leq t \leq k \delta} \mid \int_{(k-1) \delta}^{k \delta} \theta(1+U(s))^{\theta-1} U^{2}\right. \\
& \left.\times(s) x^{T}(s) \sigma(r(s)) d w(s) \mid\right\} \\
& \leq 3 E\left\{\int_{(k-1) \delta}^{k \delta}\left[\theta(1+U(s))^{\theta-1} U^{2}(s)\right]^{2}\left|x^{T} \sigma(r(s))\right|^{2}\right\}^{1 / 2} \\
& \leq 3 \theta E\left\{\int_{(k-1) \delta}^{k \delta}(1+U(s))^{2(\theta-1)} U^{2}(s) \frac{|x(s)|^{2}|\sigma(r(s))|^{2}}{|x(s)|^{2}}\right\}^{1 / 2} \\
& \leq 3 \theta \max _{k \in S}|\sigma(k)| E\left\{\int_{(k-1) \delta}^{k \delta}(1+U(s))^{2 \theta} d s\right\}^{1 / 2} \\
& \leq 3 \theta \max _{k \in S}|\sigma(k)| \delta^{1 / 2} E\left\{\sup _{(k-1) \delta \leq t \leq k \delta}(1+U(s))^{2 \theta}\right\} \\
& \leq 3 \theta \max _{k \in S}|\sigma(k)| \delta^{1 / 2} E\left\{\sup _{(k-1) \delta \leq t \leq k \delta}(1+U(s))^{\theta}\right\} \tag{63}
\end{align*}
$$

Substituting this and (62) into (61) gives

$$
\begin{align*}
& E\left[\sup _{(k-1) \delta \leq t \leq k \delta}(1+U(t))^{\theta}\right] \\
& \quad \leq \\
& \quad E\left[(1+U((k-1) \delta))^{\theta}\right]  \tag{64}\\
& \\
& \quad+\theta\left\{\left[\widehat{\alpha}+\frac{1}{2} \theta \check{\sigma}^{2}+K_{1}\right] \delta+3 \max _{k \in S}|\sigma(k)| \delta^{1 / 2}\right\} \\
& \\
& \quad \times E\left\{\sup _{(k-1) \delta \leq t \leq k \delta}(1+U(s))^{\theta}\right\} .
\end{align*}
$$

Making use of (59) and (60), we obtain that

$$
\begin{equation*}
\sup _{(k-1) \delta \leq t \leq k \delta} E\left[(1+U(t))^{\theta}\right] \leq 2 M \quad \text { on } t \geq 0 . \tag{65}
\end{equation*}
$$

Let $\varepsilon>0$ be arbitrary. Then, by Chebyshev inequality, we have

$$
\begin{align*}
& P\left\{\omega: \sup _{(k-1) \delta \leq t \leq k \delta}(1+U(t))^{\theta}>(k \delta)^{1+\varepsilon}\right\}  \tag{66}\\
& \quad \leq \frac{2 M}{(k \delta)^{1+\varepsilon}}, \quad k=1,2, \ldots
\end{align*}
$$

Applying the Borel-Cantelli lemma, we obtain that for almost all $\omega \in \Omega$

$$
\begin{equation*}
\sup _{(k-1) \delta \leq t \leq k \delta}(1+U(t))^{\theta} \leq(k \delta)^{1+\varepsilon} \tag{67}
\end{equation*}
$$

holds for all but finitely many $k$. Hence, there exists an integer $k_{0}(\omega)>1 / \delta+2$, for almost all $\omega \in \Omega$, for which (67) holds whenever $k \geq k_{0}$. Consequently, for almost all $\omega \in \Omega$, if $k \geq$ $k_{0}$ and $(k-1) \delta \leq t \leq k \delta$,

$$
\begin{equation*}
\frac{\log (1+U(t))^{\theta}}{\log t} \leq \frac{(1+\varepsilon) \log (k \delta)}{\log ((k-1) \delta)} \leq 1+\varepsilon . \tag{68}
\end{equation*}
$$

Therefore $\lim \sup _{t \rightarrow \infty}\left(\log (1+U(t))^{\theta} / \log t\right) \leq 1+\varepsilon$ a.s. Let $\varepsilon \rightarrow 0$, we obtain the desired assertion

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log (1+U(t))^{\theta}}{\log t} \leq 1 \quad \text { a.s. } \tag{69}
\end{equation*}
$$

Recalling the definition of $U(t)$, this yields $\lim \sup _{t \rightarrow \infty}\left(\log |x(t)|^{-\theta} / \log t\right) \leq 1$ a.s., which further implies

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{\log (|x(t)|)}{\log t} \geq-\frac{1}{\theta} \quad \text { a.s. } \tag{70}
\end{equation*}
$$

This is our required assertion (57).
Assumption 8. Assume that there exist positive numbers $c_{1}, \ldots, c_{n}$ such that

$$
\begin{align*}
-\lambda= & |c|^{-1} \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right]\right\}  \tag{71}\\
& +\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\|<0,
\end{align*}
$$

where $\bar{C}=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$ and $\widehat{c}=\min _{1 \leq i \leq n} c_{i}$. Moreover for each $k \in S, B(k) \geq 0$.

Theorem 12. Under Assumptions 4, 5, and 8, for any given initial data $\{x(t):-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}\right)$, the solution $x(t)$ of (4) obeys

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t}|x(s)| d s \leq \frac{1}{\lambda} \sum_{i=k}^{N} \pi_{k} \beta(k) \quad \text { a.s. } \\
& \liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t}|x(s)| d s \geq \frac{2 \widehat{c}}{\widehat{\lambda}} \sum_{k=1}^{n} \pi_{k} \alpha(k) \quad \text { a.s., } \tag{72}
\end{align*}
$$

where $-\hat{\lambda}=\min _{k \in S}\left[\lambda_{\text {min }}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right]<0$.
Proof. By Theorem 1(1), the solution $x(t)$ will remain in $R_{+}$for all $t \geq-\tau$ with probability 1 . Define $V(x)=c^{T} x=\sum_{i=1}^{n} c_{i} x_{i}$, for $x \in R_{+}^{n}$. By generalized Itô formula, one has

$$
\begin{align*}
d V(x(t))=x^{T}(t) \bar{C}[( & b(r(t))+A(r(t)) x(t) \\
& +B(r(t)) x(t-\tau)) d t  \tag{73}\\
& +\sigma(r(t)) d w(t)]
\end{align*}
$$

From Lemmas 5, 10, and 11, it follows that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\log (V(x(t)))}{t}=0 \quad \text { a.s. } \tag{74}
\end{equation*}
$$

By (73), it has

$$
\begin{aligned}
& d \log V(x(t)) \\
& \qquad \begin{array}{r}
=V^{-1}(x(t)) x^{T}(t) \bar{C}[(b(r(t))+A(r(t)) x(t) \\
+B(r(t)) x(t-\tau)) d t \\
+ \\
+\sigma(r(t)) d w(t)] \\
\quad-0.5 V^{-2}(x(t))\left|x^{T}(t) \bar{C} \sigma(r(t))\right|^{2} d t .
\end{array}
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
& \alpha(r(t)) \leq V^{-1}(x(t)) \bar{C} b(r(t)) \\
& \quad-0.5 V^{-2}(x(t))\left|x^{T}(t) \bar{C} \sigma(r(t))\right|^{2} \\
& \leq \beta(r(t)), \\
& -0.5 \widehat{c}^{-1} \widehat{\lambda}|x(t)| \\
& \leq V^{-1}(x(t)) x^{T}(t) \bar{C}(A(r(t)) x(t) \\
& \quad+B(r(t)) x(t-\tau)) \\
& \leq|c|^{-1} \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right]\right\} \\
& \quad+\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\||x(t)|+0.5 \widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\| \\
& \quad \times(-|x(t)|+|x(t-\tau)|) .
\end{aligned}
$$

Substituting (76) into (75) yields

$$
\begin{align*}
d \log V(x(t)) \leq & \beta(r(t)) d t-\lambda|x(t)| d t \\
& +0.5 \widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\|(-|x(t)|+|x(t-\tau)|) \\
& +V^{-1}(x(t)) x^{T}(t) \bar{C} \sigma(r(t)) d w(t) . \tag{77}
\end{align*}
$$

Hence,

$$
\begin{align*}
& \log V(x(t))+\lambda \int_{0}^{t}|x(s)| d s \\
& \leq \log V(x(0))+\int_{0}^{t} \beta(r(s)) d s+\int_{-\tau}^{0}|x(s)| d s  \tag{78}\\
&+\int_{0}^{t} V^{-1}(x(s)) x^{T}(s) \bar{C} \sigma(r(s)) d w(s)
\end{align*}
$$

Applying the strong law of large numbers for martingales, we have

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} V^{-1}(x(s)) x^{T}(s) \bar{C} \sigma(r(s)) d w(s)=0 \quad \text { a.s. } \tag{79}
\end{equation*}
$$

Dividing both sides of (78) by $t$ and letting $t \rightarrow \infty$, we obtain that

$$
\begin{equation*}
\lambda \limsup _{t \rightarrow+\infty} \frac{1}{t} \int_{0}^{t}|x(s)| d s \leq \sum_{k=1}^{N} \pi_{k} \beta(k) \quad \text { a.s. } \tag{80}
\end{equation*}
$$

which implies the required assertion (72).
On the other hand, it is observed from (75)-(76) that

$$
\begin{align*}
d \log V(x(t)) \geq & \alpha(r(t)) d t-0.5 \widehat{c}^{-1} \widehat{\lambda}|x(t)| d t \\
& +V^{-1}(x(t)) x^{T}(t) \bar{C} \sigma(r(t)) d w(t) \tag{81}
\end{align*}
$$

Hence,

$$
\begin{align*}
& \log V(x(t))+0.5 \widehat{c}^{-1} \hat{\lambda} \int_{0}^{t}|x(s)| d s \\
& \quad \geq \log V(x(0))+\int_{0}^{t} \alpha(r(s)) d s  \tag{82}\\
& \quad+\int_{0}^{t} V^{-1}(x(s)) x^{T}(s) \bar{C} \sigma(r(s)) d w(s)
\end{align*}
$$

Consequently, one gets that

$$
\begin{equation*}
0.5 \widehat{c}^{-1} \widehat{\lambda} \liminf _{t \rightarrow+\infty} \frac{1}{t} \int_{0}^{t}|x(s)| d s \geq \sum_{k=1}^{N} \pi_{k} \alpha(k) \quad \text { a.s. } \tag{83}
\end{equation*}
$$

which implies the other required assertion (4.12).
Similarly, using Lemmas 6, 10, and 11, we can show the following

Theorem 13. Under Assumptions 5 and 8, for any given initial data $\{x(t):-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}\right)$, the solution $x(t)$ of (4) obeys

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t}|x(s)| d s \leq \frac{1}{\lambda} \sum_{i=k}^{N} \pi_{k} \beta(k) \quad \text { a.s., }  \tag{84}\\
& \liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t}|x(s)| d s \geq \frac{2 \widehat{c}}{\widehat{\lambda}} \sum_{k=1}^{n} \pi_{k} \alpha(k) \quad \text { a.s. }
\end{align*}
$$

where $-\hat{\lambda}=\min _{k \in S}\left[\lambda_{\text {min }}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right]<0$.

## 5. Examples

In this section, an example is given to illustrate our main results.

Example 1. Consider the two-species Lotka-Volterra system with regime switching described by

$$
\begin{align*}
d x(t)= & \operatorname{diag}\left(x_{1}(t), x_{2}(t)\right) \\
& \times[(b(r(t))+A(r(t)) x(t)  \tag{85}\\
& +B(r(t)) x(t-\tau)) d t+\sigma(r(t)) d w(t)]
\end{align*}
$$

where $\left.x(t)=\left(x_{1}(t), x_{2}(t)\right)^{T}, b(r(t))=\left(b_{1} r(t)\right), b_{2}(r(t))\right)^{T}$, $\sigma(r(t))=\left(\sigma_{1}(r(t)), \sigma_{2}(r(t))\right)^{T}$,

$$
\begin{align*}
A(r(t)) & =\left(\begin{array}{ll}
a_{11}(r(t)) & a_{12}(r(t)) \\
a_{21}(r(t)) & a_{22}(r(t))
\end{array}\right),  \tag{86}\\
B(r(t)) & =\left(\begin{array}{ll}
b_{11}(r(t)) & b_{12}(r(t)) \\
b_{21}(r(t)) & b_{22}(r(t))
\end{array}\right),
\end{align*}
$$

and $r(t)$ is a right-continuous Markov chain taking values in $S=\{1,2\}$, and $r(t)$ and $w(t)$ are independent. Here

$$
\begin{array}{ccc}
b_{1}(1)=5, & a_{11}(1)=-5, & a_{12}(1)=\sqrt{10}, \\
b_{2}(1)=8, & a_{21}(1)=\sqrt{10}, & a_{22}(1)=-5, \\
b_{1}(2)=4, & a_{11}(2)=-3, & a_{12}(2)=\sqrt{2}, \\
b_{2}(2)=5, & a_{21}(2)=\sqrt{2}, & a_{22}(2)=-3, \\
b_{11}(1)=0, & b_{12}(1)=\frac{1}{2}, & \sigma_{1}(1)=\sqrt{2}, \\
b_{21}(1)=1, & b_{22}(1)=0, & \sigma_{2}(1)=2, \\
b_{11}(2)=0, & b_{12}(2)=\frac{\sqrt{2}}{2}, & \sigma_{1}(2)=\sqrt{14} \\
b_{21}(2)=\frac{\sqrt{2}}{2}, & b_{22}(2)=0, & \sigma_{2}(2)=4,
\end{array}
$$

Let $\bar{C}=I \in R^{2 \times 2}$. It is easy to compute that

$$
\begin{gather*}
|c|=\sqrt{2}, \quad \widehat{c}=1 \\
\max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right]\right\} \leq-3+\sqrt{2}  \tag{88}\\
-\hat{\lambda}=-5-\sqrt{10}<0, \quad \max _{k \in S}\|\bar{C} B(k)\| \leq \frac{\sqrt{5}}{2}
\end{gather*}
$$

Then

$$
\begin{align*}
-\lambda= & |c|^{-1} \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right]\right\}  \tag{89}\\
& +\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\|<0
\end{align*}
$$

Moreover, $\alpha(1)=3, \alpha(2)=-4, \beta(1)=7$, and $\beta(2)=-2$.
By Theorem 1(1), the solution $x(t)$ of (85) will remain in $R_{+}$for all $t \geq-\tau$ with probability 1 . Let the generator of the Markov chain $r(t)$ be

$$
\Gamma=\left(\begin{array}{rr}
-1 & 1  \tag{90}\\
2 & -2
\end{array}\right)
$$

By solving the linear equation $\pi \Gamma=0$, we obtain the unique stationary (probability) distribution $\pi=\left(\pi_{1}, \pi_{2}\right)=$ $(2 / 3,1 / 3)$. Then

$$
\begin{equation*}
\sum_{1=1}^{2} \pi_{k} \alpha(k)=\frac{2}{3}>0, \quad \sum_{1=1}^{2} \pi_{k} \beta(k)=4>0 \tag{91}
\end{equation*}
$$

Therefore, by Theorems 8 and 12, (85) is stochastically permanent and the solutions have the following properties:

$$
\begin{align*}
\frac{2}{\hat{\lambda}} & \leq \liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t}|x(s)| d s  \tag{92}\\
& \leq \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t}|x(s)| d s \leq \frac{4}{\lambda} \quad \text { a.s. }
\end{align*}
$$

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# Razumikhin-Type Theorems on Exponential Stability of SDDEs Containing Singularly Perturbed Random Processes 

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#### Abstract

This paper concerns Razumikhin-type theorems on exponential stability of stochastic differential delay equations with Markovian switching, where the modulating Markov chain involves small parameters. The smaller the parameter is, the rapider switching the system will experience. In order to reduce the complexity, we will "replace" the original systems by limit systems with a simple structure. Under Razumikhin-type conditions, we establish theorems that if the limit systems are $p$ th-moment exponentially stable; then, the original systems are $p$ th-moment exponentially stable in an appropriate sense.


## 1. Introduction

The stability of time delay systems is a field of intense research [1,2]. In [2], the global uniform exponential stability independent of time delay linear and time invariant systems subjected to point and distributed delays was studied. Moreover, noise and time delay are often the sources of instability, and they may destabilize the systems if they exceed their limits [3].

Hybrid delay systems driven by continuous-time Markov chains have been used to model many practical systems in which abrupt changes may be experienced in the structure and parameters caused by phenomena such as component failures or repairs. An area of particular interest has been the automatic control of the underlying systems, with consequent emphasis on the analysis of stability of the stochastic models. For systems with time delay, there are two approaches to proving stability that correspond to the conventional Lyapunov stability theory. The first is based on Lyapunov-Krasovski functionals, the second on LyapunovRazumikhin functions. The latter one originated with Razumikhin [4] for the ordinary differential delay equation which is called Razumikhin-type theorem and was developed by several people [5]. In his paper, Mao [6] was the first who established a Razumikhin-type theorem for stochastic functional differential equations (SFDEs). Roughly speaking,
a Razumikhin-type theorem states that if the derivative of a Lyapunov function along trajectories is negative whenever the current value of the function dominates other values over the interval of time delay; then, the Lyapunov function along trajectories will converge to zero. The Razumikhin methods have been widely used in the study of stability for functional and differential-delay systems. In this work, we shall investigate stochastic differential delay equations with Markovian switching (SDDEwMSs). The switching we shall use will be a finite-state Markov chain, which incorporates various considerations into the models and often results in the underlying Markov chain having a large state space. To overcome the difficulties and to reduce the computational complexity, much effort has been devoted to the modeling and analysis of such systems, in which one of the main ideas is to split a large-scale system into several classes and lumping the states in each class into one state; see [7-9]. Starting from the work [10], by introducing a small parameter $\varepsilon>0$, a number of asymptotic properties of the Markov chain $r^{\varepsilon}(\cdot)$ have been established. One of the main results in [9] is that a complicated system can be replaced by the corresponding limit system having a much simpler structure. In [11, 12], long-term behavior of SDEwMSs and SDDEwMSs was investigated, respectively, while in $[13,14]$ the stability of random
delay system with two-time-scale Markovian switching was studied. Using the stability of the limit system as a bridge, the desired asymptotic properties of the original system is obtained using perturbed Lyapunov function methods. In this work, we shall establish a Razumikhin-type theorem for SDDEwMSs.

The remainder of this work is organised as follows: in the next section, we shall begin with the formulation of the problem. Section 3 investigates the Razumikhin-type theorem for SDDEs driven by Brownian motion. The exponential stability for SDDEs driven by pure jumps is discussed in Section 4.

## 2. Preliminaries

Let $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, \mathbb{P}\right)$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous, and $\mathscr{F}_{0}$ contains all $\mathbb{P}$-null sets). Throughout the paper, we let $B(t)=\left(B_{1}(t), \ldots, B_{m}(t)\right)^{T}$ be an $m$-dimensional Brownian motion defined on the probability space $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, \mathbb{P}\right)$. If $A$ is a vector or matrix, its transpose is denoted by $A^{T}$. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^{n}$ as well as the trace norm of a matrix. For $\tau>$ $0, C\left([-\tau, 0] ; \mathbb{R}^{n}\right)$ denotes the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^{n}$ with the norm $\|\varphi\|=\sup _{-\tau \leq \theta \leq 0}|\varphi(\theta)|$. Denote by $C_{\mathscr{F}}^{b}\left([-\tau, 0] ; \mathbb{R}^{n}\right)$ the family of all $\mathscr{F}$ measurable and bounded $C\left([-\tau, 0] ; \mathbb{R}^{n}\right)$-valued random variable. We will denote the indicator function of a set $G$ by $I_{G}$.

Let $r(t)(t \geq 0)$ be a right-continuous Markov chain on $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, \mathbb{P}\right)$ taking values in a finite state space $\mathbb{S}=$ $\{1,2, \ldots, N\}$ with the generator $\Gamma=\left(\gamma_{i j}\right)_{N \times N}$ given by

$$
\begin{align*}
& \mathbb{P}\{r(t+\delta)=j \mid r(t)=i\} \\
& \quad= \begin{cases}\gamma_{i j} \delta+\circ(\delta), & \text { if } i \neq j, \\
1+\gamma_{i i} \delta+\circ(\delta), & \text { if } i=j,\end{cases} \tag{1}
\end{align*}
$$

where $\delta>0$ and $\gamma_{i j}$ is the transition rate from $i$ to $j$ satisfying $\gamma_{i j}>0$ if $i \neq j$ and $\gamma_{i i}=-\sum_{i \neq j} \gamma_{i j}$. We assume the Markov $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path $r(\cdot)$ is a rightcontinuous step function with finite number of simple jumps in any finite subinterval of $\mathbb{R}_{+}:=[0, \infty)$. As a standing hypothesis, we assume that the Markov chain is irreducible. This is equivalent to the condition that for any $i, j \in \mathbb{S}$, we can find $i_{1}, i_{2}, \ldots, i_{k} \in \mathbb{S}$ such that

$$
\begin{equation*}
\gamma_{i, i_{1}} \gamma_{i_{1}, i_{2}} \ldots \gamma_{i_{k}, j}>0 \tag{2}
\end{equation*}
$$

Thus, $\Gamma$ always has an eigenvalue 0 . The algebraic interpretation of irreducibility is $\operatorname{rank}(\Gamma)=N-1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi \Gamma=0$, subject to $\sum_{j=1}^{N} \pi_{j}=1$ and $\pi_{j}>0$ for all $j \in \mathbb{S}$. For a real valued function $\sigma(\cdot)$ defined on $\mathbb{S}$, we define

$$
\begin{align*}
\Gamma \sigma(\cdot)(\kappa) & :=\sum_{\ell \in \mathbb{S}} \gamma_{\kappa \ell} \sigma(\ell)  \tag{3}\\
& =\sum_{\ell \neq \kappa} \gamma_{\kappa \ell}(\sigma(\ell)-\sigma(\kappa)),
\end{align*}
$$

for each $\kappa \in \mathbb{S}$.

Consider the following stochastic delay system with Markovian swtching:

$$
\begin{align*}
d x(t)= & f(x(t), x(t-\tau), r(t)) d t \\
& +g(x(t), x(t-\tau), r(t)) d B(t),  \tag{4}\\
x_{0}=\xi \in & C\left([-\tau, 0] ; \mathbb{R}^{n}\right), \quad r(0) \in \mathbb{S},
\end{align*}
$$

where $f: \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{S} \rightarrow \mathbb{R}^{n}$ and $g: \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{S} \rightarrow \mathbb{R}^{n \times m}$.
To highlight the fast and slow motions, we introduce a parameter $\varepsilon>0$ and rewrite the Markov chain $r(t)$ as $r^{\varepsilon}(t)$ and the generator $\Gamma$ as $\Gamma^{\varepsilon}$. $\Gamma^{\varepsilon}$ is given by

$$
\begin{equation*}
\Gamma^{\varepsilon}=\frac{1}{\varepsilon} \widetilde{\Gamma}+\widehat{\Gamma} \tag{5}
\end{equation*}
$$

where $\widetilde{\Gamma} / \varepsilon$ represents the fast varying motions, and $\widehat{\Gamma}$ represents the slowly changing dynamics. We denote $\Gamma^{\varepsilon}=\left(\gamma_{i j}^{\varepsilon}\right)_{N \times N}$, $\widetilde{\Gamma}=\left(\widetilde{\gamma}_{i j}\right)_{N \times N}$, and $\widehat{\Gamma}=\left(\widehat{\gamma}_{i j}\right)_{N \times N}$. To the reduction of complexity, $\widetilde{\Gamma}$ needs to have a certain structure. Suppose that

$$
\begin{equation*}
\mathbb{S}=\mathbb{S}^{1} \cup \mathbb{S}^{2} \cup \cdots \cup \mathbb{S}^{l} \tag{6}
\end{equation*}
$$

with $\mathbb{S}^{i}=\left\{s_{i 1}, \ldots, s_{i N_{i}}\right\}$ and $N=N_{1}+N_{2}+\cdots+N_{l}$, and that

$$
\begin{equation*}
\widetilde{\Gamma}=\operatorname{diag}\left(\widetilde{\Gamma}^{1}, \ldots, \widetilde{\Gamma}^{l}\right) \tag{7}
\end{equation*}
$$

where for each $k \in\{1, \ldots, l\}$ and $\widetilde{\Gamma}^{k}$ is a generator of a Markov chain taking values in $\mathbb{S}^{k}$. We impose the following hypothesis:
(H1) For each $k \in\{1, \ldots, l\}, \widetilde{\Gamma}^{k}$ is irreducible.
To highlight the effect of the fast switching, we rewrite the system (4) as

$$
\begin{align*}
d x^{\varepsilon}(t)= & f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) d t \\
& +g\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) d w(t),  \tag{8}\\
x_{0}^{\varepsilon}= & \xi \in C\left([-\tau, 0] ; \mathbb{R}^{n}\right), \quad r^{\varepsilon}=r_{0} .
\end{align*}
$$

To assure the existence and uniqueness of the solution, we give the following standard assumptions.
(H2) For any integer $R$, there is a constant $h_{R}>0$, such that

$$
\begin{align*}
& \left|f(x, y, \kappa)-f\left(x_{1}, y_{1}, \kappa\right)\right| \vee\left|g(x, y, \kappa)-g\left(x_{1}, y_{1}, \kappa\right)\right| \\
& \quad \leq h_{R}\left(\left|x-x_{1}\right|+\left|y-y_{1}\right|\right) \tag{9}
\end{align*}
$$

for all $\kappa \in \mathbb{S}$ and those $x, x_{1}, y, y_{1} \in \mathbb{R}^{n}$ with $|x| \vee\left|x_{1}\right| \vee|y| \vee$ $\left|y_{1}\right| \leq R$.
(H3) There is an $h>0$, such that for any $x, y \in \mathbb{R}^{n}, \kappa \in \mathbb{S}$,

$$
\begin{align*}
|f(x, y, \kappa)| & \vee|g(x, y, \kappa)| \leq h(1+|x|+|y|)  \tag{10}\\
f(0,0, \kappa) \equiv 0, & g(0,0, \kappa) \equiv 0 .
\end{align*}
$$

Under the assumptions (H2) and (H3), system (8) has a unique solution denoted by $x^{\varepsilon, \xi, \ell}(t)$ on $t \geq-\tau$, where the notation $x^{\varepsilon, \xi, \ell}$ emphasizes the dependence on the initial data
$(\xi, \ell)$. Moreover, for every $p>0$ and any compact subset $K$ of $C\left([-\tau, 0] ; \mathbb{R}^{n}\right)$, there exists a positive constant $H$ which is independent of $\varepsilon$ such that

$$
\begin{equation*}
\sup _{(\xi, \ell) \in K \times \mathbb{S}} E\left[\sup _{-\tau \leq s \leq t}\left|x^{\varepsilon, \xi, \ell}(s)\right|^{p}\right] \leq H, \quad \text { on } t \geq 0 . \tag{11}
\end{equation*}
$$

We will consider the stability of system (8), but the state space of the Markov chain is large, and it is difficult to handle (8) directly. So we will consider the average system of (8). To proceed, lump the states in each $\mathbb{S}^{k}$ into a single state and define an aggregated process $\bar{r}^{\varepsilon}(\cdot)$ as

$$
\begin{equation*}
\bar{r}^{\varepsilon}(t)=k, \quad \text { if } r^{\varepsilon}(t) \in \mathbb{S}^{k} \tag{12}
\end{equation*}
$$

Denote the state space of $\bar{r}^{\varepsilon}(t)$ by $\overline{\mathbb{S}}=\{1, \ldots, l\}$, the stationary distribution $\widetilde{\Gamma}^{k}$ by $\mu^{k}=\left(\mu_{1}^{k}, \ldots, \mu_{N_{k}}^{k}\right) \in \mathbb{R}^{1 \times N_{k}}$ and $\widetilde{\mu}=$ $\operatorname{diag}\left(\mu^{1}, \ldots, \mu^{l}\right) \in \mathbb{R}^{l \times N}$. Define

$$
\begin{equation*}
\bar{\Gamma}=\left(\bar{\gamma}_{i j}\right)_{l \times l}=\tilde{\mu} \widehat{\Gamma} 1 \tag{13}
\end{equation*}
$$

with $\mathbf{1}=\operatorname{diag}\left(\mathbf{1}_{N_{1}}, \ldots, \mathbf{1}_{N_{l}}\right)$ and $\mathbf{1}_{N_{k}}=(1, \ldots, 1)^{T} \in \mathbb{R}^{N_{k} \times 1}$, $k=1, \ldots, l$. It has been known that $\vec{r}^{\varepsilon}(\cdot)$ converges weakly to $\bar{r}(\cdot)$ as $\varepsilon \rightarrow 0$, where $\bar{r}(\cdot)$ is a continuous-time Markov chain with generator $\bar{\Gamma}$ and state space $\overline{\mathbb{S}}$ (cf. [9]).

Define

$$
\begin{gather*}
\bar{f}(x, y, i)=\sum_{j=1}^{N_{i}} \mu_{j}^{i} f\left(x, y, s_{i j}\right)  \tag{14}\\
\bar{g}(x, y, i) \bar{g}^{T}(x, y, i)=\sum_{j=1}^{N_{i}} \mu_{j}^{i} g\left(x, y, s_{i j}\right) g^{T}\left(x, y, s_{i j}\right) \tag{15}
\end{gather*}
$$

for each $s_{i j} \in \mathbb{S}^{i}$ with $i \in\{1, \ldots, l\}$ and $j \in\left\{1, \ldots, N_{i}\right\}$. It is easily seen that $\bar{f}(x, y, i)$ and $\bar{g}(x, y, i)$ are the averages with respect to the stationary distribution of the Markov chain. Note that for any $(x, y) \neq(0,0), g\left(x, y, s_{i j}\right) g^{T}\left(x, y, s_{i j}\right)$ are nonnegative definite matrices, so we find its "square root" of (15), which is denoted by $\bar{g}(x, y, i)$. For degenerate diffusions, we can see the argument in [15].

The averaged system of (8) is defined as follows:

$$
\begin{gather*}
d \bar{x}(t)=\bar{f}(\bar{x}(t), \bar{x}(t-\tau), \bar{r}(t)) d t \\
+\bar{g}(\bar{x}(t), \bar{x}(t-\tau), \bar{r}(t)) d w(t),  \tag{16}\\
\bar{x}_{0}=\xi, \quad \bar{r}=\bar{r}_{0} .
\end{gather*}
$$

## 3. Moment Exponential Stability

In this section, we shall establish the Razumikhin-type theorem on the exponential stability for (8).

Let $C^{p}\left(\mathbb{R}^{n} \times \overline{\mathbb{S}} ; \mathbb{R}_{+}\right)$be the class of nonnegative real-valued functions defined on $\mathbb{R}^{n} \times \overline{\mathbb{S}}$ that are $p$-times continuously differentiable with respect to $x$. We give the following assumption about $V(x, i) \in C^{p}\left(\mathbb{R}^{n} \times \overline{\mathbb{S}} ; \mathbb{R}_{+}\right)$for some $p \geq 4$.
(H4) For each $i \in \overline{\mathbb{S}}, V(x, i) \rightarrow \infty$ as $|x| \rightarrow \infty$. Moreover, $\partial^{p} V(x, i)=O(1), \partial^{\ell} V(x, i)\left(|x|^{\ell}+|y|^{\ell}\right) \leq K\left(|x|^{p}+\right.$ $\left.|y|^{p}+1\right)$ for $1 \leq \ell \leq p-1$, where $\partial^{\ell} V(x, i)$ denotes the $\ell$ th derivative of $V(x, i)$ with respect to $x$ and $O(y)$ denotes the function of $y$ satisfying $\sup _{y}|O(y)| / y<\infty$.

Theorem 1. Let (H1)-(H3) hold; there is a function $V(x, i) \in$ $C^{p}\left(\mathbb{R}^{n} \times \overline{\mathbb{S}} ; \mathbb{R}_{+}\right)$satisfying (H4), and there are positive constants $\lambda, c_{1}, c_{2}$, and $q>1$ such that
(i) $c_{1}|x|^{p} \leq V(x, i) \leq c_{2}|x|^{p}$,
(ii) $\mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} \mathscr{L} V(x(t), x(t-\tau), i)\right] \leq-\lambda \mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} V(x(t)\right.$, $i)$ provided $\mathbb{E}\left[\min _{i \in \overline{\mathbb{S}}} V(x(t+\theta), i)\right]<q \mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}}\right.$ $V(x(t), i)],-\tau \leq \theta \leq 0$,
where

$$
\begin{align*}
\mathscr{L} V(x, y, i)= & V_{x}(x, i) \bar{f}(x, y, i) \\
& +\frac{1}{2} \operatorname{trace}\left[V_{x x}(x, i) \bar{g}(x, y, i) \bar{g}^{T}(x, y, i)\right] \\
& +\sum_{j=1}^{l} \bar{\gamma}_{i j} V(x, j) . \tag{17}
\end{align*}
$$

Then, for all $\xi \in C\left([-\tau, 0] ; \mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\limsup _{\varepsilon \rightarrow 0} \mathbb{E}\left|x^{\varepsilon}(t)\right|^{p} \leq v_{2} e^{-v_{1} t} \tag{18}
\end{equation*}
$$

where

$$
\nu_{1}=\min \left\{\lambda, \frac{\log q}{\tau}\right\}
$$

$$
\begin{equation*}
v_{2} \text { is a fixed constant such that } \tag{19}
\end{equation*}
$$

$$
v_{2}=\frac{c_{2}}{c_{1}} \sup _{-\tau \leq \theta \leq 0} \mathbb{E}|\xi|^{p}
$$

Remark 2. Note that the conditions of Theorem 1 are sufficient conditions for the average system (16) $\bar{x}(t)$ (or the limit process $\bar{x}(t)$ ). However the conclusion of Theorem 1 is about the process $x^{\varepsilon}(t)$. Since the structure of the the average system (16) is much simpler than that of $x^{\varepsilon}(t)$, this theorem has reduced the computational complexity for the system (8).

Remark 3. $\lim \sup _{\varepsilon \rightarrow 0} \mathbb{E}\left|x^{\varepsilon}(t)\right|^{p}$ does exist by (11).
Proof of Theorem 1. Define

$$
\begin{equation*}
\bar{V}(x, \zeta)=\sum_{i=1}^{l} V(x, i) I_{\left\{\zeta \in \mathbb{S}^{i}\right\}}=V(x, i), \quad \text { if } \zeta \in \mathbb{S}^{i} \tag{20}
\end{equation*}
$$

Note that

$$
\begin{gather*}
\bar{V}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right)=V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
\sum_{\kappa=1}^{N} \widetilde{\gamma}_{l \kappa} \bar{V}(x, \kappa)=\sum_{\kappa=1}^{N} \widetilde{\gamma}_{l \kappa} \sum_{i=1}^{l} V(x, i) I_{\left\{k \in \mathbb{S}^{i}\right\}}=0 . \tag{21}
\end{gather*}
$$

We extend $r(t)$ to $[-\tau, 0$ ] by setting $r(t)=r(0)$; then, $\mathbb{E} \bar{V}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right)$ is right continuous on $t \geq-\tau$.

Let $\bar{\nu} \in\left(0, v_{1}\right)$ be arbitrary, and define

$$
\begin{align*}
U(t) & :=e^{\bar{\nu} t} \limsup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& =e^{\bar{\nu} t} \limsup _{\varepsilon \rightarrow 0} \mathbb{E} \bar{V}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) . \tag{22}
\end{align*}
$$

If we can show that $U(t) \leq c_{1} v_{2}$, then the proof is completed.
If $t \in[-\tau, 0]$, by condition (i),

$$
\begin{align*}
U(t) & \leq \lim _{\varepsilon \rightarrow 0} \mathbb{E} \bar{V}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right)=\mathbb{E} V(\xi, 0) \leq c_{2} \mathbb{E}|\xi(0)|^{p} \\
& \leq c_{2} \sup _{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^{p}=c_{1} v_{2} . \tag{23}
\end{align*}
$$

If $t \geq 0$, we will prove that $U(t) \leq c_{1} v_{2}$. Otherwise, there exists the smallest $\rho \in(0, \infty)$ such that all $t \in[-\tau, \rho), U(t) \leq$ $c_{1} \nu_{2}$ and $U(\rho) \geq c_{1} \nu_{2}$ as well as $U(\rho+\bar{\delta})>U(\rho)$ for all suffieciently small $\bar{\delta}$.

For $t \in[\rho-\tau, \rho)$,

$$
\begin{array}{rl}
\limsup _{\varepsilon \rightarrow 0} & \mathbb{E} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& =e^{-\bar{\nu} t} U(t) \\
& \leq e^{-\bar{\nu} t} U(\rho)=e^{-\bar{\nu} t} e^{\bar{\nu} \rho} \limsup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(\rho), \bar{r}^{\varepsilon}(\rho)\right)  \tag{24}\\
& \leq e^{\bar{\nu} \tau} \limsup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(\rho), \bar{r}^{\varepsilon}(\rho)\right) .
\end{array}
$$

If $\lim \sup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(\rho), \bar{r}^{\varepsilon}(\rho)\right)=0$, then $\lim \sup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)=0, t \in[\rho-\tau, \rho)$.

Since $\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)$ converges to $(\bar{x}(t), \bar{r}(t))$ with probability one (see Lemma 2.3 in [12]), by condition (i), we can derive

$$
\begin{equation*}
\bar{x}(t)=0, \quad t \in[\rho-\tau, \rho) . \tag{25}
\end{equation*}
$$

Recalling the fact $\bar{f}(0,0, i) \equiv 0, \bar{g}(0,0, i) \equiv 0$ and using the uniqueness of the equation, we then have $\bar{x}(t)=0$, a.e. $t>0$. Therefore we have

$$
\begin{equation*}
\limsup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)=0, \quad t>0 . \tag{26}
\end{equation*}
$$

Then $U(\rho)=0$, which is a contradiction. Hence we see that $\lim _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(\rho), \bar{r}^{\varepsilon}(\rho)\right) \neq 0$. For $t \in[\rho-\tau, \rho)$, there exists a $q>1$ such that

$$
\begin{align*}
& \limsup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& \quad \leq q \limsup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(\rho), \bar{r}^{\varepsilon}(\rho)\right), \bar{\nu}<\frac{\log q}{\tau} . \tag{27}
\end{align*}
$$

Consequently, there exists a sufficiently small $\varepsilon_{0}>0$, such that, for any $\varepsilon \in\left(0, \varepsilon_{0}\right)$,

$$
\begin{align*}
& \mathbb{E}\left[\min _{i \in \overline{\mathbb{S}}} V\left(x^{\varepsilon}(\rho+\theta), i\right)\right] \\
& \quad \leq q \mathbb{E}\left[\max _{i \in \mathbb{\mathbb { S }}} V\left(x^{\varepsilon}(\rho), i\right)\right], \quad \theta \in[-\tau, 0] . \tag{28}
\end{align*}
$$

By condition (ii),

$$
\begin{equation*}
\mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} \mathscr{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), i\right)\right] \leq-\lambda \mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} V\left(x^{\varepsilon}(t), i\right)\right] \tag{29}
\end{equation*}
$$

then,

$$
\begin{equation*}
\mathbb{E}\left[\mathscr{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] \leq-\lambda \mathbb{E}\left[V\left(x^{\varepsilon}(t), \bar{r}(t)\right)\right] \tag{30}
\end{equation*}
$$

Noting that $\bar{\nu}<\nu \leq \lambda$, we have

$$
\begin{equation*}
\mathbb{E}\left[\mathscr{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] \leq-\bar{\nu} \mathbb{E}\left[V\left(x^{\varepsilon}(t), \bar{r}(t)\right)\right] . \tag{31}
\end{equation*}
$$

We now consider

$$
\begin{align*}
& U(\rho+\bar{\delta})-U(\rho) \\
&= \limsup _{\varepsilon \rightarrow 0}\left[e^{\bar{\nu}(\rho+\bar{\delta})} \mathbb{E}\left[V\left(x^{\varepsilon}(\rho+\bar{\delta}), \bar{r}^{\varepsilon}(\rho+\bar{\delta})\right)\right]\right. \\
&\left.-e^{\bar{\nu} \rho} \mathbb{E}\left[V\left(x^{\varepsilon}(\rho), \bar{r}^{\varepsilon}(\rho)\right)\right]\right] \\
&=\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\overline{\bar{v}} t}\left[\mathscr{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right. \\
&\left.+\bar{\nu} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right] d t \\
&=\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{v} t}\left[\mathscr{L} \bar{V}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\
&\left.+\bar{\nu} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right] d t . \tag{32}
\end{align*}
$$

By the definition of operator $\mathscr{L}$, we have

$$
\begin{aligned}
\mathscr{L} \bar{V} & \left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
= & \bar{V}_{x}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
& +\frac{1}{2} \operatorname{trace}\left[\bar{V}_{x x}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right)\right.
\end{aligned}
$$

$$
\begin{align*}
& \times g\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
& \left.\times g^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right] \\
& +\sum_{\kappa=1}^{N} \gamma_{r^{\varepsilon}(t) \kappa}^{\varepsilon} \bar{V}\left(x^{\varepsilon}(t), \kappa\right) \\
& =\bar{V}_{x}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
& +\frac{1}{2} \operatorname{trace}\left[\bar{V}_{x x}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right)\right. \\
& \times g\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
& \left.\times g^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right] \\
& +\sum_{\kappa=1}^{N} \widehat{\gamma}_{r^{\varepsilon}(t) \kappa} \bar{V}\left(x^{\varepsilon}(t), \kappa\right)  \tag{33}\\
& =V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \bar{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right) \\
& +\frac{1}{2} \operatorname{trace}\left[V_{x x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right. \\
& \times \bar{g}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right) \\
& \left.\times \bar{g}^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] \\
& +\sum_{j=1}^{l} \bar{\gamma}_{\tilde{r}^{\varepsilon}(t) j}^{\varepsilon} V\left(x^{\varepsilon}(t), j\right) \\
& +V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\
& \left.-\bar{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] \\
& +\frac{1}{2} \operatorname{trace}\left[V_{x x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right. \\
& \times\left(g\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\
& \times g^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
& -\bar{g}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right) \\
& \left.\left.\times \bar{g}^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right)\right] \\
& +\sum_{\kappa=1}^{N} \widehat{\gamma}_{r^{\varepsilon}(t) \kappa} \bar{V}\left(x^{\varepsilon}(t), \kappa\right) \\
& -\sum_{j=1}^{l} \bar{\gamma}_{\vec{r}^{\varepsilon}(t) j} V\left(x^{\varepsilon}(t), j\right) \\
& =\mathscr{L V}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right) \\
& +V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& \times\left[f \left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau(t)),\right.\right. \\
& \left.\left.r^{\varepsilon}(t)\right)-\bar{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau(t)), \bar{r}^{\varepsilon}(t)\right)\right] \tag{34}
\end{align*}
$$

$$
\begin{aligned}
&+\frac{1}{2} \operatorname{trace}[ V_{x x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& \times\left(g\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\
& \times g^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
&-\bar{g}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right) \\
&\left.\left.\times \bar{g}^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right)\right] \\
&+\sum_{\kappa=1}^{N} \widehat{\gamma}_{r^{\varepsilon}(t) \kappa} \bar{V}\left(x^{\varepsilon}(t), \kappa\right) \\
&-\sum_{j=1}^{l} \bar{\gamma}_{\bar{r}^{\varepsilon}(t) j} V\left(x^{\varepsilon}(t), j\right)
\end{aligned}
$$

$$
\begin{aligned}
& \text { So } \\
& U(\rho+\bar{\delta})-U(\rho) \\
& =\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t}\left[\mathscr { L } V \left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau),\right.\right. \\
& \left.\left.\bar{r}^{\varepsilon}(t)\right)+\bar{\nu} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right] d t \\
& +\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& \times\left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\
& \left.-\bar{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] d t \\
& +\frac{1}{2} \limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{v} t} \text { trace } \\
& \times\left[V_{x x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right. \\
& \times\left(g\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\
& \times g^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
& -\bar{g}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau(t)), \bar{r}^{\varepsilon}(t)\right) \\
& \left.\times \bar{g}^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] d t \\
& +\underset{\varepsilon \rightarrow 0}{\limsup \mathbb{E}} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t}\left(\sum_{\kappa=1}^{N} \widehat{\gamma}_{r^{\varepsilon}(t) \kappa}^{\varepsilon} \bar{V}\left(x^{\varepsilon}(t), \kappa\right)\right. \\
& \left.-\sum_{j=1}^{l} \bar{\gamma}_{\tilde{r}^{\varepsilon}(t) j}^{\varepsilon} V\left(x^{\varepsilon}(t), j\right)\right) d t \\
& =: I_{1}+I_{2}+I_{3}+I_{4} .
\end{aligned}
$$

By the definition of $\bar{f}$,

$$
\begin{align*}
& f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)-\bar{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right) \\
& =\sum_{i=1}^{l} \sum_{j=1}^{N_{i}} f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), s_{i j}\right) \\
& \times\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{r^{\varepsilon}(t)=i\right\}}\right] . \tag{35}
\end{align*}
$$

This, together with assumption (H2), implies

$$
\begin{align*}
& \lim _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& \times\left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\
& \left.-\bar{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] d t \\
& \leq \lim _{\varepsilon \rightarrow 0}\left[\mathbb{E} \mid \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right. \\
& \times\left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\
& \left.\left.-\bar{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right]\left.d t\right|^{2}\right]^{1 / 2} \\
& =\lim _{\varepsilon \rightarrow 0}\left[\mathbb{E} \mid \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right. \\
& \times \sum_{i=1}^{l} \sum_{j=1}^{N_{i}} f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), s_{i j}\right) \\
& \left.\times\left.\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{\vec{r}^{e}(t)=i\right\}}\right] d t\right|^{2}\right]^{1 / 2} \\
& \leq \lim _{\varepsilon \rightarrow 0}\left[\mathbb{E} \mid \int_{\rho}^{\rho+\bar{\delta}} \sum_{i=1}^{l} \sum_{j=1}^{N_{i}} e^{\bar{t} t} h\left(1+\left|x^{\varepsilon}(t)\right|^{p}+\left|x^{\varepsilon}(t-\tau)\right|^{p}\right)\right. \\
& \left.\times\left.\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{\bar{r}^{\varepsilon}(t)=i\right\}}\right] d t\right|^{2}\right]^{1 / 2} . \tag{36}
\end{align*}
$$

By the argument of Lemma 7.14 in [9], the right side of above inequality is equivalent to to 0 ; that is, $I_{2}=0$. Similarly, we can show

$$
\begin{aligned}
I_{3}=\frac{1}{2} \limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} & e^{\bar{\nu} t} \\
& \operatorname{trace} \times\left[V_{x x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right. \\
& \times\left(g\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right.
\end{aligned}
$$

$$
\begin{align*}
& \times g^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
& -\bar{g}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right) \\
& \left.\left.\times \bar{g}^{T}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right)\right] d t=0 \tag{37}
\end{align*}
$$

By the definition of $\widehat{\Gamma}$ and $\bar{\Gamma}$, we have

$$
\begin{align*}
& \sum_{\kappa=1}^{N} \widehat{\gamma}_{r^{\varepsilon}(t) \kappa} \bar{V}\left(x^{\varepsilon}(t), \kappa\right)=\widehat{\Gamma} \bar{V}\left(x^{\varepsilon}(t), \cdot\right)\left(r^{\varepsilon}(t)\right), \\
& \sum_{j=1}^{l} \bar{\gamma}_{\vec{r}^{\varepsilon}(t) j} V\left(x^{\varepsilon}(t), j\right)=\bar{\Gamma} V\left(x^{\varepsilon}(t), \cdot\right)\left(\bar{r}^{\varepsilon}(t)\right), \tag{38}
\end{align*}
$$

hence

$$
\begin{align*}
& I_{4}=\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t}\left(\sum_{\kappa=1}^{N} \widehat{\gamma}_{r^{\varepsilon}(t) \kappa}^{\varepsilon} \bar{V}\left(x^{\varepsilon}(t), \kappa\right)\right. \\
& \left.-\sum_{j=1}^{l} \bar{\gamma}_{\vec{r}^{\varepsilon}(t) j}^{\varepsilon} V\left(x^{\varepsilon}(t), j\right)\right) d t \\
& =\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{v} t}\left(\widehat{\Gamma} \bar{V}\left(x^{\varepsilon}(t), \cdot\right)\left(r^{\varepsilon}(t)\right)\right. \\
& \left.-\bar{\Gamma} V\left(x^{\varepsilon}(t), \cdot\right)\left(\bar{r}^{\varepsilon}(t)\right)\right) d t \\
& =\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} \sum_{i=1}^{l} \sum_{j=1}^{N_{i}} \widehat{\Gamma} \bar{V}\left(x^{\varepsilon}(t), \cdot\right)\left(s_{i j}\right) \\
& \times\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{\vec{r}^{e}(t)=i\right\}}\right] d t \\
& \leq \limsup _{\varepsilon \rightarrow 0}\left[\mathbb{E} \mid \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} \sum_{i=1}^{l} \sum_{j=1}^{N_{i}} \widehat{\Gamma} \bar{V}\left(x^{\varepsilon}(t), \cdot\right)\left(s_{i j}\right)\right. \\
& \left.\times\left.\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{r^{\varepsilon}(t)=i\right\}}\right]\right|^{2}\right]^{1 / 2} . \tag{39}
\end{align*}
$$

By assumption (H4) and the argument of Lemma 7.14 in [9], we have the right side of above inequality is equivalent to 0 , that is, $I_{4}=0$.

Therefore by the condition (ii)

$$
\begin{align*}
& U(\rho+\bar{\delta})-U(\rho) \\
& =\lim _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t}\left[\mathscr{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right.  \tag{40}\\
& \\
& \left.+\bar{\gamma} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right] d t \leq 0
\end{align*}
$$

this is

$$
\begin{equation*}
U(\rho+\bar{\delta}) \leq U(\rho) \tag{41}
\end{equation*}
$$

This contradicts the definition of $\rho$. The proof is now completed.

Example 4. Let $r^{\varepsilon}(\cdot)$ be a Markov chain generated by $\Gamma^{\varepsilon}$ given in (5) with

$$
\begin{align*}
& \widetilde{\Gamma}=\left(\begin{array}{ccccc}
-2 & 2 & 0 & 0 & 0 \\
2 & -2 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 3 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & -1
\end{array}\right),  \tag{42}\\
& \widehat{\Gamma}=\left(\begin{array}{ccccc}
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1
\end{array}\right) .
\end{align*}
$$

The generator $\widetilde{\Gamma}$ consists of two irreducible blocks. The stationary distributions are $\mu^{1}=(0.5,0.5), \mu^{2}=(1 / 7,2 / 7,4 / 7)$, and

$$
\bar{\Gamma}=\left(\begin{array}{cc}
-1 & 1  \tag{43}\\
\frac{6}{7} & -\frac{6}{7}
\end{array}\right)
$$

Consider a one-dimensional equation

$$
\begin{equation*}
d x^{\varepsilon}(t)=f\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) d t+g\left(x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) d w(t) \tag{44}
\end{equation*}
$$

with

$$
\begin{gather*}
f\left(x, s_{11}\right)=\frac{x}{8}, \quad f\left(x, s_{12}\right)=\frac{x}{8} \\
g\left(x, s_{11}\right)=\frac{x \cos x}{8 \sqrt{2}}, \quad g\left(x, s_{12}\right)=\frac{x \sin x}{8 \sqrt{2}} \\
f\left(x, s_{21}\right)=-28(x+\sin x), \\
f\left(x, s_{22}\right)=7 x+14 \sin x, \quad f\left(x, s_{23}\right)=-\frac{7}{4} x  \tag{45}\\
g\left(x, s_{21}\right)=\frac{\sqrt{7}}{4} x \sin x, \\
g\left(x, s_{22}\right)=-\frac{\sqrt{7}}{4} x \cos x, \quad g\left(x, s_{23}\right)=\frac{\sqrt{7}}{8} x .
\end{gather*}
$$

Then the limit equation is

$$
\begin{equation*}
d \bar{x}(t)=f(\bar{x}(t), \bar{r}(t)) d t+g(\bar{x}(t-\tau), \bar{r}(t)) d w(t) \tag{46}
\end{equation*}
$$

where $\bar{r}$ is the Markov chain generated by $\bar{\Gamma}$ and

$$
\begin{array}{ll}
\bar{f}(x, 1)=\frac{x}{8}, & \bar{f}(x, 2)=-3 x  \tag{47}\\
\bar{g}(x, 1)=\frac{x}{16}, & \bar{g}(x, 2)=\frac{x}{4}
\end{array}
$$

Let $V(x, 1)=2 x^{2}, V(x, 2)=x^{2}$; then,

$$
\begin{align*}
& \mathscr{L} V(x, y, 1) \leq-\frac{1}{2}|x|^{2}+\frac{|y|^{2}}{128}  \tag{48}\\
& \mathscr{L} V(x, y, 2) \leq-\frac{36}{7}|x|^{2}+\frac{|y|^{2}}{16}
\end{align*}
$$

Consequently

$$
\begin{array}{rl}
\max _{i=1,2} & \mathscr{L} V(x, y, i) \leq-\frac{1}{2}|x|^{2}+\frac{1}{16}|y|^{2} \\
& =-\frac{1}{4}\left[\max _{i=1,2} V(x, i)\right]+\frac{1}{16}\left[\min _{i=1,2} V(y, i)\right] . \tag{50}
\end{array}
$$

It is easy to see that we can find a $q>1$ such that $(1 / 4)-$ $(q / 16)>0$. Therefore, for any $\phi \in L_{\mathscr{F}_{t}}^{2}\left([-\tau, 0] ; \mathbb{R}^{n}\right)$ satisfying $\mathbb{E}\left[\min _{i \in \bar{\Phi}} \phi(\theta)\right] \leq q \mathbb{E}\left[\max _{i \in \bar{\Phi}} \phi(0)\right]$ on $-\tau \leq \theta \leq 0$, (49) yields

$$
\begin{equation*}
\mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} \mathscr{L} V(x, y, i)\right] \leq-\left(\frac{1}{4}-\frac{q}{16}\right) \mathbb{E}\left[\max _{i=1,2} V(x, i)\right] . \tag{51}
\end{equation*}
$$

Hence, by Theorem 1, the solution $x^{\varepsilon}(t)$ is mean square stable when $\varepsilon$ is sufficient small.

## 4. Stochastic Delay System with Pure Jumps

In this section we discuss the stability of the following stochastic delay system with pure jumps:

$$
\begin{align*}
& d x^{\varepsilon}(t) \\
& =f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) d t \\
& \quad+\int_{\mathbb{R}^{m}} b\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z\right) \widetilde{N}(d t, d z), \\
& \quad x_{0}=\xi \in C\left([-\tau, 0] ; \mathbb{R}^{n}\right), \quad r(0) \in \mathbb{S}, \tag{52}
\end{align*}
$$

where $x^{\varepsilon}(t-)=\lim _{s \uparrow t} x^{\varepsilon}(s), b: \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{S} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n \times m}$. We assume that the each column $b^{(l)}$ of the $n \times m$ matrix $b=\left[b_{i j}\right]$ depends on $z$ only through the $l$ th coordinate $z_{l}$; that is,

$$
\begin{align*}
& b^{(k)}(x, y, \kappa, z)=b^{(k)}\left(x, y, \kappa, z_{k}\right)  \tag{53}\\
& \quad z=\left(z_{1}, \ldots, z_{m}\right) \in \mathbb{R}^{m}, \quad \kappa \in \mathbb{S} .
\end{align*}
$$

$N(t, z)$ is a $m$-dimensional Poisson process, and the compensated Poisson, process is defined by

$$
\begin{array}{r}
\widetilde{N}(d t, d z)=\left(\widetilde{N}_{1}\left(d t, d z_{1}\right), \ldots, \widetilde{N}_{d}\left(d t, d z_{m}\right)\right) \\
=\left(N_{1}\left(d t, d z_{1}\right)-\lambda_{1}\left(d z_{1}\right) d t, \ldots\right.  \tag{54}\\
\\
\left.\quad N_{m}\left(d t, d z_{m}\right)-\lambda_{m}\left(d z_{m}\right) d t\right)
\end{array}
$$

where $\left\{N_{j}, j=1, \ldots, m\right\}$ are independent one-dimensional Poisson random measures with characteristic measure $\left\{\lambda_{j}\right.$, $j=1, \ldots, m\}$ coming from $m$ independent one-dimensional Poisson point processes.

The averaged system of (18) is defined as follows:

$$
\begin{align*}
d \bar{x}(t)= & \bar{f}(\bar{x}(t), \bar{x}(t-\tau), \bar{r}(t)) d t \\
& +\int_{\mathbb{R}^{m}} \bar{b}(\bar{x}(t-), \bar{x}((t-\tau)-), \bar{r}(t), z)  \tag{55}\\
& \times \widetilde{N}(d t, d z), \\
\bar{x}_{0}= & \xi \in C\left([-\tau, 0] ; \mathbb{R}^{n}\right), \quad \bar{r}(0) \in \mathbb{S}
\end{align*}
$$

where $\bar{x}(t-)=\lim _{s \uparrow t} \bar{x}(s), \bar{b}: \mathbb{R}^{n} \times \mathbb{R}^{n} \times \overline{\mathbb{S}} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n \times m}$. Similar to the definition of $\bar{f}$, we define

$$
\begin{equation*}
\bar{b}(x, y, i, z)=\sum_{j=1}^{N_{i}} \mu_{j}^{i} b\left(x, y, s_{i j}, z\right) \tag{56}
\end{equation*}
$$

For each $s_{i j} \in \mathbb{S}^{i}$ with $i \in\{1, \ldots, l\}$ and $j \in\left\{1, \ldots, N_{i}\right\}$.
To assure the existence and uniqueness of the solution of (52), we also give the following standard assumptions.
$\left(\mathrm{H} 2^{\prime}\right)$ For any integer $R$, there is a constant $h_{R}>0$, such that

$$
\begin{align*}
& \left|f(x, y, i)-f\left(x_{1}, y_{1}, i\right)\right| \\
& \quad \vee \sum_{k=1}^{m} \int_{\mathbb{R}}\left|b^{(k)}\left(x_{2}, y_{2}, \kappa, z_{k}\right)-b^{(k)}\left(x_{1}, y_{1}, \kappa, z_{k}\right)\right| \lambda_{k}\left(d z_{k}\right) \\
& \quad \leq h_{R}\left(\left|x_{2}-x_{1}\right|+\left|y_{2}-y_{1}\right|\right) \tag{57}
\end{align*}
$$

for all $i \in \mathbb{S}$ and those $x_{1}, x_{2}, y_{1}, y_{2} \in \mathbb{R}^{n}$ with $\left|x_{1}\right| \vee\left|x_{2}\right| \vee$ $\left|y_{1}\right| \vee\left|y_{2}\right| \leq R$.
$\left(\mathrm{H}^{\prime}\right)$ There is an $h>0$, such that for any $x, y \in \mathbb{R}^{n}, i \in \mathbb{S}$,

$$
\begin{align*}
& |f(x, y, i)| \vee \sum_{k=1}^{m} \int_{\mathbb{R}}\left|b^{(k)}\left(x, y, \kappa, z_{k}\right)\right| \lambda_{k}\left(d z_{k}\right) \\
& \quad \leq h(1+|x|+|y|), \quad f(0,0, \kappa) \equiv 0, \quad b(0,0, \kappa, z) \equiv 0 . \tag{58}
\end{align*}
$$

Given $V \in C^{p}\left(\mathbb{R}^{n} \times \mathbb{S}\right.$; $\left.\mathbb{R}_{+}\right)$, we define the operator $\mathbb{L} V$ by

$$
\begin{align*}
& \mathbb{L} V(x, y, i) \\
& =V_{x}(x, i) f(x, y, i)+\sum_{j=1}^{N} \gamma_{i j} V(x, j)  \tag{59}\\
& +\int_{\mathbb{R}} \sum_{k=1}^{m}\left\{V\left(x+b^{(k)}\left(x, y, \kappa, z_{k}\right), \kappa\right)-V(x, i)\right. \\
& \left.-V_{x}(x, i) b^{(k)}\left(x, y, \kappa, z_{k}\right)\right\} \lambda_{k}\left(d z_{k}\right),
\end{align*}
$$

where

$$
\begin{equation*}
V_{x}(x, i)=\left(\frac{\partial V(x, i)}{\partial x_{1}}, \ldots, \frac{\partial V(x, i)}{\partial x_{m}}\right) \tag{60}
\end{equation*}
$$

We need the following lemma, for details see [16].
Lemma 5. Let (H1) and $\left(H 2^{\prime}\right),\left(H 3^{\prime}\right)$ hold, as $\varepsilon \rightarrow 0$; then, $\left(x^{\varepsilon}(\cdot), \bar{r}^{\varepsilon}(\cdot)\right)$ converges weakly to $(\bar{x}(\cdot), \bar{r}(\cdot))$ in $D\left([0, \infty), \mathbb{R}^{n} \times\right.$ $\overline{\mathbb{S}})$, where $D\left([0, \infty), \mathbb{R}^{n} \times \overline{\mathbb{S}}\right)$ is the space of functions defined on $[0, \infty)$ that are right continuous and have left limits taking values in $\mathbb{R}^{n} \times \overline{\mathbb{S}}$ and endowed with the Skorohod topology.

We now state our main result in this section.
Theorem 6. Let (H1) and $\left(H 2^{\prime}\right),\left(H 3^{\prime}\right)$ hold; there is a function $V(x, i) \in C^{p}\left(\mathbb{R}^{n} \times \overline{\mathbb{S}} ; \mathbb{R}_{+}\right)$satisfying $(H 4)$, and there are positive constants $\lambda, c_{1}, c_{2}$, and $q>1$ such that
(i) $c_{1}|x|^{p} \leq V(x, i) \leq c_{2}|x|^{p}$,
(ii) $\mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} \mathbb{L} V(x(t), x(t-\tau), i)\right] \leq-\lambda \mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} V(x(t)\right.$, $i)]$ provided $\mathbb{E}\left[\min _{i \in \overline{\mathbb{S}}} V(x(t+\theta), i)\right]<q \mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} V(x\right.$ $(t), i)],-\tau \leq \theta \leq 0$,
Then, for all $\xi \in C\left([-\tau, 0] ; \mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\limsup _{\varepsilon \rightarrow 0} \mathbb{E}\left|x^{\varepsilon}(t)\right|^{p} \leq v_{4} e^{-v_{3} t}, \tag{61}
\end{equation*}
$$

where

$$
\begin{align*}
& \nu_{3}=\min \left\{\lambda, \frac{\log q}{\tau}\right\}, \text { and } \\
& \nu_{4} \text { is a fixed constant such that }  \tag{62}\\
& \nu_{4}=\frac{c_{2}}{c_{1}} \sup _{\tau \leq \theta \leq 0} E|\xi|^{p} .
\end{align*}
$$

Proof. As the proof of Theorem 1, define

$$
\begin{equation*}
\bar{V}(x, \zeta)=\sum_{i=1}^{l} V(x, i) I_{\left\{\zeta \in \mathbb{S}^{i}\right\}}=V(x, i) \quad \text { if } \zeta \in \mathbb{S}^{i} \tag{63}
\end{equation*}
$$

We extend $r(t)$ to $[-\tau, 0$ ] by setting $r(t)=r(0)$. Then, $\mathbb{E} \bar{V}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right)$ is right continuous on $t \geq-\tau$.

Let $\bar{\nu} \in\left(0, v_{3}\right)$ be arbitrary, and define

$$
\begin{align*}
U(t) & :=e^{\bar{\nu} t} \limsup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)  \tag{64}\\
& =e^{\bar{\nu} t} \limsup _{\varepsilon \rightarrow 0} \mathbb{E} \bar{V}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) .
\end{align*}
$$

If we can show that $U(t) \leq c_{1} v_{4}$, then the proof is completed.
If $t \in[-\tau, 0]$, by condition (i), is the same as the proof of Theorem 1, we have $U(t) \leq c_{1} \nu_{4}$.

In the following we shall prove that $U(t) \leq c_{1} \nu_{4}$ if $t \geq 0$. Otherwise, there exists the smallest $\rho \in(0, \infty)$ such that all $t \in[-\tau, \rho), U(t) \leq c_{1} v_{4}$, and $U(\underline{\rho}) \geq c_{1} \nu_{4}$ as well as $U(\rho+\bar{\delta})>$ $U(\rho)$ for all suffieciently small $\bar{\delta}$.

As the same in the proof of Theorem 1 we can have that $\lim _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(\rho), \bar{r}^{\varepsilon}(\rho)\right) \neq 0$. Hence for $t \in[\rho-\tau, \rho)$, there exists a $q$ such that

$$
\begin{align*}
& \limsup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& \quad<q \limsup _{\varepsilon \rightarrow 0} \mathbb{E} V\left(x^{\varepsilon}(\rho), \bar{r}^{\varepsilon}(\rho)\right), \bar{\nu}<\frac{\log q}{\tau} . \tag{65}
\end{align*}
$$

Consequently, there exists a sufficiently small $\varepsilon_{0}>0$, such that for any $\varepsilon \in\left(0, \varepsilon_{0}\right)$,

$$
\begin{array}{r}
\mathbb{E}\left[\min _{i \in \overline{\mathbb{S}}} V\left(x^{\varepsilon}(\rho+\theta), i\right)\right] \leq q \mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} V\left(x^{\varepsilon}(\rho), i\right)\right],  \tag{66}\\
\theta \in[-\tau, 0] .
\end{array}
$$

By condition (ii),

$$
\begin{equation*}
\mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} \mathbb{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), i\right)\right] \leq-\lambda \mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} V\left(x^{\varepsilon}(t), i\right)\right], \tag{67}
\end{equation*}
$$

we then have for $\bar{\nu}<\nu \leq \lambda$,

$$
\begin{equation*}
\mathbb{E}\left[\mathbb{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] \leq-\bar{\nu} \mathbb{E}\left[V\left(x^{\varepsilon}(t), \bar{r}(t)\right)\right] \tag{68}
\end{equation*}
$$

We now consider

$$
\begin{aligned}
& U(\rho+\bar{\delta})-U(\rho) \\
&=\limsup _{\varepsilon \rightarrow 0} {\left[e^{\bar{\nu}(\rho+\bar{\delta})} \mathbb{E}\left[V\left(x^{\varepsilon}(\rho+\bar{\delta}), \bar{r}^{\varepsilon}(\rho+\bar{\delta})\right)\right]\right.} \\
&\left.-e^{\bar{\nu} \rho} \mathbb{E} V\left[\left(x^{\varepsilon}(\rho), \bar{r}^{\varepsilon}(\rho)\right)\right]\right] \\
&=\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t}[ {\left[\mathbb{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right.} \\
&\left.+\bar{\nu} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right] d t \\
&=\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t}[ {\left[\mathbb{V}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right.} \\
&\left.+\bar{\nu} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right] d t .
\end{aligned}
$$

By the definition of the operator $\mathbb{L}$ similar to that of the proof of Theorem 1, we have

$$
\begin{aligned}
& \mathbb{L} \bar{V}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
& \left.\begin{array}{rl}
= & \bar{V}_{x}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
+ & \sum_{k=1}^{m} \int_{\mathbb{R}}\left\{\overline { V } \left(x^{\varepsilon}(t)+b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right.\right.\right. \\
\left.\left.r^{\varepsilon}(t), z_{k}\right), r^{\varepsilon}(t)\right) \\
& \quad \bar{V}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right)-\bar{V}_{x}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) b^{(k)} \\
& \left.\times\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_{k}\right)\right\} \\
& \times \lambda_{k}\left(d z_{k}\right) \\
+\sum_{j=1}^{N} \gamma_{r^{\varepsilon}(t) j}^{\varepsilon} \bar{V}\left(x^{\varepsilon}(t), j\right) \\
=\mathbb{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\
+ & V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
\quad \times & {\left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right.} \\
\left.\quad-\bar{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] \\
\quad+\sum_{k=1}^{m} \int_{\mathbb{R}}\left\{V \left(x^{\varepsilon}(t)+b^{(k)}\right.\right. \\
\quad
\end{array} \quad \times\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_{k}\right), \bar{r}^{\varepsilon}(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
& -V\left(x^{\varepsilon}(t)+\overline{\mathrm{b}}^{(k)}\right. \\
& \times\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), \bar{r}^{\varepsilon}(t), z_{k}\right) \\
& \left.\left.\quad \quad^{\varepsilon}(t)\right)\right\} \lambda_{k}\left(d z_{k}\right)
\end{aligned}
$$

$$
\begin{align*}
& -\sum_{k=1}^{m} \int_{\mathbb{R}}\left\{V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right. \\
& \quad \times\left(b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_{k}\right)\right) \\
& \quad-\bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right. \\
& \left.\left.\quad \quad^{\varepsilon}(t), z_{k}\right)\right\} \lambda_{k}\left(d z_{k}\right) \\
& +\sum_{j=1}^{N} \widehat{\gamma}_{r^{\varepsilon}(t) j} \bar{V}\left(x^{\varepsilon}(t), j\right)-\sum_{j=1}^{l} \bar{\gamma}_{\bar{r}^{\varepsilon}(t) j} V\left(x^{\varepsilon}(t), j\right) \tag{70}
\end{align*}
$$

This implies

$$
\begin{aligned}
& U(\rho+\bar{\delta})-U(\rho) \\
& =\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t}\left[\mathscr{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right. \\
& \left.+\bar{\gamma} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right] d t \\
& +\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& \times\left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\
& \left.-\bar{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right] d t \\
& +\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} \\
& \times\left\{\sum _ { k = 1 } ^ { m } \int _ { \mathbb { R } } \left\{V \left(x^{\varepsilon}(t)\right.\right.\right. \\
& +b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right. \\
& \left.\left.r^{\varepsilon}(t), z_{k}\right), \bar{r}^{\varepsilon}(t)\right) \\
& -V\left(x^{\varepsilon}(t)+\bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right.\right. \\
& \left.\left.\left.\left.\bar{r}^{\varepsilon}(t), z_{l}\right), \bar{r}^{\varepsilon}(t)\right)\right\} \lambda_{k}\left(d z_{k}\right)\right\} d t \\
& -\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} \\
& \times\left\{\sum _ { k = 1 } ^ { m } \int _ { \mathbb { R } } \left\{V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right.\right.
\end{aligned}
$$

$$
\begin{array}{r}
\times\left(b ^ { ( k ) } \left(x^{\varepsilon}(t-),\right.\right. \\
x^{\varepsilon}((t-\tau)-), \\
\left.\left.r^{\varepsilon}(t), z_{k}\right)\right) \\
-\bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right. \\
\left.\left.\left.\bar{r}^{\varepsilon}(t), z_{k}\right)\right)\right\} \\
\left.\times \lambda_{k}\left(d z_{k}\right)\right\} d t \\
+\limsup _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{v} t}\left(\sum_{j=1}^{N} \widehat{\gamma}_{r^{\varepsilon}(t) j} \bar{V}\left(x^{\varepsilon}(t), j\right)\right. \\
\left.-\sum_{j=1}^{l} \bar{\gamma}_{\bar{r}^{\varepsilon}(t) j} V\left(x^{\varepsilon}(t), j\right)\right) d t \\
=: J_{1}+J_{2}+J_{3}+J_{4}+J_{5} . \tag{71}
\end{array}
$$

By the definition of $\bar{b}$,

$$
\begin{align*}
& b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_{k}\right) \\
& \quad-\bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), \bar{r}^{\varepsilon}(t), z_{k}\right) \\
& =\sum_{i=1}^{l} \sum_{j=1}^{N_{i}} b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), s_{i j}, z_{k}\right)  \tag{72}\\
& \quad \times\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{\tilde{r}^{\varepsilon}(t)=i\right\}}\right] .
\end{align*}
$$

By assumption ( $\mathrm{H} 2^{\prime}$ ), we have

$$
\begin{gathered}
J_{4}=\limsup _{\varepsilon \rightarrow 0} \sum_{k=1}^{m} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{v} t} V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
\times \int_{\mathbb{R}}\left[b ^ { ( k ) } \left(x^{\varepsilon}(t-),\right.\right. \\
\left.x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_{k}\right) \\
-\bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-)\right. \\
\left.\left.\quad \bar{r}^{\varepsilon}(t), z_{k}\right)\right] \\
\times \lambda_{k}\left(d z_{k}\right) d t
\end{gathered} \quad \begin{array}{r}
\limsup _{\varepsilon \rightarrow 0} \sum_{k=1}^{m} \sum_{i=1}^{l} \sum_{j=1}^{N_{i}} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} V_{x}\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
\\
\times \int_{\mathbb{R}} b^{(k)} \\
\times\left(x^{\varepsilon}(t-),\right. \\
\\
\left.x^{\varepsilon}((t-\tau)-), s_{i j}, z_{k}\right)
\end{array}
$$

$$
\begin{align*}
& \times\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{\bar{r}^{\varepsilon}(t)=i\right\}}\right] \\
& \times \limsup _{\varepsilon \rightarrow 0} \sum_{k=1}^{m} \sum_{i=1}^{l} \sum_{j=1}^{N_{i}}\left[z_{k}\right) d t \\
& \times \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} V_{x} \\
& \times\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right) \\
& \times \int_{\mathbb{R}} b^{(k)} \\
& \times\left(x^{\varepsilon}(t-),\right. \\
&\left.x^{\varepsilon}((t-\tau)-), s_{i j}, z_{k}\right) \\
& \times\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{\bar{r}^{\varepsilon}(t)=i\right\}}\right] \\
&\left.\times\left.\lambda_{k}\left(d z_{k}\right) d t\right|^{2}\right]^{1 / 2} \tag{73}
\end{align*}
$$

By the argument of Lemma 7.14 in [9], the right side of the inequality above is equivalent to 0 , that is, $J_{4}=0$. Similarly, by mean-value theorem, we can show that there exists $\eta^{(k)}(t)$ which is between $x^{\varepsilon}(t)+b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_{k}\right)$ and $x^{\varepsilon}(t)+\bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_{k}\right)$ such that

$$
\begin{aligned}
& J_{3}=\lim _{\varepsilon \rightarrow 0} \sum_{k=1}^{m} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} \\
& \times\left\{\int _ { \mathbb { R } } \left\{V_{x}(\eta(t))\right.\right. \\
& \times\left[b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_{k}\right)\right. \\
& -\bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right. \\
& \left.\left.\left.\bar{r}^{\varepsilon}(t), z_{k}\right]\right\} \lambda_{k}\left(d z_{k}\right)\right\} d t \\
& =\lim _{\varepsilon \rightarrow 0} \sum_{k=1}^{m} \sum_{i=1}^{l} \sum_{j=1}^{N_{i}} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} \\
& \times V_{x}(\eta(t)) \int_{\mathbb{R}} b^{(k)}\left(x^{\varepsilon}(t-),\right. \\
& \left.x^{\varepsilon}((t-\tau)-), s_{i j}, z_{k}\right) \\
& \times\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{r^{\varepsilon}(t)=i\right\}}\right] \lambda_{k}\left(d z_{k}\right) d t \\
& \leq \lim _{\varepsilon \rightarrow 0} \sum_{k=1}^{m} \sum_{i=1}^{l} \sum_{j=1}^{N_{i}}\left[\mathbb{E} \mid \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t} V_{x}(\eta(t))\right. \\
& \times \int_{\mathbb{R}} b^{(k)}\left(x^{\varepsilon}(t-),\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.\quad x^{\varepsilon}((t-\tau)-), s_{i j}, z_{k}\right) \\
& \times\left[I_{\left\{r^{\varepsilon}(t)=s_{i j}\right\}}-\mu_{j}^{i} I_{\left\{r^{\varepsilon}(t)=i\right\}}\right] \\
& \left.\times\left.\lambda_{k}\left(d z_{k}\right) d t\right|^{2}\right]^{1 / 2} . \tag{74}
\end{align*}
$$

By the argument of Lemma 7.14 in [9], we have $J_{3}=0$. Similar to the proof of Theorem 1, we can derive $J_{2}=0, J_{5}=0$.

Therefore we arrive at

$$
\begin{align*}
& U(\rho+\bar{\delta})-U(\rho) \\
& =\lim _{\varepsilon \rightarrow 0} \mathbb{E} \int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\nu} t}\left[\mathbb{L} V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \bar{r}^{\varepsilon}(t)\right)\right.  \tag{75}\\
& \left.\quad+\bar{\nu} V\left(x^{\varepsilon}(t), \bar{r}^{\varepsilon}(t)\right)\right] d t \leq 0
\end{align*}
$$

then,

$$
\begin{equation*}
U(\rho+\bar{\delta}) \leq U(\rho) \tag{76}
\end{equation*}
$$

This contradicts the definition of $\rho$. The proof is therefore completed.

We shall give an example to illustrate our theory:
Example 7. Let $r^{\varepsilon}(\cdot)$ be a Markov chain generated by

$$
\Gamma^{\varepsilon}=\frac{1}{\varepsilon} \widetilde{\Gamma}+\widehat{\Gamma}=\frac{1}{\varepsilon}\left(\begin{array}{cccc}
-1 & 0 & 1 & 0  \tag{77}\\
\frac{1}{2} & -1 & 0 & \frac{1}{2} \\
0 & 2 & -2 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & -1
\end{array}\right)
$$

here we set $\widehat{\Gamma}=0$. The stationary distribution is $\mu=$ ( $4 / 19,8 / 19,3 / 19,4 / 19$ ). Consider a one-dimensional equation

$$
\begin{align*}
d x^{\varepsilon}(t)= & f\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) d t \\
& +\int_{0}^{\infty} \sigma\left(r^{\varepsilon}(t), z\right) x^{\varepsilon}((t-\tau)-) \widetilde{N}(d t, d z) \tag{78}
\end{align*}
$$

with

$$
\begin{array}{ll}
f(x, 1)=2 \sin x, & f(x, 2)=-\frac{19}{8} x  \tag{79}\\
f(x, 3)=-\frac{19}{6} x, & f(x, 4)=-2 \sin x
\end{array}
$$

Let

$$
\begin{gather*}
\beta(z)=\frac{4}{19} \sigma(1, z)+\frac{8}{19} \sigma(2, z)+\frac{3}{19} \sigma(3, z)+\frac{4}{19} \sigma(4, z), \\
\int_{0}^{\infty} \beta^{2}(z) \lambda(d z)<2 \tag{80}
\end{gather*}
$$

Then the limit equation is

$$
\begin{equation*}
d \bar{x}(t)=-\frac{3}{2} \bar{x}(t) d t+\int_{0}^{\infty} \beta(z) \bar{x}((t-\tau)-) \widetilde{N}(d t, d z) . \tag{81}
\end{equation*}
$$

Let $V(x)=x^{2}$; then,

$$
\begin{equation*}
\mathbb{L V}(x, y) \leq-3|x|^{2}+\int_{0}^{\infty} \beta^{2}(z) \lambda(d z)|y|^{2} \tag{82}
\end{equation*}
$$

We can find a $q>1$ such that $3-2 q>0$. Therefore, for any $\phi \in$ $L_{\mathscr{F}_{t}}^{2}\left([-\tau, 0] ; \mathbb{R}^{n}\right)$ satisfying $\mathbb{E}\left[\min _{i \in \overline{\mathbb{S}}} \phi(\theta)\right] \leq q \mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} \phi(0)\right]$ on $-\tau \leq \theta \leq 0$, (49) yields

$$
\begin{equation*}
\mathbb{E}\left[\max _{i \in \overline{\mathbb{S}}} \mathscr{L} V(x, y, i)\right] \leq-(3-2 q) \mathbb{E}\left[\max _{i=1,2} V(x, i)\right] \tag{83}
\end{equation*}
$$

Hence, by Theorem 6, the solution $x^{\varepsilon}(t)$ is mean square stable.

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## Research Article

# Delay-Dependent Dynamics of Switched Cohen-Grossberg Neural Networks with Mixed Delays 

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#### Abstract

This paper aims at studying the problem of the dynamics of switched Cohen-Grossberg neural networks with mixed delays by using Lyapunov functional method, average dwell time (ADT) method, and linear matrix inequalities (LMIs) technique. Some conditions on the uniformly ultimate boundedness, the existence of an attractors, the globally exponential stability of the switched Cohen-Grossberg neural networks are developed. Our results extend and complement some earlier publications.


## 1. Introduction

In recent years, much attention has been devoted to the study of neural networks due to the fact that they have been fruitfully applied in classification of patterns and associative memories, image processing, parallel computation, optimization, and so on $[1-3]$. These applications rely crucially on the analysis of the dynamical behavior [4-7]. Various neural networks, such as Hopfield neural networks, cellular neural networks, bidirectional associative neural networks, and CohenGrossberg neural networks, have been successfully applied. Among them, the Cohen-Grossberg neural network (CGNN) [8] is an important one, which can be described as follows:

$$
\begin{equation*}
\dot{x}_{i}(t)=-\widehat{\alpha}_{i}\left(x_{i}(t)\right)\left[\widehat{\beta}_{i}\left(x_{i}(t)\right)-\sum_{j=1}^{n} a_{i j} f_{j}\left(x_{j}(t)\right)\right]+J_{i} \tag{1}
\end{equation*}
$$

where $t \geq 0, n \geq 2 ; n$ corresponds to the number of units in a neural network; $x_{i}(t)$ denotes the potential (or voltage) of cell $i$ at time $t ; f_{j}(\cdot)$ denotes a nonlinear output function; $\widehat{\alpha}_{i}(\cdot)>$ 0 represents an amplification function; $\widehat{\beta}_{i}(\cdot)$ represents an appropriately behaved function; the $n \times n$ connection matrix
$A=\left(a_{i j}\right)_{n \times n}$ denotes the strengths of connectivity between cells, and if the output from neuron $j$ excites (resp., inhibits) neuron $i, a_{i j} \geq 0$ (resp., $a_{i j} \leq 0$ ); $J_{i}$ denotes an external input source.

Neural network is nonlinearity; in the real world, nonlinear problems are not exceptional, but regular phenomena. Nonlinearity is the nature of matter and its development [ 9,10$]$. In many practical cases, time delays are common phenomenon encountered in the implementation of neural networks, and they may cause the undesirable dynamic behaviors such as oscillation, divergence, or other poor performances. Time delay exists due to the finite speeds of the switching and transmission of signals in a network, which is degenerate to the instability of networks furthermore. For model (1), Ye et al. [11] introduced delays by considering the following differential equation:

$$
\begin{array}{r}
\dot{x}_{i}(t)=-\widehat{\alpha}_{i}\left(x_{i}(t)\right)\left[\widehat{\beta}_{i}\left(x_{i}(t)\right)-\sum_{j=1}^{n} a_{i j} f_{j}\left(x_{j}\left(t-\tau_{j}\right)\right)\right]+J_{i} \\
i=1, \ldots, n \tag{2}
\end{array}
$$

Then, the dynamics of delayed neural networks has been widely studied; see [1, 11-18] for some recent results concerning mixed delays. The CGNN models with discrete delays and distributed delays can be characterized as follows:

$$
\begin{align*}
\dot{x}_{i}(t)= & -\widehat{\alpha}_{i}\left(x_{i}(t)\right) \\
& \times\left[\widehat{\beta}_{i}\left(x_{i}(t)\right)-\sum_{j=1}^{n} a_{i j} f_{j}\left(x_{j}(t)\right)\right. \\
& -\sum_{j=1}^{n} b_{i j} f_{j}\left(x_{j}\left(t-\tau_{j}\right)\right)  \tag{3}\\
& \left.\quad-\sum_{j=1}^{n} c_{i j} \int_{t-h_{j}}^{t} f_{j}\left(x_{j}(s)\right) \mathrm{d} s-J_{i}\right]
\end{align*}
$$

System (3) for convenience can be rewritten as the following compact matrix form:

$$
\begin{align*}
\dot{x}(t)= & -\widehat{\alpha}(x(t)) \\
\times & {[\widehat{\beta}(x(t))-A F(x(t))-B F(x(t-\tau))}  \tag{4}\\
& \left.\quad-C \int_{t-h}^{t} F(x(s)) \mathrm{d} s-J\right],
\end{align*}
$$

where $x(t)=\left(x_{1}(t), \ldots, x_{n}(t)\right)^{T} \in R^{n}$ is the neural state vector, $\widehat{\alpha}(x(t))=\operatorname{diag}\left(\widehat{\alpha}_{1}\left(x_{1}(t)\right), \ldots, \widehat{\alpha}_{n}\left(x_{n}(t)\right)\right) \in R^{n \times n}$, $\widehat{\beta}(x(t)), F(x(t))$ are appropriate dimensions functions, $\tau=$ $\left(\tau_{1}, \ldots, \tau_{n}\right)^{T}, h=\left(h_{1}, \ldots, h_{n}\right)^{T}$, and $J=\left(J_{1}, \ldots, J_{n}\right)^{T}$ is the constant external input.

With the rapid development of intelligent control, hybrid systems have been investigated extensively for their significance, which is in theory and application. As one of the most important classes of hybrid systems, switched systems have drawn increasing attention in the last decade [19-21]. A typical switched systems are composed of a set of subsystems and a logical switching rule whose subsystem will be activated at each instant of time and orchestrates the switching among the subsystems [22]. In general, the switching rule is a piecewise constant function dependent on the state or time. The logical rule that orchestrates switching between these subsystems generates switching signals [23]. Recently, many results on the stability of switched system with time delay and parametric uncertainties have been reported [24, 25]. Switched system in which all subsystems are stable was studied in [26], and Hu and Michel used dwell time approach to analyse the local asymptotic stability of non-linear switched systems in [27]. Hespanha and Morse [28] extended this concept to develop the average dwell time approach subsequently. Furthermore, in [29], the stability results of switched system extended to the case when subsystems are both stable and unstable have been reported, and therefore we derive less conservative results. So, average dwell time (ADT) approach turns out to be an effective tool to study the stability of switched systems [28] and specially when not all subsystems are stable [29].

Meanwhile, neural networks as a special kind of complex networks, the connection topology of networks may change frequently and often lead to link failure or new link creation during the hardware implemtation. Hence, the abrupt changes in the network structure often occur, and switchings between some different topologies are inevitable [30]. Thus, the switched neural network was proposed and has successful applications in the field of high-speed signal processing and artificial intelligence [31, 32], and switched neural networks are also used to perform the gene selection in a DNA microarray analysis in [33]. Thus, it is of great meaning to discuss the switched neural networks. Recently, the stability of switching neural networks has been intensively investigated [34-36]. Robust exponential stability and $H_{\infty}$ control for switched neutral-type neural networks were discussed in [34].

In [35], delay-dependent stability analysis for switched neural networks with time-varying delay was analyzed. In [36], by employing nonlinear measure and LMI techniques, some new sufficient conditions were obtained to ensure global robust asymptotical stability and global robust exponential stability of the unique equilibrium for a class of switched recurrent neural networks with time-varying delay.

By combining the theories of switched systems and Cohen-Grossberg neural networks, the mathematical model of the switched Cohen-Grossberg neural networks is discussed in detail, which can be written as follows:

$$
\begin{align*}
\dot{x}(t)= & -\widehat{\alpha}(x(t)) \\
\times & {\left[\widehat{\beta}(x(t))-A_{\sigma}(t) F(x(t))-B_{\sigma} F(t)(x(t-\tau))\right.} \\
& \left.\quad-C_{\sigma}(t) \int_{t-h}^{t} F(x(s)) \mathrm{d} s-J\right] . \tag{5}
\end{align*}
$$

The function $\sigma(t):\left[t_{0},+\infty\right) \rightarrow \underline{N}=\{1,2 \ldots, N\}$ is a piece-wise constant function of time, called a switching signal, which specifies that subsystem will be activated. $N$ denotes the number of subsystems. The switched sequence can be described as $\left\{\sigma(t):\left(t_{0}, \sigma\left(t_{0}\right)\right), \ldots,\left(t_{k}, \sigma\left(t_{k}\right)\right), \ldots, \mid\right.$ $\left.\sigma\left(t_{k}\right) \in \underline{N}, k=0,1 \cdots\right\}$, where $t_{0}$ denotes the initial time and $t_{k}$ is the $k$ th switching instant. Moreover, $\sigma(t)=i$ means that the $i$ th subsystem is activated. For any $i \in \underline{N}$, this means that the matrices $\left(A_{\sigma}, B_{\sigma}, C_{\sigma}\right)$ can taken values in the finite set $\left\{\left(A_{1}, B_{1}, C_{1}\right), \ldots,\left(A_{N}, B_{N}, C_{N}\right)\right\}$. Meanwhile, we assume that the state of the switched CGNN does not jump at the switching instants; that is, the trajectory $x_{t}$ is everywhere continuous.

However, these available literatures mainly consider the stability property of switching neural networks. In fact, except for stability property, boundedness and attractor are also foundational concepts of dynamical neural networks, which play an important role in investigation of the uniqueness of equilibrium point (periodic solutions), stability and synchronization and so on [13, 14]. To the best of the author's knowledge, few authors have considered the uniformly ultimate boundedness and attractors for switched CGNN with discrete delays and distributed delays.

As is well known, compared with linear matrix inequalities (LMIs) result, algebraic result is more conservative, and criteria in terms of LMI can be easily checked by using the powerful Matlab LMI toolbox. This motivates us to investigate the problems of the uniformly ultimate boundedness and the existence of an attractor for switched CGNN in this paper. The illustrative examples are given to demonstrate the validity of the theoretical results.

The paper is organized as follows. In Section 2, preliminaries and problem formulation are introduced. Section 3 gives the sufficient conditions of uniformly ultimate boundedness (UUB) and the existence of an attractor for switched CGNN. It is the main result of this paper. In Section 4, an example is given to illustrate the effectiveness of the proposed approach. The conclusions are summarized in Section 5.

## 2. Problem Formulation

Throughout this paper, we use the following notations. The superscript " $T$ " stands for matrix transposition; $R^{n}$ denotes the $n$-dimensional Euclidean space; the notation $P>0$ means that $P$ is real symmetric and positive definite; $I$ and $O$ represent the identity matrix and a zero matrix; $\operatorname{diag}\{\cdots\}$ stands for a block-diagonal matrix; $\lambda_{\text {min }}(P)$ denotes the minimum eigenvalue of symmetric matrix $P$; in symmetric block matrices or long matrix expressions, "*" is used to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

Consider the following Cohen-Grossberg neural network model with mixed delays (discrete delays and distributed delays):

$$
\begin{align*}
\dot{x}(t)= & -\widehat{\alpha}(x(t)) \\
\times & {[\widehat{\beta}(x(t))-A F(x(t))-B F(x(t-\tau))}  \tag{6}\\
& \left.-C \int_{t-h}^{t} F(x(s)) \mathrm{d} s-J\right] \triangleq-\widehat{\alpha}(x(t)) H(t),
\end{align*}
$$

where

$$
\begin{align*}
H(t)= & \widehat{\beta}(x(t))-A F(x(t))-B F(x(t-\tau)) \\
& -C \int_{t-h}^{t} F(x(s))-J \tag{7}
\end{align*}
$$

The discrete delays and distributed delays are bounded as follows:

$$
\begin{gather*}
0 \leq \tau_{i}, \quad \tau^{*}=\max _{1 \leq i \leq n}\left\{\tau_{i}\right\} ; \\
0 \leq h_{i}, \quad h^{*}=\max _{1 \leq i \leq n}\left\{h_{i}\right\} ; \quad \delta=\max \left\{\tau^{*}, h^{*}\right\}, \tag{8}
\end{gather*}
$$

where $\tau^{*}, h^{*}, \delta$ are scalars.
As usual, the initial conditions associated with system (6) are given in the form

$$
\begin{equation*}
x(t)=\varphi(t), \quad-\delta \leq t \leq 0 \tag{9}
\end{equation*}
$$

where $\varphi(t)$ is a differentiable vector-valued function.

Throughout this paper, we make the following assumptions.
$\left(\mathrm{H}_{1}\right)$ For any $j \in\{1,2, \ldots, n\}$, there exist constants $l_{j}$ and $L_{j}$, such that

$$
\begin{equation*}
l_{j} \leq \frac{f_{j}(x)-f_{j}(y)}{x-y} \leq L_{j}, \quad \forall x, y \in R, x \neq y \tag{10}
\end{equation*}
$$

Remark 1. The constants $l_{j}$ and $L_{j}$ can be positive, negative, or zero. Therefore, the activation functions $f(\cdot)$ are more general than the forms $\left|f_{j}(u)\right| \leq K_{j}|u|, K_{j}>0, j=1,2, \ldots, n$.
$\left(\mathrm{H}_{2}\right)$ For continuously bounded function $\alpha_{j}(\cdot)$, there exist positive constants $\underline{\alpha}_{j}, \bar{\alpha}_{j}$, such that

$$
\begin{equation*}
\underline{\alpha}_{j} \leq a_{j}\left(x_{j}(t)\right) \leq \bar{\alpha}_{j} . \tag{11}
\end{equation*}
$$

$\left(\mathrm{H}_{3}\right)$ There exist positive constants $b_{j}$, such that

$$
\begin{equation*}
x_{j}(t) \widehat{\beta}_{j}\left(x_{j}(t)\right) \geq b_{j} x_{j}^{2}(t) \tag{12}
\end{equation*}
$$

So, to obtain main results of this paper, the following definitions and lemmas are introduced.

Definition 2 (see [15]). System (6) is uniformly ultimately bounded; if there is $\widetilde{B}>0$, for any constant $\varrho>0$, there is $t^{\prime}=t^{\prime}(\varrho)>0$, such that $\left\|x\left(t, t_{0}, \varphi\right)\right\|<\widetilde{B}$ for all $t \geq t_{0}+t^{\prime}$, $t_{0}>0,\|\varphi\|<\varrho$, where the supremum norm $\left\|x\left(t, t_{0}, \varphi\right)\right\|=$ $\max _{1 \leq i \leq n} \sup _{-\delta \leq s \leq 0}\left|x_{i}\left(t+s, t_{0}, \varphi\right)\right|$.

Definition 3 (see [37]). The nonempty closed set $A \subset R^{n}$ is called the attractor for the solution $x(t ; \varphi)$ of system (6) if the following formula holds:

$$
\begin{equation*}
\lim _{t \rightarrow \infty} d(x(t ; \varphi) ; \mathbb{A})=0 \quad \text { a.s. } \tag{13}
\end{equation*}
$$

in which $\varphi \in C\left([-\delta, 0], R^{n}\right), d(x, \mathbb{A})=\inf _{y \in \mathbb{A}}\|x-y\|$.
Definition 4 (see [28]). For a switching signal $\sigma(t)$ and each $T>t \geq 0$, let $N_{\sigma}(t, T)$ denote the number of discontinuities of $\sigma(t)$ in the interval $(t, T)$. If there exist $N_{0}>0$ and $T_{a}>0$ such that $N_{\sigma}(t, T) \leq N_{0}+(T-t) / T_{a}$ holds, then $T_{a}$ is called the average dwell time. $N_{0}$ is the chatter bound.

Remark 5. In Definition 4, it is obvious that there exists a positive number $T_{a}$ such that a switched signal has the ADT property, which means that the average dwell time between any two consecutive switchings is no less than a specified constant $T_{a}$, Hespanha and Morse have proved that if $T_{a}$ is sufficiently large, then the switched system is exponentially stable. In addition, in [18], one can choose $N_{0}=0$, but in our paper, we assume that $N_{0}>0$, this is more preferable.

Lemma 6 (see [16]). For any positive definite constant matrix $W \in R^{n \times n}$, scalar $r>0$, and vector function $u(t):[t-r, t] \rightarrow$ $R^{n}, t \geq 0$, then

$$
\begin{equation*}
\left(\int_{0}^{r} u(s) \mathrm{d} s\right)^{T} W\left(\int_{0}^{r} u(s) \mathrm{d} s\right) \leq r \int_{0}^{r} u^{T}(s) W u(s) \mathrm{d} s . \tag{14}
\end{equation*}
$$

Lemma 7 (see [38]). For any given symmetric positive definite matrix $X \in R^{n \times n}$ and scalars $\alpha>0,0 \leq d_{1}<d_{2}$, if there exists a vector function $\dot{x}(t):\left[-d_{2}, 0\right] \rightarrow R^{n}$ such that the following integration is well defined, then

$$
\begin{align*}
& -\int_{-d_{2}}^{-d_{1}} \dot{x}^{T}(t+\theta) e^{\alpha \theta} X \dot{x}(t+\theta) \mathrm{d} \theta \\
& \quad \leq \frac{\alpha}{e^{\alpha d_{1}}-e^{\alpha d_{2}}}\left[\begin{array}{l}
x\left(t-d_{1}\right) \\
x\left(t-d_{2}\right)
\end{array}\right]^{T}\left[\begin{array}{cc}
X & -X \\
-X & X
\end{array}\right]  \tag{15}\\
& \quad \times\left[\begin{array}{c}
x\left(t-d_{1}\right) \\
x\left(t-d_{2}\right)
\end{array}\right]
\end{align*}
$$

## 3. Main Results

Theorem 8. For a given constant $a>0$, if there is positive definite matrix $P=\operatorname{diag}\left(p_{1}, p_{2} \ldots, p_{n}\right), D_{i}=$ $\operatorname{diag}\left(D_{i 1}, D_{i 2} \ldots, D_{\text {in }}\right), i=1,2, \ldots, Q, S^{i}$, such that the following condition holds:

$$
\Delta=\left[\begin{array}{ccccccc}
\Phi_{11} & \Phi_{12} & \Phi_{13} & P B & 0 & P C & 0  \tag{16}\\
* & \Phi_{22} & 0 & \Phi_{24} & 0 & 0 & 0 \\
* & * & \Phi_{33} & 0 & 0 & 0 & 0 \\
* & * & * & \Phi_{44} & 0 & 0 & 0 \\
* & * & * & * & \Phi_{55} & \Phi_{56} & 0 \\
* & * & * & * & * & \Phi_{66} & 0 \\
* & * & * & * & * & * & \Phi_{77}
\end{array}\right]<0
$$

where

$$
\begin{aligned}
& Q=\left(\begin{array}{cc}
Q_{11} & Q_{12} \\
* & Q_{22}
\end{array}\right) \geq 0, \quad D_{i} \geq 0, \quad i=1,2, \\
& \Phi_{11}= a \Omega_{1} P-2 a \Omega_{2} P+P-\frac{\delta e^{-a \tau^{*}}}{\tau^{*}} S^{(2)} \\
&-\Omega_{3} D_{1}+\frac{1}{4 a^{2}} I+S^{(1)}-e^{-a \tau^{*}} S^{(1)} \\
&+h^{*} Q_{11}+\frac{\delta a}{1-e^{a \tau^{*}}} S^{(2)}, \\
& \Phi_{12}=-\frac{\delta a}{1-e^{a \tau^{*}}} S^{(2)}, \\
& \Phi_{13}= P A+\Omega_{4} D_{1}+h^{*} Q_{12}, \\
& \Phi_{22}=-\Omega_{3} D_{2}-\frac{\delta e^{-a \tau^{*}}}{\tau^{*}} S^{(2)} \\
&-e^{-a \tau^{*}} S^{(1)}+\frac{\delta a}{1-e^{a \tau^{*}}} S^{(2)}+\frac{1}{4 a^{2}} I, \\
& \Phi_{24}= \Omega_{4} D_{2}, \quad \Phi_{33}=-D_{1}+\frac{1}{a^{2}} I+h^{*} Q_{22}, \\
& \Phi_{44}=-D_{2}+\frac{1}{a^{2}} I, \quad \Phi_{55}=-\frac{e^{-a h^{*}}}{h^{*}} Q_{11},
\end{aligned}
$$

$$
\begin{align*}
& \Phi_{56}=-\frac{e^{-a h^{*}}}{h^{*}} Q_{12}, \quad \Phi_{66}=-\frac{e^{-a h^{*}}}{h^{*}} Q_{22} \\
& \Phi_{77}=\delta \tau^{*} \alpha^{2} S^{(2)}, \\
& \Omega_{1}=\operatorname{diag}\left\{\frac{1}{\underline{\alpha}_{1}}, \frac{1}{\underline{\alpha}_{2}}, \ldots, \frac{1}{\underline{\alpha}_{n}}\right\}, \\
& \Omega_{2}=\operatorname{diag}\left\{b_{1}, b_{2}, \ldots, b_{n}\right\}, \\
& \Omega_{3}=\operatorname{diag}\left\{l_{1} L_{1}, l_{2} L_{2}, \ldots, l_{n} L_{n}\right\}, \\
& \Omega_{4}=\operatorname{diag}\left\{\frac{l_{1}+L_{1}}{2}, \frac{l_{2}+L_{2}}{2}, \ldots, \frac{l_{n}+L_{n}}{2}\right\}, \\
& \rho=\max _{1 \leq i \leq n}\left\{\left|l_{i}^{2}\right|,\left|L_{i}^{2}\right|\right\}+1 ; \tag{17}
\end{align*}
$$

the symbol "*" within the matrix represents the symmetric term of the matrix, and then system (6) is uniformly ultimately bounded.

Proof. Let us consider the following Lyapunov-Krasovskii functional:

$$
\begin{equation*}
V(t)=V_{1}(t)+V_{2}(t)+V_{3}(t) \tag{18}
\end{equation*}
$$

where

$$
\begin{align*}
V_{1}(t)= & \sum_{j=1}^{n} e^{a t} p_{j} \int_{0}^{x_{j}(t)} \frac{2 s}{\alpha_{j}(s)} \mathrm{d} s \\
V_{2}(t)= & \int_{t-h}^{t} e^{a s}(s-(t-h)) \xi^{T}(s) Q \xi(s) \mathrm{d} s \\
\xi(t)= & {\left[x^{T}(t), F^{T}(x(t))\right]^{T} }  \tag{19}\\
V_{3}(t)= & \int_{t-\tau}^{t} e^{a s} x^{T}(s) S^{(1)} x(s) \mathrm{d} s \mathrm{~d} \theta \\
& +\delta \int_{-\tau}^{0} \int_{t+\theta}^{t} e^{a s} \dot{x}^{T}(s) S^{(2)} \dot{x}(s) \mathrm{d} s \mathrm{~d} \theta
\end{align*}
$$

We proceed to evaluate the time derivative of $V_{1}(t)$ along the trajectory of system (6), and one can get

$$
\begin{align*}
\dot{V}_{1}(t)= & \sum_{j=1}^{n} 2\left[a p_{j} e^{a t} \int_{0}^{x_{j}(t)} \frac{s}{\alpha_{j}(s)} \mathrm{d} s-p_{j} e^{a t} x_{j}(t) \beta_{j}\left(x_{j}(t)\right)\right] \\
& +\left[2 x^{T}(t) P A F(x(t))+2 x^{T}(t) P J\right. \\
& +2 x^{T}(t) P B F(x(t-\tau)) \\
& \left.+2 x^{T}(t) P C \int_{t-h}^{t} F(x(s)) \mathrm{d} s\right] e^{a t} . \tag{20}
\end{align*}
$$

According to assumption $\left(\mathrm{H}_{2}\right)$, we obtain the following inequality:

$$
\begin{equation*}
2 a p_{j} \int_{0}^{x_{j}(t)} \frac{s}{\alpha_{j}(s)} \mathrm{d} s \leq \frac{a}{\underline{\alpha}_{j}} p_{j} x_{j}^{2}(t) \tag{21}
\end{equation*}
$$

From assumption $\left(\mathrm{H}_{3}\right)$ and inequality (21), we obtain

$$
\begin{align*}
\dot{V}_{1}(t) \leq a e^{a t} & {\left[x^{T}(t) \Omega_{1} P x(t)-2 x^{T}(t) \Omega_{2} P x(t)\right] } \\
+ & {\left[2 x^{T}(t) P A F(x(t))+2 x^{T}(t) P B F(x(t-\tau))\right.} \\
& +2 x^{T}(t) P C \int_{t-h}^{t} F(x(s)) \mathrm{d} s+x^{T}(t) P x(t)  \tag{25}\\
& \left.+J^{T} P J\right] e^{a t}
\end{align*}
$$

Similarly, taking the time derivative of $V_{2}(t)$ along the trajectory of system (6), we obtain

$$
\begin{align*}
\dot{V}_{2}(t)= & t e^{a t} \xi^{T}(t) Q \xi(t) \\
& -(t-h) e^{a(t-h)} \xi^{T}(t-h) \times Q \xi(t-h) \\
- & {\left[\int_{t-h}^{t} e^{a s} \xi^{T}(s) Q \xi(s) \mathrm{d} s+t e^{a t} \xi^{T}(t) Q \xi(t)\right.} \\
& \left.\quad-t e^{a(t-h)} \xi^{T}(t-h) \times Q \xi(t-h)\right] \\
+ & h e^{a t} \xi^{T}(t) Q \xi(t)-h e^{a(t-h)} \xi^{T}(t-h) Q \xi(t-h)  \tag{27}\\
\leq & h^{*} e^{a t} \\
\times & {\left[x^{T}(t) Q_{11} x(t)+F^{T}(x(t)) Q_{12}^{T} x(t)\right.} \\
& \left.\quad+x^{T}(t) Q_{12} F(x(t))+F^{T}(x(t)) Q_{22} F(x(t))\right] \\
- & e^{a\left(t-h^{*}\right)} \int_{t-h}^{t} \xi^{T}(s) Q \xi(s) \mathrm{d} s . \tag{28}
\end{align*}
$$

According to Lemma 6, we can conclude that

$$
\begin{align*}
& -e^{a\left(t-h^{*}\right)} \int_{t-h}^{t} \xi^{T}(s) Q \xi(s) \mathrm{d} s  \tag{29}\\
& \quad \leq-\frac{e^{a\left(t-h^{*}\right)}}{h^{*}}\left(\int_{t-h}^{t} \xi(s) \mathrm{d} s\right)^{T} Q\left(\int_{t-h}^{t} \xi(s) \mathrm{d} s\right)  \tag{24}\\
& \quad \leq-\frac{e^{-a h^{*}}}{h^{*}}\left(\int_{t-h}^{t} \xi(s) \mathrm{d} s\right)^{T} Q\left(\int_{t-h}^{t} \xi(s) \mathrm{d} s\right)
\end{align*}
$$

Computing the derivative of $V_{3}(t)$ along the trajectory of system (6) turns out to be

$$
\begin{aligned}
\dot{V}_{3}(t)= & e^{a t} x^{T}(t) S^{(1)} x(t)-e^{a(t-\tau)} x^{T}(t-\tau) S^{(1)} x(t-\tau) \\
+ & \delta \int_{-\tau}^{0}\left[e^{a t} \dot{x}^{T}(t) S^{(2)} \dot{x}(t)\right. \\
& \left.-e^{a(t+\theta)} \dot{x}^{T}(t+\theta) S^{(2)} \dot{x}(t+\theta)\right] \mathrm{d} \theta \\
\leq & e^{a t} x^{T}(t) S^{(1)} x(t)-e^{-a \tau^{*}} x^{T}(t-\tau) S^{(1)} x(t-\tau) \\
& +\delta \tau^{*} e^{a t} \dot{x}^{T}(t) S^{(2)} \dot{x}(t) \\
& -\delta \int_{-\tau}^{0} e^{a(t+\theta)} \dot{x}^{T}(t+\theta) S^{(2)} \dot{x}(t+\theta) \mathrm{d} \theta
\end{aligned}
$$

where $\tau^{*}=\max _{1 \leq i \leq n}\left\{\tau_{i}\right\}$.
Denoting that $\alpha=\max \left\{\bar{\alpha}_{1}, \bar{\alpha}_{2}, \ldots, \bar{\alpha}_{n}\right\}$, we obtain

$$
\begin{align*}
& \delta \tau^{*} e^{a t} \dot{x}^{T}(t) S^{(2)} \dot{x}(t)  \tag{22}\\
&=\delta \tau^{*} e^{a t}[\widehat{\alpha}(x(t)) H(t)]^{T} S^{(2)} \widehat{\alpha}(x(t)) H(t)  \tag{26}\\
& \leq \delta \tau^{*} \alpha^{2} e^{a t} H^{T}(t) S^{(2)} H(t)
\end{align*}
$$

Using Lemma 7, the following inequality is easily obtained:

$$
\begin{aligned}
& -\delta \int_{-\tau}^{0} e^{a \theta} \dot{x}^{T}(t+\theta) S^{(2)} \dot{x}(t+\theta) \mathrm{d} \theta \\
& \quad \leq \frac{\delta a}{1-e^{a \tau}}\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right]^{T}\left[\begin{array}{cc}
S^{(2)} & -S^{(2)} \\
-S^{(2)} & S^{(2)}
\end{array}\right] \times\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right] \\
& \quad \leq \frac{\delta a}{1-e^{a \tau^{*}}}\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right]^{T}\left[\begin{array}{cc}
S^{(2)} & -S^{(2)} \\
-S^{(2)} & S^{(2)}
\end{array}\right] \times\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right] .
\end{aligned}
$$

From assumption $\left(\mathrm{H}_{1}\right)$, it follows that, for $j=1,2, \ldots, n$,

$$
\begin{align*}
& {\left[F_{j}\left(x_{j}(t)\right)-F_{j}(0)-l_{j} x_{j}(t)\right]} \\
& \quad \times\left[F_{j}\left(x_{j}(t)\right)-F_{j}(0)-L_{j} x_{j}(t)\right] \leq 0 \\
& {\left[F_{j}\left(x_{j}(t-\tau)\right)-F_{j}(0)-l_{j} x_{j}(t-\tau)\right]} \\
& \quad \times\left[F_{j}\left(x_{j}(t-\tau)\right)-F_{j}(0)-L_{j} x_{j}(t-\tau)\right] \leq 0 \tag{23}
\end{align*}
$$

Then, let

$$
\begin{aligned}
& \Upsilon_{1}=-\sum_{j=1}^{n} D_{1 j}\left[F_{j}\left(x_{j}(t)\right)-F_{j}(0)-l_{j} x_{j}(t)\right] \\
& \times\left[F_{j}\left(x_{j}(t)\right)-F_{j}(0)-L_{j} x_{j}(t)\right] \\
& \Upsilon_{2}=-\sum_{j=1}^{n} D_{2 j}\left[F_{j}\left(x_{j}(t-\tau)\right)-F_{j}(0)-l_{j} x_{j}(t-\tau)\right] \\
& \times\left[F_{j}\left(x_{j}(t-\tau)\right)-F_{j}(0)-L_{j} x_{j}(t-\tau)\right]
\end{aligned}
$$

So,

$$
\begin{aligned}
& \Upsilon_{1}+\Upsilon_{2}=- \sum_{j=1}^{n} D_{1 j}\left[F_{j}\left(x_{j}(t)\right)-F_{j}(0)-l_{j} x_{j}(t)\right] \\
& \times\left[F_{j}\left(x_{j}(t)\right)-F_{j}(0)-L_{j} x_{j}(t)\right] \\
&-\sum_{j=1}^{n} D_{2 j}\left[F_{j}\left(x_{j}(t-\tau)\right)-F_{j}(0)-l_{j} x_{j}(t-\tau)\right] \\
& \times\left[F_{j}\left(x_{j}(t-\tau)\right)-F_{j}(0)-L_{j} x_{j}(t-\tau)\right] \\
&=-\sum_{j=1}^{n} D_{1 j}\left[F_{j}\left(x_{j}(t)\right)-l_{j} x_{j}(t)\right] \\
& \quad \times\left[F_{j}\left(x_{j}(t)\right)-L_{j} x_{j}(t)\right] \\
&-\sum_{j=1}^{n} D_{2 j}\left[F_{j}\left(x_{j}(t-\tau)\right)-l_{j} x_{j}(t-\tau)\right] \\
& \quad \times\left[F_{j}\left(x_{j}(t-\tau)\right)-L_{j} x_{j}(t-\tau)\right] \\
&- \sum_{j=1}^{n} D_{1 j} F_{j}^{2}(0) \\
&+ \sum_{j=1}^{n} D_{1 j} F_{j}(0)\left[2 F_{j}\left(x_{j}(t)\right)-\left(L_{j}+l_{j}\right) x_{j}(t)\right] \\
&- \sum_{j=1}^{n} D_{2 j} F_{j}^{2}(0) \\
&+\sum_{j=1}^{n} D_{2 j} F_{j}(0)\left[2 F_{j}\left(x_{j}(t-\tau)\right)-\left(L_{j}+l_{j}\right) x_{j}(t-\tau)\right]
\end{aligned}
$$

$$
\leq-\sum_{j=1}^{n} D_{1 j}\left[F_{j}\left(x_{j}(t)\right)-l_{j} x_{j}(t)\right]
$$

$$
\times\left[F_{j}\left(x_{j}(t)\right)-L_{j} x_{j}(t)\right]
$$

$$
-\sum_{j=1}^{n} D_{2 j}\left[F_{j}\left(x_{j}(t-\tau)\right)-l_{j} x_{j}(t-\tau)\right]
$$

$$
\times\left[F_{j}\left(x_{j}(t-\tau)\right)-L_{j} x_{j}(t-\tau)\right]
$$

$$
+\sum_{j=1}^{n}\left[\frac{1}{a^{2}} F_{j}^{2}\left(x_{j}(t)\right)+a^{2} D_{1 j}^{2} F_{j}^{2}(0)\right.
$$

$$
\left.+\frac{1}{4 a^{2}} x_{j}^{2}(t)+a^{2} D_{1 j}^{2} F_{j}^{2}(0)\left(L_{j}+l_{j}\right)^{2}\right]
$$

$$
+\sum_{j=1}^{n}\left[\frac{1}{a^{2}} F_{j}^{2}\left(x_{j}(t-\tau)\right)+a^{2} D_{2 j}^{2} F_{j}^{2}(0)\right.
$$

$$
\begin{equation*}
\left.+\frac{1}{4 a^{2}} x_{j}^{2}(t-\tau)+a^{2} D_{2 j}^{2} F_{j}^{2}(0)\left(L_{j}+l_{j}\right)^{2}\right] . \tag{30}
\end{equation*}
$$

Using (20)-(27) and adding (29), we can derive

$$
\begin{align*}
& \dot{V}(t) \leq \dot{V}_{1}(t)+\dot{V}_{2}(t)+\dot{V}_{3}(t)+\dot{V}_{4}(t)+e^{a t}\left(\Upsilon_{1}+\Upsilon_{2}\right) \\
& \leq e^{a t} M^{T}(t) \Delta_{1} M(t)+e^{a t} J^{T} P J \\
& \quad+e^{a t} \sum_{j=1}^{n}\left[a^{2} D_{1 j}^{2} F_{j}^{2}(0)+a^{2} D_{1 j}^{2} F_{j}^{2}(0)\left(L_{j}+l_{j}\right)^{2}\right. \\
&  \tag{31}\\
& \left.\quad+a^{2} D_{2 j}^{2} F_{j}^{2}(0)+a^{2} D_{2 j}^{2} F_{j}^{2}(0)\left(L_{j}+l_{j}\right)^{2}\right] .
\end{align*}
$$

Denote that

$$
\begin{align*}
M^{T}(t)= & \left(x^{T}(t), x^{T}(t-\tau), F^{T}(x(t)), F^{T}(x(t-\tau))\right. \\
& \left.\left(\int_{t-h}^{t} x(s) \mathrm{d} s\right)^{T},\left(\int_{t-h}^{t} F(x(s)) \mathrm{d} s\right)^{T}, H^{T}(x(t))\right)^{T} \tag{32}
\end{align*}
$$

By integrating both sides of (31) in time interval $t \in\left[t_{0}, t\right]$, then we can obtain

$$
\begin{align*}
& K e^{a t}\|x(t)\|^{2} \leq V(x(t)) \\
& \qquad \begin{array}{l}
\leq V\left(x\left(t_{0}\right)\right)+a^{-1} e^{a t} J^{T} P J \\
\\
\quad+e^{a t} \sum_{j=1}^{n}\left[a D_{1 j}^{2} F_{j}^{2}(0)+a D_{1 j}^{2} F_{j}^{2}(0)\left(L_{j}+l_{j}\right)^{2}\right. \\
\\
\left.\quad+a D_{2 j}^{2} F_{j}^{2}(0)+a D_{2 j}^{2} F_{j}^{2}(0)\left(L_{j}+l_{j}\right)^{2}\right]
\end{array}
\end{align*}
$$

which implies that

$$
\begin{equation*}
\|x(t)\|^{2} \leq \frac{e^{-a t} V\left(x\left(t_{0}\right)\right)+a^{-1} J^{T} P J+\Upsilon}{K} \tag{34}
\end{equation*}
$$

where $K=\min _{1 \leq i \leq n}\left\{\lambda_{\text {min }}(P) / \underline{\alpha}_{i}\right\}$, and

$$
\begin{align*}
\Upsilon=\sum_{j=1}^{n} & {\left[a D_{1 j}^{2} F_{j}^{2}(0)+a D_{1 j}^{2} F_{j}^{2}(0)\left(L_{j}+l_{j}\right)^{2}\right.}  \tag{35}\\
& \left.+a D_{2 j}^{2} F_{j}^{2}(0)+a D_{2 j}^{2} F_{j}^{2}(0)\left(L_{j}+l_{j}\right)^{2}\right]
\end{align*}
$$

If one chooses $\widetilde{B}=\sqrt{\left(1+a^{-1} J^{T} P J+\Upsilon\right) / K}>0$, then for any constant $\varrho>0$ and $\|\varphi\|<\varrho$, there is $t^{\prime}=t^{\prime}(\varrho)>0$, such that $e^{-a t} V\left(x\left(t_{0}\right)\right)^{2}<1$ for all $t \geq t^{\prime}$. According to Definition 2 , we have $\left\|x\left(t, x\left(t_{0}\right), \varphi\right)\right\|<\widetilde{B}$ for all $t \geq t^{\prime}$. That is to say, system (6) is uniformly ultimately bounded. This completes the proof.

From (18), we know that there is a positive constant $L_{0}$, such that

$$
\begin{equation*}
V\left(x\left(t_{0}\right)\right) \leq L_{0}\left\|x\left(t_{0}\right)\right\|^{2} e^{-a t_{0}} \tag{36}
\end{equation*}
$$

Thus, considering (34) and (36), we have the following result:

$$
\begin{align*}
\|x(t)\|^{2} & \leq \frac{e^{-a t} V\left(x\left(t_{0}\right)\right)+a^{-1} J^{T} P J+\Upsilon}{K} \\
& =\frac{e^{-a t} V\left(x\left(t_{0}\right)\right)}{K}+\frac{a^{-1} J^{T} P J+\Upsilon}{K}  \tag{37}\\
& \leq \frac{L_{0}\left\|x\left(t_{0}\right)\right\|^{2} e^{-a\left(t-t_{0}\right)}}{K}+N,
\end{align*}
$$

where $N=\left(a^{-1} J^{T} P J+\Upsilon\right) / K$.
Theorem 9. If all of the conditions of Theorem 8 hold, then there exists an attractor $\widetilde{\mathbb{A}}_{\widetilde{B}}$ for the solutions of system (6), where $\widetilde{\mathbb{A}}_{\widetilde{B}}=\left\{x(t):\|x(t)\| \leq \widetilde{B}, t \geq t_{0}\right\}$.

Proof. If one chooses $\widetilde{B}=\sqrt{\left(1+a^{-1} J^{T} P J+\Upsilon\right) / K}>0$, Theorem 8 shows that for any $\phi$, there is $t^{\prime}>0$, such that $\left\|x\left(t, t_{0}, \phi\right)\right\|<\widetilde{B}$ for all $t \geq t^{\prime}$. Let $\widetilde{\mathbb{A}}_{\widetilde{B}}=\{x(t)$ : $\left.\|x(t)\| \leq \widetilde{B}, t \geq t_{0}\right\}$. Clearly, $\widetilde{A}_{\widetilde{B}}$ is closed, bounded, and invariant. Furthermore, $\lim _{t \rightarrow \infty} \sup _{\inf }^{y \in \widetilde{\mathbb{A}}}{ }_{\widetilde{\mathcal{B}}}\left\|x\left(t ; t_{0}, \phi\right)-y\right\|=$ 0 . Therefore, $\widetilde{\mathbb{A}}_{\tilde{B}}$ is an attractor for the solutions of system (6).

Corollary 10. In addition to the fact that all of the conditions of Theorem 8 hold, if $J=0$, and $F_{j}(0)=0$, then system (6) has a trivial solution $x(t) \equiv 0$, and the trivial solution of system (6) is globally exponentially stable.

Proof. If $J=0$, and $F_{j}(0)=0$, then it is obvious that system (6) has a trivial solution $x(t) \equiv 0$. From Theorem 8 , one has

$$
\begin{equation*}
\|x(t ; 0, \phi)\|^{2} \leq K_{1} e^{-a t}, \quad \forall \phi, \tag{38}
\end{equation*}
$$

where $K_{1}=V(x(0)) / K$.
Therefore, the trivial solution of system (6) is globally exponentially stable. This completes the proof.

In this section, we will present conditions for uniformly ultimate boundedness and the existence of an attractor of the switching CGNN by applying the average dwell time.

Now, we can consider the switched Cohen-Grossberg neural networks with discrete delays and distributed delays as follows:

$$
\begin{align*}
\dot{x}(t)=- & \widehat{\alpha}(x(t)) \\
\times & {\left[\widehat{\beta}(x(t))-A_{\sigma}(t) F(x(t))-B F_{\sigma}(t)(x(t-\tau))\right.} \\
& \left.\quad-C_{\sigma}(t) \int_{t-h}^{t} F(x(s)) \mathrm{d} s+J\right], \tag{39}
\end{align*}
$$

$$
\begin{equation*}
x(t)=\varphi(t), \quad \text { when } t \in[-\delta, 0] \tag{40}
\end{equation*}
$$

Theorem 11. For a given constant $a>0$, if there is positive definite matrix $P=\operatorname{diag}\left(p_{i 1}, p_{i 2} \ldots, p_{i n}\right), D_{i}=\operatorname{diag}\left(D_{i 1}\right.$, $\left.D_{i 2} \ldots, D_{i n}\right), i=1,2, Q_{i}, S_{i}^{(i)}$, such that the following condition holds:

$$
\triangle=\left[\begin{array}{ccccccc}
\Phi_{i 11} & \Phi_{i 12} & \Phi_{i 13} & P_{i} B_{i} & 0 & P_{i} C_{i} & 0  \tag{41}\\
* & \Phi_{i 22} & 0 & \Phi_{i 24} & 0 & 0 & 0 \\
* & * & \Phi_{i 33} & 0 & 0 & 0 & 0 \\
* & * & * & \Phi_{i 44} & 0 & 0 & 0 \\
* & * & * & * & \Phi_{i 55} & \Phi_{i 56} & 0 \\
* & * & * & * & * & \Phi_{i 66} & 0 \\
* & * & * & * & * & * & \Phi_{i 77}
\end{array}\right]<0,
$$

where

$$
\begin{align*}
& Q=\left(\begin{array}{cc}
Q_{i 11} & Q_{i 12} \\
* & Q_{i 22}
\end{array}\right) \geq 0, \quad D_{i} \geq 0, \quad i=1,2, \\
& \Phi_{i 11}= a \Omega_{1} P_{i}-2 a \Omega_{2} P_{i}+P_{i}-\frac{\delta e^{-a \tau^{*}}}{\tau^{*}} S_{i}^{(2)} \\
&-\Omega_{3} D_{1}+\frac{1}{4 a^{2}} I+S_{i}^{(1)}-e^{-a \tau^{*}} S_{i}^{(1)} \\
&+h^{*} Q_{i 11}+\frac{\delta a}{1-e^{a \tau^{*}}} S_{i}^{(2)}, \\
& \Phi_{i 12}=-\frac{\delta a}{1-e^{a \tau^{*}} S_{i}^{(2)},} \\
& \Phi_{i 13}= P_{i} A_{i}+h^{*^{2}} R_{i 12}+\Omega_{4} D_{1}+h^{*} Q_{i 12},  \tag{42}\\
& \Phi_{i 22}=-\Omega_{3} D_{2}-\frac{\delta e^{-a \tau^{*}}}{\tau^{*}} S_{i}^{(2)}-e^{-a \tau^{*}} S_{i}^{(1)} \\
&+\frac{\delta a}{1-e^{a t^{*}}} S_{i}^{(2)}+\frac{1}{4 a^{2}} I, \\
& \Phi_{i 24}= \Omega_{4} D_{2}, \quad \Phi_{i 33}=-D_{1}+\frac{1}{a^{2}} I+h^{*} Q_{i 22}, \\
& \Phi_{i 44}=-D_{2}+\frac{1}{a^{2}} I, \quad \Phi_{i 55}=-\frac{e^{-a h^{*}}}{h^{*}} Q_{i 11}, \\
& \Phi_{i 56}=-\frac{e^{-a h^{*}}}{h^{*}} Q_{i 12}, \quad \Phi_{i 66}=-\frac{e^{-a h^{*}}}{h^{*}} Q_{i 22}, \\
& \Phi_{i 77}= \delta \tau^{*} \alpha^{2} S_{i}^{(2)} . \quad
\end{align*}
$$

Then, system (39) is uniformly ultimately bounded for any switching signal with average dwell time satisfying

$$
\begin{equation*}
T_{a}>T_{a}^{*}=\frac{\ln \bar{M}_{\mathrm{max}}}{a} \tag{43}
\end{equation*}
$$

where $\bar{M}_{\max }=L_{\max } / K_{\min }, L_{\max }=\max _{k \in \bar{N}, 1 \leq i \leq n}\left\{L_{i_{k}}\right\}, K_{\min }=$ $\min _{i_{k}}\left\{K_{i_{k}}\right\}$.

Proof. Define the Lyapunov functional candidate of the form

$$
\begin{align*}
V_{\sigma}(t)= & \sum_{j=1}^{n} 2 p_{\sigma(t)} e^{a t} \int_{0}^{x_{j}(t)} \frac{s}{\alpha_{j}(s)} \mathrm{d} s \\
& +\int_{t-h}^{t} e^{a s}(s-(t-h)) \xi^{T}(s) Q_{\sigma(t)} \xi(s) \mathrm{d} s  \tag{44}\\
& +\int_{t-\tau}^{t} e^{a s} \dot{x}^{T}(s) S_{\sigma(t)}^{(1)} \dot{x}(s) \mathrm{d} s \mathrm{~d} \theta \\
& +\delta \int_{-\tau}^{0} \int_{t+\theta}^{t} e^{a s} \dot{x}^{T}(s) S_{\sigma(t)}^{(2)} \dot{x}(s) \mathrm{d} s \mathrm{~d} \theta
\end{align*}
$$

When $t \in\left[t_{k}, t_{k+1}\right)$, the $i_{k}$ th subsystem is activated, and from Theorem 8 and (34), we can conclude that there is a positive constant $L_{i_{k}}$, such that

$$
\begin{align*}
\|x(t)\|^{2} & \leq \frac{L_{i_{k}}\left\|x\left(t_{k}\right)\right\|^{2} e^{-a\left(t-t_{k}\right)}+a^{-1} J^{T} P J+\Upsilon}{K_{i_{k}}}  \tag{45}\\
& =\bar{M}_{i_{k}}\left\|x\left(t_{k}\right)\right\|^{2} e^{-a\left(t-t_{k}\right)}+N_{i_{k}},
\end{align*}
$$

where

$$
\begin{gather*}
\bar{M}_{i_{k}}=\frac{L_{i_{k}}}{K_{i_{k}}}, \quad K_{i_{k}}=\min _{k \in \bar{N}, 1 \leq i \leq n}\left\{\frac{\lambda_{\min }\left(P_{i}\right)}{\underline{\alpha}_{i}}\right\},  \tag{46}\\
N_{i_{k}}=\frac{a^{-1} J^{T} P J+\Upsilon}{K_{i_{k}}} .
\end{gather*}
$$

The system state is continuous. Therefore, it follows that

$$
\begin{aligned}
\|x(t)\|^{2} \leq & \frac{L_{i_{k}}\left\|x\left(t_{k}\right)\right\|^{2} e^{-a\left(t-t_{k}\right)}+a^{-1} J^{T} P J+\Upsilon}{K_{i_{k}}} \\
= & M_{i_{k}}\left\|x\left(t_{k}\right)\right\|^{2} e^{-a\left(t-t_{k}\right)}+N_{i_{k}} \\
\leq & \cdots \leq e^{\sum_{v=0}^{k} \ln \bar{M}_{i_{v}}-a\left(t-t_{0}\right)}\left\|x\left(t_{0}\right)\right\|^{2} \\
& +\left[\bar{M}_{i_{k}} e^{-a\left(t-t_{k}\right)} N_{i_{k}}+\bar{M}_{i_{k}} \bar{M}_{i_{k-1}} e^{-a\left(t-t_{k-1}\right)} N_{i_{k-1}}\right. \\
& +\bar{M}_{i_{k}} \bar{M}_{i_{k-1}} \bar{M}_{i_{k-2}} e^{-a\left(t-t_{k-2}\right)} N_{i_{k-2}}+\cdots \\
& \left.+\bar{M}_{i_{k}} \bar{M}_{i_{k-1}} \bar{M}_{i_{k-2}} \cdots \bar{M}_{i_{1}} e^{-a\left(t-t_{1}\right)} N_{i_{1}}+N_{i_{k}}\right] \\
\leq & e^{(k+1) \ln \bar{M}_{\max }-a\left(t-t_{0}\right)}\left\|x\left(t_{0}\right)\right\|^{2} \\
& +\left[\bar{M}_{\max }^{k} N_{\max }+\bar{M}_{\max }^{(k-1)} N_{\max }+\bar{M}_{\max }^{(k-2)} N_{\max }\right. \\
& \left.\quad+\cdots+\bar{M}_{\max }^{2} N_{\max }+\bar{M}_{\max } N_{\max }+N_{\max }\right] \\
\leq & \bar{M}_{\max } e^{k \ln \bar{M}_{\max }-a\left(t-t_{0}\right)}\left\|x\left(t_{0}\right)\right\|^{2}
\end{aligned}
$$

$$
\begin{align*}
& +\frac{N_{\max }\left(1-\bar{M}_{\max }^{(k+1)}\right)}{1-\bar{M}_{\max }} \\
\leq & \bar{M}_{\max } e^{\ln \bar{M}_{\max } N_{\sigma}\left(t-t_{0}\right)-a\left(t-t_{0}\right)}\left\|x\left(t_{0}\right)\right\|^{2} \\
& +\frac{N_{\max }\left(1-\bar{M}_{\max }^{(k+1)}\right)}{1-\bar{M}_{\max }} \\
\leq & \bar{M}_{\max } e^{N_{0} \ln \bar{M}_{\max }-\left(a-\ln M_{\max } / T_{a}\right)\left(t-t_{0}\right)}\left\|x\left(t_{0}\right)\right\|^{2} \\
& +\frac{N_{\max }\left(1-\bar{M}_{\max }^{(k+1)}\right)}{1-\bar{M}_{\max }} \\
\leq & \frac{L_{\max } e^{N_{0} \ln \bar{M}_{\max }-\left(a-\ln M_{\max } / T_{a}\right)\left(t-t_{0}\right)}\left\|x\left(t_{0}\right)\right\|^{2}}{k_{\min }} \\
& +\frac{\left(a^{-1} J^{T} P J+\Upsilon\right)\left(1-L_{\max }^{(k+1)} / K_{\min }^{(k+1)}\right)}{K_{\min }-L_{\max }} . \tag{47}
\end{align*}
$$

$\sqrt{\text { If }} \sqrt{\text { one chooses }} \frac{\widetilde{B}}{1 / K_{\min }+\left(a^{-1} J^{T} P J+\Upsilon\right)\left(1-L_{\max }^{(k+1)} / K_{\min }^{(k+1)}\right) /\left(K_{\min }-L_{\max }\right)}>$ 0 , then for any constant $\varrho>0$ and $\|\varphi\|<\varrho$, there is $t^{\prime}=t^{\prime}(\varrho)>$ 0 , such that $L_{\text {max }} e^{N_{0} \ln \bar{M}_{\text {max }}-\left(a-\ln M_{\text {max }} / T_{a}\right)\left(t-t_{0}\right)}\left\|x\left(t_{0}\right)\right\|^{2}<1$ for all $t \geq t^{\prime}$. According to Definition 2, we have $\left\|x\left(t, t_{0}, \varphi\right)\right\|<\widetilde{B}$ for all $t \geq t^{\prime}$. That is to say, the switched Cohen-Grossberg neural networks system (39) is uniformly ultimately bounded. This completes the proof.

Theorem 12. If all of the conditions of Theorem 11 hold, then there exists an attractor $\mathbb{A}_{\tilde{B}}$ for the solutions of system (39), where $\mathbb{A}_{\widetilde{B}}=\left\{x(t):\|x(t)\| \leq \widetilde{B}, t \geq t_{0}\right\}$.

Proof. If we choose $\widetilde{B}=$ $\sqrt{1 / K_{\min }+\left(a^{-1} J^{T} P J+\Upsilon\right)\left(1-L_{\max }^{(k+1)} / K_{\min }^{(k+1)}\right) /\left(K_{\min }-L_{\max }\right)}>$ 0 , Theorem 11 shows that for any $\phi$, there is $t^{\prime}>0$, such that $\left\|x\left(t, t_{0}, \phi\right)\right\|<\widetilde{B}$ for all $t \geq t^{\prime}$. Let $\mathbb{A}_{\widetilde{B}}=\{x(t):\|x(t)\| \leq$ $\left.\widetilde{B}, t \geq t_{0}\right\}$. Clearly, $\mathbb{A}_{\widetilde{B}}$ is closed, bounded, and invariant. Furthermore, $\lim _{t \rightarrow \infty} \sup _{\inf }^{y \in \mathbb{A}_{\tilde{B}}}\left\|x\left(t ; t_{0}, \phi\right)-y\right\|=0$.

Therefore, $\mathbb{A}_{\tilde{B}}$ is an attractor for the solutions of system (39).

Corollary 13. In addition to the fact that all of the conditions of Theorem 8 hold, if $J=0$ and $F_{i}(0)=0$, then system (39) has a trivial solution $x(t) \equiv 0$, and the trivial solution of system (39) is globally exponentially stable.

Proof. If $J=0$ and $F_{i}(0)=0$, then it is obvious that the switched system (39) has a trivial solution $x(t) \equiv 0$. From Theorem 8, one has

$$
\begin{equation*}
\left\|x\left(t ; t_{0}, \phi\right)\right\|^{2} \leq K_{2} e^{-a\left(t-t_{0}\right)}, \quad \forall \phi, \tag{48}
\end{equation*}
$$

where $K_{2}=\bar{M}_{\text {max }} e^{N_{0} \ln \bar{M}_{\max }-\left(a-\ln M_{\max } / T_{\alpha}\right)\left(t-t_{0}\right)}\left\|x\left(t_{0}\right)\right\|^{2}$.

It means that the trivial solution of the switched CohenGrossberg neural networks (39) is globally exponentially stable. This completes the proof.

Remark 14. It is noted that common Lyapunov function method requires all the subsystems of the switched system to share a positive definite radially unbounded common Lyapunov function. Generally speaking, this requirement is difficult to achieve. So, in this paper, we select a novel multiple Lyapunov function to study the uniformly ultimate boundedness and the existence of an attractor for switched Cohen-Grossberg neural networks. furthermore, this type of Lyapunov function enables us to establish less conservative results.

Remark 15. When $N=1$, we have $P_{i}=P_{j}, Q_{i}=Q_{j}$, $S_{i}^{(1)}=S_{j}^{(1)}, S_{i}^{(2)}=S_{j}^{(2)}, i, j \in \Sigma$, then the switched CohenGrossberg neural networks (4) degenerates into a general Cohen-Grossberg neural networks with time-delay [15, 17]. Obviously, our result generalizes the previous result.

Remark 16. It is easy to see that $\tau_{a}=0$ is equivalent to existence of a common function for all subsystems, which implies that switching signals can be arbitrary. Hence, the results reported in this paper are more effective than arbitrary switching signal in the previous literature [16].

Remark 17. The constants $l_{i}, L_{i}$ in assumption $\left(\mathrm{H}_{1}\right)$ are allowed to be positive, negative, or zero, whereas the constant $l_{i}$ is restricted to be the zero in $[1,15]$, and the non-linear output function in $[5,18,34-37]$ is required to satisfy $F_{j}(0)=$ 0 . However, in our paper, the assumption condition was deleted. Therefore, assumption $\left(\mathrm{H}_{1}\right)$ of this paper is weaker than those given in $[1,5,15,18,34-37]$.

## 4. Illustrative Examples

In this section, we present an example to show the effectiveness and advantages of the proposed method and consider the switched neural networks with two subsystems.

Example. Consider the following switched Cohen-Grossberg neural network with discrete delays and distributed delays:

$$
\begin{align*}
\dot{x}(t)= & -\widehat{\alpha}(x(t)) \\
\times & {\left[\widehat{\beta}(x(t))-A_{\sigma}(t) F(x(t))-B_{\sigma} F(t)(x(t-\tau))\right.} \\
& \left.\quad-C_{\sigma}(t) \int_{t-h}^{t} F(x(s)) \mathrm{d} s-J\right] \tag{49}
\end{align*}
$$

where the behaved function is described by $\widehat{\beta}_{i}\left(x_{i}(t)\right)=x_{i}(t)$, and $F_{i}\left(x_{i}(t)\right)=0.5 \tanh \left(x_{i}(t)\right)(i=1,2)$; let

$$
\widehat{\alpha}(x(t))=\left(\begin{array}{cc}
1+\sin ^{2}\left(x_{1}(t)\right) & 0  \tag{50}\\
0 & 1+\cos ^{2}\left(x_{1}(t)\right)
\end{array}\right)
$$

Take the parameters as follows:

$$
\begin{array}{ll}
A_{1}=\left(\begin{array}{cc}
-0.1 & -0.4 \\
0.2 & -0.5
\end{array}\right), & B_{1}=\left(\begin{array}{cc}
-0.1 & -1 \\
1.4 & -0.4
\end{array}\right), \\
C_{1}=\left(\begin{array}{cc}
-0.1 & -0.2 \\
0.2 & -0.1
\end{array}\right), & A_{2}=\left(\begin{array}{cc}
-0.3 & -0.5 \\
0.2 & -0.1
\end{array}\right),  \tag{51}\\
B_{2}=\left(\begin{array}{cc}
-0.25 & -0.7 \\
0.9 & -0.5
\end{array}\right), & C_{2}=\left(\begin{array}{cc}
-0.15 & -0.3 \\
1.6 & 0.25
\end{array}\right) .
\end{array}
$$

From assumptions $\mathrm{H}_{1}, \mathrm{H}_{2}$, we can gain $l_{i}=0.5, L_{i}=$ $1, \underline{\alpha}=1, \alpha=1.5, \bar{\alpha}=2, b_{i}=1.2, \tau^{*}=0.15, h^{*}=$ 0.3 and $\delta=0.3$ and $i=1,2$.

Therefore, for $a=2$ and $F_{i}(0)=0$, by solving the inequality (41), we get

$$
\begin{gather*}
P_{1}=\left(\begin{array}{cc}
7.2667 & 0 \\
0 & 7.2667
\end{array}\right), \\
S_{1}^{(1)}=\left(\begin{array}{ccc}
56.5921 & 0.2054 \\
0.2054 & 56.1324
\end{array}\right), \\
S_{1}^{(2)}=\left(\begin{array}{ccc}
12.2582 & -0.1936 \\
-0.1936 & 11.9901
\end{array}\right), \\
P_{2}=\left(\begin{array}{ccc}
7.3794 & 0 \\
0 & 7.3794
\end{array}\right), \\
S_{2}^{(1)}=\left(\begin{array}{ccc}
55.5355 & -0.0809 \\
-0.0809 & 55.8300
\end{array}\right),  \tag{52}\\
S_{2}^{(2)}=\left(\begin{array}{ccc}
11.4579 & -0.5681 \\
-0.5681 & 13.4627
\end{array}\right), \\
Q_{2}=\left(\begin{array}{ccc}
17.8905 & -0.1476 & -3.0462 \\
-0.1476 & 18.0434 & 0.0962
\end{array}-2.0962\right. \\
-3.0462 \\
0.0962 \\
\hline 0.0962
\end{gather*}-2.911700 .1066
$$

Using (41), we can get the average dwell time $\tau_{a}>\tau_{a}^{*}=$ 2.0889 .

## 5. Conclusion

In this paper, the dynamics of switched Cohen-Grossberg neural networks with average dwell time is investigated. A novel multiple Lyapunov-Krasovskii functional is designed to get new sufficient conditions guaranteeing the uniformly ultimate boundedness, the existence of an attractor, and the globally exponential stability. The derived conditions are expressed in terms of LMIs, which are more relaxed than algebraic formulation and can be easily checked by the effective LMI toolbox in Matlab in practice. Based on the method provided in this paper, stochastic disturbance, impulse, and reaction diffusion for switched systems will be considered in the future works.

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# Asymptotic Behavior of Switched Stochastic Delayed Cellular Neural Networks via Average Dwell Time Method 

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#### Abstract

The asymptotic behavior of a class of switched stochastic cellular neural networks (CNNs) with mixed delays (discrete timevarying delays and distributed time-varying delays) is investigated in this paper. Employing the average dwell time approach (ADT), stochastic analysis technology, and linear matrix inequalities technique (LMI), some novel sufficient conditions on the issue of asymptotic behavior (the mean-square ultimate boundedness, the existence of an attractor, and the mean-square exponential stability) are established. A numerical example is provided to illustrate the effectiveness of the proposed results.


## 1. Introduction

Since Chua and Yang's seminal work on cellular neural networks (CNNs) in 1988 [1, 2], it has witnessed the successful applications of CNN in various areas such as signal processing, pattern recognition, associative memory, and optimization problems (see, e.g., [3-5]). From a practical point of view, both in biological and man-made neural networks, processing of moving images and pattern recognition problems require the introduction of delay in the signals transmitted among the cells [6, 7]. After the widely use of discrete delays, distributed delays arise because that neural networks usually have a spatial extent due to the presences of a multitude of parallel pathway with a variety of axon sizes and lengths. The mathematical model can be described by the following differential equations:

$$
\begin{aligned}
d x_{i}(t)= & -d_{i} x_{i}(t)+\sum_{j=1}^{n} a_{i j} f_{j}\left(x_{j}(t)\right) \\
& +\sum_{j=1}^{n} b_{i j} f_{j}\left(x_{j}\left(t-\tau_{i}(t)\right)\right)
\end{aligned}
$$

$$
\begin{array}{r}
+\sum_{j=1}^{n} c_{i j} \int_{t-h_{i}(t)}^{t} f_{j}\left(x_{j}(s)\right) \mathrm{d} s+J_{i} \\
i=1, \ldots, n \tag{1}
\end{array}
$$

where $t \geq 0, n \geq 2$ corresponds to the number of units in a neural network; $x_{i}(t)$ denotes the potential (or voltage) of cell $i$ at time $t ; f_{j}(\cdot)$ denotes a nonlinear output function; $d_{i}>0$ denotes the rate with which the cell $i$ resets its potential to the resting state when isolated from other cells and external inputs; $a_{i j}, b_{i j}, c_{i j}$ denote the strengths of connectivity between cell $i$ and $j$ at time $t$, respectively; $\tau_{i}(t)$ and $h_{i}(t)$ correspond to the discrete time-varying delays and distributed time-varying delays, respectively.

Neural network is nonlinearity; in the real world, nonlinear problems are not exceptional, but regular phenomena. Nonlinearity is the nature of matter and its development [8, 9]. Although discrete delays combined with distributed delays can usually provide a good approximation for prime model, most real models are often affected by so many external perturbations which are of great uncertainty. For
instance, in electronic implementations, it was realized that stochastic disturbances are mostly inevitable owing to thermal noise. Just as Haykin [10] point out that in real nervous systems, synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. Consequently, noise is unavoidable and should be taken into consideration in modeling. Moreover, it has been well recognized that a CNN could be stabilized or destabilized by certain stochastic inputs. Therefore, it is of significant importance to consider stochastic effects to the delayed neural networks. One approach to the mathematical incorporation of such effects is to use probabilistic threshold models. However, the previous literatures all focus on the stability of stochastic neural networks with delays [11-14]. Actually, studies on dynamical systems involve not only a discussion of the stability property, but also other dynamic behaviors such as the ultimate boundedness and attractor. However, there are very few results on the ultimate boundedness and attractor for stochastic neural networks [15-17]. Hence, discussing the asymptotic behavior of neural networks with mixed delays is valuable and meaningful.

On the other hand, neural networks often exhibit a special characteristic of network mode switching; that is, a neural network sometimes has finite modes that switch from one to another at different times according to a switching law generated from a switching logic. As an important class of hybrid systems, switched systems arise in many practical processes. In current papers, the analysis of switched systems has drawn considerable attention since they have numerous applications in control of mechanical systems, computer communities, automotive industry, electric power systems and many other fields [18-22]. Most recently, the stability analysis of switched neural systems has been further investigated which was mainly based on Lyapunov functions [23, 24]. It is worth noting that the average dwell time (ADT) approach is an effective method for the switched systems, which avoid the common Lyapunov function and can be adopted to obtain less conservative stability conditions. For instance, based on the average dwell time method, the problems of stability have been discussed for uncertain switched CohenGrossberg neural networks with interval time-varying delay and distributed time-varying delay in [25]. In [26], the average dwell time method has been utilized to get some sufficient conditions for the exponential stability and the weighted $L_{2}$ gain for a class of switched systems.

However, it is worth emphasizing that when the activation functions are unbounded in some special applications, the existence of equilibrium point cannot be guaranteed [27]. Therefore, in these circumstances, the discussing of stability of equilibrium point for switched neural networks turns to be unreachable, which motivated us to consider the ultimate boundedness and attractor for the switched neural networks. Unfortunately, as far as we know, the issue of asymptotic behavior of switched systems with mixed time delays has not been investigated yet, let alone studying the asymptotic behavior of switched stochastic systems. Therefore, these researches are challenging and interesting since they integrate the switched hybrid system into the stochastic system and are
thus theoretically and practically significant. Notice that the asymptotic behavior of switched stochastic neural networks with mixed delays should be studied intensively.

Motivated by the above analysis, the main purpose of this paper is to get sufficient conditions on the asymptotic behavior (the mean-square ultimate boundedness, the existence of an attractor, and mean-square exponential stability) for the switched stochastic system. This paper is organized as follows. In Section 2, the considered model of switched stochastic CNN with mixed delays is presented. Some necessary assumptions, definitions and lemmas are also given in this section. In Section 3, mean-square ultimate boundedness and attractor for the proposed model are studied. A numerical example is arranged to demonstrate the effectiveness of the theoretical results in Section 4, and we conclude this paper in Section 5.

## 2. Problem Formulation

In general, a stochastic cellular neural network with mixed delays can be described as follows:

$$
\begin{align*}
d x(t)= & {[-D x(t)+A F(x(t))+B F(x(t-\tau(t)))} \\
& \left.+C \int_{t-h(t)}^{t} F(x(s)) \mathrm{d} s+J\right] d t  \tag{2}\\
& +G(x(t), x(t-\tau(t))) d w(t)
\end{align*}
$$

where $x(t)=\left(x_{1}(t), \ldots, x_{n}(t)\right)^{T} \in R^{n}, F(x(t))=\left(f_{1}\left(x_{1}(t)\right)\right.$, $\left.\ldots, f_{n}\left(x_{n}(t)\right)\right)^{T}, D=\operatorname{diag}\left(d_{1}, \ldots, d_{n}\right), A=\left(a_{i j}\right)_{n \times n}, B=$ $\left(b_{i j}\right)_{n \times n}, C=\left(c_{i j}\right)_{n \times n}, J=\left(J_{1}, \ldots, J_{n}\right)^{T}, \tau(t)=\left(\tau_{1}(t), \ldots\right.$, $\left.\tau_{n}(t)\right)^{T}, h(t)=\left(h_{1}(t), \ldots, h_{n}(t)\right)^{T}, G(\cdot, \cdot)$ is a $n \times n$ matrix valued function, and $w(t)=\left(w_{1}(t), \ldots, w_{n}(t)\right)^{T}$ is an $n$-dimensional Brownian motion defined on a complete probability space $(\boldsymbol{\Omega}, \mathscr{F}, \mathbf{P})$ with a natural filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ (i.e., $\mathscr{F}_{t}=$ $\sigma\{w(s): 0 \leq s \leq t\}$ ).

By introducing switching signal into the system (2) and taking a set of neural networks as the individual subsystems, the switched system can be obtained, which is described as

$$
\begin{align*}
d x(t)= & {\left[-D_{\sigma(t)} x(t)+A_{\sigma(t)} F(x(t))+B_{\sigma(t)} F(x(t-\tau(t)))\right.} \\
& \left.+C_{\sigma(t)} \int_{t-h(t)}^{t} F(x(s)) \mathrm{d} s+J\right] d t \\
& +G_{\sigma(t)}(x(t), x(t-\tau(t))) d w(t) \tag{3}
\end{align*}
$$

where $\sigma(t):[0,+\infty) \rightarrow \boldsymbol{\Sigma}=\{1,2 \ldots m\}$ is the switching signal. At each time instant $t$, the index $\sigma(t) \in \mathbf{\Sigma}$ (i.e., $\sigma(t)=$ $i)$ of the active subsystem means that the $i$ th subsystem is activated.

For the convenience of discussion, it is necessary to introduce some notations. $R^{n}$ denotes the $n$-dimensional Euclidean space. $X \leq Y(X<Y)$ means that each pair of corresponding elements of $X$ and $Y$ satisfies the inequality " $\leq$
(<)". $X$ is especially called a positive (negative) matrix if $X>0$ $(<0) . X^{T}$ denotes the transpose of any square matrix $X$, and the symbol "*" within the matrix represents the symmetric term of the matrix. $\lambda_{\text {min }}(X)$ means the minimum eigenvalue of matrix $X$, and $\lambda_{\max }(X)$ means the maximum eigenvalue of matrix $X$. I denotes unit matrix.

Let $\mathscr{C}\left(\left[-\tau^{*}, 0\right], R^{n}\right)$ denote the Banach space of continuous functions which mapping from $\left[-\tau^{*}, 0\right]$ to $R^{n}$ with is the topology of uniform convergence. For any $\|\varphi\| \in$ $\mathscr{C}\left(\left[-\tau^{*}, 0\right], R^{n}\right)$, we define $\|\varphi\|=\max _{1 \leq i \leq n} \sup _{t-\tau_{*}<s \leq t}\left|\varphi_{i}(s)\right|$.

The initial conditions for system (3) are given in the form:

$$
\begin{equation*}
x(t)=\varphi, \quad \varphi \in \mathscr{C}_{\mathscr{F}_{0}}\left(\left[-\tau^{*}, 0\right], R^{n}\right) \tag{4}
\end{equation*}
$$

where $\mathscr{C}_{\mathscr{F}_{0}}\left(\left[-\tau^{*}, 0\right], R^{n}\right)$ is the family of all $\mathscr{F}_{0}$-measurable bounded $\mathscr{C}\left(\left[-\tau^{*}, 0\right], R^{n}\right)$-valued random variables.

Throughout this paper, we assume the following assumptions are always satisfied.
$\left(H_{1}\right)$ The discrete time-varying delay $\tau(t)$ and distributed time-varying delay $h(t)$ are satisfying

$$
\begin{equation*}
0 \leq \tau(t) \leq \tau, \quad 0 \leq h(t) \leq h, \quad \tau^{*}=\max _{1 \leq i \leq n}\{\tau, h\} \tag{5}
\end{equation*}
$$

where $\tau, h, \tau^{*}$ are scalars.
$\left(H_{2}\right)$ There exist constants $l_{j}$ and $L_{j}, i=1,2, \ldots, n$, such that

$$
\begin{equation*}
l_{j} \leq \frac{f_{j}(x)-f_{j}(y)}{x-y} \leq L_{j}, \quad \forall x, y \in R, x \neq y \tag{6}
\end{equation*}
$$

Moreover, we define

$$
\begin{gather*}
\Sigma_{1}=\operatorname{diag}\left\{l_{1} L_{1}, l_{2} L_{2}, \ldots, l_{n} L_{n}\right\}, \\
\Sigma_{2}=\operatorname{diag}\left\{l_{1}+L_{1}, l_{2}+L_{2}, \ldots, l_{n}+L_{n}\right\} . \tag{7}
\end{gather*}
$$

$\left(H_{3}\right)$ We assume that $G(t, x, y): R^{+} \times R^{n} \times R^{n} \rightarrow$ $R^{n \times m}$ is locally Lipschitz continuous and satisfies the following condition:
$\operatorname{trace}\left[G^{T}(t, x, y) G(t, x, y)\right] \leq x^{T} U_{1}^{T} U_{1} x$

$$
\begin{equation*}
+y^{T} U_{2}^{T} U_{2} y+2 x^{T} U_{1}^{T} U_{2} y \tag{8}
\end{equation*}
$$

where $U_{1}>0, U_{2}>0$ are constant matrices with appropriate dimensions.
Some definitions and lemmas are introduced as follows.
Definition 1 (see [15]). System (2) is called mean-square ultimate boundedness if there exists a constant vector $\widetilde{B}>0$, such that, for any initial value $\varphi \in \mathscr{C}_{\mathscr{F}_{0}}$, there is a $t^{\prime}=t^{\prime}(\varphi)>$ 0 , for all $t \geq t^{\prime}$, the solution $x(t, \varphi)$ of system (2) satisfies

$$
\begin{equation*}
E\|x(t, \varphi)\|^{2} \leq \widetilde{B} \tag{9}
\end{equation*}
$$

In this case, the set $A=\left\{\varphi \in \mathscr{C}_{\widetilde{F}_{0}} \mid E\|\varphi(s)\|^{2} \leq \widetilde{B}\right\}$ is said to be an attractor of system (2) in mean square sense.

Clearly, proposition above equals to $\lim _{t \rightarrow \infty} \sup E\|x(t)\|^{2} \leq \widetilde{B}$.

Definition 2 (see [28]). For any switching signal $\sigma(t)$, corresponding a switching sequence $\left\{\left(\sigma\left(t_{0}\right), t_{0}\right), \ldots\left(\sigma\left(t_{k}\right), t_{k}\right), \ldots, \mid\right.$ $k=0,1, \ldots\}$, where $\left(\sigma\left(t_{k}\right), t_{k}\right)$ means the $\sigma\left(t_{k}\right)$ th subsystem, is activated during $t \in\left[t_{k}, t_{k-1}\right)$, and $k$ denotes the switching ordinal number. Given any finite constants $T_{1}, T_{2}$ satisfying $T_{2}>T_{1} \geq 0$ denotes the number of discontinuity of a switching signal $\sigma(t)$ over the time interval $\left(T_{1}, T_{2}\right)$ by $N_{\sigma}\left(T_{1}, T_{2}\right)$. If $N_{\sigma}\left(T_{1}, T_{2}\right) \leq N_{0}+\left(T_{2}-T_{1}\right) / T_{\alpha}$ holds for $T_{\alpha}>0$, $N_{0}>0$, then $T_{\alpha}>0$ is called the average dwell time. $N_{0}$ is the chatter bound.

Lemma 3. Let $X$ and $Y$ be any n-dimensional real vectors, $P$ be a positive semidefinite matrix and a scalar $\varepsilon>0$. Then the following inequality holds:

$$
\begin{equation*}
2 X^{T} P Y \leq \varepsilon X^{T} P X+\varepsilon^{-1} Y^{T} P Y \tag{10}
\end{equation*}
$$

Lemma 4 (see [29]). For any positive definite constant matrix $M \in \mathscr{R}^{n \times n}$, and a scalar $r$, if there exists a vector function $\eta:[0, r] \rightarrow \mathscr{R}^{n}$ such that the integrals $\int_{0}^{r} \eta^{T}(s) M \eta(s) d s$ and $\int_{0}^{r} \eta(s) d s$ are well defined, then

$$
\begin{equation*}
\int_{0}^{r} \eta^{T}(s) M \eta(s) d s \geq \frac{1}{r} \int_{0}^{r} \eta^{T}(s) d s M \int_{0}^{r} \eta(s) d s \tag{11}
\end{equation*}
$$

## 3. Main Results

Let $\mathscr{C}^{2,1}:\left(R^{n} \times R^{+} ; R+\right)$ denote the family of all nonnegative functions $V(t, x)$ on $R^{n} \times R^{+}$which are continuously twice differentiable in $x$ and once differentiable in $t$. If $V \in$ $\mathscr{C}^{2,1}:\left(R^{n} \times R^{+} ; R+\right)$, define an operator $\mathscr{L} V$ associated with general stochastic system $d x(t)=f(x(t), t) d t+G(x(t), x(t-$ $\tau(t))) d w(t)$ as

$$
\begin{align*}
& \mathscr{L} V(t, x)=V_{t}(t, x)+V_{x}(t, x) f(x(t), t) \\
& +\frac{1}{2} \operatorname{trace}\left\{G^{T}(x(t), x(t-\tau(t))) V_{x x}(t, x)\right. \\
&  \tag{12}\\
& \quad \times G(x(t), x(t-\tau(t)))\},
\end{align*}
$$

where

$$
\begin{gather*}
V_{t}(t, x)=\frac{\partial V(t, x)}{\partial t}, \quad V_{x}(t, x)=\left(\frac{\partial V(t, x)}{\partial x_{1}}, \ldots, \frac{\partial V(t, x)}{\partial x_{n}}\right)^{T} \\
V_{x x}(t, x)=\left(\frac{\partial V(t, x)}{\partial x_{i} \partial x_{j}}\right)_{n \times n} . \tag{13}
\end{gather*}
$$

Theorem 5. If there are constants $\mu, \nu$ such that $\dot{\tau}(t) \leq \mu$, $\dot{h}(t) \leq \nu$, we denote $g(\mu), k(\nu)$ as:

$$
\begin{align*}
& g(\mu)= \begin{cases}(1-\mu) e^{-\alpha \tau}, & \mu \leq 1 \\
1-\mu, & \mu \geq 1\end{cases} \\
& k(v)= \begin{cases}(1-v) e^{-\alpha h}, & v \leq 1 \\
1-v, & v \geq 1\end{cases} \tag{14}
\end{align*}
$$

For a given constant $\alpha>0$, if there exist positive definite matrixes $P=\operatorname{diag}\left(p_{1}, p_{2}, \ldots, p_{n}\right), Q, R, S, Z, U_{1}, U_{2}, Y_{i}=$ $\operatorname{diag}\left(Y_{i 1}, Y_{i 2}, \ldots, Y_{i n}\right), i=1,2$, such that the following condition holds:

$$
\begin{gather*}
\Delta_{1}=\left[\begin{array}{cccccc}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & 0 & \Phi_{16} \\
* & \Phi_{22} & 0 & \Phi_{24} & 0 & 0 \\
* & * & \Phi_{33} & 0 & 0 & 0 \\
* & * & * & \Phi_{44} & \Phi_{55} & 0 \\
* & * & * & * & 0 & \Phi_{66}
\end{array}\right]<0, \\
\Phi_{11}=2 \alpha P-2 D P+Q+\tau^{2} S-2 \Sigma_{1} Y_{1}+\alpha I+U_{1}^{T} P U_{1}, \\
\Phi_{12}=U_{1}^{T} P U_{2}, \quad \Phi_{13}=P A+\Sigma_{2} Y_{1}  \tag{15}\\
\Phi_{14}=P B, \quad \Phi_{16}=P C \\
\Phi_{22}=-g(\mu) Q-2 \Sigma_{1} Y_{2}+\alpha I+U_{2}^{T} P U_{2} \\
\Phi_{24}=\Sigma_{2} Y_{2}, \quad \Phi_{33}=R+h^{2} Z-2 Y_{1}+\alpha I \\
\Phi_{44}=-k(\nu) R-2 Y_{2}+\alpha I \\
\Phi_{55}=-g(\mu) S, \quad \Phi_{66}=-k(\nu) Z,
\end{gather*}
$$

then system (2) is mean-square ultimate boundedness.
Proof. Consider the positive definite Lyapunov functional as follows:

$$
\begin{equation*}
V(t)=V_{1}(t)+V_{2}(t)+V_{3}(t)+V_{4}(t)+V_{5}(t), \tag{16}
\end{equation*}
$$

where

$$
\begin{align*}
& V_{1}(t)=e^{\alpha t} x^{T}(t) P x(t) \\
& V_{2}(t)=\int_{t-\tau(t)}^{t} x^{T}(s) Q e^{\alpha s} x(s) \mathrm{d} s \\
& V_{3}(t)=\int_{t-h(t)}^{t} F^{T}(x(s)) R e^{\alpha s} F(x(s)) \mathrm{d} s  \tag{17}\\
& V_{4}(t)=\tau \int_{-\tau(t)}^{0} \int_{t+\theta}^{t} x^{T}(s) S e^{\alpha s} x(s) \mathrm{d} s \mathrm{~d} \theta \\
& V_{5}(t)=h \int_{-h(t)}^{0} \int_{t+\theta}^{t} F^{T}(x(s)) Z e^{\alpha s} F(x(s)) \mathrm{d} s \mathrm{~d} \theta
\end{align*}
$$

Then, by Ito's formula, the stochastic derivative of $V(x, t)$ is

$$
\begin{align*}
d V(x, t)= & \mathscr{L} V(x, t) d t \\
& +V_{x}(x, t) G(x(t), x(t-\tau(t))) d w(t) \tag{18}
\end{align*}
$$

the operator $\mathscr{L V}$ along the trajectory of system (2) can be obtained

$$
\begin{align*}
& \mathscr{L} V_{1}(t)= \frac{\partial V_{1}(x(t), t)}{\partial t}+\frac{\partial V_{1}(x(t), t)}{\partial x} \\
& \times[-D x(t)+A F(x(t))+B F(x(t-\tau(t))) \\
&\left.+C \int_{t-h(t)}^{t} F(x(s)) \mathrm{d} s+J\right] \\
&+\frac{1}{2} \operatorname{trace}\left[G^{T}(x(t), x(t-\tau(t))) \frac{\partial^{2} V_{1}(x(t), t)}{\partial x^{2}}\right. \\
& \times\times G(x(t), x(t-\tau(t)))] \\
& \times {\left[-D e^{\alpha t} x^{T}(t) P x(t)+2 e^{\alpha t} x^{T}(t) P\right.} \\
&\left.+C \int_{t-h(t)}^{t} F(x(x)) \mathrm{d} s+J\right] \\
&+e^{\alpha t} \operatorname{trace}\left[G^{T}(x(t), x(t-\tau(t))) P\right. \\
&\quad \times G(x(t), x(t-\tau(t)))] .
\end{align*}
$$

From Assumption $\left(H_{3}\right)$, Lemma 3, and (19), we can get

$$
\begin{align*}
\mathscr{L} V_{1}(t) \leq & 2 \alpha e^{\alpha t} x^{T}(t) P x(t)+2 e^{\alpha t} x^{T}(t) P \\
\times & \times[-D x(t)+A F(x(t))+B F(x(t-\tau(t))) \\
& \left.+C \int_{t-h(t)}^{t} F(x(s)) \mathrm{d} s\right]+e^{\alpha t} \alpha^{-1} J^{T} P J \\
& +e^{\alpha t} x^{T}(t) U_{1}^{T} P U_{1} x(t) \\
& +x^{T}(t-\tau(t)) U_{2}^{T} P U_{2} x(t-\tau(t)) \\
& +2 x^{T}(t) U_{1}^{T} P U_{2} x(t-\tau(t)) \tag{20}
\end{align*}
$$

Similarly, calculating the operator $\mathscr{L} V_{i}(i=2,3,4,5)$, along the trajectory of system (2), one can get

$$
\begin{aligned}
\mathscr{L} V_{2}= & e^{\alpha t} x^{T}(t) \mathrm{Q} x(t) \\
& -(1-\dot{\tau}(t)) e^{\alpha(t-\tau(t))} x^{T}(t-\tau(t)) \mathrm{Q} x(t-\tau(t))
\end{aligned}
$$

$$
\begin{align*}
\leq & e^{\alpha t} x^{T}(t) Q x(t) \\
& -(1-\mu) e^{\alpha(t-\tau)} x^{T}(t-\tau(t)) Q x(t-\tau(t)) \\
\leq & e^{\alpha t} x^{T}(t) Q x(t) \\
& -g(\mu) e^{\alpha t} x^{T}(t-\tau(t)) Q x(t-\tau(t)), \\
\mathscr{L} V_{3} \leq & e^{\alpha t} F^{T}(x(t)) R F(x(t)) \\
& -k(v) e^{\alpha t} F^{T}(x(t-\tau(t))) R F(x(t-\tau(t))), \\
\mathscr{L} V_{4}= & \tau\left[\tau(t) e^{\alpha t} x^{T}(t) S x(t)\right. \\
& \left.\quad-(1-\dot{\tau}(t)) e^{\alpha(t-\tau(t))} \int_{t-\tau(t)}^{t} x^{T}(s) S x(s) \mathrm{d} s\right] \\
\leq & \tau^{2} e^{\alpha t} x^{T}(t) S x(t) \\
& -\tau g(\mu) e^{\alpha t} \int_{t-\tau(t)}^{t} x^{T}(s) S x(s) \mathrm{d} s, \\
\mathscr{L} V_{5} \leq & h^{2} e^{\alpha t} F^{T}(x(t)) Z F(x(t)) \\
& -h k(v) e^{\alpha t} \int_{t-h(t)}^{t} F^{T}(x(s)) Z F(x(s)) \mathrm{d} s . \tag{21}
\end{align*}
$$

According to Lemma 4, the following inequalities can be obtained:

$$
\begin{align*}
& \int_{t-\tau(t)}^{t} x^{T}(s) S x(s) \mathrm{d} s \\
& \quad \geq \frac{1}{\tau} \int_{t-\tau(t)}^{t} x^{T}(s) \mathrm{d} s S \int_{t-\tau(t)}^{t} x(s) \mathrm{d} s  \tag{22}\\
& \int_{t-h(t)}^{t} F^{T}(x(s)) Z F(x(s)) \mathrm{d} s \\
& \quad \geq \frac{1}{h} \int_{t-h(t)}^{t} F^{T}(x(s)) \mathrm{d} s Z \int_{t-h(t)}^{t} F(x(s)) \mathrm{d} s
\end{align*}
$$

Then, we can get

$$
\begin{align*}
\mathscr{L} V_{4} \leq & \tau^{2} e^{\alpha t} x^{T}(t) S x(t) \\
& -g(\mu) e^{\alpha t} \int_{t-\tau(t)}^{t} x^{T}(s) \mathrm{d} s S \int_{t-\tau(t)}^{t} x(s) \mathrm{d} s \\
\mathscr{L} V_{5} \leq & h^{2} e^{\alpha t} F^{T}(x(t)) Z F(x(t)) \\
& -k(\nu) e^{\alpha t} \int_{t-h(t)}^{t} F^{T}(x(s)) \mathrm{d} s Z \int_{t-h(t)}^{t} F(x(s)) \mathrm{d} s \tag{23}
\end{align*}
$$

On the other hand, it follows from Assumption $\left(\mathrm{H}_{2}\right)$ that we can easily obtain

$$
\begin{align*}
& {\left[f_{i}\left(x_{i}(t)\right)-f_{i}(0)-L_{i} x_{i}(t)\right]} \\
& \quad \times\left[f_{i}\left(x_{i}(t)\right)-f_{i}(0)-l_{i} x_{i}(t)\right] \leq 0, \\
& {\left[f_{i}\left(x_{i}(t-\tau(t))\right)-f_{i}(0)-L_{i} x_{i}(t-\tau(t))\right]} \\
& \quad \times\left[f_{i}\left(x_{i}(t-\tau(t))\right)-f_{i}(0)-l_{i} x_{i}(t-\tau(t))\right] \leq 0, \\
& \quad i=1,2, \ldots, n . \tag{24}
\end{align*}
$$

Then we obtain

$$
\begin{align*}
& 0 \leq \delta_{1}=-2 \sum_{i=1}^{n} y_{1 i}\left[f_{i}\left(x_{i}(t)\right)-f_{i}(0)-L_{i} x_{i}(t)\right] \\
& \times\left[f_{i}\left(x_{i}(t)\right)-f_{i}(0)-l_{i} x_{i}(t)\right] \\
& 0 \leq \delta_{2}=-2 \sum_{i=1}^{n} y_{2 i}\left[f_{i}\left(x_{i}(t-\tau(t))\right)-f_{i}(0)\right. \\
&\left.-L_{i} x_{i}(t-\tau(t))\right] \\
& \times\left[f_{i}\left(x_{i}(t-\tau(t))\right)-f_{i}(0)-l_{i} x_{i}(t-\tau(t))\right] \\
& \delta_{1}=- 2 \sum_{i=1}^{n} y_{1 i}\left[f_{i}\left(x_{i}(t)\right)-L_{i} x_{i}(t)\right]\left[f_{i}\left(x_{i}(t)\right)-l_{i} x_{i}(t)\right] \\
&-2 \sum_{i=1}^{n} y_{1 i} f_{i}^{2}(0) \\
&+2 \sum_{i=1}^{n} y_{1 i} f_{i}(0)\left[2 f_{i}\left(x_{i}(t)\right)-\left(L_{i}+l_{i}\right) x_{i}(t)\right] \\
& \leq-2 \sum_{i=1}^{n} y_{1 i}\left[f_{i}\left(x_{i}(t)\right)-L_{i} x_{i}(t)\right]\left[f_{i}\left(x_{i}(t)\right)-l_{i} x_{i}(t)\right] \\
&+\sum_{i=1}^{n}\left[\alpha f_{i}^{2}\left(x_{i}(t)\right)+4 \alpha^{-1} f_{i}^{2}(0) y_{1 i}^{2}+\alpha x_{i}^{2}(t)\right. \\
&\left.+\alpha^{-1} f_{i}^{2}(0) y_{1 i}^{2}\left(L_{i}+l_{i}\right)^{2}\right] \tag{25}
\end{align*}
$$

Similarly, one can get

$$
\begin{align*}
\delta_{2} \leq-2 \sum_{i=1}^{n} y_{2 i} & {\left[f_{i}\left(x_{i}(t-\tau(t))\right)-L_{i} x_{i}(t-\tau(t))\right] } \\
\times & {\left[f_{i}\left(x_{i}(t-\tau(t))\right)-l_{i} x_{i}(t-\tau(t))\right] }  \tag{26}\\
+ & {\left[\alpha f_{i}^{2}\left(x_{i}(t-\tau(t))\right)+4 \alpha^{-1} f_{i}^{2}(0) y_{2 i}^{2}\right.} \\
+ & \left.\alpha x_{i}^{2}(t-\tau(t))+\alpha^{-1} f_{i}^{2}(0) y_{2 i}^{2}\left(L_{i}+l_{i}\right)^{2}\right]
\end{align*}
$$

Denote

$$
\begin{align*}
\zeta(t)= & {\left[x^{T}(t), x^{T}(t-\tau(t)), F^{T}(x(t)),\right.} \\
& F^{T}(x(t-\tau(t))),\left(\int_{t-\tau(t)}^{t} x(s) \mathrm{d} s\right)^{T},  \tag{27}\\
& \left.\left(\int_{t-h(t)}^{t} F(x(s)) \mathrm{d} s\right)^{T}\right]^{T},
\end{align*}
$$

and combing with (16)-(26), we can get

$$
\begin{align*}
d V= & \mathscr{L} V_{1} d t+\mathscr{L} V_{2} d t+\mathscr{L} V_{3} d t+\mathscr{L} V_{4} d t+\mathscr{L} V_{5} d t \\
& +2 P e^{\alpha t} x^{T}(t) G(x(t), x(t-\tau(t))) d w(t) \\
\leq & e^{\alpha t} \zeta^{T}(t) \Delta_{1} \zeta(t) d t+e^{\alpha t} \mathcal{N}_{1} d t  \tag{28}\\
& +2 P e^{\alpha t} x(t) G(x(t), x(t-\tau(t))) d w(t),
\end{align*}
$$

where

$$
\begin{align*}
\mathcal{N}_{1}= & \alpha^{-1} J^{T} P J \\
& +\sum_{i=1}^{n}\left[4 \alpha^{-1} f_{i}^{2}(0) y_{2 i}^{2}+\alpha^{-1} f_{i}^{2}(0) y_{1 i}^{2}\left(L_{i}+l_{i}\right)^{2}\right. \\
& \left.+4 \alpha^{-1} f_{i}^{2}(0) y_{2 i}^{2}+\alpha^{-1} f_{i}^{2}(0) y_{2 i}^{2}\left(L_{i}+l_{i}\right)^{2}\right] \tag{29}
\end{align*}
$$

By integrating both sides of (28) in time interval $t \in\left[t_{0}, t\right]$ and then taking expectation results in

$$
\begin{align*}
K e^{\alpha t}\|x(t)\|^{2} \leq & V(x(t)) \leq V\left(x\left(t_{0}\right)\right)+\alpha^{-1} e^{\alpha t} \mathcal{N}_{1} \\
& +\int_{t_{0}}^{t} 2 P e^{\alpha t} x(s) G(x(s), x(s-\tau(s))) \mathrm{d} w(s), \tag{30}
\end{align*}
$$

where $K=\lambda_{\text {min }}(P)$.
Therefore, one obtains

$$
\begin{equation*}
E\{V(x(t))\} \leq E\left\{V\left(x\left(t_{0}\right)\right)\right\}+E\left\{\alpha^{-1} e^{\alpha t} \mathscr{N}_{1}\right\}, \tag{31}
\end{equation*}
$$

which implies

$$
\begin{equation*}
E\|x(t)\|^{2} \leq \frac{e^{-\alpha t} E\left\{V\left(x\left(t_{0}\right)\right)\right\}+\alpha^{-1} \mathcal{N}_{1}}{K} . \tag{32}
\end{equation*}
$$

If one chooses $\widetilde{B}=\left(1+\alpha^{-1} \mathcal{N}_{1}\right) / K>0$, then, for initial value $\varphi \in \mathscr{C}_{\mathscr{F}_{0}}$, there is $t^{\prime}=t^{\prime}(\varphi)>0$, such that $e^{-\alpha t} E\left\{V\left(x\left(t_{0}\right)\right)\right\} \leq 1$ for all $t \geq t^{\prime}$. According to Definition 1, we have $E\|x(t, \varphi)\|^{2} \leq \widetilde{B}$ for all $t \geq t^{\prime}$. That is to say, system (2) is mean-square ultimate boundedness. This completes the proof.

Theorem 6. If all of the conditions of Theorem 5 hold, then there exists an attractor $\mathbb{A}_{\widetilde{B}}=\left\{\varphi \in \mathscr{C}_{\mathscr{F}_{0}} \mid E\|\varphi(s)\|^{2} \leq \widetilde{B}\right\}$ for the solutions of system (2).

Proof. If one chooses $\widetilde{B}=\left(1+\alpha^{-1} \mathcal{N}_{1}\right) / K>0$, Theorem 5 shows that, for any $\varphi$, there is $t^{\prime}>0$, such that $E\|x(t, \varphi)\|^{2} \leq$ $\widetilde{B}$ for all $t \geq t^{\prime}$. Let $\mathbb{A}_{\widetilde{B}}$ denote by $\mathbb{A}_{\widetilde{B}}=\{\varphi \in$ $\left.\mathscr{C}_{\mathscr{F}_{0}} \mid E\|\varphi(s)\|^{2} \leq \widetilde{B}\right\}$. Clearly, $A_{\widetilde{B}}$ is closed, bounded, and invariant. Furthermore, $\lim _{t \rightarrow \infty} \sup _{\operatorname{Sinf}}^{y \in \mathbb{A}_{\tilde{B}}} \mid\|x(t, \varphi)-y\|=0$. Therefore, $\mathbb{A}_{\tilde{B}}$ is an attractor for the solutions of system (2). This completes the proof.

Corollary 7. In addition to that all of the conditions of Theorem 5 hold, if $J=0, G(t, 0,0)=0$, and $f_{i}(0)=0$ for all $i=1,2, \ldots, n$, then system (2) has a trivial solution $x(t) \equiv 0$, and the trivial solution of system (2) is mean-square exponentially stable.

Proof. If $J=0$ and $f_{i}(0)=0(i=1,2, \ldots, n)$, then $\mathcal{N}_{1}=0$, and it is obvious that system (2) has a trivial solution $x(t) \equiv 0$. From Theorem 5, one has

$$
\begin{equation*}
E\|x(t, \varphi)\|^{2} \leq K^{*} e^{-\alpha t}, \quad \forall \varphi, \tag{33}
\end{equation*}
$$

where $K^{*}=E\left\{V\left(x\left(t_{0}\right)\right)\right\} / K$. Therefore, the trivial solution of system (2) is mean-square exponentially stable. This completes the proof.

According to Theorem 5-Corollary 7, we will present conditions of mean-square ultimate boundedness for the switched systems (3) by applying the average dwell time method in the follow-up studies.

Theorem 8. If there are constants $\mu, \nu$ such that $\dot{\tau}(t) \leq \mu$, $\dot{h}(t) \leq \nu$, we denote $g(\mu), k(\nu)$ as

$$
\begin{align*}
& g(\mu)= \begin{cases}(1-\mu) e^{-\alpha \tau}, & \mu \leq 1 \\
1-\mu, & \mu \geq 1\end{cases} \\
& k(v)= \begin{cases}(1-v) e^{-\alpha h}, & v \leq 1 \\
1-v, & v \geq 1\end{cases} \tag{34}
\end{align*}
$$

For a given constant $\alpha>0$, if there exist positive definite matrixs $Q_{i}, R_{i}, S_{i}, Z_{i}, U_{1_{i}}, U_{2_{i}}, P_{i}=\operatorname{diag}\left(p_{i 1}, p_{i 2}, \ldots, p_{i n}\right)$, $Y_{i}=\operatorname{diag}\left(Y_{i 1}, Y_{i 2}, \ldots, Y_{i n}\right), i=1,2$, such that the following condition holds

$$
\Delta_{i 1}=\left[\begin{array}{cccccc}
\Phi_{i 11} & \Phi_{i 12} & \Phi_{i 13} & \Phi_{i 14} & 0 & \Phi_{i 16}  \tag{35}\\
* & \Phi_{i 22} & 0 & \Phi_{i 24} & 0 & 0 \\
* & * & \Phi_{i 33} & 0 & 0 & 0 \\
* & * & * & \Phi_{i 44} & \Phi_{i 55} & 0 \\
* & * & * & * & 0 & \Phi_{i 66}
\end{array}\right]<0
$$

where

$$
\begin{gather*}
\Phi_{i 11}=2 \alpha P_{i}-2 D P_{i}+Q_{i}+\tau^{2} S_{i}-2 \Sigma_{1} Y_{1}+\alpha I+U_{1_{i}}^{T} P U_{1_{i}} \\
\Phi_{i 12}=U_{1_{i}}^{T} P U_{2_{i}}, \quad \Phi_{i 13}=P_{i} A_{i}+\Sigma_{2} Y_{1} \\
\Phi_{i 14}=P_{i} B_{i}, \quad \Phi_{16}=P_{i} C_{i}, \\
\Phi_{i 22}=-g(\mu) Q_{i}-2 \Sigma_{1} Y_{2}+\alpha I+U_{2_{i}}^{T} P U_{2_{i}} \\
\Phi_{i 24}=\Sigma_{2} Y_{2}, \quad \Phi_{i 33}=R_{i}+h^{2} Z_{i}-2 Y_{1}+\alpha I \\
\Phi_{i 44}=-k(\nu) R_{i}-2 Y_{2}+\alpha I, \quad \Phi_{i 55}=-g(\mu) S_{i}, \\
\Phi_{i 66}=-k(\nu) Z_{i} . \tag{36}
\end{gather*}
$$

Then system (3) is mean-square ultimate boundedness for any switching signal with average dwell time satisfying

$$
\begin{equation*}
T_{\alpha}>T_{\alpha}^{*}=\frac{\ln \mathscr{R}_{\max }}{\alpha} \tag{37}
\end{equation*}
$$

where $\mathscr{R}_{\max }=\max _{k \in \Sigma, 1 \leq i \leq n}\left\{\mathscr{R}_{i_{k}}\right\}$.
Proof. Define the Lyapunov functional candidate

$$
\begin{align*}
V_{\sigma(t)}= & e^{\alpha t} x^{T}(t) P_{\sigma(t)} x(t) \\
& +\int_{t-\tau(t)}^{t} x^{T}(s) Q_{\sigma(t)} e^{\alpha s} x(s) \mathrm{d} s \\
& +\int_{t-h(t)}^{t} F^{T}(x(s)) R_{\sigma(t)} e^{\alpha s} F(x(s)) \mathrm{d} s  \tag{38}\\
& +\tau \int_{-\tau(t)}^{0} \int_{t+\theta}^{t} x^{T}(s) S_{\sigma(t)} e^{\alpha s} x(s) \mathrm{d} s \mathrm{~d} \theta \\
& +h \int_{-h(t)}^{0} \int_{t+\theta}^{t} F^{T}(x(s)) Z_{\sigma(t)} e^{\alpha s} F(x(s)) \mathrm{d} s \mathrm{~d} \theta
\end{align*}
$$

From (16) and (32), we have the following result:

$$
\begin{equation*}
E\|x(t)\|^{2} \leq \frac{\mathscr{R}_{0} E\left\|x\left(t_{0}\right)\right\|^{2} e^{-\alpha\left(t-t_{0}\right)}}{K}+\frac{\Lambda}{K}, \tag{39}
\end{equation*}
$$

where $\Lambda=\alpha^{-1} \mathcal{N}_{1}, \mathscr{R}_{0}$ is a positive constant.
When $t \in\left[t_{k}, t_{k+1}\right]$, the $i_{k}$ th subsystem is activated; from (39) and Theorem 5, we can get

$$
\begin{align*}
E\|x(t)\|^{2} & \leq \frac{\mathscr{R}_{i_{k}} E\left\|x\left(t_{k}\right)\right\|^{2} e^{-\alpha\left(t-t_{k}\right)}}{K_{i_{k}}}+\frac{\Lambda}{K_{i_{k}}}  \tag{40}\\
& =\bar{H}_{i_{k}} E\left\|x\left(t_{k}\right)\right\|^{2} e^{-\alpha\left(t-t_{k}\right)}+\bar{J}_{i_{k}}
\end{align*}
$$

where $\mathscr{R}_{i_{k}}$ is a positive constant, $K_{i_{k}}=\lambda_{\min }\left(P_{i}\right), \bar{H}_{i_{k}}=$ $\mathscr{R}_{i_{k}} / K_{i_{k}}, \bar{J}_{i_{k}}=\Lambda / K_{i_{k}}$.

Since the system state is continuous, it follows from (40) that

$$
\begin{align*}
& E\|x(t)\|^{2} \leq \frac{\mathscr{R}_{i_{k}}\left\|x\left(t_{k}\right)\right\|^{2} e^{-\alpha\left(t-t_{k}\right)}}{K_{i_{k}}}+\frac{\Lambda}{K_{i_{k}}} \\
& =\bar{H}_{i_{k}} E\left\|x\left(t_{k}\right)\right\|^{2} e^{-\alpha\left(t-t_{k}\right)}+\bar{J}_{i_{k}} \leq \cdots \\
& \leq e^{\sum_{v=0}^{k} \ln \bar{H}_{i_{v}}-\alpha\left(t-t_{0}\right)} E\left\|x\left(t_{0}\right)\right\|^{2} \\
& +\left[\bar{H}_{i_{k}} e^{-\alpha\left(t-t_{k}\right)} \bar{J}_{i_{k}}+\bar{H}_{i_{k}} \bar{H}_{i_{k-1}} e^{-\alpha\left(t-t_{k-1}\right)} \bar{J}_{i_{k-1}}\right. \\
& +\bar{H}_{i_{k}} \bar{H}_{i_{k-1}} \bar{H}_{i_{k-2}} e^{-\alpha\left(t-t_{k-2}\right)} \bar{J}_{i_{k-2}}+\cdots \\
& \left.+\bar{H}_{i_{k}} \bar{H}_{i_{k-1}} \bar{H}_{i_{k-2}} \cdots \bar{H}_{i_{1}} e^{-\alpha\left(t-t_{1}\right)} \bar{J}_{i_{1}}+\bar{J}_{i_{k}}\right] \\
& \leq e^{(k+1) \ln \bar{H}_{\max }-\alpha\left(t-t_{0}\right)} E\left\|x\left(t_{0}\right)\right\|^{2} \\
& +\left[\bar{H}_{\max }^{k} \bar{J}_{\text {max }}+\bar{H}_{\max }^{k-1} \bar{J}_{\text {max }}+\bar{H}_{\max }^{k-2} \bar{J}_{\text {max }}\right. \\
& \left.+\cdots+\bar{H}_{\max }^{2} \bar{J}_{\max }+\bar{H}_{\max } \bar{J}_{\max }+\bar{J}_{\max }\right] \\
& \leq \bar{H}_{\max } e^{k \ln \bar{H}_{\max }-\alpha\left(t-t_{0}\right)} E\left\|x\left(t_{0}\right)\right\|^{2} \\
& +\frac{\bar{J}_{\text {max }}}{\bar{H}_{\text {max }}-1}\left[\bar{H}_{\text {max }}^{k+1}-1\right] \\
& \leq \bar{H}_{\max } e^{\ln \bar{H}_{\max } N_{\sigma}\left(t_{0}, t\right)-\alpha\left(t-t_{0}\right)} E\left\|x\left(t_{0}\right)\right\|^{2} \\
& +\frac{\bar{J}_{\text {max }}}{\bar{H}_{\text {max }}-1}\left[\bar{H}_{\text {max }}^{k+1}-1\right] \\
& \leq \frac{\mathscr{R}_{\max } e^{N_{0} \ln \mathscr{R}_{\max }-\left(\alpha-\left(\ln \mathscr{R}_{\max } / T_{\alpha}\right)\right)\left(t-t_{0}\right)}}{K_{\min }^{k+1}} E\left\|x\left(t_{0}\right)\right\|^{2} \\
& +\frac{\Lambda\left[\left(\mathscr{R}_{\max }^{n+1} / K_{\min }^{n+1}\right)-1\right]}{\mathscr{R}_{\max }-K_{\min }}, \tag{41}
\end{align*}
$$

where $K_{\min }=\min _{i_{k}}\left\{K_{\widetilde{C}_{k}}\right\}, \bar{H}_{\max }=\max _{i_{k}}\left\{\bar{H}_{i_{k}}\right\}$.
If one chooses $\widetilde{B}=\left(1 / K_{\min }\right)+\Lambda\left[\left(\mathscr{R}_{\max }^{n+1} / K_{\min }^{n+1}\right)-\right.$ $1] /\left(\mathscr{R}_{\max }-K_{\min }\right)>0$, then, for initial value $\varphi \in \mathscr{C}_{\mathscr{F}_{0}}$, there is $t^{\prime}=t^{\prime}(\varphi)>0$, such that $\mathscr{R}_{\max } e^{N_{0} \ln \mathscr{R}_{\max }-\left(\alpha-\left(\ln \mathscr{R}_{\max } / T_{\alpha}\right)\right)\left(t-t_{0}\right)} E\left\|x\left(t_{0}\right)\right\|^{2} \leq 1$ for all $t \geq t^{\prime}$. According to Definition 1, we have $E\|x(t, \varphi)\|^{2} \leq \widetilde{B}$ for all $t \geq t^{\prime}$. That is to say, system (3) is mean-square ultimate boundedness, and the proof is completed.

Remark 9. In this paper, we construct two piecewise functions $g(\mu), k(\nu)$ to remove the restrictive condition $\mu<1$ and $\nu<1$ in the results, which have reduced the conservatism of the obtained results and also avoid the computational complexity.

Remark 10. The condition (35) is given as in the form of linear matrix inequalities, which are more relaxing than the algebraic formulation. Furthermore, by using the MATLAB LMI
toolbox, we can check the feasibility of (35) straightforward without tuning any parameters.

Theorem 11. If all of the conditions of Theorem 8 hold, then there exists an attractor $\mathbb{A}_{\tilde{B}}^{\prime}$ for the solutions of system (3), where $\mathbb{A}_{\widetilde{B}}^{\prime}=\left\{\varphi \in \mathscr{C}_{\mathscr{F}_{0}} \mid E\|\varphi(s)\|^{2} \leq \widetilde{B}\right\}$.

Proof. If one chooses $\widetilde{B}=\left(1 / K_{\min }\right)+\Lambda\left[\left(\mathscr{R}_{\max }^{n+1} / K_{\min }^{n+1}\right)-\right.$ $1] /\left(\mathscr{R}_{\max }-K_{\min }\right)>0$, Theorem 8 shows that, for any $\varphi$, there is $t^{\prime}>0$, such that $E\|x(t, \varphi)\|^{2} \leq \widetilde{B}$ for all $t \geq t^{\prime}$. Let $\mathbb{A}_{\widetilde{B}}^{\prime}$ denote by $\mathbb{A}_{\widetilde{B}}^{\prime}=\left\{\varphi \in \mathscr{C}_{\mathscr{F}_{0}} \mid E\|\varphi(s)\|^{2} \leq \widetilde{B}\right\}$. Clearly, $\mathbb{A}_{\widetilde{B}}^{\prime}$ is closed, bounded, and invariant. Furthermore, $\lim _{t \rightarrow \infty} \sup _{\inf }^{y \in \mathbb{A}_{\tilde{B}}^{\prime}} 1\|x(t, \varphi)-y\|=0$. Therefore, $\mathbb{A}_{\widetilde{B}}^{\prime}$ is an attractor for the solutions of system (3). This completes the proof.

Corollary 12. In addition to all that of the conditions of Theorem 8 hold, if $J=0, G(t, 0,0)=0$ and $f_{i}(0)=0$ for all $i=1,2, \ldots, n$, then system (3) has a trivial solution $x(t) \equiv 0$, and the trivial solution of system (3) is mean-square exponentially stable.

Proof. If $J=0$ and $f_{i}(0)=0$ for all $i=1,2, \ldots, n$, then it is obvious that system (3) has a trivial solution $x(t) \equiv 0$. From Theorem 8, one has

$$
\begin{equation*}
E\|x(t, \varphi)\|^{2} \leq \widetilde{K}^{*} e^{-\alpha t}, \quad \forall \varphi, \tag{42}
\end{equation*}
$$

where $\widetilde{K}^{*}=\left(\mathscr{R}_{\max } e^{N_{0} \ln \mathscr{R}_{\max }-\left(\alpha-\left(\ln \mathscr{R}_{\max } / T_{\alpha}\right)\right)\left(t-t_{0}\right)} E\left\|x\left(t_{0}\right)\right\|^{2} /\right.$ $K_{\min }^{k+1}$. Therefore, the trivial solution of system (3) is meansquare exponentially stable. This completes the proof.

Remark 13. Assumption $\left(H_{3}\right)$ is less conservative than that in [17] since the constants $l_{j}$ and $L_{j}$ are allowed to be positive, negative, or zero. Hence, the resulting activation functions $f(\cdot)$ could be nonmonotonic and are more general than the usual forms $\left|f_{j}(u)\right| \leq K_{j}|u|, K_{j}>0, j=1,2, \ldots, n$. Moreover, unlike the bounded case, there will be no equilibrium point for the switched system (3) under the assumption $\left(H_{3}\right)$. For this reason, to investigate the asymptotic behavior (the ultimate boundedness and the existence of attractor) of switched system that contains mixed delays is more complex and challenge.

Remark 14. In this paper, the chatter bound $N_{0}$ is a positive integer, which is more practical in significance and can include the model $N_{0}=0$ in $[16,25,26]$ as a special case.

Remark 15. If $\boldsymbol{\Sigma}=0$, which implies that the switched delay system (3) reduces to the usual stochastic CNN with delays. In this case, attractor and ultimate boundedness are discussed in [17]. And when $U_{1}=U_{2}=0$, the model in our paper turns out to be a switched CNN with mixed delays; to the best of our knowledge, there are no published results in this aspect yet. Thus, the main results of this paper are novel. Moreover, when uncertainties appear in the switched stochastic CNN system (3), we can obtain the corresponding results, by applying the similar method as in [25].

## 4. Illustrative Examples

In this section, we shall give a numerical example to demonstrate the validity and effectiveness of our results. Consider the switched cellular neural networks with two subsystems.

Consider the switched stochastic cellular neural network system (3) with $f_{i}\left(x_{i}(t)\right)=0.5 \tanh \left(x_{i}(t)\right), f_{i}(0)=0(i=1,2)$, $\tau(t)=0.25 \sin ^{2}(t), h(t)=0.3 \sin ^{2}(t)$, and the connection weight matrices as follows:

$$
\begin{align*}
A_{1}=\left(\begin{array}{cc}
0.3 & 0.1 \\
0.2 & 0.2
\end{array}\right), & B_{1}=\left(\begin{array}{cc}
0.2 & 0 \\
0.3 & 0.5
\end{array}\right), \\
C_{1}=\left(\begin{array}{cc}
0.2 & -0.1 \\
0.3 & 0.1
\end{array}\right), & U_{1_{1}}=\left(\begin{array}{cc}
0.1 & 0 \\
-0.1 & 0.2
\end{array}\right), \\
U_{2_{1}}=\left(\begin{array}{cc}
0.2 & 0.1 \\
0 & 0.1
\end{array}\right), & A_{2}=\left(\begin{array}{cc}
0.2 & 0.4 \\
0.1 & 0.3
\end{array}\right),  \tag{43}\\
B_{2}=\left(\begin{array}{cc}
0.1 & 0 \\
-0.1 & 0.2
\end{array}\right), & C_{2}=\left(\begin{array}{cc}
0.3 & 0.2 \\
0.1 & 0.2
\end{array}\right), \\
U_{1_{2}}=\left(\begin{array}{cc}
0.2 & 0.1 \\
0 & 0.3
\end{array}\right), & U_{2_{2}}=\left(\begin{array}{cc}
0.1 & 0 \\
0.2 & 0.1
\end{array}\right),
\end{align*}
$$

From assumptions $\left(H_{1}\right)-\left(H_{3}\right)$, we can gain $d_{i}=1, l_{i}=$ $0, L_{i}=0.5,(i=1,2), \tau=0.25, h=0.3$, and $\mu=0.5, \nu=0.6$.

Therefore, for $\alpha=0.5$, by solving LMIs (35), we get

$$
\begin{array}{cc}
P_{1}=\left(\begin{array}{cc}
1.4968 & 0 \\
0 & 1.4851
\end{array}\right), & Q_{1}=\left(\begin{array}{cc}
1.6073 & -0.0528 \\
-0.0528 & 1.4567
\end{array}\right), \\
R_{1}=\left(\begin{array}{ll}
1.8642 & 0.4698 \\
0.4698 & 1.5241
\end{array}\right), & S_{1}=\left(\begin{array}{cc}
2.7467 & 0.0225 \\
0.0225 & 1.9941
\end{array}\right), \\
Z_{1}=\left(\begin{array}{ll}
5.4373 & 0.0644 \\
0.0644 & 4.5969
\end{array}\right), & P_{2}=\left(\begin{array}{cc}
1.4316 & 0 \\
0 & 1.4528
\end{array}\right), \\
Q_{2}=\left(\begin{array}{ll}
1.6541 & 0.0229 \\
0.0229 & 1.8391
\end{array}\right), & R_{2}=\left(\begin{array}{cc}
1.0837 & 0.4540 \\
0.4540 & 1.2710
\end{array}\right), \\
S_{2}=\left(\begin{array}{ll}
1.6888 & 0.4356 \\
0.4356 & 1.6165
\end{array}\right), & Z_{2}=\left(\begin{array}{ll}
4.5736 & 0.5698 \\
0.5698 & 4.4524
\end{array}\right) \tag{44}
\end{array}
$$

Using (37), we can get the average dwell time $T_{a}^{*}=1.3445$.

## 5. Conclusions

In this paper, we studied the switched stochastic cellular neural networks with discrete time-varying delays and distributed time-varying delays. With the help of the average dwell time approach, the novel multiple Lyapunov-Krasovkii functionals methods, and some inequality techniques, we obtain the new sufficient conditions guaranteeing the meansquare ultimate boundedness, the existence of an attractor, and the mean-square exponential stability. A numerical example is also given to demonstrate our results. Furthermore, our derived conditions are presented in the forms of LMIs, which are more relaxing than the algebraic formulation and can be easily checked in practice by the effective LMI toolbox in MATLAB.

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## Research Article

# Stochastic Delay Population Dynamics under Regime Switching: Global Solutions and Extinction 

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#### Abstract

This paper is concerned with a delay Lotka-Volterra model under regime switching diffusion in random environment. By using generalized Itô formula, Gronwall inequality and Young's inequality, some sufficient conditions for existence of global positive solutions and stochastically ultimate boundedness are obtained, respectively. Finally, an example is given to illustrate the main results.


## 1. Introduction

The delay differential equation

$$
\begin{equation*}
\frac{d x(t)}{d t}=x(t)(a-b x(t)+c x(t-\tau)) \tag{1}
\end{equation*}
$$

has been used to model the population growth of certain species and is known as the delay Lotka-Volterra model or the delay logistic equation. The delay Lotka-Volterra model for $n$ interacting species is described by the $n$-dimensional delay differential equation

$$
\begin{equation*}
\frac{d x(t)}{d t}=\operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right)(b+A x(t)+B x(t-\tau)) \tag{2}
\end{equation*}
$$

where $x=\left(x_{1}, \ldots, x_{n}\right)^{T} \in R^{n}, b=\left(b_{1}, \ldots, b_{n}\right)^{T} \in R_{+}^{n}$, $A=\left(a_{i j}\right)_{n \times n} \in R^{n \times n}$, and $B=\left(b_{i j}\right)_{n \times n} \in R^{n \times n}$. There is an extensive literature concerned with the dynamics of this delay model and have had lots of nice results. We here only mention Ahmad and Rao [1], Bereketoglu and Győri [2], Freedman and Ruan [3], and in particular, the books by Gopalsamy [4], Kolmanovskiĭ and Myshkis [5], and Kuang [6], among many others.

In the equations above, the state $x(t)$ denotes the population sizes of the species. Naturally, we focus on the positive solutions and also require the solutions not to explode at a finite time. To guarantee the positive solutions without
explosion (i.e., the global positive solutions), some conditions are in general needed to impose on the system parameters. For example, it is generally assumed that $a>0, b>0$, and $c<b$ for (1) while much more complicated conditions are required on matrices $A$ and $B$ for (2) [7] (and the references cited therein).

On the other hand, population systems are often subject to environmental noise, and the system will change significantly, which may change the dynamics of solutions significantly [8, 9]. It is therefore necessary to reveal how the noise affects the dynamics of solutions for the delay population systems. In fact, many authors have discussed population systems subject to white noise [7-18]. Recall that the parameter $b_{i}$ in (2) represents the intrinsic growth rate of species $i$. In practice we usually estimate it by an average value plus an error term. According to the well-known central limit theorem, the error term follows a normal distribution. In term of mathematics, we can therefore replace the rate $b_{i}$ by $b_{i}+\sigma_{i} \dot{w}(t)$, where $\dot{w}(t)$ is a white noise (i.e., $w(t)$ is a Brownian motion) and $\sigma_{i} \geq 0$ represents the intensity of noise. As a result, (2) becomes a stochastic differential equation (SDE, in short)

$$
\begin{align*}
d x(t)= & \operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right) \\
& \times[(b+A x(t)+B x(t-\tau)) d t+\sigma d w(t)] \tag{3}
\end{align*}
$$

where $\sigma=\left(\sigma_{1}, \ldots, \sigma_{n}\right)^{T}$. We refer to [7] for more details.

To our knowledge, much of the attention paid to environmental noise is focused on white noise. But another type of environmental noise, namely, color noise, say telegraph noise, has been studied by many authors ([19-25] and the references cited therein). In this context, telegraph noise can be described as a random switching between two or more environmental regimes, which differ in terms of factors such as nutrition or rain falls [23, 24]. Usually, the switching between different environments is memoryless and the waiting time for the next switch has an exponential distribution. This indicates that we may model the random environments and other random factors in the system by a continuoustime Markov chain $r(t), t \geq 0$ with a finite state space $S=\{1,2, \ldots, N\}$. Therefore stochastic delay population system (3) in random environments can be described by the following stochastic model with regime switching:

$$
\begin{align*}
d x(t)= & \operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right) \\
& \times[(b(r(t))+A(r(t)) x(t)+B(r(t)) x(t-\tau)) d t \\
& +\sigma(r(t)) d w(t)] . \tag{4}
\end{align*}
$$

The mechanism of ecosystem described by (4) can be explained as follows. Assume that initially, the Markov chain $r(0)=\iota \in S$. Then the ecosystem (4) obeys the SDE

$$
\begin{align*}
& d x(t) \\
& =\operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right) \\
& \quad \times[(b(\iota)+A(\iota) x(t)+B(\iota) x(t-\tau)) d t+\sigma(\iota) d w(t)] \tag{5}
\end{align*}
$$

until the Markov chain $r(t)$ jumps to another state, say, $\varsigma$. Therefore, the ecosystem (4) satisfies the SDE

$$
\begin{align*}
& d x(t) \\
& =\operatorname{diag}\left(x_{1}(t), \ldots, x_{n}(t)\right) \\
& \quad \times[(b(\varsigma)+A(\varsigma) x(t)+B(\varsigma) x(t-\tau)) d t+\sigma(\varsigma) d w(t)], \tag{6}
\end{align*}
$$

for a random amount of time until the Markov chain $r(t)$ jumps to a new state again.

It should be pointed out that the stochastic population systems under regime switching have received much attention lately. For instance, the stochastic permanence and extinction of a logistic model under regime switching were considered in $[20,24]$, asymptotic results of a competitive Lotka-Volterra model in random environment are obtain in [25], a new single-species model disturbed by both white noise and colored noise in a polluted environment was developed and analyzed in [26], and a general stochastic logistic system under regime switching was proposed and was treated in [27].

Equation (4) describes the dynamics of populations. This paper is concerned with the positive global solutions, ultimate boundedness and extinction. The stochastic permanence and
asymptotic estimations of solutions will be investigated in the next note [28].

This paper is organized as follows. In the next section, some sufficient conditions for global positive solutions for any initial positive value are given by using generalized Itô formula, Gronwall inequality, and $V$-function techniques. In Section 3, the stochastically ultimate boundedness of solutions is obtained by virtue of Young's inequality. Section 4 is devoted to the extinction of solutions. Finally, an example and its numerical simulation are given to illustrate our main results.

## 2. Global Positive Solution

Throughout this paper, unless otherwise specified, let $(\Omega, \mathscr{F}$, $\left.\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, P\right)$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_{0}$ contains all $P$-null sets). Let $w(t), t \geq 0$, be a scalar standard Brownian motion defined on this probability space. We also denote by $R_{+}^{n}$ the positive cone in $R^{n}$, that is $R_{+}^{n}=\left\{x \in R^{n}: x_{i}>0\right.$ for all $\left.1 \leq i \leq n\right\}$, and denote by $\bar{R}_{+}^{n}$ the nonnegative cone in $R^{n}$, that is $\bar{R}_{+}^{n}=\left\{x \in R^{n}: x_{i} \geq\right.$ 0 for all $1 \leq i \leq n\}$. If $A$ is a vector or matrix, its transpose is denoted by $A^{T}$. If $A$ is a matrix, its trace norm is denoted by $|A|=\sqrt{\operatorname{trace}\left(A^{T} A\right)}$, and its operator norm is denoted by $\|A\|=\sup \{|A x|:|x|=1\}$. Moreover, let $\tau>0$ and denote by $C\left([-\tau, 0] ; R_{+}^{n}\right)$ the family of continuous functions from $[-\tau, 0]$ to $R_{+}^{n}$.

In this paper we will use a lot of quadratic functions of the form $x^{T} A x$ for the state $x \in R_{+}^{n}$ only. Therefore, for a symmetric $n \times n$ matrix $A$, we naturally introduce the following definition

$$
\begin{equation*}
\lambda_{\max }^{+}(A)=\sup _{x \in R_{+}^{n}|x|=1} x^{T} A x \tag{7}
\end{equation*}
$$

For more properties of $\lambda_{\max }^{+}(A)$, refer to the appendix in [7].
Let $r(t)$ be a right-continuous Markov chain on the probability space, taking values in a finite state space $S=$ $\{1,2, \ldots, N\}$, with the generator $\Gamma=\left(\gamma_{u v}\right)$ given by

$$
P\{r(t+\delta)=v \mid r(t)=u\}= \begin{cases}\gamma_{u v} \delta+o(\delta), & \text { if } u \neq v  \tag{8}\\ 1+\gamma_{u v} \delta+o(\delta), & \text { if } u=v\end{cases}
$$

where $\delta>0, \gamma_{u v}$ is the transition rate from $u$ to $v$, and $\gamma_{u v} \geq 0$ if $u \neq v$, while $\gamma_{u u}=-\sum_{v \neq u} \gamma_{u v}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a rightcontinuous step function with a finite number of jumps in any finite subinterval of $\bar{R}_{+}$. As a standing hypothesis we assume in this paper that the Markov chain $r(t)$ is irreducible. This is a very reasonable assumption as it means that the system can switch from any regime to any other regime. This is equivalent to the condition that for any $u, v \in S$, one can find finite numbers $i_{1}, i_{2}, \ldots, i_{k} \in S$ such that $\gamma_{u i_{1}} \gamma_{i_{1} i_{2}} \cdots \gamma_{i_{k} v}>0$. Under this condition, the Markov chain has a unique stationary
(probability) distribution $\pi=\left(\pi_{1}, \pi_{2}, \ldots, \pi_{N}\right) \in R^{1 \times N}$ which can be determined by solving the following linear equation:

$$
\begin{equation*}
\pi \Gamma=0 \tag{9}
\end{equation*}
$$

subject to

$$
\begin{equation*}
\sum_{i=1}^{N} \pi_{i}=1, \quad \pi_{i}>0, \forall i \in S \tag{10}
\end{equation*}
$$

We refer to $[12,29]$ for the fundamental theory of stochastic differential equations.

For convenience and simplicity in the following discussion, for any constant sequence $f_{i}(k),(1 \leq i \leq n, k \in S)$ let

$$
\begin{array}{ll}
\check{f}=\max _{1 \leq i \leq n, k \in S} f_{i}(k), & \check{f}(k)=\max _{1 \leq i \leq n} f_{i}(k),  \tag{11}\\
\widehat{f}=\min _{1 \leq i \leq n, k \in S} f_{i}(k), & \widehat{f}(k)=\min _{1 \leq i \leq n} f_{i}(k)
\end{array}
$$

As $x(t)$ in system (4) denotes populations size at time $t$, it should be nonnegative. Thus for further study, we must give some condition under which (4) has a unique global positive solution.

Theorem 1. Assume that there are positive numbers $c_{1}, \ldots, c_{n}$ and $\theta$ such that

$$
\begin{align*}
\max _{k \in S}\left\{\lambda_{\max }^{+}[ \right. & \frac{1}{2} \bar{C}\left(A(k)+A^{T}(k)\right) \bar{C}  \tag{12}\\
& \left.\left.+\frac{1}{4 \theta} \bar{C} B(k) B^{T}(k) \bar{C}+\theta I\right]\right\} \leq 0
\end{align*}
$$

where $\bar{C}=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$. Then for any given initial data $\{x(t):-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}^{n}\right)$, there is a unique solution $x(t)$ to (4) on $t \geq-\tau$ and the solution will remain in $R_{+}^{n}$ with probability 1, namely, $x(t) \in R_{+}^{n}$ for all $t \geq-\tau$ a.s.

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial data $\{x(t):-\tau \leq$ $t \leq 0\} \in C\left([-\tau, 0] ; R_{+}^{n}\right)$, there is a unique maximal local solution $x(t)$ on $t \in\left[-\tau, \tau_{e}\right)$, where $\tau_{e}$ is the explosion time. To show that the solution is global, we need to show that $\tau_{e}=\infty$ a.s.

Let $k_{0}>0$ be sufficiently lager such that

$$
\begin{equation*}
\frac{1}{k_{0}} \leq \min _{-\tau \leq t \leq 0}|x(t)| \leq \max _{-\tau \leq t \leq 0}|x(t)| \leq k_{0} \tag{13}
\end{equation*}
$$

For each integer $k \geq k_{0}$, define the stopping time

$$
\begin{array}{r}
\tau_{k}=\inf \left\{t \in\left[0, \tau_{e}\right): x_{i}(t) \notin\left(\frac{1}{k}, k\right)\right.  \tag{14}\\
\text { for some } i=1,2, \ldots, n\}
\end{array}
$$

where throughout this paper we set $\inf \emptyset=\infty$ (as usual $\emptyset$ denotes the empty set). Clearly, $\tau_{k}$ is increasing as $k \rightarrow \infty$. Set $\tau_{\infty}=\lim _{k \rightarrow \infty} \tau_{k}$, where $\tau_{\infty} \leq \tau_{e}$ a.s. If $\tau_{\infty}=\infty$ a.s., then $\tau_{e}=\infty$ a.s. and $x(t) \in R_{+}^{n}$ a.s. for all $t \geq 0$. In other words, to
complete the proof, one should show that $\tau_{\infty}=\infty$ a.s. Define $V: R_{+}^{n} \rightarrow R_{+}$by

$$
\begin{equation*}
V(x)=\sum_{i=1}^{n} c_{i}\left(x_{i}-1-\log x_{i}\right) \tag{15}
\end{equation*}
$$

The nonnegativity of this function can be seen from $u-1-$ $\log u \geq 0$ on $u>0$. Let $k \geq k_{0}$ and $T>0$ be arbitrary. For $0 \leq t \leq \tau_{k} \wedge T$, it is easy to see by the generalized Itô formula that

$$
\begin{align*}
E V\left(x\left(\tau_{k} \wedge t\right)\right)= & V(x(0)) \\
& +E \int_{0}^{\tau_{k} \wedge t} L V(x(s), x(s-\tau), r(s)) d s \tag{16}
\end{align*}
$$

where $L V: R_{+}^{n} \times R_{+}^{n} \times S \rightarrow R$ is defined by

$$
\begin{align*}
L V(x, y, k)= & x^{T} \bar{C} b(k)+x^{T} \bar{C} A(k) x+x^{T} \bar{C} B(k) y \\
& -c^{T}(b(k)+A(k) x+B(k) y)  \tag{17}\\
& +\frac{1}{2} \sigma^{T}(k) \bar{C} \sigma(k)
\end{align*}
$$

and $c=\left(c_{1}, \ldots, c_{n}\right)^{T}$. Using condition (12) we compute

$$
\begin{align*}
& x^{T} \bar{C} A(k) x+x^{T} \bar{C} B(k) y \\
& \leq \frac{1}{2} x^{T}(\bar{C} A(k)+A(k) \bar{C}) x \\
&+\frac{1}{4 \theta} x^{T} \bar{C} B(k) B^{T}(k) \bar{C} x+\theta|y|^{2}  \tag{18}\\
&= x^{T}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right. \\
& \quad\left.+\frac{1}{4 \theta} \bar{C} B(k) B^{T}(k) \bar{C}+\theta I\right] x-\theta|x|^{2}+\theta|y|^{2} \\
& \leq-\theta|x|^{2}+\theta|y|^{2} .
\end{align*}
$$

Moreover, there is a constant $K_{1}>0$ such that

$$
\begin{align*}
\max _{k \in S}( & x^{T} \bar{C} b(k)+c^{T} A(k) x+c^{T} B(k) y-c^{T} b(k) \\
& \left.+\frac{1}{2} \sigma^{T}(k) \bar{C} \sigma(k)\right)  \tag{19}\\
& \leq K_{1}(1+|x|+|y|) .
\end{align*}
$$

Substituting these inequalities into (17) yields

$$
\begin{equation*}
L V(x, y, i) \leq K_{1}(1+|x|+|y|)-\theta|x|^{2}+\theta|y|^{2} \tag{20}
\end{equation*}
$$

Noticing that $u \leq 2(u-1-\log u)+2$ on $u>0$, we compute

$$
\begin{align*}
|x| & \leq \sum_{i=1}^{n} x_{i} \leq \sum_{i=1}^{n}\left[2\left(x_{i}-1-\log x_{i}\right)+2\right] \\
& \leq 2 n+\frac{2}{\hat{c}} \sum_{i=1}^{n} c_{i}\left(x_{i}-1-\log x_{i}\right)  \tag{21}\\
& =2 n+\frac{2}{\hat{c}} V(x) .
\end{align*}
$$

It follows from (20) and (21) that

$$
\begin{equation*}
L V(x, y, k) \leq K_{2}(1+V(x)+V(y))-\theta|x|^{2}+\theta|y|^{2} \tag{22}
\end{equation*}
$$

where $K_{2}$ is a positive constant. Substituting this inequality into (16) yields

$$
\begin{align*}
& E V\left(x\left(\tau_{k} \wedge t\right)\right) \\
& \leq V(x(0))+K_{2} E \int_{0}^{\tau_{k} \wedge t}[1+V(x(s))+V(x(s-\tau))] d s \\
& \quad+E \int_{0}^{\tau_{k} \wedge t}\left[-\theta x^{2}(s)+\theta x^{2}(s-\tau)\right] d s \tag{23}
\end{align*}
$$

Compute

$$
\begin{align*}
& E \int_{0}^{\tau_{k} \wedge t} V(x(s-\tau)) d s \\
& \quad=E \int_{-\tau}^{\tau_{k} \wedge(t-\tau)} V(x(s)) d s  \tag{24}\\
& \quad \leq \int_{-\tau}^{0} V(x(s)) d s+E \int_{0}^{\tau_{k} \wedge t} V(x(s)) d s
\end{align*}
$$

and, similarly

$$
\begin{equation*}
E \int_{0}^{\tau_{k} \wedge t}|x(s-\tau)|^{2} d s \leq \int_{-\tau}^{0}|x(s)|^{2} d s+E \int_{0}^{\tau_{k} \wedge t}|x(s)|^{2} d s \tag{25}
\end{equation*}
$$

Substituting these inequalities into (23) gives

$$
\begin{align*}
E V\left(x\left(\tau_{k} \wedge t\right)\right) & \leq K_{3}+2 K_{2} E \int_{0}^{\tau_{k} \wedge t} V(x(s)) d s \\
& \leq K_{3}+2 K_{2} E \int_{0}^{t} V\left(x\left(\tau_{k} \wedge s\right)\right) d s  \tag{26}\\
& \leq K_{3}+2 K_{2} \int_{0}^{t} E V\left(x\left(\tau_{k} \wedge s\right)\right) d s
\end{align*}
$$

where $K_{3}=V(x(0))+K_{2} T+K_{2} \int_{-\tau}^{0} V(x(s)) d s+$ $\theta \int_{-\tau}^{0}|x(s)|^{2} d s$.

By the Gronwall inequality, we obtain that

$$
\begin{equation*}
E V\left(x\left(\tau_{k} \wedge T\right)\right) \leq K_{3} e^{2 T K_{2}} \tag{27}
\end{equation*}
$$

Noting that for every $\omega \in\left\{\tau_{k} \leq T\right\}$,

$$
\begin{equation*}
V\left(x\left(\tau_{k}, \omega\right)\right) \geq \widehat{c}[(k-1-\log k) \wedge(1 / k-1+\log k)] \tag{28}
\end{equation*}
$$

one has by (27) that

$$
\begin{aligned}
K_{3} e^{2 T K_{2}} \geq & E V\left(x\left(\tau_{k} \wedge T\right)\right) \\
& \geq E\left[1_{\left\{\tau_{k} \leq T\right\}}(\omega) V\left(x\left(\tau_{k} \wedge T, \omega\right)\right)\right] \\
= & E\left[\mathbf{1}_{\left\{\tau_{k} \leq T\right\}}(\omega) V\left(x\left(\tau_{k}, \omega\right)\right)\right] \\
\geq & \widehat{c} P\left\{\tau_{k} \leq T\right\} \\
& \times[(k-1-\log k) \wedge(1 / k-1+\log k)]
\end{aligned}
$$

where $\mathbf{1}_{\left\{\tau_{k} \leq T\right\}}$ is the indicator function of $\left\{\tau_{k} \leq T\right\}$. Letting $k \rightarrow \infty$ gives $\lim _{k \rightarrow \infty} P\left\{\tau_{k} \leq T\right\}=0$ and hence $P\left\{\tau_{\infty} \leq T\right\}=$ 0 . Since $T>0$ is arbitrary, we must have $P\left\{\tau_{\infty}<\infty\right\}=0$, so $P\left\{\tau_{\infty}=\infty\right\}=1$ as required.

Assumption 2. Assume that there exist positive numbers $c_{1}, \ldots, c_{n}$ such that

$$
\begin{equation*}
\max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right]\right\}+\max _{k \in S}\|\bar{C} B(k)\| \leq 0 \tag{30}
\end{equation*}
$$

where $\bar{C}=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$.
The following theorem is easy to verify in applications, which will be used in the sections below.

Theorem 3. Under Assumption 2, for any given initial data $\{x(t):-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}^{n}\right)$, there is a unique solution $x(t)$ to (4) on $t \geq-\tau$ and the solution will remain in $R_{+}^{n}$ with probability 1 , namely, $x(t) \in R_{+}^{n}$ for all $t \geq-\tau$ a.s.

Proof. Define $V: R_{+}^{n} \rightarrow R_{+}$by $V(x)=\sum_{i=1}^{n} c_{i}\left(x_{i}-1-\log x_{i}\right)$. The non-negativity of this function can be seen from $u-1-$ $\log u \geq 0$ on $u>0$, and then we have (16) and (17).

If $B(k) \neq 0, k \in S$, then $\|\bar{C} B(k)\| \neq 0$. Consequently

$$
\begin{align*}
& x^{T} \bar{C} A(k) x+x^{T} \bar{C} B(k) y \\
& \leq \frac{1}{2} x^{T}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right) x \\
&+\frac{1}{2\|\bar{C} B(k)\|} x^{T} \bar{C} B(k) B^{T}(k) \bar{C} x \\
&+\frac{1}{2}\|\bar{C} B(k)\||y|^{2}  \tag{31}\\
&= \frac{1}{2} x^{T}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right) x \\
&+\frac{1}{2}\|\bar{C} B(k)\||x|^{2}+\frac{1}{2}\|\bar{C} B(k)\||y|^{2}
\end{align*}
$$

Otherwise $\|\bar{C} B(k)\|=0$ for $B(k)=0, k \in S$. In this case, we also have that

$$
\begin{align*}
x^{T} & \bar{C} A(k) x+x^{T} \bar{C} B(k) y \\
\leq & \frac{1}{2} x^{T}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right) x+\frac{1}{2}\|\bar{C} B(k)\||x|^{2}  \tag{32}\\
& +\frac{1}{2}\|\bar{C} B(k)\||y|^{2} .
\end{align*}
$$

Thus,

$$
\begin{align*}
& x^{T} \bar{C} A(k) x+x^{T} \bar{C} B(k) y \\
& \leq \frac{1}{2} x^{T}(\bar{C} A(k)+A(k) \bar{C}) x+\frac{1}{2}\|\bar{C} B(k)\||x|^{2}  \tag{33}\\
&+\frac{1}{2}\|\bar{C} B(k)\||y|^{2} .
\end{align*}
$$

Denote $\eta=\max _{k \in S}\|\bar{C} B(k)\|$. By (33) and Assumption 2, one has

$$
\begin{align*}
& x^{T} \bar{C} A(k) x+x^{T} \bar{C} B(k) y \\
& \leq \frac{1}{2} x^{T}(\bar{C} A(k)+A(k) \bar{C}) x+\frac{1}{2} \eta|x|^{2}+\frac{1}{2} \eta|y|^{2} \\
& \leq \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right]\right\}|x|^{2}  \tag{34}\\
&+\frac{1}{2} \eta|x|^{2}+\frac{1}{2} \eta|y|^{2} \\
& \leq-\frac{1}{2} \eta|x|^{2}+\frac{1}{2} \eta|y|^{2} .
\end{align*}
$$

The rest of the proof is similar to that of Theorem 1 and omitted.

## 3. Ultimate Boundness

Theorem 3 shows that solutions of the SDE (4) will remain in the positive cone $R_{+}^{n}$. This nice property provides us with a great opportunity to discuss how solutions vary in $R_{+}^{n}$ in detail. In this section, we give the definitions of stochastically ultimate boundedness of the SDE (4) and some sufficient conditions under which solutions of SDE (4) are stochastically ultimate bounded.

Definition 4. The solutions of (4) are called stochastically ultimately bounded, if for any $\varepsilon \in(0,1)$, there exists a positive constant $H=H(\varepsilon)$, such that the solutions of (4) with any positive initial value have the property that

$$
\begin{equation*}
\limsup _{t \rightarrow+\infty} P\{|x(t)|>H\}<\varepsilon . \tag{35}
\end{equation*}
$$

Assumption 5. Assume that there exist positive numbers $c_{1}, \ldots, c_{n}$ such that

$$
\begin{align*}
-\lambda= & \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right]\right\}  \tag{36}\\
& +\max _{k \in S}\|\bar{C} B(k)\|<0,
\end{align*}
$$

where $\bar{C}=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$.
Theorem 6. Under Assumption 5, for any given initial data $\{x(t):-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}^{n}\right)$ and any given positive constant $p$, there are two positive constants $K_{1}(p)$ and $K_{2}(p)$, such that the solution $x(t)$ of (4) has the properties that

$$
\begin{gather*}
\limsup _{t \rightarrow \infty} E|x(t)|^{p} \leq K_{1}(p)  \tag{37}\\
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} E|x(s)|^{p+1} d s \leq K_{2}(p) . \tag{38}
\end{gather*}
$$

Proof. By Theorem 3, the solution $x(t)$ will remain in $R_{+}^{n}$ for all $t \geq-\tau$ with probability 1 . If $\max _{k \in S}\|\bar{C} B(k)\|>0$, we let $\eta=(p+1)^{-1} \max _{k \in S}\|\bar{C} B(k)\|$ and $\gamma=\tau^{-1} \log ((\lambda+2 \eta) / 2 \eta)>$

0 . Define $V(x, t)=e^{\gamma t}\left(\sum_{i=1}^{n} c_{i} x_{i}\right)^{p}=e^{\gamma t}\left(c^{T} x\right)^{p}$. It has by the generalized Itô formula that

$$
\begin{align*}
d V(x(t), t)= & L V(x(t), x(t-\tau), t, r(t)) d t \\
& +p e^{\gamma t}\left(c^{T} x(t)\right)^{p-1} x^{T}(t) \bar{C} \sigma(r(t)) d w(t), \tag{39}
\end{align*}
$$

where $L V: R_{+}^{n} \times R_{+}^{n} \times R_{+} \times S \rightarrow R$ is defined by

$$
\begin{align*}
& \operatorname{LV}(x, y, t, k) \\
& \begin{aligned}
=e^{\gamma t}\{ & \gamma\left(c^{T} x\right)^{p}+p\left(c^{T} x\right)^{p-1} x^{T} \bar{C}(b(k)+A(k) x+B(k) y) \\
& \left.+\frac{1}{2} p(p-1)\left(c^{T} x\right)^{p-2}\left(x^{T} \bar{C} \sigma(k)\right)^{2}\right\} .
\end{aligned}
\end{align*}
$$

Meanwhile, by Assumption 5 and Young's inequality, one gets

$$
\begin{align*}
& L V(x, y, t, k) \\
& \leq e^{\gamma t}\left[\gamma|c|^{p}|x|^{p}+p|c|^{p}|b(k)||x|^{p}\right. \\
& +\frac{1}{2} p(p-1)|c|^{p}|\sigma(k)|^{2}|x|^{p} \\
& \left.+p\left(c^{T} x\right)^{p-1} x^{T} \bar{C}(A(k) x+B(k) y)\right] \\
& \leq e^{\gamma t}\left\{K(p)|x|^{p}+\frac{1}{2} p\left(c^{T} x\right)^{p-1} x\right. \\
& \times\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right) x^{T} \\
& \left.+p\left(c^{T} x\right)^{p-1}\|\bar{C} B(k)\||x||y|\right\} \\
& \leq e^{\gamma t} K(p)|x|^{p}+e^{\gamma t} p|c|^{p-1} \\
& \times \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right]\right\}|x|^{p+1} \\
& +e^{\gamma t} p|c|^{p-1}\|\bar{C} B(k)\||x|^{p}|y| \\
& \leq e^{\gamma t} K(p)|x|^{p}+e^{\gamma t} p|c|^{p-1} \\
& \times \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right]\right\}|x|^{p+1} \\
& +e^{\gamma t} p|c|^{p-1}\|\bar{C} B(k)\|\left(\frac{p}{p+1}|x|^{p+1}+\frac{1}{p+1}|y|^{p+1}\right) \\
& \leq e^{\gamma t}\left\{K(p)|x|^{p}+p|c|^{p-1}\left[-(\lambda+\eta)|x|^{p+1}+\eta|y|^{p+1}\right]\right\} \\
& \leq e^{\gamma t}\left\{K(p)|x|^{p}-\frac{1}{2} p \lambda|c|^{p-1}|x|^{p+1}+p|c|^{p-1}\right. \\
& \left.\times\left[-\left(\frac{1}{2} \lambda+\eta\right)|x|^{p+1}+\eta|y|^{p+1}\right]\right\} \\
& \leq H(p) e^{\gamma t}+p \eta|c|^{p-1} e^{\gamma t}\left(-e^{\gamma \tau}|x|^{p+1}+|y|^{p+1}\right) \text {, } \tag{41}
\end{align*}
$$

where

$$
\begin{gather*}
K(p)=\max _{k \in S}\left[\gamma|c|^{p}+p|c|^{p}|b(k)|+\frac{1}{2} p(p-1)|c|^{p}|\sigma(k)|^{2}\right], \\
H(p)=\sup _{x \in R_{+}}\left(K(p)|x|^{p}-\frac{1}{2} p \lambda|c|^{p-1}|x|^{p+1}\right) \vee 1 . \tag{42}
\end{gather*}
$$

On the other hand,

$$
\begin{align*}
& \int_{0}^{t} e^{\gamma s}|x(s-\tau)|^{p+1} d s \\
& \quad=e^{\gamma \tau} \int_{0}^{t} e^{\gamma(s-\tau)}|x(s-\tau)|^{p+1} d s  \tag{43}\\
& \quad=e^{\gamma \tau} \int_{-\tau}^{t-\tau} e^{\gamma s}|x(s)|^{p+1} d s \\
& \quad \leq e^{\gamma \tau} \int_{-\tau}^{0}|x(s)|^{p+1} d s+e^{\gamma \tau} \int_{0}^{t} e^{\gamma s}|x(s)|^{p+1} d s
\end{align*}
$$

by (41) and (43), we obtain that

$$
\begin{align*}
& e^{\gamma t} E[V(x(t))] \\
& \leq V(x(0))+\int_{0}^{t} H(p) e^{\gamma s} d s+p|c|^{p-1} \eta \\
& \quad \times \int_{0}^{t} e^{\gamma s}\left(-e^{\gamma \tau}|x(s)|^{p+1}+|x(s-\tau)|^{p+1}\right) d s \\
& \leq V(x(0))+\frac{H(p)}{\gamma}\left(e^{\gamma t}-1\right)+p|c|^{p-1} \eta e^{\gamma \tau} \int_{-\tau}^{0}|x(s)|^{p+1} d s \tag{44}
\end{align*}
$$

which yields

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E V(x(t)) \leq \frac{H(p)}{\gamma} \tag{45}
\end{equation*}
$$

Since $|x(t)| \leq \sum_{i=1}^{n} x_{i}(t) \leq V(x(t)) / \hat{c}$, it has $\lim \sup _{t \rightarrow \infty}$ $E|x(t)|^{p} \leq H(p) / \widehat{c} \gamma$ and the desired assertion (37) follows by setting $K_{1}(p)=H(p) / \widehat{c} \gamma$. It is easy to verify this result, if $\max _{k \in S}\|\bar{C} B(k)\|=0$. We omit its proof here.

Define $\bar{V}(x)=\left(c^{T} x\right)^{p}$. By the generalized Itô formula, it follows

$$
\begin{align*}
d \bar{V}(x(t))= & L \bar{V}(x(t), x(t-\tau), r(t)) d t \\
& +p\left(c^{T} x(t)\right)^{p-1} x^{T}(t) \bar{C} \sigma(r(t)) d w(t) \tag{46}
\end{align*}
$$

where $L \bar{V}: R_{+}^{n} \times R_{+}^{n} \times S \rightarrow R$ is defined by

$$
\begin{align*}
L \bar{V}(x, y, k)= & p\left(c^{T} x\right)^{p-1} x^{T} \bar{C}[b(k)+A(k) x+B(k) y] \\
& +\frac{1}{2} p(p-1)\left(c^{T} x\right)^{p-2}\left(x^{T} \bar{C} \sigma(k)\right)^{2} \tag{47}
\end{align*}
$$

By Assumption 5 and Young's inequality again,

$$
\begin{align*}
& L \bar{V}(x, y, k) \\
& \leq p|c|^{p}|b(k)||x|^{p}+\frac{1}{2} p(p-1)|c|^{p}|\sigma(k)|^{2}|x|^{p} \\
&+p\left(c^{T} x\right)^{p-1} x^{T} \bar{C}(A(k) x+B(k)) y  \tag{48}\\
& \leq p|c|^{p}|b(k)||x|^{p}+\frac{1}{2} p(p-1)|c|^{p}|\sigma(k)|^{2}|x|^{p} \\
&+p|c|^{p-1}\left[-(\lambda+\eta)|x|^{p+1}+\eta|y|^{p+1}\right] .
\end{align*}
$$

It is easy to compute

$$
\begin{align*}
0 \leq E \bar{V}(x & (0)) \\
& +E \int_{0}^{t}
\end{aligned} \begin{aligned}
& {\left[p \check{b}|c|^{p}|x(s)|^{p}\right.} \\
& +\frac{1}{2} p|p-1||c|^{p} \check{\sigma}^{2}|x(s)|^{p}  \tag{49}\\
& \quad-p|c|^{p-1}(\lambda+\eta)|x(s)|^{p+1} \\
& \left.+p|c|^{p-1} \eta|x(s-\tau)|^{p+1}\right] d s .
\end{align*}
$$

Moreover,

$$
\begin{equation*}
\int_{0}^{t}|x(s-\tau)|^{p+1} d s \leq \int_{-\tau}^{0}|x(s)|^{p+1} d s+\int_{0}^{t}|x(s)|^{p+1} d s \tag{50}
\end{equation*}
$$

hence, we get

$$
\begin{align*}
& \begin{aligned}
& \frac{1}{2} \lambda p|c|^{p-1} E \int_{0}^{t}|x(s)|^{p+1} d s \\
& \leq E \bar{V}(x(0)) \\
&+E \int_{0}^{t} \\
& {\left[p|c|^{p \check{b}|x(s)|^{p}}\right.} \\
&+\frac{1}{2} p|p-1||c|^{p} \check{\sigma}^{2}|x(s)|^{p} \\
& \quad-p\left(\frac{\lambda}{2}+\eta\right)|c|^{p-1}|x(s)|^{p+1} \\
&\left.\quad+p \eta|c|^{p-1}|x(s-\tau)|^{p+1}\right] d s \\
& \leq E \bar{V}(x(0))+p \eta|c|^{p-1} \int_{-\tau}^{0}|x(s)|^{p+1} d s \\
& \quad E \int_{0}^{t}\left(p|c|^{p \check{b}|x(s)|^{p}+\frac{1}{2} p|p-1||c|^{p} \check{\sigma}^{2}|x(s)|^{p}}\right. \\
&\left.\quad-\frac{1}{2} p \lambda|c|^{p-1}|x(s)|^{p+1}\right) d s \\
& \leq E \bar{V}(x(0))+p \eta|c|^{p-1} \\
& \quad \times \int_{-\tau}^{0}|x(s)|^{p+1} d s+H(p) t
\end{aligned}
\end{align*}
$$

where $H(p)=\sup _{x \in R_{+}}\left(p|c|^{p} \check{b}|x|^{p}+(1 / 2) p|p-1||c|^{p} \check{\sigma}^{2}\right.$ $\left.|x(s)|^{p}-(1 / 2) \lambda p|c|^{p-1}|x|^{p+1}\right)$. This implies immediately that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} E \int_{0}^{t}|x(s)|^{p+1} d s \leq \frac{2 H(p)}{p \lambda|c|^{p-1}} \tag{52}
\end{equation*}
$$

and the desired assertion (38) follows by setting $K_{2}(p)=$ $2 H(p) / p \lambda|c|^{p-1}$.

Remark 7. From (37) of Theorem 6, there is a $T>0$ such that

$$
\begin{equation*}
E|x(t)|^{p} \leq 2 K_{1}(p), \quad \forall t \geq T \tag{53}
\end{equation*}
$$

Since $E|x(t)|^{p}$ is continuous, there is a $\bar{K}_{1}\left(p, x_{0}\right)$ such that

$$
\begin{equation*}
E|x(t)|^{p} \leq \bar{K}_{1}\left(p, x_{0}\right) \quad \text { for } t \in[0, T] . \tag{54}
\end{equation*}
$$

Let $L\left(p, x_{0}\right)=\max \left(2 K_{1}(p), \bar{K}_{1}\left(p, x_{0}\right)\right)$, we have

$$
\begin{equation*}
E|x(t)|^{p} \leq L\left(p, x_{0}\right), \quad \forall t \in[0, \infty) \tag{55}
\end{equation*}
$$

which implies that the $p$ th moment of any positive solution of (4) is bounded.

Remark 8. Conclusion (38) of Theorem 6 shows that the average in time of the $p$ th $(p>1)$ moment of solutions of (4) will be bounded.

Theorem 9. Solutions of (4) are stochastically ultimately bounded under Assumption 5.

Proof. This can be easily verified by Chebyshev's inequality and Theorem 6.

## 4. Extinction

Assumption 10. Assume that there exist positive numbers $c_{1}, \ldots, c_{n}$ such that

$$
\begin{align*}
& |c|^{-1} \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right]\right\} \\
& \quad+\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\| \leq 0 \tag{56}
\end{align*}
$$

where $\bar{C}=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$ and $\widehat{c}=\min _{1 \leq i \leq n} c_{i}$.
Theorem 11. Under Assumption 10, for any given initial data $\{x(t):-\tau \leq t \leq 0\} \in C\left([-\tau, 0] ; R_{+}^{n}\right)$, the solution $x(t)$ of (4) has the properties that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log |x(t)| \leq \sum_{i=1}^{n} \pi_{k} \beta(k) \quad \text { a.s. } \tag{57}
\end{equation*}
$$

where $\beta(k)=\check{b}(k)-(1 / 2) \widehat{\sigma}^{2}(k)$. Particularly, if $\sum_{k=1}^{N} \pi_{k} \beta(k)<$ 0 , then

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log |x(t)|<0 \quad \text { a.s. } \tag{58}
\end{equation*}
$$

That is, the population will become extinct exponentially with probability 1.

Proof. By Theorem 3, the solution $x(t)$ will remain in $R_{+}^{n}$ for all $t \geq-\tau$ with probability 1 . Define

$$
\begin{equation*}
V(x)=c^{T} x=\sum_{i=1}^{n} c_{i} x_{i} \quad \text { on } x \in R_{+}^{n}, \tag{59}
\end{equation*}
$$

where $c=\left(c_{1}, \ldots, c_{n}\right)^{T}$. Then

$$
\begin{align*}
& d V(x(t))=x^{T}(t) \bar{C}[(b(r(t)+A(r(t)) x(t) \\
&+B(r(t)) x(t-\tau))) d t  \tag{60}\\
&+\sigma(r(t)) d w(t)]
\end{align*}
$$

By the generalized Itô formula,

$$
\begin{align*}
d & \log V(x(t)) \\
= & \frac{1}{V(x(t))} d V(x(t))-\frac{1}{2 V^{2}(x(t))}(d V(x(t)))^{2} \\
= & \frac{1}{V(x(t))} x^{T}(t) \bar{C}  \tag{61}\\
& \times[(b(r(t)+A(r(t)) x(t)+B(r(t)) x(t-\tau))) d t \\
& \quad+\sigma(r(t)) d w(t)] \\
& -\frac{1}{2 V^{2}(x(t))}\left|x^{T}(t) \bar{C} \sigma(r(t))\right|^{2} d t .
\end{align*}
$$

It is computed

$$
\begin{align*}
& \frac{x^{T}(t) \bar{C} A(r(t)) x(t)}{V(x(t))}+\frac{x^{T}(t) \bar{C} B(r(t)) x(t-\tau)}{V(x(t))} \\
& \leq \frac{x^{T}(t)\left(\bar{C} A(r(t))+A^{T}(r(t)) \bar{C}\right) x(t)}{2 V(x(t))} \\
& \quad+\frac{\|\bar{C} B(r(t))\||x(t-\tau)|}{\widehat{c}}  \tag{62}\\
& \leq\left(|c|^{-1} \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}(\bar{C} A(k)+A(k) \bar{C})\right]\right\}\right. \\
& \left.\quad+\widehat{c}^{-1} \max _{k \in S}(\|\bar{C} B(k)\|)\right)|x(t)| \\
& \quad+\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\|(-|x(t)|+|x(t-\tau)|), \\
& \frac{x^{T}(t) \bar{C} b(r(t))}{V(x(t))}-\frac{\left|x^{T}(t) \bar{C} \sigma(r(t))\right|^{2}}{2 V^{2}(t)}  \tag{63}\\
& \quad \leq \check{b}(r(t))-\frac{1}{2} \widehat{\sigma}^{2}(r(t))=\beta(r(t)) .
\end{align*}
$$

Substituting these two inequalities into (61) yields

$$
\begin{align*}
& \log V(x(t)) \\
& \leq \log V(x(0))+\int_{0}^{t} \beta(r(s)) d s+\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\| \\
& \quad \times \int_{0}^{t}[-|x(s)|+|x(s-\tau)|] d s+M(t)  \tag{64}\\
& \leq \log V(x(0))+\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\| \int_{-\tau}^{0} x(s) d s \\
& \quad+\int_{0}^{t} \beta(r(s)) d s+M(t)
\end{align*}
$$

where $M(t)$ is a martingale defined by

$$
\begin{equation*}
M(t)=\int_{0}^{t} \frac{x^{T}(s) \bar{C} \sigma(r(s))}{V(x(s))} d w(t) \tag{65}
\end{equation*}
$$

The quadratic variation of this martingale is

$$
\begin{equation*}
\langle M, M\rangle_{t}=\int_{0}^{t} \frac{\left|x^{T}(s) \bar{C} \sigma(r(s))\right|^{2}}{V^{2}(x(s))} d s \leq \check{\sigma}^{2} t \tag{66}
\end{equation*}
$$

hence

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\langle M, M\rangle_{t}}{t} \leq \check{\sigma}^{2} \quad \text { a.s. } \tag{67}
\end{equation*}
$$

Applying the strong law of large numbers for martingales [29], we therefore have

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{M(t)}{t}=0 \quad \text { a.s. } \tag{68}
\end{equation*}
$$

It finally follows from (64) by dividing $t$ on the both sides and then letting $t \rightarrow \infty$ that

$$
\begin{align*}
\limsup _{t \rightarrow \infty} \frac{\log V(x(t))}{t} & \leq \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \beta(r(s)) d s \\
& =\sum_{k=1}^{N} \pi_{k} \beta(k) \quad \text { a.s., } \tag{69}
\end{align*}
$$

which is the required assertion (57).
Similarly, we can prove the following conclusions.
Theorem 12. Assume that Assumption 10 holds. Assume moreover that the noise intensities $\sigma(i)$ are sufficiently large in the sense that

$$
\begin{gather*}
\sigma_{i}(k) \sigma_{j}(k)-b_{i}(k)-b_{j}(k)>0,  \tag{70}\\
1 \leq i, j \leq n, \text { for each } k \in S
\end{gather*}
$$

then for any given initial data $\{x(t):-\tau \leq t \leq 0\} \in$ $C\left([-\tau, 0] ; R_{+}^{n}\right)$, the solution $x(t)$ of (4) has the properties that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log |x(t)| \leq-\frac{1}{2} \sum_{k=1}^{N} \pi_{k} \varphi(k) \quad \text { a.s., } \tag{71}
\end{equation*}
$$

where

$$
\begin{equation*}
\varphi(k)=\min _{1 \leq i, j \leq n}\left(\sigma_{i}(k) \sigma_{j}(k)-b_{i}(k)-b_{j}(k)\right)>0 \tag{72}
\end{equation*}
$$

That is, the population will become extinct exponentially with probability 1.

Proof. Let $V: R_{+}^{n} \rightarrow R_{+}$be the same as defined in the proof of Theorem 11, so we have (60), (61), and (62). It is also computed

$$
\begin{align*}
& \frac{x^{T}(t) \bar{C} b(r(t))}{V(x(t))}-\frac{\left|x^{T}(x(t)) \bar{C} \sigma(r(t))\right|^{2}}{2 V^{2}(x(t))} \\
&= \frac{2 x^{T}(t) \bar{C} b(r(t)) c^{T} x(t)}{2 V^{2}(x(t))} \\
&-\frac{x^{T}(t) \bar{C} \sigma(r(t)) \sigma^{T}(r(t)) \bar{C} x(t)}{2 V^{2}(x(t))} \\
&= \frac{2 x^{T}(t) \bar{C} b(r(t)) \overrightarrow{1} \bar{C} x(t)}{2 V^{2}(x(t))}  \tag{73}\\
&= \frac{x^{T}(t) \bar{C} b(r(t)) \overrightarrow{1}+\overrightarrow{1}^{T} b^{T}(r(t)) \bar{C} x(t)}{2 V^{2}(x(t))} \\
& 2 V^{2}(x(t)) \\
&=-\frac{x^{T}(t) \bar{C} \sigma(r(t)) \sigma^{T}(r(t)) \bar{C} x(t)}{2 V^{2}(x(t))} \\
& 2 V^{2}(x(t)) \\
& x^{T}(t) \bar{C} Q(r(t) \bar{C} x(t) \\
&
\end{align*}
$$

where $\overrightarrow{1}=(1, \ldots, 1)$ and $Q(k)=\sigma(k) \sigma^{T}(k)-(b(k) \overrightarrow{1}+$ $\overrightarrow{1}^{T} b^{T}(k)$ ). Substituting (62) and (73) into (61) yields

$$
\begin{align*}
& \log V(x(t)) \\
& \leq \log V(x(0))-\int_{0}^{t} \frac{x^{T}(s) \bar{C} Q(r(s)) \bar{C} x(s)}{2 V^{2}(x(s))} d s \\
& \quad+\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\| \int_{0}^{t}[-|x(s)|+|x(s-\tau)|] d s+M(t) . \tag{74}
\end{align*}
$$

Note that $\sigma_{i}(k) \sigma_{j}(k)-b_{i}(k)-b_{j}(k)$, the $i j$ th element of the matrix $Q(k)$ is positive by (70). It is therefore easy to verify

$$
\begin{equation*}
x^{T}(t) \bar{C} Q(k) \bar{C} x(t) \geq \varphi(k) V^{2}(x(t)) \tag{75}
\end{equation*}
$$

where $\varphi(\cdot)$ has been defined in the statement of the theorem. Substituting this inequality into (74) yields

$$
\begin{align*}
& \log V(x(t)) \\
& \leq \log V(x(0))-\int_{0}^{t} \frac{1}{2} \varphi(k) d s+\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\|  \tag{76}\\
& \quad \times \int_{0}^{t}(-|x(s)|+|x(s-\tau)|) d s+M(t)
\end{align*}
$$

The rest of the proof is similar to that of Theorem 11 and omitted.

## 5. Examples

In this section, an example and corresponding numerical simulations are given to illustrate our main results.

Example 13. Consider the two-species Lotka-Volterra system with regime switching described by

$$
\begin{align*}
d x(t)= & \operatorname{diag}( \\
( & \left.(t), x_{2}(t)\right) \\
& \times[(b(r(t))+A(r(t)) x(t)  \tag{77}\\
& +B(r(t)) x(t-\tau)) d t+\sigma(r(t)) d w(t)]
\end{align*}
$$

where $x(t)=\left(x_{1}(t), x_{2}(t)\right)^{T}, b(r(t))=\left(b_{1}(r(t)), b_{2}(r(t))\right)^{T}$, $\sigma(r(t))=\left(\sigma_{1}(r(t)), \sigma_{2}(r(t))\right)^{T}$,

$$
\begin{align*}
A(r(t)) & =\left(\begin{array}{ll}
a_{11}(r(t)) & a_{12}(r(t)) \\
a_{21}(r(t)) & a_{22}(r(t))
\end{array}\right), \\
B(r(t)) & =\left(\begin{array}{ll}
b_{11}(r(t)) & b_{12}(r(t)) \\
b_{21}(r(t)) & b_{22}(r(t))
\end{array}\right) \tag{78}
\end{align*}
$$

and $r(t)$ is a right-contiuous Markov chain taking values in $S=\{1,2\}$, and $r(t)$ and $w(t)$ are independent. Here

$$
\begin{array}{lll}
b_{1}(1)=5, & a_{11}(1)=-5, & a_{12}(1)=3, \\
b_{11}(1)=0, & b_{12}(1)=\frac{1}{2}, & \sigma_{1}(1)=\sqrt{2}, \\
b_{2}(1)=8, & a_{21}(1)=3, & a_{22}(1)=-5, \\
b_{21}(1)=1, & b_{22}(1)=0, & \sigma_{2}(1)=2,  \tag{79}\\
b_{1}(2)=4, & a_{11}(2)=-3, & a_{12}(2)=1, \\
b_{11}(2)=0, & b_{12}(2)=1, & \sigma_{1}(2)=\sqrt{14}, \\
b_{2}(2)=5, & a_{21}(2)=1, & a_{22}(2)=-3, \\
b_{21}(2)=\frac{1}{2}, & b_{22}(2)=0, & \sigma_{2}(2)=4,
\end{array}
$$

Let $\bar{C}=I \in R^{2 \times 2}$, the identity matrix. It is easy to compute

$$
\begin{gather*}
|c|=\sqrt{2}, \quad \hat{c}=1, \quad \beta(1)=7, \quad \beta(2)=-2, \\
\max _{k \in S} \lambda_{\max }^{+}\left[\frac{1}{2}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right] \leq-2,  \tag{80}\\
\max _{k \in S}\|\bar{C} B(k)\| \leq \frac{\sqrt{5}}{2} .
\end{gather*}
$$

Then

$$
\begin{align*}
& |c|^{-1} \max _{k \in S}\left\{\lambda_{\max }^{+}\left[\frac{1}{2}\left(\bar{C} A(k)+A^{T}(k) \bar{C}\right)\right]\right\}  \tag{81}\\
& \quad+\widehat{c}^{-1} \max _{k \in S}\|\bar{C} B(k)\|<0
\end{align*}
$$



Figure 1

By Theorems 3 and 9, the solutions of (77) will remain in $R_{+}^{2}$ for all $t \geq-\tau$ with probability 1 and are stochastically ultimately bounded.

Let the generator of the Markov chain $r(t)$ be

$$
\Gamma=\left(\begin{array}{rr}
-4 & 4  \tag{82}\\
1 & -1
\end{array}\right)
$$

By solving the linear equation $\pi \Gamma=0$, we obtain the unique stationary (probability) distribution $\pi=\left(\pi_{1}, \pi_{2}\right)=$ $(1 / 5,4 / 5)$. Then $\sum_{k=1}^{2} \pi_{k} \beta(k)=-1 / 5<0$. Therefore, by Theorems 11, (77) is extinctive, shown in Figure 1.

In Figure 1, for numerical solutions of (77), step size $\Delta t=$ 0.001 , delay $\tau=1$. Initial datum of $\left(x_{1}(t), x_{2}(t)\right)$ are random numbers in $[1,200] \times[1,600]$. Initial datum are not shown in Figure 1.

## 6. Conclusion

This work is concerned with delay Lotka-Volterra model under regime switching diffusion in random environment. It should be pointed out that (77) is more difficult to handle than (3) in [23]. Fortunately, the difficulties caused by delay term are overcome by using Young's inequality. The model in [7] is similar to (4), while the coefficients in (4) are varied with
regime switching. Similar results are technically obtained by making use of comparison principle.

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## Research Article

# Global Attractivity of a Diffusive Nicholson's Blowflies Equation with Multiple Delays 

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The present paper considers a diffusive Nicholson's blowflies model with multiple delays under a Neumann boundary condition. Delay independent conditions are derived for the global attractivity of the trivial equilibrium and the positive equilibrium, respectively. Two open problems concerning the stability of positive equilibrium and the occurrence of Hopf bifurcation are proposed.

## 1. Introduction

Since blowflies are important parasites of the sheep industry in some countries such as Australia, based on the experimental data of Nicholson [1, 2], Gurney et al. [3] first proposed Nicholson's blowflies equation

$$
\begin{equation*}
\dot{N}(t)=-\delta N(t)+p N(t-\tau) e^{-a N(t-\tau)}, \quad t>0 \tag{1}
\end{equation*}
$$

where $N(t)$ is the size of the adult blowflies population at time $t ; p$ is the maximum per capita daily egg production rate; $1 / a$ is the size at which the blowflies population reproduces at its maximum rate; $\delta$ is the per capita daily adult death rate; $\tau$ is the generation time. For this equation, global attractivity and oscillation of solutions have been investigated by several authors (see [4-9]).

It is impossible that the size of the adult blowflies population is independent of a spatial variable; therefore, Yang and So [10] investigated both temporal and spatial variations of the diffusive Nicholson's blowflies equation

$$
\begin{align*}
\frac{\partial N(t, x)}{\partial t}= & \Delta N(t, x)-\delta N(t, x) \\
& +p N(t-\tau, x) e^{-a N(t-\tau, x)}  \tag{2}\\
& \quad \text { in } D \triangleq(0, \infty) \times \Omega
\end{align*}
$$

under Neumann boundary condition and gave the similar sufficient conditions for oscillation of all positive solutions
about the positive steady state. Whereafter, many authors studied the various dynamical behaviors for this equation; we refer to Lin and Mei [11], Saker [12], Wang and Li [13], and Yi and Zou [14].

Meanwhile, one can consider a nonlinear equation with several delays because of variability of the generation time; for this purpose, Györi and Ladas [15] and Kulenović and Ladas [6] proposed the following generalized Nicholson's blowflies model:

$$
\begin{equation*}
N^{\prime}(t)=-\delta N(t)+\sum_{i=1}^{n} p_{i} N\left(t-\tau_{i}\right) e^{-a_{i} N\left(t-\tau_{i}\right)}, \quad t>0 \tag{3}
\end{equation*}
$$

Luo and Liu [16] studied the global attractivity of the nonnegative equilibria of (3).

It is of interest to investigate both several temporal and spatial variations of the blowflies population using mathematical models. Hereby, in this paper, we consider the following system:

$$
\begin{align*}
& \frac{\partial N(t, x)}{\partial t} \\
& \quad=\Delta N(t, x)-\delta N(t, x)  \tag{4}\\
& \quad+\sum_{i=1}^{n} p_{i} N\left(t-\tau_{i}, x\right) e^{-a_{i} N\left(t-\tau_{i}, x\right)}, \quad \text { in } D
\end{align*}
$$

with Neumann boundary condition

$$
\begin{equation*}
\frac{\partial N(t, x)}{\partial v}=0, \quad \text { on } \Gamma \triangleq(0, \infty) \times \partial \Omega, \tag{5}
\end{equation*}
$$

and initial condition

$$
\begin{equation*}
N(\theta, x)=\psi(\theta, x) \geq 0, \quad \text { in } D_{\tau} \triangleq[-\tau, 0] \times \bar{\Omega} \tag{6}
\end{equation*}
$$

where $\tau_{i} \geq 0, \tau=\max _{1 \leq i \leq n}\left\{\tau_{i}\right\}, p_{i}$ and $a_{i}=a, i=1,2, \ldots, n$, are all positive constants, $\Omega \subset \mathbb{R}^{m}$ is a bounded domain with a smooth boundary $\partial \Omega, \Delta N(t, x)=\sum_{i=1}^{m}\left(\left(\partial_{i}^{2} N(t, x)\right) /\left(\partial x_{i}^{2}\right)\right)$, $(\partial / \partial \nu)$ denotes the exterior normal derivative on $\partial \Omega$, and $\psi(\theta, x)$ is Hölder continuous in $D_{\tau}$ with $\psi(0, x) \in C^{1}(\bar{\Omega})$.

Though the global attractivity of the nonnegative equilibria of (2) has been studied by Yang and So [10] and Wang and Li [13, 17], they just gave some sufficient conditions. Furthermore, as far as we know, the stability for partial functional differential equations with several delays was investigated by few papers. Motivated by the above excellent works, in this paper, we consider the global attractivity of the nonnegative equilibria of the systems (4)-(6) and present some conditions which depend on coefficients of the systems (4)-(6). When $n=1$, our results complement those in Yang and So [10] and Wang and Li [13].

It is not difficult to see that if $\sum_{i=1}^{n} p_{i} \leq \delta$, then (4) has a unique nonnegative equilibrium $N_{0} \equiv 0$ and if $\sum_{i=1}^{n} p_{i}>\delta$, then (4) has a unique positive equilibrium $N^{*}=$ $(1 / a) \ln \left(\left(\sum_{i=1}^{n} p_{i}\right) / \delta\right)$.

The rest of the paper is organized as follows. We give some lemmas and definitions in Section 2 and state and prove our main results in Section 3. In Section 4, several simulations are obtained to testify our results, and some unsolved problems are discussed.

## 2. Preliminaries

In this section, we will give some lemmas which can be proved by using the similar methods as those in Yang and So [10].

Lemma 1. (i) The solution $N(t, x)$ of (4)-(6) satisfies $N(t, x) \geq 0$ for $(t, x) \in(0, \infty) \times \bar{\Omega}$.
(ii) If $\psi(\theta, x) \not \equiv 0$ on $D_{\tau}$, then the solution $N(t, x)$ of (4)(6) satisfies $N(t, x)>0$ for $(t, x) \in(\tau, \infty) \times \bar{\Omega}$.

Next, we will introduce the concept of lower-upper solution due to Redlinger [18] as adapted to (4)-(6).

Definition 2. A lower-upper solution pair for (4)-(6) is a pair of suitably smooth function $v$ and $w$ such that
(i) $v \leq w$ in $\bar{D}$,
(ii) $v$ and $w$ satisfy

$$
\begin{aligned}
& \frac{\partial w}{\partial t} \geq \Delta w(t, x)-\delta w+\sum_{i=1}^{n} p_{i} \varphi\left(t-\tau_{i}, x\right) e^{-a \varphi\left(t-\tau_{i}, x\right)}, \quad(t, x) \in D, \\
& \frac{\partial w}{\partial v} \geq 0, \quad(t, x) \in \Gamma,
\end{aligned}
$$

$$
\begin{align*}
& \frac{\partial v}{\partial t} \leq \Delta v(t, x)-\delta v+\sum_{i=1}^{n} p_{i} \varphi\left(t-\tau_{i}, x\right) e^{-a \varphi\left(t-\tau_{i}, x\right)}, \quad(t, x) \in D \\
& \frac{\partial v}{\partial v} \leq 0, \quad(t, x) \in \Gamma \tag{7}
\end{align*}
$$

for all $\varphi \in C\left(D_{\tau} \cup \bar{D}\right)$ with $v \leq \varphi \leq w,(t, x) \in D_{\tau} \cup \bar{D}$, and
(iii) $v(\theta, x) \leq \varphi(\theta, x) \leq w(\theta, x),(\theta, x) \in D_{\tau}$.

The following lemma is a special case of Redlinger [19].
Lemma 3. Let $(v, w)$ be a lower-upper solution pair for the initial boundary value problem (4)-(6). Then, there exists a unique regular solution $N(t, x)$ of (4)-(6) such that $v \leq N \leq w$ on $D_{\tau} \cup \bar{D}$.

The following lemma gives us boundedness of the solution $N(t, x)$.

Lemma 4. (i) The solution $N(t, x)$ of (4)-(6) satisfies

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} N(t, x) \leq \sum_{i=1}^{n} \frac{p_{i}}{a e \delta}, \quad \text { uniformly in } x . \tag{8}
\end{equation*}
$$

(ii) There exists a constant $K=K(\psi) \geq 0$ such that $N(t, x) \leq K$ on $D_{\tau} \cup \bar{D}$.

Proof. Let $w(t)$ be the solution of the following Cauchy problem:

$$
\begin{gather*}
\frac{d w}{d t}=-\delta w+\sum_{i=1}^{n} \frac{p_{i}}{a e}, \quad t>0,  \tag{9}\\
w(0)=\max _{(\theta, x) \in D_{\tau}} \psi(\theta, x)
\end{gather*}
$$

Solving the equation, we have

$$
\begin{equation*}
w(t)=\sum_{i=1}^{n} \frac{p_{i}}{a e \delta}+e^{-\delta t}\left(w(0)-\sum_{i=1}^{n} \frac{p_{\mathrm{i}}}{a e \delta}\right), \quad t \geq 0 . \tag{10}
\end{equation*}
$$

Taking

$$
\bar{w}(t)= \begin{cases}w(0), & t \in[-\tau, 0]  \tag{11}\\ w(t), & t>0\end{cases}
$$

then $(\bar{w}(t), 0)$ is a lower-upper solution pair for (4)-(6). In fact, for any $\varphi \in C\left(D_{\tau} \cup \bar{D}\right)$ with $0 \leq \varphi \leq \bar{w}(t),(t, x) \in D_{\tau} \cup \bar{D}$, one can get

$$
\begin{align*}
\frac{\partial \bar{w}(t)}{\partial t} & -\Delta \bar{w}(t)+\delta \bar{w}(t)-\sum_{i=1}^{n} p_{i} \varphi\left(t-\tau_{i}, x\right) e^{-a \varphi\left(t-\tau_{i}, x\right)} \\
& \geq \frac{\partial \bar{w}(t)}{\partial t}+\delta \bar{w}(t)-\sum_{i=1}^{n} \frac{p_{i}}{a e}  \tag{12}\\
& =\frac{d w}{d t}+\delta w-\sum_{i=1}^{n} \frac{p_{i}}{a e}=0 .
\end{align*}
$$

By Lemma 3, there is a unique regular solution $N(t, x)$ such that

$$
\begin{equation*}
0 \leq N(t, x) \leq \bar{w}(t), \quad(t, x) \in D_{\tau} \cup \bar{D} . \tag{13}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \bar{w}(t)=\lim _{t \rightarrow+\infty} w(t)=\sum_{i=1}^{n} \frac{p_{i}}{\text { ae } \delta} . \tag{14}
\end{equation*}
$$

Therefore, the formula (8) is correct, and there exists one $K(\psi)>0$ such that $\bar{w}(t) \leq K(\psi)$ for any $t \in(-\tau, \infty)$ and

$$
\begin{equation*}
0 \leq N(t, x) \leq K(\psi), \quad(t, x) \in D_{\tau} \cup \bar{D} . \tag{15}
\end{equation*}
$$

So we complete Lemma 4.

## 3. Main Results and Proofs

Theorem 5. Assume that $\sum_{i=1}^{n} p_{i} \leq \delta$, then every solution $N(t, x)$ of (4)-(6) tends to $N_{0}=0$ (uniformly in $x$ ) as $t \rightarrow$ $+\infty$.

Proof. By Lemma 4, without loss of generality, let $0<$ $N(t, x) \leq \sum_{i=1}^{n}\left(p_{i} / a e \delta\right)$ for $(t, x) \in D_{\tau} \cup \bar{D}$. Under the condition $\sum_{i=1}^{n} p_{i} \leq \delta$, we can get

$$
\begin{equation*}
0<N(t, x) \leq \frac{1}{a e}<\frac{1}{a} \quad \text { for }(t, x) \in D_{\tau} \cup \bar{D} \tag{16}
\end{equation*}
$$

Define $m(t)$ and $y(t)$ to be the solutions of the following two delay equations, respectively:

$$
\begin{align*}
& m^{\prime}(t)=-\delta m(t)+\sum_{i=1}^{n} p_{i} m\left(t-\tau_{i}\right) e^{-a m\left(t-\tau_{i}\right)}, \quad t>0 \\
& m(\theta)=\min _{x \in \bar{\Omega}} \psi(\theta, x), \quad \theta \in[-\tau, 0]  \tag{17}\\
& y^{\prime}(t)=-\delta y(t)+\sum_{i=1}^{n} p_{i} y\left(t-\tau_{i}\right) e^{-a y\left(t-\tau_{i}\right)}, \quad t>0 \\
& y(\theta)=\max _{x \in \bar{\Omega}} \psi(\theta, x), \quad \theta \in[-\tau, 0] .
\end{align*}
$$

By using the similar methods to prove Lemma 4, we can get that
$\limsup _{t \rightarrow \infty} m(t) \leq \sum_{i=1}^{n} \frac{p_{i}}{a e \delta}<\frac{1}{a}, \quad \limsup _{t \rightarrow \infty} y(t) \leq \sum_{i=1}^{n} \frac{p_{i}}{a e \delta}<\frac{1}{a}$
under the condition $\sum_{i=1}^{n} p_{i} \leq \delta$, and here $m(t)$ and $y(t)$ are the solutions of (17).

Because of $N(t, x)<1 / a$, for any $\varphi \in C\left(D_{\tau} \cup \bar{D}\right), m(t) \leq$ $\varphi \leq y(t)<1 / a$, one can get

$$
\begin{align*}
& \frac{\partial m(t)}{\partial t}-\Delta m(t)+\delta m(t)-\sum_{i=1}^{n} p_{i} \varphi\left(t-\tau_{i}, x\right) e^{-a \varphi\left(t-\tau_{i}, x\right)} \\
& \quad \leq \frac{\partial m(t)}{\partial t}+\delta m(t)-\sum_{i=1}^{n} p_{i} m\left(t-\tau_{i}\right) e^{-a m\left(t-\tau_{i}\right)} \\
& \quad=0  \tag{19}\\
& \frac{\partial y(t)}{\partial t}-\Delta y(t)+\delta y(t)-\sum_{i=1}^{n} p_{i} \varphi\left(t-\tau_{i}, x\right) e^{-a \varphi\left(t-\tau_{i}, x\right)} \\
& \quad \geq \frac{\partial y(t)}{\partial t}+\delta y(t)-\sum_{i=1}^{n} p_{i} y\left(t-\tau_{i}\right) e^{-a y\left(t-\tau_{i}\right)} \\
& \quad=0
\end{align*}
$$

Therefore, from Definition 2, $(m(t), y(t))$ is a lower-upper pair of (4)-(5) with initial condition $m(\theta) \leq \psi(\theta, x) \leq y(\theta)$ on $D_{\tau}$. Consequently, by Lemma 3, we have

$$
\begin{equation*}
m(t) \leq N(t, x) \leq y(t) \quad \text { on }[-\tau,+\infty) \times \bar{\Omega} \tag{20}
\end{equation*}
$$

By Theorem 1 of Luo and Liu [16], it follows from $\sum_{i=1}^{n} p_{i} \leq \delta$ that the solutions $m(t)$ and $y(t)$ of (17) both satisfy

$$
\begin{equation*}
\lim _{t \rightarrow \infty} m(t)=0, \quad \lim _{t \rightarrow \infty} y(t)=0 \tag{21}
\end{equation*}
$$

Hence, we complete the proof of Theorem 5.
Theorem 6. If $1<\sum_{i=1}^{n}\left(p_{i} / \delta\right) \leq e$, then every nontrivial solution $N(t, x)$ of (4)-(6) satisfies

$$
\begin{equation*}
\lim _{t \rightarrow \infty} N(t, x)=N^{*}, \quad \text { uniformly in } x . \tag{22}
\end{equation*}
$$

Proof. Let $f(x)=x e^{-a x}$, then the function $f(x)$ is increasing on $(0,(1 / a))$ and decreasing on $((1 / a),+\infty), f(1 / a)=$ $\max _{x \in[0, \infty)} f(x), N^{*}=(1 / a) \ln \left(\sum_{i=1}^{n}\left(p_{i} / \delta\right)\right) \leq 1 / a$ for $1<$ $\sum_{i=1}^{n}\left(p_{i} / \delta\right) \leq e$. Let $g(y)=\sum_{i=1}^{n} p_{i} f(y)$, then it is not difficult to verify that the function $g(y)$ satisfies the following conditions:
$\left(g_{1}\right)$ the function $g(y)$ is increasing on $(0,(1 / a))$ and decreasing on $((1 / a),+\infty), \max _{x \in[0, \infty)} g(x)=$ $g(1 / a)=\sum_{i=1}^{n}\left(p_{i} / a e\right)$,
$\left(g_{2}\right) g(y)>\delta y$ for $y \in\left(0, N^{*}\right)$ and $g(y)<\delta y$ for $y \in$ $\left(N^{*},+\infty\right)$.

There are now two possible cases to consider.
Case 1 ( $N^{*}<1 / a$ ). In view of Lemma 4, we may also assume without loss of generality that every solution $N(t, x)$ of (4)(6) satisfies

$$
\begin{equation*}
0 \leq N(t, x) \leq \frac{g(1 / a)}{\delta}=\sum_{i=1}^{n} \frac{p_{i}}{a e \delta}<\frac{1}{a}, \quad \text { on } D_{\tau} \cup \bar{D} \tag{23}
\end{equation*}
$$

Let $\underline{N}(t)=\min _{x \in \bar{\Omega}} N(t, x), \bar{N}(t)=\max _{x \in \bar{\Omega}} N(t, x), \underline{N}=$ $\liminf _{t \rightarrow \infty} \underline{N}(t)$ and $\bar{N}=\lim \sup _{t \rightarrow \infty} \bar{N}(t)$. By (23), we have

$$
\begin{equation*}
0 \leq \underline{N} \leq \bar{N} \leq \frac{g(1 / a)}{\delta}=\sum_{i=1}^{n} \frac{p_{i}}{a e \delta}<\frac{1}{a} \tag{24}
\end{equation*}
$$

From Lemma 1(ii), let

$$
\begin{gather*}
z_{0}=\min \left\{\min _{(t, x) \in[2 \tau, \infty) \times \bar{\Omega}} N(t, x), N^{*}\right\}>0,  \tag{25}\\
y_{0}=\frac{1}{a} .
\end{gather*}
$$

Let $I_{\infty}=\{1,2, \ldots\}$. Now, we define two sequences $\left\{z_{k}\right\}$ and $\left\{y_{k}\right\}$ to satisfy, respectively,

$$
\begin{array}{ll}
z_{k}=\frac{g\left(z_{k-1}\right)}{\delta}, & k \in I_{\infty} \\
y_{k}=\frac{g\left(y_{k-1}\right)}{\delta}, & k \in I_{\infty} . \tag{26}
\end{array}
$$

We prove that $\left\{z_{k}\right\}$ and $\left\{y_{k}\right\}$ are monotonic and bounded. First of all, we prove that $\left\{z_{k}\right\}$ is monotonically increasing, and $N^{*}$ is the least upper bounded. Note $\left(g_{1}\right)$ and $\left(g_{2}\right)$, we have

$$
\begin{equation*}
z_{1}=\frac{g\left(z_{0}\right)}{\delta}>z_{0}, \quad z_{1}=\frac{g\left(z_{0}\right)}{\delta}<\frac{g\left(N^{*}\right)}{\delta}=N^{*} \tag{27}
\end{equation*}
$$

By induction and direct computation, we have

$$
\begin{equation*}
0<z_{0}<z_{1}<\cdots<\lim _{k \rightarrow \infty} z_{k}=N^{*} \tag{28}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
0>y_{0}>y_{1}>\cdots>\lim _{k \rightarrow \infty} y_{k}=N^{*} \tag{29}
\end{equation*}
$$

Define $v_{1}(t)$ and $w_{1}(t)$ to be the solutions of the following differential equations, respectively:

$$
\begin{align*}
& v_{1}^{\prime}(t)=-\delta\left[v_{1}(t)-z_{1}\right], \quad t \geq 3 \tau \\
& v_{1}(\theta)=z_{0}<N^{*}, \quad \theta \in[2 \tau, 3 \tau] \\
& w_{1}^{\prime}(t)=-\delta\left[w_{1}(t)-y_{1}\right], \quad t \geq 3 \tau  \tag{30}\\
& w_{1}(\theta)=y_{0}>N^{*}, \quad \theta \in[2 \tau, 3 \tau]
\end{align*}
$$

It follows from (24) and (25) that $z_{0} \leq N(t, x) \leq y_{0}$ for any $(t, x) \in[2 \tau, \infty) \times \bar{\Omega}$. Consider (30), for any $(t, x) \in[2 \tau, \infty] \times$ $\bar{\Omega}$, we have

$$
\begin{align*}
\frac{\partial v_{1}(t)}{\partial t} & =\Delta v_{1}(t)-\delta v_{1}(t)+g\left(z_{0}\right) \\
& \leq \Delta v_{1}(t)-\delta v_{1}(t)+g(N(t-\tau, x)) \\
\frac{\partial w_{1}(t)}{\partial t} & =\Delta w_{1}(t)-\delta w_{1}(t)+g\left(y_{0}\right)  \tag{31}\\
& \geq \Delta w_{1}(t)-\delta w_{1}(t)+g(N(t-\tau, x))
\end{align*}
$$

Therefore, from Definition 2, $\left(v_{1}(t), w_{1}(t)\right)$ is a lower-upper pair of (4)-(5) with initial condition $z_{0} \leq N(t, x) \leq y_{0}$ on $[2 \tau, 3 \tau] \times \bar{\Omega}$. Consequently, by Lemma 3, we have

$$
\begin{equation*}
v_{1}(t) \leq N(t, x) \leq \omega_{1}(t) \quad \text { on }[2 \tau, \infty] \times \bar{\Omega} . \tag{32}
\end{equation*}
$$

Note that $w_{1}(t)$ is monotonically decreasing for $t \geq$ $3 \tau$ and $\lim _{t \rightarrow \infty} w_{1}(t)=y_{1}$, while $v_{1}(t)$ is monotonically increasing for $t \geq 3 \tau$ and $\lim _{t \rightarrow \infty} v_{1}(t)=z_{1}$. Hence,

$$
\begin{equation*}
z_{1}=\lim _{t \rightarrow \infty} v_{1}(t) \leq \underline{N} \leq \bar{N} \leq \lim _{t \rightarrow \infty} w_{1}(t)=y_{1} \tag{33}
\end{equation*}
$$

Define $v_{n}(t)$ and $w_{n}(t)$ to be the solutions of the following differential equations, respectively:

$$
\begin{align*}
& v_{n}^{\prime}(t)=-\delta\left[v_{n}(t)-z_{n}\right], \quad t \geq 3 \tau \\
& v_{n}(\theta)=z_{n-1}<N^{*}, \quad \theta \in[2 \tau, 3 \tau]  \tag{34}\\
& w_{n}^{\prime}(t)=-\delta\left[w_{n}(t)-y_{n}\right], \quad t \geq 3 \tau \\
& w_{n}(\theta)=w_{n-1}<N^{*}, \quad \theta \in[2 \tau, 3 \tau] .
\end{align*}
$$

Repeating the above procedure, we have the following relation:

$$
\begin{equation*}
z_{1}<z_{2}<\cdots<z_{n} \leq \underline{N} \leq \bar{N} \leq y_{n}<\cdots<y_{2}<y_{1} . \tag{35}
\end{equation*}
$$

By (28) and (29), and taking limits on both sides of (35), we have

$$
\begin{equation*}
N^{*}=\lim _{n \rightarrow \infty} z_{n} \leq \underline{N} \leq \bar{N} \leq \lim _{n \rightarrow \infty} y_{n}=N^{*} \tag{36}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\lim _{t \rightarrow \infty} N(t, x)=N^{*}, \quad \text { uniformly in } x \tag{37}
\end{equation*}
$$

Case $2\left(N^{*}=y_{0}\right)$. Similarly, let $y_{k}=N^{*}$ and $z_{k}$ be the same as in the proof of Case 1; we can also get (35). Hence, the proof of Theorem 6 is complete.

Remark 7. Our main results are also valid when $N$ does not depend on a spatial variable $x \in \Omega$ in (4).

## 4. Numerical Simulations and Discussion

In this section, we will give some numerical simulations to verify our main results in Section 3 and present several interesting phenomena by simulations that we cannot give a theoretical proof. We just consider the case $n=2$ in (4).
4.1. Numerical Simulations. Different parameters will be used for simulations, and some data come from [20]. Figure 1 corresponds to the case with $\delta=0.4, p_{1}=0.1, p_{2}=0.15$, $a=0.1, \tau_{1}=12$, and $\tau_{2}=15$, and under the above conditions, we have $0<\left(p_{1}+p_{2}\right) / \delta=0.625<1$. We choose the initial condition $\psi(\theta, x)=1,(\theta, x) \in[-15,0] \times[0,1]$, and the solution $N(t, x)$ is decreasing and almost zero at time 160.

Figure 2 corresponds to the case with $\delta=0.1, p_{1}=0.1$, $p_{2}=0.15, a=0.2, \tau_{1}=12$, and $\tau_{2}=15$, and under the above


Figure 1: Parameters: $\delta=0.4, p_{1}=0.1, p_{2}=0.15, a=0.1, \tau_{1}=12$, and $\tau_{2}=15$. Initial condition is $\psi(\theta, x)=1,(\theta, x) \in[-15,0] \times[0,1]$.
conditions, we have $1<\left(p_{1}+p_{2}\right) / \delta=2.5<e$ and $N^{*}=$ 4.58145. Choose the initial condition $\psi(\theta, x)=4+\sin \theta$, $(\theta, x) \in[-15,0] \times[0,1]$. From Figure 2, we can observe that the solution $N(t, x)$ oscillates around 13 and 14 days; however, $N(t, x)$ tends to $N^{*}$ as time $t$ tends to 100 days. Therefore, Figures 1 and 2 support our main results (Theorems 5 and 6).
4.2. Discussion. In Section 3, we obtain two main results under the conditions $\sum_{i=1}^{n}\left(p_{i} / \delta\right) \leq 1$ and $1<\sum_{i=1}^{n}\left(p_{i} / \delta\right) \leq e$, which are independent of the delays $\tau_{i}, i=1,2, \ldots, n$. A natural problem is what will happen when $\sum_{i=1}^{n}\left(p_{i} / \delta\right)>e$ and the delays $\tau_{i}, i=1,2, \ldots, n$ are changed.

It is similar to Theorem 3 in Luo and Liu [16]; we present the following open problems.

Open Problem 1. If $\sum_{i=1}^{n}\left(p_{i} / \delta\right)>e$ and $a N^{*}\left(e^{\delta \tau}-1\right) \leq 1$, then every nontrivial solution $N(t, x)$ of (4)-(6) satisfies

$$
\begin{equation*}
\lim _{t \rightarrow \infty} N(t, x)=N^{*}, \quad \text { uniformly in } x . \tag{38}
\end{equation*}
$$

Figure 3 corresponds to the case with $\delta=0.01, p_{1}=0.5$, $p_{2}=0.5, a=0.2, \tau_{1}=12, \tau_{2}=15$, and $N^{*}=23.0259$, and initial condition is $\psi(\theta, x)=10+\sin \theta,(\theta, x) \in[-15,0] \times$ $[0,1]$. Under the above conditions, we have $\left(p_{1}+p_{2}\right) / \delta=$ $100>e$ and $a N^{*}\left(e^{\delta \tau}-1\right)=0.745274<1$. Sufficient conditions are dependent on coefficients and delay for the global attractivity of equilibria $N^{*}$, and Figure 3 shows that the Open Problem 1 is right, but we cannot prove that.

From Figure 4, we have $\left(\left(p_{1}+p_{2}\right) / \delta\right)=5>e$ and $a N^{*}\left(e^{\delta \tau}-1\right)=30.717>1$. The condition is not satisfied, but $N^{*}$ is still globally attractive.

From Figure 5, we have $\left(\left(p_{1}+p_{2}\right) / \delta\right)=50>e$ and $a N^{*}\left(e^{\delta \tau}-1\right)=13.6204>1$. The condition is not satisfied, but the global attractivity $N^{*}$ is not true. Moreover, Figure 5 shows that there is a periodic solution, which is very interesting. We guess that the reason is that the system brings Hopf bifurcation as the parameters change. Therefore, we state the following open problem.


Figure 2: Parameters: $\delta=0.1, p_{1}=0.1, p_{2}=0.15, a=0.2, \tau_{1}=12$, $\tau_{2}=15$, and $N^{*}=4.58145$. Initial condition is $\psi(\theta, x)=4+\sin \theta$, $(\theta, x) \in[-15,0] \times[0,1]$.


Figure 3: Parameters: $\delta=0.01, p_{1}=0.5, p_{2}=0.5, a=0.2, \tau_{1}=12$, $\tau_{2}=15$, and $N^{*}=23.0259$. Initial condition is $\psi(\theta, x)=10+\sin \theta$, $(\theta, x) \in[-15,0] \times[0,1]$.


Figure 4: Parameters: $\delta=0.2, p_{1}=0.5, p_{2}=0.5, a=0.2, \tau_{1}=12$, $\tau_{2}=15$, and $N^{*}=8.04719$. Initial condition is $\psi(\theta, x)=9+\sin \theta$, $(\theta, x) \in[-15,0] \times[0,1]$.


Figure 5: Parameters: $\delta=0.1, p_{1}=3, p_{2}=2, a=0.2, \tau_{1}=12$, $\tau_{2}=15$, and $N^{*}=19.5601$. Initial condition is $\psi(\theta, x)=10+\sin \theta$, $(\theta, x) \in[-15,0] \times[0,1]$.

Open Problem 2. Under suitable conditions, the systems (4)(6) will lead to Hopf bifurcation.

Remark 8. Now, we have not intensively studied these two problems. Because the nonmonotonicity of the nonlinear term in (4) makes it very difficult for us to solve Open Problem 1, and we cannot prove Open Problem 2 because of multiple delays.

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# Almost Automorphic Solutions to Nonautonomous Stochastic Functional Integrodifferential Equations 

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#### Abstract

This paper concerns the square-mean almost automorphic solutions to a class of abstract semilinear nonautonomous functional integrodifferential stochastic evolution equations in real separable Hilbert spaces. Using the so-called "Acquistapace-Terreni" conditions and Banach contraction principle, the existence, uniqueness, and asymptotical stability results of square-mean almost automorphic mild solutions to such stochastic equations are established. As an application, square-mean almost automorphic solution to a concrete nonautonomous integro-differential stochastic evolution equation is analyzed to illustrate our abstract results.


## 1. Introduction

Stochastic differential equations in both finite and infinite dimensions, which are important from the viewpoint of applications since they incorporate natural randomness into the mathematical description of the phenomena and hence provide a more accurate description of it, have received considerable attention. Based on this viewpoint, there has been an increasing interest in extending certain classical deterministic results to stochastic differential equations in recent years. As a good case in point, the existence of almost periodic or pseudo-almost periodic solutions to stochastic evolution equations has been extensively considered in many publications; see $[1-8]$ and the references therein.

Integrodifferential equations are used to describe lots of phenomena arising naturally from many fields such as fluid dynamics, number reactor dynamics, population dynamics, electromagnetic theory, and biological models, most of which cannot be described by classical differential equations, and hence they have attracted more and more attention in recent years; see [1, 9-12] for more details.

Recently, Keck and McKibben [9, 10] proposed a general abstract model for semilinear functional stochastic integrodifferential equations and studied the existence and uniqueness of mild solutions to these equations. Based on their works, the existence and uniqueness of square-mean almost periodic solutions to some functional
integrodifferential stochastic evolution equations was carefully investigated in [1] for the autonomous case and in our forthcoming paper for the nonautonomous case. In a very recent paper, as a natural generalization of the notion of square-mean almost periodicity, a new concept of squaremean almost automorphic stochastic process was introduced by Fu and Liu [13], and the existence results of squaremean almost automorphic mild solutions to some linear and semilinear autonomous stochastic differential equations were formulated, while paper [14] investigated the same issue for nonautonomous stochastic differential equations. Under some suitable assumptions, the authors established in a forthcoming paper the existence and uniqueness of squaremean almost automorphic solutions to a class of autonomous functional integrodifferential stochastic evolution equations.

In this paper, we are concerned with a class of semilinear nonautonomous functional stochastic integrodifferential equations in a real separable Hilbert space in the abstract form:

$$
\begin{align*}
X^{\prime}(t)= & A(t) X(t)+\int_{-\infty}^{t} C(t-s) G(s, X(s)) d W(s) \\
& +\int_{-\infty}^{t} B(t-s) F_{2}(s, X(s)) d s  \tag{1}\\
& +F_{1}(t, X(t)), \quad t \in \mathbb{R},
\end{align*}
$$

where $A(t): D(A(t)) \subset \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}) \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ is a family of densely defined closed (possibly unbounded) linear operator satisfying the so-called "Acquistapace-Terreni" conditions introduced in [15], $B$ and $C$ are convolution-type kernels in $\mathscr{L}^{1}(0, \infty)$ and $\mathscr{L}^{2}(0, \infty)$, respectively, satisfying Assumption 3.2 in [16], $W(t)$ is a two-sided standard onedimensional Brownian motion defined on the filtered probability space $\left(\Omega, \mathscr{F}, \mathbf{P}, \mathscr{F}_{t}\right)$, where $\mathscr{F}_{t}=\sigma\{W(u)-W(v) ; u, v \leq$ $t\}$. Here $F_{1}, F_{2}, G: \mathbb{R} \times \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}) \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ are jointly continuous functions satisfying some additional conditions to be specified later in Section 3.

Motivated by the aforementioned works [1, 13, 14], we investigate in this paper the existence and uniqueness of square-mean almost automorphic solutions to nonautonomous equation (1). The main tools employed here are Banach contraction principle and an estimate on the Ito integral. The obtained results can be seen as a contribution to this emerging field.

The paper is organized as follows. In Section 2, we review some basic definitions and preliminary facts on square-mean almost automorphic processes which will be used throughout this paper. Section 3 is devoted to establish the existence, uniqueness, and the asymptotical stability of square-mean almost automorphic mild solution to (1). As an illustration of our abstract result, square-mean almost automorphic solution to a concrete nonautonomous integrodifferential stochastic evolution equation is investigated in Section 4.

## 2. Preliminaries

To begin this paper, we recall some primary definitions, notations, lemmas, and technical results which will be used in the sequel. For more details on almost automorphy and stochastic differential equations, the readers are referred to [ $13,17-23$ ] and the references therein.

Throughout this paper, we assume that $(\mathbb{H},\|\cdot\|)$ is a real separable Hilbert space, $(\Omega, \mathscr{F}, \mathbf{P})$ is a probability space, and $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ stands for the space of all $\mathbb{H}$-valued random variables $X$ such that

$$
\begin{equation*}
\mathbf{E}\|X\|^{2}=\int_{\Omega}\|X\|^{2} d \mathbf{P}<\infty . \tag{2}
\end{equation*}
$$

For $X \in \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$, let

$$
\begin{equation*}
\|X\|_{2}:=\left(\int_{\Omega}\|X\|^{2} d \mathbf{P}\right)^{1 / 2} \tag{3}
\end{equation*}
$$

It is routine to check that $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ is a Banach space equipped with the norm $\|\cdot\|_{2}$.

It is well-known that Brownian motion plays a key role in the construction of stochastic integrals. Throughout this paper, $W(t)$ denotes a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $\left(\Omega, \mathscr{F}, \mathbf{P}, \mathscr{F}_{t}\right)$, where $\mathscr{F}_{t}=\sigma\{W(u)-W(v) ; u, v \leq t\}$.

Definition 1. A standard one-dimensional Brownian motion is a continuous, adapted real-valued stochastic process $(W(t), t \geq 0)$ such that
(i) $W(0)=0$ a.s.;
(ii) $W(t)-W(s)$ is independent of $\mathscr{F}_{s}$ for all $0 \leq s<t$;
(iii) $W(t)-W(s)$ is $N(0, t-s)$ distributed for all $0 \leq s \leq t$.

The following definitions and lemmas concerning squaremean almost automorphic functions can be found in [13, 14].

Definition 2. A stochastic process $X: \mathbb{R} \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ is said to be stochastically continuous if

$$
\begin{equation*}
\lim _{t \rightarrow s} \mathbf{E}\|X(t)-X(s)\|^{2}=0 \tag{4}
\end{equation*}
$$

Definition 3. A stochastically continuous stochastic process $X: \mathbb{R} \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ is said to be square-mean almost automorphic if for every sequence of real numbers $\left\{s_{n}^{\prime}\right\}$ there exists a subsequence $\left\{s_{n}\right\}$ and a stochastic process $Y: \mathbb{R} \rightarrow$ $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ such that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|X\left(t+s_{n}\right)-Y(t)\right\|^{2}=0  \tag{5}\\
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|X\left(t+s_{n}\right)-Y(t)\right\|^{2}=0
\end{align*}
$$

hold for each $t \in \mathbb{R}$. The collection of all square-mean almost automorphic stochastic processes is denoted by $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$.

Lemma 4. If $X, X_{1}$, and $X_{2}$ are all square-mean almost automorphic stochastic processes, then the following statements hold true:
(i) $X_{1}+X_{2}$ is square-mean almost automorphic;
(ii) $\lambda X$ is square-mean almost automorphic for every scalar $\lambda$;
(iii) There exists a constant $M>0$ such that $\sup _{t \in \mathbb{R}}\|X(t)\|_{2}$ $\leq M$. That is, $X$ is bounded in $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$.

Lemma 5. $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$ is a Banach space when it is equipped with the norm

$$
\begin{equation*}
\|X\|_{\infty}:=\sup _{t \in \mathbb{R}}\|X(t)\|_{2}=\sup _{t \in \mathbb{R}}\left(\mathbf{E}\|X(t)\|^{2}\right)^{1 / 2} \tag{6}
\end{equation*}
$$

for $X \in A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$.
Definition 6. A jointly continuous function $F: \mathbb{R} \times$ $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}) \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}),(t, X) \mapsto F(t, X)$ is said to be square-mean almost automorphic in $t \in \mathbb{R}$ for each $X \in$ $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ if for every sequence of real numbers $\left\{s_{n}^{\prime}\right\}$ there exists a subsequence $\left\{s_{n}\right\}$ and a stochastic process $G: \mathbb{R} \times$ $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}) \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ such that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|F\left(t+s_{n}, X\right)-G(t, X)\right\|^{2}=0 \\
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|G\left(t-s_{n}, X\right)-F(t, X)\right\|^{2}=0 \tag{7}
\end{align*}
$$

hold for each $t \in \mathbb{R}$ and each $X \in \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$.

Lemma 7. Let $f: \mathbb{R} \times \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}) \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}),(t, X) \mapsto$ $f(t, X)$ be square-mean almost automorphic in $t \in \mathbb{R}$ for each $X \in \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$, and assume that $f$ satisfies a Lipschitz condition in the following sense:

$$
\begin{equation*}
\mathbf{E}\|f(t, \varphi)-f(t, \psi)\|^{2} \leq L \mathbf{E}\|\varphi-\psi\|^{2} \tag{8}
\end{equation*}
$$

for all $\varphi, \psi \in \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ and each $t \in \mathbb{R}$, where $L>0$ is independent of $t$. Then for any square-mean almost automorphic stochastic process $X: \mathbb{R} \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$, the stochastic process $F: \mathbb{R} \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ given by $F(t):=f(t, X(t))$ is square-mean almost automorphic.

The Acquistapace-Terreni conditions (ATCs, for short) play an important role in the study of nonautonomous stochastic differential equations. We state it below for the readers' convenience.

ATCs. There exist constants $\lambda_{0} \geq 0, \theta \in(\pi / 2, \pi), L, K \geq 0$, and $\alpha, \beta \in(0,1]$ with $\alpha+\beta>1$ such that

$$
\begin{gather*}
\Sigma_{\theta} \cup\{0\} \subset \rho\left(A(t)-\lambda_{0}\right), \\
\left\|R\left(\lambda, A(t)-\lambda_{0}\right)\right\| \leq \frac{K}{1+|\lambda|},  \tag{9}\\
\|\left(A(t)-\lambda_{0}\right) R\left(\lambda, A(t)-\lambda_{0}\right) \\
\times\left[R\left(\lambda_{0}, A(t)\right)-R\left(\lambda_{0}, A(s)\right)\right] \| \leq L|t-s|^{\alpha}|\lambda|^{\beta}
\end{gather*}
$$

for $t, s \in \mathbb{R}, \lambda \in \Sigma_{\theta}:=\{\lambda \in \mathbb{C}-\{0\}:|\arg \lambda| \leq \theta\}$.
The following lemma can be found in [15, 24, 25].
Lemma 8. Suppose that the ATCs are satisfied, and then there exists a unique evolution family $\{U(t, s)\}_{-\infty<s \leq t<\infty}$ on $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$, which governs the linear part of (1).

## 3. Main Results

In this section, we investigate the existence and uniqueness of square-mean almost automorphic solution to the nonautonomous stochastic integrodifferential evolution equation (1). The following assumptions are imposed on (1) which will be assumed throughout the manuscript.
(H1) The operator $A(t): D(A(t)) \quad \subset \quad \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}) \quad \rightarrow$ $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ is a family of densely defined closed linear operators satisfying the ATCs, and the generated evolution family $U(t, s)$ is uniformly exponentially stable; that is, there exist constants $M \geq 1$ and $\delta>0$ such that

$$
\begin{equation*}
\|U(t, s)\| \leq M e^{-\delta(t-s)}, \quad \forall t \geq s \tag{10}
\end{equation*}
$$

(H2) The evolution family $\{U(t, s), t \geq s\}$ generated by $A(t)$ satisfies the following condition: from every sequence of real numbers $\left\{s_{n}^{\prime}\right\}_{n \in \mathbb{N}}$, we can extract a subsequence
$\left\{s_{n}\right\}_{n \in \mathbb{N}}$ such that, for any $\varepsilon>0$, there exists an $N \in \mathbb{N}$ such that $n>N$ implies that

$$
\begin{align*}
& \left\|U\left(t+s_{n}, s+s_{n}\right)-U(t, s)\right\| \leq \varepsilon e^{-\delta(t-s)} \\
& \left\|U\left(t-s_{n}, s-s_{n}\right)-U(t, s)\right\| \leq \varepsilon e^{-\delta(t-s)} \tag{11}
\end{align*}
$$

for all $t \geq s$, where $\delta>0$ is the constant required in (H1).
(H3) The functions $F_{i}: \mathbb{R} \times \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}) \quad \rightarrow$ $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}),(t, X) \mapsto F_{i}(t, X)(i=1,2)$, and $G: \mathbb{R} \times \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}) \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H}),(t, X) \mapsto G(t, X)$ are square-mean almost automorphic in $t \in \mathbb{R}$ for each $X \in \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$. Moreover, $F_{1}, F_{2}$, and $G$ are Lipschitz in $X$ uniformly for $t$ in the following sense: there exist constants $K_{i}>0(i=1,2,3)$ such that

$$
\begin{gather*}
\mathbf{E}\left\|F_{i}(t, X)-F_{i}(t, Y)\right\|^{2} \leq K_{i} \mathbf{E}\|X-Y\|^{2}, \quad i=1,2,  \tag{12}\\
\mathbf{E}\|G(t, X)-G(t, Y)\|^{2} \leq K_{3} \mathbf{E}\|X-Y\|^{2}
\end{gather*}
$$

for all stochastic processes $X, Y \in \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ and $t \in$ $\mathbb{R}$.

Definition 9. An $\mathscr{F}_{t}$ progressively measurable process $(X(t))_{t \in \mathbb{R}}$ is called a mild solution of (1) if it satisfies the corresponding stochastic integral equation:

$$
\begin{align*}
X(t)= & U(t, a) X(a) \\
& +\int_{a}^{t} U(t, \sigma) \int_{a}^{\sigma} C(\sigma-s) G(s, X(s)) d W(s) d \sigma \\
& +\int_{a}^{t} U(t, \sigma) \int_{a}^{\sigma} B(\sigma-s) F_{2}(s, X(s)) d s d \sigma \\
& +\int_{a}^{t} U(t, s) F_{1}(s, X(s)) d s \tag{13}
\end{align*}
$$

for all $t \geq a$ and each $a \in \mathbb{R}$.
Now we are in a position to show the existence and uniqueness of square-mean almost automorphic solution to (1).

Theorem 10. Assume that conditions (H1)-(H3) are satisfied, then (1) has a unique square-mean almost automorphic mild solution $X(\cdot) \in A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$ which can be explicitly expressed as

$$
\begin{align*}
X(t)= & \int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} C(\sigma-s) G(s, X(s)) d W(s) d \sigma \\
& +\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} B(\sigma-s) F_{2}(s, X(s)) d s d \sigma \\
& +\int_{-\infty}^{t} U(t, s) F_{1}(s, X(s)) d s \tag{14}
\end{align*}
$$

provided that

$$
\begin{equation*}
\Theta:=3 \frac{M^{2}}{\delta^{2}}\left[K_{1}+K_{2} \cdot\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2}+K_{3} \cdot\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2}\right]<1 . \tag{15}
\end{equation*}
$$

Proof. First of all, it is not difficult to verify that the stochastic process

$$
\begin{align*}
X(t)= & \int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} C(\sigma-s) G(s, X(s)) d W(s) d \sigma \\
& +\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} B(\sigma-s) F_{2}(s, X(s)) d s d \sigma \\
& +\int_{-\infty}^{t} U(t, s) F_{1}(s, X(s)) d s \tag{16}
\end{align*}
$$

is well defined and satisfies

$$
\begin{align*}
X(t)= & U(t, a) X(a) \\
& +\int_{a}^{t} U(t, \sigma) \int_{a}^{\sigma} C(\sigma-s) G(s, X(s)) d W(s) d \sigma \\
& +\int_{a}^{t} U(t, \sigma) \int_{a}^{\sigma} B(\sigma-s) F_{2}(s, X(s)) d s d \sigma \\
& +\int_{a}^{t} U(t, s) F_{1}(s, X(s)) d s \tag{17}
\end{align*}
$$

for all $t \geq a$ and each $a \in \mathbb{R}$, and hence it is a mild solution of the original (1).

To seek the square-mean almost automorphic mild solution to (1), let us consider the nonlinear operator $\mathcal{S}$ acting on the Banach space $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$ given by

$$
\begin{align*}
(\mathcal{S} X)(t):= & \int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} C(\sigma-s) G(s, X(s)) d W(s) d \sigma \\
& +\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} B(\sigma-s) F_{2}(s, X(s)) d s d \sigma \\
& +\int_{-\infty}^{t} U(t, s) F_{1}(s, X(s)) d s . \tag{18}
\end{align*}
$$

If we can show that the operator $\mathcal{S}$ maps $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$ into itself and it is a contraction mapping, then, by Banach contraction principle, we can conclude that there is a unique square-mean almost automorphic mild solution to (1).

Now define three nonlinear operators acting on the Banach space $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$ as follows:

$$
\begin{gather*}
\left(\mathcal{S}_{1} X\right)(t):=\int_{-\infty}^{t} U(t, s) F_{1}(s, X(s)) d s \\
\left(\mathcal{S}_{2} X\right)(t):=\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} B(\sigma-s) F_{2}(s, X(s)) d s d \sigma \\
\left(\mathcal{S}_{3} X\right)(t):=\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} C(\sigma-s) G(s, X(s)) d W(s) d \sigma \tag{19}
\end{gather*}
$$

We show first that $\mathcal{S}_{1} X$ is square-mean almost automorphic whenever $X$ is. Indeed, assuming that $X$ is square-mean almost automorphic, then Lemma 7 implies that $f_{1}(\cdot)$ := $F_{1}(\cdot, X(\cdot))$ is also square-mean almost automorphic. Hence, for any sequence of real numbers $\left\{s_{n}^{\prime}\right\}$, there exists a subsequence $\left\{s_{n}\right\}$ of $\left\{s_{n}^{\prime}\right\}$ and a stochastic process $\widetilde{f_{1}}: \mathbb{R} \rightarrow$ $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ such that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|f_{1}\left(t+s_{n}\right)-\widetilde{f}_{1}(t)\right\|^{2}=0, \\
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|\widetilde{f}_{1}\left(t-s_{n}\right)-f_{1}(t)\right\|^{2}=0 \tag{20}
\end{align*}
$$

hold for each $t \in \mathbb{R}$. By assumption (H2), for any $\varepsilon>0$, there exists $N_{1}=N_{1}(\varepsilon) \in \mathbb{N}$ such that $n>N_{1}$ implies that

$$
\begin{gather*}
\left\|U\left(t+s_{n}, s+s_{n}\right)-U(t, s)\right\| \leq \varepsilon e^{-\delta(t-s)} \quad \forall t \geq s \\
\mathbf{E}\left\|f_{1}\left(t+s_{n}\right)-\widetilde{f}_{1}(t)\right\|^{2}<\varepsilon  \tag{21}\\
\mathbf{E}\left\|\widetilde{f}_{1}\left(t-s_{n}\right)-f_{1}(t)\right\|^{2}<\varepsilon \quad \forall t \in \mathbb{R} .
\end{gather*}
$$

Now, define functions on $\mathbb{R}$ as follows:

$$
\begin{align*}
u(t) & :=\int_{-\infty}^{t} U(t, s) f_{1}(s) d s \\
\widetilde{u}(t) & :=\int_{-\infty}^{t} U(t, s) \widetilde{f}_{1}(s) d s \tag{22}
\end{align*}
$$

and then the above observation together with (3.1) and Cauchy-Schwarz inequality implies that for any $\varepsilon>0$ and for the aforementioned $N_{1}=N_{1}(\varepsilon) \in \mathbb{N}$ if $n>N_{1}$ it yields that

$$
\begin{aligned}
& \mathbf{E}\left\|u\left(t+s_{n}\right)-\widetilde{u}(t)\right\|^{2} \\
&= \mathbf{E}\left\|\int_{-\infty}^{t+s_{n}} U\left(t+s_{n}, s\right) f_{1}(s) d s-\int_{-\infty}^{t} U(t, s) \widetilde{f}_{1}(s) d s\right\|^{2} \\
&= \mathbf{E} \| \int_{-\infty}^{t} U\left(t+s_{n}, s+s_{n}\right) f_{1}\left(s+s_{n}\right) d s \\
& \quad-\int_{-\infty}^{t} U(t, s) \widetilde{f}_{1}(s) d s \|^{2} \\
& \leq 2 \mathbf{E}\left\|\int_{-\infty}^{t}\left[U\left(t+s_{n}, s+s_{n}\right)-U(t, s)\right] f_{1}\left(s+s_{n}\right) d s\right\|^{2} \\
&+2 \mathbf{E}\left\|\int_{-\infty}^{t} U(t, s)\left[f_{1}\left(s+s_{n}\right)-\widetilde{f}_{1}(s)\right] d s\right\|^{2}
\end{aligned}
$$

$$
\begin{align*}
\leq & 2 \varepsilon^{2} \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\left\|f_{1}\left(s+s_{n}\right)\right\| d s\right)^{2} \\
& +2 M^{2} \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\left\|f_{1}\left(s+s_{n}\right)-\widetilde{f}_{1}(s)\right\| d s\right)^{2} \\
\leq & 2 \varepsilon^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-s)} d s\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)} \mathbf{E}\left\|f_{1}\left(s+s_{n}\right)\right\|^{2} d s\right) \\
& +2 M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-s)} d s\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-s)} \mathbf{E}\left\|f_{1}\left(s+s_{n}\right)-\widetilde{f}_{1}(s)\right\|^{2} d s\right) \\
\leq & 2 \varepsilon^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-s)} d s\right)^{2} \sup _{t \in \mathbb{R}}\left\|f_{1}\left(t+s_{n}\right)\right\|^{2} \\
& +2 M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-s)} d s\right)^{2} \sup _{t \in \mathbb{R}} \mathbb{E}\left\|f_{1}\left(t+s_{n}\right)-\widetilde{f}_{1}(t)\right\|^{2} \\
\leq & \frac{2 M_{1}}{\delta^{2}} \cdot \varepsilon^{2}+\frac{2 M^{2}}{\delta^{2}} \cdot \varepsilon, \tag{23}
\end{align*}
$$

where $M_{1}:=\sup _{t \in \mathbb{R}} \mathbf{E}\left\|f_{1}(t)\right\|^{2}<+\infty$. Hence,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbf{E}\left\|u\left(t+s_{n}\right)-\widetilde{u}(t)\right\|^{2}=0 \tag{24}
\end{equation*}
$$

for each $t \in \mathbb{R}$. And we can show in a similar way that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbf{E}\left\|\tilde{u}\left(t-s_{n}\right)-u(t)\right\|^{2}=0 \tag{25}
\end{equation*}
$$

for each $t \in \mathbb{R}$. Thus, we conclude that $u=\mathcal{S}_{1} X \in$ $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$.

In an analogous way, assuming that $X$ is square-mean almost automorphic and using Lemma 7, one can easily see that $f_{2}(\cdot):=F_{2}(\cdot, X(\cdot))$ is also square-mean almost automorphic. Let $\left\{s_{n}^{\prime}\right\}$ be an arbitrary sequence of real numbers, and then there exists a subsequence $\left\{s_{n}\right\}$ of $\left\{s_{n}^{\prime}\right\}$ and a stochastic process $\widetilde{f_{2}}: \mathbb{R} \rightarrow \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ such that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|f_{2}\left(t+s_{n}\right)-\widetilde{f}_{2}(t)\right\|^{2}=0  \tag{26}\\
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|\widetilde{f_{2}}\left(t-s_{n}\right)-f_{2}(t)\right\|^{2}=0
\end{align*}
$$

hold for each $t \in \mathbb{R}$. Now define functions on $\mathbb{R}$ as follows:

$$
\begin{align*}
& v(t):=\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} B(\sigma-s) f_{2}(s) d s d \sigma  \tag{27}\\
& \widetilde{v}(t):=\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} B(\sigma-s) \widetilde{f}_{2}(s) d s d \sigma
\end{align*}
$$

and then, by making change of variables $\tau=\sigma-s_{n}$ and $r=$ $s-s_{n}$, we obtain that

$$
\begin{align*}
& \mathbf{E}\left\|v\left(t+s_{n}\right)-\widetilde{v}(t)\right\|^{2} \\
& =\mathbf{E} \| \int_{-\infty}^{t+s_{n}} U\left(t+s_{n}, \sigma\right) \int_{-\infty}^{\sigma} B(\sigma-s) f_{2}(s) d s d \sigma \\
& -\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} B(\sigma-s) \widetilde{f}_{2}(s) d s d \sigma \|^{2} \\
& =\mathbf{E} \| \int_{-\infty}^{t} U\left(t+s_{n}, \sigma+s_{n}\right) \\
& \times \int_{-\infty}^{\sigma} B(\sigma-s) f_{2}\left(s+s_{n}\right) d s d \sigma \\
& -\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} B(\sigma-s) \widetilde{f}_{2}(s) d s d \sigma \|^{2} \\
& \leq 2 \mathrm{E} \| \int_{-\infty}^{t}\left[U\left(t+s_{n}, \sigma+s_{n}\right)-U(t, \sigma)\right] \\
& \times \int_{-\infty}^{\sigma} B(\sigma-s) f_{2}\left(s+s_{n}\right) d s d \sigma \|^{2} \\
& +2 \mathbf{E} \| \int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} B(\sigma-s) \\
& \times\left[f_{2}\left(s+s_{n}\right)-\widetilde{f_{2}}(s)\right] d s d \sigma \|^{2} \\
& \leq 2 \varepsilon^{2} \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \left.\times\left\|\int_{-\infty}^{\sigma} B(\sigma-s) f_{2}\left(s+s_{n}\right) d s\right\| d \sigma\right)^{2} \\
& +2 M^{2} \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \times \| \int_{-\infty}^{\sigma} B(\sigma-s) \\
& \left.\times\left[f_{2}\left(s+s_{n}\right)-\widetilde{f}_{2}(s)\right] d s \| d \sigma\right)^{2} . \tag{28}
\end{align*}
$$

Let us evaluate the first term of the right-handed side by using Cauchy-Schwarz inequality:

$$
\begin{aligned}
& \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\left\|\int_{-\infty}^{\sigma} B(\sigma-s) f_{2}\left(s+s_{n}\right) d s\right\| d \sigma\right)^{2} \\
& \quad \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \quad \times \int_{-\infty}^{t} e^{-\delta(t-\sigma)} \mathbf{E}\left\|\int_{-\infty}^{\sigma} B(\sigma-s) f_{2}\left(s+s_{n}\right) d s\right\|^{2} d \sigma
\end{aligned}
$$

$$
\begin{align*}
\leq & \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\left\{\int_{-\infty}^{\sigma}\|B(\sigma-s)\| d s\right\}\right. \\
& \left.\times\left\{\int_{-\infty}^{\sigma}\|B(\sigma-s)\| \mathbf{E}\left\|f_{2}\left(s+s_{n}\right)\right\|^{2} d s\right\} d \sigma\right) \\
\leq & \left\{\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right\}^{2} \cdot\left\{\int_{0}^{+\infty}\|B(u)\| d u\right\}^{2} \\
& \cdot \sup _{s \in \mathbb{R}} \mathbf{E}\left\|f_{2}\left(s+s_{n}\right)\right\|^{2} \leq \frac{\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2}}{\delta^{2}} \cdot \sup _{t \in \mathbb{R}} \mathbf{E}\left\|f_{2}(t)\right\|^{2} \tag{29}
\end{align*}
$$

As to the second term, in a similar manner as above, we have the following observation:

$$
\begin{aligned}
& \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\left\|\int_{-\infty}^{\sigma} B(\sigma-s)\left[f_{2}\left(s+s_{n}\right)-\widetilde{f_{2}}(s)\right] d s\right\| d \sigma\right)^{2} \\
& \leq \mathbf{E}\left[\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right)\right. \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \times \| \int_{-\infty}^{\sigma} B(\sigma-s) \\
& \left.\left.\times\left[f_{2}\left(s+s_{n}\right)-\widetilde{f}_{2}(s)\right] d s \|^{2} d \sigma\right)\right] \\
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \times \mathbf{E} \| \int_{-\infty}^{\sigma} B(\sigma-s) \\
& \left.\times\left[f_{2}\left(s+s_{n}\right)-\widetilde{f}_{2}(s)\right] d s \|^{2} d \sigma\right) \\
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\left\{\int_{-\infty}^{\sigma}\|B(\sigma-s)\| d s\right\}\right. \\
& \times\left\{\int_{-\infty}^{\sigma}\|B(\sigma-s)\|\right. \\
& \left.\left.\times \mathbf{E}\left\|f_{2}\left(s+s_{n}\right)-\widetilde{f_{2}}(s)\right\|^{2} d s\right\} d \sigma\right) \\
& \leq\left\{\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right\}^{2} \cdot\left\{\int_{0}^{+\infty}\|B(u)\| d u\right\}^{2}
\end{aligned}
$$

$$
\begin{align*}
& \cdot \sup _{t \in \mathbb{R}} \mathbb{E}\left\|f_{2}\left(t+s_{n}\right)-\widetilde{f_{2}}(t)\right\|^{2} \\
\leq & \frac{1}{\delta^{2}} \cdot\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2} \cdot \sup _{t \in \mathbb{R}} \mathbf{E}\left\|f_{2}\left(t+s_{n}\right)-\widetilde{f_{2}}(t)\right\|^{2} \\
\leq & \frac{\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2}}{\delta^{2}} \cdot \varepsilon . \tag{30}
\end{align*}
$$

Based on the above argument, for arbitrary $\varepsilon>0$, thanks to the boundedness and square-mean almost automorphy of $f_{2}$, there exists certain $N_{2}=N_{2}(\varepsilon) \in \mathbb{N}$ such that $n>N_{2}$ implies that

$$
\begin{align*}
& \mathbf{E}\left\|v\left(t+s_{n}\right)-\widetilde{v}(t)\right\|^{2} \\
& \leq 2 \varepsilon^{2} \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \left.\times\left\|\int_{-\infty}^{\sigma} B(\sigma-s) f_{2}\left(s+s_{n}\right) d s\right\| d \sigma\right)^{2} \\
& +2 M^{2} \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \times \| \int_{-\infty}^{\sigma} B(\sigma-s) \\
& \left.\times\left[f_{2}\left(s+s_{n}\right)-\widetilde{f}_{2}(s)\right] d s \| d \sigma\right)^{2} \\
& \leq \frac{2\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2} \cdot \sup _{t \in \mathbb{R}} \mathbf{E}\left\|f_{2}(t)\right\|^{2}}{\delta^{2}} \\
& \cdot \varepsilon^{2}+\frac{2 M^{2} \cdot\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2}}{\delta^{2}} \cdot \varepsilon, \tag{31}
\end{align*}
$$

where we use the fact that, for arbitrary $\varepsilon>0$, there exists $N_{2}=N_{2}(\varepsilon) \in \mathbb{N}$ such that, for all $n>N_{2}$,

$$
\begin{equation*}
\mathbf{E}\left\|f_{2}\left(t+s_{n}\right)-\widetilde{f_{2}}(t)\right\|^{2}<\varepsilon \tag{32}
\end{equation*}
$$

holds for all $t \in \mathbb{R}$. This indicates that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbf{E}\left\|v\left(t+s_{n}\right)-\widetilde{v}(t)\right\|^{2}=0 \tag{33}
\end{equation*}
$$

for each $t \in \mathbb{R}$. In an analogous way, we can show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbf{E}\left\|\widetilde{v}\left(t-s_{n}\right)-v(t)\right\|^{2}=0 \tag{34}
\end{equation*}
$$

for each $t \in \mathbb{R}$. Combining (33) with (34), we obtain that $v=$ $\mathcal{S}_{2} X$ is square-mean almost automorphic.

Assuming that $X \in A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$, then similar argument as above ensures that $g(\cdot):=G(\cdot, X(\cdot)) \quad \epsilon$ $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$. As a consequence, for every sequence of real numbers $\left\{s_{n}^{\prime}\right\}$ there exist a subsequence $\left\{s_{n}\right\} \subset\left\{s_{n}^{\prime}\right\}$ and a stochastic process $\tilde{g}$ such that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|g\left(t+s_{n}\right)-\tilde{g}(t)\right\|^{2}=0 \\
& \lim _{n \rightarrow \infty} \mathbf{E}\left\|\tilde{g}\left(t-s_{n}\right)-g(t)\right\|^{2}=0 \tag{35}
\end{align*}
$$

hold for each $t \in \mathbb{R}$. Hence, for arbitrary $\varepsilon>0$, there exists certain $N_{3}=N_{3}(\varepsilon) \in \mathbb{N}$ such that, for all $n>N_{3}$, there holds

$$
\begin{equation*}
\mathbf{E}\left\|g\left(t+s_{n}\right)-\tilde{g}(t)\right\|^{2}<\varepsilon \tag{36}
\end{equation*}
$$

for all $t \in \mathbb{R}$.
The next step aims to prove the square-mean almost automorphy of $\mathcal{S}_{3} X$. This is more complicated because the involvement of the Brownian motion W. Consider the Brownian motion $\widetilde{W}$ defined by

$$
\begin{equation*}
\widetilde{W}(s)=W\left(s+s_{n}\right)-W\left(s_{n}\right) \tag{37}
\end{equation*}
$$

for each $s \in \mathbb{R}$, which has the same distribution as $W$. Define two functions on $\mathbb{R}$ as below:

$$
\begin{align*}
& \omega(t):=\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} C(\sigma-s) g(s) d W(s) d \sigma \\
& \widetilde{\omega}(t):=\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} C(\sigma-s) \tilde{g}(s) d W(s) d \sigma \tag{38}
\end{align*}
$$

By making change of variables $\tau=\sigma-s_{n}$ and $r=s-s_{n}$, and then using Cauchy-Schwarz inequality, we obtain that

$$
\begin{aligned}
& \mathbf{E}\left\|\omega\left(t+s_{n}\right)-\widetilde{\omega}(t)\right\|^{2} \\
& =\mathbf{E} \| \int_{-\infty}^{t+s_{n}} U\left(t+s_{n}, \sigma\right) \int_{-\infty}^{\sigma} C(\sigma-s) g(s) d W(s) d \sigma \\
& -\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} C(\sigma-s) \tilde{g}(s) d W(s) d \sigma \|^{2} \\
& =\mathrm{E} \| \int_{-\infty}^{t} U\left(t+s_{n}, \sigma+s_{n}\right) \\
& \times \int_{-\infty}^{\sigma} C(\sigma-s) g\left(s+s_{n}\right) d W\left(s+s_{n}\right) d \sigma \\
& -\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} C(\sigma-s) \tilde{g}(s) d W(s) d \sigma \|^{2} \\
& =\mathbf{E} \| \int_{-\infty}^{t} U\left(t+s_{n}, \sigma+s_{n}\right) \\
& \times \int_{-\infty}^{\sigma} C(\sigma-s) g\left(s+s_{n}\right) d \widetilde{W}(s) d \sigma \\
& -\int_{-\infty}^{t} U(t, \sigma) \int_{-\infty}^{\sigma} C(\sigma-s) \widetilde{g}(s) d \widetilde{W}(s) d \sigma \|^{2} \\
& \leq 2 \mathrm{E} \| \int_{-\infty}^{t}\left[U\left(t+s_{n}, \sigma+s_{n}\right)-U(t, \sigma)\right] \\
& \times \int_{-\infty}^{\sigma} C(\sigma-s) g\left(s+s_{n}\right) d \widetilde{W}(s) d \sigma \|^{2}
\end{aligned}
$$

$$
\begin{align*}
& +2 \mathbf{E} \| \int_{-\infty}^{t} U(t, \sigma) \\
& \times \int_{-\infty}^{\sigma} C(\sigma-s) \\
& \times\left[g\left(s+s_{n}\right)-\widetilde{g}(s)\right] d \widetilde{W}(s) d \sigma \|^{2} \\
& \leq 2 \varepsilon^{2} \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \left.\times\left\|\int_{-\infty}^{\sigma} C(\sigma-s) g\left(s+s_{n}\right) d \widetilde{W}(s)\right\| d \sigma\right)^{2} \\
& +2 M^{2} \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \times \| \int_{-\infty}^{\sigma} C(\sigma-s) \\
& \times\left[g\left(s+s_{n}\right)\right. \\
& -\widetilde{g}(s)] d \widetilde{W}(s) \| d \sigma)^{2} \\
& \leq 2 \varepsilon^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \left.\times \mathbf{E}\left\|\int_{-\infty}^{\sigma} C(\sigma-s) g\left(s+s_{n}\right) d \widetilde{W}(s)\right\|^{2} d \sigma\right) \\
& +2 M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \times \mathbf{E} \| \int_{-\infty}^{\sigma} C(\sigma-s) \\
& \left.\times\left[g\left(s+s_{n}\right)-\widetilde{g}(s)\right] d \widetilde{W}(s) \|^{2} d \sigma\right) . \tag{39}
\end{align*}
$$

For the above-mentioned $\varepsilon>0$, by using an estimate on Ito integral established in [26], it follows that once $n>N_{3}$, then the first term of the above inequality can be evaluated as

$$
\begin{aligned}
& \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \quad \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \left.\quad \times \mathrm{E}\left\|\int_{-\infty}^{\sigma} C(\sigma-s) g\left(s+s_{n}\right) d \widetilde{W}(s)\right\|^{2} d \sigma\right)
\end{aligned}
$$

$$
\begin{align*}
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
&\left.\times\left\{\int_{-\infty}^{\sigma} \mathbf{E}\left\|C(\sigma-s) g\left(s+s_{n}\right)\right\|^{2} d s\right\} d \sigma\right) \\
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
&\left.\quad \times\left\{\int_{-\infty}^{\sigma}\|C(\sigma-s)\|^{2} \mathbf{E}\left\|g\left(s+s_{n}\right)\right\|^{2} d s\right\} d \sigma\right) \\
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\left\{\int_{-\infty}^{\sigma}\|C(\sigma-s)\|^{2} d s\right\} d \sigma\right) \\
& \cdot \sup _{s \in \mathbb{R}} \mathbf{E}\left\|g\left(s+s_{n}\right)\right\|^{2} \\
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right)^{2} \cdot\left(\int_{0}^{\infty}\|C(u)\|^{2} d u\right) \\
& \cdot \sup _{t \in \mathbb{R}} \mathbf{E}\|g(t)\|^{2} \\
& \leq \frac{C \|_{\mathscr{L}^{2}(0, \infty)}^{2}}{\delta^{2}} \cdot \sup \left\|_{t \in \mathbb{R}}\right\| g(t) \|^{2},  \tag{40}\\
&
\end{align*}
$$

and the second term can be evaluated analogously as

$$
\begin{aligned}
& \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \quad \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \mathbf{E} \| \int_{-\infty}^{\sigma} C(\sigma-s)\right. \\
& \left.\times\left[g\left(s+s_{n}\right)-\tilde{g}(s)\right] d \widetilde{W}(s) \|^{2} d \sigma\right) \\
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \quad \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\left\{\int_{-\infty}^{\sigma} \mathbf{E}\left\|C(\sigma-s)\left[g\left(s+s_{n}\right)-\widetilde{g}(s)\right]\right\|^{2} d s\right\} d \sigma\right) \\
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \quad \times\left(\int _ { - \infty } ^ { t } e ^ { - \delta ( t - \sigma ) } \left\{\int_{-\infty}^{\sigma}\|C(\sigma-s)\|^{2} \mathbf{E}\right.\right. \\
& \left.\left.\times\left\|g\left(s+s_{n}\right)-\widetilde{g}(s)\right\|^{2} d s\right\} d \sigma\right)
\end{aligned}
$$

$$
\begin{align*}
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right)\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \left.\quad \times\left\{\int_{-\infty}^{\sigma}\|C(\sigma-s)\|^{2} d s\right\} d \sigma\right) \\
& \cdot \sup _{t \in \mathbb{R}}\left\|g\left(s+s_{n}\right)-\tilde{g}(s)\right\|^{2} \\
& \leq\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right)^{2} \\
& \quad \cdot\left(\int_{0}^{\infty}\|C(u)\|^{2} d u\right) \cdot \sup _{t \in \mathbb{R}}\left\|g\left(s+s_{n}\right)-\tilde{g}(s)\right\|^{2} \\
& \leq \frac{\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2}}{\delta^{2}} \cdot \varepsilon . \tag{41}
\end{align*}
$$

The above argument yields that

$$
\begin{align*}
& \mathbf{E} \| \omega(t+\left.s_{n}\right)-\widetilde{\omega}(t) \|^{2} \\
& \leq 2 \varepsilon^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
&\left.\quad \times \mathbf{E}\left\|\int_{-\infty}^{\sigma} C(\sigma-s) g\left(s+s_{n}\right) d \widetilde{W}(s)\right\|^{2} d \sigma\right) \\
& \quad 2 M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \quad \times \mathbf{E} \| \int_{-\infty}^{\sigma} C(\sigma-s) \\
& \leq \frac{2\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2} \cdot \sup _{t \in \mathbb{R}} \mathbf{E}\|g(t)\|^{2}}{\delta^{2}} \cdot \varepsilon^{2} \\
& \quad+\frac{2 M^{2} \cdot\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2}}{\delta^{2}} \cdot \varepsilon,
\end{align*}
$$

which implies that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbf{E}\left\|\omega\left(t+s_{n}\right)-\widetilde{\omega}(t)\right\|^{2}=0 \tag{43}
\end{equation*}
$$

for each $t \in \mathbb{R}$. Analogously, we can show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbf{E}\left\|\widetilde{\omega}\left(t-s_{n}\right)-\omega(t)\right\|^{2}=0 \tag{44}
\end{equation*}
$$

for each $t \in \mathbb{R}$. Combining (43) with (44), we obtain that $\omega=\delta_{3} X$ is square-mean almost automorphic.

In view of the above arguments, it follows from (18) that the nonlinear operator $\mathcal{S}=\mathcal{S}_{1}+\mathcal{S}_{2}+\mathcal{S}_{3}$ maps $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{W})\right)$ into itself. To complete the proof, it suffices to show that $\mathcal{S}$ is a contraction mapping on $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$. Indeed, for each $X, Y \in A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$, thanks to the fact that $(a+b+c)^{2} \leq 3 a^{2}+3 b^{2}+3 c^{2}$, we have the following observation:
$\mathbf{E}\|(\mathcal{S} X)(t)-(\mathcal{S} Y)(t)\|^{2}$

$$
\begin{aligned}
& \leq 3 \mathrm{E}\left\|\int_{-\infty}^{t} U(t, s)\left[F_{1}(s, X(s))-F_{1}(s, Y(s))\right] d s\right\|^{2} \\
& +3 \mathrm{E} \| \int_{-\infty}^{t} U(t, \sigma) \\
& \quad \times \int_{-\infty}^{\sigma} B(\sigma-s)
\end{aligned}
$$

$$
\times\left[F_{2}(s, X(s))-F_{2}(s, Y(s))\right] d s d \sigma \|^{2}
$$

$$
+3 \mathbf{E} \| \int_{-\infty}^{t} U(t, \sigma)
$$

$$
\times \int_{-\infty}^{\sigma} C(\sigma-s)
$$

$$
\times[G(s, X(s))-G(s, Y(s))] d W(s) d \sigma \|^{2}
$$

$$
\leq 3 \mathrm{E}\left(\int_{-\infty}^{t}\|U(t, s)\|\left\|F_{1}(s, X(s))-F_{1}(s, Y(s))\right\| d s\right)^{2}
$$

$$
+3 \mathrm{E}\left(\int_{-\infty}^{t}\|U(t, \sigma)\|\right.
$$

$$
\times \int_{-\infty}^{\sigma} \| B(\sigma-s)
$$

$$
\left.\times\left[F_{2}(s, X(s))-F_{2}(s, Y(s))\right] \| d s d \sigma\right)^{2}
$$

$$
+3 \mathbf{E}\left(\int_{-\infty}^{t}\|U(t, \sigma)\|\right.
$$

$$
\times \| \int_{-\infty}^{\sigma} C(\sigma-s)[G(s, X(s))
$$

$$
\begin{equation*}
-G(s, Y(s))] d W(s) \| d \sigma)^{2} \tag{45}
\end{equation*}
$$

Now, we evaluate the first term of the right-hand side with the help of Cauchy-Schwarz inequality as follows:

$$
\begin{aligned}
& 3 \mathrm{E}\left(\int_{-\infty}^{t}\|U(t, s)\|\left\|F_{1}(s, X(s))-F_{1}(s, Y(s))\right\| d s\right)^{2} \\
& \quad \leq 3 M^{2} \mathbf{E}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\left\|F_{1}(s, X(s))-F_{1}(s, Y(s))\right\| d s\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
\leq & 3 M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-s)} d s\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-s)} \mathbf{E}\left\|F_{1}(s, X(s))-F_{1}(s, Y(s))\right\|^{2} d s\right) \\
\leq & 3 M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-s)} d s\right)^{2} \\
& \times \sup _{s \in \mathbb{R}} \mathrm{E}\left\|F_{1}(s, X(s))-F_{1}(s, Y(s))\right\|^{2} \\
\leq & 3 K_{1} M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-s)} d s\right)^{2} \sup _{t \in \mathbb{R}} \mathbf{E}\|X(t)-Y(t)\|^{2} \\
= & \frac{3 K_{1}}{\delta^{2}} \cdot M^{2} \cdot \sup _{t \in \mathbb{R}} \mathrm{E}\|X(t)-Y(t)\|^{2} .
\end{aligned}
$$

As to the second term, we can proceed in the same manner as above and obtain

$$
\begin{align*}
& 3 \mathrm{E}\left(\int_{-\infty}^{t}\|U(t, \sigma)\|\right. \\
& \left.\quad \times \int_{-\infty}^{\sigma}\left\|B(\sigma-s)\left[F_{2}(s, X(s))-F_{2}(s, Y(s))\right]\right\| d s d \sigma\right)^{2} \\
& \quad \leq \frac{3 K_{2}}{\delta^{2}} \cdot M^{2} \cdot\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2} \cdot \sup _{t \in \mathbb{R}}\|X(t)-Y(t)\|^{2} \tag{47}
\end{align*}
$$

As far as the last term of the right-hand side is concerned, we use again the estimate on the Ito integral to obtain

$$
\begin{aligned}
& 3 \mathbf{E}\left(\int_{-\infty}^{t}\|U(t, \sigma)\|\right. \\
& \times \| \int_{-\infty}^{\sigma} C(\sigma-s) \\
& \times
\end{aligned}
$$

$$
\begin{align*}
& \leq 3 M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \times \mathbf{E} \| \int_{-\infty}^{\sigma} C(\sigma-s) \\
& \left.\times[G(s, X(s))-G(s, Y(s))] d W(s) \|^{2} d \sigma\right) \\
& \leq 3 M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right) \\
& \times\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\right. \\
& \times \int_{-\infty}^{\sigma}\|C(\sigma-s)\|^{2} \\
& \left.\times \mathbf{E}\|G(s, X(s))-G(s, Y(s))\|^{2} d s d \sigma\right) \\
& \leq 3 M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right)^{2} \cdot\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2} \\
& \cdot \sup _{s \in \mathbb{R}} \mathrm{E}\|G(s, X(s))-G(s, Y(s))\|^{2} \\
& \leq 3 K_{3} M^{2}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d \sigma\right)^{2} \cdot\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2} \\
& \cdot \sup _{t \in \mathbb{R}} \mathbf{E}\|X(t)-Y(t)\|^{2} \\
& =\frac{3 K_{3}}{\delta^{2}} \cdot M^{2} \cdot\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2} \cdot \sup _{t \in \mathbb{R}} \mathrm{E}\|X(t)-Y(t)\|^{2} . \tag{48}
\end{align*}
$$

Thus, by combining the three inequalities together, we obtain that, for each $t \in \mathbb{R}$,

$$
\begin{align*}
& \mathbf{E}\|(\mathcal{S} X)(t)-(\mathcal{S} Y)(t)\|^{2} \\
& \leq\left\{3 \frac{M^{2}}{\delta^{2}}\left[K_{1}+K_{2} \cdot\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2}+K_{3} \cdot\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2}\right]\right\} \\
& \cdot \sup _{t \in \mathbb{R}} \mathbf{E}\|X(t)-Y(t)\|^{2} . \tag{49}
\end{align*}
$$

That is,

$$
\begin{equation*}
\|(\mathcal{S} X)(t)-(\mathcal{S} Y)(t)\|_{2}^{2} \leq \Theta \sup _{t \in \mathbb{R}}\|X(t)-Y(t)\|_{2}^{2} \tag{50}
\end{equation*}
$$

where $\Theta:=3\left(M^{2} / \delta^{2}\right)\left[K_{1}+K_{2} \cdot\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2}+K_{3} \cdot\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2}\right]$. Notice that

$$
\begin{equation*}
\sup _{t \in \mathbb{R}}\|X(t)-Y(t)\|_{2}^{2} \leq\left(\sup _{t \in \mathbb{R}}\|X(t)-Y(t)\|_{2}\right)^{2} . \tag{51}
\end{equation*}
$$

As a result, (50) together with (51) gives that, for each $t \in \mathbb{R}$,

$$
\begin{align*}
& \|(\mathcal{S} X)(t)-(\mathcal{S} Y)(t)\|_{2} \leq \sqrt{\Theta} \sup _{t \in \mathbb{R}}\|X(t)-Y(t)\|_{2}  \tag{52}\\
& =\sqrt{\Theta}\|X-Y\|_{\infty} .
\end{align*}
$$

It follows that

$$
\begin{align*}
\| \mathcal{S} X & -\mathcal{S} Y \|_{\infty} \\
& =\sup _{t \in \mathbb{R}}\|(\mathcal{S} X)(t)-(\mathcal{S} Y)(t)\|_{2} \leq \sqrt{\Theta}\|X-Y\|_{\infty}, \tag{53}
\end{align*}
$$

which implies that $\mathcal{S}$ is a contraction mapping by the assumption (15) imposed on $\Theta$. Therefore, by the Banach contraction principle, we conclude that there exists a unique fixed point $X$ for $\mathcal{S}$ in $A A\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})\right)$, which is the unique squaremean almost automorphic mild solution to the functional integrodifferential semilinear stochastic evolution equation (1) as we have claimed. The proof is complete.

Remark 11. If the functions $F_{1}, F_{2}, G$ in (1) are square-mean almost periodic in $t$, then the unique square-mean almost automorphic solution obtained in Theorem 10 is actually square-mean almost periodic; see paper [27].

Now we are in a position to show the asymptotically stable property of the unique square-mean almost automorphic solution to (1). Recall that the unique square-mean almost automorphic solution $X^{*}(t)$ of (1) is said to be stable in square-mean sense if, for arbitrary $\epsilon>0$, there exists $\delta>0$ such that

$$
\begin{equation*}
\mathbf{E}\left\|X(t)-X^{*}(t)\right\|^{2}<\epsilon, \quad t \geq 0 \tag{54}
\end{equation*}
$$

whenever $\mathbf{E}\left\|X(0)-X^{*}(0)\right\|^{2}<\delta$, where $X(t)$ stands for a solution of (1) with initial value $X(0)$. The solution $X^{*}(t)$ is said to be asymptotically stable in square-mean sense if it is stable in square-mean sense and

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \mathbf{E}\left\|X(t)-X^{*}(t)\right\|^{2}=0 \tag{55}
\end{equation*}
$$

The following Gronwall-type inequality is proved to be useful in our asymptotic stability analysis.

Lemma 12. Let $u(t), b(t)$ be nonnegative continuous functions for $t \geq a$, and $\alpha, \gamma$ be some positive constants. If

$$
\begin{equation*}
u(t) \leq \alpha e^{-\beta(t-a)}+\int_{a}^{t} e^{-\beta(t-s)} b(s) u(s) d s, \quad t \geq a \tag{56}
\end{equation*}
$$

then

$$
\begin{equation*}
u(t) \leq \alpha \exp \left\{-\beta(t-a)+\int_{a}^{t} b(s) d s\right\} . \tag{57}
\end{equation*}
$$

Theorem 13. Let all the assumptions in Theorem 10 hold and assume that

$$
\begin{equation*}
\frac{M^{2}}{\delta^{2}}\left[K_{1}+K_{2} \cdot\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2}+K_{3} \cdot\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2}\right]<\frac{1}{4} . \tag{58}
\end{equation*}
$$

Then the unique square-mean almost automorphic mild solution $X^{*}(t)$ of $(1)$ is asymptotically stable in square-mean sense.

Proof. Let $X(t)$ be any mild solution of (1) with initial value $X(0)$. Then, on account of (H1)-(H2) and the assumptions imposed on $B$ and $C$, along the same line as in [9] we could show that, for any $t \geq 0$,

$$
\begin{align*}
& \mathbf{E}\left\|X(t)-X^{*}(t)\right\|^{2} \\
& =\mathbf{E} \| U(t, 0)\left[X(0)-X^{*}(0)\right] \\
& +\int_{0}^{t} U(t, s) \\
& \times\left[F_{1}(s, X(s))-F_{1}\left(s, X^{*}(s)\right)\right] d s \\
& +\int_{0}^{t} U(t, \sigma) \\
& \times \int_{0}^{\sigma} B(\sigma-s) \\
& \times\left[F_{2}(s, X(s))-F_{2}\left(s, X^{*}(s)\right)\right] d s d \sigma \\
& +\int_{0}^{t} U(t, \sigma) \\
& \times \int_{0}^{\sigma} C(\sigma-s) \\
& \times\left[G(s, X(s))-G\left(s, X^{*}(s)\right)\right] d W(s) d \sigma \|^{2} \\
& \leq 4 \mathrm{E}\left\|U(t, 0)\left[X(0)-X^{*}(0)\right]\right\|^{2} \\
& +4 \mathrm{E}\left\|\int_{0}^{t} U(t, s)\left[F_{1}(s, X(s))-F_{1}\left(s, X^{*}(s)\right)\right]\right\|^{2} \\
& +4 \mathbf{E} \| \int_{0}^{t} U(t, \sigma) \\
& \times \int_{0}^{\sigma} B(\sigma-s) \\
& \times\left[F_{2}(s, X(s))-F_{2}\left(s, X^{*}(s)\right)\right] d s d \sigma \|^{2} \\
& +4 \mathbf{E} \| \int_{0}^{t} U(t, \sigma) \\
& \times \int_{0}^{\sigma} C(\sigma-s) \\
& \times\left[G(s, X(s))-G\left(s, X^{*}(s)\right)\right] d W(s) d \sigma \|^{2} \\
& \leq 4 M^{2} e^{-\delta t} \mathbf{E}\left\|X(0)-X^{*}(0)\right\|^{2} \\
& +\kappa \int_{0}^{t} e^{-\delta(t-s)} \mathbf{E}\left\|X(s)-X^{*}(s)\right\|^{2} d s, \tag{59}
\end{align*}
$$

where $\kappa:=\left(4 M^{2}\left(K_{1}+K_{2} \cdot\|B\|_{\mathscr{L}^{1}(0, \infty)}^{2}+K_{3} \cdot\|C\|_{\mathscr{L}^{2}(0, \infty)}^{2}\right)\right) / \delta$.

Define $Y(t):=\mathbf{E}\left\|X(t)-X^{*}(t)\right\|^{2}$, and it yields that

$$
\begin{equation*}
Y(t) \leq 4 M^{2} Y(0) e^{-\delta t}+\kappa \int_{0}^{t} e^{-\delta(t-s)} Y(s) d s \tag{60}
\end{equation*}
$$

Hence, it follows from Lemma 12 that

$$
\begin{equation*}
Y(t) \leq 4 M^{2} Y(0) \exp \{(-\delta+\kappa) t\} \tag{61}
\end{equation*}
$$

Straightforwardly, we obtain that $Y(t)$ converges to 0 exponentially fast if $-\delta+\kappa<0$, which is equivalent to our condition (58). Thus, we come to the conclusion that the unique square-mean almost automorphic mild solution $X^{*}(t)$ of (1) is asymptotically stable in square-mean sense. The proof is completed.

## 4. Applications

To illustrate the applications of our abstract results, let us consider the following nonautonomous functional integrodifferential stochastic partial differential equation:

$$
\begin{align*}
\frac{\partial X}{\partial t}= & \frac{\partial^{2} X}{\partial x^{2}}+\int_{-\infty}^{t} C(t-s) G(s, X(s, x)) d W(s) \\
& +\int_{-\infty}^{t} B(t-s) F_{2}(s, X(s, x)) d s+F_{1}(t, X(t, x)) \tag{62}
\end{align*}
$$

for $t \in \mathbb{R}$ and $x \in \Omega$, where $\Omega \subset \mathbb{R}^{n}$ is a bounded subset whose boundary $\partial \Omega$ is both of class $C^{2}$ and locally on one side of $\Omega$. Suppose further that (62) satisfies the following boundary conditions:

$$
\begin{equation*}
\sum_{i, j=1}^{n} n_{i}(x) a_{i j}(t, x) \frac{d X(t, x)}{d x_{i}}=0, \quad t \in \mathbb{R}, x \in \partial \Omega \tag{63}
\end{equation*}
$$

where $n(x)=\left(n_{1}(x), n_{2}(x), \ldots, n_{n}(x)\right)$ is the outer unit normal vector. A family of operators $A(t, x)$ defined by $\partial^{2} X / \partial x^{2}=A(t, x) X(t, x)$ is formally assigned to be

$$
\begin{equation*}
A(t, x)=\sum_{i, j=1}^{n} \frac{\partial}{\partial x_{i}}\left(a_{i j}(t, x) \frac{\partial}{\partial x_{j}}\right)+c(t, x), \quad t \in \mathbb{R}, x \in \Omega, \tag{64}
\end{equation*}
$$

where $a_{i j}(i, j=1,2, \ldots, n)$ and $c$ satisfy the following conditions.
(H4) The coefficients $a_{i j}(i, j=1,2, \ldots, n)$ are symmetric, that is, $a_{i j}=a_{j i}$ for all $i, j=1,2, \ldots, n$. In addition,

$$
\begin{align*}
a_{i j} \in & C_{b}^{\mu}\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; C(\bar{\Omega}))\right) \\
& \cap C_{b}\left(\mathbb{R} ; \mathscr{L}^{2}\left(\mathbf{P} ; C^{1}(\bar{\Omega})\right)\right)  \tag{65}\\
& \cap A A\left(\mathbb{R} ; \mathscr{L}^{2}\left(\mathbf{P} ; \mathscr{L}^{2}(\Omega)\right)\right)
\end{align*}
$$

for all $i, j=1,2, \ldots, n$ and

$$
\begin{align*}
c \in & C_{b}^{\mu}\left(\mathbb{R} ; \mathscr{L}^{2}\left(\mathbf{P} ; \mathscr{L}^{2}(\Omega)\right)\right) \cap C_{b}\left(\mathbb{R} ; \mathscr{L}^{2}(\mathbf{P} ; C(\bar{\Omega}))\right) \\
& \cap A A\left(\mathbb{R} ; \mathscr{L}^{2}\left(\mathbf{P} ; \mathscr{L}^{1}(\Omega)\right)\right) \tag{66}
\end{align*}
$$

for some $\mu \in(1 / 2,1]$, where $\bar{\Omega}$ means the closure of $\Omega$.
(H5) There exists $\delta_{0}>0$ such that

$$
\begin{equation*}
\sum_{i, j=1}^{n} a_{i j}(t, x) \eta_{i} \eta_{j} \geq \delta_{0}|\eta|^{2} \tag{67}
\end{equation*}
$$

for all $(t, x) \in \mathbb{R} \times \bar{\Omega}$ and $\eta \in \mathbb{R}^{n}$.
Now, let $\mathbb{H}=\mathscr{L}^{2}(\Omega)$ and let $\mathscr{H}^{2}(\Omega)$ be the Sobolev space of order 2 on $\Omega$. For each $t \in \mathbb{R}$, define an operator $A(t)$ on $\mathscr{L}^{2}(\mathbf{P} ; \mathbb{H})$ by

$$
\begin{equation*}
A(t) X=A(t, x) X \quad \forall X \in \mathscr{D}(A(t)), \tag{68}
\end{equation*}
$$

where

$$
\begin{align*}
\mathscr{D}(A(t))= & \left\{X \in \mathscr{L}^{2}\left(\mathbf{P}, \mathscr{H}^{2}(\Omega)\right):\right. \\
& \left.\sum_{i, j=1}^{n} n_{i}(\cdot) a_{i j}(t, \cdot) \frac{d X(t, \cdot)}{d x_{i}}=0 \text { on } \partial \Omega\right\} . \tag{69}
\end{align*}
$$

Under assumptions (H4)-(H5), the existence of an evolution family $U(t, s)$ satisfying (H1) is guaranteed; see, for example, [28].

And thus, as an immediate consequence of Theorem 10, it yields the following.

Theorem 14. Under assumptions (H2), (H3), (H4), and (H5), the nonautonomous integrodifferential stochastic evolution equation (62)-(63) has a unique mild solution which is squaremean almost automorphic provided that (15) holds. If, in addition, (58) is valid, then the unique almost automorphic solution is asymptotically stable in square-mean sense.

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## Research Article

# Stability and Permanence of a Pest Management Model with Impulsive Releasing and Harvesting 

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#### Abstract

We formulate a pest management model with periodically releasing infective pests, immature and mature natural enemies, and harvesting pests and crops at two different fixed moments. Sufficient conditions ensuring the locally and globally asymptotical stability of the susceptible pest-eradication period solution are found by means of Floquet theory, small amplitude perturbation techniques, and multicomparison results. Furthermore, the permanence of system is also derived. By numerical analysis, we also show that impulsive releasing and harvesting at two different fixed moments can bring obvious effects on the dynamics of system, which also corroborates our theoretical results.


## 1. Introduction

As is known to all, pest outbreaks often cause serious ecological and economic problems. Therefore, how to effectively control insects and other arthropods has become an increasingly complex issue. Usually, chemical pesticides were taken as a relatively simple way to solve the pest-related problems, and some mathematical models on pest management with toxin (pesticide) input were studied in [1-4]. However, the overuse of chemical pesticides may create new ecological and sociological harm such as pesticide pollution and pesticideresistant pest varieties and inflicts harmful effects on humans and so forth. Therefore, nonchemical use instead for pest control has become a hot topic in order to reduce pest density to tolerable levels and minimize the damage caused. For instance, biological control methods by periodically releasing infective pests or their natural enemies are often taken due to their advantage in the aspects of self-sustainable mechanism, lower environmental impact, and cost effectiveness.

Recently, some biocontrol models on pest management described by impulsive differential equation were proposed and the dynamics such as stability, permanence, periodicity, and bifurcation are deeply investigated (see also, e.g., [2-12]). In [5], an impulsive system to model the process of periodic releasing natural enemies and harvesting pest at different fixed time for pest control is considered, and the sufficient conditions on the existence and global stability of the periodic
solution are derived for the given model. Georgescu et al. $[6,7]$ construct an integrated pest management model which relies on the simultaneous periodic release of infective pest individuals and of natural predators with age structure and obtain some sufficient conditions on the local and global stability, permanence, and bifurcation of the systems. However, most of the existing models on pest management scarcely take into account the factor on the relation between pest and its food (e.g., crop). In fact, farmers may harvest crops several times in process of its growth, which should cause a great impact on the density of the pest.

Motived by the above discussion, we construct a model of pest control by periodically releasing infective pests, immature and mature natural enemies, and harvesting pests and crops. To account for the discontinuity of release and harvest at different fixed moments, our model is based on impulsive differential equations. We analyze the dynamical behavior of the system by using the theory of impulsive differential equation introduced in [13-15].

The rest of this paper is organized as follows. A pest management model with impulsive releasing and harvesting is introduced in Section 2 and some useful preliminaries are given in Section 3. Section 4 deals with stability and permanence analysis of system. In this section, two sufficient conditions are deduced including the locally and globally asymptotical stability of the susceptible pest-eradication
period solution, the permanence of system is also discussed. A simple example and conclusions are given in Section 5.

## 2. Model Description

In the following, to establish our pest management model, we rely on the following biological assumptions.
(A1) The pest population is divided into two classes, the susceptible and infective. The infective pests neither recover nor reproduce and infective pests cannot damage crops. The disease is transmitted from infective pests to susceptible pests and does not propagate to predators.
(A2) In the absence of susceptible pests, the crops have a logistic growth rate with intrinsic birth rate $r$ and carrying capacity $K$.
(A3) The predators (natural enemies) have an age structure, that is, immature and mature. Only the mature predators have the ability to feed on susceptible pests, but do not prey on infective pests and crops.
(A4) The functional response of the susceptible pest is described by the abstract function $P_{1}$, the functional response of the mature predator is described by the abstract function $P_{2}$, and the infection rate is described by the abstract function $g$, where $P_{1}, P_{2}$, and $g$ satisfy certain assumptions outlined below.

On the basis of the above assumptions, we establish the following impulsively controlled system:

$$
\begin{aligned}
& x^{\prime}(t)= r x(t)\left(1-\frac{x(t)}{K}\right)-P_{1}(x(t)) S(t), \\
& S^{\prime}(t)= \beta P_{1}(x(t)) S(t)-g(I(t)) S(t) \\
&-P_{2}(S(t)) y_{M}(t)-d_{S} S(t), \\
& I^{\prime}(t)= g(I(t)) S(t)-d_{I} I(t), \\
& y_{J}^{\prime}(t)= \lambda P_{2}(S(t)) y_{M}(t)-d_{J} y_{J}(t)-m y_{J}(t), \\
& y_{M}^{\prime}(t)= m y_{J}(t)-d_{M} y_{M}(t), \\
& \Delta x(t)=-\delta x(t), \\
& \Delta S(t)=-P_{S} S(t), \\
& \Delta I(t)=-P_{I} I(t), \\
& \Delta y_{J}(t)=-P_{J} y_{J}(t), \\
& \Delta y_{M}(t)=-P_{M} y_{M}(t), \\
& t=(n+\tau-1) T,
\end{aligned}
$$

$$
\begin{align*}
& \Delta x(t)=0, \\
& \Delta S(t)=0, \\
& \Delta I(t)=\delta_{I}, \\
& \Delta y_{J}(t)=\delta_{J}, \\
& \Delta y_{M}(t)=\delta_{M}, \\
& t=n T, n \in \mathbb{N}, \tag{1}
\end{align*}
$$

where $x(t)$ represents the density of the crop at time $t, S(t)$ represents the density of the susceptible pest at time $t, I(t)$ represents the density of the infective pest at time $t, y_{J}(t)$ and $y_{M}(t)$ represent the density of the immature and mature predator at time $t$, respectively; $r$ is the logistic intrinsic growth rate of the crop in the absence of the susceptible pest, $K$ is its carrying capacity; $0<\beta, \lambda \leq 1$ represent the conversion rate at which ingested preys in excess of what is needed for maintenance is translated into predator population increase; $m$ is the rate at which the immature predators become the mature predators. $d_{S}, d_{I}, d_{J}, d_{M}>0$ are the death rates of the susceptible pest population, infective pest population, and of the immature and mature predator population, respectively; $\Delta x(t)=x\left(t^{+}\right)-x(t), \Delta S(t)=$ $S\left(t^{+}\right)-S(t), \Delta I(t)=I\left(t^{+}\right)-I(t), \Delta y_{J}(t)=y_{J}\left(t^{+}\right)-y_{J}(t)$, $\Delta y_{M}(t)=y_{M}\left(t^{+}\right)-y_{M}(t) ; T$ is the period of the impulsive effect; $\delta\left(0 \leq \delta<\left(1-e^{-r T}\right) / 2\right)$ is the harvesting rate of crop population; $0 \leq P_{S}, P_{I}, P_{J}, P_{M}<1$ denote the transfer rate of susceptible pest population, infective pest population, immature and mature predator population at every impulsive period $(n+\tau-1) T(n \in \mathbb{N}, 0<\tau<1)$, respectively; $\delta_{I}, \delta_{J}$, $\delta_{M}>0$ represents the amount of infective pests, immature and mature predators, respectively, which are released at every impulsive period $n T(n \in \mathbb{N})$, respectively; Also, $P_{1}(\cdot)$, $P_{2}(\cdot), g(\cdot) \in H$, here $H=\left\{f: R \rightarrow R \mid f(0)=0, f^{\prime}(x)>0\right.$ and $f^{\prime \prime}(x) \leq 0$ for all $\left.x>0\right\}$.

Some familiar examples of functions $f \in H$ in the biological literature include
(F1) $f_{1}(x)=a x$, with $a>0$;
(F2) $f_{2}(x)=a x /(1+b x)$, with $a, b>0$;
(F3) $f_{3}(x)=a\left(1-e^{-b x}\right)$, with $a, b>0$,
where functions (F1) and (F2) are known as Holling type functional responses (see, [16-26]), and (F3) belongs to Ivlev type functional responses (see, [27-30]).

## 3. Preliminaries

In this section, we will give some definitions and lemmas, which will be useful for our main results. Let $\mathbb{R}_{+}=$ $[0, \infty)$ and $\mathbb{R}_{+}^{5}=\left\{X=\left(x(t), S(t), I(t), y_{J}(t), y_{M}(t)\right) \in\right.$ $\left.\mathbb{R}^{5} \mid x(t), S(t), I(t), y_{J}(t), y_{M}(t) \geq 0\right\}$. Denote $f=$ $\left(f_{x}, f_{S}, f_{I}, f_{J}, f_{M}\right)^{T}$ the map defined by the right hand of the first five equations in system (1). Let $V: \mathbb{R}_{+} \times \mathbb{R}_{+}^{5} \rightarrow \mathbb{R}_{+}$, then $V \in V_{0}$ if
(1) $V$ is continuous in $((n-1) T,(n+\tau-1) T] \times \mathbb{R}_{+}^{5}$, $((n+\tau-1) T, n T] \times \mathbb{R}_{+}^{5}$ and for each $x \in R_{+}^{5}, n \in \mathbb{N}$, $\lim _{(t, y) \rightarrow\left((n+\tau-1) T^{+}, x\right)} V(t, y)=V\left((n+\tau-1) T^{+}, x\right)$ and $\lim _{(t, y) \rightarrow\left(n T^{+}, x\right)} V(t, y)=V\left(n T^{+}, x\right)$ exist.
(2) $V$ is locally Lipschitzian $x$.

Definition 1. Letting $V \in V_{0}$, one defines the upper right derivative of $V$ with respect to the impulsive differential system (1) at $(t, x) \in((n-1) T,(n+\tau-1) T] \times \mathbb{R}_{+}^{5}$ and $((n+\tau-1) T, n T] \times \mathbb{R}_{+}^{5}$ by

$$
\begin{equation*}
D^{+} V(t, x)=\limsup _{h \rightarrow 0^{+}} \frac{1}{h}[V(t+h, x+h f(t, x))-V(t, x)] \tag{2}
\end{equation*}
$$

Definition 2. The system (1) is said to be permanent if there are positive constants $m, M>0$ and a finite time $T_{0}$ such that all solutions of (1) with initial values $x\left(0^{+}\right), S\left(0^{+}\right), I\left(0^{+}\right)$, $y_{J}\left(0^{+}\right), y_{M}\left(0^{+}\right), m \leq x(t), S(t), I(t), y_{J}(t), y_{M}(t) \leq M$ hold for all $t \geq T_{0}$, where $m$ and $M$ are independent of initial value, $T_{0}$ may depend on initial value.

Remark 3. The global existence and uniqueness of system (1) is guaranteed by the smoothness properties of $f$ (for details, see [13, 14]).

Lemma 4 (see [15]). Let $V: \mathbb{R}_{+} \times \mathbb{R}^{n} \rightarrow \mathbb{R}_{+}^{m}$ satisfy $V_{i} \in$ $V_{0}, i=1,2, \ldots, m$, and assume that

$$
\begin{gather*}
D^{+} V(t, x(t)) \leq(\geq) g(t, V(t, x(t))), \quad t \neq(k+\tau-1) T, k T, \\
V\left(t, x\left(t^{+}\right)\right) \leq(\geq) \psi_{k}^{\tau}(V(t, x(t))), \quad t=(k+\tau-1) T, \\
V\left(t, x\left(t^{+}\right)\right) \leq(\geq) \psi_{k}(V(t, x(t))), \quad t=k T, k \in \mathbb{N}, \\
x\left(0^{+}\right)=x_{0}, \tag{3}
\end{gather*}
$$

where $g: \mathbb{R}_{+} \times \mathbb{R}_{+}^{m} \rightarrow \mathbb{R}_{+}^{m}$ is continuous in $((k-1) T,(k+$ $\tau-1) T] \times \mathbb{R}^{m}$ and $((k+\tau-1) T, k T] \times \mathbb{R}^{m}$, for each $p \in$ $\mathbb{R}^{m}, k=1,2, \ldots$, the limit $\lim _{(t, q) \rightarrow\left((k+\tau-1) T^{+}, p\right)} g(t, q)=g((k+$ $\left.\tau-1) T^{+}, p\right)$ and $\lim _{(t, q) \rightarrow\left((k-1) T^{+}, p\right)} g(t, q)=g\left((k-1) T^{+}, p\right)$ exists. $g(t, q)$ is quasimonotone nondecreasing in $q . \psi_{k}^{\tau}, \psi_{k}$ : $\mathbb{R}_{+}^{m} \rightarrow \mathbb{R}_{+}^{m}$ is nondecreasing for all $k \in \mathbb{N}$. Let $\theta(t)$ be the maximal (minimal) solution of the following impulsive differential equation on $[0, \infty)$ :

$$
\begin{gather*}
w^{\prime}(t)=g(t, w(t)), \quad t \neq(k+\tau-1) T, k T, \\
w\left(t^{+}\right)=\psi_{k}^{\tau}(w(t)), \quad t=(k+\tau-1) T, \\
w\left(t^{+}\right)=\psi_{k}(w(t)), \quad t=k T, k \in \mathbb{N},  \tag{4}\\
w\left(0^{+}\right)=w_{0} .
\end{gather*}
$$

Then for any solution $x(t)$ of the system (3), $V\left(0^{+}\right.$, $\left.x_{0}\right) \leq(\geq) w_{0}$ implies that $V(t, x(t)) \leq(\geq) \theta(t)$ for all $t \geq 0$.

Lemma 5 (see [13, 15]). Consider the following system:

$$
\begin{gather*}
v^{\prime}(t) \leq(\geq) p(t) v(t)+q(t), \quad t \neq t_{k}, \\
v\left(t_{k}^{+}\right) \leq(\geq) d_{k} v\left(t_{k}\right)+b_{k}, \quad t=t_{k}, \quad k \in \mathbb{N},  \tag{5}\\
v\left(0^{+}\right) \leq(\geq) v_{0},
\end{gather*}
$$

where $p, q \in P C\left(\mathbb{R}_{+}, \mathbb{R}\right)$ and $d_{k} \geq 0, v_{0}$ and $b_{k}$ are constants. Suppose that
(A1) the sequence $t_{k}$ satisfies $0 \leq t_{1} \leq t_{2}<\cdots$, with $\lim _{t \rightarrow \infty} t_{k}=\infty$;
(A2) $v \in P C^{\prime}\left(\mathbb{R}_{+}, \mathbb{R}\right)$ and $v(t)$ is left-continuous at $t_{k}, k \in$ $\mathbb{N}$.

Then, for $t>0$,

$$
\begin{align*}
v(t) \leq & (\geq) v_{0} e^{\int_{0}^{t} p(s) d s} \prod_{0<t_{k}<t} d_{k} \\
& +\sum_{0<t_{k}<t}\left(\prod_{t_{k}<t_{j}<t} d_{j} e^{\int_{t_{k}}^{t} p(s) d s}\right) b_{k}  \tag{6}\\
& +\int_{0}^{t}\left(\prod_{s<t_{k}<t} d_{k}\right) e^{\int_{s}^{t} p(\tau) d \tau} q(s) d s .
\end{align*}
$$

Lemma 6. There exists a constant $M=\max \{(1 / \lambda)((L / d)+$ $\left.\left.\left(\rho e^{d T} /\left(e^{d T}-1\right)\right)\right), K\right\}>0$, such that $x(t), S(t), I(t), y_{J}(t)$, $y_{M}(t) \leq M$ for each solution of (1) with $t$ large enough.

Proof. Since $x^{\prime}(t) \leq r x(1-(x(t) / K))$, then $\left.x^{\prime}(t)\right|_{x(t)=K} \leq 0$, and $x\left((n+\tau-1) T^{+}\right) \leq x((n+\tau-1) T)(0<\delta<1)$, so $x(t) \leq K$ for $t$ large enough. Let us define $V(t) \in V_{0}$ by $V(t)=\lambda \beta x(t)+\lambda S(t)+\lambda I(t)+y_{J}(t)+y_{M}(t)$ and denote $d=\min \left\{d_{S}, d_{I}, d_{J}, d_{M}\right\}$. Then, it is obvious that

$$
\begin{array}{r}
\frac{d V(t)}{d t}+d V(t) \leq \lambda \beta(r+d) x(t)-\frac{\lambda \beta r x^{2}(t)}{K}  \tag{7}\\
t \neq(n+\tau-1) T, t \neq n T
\end{array}
$$

Since the right-hand side (7) is bounded from above by $L=$ $K \lambda \beta(r+d)^{2} / 4 r$, it follows that

$$
\begin{equation*}
\frac{d V(t)}{d t}+d V(t) \leq L, \quad t \neq(n+\tau-1) T, \quad t \neq n T \tag{8}
\end{equation*}
$$

When $t=(n+\tau-1) T$ and $t=n T$, it is easy to obtain that

$$
\begin{align*}
& V\left((n+\tau-1) T^{+}\right) \leq V((n+\tau-1) T)  \tag{9}\\
& V\left(n T^{+}\right)=V(n T)+\left(\lambda \delta_{I}+\delta_{J}+\delta_{M}\right)
\end{align*}
$$

Then, by Lemma 5, we can obtain that

$$
\begin{align*}
V(t) \leq & V(0) e^{-d t}+\int_{0}^{t} L e^{-d(t-s)} d s \\
& +\sum_{0<k T<t} \rho e^{-d(t-k T)} \longrightarrow \frac{L}{d}+\frac{\rho e^{d T}}{e^{d T}-1}, \quad t \longrightarrow \infty \tag{10}
\end{align*}
$$

where $\rho=\lambda \delta_{I}+\delta_{J}+\delta_{M}$. So it follows that $V(t)$ is uniformly bounded on $[0, \infty)$. The proof is completed.

Lemma 7 (see [31]). Let one consider the following impulsive control subsystem:

$$
\begin{gather*}
x^{\prime}(t)=r x(t)\left(1-\frac{x(t)}{K}\right), \quad t \neq(n+\tau-1) T,  \tag{11}\\
\Delta x(t)=-\delta x(t), t=(n+\tau-1) T .
\end{gather*}
$$

Suppose $\delta_{0}^{*}=1-e^{-r T}$. Then one has the following results.
(1) If $\delta>\delta_{0}^{*}$, then the trivial periodic solution of system (11) is locally asymptotically stable.
(2) If $\delta<\delta_{0}^{*}$, then the system (11) has a unique positive periodic solution $x^{*}(t)$, which is globally asymptotically stable, where

$$
\begin{gather*}
x^{*}(t)=\frac{K\left(1-\delta-e^{-r T}\right)}{1-\delta-e^{-r T}+\delta e^{-r(t-(n+\tau-1) T)}}, \\
t \in((n+\tau-1) T,(n+\tau) T], n \in \mathbb{N}, \\
x^{*}\left(0^{+}\right)=x^{*}\left(n T^{+}\right)=\frac{K\left((1-\delta) e^{r T}-1\right)}{\left(e^{r \tau T}-1\right)+(1-\delta)\left(e^{r T}-e^{r \tau T}\right)} . \tag{12}
\end{gather*}
$$

Remark 8. From Lemma 7, we have
(1a) if $\delta_{0}^{*}>2 \delta$, then $x^{*}(t)>K / 2$ for all $t \geq 0$;
(2a) if $t \in((n-1) T, n T], n \in \mathbb{N}$, then the periodic solution $x^{*}(t)$ can be rewritten in the form

$$
x^{*}(t)=\left\{\begin{array}{c}
\frac{K\left(1-\delta-e^{-r T}\right)}{1-\delta-e^{-r T}+\delta e^{-r(T-\tau T)} e^{-r(t-(n-1) T)}},  \tag{13}\\
t \in((n-1) T,(n+\tau-1) T] \\
\frac{K\left(1-\delta-e^{-r T}\right)}{1-\delta-e^{-r T}+\delta e^{-r(t-(n+\tau-1) T)}}, \\
t \in((n+\tau-1) T, n T], n \in \mathbb{N} .
\end{array}\right.
$$

Lemma 9. Let one consider the following impulsive control subsystem:

$$
\begin{gather*}
z^{\prime}(t)=a(t)-d z(t), \quad t \neq(n+\tau-1) T, t \neq n T, \\
\Delta z(t)=-p z(t), \quad t=(n+\tau-1) T, \\
\Delta z(t)=\delta, \quad t=n T, n \in \mathbb{N}, \tag{14}
\end{gather*}
$$

$$
z\left(0^{+}\right)=z_{0}
$$

where $a(t)$ is a $T$-periodic $P C\left(\mathbb{R}_{+}, \mathbb{R}\right)$ function. $p, d$ are the positive real constants and $p<1$. Then system (14) has a unique T-periodic solution $z^{*}(t)$, and for each solution $z(t)$ of (14), $z(t) \rightarrow z^{*}(t)$ as $t \rightarrow \infty$, where

$$
\begin{align*}
& \begin{array}{l}
z^{*}(t)=e^{-d(t-(n-1) T)}\left(z^{*}\left(0^{+}\right)+\int_{0}^{t-(n-1) T} a(s) e^{d s} d s\right), \\
t \in((n-1) T,(n+\tau-1) T], \\
z^{*}(t)=e^{-d(t-(n-1) T)}\left(z^{*}\left(\tau T^{+}\right) e^{d \tau T}+\int_{\tau T}^{t-(n-1) T} a(s) e^{d s} d s\right), \\
t \in((n+\tau-1) T, n T], \\
z^{*}\left(0^{+}\right) \\
=\frac{\left[(1-p) \int_{0}^{\tau T} a(s) e^{d s} d s+\int_{\tau T}^{T} a(s) e^{d s} d s\right] e^{-d T}+\delta}{1-(1-p) e^{-d T}}, \\
z^{*}\left(\tau T^{+}\right) \\
=\frac{(1-p)\left[\int_{0}^{\tau T} a(s) e^{d s} d s+e^{-d T} \int_{\tau T}^{T} a(s) e^{d s} d s+\delta\right] e^{-d \tau T}}{1-(1-p) e^{-d T}}
\end{array} .
\end{align*}
$$

Proof. First, it is easy to obtain that

$$
\begin{gather*}
z(t)=e^{-d t}\left(z\left(0^{+}\right)+\int_{0}^{t} a(s) e^{d s} d s\right), \quad t \in(0, \tau T], \\
z(t)=e^{-d(t-\tau T)} z\left(\tau T^{+}\right)+e^{-d t} \int_{\tau T}^{t} a(s) e^{d s} d s, \quad t \in(\tau T, T] \tag{16}
\end{gather*}
$$

Since the $T$-periodicity requirement, we have

$$
\begin{align*}
& z^{*}\left(\tau T^{+}\right)=e^{-d \tau T}\left(z^{*}\left(0^{+}\right)+\int_{0}^{\tau T} a(s) e^{d s} d s\right)(1-p), \\
& z^{*}\left(0^{+}\right)=e^{-d(T-\tau T)} z^{*}\left(\tau T^{+}\right)+e^{-d T} \int_{\tau T}^{T} a(s) e^{d s} d s+\delta \tag{17}
\end{align*}
$$

By (17), we can obtain that

$$
\begin{align*}
& z^{*}\left(0^{+}\right) \\
& \quad=\frac{\left[(1-p) \int_{0}^{\tau T} a(s) e^{d s} d s+\int_{\tau T}^{T} a(s) e^{d s} d s\right] e^{-d T}+\delta}{1-(1-p) e^{-d T}}, \\
& z^{*}\left(\tau T^{+}\right) \\
& =\frac{(1-p)\left[\int_{0}^{\tau T} a(s) e^{d s} d s+e^{-d T} \int_{\tau T}^{T} a(s) e^{d s} d s+\delta\right] e^{-d \tau T}}{1-(1-p) e^{-d T}} . \tag{18}
\end{align*}
$$

So, we will obtain the $T$-periodic solution of (14):

$$
\begin{array}{r}
z^{*}(t)=e^{-d(t-(n-1) T)}\left(z^{*}\left(0^{+}\right)+\int_{0}^{t-(n-1) T} a(s) e^{d s} d s\right) \\
t \in((n-1) T,(n+\tau-1) T] \\
z^{*}(t)=e^{-d(t-(n-1) T)}\left(z^{*}\left(\tau T^{+}\right) e^{d \tau T}+\int_{\tau T}^{t-(n-1) T} a(s) e^{d s} d s\right) \\
t \in((n+\tau-1) T, n T] \tag{19}
\end{array}
$$

Let $Z(t)=z(t)-z^{*}(t)$. Substituting $Z(t)$ into (14), we have

$$
\begin{gather*}
Z^{\prime}(t)=-d Z(t), \quad t \neq(n+\tau-1) T, t \neq n T, \\
\Delta Z(t)=-p Z(t), \quad t=(n+\tau-1) T,  \tag{20}\\
\Delta Z(t)=0, \quad t=n T, n \in \mathbb{N}, \\
Z\left(0^{+}\right)=z_{0}-z^{*}\left(0^{+}\right) .
\end{gather*}
$$

Then, $Z(t)=Z\left(0^{+}\right) e^{-d t} \prod_{0<(n+\tau-1) T<t}(1-p) \rightarrow 0$, as $t \rightarrow \infty$. The proof is completed.

## 4. Main Results

4.1. Local and Global Stability. In this section, we will study the existence and stability of the system (1) susceptible pesteradication periodic solution $\left(x^{*}(t), 0, I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$. To this purpose, it is seen first that when $S(t)=0$, system (1) can be rewritten in the form

$$
\begin{align*}
& x^{\prime}(t)=r x(t)\left(1-\frac{x(t)}{K}\right), \\
& I^{\prime}(t)=-d_{I} I(t), \\
& y_{J}^{\prime}(t)=-\left(d_{J}+m\right) y_{J}(t), \\
& y_{M}^{\prime}(t)=m y_{J}(t)-d_{M} y_{M}(t), \\
& t \neq(n+\tau-1) T, \\
& t \neq n T \text {, } \\
& \Delta x(t)=-\delta x(t), \\
& \Delta I(t)=-P_{I} I(t), \\
& \Delta y_{J}(t)=-P_{J} y_{J}(t), \\
& \Delta y_{M}(t)=-P_{M} y_{M}(t), \\
& t=(n+\tau-1) T, \\
& \Delta x(t)=0, \\
& \Delta I(t)=\delta_{I}, \\
& \Delta y_{J}(t)=\delta_{J}, \\
& \Delta y_{M}(t)=\delta_{M}, \\
& t=n T, \tag{21}
\end{align*}
$$

which describes the dynamics of system (1) in the absence of the susceptible pest population. So, when $t \in \quad((n-$ 1) $T, n T](n \in \mathbb{N})$, we can calculate the $T$-periodic solution of (21) by Lemmas 7 and 9. It is seen that

$$
\begin{align*}
& x^{*}(t)=\left\{\begin{array}{c}
\frac{K\left(1-\delta-e^{-r T}\right)}{\frac{1-\delta-e^{-r T}+\delta e^{-r(T-\tau T)} e^{-r(t-(n-1) T)}}{t \in((n-1) T,(n+\tau-1) T]},}, \\
\frac{K\left(1-\delta-e^{-r T}\right)}{1-\delta-e^{-r T}+\delta e^{-r(t-(n+\tau-1) T)}}, \\
t \in((n+\tau-1) T, n T],
\end{array}\right.  \tag{22}\\
& I^{*}(t)=\left\{\begin{array}{l}
\frac{\delta_{I} e^{-d_{I}(t-(n-1) T)}}{1-\left(1-P_{I}\right) e^{-d_{I} T}}, \\
t \in((n-1) T,(n+\tau-1) T], \\
\frac{\delta_{I}\left(1-P_{I}\right) e^{-d_{I}(t-(n-1) T)}}{1-\left(1-P_{I}\right) e^{-d_{I} T}}, \\
t \in((n+\tau-1) T, n T],
\end{array}\right.  \tag{23}\\
& y_{J}^{*}(t)=\left\{\begin{array}{c}
\frac{\delta_{J} e^{-\left(m+d_{J}\right)(t-(n-1) T)}}{1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}}, \\
t \in((n-1) T,(n+\tau-1) T], \\
\frac{\delta_{J}\left(1-P_{J}\right) e^{-\left(m+d_{J}\right)(t-(n-1) T)}}{1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}}, \\
t \in((n+\tau-1) T, n T],
\end{array}\right. \tag{24}
\end{align*}
$$

$$
\begin{align*}
& y_{M}^{*}(t) \\
& =\left\{\begin{array}{c}
e^{-d_{M}(t-(n-1) T)}\left(y_{M}^{*}\left(0^{+}\right)+A(t-(n-1) T)\right), \\
t \in((n-1) T,(n+\tau-1) T], \\
e^{-d_{M}(t-(n-1) T)}\left(y_{M}^{*}\left(\tau T^{+}\right) e^{d_{M} \tau T}+B(t-(n-1) T)\right), \\
t \in((n+\tau-1) T, n T],
\end{array}\right.  \tag{25}\\
& y_{M}^{*}\left(0^{+}\right)=\frac{\left[\left(1-P_{M}\right) A(\tau T)+B(T)\right] e^{-d_{M} T}+\delta_{M}}{1-\left(1-P_{M}\right) e^{-d_{M} T}},  \tag{26}\\
& y_{M}^{*}\left(\tau T^{+}\right)=\frac{\left(1-P_{M}\right)\left[A(\tau T)+e^{-d_{M} T} B(T)+\delta_{M}\right] e^{-d_{M} \tau T}}{1-\left(1-P_{M}\right) e^{-d_{M} T}}, \tag{27}
\end{align*}
$$

where

$$
\begin{array}{r}
A(t)=\frac{m \delta_{J}\left(e^{\left(d_{M}-\left(m+d_{J}\right)\right) t}-1\right)}{\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)\left(d_{M}-\left(m+d_{J}\right)\right)}, \\
t \in(0, \tau T], \\
B(t)=\frac{m \delta_{J}\left(1-P_{J}\right)\left(e^{\left(d_{M}-\left(m+d_{J}\right)\right) t}-e^{\left(d_{M}-\left(m+d_{J}\right)\right) \tau T}\right)}{\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)\left(d_{M}-\left(m+d_{J}\right)\right)}, \\
t \in(\tau T, T] . \tag{28}
\end{array}
$$

To discuss the locally asymptotical stability of the susceptible pest-eradication periodic solution, we now introduce the Floquet theory for a linear impulsive control system:

$$
\begin{gather*}
\omega^{\prime}(t)=A(t) \omega(t), \quad t \neq \tau_{k},  \tag{29}\\
\Delta \omega(t)=B_{k} \omega(t), \quad t=\tau_{k}, \quad k \in \mathbb{N},
\end{gather*}
$$

under the following conditions:
H1: $A(\cdot) \in \operatorname{PC}\left(\mathbb{R}, M_{n}(\mathbb{R})\right)$ and $A(t+T)=A(t)$ for $t \geq 0$.
$H 2: B_{k} \in M_{n}, \operatorname{det}\left(I_{n}+B_{k}\right) \neq 0, \tau_{k}<\tau_{k+1}$ for $k \in \mathbb{N}$, and $I_{n}$ denotes the $n \times n$ real identity matrix.
H3: There is a $q \in \mathbb{N}$ such that $B_{k+q}=B_{k}, \tau_{k+q}=\tau_{k}+T$ for $k \in \mathbb{N}$.

Let $\Psi(t)$ be a fundamental matrix of (29), then there is a unique reversible matrix $M \in M_{n}(\mathbb{R})$ such that $\Psi(t+T)=$ $\Psi(t) M$ for all $t \in \mathbb{R}$, which is called the monodromy matrix of (29) corresponding to $\Psi$. All monodromy matrices of (29) are similar and they have the same eigenvalues $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$, which are called the Floquet multipliers of (29).

Lemma 10 (see [13] (Floquet theory)). Let the conditions H1H3 hold. Then system (29) have the following properties
(1) stable if and only if all Floquet multipliers $\lambda_{i}(1 \leq i \leq$ $n$ ) of (29) satisfy $\left|\lambda_{i}\right| \leq 1$ and moreover, to those $\lambda_{i}$ for which $\left|\lambda_{i}\right|=1$, there correspond simple elementary divisors;
(2) asymptotically stable if and only if all Floquet multipliers $\lambda_{i}(1 \leq i \leq n)$ of $(29)$ satisfy $\left|\lambda_{i}\right|<1$;
(3) unstable if there is a Floquet multipliers $\lambda_{i}(1 \leq i \leq n)$ of (29) such that $\left|\lambda_{i}\right|>1$.

In the following, we present two main results with the locally and globally asymptotical stability of the susceptible pest-eradication periodic solution $\left(x^{*}(t), 0, I^{*}(t), P_{J}^{*}(t)\right.$, $\left.P_{M}^{*}(t)\right)$.

Theorem 11. If

$$
\begin{align*}
\beta \int_{0}^{T} & P_{1}\left(x^{*}(t)\right) d t-\int_{0}^{T} g\left(I^{*}(t)\right) d t  \tag{30}\\
& \quad-P_{2}^{\prime}(0) \int_{0}^{T} y_{M}^{*}(t) d t-d_{S} T<\ln \frac{1}{1-P_{S}}
\end{align*}
$$

then the susceptible pest-eradication periodic solution $\left(x^{*}(t), 0\right.$, $\left.I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$ of system (1) is locally asymptotically stable.

Proof. Let $\left(x(t), S(t), I(t), y_{J}(t), y_{M}(t)\right)$ be any solution of system (1). We define error $e_{1}(t)=x(t)-x^{*}(t), e_{2}(t)=$ $S(t), e_{3}(t)=I(t)-I^{*}(t), e_{4}(t)=y_{J}(t)-y_{J}^{*}(t), e_{5}(t)=$ $y_{M}(t)-y_{M}^{*}(t)$. The linearized system of (1) at $\left(x^{*}(t)\right.$, $\left.0, I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$ is

$$
\begin{aligned}
& e_{1}^{\prime}(t)=\left(r-2 r \frac{x^{*}(t)}{K}\right) e_{1}(t)-P_{1}\left(x^{*}(t)\right) e_{2}(t), \\
& e_{2}^{\prime}(t)=\left(\beta P_{1}\left(x^{*}(t)\right)-g\left(I^{*}(t)\right)-P_{2}^{\prime}(0) y_{M}^{*}(t)-d_{S}\right) e_{2}(t), \\
& e_{3}^{\prime}(t)=g\left(I^{*}(t)\right) e_{2}(t)-d_{I} e_{3}(t), \\
& e_{4}^{\prime}(t)=\lambda P_{2}^{\prime}(0) y_{M}^{*}(t) e_{2}(t)-\left(d_{J}+m\right) e_{4}(t), \\
& e_{5}^{\prime}(t)=m e_{4}(t)-d_{M} e_{5}(t), \\
& t \neq(n+\tau-1) T, \\
& t \neq n T
\end{aligned}
$$

$$
\Delta e_{1}(t)=-\delta e_{1}(t)
$$

$$
\Delta e_{2}(t)=-P_{S} e_{2}(t)
$$

$$
\Delta e_{3}(t)=-P_{I} e_{3}(t)
$$

$$
\Delta e_{4}(t)=-P_{J} e_{4}(t)
$$

$$
\Delta e_{5}(t)=-P_{M} e_{5}(t),
$$

$$
t=(n+\tau-1) T
$$

$\Delta e_{1}(t)=\Delta e_{2}(t)=\Delta e_{3}(t)=\Delta e_{4}(t)=\Delta e_{5}(t)=0$,

$$
\begin{equation*}
t=n T . \tag{31}
\end{equation*}
$$

Let $\Psi(t)$ be the fundamental matrix of (31), then $\Psi(t)$ satisfies

$$
\frac{d \Psi(t)}{d t}=\left(\begin{array}{ccccc}
r-2 r \frac{x^{*}(t)}{K} & P_{1}\left(x^{*}(t)\right) & 0 & 0 & 0  \tag{32}\\
0 & \beta P_{1}\left(x^{*}(t)\right)-g\left(I^{*}(t)\right)-P_{2}^{\prime}(0) y_{M}^{*}(t)-d_{S} & 0 & 0 & 0 \\
0 & g\left(I^{*}(t)\right) & -d_{I} & 0 & 0 \\
0 & \lambda P_{2}^{\prime}(0) y_{M}^{*}(t) & 0 & -\left(d_{J}+m\right) & 0 \\
0 & 0 & 0 & m & -d_{M}
\end{array}\right) \Psi(t)
$$

Then, a fundamental matrix $\Psi(t)\left(\Psi(0)=I_{4}\right)$ of (31) is

$$
\Psi(t)=\left(\begin{array}{ccccc}
e^{\int_{0}^{t}\left(r-2 r\left(x^{*}(s) / K\right)\right) d s} & \phi_{12}(t) & 0 & 0 & 0  \tag{33}\\
0 & e^{\int_{0}^{t}\left(\beta P_{1}\left(x^{*}(s)\right)-g\left(I^{*}(s)\right)-P_{2}^{\prime}(0) y_{M}^{*}(s)-d_{S}\right) d s} & 0 & 0 & 0 \\
0 & \phi_{32}(t) & e^{-d_{I} t} & 0 & 0 \\
0 & \phi_{42}(t) & 0 & e^{-\left(d_{J}+m\right) t} & 0 \\
0 & \phi_{52}(t) & 0 & \phi_{54}(t) & e^{-d_{M} t}
\end{array}\right),
$$

where

$$
\begin{gather*}
\phi_{12}(t)=-e^{-\int_{0}^{t}\left(r-2 r\left(x^{*}(s) / K\right)\right) d s} \\
\times \int_{0}^{t} P_{1}\left(x^{*}(s)\right) e^{\int_{0}^{s}\left(\beta P_{1}\left(x^{*}(\xi)\right)-g\left(I^{*}(\xi)\right)-P_{2}^{\prime}(0) y_{M}^{*}(\xi)-d_{S}\right) d \xi} \\
\times e^{\int_{0}^{s}\left(r-2 r\left(x^{*}(\xi) / K\right)\right) d \xi} d s, \\
\phi_{32}(t) \quad e^{-d_{I} t} \int_{0}^{t} g\left(I^{*}(s)\right) e^{\int_{0}^{s}\left(\beta P_{1}\left(x^{*}(\xi)\right)-g\left(I^{*}(\xi)\right)-P_{2}^{\prime}(0) y_{M}^{*}(\xi)-d_{S}\right) d \xi} \\
\phi_{42}(t) \quad e^{d_{I} s} d s, \\
=e^{-d_{J} t} \int_{0}^{t} \lambda P_{2}^{\prime}(0) y_{M}^{*}(s) e^{\int_{0}^{s}\left(\beta P_{1}\left(x^{*}(\xi)\right)-g\left(I^{*}(\xi)\right)-P_{2}^{\prime}(0) y_{M}^{*}(\xi)-d_{S}\right) d \xi} \\
\times e^{d_{J} s} d s, \\
\phi_{52}(t)=e^{-d_{M} t} \int_{0}^{t} m \phi_{42}(s) e^{d_{M} s} d s, \\
\phi_{54}(t)=\frac{m\left(e^{-\left(d_{J}+m\right) t}-e^{-d_{M} t}\right)}{d_{M}-\left(d_{J}+m\right)} .
\end{gather*}
$$

The resetting impulsive condition of (31) becomes

$$
\begin{align*}
& \left(\begin{array}{l}
e_{1}\left((n+\tau-1) T^{+}\right) \\
e_{2}\left((n+\tau-1) T^{+}\right) \\
e_{3}\left((n+\tau-1) T^{+}\right) \\
e_{4}\left((n+\tau-1) T^{+}\right) \\
e_{5}\left((n+\tau-1) T^{+}\right)
\end{array}\right) \\
& \quad=\left(\begin{array}{ccccc}
1-\delta & 0 & 0 & 0 & 0 \\
0 & 1-P_{S} & 0 & 0 & 0 \\
0 & 0 & 1-P_{I} & 0 & 0 \\
0 & 0 & 0 & 1-P_{J} & 0 \\
0 & 0 & 0 & 0 & 1-P_{M}
\end{array}\right) \\
& \quad \times\left(\begin{array}{l}
e_{1}((n+\tau-1) T) \\
e_{2}((n+\tau-1) T) \\
e_{3}((n+\tau-1) T) \\
e_{4}((n+\tau-1) T) \\
e_{5}((n+\tau-1) T)
\end{array}\right),  \tag{35}\\
& \left(\begin{array}{l}
e_{1}\left(n T^{+}\right) \\
e_{2}\left(n T^{+}\right) \\
e_{3}\left(n T^{+}\right) \\
e_{4}\left(n T^{+}\right) \\
e_{5}\left(n T^{+}\right)
\end{array}\right)=\left(\begin{array}{lllll}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right)\left(\begin{array}{l}
e_{1}(n T) \\
e_{2}(n T) \\
e_{3}(n T) \\
e_{4}(n T) \\
e_{5}(n T)
\end{array}\right)
\end{align*}
$$

$$
M=\left(\begin{array}{ccccc}
1-\delta & 0 & 0 & 0 & 0 \\
0 & 1-P_{S} & 0 & 0 & 0 \\
0 & 0 & 1-P_{I} & 0 & 0 \\
0 & 0 & 0 & 1-P_{J} & 0 \\
0 & 0 & 0 & 0 & 1-P_{M}
\end{array}\right)
$$

$$
\times\left(\begin{array}{lllll}
1 & 0 & 0 & 0 & 0  \tag{36}\\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right) \Psi(T)
$$

We have $\lambda_{1}=(1-\delta) e^{\int_{0}^{T}\left(r-2 r\left(x^{*}(s) / K\right)\right) d s}, \lambda_{2}=\left(1-P_{S}\right)$ $e^{\int_{0}^{T}\left(\beta P_{1}\left(x^{*}(s)\right)-g\left(I^{*}(s)\right)-P_{2}^{\prime}(0) y_{M}^{*}(s)-d_{S}\right) d s}, \lambda_{3}=\left(1-P_{I}\right) e^{-d_{I} T}<1$, $\lambda_{4}=\left(1-P_{J}\right) e^{-\left(d_{J}+m\right) T}<1$ and $\lambda_{5}=\left(1-P_{M}\right) e^{-d_{M} T}<1$. Since $x^{*}(t)>(K / 2)$, so $\lambda_{1}<1$. By the condition (30), we have $\lambda_{2}<$ 1. Therefore, according to Lemma 10 , the susceptible pesteradication periodic solution $\left(x^{*}(t), 0, I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$ of system (1) is locally asymptotically stable. The proof is completed.

Theorem 12. If

$$
\begin{align*}
& \beta \int_{0}^{T} P_{1}\left(x^{*}(t)\right) d t-\int_{0}^{T} g\left(I^{*}(t)\right) d t \\
& \quad-\min _{0 \leq \omega \leq U_{S}} P_{2}^{\prime}(\omega) \int_{0}^{T} y_{M}^{*}(t) d t-d_{S} T<\ln \frac{1}{1-P_{S}} \tag{37}
\end{align*}
$$

where $U_{S}$ is an ultimate boundedness constant for $S$, then the susceptible pest-eradication periodic solution $\left(x^{*}(t), 0\right.$, $\left.I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$ of system (1) is globally asymptotically stable.

Proof. Since

$$
\begin{align*}
& \beta \int_{0}^{T} P_{1}\left(x^{*}(t)\right) d t-\int_{0}^{T} g\left(I^{*}(t)\right) d t \\
& \quad-\min _{0 \leq \omega \leq U_{S}} P_{2}^{\prime}(\omega) \int_{0}^{T} y_{M}^{*}(t) d t-d_{S} T<\ln \frac{1}{1-P_{S}}, \tag{38}
\end{align*}
$$

we can choose an $\varepsilon$ small enough such that

$$
\begin{align*}
& \beta \int_{0}^{T} \quad P_{1}\left(x^{*}(t)+\varepsilon\right) d t-\int_{0}^{T} g\left(I^{*}(t)-\varepsilon\right) d t \\
& \quad-\min _{0 \leq \omega \leq U_{S}} P_{2}^{\prime}(\omega) \int_{0}^{T}\left(\overline{y_{M}^{*}(t)}-\varepsilon\right) d t-d_{S} T  \tag{39}\\
& \quad-\ln \frac{1}{1-P_{S}}=\epsilon<0
\end{align*}
$$

where $\overline{y_{M}^{*}(t)}$ is defined in the following. According to system (1), we have

$$
\begin{aligned}
x^{\prime}(t) & \leq r x(t)\left(1-\frac{x(t)}{K}\right) \\
I^{\prime}(t) & \geq-d_{I} I(t) \\
y_{J}^{\prime}(t) & \geq-d_{J} y_{J}(t)-m y_{J}(t)
\end{aligned}
$$

$$
\begin{align*}
& y_{M}^{\prime}(t)=m y_{J}(t)-d_{M} y_{M}(t), \\
& t \neq(n+\tau-1) T, \\
& \Delta x(t)=-\delta x(t), \\
& \Delta I(t)=-P_{I} I(t), \\
& \Delta y_{J}(t)=-P_{J} y_{J}(t), \\
& \Delta y_{M}(t)=-P_{M} y_{M}(t), \\
& \quad t=(n+\tau-1) T, \\
& \Delta x(t)=0, \\
& \Delta I(t)=\delta_{I}, \\
& \Delta y_{J}(t)=\delta_{J}, \\
& \Delta y_{M}(t)=\delta_{M}, \\
& t=n T, n \in \mathbb{N} .
\end{align*}
$$

From the first equation of system (40), we obtain the following comparison system:

$$
\begin{gather*}
\nu^{\prime}(t)=r v(t)\left(1-\frac{v(t)}{K}\right), \quad t \neq(n+\tau-1) T  \tag{41}\\
\Delta v(t)=-\delta v(t), \quad t=(n+\tau-1) T
\end{gather*}
$$

By Lemma 7, system (41) has a positive periodic solution $\nu^{*}(t)$, and for any solution $\nu(t)$ of (41), $\nu(t) \rightarrow \nu^{*}(t)$ as $t$ large enough, where $\nu^{*}(t)=x^{*}(t)$. Then, according to Lemmas 4 and 7, there exists a positive constant $n^{*}$ such that for all $t \geq n^{*} T$

$$
\begin{equation*}
x(t) \leq x^{*}(t)+\varepsilon \tag{42}
\end{equation*}
$$

Let us define $V(t)=\left(V_{1}(t), V_{2}(t)\right)^{T} \in C\left[\mathbb{R}_{+} \times \mathbb{R}^{2}, \mathbb{R}_{+}^{2}\right]$ and $V_{i}(t) \in V_{0},(i=1,2)$, where $V_{1}(t)=I(t), V_{2}(t)=y_{J}(t)$. Then, we have

$$
\begin{align*}
V^{\prime}(t) \geq\binom{-d_{I} I(t)}{-\left(d_{J}+m\right) y_{J}(t)}= & \left(\begin{array}{cc}
-d_{I} & 0 \\
0 & -\left(d_{J}+m\right)
\end{array}\right) V(t), \\
& t \neq(n+\tau-1) T, t \neq n T \tag{43}
\end{align*}
$$

$$
\begin{align*}
V\left((n+\tau-1) T^{+}\right) & =\left(\begin{array}{cc}
1-P_{I} & 0 \\
0 & 1-P_{J}
\end{array}\right) V((n+\tau-1) T) \\
V\left(n T^{+}\right) & =\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) V(n T)+\binom{\delta_{I}}{\delta_{J}} \\
V\left(0^{+}\right) & =\left(I\left(0^{+}\right), y_{J}\left(0^{+}\right)\right) \tag{44}
\end{align*}
$$

Then, the multicomparison system of (43) is

$$
\begin{gather*}
w^{\prime}(t)=A w(t), \quad t \neq(n+\tau-1) T, n T, \\
w\left(t^{+}\right)=B w(t), \quad t=(n+\tau-1) T \\
w\left(t^{+}\right)=I_{2} w(t)+C, \quad t=n T  \tag{45}\\
w\left(0^{+}\right)=V\left(0^{+}\right),
\end{gather*}
$$

where $A=\left(\begin{array}{cc}-d_{I} & 0 \\ 0 & -\left(d_{J}+m\right)\end{array}\right), \quad B=\left(\begin{array}{cc}1-P_{I} & 0 \\ 0 & 1-P_{J}\end{array}\right)$, and $C=\binom{\delta_{I}}{\delta_{J}}$.
By Lemma 9, it is easy to obtain a periodic solution $\left(I^{*}(t), y_{J}^{*}(t)\right)^{T}$ of system (45). Then, according to Lemmas 4 and 9 , one may find $n_{0}^{*}\left(n_{0}^{*}>n^{*}\right)$ such that for all $t \geq n_{0}^{*} T$

$$
\begin{equation*}
I(t) \geq I^{*}(t)-\varepsilon, \quad y_{J}(t) \geq y_{J}^{*}(t)-\varepsilon \tag{46}
\end{equation*}
$$

From the fourth equation of system (40), we have $y_{M}^{\prime}(t) \geq$ $m\left(y_{J}^{*}(t)-\varepsilon\right)-d_{M} y_{M}(t)$, by Lemmas 4 and 9 , there exists $n_{1}^{*}\left(n_{1}^{*}>n_{0}^{*}\right)$ such that for all $t \geq n_{1}^{*} T$

$$
\begin{equation*}
y_{M}(t) \geq \overline{y_{M}^{*}(t)}-\varepsilon \tag{47}
\end{equation*}
$$

where

$$
\begin{align*}
& \overline{y_{M}^{*}(t)}=\left\{\begin{array}{l}
e^{-d_{M}(t-(n-1) T)} \\
\times\left(\overline{y_{M}^{*}\left(0^{+}\right)}\right. \\
\left.\quad+m \int_{0}^{t-(n-1) T}\left(y_{J}^{*}(t)-\varepsilon\right) e^{d_{M} s} d s\right), \\
\quad t \in((n-1) T,(n+\tau-1) T], \\
e^{-d_{M}(t-(n-1) T)} \\
\times\left(\overline{y_{M}^{*}\left(\tau T^{+}\right)} e^{d_{M} \tau T}\right. \\
\left.\quad+m \int_{\tau T}^{t-(n-1) T}\left(y_{J}^{*}(t)-\varepsilon\right) e^{d_{M} s} d s\right), \\
\quad \begin{array}{l}
\quad t((n+\tau-1) T, n T],
\end{array},
\end{array}\right.  \tag{48}\\
& \overline{y_{M}^{*}\left(0^{+}\right)}=\left(m \left[\left(1-P_{M}\right) \int_{0}^{\tau T}\left(y_{J}^{*}(t)-\varepsilon\right) e^{d_{M} s} d s\right.\right. \\
& \left.\left.+\int_{\tau T}^{T}\left(y_{J}^{*}(t)-\varepsilon\right) e^{d_{M} s} d s\right] e^{-d_{M} T}+\delta_{M}\right) \\
& \times\left(1-\left(1-P_{M}\right) e^{-d_{M} T}\right)^{-1}, \\
& \overline{y_{M}^{*}\left(\tau T^{+}\right)} \\
& =\left(( 1 - P _ { M } ) \left[m \int_{0}^{\tau T}\left(y_{J}^{*}(t)-\varepsilon\right) e^{d_{M} s} d s\right.\right. \\
& \left.+m e^{-d_{M} T} \int_{\tau T}^{T}\left(y_{J}^{*}(t)-\varepsilon\right) e^{d_{M} s} d s+\delta_{M}\right] \\
& \left.\times e^{-d_{M} \tau T}\right) \\
& \times\left(1-\left(1-P_{M}\right) e^{-d_{M} T}\right)^{-1} . \tag{49}
\end{align*}
$$

Therefore,

$$
\begin{align*}
& S^{\prime}(t) \leq\left[\beta P_{1}\left(x^{*}(t)+\varepsilon\right)-g\left(I^{*}(t)-\varepsilon\right)\right. \\
& \left.-\min _{0 \leq \omega \leq U_{S}} P_{2}^{\prime}(\omega)\left(\overline{y_{M}^{*}(t)}-\varepsilon\right)-d_{S}\right] S(t), \\
&  \tag{50}\\
& t \neq(n+\tau-1) T, \\
& t \neq n T
\end{align*}
$$

$$
\Delta S(t)=-P_{S} S(t), \quad t=(n+\tau-1) T
$$

$$
\Delta S(t)=0, \quad t=n T,
$$

for $t \geq n_{1}^{*} T$. Let $N \in \mathbb{N}$ and $(N+\tau-1) \geq n_{1}^{*}$. Integrating (50) on $((n+\tau-1) T,(n+\tau) T], n \geq N$, we have

Then $S(t) \leq S((n+\tau) T) e^{k \epsilon}$ for $t \in((n+\tau+k) T,(n+\tau+k+1) T]$. Since $\epsilon<0$, we can easily get $S(t) \rightarrow 0$, as $t \rightarrow \infty$. In the following, we prove $x(t) \rightarrow x^{*}(t), I(t) \rightarrow I^{*}(t), y_{J}(t) \rightarrow$ $y_{J}^{*}(t), y_{M}(t) \rightarrow y_{M}^{*}(t)$, as $t \rightarrow \infty$. Give $\varepsilon_{0}>0$ small enough $\left(\varepsilon_{0}<\left(r / P_{1}^{\prime}(0)\right)\right)$, there must exist $n_{2}^{*}\left(n_{2}^{*}>n_{1}^{*}\right)$ such that $S(t)<\varepsilon_{0}$, for $t \geq n_{2}^{*} T$. Then, we have

$$
\begin{aligned}
& x^{\prime}(t) \geq\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right) x(t)\left(1-\frac{r x(t)}{K\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right)}\right), \\
& I^{\prime}(t) \leq-\left(d_{I}-g^{\prime}(0) \varepsilon_{0}\right) I(t), \\
& y_{J}^{\prime}(t) \leq \lambda P_{2}\left(\varepsilon_{0}\right) M-\left(d_{J}+m\right) y_{J}(t), \\
& y_{M}^{\prime}(t)=m y_{J}(t)-d_{M} y_{M}(t), \\
& \\
& \Delta x(t)=-\delta x(t), \\
& \Delta I(t)=-P_{I} I(t), \\
& \Delta y_{J}(t)=-P_{J} y_{J}(t), \\
& \Delta y_{M}(t)=-P_{M} y_{M}(t), \\
& t=(n+\tau-1) T, \\
& \Delta x(t)=0, \\
& \Delta I(t)=\delta_{I},
\end{aligned}
$$

$$
\begin{aligned}
& S((n+\tau) T) \\
& \leq S((n+\tau-1) T)\left(1-P_{S}\right) \\
& \times e^{\left.\int_{(n+t-1) T}^{(n+\tau) T}\left(\beta P_{1} \widetilde{\left(x^{*}(t)\right.}-\varepsilon\right)-g\left(\widetilde{I^{*}(t)}+\varepsilon\right)-\min _{0 \leq \Omega \leq U_{S}} P_{2}^{\prime}(\omega)\left(\widetilde{y_{M}^{*}(t)}+\varepsilon\right)-d_{S}\right) d t} \\
& =S((n+\tau-1) T) e^{\epsilon} .
\end{aligned}
$$

$$
\begin{align*}
\Delta y_{J}(t) & =\delta_{J} \\
\Delta y_{M}(t) & =\delta_{M} \\
t & =n T . \tag{52}
\end{align*}
$$

Analyzing (52) with similarity as (40), there exists $n_{3}^{*}\left(n_{3}^{*}>\right.$ $n_{2}^{*}$ ) such that for all $t \geq n_{3}^{*} T$

$$
\begin{gather*}
x(t) \geq \widehat{x^{*}(t)}-\varepsilon, \quad I(t) \leq \widehat{I^{*}(t)}+\varepsilon \\
y_{J}(t) \leq \widehat{y_{J}^{*}(t)}+\varepsilon, \quad y_{M}(t) \leq \widehat{y_{M}^{*}(t)}+\varepsilon \tag{53}
\end{gather*}
$$

where
$\widehat{x^{*}(t)}$

$$
=\left\{\begin{array}{c}
K\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right)\left(1-\delta-e^{-\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right) T}\right) \\
\times\left(r \left[1-\delta-e^{-\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right) T}\right.\right. \\
\left.\left.+\delta e^{-\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right)(T-\tau T)} e^{-\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right)(t-(n-1) T)}\right]\right)^{-1} \\
t \in((n-1) T,(n+\tau-1) T] \\
\frac{K\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right)\left(1-\delta-e^{-\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right) T}\right)}{r\left[1-\delta-e^{-\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right) T}+\delta e^{-\left(r-P_{1}^{\prime}(0) \varepsilon_{0}\right)(t-(n+\tau-1) T)}\right]} \\
t \in((n+\tau-1) T, n T]
\end{array}\right.
$$

$\widehat{I^{*}(t)}$

$$
=\left\{\begin{array}{l}
\frac{\delta_{I} e^{-\left(d_{I}-g^{\prime}(0) \varepsilon_{0}\right)(t-(n-1) T)}}{1-\left(1-P_{I}\right) e^{-\left(d_{I}-g^{\prime}(0) \varepsilon_{0}\right) T}} \\
t \in((n-1) T,(n+\tau-1) T] \\
\frac{\delta_{I}\left(1-P_{I}\right) e^{-\left(d_{I}-g^{\prime}(0) \varepsilon_{0}\right)(t-(n-1) T)}}{1-\left(1-P_{I}\right) e^{-\left(d_{I}-g^{\prime}(0) \varepsilon_{0}\right) T}} \\
t \in((n+\tau-1) T, n T]
\end{array}\right.
$$

$\widehat{y_{J}^{*}(t)}$

$$
\widehat{y_{J}^{*}\left(0^{+}\right)}
$$

$$
=\left(\lambda P_{2}\left(\varepsilon_{0}\right) M\left[\left(1-P_{J}\right) \int_{0}^{\tau T} e^{\left(m+d_{J}\right) s} d s+\int_{\tau T}^{T} e^{\left(m+d_{J}\right) s} d s\right]\right.
$$

$$
\left.\times e^{-\left(m+d_{J}\right) T}+\delta_{J}\right)\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)^{-1}
$$

$$
\begin{aligned}
& \widehat{y_{J}^{*}\left(\tau T^{+}\right)} \\
& =\left(( 1 - P _ { J } ) \left[\lambda P_{2}\left(\varepsilon_{0}\right) M \int_{0}^{\tau T} e^{\left(m+d_{J}\right) s} d s\right.\right. \\
& \left.+e^{-\left(m+d_{J}\right) T} \lambda P_{2}\left(\varepsilon_{0}\right) M \int_{\tau T}^{T} e^{\left(m+d_{J}\right) s} d s+\delta_{J}\right] \\
& \left.\times e^{-\left(m+d_{J}\right) \tau T}\right)\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)^{-1}, \\
& \widehat{y_{M}^{*}(t)}
\end{aligned}
$$

$$
\begin{align*}
& \widehat{y_{M}^{*}\left(0^{+}\right)} \\
& =\left(m \left[\left(1-P_{M}\right) \int_{0}^{\tau T}\left(\widehat{y_{J}^{*}(t)}+\varepsilon\right) e^{d_{M} s} d s\right.\right. \\
& \left.\left.+\int_{\tau T}^{T}\left(\widetilde{y_{J}^{*}(t)}+\varepsilon\right) e^{d_{M} s} d s\right] e^{-d_{M} T}+\delta_{M}\right) \\
& \times\left(1-\left(1-P_{M}\right) e^{-d_{M} T}\right)^{-1}, \\
& \widehat{y_{M}^{*}\left(\tau T^{+}\right)} \\
& =\left(\left(1-P_{M}\right)\right. \\
& \times\left[m \int_{0}^{\tau T}\left(\widehat{y_{J}^{*}(t)}+\varepsilon\right) e^{d_{M} s} d s+m e^{-d_{M} T}\right. \\
& \left.\left.\times \int_{\tau T}^{T}\left(\widetilde{y_{J}^{*}(t)}+\varepsilon\right) e^{d_{M} s} d s+\delta_{M}\right] e^{-d_{M} \tau T}\right) \\
& \times\left(1-\left(1-P_{M}\right) e^{-d_{M} T}\right)^{-1} \text {. } \tag{54}
\end{align*}
$$

Letting $\varepsilon, \varepsilon_{0} \rightarrow 0$, we have $\overline{y_{M}^{*}(t)} \rightarrow y_{M}^{*}(t), \widehat{x^{*}(t)} \rightarrow x^{*}(t)$, $\widehat{I^{*}(t)} \rightarrow I^{*}(t), \widehat{y_{J}^{*}(t)} \rightarrow y_{J}^{*}(t), \widehat{y_{M}^{*}(t)} \rightarrow y_{M}^{*}(t)$. Together with (42), (46), (47), and (53), we get $x(t) \rightarrow x^{*}(t), I(t) \rightarrow$ $I^{*}(t), y_{J}(t) \rightarrow y_{J}^{*}(t), y_{M}(t) \quad \rightarrow \quad y_{M}^{*}(t)$ as $t \quad \rightarrow \quad \infty$. Therefore, the susceptible pest-eradication periodic solution $\left(x^{*}(t), 0, I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$ is globally attractive. Since (37) implies (30), it follows from Theorem 11 that $\left(x^{*}(t), 0, I^{*}(t)\right.$,
$\left.y_{J}^{*}(t)\right)$ is locally asymptotically stable. So, the susceptible pesteradication periodic solution $\left(x^{*}(t), 0, I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$ of system (1) is globally asymptotically stable. The proof is completed.
4.2. Permanence. Next, we will discuss the permanence of system (1). In order to facilitate discussion, we give one lemma.

Lemma 13. There exists a constant $m_{4}>0$, such that $x(t)$, $I(t), y_{J}(t), y_{M}(t) \geq m_{4}$ for each solution of (1) with $t$ large enough.

Proof. First, we discuss $x(t)$. Since $S(t) \leq M$, by the first equation of system (1), we have

$$
\begin{gather*}
x^{\prime}(t) \geq\left(r-P_{1}^{\prime}(0) M\right) x(t)\left(1-\frac{r x(t)}{K\left(r-P_{1}^{\prime}(0) M\right)}\right), \\
t \neq(n+\tau-1) T, t \neq n T, \\
\Delta x(t)=-\delta x(t), \quad t=(n+\tau-1) T, \\
\Delta x(t)=0, \quad t=n T . \tag{55}
\end{gather*}
$$

Then, we obtain the following comparison system:

$$
\begin{gather*}
\chi^{\prime}(t)=\left(r-P_{1}^{\prime}(0) M\right) \chi(t)\left(1-\frac{r \chi(t)}{K\left(r-P_{1}^{\prime}(0) M\right)}\right), \\
t \neq(n+\tau-1) T, t \neq n T, \\
\Delta \chi(t)=-\delta \chi(t), \quad t=(n+\tau-1) T, \\
\Delta \chi(t)=0, \quad t=n T . \tag{56}
\end{gather*}
$$

Letting $r>P_{1}^{\prime}(0) M$ and $(1-\delta) e^{\left(r-P_{1}^{\prime}(0) M\right) T}>1$, by Lemma 7, the system (56) has a positive periodic solution $\chi^{*}(t)$, and for any solution $\chi(t)$ of $(56), \chi(t) \rightarrow \chi^{*}(t)$ as $t$ large enough, where

$$
\chi^{*}(t)=\left\{\begin{array}{c}
K\left(r-P_{1}^{\prime}(0) M\right)\left(1-\delta-e^{-\left(r-P_{1}^{\prime}(0) M\right) T}\right)  \tag{57}\\
\times\left(r \left[1-\delta-e^{-\left(r-P_{1}^{\prime}(0) M\right) T}\right.\right. \\
+\delta e^{-\left(r-P_{1}^{\prime}(0) M\right)(T-\tau T)} \\
\left.\left.\times e^{-\left(r-P_{1}^{\prime}(0) M\right)(t-(n-1) T)}\right]\right)^{-1} \\
t \in((n-1) T,(n+\tau-1) T] \\
K\left(r-P_{1}^{\prime}(0) M\right)\left(1-\delta-e^{-\left(r-P_{1}^{\prime}(0) M\right) T}\right) \\
\times\left(r \left[1-\delta-e^{-\left(r-P_{1}^{\prime}(0) M\right) T}\right.\right. \\
\left.\left.+\delta e^{-\left(r-P_{1}^{\prime}(0) M\right)(t-(n+\tau-1) T)}\right]\right)^{-1} \\
t \in((n+\tau-1) T, n T]
\end{array}\right.
$$

According to Lemmas 4 and 7, one may find $n_{4}^{*} \in \mathbb{N}$ such that $x(t) \geq \chi^{*}(t)-\varepsilon$ for $t \geq n_{4}^{*} T$. Since $\chi^{*}(t)-\varepsilon \geq(K(r-$ $\left.\left.P_{1}^{\prime}(0) M\right)\left(1-\delta-e^{-\left(r-P_{1}^{\prime}(0) M\right) T}\right) / r\left(1-e^{-\left(r-P_{1}^{\prime}(0) M\right) T}\right)\right)-\varepsilon=m_{0}>0$,
so $x(t) \geq m_{0}$ for $t \geq n_{4}^{*} T$. Next, we will discuss the rest of parts.

From (46) and (47), we know that there exists $n_{5}^{*}\left(n_{5}^{*}>\right.$ $\left.\max \left\{n_{1}^{*}, n_{4}^{*}\right\}\right)$ such that $I(t) \geq I^{*}(t)-\varepsilon, y_{J}(t) \geq y_{J}^{*}(t)-\varepsilon$, and $y_{M}(t) \geq \overline{y_{M}^{*}(t)}-\varepsilon$ for all $t \geq n_{5}^{*} T$. By (22), (23), and (48), we have $I(t) \geq\left(\delta_{I}\left(1-P_{I}\right) e^{-d_{I} T} /\left(1-\left(1-P_{I}\right) e^{-d_{I} T}\right)\right)-\varepsilon=m_{1}>0$, $y_{J}(t) \geq\left(\delta_{J}\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T} /\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)\right)-\varepsilon=m_{2}>$ 0 , and $y_{M}(t) \geq m_{M}-\varepsilon=m_{3}>0$, where $m_{M}=\min _{0<t \leq T} \overline{y_{M}^{*}(t)}$. Let $m_{4}=\min \left\{m_{0}, m_{1}, m_{2}, m_{3}\right\}$, then $x(t), I(t), y_{J}(t), y_{M}(t) \geq$ $m_{4}$ for $t \geq n_{5}^{*} T$. The proof is completed.

Theorem 14. If

$$
\begin{align*}
& \beta \int_{0}^{T} P_{1}\left(x^{*}(t)\right) d t-\int_{0}^{T} g\left(I^{*}(t)\right) d t-P_{2}^{\prime}(0) \\
& \quad \times \int_{0}^{T} y_{M}^{*}(t) d t-d_{S} T>\ln \frac{1}{1-P_{S}} \tag{58}
\end{align*}
$$

then system (1) is permanent.
Proof. By Lemmas 6 and 13, we have already known that there exist two constants $m_{4}, M>0$, such that $x(t), I(t), y_{J}(t), y_{M}(t) \geq m_{4}$ and $x(t), S(t), I(t), y_{J}(t), y_{M}(t) \leq$ $M$ for $t$ large enough. Thus, we only need to find $m^{*}>0$ such that $S(t) \geq m^{*}$ for $t$ large enough. We will do this in the following two steps.
Step 1. Let $m_{5}>0$ and $\varepsilon_{1}>0$ small enough, so that $m_{5}<$ $\min \left\{\left(r / P_{1}^{\prime}(0)\right),\left(d_{I} / g^{\prime}(0)\right), M\right\}$ and

$$
\begin{align*}
& \beta \int_{0}^{T} P_{1}\left(\widetilde{x^{*}(t)}-\varepsilon_{1}\right) d t-\int_{0}^{T} g\left(\widetilde{I^{*}(t)}+\varepsilon_{1}\right) d t-P_{2}^{\prime}(0) \\
& \quad \times \int_{0}^{T}\left(\widetilde{y_{M}^{*}(t)}+\varepsilon_{1}\right) d t-d_{S} T-\ln \frac{1}{1-P_{S}}=\eta>0 \tag{59}
\end{align*}
$$

where

$$
\begin{aligned}
& \left(\begin{array}{c}
K\left(r-P_{1}^{\prime}(0) m_{5}\right)\left(1-\delta-e^{-\left(r-P_{1}^{\prime}(0) m_{5}\right) T}\right) \\
\times\left(r \left[1-\delta-e^{-\left(r-P^{\prime}{ }_{1}(0) m_{5}\right) T}\right.\right. \\
+\delta e^{-\left(r-P_{1}^{\prime}(0) m_{5}\right)(T-\tau T)}
\end{array}\right. \\
& \left.\left.\times e^{-\left(r-P_{1}^{\prime}(0) m_{5}\right)(t-(n-1) T)}\right]\right)^{-1} \text {, } \\
& \widetilde{x^{*}(t)}=\left\{\begin{array}{l}
\quad t \in((n-1) T,(n+\tau-1) T], \\
K\left(r-P_{1}^{\prime}(0) m_{5}\right)\left(1-\delta-e^{\prime}\left(r-p_{1}(n)\right.\right.
\end{array}\right. \\
& K\left(r-P_{1}^{\prime}(0) m_{5}\right)\left(1-\delta-e^{-\left(r-P_{1}^{\prime}(0) m_{5}\right) T}\right) \\
& \times\left(r \left[1-\delta-e^{-\left(r-P_{1}^{\prime}(0) m_{5}\right) T}\right.\right. \\
& \begin{array}{c}
\left.\left.+\delta e^{-\left(r-P_{1}^{\prime}(0) m_{5}\right)(t-(n+\tau-1) T)}\right]\right)^{-1}, \\
t \in((n+\tau-1) T, n T],
\end{array} \\
& \widetilde{I^{*}(t)}=\left\{\begin{array}{c}
\frac{\delta_{I} e^{-\left(d_{I}-g^{\prime}(0) m_{5}\right)(t-(n-1) T)}}{1-\left(1-P_{I}\right) e^{-\left(d_{I}-g^{\prime}(0) m_{5}\right) T}}, \\
t \in((n-1) T,(n+\tau-1) T], \\
\frac{\delta_{I}\left(1-P_{I}\right) e^{-\left(d_{I}-g^{\prime}(0) m_{5}\right)(t-(n-1) T)}}{1-\left(1-P_{I}\right) e^{-\left(d_{I}-g^{\prime}(0) m_{5}\right) T}}, \\
t \in((n+\tau-1) T, n T],
\end{array}\right.
\end{aligned}
$$

$\widetilde{y_{J}^{*}(t)}$

$$
\widetilde{y_{M}^{*}(t)}
$$

$$
=\left\{\begin{array}{l}
e^{-d_{M}(t-(n-1) T)} \\
\quad \times\left(\widetilde{y_{M}^{*}\left(0^{+}\right)}+m \int_{0}^{t-(n-1) T}\right. \\
\left.\quad \times\left(\widetilde{y_{J}^{*}(t)}+\varepsilon_{1}\right) e^{d_{M} s} d s\right) \\
\quad t \in((n-1) T,(n+\tau-1) T] \\
e^{-d_{M}(t-(n-1) T)} \\
\quad \times\left(\widetilde{y_{M}^{*}\left(\tau T^{+}\right)} e^{d_{M} \tau T}\right. \\
\left.\quad+m \int_{\tau T}^{t-(n-1) T}\left(\widetilde{y_{J}^{*}(t)}+\varepsilon_{1}\right) e^{d_{M} s} d s\right) \\
\quad t \in((n+\tau-1) T, n T]
\end{array}\right.
$$

$$
\widetilde{y_{M}^{*}\left(0^{+}\right)}
$$

$$
=\left(m \left[\left(1-P_{M}\right) \int_{0}^{\tau T}\left(\widetilde{y_{J}^{*}(t)}+\varepsilon_{1}\right) e^{d_{M} s} d s\right.\right.
$$

$$
\left.+\int_{\tau T}^{T}\left(\widetilde{y_{J}^{*}(t)}+\varepsilon_{1}\right) e^{d_{M^{s}}} d s\right]
$$

$$
\begin{aligned}
& =\left\{\begin{array}{l}
e^{-\left(m+d_{J}\right)(t-(n-1) T)} \\
\times\left(\widetilde{y_{J}^{*}\left(0^{+}\right)}+\lambda P_{2}\left(m_{5}\right) M \int_{0}^{t-(n-1) T} e^{\left(m+d_{J}\right) s} d s\right), \\
\quad t \in((n-1) T,(n+\tau-1) T], \\
e^{-\left(m+d_{J}\right)(t-(n-1) T)} \\
\times\left(\widetilde{y_{J}^{*}\left(\tau T^{+}\right.}\right) e^{\left(m+d_{J}\right) \tau T} \\
\left.\quad+\lambda P_{2}\left(m_{5}\right) M \int_{\tau T}^{t-(n-1) T} e^{\left(m+d_{J}\right) s} d s\right), \\
t \in((n+\tau-1) T, n T],
\end{array}\right. \\
& \widetilde{y_{J}^{*}\left(0^{+}\right)} \\
& =\left(\lambda P_{2}\left(m_{5}\right) M\left[\left(1-P_{J}\right) \int_{0}^{\tau T} e^{\left(m+d_{J}\right) s} d s+\int_{\tau T}^{T} e^{\left(m+d_{J}\right) s} d s\right]\right. \\
& \left.\times e^{-\left(m+d_{J}\right) T}+\delta_{J}\right)\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)^{-1}, \\
& y_{J}^{*}\left(\tau T^{+}\right) \\
& =\left(\left(1-P_{J}\right)\right. \\
& \times\left[\lambda P_{2}\left(m_{5}\right) M \int_{0}^{\tau T} e^{\left(m+d_{J}\right) s} d s+e^{-\left(m+d_{J}\right) T}\right. \\
& \left.\times \lambda P_{2}\left(m_{5}\right) M \int_{\tau T}^{T} e^{\left(m+d_{J}\right) s} d s+\delta_{J}\right] \\
& \left.\times e^{-\left(m+d_{J}\right) \tau T}\right)\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)^{-1},
\end{aligned}
$$

$$
\begin{align*}
& \left.\times e^{-d_{M} T}+\delta_{M}\right)\left(1-\left(1-P_{M}\right) e^{-d_{M} T}\right)^{-1}, \\
& \left.\widetilde{y_{M}^{*}\left(\tau T^{+}\right.}\right) \\
& =\left(\left(1-P_{M}\right)\right. \\
& \quad \times\left[m \int_{0}^{\tau T}\left(\widetilde{y_{J}^{*}(t)}+\varepsilon_{1}\right) e^{d_{M} s} d s\right. \\
& \left.\quad+m e^{-d_{M} T} \int_{\tau T}^{T}\left(\widetilde{y_{J}^{*}(t)}+\varepsilon_{1}\right) e^{d_{M} s} d s+\delta_{M}\right] \\
& \left.\quad \times e^{-d_{M} \tau T}\right)\left(1-\left(1-P_{M}\right) e^{-d_{M} T}\right)^{-1} . \tag{60}
\end{align*}
$$

We shall prove that one cannot have $S(t)<m_{5}$ for all $t>0$, otherwise

$$
\begin{align*}
& x^{\prime}(t) \geq\left(r-P_{1}^{\prime}(0) m_{5}\right) x(t)\left(1-\frac{r x(t)}{K\left(r-P_{1}^{\prime}(0) m_{5}\right)}\right), \\
& I^{\prime}(t) \leq-\left(d_{I}-g^{\prime}(0) m_{5}\right) I(t), \\
& y_{J}^{\prime}(t) \leq \lambda P_{2}\left(m_{5}\right) M-\left(d_{J}+m\right) y_{J}(t), \\
& y_{M}^{\prime}(t)=m y_{J}(t)-d_{M} y_{M}(t), \\
& \Delta \neq(n+\tau-1) T, \\
& \Delta x(t)=-\delta x(t), \\
& \Delta I(t)=-P_{I} I(t), \\
& \Delta y_{J}(t)=-P_{J} y_{J}(t),  \tag{61}\\
& \Delta y_{M}(t)=-P_{M} y_{M}(t), \\
& t=(n+\tau-1) T, \\
& \Delta x(t)=0, \\
& \Delta I(t)=\delta_{I}, \\
& \Delta y_{J}(t)=\delta_{J}, \\
& \Delta y_{M}(t)=\delta_{M}, \\
& t=n T .
\end{align*}
$$

Analyzing (61) with similarity as (40), it is easy to obtain that there exists a positive constant $n_{6}^{*}$, such that $x(t) \geq \widetilde{x^{*}(t)}-$ $\varepsilon_{1}, I(t) \leq \widetilde{I^{*}(t)}+\varepsilon_{1}, y_{J}(t) \leq \widetilde{y_{J}^{*}(t)}+\varepsilon_{1}, y_{M}(t) \leq \widetilde{y_{M}^{*}(t)}+\varepsilon_{1}$ for $t \geq n_{6}^{*} T$. Therefore,

$$
\begin{aligned}
S^{\prime}(t) \geq & \beta P_{1}\left(\widetilde{x^{*}(t)}-\varepsilon_{1}\right) S(t)-g\left(\widetilde{I^{*}(t)}+\varepsilon_{1}\right) S(t) \\
& -P_{2}^{\prime}(0)\left(\widetilde{y_{M}^{*}(t)}+\varepsilon_{1}\right) S(t)-d_{S} S(t)
\end{aligned}
$$

$$
\begin{gather*}
t \neq(n+\tau-1) T, t \neq n T \\
\Delta S(t)=-P_{S} S(t), \quad t=(n+\tau-1) T \\
\Delta S(t)=0, \quad t=n T \tag{62}
\end{gather*}
$$

for $t \geq n_{6}^{*} T$. Let $N_{0} \in \mathbb{N}$ and $\left(N_{0}+\tau-1\right) \geq n_{6}^{*}$. Integrating (62) on $((n+\tau-1) T,(n+\tau) T], n \geq N_{0}$, we have

$$
\begin{align*}
& S((n+\tau) T) \\
& \qquad \geq S((n+\tau-1) T)\left(1-P_{S}\right) \\
& \quad \times e^{\left.\iint_{(n+\tau-1) T}^{(n+\tau) T}\left(\beta P_{1}\left(\widetilde{x^{*}(t)}\right)-\varepsilon_{1}\right)-g\left(\overline{I^{*}(t)}+\varepsilon_{1}\right)-P_{2}^{\prime}(0)\left(\overline{y_{M}^{*}(t)}+\varepsilon_{1}\right)-d_{S}\right) d t}  \tag{63}\\
& \quad=S((n+\tau-1) T) e^{\eta} .
\end{align*}
$$

Then $S\left(\left(N_{0}+\tau+k\right) T\right) \geq S\left(\left(N_{0}+\tau\right) T\right) e^{k \eta} \rightarrow \infty$ as $k \rightarrow \infty$, which is a contradiction. So there exists a $t_{1}\left(t_{1}>n_{6}^{*} T\right)$ such that $S\left(t_{1}\right) \geq m_{5}$.
Step 2. If $S(t) \geq m_{5}$ for all $t \geq t_{1}$, then Our purpose is obtained. If not, let $t_{2}=\inf \left\{t>t_{1} \mid S(t)<m_{5}\right\}$. Then $S(t) \geq m_{5}$ for $t \in\left[t_{1}, t_{2}\right)$ and $S\left(t_{2}\right)=m_{5}$. In this step, we consider two possible cases for $t_{2}$.

Case 1. $t_{2}=\left(n_{1}+\tau-1\right) T, n_{1} \in \mathbb{N}$. Then $S\left(t_{2}^{+}\right)=(1-$ $\left.P_{S}\right) S\left(t_{2}\right)<m_{5}$. Select $n_{2}, n_{3} \in \mathbb{N}$ such that $\left(n_{2}-1\right) \geq n_{6}^{*}$ and $\left(1-P_{S}\right)^{n_{2}} e^{n_{3} \eta+n_{2} \sigma T}>\left(1-P_{S}\right)^{n_{2}} e^{n_{3} \eta+\left(n_{2}+1\right) \sigma T}>1$, where $\sigma=\beta P_{1}\left(m_{4}\right)-g(M)-P_{2}^{\prime}(0) M-d_{S}<0$. Let $\widetilde{T}=\left(n_{2}+n_{3}\right) T$, then we have the claim: there exists $t_{3} \in\left(t_{2}, t_{2}+\widetilde{T}\right]$ such that $S\left(t_{3}\right) \geq m_{5}$. If the claim is false, we will get a contradiction in the following.

According to Step 1, we have $x(t) \geq \widetilde{x^{*}(t)}-\varepsilon_{1}, I(t) \leq$ $\widetilde{I^{*}(t)}+\varepsilon_{1}, y_{J}(t) \leq \widetilde{y_{J}^{*}(t)}+\varepsilon_{1}, y_{M}(t) \leq \widetilde{y_{M}^{*}(t)}+\varepsilon_{1}$ for $t \geq$ $\left(n_{1}+n_{2}-1\right) T$. Then, we have

$$
\begin{gather*}
S^{\prime}(t) \geq \beta P_{1}\left(\widetilde{x^{*}(t)}-\varepsilon_{1}\right) S(t)-g\left(\widetilde{I^{*}(t)}+\varepsilon_{1}\right) S(t) \\
-P_{2}^{\prime}(0)\left(\widetilde{y_{M}^{*}(t)}+\varepsilon_{1}\right) S(t)-d_{S} S(t) \\
t \neq(n+\tau-1) T, t \neq n T  \tag{64}\\
\Delta S(t)=-P_{S} S(t), \quad t=(n+\tau-1) T \\
\Delta S(t)=0, \quad t=n T
\end{gather*}
$$

for $t \in\left[t_{2}+n_{2} T, t_{2}+\widetilde{T}\right]$. As in Step 1, we have

$$
\begin{equation*}
S\left(t_{2}+\widetilde{T}\right) \geq S\left(t_{2}+n_{2} T\right) e^{n_{3} \eta} \tag{65}
\end{equation*}
$$

Since $x(t) \geq m_{4}, I(t) \leq M, y_{M}(t) \leq M$ and $P_{2}(S(t))<$ $P_{2}^{\prime}(0) S(t)$, we have
$S^{\prime}(t)$

$$
\begin{gather*}
\geq\left(\beta P_{1}\left(m_{4}\right)-g(M)-P_{2}^{\prime}(0) M-d_{S}\right) S(t)=\sigma S(t), \\
\quad t \neq(n+\tau-1) T, \quad t \neq n T \\
\Delta S(t)=-P_{S} S(t), \quad t=(n+\tau-1) T \\
\Delta S(t)=0, \quad t=n T \tag{66}
\end{gather*}
$$

for $t \in\left[t_{2}, t_{2}+n_{2} T\right]$. Integrating (66) on $\left[t_{2}, t_{2}+n_{2} T\right]$, we have

$$
\begin{align*}
S\left(t_{2}+n_{2} T\right) & \geq S\left(t_{2}^{+}\right) e^{n_{2} \sigma T} \\
& =\left(1-P_{S}\right) m_{5} e^{n_{2} \sigma T}  \tag{67}\\
& \geq\left(1-P_{S}\right)^{n_{2}} m_{5} e^{n_{2} \sigma T}
\end{align*}
$$

Thus, by (65) and (67), we have $S\left(t_{2}+\widetilde{T}\right) \geq(1-$ $\left.P_{S}\right)^{n_{2}} m_{5} e^{n_{3} \eta+n_{2} \sigma T}>m_{5}$, which is a contradiction. Let $t_{4}=$ $\inf \left\{t>t_{2} \mid S(t) \geq m_{5}\right\}$, then for $t \in\left[t_{2}, t_{4}\right), S(t)<m_{5}$ and $S\left(t_{4}\right)=m_{5}$. So, $S(t) \geq S\left(t_{2}^{+}\right) e^{\sigma\left(t-t_{2}\right)}=\left(1-P_{S}\right) m_{5} e^{\sigma\left(t-t_{2}\right)} \geq$ $\left(1-P_{S}\right)^{n_{2}+n_{3}} m_{5} e^{\sigma\left(n_{2}+n_{3}\right) T}=\widetilde{m_{1}}$ for $t \in\left[t_{2}, t_{4}\right)$.
Case $2\left(t_{2} \neq\left(n_{1}+\tau-1\right) T, n_{1} \in \mathbb{N}\right)$. Suppose that $t_{2} \in\left(\left(n_{1}^{\prime}+\right.\right.$ $\left.\tau-1) T,\left(n_{1}^{\prime}+\tau\right) T\right), n_{1}^{\prime} \in \mathbb{N} . S(t) \geq m_{5}$ for $t \in\left[t_{1}, t_{2}\right)$ and $S\left(t_{2}\right)=m_{5}$. There are two possible cases for $t \in\left(t_{2},\left(n_{1}^{\prime}+\tau\right) T\right)$.
Case 2a. If $S(t) \leq m_{5}$ for all $t \in\left(t_{2},\left(n_{1}^{\prime}+\tau\right) T\right)$, similar to Case 1, we can prove there exists a $t_{3}^{\prime} \in\left(\left(n_{1}^{\prime}+\tau\right) T,\left(n_{1}^{\prime}+\tau\right) T+\widetilde{T}\right]$ such that $S\left(t_{3}^{\prime}\right) \geq m_{5}$. Let $t_{4}^{\prime}=\inf \left\{t>t_{2} \mid S(t) \geq m_{5}\right\}$, then for $t \in\left[t_{2}, t_{4}^{\prime}\right), S(t)<m_{5}$ and $S\left(t_{4}^{\prime}\right)=m_{5}$. So, $S(t) \geq S\left(t_{2}\right) e^{\sigma\left(t-t_{2}\right)}=$ $m_{5} e^{\sigma\left(t-t_{2}\right)} \geq\left(1-P_{S}\right)^{n_{2}+n_{3}} m_{5} e^{\sigma\left(n_{2}+n_{3}+1\right) T}=m^{*}<\widetilde{m_{1}}$ for all $t \in\left[t_{2}, t_{4}^{\prime}\right)$.

Case $2 b$. If there exists a $t \in\left(t_{2},\left(n_{1}^{\prime}+\tau\right) T\right)$ such that $S(t) \geq m_{5}$. Let $\overline{t_{4}^{\prime}}=\inf \left\{t>t_{2} \mid S(t) \geq m_{5}\right\}$, then for $t \in\left[t_{2}, \overline{t_{4}^{\prime}}\right), S(t)<$ $m_{5}$ and $S\left(\overline{t_{4}^{\prime}}\right)=m_{5}$. So, $S(t) \geq S\left(t_{2}\right) e^{\sigma\left(t-t_{2}\right)}=m_{5} e^{\sigma\left(t-t_{2}\right)} \geq$ $m_{5} e^{\sigma T}>m^{*}$ for all $t \in\left[t_{2}, \overline{t_{4}^{\prime}}\right)$.

Since $S(t) \geq m_{5}$ for some $t \geq t_{1}$, in both cases a similar discussion can be continued. The proof is completed.

## 5. Numerical Simulations and Conclusions

In this section, we will give an example and its simulations to show the efficiency of the criteria derived in Section 4.

In system (1), let $P_{1}(x(t))=a x(t), g(I(t))=b I(t)$, and $P_{2}(S(t))=h\left(1-e^{-c S(t)}\right), a, b, c, h>0$. Namely, $P_{1}(x(t))$ describes an Holling type-I functional response of the pest, $P_{2}(S(t))$ describes a Ivlev-type functional response of the pest's natural predator. Therefore, we consider the pest management model with impulsive releasing and harvesting at two different fixed moments:

$$
\begin{aligned}
& x^{\prime}(t)=r x(t)\left(1-\frac{x(t)}{K}\right)-a x(t) S(t), \\
& S^{\prime}(t)=\beta a x(t) S(t)-b I(t) S(t) \\
&-h\left(1-e^{-c S(t)}\right) y_{M}(t)-d_{S} S(t), \\
& I^{\prime}(t)= b I(t) S(t)-d_{I} I(t), \\
& y_{J}^{\prime}(t)= \lambda h\left(1-e^{-c S(t)}\right) y_{M}(t)-d_{J} y_{J}(t)-m y_{J}(t), \\
& y_{M}^{\prime}(t)=m y_{J}(t)-d_{M} y_{M}(t), \\
& t \neq(n+\tau-1) T, \\
& t \neq n T,
\end{aligned}
$$



Figure 1: Time series of the system (68) with $r=8, K=10, a=0.8, \beta=0.5, b=0.3, h=8, c=0.2, \lambda=0.6, m=2, d_{S}=0.2, d_{I}=0.5$, $d_{J}=0.4, d_{M}=0.2, \delta=0.4, P_{S}=P_{I}=P_{J}=P_{M}=0.2, \delta_{I}=0.2, \delta_{J}=0.3, \delta_{M}=0.5, \tau=0.3, T=0.5, x(0)=20, S(0)=2, I(0)=2, y_{J}(0)=0.5$, $y_{M}(0)=0.5$.

$$
\begin{align*}
& \Delta x(t)=-\delta x(t) \\
& \Delta S(t)=-P_{S} S(t) \\
& \Delta I(t)=-P_{I} I(t) \tag{68}
\end{align*}
$$

$$
\begin{aligned}
\Delta y_{M}(t) & =\delta_{M} \\
t & =n T
\end{aligned}
$$

So, by (22), (23), and (25), we have

$$
\begin{aligned}
& \beta \int_{0}^{T} P_{1}\left(x^{*}(t)\right) d t=\frac{\beta a K}{r}(r T+\ln (1-\delta))=\theta_{1}, \\
& \int_{0}^{T} g\left(I^{*}(t)\right) d t \\
& \quad=\frac{b \delta_{I}\left[1-e^{-d_{I} \tau T}+\left(1-P_{I}\right)\left(e^{-d_{I} \tau T}-e^{-d_{I} T}\right)\right]}{d_{I}\left(1-\left(1-P_{I}\right) e^{-d_{I} T}\right)} \\
& \quad=\theta_{2},
\end{aligned}
$$



Figure 2: Phase portrait of the system (68) with $r=8, K=10, a=0.8, \beta=0.5, b=0.3, h=8, c=0.2, \lambda=0.6, m=2, d_{S}=0.2, d_{I}=0.5$, $d_{J}=0.4, d_{M}=0.2, \delta=0.4, P_{S}=P_{I}=P_{J}=P_{M}=0.2, \delta_{I}=0.2, \delta_{J}=0.3, \delta_{M}=0.5, \tau=0.3, T=0.5, x(0)=20, S(0)=2, I(0)=2, y_{J}(0)=0.5$, $y_{M}(0)=0.5$.

$$
\begin{align*}
& \int_{0}^{T} y_{M}^{*}(t) d t \\
& = \\
& \quad \frac{y_{M}^{*}\left(0^{+}\right)}{d_{M}}\left(1-e^{-d_{M} \tau T}\right) \\
& \quad+\frac{m \delta_{J}}{\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)\left(d_{M}-\left(m+d_{J}\right)\right)} \\
& \quad \times\left(\frac{1-e^{-\left(m+d_{J}\right) \tau T}}{m+d_{J}}-\frac{1-e^{-d_{M} \tau T}}{d_{M}}\right) \\
& \quad+\frac{y_{M}^{*}\left(\tau T^{+}\right) e^{d_{M} \tau T}}{d_{M}}\left(e^{-d_{M} \tau T}-e^{d_{M} T}\right) \\
& \quad+\frac{m \delta_{J}\left(1-P_{J}\right)}{\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)\left(d_{M}-\left(m+d_{J}\right)\right)} \\
& \quad \times\left(\frac{e^{-\left(m+d_{J}\right) \tau T}-e^{-\left(m+d_{J}\right) T}}{m+d_{J}}\right. \\
& = \tag{69}
\end{align*}
$$

where

$$
\begin{gathered}
y_{M}^{*}\left(0^{+}\right)=\frac{\left[\left(1-P_{M}\right) A(\tau T)+B(T)\right] e^{-d_{M} T}+\delta_{M}}{1-\left(1-P_{M}\right) e^{-d_{M} T}}, \\
y_{M}^{*}\left(\tau T^{+}\right)=\frac{\left(1-P_{M}\right)\left[A(\tau T)+e^{-d_{M} T} B(T)+\delta_{M}\right] e^{-d_{M} \tau T}}{1-\left(1-P_{M}\right) e^{-d_{M} T}}, \\
A(\tau T)=\frac{m \delta_{J}\left(e^{\left(d_{M}-\left(m+d_{J}\right)\right) \tau T}-1\right)}{\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)\left(d_{M}-\left(m+d_{J}\right)\right)}, \\
B(T)=\frac{m \delta_{J}\left(1-P_{J}\right)\left(e^{\left(d_{M}-\left(m+d_{J}\right)\right) T}-e^{\left(d_{M}-\left(m+d_{J}\right)\right) \tau T}\right)}{\left(1-\left(1-P_{J}\right) e^{-\left(m+d_{J}\right) T}\right)\left(d_{M}-\left(m+d_{J}\right)\right)} .
\end{gathered}
$$

Then, by Theorems 11 and 14 , we have the following.
(T1) If $\theta_{1}-\theta_{2}-\theta_{3}-d_{S} T<\ln \left(1 /\left(1-P_{S}\right)\right)$, then the susceptible pest-eradication periodic solution $\left(x^{*}(t)\right.$, $\left.0, I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$ of system (68) is locally asymptotically stable.
(T2) If $\theta_{1}-\theta_{2}-h c e^{-c M} \theta_{3}-d_{S} T<\ln \left(1 /\left(1-P_{S}\right)\right)$, then the susceptible pest-eradication periodic solution $\left(x^{*}(t), 0, I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$ of system (68) is globally asymptotically stable, where $M$ is defined in Lemma 6.
(T3) If $\theta_{1}-\theta_{2}-h c \theta_{3}-d_{S} T>\ln \left(1 /\left(1-P_{S}\right)\right.$, then system (68) is permanent.

In the following, we analyze the locally asymptotical stability of the susceptible pest-eradication periodic solution and permanence of system (68).


Figure 3: Time series of the system (68) with $r=8, K=10, a=0.8, \beta=0.5, b=0.3, h=8, c=0.2, \lambda=0.6, m=2, d_{S}=0.2, d_{I}=0.5$, $d_{J}=0.4, d_{M}=0.2, \delta=0.4, P_{S}=P_{I}=P_{J}=P_{M}=0.2, \delta_{I}=0.2, \delta_{J}=0.3, \delta_{M}=0.5, \tau=0.3, T=1, x(0)=20, S(0)=0.2, I(0)=2, y_{J}(0)=2$, $y_{M}(0)=2$.

Assume that $x(0)=20, S(0)=2, I(0)=2, y_{J}(0)=0.5$, $y_{M}(0)=0.5, r=8, K=10, a=0.8, \beta=0.5, b=0.3, h=8$, $c=0.2, \lambda=0.6, m=2, d_{S}=0.2, d_{I}=0.5, d_{J}=0.4, d_{M}=0.2$, $\delta=0.4, P_{S}=P_{I}=P_{J}=P_{M}=0.2, \delta_{I}=0.2, \delta_{J}=0.3$, $\delta_{M}=0.5, \tau=0.3, T=0.5$. Obviously, the condition of ( $T 1$ ) is satisfied, then the susceptible pest-eradication periodic solution of system (68) is locally asymptotically stable, which can be seen from the numerical simulation in Figures 1 and 2.

Assume that $x(0)=20, S(0)=0.2, I(0)=2, y_{J}(0)=2$, $y_{M}(0)=2, r=8, K=10, a=0.8, \beta=0.5, b=0.3, h=8$, $c=0.2, \lambda=0.6, m=2, d_{S}=0.2, d_{I}=0.5, d_{J}=0.4, d_{M}=0.2$, $\delta=0.4, P_{S}=P_{I}=P_{J}=P_{M}=0.2, \delta_{I}=0.2, \delta_{J}=0.3$, $\delta_{M}=0.5, \tau=0.3, T=1$. Obviously, the condition of (T3) is satisfied. Then, system (68) is permanent, which can also be seen from Figures 3 and 4.

From results of the numerical simulation, we know that there exists an impulsive harvesting(or releasing) periodic
threshold $T^{*}$, which satisfies $0.5<T^{*}<1$. If $T<T^{*}$ and the other parameters are fixed $(r=8, K=10, a=0.8$, $\beta=0.5, b=0.3, h=8, c=0.2, \lambda=0.6, m=2$, $d_{S}=0.2, d_{I}=0.5, d_{J}=0.4, d_{M}=0.2, \delta=0.4, P_{S}=$ $P_{I}=P_{J}=P_{M}=0.2, \delta_{I}=0.2, \delta_{J}=0.3, \delta_{M}=0.5, \tau=$ $0.3, T=0.5$.), then the susceptible pest-eradication periodic solution $\left(x^{*}(t), 0, I^{*}(t), y_{J}^{*}(t), y_{M}^{*}(t)\right)$ of system (68) is locally asymptotically stable. If $T>T^{*}$ and the other parameters are fixed $(r=8, K=10, a=0.8, \beta=0.5, b=0.3, h=8, c=0.2$, $\lambda=0.6, m=2, d_{S}=0.2, d_{I}=0.5, d_{J}=0.4, d_{M}=0.2$, $\delta=0.4, P_{S}=P_{I}=P_{J}=P_{M}=0.2, \delta_{I}=0.2, \delta_{J}=0.3$, $\delta_{M}=0.5, \tau=0.3, T=0.5$.), then system (68) is permanent. The same discussion can be applied to other parameters.

In this paper, we proposed a pest management model with impulsive releasing (periodic infective pests, immature and mature natural enemies releasing) and harvesting (periodic crops harvesting) at two different fixed moments. By means


Figure 4: Phase portrait of the system (68) with $r=8, K=10, a=0.8, \beta=0.5, b=0.3, h=8, c=0.2, \lambda=0.6, m=2, d_{S}=0.2, d_{I}=0.5$, $d_{J}=0.4, d_{M}=0.2, \delta=0.4, P_{S}=P_{I}=P_{J}=P_{M}=0.2, \delta_{I}=0.2, \delta_{J}=0.3, \delta_{M}=0.5, \tau=0.3, T=1, x(0)=20, S(0)=0.2, I(0)=2, y_{J}(0)=2$, $y_{M}(0)=2$.
of Floquet theory and multicomparison results for impulsive differential equations, two sufficient conditions ensuring the locally and globally asymptotical stability of the susceptible pest-eradication period solution and permanence of the system are derived.

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## Research Article

# Binary Tree Pricing to Convertible Bonds with Credit Risk under Stochastic Interest Rates 

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#### Abstract

The convertible bonds usually have multiple additional provisions that make their pricing problem more difficult than straight bonds and options. This paper uses the binary tree method to model the finance market. As the underlying stock prices and the interest rates are important to the convertible bonds, we describe their dynamic processes by different binary tree. Moreover, we consider the influence of the credit risks on the convertible bonds that is described by the default rate and the recovery rate; then the two-factor binary tree model involving the credit risk is established. On the basis of the theoretical analysis, we make numerical simulation and get the pricing results when the stock prices are CRR model and the interest rates follow the constant volatility and the time-varying volatility, respectively. This model can be extended to other financial derivative instruments.


## 1. Introduction

Convertible bonds with the characteristic of bonds and stock are a complex financial derivative. They provide the right to holders that they give up the future dividend to obtain some stock with specified quantity. The pricing of convertible bonds is more difficult than straight bonds and options; the main reason is that it not only has value of bonds, but also involves all kinds of embedded option value brought by provisions of conversion, call, put, and so on. What is more, the embedded options in most times are American options. So, generally speaking, the pricing of convertible bonds cannot get closed-form solution; in most conditions, numerical method was adopted, for example, binary tree method, Monte Carlo method, finite difference method, and so on. As for Monte Carlo method, firstly it uses different stochastic differential equations to describe the pricing factor models in the market for simulation, then it makes pricing based on the characteristic of convertible bonds, for example, the boundary conditions acquired by all kinds of provisions, to make pricing for conversion bonds (Ammann et al. [1], Guzhva et al. [2], Kimura and Shinohara [3], Yang et al. [4], and Siddiqi [5]). But because the supposed parameters of
stochastic differential equations are exogenous, this method not necessarily makes a better fitting for the existing market conditions. The binary tree method can solve the above problems.

The binary tree method is firstly put forward by Cox et al. [6], Cox-Ross-Rubinstein (CRR) binomial option pricing model. After that, many researchers revised and popularized it. Cheung and Nelken [7] firstly apply binary tree to convertible bonds pricing and obtain the pricing solution of a twofactor model which is based on stock prices and interest rates. Carayannopoulos and Kalimipalli [8] apply the trigeminal tree pricing model to convertible bonds pricing research with single factor. Hung and Wang [9] also apply binary tree model to convertible bonds pricing which embodies default risk and considers the influences of stock prices and interest rates. Chambers and Lu [10] further considered the correlation of stock prices and interest rates and expanded the model of Hung and Wang. Binary tree model has been widely used in the pricing of contingent claims such as stock options, currency options, stock index options, and future options. Xu [11] proposes a trinomial lattice model to price convertible bonds and asset swaps with market risk and counterparty risk.

Interest rate is a very important factor in the financial market; all security prices and yields are related to it. The interest rates model has equilibrium model, no-arbitrage model, and so on. In the equilibrium model, interest rates are generally described by some stochastic process, which is mean-reversion, to make interest rates show the trend of convergence to a long-term average with the passing of time, including Vasicek model, Rendleman-Bartter model, and CIR model. The parameters of these models should be estimated with history data, by selecting parameters purposely, but generally this fitting is not accurate and even reasonable fitting formula cannot be found. No-arbitrage models make initial term structure as model input and construct a binary tree such as CRR model for interest rate process, so that the term structure can fit the reality better and more concise. The relatively wide application no-arbitrage models include Holee model, Hull-White model, Black-Derman-Toy model, and Heath-Jarrow-Morton model. Different from the interest rate model of Hung and Wang and Chambers and Lu, this paper uses constant volatility and time-varying volatility binary tree model to describe short interest rates which are more intuitive and convenient.

This paper makes pricing research of convertible bonds with the call and put provisions and uses binary tree method for the modeling of state variables in the financial market. As the duration of convertible bonds is relatively longer than straight bonds, their prices are subject to the impact of interest rates. Moreover, as one of the corporate bonds, convertible bonds may have credit risk. So this paper uses different binary trees to model the process of stock prices and interest rates, and considering the impact of stock dividends and credit risk to convertible bonds, it adopts default rate and recovery rate to describe credit risk and get two-factor binary tree model involving credit risk; on this basis, with example simulation to get the convertible bonds, pricing results under the condition of stock price obey CRR model, constant volatility interest rate binary tree model, and timevarying volatility interest rate binary tree model.

## 2. Market Model

### 2.1. Interest Rates Binary Tree

2.1.1. Constant Volatility Binary Tree Model. Ritchken [12] deduces the continuous form of short-term $r(t)$ in HoLee model [13] meeting the following stochastic differential equations:

$$
\begin{equation*}
d r(t)=\mu(t) d t+\sigma(t) d z(t), \quad t>0 \tag{1}
\end{equation*}
$$

where $\mu(t)$ is the drift, $\sigma(t)$ is the instantaneous volatility, both can be the function of time $t$, and $z(t)$ is Brownian motion.

Grant and Vora [14] get the discrete form of (1) as follows:

$$
\begin{equation*}
\Delta r(t)=\mu(t) \Delta t+\sigma(t) \Delta z(t), \quad t \geq 0 \tag{2}
\end{equation*}
$$

Make $f(j)$ to be the forward interest rate in the interval $[j, j+$ 1]. Then get

$$
\begin{gather*}
\mu(0) \Delta t=f(1)-r(0)+\frac{\Delta t}{2} \sigma^{2}(r(1)), \\
\mu(1) \Delta t=f(2)-f(1)+\frac{\Delta t}{2} \sigma^{2}\left(\sum_{j=1}^{2} r(j)\right)-\Delta t \cdot \sigma^{2}(r(1)), \\
\mu(t-1) \Delta t=f(t)-f(t-1)+\frac{\Delta t}{2} \sigma^{2}\left(\sum_{j=1}^{t} r(j)\right) \\
-\Delta t \cdot \sigma^{2}\left(\sum_{j=1}^{t-1} r(j)\right)+\frac{\Delta t}{2} \sum_{n=1}^{t-2} \sigma^{2}\left(\sum_{j=1}^{n} r(j)\right), \\
\forall t \geq 3 \\
\sum_{n=0}^{t} \mu(n) \Delta t=f(t+1)-r(0)+\sum_{n=1}^{t} \delta(n), \quad \forall t \geq 1, \tag{3}
\end{gather*}
$$

where

$$
\begin{align*}
& \sum_{n=0}^{t} \delta(n)=\frac{\Delta t}{2} \sigma^{2}\left(\sum_{j=1}^{t+1} r(j)\right)-\frac{\Delta t}{2} \sigma^{2}\left(\sum_{j=1}^{t} r(j)\right), \\
& \forall t \geq 1 \\
& \sigma^{2}\left(\sum_{j=1}^{t} r(j)\right)=\sigma^{2}\left(\sum_{j=1}^{t}(t-j+1) \sigma_{j-1} \Delta z_{j-1}\right)  \tag{4}\\
&=\sum_{j=1}^{t}(t-j+1)^{2} \sigma_{j-1}^{2} \Delta t .
\end{align*}
$$

Suppose that volatility is constant; that is, $\sigma(t)=\sigma_{c}, \forall t>$ 0 , and then

$$
\begin{align*}
\sigma^{2}\left(\sum_{j=1}^{t} r(j)\right) & =\sigma^{2}\left(\sum_{j=1}^{t}(t-j+1) \sigma_{c} \Delta z_{j-1}\right)  \tag{5}\\
& =\sigma_{c}^{2} \sum_{j=1}^{t}(t-j+1)^{2} \Delta t
\end{align*}
$$

And get the constant volatility interest rates binary tree as shown in Figure 1.
2.1.2. Time Varying Volatility Binary Tree Model. Jarrow and Turnbull [15] supposed that the volatility of short-term is changeable in different intervals, but is constant in the same time interval. Let $\Delta t=1$, and then the discrete form of interest rates can meet

$$
\begin{equation*}
r(t)=r(0)+\sum_{j=0}^{t-1} \mu(j)+\sigma(t-1) \sum_{j=0}^{t-1} \Delta z(j) . \tag{6}
\end{equation*}
$$



Figure 1: 4-period constant volatility interest rates binary tree.

The variance formula of the sum of short-term interest rate is

$$
\begin{equation*}
\sigma^{2}\left(\sum_{j=1}^{t} r_{j}\right)=\sum_{j=0}^{t-1}(j+1) \sigma_{j}^{2}+\sum_{j=0}^{t-2} \sum_{k=j+1}^{t-1} 2(j+1) \sigma_{j} \sigma_{k} \tag{7}
\end{equation*}
$$

And then get time-varying interest rates tree as shown in Figure 2.
2.2. Stock Price Binary Tree. Suppose that the current moment is 0 and the expiration date of convertible bonds is $T$. According to interval $\Delta t$, we divide the period $[0, T]$ to $L$ subintervals: $\left[t_{i}, t_{i+1}\right], 0 \leq i \leq L, t_{0}=0, t_{L}=T, T=L \Delta t$. In each interval $\left[t_{i}, t_{i+1}\right]$, there are two possible states in the market, up or down. The change of every market state is independent. $U$ means the up state and $D$ means the down state.

The stock prices will have two states; $p$ means the probability of market up, and then the probability of market down is $1-p$. If the current stock price is $S$, then the stock price of later period may have two possibilities: $S_{u}, S_{d}$, and $S_{u}=S \times u, S_{d}=S \times d ; u, d$ separately mean the magnitude of up and down. If the initial price of stock is known, then the stock price tree can be determined by the given model parameters $p, u$, and $d$.

Model parameters $p, u$, and $d$ will directly impact the results of binary tree; the selection of them should follow no-arbitrage principle. Generally speaking, there are two selections: CRR model [4], equal-probability binomial model (Roman [16], Hull [17]). This paper adopts CRR model to describe pricing process of stock.

CRR model selects parameters as follows:

$$
\begin{gather*}
u=e^{\sigma_{s} \sqrt{\Delta t}}, \quad d=u^{-1}=e^{-\sigma_{s} \sqrt{\Delta t}}, \\
p=\frac{1}{2}\left[1+\frac{\mu_{s}}{\sigma_{s}} \sqrt{\Delta t}\right] . \tag{8}
\end{gather*}
$$

Especially, if the actual financial market is changed to risk neutral market. Then the expected profit $\mu_{s}$ of stock will
change to risk-free interest rate $r$, but the volatility $\sigma_{S}$ is the same. The probability of price up in this model is $p=\left(e^{r \Delta t}-\right.$ $d) /(u-d)$; among them, $r$ is risk-free interest rate. Under this condition, the pricing result is no arbitrage.
2.3. Credit Risk. Consider convertible bonds with credit risk. We adopt the method of Jarrow and Turnbull [18] to model the credit risk of convertible bonds. Suppose that the probability of default risk in time interval $\left[t_{i-1}, t_{i}\right]$ is $\lambda_{i}$ and the rate of recovery is $\xi_{i}$ when default. If there are serials different deadline risk-free zero-coupon bonds in the financial market and the prices are $\{P(1), P(2), P(3), \ldots, P(n)\}$, the serials of different deadline risk company zero-coupon bonds and the prices are $\{D(1), D(2), D(3), \ldots, D(n)\}$. We can get the riskfree interest term structure and risk interest term structure from them. If the recovery rate $\xi_{i}$ is already known, then the rate of risk $\lambda_{i}$, in number $i$ period of bonds, can be acquired. The detail analysis process is as follows.

If the risk-free interest rate of one-year period is $r_{0}$ and risk interest rate is $r_{1}^{*}$, then

$$
\begin{equation*}
e^{-r_{1}^{*}}=\left[1 \cdot\left(1-\lambda_{1}\right)+\xi_{1} \cdot \lambda_{1}\right] e^{-r_{0}}, \text { and get } \lambda_{1}=\frac{1-e^{r_{0}-r_{1}^{*}}}{1-\xi_{1}} \tag{9}
\end{equation*}
$$

If the risk interest rate of two-year period is $r_{2}^{*}$, then

$$
\begin{align*}
e^{-2 r_{2}^{*}}=\{ & {\left[1\left(1-\lambda_{2}\right)+\xi_{2} \lambda_{2}\right] e^{-r_{u}} \pi\left(1-\lambda_{1}\right) } \\
& +\left[1\left(1-\lambda_{2}\right)+\xi_{2} \lambda_{2}\right] e^{-r_{d}}(1-\pi)\left(1-\lambda_{1}\right)  \tag{10}\\
& \left.+\xi_{1} \lambda_{1}\right\} e^{-r_{0}} .
\end{align*}
$$

When $\lambda_{1}, \lambda_{2}$ can be got by the above formula, with the same method, we can get the risk interest rate $\left\{\lambda_{i}, i \geq 1\right\}$ of each period.


Figure 2: 4-period time-varying interest rate tree.

## 3. Stock and Interest Rate Binary Tree Model with Credit Risk

For convertible bonds with credit risk, suppose that the underlying stock price and risk-free interest rate process are random, and the underlying stock price process is described by CRR model, where the stock magnitude of up and down is $u=e^{\sigma_{S} \sqrt{\Delta t}}, d=e^{-\sigma_{S} \sqrt{\Delta t}}$ respectively. Suppose that the stock price is 0 when default, and then the possible price of stock is $0, S_{u}, S_{d}$.

In risk-neutral world, the expected yield rate is risk-free interest rate $r$, and the stock continuous dividends yield is $q$, then the expected yield rate is $r-q$; so to meet the no-arbitrage condition, there is

$$
\begin{equation*}
S e^{(r-q) \Delta t}=p(1-\lambda) S u+(1-p)(1-\lambda) S d+0 \cdot \lambda . \tag{11}
\end{equation*}
$$

Then get $p=\left(e^{(r-q) \Delta t} /((1-\lambda)-d)\right) /(u-d) . p$ is the up probability of stock with credit risk. As the risk-free interest rate of all periods is random, suppose that the risk-free interest rate of number $i$ period is $r_{i}$, the volatility of stock is constant $\sigma_{S}$, and dividends rate is $q_{i}$, so the parameters of stock price in all periods can be generally presented as

$$
\begin{align*}
& u_{i}=e^{\sigma_{S} \sqrt{\Delta t}}, \quad d_{i}=e^{-\sigma_{S} \sqrt{\Delta t}}, \\
& p_{i}=\frac{e^{\left(r_{i}-q_{i}\right) \Delta t} /\left(1-\lambda_{i}\right)-d}{u-d} . \tag{12}
\end{align*}
$$

Suppose risk-free interest rates are stochastic and described by binary tree model, then the stock tree and interest rate tree involving credit risk are combined as shown in Figure 3. In this paper, we suppose that the correlation coefficient of interest rate and stock price is 0 .

After obtaining the process of stock prices and risk-free interest rates, the value of convertible bonds can be got by backward induction. We divide the value of convertible bonds into two parts; one is the value of equity got by converting to stock or exercise embedded options; the other is bonds value


Figure 3: 4-period two-factor binary tree with credit risk added.
that is the present value of bonds when repaying capital and interest and the present value of the residual value.

Suppose that the default probability of convertible bonds in the interval $\left[t_{k-1}, t_{k}\right]$ is $\lambda_{k}$, recovery value is $\xi_{k}$, and then the holding value at $t_{k}$ time is $E V_{k}$, given by
$E V_{k}=$ (the expected equity value at $t_{k+1}$-time

+ the expected bonds value at

$$
\begin{equation*}
t_{k+1} \text {-time) } \cdot e^{-r_{k} \cdot \Delta t} \tag{13}
\end{equation*}
$$

So the value of convertible value at time- $t_{k}$ is

$$
\begin{array}{r}
V_{k}=\max [\min (\text { the holding value at } \\
\left.\qquad t_{k} \text {-time, call value }\right), \\
\text { conversion value, put value }]  \tag{14}\\
=\max \left[\min \left(E V_{k}, V_{k}^{\text {call }}\right), V_{k}^{\text {con }}, V_{k}^{\text {put }}\right] .
\end{array}
$$

## 4. Numerical Examples

We take a four-period binary tree model as an example to expound the convertible bonds pricing process with call provision and put provision in the above models and compare the results under the constant volatility interest rate model and the time-varying volatility interest rate model.
4.1. Process of Interest Rate and Stock. The initial parameters of the convertible bond are all the same to the constant volatility interest rate model and the time-varying volatility interest rate model. Suppose that time interval $\Delta t=1$, the up probability of interest rates is $\pi=1 / 2$, and the $1-4$ year period yields of risk-free zero-coupon bonds are 6.145\%, $6.366 \%, 6.837 \%$, and $6.953 \%$, respectively; the volatility of short-term interest rates is $2.5 \%$. So the other binary tree parameters of constant volatility interest rate can be got as shown in Table 1. In the same way, the other binary tree parameters of time-varying volatility interest rate can be got as shown in Table 2. Then we get the two interest rate binary trees.

The process of underlying stock prices uses CRR model to describe. The selected parameters 8 are as follows: $S_{0}=25$, $\sigma_{S}=0.185, \Delta t=1, T=4$, and $q=0.04$. So all the parameters under constant volatility interest rate are

$$
\begin{align*}
& u=1.2032, \quad d=0.8311, \\
& p_{r_{0}}=0.5887, \quad p_{r(1)_{u}}=0.6841, \\
& p_{r(1)_{d}}=0.5923, \quad p_{r(2)_{u u}}=0.7517, \\
& p_{r(2)_{u d}}=0.6577, \quad p_{r(2)_{d d}}=0.5667,  \tag{15}\\
& p_{r(3)_{u u u}}=0.9162, \\
& p_{r(3)_{u d d}}=0.7209,
\end{align*} p_{r(3)_{u u d}=0.8170,} \quad p_{r(3)_{d d d}}=0.6279 .
$$

In the same way, the parameters under time-varying volatility interest rate model are

$$
\begin{align*}
& u=1.2032, \quad d=0.8311, \\
& p_{r_{0}}=0.5887, \quad p_{r(1)_{u}}=0.6826, \\
& p_{r(1)_{d}}=0.5908, \quad p_{r(2)_{u u}}=0.7475, \\
& p_{r(2)_{u d}}=0.6594, \quad p_{r(2)_{d d}}=0.5739,  \tag{16}\\
& p_{r(3)_{u u u}}=0.8764, \quad p_{r(3)_{u u d}}=0.8027, \\
& p_{r(3)_{u d d}}=0.7307, \quad p_{r(3)_{d d d}}=0.6604 .
\end{align*}
$$

Then we get the four-period stock prices binary tree.
4.2. Default Rates. We take the corporate bonds as reference risk bonds; suppose that the $1-4$-year period yields of corporate zero-coupon bonds are $7.645 \%, 8.155 \%, 8.557 \%$, and $9.128 \%$, respectively, and the recovery rate of convertible bonds is constant $\xi=45 \%$, and one-year risk-free interest rate $r_{0}=6.145 \%$.

Interest rate binary tree indicates that the branch point of number $n$ period is $n$. If the interest rate of number $i$ branch point in number $n$ period is $r(n-1)_{\omega}, \omega$ is the interest rates state from start to current, and then the derived interest rate branch point of number $n+1$ period is $r(n)_{\omega u}, r(n)_{\omega d}$. As $\pi=$ $1 / 2=1-\pi$, to meet the no-arbitrage principle, parameters $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$ meet the following four equations:

$$
\begin{gather*}
e^{-r_{1}^{*}+r_{0}}=\left(1-\lambda_{1}\right)+\xi \cdot \lambda_{1}, \\
e^{-2 r_{2}^{*}+r_{0}}=\pi\left(1-\lambda_{1}\right)\left(1-\lambda_{2}+\xi \lambda_{2}\right) \\
\times\left(e^{-r(1)_{u}}+e^{-r(1)_{d}}\right)+\xi \lambda_{1}, \\
e^{-3 r_{3}^{*}+r_{0}}=\pi^{2}\left(1-\lambda_{1}\right)\left(1-\lambda_{2}\right)\left(1-\lambda_{3}+\xi \lambda_{3}\right) \\
\times\left[e^{-r(1)_{u}}\left(e^{-r(2)_{u u}}+e^{-r(2)_{u d}}\right)\right. \\
\left.+e^{-r(1)_{d}} \cdot\left(e^{-r(2)_{u d}}+e^{-r(2)_{d d}}\right)\right] \\
e^{-4 r_{4}^{*}+r_{0}}=\pi^{3}\left(1-\lambda_{1}\right)\left(1-\lambda_{2}\right)\left(1-\lambda_{3}\right)\left(1-\lambda_{4}+\xi \lambda_{4}\right) \\
\times\left\{e ^ { - r ( 1 ) _ { u } } \left[e^{-r(2)_{u u}}\left(e^{-r(3)_{u u u}}+e^{-r(3)_{u u d}}\right)\right.\right. \\
\left.+e^{-r(2)_{u d}}\left(e^{-r(3)_{u u d}}+e^{-r(3)_{u d d}}\right)\right] \\
\\
+e^{-r(1)_{d}}\left[e^{-r(2)_{u d}}\left(e^{-r(3)_{u u d}}+e^{-r(3)_{u d d}}\right)\right. \\
\left.\left.+e^{-r(2)_{d d}}\left(e^{-r(3)_{u d d}}+e^{-r(3)_{d d d}}\right)\right]\right\} \\
+
\end{gather*}
$$

By the above equations and the constant volatility interest rate binary tree, the default rates of bonds in every period are shown in Table 3.

Similarly, the default rates of corporate bonds in every period under time-varying volatility interest rate binary tree are shown in Table 4.
4.3. Price Process of Convertible Bonds. The convertible bond contains call and put provisions, the duration is $T=4$, the face value got in maturity date is 100 , conversion rate is 3 , callable price is $V_{\text {call }}=106$, and puttable price is $V_{\text {put }}=80$. We suppose that the investors can exercise the puttable right after one year.

Now we take the convertible bond under time-varying volatility interest rate binary tree to explain its pricing process. Take four points A, B, C, and D in pricing tree of Figure 4 into consideration; among them, C, D are at the end of period, $4, B$ are at the end of period 3 , and $A$ is at

Table 1: Constant volatility interest rate parameters.

| Deadline year(s) $t$ | Price of bonds (yuan) $P(t)$ | Volatility of short-term interest rate (\%) $\sigma(t)$ | Annual profit rate of bonds (\%) $y(t)$ | 1-year long-term interest rate $(\%)$ $f(t)$ | $\begin{gathered} \text { Variance } \\ \sigma^{2}\left(\sum_{j=1}^{t} r(j)\right) \end{gathered}$ | Sum of Delta (\%) $\sum_{j=1}^{t-1} \delta(j)$ | Delta <br> (\%) <br> $\delta(t)$ | Drift item <br> (\%) <br> $\mu(t)$ | Sum of drift items (\%) $\sum_{j=0}^{t-1} \mu(j)$ | Expectation <br> (\%) $E_{0}^{Q}[r(t)]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 |  | 1.6 |  | 6.145 |  | 0.0128 | 0.0128 | 0.4548 | 0.4548 | 6.145 |
| 1 | 0.9404 |  | 6.145 | 6.587 | 0.000256 | 0.0512 | 0.0384 | 1.2304 | 1.6852 | 6.5998 |
| 2 | 0.8805 |  | 6.366 | 7.779 | 0.00128 | 0.1152 | 0.064 | -0.414 | 1.2712 | 7.8302 |
| 3 | 0.8146 |  | 6.837 | 7.301 | 0.003584 | 0.2048 | 0.0896 |  |  | 7.4162 |
| 4 | 0.7572 |  | 6.953 |  | 0.00768 |  |  |  |  |  |

Table 2: Time-varying volatility interest rate parameters.

| $\begin{aligned} & \text { Deadline } \\ & \text { year(s) } \\ & t \end{aligned}$ | Price of bonds (yuan) $P(t)$ | Volatility of short-term interest rate (\%) $\sigma(t)$ | Annual profit rate of bonds (\%) $y(t)$ | 1-year long-term interest rate <br> (\%) $f(t)$ | Variance $\sigma^{2}\left(\sum_{j=1}^{t} r(j)\right)$ | Sum of Delta (\%) $\sum_{j=1}^{t-1} \delta(j)$ | Delta <br> (\%) <br> $\delta(t)$ | Drift item <br> (\%) <br> $\mu(t)$ | $\begin{aligned} & \begin{array}{l} \text { Sum of drift } \\ \text { items }(\%) \\ \sum_{j=0}^{t-1} \mu(j) \end{array} \end{aligned}$ | $\begin{gathered} \text { Expectation } \\ (\%) \\ E_{0}^{Q}[r(t)] \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 |  | 1.6 |  | 6.145 |  | 0.0128 | 0.0128 | 0.4548 | 0.4548 | 6.145 |
| 1 | 0.9404 | 1.5 | 6.145 | 6.587 | 0.000256 | 0.0465 | 0.0337 | 1.2257 | 1.6805 | 6.5998 |
| 2 | 0.8805 | 1.2 | 6.366 | 7.779 | 0.001186 | 0.0768 | 0.0303 | -0.4477 | 1.2328 | 7.8255 |
| 3 | 0.8146 | 1.3 | 6.837 | 7.301 | 0.002722 | 0.1404 | 0.0636 |  |  | 7.3778 |
| 4 | 0.7572 |  | 6.953 |  | 0.00553 |  |  |  |  |  |

Table 3: Default rate of corporate bonds under constant volatility interest rate model.

| Time period | $0-1\left(\lambda_{1}\right)$ | $1-2\left(\lambda_{2}\right)$ | $2-3\left(\lambda_{3}\right)$ | $3-4\left(\lambda_{4}\right)$ |
| :--- | :---: | :---: | :---: | :---: |
| Default rate | 0.0271 | 0.0394 | 0.0342 | 0.0737 |

Table 4: Default rate of corporate bonds under time-varying volatility interest rate model.

| Time period | $0-1\left(\lambda_{1}\right)$ | $1-2\left(\lambda_{2}\right)$ | $2-3\left(\lambda_{3}\right)$ | $3-4\left(\lambda_{4}\right)$ |
| :--- | :---: | :---: | :---: | :---: |
| Default rate | 0.0271 | 0.0389 | 0.0348 | 0.0734 |

the end of period 2 . At point $C$, as the convertible value is 108.57 that is larger than the face value 100 , the value of convertible bond is 108.57 , while the bonds value is 0 , so write it as [108.57, 0]; the first number means the equity value and the second number means the bonds value. In the same way, we can get the convertible value of $D$ point that can be written as $[0,100]$. And the up probability of $B$ point is $p_{r(3) u u d}\left(1-\lambda_{4}\right)=0.8027 \times(1-0.0734)=0.7438$. At the same way, the down probability is 0.1828 . Then the equity value of $B$ point is $108.57 \times 0.7438 \times e^{-0.08578}=74.12$. The bonds value of B point is $(45 \times 0.0734+100 \times 0.1828) e^{-0.08578}=19.81$. Then the holding value $E V_{B}$ is 93.93 . And the convertible value at B point is 90.24 , so the value of convertible bonds in $B$ point is

$$
\begin{align*}
V_{B} & =\max \left[\min \left(E V_{B}, V_{\text {call }}\right), V_{B}^{\text {con }}, V_{\text {put }}\right]  \tag{18}\\
& =\max [\min (93.93,108), 90.24,80]=93.93,
\end{align*}
$$

written as [74.12, 19.81].

In the same way, we can calculate the value of other three branch points E, F, and G at the end of period 3 that are [125.51, 2.96], [125.51, 3.03], and [79.00, 13.22]. So the equity value at A point is

$$
\begin{align*}
& (125.51 \times 0.3607+125.51 \times 0.3607+79.00 \\
& \quad \times 0.1219+74.12 \times 0.1219) \times e^{-0.10826}=98.00 \tag{19}
\end{align*}
$$

The bonds value at A point is

$$
\begin{align*}
& (45 \times 0.0348+2.96 \times 0.3607+3.03 \times 0.3607 \\
& \quad+13.22 \times 0.1219+19.81 \times 0.1219) e^{-0.10826}=6.96 \tag{20}
\end{align*}
$$

Then the holding value $E V_{A}$ is 104.96 . And the convertible value at A point is 108.57, so the value of convertible bonds in A point is

$$
\begin{align*}
V_{A} & =\max \left[\min \left(E V_{A}, V_{\text {call }}\right), V_{A}^{\text {con }}, V_{\text {put }}\right] \\
& =\max [\min (104.96,108), 108.57,80]=108.57, \tag{21}
\end{align*}
$$

written as [108.57, 0]. The other branch point in the pricing binary tree of convertible bonds can be got in the same way.

At last, under time-varying volatility interest rate binary tree model, we can get the price of convertible bond which contains credit risk that is 79.32 at the time of $t=0$. Similarly, under constant volatility interest rate binary tree model, we can get the price of convertible bond which contains credit risk that is 78.52 at the time of $t=0$; this is less than the former.


Figure 4: Constructing pricing tree under time-varying volatility interest rate model.

## 5. Conclusions

Binary tree method is a classical pricing method, by constructing the binary tree of state variable to describe the possible paths of state variable in the duration of contingent claims and then to make pricing research. Binary tree method can effectively solve the path-dependent options pricing, intuitive and easy to operate. As the embedded options in the convertible bonds are all American options, binary tree method becomes one of the main pricing methods of convertible bonds. Interest rate is the main factor which impacts the price of convertible bonds; the description of its binary tree model is the main problem of convertible bonds pricing. This paper adopts constant volatility and timevarying volatility binary tree model to describe interest rates and further consider the impact of stock dividends and credit risk to the price of convertible bonds, adopt default rate and recovery rate to describe the credit risk, and get the twofactor binary tree model with credit risk added. Based on this, we make a numerical example and get the convertible bonds pricing result under the stock prices obeying CRR model and the constant and time-varying volatility interest rate binary tree model. The model can be popularized to the pricing of convertible bonds with more complex provisions and other financial derivatives such as bond options, catastrophe bonds, and mortgage-backed security.

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## Research Article

# Ground State Solutions for the Periodic Discrete Nonlinear Schrödinger Equations with Superlinear Nonlinearities 

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We consider the periodic discrete nonlinear Schrödinger equations with the temporal frequency belonging to a spectral gap. By using the generalized Nehari manifold approach developed by Szulkin and Weth, we prove the existence of ground state solutions of the equations. We obtain infinitely many geometrically distinct solutions of the equations when specially the nonlinearity is odd. The classical Ambrosetti-Rabinowitz superlinear condition is improved.

## 1. Introduction

The following discrete nonlinear Schrödinger equation (DLNS):

$$
\begin{equation*}
i \dot{\psi}_{n}=-\Delta \psi_{n}+\varepsilon_{n} \psi_{n}-\sigma \chi_{n} f_{n}\left(\psi_{n}\right), \quad n \in \mathbb{Z} \tag{1}
\end{equation*}
$$

where $\sigma= \pm 1$ and

$$
\begin{equation*}
\Delta \psi_{n}=\psi_{n+1}+\psi_{n-1}-2 \psi_{n} \tag{2}
\end{equation*}
$$

is the discrete Laplacian operator, appears in many physical problems, like polarons, energy transfer in biological materials, nonlinear optics, and so forth (see [1]). The parameter $\sigma$ characterizes the focusing properties of the equation: if $\sigma=1$, the equation is self-focusing, while $\sigma=-1$ corresponds to the defocusing equation. The given sequences $\left\{\varepsilon_{n}\right\}$ and $\left\{\chi_{n}\right\}$ are assumed to be $T$-periodic in $n$, that is, $\varepsilon_{n+T}=\varepsilon_{n}$ and $\chi_{n+T}=$ $\chi_{n}$. Moreover, $\left\{\chi_{n}\right\}$ is a positive sequence. Here, $T$ is a positive integer. We assume that $f_{n}(0)=0$ and the nonlinearity $f_{n}(u)$ is gauge invariant, that is,

$$
\begin{equation*}
f_{n}\left(e^{i \theta} u\right)=e^{i \theta} f_{n}(u), \quad \theta \in \mathbb{R} \tag{3}
\end{equation*}
$$

We are interested in the existence of solitons of (1), that is, solutions which are spatially localized time-periodic and decay to zero at infinity. Thus, $\psi_{n}$ has the form

$$
\begin{align*}
& \psi_{n}=u_{n} e^{-i \omega t} \\
& \lim _{|n| \rightarrow \infty} \psi_{n}=0 \tag{4}
\end{align*}
$$

where $\left\{u_{n}\right\}$ is a real-valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then, (1) becomes

$$
\begin{gather*}
-\Delta u_{n}+\varepsilon_{n} u_{n}-\omega u_{n}=\sigma \chi_{n} f_{n}\left(u_{n}\right), \quad n \in \mathbb{Z}  \tag{5}\\
\lim _{|n| \rightarrow \infty} u_{n}=0 \tag{6}
\end{gather*}
$$

holds. Naturally, if we look for solitons of (1), we just need to get the solutions of (5) satisfying (6).

Actually, we consider a more general equation:

$$
\begin{equation*}
L u_{n}-\omega u_{n}=\sigma \chi_{n} f_{n}\left(u_{n}\right), \quad n \in \mathbb{Z} \tag{7}
\end{equation*}
$$

with the same boundary condition (6). Here, $L$ is a secondorder difference operator

$$
\begin{equation*}
L u_{n}=a_{n} u_{n+1}+a_{n-1} u_{n-1}+b_{n} u_{n} \tag{8}
\end{equation*}
$$

where $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ are real-valued $T$-periodic sequences. When $a_{n} \equiv-1$ and $b_{n}=2+\varepsilon_{n}$, we obtain (5).

We consider (7) as a nonlinear equation in the space $l^{2}$ of two-sided infinite sequences. Note that every element of $l^{2}$ automatically satisfies (6).

As it is well known, the operator $L$ is a bounded and selfadjoint operator in $l^{2}$. The spectrum $\sigma(L)$ is a union of a finite number of closed intervals, and the complement $\mathbb{R} \backslash \sigma(L)$ consists of a finite number of open intervals called spectral gaps. Two of them are semi-infinite (see [2]). If $T=1$, then finite gaps do not exist. However, in general, finite gaps exist, and the most interesting case in (7) is when the frequency $\omega$ belongs to a finite spectral gap. Let us fix any spectral gap and denote it by $(\alpha, \beta)$.

DNLS equation is one of the most important inherently discrete models. DNLS equation plays a crucial role in the modeling of a great variety of phenomena, ranging from solid state and condensed matter physics to biology (see [1, 3-6] and references therein). In the past decade, solitons of the periodic DNLS have become a hot topic. The existence of solitons for the periodic DNLS equations with superlinear nonlinearity [7-10] and with saturable nonlinearity [11-13] has been studied, respectively. If $\omega$ is below or above the spectrum of the difference operator $-\Delta+\varepsilon_{n}$, solitons were shown by using the Nehari manifold approach and a discrete version of the concentration compactness principle in [14]. If $\omega$ is a lower edge of a finite spectral gap, the existence of solitons was obtained by using variant generalized weak linking theorem in [10]. If $\omega$ lies in a finite spectral gap, the existence of solitons was proved by using periodic approximations in combination with the linking theorem in [8] and the generalized Nehari manifold approach in [9], respectively. The results were extended by Chen and Ma in [7]. In this paper, we employ the generalized Nehari manifold approach instead of periodic approximation technique to obtain the existence of a kind of special solitons of (7), which called ground state solutions, that is, nontrivial solutions with least possible energy in $l^{2}$. We should emphasize that the results are obtained under more general super nonlinearity than the classical Ambrosetti-Rabinowitz superlinear condition [ $8,9,15]$.

This paper is organized as follows. In Section 2, we first establish the variational framework associated with (7) and transfer the problem on the existence of solutions in $l^{2}$ of (7) into that on the existence of critical points of the corresponding functional. We then present the main results of this paper and compare them with existing ones. Section 3 is devoted to the proofs of the main results.

## 2. Preliminaries and Main Results

The following are the basic hypotheses to establish the main results of this paper:

$$
\left(V_{1}\right) \omega \in(\alpha, \beta)
$$

$\left(f_{1}\right) f_{n} \in C(\mathbb{R}, \mathbb{R})$ and $f_{n+T}(u)=f_{n}(u)$, and there exist $a>0$ and $p \in(2, \infty)$ such that

$$
\begin{equation*}
\left|f_{n}(u)\right| \leq a\left(1+|u|^{p-1}\right) \quad \forall n \in \mathbb{Z}, u \in \mathbb{R} \tag{9}
\end{equation*}
$$

$\left(f_{2}\right) f_{n}(u)=o(|u|)$ as $u \rightarrow 0$,
$\left(f_{3}\right) \lim _{|u| \rightarrow \infty} F_{n}(u) / u^{2}=\infty$, where $F_{n}(u)$ is the primitive function of $f_{n}(u)$, that is,

$$
\begin{equation*}
F_{n}(u)=\int_{0}^{u} f_{n}(s) d s \tag{10}
\end{equation*}
$$

$\left(f_{4}\right) u \mapsto f_{n}(u) /|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.
To state our results, we introduce some notations. Let

$$
\begin{equation*}
A=L-\omega, \quad E=l^{2}(\mathbb{Z}) \tag{11}
\end{equation*}
$$

Consider the functional $J$ defined on $E$ by

$$
\begin{equation*}
J(u)=\frac{1}{2}(A u, u)_{E}-\sigma \sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(u_{n}\right), \tag{12}
\end{equation*}
$$

where $(\cdot, \cdot)_{E}$ is the inner product in $E$ and $\|\cdot\|_{E}$ is the corresponding norm in $E$. The hypotheses on $f_{n}(u)$ imply that the functional $J \in C^{1}(E, \mathbb{R})$ and (7) is easily recognized as the corresponding Euler-Lagrange equation for $J$. Thus, to find nontrivial solutions of (7), we need only to look for nonzero critical points of $J$ in $E$.

For the derivative of $J$, we have the following formula:

$$
\begin{equation*}
\left(J^{\prime}(u), v\right)=(A u, v)_{E}-\sigma \sum_{n \in \mathbb{Z}} \chi_{n} f_{n}\left(u_{n}\right) v_{n}, \quad \forall v \in E \tag{13}
\end{equation*}
$$

$\operatorname{By}\left(V_{1}\right)$, we have $\sigma(A) \subset \mathbb{R} \backslash(\alpha-\omega, \beta-\omega)$. So, $E=E^{+} \oplus E^{-}$ corresponds to the spectral decomposition of $A$ with respect to the positive and negative parts of the spectrum, and

$$
\begin{array}{ll}
(A u, u)_{E} \geq(\beta-\omega)\|u\|_{E}^{2}, & u \in E^{+}  \tag{14}\\
(A u, u)_{E} \leq(\alpha-\omega)\|u\|_{E}^{2}, & u \in E^{-}
\end{array}
$$

For any $u, v \in E$, letting $u=u^{+}+u^{-}$with $u^{ \pm} \in E^{ \pm}$and $v=v^{+}+v^{-}$with $v^{ \pm} \in E^{ \pm}$, we can define an equivalent inner product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$ on $E$ by

$$
\begin{equation*}
(u, v)=\left(A u^{+}, v^{+}\right)_{E}-\left(A u^{-}, v^{-}\right)_{E}, \quad\|u\|=(u, u)^{1 / 2} \tag{15}
\end{equation*}
$$

respectively. So, $J$ can be rewritten as

$$
\begin{equation*}
J(u)=\frac{1}{2}\left\|u^{+}\right\|^{2}-\frac{1}{2}\left\|u^{-}\right\|^{2}-\sigma \sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(u_{n}\right) \tag{16}
\end{equation*}
$$

We define for $u \in E \backslash E^{-}$, the subspace

$$
\begin{equation*}
E(u):=\mathbb{R} u+E^{-}=\mathbb{R} u^{+} \oplus E^{-} \tag{17}
\end{equation*}
$$

and the convex subset

$$
\begin{equation*}
\widehat{E}(u):=\mathbb{R}^{+} u+E^{-}=\mathbb{R}^{+} u^{+} \oplus E^{-} \tag{18}
\end{equation*}
$$

of $E$, where, as usual, $\mathbb{R}^{+}=[0, \infty)$. Let

$$
\begin{gather*}
\mathscr{M}=\left\{u \in E \backslash E^{-}: J^{\prime}(u) u=0, J^{\prime}(u) v=0 \forall v \in E^{-}\right\},  \tag{19}\\
c=\inf _{u \in \mathscr{M}} J(u) \tag{20}
\end{gather*}
$$

In this paper, we also consider the multiplicity of solutions of (7).

For each $\ell \in \mathbb{Z}$, let

$$
\begin{equation*}
\ell * u=\left(u_{n+\ell T}\right)_{n \in \mathbb{Z}}, \quad \forall u=\left(u_{n}\right)_{n \in \mathbb{Z}}, \tag{21}
\end{equation*}
$$

which defines a $\mathbb{Z}$-action on $E$. By the periodicity of the coefficients, we know that both $J$ and $J^{\prime}$ are $\mathbb{Z}$-invariants. Therefore, if $u \in E$ is a critical point of $J$, so is $\ell * u$. Two critical points $u_{1}, u_{2} \in E$ of $J$ are said to be geometrically distinct if $u_{1} \neq \ell * u_{2}$ for all $\ell \in \mathbb{Z}$.

Now, we are ready to state the main results.
Theorem 1. Suppose that conditions $\left(V_{1}\right),\left(f_{1}\right)-\left(f_{4}\right)$ are satisfied. Then, one has the following conclusions.
(1) If either $\sigma=1$ and $\beta \neq \infty$ or $\sigma=-1$ and $\alpha \neq-\infty$, then (7) has at least a nontrivial ground state solution.
(2) If either $\sigma=1$ and $\beta=\infty$ or $\sigma=-1$ and $\alpha=-\infty$, then (7) has no nontrivial solution.

Theorem 2. Suppose that conditions $\left(V_{1}\right),\left(f_{1}\right)-\left(f_{4}\right)$ are satisfied and $f_{n}$ is odd in $u$. If either $\sigma=1$ and $\beta \neq \infty$ or $\sigma=-1$ and $\alpha \neq-\infty$, then (7) has infinitely many pairs of geometrically distinct solutions.

In what follows, we always assume that $\sigma=1$. The other case can be reduced to $\sigma=1$ by switching $L$ to $-L$ and $\omega$ to $-\omega$.

Remark 3. In [8], the author considered (7) with $f_{n}$ defined by

$$
\begin{equation*}
f_{n}(u)=|u|^{2} u \tag{22}
\end{equation*}
$$

which obviously satisfies $\left(f_{1}\right)-\left(f_{4}\right)$; the author also discussed the case where $f$ satisfies the Ambrosetti-Rabinowitz condition; that is, there exists $\mu>2$ such that

$$
\begin{equation*}
0<\mu F_{n}(u) \leq f_{n}(u) u, \quad u \neq 0 \tag{23}
\end{equation*}
$$

Clearly, (23) implies that $F_{n}(u) \geq c|u|^{\mu}>0$ for $|u| \geq 1$. So, it is a stronger condition than $\left(f_{3}\right)$.

Remark 4. In [9], the author assumed that $f_{n}$ satisfies the following condition: there exists $\theta \in(0,1)$ such that

$$
\begin{equation*}
0<u^{-1} f_{n}(u) \leq \theta f_{n}^{\prime}(u), \quad u \neq 0 \tag{24}
\end{equation*}
$$

Obviously, (24) implies (23) with $\mu=1+(1 / \theta)$, so it is a stronger condition than the Ambrosetti-Rabinowitz condition. In our paper, the nonlinearities satisfy more general superlinear assumptions instead of (24) which also implies $\left(f_{4}\right)$. However, we do not assume that $f_{n}$ is differentiable and satisfies (24), $\mathscr{M}$ is not a $C^{1}$ manifold of $E$, and the minimizers on $\mathscr{M}$ may not be critical points of $J$. Hence, the method of [9] does not apply any more. Nevertheless, $\mathscr{M}$ is still a topological manifold, naturally homeomorphic to the unit sphere in $E^{+}$ (see in detail in Section 3). We use the generalized Nehari manifold approach developed by Szulkin and Weth which is based on reducing the strongly indefinite variational problem to a definite one and prove that the minimizers of $J$ on $\mathscr{M}$ are indeed critical points of $J$.

Remark 5. In [7], it is shown that (7) has at least a nontrivial solution $u \in l^{2}$ if $f$ satisfies $\left(V_{1}\right),\left(f_{2}\right),\left(f_{3}\right)$, and the following conditions:
$\left(B_{1}\right) F_{n}(u) \geq 0$ for any $u \in \mathbb{R}$ and $H_{n}(u):=(1 / 2) f_{n}(u) u-$ $F_{n}(u)>0$ if $u \neq 0$,
$\left(B_{2}\right) H_{n}(u) \rightarrow \infty$ as $|u| \rightarrow \infty$, and there exist $r_{0}>0$ and $\gamma>1$ such that $\left|f_{n}(u)\right|^{\gamma} /|u|^{\gamma} \leq c_{0} H_{n}(u)$ if $|u| \geq r_{0}$, where $c_{0}$ is a positive constant,

In our paper, we use (9) and $\left(f_{4}\right)$ instead of $\left(B_{1}\right)$ and $\left(B_{2}\right)$.

## 3. Proofs of Main Results

We assume that $\left(V_{1}\right)$ and $\left(f_{1}\right)-\left(f_{4}\right)$ are satisfied from now on.
Lemma 6. $F_{n}(u)>0$ and $(1 / 2) f_{n}(u) u>F_{n}(u)$ for all $u \neq 0$.
Proof. By $\left(f_{2}\right)$ and $\left(f_{4}\right)$, it is easy to get that

$$
\begin{equation*}
F_{n}(u)>0 \quad \forall u \neq 0 . \tag{25}
\end{equation*}
$$

Set $H_{n}(u)=(1 / 2) f_{n}(u) u-F_{n}(u)$. It follows from $\left(f_{4}\right)$ that

$$
\begin{align*}
H_{n}(u) & =\frac{u}{2} f_{n}(u)-\int_{0}^{u} f_{n}(s) d s  \tag{26}\\
& >\frac{u}{2} f_{n}(u)-\frac{f_{n}(u)}{u} \int_{0}^{u} s d s=0 .
\end{align*}
$$

So, $(1 / 2) f_{n}(u) u>F_{n}(u)$ for all $u \neq 0$.

To continue the discussion, we need the following proposition.

Proposition 7 (see $[16,17]$ ). Let $u, s, v \in \mathbb{R}$ be numbers with $s \geq-1$ and $w:=s u+v \neq 0$. Then,

$$
\begin{equation*}
f_{n}(u)\left[s\left(\frac{s}{2}+1\right) u+(1+s) v\right]+F_{n}(u)-F_{n}(u+w)<0 \tag{27}
\end{equation*}
$$

Lemma 8. If $u \in \mathscr{M}$, then

$$
\begin{align*}
& J(u+w)<J(u) \quad \text { for every } w \in U \\
& :=\left\{s u+v: s \geq-1, v \in E^{-}\right\}, \quad w \neq 0 . \tag{28}
\end{align*}
$$

Hence, $u$ is the unique global maximum of $\left.J\right|_{\widehat{E}(u)}$.
Proof. We rewrite $J$ by

$$
\begin{equation*}
J(u)=\frac{1}{2}\left(A u^{+}, u^{+}\right)_{E}+\frac{1}{2}\left(A u^{-}, u^{-}\right)_{E}-\sigma \sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(u_{n}\right) . \tag{29}
\end{equation*}
$$

Since $u \in \mathscr{M}$, we have

$$
\begin{align*}
0= & \left(J^{\prime}(u), \frac{2 s+s^{2}}{2} u+(1+s) v\right) \\
= & \frac{2 s+s^{2}}{2}\left(A u^{+}, u^{+}\right)_{E}+\frac{2 s+s^{2}}{2}\left(A u^{-}, u^{-}\right)_{E}  \tag{30}\\
& +(1+s)\left(A u^{-}, v\right)_{E} \\
& -\sum_{n \in \mathbb{Z}} \chi_{n} f_{n}\left(u_{n}\right)\left(\frac{2 s+s^{2}}{2} u_{n}+(1+s) v_{n}\right) .
\end{align*}
$$

Together with Proposition 7, we know that

$$
\begin{align*}
J(u+ & w)-J(u) \\
= & \frac{1}{2}\left\{\left(A(1+s) u^{+},(1+s) u^{+}\right)_{E}-\left(A u^{+}, u^{+}\right)_{E}\right\} \\
& +\frac{1}{2}\left\{\left(A\left((1+s) u^{-}+v\right),(1+s) u^{-}+v\right)_{E}-\left(A u^{-}, u^{-}\right)_{E}\right\} \\
& +\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(u_{n}\right)-\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(u_{n}+w_{n}\right) \\
= & \frac{2 s+s^{2}}{2}\left(A u^{+}, u^{+}\right)_{E}+\frac{2 s+s^{2}}{2}\left(A u^{-}, u^{-}\right)_{E}+\frac{1}{2}(A v, v)_{E} \\
& +(1+s)\left(A u^{-}, v\right)_{E}+\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(u_{n}\right)-\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(u_{n}+w_{n}\right) \\
= & \frac{1}{2}(A v, v)_{E}+\sum_{n \in \mathbb{Z}} \chi_{n}\left\{f_{n}\left(u_{n}\right)\left[s\left(\frac{s}{2}+1\right) u_{n}+(1+s) v_{n}\right]\right. \\
& \left.+F_{n}\left(u_{n}\right)-F_{n}\left(u_{n}+w_{n}\right)\right\}<0 . \tag{31}
\end{align*}
$$

The proof is complete.
Lemma 9. (a) There exists $\alpha>0$ such that $c:=\inf _{\mathscr{M}} J(u) \geq$ $\inf _{S_{\alpha}} J(u)>0$, where $S_{\alpha}:=\left\{u \in E^{+}:\|u\|=\alpha\right\}$.
(b) $\left\|u^{+}\right\| \geq \max \left\{\left\|u^{-}\right\|, \sqrt{2 c}\right\}$ for every $u \in \mathscr{M}$.

Proof. (a) By $\left(f_{1}\right)$ and $\left(f_{2}\right)$, it is easy to show that for any $\varepsilon>$ 0 , there exists $c_{\varepsilon}>0$ such that

$$
\begin{equation*}
\left|f_{n}(u)\right| \leq \varepsilon|u|+c_{\varepsilon}|u|^{p-1}, \quad\left|F_{n}(u)\right| \leq \varepsilon|u|^{2}+c_{\varepsilon}|u|^{p} . \tag{32}
\end{equation*}
$$

$\|\cdot\|$ is equivalent to the $E$ norm on $E^{+}$and $E \subset l^{q}$ for $2 \leq q \leq \infty$ with $\|u\|_{l^{9}} \leq\|u\|_{E}$. Hence, for any $\varepsilon \in(0,1 / 2)$ and $u \in E^{+}$, we have

$$
\begin{equation*}
J(u) \geq \frac{1}{2}\|u\|^{2}-\varepsilon\|u\|^{2}-c_{\varepsilon} \bar{\chi}\|u\|^{p} \tag{33}
\end{equation*}
$$

which implies $\inf _{S_{\alpha}} J(u)>0$ for some $\alpha>0$ (small enough), where $\bar{\chi}=\max \left\{\chi_{n}\right\}$.

The first inequality is a consequence of Lemma 8 since for every $u \in \mathscr{M}$, there is $s>0$ such that $s u^{+} \in \widehat{E}(u) \cap S_{\alpha}$.
(b) For $u \in \mathscr{M}$, by (25), we have

$$
\begin{align*}
c & \leq \frac{1}{2}\left\|u^{+}\right\|^{2}-\frac{1}{2}\left\|u^{-}\right\|^{2}-\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(u_{n}\right)  \tag{34}\\
& \leq \frac{1}{2}\left(\left\|u^{+}\right\|^{2}-\left\|u^{-}\right\|^{2}\right) .
\end{align*}
$$

Hence, $\left\|u^{+}\right\| \geq \max \left\{\left\|u^{-}\right\|, \sqrt{2 c}\right\}$.
Lemma 10. Let $\mathscr{W} \subset E^{+} \backslash\{0\}$ be a compact subset. Then, there exists $R>0$ such that $J \leq 0$ on $E(u) \backslash B_{R}(0)$ for every $u \in \mathscr{W}$, where $B_{R}(0)$ denotes the open ball with radius $R$ and center 0 .

Proof. Suppose by contradiction that there exist $u^{(k)} \in \mathscr{W}$ and $w^{(k)} \in E\left(u^{(k)}\right), k \in \mathbb{N}$, such that $J\left(w^{(k)}\right)>0$ for all $k$ and $\left\|w^{(k)}\right\| \rightarrow \infty$ as $k \rightarrow \infty$. Without loss of generality, we may assume that $\left\|u^{(k)}\right\|=1$ for $k \in \mathbb{Z}$. Then, there exists a subsequence, still denoted by the same notation, such that $u^{(k)} \rightarrow u \in E^{+}$. Set $v^{(k)}=w^{(k)} /\left\|w^{(k)}\right\|=s^{(k)} u^{(k)}+v^{(k)-}$. Then,

$$
\begin{align*}
& 0<\frac{J\left(w^{(k)}\right)}{\left\|w^{(k)}\right\|^{2}}=\frac{1}{2}\left(\left(s^{(k)}\right)^{2}-\left\|v^{(k)-}\right\|^{2}\right) \\
&-\sum_{n \in \mathbb{Z}} \chi_{n} \frac{F_{n}\left(w_{n}^{(k)}\right)}{\left(w_{n}^{(k)}\right)^{2}}\left(v_{n}^{(k)}\right)^{2} \tag{35}
\end{align*}
$$

By (25), we have

$$
\begin{equation*}
\left\|v^{(k)-}\right\|^{2} \leq\left(s^{(k)}\right)^{2}=1-\left\|v^{(k)-}\right\|^{2} \tag{36}
\end{equation*}
$$

Consequently, we know that $\left\|v^{(k)-}\right\| \leq 1 / \sqrt{2}$ and $1 / \sqrt{2} \leq$ $s^{(k)} \leq 1$. Passing to a subsequence if necessary, we assume that $s^{(k)} \rightarrow s \in[1 / \sqrt{2}, 1], v^{(k)} \rightharpoonup v, v^{(k)-} \rightharpoonup v_{*}^{-} \in E^{-}$, and $v_{n}^{(k)} \rightarrow v_{n}$ for every $n$. Hence, $v=s u+v_{*}^{-} \neq 0$ and $v_{*}^{-}=v^{-}$. It follows that for $n_{0} \in \mathbb{Z}$ with $v_{n_{0}} \neq 0,\left|w_{n_{0}}^{(k)}\right|=\left\|w^{(k)}\right\| \cdot\left|v_{n_{0}}^{(k)}\right| \rightarrow$ $\infty$, as $k \rightarrow \infty$. Then, by $\left(f_{3}\right)$, we have

$$
\begin{equation*}
\sum_{n \in \mathbb{Z}} \chi_{n} \frac{F_{n}\left(w_{n}^{(k)}\right)}{\left(w_{n}^{(k)}\right)^{2}}\left(v_{n}^{(k)}\right)^{2} \rightarrow \infty \tag{37}
\end{equation*}
$$

which contradicts with (35).
Lemma 11. For each $u \in E^{+} \backslash\{0\}$, the set $\mathscr{M} \cap \widehat{E}(u)$ consists of precisely one point which is the unique global maximum of $\left.J\right|_{\widehat{E}(u)}$.

Proof. By Lemma 8, it suffices to show that $\mathscr{M} \cap \widehat{E}(u) \neq \emptyset$. Since $\widehat{E}(u)=\widehat{E}\left(u^{+} /\left\|u^{+}\right\|\right)$, we may assume that $u \in S^{+}$. By Lemma 10, there exists $R>0$ such that $J \leq 0$ on $E(u) \backslash B_{R}(0)$ provided that $R$ is large enough. By Lemma $9(\mathrm{a}), J(t u)>0$ for small $t>0$. Moreover, $J \leq 0$ on $\widehat{E}(u) \backslash B_{R}(0)$. Hence, $0<\sup _{\widehat{E}(u)} J<\infty$.

Let $v^{(k)} \rightharpoonup v$ in $\widehat{E}(u)$. Then, $v_{n}^{(k)} \rightarrow v_{n}$ as $k \rightarrow \infty$ for all $n$ after passing to a subsequence if necessary. Hence, $F_{n}\left(v_{n}^{(k)}\right) \rightarrow$ $F_{n}\left(v_{n}\right)$. Let $\varphi(v)=\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(v_{n}\right)$. Then,

$$
\begin{align*}
\varphi(v) & =\sum_{n \in \mathbb{Z}^{k} \rightarrow \infty} \lim _{n} \chi_{n}\left(v_{n}^{(k)}\right) \\
& \leq \liminf _{k \rightarrow \infty} \sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(v_{n}^{(k)}\right)  \tag{38}\\
& =\liminf _{k \rightarrow \infty} \varphi\left(v^{(k)}\right)
\end{align*}
$$

that is, $\varphi$ is a weakly lower semicontinuous. From the weak lower semi-continuity of the norm, it is easy to see that $J$ is weakly upper semicontinuous on $\widehat{E}(u)$. Therefore, $J\left(u_{0}\right)=$ $\sup _{\widehat{E}(u)} J$ for some $u_{0} \in \widehat{E}(u) \backslash\{0\}$. By the proof of Lemma 10, $u_{0}$ is a critical point of $\left.J\right|_{\widehat{\mathrm{E}}(u)}$. It follows that $\left(J^{\prime}\left(u_{0}\right), u_{0}\right)=$ $\left(J^{\prime}\left(u_{0}\right), z\right)=0$ for all $z \in E$ and hence $u_{0} \in \mathscr{M}$. To summarize, $u_{0} \in \mathscr{M} \cap \widehat{E}(u)$.

According to Lemma 11 , for each $u \in E^{+} \backslash\{0\}$, we may define the mapping $\widehat{m}: E^{+} \backslash\{0\} \rightarrow \mathcal{M}, u \mapsto \widehat{m}(u)$, where $\widehat{m}(u)$ is the unique point of $\mathscr{M} \cap \widehat{E}(u)$.

Lemma 12. J is coercive on $\mathscr{M}$; that is, $J(u) \rightarrow \infty$ as $\|u\| \rightarrow$ $\infty, u \in \mathscr{M}$.

Proof. Suppose, by contradiction, that there exists a sequence $\left\{u^{(k)}\right\} \subset \mathscr{M}$ such that $\left\|u^{(k)}\right\| \rightarrow \infty$ and $J\left(u^{(k)}\right) \leq d$ for some $d \in[c, \infty)$. Let $v^{(k)}=u^{(k)} /\left\|u^{(k)}\right\|$. Then, there exists a subsequence, still denoted by the same notation, such that $v^{(k)} \rightharpoonup v$ and $v_{n}^{(k)} \rightarrow v_{n}$ for every $n$ as $k \rightarrow \infty$.

First, we know that there exist $\delta>0$ and $n_{k} \in \mathbb{Z}$ such that

$$
\begin{equation*}
\left|v_{n_{k}}^{(k)+}\right| \geq \delta . \tag{39}
\end{equation*}
$$

Indeed, if not, then $v^{(k)+} \rightarrow 0$ in $l^{\infty}$ as $k \rightarrow \infty$. By Lemma 9 (b), $1 / 2 \leq\left\|v^{(k)+}\right\|^{2} \leq 1$, which means that $\left\|v^{(k)+}\right\|_{l^{2}}$ is bounded. For $q>2$,

$$
\begin{equation*}
\left\|v^{(k)+}\right\|_{l^{9}}^{q} \leq\left\|v^{(k)+}\right\|_{l^{\infty}}^{q-2}\left\|v^{(k)+}\right\|_{l^{2}}^{2} . \tag{40}
\end{equation*}
$$

Then, $v^{(k)+} \rightarrow 0$ in all $l^{q}, q>2$. By (32), for any $s \in \mathbb{R}$,

$$
\begin{equation*}
\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(s v_{n}^{(k)+}\right) \leq \varepsilon s^{2} \bar{\chi}\left\|v^{(k)+}\right\|_{l^{2}}^{2}+c_{\varepsilon} s^{p} \bar{\chi}\left\|v^{(k)+}\right\|_{l^{p}}^{q} \tag{41}
\end{equation*}
$$

which implies that $\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(s v_{n}^{(k)+}\right) \rightarrow 0$ as $k \rightarrow \infty$.
Since $s v^{(k)+} \in \widehat{E}\left(u^{(k)}\right)$ for $s \geq 0$, Lemma 8 implies that

$$
\begin{align*}
d \geq J\left(u^{(k)}\right) & \geq J\left(s v^{(k)+}\right) \\
& =\frac{s^{2}}{2}\left\|v^{(k)+}\right\|^{2}-\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(s v_{n}^{(k)+}\right)  \tag{42}\\
& \geq \frac{s^{2}}{4}-\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(s v_{n}^{(k)+}\right) \longrightarrow \frac{s^{2}}{4},
\end{align*}
$$

as $k \rightarrow \infty$. This is a contradiction if $s>\sqrt{4 d}$.

Due to the periodicity of coefficients, both $J$ and $\mathscr{M}$ are invariant under $T$-translation. Making such shifts, we can assume that $1 \leq n_{k} \leq T-1$ in (39). Moreover, passing to a subsequence if needed, we can assume that $n_{k}=n_{0}$ is independent of $k$. Next, we may extract a subsequence, still denoted by $\left\{v^{(k)}\right\}$, such that $v_{n}^{(k)+} \rightarrow v_{n}^{+}$for all $n \in \mathbb{Z}$. In particular, for $n=n_{0}$, inequality (39) shows that $\left|v_{n_{0}}^{+}\right| \geq \delta$ and hence $v^{+} \neq 0$.

Since $\left|u_{n}^{(k)}\right| \rightarrow \infty$ as $k \rightarrow \infty$, it follows again from $\left(f_{3}\right)$ and Fatou's lemma that

$$
\begin{align*}
0 \leq \frac{J\left(u^{(k)}\right)}{\left\|u^{(k)}\right\|^{2}}= & \frac{1}{2}\left(\left\|v^{(k)+}\right\|^{2}-\left\|v^{(k)-}\right\|^{2}\right) \\
& -\sum_{n \in \mathbb{Z}} \chi_{n} \frac{F_{n}\left(u_{n}^{(k)}\right)}{\left(u_{n}^{(k)}\right)^{2}}  \tag{43}\\
& \times\left(v_{n}^{(k)}\right)^{V} \longrightarrow-\infty \quad \text { as } k \longrightarrow \infty
\end{align*}
$$

a contradiction again. The proof is finished.
Lemma 13. (a) The mapping $\widehat{m}: E^{+} \backslash\{0\} \rightarrow \mathscr{M}$ is continuous.
(b) The mapping $m=\left.\widehat{m}\right|_{S^{+}}: S^{+} \rightarrow \mathcal{M}$ is a homeomorphism between $S^{+}$and $\mathscr{M}$, and the inverse of $m$ is given by $m^{-1}(u)=u^{+} /\left\|u^{+}\right\|$, where $S^{+}:=\left\{u \in E^{+}:\|u\|=1\right\}$.
(c) The mapping $\mathrm{m}^{-1}: \mathscr{M} \mapsto S^{+}$is the Lipschitz continuous.

Proof. (a) Let $\left(u^{(k)}\right) \subset E^{+} \backslash\{0\}$ be a sequence with $u^{(k)} \rightarrow$ $u$. Since $\widehat{m}(w)=\widehat{m}\left(w^{+} /\left\|w^{+}\right\|\right)$, without loss of generality, we may assume that $\left\|u^{(k)}\right\|=1$ for all $k$. Then, $\widehat{m}\left(u^{(k)}\right)=$ $\left\|\widehat{m}\left(u^{(k)}\right)^{+}\right\| u^{(k)}+\widehat{m}\left(u^{(k)}\right)^{-}$. By Lemma 10, there exists $R>0$ such that

$$
\begin{align*}
J\left(\widehat{m}\left(u^{(k)}\right)\right) & =\sup _{E\left(u^{(k)}\right)} J \leq \sup _{B_{R}(0)} J \\
& \leq \sup _{u \in B_{R}(0)}\left\|u^{+}\right\|^{2}=R^{2} \quad \text { for every } k . \tag{44}
\end{align*}
$$

It follows from Lemma 12 that $\widehat{m}\left(u^{(k)}\right)$ is bounded. Passing to a subsequence if needed, we may assume that

$$
\begin{gather*}
t^{(k)}:=\left\|\widehat{m}\left(u^{(k)}\right)^{+}\right\| \longrightarrow t  \tag{45}\\
\widehat{m}\left(u^{(k)}\right)^{-} \rightharpoonup u_{*}^{-} \quad \text { in } E \text { as } k \rightarrow \infty,
\end{gather*}
$$

where $t \geq \sqrt{2 c}>0$ by Lemma 9(b). Moreover, by Lemma 11,

$$
\begin{aligned}
J\left(\widehat{m}\left(u^{(k)}\right)\right) & \geq J\left(t^{(k)} u^{(k)}+\widehat{m}(u)^{-}\right) \longrightarrow J\left(t u+\widehat{m}(u)^{-}\right) \\
& =J(\widehat{m}(u))
\end{aligned}
$$

Therefore, using the weak lower semicontinuity of the norm and $\varphi$ (defined in Lemma 11), we get

$$
\begin{align*}
J(\widehat{m}(u)) \leq & \lim _{k \rightarrow \infty} J\left(\widehat{m}\left(u^{(k)}\right)\right) \\
= & \lim _{k \rightarrow \infty}\left(\frac{1}{2}\left(t^{(k)}\right)^{2}-\frac{1}{2}\left\|\widehat{m}\left(u^{(k)}\right)^{-}\right\|^{2}\right. \\
& \left.\quad-\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(\widehat{m}\left(u_{n}^{(k)}\right)\right)\right)  \tag{47}\\
\leq & \frac{1}{2} t^{2}-\frac{1}{2}\left\|u_{*}^{-}\right\|^{2}-\sum_{n \in \mathbb{Z}} \chi_{n} F_{n}\left(t u_{n}+u_{*, n}^{-}\right) \\
= & J\left(t u+u_{*}^{-}\right) \leq J(\widehat{m}(u)),
\end{align*}
$$

which implies that all inequalities above must be equalities and $\widehat{m}\left(u^{(k)}\right)^{-} \rightarrow u_{*}^{-}$. By Lemma 11, $u_{*}^{-}=\widehat{m}(u)^{-}$and hence $\widehat{m}\left(u^{(k)}\right) \rightarrow \widehat{m}(u)$.
(b) This is an immediate consequence of (a).
(c) For $u, v \in \mathscr{M}$, by (b), we have

$$
\begin{align*}
\left\|m^{-1}(u)-m^{-1}(v)\right\| & =\left\|\frac{u^{+}}{\left\|u^{+}\right\|}-\frac{v^{+}}{\left\|v^{+}\right\|}\right\| \\
& =\left\|\frac{u^{+}-v^{+}}{\left\|u^{+}\right\|}+\frac{\left(\left\|v^{+}\right\|-\left\|u^{+}\right\|\right) v^{+}}{\left\|u^{+}\right\|\left\|v^{+}\right\|}\right\|  \tag{48}\\
& \leq \frac{2}{\left\|u^{+}\right\|}\left\|(u-v)^{+}\right\| \leq \sqrt{\frac{2}{c}}\|u-v\| .
\end{align*}
$$

We will consider the functional $\widehat{\Psi}: E^{+} \backslash\{0\} \rightarrow \mathbb{R}$ and $\Psi: S^{+} \rightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
\widehat{\Psi}:=J(\widehat{m}(w)), \quad \Psi:=\left.\widehat{\Psi}\right|_{S^{+}} \tag{49}
\end{equation*}
$$

Lemma 14. (a) $\widehat{\Psi} \in C^{1}\left(E^{+} \backslash\{0\}, \mathbb{R}\right)$ and

$$
\begin{equation*}
\widehat{\Psi}^{\prime}(w) z=\frac{\left\|\widehat{m}(w)^{+}\right\|}{\|w\|} J^{\prime}(\widehat{m}(w)) z \quad \forall w, z \in E^{+}, w \neq 0 \tag{50}
\end{equation*}
$$

(b) $\Psi \in C^{1}\left(S^{+}, \mathbb{R}\right)$ and

$$
\begin{align*}
\Psi^{\prime}(w) z & =\left\|\widehat{m}(w)^{+}\right\| J^{\prime}(m(w)) z \quad \forall z \in T_{w} S^{+} \\
& =\left\{v \in E^{+}:(w, v)=0\right\} . \tag{51}
\end{align*}
$$

(c) $\left\{w_{n}\right\}$ is a Palais-Smale sequence for $\Psi$ if and only if $\left\{m\left(w_{n}\right)\right\}$ is a Palais-Smale sequence for $J$.
(d) $w \in S^{+}$is a critical point of $\Psi$ if and only ifm $(w) \in \mathscr{M}$ is a nontrivial critical point of J. Moreover, the corresponding values of $\Psi$ and $J$ coincide and $\inf _{S^{+}} \Psi=\inf _{\mu} J=c$.

Proof. (a) We put $u=\widehat{m}(w) \in \mathscr{M}$, so we have $u=$ $\left(\left\|u^{+}\right\| /\|w\|\right) w+u^{-}$. Let $z \in E^{+}$. Choose $\delta>0$ such that $w_{t}:=$ $w+t z \in E^{+} \backslash\{0\}$ for $|t|<\delta$ and put $u_{t}=\widehat{m}\left(w_{t}\right) \in \mathscr{M}$. We may write $u_{t}=s_{t} w_{t}+u_{t}^{-}$with $s_{t}>0$. From the proof of Lemma 13,
the function $t \mapsto s_{t}$ is continuous. Then, $s_{0}=\left\|u^{+}\right\| /\|w\|$. By Lemma 11 and the mean value theorem, we have

$$
\begin{align*}
\widehat{\Psi}\left(w_{t}\right)-\widehat{\Psi}(w) & =J\left(u_{t}\right)-J(u) \\
& =J\left(s_{t} w_{t}+u_{t}^{-}\right)-J\left(s_{0} w+u^{-}\right)  \tag{52}\\
& \leq J\left(s_{t} w_{t}+u_{t}^{-}\right)-J\left(s_{t} w+u_{t}^{-}\right) \\
& =J^{\prime}\left(s_{t}\left[w+\eta_{t}\left(w_{t}-w\right)\right]+u_{t}^{-}\right) s_{t} t z
\end{align*}
$$

with some $\eta_{t} \in(0,1)$. Similarly,

$$
\begin{align*}
\widehat{\Psi}\left(w_{t}\right)-\widehat{\Psi}(w) & =J\left(s_{t} w_{t}+u_{t}^{-}\right)-J\left(s_{0} w+u^{-}\right) \\
& \geq J\left(s_{0} w_{t}+u^{-}\right)-J\left(s_{0} w+u^{-}\right)  \tag{53}\\
& =J^{\prime}\left(s_{0}\left[w+\tau_{t}\left(w_{t}-w\right)\right]+u^{-}\right) s_{0} t z
\end{align*}
$$

with some $\tau_{t} \in(0,1)$. Combining these inequalities and the continuity of function $t \mapsto s_{t}$, we have

$$
\begin{equation*}
\lim _{t \rightarrow 0} \frac{\widehat{\Psi}\left(w_{t}\right)-\widehat{\Psi}(w)}{t}=s_{0} J^{\prime}(u) z=\frac{\left\|\widehat{m}(w)^{+}\right\|}{\|w\|} J^{\prime}(\widehat{m}(w)) z \tag{54}
\end{equation*}
$$

Hence, the Gâteaux derivative of $\widehat{\Psi}$ is bounded linear in $z$ and continuous in $w$. It follows that $\widehat{\Psi}$ is of class $C^{1}$ (see [15]).
(b) It follows from (a) by noting that $m(w)=\widehat{m}(w)$ since $w \in S^{+}$.
(c) Let $\left\{w_{n}\right\}$ be a Palais-Smale sequence for $\Psi$, and let $u_{n}=$ $m\left(w_{n}\right) \in \mathscr{M}$. Since for every $n \in \mathbb{Z}$, we have an orthogonal splitting $E=T_{w_{n}} S^{+} \oplus E\left(w_{n}\right)$; using (b), we have

$$
\begin{align*}
\left\|\Psi^{\prime}\left(w_{n}\right)\right\| & =\sup _{\substack{z \in T_{w_{n}} S^{+} \\
\|z\|=1}} \Psi^{\prime}\left(w_{n}\right) z \\
& =\left\|m\left(w_{n}\right)^{+}\right\| \sup _{\substack{z \in T_{w_{n}} S^{+} \\
\|z\|=1}} J^{\prime}\left(m\left(w_{n}\right)\right) z  \tag{55}\\
& =\left\|u_{n}^{+}\right\| \sup _{\substack{z \in T_{w_{0}} S^{+} \\
\|z\|=1}} J^{\prime}\left(u_{n}\right) z
\end{align*}
$$

because $J^{\prime}\left(u_{n}\right) v=0$ for all $v \in E\left(w_{n}\right)$ and $E\left(w_{n}\right)$ is orthogonal to $T_{w_{n}} S^{+}$. Using (b) again, we have

$$
\begin{align*}
\left\|\Psi^{\prime}\left(w_{n}\right)\right\| & \leq\left\|u_{n}^{+}\right\|\left\|J^{\prime}\left(u_{n}\right)\right\| \\
& =\left\|u_{n}^{+}\right\| \sup _{\substack{z \in T_{w_{n}} S^{+}, v \in E\left(w_{n}\right) \\
z+v \neq 0}} \frac{J^{\prime}\left(u_{n}\right)(z+v)}{\|z+v\|}  \tag{56}\\
& \leq\left\|u_{n}^{+}\right\| \sup _{z \in T_{w_{n}} S^{+} \backslash\{0\}} \frac{J^{\prime}\left(u_{n}\right)(z)}{\|z\|}=\left\|\Psi^{\prime}\left(w_{n}\right)\right\| .
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\left\|\Psi^{\prime}\left(w_{n}\right)\right\|=\left\|u_{n}^{+}\right\|\left\|J^{\prime}\left(u_{n}\right)\right\| . \tag{57}
\end{equation*}
$$

According to Lemma 9(b) and Lemma 12, $\sqrt{2 c} \leq\left\|u_{n}^{+}\right\| \leq$ $\sup _{n}\left\|u_{n}^{+}\right\|<\infty$. Hence, $\left\{w_{n}\right\}$ is a Palais-Smale sequence for $\Psi$ if and only if $\left\{u_{n}\right\}$ is a Palais-Smale sequence for $J$.
(d) By (57), $\Psi^{\prime}(w)=0$ if and only if $J^{\prime}(m(w))=0$. The other part is clear.

Proof of Theorem 1. (1) We know that $c>0$ by Lemma 9(a). If $u_{0} \in \mathscr{M}$ satisfies $J\left(u_{0}\right)=c$, then $m^{-1}\left(u_{0}\right) \in S^{+}$is a minimizer of $\Psi$ and therefore a critical point of $\Psi$ and also a critical point of $J$ by Lemma 14 . We shall show that there exists a minimizer $u \in \mathscr{M}$ of $\left.J\right|_{\mathscr{M}}$. Let $\left\{w^{(k)}\right\} \subset S^{+}$be a minimizing sequence for $\Psi$. By Ekeland's variational principle, we may assume that $\Psi\left(w^{(k)}\right) \rightarrow c$ and $\Psi^{\prime}\left(w^{(k)}\right) \rightarrow 0$ as $k \rightarrow \infty$. Then, $J\left(u^{(k)}\right) \rightarrow$ $c$ and $J^{\prime}\left(u^{(k)}\right) \rightarrow 0$ as $k \rightarrow \infty$ by Lemma 14(c), where $u^{(k)}:=$ $m\left(w^{(k)}\right) \in \mathscr{M}$. By Lemma $12,\left\{u^{(k)}\right\}$ is bounded, and hence $\left\{u^{(k)}\right\}$ has a weakly convergent subsequence.

First, we show that there exist $\delta>0$ and $n_{k} \in \mathbb{Z}$ such that

$$
\begin{equation*}
\left|u_{n_{k}}^{(k)}\right| \geq \delta \tag{58}
\end{equation*}
$$

Indeed, if not, then $u^{(k)} \rightarrow 0$ in $l^{\infty}$ as $k \rightarrow \infty$. From the simple fact that for $q>2$,

$$
\begin{equation*}
\left\|u^{(k)}\right\|_{l^{9}}^{q} \leq\left\|u^{(k)}\right\|_{l^{\infty}}^{q-2}\left\|u^{(k)}\right\|_{l^{2}}^{2}, \tag{59}
\end{equation*}
$$

we have $u^{(k)} \rightarrow 0$ in all $l^{q}, q>2$. By (32), we know that

$$
\begin{align*}
\sum_{n \in \mathbb{Z}} \chi_{n} f_{n}\left(u_{n}^{(k)}\right) u_{n}^{(k)+} \leq & \varepsilon \bar{\chi} \sum_{n \in \mathbb{Z}}\left|u_{n}^{(k)}\right| \cdot\left|u_{n}^{(k)+}\right| \\
& +c_{\varepsilon} \bar{\chi} \sum_{n \in \mathbb{Z}}\left|u_{n}^{(k)}\right|^{p-1} \cdot\left|u_{n}^{(k)+}\right| \\
\leq & \varepsilon \bar{\chi}\left\|u^{(k)}\right\|_{l^{2}} \cdot\left\|u^{(k)+}\right\|_{l^{2}}  \tag{60}\\
& +c_{\varepsilon} \bar{\chi}\left\|u^{(k)}\right\|_{l^{p}}^{p-1} \cdot\left\|u^{(k)+}\right\|_{l^{p}} \\
\leq & \varepsilon \bar{\chi}\left\|u^{(k)}\right\|_{l^{2}} \cdot\left\|u^{(k)+}\right\| \\
& +c_{\varepsilon} \bar{\chi}\left\|^{(k)}\right\|_{l^{p}}^{p-1} \cdot\left\|u^{(k)+}\right\|
\end{align*}
$$

which implies that $\sum_{n \in \mathbb{Z}} \chi_{n} f_{n}\left(u_{n}^{(k)}\right) u_{n}^{(k)+}=o\left(\left\|u^{(k)+}\right\|\right)$ as $k \rightarrow$ $\infty$. Therefore,

$$
\begin{align*}
o\left(\left\|u^{(k)+}\right\|\right) & =\left(J^{\prime}\left(u^{(k)}\right), u^{(k)+}\right) \\
& =\left\|u^{(k)+}\right\|^{2}-\sum_{n \in \mathbb{Z}} \chi_{n} f_{n}\left(u_{n}^{(k)}\right) u_{n}^{(k)+}  \tag{61}\\
& =\left\|u^{(k)+}\right\|^{2}-o\left(\left\|u^{(k)+}\right\|\right) .
\end{align*}
$$

Then, $\left\|u^{(k)+}\right\|^{2} \rightarrow 0$ as $k \rightarrow \infty$, contrary to Lemma 9(b).
From the periodicity of the coefficients, we know that $J$ and $J^{\prime}$ are both invariant under $T$-translation. Making such shifts, we can assume that $1 \leq n_{k} \leq T-1$ in (58). Moreover, passing to a subsequence, we can assume that $n_{k}=n_{0}$ is independent of $k$.

Next, we may extract a subsequence, still denoted by $\left\{u^{(k)}\right\}$, such that $u^{(k)} \rightharpoonup u$ and $u_{n}^{(k)} \rightarrow u_{n}$ for all $n \in \mathbb{Z}$. Particularly, for $n=n_{0}$, inequality (58) shows that $\left|u_{n_{0}}\right| \geq \delta$, so $u \neq 0$. Moreover, we have

$$
\begin{equation*}
\left(J^{\prime}(u), v\right)=\lim _{k \rightarrow \infty}\left(J^{\prime}\left(u^{(k)}\right), v\right)=0, \quad \forall v \in E ; \tag{62}
\end{equation*}
$$

that is, $u$ is a nontrivial critical point of $J$.
Finally, we show that $J(u)=c$. By Lemma 6 and Fatou's lemma, we have

$$
\begin{align*}
c & =\lim _{k \rightarrow \infty}\left(J\left(u^{(k)}\right)-\frac{1}{2} J^{\prime}\left(u^{(k)}\right) u^{(k)}\right) \\
& =\lim _{k \rightarrow \infty} \sum_{n \in \mathbb{Z}} \chi_{n}\left(\frac{1}{2} f_{n}\left(u_{n}^{(k)}\right) u_{n}^{(k)}-F_{n}\left(u_{n}^{(k)}\right)\right)  \tag{63}\\
& \geq \sum_{n \in \mathbb{Z}} \chi_{n}\left(\frac{1}{2} f_{n}\left(u_{n}\right) u_{n}-F_{n}\left(u_{n}\right)\right) \\
& =J(u)-\frac{1}{2} J^{\prime}(u) u=J(u) \geq c .
\end{align*}
$$

Hence, $J(u)=c$. That is, $u$ is a nontrivial ground state solution of (7).
(2) If $\beta=\infty$, by way of contradiction, we assume that (7) has a nontrivial solution $u \in E$. Then, $u$ is a nonzero critical point of $J$ in $E$. Thus, $J^{\prime}(u)=0$. But by Lemma 6,

$$
\begin{equation*}
\left(J^{\prime}(u), u\right)=((L-\omega) u, u)-\sum_{n \in \mathbb{Z}} \chi_{n} f_{n}\left(u_{n}\right) u_{n}<0 \tag{64}
\end{equation*}
$$

This is a contradiction, so the conclusion holds.
This completes the proof of Theorem 1.
Now, we are ready to prove Theorem 2. From now on, we always assume that $f_{n}$ is odd in $u$. We need some notations. For $a \geq b \geq c$, denote

$$
\begin{gather*}
J^{a}=\{u \in \mathscr{M}: J(u) \leq a\} \\
J_{b}:=\{u \in \mathscr{M}: J(u) \geq b\} \\
J_{b}^{a}=J^{a} \cap J_{b} \\
\Psi^{a}=\left\{w \in S^{+}: \Psi(w) \leq a\right\} \\
\Psi_{b}:=\left\{w \in S^{+}: \Psi(w) \geq b\right\},  \tag{65}\\
\\
\Psi_{b}^{a}=\Psi^{a} \cap \Psi_{b} \\
K=\left\{w \in S^{+}: \Psi^{\prime}(w)=0\right\} \\
K_{a}=\{w \in K: \Psi(w)=a\}, \\
\nu(a)=\sup \left\{\|u\|: u \in J^{a}\right\}
\end{gather*}
$$

It is easy to see that $\nu(a)<\infty$ for every $a$ by Lemma 12.
Proof of Theorem 2. It is easy to see that mappings $m, m^{-1}$ are equivariant with respect to the $\mathbb{Z}$-action by Lemma 13; hence, the orbits $\mathcal{O}(u) \subset \mathscr{M}$ consisting of critical points of $J$ are in 1-1 correspondence with the orbits $\mathcal{O}(w) \subset S^{+}$consisting of
critical points of $\Psi$ by Lemma 14(d). Next, we may choose a subset $\mathscr{F} \subset K$ such that $\mathscr{F}=-\mathscr{F}$ and $\mathscr{F}$ consists of a unique representative of $\mathbb{Z}$-orbits. So, we only need to prove that the set $\mathscr{F}$ is infinite. By contradiction, we assume that

$$
\begin{equation*}
\mathscr{F} \text { is a finite set. } \tag{66}
\end{equation*}
$$

Let

$$
\begin{equation*}
\Gamma_{j}=\left\{A \subset S^{+}: A=-A, A \text { is closed and } \gamma(A) \geq j\right\} \tag{67}
\end{equation*}
$$

where $\gamma$ denotes genus and $j \in \mathbb{N}$. We consider the sequence of the Lusternik-Schnirelmann values of $\Psi$ defined by

$$
\begin{equation*}
c_{k}=\inf \left\{d \in \mathbb{R}: \gamma\left(\Psi^{d}\right) \geq k, k \in \mathbb{N}\right\} \tag{68}
\end{equation*}
$$

Now, we claim that

$$
\begin{equation*}
K_{c_{k}} \neq \emptyset, \quad c_{k}<c_{k+1} . \tag{69}
\end{equation*}
$$

Firstly, we show that

$$
\begin{equation*}
\mathcal{\kappa}=\inf \{\|v-w\|: v, w \in K, v \neq w\}>0 . \tag{70}
\end{equation*}
$$

In fact, there exist $v^{(k)}, w^{(k)} \in \mathscr{F}$, and $g_{k}, l_{k} \in \mathbb{Z}$ such that $v^{(k)} * g_{k} \neq w^{(k)} * l_{k}$ for all $k$ and

$$
\begin{equation*}
\left\|v^{(k)} * g_{k}-w^{(k)} * l_{k}\right\| \longrightarrow \kappa \quad \text { as } k \longrightarrow \infty \tag{71}
\end{equation*}
$$

Let $m_{k}=g_{k}-l_{k}$. Passing to a subsequence, $v^{(k)}=v \in \mathscr{F}$, $w^{(k)}=w \in \mathscr{F}$, and either $m_{k}=m \in \mathbb{Z}$ for all $k$ or $\left|m_{k}\right| \rightarrow \infty$. In the first case, $0<\left\|v^{(k)} * g_{k}-w^{(k)} * l_{k}\right\|=\|v-w * m\|=\kappa$ for all $k$. In the second case, $w * m_{k} \rightharpoonup 0$ and therefore $\kappa=$ $\lim _{k \rightarrow \infty}\left\|v-w * m_{k}\right\| \geq\|v\|=1$. By $(70), \gamma\left(K_{c_{k}}\right)=0$ or 1 .

Next, we consider a pseudogradient vector field of $\Psi$ [18]; that is, there exists a Lipschitz continuous map $V: S^{+} \backslash K \rightarrow$ $T_{w} S^{+}$and for all $w \in S^{+} \backslash K$,

$$
\begin{gather*}
\|V(w)\|<2\left\|\Psi^{\prime}(w)\right\| \\
\left\langle V(w), \Psi^{\prime}(w)\right\rangle>\frac{1}{2}\left\|\Psi^{\prime}(w)\right\|^{2} \tag{72}
\end{gather*}
$$

Let $\eta: \mathscr{D} \rightarrow S^{+} \backslash K$ be the corresponding $\Psi$-decreasing flow defined by

$$
\begin{gather*}
\frac{d}{d t} \eta(t, w)=-V(\eta(t, w))  \tag{73}\\
\eta(0, w)=w
\end{gather*}
$$

where $\mathscr{D}=\left\{(t, w): w \in S^{+} \backslash K, T^{-}(w)<t<T^{+}(w)\right\} \subset \mathbb{R} \times$ $\left(S^{+} \backslash K\right)$, and $T^{-}(w)<0, T^{+}(w)>0$ are the maximal existence times of the trajectory $t \rightarrow \eta(t, w)$ in negative and positive direction. By the continuity property of the genus, there exists $\delta>0$ such that $\gamma(\bar{U})=\gamma\left(K_{c_{k}}\right)$, where $U=N_{\delta}\left(K_{c_{k}}\right):=\{w \in$ $\left.S^{+}: \operatorname{dist}\left(w, K_{c_{k}}\right)<\delta\right\}$ and $\delta<\kappa / 2$. Following the deformation $\operatorname{argument}$ (Lemma A.3), we choose $\varepsilon=\varepsilon(\delta)>0$ such that

$$
\begin{equation*}
\lim _{t \rightarrow T^{+}(w)} \Psi(\eta(t, w))<c_{k}-\varepsilon \quad \text { for } w \in \Psi^{c_{k}+\varepsilon} \backslash U \tag{74}
\end{equation*}
$$

Then, for every $w \in \Psi^{\mathcal{c}_{k}+\varepsilon} \backslash U$, there exists $t \in\left[0, T^{+}(w)\right)$ such that $\Psi(\eta(t, w))<c_{k}-\varepsilon$. Hence, we may define the entrance time map

$$
\begin{gather*}
r: w \in \Psi^{c_{k}+\varepsilon} \backslash U \longrightarrow[0, \infty) \\
r(w)=\inf \left\{t \in\left[0, T^{+}(w)\right): \Psi(\eta(t, w)) \leq c_{k}-\varepsilon\right\}, \tag{75}
\end{gather*}
$$

which satisfies $r(w)<T^{+}(w)$ for every $w \in \Psi^{c_{k}+\varepsilon} \backslash U$. Since $c_{k}-\varepsilon$ is not a critical value of $\Psi$ by (74), it is easy to see that $r$ is a continuous and even map. It follows that the map

$$
\begin{equation*}
g: \Psi^{c_{k}+\varepsilon} \backslash U \longrightarrow \Psi^{c_{k}-\varepsilon}, \quad g(w)=\eta(r(w), w) \tag{76}
\end{equation*}
$$

is odd and continuous. Then, $\gamma\left(\Psi^{c_{k}+\varepsilon} \backslash U\right) \leq \gamma\left(\Psi^{c_{k}-\varepsilon}\right) \leq k-1$, and consequently,

$$
\begin{equation*}
\gamma\left(\Psi^{c_{k}+\varepsilon}\right) \leq \gamma(\bar{U})+k-1=\gamma\left(K_{c_{k}}\right)+k-1 . \tag{77}
\end{equation*}
$$

So, $\gamma\left(K_{c_{k}}\right) \geq 1$. Therefore, $K_{c_{k}} \neq \emptyset$. Moreover, the definition of $c_{k}$ and of $c_{k+1}$ implies that $\gamma\left(K_{c_{k}}\right) \geq 1$ if $c_{k}<c_{k+1}$ and $\gamma\left(K_{c_{k}}\right)>1$ if $c_{k}=c_{k+1}$. Since $\gamma(\mathscr{F})=\gamma\left(K_{c_{k}}\right) \leq 1, c_{k}<$ $c_{k+1}$. Therefore, there is an infinite sequence $\left\{ \pm w_{k}\right\}$ of pairs of geometrically distinct critical points of $\Psi$ with $\Psi\left(w_{k}\right)=c_{k}$, which contradicts with (66). Therefore, the set $\mathscr{F}$ is infinite.

This completes the proof of Theorem 2.

## Appendix

Here, we give a proof of (74). We state the discrete property of the Palais-Smale sequences. It yields nice properties of the corresponding pseudogradient flow.

Lemma A.1. Let $d \geq c$. If $\left\{w_{1}^{(k)}\right\},\left\{w_{2}^{(k)}\right\} \subset \Psi^{d}$ are two PalaisSmale sequences for $\Psi$, then either $\left\|w_{1}^{(k)}-w_{2}^{(k)}\right\| \rightarrow 0$ as $k \rightarrow \infty$ or $\lim \sup _{k \rightarrow \infty}\left\|w_{1}^{(k)}-w_{2}^{(k)}\right\| \geq \varrho(d)>0$, where $\varrho(d)$ depends on $d$ but not on the particular choice of the PalaisSmale sequences.

Proof. Set $u_{1}^{(k)}=m\left(w_{1}^{(k)}\right)$ and $u_{2}^{(k)}=m\left(w_{2}^{(k)}\right)$. Then, $\left\{u_{1}^{(k)}\right\}$, $\left\{u_{2}^{(k)}\right\} \subset J^{d}$ are the bounded Palais-Smale sequences for $J$. We fix $p$ in $\left(f_{2}\right)$ and consider the following two cases.
(i) $\left\|u_{1}^{(k)}-u_{2}^{(k)}\right\|_{l^{p}} \rightarrow 0$ as $k \rightarrow \infty$.

By a straightforward calculation and (32), for any $\varepsilon>0$, there exist $C_{1}, C_{2}>0$, and $k_{0}$ such that for all $k \geq k_{0}$,

$$
\begin{aligned}
& \left\|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}\right\|^{2} \\
& =J^{\prime}\left(u_{1}^{(k)}\right)\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}-J^{\prime}\left(u_{2}^{(k)}\right)\left(u_{2}^{(k)}-u_{2}^{(k)}\right)^{+} \\
& \quad+\sum_{n \in \mathbb{Z}} x_{n}\left[f_{n}\left(u_{1 n}^{(k)}\right)-f_{n}\left(u_{2 n}^{(k)}\right)\right]\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+} \\
& \quad \leq \varepsilon\left\|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}\right\|^{2}
\end{aligned}
$$

$$
\begin{align*}
& \quad+\bar{\chi} \sum_{n \in \mathbb{Z}}\left[\varepsilon\left(\left|u_{1 n}^{(k)}\right|+\left|u_{2 n}^{(k)}\right|\right)\right. \\
& \left.\quad+c_{\varepsilon}\left(\left|u_{1 n}^{(k)}\right|^{p-1}+\left|u_{2 n}^{(k)}\right|^{p-1}\right)\right] \\
& \quad \times\left|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}\right| \\
& \leq \\
& \varepsilon\left\|\left\|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}\right\|\right. \\
& \\
& +\bar{\chi} \varepsilon\left(\left\|u_{1}^{(k)}\right\|+\left\|u_{2}^{(k)}\right\|\right)\left\|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}\right\| \\
&  \tag{A.1}\\
& +\bar{\chi} c_{\varepsilon}\left(\left\|u_{1}^{(k)}\right\|_{l^{p}}^{p-1}+\left\|u_{2}^{(k)}\right\|_{l^{p}}^{p-1}\right)\left\|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}\right\|_{l^{p}} \\
& \leq \\
& \varepsilon
\end{align*}
$$

This implies $\lim \sup _{k \rightarrow \infty}\left\|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}\right\|^{2} \leq \lim \sup _{k \rightarrow \infty}(1+$ $\left.\bar{\chi} C_{1}\right) \varepsilon\left\|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}\right\|$. Hence, $\left\|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{+}\right\| \rightarrow 0$. Similarly, $\left\|\left(u_{1}^{(k)}-u_{2}^{(k)}\right)^{-}\right\| \rightarrow 0$. Therefore, $\left\|u_{1}^{(k)}-u_{2}^{(k)}\right\| \rightarrow 0$ as $k \rightarrow$ $\infty$. By Lemma 13(c), we have $\left\|w_{1}^{(k)}-w_{2}^{(k)}\right\|=\| m^{-1}\left(u_{1}^{(k)}\right)-$ $m^{-1}\left(u_{2}^{(k)}\right) \| \rightarrow 0$ as $k \rightarrow \infty$.
(ii) $\left\|u_{1}^{(k)}-u_{2}^{(k)}\right\|_{l^{p}} \rightarrow 0$ as $k \rightarrow \infty$.

There exist $\delta>0$ and $n_{k} \in \mathbb{Z}$ such that

$$
\begin{equation*}
\left|u_{1 n_{k}}^{(k)}-u_{2 n_{k}}^{(k)}\right| \geq \delta . \tag{A.2}
\end{equation*}
$$

For bounded sequences $\left\{u_{1}^{(k)}\right\},\left\{u_{2}^{(k)}\right\}$, we may pass to subsequences so that

$$
\begin{equation*}
u_{1}^{(k)} \rightharpoonup u_{1} \in E, \quad u_{2}^{(k)} \rightharpoonup u_{2} \in E \tag{A.3}
\end{equation*}
$$

where $u_{1} \neq u_{2}$ by (A.2) and $J^{\prime}\left(u_{1}\right)=J^{\prime}\left(u_{2}\right)=0$, and

$$
\begin{equation*}
\left\|\left(u_{1}^{(k)}\right)^{+}\right\| \rightarrow \alpha_{1}, \quad\left\|\left(u_{1}^{(k)}\right)^{+}\right\| \longrightarrow \alpha_{2} \tag{A.4}
\end{equation*}
$$

where $\sqrt{2 c} \leq \alpha_{i} \leq \nu(d), i=1,2$ by Lemma 9 (b).
If $u_{1} \neq 0$ and $u_{2} \neq 0$. Then, $u_{1}, u_{2} \in \mathscr{M}$ and $w_{1}=m^{-1}\left(u_{1}\right) \in$ $K, w_{2}=m^{-1}\left(u_{2}\right) \in K, w_{1} \neq w_{2}$. Therefore,

$$
\begin{align*}
\liminf _{k \rightarrow \infty}\left\|w_{1}^{(k)}-w_{2}^{(k)}\right\| & =\liminf _{k \rightarrow \infty}\left\|\frac{\left(u_{1}^{(k)}\right)^{+}}{\left\|\left(u_{1}^{(k)}\right)^{+}\right\|}-\frac{\left(u_{2}^{(k)}\right)^{+}}{\left\|\left(u_{2}^{(k)}\right)^{+}\right\|}\right\| \\
& \geq\left\|\frac{u_{1}^{+}}{\alpha_{1}}-\frac{u_{2}^{+}}{\alpha_{2}}\right\|=\left\|\beta_{1} w_{1}-\beta_{2} w_{2}\right\| \tag{A.5}
\end{align*}
$$

where $\beta_{1}=\left\|u_{1}^{+}\right\| / \alpha_{1} \geq \sqrt{2 c} / \nu(d)$ and $\beta_{2}=\left\|u_{2}^{+}\right\| / \alpha_{2} \geq$ $\sqrt{2 c} / \nu(d)$. Since $\left\|w_{1}\right\|=\left\|w_{2}\right\|=1$, we have

$$
\begin{align*}
\liminf _{k \rightarrow \infty}\left\|w_{1}^{(k)}-w_{2}^{(k)}\right\| & \geq\left\|\beta_{1} w_{1}-\beta_{2} w_{2}\right\| \\
& \geq \min \left\{\beta_{1}, \beta_{2}\right\}\left\|w_{1}-w_{2}\right\| \geq \frac{\sqrt{2 c} \kappa}{\nu(d)} \tag{A.6}
\end{align*}
$$

If $u_{1}=0$, then $u_{2} \neq 0$ and

$$
\begin{gather*}
\liminf _{k \rightarrow \infty}\left\|w_{1}^{(k)}-w_{2}^{(k)}\right\|=\liminf _{k \rightarrow \infty} \| \frac{\left(u_{1}^{(k)}\right)^{+}}{\left\|\left(u_{1}^{(k)}\right)^{+}\right\|}-\frac{\left(u_{2}^{(k)}\right)^{+}}{\left\|\left(u_{2}^{(k)}\right)^{+}\right\| \|} \\
\geq \frac{\left\|u_{2}^{+}\right\|}{\alpha_{2}} \geq \frac{\sqrt{2 c}}{v(d)} \tag{A.7}
\end{gather*}
$$

Similarly, if $u_{2}=0$, then $u_{1} \neq 0$ and $\liminf _{k \rightarrow \infty}\left\|w_{1}^{(k)}-w_{2}^{(k)}\right\| \geq$ $\sqrt{2 c} / \nu(d)$.

The proof is complete.
Lemma A.2. For every $w \in S^{+}$, the limit $\lim _{t \rightarrow T^{+}(w)} \eta(t, w)$ exists and is a critical point of $\Psi$.

Proof. Fix $w \in S^{+}$and set $d=\Psi(w)$. We distinguish two cases to finish the proof.

Case $1\left(T^{+}(w)<\infty\right)$. For $0 \leq s<t<T^{+}(w)$, by (72) and (73), we have

$$
\begin{align*}
& \|\eta(t, w)-\eta(s, w)\| \\
& \quad \leq \int_{s}^{t}\|V(\eta(\tau, w))\| d \tau \\
& \quad \leq 2 \sqrt{2} \int_{s}^{t} \sqrt{\left\langle\Psi^{\prime}(\eta(\tau, w)), V(\eta(\tau, w))\right\rangle} d \tau \\
& \quad \leq 2 \sqrt{2(t-s)}\left(\int_{s}^{t}\left\langle\Psi^{\prime}(\eta(\tau, w)), V(\eta(\tau, w))\right\rangle d \tau\right)^{1 / 2} \\
& \quad=2 \sqrt{2(t-s)}[\Psi(\eta(s, w))-\Psi(\eta(t, w))]^{1 / 2} \\
& \quad \leq 2 \sqrt{2(t-s)}[\Psi(w)-c]^{1 / 2} \tag{A.8}
\end{align*}
$$

Since $T^{+}(w)<\infty$, this implies that $\lim _{t \rightarrow T^{+}(w)} \eta(t, w)$ exists and is a critical point of $\Psi$, otherwise the trajectory $t \rightarrow$ $\eta(t, w)$ could be continued beyond $T^{+}(w)$.

Case $2\left(T^{+}(w)=\infty\right)$. To prove that $\lim _{t \rightarrow T^{+}(w)} \eta(t, w)$ exists, we claim that for every $\varepsilon>0$, there exists $t_{\varepsilon}>0$ such that $\left\|\eta\left(t_{\varepsilon}, w\right)-\eta(t, w)\right\|<\varepsilon$ for $t \geq t_{\varepsilon}$. If not, then there exist $0<\varepsilon_{0}<(1 / 2) \varrho(d)(\varrho(d)$ is the same number in Lemma A.1) and a sequence $\left\{t_{n}\right\} \subset[0, \infty)$ with $t_{n} \rightarrow \infty$ such that $\left\|\eta\left(t_{n}, w\right)-\eta\left(t_{n+1}, w\right)\right\|=\varepsilon_{0}$ for every $n$. Choose the smallest $t_{n}^{1} \in\left(t_{n}, t_{n+1}\right)$ such that $\left\|\eta\left(t_{n}, w\right)-\eta\left(t_{n}^{1}, w\right)\right\|=\varepsilon_{0} / 3$. Let $\left.\iota_{n}=\min _{s \in\left[t_{n}, t_{n}^{1}\right]}\right] \mid \Psi^{\prime}(\eta(s, w)) \|$. By (72) and (73), we have

$$
\begin{aligned}
\frac{\varepsilon_{0}}{3} & =\left\|\eta\left(t_{n}^{1}, w\right)-\eta\left(t_{n}, w\right)\right\| \\
& \leq \int_{t_{n}}^{t_{n}^{1}}\|V(\eta(\tau, w))\| d \tau \\
& \leq 2 \int_{t_{n}}^{t_{n}^{1}}\left\|\Psi^{\prime}(\eta(\tau, w))\right\| d \tau
\end{aligned}
$$

$$
\begin{align*}
& \leq \frac{2}{l_{n}} \int_{t_{n}}^{t_{n}^{1}}\left\|\Psi^{\prime}(\eta(\tau, w))\right\|^{2} d \tau \\
& \leq \frac{4}{l_{n}} \int_{t_{n}}^{t_{n}^{1}}\left\langle\Psi^{\prime}(\eta(\tau, w)), V(\eta(\tau, w))\right\rangle d \tau \\
& =\frac{4}{l_{n}}\left(\Psi\left(\eta\left(t_{n}, w\right)\right)-\Psi\left(\eta\left(t_{n}^{1}, w\right)\right)\right) . \tag{A.9}
\end{align*}
$$

Since $\Psi\left(\eta\left(t_{n}, w\right)\right)-\Psi\left(\eta\left(t_{n}^{1}, w\right)\right) \rightarrow 0$ as $n \rightarrow \infty, \iota_{n} \rightarrow 0$ and there exist $\tilde{t}_{n}^{1} \in\left[t_{n}, t_{n}^{1}\right]$ such that $\Psi^{\prime}\left(w_{n}^{1}\right) \rightarrow 0$, where $w_{n}^{1}=\eta\left(\tilde{t}_{n}^{1}, w\right)$. Similarly, we choose the largest $t_{n}^{2} \in\left(t_{n}^{1}, t_{n+1}\right)$ such that $\left\|\eta\left(t_{n+1}, w\right)-\eta\left(t_{n}^{2}, w\right)\right\|=\varepsilon_{0} / 3$. Then, there exist $\widetilde{t}_{n}^{2} \in$ $\left[t_{n}^{2}, t_{n+1}\right]$ such that $\Psi^{\prime}\left(w_{n}^{2}\right) \rightarrow 0$, where $w_{n}^{2}=\eta\left(\widetilde{t}_{n}^{2}, w\right)$. Since $\left\|w_{n}^{1}-\eta\left(t_{n}, w\right)\right\| \leq \varepsilon_{0} / 3$ and $\left\|w_{n}^{2}-\eta\left(t_{n+1}, w\right)\right\| \leq \varepsilon_{0} / 3,\left\{w_{n}^{1}\right\},\left\{w_{n}^{2}\right\}$ are two the Palais-Smale sequences such that

$$
\begin{align*}
\frac{\varepsilon_{0}}{3} & \leq\left\|w_{n}^{1}-w_{n}^{2}\right\| \\
\leq & \left\|w_{n}^{1}-\eta\left(t_{n}, w\right)\right\|  \tag{A.10}\\
& +\left\|\eta\left(t_{n}, w\right)-\eta\left(t_{n+1}, w\right)\right\|+\left\|w_{n}^{2}-\eta\left(t_{n+1}, w\right)\right\| \\
& \leq 2 \varepsilon_{0}<\varrho(d)
\end{align*}
$$

which contradicts with Lemma A.1. This proves the claim. Therefore, $\lim _{t \rightarrow T^{+}(w)} \eta(t, w)$ exists, and, obviously, it must be a critical point of $\Psi$. This completes the proof.

Lemma A.3. Let $d \geq c$. Then, for every $\delta>0$, there exists $\varepsilon=\varepsilon(\delta)>0$ such that
(a) $\Psi_{d-\varepsilon}^{d+\varepsilon} \cap K=K_{d}$,
(b) $\lim _{t \rightarrow T^{+}(w)} \Psi(\eta(t, w))<d-\varepsilon$ for $w \in \Psi^{d+\varepsilon} \backslash N_{\delta}\left(K_{d}\right)$.

Proof. (a) According to (66), for $\varepsilon>0$ small enough, it is easy to see that (a) is satisfied.
(b) Without loss of generality, we may assume that $N_{\delta}\left(K_{d}\right) \subset \Psi^{d+1}$ and $\delta<\varrho(d+1)$. Set

$$
\begin{equation*}
\tau=\inf \left\{\left\|\Psi^{\prime}(w)\right\|: w \in N_{\delta}\left(K_{d}\right) \backslash N_{\delta / 2}\left(K_{d}\right)\right\} \tag{A.11}
\end{equation*}
$$

We claim that $\tau>0$. Indeed, if not, then there exists a sequence $\left\{w_{1}^{(k)}\right\} \subset N_{\delta}\left(K_{d}\right) \backslash N_{\delta / 2}\left(K_{d}\right)$ such that $\Psi^{\prime}\left(w_{1}^{(k)}\right) \rightarrow$ 0 . By the $\mathbb{Z}$-invariance of $\Psi$ and assumption (66), we may assume $w_{1}^{(k)} \in N_{\delta}\left(w_{0}\right) \backslash N_{\delta / 2}\left(w_{0}\right)$ for some $w_{0} \in K_{d}$ after passing to a subsequence. Let $w_{2}^{(k)} \rightarrow w_{0}$. Then, $\Psi^{\prime}\left(w_{2}^{(k)}\right) \rightarrow$ 0 and

$$
\begin{equation*}
\frac{\delta}{2} \leq \limsup _{n \rightarrow \infty}\left\|w_{1}^{(k)}-w_{2}^{(k)}\right\| \leq \delta<\varrho(d+1) \tag{A.12}
\end{equation*}
$$

which contradicts with Lemma A.1. This proves the claim.
Let

$$
\begin{equation*}
M=\sup \left\{\left\|\Psi^{\prime}(w)\right\|: w \in N_{\delta}\left(K_{d}\right) \backslash N_{\delta / 2}\left(K_{d}\right)\right\} \tag{A.13}
\end{equation*}
$$

Choose $\varepsilon<\delta \tau^{2} / 8 M$ such that (a) holds. By Lemma A. 1 and (a), the only way that (b) can fail is that $\eta(t, w) \rightarrow \widetilde{w} \in K_{d}$ as $t \rightarrow T^{+}(w)$ for some $w \in \Psi^{d+\varepsilon} \backslash N_{\delta}\left(K_{d}\right)$. In this case, we let

$$
\begin{gather*}
t_{1}=\sup \left\{t \in\left[0, T^{+}(w)\right): \eta(t, w) \notin N_{\delta}(\widetilde{w})\right\}  \tag{A.14}\\
t_{2}=\inf \left\{t \in\left(t_{1}, T^{+}(w)\right): \eta(t, w) \in N_{\delta / 2}(\widetilde{w})\right\}
\end{gather*}
$$

Then,

$$
\begin{align*}
& \frac{\delta}{2}=\left\|\eta\left(t_{1}, w\right)-\eta\left(t_{2}, w\right)\right\| \\
& \leq \int_{t_{1}}^{t_{2}}\|V(\eta(\tau, w))\| d \tau \\
& \leq 2 \int_{t_{1}}^{t_{2}}\left\|\Psi^{\prime}(\eta(\tau, w))\right\| d \tau \\
& \leq 2 M\left(t_{2}-t_{1}\right), \\
& \Psi\left(\eta\left(t_{2}, w\right)\right)-\Psi\left(\eta\left(t_{1}, w\right)\right)  \tag{A.15}\\
&=-\int_{t_{1}}^{t_{2}}\left\langle\Psi^{\prime}(\eta(\tau, w)), V(\eta(\tau, w))\right\rangle d s \\
& \leq-\frac{1}{2} \int_{t_{1}}^{t_{2}}\left\|\Psi^{\prime}(\eta(s, w))\right\|^{2} d s \\
& \leq-\frac{1}{2} \tau^{2}\left(t_{2}-t_{1}\right) \leq-\frac{\delta \tau^{2}}{8 M}
\end{align*}
$$

It follows that $\Psi\left(\eta\left(t_{2}, w\right)\right) \leq d+\varepsilon-\left(\delta \tau^{2} / 8 M\right)<d$ and therefore $\eta\left(t_{2}, w\right) \nrightarrow \widetilde{w}$, a contradiction again. This completes the proof.

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## Research Article

# Dynamical Behaviors of Stochastic Hopfield Neural Networks with Both Time-Varying and Continuously Distributed Delays 

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#### Abstract

This paper investigates dynamical behaviors of stochastic Hopfield neural networks with both time-varying and continuously distributed delays. By employing the Lyapunov functional theory and linear matrix inequality, some novel criteria on asymptotic stability, ultimate boundedness, and weak attractor are derived. Finally, an example is given to illustrate the correctness and effectiveness of our theoretical results.


## 1. Introduction

Hopfield neural networks [1] have been extensively studied in the past years and found many applications in different areas such as pattern recognition, associative memory, and combinatorial optimization. Such applications heavily depend on the dynamical behaviors such as stability, uniform boundedness, ultimate boundedness, attractor, bifurcation, and chaos. As it is well known, time delays are unavoidably encountered in the implementation of neural networks. Since time delays as a source of instability and bad performance always appear in many neural networks owing to the finite speed of information processing, the stability analysis for the delayed neural networks has received considerable attention. However, in these recent publications, most research on delayed neural networks has been restricted to simple cases of discrete delays. Since a neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths, it is desired to model them by introducing distributed delays. Therefore, both discrete and distributed delays should be taken into account when modeling realistic neural networks $[2,3]$.

On the other hand, it has now been well recognized that stochastic disturbances are also ubiquitous owing to thermal noise in electronic implementations. Therefore, it
is important to understand how these disturbances affect the networks. Many results on stochastic neural networks have been reported in [4-24]. Some sufficient criteria on the stability of uncertain stochastic neural networks were derived in [4-7]. Almost sure exponential stability of stochastic neural networks was studied in [8-10]. In [11-16], mean square exponential stability and $p$ th moment exponential stability of stochastic neural networks were discussed. The stability of stochastic impulsive neural networks was discussed in [17-19]. The stability of stochastic neural networks with the Markovian jumping parameters was investigated in [20-22]. The passivity analysis for stochastic neural networks was discussed in [23, 24]. These references mainly considered the stability of equilibrium point of stochastic delayed neural networks. What do we study to understand the asymptotic behaviors when the equilibrium point does not exist?

Except for the stability property, boundedness and attractor are also foundational concepts of dynamical systems. They play an important role in investigating the uniqueness of equilibrium, global asymptotic stability, global exponential stability, and the existence of periodic solution, its control, and synchronization [25]. Recently, ultimate boundedness and attractor of several classes of neural networks with time delays have been reported in [26-33]. Some sufficient criteria were derived in [26, 27], but these results hold only
under constant delays. Following, in [28], the globally robust ultimate boundedness of integrodifferential neural networks with uncertainties and varying delays were studied. After that, some sufficient criteria on the ultimate boundedness of neural networks with both varying and unbounded delays were derived in [29], but the concerned systems are deterministic ones. In [30, 31], a series of criteria on the boundedness, global exponential stability, and the existence of periodic solution for nonautonomous recurrent neural networks were established. In [32, 33], the ultimate boundedness and weak attractor of stochastic neural networks with time-varying delays were discussed. To the best of our knowledge, for stochastic neural networks with mixed time delays, there are few published results on the ultimate boundedness and weak attractor. Therefore, the arising questions about the ultimate boundedness, weak attractor, and asymptotic stability of the stochastic Hopfield neural networks with mixed time delays are important and meaningful.

The left paper is organized as follows. Some preliminaries are in Section 2, main results are presented in Section 3, a numerical example is given in Section 4, and conclusions are drawn in Section 5.

## 2. Preliminaries

Consider the following stochastic Hopfield neural networks with both time-varying and continuously distributed delays:

$$
\begin{align*}
d x(t)=[ & -C x(t)+A f(x(t))+B f(x(t-\tau(t))) \\
& \left.+D \int_{-\infty}^{t} K(t-s) g(x(s)) d s+J\right] d t  \tag{1}\\
& +\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right] d w(t)
\end{align*}
$$

in which $x=\left(x_{1}, \ldots, x_{n}\right)^{T}$ is a state vector associated with the neurons; $C=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right), c_{i}>0$ represents the rate with which the $i$ th unit will reset its potential to the resting state in isolation when being disconnected from the network and the external stochastic perturbation; $A=$ $\left(a_{i j}\right)_{n \times n}, B=\left(b_{i j}\right)_{n \times n}$, and $D=\left(d_{i j}\right)_{n \times n}$ represent the connection weight matrix, the delayed connection weight matrix, and the distributively delayed connection weight matrix, respectively; $J=\left(J_{1}, \ldots, J_{n}\right)^{T}$, $J_{i}$ denotes the external bias on the ith unit; $f(x(t))=\left(f_{1}\left(x_{1}(t)\right), \ldots, f_{n}\left(x_{n}(t)\right)\right)^{T}$, $g(x(t))=\left(g_{1}\left(x_{1}(t)\right), \ldots, g_{n}\left(x_{n}(t)\right)\right)^{T}, f_{j}$ and $g_{j}$ denote activation functions, $K(t)=\operatorname{diag}\left(k_{1}(t), \ldots, k_{n}(t)\right)$, and the delay kernel $k_{j}(t)$ is a real-valued nonnegative continuous function defined on $[0, \infty) ; \sigma_{1}$ and $\sigma_{2}$ are diffusion coefficient matrices; $w(t)$ is a one-dimensional Brownian motion or Winner process, which is defined on a complete probability space $(\Omega, \mathscr{F}, P)$ with a natural filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ generated by $\{w(s): 0 \leq s \leq t\} ; \tau(t)$ is the transmission delay, and the initial conditions associated with system (1) are of the following forms: $x(t)=\xi(t),-\infty<t \leq 0$, where $\xi(\cdot)$ is a $\mathscr{F}_{0}$-measurable, bounded, and continuous $R^{n}$-valued random variable defined on $(-\infty, 0]$.

Throughout this paper, one always supposes that the following condition holds.
(A1) For $f(x(t))$ and $g(x(t))$ in (1), there are always constants $l_{i}^{-}, l_{i}^{+}, m_{i}^{-}$, and $m_{i}^{+}$such that

$$
\begin{gather*}
l_{i}^{-} \leq \frac{f_{i}(x)-f_{i}(y)}{x-y} \leq l_{i}^{+} \\
m_{i}^{-} \leq \frac{g_{i}(x)-g_{i}(y)}{x-y} \leq m_{i}^{+}, \quad \forall x, y \in R \tag{2}
\end{gather*}
$$

Moreover, there exist constant $v>0$ and matrix $\bar{K}(\nu)=\operatorname{diag}\left(\bar{k}_{1}(\nu), \ldots, \bar{k}_{n}(\nu)\right)>0$ such that

$$
\begin{equation*}
\int_{0}^{\infty} k_{j}(\theta) d \theta=1, \quad \int_{0}^{\infty} k_{j}(\theta) e^{\nu \theta} d \theta=\bar{k}_{j}(\nu)<\infty \tag{3}
\end{equation*}
$$

Following, $A>0$ (resp., $A \geq 0$ ) means that matrix $A$ is a symmetric positive definite (resp., positive semi-definite). $A^{T}$ and $A^{-1}$ denote the transpose and inverse of the matrix $A$. $\lambda_{\text {max }}(A)$ and $\lambda_{\text {min }}(A)$ represent the maximum and minimum eigenvalues of matrix $A$, respectively.

Definition 1. System (1) is said to be stochastically ultimately bounded if, for any $\varepsilon \in(0,1)$, there is a positive constant $C=$ $C(\varepsilon)$ such that the solution $x(t)$ of system (1) satisfies

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} P\{\|x(t)\| \leq C\} \geq 1-\varepsilon \tag{4}
\end{equation*}
$$

Lemma 2 (see [34]). Let $Q(x)=Q^{T}(x), R(x)=R^{T}(x)$ and $S(x)$ depends affinely on $x$. Then, linear matrix inequality

$$
\left(\begin{array}{cc}
Q(x) & S(x)  \tag{5}\\
S^{T}(x) & R(x)
\end{array}\right)>0
$$

is equivalent to
(1) $R(x)>0, Q(x)-S(x) R^{-1}(x) S^{T}(x)>0$,
(2) $Q(x)>0, R(x)-S^{T}(x) Q^{-1}(x) S(x)>0$.

## 3. Main Results

Theorem 3. System (1) is stochastically ultimately bounded provided that $\tau(t)$ satisfies $0 \leq \tau(t) \leq \tau, \dot{\tau}(t) \leq \mu \leq 1$, and there exist some matrices $P>0, Q_{i} \geq 0, U_{i}=\operatorname{diag}\left(u_{i 1}, \ldots, u_{i n}\right) \geq 0$ ( $i=1,2,3,4$ ) such that the following linear matrix inequality holds:

$$
\Sigma=\left(\begin{array}{ccccccc}
\Sigma_{11} & 0 & P A+L_{2} U_{1} & P B & M_{2} U_{4} & P D & \sigma_{1}^{T} P \\
* & \Sigma_{22} & 0 & L_{2} U_{2} & 0 & 0 & \sigma_{2}^{T} P \\
* & * & \Sigma_{33} & 0 & 0 & 0 & 0 \\
* & * & * & \Sigma_{44} & 0 & 0 & 0 \\
* & * & * & * & \Sigma_{55} & 0 & 0 \\
* & * & * & * & * & -U_{3} & 0 \\
& * & * & * & * & * & -P
\end{array}\right)
$$

$<0$,
where $*$ denotes the corresponding symmetric terms,

$$
\begin{gather*}
\Sigma_{11}=Q_{1}+\tau Q_{2}-P C-C P-2 L_{1} U_{1}-2 M_{1} U_{4} \\
\Sigma_{22}=-(1-\mu) Q_{1}-2 L_{1} U_{2}, \quad \Sigma_{33}=Q_{3}+\tau Q_{4}-2 U_{1} \\
\Sigma_{44}=-(1-\mu) Q_{3}-2 U_{2}, \quad \Sigma_{55}=U_{3} \bar{K}(v)-2 U_{4} \\
L_{1}=\operatorname{diag}\left(l_{1}^{-} l_{1}^{+}, \ldots, l_{n}^{-} l_{n}^{+}\right) \\
L_{2}=\operatorname{diag}\left(l_{1}^{-}+l_{1}^{+}, \ldots, l_{n}^{-}+l_{n}^{+}\right) \\
M_{1}=\operatorname{diag}\left(m_{1}^{-} m_{1}^{+}, \ldots, m_{n}^{-} m_{n}^{+}\right) \\
M_{2}=\operatorname{diag}\left(m_{1}^{-}+m_{1}^{+}, \ldots, m_{n}^{-}+m_{n}^{+}\right) \tag{7}
\end{gather*}
$$

Proof. The key of the proof is to prove that there exists a positive constant $C_{*}$, which is independent of the initial data, such that

$$
\begin{equation*}
\underset{t \rightarrow \infty}{\limsup } E\|x(t)\|^{2} \leq C_{*} . \tag{8}
\end{equation*}
$$

If (8) holds, then it follows from Chebyshev's inequality that for any $\varepsilon>0$ and $C=\sqrt{C_{*} / \varepsilon}$,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} P\{\|x(t)\|>C\} \leq \frac{\lim \sup _{t \rightarrow \infty} E\|x(t)\|^{2}}{C^{2}}=\varepsilon \tag{9}
\end{equation*}
$$

which implies that (4) holds. Now, we begin to prove that (8) holds.

From $\Sigma<0$ and Lemma 2, one may obtain

$$
\begin{gather*}
\left(\begin{array}{cccccc}
\Sigma_{11} & 0 & P A+L_{2} U_{1} & P B & M_{2} U_{4} & P D \\
* & \Sigma_{22} & 0 & L_{2} U_{2} & 0 & 0 \\
* & * & \Sigma_{33} & 0 & 0 & 0 \\
* & * & * & \Sigma_{44} & 0 & 0 \\
* & * & * & * & \Sigma_{55} & 0 \\
* & * & * & * & * & -U_{3}
\end{array}\right) \\
 \tag{10}\\
\quad+\left(\begin{array}{c}
\sigma_{1}^{T} P \\
\sigma_{2}^{T} P \\
0 \\
0 \\
0 \\
0
\end{array}\right) P^{-1}\left(\begin{array}{c}
\sigma_{1}^{T} P \\
\sigma_{2}^{T} P \\
0 \\
0 \\
0 \\
0
\end{array}\right)
\end{gather*}
$$

Hence, there exists a sufficiently small $\lambda \in(0, \nu)$ such that

$$
\begin{align*}
\Gamma= & \left(\begin{array}{cccccc}
\Gamma_{11} & 0 & P A+L_{2} U_{1} & P B & M_{2} U_{4} & P D \\
* & \Gamma_{22} & 0 & L_{2} U_{2} & 0 & 0 \\
* & * & \Gamma_{33} & 0 & 0 & 0 \\
* & * & * & \Gamma_{44} & 0 & 0 \\
* & * & * & * & \Gamma_{55} & 0 \\
* & * & * & * & * & -U_{3}
\end{array}\right) \\
& +\left(\begin{array}{c}
\sigma_{1}^{T} P \\
\sigma_{2}^{T} P \\
0 \\
0 \\
0 \\
0
\end{array}\right) P^{-1}\left(\begin{array}{c}
\sigma_{1}^{T} P \\
\sigma_{2}^{T} P \\
0 \\
0 \\
0 \\
0
\end{array}\right)<0, \tag{11}
\end{align*}
$$

where $I$ is identity matrix,

$$
\begin{gather*}
\Gamma_{11}=e^{\lambda \tau} Q_{1}+\tau Q_{2}-P C-C P-2 L_{1} U_{1} \\
-2 M_{1} U_{4}+2 \lambda P+2 \lambda I \\
\Gamma_{22}=\Sigma_{22}+\lambda I, \quad \Gamma_{33}=\lambda I+e^{\lambda \tau} Q_{3}+\tau Q_{4}-2 U_{1},  \tag{12}\\
\Gamma_{44}=\Sigma_{44}+\lambda I, \quad \Gamma_{55}=\Sigma_{55}+\lambda I .
\end{gather*}
$$

Consider the Lyapunov-Krasovskii function as follows:

$$
\begin{align*}
V(x(t), t)= & V_{1}(x(t), t)+V_{2}(x(t), t) \\
& +V_{3}(x(t), t)+V_{4}(x(t), t), \tag{13}
\end{align*}
$$

where

$$
\begin{aligned}
& V_{1}(x(t), t)=e^{\lambda t} x^{T}(t) P x(t), \\
& V_{2}(x(t), t)=\sum_{j=1}^{n} u_{3 j} \int_{0}^{\infty} k_{j}(\theta) \int_{t-\theta}^{t} e^{\lambda(s+\theta)} g_{j}^{2}\left(x_{j}(s)\right) d s d \theta, \\
& V_{3}(x(t), t) \\
& \quad=\int_{t-\tau(t)}^{t} e^{\lambda(s+\tau)}\left[x^{T}(s) Q_{1} x(s)+f^{T}(x(s)) Q_{3} f(x(s))\right] d s,
\end{aligned}
$$

$$
V_{4}(x(t), t)
$$

$$
\begin{align*}
=\int_{t-\tau(t)}^{t} \int_{s}^{t} e^{\lambda \theta}[ & x^{T}(\theta) Q_{2} x(\theta) \\
& \left.+f^{T}(x(\theta)) Q_{4} f(x(\theta))\right] d \theta d s \tag{14}
\end{align*}
$$

Then, it can be obtained by Ito's formula in [35] that

$$
\begin{align*}
& d V(x(t), t) \\
&= e^{\lambda t} 2 x^{T}(t) P\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right] d w(t) \\
&+\left\{\mathscr{L} V_{1}(x(t), t)\right. \\
&\left.+\frac{\partial\left[V_{2}(x(t), t)+V_{3}(x(t), t)+V_{4}(x(t), t)\right]}{\partial t}\right\} d t \tag{15}
\end{align*}
$$

where

$$
\mathscr{L} V_{1}(x(t), t)
$$

$$
=\frac{\partial V_{1}(x(t), t)}{\partial t}+\frac{\partial V_{1}(x(t), t)}{\partial x}
$$

$$
\times[-C x(t)+A f(x(t))+B f(x(t-\tau(t)))
$$

$$
\left.+D \int_{-\infty}^{t} K(t-s) g(x(s)) d s+J\right]
$$

$$
+\operatorname{trace}\left(\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right]^{T} \frac{\partial^{2} V_{1}(x(t), t)}{\partial x^{2}}\right.
$$

$$
\left.\times\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right]\right)
$$

$$
\begin{aligned}
& -e^{\lambda t} \sum_{j=1}^{n} u_{3 j}\left[\int_{0}^{\infty} k_{j}(\theta) g_{j}\left(x_{j}(t-\theta)\right) d \theta\right]^{2} \\
& =e^{\lambda t} g^{T}(x(t)) U_{3} \bar{K}(\nu) g(x(t)) \\
& -e^{\lambda t}\left(\int_{-\infty}^{t} K(t-s) g(x(s)) d s\right)^{T} \\
& \times U_{3}\left(\int_{-\infty}^{t} K(t-s) g(x(s)) d s\right), \\
& \frac{\partial V_{3}(x(t), t)}{\partial t} \\
& =e^{\lambda(t+\tau)}\left[x^{T}(t) Q_{1} x(t)+f^{T}(x(t)) Q_{3} f(x(t))\right] \\
& -(1-\dot{\tau}(t)) e^{\lambda(t-\tau(t)+\tau)} \\
& \times\left[x^{T}(t-\tau(t)) Q_{1} x(t-\tau(t))\right. \\
& \left.+f^{T}(x(t-\tau(t))) Q_{3} f(x(t-\tau(t)))\right] \\
& \leq e^{\lambda(t+\tau)}\left[x^{T}(t) Q_{1} x(t)+f^{T}(x(t)) Q_{3} f(x(t))\right] \\
& -(1-\mu) e^{\lambda t} \\
& \times\left[x^{T}(t-\tau(t)) Q_{1} x(t-\tau(t))\right. \\
& \left.+f^{T}(x(t-\tau(t))) Q_{3} f(x(t-\tau(t)))\right], \\
& \frac{\partial V_{4}(x(t), t)}{\partial t} \\
& =e^{\lambda t} \tau(t)\left[x^{T}(t) Q_{2} x(t)+f^{T}(x(t)) Q_{4} f(x(t))\right] \\
& -(1-\dot{\tau}(t)) \\
& \times \int_{t-\tau(t)}^{t} e^{\lambda s}\left[x^{T}(s) Q_{2} x(s)+f^{T}(x(s)) Q_{4} f(x(s))\right] d s \\
& \leq e^{\lambda t} \tau\left[x^{T}(t) Q_{2} x(t)+f^{T}(x(t)) Q_{4} f(x(t))\right],
\end{aligned}
$$

$$
\begin{align*}
&= \lambda e^{\lambda t} x^{T}(t) P x(t) \\
&+e^{\lambda t} 2 x^{T}(t) P[-C x(t)+A f(x(t)) \\
&+B f(x(t-\tau(t))) \\
&\left.+D \int_{-\infty}^{t} K(t-s) g(x(s)) d s+J\right] \\
&+e^{\lambda t}\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right]^{T} \\
& \times P\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right] \\
& \leq \lambda e^{\lambda t} x^{T}(t) P x(t) \\
&+e^{\lambda t} 2 x^{T}(t) P[-C x(t)+A f(x(t)) \\
&+B f(x(t-\tau(t))) \\
&\left.+D \int_{-\infty}^{t} K(t-s) g(x(s)) d s\right] \\
&+e^{\lambda t}\left[\lambda x^{T}(t) P x(t)+\lambda^{-1} J^{T} P J\right] \\
&+e^{\lambda t}\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right]^{T} \\
& \times P\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right] \tag{17}
\end{align*}
$$

in which the following inequality is used:

$$
\begin{align*}
2 x^{T} P J & \leq \lambda x^{T} P x+\lambda^{-1} J^{T} P^{T} P^{-1} P J \\
& =\lambda x^{T} P x+\lambda^{-1} J^{T} P J, \quad \text { for } P>0, \lambda>0 \tag{18}
\end{align*}
$$

On the other hand, it follows from (A1) that for $i=$ $1, \ldots, n$,

$$
\begin{align*}
& {\left[f_{i}\left(x_{i}(t)\right)-f_{i}(0)-l_{i}^{+} x_{i}(t)\right]} \\
& \quad \times\left[f_{i}\left(x_{i}(t)\right)-f_{i}(0)-l_{i}^{-} x_{i}(t)\right] \leq 0 \\
& {\left[f_{i}\left(x_{i}(t-\tau(t))\right)-f_{i}(0)-l_{i}^{+} x_{i}(t-\tau(t))\right]}  \tag{19}\\
& \times\left[f_{i}\left(x_{i}(t-\tau(t))\right)-f_{i}(0)-l_{i}^{-} x_{i}(t-\tau(t))\right] \leq 0, \\
& 0 \leq e^{\lambda t}\left\{-2 \sum_{i=1}^{n} u_{1 i}\left[f_{i}\left(x_{i}(t)\right)-f_{i}(0)-l_{i}^{+} x_{i}(t)\right]\right. \\
& \quad \times\left[f_{i}\left(x_{i}(t)\right)-f_{i}(0)-l_{i}^{-} x_{i}(t)\right] \\
& -2 \sum_{i=1}^{n} u_{2 i}\left[f_{i}\left(x_{i}(t-\tau(t))\right)\right. \\
& \left.\quad-f_{i}(0)-l_{i}^{+} x_{i}(t-\tau(t))\right]
\end{align*}
$$

$$
\begin{align*}
& \times\left[f_{i}\left(x_{i}(t-\tau(t))\right)-f_{i}(0)\right. \\
& \left.\left.-l_{i}^{-} x_{i}(t-\tau(t))\right]\right\} \\
& =e^{\lambda t}\left\{-2 \sum_{i=1}^{n} u_{1 i}\left[f_{i}\left(x_{i}(t)\right)-l_{i}^{+} x_{i}(t)\right]\right. \\
& \times\left[f_{i}\left(x_{i}(t)\right)-l_{i}^{-} x_{i}(t)\right] \\
& -2 \sum_{i=1}^{n} u_{2 i}\left[f_{i}\left(x_{i}(t-\tau(t))\right)-l_{i}^{+} x_{i}(t-\tau(t))\right] \\
& \times\left[f_{i}\left(x_{i}(t-\tau(t))\right)-l_{i}^{-} x_{i}(t-\tau(t))\right] \\
& -2 \sum_{i=1}^{n} u_{1 i} f_{i}^{2}(0) \\
& +2 \sum_{i=1}^{n} u_{1 i} f_{i}(0)\left[2 f_{i}\left(x_{i}(t)\right)-\left(l_{i}^{+}+l_{i}^{-}\right) x_{i}(t)\right] \\
& -2 \sum_{i=1}^{n} u_{2 i} f_{i}^{2}(0)  \tag{20}\\
& +2 \sum_{i=1}^{n} u_{2 i} f_{i}(0)\left[2 f_{i}\left(x_{i}(t-\tau(t))\right)\right. \\
& \left.\left.-\left(l_{i}^{+}+l_{i}^{-}\right) x_{i}(t-\tau(t))\right]\right\} \\
& \leq e^{\lambda t}\left\{-2 \sum_{i=1}^{n} u_{1 i}\left[f_{i}\left(x_{i}(t)\right)-l_{i}^{+} x_{i}(t)\right]\right. \\
& \times\left[f_{i}\left(x_{i}(t)\right)-l_{i}^{-} x_{i}(t)\right] \\
& -2 \sum_{i=1}^{n} u_{2 i}\left[f_{i}\left(x_{i}(t-\tau(t))\right)-l_{i}^{+} x_{i}(t-\tau(t))\right] \\
& \times\left[f_{i}\left(x_{i}(t-\tau(t))\right)-l_{i}^{-} x_{i}(t-\tau(t))\right] \\
& +\sum_{i=1}^{n}\left[\left|4 u_{1 i} f_{i}(0) f_{i}\left(x_{i}(t)\right)\right|\right.  \tag{21}\\
& \left.+\left|2 u_{1 i} f_{i}(0)\left(l_{i}^{+}+l_{i}^{-}\right) x_{i}(t)\right|\right] \\
& +\sum_{i=1}^{n}\left[\left|4 u_{2 i} f_{i}(0) f_{i}\left(x_{i}(t-\tau(t))\right)\right|\right. \\
& \left.\left.+\left|2 u_{2 i} f_{i}(0)\left(l_{i}^{+}+l_{i}^{-}\right) x_{i}(t-\tau(t))\right|\right]\right\} \tag{22}
\end{align*}
$$

$$
\begin{aligned}
\leq e^{\lambda t}\left\{-2 \sum_{i=1}^{n} u_{1 i}\right. & {\left[f_{i}\left(x_{i}(t)\right)-l_{i}^{+} x_{i}(t)\right] } \\
& \times\left[f_{i}\left(x_{i}(t)\right)-l_{i}^{-} x_{i}(t)\right] \\
-2 \sum_{i=1}^{n} u_{2 i} & {\left[f_{i}\left(x_{i}(t-\tau(t))\right)-l_{i}^{+} x_{i}(t-\tau(t))\right] } \\
& \times\left[f_{i}\left(x_{i}(t-\tau(t))\right)-l_{i}^{-} x_{i}(t-\tau(t))\right] \\
+\sum_{i=1}^{n}[ & \lambda f_{i}^{2}\left(x_{i}(t)\right)+4 \lambda^{-1} f_{i}^{2}(0) u_{1 i}^{2} \\
& \left.+\lambda x_{i}^{2}(t)+\lambda^{-1} f_{i}^{2}(0) u_{1 i}^{2}\left(l_{i}^{+}+l_{i}^{-}\right)^{2}\right] \\
+\sum_{i=1}^{n} & {\left[\lambda f_{i}^{2}\left(x_{i}(t-\tau(t))\right)+4 \lambda^{-1} f_{i}^{2}(0) u_{2 i}^{2}\right.} \\
+ & \lambda x_{i}^{2}(t-\tau(t)) \\
+ & \left.\left.\lambda^{-1} f_{i}^{2}(0) u_{2 i}^{2}\left(l_{i}^{+}+l_{i}^{-}\right)^{2}\right]\right\} .
\end{aligned}
$$

Similarly, one may obtain

$$
\begin{aligned}
& 0 \leq e^{\lambda t}\left\{-2 \sum_{i=1}^{n} u_{4 i}\left[g_{i}\left(x_{i}(t)\right)-g_{i}(0)-m_{i}^{+} x_{i}(t)\right]\right. \\
& \left.\times\left[g_{i}\left(x_{i}(t)\right)-g_{i}(0)-m_{i}^{-} x_{i}(t)\right]\right\} \\
& \leq e^{\lambda t}\left\{-2 \sum_{i=1}^{n} u_{4 i}\left[g_{i}\left(x_{i}(t)\right)-m_{i}^{+} x_{i}(t)\right]\right. \\
& \times\left[g_{i}\left(x_{i}(t)\right)-m_{i}^{-} x_{i}(t)\right] \\
& +\sum_{i=1}^{n}\left[\lambda g_{i}^{2}\left(x_{i}(t)\right)+4 \lambda^{-1} g_{i}^{2}(0) u_{4 i}^{2}\right. \\
& \left.\left.+\lambda x_{i}^{2}(t)+\lambda^{-1} g_{i}^{2}(0) u_{4 i}^{2}\left(m_{i}^{+}+m_{i}^{-}\right)^{2}\right]\right\} .
\end{aligned}
$$

Therefore, from (15)-(21), it follows that

$$
\begin{aligned}
d V(x(t), t) \leq & e^{\lambda t} 2 x^{T}(t) P\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right] d w(t) \\
& +e^{\lambda t} \eta^{T}(t) \Gamma \eta(t) d t+e^{\lambda t} C_{1} d t \\
\leq & e^{\lambda t} 2 x^{T}(t) P\left[\sigma_{1} x(t)+\sigma_{2} x(t-\tau(t))\right] d w(t) \\
& +e^{\lambda t} C_{1} d t,
\end{aligned}
$$

where

$$
\begin{gather*}
\eta(t)=\left(x^{T}(t), x^{T}(t-\tau(t)), f^{T}(x(t)), f^{T}(x(t-\tau(t))),\right. \\
\left.g^{T}(x(t)),\left(\int_{-\infty}^{t} K(t-s) g(x(s)) d s\right)^{T}\right)^{T}, \tag{23}
\end{gather*}
$$

$$
C_{1}=\lambda^{-1} J^{T} P J
$$

$$
+\sum_{i=1}^{n}\left[4 \lambda^{-1} f_{i}^{2}(0) u_{1 i}^{2}+\lambda^{-1} f_{i}^{2}(0) u_{1 i}^{2}\left(l_{i}^{+}+l_{i}^{-}\right)^{2}\right.
$$

$$
\begin{equation*}
+4 \lambda^{-1} f_{i}^{2}(0) u_{2 i}^{2} \tag{24}
\end{equation*}
$$

$$
+\lambda^{-1} f_{i}^{2}(0) u_{2 i}^{2}\left(l_{i}^{+}+l_{i}^{-}\right)^{2}
$$

$$
+4 \lambda^{-1} g_{i}^{2}(0) u_{4 i}^{2}
$$

$$
\left.+\lambda^{-1} g_{i}^{2}(0) u_{4 i}^{2}\left(m_{i}^{+}+m_{i}^{-}\right)^{2}\right] .
$$

Thus, one may obtain that

$$
\begin{align*}
V(x(t), t) \leq & V(x(0), 0)+e^{\lambda t} \lambda^{-1} C_{1} \\
& +\int_{0}^{t} 2 x^{T}(s) P\left[\sigma_{1} x(s)+\sigma_{2} x(s-\tau(s))\right] d w(s), \tag{25}
\end{align*}
$$

$$
\begin{align*}
E\|x(t)\|^{2} & \leq \frac{e^{-\lambda t} E V(x(0), 0)+\lambda^{-1} C_{1}}{\lambda_{\min }(P)}  \tag{26}\\
& =\frac{e^{-\lambda t} E V(x(0), 0)}{\lambda_{\min }(P)}+C_{*}
\end{align*}
$$

where $C_{*}=\lambda^{-1} C_{1} / \lambda_{\text {min }}(P)$. Equation (26) implies that (8) holds. The proof is completed.

Theorem 3 shows that there exists $t_{0}>0$ such that for any $t \geq t_{0}, P\{\|x(t)\| \leq C\} \geq 1-\delta$. Let $B_{C}$ be denoted by

$$
\begin{equation*}
B_{C}=\left\{x \in R^{n} \mid\|x(t)\| \leq C, t \geq t_{0}\right\} . \tag{27}
\end{equation*}
$$

Clearly, $B_{C}$ is closed, bounded, and invariant. Moreover,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \inf _{y \in B_{C}}\|x(t)-y\|=0 \tag{28}
\end{equation*}
$$

with no less than probability $1-\delta$, which means that $B_{C}$ attracts the solutions infinitely many times with no less than probability $1-\delta$; so we may say that $B_{C}$ is a weak attractor for the solutions.

Theorem 4. Suppose that all conditions of Theorem 3 hold, then there exists a weak attractor $B_{C}$ for the solutions of system (1).

Theorem 5. Suppose that all conditions of Theorem 3 hold and $f(0)=g(0)=J=0$; then, zero solution of system (1) is mean square exponential stability and almost sure exponential stability.

Proof. If $f(0)=g(0)=J=0$, then $C_{1}=C_{*}=0$. By (25) and the semimartingale convergence theorem used in [35], zero solution of system (1) is almost sure exponential stability. It follows from (26) that zero solution of system (1) is mean square exponential stability.

Remark 6. If one takes $Q_{2}=Q_{4}=0$ in Theorem 3, then it is not required that $\mu \leq 1$. Furthermore, If one takes $Q_{1}=$ $Q_{2}=Q_{3}=Q_{4}=0$, then $\tau(t)$ may be nondifferentiable or the boundedness of $\dot{\tau}(t)$ is unknown.

Remark 7. Assumption (A1) is less conservative than that in [32] since the constants $l_{i}^{-}, l_{i}^{+}, m_{i}^{-}$, and $m_{i}^{+}$are allowed to be positive, negative, or zero. System (1) includes mixed timedelays, which is more complex than that in [33]. The systems concerned in [26-31] are deterministic, so the stochastic system studied in this paper is more complex and realistic.

When $\sigma_{1}=\sigma_{2}=0$, system (1) becomes the following determined system:

$$
\begin{align*}
\frac{d x(t)}{d t}= & -C x(t)+A f(x(t))+B f(x(t-\tau(t))) \\
& +D \int_{-\infty}^{t} K(t-s) g(x(s)) d s+J \tag{29}
\end{align*}
$$

Definition 8. System (29) is said to be uniformly bounded, if, for each $H>0$, there exists a constant $M=M(H)>0$ such that $\left[t_{0} \in R, \phi \in C(-\infty, 0],\|\phi\| \leq H, t>t_{0}\right]$ imply $\left\|x\left(t, t_{0}, \phi\right)\right\| \leq M$, where $\|\phi\|=\sup _{t \leq 0}\|\phi(t)\|$.

Theorem 9. System (29) is uniformly bounded provided that $\tau(t)$ satisfies $0 \leq \tau(t) \leq \tau, \dot{\tau}(t) \leq \mu \leq 1$, and there exist some matrices $P>0, Q_{i} \geq 0$, and $U_{i}=\operatorname{diag}\left(u_{i 1}, \ldots, u_{i n}\right) \geq$ $0(i=1,2,3,4)$ such that the following linear matrix inequality holds:

$$
\Sigma=\left(\begin{array}{cccccc}
\Sigma_{11} & 0 & P A+L_{2} U_{1} & P B & M_{2} U_{4} & P D  \tag{30}\\
* & \Sigma_{22} & 0 & L_{2} U_{2} & 0 & 0 \\
* & * & \Sigma_{33} & 0 & 0 & 0 \\
* & * & * & \Sigma_{44} & 0 & 0 \\
* & * & * & * & \Sigma_{55} & 0 \\
* & * & * & * & * & -U_{3}
\end{array}\right)
$$

$$
<0,
$$

where $*$ denotes the corresponding symmetric terms, $\Sigma_{i i}(i=$ $1,2,3,4,5), L_{1}, L_{2}, M_{1}, M_{2}$ are the same as in Theorem 3.

Proof. From $\Sigma<0$, there exists a sufficiently small $\lambda \in(0, \nu)$ such that

$$
\Gamma=\left(\begin{array}{cccccc}
\Gamma_{11} & 0 & P A+L_{2} U_{1} & P B & M_{2} U_{4} & P D  \tag{31}\\
* & \Gamma_{22} & 0 & L_{2} U_{2} & 0 & 0 \\
* & * & \Gamma_{33} & 0 & 0 & 0 \\
* & * & * & \Gamma_{44} & 0 & 0 \\
* & * & * & * & \Gamma_{55} & 0 \\
* & * & * & * & * & -U_{3}
\end{array}\right)
$$

$$
<0
$$

where $I$ is identity matrix, $\Gamma_{11}=e^{\lambda \tau} Q_{1}+\tau Q_{2}-P C-C P-$ $2 L_{1} U_{1}-2 M_{1} U_{4}+2 \lambda P+2 \lambda I, \Gamma_{22}=\Sigma_{22}+\lambda I, \Gamma_{33}=\lambda I+$ $e^{\lambda \tau} Q_{3}+\tau Q_{4}-2 U_{1}, \Gamma_{44}=\Sigma_{44}+\lambda I$, and $\Gamma_{55}=\Sigma_{55}+\lambda I$.

We still consider the Lyapunov-Krasovskii functional $V(x(t))$ in (13). From (16)-(21), one may obtain

$$
\begin{align*}
&\|x(t)\|^{2} \leq \lambda_{\min }^{-1}(P)\left(e^{-\lambda t} V(x(0), 0)+\lambda^{-1} C_{1}\right) \\
& \leq \lambda_{\min }^{-1}(P)\left(V(x(0), 0)+\lambda^{-1} C_{1}\right) \\
& \leq \lambda_{\min }^{-1}(P)\left[\lambda_{\max }(P)\|\xi\|^{2}\right. \\
&+\sum_{j=1}^{n} u_{3 j} \int_{0}^{\infty} k_{j}(\theta) \\
& \times \int_{-\theta}^{0} e^{\lambda(s+\theta)} g_{j}^{2}\left(x_{j}(s)\right) d s d \theta \\
&+\int_{-\tau}^{0} e^{\lambda(s+\tau)} \\
& \quad \times\left[x^{T}(s) Q_{1} x(s)\right. \\
&+\int_{-\tau}^{0} \int_{s}^{0} e^{\lambda \theta} \\
& \quad \times\left[x^{T}(\theta(s)) Q_{3} f(x(s))\right] d s
\end{align*}
$$

where $\|\xi\|^{2}=\sup _{t \leq 0}\|x(t)\|^{2}, C_{1}$ is the same as in (24). Note that

$$
\begin{aligned}
& \sum_{j=1}^{n} u_{3 j} \int_{0}^{\infty} k_{j}(\theta) \int_{-\theta}^{0} e^{\lambda(s+\theta)} g_{j}^{2}\left(x_{j}(s)\right) d s d \theta \\
& \leq \sum_{j=1}^{n} 2 u_{3 j} \int_{0}^{\infty} k_{j}(\theta) \int_{-\theta}^{0} e^{\lambda(s+\theta)}\left[\max _{1 \leq j \leq n}\left\{\left(m_{j}^{-}\right)^{2},\left(m_{j}^{+}\right)^{2}\right\} x_{j}^{2}(s)\right.
\end{aligned}
$$

$$
\left.+g_{j}^{2}(0)\right] d s d \theta
$$

$$
\begin{align*}
& \leq \sum_{j=1}^{n} 2 u_{3 j} \int_{0}^{\infty} k_{j}(\theta) \int_{-\theta}^{0} e^{\lambda(s+\theta)}\left[\max _{1 \leq j \leq n}\left\{\left(m_{j}^{-}\right)^{2},\left(m_{j}^{+}\right)^{2}\right\}\right. \\
& \left.\times\|x(s)\|^{2}+\|g(0)\|^{2}\right] d s d \theta \\
& \leq\left(\sum_{j=1}^{n} 2 u_{3 j} \lambda^{-1} \bar{k}_{j}(\nu)\right) \\
& \times\left[\max _{1 \leq j \leq n}\left\{\left(m_{j}^{-}\right)^{2},\left(m_{j}^{+}\right)^{2}\right\}\|\xi\|^{2}+\|g(0)\|^{2}\right], \\
& \int_{-\tau}^{0} e^{\lambda(s+\tau)}\left[x^{T}(s) Q_{1} x(s)+f^{T}(x(s)) Q_{3} f(x(s))\right] d s \\
& \leq \int_{-\tau}^{0} e^{\lambda(s+\tau)}\left[\lambda_{\max }\left(Q_{1}\right)\|x(s)\|^{2}\right. \\
& \left.+\lambda_{\max }\left(Q_{3}\right)\|f(x(s))\|^{2}\right] d s \\
& \leq \int_{-\tau}^{0} e^{\lambda(s+\tau)}\left[\lambda_{\max }\left(Q_{1}\right)\|\xi\|^{2}+2 \lambda_{\max }\left(Q_{3}\right)\right. \\
& \times\left(\|f(0)\|^{2}\right. \\
& \left.\left.+\max _{1 \leq i \leq n}\left\{\left(l_{i}^{-}\right)^{2},\left(l_{i}^{+}\right)^{2}\right\}\|\xi\|^{2}\right)\right] d s \\
& \leq \lambda^{-1}\left[\lambda_{\max }\left(Q_{1}\right)\|\xi\|^{2}+2 \lambda_{\text {max }}\left(Q_{3}\right)\right. \\
& \left.\times\left(\|f(0)\|^{2}+\max _{1 \leq i \leq n}\left\{\left(l_{i}^{-}\right)^{2},\left(l_{i}^{+}\right)^{2}\right\}\|\xi\|^{2}\right)\right] d s, \\
& \int_{-\tau}^{0} \int_{s}^{0} e^{\lambda \theta}\left[x^{T}(\theta) Q_{2} x(\theta)+f^{T}(x(\theta)) Q_{4} f(x(\theta))\right] d \theta d s \\
& \leq \int_{-\tau}^{0} \int_{s}^{0} e^{\lambda \theta}\left[\lambda_{\text {max }}\left(Q_{2}\right)\|\xi\|^{2}+2 \lambda_{\text {max }}\left(Q_{4}\right)\right. \\
& \times\left(\|f(0)\|^{2}\right. \\
& \left.\left.+\max _{1 \leq i \leq n}\left\{\left(l_{i}^{-}\right)^{2},\left(l_{i}^{+}\right)^{2}\right\}\|\xi\|^{2}\right)\right] d \theta d s \\
& \leq \tau \lambda^{-1}\left[\lambda_{\max }\left(Q_{2}\right)\|\xi\|^{2}+2 \lambda_{\max }\left(Q_{4}\right)\right. \\
& \left.\times\left(\|f(0)\|^{2}+\max _{1 \leq i \leq n}\left\{\left(l_{i}^{-}\right)^{2},\left(l_{i}^{+}\right)^{2}\right\}\|\xi\|^{2}\right)\right] . \tag{33}
\end{align*}
$$



Figure 1: Time trajectories (a) as well as the phase portrait (b) for the system in Example 1.

## 4. One Example

Example 1. Consider system (1) with $J=(0,1)^{T}, K(t)=\operatorname{diag}$ $\left(e^{-t}, 2 e^{-2 t}\right), \nu=0.5$, and

$$
\begin{gathered}
A=\left(\begin{array}{cc}
-0.1 & 0.4 \\
0.2 & -0.5
\end{array}\right), \quad B=\left(\begin{array}{cc}
0.1 & -1 \\
-1.4 & 0.4
\end{array}\right), \\
C=\left(\begin{array}{cc}
1.4 & 0 \\
0 & 1.65
\end{array}\right), \\
D=\left(\begin{array}{cc}
-0.6 & 0.7 \\
1 & 1.15
\end{array}\right), \quad \sigma_{1}=\left(\begin{array}{cc}
0.23 & 0.1 \\
0.3 & 0.2
\end{array}\right), \\
\sigma_{2}=\left(\begin{array}{cc}
0.1 & -0.2 \\
0.2 & 0.3
\end{array}\right) .
\end{gathered}
$$

The activation functions $f_{i}\left(x_{i}\right)=x_{i}+\sin \left(x_{i}\right), g_{i}\left(x_{i}\right)=$ $\tanh \left(x_{i}\right)(i=1,2)$ satisfy, $l_{i}^{-}=m_{i}^{-}=0, l_{i}^{+}=m_{i}^{+}=1$. Then, one computes $L_{1}=M_{1}=0, L_{2}=M_{2}=\operatorname{diag}(1,1)$, and $\bar{K}(0.5)=$ $\operatorname{diag}(2,4 / 3)$. By using MATLAB's LMI Control Toolbox [34], for $\mu=0.0035$ and $\tau=1$, based on Theorem 3, such system is stochastically ultimately bounded when $P, U_{i}$, and $Q_{i}(i=$ $1,2,3,4)$ satisfy

$$
\begin{align*}
& P=\left(\begin{array}{ll}
1.7748 & 0.2342 \\
0.2342 & 1.9398
\end{array}\right), \quad U_{1}=\left(\begin{array}{cc}
1.3091 & 0 \\
0 & 2.1602
\end{array}\right), \\
& U_{2}=\left(\begin{array}{cc}
218.3215 & 0 \\
0 & 274.1607
\end{array}\right), \quad U_{3}=\left(\begin{array}{cc}
1.0825 & 0 \\
0 & 1.4021
\end{array}\right), \\
& U_{4}=\left(\begin{array}{cc}
2.1620 & 0 \\
0 & 1.8493
\end{array}\right), \quad Q_{1}=\left(\begin{array}{ll}
275.3435 & 104.0781 \\
104.0781 & 397.1452
\end{array}\right), \\
& Q_{2}=\left(\begin{array}{ll}
31.3129 & 31.4377 \\
31.4377 & 74.4756
\end{array}\right), \quad Q_{3}=\left(\begin{array}{cc}
1.0857 & -1.2164 \\
-1.2164 & 2.9090
\end{array}\right), \\
& Q_{4}=\left(\begin{array}{cc}
189.3661 & 16.3951 \\
16.3951 & 92.8987
\end{array}\right) . \tag{35}
\end{align*}
$$

From Figure 1, it is easy to see that $x(t)$ is stochastically ultimately bounded.

## 5. Conclusions

A proper Lyapunov functional and linear matrix inequalities are employed to investigate the ultimate boundedness, stability, and weak attractor of stochastic Hopfield neural networks with both time-varying and continuously distributed delays. New results and sufficient criteria are derived after extensive deductions. From the proposed sufficient conditions, one can easily prove that zero solution of such network is mean square exponentially stable and almost surely exponentially stable by applying the semimartingale convergence theorem.

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## Research Article

# The Asymptotic Behavior for a Class of Impulsive Delay Differential Equations 

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#### Abstract

This paper is concerned with asymptotical behavior for a class of impulsive delay differential equations. The new criteria for determining attracting sets and attracting basin of the impulsive system are obtained by developing the properties of quasi-invariant sets. Examples and numerical simulations are given to illustrate the effectiveness of our results. In addition, we show that the impulsive effects may play a key role to these asymptotical properties even though the solutions of corresponding nonimpulsive systems are unbounded.


## 1. Introduction

Impulsive delay differential equations have attracted increasing interests since time delays and impulsive effects commonly exist in many fields such as population dynamics, automatic control, drug administration, and communication networks [1-4]. In past two decades, its asymptotical behaviors such as stability and attractivity of the equilibrium point or periodical solutions have been deeply studied for impulsive functional differential equations (see, [5-18]).

However, under impulsive perturbation, the solutions may not be attracted to an equilibrium point or periodical trajectory but to some bounded region. In this case, it is interesting to investigate the attracting set and attracting basin, that is, the region attracting the solutions and the range in which initial values vary when remaining the attractivity for impulsive delay differential equations. In [19], Xu and Yang first give the method to estimate global attracting set and invariant set for impulsive delayed systems by developing delayed differential inequalities. The techniques are further developed to study global attractivity for some complex impulsive systems such as impulsive neutral differential equations [20, 21] and impulsive stochastic systems [22]. But the techniques and methods given in the existing publications are invalid for determining locally attracting set and attracting basin for impulsive delay differential equations.

In this paper, our objective is to mainly discuss the asymptotical behavior on (locally) attracting set and its attracting basin for a class of impulsive delay differential equations. Based on the quasi-invariant properties, we estimate the existence range of attracting set and attracting basin of the impulsive delay systems by solving algebraic equations and employing differential inequality technique. Examples are given to illustrate the effectiveness of our method and show that the asymptotic behavior of the impulsive systems may be different from one of the corresponding continuous systems.

## 2. Preliminaries

Let $N$ be the set of all positive integers, $R^{n}$ the space of $n$ dimensional real column vectors, and $R^{m \times n}$ the set of $m \times n$ real matrices. For $A, B \in R^{m \times n}$ or $A, B \in R^{n}, A \geq B(A \leq$ $B, A>B, A<B)$ means that each pair of corresponding elements of $A$ and $B$ satisfies the inequality " $\geq(\leq,>,<)$." $R_{+}^{n}=$ $\left\{x \in R^{n} \mid x \geq 0\right\}, E=(1,1, \ldots, 1)^{T} \in R^{n}$, and $I$ denotes an $n \times n$ unit matrix.

Let $\tau>0$ and $t_{0}<t_{1}<t_{2}<\cdots$ be the fixed points with $\lim _{k \rightarrow \infty} t_{k}=\infty$ (called impulsive moments).
$C[X, Y]$ denotes the space of continuous mappings from the topological space $X$ to the topological space $Y$. Let $C \stackrel{\Delta}{=}$ $C\left[[-\tau, 0], R^{n}\right]$ especially.

Morever, PC $\stackrel{\Delta}{=}\left\{\phi:[-\tau, 0] \rightarrow R^{n} \mid \phi\left(t^{+}\right)=\phi(t)\right.$ for $t \in[-\tau, 0), \phi\left(t^{-}\right)$exists for $t \in(-\tau, 0], \phi\left(t^{-}\right)=\phi(t)$ for all but at most a finite number of points $t \in(-\tau, 0]\}$. PC is a space of piecewise right-hand continuous functions which is a nature extension of the phrase space $C$.

We define PC[[t $\left.\left.t_{0}, \infty\right), R^{n}\right] \stackrel{\Delta}{=}\left\{\psi:\left[t_{0}, \infty\right) \rightarrow R^{n} \mid \psi(t)\right.$ is continuous at $t \neq t_{k}, \psi\left(t_{k}^{+}\right)$and $\psi\left(t_{k}^{-}\right)$exist, $\psi\left(t_{k}\right)=\psi\left(t_{k}^{+}\right)$, for $k \in N\}$.

For $x \in R^{n}, A \in R^{n \times n}, \phi \in C$ or $\phi \in \mathrm{PC}$, we define

$$
\begin{gather*}
{[x]^{+}=\left(\left|x_{1}\right|,\left|x_{2}\right|, \ldots,\left|x_{n}\right|\right)^{T},} \\
{[A]^{+}=\left(\left|a_{i j}\right|\right)_{n \times n}}  \tag{1}\\
{[\phi]_{\tau}^{+}=\left(\left\|\phi_{1}\right\|_{\tau},\left\|\phi_{2}\right\|_{\tau}, \ldots,\left\|\phi_{n}\right\|_{\tau}\right),}
\end{gather*}
$$

where $\left\|\phi_{i}\right\|_{\tau}=\sup _{s \in[-\tau, 0]}\left\|\phi_{i}(s)\right\|$ and $\|\cdot\|$ is an norm in $R^{n}$.
In this paper, we will consider a impulsive delay differential equations:

$$
\begin{align*}
\dot{x}(t) & =A x(t)+f\left(t, x_{t}\right), \quad t \neq t_{k}, t \geq t_{0}  \tag{2}\\
\Delta x & =B x\left(t_{k}^{-}\right)+I_{k}\left(t_{k}^{-}, x\left(t_{k}^{-}\right)\right), \quad k \in N
\end{align*}
$$

where $\dot{x}(t)$ denotes the right-hand derivative of $x(t), \Delta x=$ $x\left(t_{k}^{+}\right)-x\left(t_{k}^{-}\right), x\left(t_{k}\right)=x\left(t_{k}^{+}\right), A=\operatorname{diag}\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}, B=$ $\operatorname{diag}\left\{b_{1}, b_{2}, \ldots, b_{n}\right\}, f \in C\left[\left[t_{k-1}, t_{k}\right) \times \mathrm{PC}, R^{n}\right]$, and the limit $\lim _{(t, \phi) \rightarrow\left(t_{k}^{-}, \varphi\right)} f(t, \phi)=f\left(t_{k}^{-}, \varphi\right)$ exists, $I_{k} \in C\left[\left[t_{0}, \infty\right) \times\right.$ $\left.R^{n}, R^{n}\right]$, and $x_{t} \in \mathrm{PC}$ is defined by $x_{t}(s)=x(t+s), s \in[-\tau, 0]$.

A function $x(t):\left[t_{0}-\tau, \infty\right) \rightarrow R^{n}$ is called to be a solution of (2) through $\left(t_{0}, \phi\right)$, if $x(t) \in \mathrm{PC}\left[\left[t_{0}, \infty\right), R^{n}\right]$ as $t \geq t_{0}$, and satisfies (2) with the initial condition

$$
\begin{equation*}
x\left(t_{0}+s\right)=\phi(s), \quad s \in[-\tau, 0], \phi \in \mathrm{PC} \tag{3}
\end{equation*}
$$

Throughout the paper, we always assume that for any $\phi \in \mathrm{PC}$, system (2) has at least one solution through $\left(t_{0}, \phi\right)$, denoted by $x\left(t, t_{0}, \phi\right)$ or $x_{t}\left(t_{0}, \phi\right)$ (simply $x(t)$ and $x_{t}$ if no confusion should occur), where $x_{t}\left(t_{0}, \phi\right)=x\left(t+s, t_{0}, \phi\right) \in \mathrm{PC}, s \in$ $[-\tau, 0]$.

In this paper, we need the following definitions involving attracting set, attracting basin, the quasi-invariant set of impulsive systems, and monotonous vector functions.

Definition 1. The set $S \subset \mathrm{PC}$ is called to be an attracting set of (2), and $D \subset$ PC is called an attraction basin of $S$, if for any initial value $\phi \in D$, the solution $x_{t}\left(t_{0}, \phi\right)$ converges to $S$ as $t \rightarrow+\infty$. That is,

$$
\begin{equation*}
\operatorname{dist}\left(x_{t}\left(t_{0}, \phi\right), S\right) \longrightarrow 0, \quad \text { as } t \longrightarrow+\infty \tag{4}
\end{equation*}
$$

where $\operatorname{dist}(\varphi, S)=\inf _{\psi \in S} \operatorname{dist}(\varphi, \psi), \operatorname{dist}(\varphi, \psi)=\sup _{s \in[-\tau, 0]}$ $\|\varphi(s)-\psi(s)\|$, for $\varphi \in \mathrm{PC}$.

Definition 2. The set $D \subset \mathrm{PC}$ is called to be a positive quasiinvariant set of (2), if there is a positive diagonal matrix $L=\operatorname{diag}\left\{l_{i}\right\}$ such that for any initial value $\phi \in D$, the solutions $x_{t}\left(t_{0}, \phi\right)$ satisfy $L x_{t}\left(t_{0}, \phi\right) \in D$, for $t \geq t_{0}$. When $L=$ $I$ (identity matrix) especially, the set $D$ is called positively invariant.

Definition 3. Let $\Omega \subset R^{n}$. The vector function $F(x): \Omega \rightarrow$ $R^{n}$ is called to be monotonically nondecreasing in $\Omega$, if for any $x^{\prime}, x^{\prime \prime} \in \Omega, x^{\prime} \leq x^{\prime \prime}$ implies $F\left(x^{\prime}\right) \leq F\left(x^{\prime \prime}\right)$.

## 3. Main Results

In this paper, we always make the following assumptions.
$\left(H_{1}\right)$ There exist nonnegative constants $\theta, \varrho$ such that $0<$ $\theta \leq t_{k}-t_{k-1} \leq \varrho$, for $k \in N$.
$\left(H_{2}\right)[f(t, \varphi)]^{+} \leq p\left([\varphi]_{\tau}^{+}\right)$for $t \geq t_{0}$ and $\varphi \in \mathrm{PC}$, where the vector function $p(\cdot): R_{+}^{n} \rightarrow R_{+}^{n}$ is continuous and monotonically nondecreasing in $R_{+}^{n}$.
$\left(H_{3}\right)\left[I_{k}(t, x)\right]^{+} \leq q\left([x]^{+}\right)$, for $t \geq t_{0}, k \in N$ and $x \in$ $R^{n}$, where the vector function $q(\cdot): R_{+}^{n} \rightarrow R_{+}^{n}$ is continuous and monotonically nondecreasing in $R_{+}^{n}$.
To obtain attractivity, we first give the quasi-invariant properties of (2).

Theorem 4. Assume that in addition to $\left(H_{1}\right)-\left(H_{3}\right)$, there is a vector $z^{*} \geq 0$ such that

$$
\begin{equation*}
p\left(M z^{*}\right)+W\left[I-e^{-W \theta}\right]^{-1} q\left(M z^{*}\right)-W z^{*}<0 \tag{5}
\end{equation*}
$$

where $W=\operatorname{diag}\left\{w_{1}, \ldots, w_{n}\right\}, M=\operatorname{diag}\left\{m_{1}, \ldots, m_{n}\right\}, w_{i}>$ $0, m_{i} \geq 1, i=1,2, \ldots, n$, are defined by

$$
\begin{align*}
& w_{i}= \begin{cases}-a_{i}-\frac{\ln \left|1+b_{i}\right|}{\varrho}, & \text { if } 0<\left|1+b_{i}\right|<1, \\
-a_{i}-\frac{\ln \left|1+b_{i}\right|}{\theta}, & \text { if }\left|1+b_{i}\right| \geq 1,\end{cases}  \tag{6}\\
& m_{i}= \begin{cases}\frac{1}{\left|1+b_{i}\right|}, & \text { if } 0<\left|1+b_{i}\right|<1, \\
1, & \text { if }\left|1+b_{i}\right| \geq 1 .\end{cases}
\end{align*}
$$

Then, the set $D=\left\{\phi \in P C \mid[\phi]_{\tau}^{+} \leq z^{*}\right\}$ is a positive quasiinvariant set of (2). When $M=I$ especially, $D$ is a positive invariant set of (2).

Proof. Let $x(t)=x\left(t, t_{0}, \phi\right)$ be a solution of (2) through $\left(t_{0}, \phi\right)$. It is easily verified that the following formula for the variation of parameters is valid:

$$
\begin{align*}
x(t)= & K\left(t, t_{0}\right) \phi(0)+\int_{t_{0}}^{t} K(t, s) f\left(s, x_{s}\right) d s  \tag{7}\\
& +\sum_{t_{0}<t_{k} \leq t} K\left(t, t_{k}\right) I_{k}\left(t_{k}^{-}, x\left(t_{k}^{-}\right)\right), \quad t \geq t_{0}
\end{align*}
$$

where $K(t, s)$ is the Cauchy matrix of linear impulsive system

$$
\begin{cases}\dot{y}(t)=A y(t), & t \neq t_{k}  \tag{8}\\ \Delta y\left(t_{k}^{+}\right)=B y\left(t_{k}^{-}\right), & k \in N\end{cases}
$$

According to the representation of the Cauchy matrix (see page 74 [2]),

$$
\begin{equation*}
K(t, s)=e^{A(t-s)} \prod_{s<t_{k} \leq t}(I+B), \quad t \geq s \geq t_{0} \tag{9}
\end{equation*}
$$

Since $0<\theta \leq t_{k}-t_{k-1} \leq \varrho$, for $k \in N$, we obtain the following estimate:

$$
\begin{align*}
& \prod_{s<t_{k} \leq t}\left|1+b_{i}\right| \\
& \quad \leq \begin{cases}\left|1+b_{i}\right|^{((t-s) / \rho)-1}=\frac{1}{\left|1+b_{i}\right|} e^{\left(\ln \left|1+b_{i}\right| / \rho\right)(t-s)}, & \text { if } 0<\left|1+b_{i}\right|<1, \\
\left|1+b_{i}\right|^{(t-s) / \theta}=e^{\left(\ln \left|1+b_{i}\right| / \theta\right)(t-s)}, & \text { if }\left|1+b_{i}\right| \geq 1 .\end{cases} \tag{10}
\end{align*}
$$

In terms of the definition of $M$ and $W$,

$$
\begin{equation*}
[K(t, s)]^{+} \leq M e^{-W(t-s)}, \quad t \geq s \geq t_{0} \tag{11}
\end{equation*}
$$

By (7) and (11) and the assumptions $\left(H_{2}\right)$ and $\left(H_{3}\right)$, then

$$
\begin{align*}
{[x(t)]^{+} \leq } & M e^{-W\left(t-t_{0}\right)}[\phi]_{\tau}^{+}+M \int_{t_{0}}^{t} e^{-W(t-s)} p\left(\left[x_{s}\right]_{\tau}^{+}\right) d s  \tag{12}\\
& +M \sum_{t_{0}<t_{k} \leq t} e^{-W\left(t-t_{k}\right)} q\left(\left[x\left(t_{k}^{-}\right)\right]^{+}\right), \quad t \geq t_{0}
\end{align*}
$$

Since $t_{k}-t_{k-1} \geq \theta>0$ and $W=\operatorname{diag}\left\{w_{1}, \ldots, w_{n}\right\}>0$, we have

$$
\begin{align*}
\sum_{t_{0}<t_{k} \leq t} e^{-W\left(t-t_{k}\right)} & =\sum_{t_{0}<t_{k} \leq t} e^{-W\left(t-t_{k}\right)}\left[I-e^{-W \theta}\right]\left[I-e^{-W \theta}\right]^{-1} \\
& \leq \sum_{t_{0}<t_{k} \leq t} e^{-W\left(t-t_{k}\right)}\left[I-e^{-W\left(t_{k}-t_{k-1}\right)}\right]\left[I-e^{-W \theta}\right]^{-1} \\
& =\sum_{t_{0}<t_{k} \leq t}\left[e^{-W\left(t-t_{k}\right)}-e^{-W\left(t-t_{k-1}\right)}\right]\left[I-e^{-W \theta}\right]^{-1} \\
& \leq\left[I-e^{-W\left(t-t_{0}\right)}\right]\left[I-e^{-W \theta}\right]^{-1} \tag{13}
\end{align*}
$$

From the strict inequality (5), there is an enough small number $\varepsilon>0$ such that

$$
\begin{gather*}
p(M z)+W\left[I-e^{-W \theta}\right]^{-1} q(M z)-W z<0 \\
z \stackrel{\Delta}{=} z^{*}+\varepsilon E>0 \tag{14}
\end{gather*}
$$

In the following, we will prove that $[\phi]_{\tau}^{+}<z$ implies

$$
\begin{equation*}
[x(t)]^{+}=\left[x\left(t, t_{0}, \phi\right)\right]^{+}<M z, \quad t \geq t_{0} \tag{15}
\end{equation*}
$$

Otherwise, from the piecewise continuity of $x(t)$, there must be an integer $i$ and $t^{*}>t_{0}$ such that

$$
\begin{gather*}
\left|x_{i}\left(t^{*}\right)\right| \geq m_{i} z_{i}  \tag{16}\\
{[x(t)]^{+} \leq M z, \quad t_{0} \leq t<t^{*}} \tag{17}
\end{gather*}
$$

By using (12), (13), (14), (17), $W>0$, and the monotonicity of $p(\cdot), q(\cdot)$, we can get

$$
\begin{align*}
{\left[x\left(t^{*}\right)\right]^{+} \leq } & e^{-W\left(t^{*}-t_{0}\right)} M[\phi]_{\tau}^{+}+M \int_{t_{0}}^{t^{*}} e^{-W\left(t^{*}-s\right)} p(M z) d s \\
& +M \sum_{t_{0}<t_{k} \leq t^{*}} e^{-W\left(t^{*}-t_{k}\right)} q(M z) \\
< & e^{-W\left(t^{*}-t_{0}\right)} M z+M\left(I-e^{-W\left(t^{*}-t_{0}\right)}\right) W^{-1} p(M z) \\
& +M\left(I-e^{-W\left(t^{*}-t_{0}\right)}\right)\left[I-e^{-W \theta}\right]^{-1} q(M z) \\
= & e^{-W\left(t^{*}-t_{0}\right)} M W^{-1} \\
& \times\left[W z-p(M z)-W\left[I-e^{-W \theta}\right]^{-1} q(M z)\right] \\
& +W W^{-1} M p(M z)+M\left[I-e^{-W \theta}\right]^{-1} q(M z) \\
< & M W^{-1}\left[W z-p(M z)-W\left[I-e^{-W \theta}\right]^{-1} q(M z)\right] \\
& +W^{-1} M p(M z)+M\left[I-e^{-W \theta}\right]^{-1} q(M z) \\
= & M z . \tag{18}
\end{align*}
$$

This contradicts (16), and so (15) holds. Letting $\varepsilon \rightarrow 0$, from (15), we have for any $\phi \in D$ (i.e., $[\phi]_{\tau}^{+} \leq z^{*}$ ),

$$
\begin{align*}
{\left[x\left(t, t_{0}, \phi\right)\right]^{+} \leq M z^{*}, \quad \text { that is, }\left[M^{-1} x\left(t, t_{0}, \phi\right)\right]^{+} } & \leq z^{*} \\
& t \geq t_{0} \tag{19}
\end{align*}
$$

Therefore, the set $D=\left\{\phi \in \mathrm{PC} \mid[\phi]_{\tau}^{+} \leq z^{*}\right\}$ is a positive quasi-invariant set of (2). When $M=I$ especially, $D$ is a positive invariant set of (2). The proof is complete.

Based on the obtained quasi-invariant set, we have the following

## Theorem 5. Let

$$
\begin{equation*}
\Delta(z)=p(z)+W\left[I-e^{-W \theta}\right]^{-1} q(z)-M^{-1} W z, \quad z \in R_{+}^{n} \tag{20}
\end{equation*}
$$

Assume that all conditions in Theorem 4 hold. Define

$$
\begin{align*}
\Omega_{1} & =\left\{z \in R_{+}^{n} \mid \Delta(M z)<0\right\} \\
\Omega_{2} & =\left\{z \in R_{+}^{n} \mid \Delta(z)<0\right\} \\
\Omega_{3} & =\left\{z \in R_{+}^{n} \mid \Delta(z) \geq 0\right\}  \tag{21}\\
\Omega_{1}^{*} & =\bigcup_{z^{*} \in \Omega_{1}}\left\{z \in R_{+}^{n} \mid z \leq z^{*}\right\} \\
\Omega_{2}^{*} & =\bigcup_{z^{*} \in \Omega_{2}}\left\{z \in R_{+}^{n} \mid z \leq z^{*}\right\}
\end{align*}
$$

Then, $S=\left\{\phi \in P C \mid[\phi]_{\tau}^{+} \in \Omega_{2}^{*} \cap \Omega_{3}\right\}$ is an attracting set of (2) and $D=\left\{\phi \in P C \mid[\phi]_{\tau}^{+} \in \Omega_{1}^{*}\right\}$ is the attracting basin of $S$.

Proof. From (5) and the definitions of the above sets, then $z^{*} \in \Omega_{1}, M z^{*} \in \Omega_{2}, 0 \in \Omega_{1}^{*}, 0 \in \Omega_{2}^{*}, 0 \in \Omega_{3}$. Obviously, $\Omega_{1}, \Omega_{2}, \Omega_{1}^{*}, \Omega_{2}^{*}, \Omega_{3}$ and $\Omega_{2}^{*} \cap \Omega_{3}$ are nonempty, and so the definitions of the sets of $S$ and $D$ are valid. For any $\phi \in D$, there is a $z^{*} \in \Omega_{1}$ satisfying $[\phi]_{\tau}^{+} \leq z^{*}$. According to Theorem 4, we obtain

$$
\begin{equation*}
[x(t)]^{+}=\left[x\left(t, t_{0}, \phi\right)\right]^{+} \leq M z^{*} \in \Omega_{2}, \quad \forall t \geq t_{0} . \tag{22}
\end{equation*}
$$

That is,

$$
\begin{equation*}
\sigma \stackrel{\Delta}{=} \limsup _{t \rightarrow \infty}[x(t)]^{+} \in \Omega_{2}^{*} \tag{23}
\end{equation*}
$$

Then, for any given $\varepsilon>0$, there is a $T_{1}>t_{0}$ such that

$$
\begin{equation*}
[x(t)]^{+} \leq \varepsilon E+\sigma, \quad t \geq T_{1} . \tag{24}
\end{equation*}
$$

In light of $W=\operatorname{diag}\left\{w_{i}\right\}>0$, for the above $\varepsilon>0$ and $T_{1}$, we can find an enough large $T_{2}>0$ such that

$$
\begin{gather*}
\int_{T_{2}}^{\infty} e^{-W s} d s \leq \varepsilon I \\
\sum_{t_{0}<t_{k} \leq T_{1}} e^{-W\left(t-t_{k}\right)} \leq \varepsilon I, \quad t>T_{2} . \tag{25}
\end{gather*}
$$

Using (12), (13), (22), (24), and (25), we have for $t \geq \tau+T_{1}+T_{2}$,

$$
\begin{aligned}
{[x(t)]^{+} \leq } & e^{-W\left(t-t_{0}\right)} M[\phi]_{\tau}^{+}+\int_{t_{0}}^{t} e^{-W(t-s)} M p\left(\left[x_{s}\right]_{\tau}^{+}\right) d s \\
& +\sum_{t_{0}<t_{k} \leq t} M e^{-W\left(t-t_{k}\right)} q\left(\left[x\left(t_{k}^{-}\right)\right]^{+}\right) \\
\leq & e^{-W\left(t-t_{0}\right)} M[\phi]_{\tau}^{+} \\
& +\left\{\int_{t_{0}}^{t-T_{2}}+\int_{t-T_{2}}^{t}\right\} e^{-W(t-s)} M p\left(\left[x_{s}\right]_{\tau}^{+}\right) d s \\
& +\left\{\sum_{t_{0}<t_{k} \leq T_{1}}+\sum_{T_{1}<t_{k} \leq t}\right\} M e^{-W\left(t-t_{k}\right)} q\left(\left[x\left(t_{k}^{-}\right)\right]^{+}\right) \\
\leq & e^{-W\left(t-t_{0}\right)} M z^{*}+\int_{T_{2}}^{\infty} e^{-W s} M p\left(M z^{*}\right) d s
\end{aligned}
$$

$$
\begin{align*}
& +\int_{t-T_{2}}^{t} e^{-W(t-s)} M p(\varepsilon E+\sigma) d s \\
& +M \sum_{t_{0}<t_{k} \leq T_{1}} e^{-W\left(t-t_{k}\right)} q\left(M z^{*}\right) \\
& +M \sum_{T_{1}<t_{k} \leq t} e^{-W\left(t-t_{k}\right)} q(\varepsilon E+\sigma) \\
\leq & e^{-W\left(t-t_{0}\right)} M z^{*}+\varepsilon M p\left(M z^{*}\right) \\
& +\left(I-e^{-W T_{2}}\right) W^{-1} M p(\varepsilon E+\sigma)+\varepsilon M q\left(M z^{*}\right) \\
& +M\left(I-e^{-W\left(t-T_{1}\right)}\right)\left[I-e^{-W \theta}\right]^{-1} q(\varepsilon E+\sigma) \\
\leq & e^{-W\left(t-t_{0}\right)} M z^{*}+\varepsilon M\left[p\left(M z^{*}\right)+q\left(M z^{*}\right)\right] \\
& +W^{-1} M p(\varepsilon E+\sigma)+M\left[I-e^{-W \theta}\right]^{-1} q(\varepsilon E+\sigma) . \tag{26}
\end{align*}
$$

This implies that

$$
\begin{align*}
\sigma= & \limsup _{t \rightarrow+\infty}[x(t)]^{+} \leq \varepsilon M\left[p\left(M z^{*}\right)+q\left(M z^{*}\right)\right]  \tag{27}\\
& +W^{-1} M p(\varepsilon E+\sigma)+M\left[I-e^{-W \theta}\right]^{-1} q(\varepsilon E+\sigma) .
\end{align*}
$$

Letting $\epsilon \rightarrow 0^{+}$, then

$$
\begin{equation*}
\sigma \leq W^{-1} M p(\sigma)+M\left[I-e^{-W \theta}\right]^{-1} q(\sigma) \tag{28}
\end{equation*}
$$

That is, $\Delta(\sigma) \geq 0$ and $\sigma \in \Omega_{3}$. Thus,

$$
\begin{equation*}
\sigma \in \Omega_{2}^{*} \cap \Omega_{3} \tag{29}
\end{equation*}
$$

From the definition of $\sigma$ and $S$, dist $\left(x_{t}\left(t_{0}, \phi\right), S\right) \rightarrow 0$ as $t \rightarrow$ $+\infty$. The proof is complete.

From the above theorems, we can obtain sufficient conditions ensuring global attractivity and stability in the following corollaries.

Corollary 6. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ hold with

$$
\begin{gather*}
p\left([\varphi]_{\tau}^{+}\right)=P[\varphi]_{\tau}^{+}+\mu, \\
P=\left(p_{i j}\right)_{n \times n} \geq 0, \quad \mu=\left(\mu_{1}, \ldots, \mu_{n}\right)^{T} \geq 0,  \tag{30}\\
q\left([x]^{+}\right)=Q[x]^{+}+\nu, \\
Q=\left(q_{i j}\right)_{n \times n} \geq 0, \quad v=\left(v_{1}, \ldots, v_{n}\right)^{T} \geq 0 .
\end{gather*}
$$

If the spectral radius

$$
\begin{equation*}
\rho(\Lambda)<1, \quad \text { where } \Lambda=W^{-1} M P+M\left[I-e^{-\theta W}\right]^{-1} Q \tag{31}
\end{equation*}
$$

then $D=\left\{\phi \in P C \mid[\phi]_{\tau}^{+} \leq Z \triangleq(I-\Lambda)^{-1} W^{-1}(\mu+W[I-\right.$ $\left.\left.\left.e^{-\theta W}\right]^{-1} \nu\right)\right\}$ is a positive quasi-invariant set of (2), and $S=\{\phi \in$ $\left.P C \mid[\phi]_{\tau}^{+} \leq(I-\Lambda)^{-1} W^{-1} M\left(\mu+W\left[I-e^{-\theta W}\right]^{-1} \nu\right)\right\}$ is a global attracting set of (2).

Proof. Since $p(z)=P z+\mu$ and $q(z)=Q z+\nu$, we directly calculate

$$
\begin{gather*}
\Delta(z)=M^{-1} W(\Lambda-I) z+\mu+W\left[I-e^{-\theta W}\right]^{-1} \nu \\
\Delta(M z)=W(\Lambda-I) z+\mu+W\left[I-e^{-\theta W}\right]^{-1} \nu \tag{32}
\end{gather*}
$$

Without loss of generality, we assume that $\mu, \nu>0$. Since $\rho(\Lambda)<1,(I-\Lambda)^{-1}$ exists and $(I-\Lambda)^{-1} \geq 0$ (see [23]), and so $Z>0$. For any $\kappa>0$, we take $z^{*}=(1+\kappa) Z>0$ in Theorem 4 and verify the condition (5):

$$
\begin{equation*}
\Delta\left(M z^{*}\right)=-\kappa\left(\mu+W\left[I-e^{-\theta W}\right]^{-1} \nu\right)<0 \tag{33}
\end{equation*}
$$

According to Theorem 4 , when $\kappa \rightarrow 0$, we deduce that $D$ is a positive quasi-invariant set of (2). Furthermore, by (33),

$$
\begin{equation*}
(1+\kappa) Z \in \Omega_{1}, \quad(1+\kappa) M Z \in \Omega_{2} \tag{34}
\end{equation*}
$$

From the arbitrariness of $\mathcal{\kappa}$, we obtain $\Omega_{1}^{*}=\Omega_{2}^{*}=R_{+}^{n}$. Moreover,

$$
\begin{align*}
\Omega_{3}= & \left\{z \in R_{+}^{n} \mid \Delta(z) \geq 0\right\} \\
= & \left\{z \in R_{+}^{n} \mid M^{-1} W(\Lambda-I) z+\mu+W\left[I-e^{-\theta W}\right]^{-1} v \geq 0\right\} \\
= & \left\{z \in R_{+}^{n} \mid(I-\Lambda) z \leq W^{-1} M\left(\mu+W\left[I-e^{-\theta W}\right]^{-1} \nu\right)\right\} \\
\subset & \left\{z \in R_{+}^{n} \mid z \leq(I-\Lambda)^{-1} W^{-1} M\right. \\
& \left.\times\left(\mu+W\left[I-e^{-\theta W}\right]^{-1} \nu\right)\right\} . \tag{35}
\end{align*}
$$

It follows from Theorem 5 that $S^{\prime}=\left\{\phi \in \mathrm{PC} \mid[\phi]_{\tau}^{+} \in \Omega_{3}\right\}$ is a global attracting set of (2) and $S$ is also a global attracting set due to $S^{\prime} \subset S$. The proof is complete.

Corollary 7. Assume that all conditions in Corollary 6 hold with $\mu=\nu=0$. Then, the zero solution $x(t)=0$ of (2) is globally asymptotically stable.

## 4. Illustrative Examples

The following illustrative examples will demonstrate the effectiveness of our results and also show the different asymptotical behaviors between the impulsive system and the corresponding continuous system.

Example 8. Consider a scalar nonlinear impulsive delay system

$$
\begin{array}{r}
\dot{x}(t)=0.2 x(t)+0.2 x^{2}(t-1)+0.1 \\
t \neq t_{k}, k \in N, t \geq t_{0}=0 \\
\Delta x=-0.6 x\left(t_{k}^{-}\right)+0.1 x^{2}\left(t_{k}^{-}\right)+0.1 \sin \left(e^{t_{k}}\right) \\
t_{k}=t_{k-1}+0.15
\end{array}
$$

According to Theorems 4 and 5, we have $\theta=\varrho=0.15, A=$ $0.2, B=-0.6, M=2.5, W \doteq 5.9086, p(z)=0.2 z^{2}+0.1$, $q(z)=0.1 z^{2}+0.1, \Delta(z) \doteq 1.2052 z^{2}-2.3634 z+1.1052$, and so

$$
\begin{gather*}
\Omega_{1}=\left\{z \in R_{+} \mid \Delta(M z)<0\right\}=(0.3079,0.4765), \\
\Omega_{1}^{*}=[0,0.4765] \\
\Omega_{2}=\left\{z \in R_{+} \mid \Delta(z)<0\right\}=(0.7698,1.1913), \\
\Omega_{2}^{*}=[0,1.1913], \\
\Omega_{3}=\left\{z \in R_{+} \mid \Delta(z) \geq 0\right\}=[0,0.7698] \cup[1.1913,+\infty), \\
\Omega_{2}^{*} \cap \Omega_{3}=[0,0.7698] . \tag{37}
\end{gather*}
$$

Thus, $S=\left\{\phi \in \mathrm{PC} \mid[\phi]_{\tau}^{+} \leq 0.7698\right\}$ is an attracting set of (36), and $D=\left\{\phi \in \mathrm{PC} \mid[\phi]_{\tau}^{+} \leq 0.4765\right\}$ is an attracting basin of $S$. However, solutions of the corresponding continuous system (i.e., $\Delta x=0$ in (36)) may be unbounded. Taking the initial condition $\phi(s)=0.2, s \in[-1,0]$, Figure 1 shows the different asymptotic behavior between the solution of (36) with no impulse and one with impulses.

Example 9. Consider a 2-dimensional impulsive delay system

$$
\begin{array}{r}
\dot{x}_{1}(t)=x_{1}(t)+0.5 \sin \left(x_{1}(t-1)\right)-0.4 x_{2}(t-1)-0.5, \\
t \geq 0, \\
\dot{x}_{2}(t)=-4 x_{2}(t)-0.5 x_{1}(t-1)+0.4 \cos \left(x_{2}(t-1)\right)+0.5, \\
t \neq t_{k}, \\
\Delta x_{1}=-0.5 x_{1}\left(t_{k}^{-}\right)+0.1 \cos \left(x_{1}\left(t_{k}^{-}\right)\right)+0.5 \sin \left(e^{t_{k}}\right), \\
t_{k}=0.1 k, \\
\Delta x_{2}=0.1 x_{2}\left(t_{k}^{-}\right)+0.2 \sin \left(x_{2}\left(t_{k}^{-}\right)\right)-0.5 \cos \left(e^{t_{k}}\right), \\
k \in N . \tag{38}
\end{array}
$$

According to Corollary 6, we have $\theta=\varrho=0.1$, $A=\operatorname{diag}\{1,-4\}, \quad B=\operatorname{diag}\{-0.5,0.1\}, \quad M=\operatorname{diag}\{2,1\}$, $W=\operatorname{diag}\{5.9315,3.0469\}, p(z)=P z+\mu, q(z)=Q z+\nu$, $\Lambda=W^{-1} M P+M\left[I-e^{-\theta W}\right]^{-1} Q$, where

$$
\begin{gather*}
P=\left(\begin{array}{ll}
0.5 & 0.4 \\
0.5 & 0.4
\end{array}\right), \quad Q=\left(\begin{array}{cc}
0.1 & 0 \\
0 & 0.2
\end{array}\right) \\
\mu=\nu=\binom{0.5}{0.5}, \quad \Lambda=\left(\begin{array}{cc}
0.6156 & 0.1349 \\
0.1641 & 0.8928
\end{array}\right), \tag{39}
\end{gather*}
$$

and so $\rho(\Lambda)=0.9575<1$. Therefore, $D=\{\phi \in$ PC | $\left.[\phi]_{\tau}^{+} \leq(1.7105,2.5228)^{T}\right\}$ is a positive quasi-invariant set of (38), and $S=\left\{\phi \in \mathrm{PC} \mid[\phi]_{\tau}^{+} \leq(1.8637,2.8720)^{T}\right\}$ is a global attracting set of (38). Figure 2 shows the asymptotic properties of solutions of (38) under the different initial conditions.


Figure 1: The trajectory of (36) with: (a) no impulse (i.e., $\Delta x=0$ ) and (b) impulses.


Figure 2: Global attracting set of (38).

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## Research Article

# Periodic Solution for Impulsive Cellar Neural Networks with Time-Varying Delays in the Leakage Terms 

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#### Abstract

This paper is concerned with impulsive cellular neural networks with time-varying delays in leakage terms. Without assuming bounded and monotone conditions on activation functions, we establish sufficient conditions on existence and exponential stability of periodic solutions by using Lyapunov functional method and differential inequality techniques. Our results are complement to some recent ones.


## 1. Introduction

It is well known that impulsive differential equations are mathematical apparatus for simulation of process and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, and so forth [1-3]. Thus, many neural networks with impulses have been studied extensively, and a great deal of literature is focused on the existence and stability of an equilibrium point [4-7]. In [8-10], the authors discussed the existence and global exponential stability of periodic solution of a class of cellular neural networks (CNNs) with impulse. Recently, Wang et al. [11] considered the following CNNs with impulses and leakage delays:

$$
\begin{array}{r}
x_{i}^{\prime}(t)=-a_{i} x_{i}\left(t-\tau_{i}\right)+\sum_{j=1}^{n} \alpha_{i j}(t) f_{j}\left(x_{j}(t)\right) \\
+\sum_{j=1}^{n} \beta_{i j}(t) f_{j}\left(x_{j}\left(t-\sigma_{i j}\right)\right)+I_{i}(t),  \tag{1}\\
t \geq 0, \quad t \neq t_{k} \\
\Delta x_{i}\left(t_{k}\right)=x_{i}\left(t_{k}^{+}\right)-x_{i}\left(t_{k}^{-}\right)=d_{i k} x_{i}\left(t_{k}\right)
\end{array}
$$

where $\Delta x_{i}\left(t_{k}\right)$ are the impulses at moments $t_{k}$ and $t_{1}<t_{2}<$ $\cdots$ is a strictly increasing sequence such that $\lim _{k \rightarrow \infty} t_{k}=$ $+\infty ; a_{i}>0$ and $\tau_{i}>0$ are constants, and $\alpha_{i j}(t), I_{i}(t)$, and $\beta_{i j}(t)$
are continuous periodic functions with period $T$. Suppose that the following conditions are satisfied.
$\left(A_{1}\right)$ There exist constants $L_{j}^{f}, j=1,2, \ldots n$, such that, for any $\alpha, \beta \in R$,

$$
\begin{equation*}
0<\frac{f_{j}(\alpha)-f_{j}(\beta)}{\alpha-\beta}<L_{j}^{f}, \quad \alpha \neq \beta, \quad j=1,2, \ldots n \tag{2}
\end{equation*}
$$

$\left(A_{2}\right) f_{i}(0)=0$ and for $i=1,2, \ldots, n$, there exists a constant $0<M_{i}<+\infty$, such that

$$
\begin{equation*}
\left|f_{i}(\alpha)\right| \leq M_{i}, \quad \alpha \in R \tag{3}
\end{equation*}
$$

By using the continuation theorem of coincidence degree theory and a suitable degenerate Lyapunov-Krasovskii functional together with model transformation technique, some results were obtained in [11] to guarantee that all solutions of system (1) converge exponentially to a periodic function. However, to the best of our knowledge, few authors have considered the existence and stability of periodic solutions of system (1) without the assumptions $\left(A_{1}\right)$ and $\left(A_{2}\right)$. Thus, it is worthwhile to continue to investigate the convergence behavior of system (1) in this case. In view of the fact that the coefficients and delays in neural networks are usually time varying in the real world, motivated by the above discussions,
in this paper, we will consider the problem on periodic solution of the following impulsive CNNs with time-varying delays in the leakage terms:

$$
\begin{gather*}
x_{i}^{\prime}(t)=-c_{i}(t) x_{i}\left(t-\eta_{i}(t)\right)+\sum_{j=1}^{n} a_{i j}(t) f_{j}\left(x_{j}(t)\right) \\
+\sum_{j=1}^{n} b_{i j}(t) g_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)+I_{i}(t)  \tag{4}\\
t \geq 0, \quad t \neq t_{k} \\
\Delta x_{i}\left(t_{k}\right)=x_{i}\left(t_{k}^{+}\right)-x_{i}\left(t_{k}^{-}\right)=d_{i k} x_{i}\left(t_{k}\right)
\end{gather*}
$$

in which $n$ corresponds to the number of units in a neural network, $x_{i}(t)$ corresponds to the state vector of the $i$ th unit at the time $t$, and $c_{i}(t)$ represents the rate with which the $i$ th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at the time $t . a_{i j}(t)$ and $b_{i j}(t)$ are the connection weights at the time $t, \eta_{i}(t)$ and $\tau_{i j}(t)$ denote the transmission delays, $I_{i}(t)$ denotes the external bias on the $i$ th unit at the time $t, f_{j}$ and $g_{j}$ are activation functions of signal transmission, $\Delta x_{i}\left(t_{k}\right)$ are the impulses at moments $t_{k}$, and $0 \leq t_{1}<t_{2}<\cdots$ is a strictly increasing sequence such that $\lim _{k \rightarrow \infty} t_{k}=+\infty$, and $i, j=$ $1,2, \ldots, n$. It is obvious that when $f=g$ and $\eta_{i}(t)$ is a constant function, (1) is a special case of (4).

The main purpose of this paper is to give the conditions for the existence and exponential stability of the periodic solutions for system (4). By applying Lyapunov functional method and differential inequality techniques, without assuming $\left(A_{1}\right)$ and ( $A_{2}$ ), we derive some new sufficient conditions ensuring the existence, uniqueness, and exponential stability of the periodic solution for system (4), which are new and complement previously known results. Moreover, an example is also provided to illustrate the effectiveness of our results.

Throughout this paper, we assume that the following conditions hold.
$\left(H_{1}\right)$ For $i, j=1,2, \ldots, n, I_{i}, a_{i j}, b_{i j}: R \rightarrow R$ and $c_{i}, \eta_{i}$, $\tau_{i j}: R \rightarrow R^{+}$are continuous periodic functions with period $T>0$, and $t-\eta_{i}(t) \geq 0$ for all $t \geq 0$. In addition, there exist constants $c_{i}^{+}, \eta_{i}^{+}, I_{i}^{+}, \tau_{i}, a_{i j}^{+}, b_{i j}^{+}$, and $\tau_{i j}^{+}$such that

$$
\begin{array}{cc}
c_{i}^{+}=\sup _{t \in R} c_{i}(t), & \eta_{i}^{+}=\sup _{t \in R} \eta_{i}(t), \\
I_{i}^{+}=\sup _{t \in R}\left|I_{i}(t)\right|, & a_{i j}^{+}=\sup _{t \in R}\left|a_{i j}(t)\right|, \\
b_{i j}^{+}=\sup _{t \in R}\left|b_{i j}(t)\right|, & \tau_{i j}^{+}=\sup _{t \in R} \tau_{i j}(t),  \tag{5}\\
\tau_{i}=\max \left\{\eta_{i}^{+}, \max _{j=1,2, \ldots, n} \tau_{j i}^{+}\right\} .
\end{array}
$$

$\left(H_{2}\right)$ The sequence of times $t_{k}(k \in N)$ satisfies $t_{k}<t_{k+1}$ and $\lim _{k \rightarrow+\infty} t_{k}=+\infty$, and $d_{i k}$ satisfies $-2 \leq d_{i k} \leq 0$ for $i \in\{1,2, \ldots, n\}$ and $k \in Z^{+}$, where $Z^{+}$denotes the set of all positive integers.
$\left(H_{3}\right)$ There exists a $q \in Z^{+}$such that

$$
\begin{equation*}
d_{i(k+q)}=d_{i k}, \quad t_{k+q}=t_{k}+T, \quad\left(k \in Z^{+}\right) \tag{6}
\end{equation*}
$$

$\left(H_{4}\right)$ For each $j \in\{1,2, \ldots, n\}$, there exist nonnegative constants $L_{j}^{f}$ and $L_{j}^{g}$ such that, for all $u, v \in R$,

$$
\begin{gather*}
g_{j}(0)=f_{j}(0)=0, \quad\left|g_{j}(u)-g_{j}(v)\right| \leq L_{j}^{g}|u-v|  \tag{7}\\
\left|f_{j}(u)-f_{j}(v)\right| \leq L_{j}^{f}|u-v|
\end{gather*}
$$

$\left(H_{5}\right)$ For all $t>0$ and $i \in\{1,2, \ldots, n\}$, there exist constants $\dot{\xi}_{i}>0$ and $\eta>0$ such that

$$
\begin{align*}
-\eta> & -\left[c_{i}(t)-c_{i}(t) \eta_{i}(t) c_{i}^{+}\right] \xi_{i} \\
& +\sum_{j=1}^{n}\left(\left|a_{i j}(t)\right|+c_{i}(t) \eta_{i}(t) a_{i j}^{+}\right) L_{j}^{f} \xi_{j}  \tag{8}\\
& +\sum_{j=1}^{n}\left(\left|b_{i j}(t)\right|+c_{i}(t) \eta_{i}(t) b_{i j}^{+}\right) L_{j}^{g} \xi_{j} .
\end{align*}
$$

For convenience, let $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)^{T} \in R^{n}$, in which " $T$ " denotes the transposition. We define $|x|=$ $\left(\left|x_{1}\right|,\left|x_{2}\right|, \ldots,\left|x_{n}\right|\right)^{T}$ and $\|x\|=\max _{1 \leq i \leq n}\left\{\left|x_{i}\right|\right\}$. As usual in the theory of impulsive differential equations, at the points of discontinuity $t_{k}$ of the solution $t \mapsto\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T}$, we assume that $\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T} \equiv\left(x_{1}(t-0), x_{2}(t-\right.$ $\left.0), \ldots, x_{n}(t-0)\right)^{T}$. It is clearly that, in general, the derivative $x_{i}^{\prime}\left(t_{k}\right)$ does not exist. On the other hand, according to system (4), there exists the limit $x_{i}^{\prime}\left(t_{k} \mp 0\right)$. In view of the above convention, we assume that $x_{i}^{\prime}\left(t_{k}\right) \equiv x_{i}^{\prime}\left(t_{k}-0\right)$.

The initial conditions associated with (4) are assumed to be of the form

$$
\begin{equation*}
x_{i}(s)=\phi_{i}(s), \quad s \in\left[-\tau_{i}, 0\right], i=1,2, \ldots, n \tag{9}
\end{equation*}
$$

where $\phi_{i}(\cdot)$ denotes a real-valued continuous function defined on $\left[-\tau_{i}, 0\right]$.

## 2. Preliminary Results

The following lemmas will be used to prove our main results in Section 3.

Lemma 1. Let $\left(H_{1}\right)-\left(H_{5}\right)$ hold. Suppose that $x(t)=\left(x_{1}(t)\right.$, $\left.x_{2}(t), \ldots, x_{n}(t)\right)^{T}$ is a solution of system (1) with the initial conditions

$$
\begin{equation*}
x_{i}(s)=\varphi_{i}(s), \quad\left|\varphi_{i}(s)\right|<\xi_{i} \frac{\gamma}{\eta}, \quad s \in\left[-\tau_{i}, 0\right] \tag{10}
\end{equation*}
$$

where $\gamma=1+\max _{i=1,2, \ldots, n}\left\{\left[c_{i}^{+} \eta_{i}^{+}+1\right] I_{i}^{+}\right\}, i=1,2, \ldots, n$. Then

$$
\begin{equation*}
\left|x_{i}(t)\right|<\xi_{i} \frac{\gamma}{\eta}, \quad \forall t \geq 0, \quad i=1,2, \ldots, n \tag{11}
\end{equation*}
$$

Proof. Assume that (11) does not hold. From $\left(H_{2}\right)$, we have

$$
\begin{equation*}
\left|x_{i}\left(t_{k}^{+}\right)\right|=\left|\left(1+d_{i k}\right)\right|\left|x_{i}\left(t_{k}\right)\right| \leq\left|x_{i}\left(t_{k}\right)\right| . \tag{12}
\end{equation*}
$$

So, if $\left|x_{i}\left(t_{k}^{+}\right)\right|>\gamma$, then $\left|x_{i}\left(t_{k}\right)\right|>\gamma$. Thus, we may assume that there exist $i \in\{1,2, \ldots, n\}$ and $t_{*} \in\left(t_{k}, t_{k+1}\right)$ such that

$$
\begin{gather*}
\left|x_{i}\left(t_{*}\right)\right|=\xi_{i} \frac{\gamma}{\eta}, \quad\left|x_{j}(t)\right|<\xi_{j} \frac{\gamma}{\eta}  \tag{13}\\
\forall t \in\left[-\tau_{i}, t_{*}\right), \quad j=1,2, \ldots, n .
\end{gather*}
$$

According to (4), we get

$$
\begin{align*}
& x_{i}^{\prime}(t)=-c_{i}(t) x_{i}\left(t-\eta_{i}(t)\right)+\sum_{j=1}^{n} a_{i j}(t) f_{j}\left(x_{j}(t)\right) \\
&+\sum_{j=1}^{n} b_{i j}(t) g_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)+I_{i}(t) \\
&=-c_{i}(t) x_{i}(t)+c_{i}(t)\left[x_{i}(t)-x_{i}\left(t-\eta_{i}(t)\right)\right] \\
&+\sum_{j=1}^{n} a_{i j}(t) f_{j}\left(x_{j}(t)\right) \\
&+\sum_{j=1}^{n} b_{i j}(t) g_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)+I_{i}(t)  \tag{14}\\
&=-c_{i}(t) x_{i}(t)+c_{i}(t) \int_{t-\eta_{i}(t)}^{t} x_{i}^{\prime}(s) d s \\
&+\sum_{j=1}^{n} a_{i j}(t) f_{j}\left(x_{j}(t)\right) \\
&+\sum_{j=1}^{n} b_{i j}(t) g_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)+I_{i}(t), \\
& \quad t>0, \quad t \neq t_{k}, \quad i=1,2, \ldots, n .
\end{align*}
$$

Calculating the upper left derivative of $\left|x_{i}(t)\right|$, together with (13), (14), ( $\mathrm{H}_{5}$ ), and

$$
\begin{equation*}
\gamma>\left[c_{i}^{+} \eta_{i}^{+}+1\right] I_{i}^{+} \tag{15}
\end{equation*}
$$

we obtain

$$
\begin{aligned}
0 \leq & D^{-}\left|x_{i}\left(t_{*}\right)\right| \\
\leq & -c_{i}\left(t_{*}\right)\left|x_{i}\left(t_{*}\right)\right|+c_{i}\left(t_{*}\right) \int_{t_{*}-\eta_{i}\left(t_{*}\right)}^{t_{*}}\left|x_{i}^{\prime}(s)\right| d s \\
& +\sum_{j=1}^{n}\left|a_{i j}\left(t_{*}\right) f_{j}\left(x_{j}\left(t_{*}\right)\right)\right| \\
& +\sum_{j=1}^{n}\left|b_{i j}\left(t_{*}\right)\right|\left|g_{j}\left(x_{j}\left(t_{*}-\tau_{i j}\left(t_{*}\right)\right)\right)\right|+\left|I_{i}\left(t_{*}\right)\right|
\end{aligned}
$$

$$
\begin{align*}
& =-c_{i}\left(t_{*}\right)\left|x_{i}\left(t_{*}\right)\right|+c_{i}\left(t_{*}\right) \\
& \times \int_{t_{*}-\eta_{i}\left(t_{*}\right)}^{t_{*}} \mid-c_{i}(s) x_{i}\left(s-\eta_{i}(s)\right) \\
& +\sum_{j=1}^{n} a_{i j}(s) f_{j}\left(x_{j}(s)\right) \\
& +\sum_{j=1}^{n} b_{i j}(s) g_{j}\left(x_{j}\left(s-\tau_{i j}(s)\right)\right) \\
& +I_{i}(s) \mid d s \\
& +\sum_{j=1}^{n}\left|a_{i j}\left(t_{*}\right) f_{j}\left(x_{j}\left(t_{*}\right)\right)\right| \\
& +\sum_{j=1}^{n}\left|b_{i j}\left(t_{*}\right)\right|\left|g_{j}\left(x_{j}\left(t_{*}-\tau_{i j}\left(t_{*}\right)\right)\right)\right|+\left|I_{i}\left(t_{*}\right)\right| \\
& \leq-\left[c_{i}\left(t_{*}\right)-c_{i}\left(t_{*}\right) \eta_{i}\left(t_{*}\right) c_{i}^{+}\right]\left|x_{i}\left(t_{*}\right)\right| \\
& +\sum_{j=1}^{n}\left(\left|a_{i j}\left(t_{*}\right)\right|+c_{i}\left(t_{*}\right) \eta_{i}\left(t_{*}\right) a_{i j}^{+}\right) L_{j}^{f} \xi_{j} \frac{\gamma}{\eta} \\
& +\sum_{j=1}^{n}\left(\left|b_{i j}\left(t_{*}\right)\right|+c_{i}\left(t_{*}\right) \eta_{i}\left(t_{*}\right) b_{i j}^{+}\right) L_{j}^{g} \xi_{j} \frac{\gamma}{\eta} \\
& +\left[c_{i}^{+} \eta_{i}^{+}+1\right] I_{i}^{+} \\
& =\left\{-\left[c_{i}\left(t_{*}\right)-c_{i}\left(t_{*}\right) \eta_{i}\left(t_{*}\right) c_{i}^{+}\right] \xi_{i}\right. \\
& +\sum_{j=1}^{n}\left(\left|a_{i j}\left(t_{*}\right)\right|+c_{i}\left(t_{*}\right) \eta_{i}\left(t_{*}\right) a_{i j}^{+}\right) L_{j}^{f} \xi_{j} \\
& \left.+\sum_{j=1}^{n}\left(\left|b_{i j}\left(t_{*}\right)\right|+c_{i}\left(t_{*}\right) \eta_{i}\left(t_{*}\right) b_{i j}^{+}\right) L_{j}^{g} \xi_{j}\right\} \frac{\gamma}{\eta} \\
& +\left[c_{i}^{+} \eta_{i}^{+}+1\right] I_{i}^{+} \\
& <-\eta \frac{\gamma}{\eta}+\left[c_{i}^{+} \eta_{i}^{+}+1\right] I_{i}^{+} \\
& <0 \text {. } \tag{16}
\end{align*}
$$

It is a contradiction and shows that (11) holds. The proof is now completed.

Remark 2. After the conditions $\left(H_{1}\right)-\left(H_{5}\right)$, the solution of system (4) always exists (see [1, 2]). In view of the boundedness of this solution, from the theory of impulsive differential equations in [1], it follows that the solution of system (4) with initial conditions (10) can be defined on $[0,+\infty)$.

Lemma 3. Suppose that $\left(H_{1}\right)-\left(H_{5}\right)$ are true. Let $x^{*}(t)=$ $\left(x_{1}^{*}(t), x_{2}^{*}(t), \ldots, x_{n}^{*}(t)\right)^{T}$ be the solution of system (4) with
initial value $\varphi^{*}=\left(\varphi_{1}^{*}(t), \varphi_{2}^{*}(t), \ldots, \varphi_{n}^{*}(t)\right)^{T}$, and let $x(t)=$ $\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T}$ be the solution of system (4) with initial value $\varphi=\left(\varphi_{1}(t), \varphi_{2}(t), \ldots, \varphi_{n}(t)\right)^{T}$. Then, there exists a positive constant $\lambda$ such that

$$
\begin{equation*}
x_{i}(t)-x_{i}^{*}(t)=O\left(e^{-\lambda t}\right), \quad i=1,2, \ldots, n . \tag{17}
\end{equation*}
$$

Proof. Let $y(t)=x(t)-x^{*}(t)$. Then, for $i \in\{1,2, \ldots, n\}$, it is followed by

$$
\begin{array}{r}
y_{i}^{\prime}(t)=-c_{i}(t)\left(x_{i}\left(t-\eta_{i}(t)\right)-x_{i}^{*}\left(t-\eta_{i}(t)\right)\right) \\
+\sum_{j=1}^{n} a_{i j}(t)\left[f_{j}\left(x_{j}(t)\right)-f_{j}\left(x_{j}^{*}(t)\right)\right] \\
+\sum_{j=1}^{n} b_{i j}(t)\left[g_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)\right.  \tag{18}\\
\left.-g_{j}\left(x_{j}^{*}\left(t-\tau_{i j}(t)\right)\right)\right] \\
t \geq 0, \quad t \neq t_{k} \\
y_{i}\left(t_{k}^{+}\right)=\left(1+d_{i k}\right) y_{i}\left(t_{k}\right), \quad k=1,2, \ldots
\end{array}
$$

Define continuous functions $\Gamma_{i}(\omega)$ by setting

$$
\begin{align*}
& \Gamma_{i}(\omega)=-\left[c_{i}(t) e^{\omega \eta_{i}(t)}-\omega\right. \\
& \left.\quad-c_{i}(t) e^{\omega \eta_{i}(t)} \eta_{i}(t)\left(\omega+c_{i}^{+} e^{\omega \eta_{i}^{+}}\right)\right] \xi_{i} \\
& +\sum_{j=1}^{n}\left(\left|a_{i j}(t)\right|+a_{i j}^{+} c_{i}(t) e^{\omega \eta_{i}(t)} \eta_{i}(t)\right) L_{j}^{f} \xi_{j}  \tag{19}\\
& +\sum_{j=1}^{n}\left(\left|b_{i j}(t)\right| e^{\omega \tau_{i j}(t)}\right. \\
& \left.\quad+b_{i j}^{+} c_{i}(t) e^{\omega \eta_{i}(t)} \eta_{i}(t) e^{\omega \tau_{i j}^{+}}\right) L_{j}^{g} \xi_{j} \\
& \quad \omega \geq 0, \quad t \geq 0, \quad i=1,2, \ldots, n
\end{align*}
$$

Then

$$
\begin{aligned}
\Gamma_{i}(0)= & -\left[c_{i}(t)-c_{i}(t) \eta_{i}(t) c_{i}^{+}\right] \xi_{i} \\
& +\sum_{j=1}^{n}\left(\left|a_{i j}(t)\right|+a_{i j}^{+} c_{i}(t) \eta_{i}(t)\right) L_{j}^{f} \xi_{j} \\
& +\sum_{j=1}^{n}\left(\left|b_{i j}(t)\right|+b_{i j}^{+} c_{i}(t) \eta_{i}(t)\right) L_{j}^{g} \xi_{j} \\
< & -\eta, \quad t \geq 0, \quad i=1,2, \ldots, n,
\end{aligned}
$$

which, together with the continuity of $\Gamma_{i}(\omega)$, implies that we can choose two positive constants $\lambda$ and $\bar{\eta}$ such that

$$
\begin{align*}
& -\bar{\eta}>\Gamma_{i}(\lambda) \\
& =-\left[c_{i}(t) e^{\lambda \eta_{i}(t)}-\lambda\right. \\
& \left.\quad-c_{i}(t) e^{\lambda \eta_{i}(t)} \eta_{i}(t)\left(\lambda+c_{i}^{+} e^{\lambda \eta_{i}^{+}}\right)\right] \xi_{i} \\
& \\
& \quad+\sum_{j=1}^{n}\left(\left|a_{i j}(t)\right|+a_{i j}^{+} c_{i}(t) e^{\lambda \eta_{i}(t)} \eta_{i}(t)\right) L_{j}^{f} \xi_{j} \\
&  \tag{21}\\
& \quad+\sum_{j=1}^{n}\left(\left|b_{i j}(t)\right| e^{\lambda \tau_{i j}(t)}+b_{i j}^{+} c_{i}(t) e^{\lambda \eta_{i}(t)}\right. \\
& \left.\quad \times \eta_{i}(t) e^{\lambda \tau_{i j}^{+}}\right) L_{j}^{g} \xi_{j}, \quad t \geq 0, \quad i=1,2, \ldots, n .
\end{align*}
$$

Let

$$
\begin{equation*}
Y_{i}(t)=y_{i}(t) e^{\lambda t}, \quad i=1,2, \ldots, n \tag{22}
\end{equation*}
$$

Then

$$
\begin{aligned}
Y_{i}^{\prime}(t)= & \lambda Y_{i}(t)-c_{i}(t) e^{\lambda t} y_{i}\left(t-\eta_{i}(t)\right) \\
& +e^{\lambda t}\left\{\sum_{j=1}^{n} a_{i j}(t)\left[f_{j}\left(x_{j}(t)\right)-f_{j}\left(x_{j}^{*}(t)\right)\right]\right. \\
& +\sum_{j=1}^{n} b_{i j}(t)\left[g_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)\right. \\
& \left.\left.\quad-g_{j}\left(x_{j}^{*}\left(t-\tau_{i j}(t)\right)\right)\right]\right\} \\
= & \lambda Y_{i}(t)-c_{i}(t) e^{\lambda n_{i}(t)} Y_{i}(t) \\
& +c_{i}(t) e^{\lambda \eta_{i}(t)}\left[Y_{i}(t)-Y_{i}\left(t-\eta_{i}(t)\right)\right]
\end{aligned}
$$

$$
\begin{gathered}
+e^{\lambda t}\left\{\sum_{j=1}^{n} a_{i j}(t)\left[f_{j}\left(x_{j}(t)\right)-f_{j}\left(x_{j}^{*}(t)\right)\right]\right. \\
+\sum_{j=1}^{n} b_{i j}(t)\left[g_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)\right.
\end{gathered}
$$

$$
\left.\left.-g_{j}\left(x_{j}^{*}\left(t-\tau_{i j}(t)\right)\right)\right]\right\}
$$

$$
\begin{align*}
& =\lambda Y_{i}(t)-c_{i}(t) e^{\lambda \eta_{i}(t)} Y_{i}(t) \\
& +c_{i}(t) e^{\lambda \eta_{i}(t)} \int_{t-\eta_{i}(t)}^{t} Y_{i}^{\prime}(s) d s \\
& +e^{\lambda t}\left\{\sum_{j=1}^{n} a_{i j}(t)\left[f_{j}\left(x_{j}(t)\right)-f_{j}\left(x_{j}^{*}(t)\right)\right]\right. \\
& +\sum_{j=1}^{n} b_{i j}(t)\left[g_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)\right. \\
& \left.\left.-g_{j}\left(x_{j}^{*}\left(t-\tau_{i j}(t)\right)\right)\right]\right\} \\
& =\lambda Y_{i}(t)-c_{i}(t) e^{\lambda \eta_{i}(t)} Y_{i}(t)+c_{i}(t) e^{\lambda \eta_{i}(t)} \\
& \times \int_{t-\eta_{i}(t)}^{t}\left\{\lambda Y_{i}(s)-c_{i}(s) e^{\lambda s} y_{i}\left(s-\eta_{i}(s)\right)\right. \\
& +e^{\lambda s} \sum_{j=1}^{n} a_{i j}(s)\left[f_{j}\left(x_{j}(s)\right)-f_{j}\left(x_{j}^{*}(s)\right)\right] \\
& +e^{\lambda s} \sum_{j=1}^{n} b_{i j}(s)\left[g_{j}\left(x_{j}\left(s-\tau_{i j}(s)\right)\right)\right. \\
& \left.\left.-g_{j}\left(x_{j}^{*}\left(s-\tau_{i j}(s)\right)\right)\right]\right\} d s \\
& +e^{\lambda t}\left\{\sum_{j=1}^{n} a_{i j}(t)\left[f_{j}\left(x_{j}(t)\right)-f_{j}\left(x_{j}^{*}(t)\right)\right]\right. \\
& +\sum_{j=1}^{n} b_{i j}(t)\left[g_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)\right. \\
& \left.\left.-g_{j}\left(x_{j}^{*}\left(t-\tau_{i j}(t)\right)\right)\right]\right\}, \\
& t \neq t_{k}, \quad i=1,2, \ldots, n, \\
& \left|Y_{i}\left(t_{k}^{+}\right)\right|=\left|1+d_{i k}\right|\left|Y_{i}\left(t_{k}\right)\right|, \quad i=1,2, \ldots, n . \tag{23}
\end{align*}
$$

We define a positive constant $M$ as follows:

$$
\begin{equation*}
M=\max _{1 \leq i \leq n}\left\{\sup _{s \in\left[-\tau_{i}, 0\right]}\left|Y_{i}(s)\right|\right\} . \tag{25}
\end{equation*}
$$

Let $K$ be a positive number such that

$$
\begin{equation*}
\left|Y_{i}(t)\right| \leq M<K \xi_{i} \quad \forall t \in\left[-\tau_{i}, 0\right], i=1,2, \ldots, n \tag{26}
\end{equation*}
$$

We claim that

$$
\begin{equation*}
\left|Y_{i}(t)\right|<K \xi_{i}, \quad \forall t>0, i=1,2, \ldots, n \tag{27}
\end{equation*}
$$

Obviously, (27) holds for $t=0$. We first prove that (27) is true for $0<t \leq t_{1}$. Otherwise, there exist $i \in\{1,2, \ldots, n\}$ and $\rho \in\left(0, t_{1}\right]$ such that one of the following two cases must occur;

$$
\begin{align*}
&(1) \quad Y_{i}(\rho)=K \xi_{i},\left|Y_{j}(t)\right|<K \xi_{j}  \tag{28}\\
& \forall t \in[0, \rho), \quad j=1,2, \ldots, n
\end{align*}
$$

(2) $Y_{i}(\rho)=-K \xi_{i}, \quad\left|Y_{j}(t)\right|<K \xi_{j}$ $\forall t \in[0, \rho), \quad j=1,2, \ldots, n$.

Now, we distinguish two cases to finish the proof.
Case (i). If (28) holds. Then, from (21), (23), and ( $H_{1}$ )$\left(H_{5}\right)$, we have

$$
\begin{aligned}
0 \leq & Y_{i}^{\prime}(\rho) \\
= & \lambda Y_{i}(\rho)-c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} Y_{i}(\rho)+c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} \\
& \times \int_{\rho-\eta_{i}(\rho)}^{\rho}\left\{\lambda Y_{i}(s)-c_{i}(s) e^{\lambda s} y_{i}\left(s-\eta_{i}(s)\right)\right.
\end{aligned}
$$

$$
+e^{\lambda s} \sum_{j=1}^{n} a_{i j}(s)\left[f_{j}\left(x_{j}(s)\right)-f_{j}\left(x_{j}^{*}(s)\right)\right]
$$

$$
+e^{\lambda s} \sum_{j=1}^{n} b_{i j}(s)\left[g_{j}\left(x_{j}\left(s-\tau_{i j}(s)\right)\right)\right.
$$

$$
\left.\left.-g_{j}\left(x_{j}^{*}\left(s-\tau_{i j}(s)\right)\right)\right]\right\} d s
$$

$$
\begin{aligned}
& +e^{\lambda \rho}\left\{\sum_{j=1}^{n} a_{i j}(\rho)\left[f_{j}\left(x_{j}(\rho)\right)-f_{j}\left(x_{j}^{*}(\rho)\right)\right]\right. \\
& +\sum_{j=1}^{n} b_{i j}(\rho)\left[g_{j}\left(x_{j}\left(\rho-\tau_{i j}(\rho)\right)\right)\right. \\
& \left.\left.-g_{j}\left(x_{j}^{*}\left(\rho-\tau_{i j}(\rho)\right)\right)\right]\right\} \\
& \leq \lambda Y_{i}(\rho)-c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} Y_{i}(\rho)+c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} \\
& \times \int_{\rho-\eta_{i}(\rho)}^{\rho}\left\{\lambda Y_{i}(s)+c_{i}^{+} e^{\lambda \eta_{i}(s)}\left|Y_{i}\left(s-\eta_{i}(s)\right)\right|\right. \\
& +\sum_{j=1}^{n} a_{i j}^{+} L_{j}^{f}\left|Y_{j}(s)\right| \\
& \left.+\sum_{j=1}^{n} b_{i j}^{+} L_{j}^{g} e^{\lambda \tau_{i j}(s)}\left|Y_{j}\left(s-\tau_{i j}(s)\right)\right|\right\} d s \\
& +\sum_{j=1}^{n}\left|a_{i j}(\rho)\right| L_{j}^{f}\left|Y_{j}(\rho)\right| \\
& +\sum_{j=1}^{n}\left|b_{i j}(\rho)\right| L_{j}^{g} e^{\lambda \tau_{i j}(\rho)}\left|Y_{j}\left(\rho-\tau_{i j}(\rho)\right)\right| \\
& \leq-\left[c_{i}(\rho) e^{\lambda \eta_{i}(\rho)}-\lambda-c_{i}(\rho) e^{\lambda n_{i}(\rho)} \eta_{i}(\rho)\right. \\
& \left.\times\left(\lambda+c_{i}^{+} e^{\lambda \eta_{i}^{+}}\right)\right] K \xi_{i} \\
& +\sum_{j=1}^{n}\left(\left|a_{i j}(\rho)\right|+a_{i j}^{+} c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} \eta_{i}(\rho)\right) L_{j}^{f} K \xi_{j} \\
& +\sum_{j=1}^{n}\left(\left|b_{i j}(\rho)\right| e^{\lambda \tau_{i j}(\rho)}+b_{i j}^{+} c_{i}(\rho) e^{\lambda \lambda_{i}(\rho)}\right. \\
& \left.\times \eta_{i}(\rho) e^{\lambda \tau_{i j}^{+}}\right) L_{j}^{g} K \xi_{j} \\
& =\left\{-\left[c_{i}(\rho) e^{\lambda \eta_{i}(\rho)}-\lambda-c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} \eta_{i}(\rho)\right.\right. \\
& \left.\times\left(\lambda+c_{i}^{+} e^{\lambda \eta_{i}^{+}}\right)\right] \xi_{i} \\
& +\sum_{j=1}^{n}\left(\left|a_{i j}(\rho)\right|+a_{i j}^{+} c_{i}(\rho) e^{\lambda n_{i}(\rho)} \eta_{i}(\rho)\right) L_{j}^{f} \xi_{j} \\
& +\sum_{j=1}^{n}\left(\left|b_{i j}(\rho)\right| e^{\lambda \tau_{i j}(\rho)}+b_{i j}^{+} c_{i}(\rho) e^{\lambda \eta_{i}(\rho)}\right. \\
& \left.\left.\times \eta_{i}(\rho) e^{\lambda \tau_{i j}^{+}}\right) L_{j}^{g} \xi_{j}\right\} K
\end{aligned}
$$

$<-\bar{\eta} K<0$.

Case (ii). If (29) holds. Then, from (21), (23), and ( $H_{1}$ )$\left(H_{5}\right)$, we get

$$
\begin{aligned}
& 0 \geq Y_{i}^{\prime}(\rho) \\
& =\lambda Y_{i}(\rho)-c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} Y_{i}(\rho)+c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} \\
& \times \int_{\rho-\eta_{i}(\rho)}^{\rho}\left\{\lambda Y_{i}(s)-c_{i}(s) e^{\lambda s} y_{i}\left(s-\eta_{i}(s)\right)\right. \\
& +e^{\lambda s} \sum_{j=1}^{n} a_{i j}(s)\left[f_{j}\left(x_{j}(s)\right)-f_{j}\left(x_{j}^{*}(s)\right)\right] \\
& +e^{\lambda s} \sum_{j=1}^{n} b_{i j}(s)\left[g_{j}\left(x_{j}\left(s-\tau_{i j}(s)\right)\right)\right. \\
& \left.\left.-g_{j}\left(x_{j}^{*}\left(s-\tau_{i j}(s)\right)\right)\right]\right\} d s \\
& +e^{\lambda \rho}\left\{\sum_{j=1}^{n} a_{i j}(\rho)\left[f_{j}\left(x_{j}(\rho)\right)-f_{j}\left(x_{j}^{*}(\rho)\right)\right]\right. \\
& +\sum_{j=1}^{n} b_{i j}(\rho)\left[g_{j}\left(x_{j}\left(\rho-\tau_{i j}(\rho)\right)\right)\right. \\
& \left.\left.-g_{j}\left(x_{j}^{*}\left(\rho-\tau_{i j}(\rho)\right)\right)\right]\right\} \\
& \geq \lambda Y_{i}(\rho)-c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} Y_{i}(\rho)+c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} \\
& \times \int_{\rho-\eta_{i}(\rho)}^{\rho}\left\{\lambda Y_{i}(s)-c_{i}^{+} e^{\lambda \eta_{i}(s)}\left|Y_{i}\left(s-\eta_{i}(s)\right)\right|\right. \\
& -\sum_{j=1}^{n} a_{i j}^{+} L_{j}^{f}\left|Y_{j}(s)\right| \\
& \left.-\sum_{j=1}^{n} b_{i j}^{+} L_{j}^{g} e^{\lambda \tau_{i j}(s)}\left|Y_{j}\left(s-\tau_{i j}(s)\right)\right|\right\} d s \\
& -\sum_{j=1}^{n}\left|a_{i j}(\rho)\right| L_{j}^{f}\left|Y_{j}(\rho)\right| \\
& -\sum_{j=1}^{n}\left|b_{i j}(\rho)\right| L_{j}^{g} e^{\lambda \tau_{i j}(\rho)}\left|Y_{j}\left(\rho-\tau_{i j}(\rho)\right)\right| \\
& \geq-\left[c_{i}(\rho) e^{\lambda n_{i}(\rho)}-\lambda-c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} \eta_{i}(\rho)\right. \\
& \left.\times\left(\lambda+c_{i}^{+} e^{\lambda \eta_{i}^{+}}\right)\right]\left(-K \xi_{i}\right) \\
& +\sum_{j=1}^{n}\left(\left|a_{i j}(\rho)\right|+a_{i j}^{+} c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} \eta_{i}(\rho)\right) L_{j}^{f}\left(-K \xi_{j}\right) \\
& +\sum_{j=1}^{n}\left(\left|b_{i j}(\rho)\right| e^{\lambda \tau_{i j}(\rho)}+b_{i j}^{+} c_{i}(\rho) e^{\lambda \eta_{i}(\rho)}\right. \\
& \left.\times \eta_{i}(\rho) e^{\lambda \tau_{i j}^{+}}\right) L_{j}^{g}\left(-K \xi_{j}\right)
\end{aligned}
$$

$$
\begin{gathered}
=\left\{-\left[c_{i}(\rho) e^{\lambda \eta_{i}(\rho)}-\lambda-c_{i}(\rho) e^{\lambda \lambda_{i}(\rho)}\right.\right. \\
\left.\times \eta_{i}(\rho)\left(\lambda+c_{i}^{+} e^{\lambda \eta_{i}^{+}}\right)\right] \xi_{i} \\
+\sum_{j=1}^{n}\left(\left|a_{i j}(\rho)\right|+a_{i j}^{+} c_{i}(\rho) e^{\lambda \eta_{i}(\rho)} \eta_{i}(\rho)\right) L_{j}^{f} \xi_{j} \\
+\sum_{j=1}^{n}\left(\left|b_{i j}(\rho)\right| e^{\lambda \tau_{i j}(\rho)}+b_{i j}^{+} c_{i}(\rho) e^{\lambda \eta_{i}(\rho)}\right. \\
\left.\left.\times \eta_{i}(\rho) e^{\lambda \tau_{i j}^{+}}\right) L_{j}^{g} \xi_{j}\right\}(-K)
\end{gathered}
$$

$>\bar{\eta} K>0$.

Therefore, (27) holds for $t \in\left[0, t_{1}\right]$. From (24) and (27), we know that

$$
\begin{array}{r}
\left|Y_{i}\left(t_{1}\right)\right|=\left|y_{i}\left(t_{1}\right)\right| e^{\lambda t_{1}}<K \xi_{i}, \quad i=1,2, \ldots, n \\
\left|Y_{i}\left(t_{1}^{+}\right)\right|=\left|1+d_{i 1}\right|\left|Y_{i}\left(t_{1}\right)\right| \leq\left|Y_{i}\left(t_{1}\right)\right|<K \xi_{i}  \tag{32}\\
i=1,2, \ldots, n
\end{array}
$$

Thus, for $t \in\left[t_{1}, t_{2}\right]$, we may repeat the above procedure and obtain

$$
\begin{equation*}
\left|Y_{i}(t)\right|=\left|y_{i}(t)\right| e^{\lambda t}<K \xi_{i}, \quad \forall t \in\left[t_{1}, t_{2}\right], i=1,2, \ldots, n . \tag{33}
\end{equation*}
$$

Further, we have

$$
\begin{equation*}
\left|Y_{i}(t)\right|=\left|y_{i}(t)\right| e^{\lambda t}<K \xi_{i}, \quad \forall t>0, \quad i=1,2, \ldots, n \tag{34}
\end{equation*}
$$

That is,

$$
\begin{equation*}
\left|x_{i}(t)-x_{i}^{*}(t)\right| \leq K \xi_{i} e^{-\lambda t}, \quad \forall t>0, i=1,2, \ldots, n \tag{35}
\end{equation*}
$$

Remark 4. If $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t), \ldots, x_{n}^{*}(t)\right)^{T}$ is the $T$-periodic solution of system (4), it follows from Lemma 3 that $x^{*}(t)$ is globally exponentially stable.

## 3. Main Results

In this section, we will study existence and exponential stability for periodic solutions of system (4).

Theorem 5. Suppose that all conditions in Lemma 3 are satisfied. Then system (4) has exactly one T-periodic solution $x^{*}(t)$. Moreover, $x^{*}(t)$ is globally exponentially stable.

Proof. Let $x(t)=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T}$ be a solution of system (4) with initial conditions (10). By Remark 2, the
solution $x(t)$ can be defined for all $t \in[0,+\infty)$. By hypothesis $\left(H_{1}\right)$, we have, for any natural number $h$,

$$
\begin{align*}
{\left[x_{i}(t+\right.} & (h+1) T)]^{\prime} \\
= & -c_{i}(t) x_{i}\left(t+(h+1) T-\eta_{i}(t)\right) \\
& +\sum_{j=1}^{n} a_{i j}(t) f_{j}\left(x_{j}(t+(h+1) T)\right)  \tag{36}\\
& +\sum_{j=1}^{n} b_{i j}(t) g_{j}\left(x_{j}\left(t+(h+1) T-\tau_{i j}(t)\right)\right) \\
& +I_{i}(t), \quad t \neq t_{k}, \quad i=1,2, \ldots, n
\end{align*}
$$

Further, by hypothesis of $\left(\mathrm{H}_{3}\right)$, we obtain

$$
\begin{align*}
x_{i}\left(\left(t_{k}\right.\right. & \left.+(h+1) T)^{+}\right) \\
& =x_{i}\left(t_{k+(h+1) q}^{+}\right)  \tag{37}\\
& =\left(1+d_{i(k+(h+1) q)}\right) x_{i}\left(t_{k+(h+1) q}\right) \\
& =\left(1+d_{i k}\right) x_{i}\left(t_{k}+(h+1) T\right), \quad k=1,2, \ldots
\end{align*}
$$

Thus, for any natural number $h$, we obtain that $x(t+(h+1) T)$ is a solution of system (4) for all $t+(h+1) T \geq 0$. Hence, $x(t+T)$ is also a solution of (4) with initial values

$$
\begin{equation*}
x_{i}(s+T), \quad s \in\left[-\tau_{i}, 0\right], i=1,2, \ldots, n . \tag{38}
\end{equation*}
$$

Then, by the proof of Lemma 3, there exists a constant $K>0$ such that for any natural number $h$,

$$
\begin{align*}
& \mid x_{i}(t+(h+1) T)-x_{i}(t+h T) \mid \\
&=\left|x_{i}(t+h T+T)-x_{i}(t+h T)\right| \\
& \leq K \xi_{i} e^{-\lambda(t+h T)} \\
&=K \xi_{i} e^{-\lambda t}\left(\frac{1}{e^{\lambda T}}\right)^{h}, \quad t+h T \geq 0 \\
& \quad t \neq t_{k}, \quad i=1,2, \ldots, n  \tag{39}\\
&\left|x_{i}\left(\left(t_{k}+(h+1) T\right)^{+}\right)-x_{i}\left(\left(t_{k}+h T\right)^{+}\right)\right| \\
&=\left(1+d_{i k}\right)\left|x_{i}\left(t_{k}+(h+1) T\right)-x_{i}\left(t_{k}+h T\right)\right| \\
& \leq K \xi_{i} e^{-\lambda\left(t_{k}+h T\right)} \\
&=K \xi_{i} e^{-\lambda t_{k}}\left(\frac{1}{e^{\lambda T}}\right)^{h}, \quad \forall k \in Z^{+}, i=1,2, \ldots, n .
\end{align*}
$$

Moreover, for any natural number $m$, we can obtain

$$
\begin{align*}
& x_{i}(t+(m+1) T) \\
& \qquad=x_{i}(t)+\sum_{h=0}^{m}\left[x_{i}(t+(h+1) T)-x_{i}(t+h T)\right] \\
& t+h T \geq 0, \quad t \neq t_{k}, \quad i=1,2, \ldots, n, \\
& \left(x_{i}\left(\left(t_{k}+(m+1) T\right)^{+}\right)\right)  \tag{40}\\
& =x_{i}(t)+\sum_{h=0}^{m}\left[x_{i}\left(\left(t_{k}+(h+1) T\right)^{+}\right)\right. \\
& \left.\quad-\left(x_{i}\left(\left(t_{k}+h T\right)^{+}\right)\right)\right] \\
& \forall k \in Z^{+}, \quad i=1,2, \ldots, n .
\end{align*}
$$

Combining (39) with (40), we know that $x(t+m T)$ will converge uniformly to a piecewise continuous function $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t), \ldots, x_{n}^{*}(t)\right)^{T}$ on any compact set of $R$.

Now we are in the position of proving that $x^{*}(t)$ is a $T$-periodic solution of system (4). It is easily known that $x^{*}(t)$ is $T$-periodic since

$$
\begin{align*}
x_{i}^{*}(t+T) & =\lim _{m \rightarrow+\infty} x_{i}(t+T+m T) \\
& =\lim _{m+1 \rightarrow+\infty} x_{i}(t+(m+1) T) \\
& =x_{i}^{*}(t), \quad t \neq t_{k},  \tag{41}\\
x_{i}^{*}\left(\left(t_{k}+T\right)^{+}\right) & =\lim _{m \rightarrow+\infty} x_{i}\left(\left(t_{k}+T+m T\right)^{+}\right) \\
& =x_{i}^{*}\left(t_{k}^{+}\right), \quad k=1,2, \ldots,
\end{align*}
$$

where $i=1,2, \ldots, n$. Noting that the right side of (4) is piecewise continuous, together with (36) and (37), we know that $\left\{x_{i}^{\prime}(t+(m+1) T)\right\}$ converges uniformly to a piecewise continuous function on any compact set of $R \backslash\left\{t_{1}, t_{2}, \ldots\right\}$. Therefore, letting $m \rightarrow+\infty$ on both sides of (36) and (37), we get

$$
\begin{align*}
& x_{i}^{* \prime}(t)=-c_{i}(t) x_{i}^{*}\left(t-\eta_{i}(t)\right) \\
&+\sum_{j=1}^{n} a_{i j}(t) f_{j}\left(x_{j}^{*}(t)\right) \\
&+\sum_{j=1}^{n} b_{i j}(t) g_{j}\left(x_{j}^{*}\left(t-\tau_{i j}(t)\right)\right)  \tag{42}\\
&+I_{i}(t), \quad t \neq t_{k}, i=1,2, \ldots, n \\
& x_{i}^{*}\left(t_{k}^{+}\right)=\left(1+d_{i k}\right) x_{i}^{*}\left(t_{k}\right) \\
& k= 1,2, \ldots, \quad i=1,2, \ldots, n .
\end{align*}
$$

Thus, $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t), \ldots, x_{n}^{*}(t)\right)^{T}$ is a $T$-periodic solution of system (4).

Finally, by Lemma 3, we can prove that $x^{*}(t)$ is globally exponentially stable. This completes the proof.

## 4. An Example

In this section, we give an example to demonstrate the results obtained in the previous sections.

Example 6. Consider the following impulsive cellar neural network consisting of two neurons with time-varying delays in the leakage terms, which is described by

$$
\begin{align*}
& x_{1}^{\prime}(t)=-3(|\sin \pi t|+1) x_{1}\left(t-\frac{\sin ^{2} \pi t}{1000}\right) \\
&+\frac{1}{16} \cos ^{2} \pi t f_{1}\left(x_{1}(t)\right) \\
&+\frac{1}{16} \sin ^{2} \pi t f_{2}\left(x_{2}(t)\right) \\
&+\frac{1}{16} \sin ^{2} \pi t g_{1}\left(x_{1}\left(t-\cos ^{2} \pi t\right)\right) \\
&+\frac{1}{16} \sin ^{2} \pi t g_{2}\left(x_{2}\left(t-2 \sin ^{2} \pi t\right)\right) \\
&+100 \cos ^{2} \pi t \quad t \neq 2 k-1, \\
& x_{2}^{\prime}(t)=-3\left(\left|\cos ^{2} \pi t\right|+1\right) x_{1}\left(t-\frac{\sin ^{4} \pi t}{1000}\right)  \tag{43}\\
&+\frac{1}{16} \cos ^{3} \pi t f_{1}\left(x_{1}(t)\right) \\
&+\frac{1}{16} \sin ^{3} \pi t f_{2}\left(x_{2}(t)\right) \\
&+\frac{1}{16} \sin ^{3} \pi t g_{1}\left(x_{1}\left(t-\cos ^{2} \pi t\right)\right) \\
&+\frac{1}{16} \sin ^{3} \pi t g_{2}\left(x_{2}\left(t-2 \sin ^{2} \pi t\right)\right) \\
&+100 \sin ^{2} \pi t \quad t \neq 2 k-1, \\
& x_{i}\left(t_{k}^{+}\right)=\left(1+d_{i k}\right) x_{i}\left(t_{k}\right), \\
& d_{i(2 s)}=-2, \quad d_{i(2 s-1)}=-1, \\
& t_{k}=k, i=1,2, \quad k, s=1,2, \ldots .
\end{align*}
$$

Here, it is assumed that the activation functions are

$$
\begin{align*}
& g_{1}(x)=g_{2}(x)=x+2 \sin x, \\
& f_{1}(x)=f_{2}(x)=x+3 \sin x . \tag{44}
\end{align*}
$$

Noting that

$$
\begin{gathered}
\eta_{1}(t)=\frac{\sin ^{2} \pi t}{1000}, \quad \eta_{2}(t)=\frac{\sin ^{4} \pi t}{1000}, \\
c_{1}(t)=3(|\sin \pi t|+1), \quad c_{2}(t)=3(|\cos \pi t|+1),
\end{gathered}
$$

$$
\begin{array}{cc}
a_{11}(t)=\frac{1}{16} \cos ^{2} \pi t, & a_{12}(t)=\frac{1}{16} \sin ^{2} \pi t \\
b_{11}(t)=\frac{1}{16} \sin ^{2} \pi t, & b_{12}(t)=\frac{1}{16} \sin ^{2} \pi t \\
a_{21}(t)=\frac{1}{16} \cos ^{3} \pi t, & a_{22}(t)=\frac{1}{16} \sin ^{3} \pi t \\
b_{21}(t)=\frac{1}{16} \sin ^{3} \pi t, & b_{22}(t)=\frac{1}{16} \sin ^{3} \pi t \\
\tau_{11}(t)=\tau_{21}(t)=\cos ^{2} \pi t \\
\tau_{12}(t)=\tau_{22}(t)=2 \sin ^{2} \pi t \tag{45}
\end{array}
$$

then we obtain

$$
\begin{align*}
& -\left[c_{i}(t)-c_{i}(t) \eta_{i}(t) c_{i}^{+}\right] \xi_{i} \\
& \quad+\sum_{j=1}^{2}\left(\left|a_{i j}(t)\right|+c_{i}(t) \eta_{i}(t) a_{i j}^{+}\right) L_{j}^{f} \xi_{j} \\
& \quad+\sum_{j=1}^{2}\left(\left|b_{i j}(t)\right|+c_{i}(t) \eta_{i}(t) b_{i j}^{+}\right) L_{j}^{g} \xi_{j} \\
& <  \tag{46}\\
& \quad-\left(3-6 \times \frac{1}{1000} \times 6\right) \\
& \quad+2\left(\frac{1}{16}+6 \times \frac{1}{1000} \times \frac{1}{16}\right) \times 3 \\
& \quad+2\left(\frac{1}{16}+6 \times \frac{1}{1000} \times \frac{1}{16}\right) \times 4 \\
& <
\end{align*}
$$

This yields that system (43) satisfies $\left(H_{1}\right)-\left(H_{5}\right)$. Hence, from Theorem 5, system (43) has exactly one 2-periodic solution. Moreover, the 2-periodic solution is globally exponentially stable.

Remark 7. Since $g_{1}(x)=g_{2}(x)=x+2 \sin x, f_{1}(x)=$ $f_{2}(x)=x+3 \sin x$ and CNNs (43) is a very simple form of CNNs with time-varying delays in the leakage terms, it is clear that the conditions $\left(A_{1}\right)$ and $\left(A_{2}\right)$ are not satisfied. Therefore, all the results in [11-19] and the references therein cannot be applicable to system (43) to obtain the existence and exponential stability of the 2-periodic solutions.

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## Research Article

# On Dimension Extension of a Class of Iterative Equations 

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This investigation aims at studying some special properties (convergence, polynomial preservation order, and orthogonal symmetry) of a class of $r$-dimension iterative equations, whose state variables are described by the following nonlinear iterative equation: $\phi^{n}(x)=T\left(\phi^{n-1}(x)\right):=\sum_{j=0}^{m} H_{j} \phi^{n-1}(2 x-k)$. The obtained results in this paper are complementary to some published results. As an application, we construct orthogonal symmetric multiwavelet with additional vanishing moments. Two examples are also arranged to demonstrate the correctness and effectiveness of the main results.

## 1. Introduction

Giving any compact supported vector-valued function $\phi^{0}(x):=\left(\phi_{1}^{0}, \ldots, \phi_{r}^{0}\right)^{\top} \in\left(L^{2}(R)\right)^{r}$, we define $r$-dimension iterative equation as follows:

$$
\begin{equation*}
\phi^{n}(x)=T\left(\phi^{n-1}(x)\right):=\sum_{j=0}^{\mu} H_{j} \phi^{n-1}(2 x-j), \tag{1}
\end{equation*}
$$

where $H_{j}$ is $r$-order real matrix, $j=0,1, \ldots, \mu, \mu \in Z^{+}$, and $n \in Z^{+}$. Let $H:=\left\{H_{j}, j=0,1, \ldots, \mu\right\}$ denote the masks of the iterative equation, then the Fourier transform of $\phi(x)=$ $\left(\phi_{1}, \ldots, \phi_{r}\right)^{\top}$, if it exists, can be defined by

$$
\begin{equation*}
\widehat{\phi}(\omega)=\left(\int_{R} \phi_{1}(x) e^{-i \omega x} d x, \ldots, \int_{R} \phi_{r}(x) e^{-i \omega x} d x\right)^{\top} \tag{2}
\end{equation*}
$$

and the discrete-time Fourier transform (DTFT) of $H=\left\{H_{j}\right\}$ can be defined by

$$
\begin{equation*}
\widehat{H}(\omega)=\frac{1}{2} \sum_{j=0}^{\mu} H_{j} e^{-i j \omega} \tag{3}
\end{equation*}
$$

where $i=\sqrt{-1}$. Therefore, iterative equation (1) takes the following frequency domain form:

$$
\begin{equation*}
\widehat{\phi}^{n}(\omega)=\widehat{H}\left(\frac{\omega}{2}\right) \widehat{\phi}^{n-1}\left(\frac{\omega}{2}\right) . \tag{4}
\end{equation*}
$$

We claim that iterative equation (1) converges to a fixed function, and if $\lim _{n \rightarrow \infty} \phi^{n}(x)$ exists, we denote it by the $r$ dimension vector function $\phi(x):=\lim _{n \rightarrow \infty} \phi^{n}(x)$, whose frequency domain form is defined as an infinite product as follows:

$$
\begin{equation*}
\widehat{\phi}(\omega)=\left[\prod_{k=1}^{\infty}\left[\widehat{H}\left(\frac{\omega}{2^{k}}\right)\right]\right] \widehat{\phi}(0) . \tag{5}
\end{equation*}
$$

Obviously, (5) is equivalent to $\phi(x)$ satisfying $r$-dimension refinement equation

$$
\begin{equation*}
\phi(x)=\sum_{j=0}^{\mu} H_{j} \phi(2 x-j) . \tag{6}
\end{equation*}
$$

Iterative equation (1) is nonlinearity; in the real world, nonlinear problems are not exceptional, but regular phenomena. Nonlinearity is the nature of matter and its development [1,2]. Recently, iterative equation (1) has attracted increasing interest due to the potential applications in the field of wavelet analysis. In fact, the limit of iterative equation (1) satisfies refinement equation (6) which is fundamental to the theory of the scaling functions, and then we can construct special properties masks $H$ of iterative equation (1) to obtain scaling vector function with special properties. For example, in order to construct the wavelet frames with high order vanishing moments, in [3], Han and Mo investigate the method to factorize a matrix mask of (1). In order to construct multiwavelets with vanishing moments of arbitrarily high order,
in [4], with the help of dimension extension and iterative scheme for revising masks of (1), Chui and Lian investigate the compactly supported orthogonal scaling function with additional polynomial preservation order (p.p.o.).

As we all know, the orthogonal symmetric scaling vector function with high p.p.o. is very important to construct symmetric multiwavelets with high vanishing moments by multiresolution analysis, and yet its construction is very difficult, especially in high dimension. The main objective of this paper is to develop an iterative equation to generate orthogonal symmetric scaling vector function as the limit of (1) and to analysis its convergence. Furthermore, we will introduce an iterative scheme by extending the dimension of iterative equation to obtain orthogonal symmetric scaling function with p.p.o increasing. As an application, we will construct compactly supported orthogonal symmetric multiwavelets to achieve any order vanishing moments.

## 2. Preliminaries

Just as shown in [5, 6], iterative equation (1) converges to a fixed point $\phi(x)$ or that the refinement equation (6) exists solution $\phi(x)$ if and only if matrix $\widehat{H}(0)$ satisfies Condition $E$ (for a matrix $A$, one says that $A$ satisfies Condition $E$ if the spectral radius of $A$ is equal to 1 , where 1 is the unique eigenvalue of $A$ on the unit circle, and it is a simple eigenvalue). The limit $\phi$ of (1) is called $r$-scaling vector function if $\phi$ is $L^{2}$-stable, meaning that $\left\{\phi_{l}(x-k): 1 \leq l \leq\right.$ $r ; k \in Z\}$ is a Riesz basis of $V_{0}$, where, for $j \in Z$,

$$
\begin{equation*}
V_{j}:=\operatorname{Clos}_{L^{2}} \operatorname{Span}\left\{2^{j / 2} \phi_{l}\left(2^{j} x-k\right): 1 \leq l \leq r ; k \in Z\right\}, \tag{7}
\end{equation*}
$$

which is also called multiresolution analysis (MRA) of $L^{2}$, provided that $L^{2}=\operatorname{Clos}_{L^{2}} \bigcup_{j \in Z} V_{j}$.

According to [6, 7], iterative equation (1) converges to an $r$-dimension scaling vector function $\phi(x)$ if and only if the matrix $\widehat{H}(0)$ and the matrix $T_{H}$ satisfy Condition $E$, where, $T_{H}:=\left(A_{2 i-j}\right)_{1-\mu \leq i, j \leq \mu-1}, A_{j}$ is the $r^{2} \times r^{2}$ matrix given by $A_{j}:=$ $\sum_{k=0}^{\mu} H_{k-j} \otimes H_{k}$, and " $\otimes$ " denotes the Kronecker product, that is, for two matrices $B=\left(b_{i j}\right)$ and $C=\left(c_{i j}\right), B \otimes C:=\left(b_{i j} C\right)$.

Definition 1 (see [4]). Let $\phi(x)=\left(\phi_{1}(x), \ldots, \phi_{r}(x)\right)^{\top}$ be $r$ dimension scaling function, and if polynomial $x^{l} \in V_{0}, l=$ $0,1, \ldots, m-1$, where $V_{0}:=\operatorname{Clos}_{L^{2}} \operatorname{Span}\left\{\phi_{l}(x-k): 1 \leq l \leq\right.$ $r ; k \in Z\}$, then we get that $\phi(x)$ has polynomial preservation order (p.p.o.) $m$.

Polynomial preservation order is a desired feature to a scaling function in application, for example, to construct multiwavelet with high-order vanishing moments. In [8], Plonka studied the polynomial preservation order properties of refinable function vectors in detail. Lian, in [9], established certain necessary and sufficient conditions for a multiscaling function $\phi(x)$ with p.p.o. $m$ in terms of the eigenvalues and their corresponding eigenvectors of masks of (1). Iterative equation (1) generates $r$-dimension scaling function $\phi(x)$ with p.p.o. $m$, if and only if the matrix masks of (1)
satisfy $m$ order sum rules; that is, there exists real vector $y_{0}, y_{1}, \ldots, y_{m-1} \in R^{r}$ with $y_{0} \neq 0$, for any $0 \leq l \leq m-1$ such that

$$
\begin{gather*}
\sum_{k=0}^{l}(-1)^{k} \frac{1}{2^{k}}\binom{l}{k} y_{l-k} \sum_{j \in Z}(2 j)^{k} H_{2 j}=\frac{1}{2^{l}} y_{l}, \\
\sum_{k=0}^{l}(-1)^{k} \frac{1}{2^{k}}\binom{l}{k} y_{l-k} \sum_{j \in Z}(2 j+1)^{k} H_{2 j+1}=\frac{1}{2^{l}} y_{l}, \tag{8}
\end{gather*}
$$

where $\binom{l}{k}=l!/ k!(l-k)!$.
A function vector $\eta=\left(\eta_{1}, \ldots, \eta_{r}\right)^{\top}$ is said to be orthogonal if it satisfies $\left\langle\eta_{l}(\cdot-k), \eta_{\hat{l}}(\cdot-\hat{k})\right\rangle=\delta_{l, l} \delta_{k, \dot{k}}, l, \dot{l}=$ $1, \ldots, r ; k, \hat{k} \in Z$. If $\phi(x)$ satisfying refinement equation is orthogonal scaling function vector, then the masks of (1) must satisfy condition

$$
\begin{equation*}
\widehat{H}(\omega) \widehat{H}^{\top}(-\omega)+\widehat{H}(\omega+\pi) \widehat{H}^{\top}(-\omega-\pi)=I_{r} \tag{9}
\end{equation*}
$$

A scaling function vector $\phi(x)$ has symmetry property if all of its components are either symmetric or antisymmetric. The symmetry of $\phi$ is decided by the masks of (1). From [4], let $\phi$ be symmetric scaling vector function generated by (1) with two-scale symbol $H(z):=(1 / 2) \sum_{j=0}^{\mu} H_{k} z^{j}, z=e^{-i \omega}$ if and only if (see [4])

$$
\begin{equation*}
S_{0} H(z) S_{0}=D_{0}\left(z^{2}\right) H\left(z^{-1}\right) D_{0}\left(z^{-1}\right) \tag{10}
\end{equation*}
$$

where $S_{0}=\operatorname{diag}\left((-1)^{i_{1}}, \ldots,(-1)^{i_{r}}\right), D_{0}(z):=\operatorname{diag}\left(z^{a_{1}+b_{1}}\right.$, $\left.\ldots, z^{a_{r}+b_{r}}\right)$ with supp $\phi_{l}=\left[a_{l}, b_{l}\right], l=1, \ldots, r ; i_{1}, \ldots, i_{r}$ being either 0 or 1 , depending on symmetry or antisymmetry of the corresponding components of $\phi$, respectively.

Let $\phi$ be $r$-dimension orthogonal symmetric scaling function which satisfies (6) with two-scale symbol $H(z)$ satisfying

$$
\begin{equation*}
z^{2 \gamma-1} S_{0} H\left(z^{-1}\right) S_{0}=H(z) \tag{11}
\end{equation*}
$$

where $S_{0}=\operatorname{diag}\left(I_{s},-I_{r-s}\right)$, for a nonnegative integer $0<$ $s<r, \gamma \in Z^{+}$. In this case, if $H(z)$ satisfies (11) and (1) generates orthogonal symmetric scaling function vectors $\phi=$ $\left(\phi_{1}, \ldots, \phi_{r}\right)^{\top}$, then $\phi_{1}, \ldots, \phi_{s}$ are symmetric about $\gamma-(1 / 2)$ and $\phi_{s+1}, \ldots, \phi_{r}$ are antisymmetric about $\gamma-(1 / 2)$.

As shown in [10], the masks of (1) or two-scale symbol $H(z)$ of (6) satisfy (9) and (11) if and only if the following formulae hold:

$$
H(z)=\frac{\sqrt{2}}{2}\left[\begin{array}{cc}
a_{0} & 0  \tag{12}\\
0 & b_{0}
\end{array}\right] V_{1}\left(z^{2}\right) \cdots V_{\gamma-1}\left(z^{2}\right) U_{0}\left[\begin{array}{c}
I_{r} \\
I_{r} z
\end{array}\right]
$$

where $V_{j}(z)$ is defined by

$$
\begin{align*}
& V_{i}(z):=\frac{1}{2}\left[\begin{array}{cc}
I_{r} & -v_{i} \\
-v_{i}^{\top} & I_{r}
\end{array}\right]+\frac{1}{2}\left[\begin{array}{cc}
I_{r} & v_{i} \\
v_{i}^{\top} & I_{r}
\end{array}\right] z, \quad v_{j} \in O(r), \\
& j=1, \ldots, \gamma-1, \tag{13}
\end{align*}
$$

where $a_{0}$ and $b_{0}$ are $s \times r$ and $(r-s) \times r$ matrices, respectively, $U_{0}=(\sqrt{2} / 2)\left[\begin{array}{cc}I_{r} & S_{0} \\ -I_{r} & S_{0}\end{array}\right]$.

If there exist matrices $a_{1}, b_{1}$, such that $\left[a_{0}^{\top}, a_{1}^{\top}\right],\left[b_{0}^{\top}, b_{1}^{\top}\right]$ being $r \times r$ orthogonal matrix, one can define $G(z)$ as follows:

$$
\begin{align*}
G(z):= & \frac{1}{2} \sum_{k=0}^{2 \gamma-1} G_{k} z^{k}=\frac{\sqrt{2}}{2}\left[\begin{array}{cc}
0 & b_{1} \\
a_{1} & 0
\end{array}\right] V_{1}\left(z^{2}\right) \cdots V_{\gamma-1}\left(z^{2}\right) \\
& \times U_{0}\left[\begin{array}{c}
I_{r} \\
I_{r} z
\end{array}\right] \tag{14}
\end{align*}
$$

and then we have

$$
\begin{gather*}
-z^{2 \gamma-1} S_{0} G\left(z^{-1}\right) S_{0}=G(z) \\
H(z) G^{\top}\left(z^{-1}\right)+H(-z) G^{\top}\left(-z^{-1}\right)=0, \quad z=e^{-i \omega} \tag{15}
\end{gather*}
$$

Let $\phi(x)$ be $r$-dimension orthogonal symmetric scaling function satisfying refinement equation (6) with two-scale symbol $H(z)$ satisfying (11). If $G(z)$ is defined by (14), define $\psi(x):=$ $\left(\psi_{1}, \ldots, \psi_{r}\right)^{\top}$ by

$$
\begin{equation*}
\psi(x):=\sum_{k=0}^{2 \gamma-1} G_{k}(2 x-k) \tag{16}
\end{equation*}
$$

and then $\psi(x)$ is $r$-dimension orthogonal symmetric multiwavelet function; that is, $\left\{2^{j / 2} \psi_{l}\left(2^{j} x-k\right) ; j, k \in Z, l=\right.$ $1,2, \ldots, r\}$ is the orthogonal basis of $L^{2}(R)$. When scaling function $\phi(x)$ has p.p.o. $=m$, one obtains that multi-wavelet function $\psi(x)$ has $m$ order vanishing moments; that is,

$$
\begin{equation*}
\int_{R} x^{k} \psi_{l}(x) d x=0, \quad k=0,1, \ldots, m-1 ; l=1,2, \ldots, r \tag{17}
\end{equation*}
$$

## 3. Main Results

At first, we give the following convergent lemma.
Lemma 2. Iterative equation (1) with any given compact supported vector-valued function $\phi^{0}(x)$ converges to a unique vector function $\phi$, if and only if the spectral radius of $\widehat{H}(0)$ is equal to 1,1 is the unique eigenvalue of $\widehat{H}(0)$ on the unit circle, and 1 is simple.

Proof. Using Fourier transform, from (1), (4), and (5), we obtain that the iterative equation (1) converges to vector function $\phi(x)$ if and only if the infinite product (5) converges. From [11], it is equivalent to the spectral radius of $\widehat{H}(0)$ which is equal to 1,1 is the unique eigenvalue of $\widehat{H}(0)$ on the unit circle, and 1 is simple. This completes the proof.

When iterative equations (1) with masks $H=\left\{H_{j}, j=\right.$ $0,1, \ldots, 2 \gamma-1\}$ generate $r$-dimension orthogonal symmetric scaling function $\phi(x)$ with p.p.o. $=m$, we can establish the following theorem to increase scaling function p.p.o. by extending the dimension of iterative equation (1).

Theorem 3. Let $\phi$ be $r$-dimension orthogonal symmetric scalingfunction with p.p.o. $=m$ generated by iterative equation (1) with mask $H=\left\{H_{j}, j=0,1, \ldots, 2 \gamma-1\right\}$, where $H(z)$ satisfying (11), and Construct $r+2$-dimension iterative equations mask $\bar{H}=\left\{\bar{H}_{j}, j=0,1, \ldots, 2 \gamma-1\right\}$ as follows:

$$
\bar{H}(z):=\frac{1}{2} \sum_{j=0}^{2 \gamma-1} \bar{H}_{j} z^{j}=\left(\begin{array}{cc}
H(z) & 0_{r \times 2}  \tag{18}\\
B G(z) & \frac{1+z^{2 \gamma-1}}{2^{m+1}} I_{2}
\end{array}\right),
$$

where $G(z)$ defined by (14). Then there exists $2 \times r$ matrix $B$ such that iterative equation (1) with mask $\bar{H}$ generates $(r+2)$-dimension orthogonal symmetric scaling function $\bar{\phi}=$ $\left(\phi^{\top}, \phi_{r+1}, \phi_{r+2}\right)^{\top}$ which has p.p.o. $\geq m+1$.

Proof. First, if matrix $H(1)$ satisfies the conditions of Lemma 2, then obviously, matrix $\bar{H}(1)$ constructed by (18) satisfies all conditions of Lemma 2. That is to say that the $r+2-$ dimension iterative equation (1) with mask $\bar{H}$ constructed by (18) converges to $\bar{\phi}$.

By applying the p.p.o. $=m$ of $\phi$ and sum rules of (8), there exist $y_{l}^{\top} \in R^{r}, l=0,1, \ldots, m-1$ with $y_{0} \neq 0_{1 \times r}$, for $0 \leq l \leq m-1$ satisfying

$$
\begin{align*}
& y_{l}\left(\sum_{j \in Z} H_{2 j}-\frac{1}{2^{l}}\right)=-\sum_{k=0}^{l-1}(-1)^{l-k} \frac{1}{2^{l-k}}\binom{l}{k} y_{k} \sum_{j \in Z}(2 j)^{l-k} H_{2 j} \\
& y_{l}\left(\sum_{j \in Z} H_{2 j+1}-\frac{1}{2^{l}}\right)=-\sum_{k=0}^{l-1}(-1)^{l-k} \frac{1}{2^{l-k}}\binom{l}{k} y_{k} \\
& \times \sum_{j \in Z}(2 j+1)^{l-k} H_{2 j+1} \tag{19}
\end{align*}
$$

When $l=m$, there is no $r$-dimension row vector $y_{m}$ which satisfies (19). Now we will show that $\bar{H}(z)$ satisfies $m+1$ order sum rules by choosing matrix $B$.

$$
\text { Let } \bar{y}_{k}=\left(y_{k}, 0,0\right), k=0, \ldots, m-1, \bar{y}_{m}=\left(y_{m}, c_{m}^{1}, c_{m}^{2}\right)
$$ $c_{m}^{1}, c_{m}^{2} \in R, y_{m}$ being some $r$-dimension vector, then it is easy to obtain that $\bar{H}(z)$ satisfies $m$ order sum rules; that is for $0 \leq$ $l \leq m-1$, we have

$$
\begin{align*}
\bar{y}_{l}\left(\sum_{j \in Z} \bar{H}_{2 j}-\frac{1}{2^{m}} I_{r}\right)= & -\sum_{k=0}^{l-1}(-1)^{l-k} \frac{1}{2^{l-k}}\binom{l}{k} \bar{y}_{k} \\
& \times \sum_{j \in Z}(2 j)^{l-k} \bar{H}_{2 j}  \tag{20}\\
\bar{y}_{l}\left(\sum_{j \in Z} \bar{H}_{2 j+1}-\frac{1}{2^{m}}\right)= & -\sum_{k=0}^{l-1}(-1)^{l-k} \frac{1}{2^{l-k}}\binom{l}{k} \bar{y}_{k} \\
& \times \sum_{j \in Z}(2 j+1)^{l-k} \bar{H}_{2 j+1}
\end{align*}
$$

When $l=m$, (20) is equivalent to

$$
\begin{align*}
& y_{m}\left(\sum_{j \in Z} H_{2 j}-\frac{1}{2^{m}} I_{r}\right)+\left(c_{m}^{1}, c_{m}^{2}\right) B \sum_{j \in Z} G_{2 j} \\
& =-\sum_{k=0}^{m-1}(-1)^{l-k} \frac{1}{2^{l-k}}\binom{l}{k} y_{k} \sum_{j \in Z}(2 j)^{l-k} H_{2 j} \\
& y_{m}\left(\sum_{j \in Z} H_{2 j+1}-\frac{1}{2^{m}}\right)+\left(c_{m}^{1}, c_{m}^{2}\right) B \sum_{j \in Z} G_{2 j+1}  \tag{21}\\
& =-\sum_{k=0}^{m-1}(-1)^{l-k} \frac{1}{2^{l-k}}\binom{l}{k} y_{k} \sum_{j \in Z}(2 j+1)^{l-k} H_{2 j+1} .
\end{align*}
$$

When $\left(c_{m}^{1}, c_{m}^{2}\right)=(0,0)$, for $\phi$ with p.p.o. $=m$, there does not exist $y_{m}$ satisfying (21), but we will show that by choosing $2 \times r$ matrix $B$ and $\left(c_{m}^{1}, c_{m}^{2}\right)$, there exists $y_{m}$ satisfying (21), thus $\bar{H}(z)$ satisfies $m+1$ order sum rules. Let

$$
\begin{gather*}
p_{0}=:-\sum_{k=0}^{m-1}(-1)^{l-k} \frac{1}{2^{l-k}}\binom{l}{k} y_{k} \sum_{j \in Z}(2 j)^{l-k} H_{2 j}, \\
p_{1}=:-\sum_{k=0}^{m-1}(-1)^{l-k} \frac{1}{2^{l-k}}\binom{l}{k} y_{k} \sum_{j \in Z}(2 j+1)^{l-k} H_{2 j+1} . \tag{22}
\end{gather*}
$$

For $\phi$ with p.p.o. $=m,(19)$ is equivalent to the following equation (23) having no solutions:

$$
\binom{\sum_{j \in Z} H_{2 j}^{\top}-\frac{1}{2^{m}} I_{r}}{\sum_{j \in Z} H_{2 j+1}^{\top}-\frac{1}{2^{m}} I_{r}}\left(\begin{array}{c}
x_{1}  \tag{23}\\
\vdots \\
x_{r}
\end{array}\right)=\binom{p_{0}^{\top}}{p_{1}^{\top}} .
$$

Because $H(z), G(z)$ are symmetric which satisfy (11) and (14), the following matrix is nonsingular:

$$
\left(\begin{array}{cc}
\sum_{j \in Z} H_{2 j}-\frac{1}{2^{m}} I_{r} & \sum_{j \in Z} H_{2 j+1}-\frac{1}{2^{m}} I_{r}  \tag{24}\\
\sum_{j \in Z} G_{2 j} & \sum_{j \in Z} G_{2 j+1}
\end{array}\right) .
$$

We claim that the following system of linear equations (25) has solutions $\left(x_{1}, x_{2}, \ldots, x_{2 r}\right)^{\top}$ :

$$
\left(\begin{array}{cc}
\sum_{j \in Z} H_{2 j}^{\top}-\frac{1}{2^{m}} I_{r} & \sum_{j \in Z} G_{2 j+1}^{\top}  \tag{25}\\
\sum_{j \in Z} H_{2 j+1}^{\top}-\frac{1}{2^{m}} I_{r} & \sum_{j \in Z} G_{2 j+1}^{\top}
\end{array}\right)\left(\begin{array}{c}
x_{1} \\
\vdots \\
x_{2 r}
\end{array}\right)=\binom{p_{0}^{\top}}{p_{1}^{\top}} .
$$

Let

$$
\begin{equation*}
y_{m}=:\left(x_{1}, \ldots, x_{r}\right), \quad B=\left(\alpha_{1}^{\top}, \alpha_{2}^{\top}\right)^{\top} \tag{26}
\end{equation*}
$$

where

$$
\begin{align*}
& \alpha_{1}=\sqrt{\frac{1-1 / 4^{m}}{x_{r+1}^{2}+\cdots+x_{r+s}^{2}}}\left(x_{r+1}, \ldots, x_{r+s}, 0_{1 \times r-s}\right),  \tag{27}\\
& \alpha_{2}=\sqrt{\frac{1-1 / 4^{m}}{x_{r+s+1}^{2}+\cdots+x_{2 r}^{2}}}\left(0_{1 \times s}, x_{r+s+1}, \ldots, x_{2 r}\right) .
\end{align*}
$$

Define

$$
\begin{equation*}
c_{m}^{1}=\sqrt{\frac{x_{r+1}^{2}+\cdots+x_{r+s}^{2}}{1-1 / 4^{m}}}, \quad c_{m}^{2}=\sqrt{\frac{x_{r+s+1}^{2}+\cdots+x_{2 r}^{2}}{1-1 / 4^{m}}} . \tag{28}
\end{equation*}
$$

For (25), (26), and (27), we have that the row vectors $y_{l}, l=$ $0,1, \ldots, m,\left(c_{m}^{1}, c_{m}^{2}\right)$ and matrix $B$ satisfy (21), that is to say that the scale symbol $\bar{H}(z)$ satisfies sum rules order $\geq m+1$.

In the following, we will show that $\bar{H}(z)$ is orthogonal symmetric two-scale symbol. For $H(z), G(z)$ satisfying (11), (14), and the defined matrix $B$, it is easy to obtain

$$
\begin{align*}
& z^{2 N-1} \operatorname{diag}\left(s_{0},-1,1\right) \bar{H}\left(z^{-1}\right) \operatorname{diag}\left(s_{0},-1,1\right)=\bar{H}(z), \\
& \bar{H}(z) \bar{H}^{\top}\left(z^{-1}\right)+\bar{H}(-z) \bar{H}^{\top}\left(-z^{-1}\right)=\left(\begin{array}{cc}
I_{r} & 0_{r \times 2} \\
0_{2 \times r} & B B^{\top}+\frac{1}{4^{m}} I_{2}
\end{array}\right) \\
&=I_{r+2} . \tag{29}
\end{align*}
$$

By (21) and (26)-(29), we have showed that $\bar{H}(z)$ is orthogonal symmetric two-scale symbol and satisfies at least $m+1$ order sum rules, that is to say that iterative function system (1) with mask $\bar{H}$ generates $r+2$-dimension orthogonal symmetric scaling function vector with p.p.o. $\geq m+1$. This completes the proof of Theorem 3.

By applying Theorem 3 to a pair of orthogonal symmetric scaling and wavelet vector functions, not only do we obtain a new scaling vector function with higher p.p.o., but also some corresponding orthogonal symmetric multi-wavelet vector function can be easily constructed. Precisely, we have the following.

Theorem 4. Let $\phi, \psi$ be r-dimension orthogonal symmetric scaling and wavelet function vectors with two-scale symbols $H(z), G(z)$ satisfying (11) and (14), respectively, where $\phi$ has p.p.o. $=m$, and then we can construct $\bar{H}(z)$ according to Theorem 3, and $r+2$-dimension corresponding two-scale symbol $\bar{G}(z)$ by

$$
\bar{G}(z)=\left(\begin{array}{cc}
A G(z) & -2^{m-1}\left(1+z^{2 \gamma-1}\right) A B^{\top}  \tag{30}\\
0_{2 \times r} & \frac{-1+z^{2 \gamma-1}}{2} I_{2}
\end{array}\right)
$$

where $A=\operatorname{diag}\left(A_{1}, A_{2}\right), A_{1}, A_{2}$ being $s$ and $r-s$ order matrices, respectively, satisfies condition

$$
\begin{equation*}
A A^{\top}+4^{m} A B^{\top} B A^{\top}=I_{r} \tag{31}
\end{equation*}
$$

Defining $\bar{\phi}=\left(\phi^{\top}, \phi_{r+1}, \phi_{r+2}\right)^{\top}, \bar{\psi}=\left(\bar{\psi}_{1}, \ldots, \bar{\psi}_{r+2}\right)^{\top}$ generated by $\bar{H}(z), \bar{G}(z)$, then $\bar{\phi}, \bar{\psi}$ are orthogonal symmetric, and $\bar{\psi}$ has at least $m+1$ vanishing moments.
Proof. First, we show that $\bar{G}(z)$ satisfies symmetry conditions. In Theorem 3, we have proved that $\bar{H}(z)$ satisfies symmetric condition

$$
\begin{equation*}
z^{2 \gamma-1} \operatorname{diag}\left(s_{0},-1,1\right) \bar{H}\left(z^{-1}\right) \operatorname{diag}\left(s_{0},-1,1\right)=\bar{H}(z) \tag{32}
\end{equation*}
$$

For matrix $B$ given by (26) and (27), $A=\operatorname{diag}\left(A_{1}, A_{2}\right)$ with $A_{1}, A_{2}$ being $s$ and $r-s$ order matrices, respectively, and $G(z)$ satisfying (14), we have that $\bar{G}(z)$ satisfies symmetric condition

$$
\begin{equation*}
z^{2 \gamma-1} \operatorname{diag}\left(-s_{0}, 1,-1\right) \bar{G}\left(z^{-1}\right) \operatorname{diag}\left(s_{0},-1,1\right)=\bar{G}(z) \tag{33}
\end{equation*}
$$

$H(z), G(z)$ are orthogonal two-scale symbols, then it is easy to get

$$
\begin{align*}
& \bar{G}(z) \bar{G}^{\top}\left(z^{-1}\right)+\bar{G}(-z) \bar{G}^{\top}\left(-z^{-1}\right) \\
& \quad=\left(\begin{array}{cc}
A A^{\top}+4^{m} A B^{\top} B A^{\top} & 0_{r \times 2} \\
0_{2 \times r} & I_{2}
\end{array}\right),  \tag{34}\\
& \bar{H}(z) \bar{G}^{\top}\left(z^{-1}\right)+\bar{H}(-z) \bar{G}^{\top}\left(-z^{-1}\right) \\
& \quad=\left(\begin{array}{cc}
0_{r \times r} & 0_{r \times 2} \\
B A^{\top}-B A^{\top} & 0_{2 \times 2}
\end{array}\right)=0_{r+2} .
\end{align*}
$$

From (34), when condition (31) is satisfied, $\bar{H}(z)$ and $\bar{G}(z)$ constructed by Theorem 3 and Theorem 4 are orthogonal symmetric two-scale symbols and wavelet functions with at least $m+1$ order vanishing moments. This completes the proof of Theorem 4.

One of the important features of the construction procedure described in Theorem 3 and Theorem 4 is that it can be applied repeatedly without increasing the support (or filter length). In Theorem 3, $2 \times r$ matrix $B$ is decided by twoscale symbol $H(z)$. In Theorem 4, matrix $A=\operatorname{diag}\left(A_{1}, A_{2}\right)$ is constructed by matrix $B$ with condition (31). How can we obtain matrix $A=\operatorname{diag}\left(A_{1}, A_{2}\right)$ satisfying (31)? Considering matrix $B$ given by (26) and (27), we have the following theorem.

Theorem 5. Let $2 \times r$ matrix $B=\left(\begin{array}{cc}b_{1} & 0_{1 \times r-s} \\ 0_{1 \times s} & b_{2}\end{array}\right)$ be given by (26) and (27), and let matrix $A=\operatorname{diag}\left(A_{1}, A_{2}\right)$ satisfy (31) with $A_{1}, A_{2}$ being $s$ and $r-s$ order matrices, respectively, and then $A_{1}, A_{2}$ can be obtained by

$$
\begin{gather*}
A_{1}=\operatorname{diag}\left(\frac{1}{2^{m}}, I_{s-1}\right)\left(b_{1}^{0 \top}, v_{1}, \ldots, v_{s-1}\right)^{\top} \\
A_{2}=\operatorname{diag}\left(\frac{1}{2^{m}}, I_{r-s-1}\right)\left(b_{2}^{0 \top}, \bar{v}_{1}, \ldots, \bar{v}_{r-s-1}\right)^{\top} \tag{35}
\end{gather*}
$$

where $b_{1}^{0}=\left(1 /\left|b_{1}\right|\right) b=\left(2^{m} / \sqrt{4^{m}-1}\right) b_{1}, v_{1}, \ldots, v_{s-1}$ are unit orthogonal complement vectors of $b_{1}^{0 \top}$, and $b_{2}^{0}=\left(1 /\left|b_{2}\right|\right) b_{2}=$ $\left(2^{m} / \sqrt{4^{m}-1}\right) b_{2}, \bar{v}_{1}, \ldots, \bar{v}_{r-s-1}$ are unit orthogonal complement vectors of $b_{2}^{0^{\top}}$.

Proof. Condition (31) is equivalent to

$$
\begin{equation*}
I_{r}+4^{m} B^{\top} B=A^{-1}\left(A^{-1}\right)^{\top} . \tag{36}
\end{equation*}
$$

For the $A=\operatorname{diag}\left(A_{1}, A_{2}\right)$ and $B=\left(\begin{array}{cc}b_{1} & 0_{1 \times r-s} \\ 0_{1 \times s} & b_{2}\end{array}\right)$, we have

$$
\begin{equation*}
I_{s}+4^{m} b_{1}^{\top} b_{1}=A_{1}^{-1}\left(A_{1}^{-1}\right) \top, \quad I_{r-s}+4^{m} b_{2}^{\top} b_{2}=A_{2}^{-1}\left(A_{2}^{-1}\right)^{\top} \tag{37}
\end{equation*}
$$

Matrix $I_{s}+4^{m} b_{1}^{\top} b_{1}$ has eigenvalue $1+4^{m}\left|b_{1}\right|^{2}$ and 1. For (26) and (27), we have $1+4^{m}\left|b_{1}\right|^{2}=4^{m}$. Characteristic unit vectors $b_{1}^{0 \top}=(1 /|b|) b^{\top}$ correspond to eigenvalue $4^{m}$ and $v_{1}, \ldots, v_{s-1}$ which are orthogonal complement vectors of $b_{1}^{0 \top}$ and characteristic unit vectors correspond to eigenvalue 1 of matrix $I_{s}+4^{m} b^{\top}$ b, and then we can get

$$
\begin{align*}
\left(I_{s}\right. & \left.+4^{m} b_{1}^{\top} b_{1}\right)\left(b_{1}^{0 \top}, v_{1}, \ldots, v_{s-1}\right) \\
& =\left(b_{1}^{0^{\top}}, v_{1}, \ldots, v_{s-1}\right) \operatorname{diag}\left(4^{m}, I_{s-1}\right), \\
I_{s} & +4^{m} b_{1}^{\top} b_{1} \\
& =\left(b_{1}^{0 \top}, v_{1}, \ldots, v_{s-1}\right) \operatorname{diag}\left(4^{m}, I_{s-1}\right)\left(b_{1}^{0^{\top}}, v_{1}, \ldots, v_{s-1}\right)^{\top} . \tag{38}
\end{align*}
$$

From (37) and (38), we obtain

$$
\begin{equation*}
A_{1}=\operatorname{diag}\left(\frac{1}{2^{m}}, I_{s-1}\right)\left(b_{1}^{0 \top}, v_{1}, \ldots, v_{s-1}\right)^{\top} \tag{39}
\end{equation*}
$$

In the same way, we have

$$
\begin{equation*}
A_{2}=\operatorname{diag}\left(\frac{1}{2^{m}}, I_{r-s-1}\right)\left(b_{2}^{0 \top}, \bar{v}_{1}, \ldots, \bar{v}_{r-s-1}\right)^{\top} \tag{40}
\end{equation*}
$$

This completes the proof of Theorem 5.

## 4. Example

Applying Theorems 3, 4, and 5, it is easy to extend the $r$ dimension orthogonal symmetric multi-wavelet to $r+2$ dimension orthogonal symmetric multi-wavelet with vanishing moments increasing. In Theorem 3, if matrix $B$ is constructed by (27) with $\alpha_{1}=0_{1 \times r}$ or $\alpha_{2}=0_{1 \times r}$, we can extend the $r$-dimension orthogonal symmetric multi-wavelet to $r+1$-dimension orthogonal symmetric multi-wavelet with vanishing moments increasing without increasing the support of wavelet functions, and then we will give two examples to show it.

Example 6. Let

$$
\begin{align*}
& H(z)=\frac{1}{2}\left(\begin{array}{cc}
1+z & 0 \\
\frac{1}{2}-\frac{1}{2} z-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2} z
\end{array}\right)  \tag{41}\\
& G(z)=\frac{1}{2}\left(\begin{array}{cc}
\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2} z & \frac{1}{2}+\frac{1}{2} z \\
0 & 1-z
\end{array}\right)
\end{align*}
$$

be two-scale symbols and satisfy

$$
\begin{gather*}
z \operatorname{diag}(1,-1) H\left(z^{-1}\right) \operatorname{diag}(1,-1)=H(z) \\
z \operatorname{diag}(-1,1) G\left(z^{-1}\right) \operatorname{diag}(1,-1) \tag{42}
\end{gather*}
$$

and then (1) generates orthogonal symmetric 2D scaling function $\phi$ with p.p.o. $=1$, and $y_{0}=(1,0)$. From Theorem 3,
let $\alpha_{1}=(\sqrt{3} / 2,0), \alpha_{2}=(0,0)$, and define $B=\alpha_{1}, A=$ $\operatorname{diag}(1 / 2,1)$, and then we obtain $\bar{H}(z), \bar{G}(z)$ as follows:

$$
\begin{gather*}
\bar{H}(z)=\frac{1}{2}\left(\begin{array}{ccc}
1+z & 0 & 0 \\
\frac{1}{2}-\frac{1}{2} z & -\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2} z & 0 \\
\frac{\sqrt{3}}{2}(\sqrt{3}-\sqrt{3}) z & \frac{\sqrt{3}}{2}(1+z) & \frac{1+z}{4}
\end{array}\right), \\
\bar{G}(z)=\frac{1}{2}\left(\begin{array}{ccc}
-\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4} z & -\frac{1}{4}-\frac{1}{4} z & \frac{\sqrt{15}}{4}(1+z) \\
0 & 1-z & 0 \\
0 & 0 & -1+z
\end{array}\right) . \tag{43}
\end{gather*}
$$

Two-scale symbols $\bar{H}(z), \bar{G}(z)$ generate 3D scaling and wavelet function vectors $\bar{\phi}=\left(\phi_{1}, \phi_{2}, \phi_{3}\right)^{\top}$ with p.p.o. $=2$, $\bar{\psi}=\left(\psi_{1}, \psi_{2}, \psi_{3}\right)^{\top}$ with vanishing moment with order 2 , and $\phi_{1}, \psi_{2}, \psi_{3}$ symmetric, $\phi_{2}, \phi_{3}, \psi_{1}$ antisymmetric. $\bar{H}(z)$ satisfies sum rules with order 2 with $y_{0}=(1,0,0), y_{1}=$ $(1 / 2,-(1 / 2(\sqrt{3}+2)),-((\sqrt{3}+1) / 3(\sqrt{3}+2)))$.

Example 7. Let $\phi$ with p.p.o. $=3, \psi$ be 2 D orthogonal symmetric scaling and wavelet function, respectively, generated by two-scale symbols [4]

$$
\begin{align*}
& H(z)=\left(\begin{array}{ll}
H_{11}(z) & H_{12}(z) \\
H_{21}(z) & H_{22}(z)
\end{array}\right),  \tag{44}\\
& G(z)=\left(\begin{array}{ll}
G_{11}(z) & G_{12}(z) \\
G_{21}(z) & G_{22}(z)
\end{array}\right), \tag{45}
\end{align*}
$$

where

$$
\begin{align*}
H_{11}(z)= & \frac{10-3 \sqrt{10}}{80}(1+z)\left(1+(38+12 \sqrt{10}) z+z^{2}\right) \\
H_{21}(z)= & \frac{5 \sqrt{6}-3 \sqrt{15}}{80}(1-z)\left(1-10(3+\sqrt{10}) z+z^{2}\right) \\
& H_{12}(z)=\frac{5 \sqrt{6}-2 \sqrt{15}}{80}(1-z)(1+z)^{2} \\
H_{22}(z)= & \frac{5-3 \sqrt{10}}{1040}(1+z)\left(13-(10+6 \sqrt{10}) z+13 z^{2}\right) \\
& G_{11}(z)=\frac{5 \sqrt{6}-2 \sqrt{15}}{80}(1-z)^{2}(1+z) \\
G_{21}(z)= & -\frac{5-3 \sqrt{10}}{1040}(1-z)\left(13+(10+6 \sqrt{10}) z+13 z^{2}\right) \\
G_{12}(z)= & -\frac{10-3 \sqrt{10}}{80}(1-z)\left(1-(38+12 \sqrt{10}) z+z^{2}\right) \\
G_{22}(z)= & \frac{5 \sqrt{6}-3 \sqrt{15}}{80}(1+z)\left(1+10(3+\sqrt{10}) z+z^{2}\right) \tag{46}
\end{align*}
$$

which satisfy

$$
\begin{gather*}
z^{3} \operatorname{diag}(1,-1) H\left(z^{-1}\right) \operatorname{diag}(1,-1)=H(z) \\
z^{3} \operatorname{diag}(-1,1) G\left(z^{-1}\right) \operatorname{diag}(1,-1) \tag{47}
\end{gather*}
$$

By applying Theorems 3-5, we obtain a new pair of orthogonal scaling function $\bar{\phi}$ with p.p.o. $=4$ and multi-wavelet $\bar{\psi}$ that can be obtained from two-scale symbols $\bar{H}(z), \bar{G}(z)$ given by

$$
\begin{align*}
\bar{H}(z) & =\left(\begin{array}{cc}
H(z) & 0_{2 \times 1} \\
B G(z) & \frac{1+z^{3}}{16}
\end{array}\right), \\
\bar{G}(z) & =\left(\begin{array}{cc}
A G(z) & \frac{1+z^{3}}{2} \xi \\
0_{1 \times 2} & \frac{1-z^{3}}{2}
\end{array}\right), \tag{48}
\end{align*}
$$

where

$$
\begin{gather*}
B=\left(0, \frac{3 \sqrt{7}}{8}\right), \quad A=\operatorname{diag}\left(\frac{1}{8}, 1\right) \\
\xi=\left(-\frac{3 \sqrt{7}}{8}, 0\right)^{\top} \tag{49}
\end{gather*}
$$

In addition, $\bar{H}(z)$ satisfies sum rules with order 4 with the four vectors given by

$$
\begin{gather*}
y_{0}=[1,0,0], \quad y_{1}=\left[\frac{3}{2}, \frac{\sqrt{15}-\sqrt{6}}{6}, 0\right], \\
y_{2}=\left[\frac{17-\sqrt{10}}{6}, \frac{\sqrt{15}-\sqrt{6}}{2}, 0\right], \\
y_{3}=\left[\frac{3(8-\sqrt{10})}{4}, \frac{\sqrt{3}(1280 \sqrt{5}-1373)}{996}, \frac{\sqrt{7}(9 \sqrt{10}-35)}{581}\right] . \tag{50}
\end{gather*}
$$

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## Research Article

# Dynamics Analysis of a Class of Delayed Economic Model 

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#### Abstract

This investigation aims at developing a methodology to establish stability and bifurcation dynamics generated by a class of delayed economic model, whose state variable is described by the scalar delay differential equation of the form $\mathrm{d}^{2} p(t) / \mathrm{d} t^{2}=$ $-\mu \delta(p(t))(\mathrm{d} p(t) / \mathrm{d} t)-\mu b p\left(t-\tau_{1}\right)-\mu\left(a_{0} p\left(t-\tau_{2}\right) /\left(a_{1}+p\left(t-\tau_{2}\right)\right)\right)+\mu\left(d_{0}-g_{0}\right)$. At appropriate parameter values, linear stability and Hopf bifurcation including its direction and stability of the economic model are obtained. The main tools to obtain our results are the normal form method and the center manifold theory introduced by Hassard. Simulations show that the theoretically predicted values are in excellent agreement with the numerically observed behavior. Our results extend and complement some earlier publications.


## 1. Introduction

Trade cycles, business cycles, and fluctuations in the price and supply of various commodities have attracted the attention of economists for well over one hundred years and possible more than thousands of years [1]. In the case of the most models discussed earlier in the literature, it is assumed that each economic agent has instantaneous information about its own as well as its rivals' behavior. Many authors often attributed these fluctuations to instantaneous information factors. This assumption is mathematically convenient but does not fully describe real economic situations in which there are always time delays between the times when information is obtained and the times when the decisions are implemented. In recent years it has been recognized in continuous-time economic dynamics that a delay differential equation is useful to describe the periodic and aperiodic behavior of economic variables [2-7]. Time delays usually cause the models to generate not only periodic cycles but also chaotic behavior for certain values of the shape parameter of the production function. With the infinite dimensionality created by a fixed-time delay, even a single first-order equation is transformed into an equation with a sufficient number of degrees of freedom to permit the occurrence of complex dynamics involving chaotic phenomena. This
finding indicates that fixed-time delay models of a dynamic economy may explain various complex dynamic behaviors of the economic variables. For decades, a lot of effort has been devoted to deepen the understanding of economic complexity including chaotic behavior, stability, and basins of attractions. In [2], Matsumoto and Szidarovszky and in [3], Akio and Ferenc investigate a continuous-time neoclassical growth model with time delay and study the dynamics of the delayed model. In [4], Bélair and Mackey develop a model of price adjustment with production delays. In [5], Zhang et al. consider a differential-algebraic biological economic system with time delay and harvesting where the dynamics is logistic with carrying capacity proportional to prey population. Howroyd and Russel detect the stability conditions of delay output adjustment processes in a general N -firm oligopoly with fixed time delays [6]. Matsumoto and Szidarovszky introduce a fixed delay in production and a mound-shaped production function into the neoclassical one-sector growth model and show the birth of complex dynamics [2]. Considering about a delay in the production process, Li establish the single commodity price-inventory control model [7].

It is well known that persistent oscillations are one of the most ubiquitous forms by which economic phenomena may be observed [8]. Limit cycles are the simplest nonlinear
phenomena, for example, they are the simplest example of how the interaction between economic forces may compel a system to abandon its steady state and start to steadily oscillate. Just as reported by Manfredi and Fanti [9], the detection of stable oscillations, for example, stable limit cycles, in continuous-time systems, is intimately related with the notion of Hopf bifurcation. Hopf bifurcation is important in economics. There are at least three reasons. First, it is always the outcome of a fully endogenous interaction between (nonlinear) economic forces. Second, it is a local bifurcation, thus much in spirit with the common belief of our science by which economic systems are generally close to their equilibrium state. Third, because it implieslocal oscillations, which are the normal route through which disequilibrium manifests itself when the equilibrating forces operating in the economy are relaxed (e.g., the adjustment process of a Walrasian market). For instance, when oscillations persist in a market normally in equilibrium (in the absence of stochastic and seasonal perturbations), it is very likely that these oscillations are the outcome of a Hopf bifurcation. So it is necessary for us to research the Hopf bifurcation in the economic system.

In the case of the most models discussed in the mentioned literature, only one delay appears in the models. Considering that the consumer memory plays an important role in the process of economic activities, just as pointed out by Li [7], bringing another delay might be better candidates for some purposes and would be of great interest. In this paper, we will generalize an economic model with the help of bringing two delays. In specifying how consumer behavior affects commodity demand, we will assume that the behavior is influenced not only by the instantaneous price, but also by the information regarding past prices. As the fact that the quantity supplied may not increase infinitely during the price increasing, similar to [7], we also assume the supply function to be fractional linear function. This investigation aims at developing a methodology to establish stability and bifurcation dynamics generated by the new delayed economic model.

The organization of the rest of this paper is as follows. In Section 2, we establish an economic model with two delays combing with the model considered in [7]. In Section 3, we take the delays as the parameters and use the distribution theory of the transcendental equation root [10] and the theory of Hopf bifurcation about functional differential equation [11] to discuss the stability of the equilibrium point for economic system and the existence of Hopf bifurcation. In Section 4, we apply the normal form theory and center manifold theory to investigate the bifurcation direction and the stability of periodic solution. In Section 5, an example with numerical simulations is arranged to illustrate the obtained results.

## 2. Characterization of a Generalized Delayed Economic Model

Considering a single commodity market, the quantity of supplied and demanded can be regarded as the function of time, namely, $G(t)$ and $D(t)$. The inventory and the level of inventory are recorded, respectively, as $S(t)$ and $\bar{S}$.

Let $p(t)$ denote the price at time $t$, so that the rate of price increase is in proportion to the difference between $S(t)$ and $\bar{S}$, namely,

$$
\begin{equation*}
\frac{\mathrm{d} p(t)}{\mathrm{d} t}=-\mu(\bar{S}-S(t)), \quad \mu>0 \tag{1}
\end{equation*}
$$

where $\mu$ is a positive real number depending on the speed of price adjustment, recording $S(t)$ as

$$
\begin{equation*}
S(t)=S_{0}+\int_{0}^{t}(G(\varepsilon)-D(\varepsilon)) d \varepsilon \tag{2}
\end{equation*}
$$

In the traditional cobweb model, demand function is a function of price. If we consider price as the only factor that influences the quantity demanded, there will be certain limitations to reflect the regularity of price change. We should consider other factors influencing the demand such as the rate of the price increase. In [7], Li assume the demand function as

$$
\begin{equation*}
D(t)=d_{0}-b p(t)-\delta(p(t)) \frac{\mathrm{d} p(t)}{\mathrm{d} t} \tag{3}
\end{equation*}
$$

where $d_{0}>0, b>0, b$ represents the sensitive degree of consumers to the increase of commodity price. $\delta(p(t))$ is the level of price relying on the rate of increase. In terms of consumer behavior, we consider a class of consumers who base buying decisions on the past prices and recent price. Therefore, in this paper, we introduce a delay in demand function $D(t)$ and set up the demand function as shown later

$$
\begin{equation*}
D(t)=d_{0}-b p\left(t-\tau_{1}\right)-\delta(p(t)) \frac{\mathrm{d} p(t)}{\mathrm{d} t} \tag{4}
\end{equation*}
$$

In generally, supply function is monotone increasing about price, but consider that as price goes up, the supply could not unlimitedly increase, one can assume supply function as a fractional linear function as the following:

$$
\begin{equation*}
G(t)=g_{0}+\frac{a_{0} p(t)}{a_{1}+p(t)}, \tag{5}
\end{equation*}
$$

where $a_{0}>0, a_{1}>0, g_{0} \geq 0$.
Noticing the delay in the production process, supply function should be a function of past price, therefore, we can introduce another delay in supply function and record it as follows:

$$
\begin{equation*}
G(t)=g_{0}+\frac{a_{0} p\left(t-\tau_{2}\right)}{a_{1}+p\left(t-\tau_{2}\right)}, \tag{6}
\end{equation*}
$$

where $a_{0}, a_{1}, g_{0}, \tau_{1}, \tau_{2}$ are constants and $a_{0}>0, a_{1}>0, g_{0} \geq 0$, $\tau_{2} \geq \tau_{1} \geq 0$.

Substituting (2) to (1), calculating the derivation of both sides about time in (1), one can get

$$
\begin{equation*}
\frac{d^{2} p(t)}{\mathrm{d} t^{2}}=-\mu(G(t)-D(t)) \tag{7}
\end{equation*}
$$

Substituting (4) and (6) to (7), we can establish a single commodity price inventory control model with two delays take the following form:

$$
\begin{align*}
\frac{d^{2} p(t)}{\mathrm{d} t^{2}}= & -\mu \delta(p(t)) \frac{\mathrm{d} p(t)}{\mathrm{d} t}-\mu b p\left(t-\tau_{1}\right) \\
& -\mu \frac{a_{0} p\left(t-\tau_{2}\right)}{a_{1}+p\left(t-\tau_{2}\right)}+\mu\left(d_{0}-g_{0}\right) \tag{8}
\end{align*}
$$

Let $C^{1}=C^{1}\left(\left[-\tau_{2}, 0\right], R^{+}\right)$denote the Banach space of continuous and differentiable mapping from $\left[-\tau_{2}, 0\right]$ into $R^{+}$equipped with the Supremum Norm $\|\phi\|=$ $\sup _{s \in\left[-\tau_{2}, 0\right]}\left\{|\phi(s)|,\left|\phi^{\prime}(s)\right|\right\}$ for $\phi \in C^{1}$. The initial condition of (8) is

$$
\begin{equation*}
p(\theta)=\psi(\theta)>0, \quad \theta \in\left[-\tau_{2}, 0\right], \psi \in C^{1} \tag{9}
\end{equation*}
$$

We will consider the following basic assumptions to further investigate the stability and bifurcation dynamics of model (8).
$\left(\mathrm{H}_{1}\right)$ The inequality hold: $d_{0}>g_{0}$.
$\left(\mathrm{H}_{2}\right)$ The inequality hold: $\delta\left(p^{*}\right)>0$, where $p^{*}$ is a positive equilibrium point.
$\left(\mathrm{H}_{3}\right)$ The inequality hold: $b<a_{0} a_{1} /\left(a_{1}+p^{*}\right)^{2}$, where $p^{*}$ is a positive equilibrium point.

## 3. Stability Analysis and the Existence of Hopf Bifurcation

At first, we will show that system (8) has only one positive equilibrium point under some assumption. We state the following theorem.

Theorem 1. If the inequality $\left(H_{1}\right)$ holds: $d_{0}>g_{0}$, then system (8) has only one positive equilibrium point.

Proof. Without loss of generality, we may assume that

$$
\begin{equation*}
\frac{\mathrm{d} p(t)}{\mathrm{d} t}=q(t) \tag{10}
\end{equation*}
$$

then model (8) can be rewritten as the following:

$$
\begin{align*}
\frac{\mathrm{d} p(t)}{\mathrm{d} t} & =q(t) \\
\frac{\mathrm{d} q(t)}{\mathrm{d} t}= & -\mu \delta(p(t)) q(t)-\mu b p\left(t-\tau_{1}\right)  \tag{11}\\
& -\mu \frac{a_{0} p\left(t-\tau_{2}\right)}{a_{1}+p\left(t-\tau_{2}\right)}+\mu\left(d_{0}-g_{0}\right) .
\end{align*}
$$

Assume $p^{*}$ to be the equilibrium point of system (8), one can show that $\left(p^{*}, 0\right)$ is the equilibrium point of system (11). Therefore, one can obtain

$$
\begin{equation*}
b\left(p^{*}\right)^{2}+\left[a_{1} b+a_{0}-\left(d_{0}-g_{0}\right)\right] p^{*}-a_{1}\left(d_{0}-g_{0}\right)=0 \tag{12}
\end{equation*}
$$

Obviously, if $d_{0}>g_{0}$, we have

$$
\begin{gather*}
\Delta=\left[a_{1} b+a_{0}-\left(d_{0}-g_{0}\right)\right]^{2}+4 a_{1} b\left(d_{0}-g_{0}\right)>0, \\
-  \tag{13}\\
-\frac{a_{1}\left(d_{0}-g_{0}\right)}{b}<0 .
\end{gather*}
$$

Then (8) has only one positive equilibrium point. This completes the proof.

In real life, $p^{*}$ is the equilibrium price. With the help of coordinate translation

$$
\begin{gather*}
x(t)=p(t)-p^{*}, \\
y(t)=q(t), \tag{14}
\end{gather*}
$$

system (11) can be further rewritten as the following form:

$$
\begin{align*}
\frac{\mathrm{d} x(t)}{\mathrm{d} t}= & y(t) \\
\frac{\mathrm{d} y(t)}{\mathrm{d} t}= & -\mu \delta\left(x(t)+p^{*}\right) y(t)-\mu b\left(x\left(t-\tau_{1}\right)+p^{*}\right) \\
& -\mu \frac{a_{0}\left(x\left(t-\tau_{2}\right)+p^{*}\right)}{a_{1}+x\left(t-\tau_{2}\right)+p^{*}}+\mu\left(d_{0}-g_{0}\right) \tag{15}
\end{align*}
$$

Then the linearized system at $(0,0)$ is

$$
\begin{align*}
\frac{\mathrm{d} x(t)}{\mathrm{d} t}= & y(t) \\
\frac{\mathrm{d} y(t)}{\mathrm{d} t}= & -\mu \delta\left(p^{*}\right) y(t)-\mu b x\left(t-\tau_{1}\right)  \tag{16}\\
& -\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} x\left(t-\tau_{2}\right)
\end{align*}
$$

The characteristic equation of the linearized system (16) at $(0,0)$ takes the following form:

$$
\begin{equation*}
\lambda^{2}+\mu \delta\left(p^{*}\right) \lambda+\mu b e^{-\lambda \tau_{1}}+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} e^{-\lambda \tau_{2}}=0 \tag{17}
\end{equation*}
$$

It is well known that the equilibrium $(0,0)$ is asymptotically stable if all roots of the characteristic equation (17) have negative real parts. Now we reach the position to study the distribution of the roots of (17). We will consider three cases as follows: (Case 1): $\tau_{1}=0, \tau_{2}=0$; (Case 2): $\tau_{1}=0, \tau_{2}>0$; and (Case 3): $\tau_{2}>\tau_{1}>0$.

Case 1. $\tau_{1}=0, \tau_{2}=0$.
Proposition 2. If $\tau_{1}=0, \tau_{2}=0$ and the inequality $\left(\mathrm{H}_{2}\right)$ hold: $\delta\left(p^{*}\right)>0$, then the equilibrium point $(0,0)$ of system $(15)$ is asymptotically stable.

Proof. As the inequality $\left(\mathrm{H}_{2}\right)$ holds, then the characteristic equation (17) turns to be

$$
\begin{equation*}
\lambda^{2}+\mu \delta\left(p^{*}\right) \lambda+\mu b+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}}=0 \tag{18}
\end{equation*}
$$

It is obvious that $\mu a_{0} a_{1} /\left(a_{1}+p^{*}\right)^{2}>0$; from Hurwitz criterion, all roots of this equation have negative real parts; therefore, the equilibrium point $(0,0)$ of system (15) is asymptotically stable.

Case $2\left(\tau_{1}=0, \tau_{2}>0\right)$. If $\tau_{1}=0, \tau_{2}>0$, then the characteristic equation (17) takes the following form:

$$
\begin{equation*}
\lambda^{2}+\mu \delta\left(p^{*}\right) \lambda+\mu b+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} e^{-\lambda \tau_{2}}=0 \tag{19}
\end{equation*}
$$

Let $E(\lambda)=\lambda^{2}+\mu \delta\left(p^{*}\right) \lambda+\mu b+\left(\mu a_{0} a_{1} /\left(a_{1}+p^{*}\right)^{2}\right) e^{-\lambda \tau_{2}}$; then we obtain the following results.

Lemma 3. If the inequality $\left(H_{3}\right)$ holds: $b<a_{0} a_{1} /\left(a_{1}+p^{*}\right)^{2}$, and $\tau_{2}=\tau_{2 n}$, then $E(\lambda)=0$ has the only pair of purely imaginary roots $\lambda= \pm i \omega$, where

$$
\begin{array}{r}
\tau_{2}=\tau_{2 n} \\
=\frac{1}{\omega}\left[\arccos \frac{\left(-\omega^{2}+\mu b\right)\left(a_{1}+p^{*}\right)^{2}}{\mu a_{0} a_{1}}+2 n \pi\right] \\
n=0,1,2 \ldots,
\end{array}
$$

$\omega$

$$
=\left\{\left(\left(2 \mu b-\mu^{2} \delta^{2}\left(p^{*}\right)\right)\right.\right.
$$

$$
\left.+\sqrt{\left(2 \mu b-\mu^{2} \delta^{2}\left(p^{*}\right)\right)^{2}-4\left(\mu^{2} b^{2}-\frac{\left(\mu a_{0} a_{1}\right)^{2}}{\left(a_{1}+p^{*}\right)^{4}}\right)}\right)
$$

$$
\begin{equation*}
\left.\times(2)^{-1}\right\}^{1 / 2} \tag{20}
\end{equation*}
$$

Proof. If $\lambda=i \omega\left(\tau_{2}\right)\left(\omega\left(\tau_{2}\right)>0\right)$ is a root of (19), then

$$
\begin{align*}
& -\omega^{2}+i \mu \delta\left(p^{*}\right) \omega+\mu b+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)}  \tag{21}\\
& \times\left(\cos \omega \tau_{2}-i \sin \omega \tau_{2}\right)+C=0
\end{align*}
$$

Separating the real and imaginary parts, we have

$$
\begin{gathered}
-\omega^{2}+\mu b+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} \cos \omega \tau_{2}=0 \\
\mu \delta\left(p^{*}\right) \omega-\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} \sin \omega \tau_{2}=0 \\
\Downarrow \\
\cos \omega \tau_{2}=\frac{\omega^{2}-\mu b}{\mu a_{0} a_{1}}\left(a_{1}+p^{*}\right)^{2} \\
\sin \omega \tau_{2}=\frac{\delta\left(p^{*}\right) \omega}{a_{0} a_{1}}\left(a_{1}+p^{*}\right)^{2}
\end{gathered}
$$

Then we can obtain

$$
\begin{equation*}
\omega^{4}+\left(\mu^{2} \delta^{2}\left(p^{*}\right)-2 \mu b\right) \omega^{2}+\mu^{2} b^{2}-\frac{\left(\mu a_{0} a_{1}\right)^{2}}{\left(a_{1}+p^{*}\right)^{4}}=0 \tag{23}
\end{equation*}
$$

The root of (23) can be expressed as follows:
$\omega$

$$
=\left\{\left(\left(2 \mu b-\mu^{2} \delta^{2}\left(p^{*}\right)\right)\right.\right.
$$

$$
\begin{align*}
& \left.+\sqrt{\left(2 \mu b-\mu^{2} \delta^{2}\left(p^{*}\right)\right)^{2}-4\left(\mu^{2} b^{2}-\frac{\left(\mu a_{0} a_{1}\right)^{2}}{\left(a_{1}+p^{*}\right)^{4}}\right)}\right) \\
& \left.\times(2)^{-1}\right\} \tag{24}
\end{align*}
$$

Then we obtain

$$
\begin{align*}
& \tau_{2}= \tau_{2 n} \\
&=\frac{1}{\omega}\left[\arccos \frac{\left(-\omega^{2}+\mu b\right)\left(a_{1}+p^{*}\right)^{2}}{\mu a_{0} a_{1}}+2 n \pi\right]  \tag{25}\\
& n=0,1,2 \ldots
\end{align*}
$$

Let $E(\lambda)=0$; if $\lambda= \pm i \omega$ is not the only pair of purely imaginary roots, then we have

$$
\begin{equation*}
\left.\frac{\partial E}{\partial \lambda}\right|_{\substack{\tau=\tau_{2 n} \\ \lambda=i \omega}}=0 . \tag{26}
\end{equation*}
$$

From (17), one can get

$$
\begin{gather*}
2 i \omega+\mu \delta\left(p^{*}\right)-\tau_{2 n} \frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} e^{-i \omega \tau_{2 n}}=0 \\
-\omega^{2}+i \mu \delta\left(p^{*}\right) \omega+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} e^{-i \omega \tau_{2 n}}+\mu b=0 \tag{27}
\end{gather*}
$$

That is to say, $\left(\omega^{2}-\mu b\right) \tau_{2 n}=\mu \delta\left(p^{*}\right)$, which contradicts (25). This completes the proof.

Denote $\lambda\left(\tau_{2}\right)=\alpha\left(\tau_{2}\right)+i \omega\left(\tau_{2}\right)$; then the root of (19) satisfies: $\alpha\left(\tau_{2 n}\right)=0, \omega\left(\tau_{2 n}\right)=\omega$.

Lemma 4. If one chooses

$$
\begin{align*}
\tau_{2} & =\tau_{2 n} \\
& =\frac{1}{\omega}\left[\arccos \frac{\left(-\omega^{2}+\mu b\right)\left(a_{1}+p^{*}\right)^{2}}{\mu a_{0} a_{1}}+2 n \pi\right], \tag{28}
\end{align*}
$$

$$
n=0,1,2 \ldots
$$

and $E(\lambda)=0$ has the only pair of purely imaginary roots $\lambda=$ $\pm i \omega$, then one has

$$
\begin{equation*}
\left.\frac{\operatorname{dRe}\left(\lambda\left(\tau_{2}\right)\right)}{\mathrm{d} \tau_{2}}\right|_{\tau_{2}=\tau_{2 n}}>0 \tag{29}
\end{equation*}
$$

Proof. Taking the derivative of $E(\lambda)=0$ with respect to $\tau_{2}$, we get

$$
\begin{align*}
& 2 \lambda \frac{\mathrm{~d} \lambda}{\mathrm{~d} \tau_{2}}+\mu \delta\left(p^{*}\right) \frac{\mathrm{d} \lambda}{\mathrm{~d} \tau_{2}}+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} e^{-\lambda \tau_{2}}\left(-\lambda-\tau_{2} \frac{\mathrm{~d} \lambda}{\mathrm{~d} \tau_{2}}\right)=0, \\
& \left.\operatorname{Re}\left(\frac{\mathrm{~d} \lambda}{\mathrm{~d} \tau_{2}}\right)^{-1}\right|_{\tau_{2}=\tau_{2 n}} \\
& \quad=\left[\left(2 \mu b-\mu^{2} \delta^{2}\left(p^{*}\right)\right)^{2}-4\left(\mu^{2} b^{2}-\frac{\left(\mu a_{0} a_{1}\right)^{2}}{\left(a_{1}+p^{*}\right)^{4}}\right)\right]^{1 / 2}>0 . \tag{30}
\end{align*}
$$

Therefore

$$
\begin{align*}
& \operatorname{sgn}\left\{\left.\operatorname{Re}\left(\frac{\mathrm{d} \lambda\left(\tau_{2}\right)}{\mathrm{d} \tau_{2}}\right)^{-1}\right|_{\tau_{2}=\tau_{2 n}}\right\}  \tag{31}\\
& \quad=\operatorname{sgn}\left\{\left.\frac{\mathrm{d} \operatorname{Re}\left(\lambda\left(\tau_{2}\right)\right)}{\mathrm{d} \tau_{2}}\right|_{\tau_{2}=\tau_{2 n}}\right\}>0 .
\end{align*}
$$

This completes the proof.

Based on the lemmas presented previous and the classical Hopf-Bifurcation-Theorem (see, [11, pages 245-249]), we have the following result.

Theorem 5. Assume that the inequalities $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{3}\right)$ hold, then one has the following result.
(1) If $\tau_{2} \in\left[0, \tau_{20}\right)$, all roots of (19) have negative real parts. Namely, the equilibrium $(0,0)$ of system $(15)$ is locally asymptotically stable.
(2) If $\tau_{2}=\tau_{20}$, (19) have a pair of purely imaginary roots $\pm i \omega$, all the other roots have negative real parts. That is to say, system (15) undergoes a Hopf bifurcation at $\tau_{2}=\tau_{20}$.
(3) If $\tau_{2}>\tau_{20}$, (19) has roots of positive real parts. Namely the equilibrium $(0,0)$ of system (15) is unstable.

Case $3\left(\tau_{2}>\tau_{1}>0\right)$. We now discuss the stability of equilibrium $(0,0)$ when $\tau_{1}>0, \tau_{2}=\tau \in\left[0, \tau_{20}\right)$.

Let $\lambda=i \omega\left(\tau_{1}\right)\left(\omega\left(\tau_{1}\right)>0\right)$ be the root of (17) and substitute it into (17), we have

$$
\begin{align*}
& -\omega^{2}+i \mu \delta\left(p^{*}\right) \omega+\mu b\left(\cos \omega \tau_{1}-i \sin \omega \tau_{1}\right) \\
& \quad+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}}(\cos \omega \tau+i \sin \omega \tau)=0 \tag{32}
\end{align*}
$$

Separating the real and imaginary parts, we have

$$
\begin{gather*}
\mu b \cos \omega \tau_{1}=\omega^{2}-\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} \cos \omega \tau \\
\mu b \sin \omega \tau_{1}=\mu \delta\left(p^{*}\right) \omega+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} \sin \omega \tau \tag{33}
\end{gather*}
$$

Then we have

$$
\begin{align*}
& \omega^{4}+\mu^{2} \delta^{2}\left(p^{*}\right) \omega^{2}+\left(\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}}\right)^{2}-\mu^{2} b^{2} \\
& \quad-2 \frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} \omega^{2} \cos \omega \tau-2 \delta\left(p^{*}\right) \frac{\mu^{2} a_{0} a_{1} \omega}{\left(a_{1}+p^{*}\right)^{2}} \sin \omega \tau=0 \tag{34}
\end{align*}
$$

If (34) has no root or negative root, all the roots of (17) have negative real part. If it has positive roots, we know that the number of positive roots is finite. Denote them to be $\omega_{i}(i=1,2,3 \ldots, N)$. From (33), we can get

$$
\begin{align*}
& \tau_{1 i}^{(j)} \\
& =\frac{1}{\omega_{i}}\left(\arccos \frac{\omega_{i}^{2}-\left(\mu a_{0} a_{1} /\left(a_{1}+p^{*}\right)^{2}\right) \cos \omega_{i} \tau}{\mu b}+2 j \pi\right) \\
& (j=0,1,2, \ldots) . \tag{35}
\end{align*}
$$

Let $\tau_{1}^{0}=\tau_{1 i_{0}}^{(0)}=\min \left\{\tau_{1 i}^{0}\right\}$, let $i \in\{0,1,2, \ldots, N\}$, and let $\omega_{0}=\omega_{i_{0}}$; if $\tau_{2}=\tau \in\left[0, \tau_{20}\right)$, (16) has a pair of purely imaginary roots $\pm i \omega_{0}$ at $\tau_{1}=\tau_{1}^{0}$, we can also prove that the purely imaginary roots $\pm i \omega_{0}$ are simple. Taking the derivative with respect to $\tau_{1}$, we can get

$$
\begin{align*}
& 2 \lambda \frac{\mathrm{~d}\left(\lambda\left(\tau_{1}\right)\right)}{\mathrm{d} \tau_{1}}+\mu \delta\left(p^{*}\right) \frac{\mathrm{d} \lambda\left(\tau_{1}\right)}{\mathrm{d} \tau_{1}} \\
& +\mu b e^{-\lambda \tau_{1}}\left(-\lambda-\tau_{1} \frac{\mathrm{~d}\left(\lambda\left(\tau_{1}\right)\right)}{\mathrm{d} \tau_{1}}\right)  \tag{36}\\
& \quad-\frac{\mu a_{0} a_{1} \tau}{\left(a_{1}+p^{*}\right)^{2}} e^{-\lambda \tau} \frac{\mathrm{d}\left(\lambda\left(\tau_{1}\right)\right)}{\mathrm{d} \tau_{1}}=0
\end{align*}
$$

Then, we have

$$
\begin{align*}
&\left.\operatorname{Re}\left(\frac{\mathrm{d} \lambda}{\mathrm{~d} \tau_{1}}\right)^{-1}\right|_{\substack{\tau=\tau_{1}^{0} \\
\lambda=i \omega_{0}}} \\
&= \frac{2 \cos \omega_{0} \tau_{1}^{0}}{\mu b}+\frac{\delta\left(p^{*}\right) \sin \omega_{0} \tau_{1}^{0}}{b \omega_{0}}  \tag{37}\\
&-\frac{\left(a_{0} a_{1} \tau /\left(a_{1}+p^{*}\right)^{2}\right) \sin \omega_{0}\left(\tau_{1}^{0}-\tau\right)}{b \omega_{0}}
\end{align*}
$$

Therefore, we have the following theorem.

Theorem 6. Assume that the inequalities $\left(H_{1}\right)-\left(H_{3}\right)$ hold, then one has the following result.
(1) If $\tau_{1} \in\left[0, \tau_{1}^{0}\right)$, then all the roots of (17) have negative real parts. One can get that the equilibrium $(0,0)$ of system (15) is locally asymptotically stable.
(2) If $\tau_{1}=\tau_{1}^{0}, \tau_{2}=\tau \in\left[0, \tau_{20}\right)$ and $2 \cos \omega_{0} \tau_{1}^{0} / \mu b-$ $\delta\left(p^{*}\right) \sin \omega_{0} \tau_{1}^{0} / b \omega_{0}+\left(\left(a_{0} a_{1} \tau /\left(a_{1}+p^{*}\right)^{2}\right) \sin \omega_{0}\left(\tau_{1}^{0}-\right.\right.$ $\tau)) / b \omega_{0} \neq 0$, then system (15) undergoes Hopf bifurcations at ( 0,0 ).

## 4. Direction and Stability of the Bifurcation

In this section, formula for determining the direction of Hopf bifurcation and the stability of bifurcation periodic solution of system (11) at $\tau_{1}=\tau_{1}^{0}, \tau_{2}=\tau \in\left[0, \tau_{20}\right)$ will be presented by employing the normal form method and center manifold theorem introduced by Hassard et al. in [12]. More precisely, we will compute the reduced system on the center manifold with the pair of conjugate complex, purely imaginary solution of the characteristic equation (17). By this reduction, we can determine the Hopf bifurcation direction, that is, to answer the question of whether the bifurcation branch of periodic solution exists locally for supercritical bifurcation or subcritical bifurcation.

Let $\tau_{1}=\tau_{1}^{0}+k$, and let $k \in R$; then $k=0$ is a critical value of Hopf bifurcation of system (11). With the translation $t \rightarrow t / \tau_{1}$, let $\mu_{1}(t)=x\left(\tau_{1} t\right), \mu_{2}(t)=y\left(\tau_{1} t\right)$, we rewrite system (15) as follows:

$$
\begin{align*}
\frac{\mathrm{d} \mu_{1}(t)}{\mathrm{d} t}= & \left(\tau_{1}^{0}+k\right) y(t) \\
\frac{\mathrm{d} \mu_{2}(t)}{\mathrm{d} t}= & \left(\tau_{1}^{0}+k\right) \\
& \times\left[-\mu \delta\left(\mu_{1}(t)+p^{*}\right) \mu_{2}(t)\right.  \tag{38}\\
& -\mu b\left(\mu_{1}(t-1)+p^{*}\right) \\
& \left.-\mu \frac{a_{0}\left(\mu_{1}\left(t-\tau^{*}\right)+p^{*}\right)}{a_{1}+\mu_{1}\left(t-\tau^{*}\right)+p^{*}}+\mu\left(d_{0}-g_{0}\right)\right]
\end{align*}
$$

where $\tau^{*}=\tau_{2} / \tau_{1}$. Furthermore, we can obtain the linear system of (38) as mentioned later

$$
\begin{align*}
& \frac{\mathrm{d} \mu_{1}(t)}{\mathrm{d} t}=\left(\tau_{1}^{0}+k\right) \mu_{2}(t) \\
& \frac{\mathrm{d} \mu_{2}(t)}{\mathrm{d} t}=\left(\tau_{1}^{0}+k\right)\left[-\mu b \mu_{1}(t-1)-\mu \delta\left(p^{*}\right) \mu_{2}(t)\right.  \tag{39}\\
& \\
& \left.\quad-\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} \mu_{1}\left(t-\tau^{*}\right)\right]
\end{align*}
$$

while the nonlinear term is

$$
\begin{align*}
f= & \left(\tau_{1}^{0}+k\right) \\
& \times\left(\begin{array}{l}
0 \\
-\mu \delta^{\prime}\left(p^{*}\right) \mu_{1}(t) \mu_{2}(t)-\frac{\mu \delta^{\prime \prime}\left(p^{*}\right)}{2!} \mu_{1}^{2}(t) \mu_{2}(t) \\
+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{3}} \mu_{1}^{2}\left(t-\tau^{*}\right)-\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{4}} \mu_{1}^{3}\left(t-\tau^{*}\right)+\cdots
\end{array}\right) . \tag{40}
\end{align*}
$$

Let $\mu(t)=\left(\mu_{1}(t), \mu_{2}(t)\right)^{T}$, let $\mu_{t}(\theta)=\mu(t+\theta)$, let $\theta \in$ $\left[-\tau^{*}, 0\right]$, let $\Phi=\left(\Phi_{1}, \Phi_{1}\right)^{T} \in C\left[-\tau^{*}, 0\right]$, and let $L_{k} \Phi=$ $A_{1}(k) \Phi(0)+A_{2}(k) \Phi(-1)+A_{3}(k) \Phi\left(-\tau^{*}\right)$, where $L_{k}$ is a linear operator,

$$
\begin{gather*}
A_{1}(k)=\left(\tau_{1}^{0}+k\right)\left(\begin{array}{cc}
0 & 1 \\
0 & -\mu \delta\left(P^{*}\right)
\end{array}\right) \\
A_{2}(k)=\left(\tau_{1}^{0}+k\right)\left(\begin{array}{cc}
0 & 0 \\
-\mu b & 0
\end{array}\right)  \tag{41}\\
A_{3}(k)=\left(\tau_{1}^{0}+k\right)\left(\begin{array}{cc}
0 & 0 \\
-\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} & 0
\end{array}\right)
\end{gather*}
$$

As $L_{k}$ is a one-parameter family of bounded linear operator in $C\left[-\tau^{*}, 0\right]$, by the Rise Representation Theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta, k), \theta \in\left[-\tau^{*}, 0\right]$ such that $L_{k} \Phi=$ $\int_{-\tau^{*}}^{0} \Phi(\theta) \mathrm{d} \eta(\theta, k), \Phi \in C\left[-\tau^{*}, 0\right]$. Actually, we can take $\eta(\theta, k)=A_{1}(k) \delta(\theta)+A_{2}(k) \delta(\theta+1)+A_{3}(k) \delta\left(\theta+\tau^{*}\right)$, where $\delta$ is Dirac delta function.

Next, we define

$$
\begin{align*}
A(k) \Phi= & \begin{cases}\frac{\mathrm{d} \Phi(\theta)}{\mathrm{d} \theta}, & \theta \in\left[-\tau^{*}, 0\right) \\
\int_{-\tau^{*}}^{0} \mathrm{~d} \eta(\xi, k) \Phi(\xi)= & L_{k} \Phi, \\
& \theta=0\end{cases}  \tag{42}\\
& R(k) \Phi= \begin{cases}0, & \theta \in\left[-\tau^{*}, 0\right) \\
f(k, \Phi), & \theta=0,\end{cases} \tag{43}
\end{align*}
$$

where $\Phi \in C\left(\left[-\tau^{*}, 0\right], R^{2}\right)$. We can rewrite (38) as

$$
\begin{equation*}
\mu_{t}^{\prime}=A(k) \mu_{t}+R(k) \mu_{t} \tag{44}
\end{equation*}
$$

where $\mu_{t}=\mu_{t}(\theta)$.
Denote $A(0)=A, R(0)=R, A_{i}(0)=A_{i}(i=1,2,3)$, $\eta(\theta, 0)=\eta(\theta)$.

For $\Psi \in C^{1}\left(\left[0, \tau^{*}\right], R^{2}\right)$, the adjoint operator $A^{*}$ of $A$ is defined as

$$
A^{*} \Psi(s)= \begin{cases}-\frac{\mathrm{d} \Psi(s)}{\mathrm{d} s}, & s \in\left(0, \tau^{*}\right]  \tag{45}\\ \int_{-\tau^{*}}^{0} \mathrm{~d} \eta^{T}(t) \Psi(-t), & s=0\end{cases}
$$

where $\eta^{T}$ is the transpose of $\eta$.
We define the bilinear form

$$
\begin{align*}
\langle\Phi(s), \Psi(\theta)\rangle= & \overline{\Psi(0)}^{T} \Phi(0) \\
& -\int_{-\tau^{*}}^{0} \int_{0}^{\theta} \overline{\Psi(\xi-\theta)}^{T} \mathrm{~d} \eta(\theta) \Phi(\xi) \mathrm{d} \xi \tag{46}
\end{align*}
$$

where $\Phi \in C\left(\left[-\tau^{*}, 0\right], C^{2}\right)$ and $\Psi \in C\left(\left[-\tau^{*}, 0\right],\left(C^{2}\right)^{*}\right)$.

We easily obtain that $\pm i \tau_{0} \omega_{0}$ are eigenvalues of (17) by the translation $t \rightarrow t / \tau_{1}$. Then we have the following lemma.

Lemma 7. $q(\theta)=(1, \alpha)^{T} e^{i \tau_{1}^{0} \omega_{0} \theta}$ is the eigenvector corresponding to $i \tau_{0} \omega_{0}$ and $q^{*}(s)=D(1, \beta)^{T} e^{i \tau_{1}^{0} \omega_{0} s}$ is the eigenvector of $A^{*}$ corresponding to $-i \tau_{0} \omega_{0} .\left\langle q^{*}(s), q(\theta)\right\rangle=1,\left\langle q^{*}(s), \overline{q^{*}(\theta)}\right\rangle=0$, where
$\alpha=\frac{\mu b e^{-i \omega_{0} \tau_{1}^{0}}+\left(\mu a_{0} a_{1} /\left(a_{1}+p^{*}\right)^{2}\right) e^{-i \omega_{0} \tau_{1}^{0} \tau^{*}}}{\mu \delta\left(p^{*}\right)+i \omega_{0}}$,
$\beta=-\frac{1}{\mu \delta\left(p^{*}\right)-i \omega_{0}}$,
$\bar{D}=\frac{1}{\bar{\beta}+\alpha-\mu b \tau_{1}^{0} e^{-i \omega_{0} \tau_{1}^{0}}-\tau_{1}^{0} \tau^{*}\left(\mu a_{0} a_{1} /\left(a_{1}+p^{*}\right)^{2}\right) e^{-i \omega_{0} \tau_{1}^{0} \tau^{*}}}$.

Proof. Letting $q(\theta)$ be the eigenvector of corresponding to $i \tau_{0} \omega_{0}$, we have

$$
\begin{equation*}
A q(\theta)=\frac{\mathrm{d} q(\theta)}{\mathrm{d} \theta}=i \tau_{1}^{0} \omega_{0} q(\theta), \quad \theta \in\left[-\tau^{*}, 0\right) \tag{48}
\end{equation*}
$$

Calculating (48), we obtain

$$
\begin{equation*}
q(\theta)=(1, \alpha)^{T} e^{i \tau_{1}^{0} \omega_{0} \theta}, \quad \theta \in\left[-\tau^{*}, 0\right) \tag{49}
\end{equation*}
$$

where $\alpha$ is constant.
From (43), we can obtain

$$
\begin{equation*}
A q(0)=i \tau_{1}^{0} \omega_{0} q(0)=\int_{-\tau^{*}}^{0} \mathrm{~d} \eta(\xi)(1, \alpha)^{T} e^{i \tau_{1}^{0} \omega_{0} \xi}, \quad \theta=0 \tag{50}
\end{equation*}
$$

Therefore, one can show that

$$
\begin{equation*}
\alpha=-\frac{\mu b e^{-i \omega_{0} \tau_{1}^{0}}+\left(\mu a_{0} a_{1} /\left(a_{1}+p^{*}\right)^{2}\right) e^{-i \omega_{0} \tau_{1}^{0} \tau^{*}}}{\mu \delta\left(p^{*}\right)+i \omega_{0}} . \tag{51}
\end{equation*}
$$

Letting $q^{*}(s)=D(1, \beta)^{T} e^{i \tau_{0} \omega_{0} s}$ be the eigenvector of $A^{*}$ corresponding to $-i \tau_{0} \omega_{0}$, based on (48), (50), we have

$$
\begin{equation*}
A^{*} q^{*}(0)=-i \tau_{1}^{0} \omega_{0} q^{*}(0)=\int_{-\tau^{*}}^{0} \mathrm{~d} \eta^{T}(t) q(-t), \quad \theta=0 \tag{52}
\end{equation*}
$$

It is easy to obtain that $\beta=-1 /\left(\mu \delta\left(p^{*}\right)-i \omega_{0}\right)$ and $q^{*}(0)=$ $D\left(1,-1 /\left(\mu \delta\left(p^{*}\right)-i \omega_{0}\right)\right)$.

Now we compute $\left\langle q^{*}(s), q(\theta)\right\rangle$ as the following:

$$
\begin{align*}
& \left\langle q^{*}(s), q(\theta)\right\rangle \\
& =\overline{q^{*}(0)} q(0)-\int_{-\tau^{*}}^{0} \int_{0}^{\theta} \overline{q^{*}(\xi-\theta)} \mathrm{d} \eta(\theta) q(\xi) \mathrm{d} \xi \\
& =\bar{D}(1, \bar{\beta})\binom{1}{\alpha}-\int_{-\tau^{*}}^{0} \bar{D}(1, \bar{\beta}) e^{i \tau_{1}^{0} \omega_{0}(\theta-\xi)} \mathrm{d} \eta(\theta) \\
& \times(1, \alpha)^{T} e^{i \tau_{1}^{0} \omega_{0} \xi} \mathrm{~d} \xi \\
& =\bar{D}(1, \bar{\beta})\binom{1}{\alpha}-\int_{-\tau^{*}}^{0} \bar{D}(1, \bar{\beta}) \mathrm{d} \eta(\theta)(1, \alpha)^{T} \theta e^{i \tau_{1} \omega_{0} \theta} \\
& =\bar{D}(1, \bar{\beta})\binom{1}{\alpha}-\left[\bar{D}(1, \bar{\beta})\left(-\tau^{*}\right) e^{-i \tau_{1}^{0} \omega_{0} \tau^{*}} \tau_{1}^{0}\right. \\
& \times\left(\begin{array}{cc}
0 & 0 \\
-\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{2}} & 0
\end{array}\right)\binom{1}{\alpha}+\bar{D}(1, \bar{\beta})(-1) e^{-i \tau_{1}^{0} \omega_{0}} \tau_{1}^{0} \\
& \times\left(\begin{array}{cc}
0 & 0 \\
-\mu b & 0
\end{array}\right)\binom{1}{\alpha} \\
& =0 \text {. } \tag{53}
\end{align*}
$$

Then we have

$$
\bar{D}
$$

$$
\begin{equation*}
=\frac{1}{\bar{\beta}+\alpha-\mu b \tau_{1}^{0} e^{-i \omega_{0} \tau_{1}^{0}}-\tau_{1}^{0} \tau^{*}\left(\mu a_{0} a_{1} /\left(a_{1}+p^{*}\right)^{2}\right) e^{-i \omega_{0} \tau_{1}^{0} \tau^{*}}} . \tag{54}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
& -i \tau_{1}^{0} \omega_{0}\left\langle q^{*}(s), \bar{q}(\theta)\right\rangle \\
& \quad=\left\langle q^{*}(s), A \bar{q}(\theta)\right\rangle=\left\langle A^{*} q^{*}(s), \bar{q}(\theta)\right\rangle  \tag{55}\\
& \quad=\left\langle-i \tau_{1}^{0} \omega_{0} q^{*}(s), \bar{q}(\theta)\right\rangle=i \tau_{1}^{0} \omega_{0}\left\langle q^{*}(s), \bar{q}(\theta)\right\rangle,
\end{align*}
$$

therefore, we have $\left\langle q^{*}(s), \bar{q}(\theta)\right\rangle=0$. This completes the proof.

In the remainder of this section, we use the same notation as in [12]. We first compute the center manifold $C_{0}$ at $k=0$.

Let $\mu_{t}$ be the solution of (44) when $k=0$, and define

$$
\begin{gather*}
W(z, \bar{z}, \theta)=\mu_{t}(\theta)-2 \operatorname{Re}\{z(t) q(\theta)\},  \tag{56}\\
z(t)=\left\langle q^{*}, \mu_{t}\right\rangle \tag{57}
\end{gather*}
$$

We have

$$
\begin{equation*}
z^{\prime}(t)=i \tau_{1}^{0} \omega_{0} z+{\overline{q^{*}(0)}}^{T} f_{0}(z, \bar{z}) \tag{58}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{0}(z, \bar{z})=f(0, W(z, \bar{z}, \theta)+2 \operatorname{Re}\{z q(\theta)\}) \tag{59}
\end{equation*}
$$

On the center manifold, we have

$$
\begin{align*}
W(t, \theta) & =W(z, \bar{z}, \theta) \\
& =W_{20}(\theta) \frac{z^{2}}{2}+W_{11}(\theta) z \bar{z}+W_{02}(\theta) \frac{\bar{z}^{2}}{2} \cdots . \tag{60}
\end{align*}
$$

In fact, $z, \bar{z}$ are local coordinates for center manifold $C_{0}$ in the direction of $q^{*}, \bar{q}^{*}$. Noting that $W$ is real if $\mu_{t}$ is real, we consider only real solutions in this paper. We rewrite (58) as follows:

$$
\begin{equation*}
z^{\prime}(t)=i \tau_{1}^{0} \omega_{0} z+g(z, \bar{z}) \tag{61}
\end{equation*}
$$

where

$$
\begin{equation*}
g(z, \bar{z})=g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+g_{21} \frac{z^{2} \bar{z}}{2}+\cdots \tag{62}
\end{equation*}
$$

From (56), we have

$$
\begin{align*}
\mu_{t}(\theta) & =W(z, \bar{z}, \theta)+2 \operatorname{Re}\{z(t) q(\theta)\}  \tag{63}\\
& =W(z, \bar{z}, \theta)+z(t) q(\theta)+\overline{z(t) q(\theta)}
\end{align*}
$$

Then we have

$$
\begin{align*}
& \binom{\mu_{1 t}(\theta)}{\mu_{2 t}(\theta)} \\
& \quad=\binom{1}{\alpha} e^{i \tau_{1}^{0} \omega_{0} \theta} z(t)+\binom{1}{\alpha} e^{-i \tau_{1}^{0} \omega_{0} \theta} \overline{z(t)} \\
& \quad+\binom{W_{20}^{(1)}(\theta)}{W_{20}^{(2)}(\theta)} \frac{z^{2}}{2}+\binom{W_{11}^{(1)}(\theta)}{W_{11}^{(2)}(\theta)} z \bar{z}  \tag{64}\\
& \quad+\binom{W_{02}^{(1)}(\theta)}{W_{02}^{(2)}(\theta)} \frac{\bar{z}^{2}}{2}+\cdots .
\end{align*}
$$

It is easy to obtain

$$
\begin{align*}
\mu_{1}(t)= & z(t)+\overline{z(t)}+W_{20}^{(1)}(0) \frac{z^{2}}{2} \\
& +W_{11}^{(1)}(0) z \bar{z}+W_{02}^{(1)}(0) \frac{\bar{z}^{2}}{2}+\cdots, \\
\mu_{1}(t-1)= & z(t) e^{-i \tau_{1}^{0} \omega_{0}}+\overline{z(t)} e^{i \tau_{1}^{0} \omega_{0}}+W_{20}^{(1)}(-1) \frac{z^{2}}{2} \\
& +W_{11}^{(1)}(-1) z \bar{z}+W_{02}^{(1)}(-1) \frac{\bar{z}^{2}}{2}+\cdots, \\
\mu_{1}\left(t-\tau^{*}\right)= & z(t) e^{-i \tau_{1}^{0} \omega_{0} \tau^{*}}+\overline{z(t)} e^{i \tau_{1}^{0} \omega_{0} \tau^{*}}+W_{20}^{(1)}\left(-\tau^{*}\right) \frac{z^{2}}{2} \\
& +W_{11}^{(1)}\left(-\tau^{*}\right) z \bar{z}+W_{02}^{(1)}\left(-\tau^{*}\right) \frac{\bar{z}^{2}}{2}+\cdots, \\
\mu_{2}(t)= & \alpha z+\overline{\alpha z}+W_{20}^{(2)}(0) \frac{z^{2}}{2} \\
& +W_{11}^{(2)}(0) z \bar{z}+W_{02}^{(2)}(0) \frac{\bar{z}^{2}}{2}+\cdots . \tag{65}
\end{align*}
$$

We can also obtain

$$
\begin{align*}
& {\overline{q^{*}(0)}}^{T} f_{0}(z, \bar{z}) \\
& \quad=\bar{D}(1, \bar{\beta}) \tau_{1}^{0} \\
& \quad \times\left(\begin{array}{l}
0 \\
-\mu \delta^{\prime}\left(p^{*}\right) \mu_{1}(t) \mu_{2}(t)-\frac{\mu \delta^{\prime \prime}\left(p^{*}\right)}{2!} \mu_{1}^{2}(t) \mu_{2}(t) \\
+\frac{\mu a_{0} a_{1}}{3!\left(a_{1}+p^{*}\right)^{3}} \mu_{1}^{2}\left(t-\tau^{*}\right)
\end{array}\right) . \tag{66}
\end{align*}
$$

Expanding (66) and comparing the coefficients, we obtain

$$
\begin{align*}
& g_{20}=2 \bar{D} \tau_{0}\left(-\mu \delta^{\prime}\left(p^{*}\right) \alpha+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{3}} e^{-2 i \tau_{1}^{0} \omega_{0} \tau^{*}}\right), \\
& g_{11}=\bar{D} \tau_{1}^{0}\left(-\mu \delta^{\prime}\left(p^{*}\right)(\alpha+\bar{\alpha})+\frac{2 \mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{3}}\right), \\
& g_{02}=2 \bar{D} \tau_{1}^{0}\left(-\mu \delta^{\prime}\left(p^{*}\right) \bar{\alpha}+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{3}} e^{2 i \tau_{1}^{0} \omega_{0} \tau^{*}}\right), \\
& g_{21}=2 \bar{D} \tau_{1}^{0}\left\{-\mu \delta^{\prime}\left(p^{*}\right)\left[\frac{1}{2} W_{20}^{(1)}(0) \bar{\alpha}+W_{11}^{(1)}(0) \alpha\right.\right. \\
&\left.+W_{11}^{(2)}(0)+\frac{W_{20}^{(2)}(0)}{2}\right] \\
&-\frac{\mu \delta^{\prime \prime}\left(p^{*}\right)}{2!}(2 \alpha+\bar{\alpha})+\frac{\mu a_{0} a_{1}}{\left(a_{1}+P^{*}\right)^{3}} \\
& \times\left[W_{20}^{(1)}\left(-\tau^{*}\right) e^{i \tau_{1}^{0} *^{*} \omega_{0}}+2 W_{11}^{(1)}\left(-\tau^{*}\right) e^{-i \tau_{1}^{0} \tau^{*} \omega_{0}}\right] \\
&\left.-\frac{3 \mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{4}} e^{-\tau_{1}^{0} \tau^{*} \omega_{0}}\right\} . \tag{67}
\end{align*}
$$

We still need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. Noticing that

$$
\begin{align*}
W^{\prime}= & \mu_{t}^{\prime}-z^{\prime} q(\theta)-\bar{z} \overline{q(\theta)} \\
= & A \mu_{t}+R \mu_{t}-\left(i \tau_{1}^{0} \omega_{0} z+\overline{q^{*}(0)} f_{0}(z, \bar{z})\right) q(\theta) \\
& -\left(-i \tau_{1}^{0} \omega_{0} z+q^{*}(0) \overline{f_{0}(z, \bar{z})}\right) \overline{q(\theta)} \\
= & A \mu_{t}+R \mu_{t}-i \tau_{1}^{0} \omega_{0} z q(\theta)-\overline{q^{*}(0)} f_{0}(z, \bar{z}) q(\theta)  \tag{68}\\
& +i \tau_{1}^{0} \omega_{0} z \overline{q(\theta)}-q^{*}(0) \overline{f_{0}(z, \bar{z}) q(\theta)} \\
= & \left\{\begin{array}{r}
A W-2 \operatorname{Re}\left\{\overline{q^{*}(0)} f_{0}(z, \bar{z}) q(\theta)\right\}, \\
A W-2 \operatorname{Re}\left\{\overline{q^{*}(0)} f_{0}(z, \bar{z}) q(\theta)+f_{0}(z, \bar{z}),\right. \\
\theta=0,
\end{array}\right.
\end{align*}
$$

we rewrite $W^{\prime}$ as the following:

$$
\begin{equation*}
W^{\prime}=A W+H(z, \bar{z}, \theta) \tag{69}
\end{equation*}
$$

where $H(z, \bar{z}, \theta)=H_{20}(\theta)\left(z^{2} / 2\right)+H_{11}(\theta) z \bar{z}+H_{02}(\theta)\left(\bar{z}^{2} / 2\right)+$

$$
\begin{align*}
H_{30}(\theta) & \left(z^{3} / 6\right)+\cdots \\
\text { As } & W^{\prime}=W_{z} z^{\prime}+W_{\bar{z}} \bar{z}^{\prime} \text {, we can get } \\
W^{\prime}= & W_{20}(\theta) z\left(i \tau_{1}^{0} \omega_{0} z+g(z, \bar{z})\right)+W_{11}(\theta) z \\
& \times\left(-i \tau_{1}^{0} \omega_{0} \bar{z}+\overline{g(z, \bar{z})}\right)+W_{11}(\theta) \bar{z}\left(i \tau_{1}^{0} \omega_{0} z+g(z, \bar{z})\right) \\
& +W_{02}(\theta) \bar{z}\left(-i \tau_{1}^{0} \omega_{0} \bar{z}+\overline{g(z, \bar{z})}\right)+\cdots \\
= & i \tau_{1}^{0} W_{20}(\theta) \omega_{0} z^{2}+W_{20}(\theta) z \\
& \times\left(g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+g_{21} \frac{z^{2} \bar{z}}{2} \cdots\right) \\
& -i \tau_{1}^{0} W_{11}(\theta) \omega_{0} z \bar{z}+W_{11}(\theta) z \\
& \times\left(\overline{g_{20}} \frac{z^{2}}{2}+\overline{g_{11}} z \bar{z}+g_{02} \frac{z^{2}}{2}+g_{21} \frac{\bar{z}^{2} z}{2} \cdots\right) \\
& +i \tau_{1}^{0} W_{11}(\theta) \omega_{0} z \bar{z}+W_{11}(\theta) \bar{z} \\
& \times\left(g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+g_{21} \frac{z^{2} \bar{z}}{2} \cdots\right) \\
& -i \tau_{1}^{0} W_{02}(\theta) \bar{z}^{2} \omega_{0}+W_{02}(\theta) \bar{z} \\
& \times\left(\frac{g_{20}}{2}+\overline{z_{11}} z \bar{z}+g_{02} \frac{z^{2}}{2}+g_{21} \frac{\bar{z}^{2} z}{2} \cdots\right) . \tag{70}
\end{align*}
$$

Comparing the coefficients with (69), we can get

$$
\begin{gather*}
\left(A-2 i \tau_{1}^{0} \omega_{0}\right) W_{20}(\theta)=-H_{20}(\theta)  \tag{71}\\
A W_{11}(\theta)=-H_{11}(\theta)
\end{gather*}
$$

From (69), $H(z, \bar{z}, \theta)=-g(z, \bar{z}) q(\theta)-\overline{g(z, \bar{z}) q(\theta)}$, where $\theta \in$ $\left[-\tau^{*}, 0\right)$. Then we have

$$
\begin{align*}
& H_{20}(\theta)=-g_{20} q(\theta)-\overline{g_{02}} \overline{q(\theta)},  \tag{72}\\
& H_{11}(\theta)=-g_{11} q(\theta)-\overline{g_{11}} \overline{q(\theta)}
\end{align*}
$$

From (43) and (71), we have

$$
\begin{gather*}
W_{20}^{\prime}(\theta)=2 i \tau_{0} \omega_{0} W_{20}(\theta)+g_{20} q(\theta)+\overline{g_{02}} \overline{q(\theta)}, \\
W_{11}^{\prime}(\theta)=g_{11} q(\theta)+\overline{g_{11}} \overline{q(\theta)}, \tag{73}
\end{gather*}
$$

where $\theta \in\left[-\tau^{*}, 0\right)$. Computing (73), we have

$$
\begin{aligned}
& W_{20}(\theta)=\frac{i g_{20} q(0)}{\tau_{0} \omega_{0}} e^{i \tau_{0} \omega_{0} \theta}+\frac{i \overline{g_{02} q(0)}}{3 \tau_{0} \omega_{0}} e^{-i \tau_{0} \omega_{0} \theta}+E_{1} e^{2 i \tau_{0} \omega_{0} \theta}, \\
& W_{11}(\theta)=-\frac{i g_{11} q(0)}{\tau_{0} \omega_{0}} e^{i \tau_{0} \omega_{0} \theta}+\frac{i \overline{g_{11}} \overline{q(0)}}{\tau_{0} \omega_{0}} e^{-i \tau_{0} \omega_{0} \theta}+E_{2},
\end{aligned}
$$

$$
\text { where } E_{1}=\binom{E_{1}^{(1)}}{E_{1}^{(2)}}, E_{2}=\binom{E_{2}^{(1)}}{E_{2}^{(2)}}
$$

$E_{1}, E_{2}$ can be determined by setting $\theta=0$ in $H(z, \bar{z}, \theta)$. As $W_{20}(\theta)$ and $W_{11}(\theta)$ are continuous on $\left[-\tau^{*}, 0\right]$, then we have

$$
\begin{align*}
& W_{20}(0)=\frac{i g_{20} q(0)}{\tau_{0} \omega_{0}}+\frac{i \overline{g_{02} q(0)}}{3 \tau_{0} \omega_{0}}+E_{1}  \tag{76}\\
& W_{11}(0)=-\frac{i g_{11} q(0)}{\tau_{0} \omega_{0}}+\frac{i \overline{g_{11}} \overline{q(0)}}{\tau_{0} \omega_{0}}+E_{2} \tag{77}
\end{align*}
$$

From (43), we obtain

$$
\begin{align*}
& A W_{20}(0)=\int_{-\tau^{*}}^{0} \mathrm{~d} \eta(\xi) W_{20}(\theta) \\
& A W_{11}(0)=\int_{-\tau^{*}}^{0} \mathrm{~d} \eta(\xi) W_{11}(\theta) \tag{78}
\end{align*}
$$

Substitute (74)-(77) to (78), we can obtain

$$
\begin{align*}
& {\left[2 i \omega_{0} \tau_{1}^{0} I-\tau_{1}^{0}\left(A_{1}+A_{2} e^{-2 i \tau_{1}^{0} \omega_{0}}+A_{3} e^{-2 i \tau_{1}^{0} \omega_{0} \tau^{*}}\right)\right] E_{1}} \\
& \quad=\binom{0}{-\mu \delta^{\prime}\left(p^{*}\right) \alpha+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{3}} e^{-2 i \tau_{1}^{0} \omega_{0} \tau^{*}}}  \tag{79}\\
& \tau_{1}^{0}\left(A_{1}+A_{2}+A_{3}\right) E_{2} \\
& \quad=\binom{0}{-\mu \delta^{\prime}\left(p^{*}\right)(\alpha+\bar{\alpha})+\frac{2 \mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{3}}}
\end{align*}
$$

Therefore, we obtain

$$
\begin{align*}
E_{1}= & {\left[2 i \omega_{0} \tau_{1}^{0} I-\tau_{1}^{0}\left(A_{1}+A_{2} e^{-2 i \tau_{1}^{0} \omega_{0}}+A_{3} e^{-2 i \tau_{1}^{0} \omega_{0} \tau^{*}}\right)\right]^{-1} } \\
& \times\binom{ 0}{-\mu \delta^{\prime}\left(p^{*}\right) \alpha+\frac{\mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{3}} e^{-2 i \tau_{1}^{0} \omega_{0} \tau^{*}}} \\
E_{2}= & \tau_{1}^{0}\left(A_{1}+A_{2}+A_{3}\right)^{-1} \\
& \times\binom{ 0}{-\mu \delta^{\prime}\left(p^{*}\right)(\alpha+\bar{\alpha})+\frac{2 \mu a_{0} a_{1}}{\left(a_{1}+p^{*}\right)^{3}}} \tag{80}
\end{align*}
$$

Based on the previous analysis,we can see that each $g_{i j}$ in (67) is determined by the parameters in system (11). Thus we can compute the following values:

$$
\begin{gather*}
C_{1}(0)=\frac{i}{2 \tau_{1}^{0} \omega_{0}}\left[g_{11} g_{20}-2\left|g_{11}\right|^{2}-\frac{\left|g_{02}\right|^{2}}{3}\right]+\frac{g_{21}}{2} \\
\mu_{2}=-\frac{\operatorname{Re}\left(C_{1}(0)\right)}{\operatorname{Re}\left(\lambda^{\prime}\left(\tau_{1}^{0}\right)\right)}  \tag{81}\\
\beta_{2}=2 \operatorname{Re}\left(C_{1}(0)\right) \\
T_{2}=-\frac{\operatorname{Im}\left(C_{1}(0)\right)+\mu_{2} \operatorname{Im}\left(\lambda^{\prime}\left(\tau_{0}\right)\right)}{\tau_{1}^{0} \omega_{0}}
\end{gather*}
$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value $k=0$; that is, we thus have the following theorem.

Theorem 8. Using (81), one can compute the values of $C_{1}(0)$, $\mu_{2}, \beta_{2}, T_{2}$. Therefore, we can answer the question of whether the bifurcation branch of periodic solution exists locally for supercritical bifurcation or subcritical bifurcation as the following.
(i) $\mu_{2}$ determines the directions of the Hopf bifurcation: if $\mu_{2}>0\left(\mu_{2}<0\right)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exists for $k=0$.
(ii) $\beta_{2}$ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_{2}<0\left(\beta_{2}>0\right)$.
(iii) $T_{2}$ determines the period of the bifurcating periodic solutions: the period increases (decreases), if $T_{2}>0$ ( $T_{2}<0$ ).

## 5. Numerical Simulation Example.

Example 9. Consider a single commodity market model as follows:

$$
\begin{align*}
\frac{\mathrm{d} x(t)}{\mathrm{d} t}= & y(t) \\
\frac{\mathrm{d} y(t)}{\mathrm{d} t}= & -\mu(2 x(t)+1) y(t)-\mu b x\left(t-\tau_{1}\right)  \tag{82}\\
& -\mu \frac{a_{0} x\left(t-\tau_{2}\right)}{a_{1}+x\left(t-\tau_{2}\right)}+\mu\left(d_{0}-g_{0}\right),
\end{align*}
$$

where $x(t)$ denotes the price at time $t$, and we choose $d_{0}=$ $125, b=2, \mu=0.08, g_{0}=85, a_{0}=180, a_{1}=12$. Using Theorem 1, a quick computation revealed that $p^{*}=x^{*}=$ 2.8292 .

If one choose $\tau_{1}=0, \tau_{2}>0$, from Lemma 3, we can calculate that $\omega=0.8875, \tau_{20}=0.7237$, and $\tau_{20}$ is the the critical value for Hopf bifurcation. It follows from Theorem 5 that if $\tau_{2} \in(0,0.7237),\left(p^{*}, 0\right)$ is asymptotically stable. System (82) undergoes a Hopf bifurcation at $\tau_{2}=0.7237$. These conclusions are verified by the numerical simulation in Figures 1, 2, 3, and 4.


Figure 1: $\tau_{1}=0, \tau_{2}=0.68,\left(p^{*}, 0\right)$ is asymptotically stable.


Figure 2: $\tau_{1}=0, \tau_{2}=0.68,\left(p^{*}, 0\right)$ is asymptotically stable.


Figure 3: $\tau_{1}=0, \tau_{2}=0.7273$, periodic solution bifurcates from $\left(p^{*}, 0\right)$.


Figure 4: $\tau_{1}=0, \tau_{2}=0.7237,\left(p^{*}, 0\right)$, periodic solution bifurcates from $\left(p^{*}, 0\right)$.

When $\tau_{2}>\tau_{1}>0$, we can choose $\tau_{2}=\tau=0.68$; from (34) and (35), we can calculate that $\omega_{0}=0.9203, \tau_{1}^{0}=0.2046$; from (37), we have $\alpha^{\prime}\left(\tau_{1}^{0}\right)=0.0618$. It follows from Theorem 6 that ( $p^{*}, 0$ ) is asymptotically stable as far as $\tau_{1}^{0} \in[0,0.2046)$ and system (82) undergoes a Hopf bifurcation when $\tau_{1}^{0}=0.2046$. Furthermore, using Lemma 7, after simple computation, we have

$$
\begin{gather*}
\alpha=0.0249+0.8772 i, \quad \beta=0.4512-0.7308 i \\
D=0.0029+0.5189 i \tag{83}
\end{gather*}
$$

Then we can obtain

$$
\begin{array}{cr}
g_{20}=-0.0256-0.0033 i, & g_{11}=0.0001-0.0108 i \\
g_{02}=0.0256-0.0030 i, & g_{21}=0.0005+0.0020 i \tag{84}
\end{array}
$$

It is easy to obtain

$$
\begin{gather*}
C_{1}(0)=-0.00061286-0.00029893 i, \quad \mu_{2}=0.0099>0, \\
\beta_{2}=-0.0012<0 . \tag{85}
\end{gather*}
$$

Therefore, the Hopf bifurcation of system (82) is supercritical, and the bifurcation periodic solutions are stable. This conclusions can be verified by the numerical simulation in Figures $5,6,7$, and 8 .

## 6. Conclusion

The delay in production process has been considered in many economic models. However, few of them considered the delay of consumer consumption. In this paper, we first establish a class of economic models with two delays; the delay in production process and consumer consumption are both considered, and then the dynamics of this system have been investigated. Specifically, stability of equilibrium point and the existence of Hopf bifurcation, in great detail, are


FIgure 5: $\tau_{1}=0.15, \tau_{2}=0.68,\left(p^{*}, 0\right)$ is asymptotically stable.


Figure 6: $\tau_{1}=0.15, \tau_{2}=0.68,\left(p^{*}, 0\right)$ is asymptotically stable.


Figure 7: $\tau_{1}=0.2046, \tau_{2}=0.68$, periodic solution bifurcates from $\left(p^{*}, 0\right)$.


Figure 8: $\tau_{1}=0.2046, \tau_{2}=0.68,\left(p^{*}, 0\right)$, periodic solution bifurcates from ( $p^{*}, 0$ ).
studied. We have derived some sufficient conditions to ensure that all the characteristic roots have negative real parts. We also show that a Hopf Bifurcation will occur once some parameters pass through critical values; that is, a family of periodic orbits bifurcates from the positive equilibrium point. At last, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem. Simulations show that the theoretically predicted values are in excellent agreement with the numerically observed behavior. Our results extend and complement some earlier publications.

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## Research Article

# Standing Wave Solutions for the Discrete Coupled Nonlinear Schrödinger Equations with Unbounded Potentials 

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We demonstrate the existence of standing wave solutions of the discrete coupled nonlinear Schrödinger equations with unbounded potentials by using the Nehari manifold approach and the compact embedding theorem. Sufficient conditions are established to show that the standing wave solutions have both of the components not identically zero.

## 1. Introduction

Consider the coupled discrete Schrödinger system

$$
\begin{align*}
& i \frac{d u_{n}}{d t}=-(\mathscr{A} u)_{n}+b_{1 n} u_{n}-a_{1}\left|u_{n}\right|^{2} u_{n}-a_{3}\left|v_{n}\right|^{2} u_{n} \\
& i \frac{d v_{n}}{d t}=-(\mathscr{A} v)_{n}+b_{2 n} v_{n}-a_{2}\left|v_{n}\right|^{2} v_{n}-a_{3}\left|u_{n}\right|^{2} v_{n} \tag{1}
\end{align*}
$$

where $a_{i}>0,\left\{b_{j n}\right\}$ are real valued sequences, $i=$ $1,2,3$, and $j=1,2 . \mathscr{A}$ is the discrete Laplacian operator defined as $(\mathscr{A} u)_{n}=u_{n+1}+u_{n-1}-2 u_{n}$.

The system (1) could be viewed as the discretization of the two-component system of time-dependent nonlinear GrossPitaevskii system (see [1] for detail)

$$
\begin{align*}
& i \hbar \partial_{t} u=-\frac{\hbar^{2}}{2 m} \Delta u+b_{1}(x) u-a_{1}|u|^{2} u-a_{3}|v|^{2} u, \\
& i \hbar \partial_{t} v=-\frac{\hbar^{2}}{2 m} \Delta v+b_{2}(x) v-a_{2}|v|^{2} v-a_{3}|u|^{2} v . \tag{2}
\end{align*}
$$

In this paper, we will study the standing wave solutions of (1), that is, solutions of the form

$$
\begin{equation*}
u_{n}=\exp \left(-i \omega_{1} t\right) \phi_{n}, \quad v_{n}=\exp \left(-i \omega_{2} t\right) \psi_{n}, \quad n \in \mathbb{Z} \tag{3}
\end{equation*}
$$

where the amplitude $\phi_{n}$ and $\psi_{n}$ are supposed to be real. Inserting the ansatz of the standing wave solutions (3) into (1), we obtain the following equivalent algebraic equations:

$$
\begin{align*}
& -(\mathscr{A} \phi)_{n}-\omega_{1} \phi_{n}+b_{1 n} \phi_{n}-a_{1}\left|\phi_{n}\right|^{2} \phi_{n}-a_{3}\left|\psi_{n}\right|^{2} \phi_{n}=0 \\
& -(\mathscr{A} \psi)_{n}-\omega_{2} \psi_{n}+b_{2 n} \psi_{n}-a_{2}\left|\psi_{n}\right|^{2} \psi_{n}-a_{3}\left|\phi_{n}\right|^{2} \psi_{n}=0 \tag{4}
\end{align*}
$$

Since Bose-Einstein condensation for a mixture of different interaction atomic species with the same mass was realized in 1997 (see [2]), this stimulated various analytical and numerical results on the standing wave solutions of the system (2). The discrete nonlinear Schrödinger equations (DNLS) have a crucial role in the modeling of a great variety of phenomena, ranging from solid-state and condensedmatter physics to biology. During the last years, there has been a growing interest in approaches to the existence problem for standing waves. We refer to the continuation methods in [3, 4], which have been proved powerful for both theoretical considerations and numerical computations (see [5]), to [6], which exploits spatial dynamics and centre manifold reduction, and to the variational methods in [7-11], which rely on critical point techniques (linking theorems and Nehari manifold).

We noticed that most works on the existence of standing waves solutions are for single discrete nonlinear

Schrödinger equation, and less is known for discrete nonlinear Schrödinger system. In the recent paper [12], the authors considered the standing wave solutions of the following system:

$$
\begin{align*}
& i \frac{d u_{n}}{d t}=-(\mathscr{A} u)_{n}+b_{1 n} u_{n}-c_{1 n} v_{n}-a_{1}\left|u_{n}\right|^{2} u_{n}-a_{3}\left|v_{n}\right|^{2} u_{n} \\
& i \frac{d v_{n}}{d t}=-(\mathscr{A} v)_{n}+b_{2 n} v_{n}-c_{2 n} u_{n}-a_{2}\left|v_{n}\right|^{2} v_{n}-a_{3}\left|u_{n}\right|^{2} v_{n} \tag{5}
\end{align*}
$$

which is more general than the system (1). However, they make a mistake to obtain the equivalent algebraic equations because $\omega_{1}$ may be different from $\omega_{2}$. Hence, there are two ways to correct this mistake. The first method is to study the special standing wave solutions (3) of the system (5) with $\omega_{1}=$ $\omega_{2}$. The second method is to study the standing wave solutions (3) of the system (5) with $c_{1 n} \equiv 0$ and $c_{2 n} \equiv 0, n \in \mathbb{Z}$. In this paper, we consider the second method. By the way, the proof of the main results in [12] is also not fully corrected.

The paper is organized as follows. In Section 2, we introduce some preliminaries and a discrete version of compact embedding theorem. Some key lemmas on the Nehari manifold are proved in Section 3. In Section 4, the main results are stated and proved.

## 2. Preliminaries

In this section we describe the functional setting needed for the treatment of the infinite nonlinear system (4). We first introduce a compact embedding theorem.

Consider the real sequence spaces

$$
\begin{gather*}
l^{p}=\left\{\phi=\left\{\phi_{n}\right\}_{n \in \mathbb{Z}}:\|\phi\|_{l^{p}}=\left(\sum_{n \in \mathbb{Z}}\left|\phi_{n}\right|^{p}\right)^{1 / p}<\infty,\right.  \tag{6}\\
\left.\phi_{n} \in \mathbb{R}, \forall n \in \mathbb{Z}\right\} .
\end{gather*}
$$

Between $l^{p}$ spaces the following elementary embedding relation holds:

$$
\begin{equation*}
l^{q} \subset l^{p}, \quad\|\phi\|_{l^{p}} \leq\|\phi\|_{l^{q}}, \quad 1 \leq q \leq p \leq \infty . \tag{7}
\end{equation*}
$$

For the case $p=2$, we need the usual Hilbert space of $l^{2}$, endowed with the real scalar product

$$
\begin{equation*}
(\phi, \psi)=\sum_{n \in \mathbb{Z}} \phi_{n} \psi_{n}, \quad \phi, \psi \in l^{2} . \tag{8}
\end{equation*}
$$

Let us point out that the spectrum of $-\mathscr{A}$ in $l^{2}$ coincides with the interval $[0,4]$. Obviously, we have

$$
\begin{equation*}
0 \leq(-\mathscr{A} \phi, \phi) \leq 4\|\phi\|_{l^{2}}^{2}, \quad \forall \phi \in l^{2} . \tag{9}
\end{equation*}
$$

Assume that the potential $V_{i}=\left\{b_{i n}\right\}_{n \in \mathbb{Z}}, i=1,2$, satisfies

$$
\begin{equation*}
\lim _{|n| \rightarrow \infty} b_{i n}=\infty, \quad i=1,2 \tag{10}
\end{equation*}
$$

Without loss of generality we assume that $V_{i} \geq 1, i=1,2$; that is $b_{\text {in }} \geq 1$ for $n \in \mathbb{Z}, i=1,2$. Let

$$
\begin{equation*}
L_{i}=-\mathscr{A}+V_{i}, \quad i=1,2 \tag{11}
\end{equation*}
$$

which are self-adjoint operators defined on $l^{2}$, and

$$
\begin{array}{r}
E_{i}=\left\{\phi \in l^{2}: L_{i}^{1 / 2} \phi \in l^{2}\right\}, \quad\|\phi\|_{E_{i}}=\left\|L_{i}^{1 / 2} \phi\right\|_{l^{2}}  \tag{12}\\
i=1,2 .
\end{array}
$$

The following lemma can be found in [9].
Lemma 1. If $V_{i}, i=1,2$, satisfy the condition (10), then for any $2 \leq p \leq \infty, E_{1}$ and $E_{2}$ are compactly embedded intol $l^{p}$ and denote the best embedding constant $\alpha_{p}=\max _{\|\phi\|_{p}=1} 1 /\|\phi\|_{E_{1}}$ and $\beta_{p}=\max _{\|\phi\|_{l p}=1} 1 /\|\phi\|_{E_{2}}$, respectively. Furthermore, the spectra $\sigma\left(L_{1}\right)$ and $\sigma\left(L_{2}\right)$ are discrete, respectively.

By (11), (4) becomes

$$
\begin{align*}
& L_{1} \phi_{n}-\omega_{1} \phi_{n}-a_{1}\left|\phi_{n}\right|^{2} \phi_{n}-a_{3}\left|\psi_{n}\right|^{2} \phi_{n}=0,  \tag{13}\\
& L_{2} \psi_{n}-\omega_{2} \psi_{n}-a_{2}\left|\psi_{n}\right|^{2} \psi_{n}-a_{3}\left|\phi_{n}\right|^{2} \psi_{n}=0 .
\end{align*}
$$

Now we can define the action functional

$$
\begin{align*}
J(\phi, \psi)= & \frac{1}{2}\left(\left(L_{1}-\omega_{1}\right) \phi, \phi\right)+\frac{1}{2}\left(\left(L_{2}-\omega_{2}\right) \psi, \psi\right) \\
& -\frac{1}{4} \sum_{n \in \mathbb{Z}}\left(a_{1} \phi_{n}^{4}+a_{2} \psi_{n}^{4}+2 a_{3} \phi_{n}^{2} \psi_{n}^{2}\right) . \tag{14}
\end{align*}
$$

By Lemma 1, it follows that the action functional $J(\phi, \psi) \in$ $C^{1}\left(E_{1} \times E_{2}, \mathbb{R}\right)$ and (13) corresponds to $J^{\prime}(\phi, \psi)=0$. So we define

$$
\begin{align*}
I(\phi, \psi)= & \left(J^{\prime}(\phi, \psi),(\phi, \psi)\right) \\
= & \left(\left(L_{1}-\omega_{1}\right) \phi, \phi\right)+\left(\left(L_{2}-\omega_{2}\right) \psi, \psi\right)  \tag{15}\\
& -\sum_{n \in \mathbb{Z}}\left(a_{1} \phi_{n}^{4}+a_{2} \psi_{n}^{4}+2 a_{3} \phi_{n}^{2} \psi_{n}^{2}\right),
\end{align*}
$$

and the Nehari manifold

$$
\begin{equation*}
N=\left\{(\phi, \psi) \in E_{1} \times E_{2}: I(\phi, \psi)=0,(\phi, \psi) \neq 0\right\} . \tag{16}
\end{equation*}
$$

## 3. Some Lemmas on the Nehari Manifold

Let

$$
\begin{equation*}
\lambda_{i}=\inf \left\{\sigma\left(L_{i}\right)\right\}, \quad i=1,2 \tag{17}
\end{equation*}
$$

To prove the main results, we need some lemmas on the Nehari manifold.

Lemma 2. Assume that $\omega_{1}<\lambda_{1}, \omega_{2}<\lambda_{2}$, and (10) holds. Then the Nehari manifold $N$ is nonempty in $E_{1} \times E_{2}$. Furthermore, for $(\phi, \psi) \in N, J(t \phi, t \psi)$ attains a unique maximum point at $t=1$.

Proof. First we show that $N \neq \emptyset$.
From (15) and (16), we rewrite

$$
\begin{align*}
J(\phi, \psi)= & \frac{1}{2}\left(\|\phi\|_{E_{1}}^{2}-\omega_{1}\|\phi\|_{l^{2}}^{2}\right) \\
& +\frac{1}{2}\left(\|\psi\|_{E_{2}}^{2}-\omega_{2}\|\psi\|_{l^{2}}^{2}\right)  \tag{18}\\
& -\frac{1}{4} \sum_{n \in \mathbb{Z}}\left(a_{1} \phi_{n}^{4}+a_{2} \psi_{n}^{4}+2 a_{3} \phi_{n}^{2} \psi_{n}^{2}\right), \\
I(\phi, \psi)= & \|\phi\|_{E_{1}}^{2}-\omega_{1}\|\phi\|_{l^{2}}^{2}+\|\psi\|_{E_{2}}^{2}-\omega_{2}\|\psi\|_{l^{2}}^{2} \\
& -\sum_{n \in \mathbb{Z}}\left(a_{1} \phi_{n}^{4}+a_{2} \psi_{n}^{4}+2 a_{3} \phi_{n}^{2} \psi_{n}^{2}\right) . \tag{19}
\end{align*}
$$

Let $(\phi, \psi) \in\left(E_{1}-\{0\}\right) \times\left(E_{2}-\{0\}\right)$; then by (19)

$$
\begin{align*}
I(t \phi, t \psi)= & t^{2}\left(\|\phi\|_{E_{1}}^{2}-\omega_{1}\|\phi\|_{l^{2}}^{2}\right) \\
& +t^{2}\left(\|\psi\|_{E_{2}}^{2}-\omega_{2}\|\psi\|_{l^{2}}^{2}\right) \\
& -t^{4} \sum_{n \in \mathbb{Z}}\left(a_{1} \phi_{n}^{4}+a_{2} \psi_{n}^{4}+2 a_{3} \phi_{n}^{2} \psi_{n}^{2}\right)  \tag{20}\\
= & t^{2}\left(\|\phi\|_{E_{1}}^{2}-\omega_{1}\|\phi\|_{l^{2}}^{2}+\|\psi\|_{E_{2}}^{2}-\omega_{2}\|\psi\|_{l^{2}}^{2}\right. \\
& \left.-t^{2} \sum_{n \in \mathbb{Z}}\left(a_{1} \phi_{n}^{4}+a_{2} \psi_{n}^{4}+2 a_{3} \phi_{n}^{2} \psi_{n}^{2}\right)\right) .
\end{align*}
$$

Notice that $\|\phi\|_{E_{1}}^{2}-\omega_{1}\|\phi\|_{l^{2}}^{2} \geq\left(\lambda_{1}-\omega_{1}\right)\|\phi\|_{1^{2}}^{2}>0$ and $\|\psi\|_{E_{2}}^{2}-\omega_{2}\|\psi\|_{l^{2}}^{2} \geq\left(\lambda_{2}-\omega_{2}\right)\|\psi\|_{l^{2}}^{2}>0$; by (20), we see that $I(t \phi, t \psi)>0$ for $t>0$ small enough and $I(t \phi, t \psi)<0$ for $t>0$ large enough. As a consequence, there exists $t_{0}>0$ such that $I\left(t_{0} \phi, t_{0} \psi\right)=0$; that is, $\left(t_{0} \phi, t_{0} \psi\right) \in N$.

Let $F(t)=J(t \phi, t \psi),(\phi, \psi) \in N$. Computing the derivative of $F$, we have

$$
\begin{equation*}
F^{\prime}(t)=t\left(1-t^{2}\right) \sum_{n \in \mathbb{Z}}\left(a_{1} \phi_{n}^{4}+a_{2} \psi_{n}^{4}+2 a_{3} \phi_{n}^{2} \psi_{n}^{2}\right) . \tag{21}
\end{equation*}
$$

This shows that $t=1$ is a unique maximum point. The proof is completed.

Lemma 3. Assume that $\omega_{1}<\lambda_{1}, \omega_{2}<\lambda_{2}$, and (10) holds. Then there exists $\eta>0$ such that $J(\phi, \psi) \geq \eta$, for all $(\phi, \psi) \in N$.

Proof. Since $\lambda_{1}$ is the smallest eigenvalue of $E_{1}$ and $\lambda_{2}$ is the smallest eigenvalue of $E_{2}$, from the definition of the constant
$\alpha_{p}$ and $\beta_{p}$, we get $\lambda_{1}=1 / \alpha_{2}^{2}$ and $\lambda_{2}=1 / \beta_{2}^{2}$. For any $(\phi, \psi) \in$ $N$, we have

$$
\begin{align*}
\|\phi\|_{E_{1}}^{2} & -\omega_{1}\|\phi\|_{l^{2}}^{2}+\|\psi\|_{E_{2}}^{2}-\omega_{2}\|\psi\|_{l^{2}}^{2} \\
& =\sum_{n \in \mathbb{Z}}\left(a_{1} \phi_{n}^{4}+a_{2} \psi_{n}^{4}+2 a_{3} \phi_{n}^{2} \psi_{n}^{2}\right) \\
& \leq a^{*}\left(\|\phi\|_{l^{4}}^{2}+\|\psi\|_{l^{4}}^{2}\right)^{2}  \tag{22}\\
& \leq a^{*}\left(\alpha_{4}^{2}\|\phi\|_{E_{1}}^{2}+\beta_{4}^{2}\|\psi\|_{E_{2}}^{2}\right)^{2} \\
& \leq a^{*} \gamma_{2}^{2}\left(\|\phi\|_{E_{1}}^{2}+\|\psi\|_{E_{2}}^{2}\right)^{2},
\end{align*}
$$

where $a^{*}=\max \left\{a_{1}, a_{2}, a_{3}\right\}$ and $\gamma_{2}=\max \left\{\alpha_{4}^{2}, \beta_{4}^{2}\right\}$.
Let

$$
\begin{equation*}
\gamma_{1}=\min \left\{1,1-\frac{\omega_{1}}{\lambda_{1}}, 1-\frac{\omega_{2}}{\lambda_{2}}\right\} . \tag{23}
\end{equation*}
$$

By (22), it is easy to see that

$$
\begin{equation*}
\gamma_{1}\left(\|\phi\|_{E_{1}}^{2}+\|\psi\|_{E_{2}}^{2}\right) \leq a^{*} \gamma_{2}^{2}\left(\|\phi\|_{E_{1}}^{2}+\|\psi\|_{E_{2}}^{2}\right)^{2} \tag{24}
\end{equation*}
$$

and this implies that

$$
\begin{equation*}
\|\phi\|_{E_{1}}^{2}+\|\psi\|_{E_{2}}^{2} \geq \frac{\gamma_{1}}{a^{*} \gamma_{2}^{2}} \tag{25}
\end{equation*}
$$

Moreover, we have

$$
\begin{align*}
J(\phi, \psi) & =J(\phi, \psi)-\frac{1}{4} I(\phi, \psi) \\
& =\frac{1}{4}\left(\|\phi\|_{E_{1}}^{2}-\omega_{1}\|\phi\|_{l^{2}}^{2}+\|\psi\|_{E_{2}}^{2}-\omega_{2}\|\psi\|_{l^{2}}^{2}\right)  \tag{26}\\
& \geq \frac{\gamma_{1}}{4}\left(\|\phi\|_{E_{1}}^{2}+\|\psi\|_{E_{2}}^{2}\right) \geq \frac{\gamma_{1}^{2}}{4 a^{*} \gamma_{2}^{2}} .
\end{align*}
$$

Let $\eta=\gamma_{1}^{2} /\left(4 a^{*} \gamma_{2}^{2}\right)$; then we get $J(\phi, \psi) \geq \eta$, for all $(\phi, \psi) \in N$. The proof is completed.

## 4. Main Results

Now we state our main results in this paper as follows.
Theorem 4. Assume that $\omega_{1}<\lambda_{1}, \omega_{2}<\lambda_{2}$, and (10) holds. Then system (13) has a nontrivial solution in $E_{1} \times E_{2}$; that is, system (1) has a nontrivial standing wave solution.

In order to prove Theorem 4, we consider the following constrained minimization problem:

$$
\begin{equation*}
d \equiv \inf _{(\phi, \psi) \in N} J(\phi, \psi) \tag{27}
\end{equation*}
$$

From the standard variational method, the proof of Theorem 4 is changed into finding a solution to the minimization problem (27). Now we are ready to prove Theorem 4.

Proof. Let $d$ be given by (27). By Lemma 2, $N$ is nonempty and there exists a sequence $\left\{\left(\phi^{(k)}, \psi^{(k)}\right)\right\} \subset N$ such that

$$
\begin{equation*}
d=\lim _{k \rightarrow \infty} J\left(\phi^{(k)}, \psi^{(k)}\right) . \tag{28}
\end{equation*}
$$

By Lemma 3, $d>0$ and $d \leq D=\max _{k}\left\{J\left(\phi^{(k)}, \psi^{(k)}\right)\right\}<$ $\infty$. By virtue of (26), we have

$$
\begin{equation*}
\left\|\phi^{(k)}\right\|_{E_{1}}^{2}+\left\|\psi^{(k)}\right\|_{E_{2}}^{2} \leq \frac{4}{\gamma_{1}} J\left(\phi^{(k)}, \psi^{(k)}\right) \leq \frac{4 D}{\gamma_{1}}<\infty . \tag{29}
\end{equation*}
$$

Thus, sequences $\left\{\phi^{(k)}\right\}$ and $\left\{\psi^{(k)}\right\}$ are bounded in Hilbert spaces $E_{1}$ and $E_{2}$, respectively. Therefore, there exist subsequences of $\left\{\phi^{(k)}\right\}$ and $\left\{\psi^{(k)}\right\}$ (denoted by itself) that weakly converge to some $\phi^{*} \in E_{1}$ and $\psi^{*} \in E_{2}$, respectively. By Lemma 1, we get, for any $2 \leq p \leq \infty$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \phi^{(k)}=\phi^{*}, \quad \lim _{k \rightarrow \infty} \psi^{(k)}=\psi^{*}, \quad \text { in } l^{p} . \tag{30}
\end{equation*}
$$

By virtue of (15) and (16), we have

$$
\begin{align*}
J\left(\phi^{(k)},\right. & \left.\psi^{(k)}\right) \\
= & \frac{1}{2}\left(\left\|\phi^{(k)}\right\|_{E_{1}}^{2}-\omega_{1}\left\|\phi^{(k)}\right\|_{l^{2}}^{2}\right)+\frac{1}{2}\left(\left\|\psi^{(k)}\right\|_{E_{2}}^{2}-\omega_{2}\left\|\psi^{(k)}\right\|_{l^{2}}^{2}\right) \\
& -\frac{1}{4} \sum_{n \in \mathbb{Z}}\left(a_{1}\left(\phi_{n}^{(k)}\right)^{4}+a_{2}\left(\psi_{n}^{(k)}\right)^{4}+2 a_{3}\left(\phi_{n}^{(k)}\right)^{2}\left(\psi_{n}^{(k)}\right)^{2}\right) \\
= & \frac{1}{4} \sum_{n \in \mathbb{Z}}\left(a_{1}\left(\phi_{n}^{(k)}\right)^{4}+a_{2}\left(\psi_{n}^{(k)}\right)^{4}+2 a_{3}\left(\phi_{n}^{(k)}\right)^{2}\left(\psi_{n}^{(k)}\right)^{2}\right) . \tag{31}
\end{align*}
$$

First, we claim that

$$
\begin{gather*}
\lim _{k \rightarrow \infty} \sum_{n \in \mathbb{Z}}\left(a_{1}\left(\phi_{n}^{(k)}\right)^{4}+a_{2}\left(\psi_{n}^{(k)}\right)^{4}+2 a_{3}\left(\phi_{n}^{(k)}\right)^{2}\left(\psi_{n}^{(k)}\right)^{2}\right) \\
=\sum_{n \in \mathbb{Z}}\left(a_{1}\left(\phi_{n}^{*}\right)^{4}+a_{2}\left(\psi_{n}^{*}\right)^{4}+2 a_{3}\left(\phi_{n}^{*}\right)^{2}\left(\psi_{n}^{*}\right)^{2}\right) . \tag{32}
\end{gather*}
$$

According to (30), it suffices to show that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \sum_{n \in \mathbb{Z}}\left(\phi_{n}^{(k)}\right)^{2}\left(\psi_{n}^{(k)}\right)^{2}=\sum_{n \in \mathbb{Z}}\left(\phi_{n}^{*}\right)^{2}\left(\psi_{n}^{*}\right)^{2} \tag{33}
\end{equation*}
$$

In fact,

$$
\begin{align*}
& \left|\sum_{n \in \mathbb{Z}}\left(\phi_{n}^{(k)}\right)^{2}\left(\psi_{n}^{(k)}\right)^{2}-\sum_{n \in \mathbb{Z}}\left(\phi_{n}^{*}\right)^{2}\left(\psi_{n}^{*}\right)^{2}\right| \\
& \quad \leq \sum_{n \in \mathbb{Z}}\left|\phi_{n}^{(k)}-\phi_{n}^{*}\right|\left|\phi_{n}^{(k)}+\phi_{n}^{*}\right|\left(\psi_{n}^{(k)}\right)^{2}  \tag{34}\\
& \quad+\sum_{n \in \mathbb{Z}}\left|\psi_{n}^{(k)}-\psi_{n}^{*}\right|\left|\psi_{n}^{(k)}+\psi_{n}^{*}\right|\left(\phi_{n}^{*}\right)^{2} .
\end{align*}
$$

Thus Hölder inequality and (30) imply the (33) holds.

Next, we show that $\left(\phi^{*}, \psi^{*}\right) \in N$ and $J\left(\phi^{*}, \psi^{*}\right)=d$. Since $E_{1}$ and $E_{2}$ are Hilbert spaces, by (32) we have

$$
\begin{align*}
& \left\|\phi^{*}\right\|_{E_{1}}^{2}+\left\|\psi^{*}\right\|_{E_{2}}^{2} \\
& =\| \text { weak }-\lim _{k \rightarrow \infty} \phi^{(k)}\left\|_{E_{1}}^{2}+\right\| \text { weak }-\lim _{k \rightarrow \infty} \psi^{(k)} \|_{E_{2}}^{2} \\
& \begin{array}{l}
\leq \liminf _{k \rightarrow \infty}\left\|\phi^{(k)}\right\|_{E_{1}}^{2}+\liminf _{k \rightarrow \infty}\left\|\psi^{(k)}\right\|_{E_{2}}^{2} \\
\begin{array}{c}
\leq \liminf _{k \rightarrow \infty}\left(\left\|\phi^{(k)}\right\|_{E_{1}}^{2}+\left\|\psi^{(k)}\right\|_{E_{2}}^{2}\right) \\
= \\
\quad \liminf _{k \rightarrow \infty}\left(\sum _ { n \in \mathbb { Z } } \left(a_{1}\left(\phi_{n}^{(k)}\right)^{4}+a_{2}\left(\psi_{n}^{(k)}\right)^{4}\right.\right. \\
\left.\quad+2 a_{3}\left(\phi_{n}^{(k)}\right)^{2}\left(\psi_{n}^{(k)}\right)^{2}\right) \\
\left.\quad+\omega_{1}\left\|\phi^{(k)}\right\|_{l_{2}}^{2}+\omega_{2}\left\|\psi^{(k)}\right\|_{l_{2}}^{2}\right)
\end{array} \\
=\sum_{n \in \mathbb{Z}}\left(a_{1}\left(\phi_{n}^{*}\right)^{4}+a_{2}\left(\psi_{n}^{*}\right)^{4}+2 a_{3}\left(\phi_{n}^{*}\right)^{2}\left(\psi_{n}^{*}\right)^{2}\right) \\
\quad+\omega_{1}\left\|\phi^{*}\right\|_{l_{2}}^{2}+\omega_{2}\left\|\psi^{*}\right\|_{l_{2}}^{2}
\end{array}
\end{align*}
$$

which implies $I\left(\phi^{*}, \psi^{*}\right)=\left\|\phi^{*}\right\|_{E_{1}}^{2}-\omega_{1}\left\|\phi^{*}\right\|_{l_{2}}^{2}+\left\|\psi^{*}\right\|_{E_{2}}^{2}-$ $\omega_{2}\left\|\psi^{*}\right\|_{l_{2}}^{2}-\sum_{n \in \mathbb{Z}}\left(a_{1}\left(\phi_{n}^{*}\right)^{4}+a_{2}\left(\psi_{n}^{*}\right)^{4}+2 a_{3}\left(\phi_{n}^{*}\right)^{2}\left(\psi_{n}^{*}\right)^{2}\right) \leq 0$. Through a similar argument to the proof of Lemma 2, we know that $I\left(t \phi^{*}, t \psi^{*}\right)$ is positive as $t$ is small enough. Therefore there exists $t^{*} \in(0,1]$ such that $I\left(t^{*} \phi^{*}, t^{*} \psi^{*}\right)=0$ which implies $\left(t^{*} \phi^{*}, t^{*} \psi^{*}\right) \in N$. Thus we have $J\left(t^{*} \phi^{*}, t^{*} \psi^{*}\right)=$ $(1 / 4) W\left(t^{*}\right)$ and by $(32), W(1)=4 d$, where

$$
\begin{equation*}
W(t)=t^{4} \sum_{n \in \mathbb{Z}}\left(a_{1}\left(\phi_{n}^{*}\right)^{4}+a_{2}\left(\psi_{n}^{*}\right)^{4}+2 a_{3}\left(\phi_{n}^{*}\right)^{2}\left(\psi_{n}^{*}\right)^{2}\right) . \tag{36}
\end{equation*}
$$

Clearly, $W(t)$ is strictly increasing on $0<t<\infty$. Therefore by (27),

$$
\begin{equation*}
d \leq J\left(t^{*} \phi^{*}, t^{*} \psi^{*}\right)=\frac{1}{4} W\left(t^{*}\right) \leq \frac{1}{4} W(1)=d \tag{37}
\end{equation*}
$$

This implies that $t^{*}=1$ and $J\left(\phi^{*}, \psi^{*}\right)=d$.
Finally, we will prove $\left(\phi^{*}, \psi^{*}\right)$ is a nontrivial solution to system (13).

Since $\left(\phi^{*}, \psi^{*}\right)$ is an energy minimizer on Nehari manifold $N$, there exists a Lagrange multiplier $\Lambda$ such that

$$
\begin{equation*}
\left(J^{\prime}\left(\phi^{*}, \psi^{*}\right)+\Lambda I^{\prime}\left(\phi^{*}, \psi^{*}\right),(\phi, \psi)\right)=0 \tag{38}
\end{equation*}
$$

for any $(\phi, \psi) \in E_{1} \times E_{2}$. Let $(\phi, \psi)=\left(\phi^{*}, \psi^{*}\right)$ in (38). $\left(J^{\prime}\left(\phi^{*}, \psi^{*}\right),\left(\phi^{*}, \psi^{*}\right)\right)=I\left(\phi^{*}, \psi^{*}\right)=0$ implies that

$$
\begin{equation*}
\Lambda\left(I^{\prime}\left(\phi^{*}, \psi^{*}\right),\left(\phi^{*}, \psi^{*}\right)\right)=0 \tag{39}
\end{equation*}
$$

but

$$
\begin{align*}
& \left(I^{\prime}\left(\phi^{*}, \psi^{*}\right),\left(\phi^{*}, \psi^{*}\right)\right) \\
& \quad=2\left(\left(L_{1}-\omega_{1}\right) \phi^{*}, \phi^{*}\right)+2\left(\left(L_{2}-\omega_{2}\right) \psi^{*}, \psi^{*}\right) \\
& \quad-4 \sum_{n \in \mathbb{Z}}\left(a_{1}\left(\phi_{n}^{*}\right)^{4}+a_{2}\left(\psi_{n}^{*}\right)^{4}+2 a_{3}\left(\phi_{n}^{*}\right)^{2}\left(\psi_{n}^{*}\right)^{2}\right) \\
& \quad=-2 \sum_{n \in \mathbb{Z}}\left(a_{1}\left(\phi_{n}^{*}\right)^{4}+a_{2}\left(\psi_{n}^{*}\right)^{4}+2 a_{3}\left(\phi_{n}^{*}\right)^{2}\left(\psi_{n}^{*}\right)^{2}\right)<0 . \tag{40}
\end{align*}
$$

Thus, $\Lambda=0$ and

$$
\begin{equation*}
\left(J^{\prime}\left(\phi^{*}, \psi^{*}\right),(\phi, \psi)\right)=0 \tag{41}
\end{equation*}
$$

for any $(\phi, \psi) \in E_{1} \times E_{2}$. Take $(\phi, \psi)=\left(e^{(k)}, 0\right)$ and $(\phi, \psi)=$ $\left(0, e^{(k)}\right)$ in (41) for $k \in \mathbb{Z}$, where

$$
e_{n}^{(k)}= \begin{cases}1, & n=k  \tag{42}\\ 0, & n \neq k\end{cases}
$$

We see that $J^{\prime}\left(\phi^{*}, \psi^{*}\right)=0$. Thus, $\left(\phi^{*}, \psi^{*}\right)$ is a nontrivial solution to system (13). The proof is completed.

By Theorem 4, the system (1) has a nontrivial solution. However, it is uncertain if two components of this solution are nonzero. Therefore, we want to find solutions of the system (1) which have both of the components not identically zero. In order to achieve this goal, we consider the system (1) with $b_{1 n}=b_{2 n}, n \in \mathbb{Z}$; that is,

$$
\begin{align*}
& i \frac{d u_{n}}{d t}=-(\mathscr{A} u)_{n}+b_{1 n} u_{n}-a_{1}\left|u_{n}\right|^{2} u_{n}-a_{3}\left|v_{n}\right|^{2} u_{n} \\
& i \frac{d v_{n}}{d t}=-(\mathscr{A} v)_{n}+b_{1 n} v_{n}-a_{2}\left|v_{n}\right|^{2} v_{n}-a_{3}\left|u_{n}\right|^{2} v_{n} \tag{43}
\end{align*}
$$

In system (43), we know that $L_{1}=L_{2}$, where $L_{i}, i=1,2$, is given by (11). By the definition of $E_{i}, i=1,2$, in Section 2 of this paper, we obtain that $E_{1}=E_{2}$. Hence, $\lambda_{1}=\lambda_{2}$. For the sake of simplicity, we let $L_{1}=L_{2}=L, E_{1}=E_{2}=E$, and $\lambda_{1}=\lambda_{2}=\lambda$. The notations in Section 2, such as $J(\phi, \psi), I(\phi, \psi)$, and $N$ are the same.

Now, we give the second result of this paper as follows.
Theorem 5. Assume that $\omega_{1}<\lambda, \omega_{2}<\lambda, a_{3}>$ $\max \left\{a_{1}, a_{2},\left(\left(\lambda-\omega_{2}\right) /\left(\lambda-\omega_{1}\right)\right) a_{1},\left(\left(\lambda-\omega_{1}\right) /\left(\lambda-\omega_{2}\right)\right) a_{2}\right\}$, and (10) holds. Then system (43) has a nontrivial standing wave solution $(\widetilde{\phi}, \widetilde{\psi})$ in $E \times E$ with $\widetilde{\phi} \neq 0$ and $\widetilde{\psi} \neq 0$.

Proof. By Theorem 4, we know that system (43) has a nontrivial standing wave solution $(\widetilde{\phi}, \widetilde{\psi})$ in $E \times E$.

Now we will prove that $\widetilde{\phi} \neq 0$ and $\widetilde{\psi} \neq 0$.
Since $(\widetilde{\phi}, \widetilde{\psi}) \in N$, we know that $(\widetilde{\phi}, \widetilde{\psi}) \neq(0,0)$. If one of the components $(\widetilde{\phi}, \widetilde{\psi})$, say $\widetilde{\psi}=0$, then $\widetilde{\phi} \neq 0$. For $\epsilon$ small enough, we consider $(\widetilde{\phi}, \epsilon \widetilde{\phi}) \in(E-\{0\}) \times(E-\{0\})$; by a similar argument to the proof of Lemma 2, we know that there exists $t^{*}$ such that $I\left(t^{*} \widetilde{\phi}, t^{*} \in \widetilde{\phi}\right)=0$; that is, $\left(t^{*} \widetilde{\phi}, t^{*} \in \widetilde{\phi}\right) \in N$.

By (20) and $I\left(t^{*} \widetilde{\phi}, t^{*} \epsilon \widetilde{\phi}\right)=0$, we have $\left(t^{*}\right)^{2}=$ $\left(H_{1}(\widetilde{\phi}, \epsilon \widetilde{\phi})\right) /\left(H_{2}(\widetilde{\phi}, \epsilon \widetilde{\phi})\right)$, where

$$
\begin{gather*}
H_{1}(\phi, \psi)=\|\phi\|_{E}^{2}-\omega_{1}\|\phi\|_{l^{2}}^{2}+\|\psi\|_{E}^{2}-\omega_{2}\|\psi\|_{l^{2}}^{2} \\
H_{2}(\phi, \psi)=\sum_{n \in \mathbb{Z}}\left(a_{1} \phi_{n}^{4}+a_{2} \psi_{n}^{4}+2 a_{3} \phi_{n}^{2} \psi_{n}^{2}\right) \tag{44}
\end{gather*}
$$

and $J\left(t^{*} \widetilde{\phi}, t^{*} \epsilon \widetilde{\phi}\right)=\left(H_{1}^{2}(\widetilde{\phi}, \epsilon \widetilde{\phi})\right) /\left(4 H_{2}(\widetilde{\phi}, \epsilon \widetilde{\phi})\right)$.
We noticed that $J(\widetilde{\phi}, 0)=\left(a_{1} / 4\right) \sum_{n \in \mathbb{Z}} \widetilde{\phi}^{4}=$ $\inf _{(\phi, \psi) \in N} J(\phi, \psi)$ and

$$
\begin{align*}
H_{2}(\widetilde{\phi}, 0) & =H_{1}(\widetilde{\phi}, 0)=\|\widetilde{\phi}\|_{E}^{2}-\omega_{1}\|\widetilde{\phi}\|_{l^{2}}^{2}  \tag{45}\\
& \geq\left(\lambda-\omega_{1}\right)\|\widetilde{\phi}\|_{l^{2}}^{2}
\end{align*}
$$

For the sake of simplicity, we let

$$
\begin{equation*}
B=\sum_{n \in \mathbb{Z}} \widetilde{\phi}_{n}^{4}, \quad D=\|\tilde{\phi}\|_{E}^{2}-\omega_{2}\|\tilde{\phi}\|_{1^{2}}^{2} . \tag{46}
\end{equation*}
$$

If $\omega_{1} \leq \omega_{2}<\lambda$, then

$$
\begin{align*}
D & =\|\tilde{\phi}\|_{E}^{2}-\omega_{2}\|\tilde{\phi}\|_{l^{2}}^{2} \\
& =\|\tilde{\phi}\|_{E}^{2}-\omega_{1}\|\tilde{\phi}\|_{l^{2}}^{2}+\left(\omega_{1}-\omega_{2}\right)\|\tilde{\phi}\|_{l^{2}}^{2}  \tag{47}\\
& =a_{1} B+\left(\omega_{1}-\omega_{2}\right)\|\tilde{\phi}\|_{l^{2}}^{2} \leq a_{1} B .
\end{align*}
$$

Thus, $a_{3}>a_{1}$ and (47) yields $a_{1} B D<a_{1} a_{3} B^{2}$.
If $\omega_{2}<\omega_{1}<\lambda$, then by (45),

$$
\begin{align*}
D & =\|\tilde{\phi}\|_{E}^{2}-\omega_{2}\|\tilde{\phi}\|_{l^{2}}^{2} \\
& =\|\tilde{\phi}\|_{E}^{2}-\omega_{1}\|\tilde{\phi}\|_{l^{2}}^{2}+\left(\omega_{1}-\omega_{2}\right)\|\tilde{\phi}\|_{l^{2}}^{2} \\
& =a_{1} B+\left(\omega_{1}-\omega_{2}\right)\|\tilde{\phi}\|_{l^{2}}^{2}  \tag{48}\\
& \leq a_{1} B+\frac{\omega_{1}-\omega_{2}}{\lambda-\omega_{1}} a_{1} B=\frac{\lambda-\omega_{2}}{\lambda-\omega_{1}} a_{1} B .
\end{align*}
$$

Thus, $a_{3}>\left(\left(\lambda-\omega_{2}\right) /\left(\lambda-\omega_{1}\right)\right) a_{1}$ and (48) yields $a_{1} B D<$ $a_{1} a_{3} B^{2}$.

From the above arguments, if $a_{3}>\max \left\{a_{1},\left(\left(\lambda-\omega_{2}\right) /(\lambda-\right.\right.$ $\left.\left.\left.\omega_{1}\right)\right) a_{1}\right\}$, then $a_{1} B D<a_{1} a_{3} B^{2}$.

For $\epsilon$ small enough, we have

$$
\begin{align*}
H_{1}^{2} & (\widetilde{\phi}, \epsilon \widetilde{\phi}) \\
& =\left(\|\widetilde{\phi}\|_{E}^{2}-\omega_{1}\|\tilde{\phi}\|_{l^{2}}^{2}+\|\epsilon \widetilde{\phi}\|_{E}^{2}-\omega_{2}\|\epsilon \widetilde{\phi}\|_{L^{2}}^{2}\right)^{2} \\
& =\left(a_{1} B+\epsilon^{2} D\right)^{2}=a_{1}^{2} B^{2}+2 a_{1} B D \epsilon^{2}+D^{2} \epsilon^{4}  \tag{49}\\
& <a_{1}^{2} B^{2}+2 a_{1} a_{3} B^{2} \epsilon^{2}+a_{1} a_{2} B^{2} \epsilon^{4} \\
& =a_{1} B\left(a_{1} B+a_{2} B \epsilon^{4}+2 a_{3} B \epsilon^{2}\right) \\
& =H_{2}(\widetilde{\phi}, 0) H_{2}(\widetilde{\phi}, \epsilon \widetilde{\phi})
\end{align*}
$$

Hence, by (49), we have

$$
\begin{align*}
J\left(t^{*} \widetilde{\phi}, t^{*} \epsilon \widetilde{\phi}\right) & =\frac{H_{1}^{2}(\widetilde{\phi}, \epsilon \widetilde{\phi})}{4 H_{2}(\widetilde{\phi}, \epsilon \widetilde{\phi})}<\frac{1}{4} H_{2}(\widetilde{\phi}, 0)  \tag{50}\\
& =J(\widetilde{\phi}, 0)=\inf _{(\phi, \psi) \in N} J(\phi, \psi) .
\end{align*}
$$

This is a contradiction. So, $\widetilde{\psi} \neq 0$.
Similarly, if $\widetilde{\psi} \neq 0$ and $a_{3}>\max \left\{a_{2},\left(\left(\lambda-\omega_{1}\right) /\left(\lambda-\omega_{2}\right)\right) a_{2}\right\}$, then $\widetilde{\phi} \neq 0$. The proof is completed.

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## Research Article

# On the Existence and Stability of Periodic Solutions for a Nonlinear Neutral Functional Differential Equation 

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#### Abstract

This paper deals with the existence and stability of periodic solutions for the following nonlinear neutral functional differential equation $(d / d t) D u_{t}=p(t)-a u(t)-a q u(t-r)-h(u(t), u(t-r))$. By using Schauder-fixed-point theorem and Krasnoselskii-fixedpoint theorem, some sufficient conditions are obtained for the existence of asymptotic periodic solutions. Main results in this paper extend the related results due to Ding (2010) and Lopes (1976).


## 1. Introduction

In recent years, the existence and stability of periodic solutions for differential equation has been extensively studied. Many researchers used the Lyapunov functional method, Hopf bifurcation techniques, and Mawhin continuation theorems to obtain the existence and stability of periodic solutions for neutral functional differential equation (NFDE); see papers $[1-14]$ and their references therein. However, researches on the existence and stability of periodic solutions for NFDE by using fixed-point theorem are relatively rare [15, 16]. The reason lies in the fact that it is difficult to construct an appropriate completely continuous operator and an appropriate bounded closed convex set.

In this paper, we will investigate the existence and stability of periodic solutions for the following nonlinear NFDE

$$
\begin{equation*}
\frac{d}{d t} D u_{t}=p(t)-a u(t)-a q u(t-r)-h(u(t), u(t-r)) \tag{1}
\end{equation*}
$$

where $D u_{t}=u(t)-q u(t-r),|q|\langle 1, a\rangle 0, h \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $p \in C(\mathbb{R}, \mathbb{R})$. Such a kind of NFDE has been used for the study of distributed networks containing a transmission line [17, 18]. For example, suppose a system consists of a long
electrical cable of length $l$, and one end of which isconnected to a power source $E(t)$ with resistance $R_{0}$, while the other end is connected to an oscillating circuit formed of a condenser $C_{0}$ and a nonlinear element, the volt-ampere characteristic of which is $i=f(u)$. Let $L, C, R$, and $G$ be the parameters of the transmission line, respectively, $Z_{0}$ the characteristic impedance of the line, $v=1 / \sqrt{L C}$ the propagation velocity and assume the losses can not be disregarded. The process of the final end volt $u(t)$ in such a system can be described by the following NFDES:

$$
\begin{align*}
u^{\prime}(t)-q u^{\prime}(t-r)= & p(t)-a u(t)-a q u(t-r) \\
& -b f(u(t))+b q f(u(t-r)) \tag{2}
\end{align*}
$$

or

$$
\begin{align*}
C_{0}\left(u^{\prime}(t)-k u^{\prime}(t-r)\right)= & p(t)-Z u(t)-Z k u(t-r)  \tag{3}\\
& -f(u(t)-k u(t-r))
\end{align*}
$$

where $a=1 / Z_{0} C_{0}, b=1 / C_{0}, q=\left(Z_{0}-R_{0}\right) / A^{2}\left(Z_{0}+R_{0}\right)$, $k=\left(1-Z R_{0}\right) /\left(1+Z R_{0}\right), p(t)=2 E(t-(r / 2)) / A\left(Z_{0}+R_{0}\right) C_{0}$, $r=2 l / v, A=e^{R l / Z_{0}}, Z=\sqrt{C / L}$, and $f(u)$ is a given nonlinear function. If $R_{0}>0$, then $|q|<1,|k|<1$. Obviously, we see that (2) (or (3)) is a special case of (1). The aim of this paper
is to establish some criteria to guarantee the existence and stability of periodic solution for (1) by using Schauder's fixedpoint theorem and Krasnoselskii's fixed-point theorem. The interesting is that main results obtained in this paper extend the related existing results. Furthermore, our results can also be applied to solve the problem of the existence and stability of periodic solutions for (2) and (3).

## 2. Main Results and Proofs

In this section, let $C^{1}\left(\mathbb{R}^{N}\right)\left(C\left(\mathbb{R}^{N}\right)\right)$ denote the set of all continuously differentiable functions (all continuous functions) $\phi: \mathbb{R}^{N} \rightarrow \mathbb{R}$, where $N=1,2 . C_{\omega}=\{\phi \mid \phi \in C(\mathbb{R}), \phi(t+$ $\omega)=\phi(t)\}$ is a Banach space with the supremum norm $\|\cdot\|_{0}$, $C_{\omega}^{1}=C^{1}(\mathbb{R}) \cap C_{\omega}$ with the norm $\|\phi\|_{1}=\|\phi\|_{0}+\left\|\phi^{\prime}\right\|_{0}$ in a period interval, and $\omega$ is a positive constant. The next lemma will be used in the sequel.

Lemma 1. If $a \neq 0, f \in C_{\omega}$, then the scalar equation $x^{\prime}(t)=$ $\operatorname{ax}(t)+f(t)$ has a unique $\omega$-periodic solution:

$$
\begin{equation*}
x(t)=\left(1-e^{a \omega}\right)^{-1} \int_{t}^{t+\omega} e^{a(t+\omega-s)} f(s) d s \tag{4}
\end{equation*}
$$

Proof. It is easy to prove. We can find it in many ODE textbooks (e.g., see Example 2 on page 35 of [19]).

Theorem 2. Suppose that $h \in C\left(\mathbb{R}^{2}\right)$ and $p \in C_{T}$. If there exists a constant $H>0$ such that

$$
\begin{equation*}
\|p\|_{0}<(1-3|q|) a H-\sup _{|x|,|y| \leq H}|h(x, y)| \tag{5}
\end{equation*}
$$

then (1) has a T-periodic solution.
Proof. According to the condition (5), we can find a sufficiently small $L>0$ such that

$$
\begin{equation*}
\left(2 L+\frac{1}{a}\right)\left[\|p\|_{0}+2 a|q| H+\sup _{|x|,|y| \leq H}|h(x, y)|\right]+|q| H \leq H \tag{6}
\end{equation*}
$$

Let $v(t)=u(L t), \tau=r / L, p_{1}(t)=p(L t)$, and $\omega=T / L$; then (1) can be rewritten as

$$
\begin{align*}
v^{\prime}(t)-q v^{\prime}(t-\tau)= & L p_{1}(t)-a L v(t)-a q L v(t-\tau) \\
& -\operatorname{Lh}(v(t), v(t-\tau)), \tag{7}
\end{align*}
$$

where $p_{1}(t) \in C_{\omega}$ with $\|p\|_{0}=\left\|p_{1}\right\|_{0}$. It suffices to prove that (7) has a $\omega$-periodic solution. Let

$$
\begin{equation*}
M=\left\{\phi \mid \phi \in C_{\omega}^{1},\|\phi\|_{1} \leq H\right\} \tag{8}
\end{equation*}
$$

Then $M$ is a bounded closed convex set of the Banach space $C^{1}(\mathbb{R})$. For any given $\phi \in M$, consider the nonhomogeneous equation:

$$
\begin{align*}
v^{\prime}(t)= & -a L v(t)+L p_{1}(t)-a q L \phi(t-\tau) \\
& -\operatorname{Lh}(\phi(t), \phi(t-\tau))+q \phi^{\prime}(t-\tau) \tag{9}
\end{align*}
$$

According to Lemma 1, (9) has a unique $\omega$-periodic solution:

$$
\begin{align*}
& v(t)=\left(1-e^{-a L \omega}\right)^{-1} \\
& \qquad \begin{aligned}
\times \int_{t}^{t+\omega} & e^{-a L(t+\omega-s)} \\
\times & {\left[L_{1}(s)-a q L \phi(s-\tau)\right.} \\
& \left.\quad-\operatorname{Lh}(\phi(s), \phi(s-\tau))+q \phi^{\prime}(s-\tau)\right] d s
\end{aligned} \tag{10}
\end{align*}
$$

Define an operator $A$ by
$(A \phi)(t)$

$$
\begin{align*}
&=\left(1-e^{-a L \omega}\right)^{-1} \\
& \times \int_{t}^{t+\omega} e^{-a L(t+\omega-s)} \\
& \times\left[L p_{1}(s)-a q L \phi(s-\tau)\right. \\
&\left.\quad-\operatorname{Lh}(\phi(s), \phi(s-\tau))+q \phi^{\prime}(s-\tau)\right] d s  \tag{11}\\
&=\left(1-e^{-a L \omega}\right)^{-1} \\
& \times \int_{t}^{t+\omega} e^{-a L(t+\omega-s)} \\
& \times\left[\operatorname{Lp_{1}}(s)-2 a q L \phi(s-\tau)\right. \\
&\quad-\operatorname{Lh}(\phi(s), \phi(s-\tau))] d s+q \phi(t-\tau) .
\end{align*}
$$

In order to prove that (7) has a periodic solution, we shall make sure that $A$ satisfies the conditions of Schauder's fixedpoint theorem (see Lemma 2.2.4 on page 40 of [20]).

Note that for any $x \in M$, we have $x(t+\omega)=x(t)$ and $\|x\|_{1} \leq H$

$$
\begin{align*}
& (A x)(t+\omega) \\
& \left.\begin{array}{l}
=\left(1-e^{-a L \omega}\right)^{-1} \\
\times \int_{t+\omega}^{t+2 \omega} e^{-a L(t+2 \omega-s)} \\
\quad \times \\
\quad\left[L p_{1}(s)-2 a q L x(s-\tau)\right. \\
\quad-L h(x(s), x(s-\tau))] d s+q x(t+\omega-\tau) \\
=\left(1-e^{-a L \omega}\right)^{-1} \\
\times \int_{t}^{t+\omega} e^{-a L(t+\omega-s)} \\
\quad \times
\end{array}\right]\left[\operatorname{Lp_{1}(s)-2aqLx(s-\tau )}\right. \\
& \quad-\operatorname{Lh}(x(s), x(s-\tau))] d s+q x(t-\tau) \\
& =(A x)(t) ;
\end{align*}
$$

Therefore, $(A x)(t+\omega)=(A x)(t)$. Meanwhile, we get

$$
\begin{align*}
& (A x)^{\prime}(t) \\
& \begin{array}{l}
=\left(1-e^{-a L \omega}\right)^{-1} \\
\times\left\{\int_{t}^{t+\omega} e^{-a L(t+\omega-s)}(-a L)\right. \\
\times\left[L p_{1}(s)-2 a q L x(s-\tau)\right. \\
\quad-\operatorname{Lh}(x(s), x(s-\tau))] d s \\
\quad+\left(1-e^{-a L \omega}\right)\left[L p_{1}(t)-2 a q L x(t-\tau)\right. \\
\quad-\operatorname{Lh}(x(t), x(t-\tau))]\}
\end{array} \\
& +q x^{\prime}(t-\tau) \\
& =\left(1-e^{-a L \omega}\right)^{-1}(-a L) \\
& \times \int_{t}^{t+\omega} e^{-a L(t+\omega-s)} \\
& \times\left[L p_{1}(s)-2 a q L x(s-\tau)\right. \\
& -\operatorname{Lh}(x(s), x(s-\tau))] d s \\
& +\left[L p_{1}(t)-2 a q L x(t-\tau)-\operatorname{Lh}(x(t), x(t-\tau))\right] \\
& +q x^{\prime}(t-\tau) \text {. } \tag{13}
\end{align*}
$$

By (6), we have
$\|A x\|_{1}$

$$
\begin{aligned}
&=\|A x\|_{0}+\left\|(A x)^{\prime}\right\|_{0} \\
& \leq \sup _{t \in \mathbb{R}} \mid\left(1-e^{-a L \omega}\right)^{-1} \\
& \times \int_{t}^{t+\omega} e^{-a L(t+\omega-s)} \\
& \times\left[L p_{1}(s)-2 a q L x(s-\tau)\right. \\
&+\sup _{t \in \mathbb{R}} \mid\left(1-e^{-a L \omega}\right)^{-1}(-a L) \\
& \times \int_{t}^{t+\omega} e^{-a L(t+\omega-s)} \\
&\times[\operatorname{Lp}(s), x(s-\tau))] d s \mid \\
&\quad-\operatorname{Lh}(x(s), x(s-\tau))] d s
\end{aligned}
$$

$$
\begin{align*}
& +\left[L p_{1}(t)-2 a q L x(t-\tau)-\operatorname{Lh}(x(t), x(t-\tau))\right] \mid \\
& +|q|\|x\|_{1} \leq\left(2+\frac{1}{a L}\right) \\
& \times\left[L\left\|p_{1}\right\|_{0}+2 a|q| L H+L \sup _{|x|,|y| \leq H}|h(x, y)|\right] \\
& +|q| H=\left(2 L+\frac{1}{a}\right) \\
& \quad \times\left[\left\|p_{1}\right\|_{0}+2 a|q| H+\sup _{|x|,|y| \leq H}|h(x, y)|\right]+|q| H \leq H \tag{14}
\end{align*}
$$

Thus, $A x \in M$.
For any $x \in M,\|A x\|_{0} \leq H,\left\|(A x)^{\prime}\right\|_{0} \leq H$. According to Arzela-Ascoli Theorem (see Theorem 4.9.6 on page 84 of [21]), the subset $A M$ of $C_{\omega}$ is a precompact set; therefore, $A$ : $M \subset C^{1}(\mathbb{R}) \rightarrow C_{\omega}$ is a compact operator.

Suppose that $\left\{x_{n}\right\} \in M, x_{n} \rightarrow x$, then $\left\|x_{n}-x\right\|_{0} \rightarrow 0$ and $\left\|x_{n}^{\prime}-x^{\prime}\right\|_{0} \rightarrow 0$ as $n \rightarrow \infty$. Also, we have

$$
\begin{aligned}
& \left\|A x_{n}-A x\right\|_{0} \\
& =\sup _{t \in \mathbb{R}} \mid\left(1-e^{-a L \omega}\right)^{-1} \\
& \times \int_{t}^{t+\omega} e^{-a L(t+\omega-s)} \\
& \times\left[-2 a q L\left(x_{n}(s-\tau)-x(s-\tau)\right)\right. \\
& -L\left(h\left(x_{n}(s), x_{n}(s-\tau)\right)\right. \\
& -h(x(s), x(s-\tau)))] d s \\
& +q\left(x_{n}(t-\tau)-x(t-\tau)\right) \mid \\
& \leq \frac{1}{a L}\left[2 a|q| L\left\|x_{n}-x\right\|_{0}\right. \\
& +L \sup _{t \in[0, \omega]} \mid h\left(x_{n}(t), x_{n}(t-\tau)\right) \\
& -h(x(t), x(t-\tau)) \mid] \\
& +|q|\left\|x_{n}-x\right\|_{0} \\
& =3|q|\left\|x_{n}-x\right\|_{0}+\frac{1}{a} \sup _{t \in[0, \omega]} \\
& \times\left|h\left(x_{n}(t), x_{n}(t-\tau)\right)-h(x(t), x(t-\tau))\right|, \\
& \left\|\left(A x_{n}\right)^{\prime}-(A x)^{\prime}\right\|_{0} \\
& =\sup _{t \in \mathbb{R}} \mid\left(1-e^{-a L \omega}\right)^{-1}(-a L)
\end{aligned}
$$

$$
\begin{gather*}
\times \int_{t}^{t+\omega} e^{-a L(t+\omega-s)} \\
\times[-2 a q L \\
\times\left(x_{n}(s-\tau)-x(s-\tau)\right) \\
-L\left(h\left(x_{n}(s), x_{n}(s-\tau)\right)\right. \\
-h(x(s), x(s-\tau)))] d s \\
-2 a q L\left(x_{n}(t-\tau)-x(t-\tau)\right)-L\left(h\left(x_{n}(t), x_{n}(t-\tau)\right)\right. \\
-h(x(t), x(t-\tau)))+q\left(x_{n}^{\prime}(t-\tau)-x^{\prime}(t-\tau)\right) \mid \\
\leq 2 L\left[2 a|q|\left\|x_{n}-x\right\|_{0}\right. \\
+\sup _{t \in[0, \omega]} \mid h\left(x_{n}(t), x_{n}(t-\tau)\right) \\
-h(x(t), x(t-\tau)) \mid]+|q|\left\|x_{n}^{\prime}-x^{\prime}\right\|_{0} . \tag{15}
\end{gather*}
$$

When $\left\|x_{n}-x\right\|_{1} \rightarrow 0$ as $n \rightarrow \infty,\left|x_{n}(t)-x(t)\right| \rightarrow 0$ for $t \in[0, \omega]$ uniformly. And since $h$ is continuous, $\| A x_{n}-$ $A x\left\|_{0} \rightarrow 0,\right\|\left(A x_{n}\right)^{\prime}-(A x)^{\prime} \|_{0} \rightarrow 0$. Consequently, $A$ is continuous.

Thus, by Schauder-fixed-point theorem (see Lemma 2.2.4 on page 40 of [20]), there is a $\phi \in M$ such that $\phi=A \phi$. Therefore, (7) has at least one $\omega$-periodic solution. Since $v(t)=u(L t)$ and $p(L t)=p_{1}(t),(1)$ has at least one $T$-periodic solution. The proof is completed.

Next, we explore the stability of this $T$-periodic solution $u^{*}(t)$ for (1). We assume that theconditions of Theorem 2 are satisfied. Therefore, (1) has at least one $T$-periodic solution $u^{*}(t)$. Let $v(t)=u(t)-u^{*}(t)$ then (1) is transformed as

$$
\begin{equation*}
\frac{d}{d t} D v_{t}=-a v(t)-a q v(t-r)-g(v(t), v(t-r)) \tag{16}
\end{equation*}
$$

where $D v_{t}=v(t)-q v(t-r)$ and $g(v(t), v(t-r))=h\left(u^{*}(t)+\right.$ $\left.v(t), u^{*}(t-r)+v(t-r)\right)-h\left(u^{*}(t), u^{*}(t-r)\right)$. Clearly, (16) has trivial solution $v(t) \equiv 0$. Now we use Krasnoselskiis-fixedpoint theorem (see [22] or [15, Lemma 2.2]) to prove that trivial solution $v(t) \equiv 0$ to (16) is asymptotically stable.

Set $S$ as the Banach space of bounded continuous function $\phi:[-r, \infty) \rightarrow R$ with the supremum norm $\|\cdot\|$. Also, Given the initial function $\psi$, denote the norm of $\psi$ by $\|\psi\|=$ $\sup _{t \in[-r, 0]}|\psi(t)|$, which should not cause confusion with the same symbol for the norm in $S$.

Theorem 3. Let $H$ be as in Theorem 2. Assume that all conditions of Theorem 2 are satisfied. Suppose that h satisfies the Lipschitz condition and

$$
\begin{equation*}
\sup _{|x|,|y| \leq H}|h(x, y)| \leq a H(1+|q|) \tag{17}
\end{equation*}
$$

If there exists $Q>0$ such that

$$
\begin{equation*}
\|\psi\| \leq \frac{Q-3|q| Q}{1+|q|}-H-\frac{1}{a(1+|q|)} \sup _{|x|,|y| \leq H+Q}|h(x, y)|, \tag{18}
\end{equation*}
$$

then the solution $v(\psi)(t)$ to (16) through $\psi$ satisfies $\lim _{t \rightarrow \infty} v(\psi)(t)=0$.

Proof. By (18), we have

$$
\begin{align*}
& (1+|q|)\|\psi\|+3|q| Q+\frac{1}{a} \sup _{|x|,|y| \leq H+Q}|h(x, y)|  \tag{19}\\
& \quad+H(1+|q|) \leq Q .
\end{align*}
$$

Given the initial function $\psi$, by [20, Theorem 12.2.3], there exists a unique solution $v(\psi)(t)$ for (16). Let

$$
\begin{align*}
& M_{\psi} \\
& \quad=\left\{\phi \mid \phi \in S,\|\phi\| \leq Q, \phi_{0}=\psi, \phi(t) \longrightarrow 0 \text { as } t \longrightarrow \infty\right\} ; \tag{20}
\end{align*}
$$

then $M_{\psi}$ is a bounded convex closed set of $S$. We write (16) as

$$
\begin{align*}
{[v(t)-q v(t-r)]^{\prime}=} & -a[v(t)-q v(t-r)]-2 a q v(t-r) \\
& -g(v(t), v(t-r)) \tag{21}
\end{align*}
$$

then we have

$$
\begin{align*}
v(t)= & {[\psi(0)-q \psi(-r)] e^{-a t}+q v(t-r) } \\
& +\int_{0}^{t}[-2 \operatorname{aqv} v(s-r)-g(v(s), v(s-r))] e^{-a(t-s)} d s . \tag{22}
\end{align*}
$$

For all $\phi \in M_{\psi}$, define the operators $A$ and $B$ by
$(A \phi)(t)$

$$
=\left\{\begin{array}{rr}
0, & t \in[-r, 0] \\
\int_{0}^{t}[-2 a q \phi(s-r)-g(\phi(s), \phi(s-r))] e^{-a(t-s)} d s \\
t \geq 0
\end{array}\right.
$$

$(B \phi)(t)$

$$
= \begin{cases}\psi(t), & t \in[-r, 0]  \tag{23}\\ (\psi(0)-q \psi(-r)) e^{-a t}+q \phi(t-r), & t \geq 0\end{cases}
$$

For any $x, y \in M_{\psi}, x(t) \rightarrow 0, y(t) \rightarrow 0$ as $t \rightarrow \infty$, and $\|x\| \leq \mathrm{Q},\|y\| \leq \mathrm{Q}$. Therefore, we have

$$
\begin{aligned}
& \lim _{t \rightarrow \infty}(A x)(t) \\
& =\lim _{t \rightarrow \infty} \frac{\int_{0}^{t}[-2 a q x(s-r)-g(x(s), x(s-r))] e^{a s} d s}{e^{a t}} \\
& \quad=\lim _{t \rightarrow \infty} \frac{1}{a}[-2 a q x(t-r)-g(x(t), x(t-r))]=0,
\end{aligned}
$$

$$
\begin{equation*}
\lim _{t \rightarrow \infty}(B y)(t)=\lim _{t \rightarrow \infty}\left[(\psi(0)-q \psi(-r)) e^{-a t}+q y(t-r)\right]=0 \tag{24}
\end{equation*}
$$

Thus, $\lim _{t \rightarrow \infty}(A x+B y)(t)=0$. Again by (17) and (19), we have

$$
\begin{align*}
\| A x & +B y \| \\
\leq & \|A x\|+\|B y\| \\
= & \sup _{t \geq 0}\left|\int_{0}^{t}[-2 a q x(s-r)-g(x(s), x(s-r))] e^{-a(t-s)} d s\right| \\
& +\sup _{t \geq-r}|(B y)(t)| \\
\leq & {\left[2 a|q| Q+\sup _{|x|,|y| \leq H+Q}|h(x, y)|+\sup _{|x|,|y| \leq H}|h(x, y)|\right] } \\
& \times \sup _{t \geq 0}\left|\int_{0}^{t} e^{-a(t-s)} d s\right| \\
& +\max _{0}\left\{\|\psi\|, \sup _{t \geq 0}\left|(\psi(0)-q \psi(-r)) e^{-a t}+q y(t-r)\right|\right\} \\
\leq & (1+|q|)\|\psi\|+3|q| Q+\frac{1}{a} \sup _{|x|,|y| \leq H+Q}|h(x, y)| \\
& +\frac{1}{a} \sup _{|x|,|y| \leq H}|h(x, y)| \\
\leq & (1+|q|)\|\psi\|+3|q| Q+\frac{1}{a} \sup _{|x|,|y| \leq H+Q}|h(x, y)| \\
& +H(1+|q|) \leq Q . \tag{25}
\end{align*}
$$

Thus, $A x+B y \in M_{\psi}$.

Since $\left|(A x)^{\prime}(t)\right|=0, t \in[-r, 0]$, and

$$
\begin{align*}
&\left|(A x)^{\prime}(t)\right| \\
&= \mid-a \int_{0}^{t}[-2 a q x(s-r)-g(x(s), x(s-r))] e^{-a(t-s)} d s \\
&-2 a q x(t-r)-g(x(t), x(t-r)) \mid \\
& \leq a\left[2 a|q| Q+\sup _{|x|,|y| \leq H+Q}|h(x, y)|+\sup _{|x|,|y| \leq H}|h(x, y)|\right] \\
& \times \sup _{t \geq 0}\left|\int_{0}^{t} e^{-a(t-s)} d s\right| \\
&+2 a|q| Q+\sup _{|x|,|y| \leq H+Q}|h(x, y)|+\sup _{|x|,|y| \leq H}|h(x, y)| \\
&= 2\left[2 a|q| Q+\sup _{|x|,|y| \leq H+Q}|h(x, y)|+\sup _{|x|,|y| \leq H}|h(x, y)|\right],
\end{align*}
$$

here, the derivative of $(A x)^{\prime}(t)$ at zero means the left-hand side derivative when $t \leq 0$ and the right-hand side derivative when $t \geq 0$, one can see $(A x)^{\prime}(t)$ is bounded for all $x \in M_{\psi}$. Therefore, $A M_{\psi}$ is a precompact set of $S$. Thus, $A$ is compact.

Suppose that $\left\{x_{n}\right\} \subset M_{\psi}, x \in S, x_{n} \rightarrow x$ as $n \rightarrow \infty$; then $\left|x_{n}(t)-x(t)\right| \rightarrow 0$ uniformly for $t \geq-r$ as $n \rightarrow \infty$. Since

$$
\begin{align*}
& \left\|A x_{n}-A x\right\| \\
& \begin{aligned}
&=\sup _{t \geq 0} \mid \int_{0}^{t}\left\{-2 a q\left[x_{n}(s-r)-x(s-r)\right]\right. \\
&-g\left(x_{n}(s), x_{n}(s-r)\right) \\
&+g(x(s), x(s-r))\} e^{-a(t-s)} d s \mid \\
& \begin{aligned}
\leq & {\left[2 a|q|\left\|x_{n}-x\right\|\right.}
\end{aligned} \\
& \quad+\sup _{t \geq 0} \mid h\left(u^{*}(t)+x_{n}(t), u^{*}(t-r)+x_{n}(t-r)\right) \\
&\left.\quad-h\left(u^{*}(t)+x(t), u^{*}(t-r)+x(t-r)\right) \mid\right] \\
& \times \sup _{t \geq 0}\left|\int_{0}^{t} e^{-a(t-s)} d s\right|=2|q|\left\|x_{n}-x\right\|
\end{aligned} \\
& \left.\quad+\frac{1}{a} \sup _{t \geq 0} \right\rvert\, h\left(u^{*}(t)+x_{n}(t), u^{*}(t-r)+x_{n}(t-r)\right) \\
& \\
& \quad-h\left(u^{*}(t)+x(t), u^{*}(t-r)+x(t-r)\right) \mid,
\end{align*}
$$

and $h$ is continuous, we have $\left\|A x_{n}-A x\right\| \rightarrow 0$ as $n \rightarrow \infty$. Thus, $A$ is continuous. Due to the fact that

$$
\begin{array}{r}
\|B x-B y\|=\sup _{t \geq 0}|q x(t-r)-q y(t-r)| \leq|q|\|x-y\|, \\
\forall x, y \in M_{\psi}, \tag{28}
\end{array}
$$

and $|q|<1$, we know that $B$ is a contractive operator.
According to Krasnoselskii's fixed-point theorem (see [22] or [15, Lemma 2.2]), there is a $\phi \in M_{\psi}$ such that $(A+$ $B) \phi=\phi$. Therefore, $\phi(t)$ is a solution for (16). Because the solution through $\psi$ for the equation is unique, the solution $\nu(\psi)(t)=\phi(t) \rightarrow 0$ as $t \rightarrow \infty$.

When $h$ satisfies the Lipschitz condition, then there is a constant $L>0$ such that

$$
\begin{align*}
& \left|h\left(u(t)+u^{*}(t), u(t-r)+u^{*}(t-r)\right)-h\left(u^{*}(t), u^{*}(t-r)\right)\right| \\
& \quad \leq L \sqrt{|u(t)|^{2}+|u(t-r)|^{2}}, \quad \forall u \in S . \tag{29}
\end{align*}
$$

Since $\phi$ satisfies

$$
\begin{align*}
\phi(t)= & {[\psi(0)-q \psi(-r)] e^{-a t}+q \phi(t-r) } \\
& +\int_{0}^{t}[-2 a q \phi(s-r)-g(\phi(s), \phi(s-r))] e^{-a(t-s)} d s, \tag{30}
\end{align*}
$$

then

$$
\begin{equation*}
\|\phi\| \leq(1+|q|)\|\psi\|+3|q|\|\phi\|+\frac{\sqrt{2} L}{a}\|\phi\| \tag{31}
\end{equation*}
$$

that is

$$
\begin{equation*}
\left(1-3|q|-\frac{\sqrt{2} L}{a}\right)\|\phi\| \leq(1+|q|)\|\psi\| \tag{32}
\end{equation*}
$$

Therefore, if $1-3|q|-(\sqrt{2 L} / a)>0$, then there clearly exists a $\delta>0$ for any $\varepsilon>0$ such that $|\phi(t)|<\varepsilon$ for all $t \geq-r$ if $\|\psi\|<\delta$. Thus, we have the following.

Theorem 4. If the Lipschitz constant $L$ for $h$ corresponding to $\mathbb{R}^{2}$ satisfies

$$
\begin{equation*}
1-3|q|-\frac{\sqrt{2} L}{a}>0 \tag{33}
\end{equation*}
$$

then the zero solution for (16) is stable.
When $p$ is constant and the equation $p-a(1+q) u=$ $h(u, u)$ has only one solution $u^{*}$, then $u^{*}$ is an equilibrium of (1) and (1) can be transformed to the following equation:

$$
\begin{equation*}
\frac{d}{d t} D u_{t}=-a u(t)-a q u(t-r)-g(u(t), u(t-r)) \tag{34}
\end{equation*}
$$

where $D u_{t}=u(t)-q u(t-r)$ and $g(u(t), u(t-r))=h\left(u^{*}+\right.$ $\left.u(t), u^{*}+u(t-r)\right)-h\left(u^{*}, u^{*}\right)$. Now, we consider the stability of the zero solution for (34).

Theorem 5. Suppose that is $h \in C^{1}\left(\mathbb{R}^{2}\right)$ and $\left(h_{x}\left(u^{*}, u^{*}\right)\right.$, $\left.h_{y}\left(u^{*}, u^{*}\right)\right)=(0,0)$; then the zero solution of (34) is exponentially asymptotically stable.

Proof. For all $\phi$ in $C=C([-r, 0], \mathbb{R})$, let

$$
\begin{gather*}
D \phi=\phi(0)-q \phi(-r), \\
L \phi=-a \phi(0)-a q \phi(-r),  \tag{35}\\
F \phi=-g(\phi(0), \phi(-r)) .
\end{gather*}
$$

Then $D$ is stable, and $D$ and $L$ are linear and continuous. Consider the equation $(d / d t) D u_{t}=L u_{t}$. Let

$$
\begin{equation*}
V(\phi)=(D \phi)^{2}+2 a q^{2} \int_{-r}^{0} \phi^{2}(\theta) d \theta \tag{36}
\end{equation*}
$$

Then
$\dot{V}(\phi)$

$$
\begin{align*}
& =2(D \phi)(-a \phi(0)-a q \phi(-r))+2 a q^{2}\left(\phi^{2}(0)-\phi^{2}(-r)\right) \\
& =-2 a\left(1-q^{2}\right) \phi^{2}(0) . \tag{37}
\end{align*}
$$

Thus, according to the last conclusion of Theorem 12.7.1 in [20, Page 297], the zero solution of $u^{\prime}(t)-q u^{\prime}(t-r)=$ $-a u(t)-a q u(t-r)$ is uniformly asymptotically stable. On the other hand, one can see that

$$
\begin{equation*}
F_{\phi} u=\left(-g_{x}(u(0), u(-r)),-g_{y}(u(0), u(-r))\right) . \tag{38}
\end{equation*}
$$

Thus, $F(0)=F_{\phi}(0)=0$. According to [20, Theorem 12.9.1], the zero solution of (34) is exponentially asymptotically stable.

## 3. Examples

In this section, we present two examples to illustrate the applicability of our main results.

Example 6 (Lopes et al. [8, 9, 13, 15, 23]). Consider the NFDE (2) which arises from a transmission line model, where $a>$ $0, b>0, r>0,|q|<1, p \in C(\mathbb{R})$, and $f$ is a given nonlinear function. Now, let $h(u(t), u(t-r))=b f(u(t))-b q f(u(t-r))$. It is not difficult to see that (2) is a special case of (1). Therefore, by Theorems $2-5$, we have the following.

Theorem 7. Suppose that $f \in C(\mathbb{R})$ and $p \in C_{T}$. If there exists a constant $H>0$ such that

$$
\begin{equation*}
\|p\|_{0}<(1-3|q|) a H-b(1+|q|) \sup _{|x| \leq H}|f(x)|, \tag{39}
\end{equation*}
$$

then (2) has a T-periodic solution.
Remark 8. Theorem 7 implies that the conditions in [15]

$$
\begin{equation*}
l<1, \quad|q|<\frac{1-l}{3+l} \tag{40}
\end{equation*}
$$

where $l=(b / a H) \sup _{|x| \leq H}|f(x)|$, are unnecessary for the existence of periodic solutions for (2).

Theorem 9. Let $H$ be as in Theorem 7. Assume that all conditions of Theorem 7 are satisfied. If $f$ satisfies the Lipschitz condition, $(b / a H) \sup _{|x| \leq H}|f(x)| \leq 1$ and there exists $Q>0$ such that

$$
\begin{equation*}
\left\|\psi-u^{*}\right\| \leq \frac{Q-3|q| Q}{1+|q|}-H-\frac{b}{a} \sup _{|x| \leq H+Q}|f(x)|, \tag{41}
\end{equation*}
$$

then the solution $u(\psi)(t)$ through $\psi$ to (2) satisfying $u(\psi)(t) \rightarrow$ $u^{*}(t)$ as $t \rightarrow \infty$, where $u^{*}(t)$ is a T-periodic solution of (2).

Remark 10. The sufficient conditions for the existence of periodic solutions in [15] are very complicated. For example, they need extra condition $Q>H, m<Q-H$ and

$$
\begin{gather*}
|q|<\frac{Q-H-m}{3 Q+H+m} \\
\left\|\psi-u^{*}\right\| \leq \frac{Q-3|q| Q}{1+|q|}-H-\frac{m}{1+|q|} \tag{42}
\end{gather*}
$$

where $m=(b / a) \sup _{|x| \leq H+Q}|f(x)|$.
Theorem 11. If all conditions of Theorem 7 are satisfied, and the Lipschitz constant $L$ for $f$ corresponding to $(-\infty,+\infty)$ satisfies $1-3|q|-(b / a)(1+|q|) L>0$, then the $T$-periodic solution $u^{*}(t)$ of (2) is stable.

Theorem 12. Suppose that $p$ is constant, the equation $p-a(1+$ q) $u=b(1-q) f(u)$ has only one solution $u^{*}, f \in C^{1}(\mathbb{R})$, and $f^{\prime}\left(u^{*}\right)=0$; then the equilibrium $u^{*}$ of (2) is exponentially asymptotically stable.

Example 13 (Lopes [9]). Consider the NFDE (3) which arises from a transmission line model, where $C_{0}>0, Z>0, r>$ $0,|k|<1, p \in C(\mathbb{R})$ and $f$ is a given nonlinear function. Let $\widetilde{p}(t)=\left(1 / C_{0}\right) p(t), \tilde{a}=Z / C_{0}$, and $\tilde{f}(u)=\left(1 / C_{0}\right) f(u)$; then (3) can be rewritten as

$$
\begin{align*}
u^{\prime}(t)-k u^{\prime}(t-r)= & \tilde{p}(t)-\tilde{a} u(t)-\tilde{a} k u(t-r) \\
& -\tilde{f}(u(t)-k u(t-r)) . \tag{43}
\end{align*}
$$

Now, let $h(u(t), u(t-r))=\widetilde{f}(u(t)-k u(t-r))$. It is not difficult to see that (43) is a special case of (1). Therefore, by Theorems $2-5$, we have the following.

Theorem 14. Suppose that $f \in C(\mathbb{R})$ and $p \in C_{T}$. If there exists a constant $H>0$ such that

$$
\begin{equation*}
\|p\|_{0}<(1-3|k|) Z H-\sup _{|x| \leq(1+|k|) H}|f(x)| \tag{44}
\end{equation*}
$$

then (3) has a T-periodic solution.
Theorem 15. Let $H$ be as in Theorem 14. Assume that all conditions of Theorem 14 are satisfied. If $f$ satisfies the Lipschitz
condition, $\sup _{|x| \leq(1+|k|) H}|f(x)| \leq Z H(1+|k|)$, and there exists $Q>0$ such that

$$
\begin{align*}
& \left\|\psi-u^{*}\right\| \\
& \quad \leq \frac{Q-3|k| Q}{1+|k|}-H-\frac{1}{Z(1+|k|)} \sup _{|x| \leq(1+|k|)(H+Q)}|f(x)|, \tag{45}
\end{align*}
$$

then the solution through $\psi$ of (3) $u(\psi)(t) \rightarrow u^{*}(t)$ as $t \rightarrow$ $\infty$, where $u^{*}(t)$ is a $T$-periodic solution of (3).

Theorem 16. If all conditions of Theorem 14 are satisfied, and the Lipschitz constant $L$ for $f$ corresponding to $(-\infty,+\infty)$ satisfies $1-3|k|-(L / Z)(1+|k|)>0$, then $T$-periodic solution $u^{*}(t)$ of (3) is stable.

Theorem 17. Suppose that $p$ is constant, the equation $p-Z(1+$ $k) u=f(u-k u)$ has only one solution $u^{*}, f \in C^{1}(\mathbb{R})$, and $f^{\prime}\left(u^{*}-k u^{*}\right)=0$, then the equilibrium $u^{*}$ of (3) is exponentially asymptotically stable.

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## Research Article

# Voltage Stability Bifurcation Analysis for AC/DC Systems with VSC-HVDC 

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#### Abstract

A voltage stability bifurcation analysis approach for modeling AC/DC systems with VSC-HVDC is presented. The steady power model and control modes of VSC-HVDC are briefly presented firstly. Based on the steady model of VSC-HVDC, a new improved sequential iterative power flow algorithm is proposed. Then, by use of continuation power flow algorithm with the new sequential method, the voltage stability bifurcation of the system is discussed. The trace of the P-V curves and the computation of the saddle node bifurcation point of the system can be obtained. At last, the modified IEEE test systems are adopted to illustrate the effectiveness of the proposed method.


## 1. Introduction

As one of the key technologies of large scale access of distributed energy resources, the HVDC transmission system has great potential for further development [1,2]. Therefore, in the past decades, the problem associated with HVDC converters connected to weak AC networks has become an important research field. The one of particular interest, and highest in consequences, is the AC voltage stability at the HVDC terminals of the AC/DC systems [3, 4].

Voltage source converter-based HVDC (VSC-HVDC) is a new generation technology of HVDC, which overcomes some of the disadvantages of the traditional thyristor-based HVDC system, with a very broad application prospect. Compared to the conventional HVDC systems, the prominent features of the VSC-HVDC system are its potential to be connected to weak AC systems, independent control of active and reactive power exchange, and so on [4-6]. Due to those characteristics, many researches have been done for the exploitation of VSC-HVDC to enhance system stability of AC/DC systems, that is, the improvement of transient stability $[7,8]$, the power oscillations damping $[9,10]$, the improvement of stability and power quality for wind farm
based on VSC-HVDC grid-connected [11, 12], the stability analysis of multi-infeed DC systems with VSC-HVDC [13], and the keeping voltage stable [3, 14-17].

In [14, 15], the voltage stability analyses of AC/DC systems with VSC-HVDC were mainly based on simulation software, and the analysis based on power flow calculation was slightly inadequate. The power flow calculation of AC/DC systems is the premise and foundation of static security analysis, transient stability, voltage stability, small signal stability analysis, and so on [18-21]. At present, there are two main types of power flow algorithms for AC/DC systems, sequential iterative method [21,22] and integrated iterative method [23, 24]. The computational practice indicates that the convergence of integrated method is good, but the inheritance of the program is relatively poor, and the writing of program code needs huge work. The sequential iterative method has better program inheritance for pure AC program, but its convergence is not good. In view of these shortcomings, a modified sequential iterative power flow algorithm is proposed in this paper.

In [16], the continuation power flow (CPF) algorithm was presented to solve available transfer capability problem, but the saddle node bifurcation point was not discussed. In [17], the PV and QV curves were used to investigate voltage


Figure 1: Simplified circuit diagram of single-phase VSC-HVDC.
stability of a weak two area AC network. Though many methods for voltage stability analysis of AC/DC systems with VSCHVDC have been proposed in different ways, few papers carefully consider the influence induced by different control modes of VSC-HVDC, which actually plays an important role on the voltage stability of the AC/DC systems. Motivated by the previous discussions, our main aim in this paper is to investigate the problem of power flow and voltage stability bifurcation for AC/DC systems with VSC-HVDC.

The rest of the paper is organized as follows. In Section 2, the steady model and control modes of AC/DC systems with VSC-HVDC are presented. In Section 3, the improved sequential iterative algorithm, the parameterization power flow and converter equations of VSC-HVDC, and the CPF strategy are presented. In Section 4, the model and method are applied to the modified IEEE 14- and IEEE 118-bus test systems with VSC-HVDC. Finally, in Section 5, the paper is completed with a conclusion.

## 2. Steady Model of VSC-HVDC

2.1. Power Flow Equations of VSC-HVDC. Figure 1 shows the single-line representation of two-terminal VSC-HVDC system. In Figure 1, $i$ is the number of VSC converters. $\mathbf{I}_{i}$ is the current flowing through transformer. $\mathbf{U}_{\mathrm{si} i}=U_{\mathrm{si}} \angle \theta_{\mathrm{si} i}$ is the AC side voltage vector of VSC converter. $\mathbf{U}_{\mathrm{c} i}=U_{\mathrm{c} i} \angle \theta_{\mathrm{c} i}$ is the output voltage vector of VSC converter. $R_{i}$ is the equivalent resistance of internal loss and converter transformer loss of VSC. $j X_{l i}$ is the impedance of converter transformer. $P_{s i}$ and $Q_{s i}$ are the power of AC system injected into converter transformer. $j X_{f}$ is the impedance of AC filter. $\mathbf{I}_{\mathrm{d}}$ and $\mathbf{U}_{\mathrm{d}}$ are the DC current vector and DC voltage vector, respectively. $P_{\mathrm{c} i}$ and $Q_{c i}$ are the power flowing through converter bridge. $P_{\mathrm{d} i}$ is the DC power. From Figure 1, $\mathbf{I}_{i}$ can be expressed as follows:

$$
\begin{equation*}
\mathbf{I}_{i}=\frac{\mathbf{U}_{s i}-\mathbf{U}_{\mathrm{c} i}}{R_{i}+j X_{l i}} \tag{1}
\end{equation*}
$$

The complex power $\widetilde{S_{\mathrm{si}}}$ of AC system is given by

$$
\begin{equation*}
\widetilde{S_{s i}}=P_{s i}+j Q_{s i}=\mathbf{U}_{s i} I_{i}^{*} \tag{2}
\end{equation*}
$$

To facilitate discussion, assume that $\delta_{i}=\theta_{s i}-\theta_{c i},\left|Y_{i}\right|=$ $1 / \sqrt{R_{i}^{2}+X_{l i}^{2}}, \alpha_{i}=\operatorname{arccot}\left(X_{l i} / R_{i}\right)$, and substituting (1) into (2) yields

$$
\begin{gather*}
P_{s i}=-\left|Y_{i}\right| U_{\mathrm{si} i} U_{\mathrm{c} i} \cos \left(\delta_{i}+\alpha_{i}\right)+\left|Y_{i}\right| U_{\mathrm{s} i}^{2} \cos \alpha_{i} \\
Q_{s i}=-\left|Y_{i}\right| U_{\mathrm{s} i} U_{\mathrm{c} i} \sin \left(\delta_{i}+\alpha_{i}\right)+\left|Y_{i}\right| U_{\mathrm{s} i}^{2} \sin \alpha_{i}+\frac{U_{\mathrm{s} i}^{2}}{X_{\mathrm{f} i}} \tag{3}
\end{gather*}
$$

Similarly, one has

$$
\begin{align*}
P_{\mathrm{c} i} & =\left|Y_{i}\right| U_{\mathrm{si}} U_{\mathrm{c} i} \cos \left(\delta_{i}-\alpha_{i}\right)-\left|Y_{i}\right| U_{\mathrm{c} i}^{2} \cos \alpha_{i}  \tag{4}\\
Q_{\mathrm{c} i} & =-\left|Y_{i}\right| U_{\mathrm{s} i} U_{\mathrm{c} i} \sin \left(\delta_{i}-\alpha_{i}\right)-\left|Y_{i}\right| U_{\mathrm{c} i}^{2} \sin \alpha_{i}
\end{align*}
$$

Since the loss of converter bridge-arm is equivalent by $R_{i}, P_{\mathrm{d} i}$ is qual to $P_{c i}$, and thus

$$
\begin{equation*}
P_{\mathrm{d} i}=U_{\mathrm{d} i} I_{\mathrm{d} i}=\left|Y_{i}\right| U_{\mathrm{si} i} U_{\mathrm{c} i} \cos \left(\delta_{i}-\alpha_{i}\right)-\left|Y_{i}\right| U_{\mathrm{c} i}^{2} \cos \alpha_{i} \tag{5}
\end{equation*}
$$

And $U_{c i}$ can be described as

$$
\begin{equation*}
U_{\mathrm{c} i}=\frac{\sqrt{6} M_{i} U_{\mathrm{d} i}}{4} \tag{6}
\end{equation*}
$$

where $M(0<M<1)$ is defined as the PWM's amplitude modulation index.

The steady model of VSC-HVDC is given by (1)-(6) in the per-unit system (p.u.).
2.2. Steady-State Control Modes of VSC-HVDC. Several regular control modes for each VSC converter are chiefly as follows:
(1) constant DC voltage control, constant AC reactive power control;
(2) constant DC voltage control, constant AC voltage control;
(3) constant AC active power control, constant AC reactive power control;
(4) constant AC active power control, constant AC voltage control.

## 3. Voltage Stability Model for AC/DC Systems with VSC-HVDC

3.1. The Modified Power Flow Algorithm Based on Sequential Iteration Method. For the previous model of AC/DC system with VSC-HVDC, the simplified power flow equations are given by

$$
\begin{gather*}
\mathbf{f}_{\mathrm{ac}}=0 \\
\mathbf{f}_{\mathrm{ac}-\mathrm{dc}}=0  \tag{7}\\
\mathbf{f}_{\mathrm{dc}}=0
\end{gather*}
$$

where $\mathbf{f}_{\mathrm{ac}}=\left[\Delta P_{\mathrm{a} 1}, \Delta Q_{\mathrm{a} 1}, \ldots, \Delta P_{\mathrm{a} n_{\mathrm{AC}}}, \Delta Q_{\mathrm{a} n_{\mathrm{AC}}}\right]^{\mathrm{T}}, \mathbf{f}_{\mathrm{ac}-\mathrm{dc}}=\left[\Delta P_{\mathrm{t} 1}\right.$, $\left.\Delta Q_{\mathrm{t} 1}, \ldots, \Delta P_{\mathrm{t} n_{\mathrm{VSC}}}, \Delta Q_{\mathrm{t} \mathrm{thvC}}\right]^{\mathrm{T}}, \mathbf{f}_{\mathrm{dc}}=\left[\Delta d_{11}, \Delta d_{12}, \Delta d_{13}, \Delta d_{14}\right.$, $\left.\ldots, \Delta d_{n_{\mathrm{VSCl}}}, \Delta d_{n_{\mathrm{VSC} 2}}, \Delta d_{n_{\mathrm{VSC} 3}}, \Delta d_{n_{\mathrm{VSC}} 4}\right]^{\mathrm{T}}$.

To expand the Taylor series of (7), the second-order item and higher-order terms are omitted, and the modified equation based on Newton-Raphson is given by

$$
\begin{equation*}
\mathbf{f}_{\mathrm{N}}=-\mathbf{J}_{\mathrm{N}} \Delta \mathbf{x}_{\mathrm{N}} \tag{8}
\end{equation*}
$$

where $\mathbf{f}_{\mathrm{N}}=\left[\mathbf{f}_{\mathrm{ac}}^{\mathrm{T}}, \mathbf{f}_{\mathrm{ac}-\mathrm{dc}}^{\mathrm{T}}, \mathbf{f}_{\mathrm{dc}}^{\mathrm{T}}\right]^{\mathrm{T}}, \Delta \mathbf{x}_{\mathrm{N}}=\left[\Delta x_{\mathrm{ac} 1}^{\mathrm{T}}, \Delta x_{\mathrm{ac} 2}^{\mathrm{T}}\right.$, $\left.\Delta x_{\mathrm{ac}-\mathrm{dc}}^{\mathrm{T}}, \Delta x_{\mathrm{dc}}^{\mathrm{T}}\right]^{\mathrm{T}}, \Delta \mathbf{x}_{\mathrm{ac} 1}=\left[\Delta U_{\mathrm{a} 1}, \Delta \theta_{\mathrm{a} 1}, \ldots, \Delta U_{\mathrm{an}}^{\mathrm{AC}}, ~ \Delta \theta_{\mathrm{an} n_{\mathrm{AC}}}\right]^{\mathrm{T}}$, $\Delta \mathbf{x}_{\mathrm{ac} 2}=\left[\Delta U_{\mathrm{t} 1}, \Delta \theta_{\mathrm{t} 1}, \ldots, \Delta U_{\mathrm{t} n_{\mathrm{VSc}}}, \Delta \theta_{\mathrm{t} n_{\mathrm{Vsc}}}\right]^{\mathrm{T}}, \Delta \mathbf{x}_{\mathrm{ac}-\mathrm{dc}}=\left[\Delta P_{\mathrm{t} 1}\right.$, $\left.\Delta Q_{\mathrm{t} 1}, \ldots, \Delta P_{\mathrm{t} n_{\mathrm{Vsc}}}, \Delta Q_{\mathrm{t} n_{\mathrm{vc}}}\right]^{\mathrm{T}}, \Delta \mathbf{x}_{\mathrm{dc}}=\left[\Delta U_{\mathrm{d} 1}, \Delta I_{\mathrm{d} 1}, \Delta \delta_{1}, \Delta M_{1}\right.$, $\left.\ldots, \Delta U_{\mathrm{d} n_{\mathrm{VSC}}}, \Delta I_{\mathrm{d} n_{\mathrm{VSC}}}, \Delta \delta_{n_{\mathrm{VSC}}}, \Delta M_{n_{\mathrm{VSC}}}\right]^{\mathrm{T}}$.

And $\mathbf{J}_{\mathrm{N}}$ is given by

$$
\begin{align*}
& \mathbf{J}_{\mathrm{N}}\left(\mathbf{x}_{\mathrm{ac} 1}, \mathbf{x}_{\mathrm{ac} 2}, \mathbf{x}_{\mathrm{ac}-\mathrm{dc}}, \mathbf{x}_{\mathrm{dc}}\right) \\
& \quad=\left[\begin{array}{cccc}
\frac{\partial \mathbf{f}_{\mathrm{ac}}}{\partial \mathbf{x}_{\mathrm{ac} 1}} & \frac{\partial \mathbf{f}_{\mathrm{ac}}}{\partial \mathbf{x}_{\mathrm{ac} 2}} & \frac{\partial \mathbf{f}_{\mathrm{ac}}}{\partial \mathbf{x}_{\mathrm{ac}-\mathrm{dc}}} & \frac{\partial \mathbf{f}_{\mathrm{ac}}}{\partial \mathbf{x}_{\mathrm{dc}}} \\
\frac{\partial \mathbf{f}_{\mathrm{ac}-\mathrm{dc}}}{\partial \mathbf{x}_{\mathrm{ac}}} & \frac{\partial \mathbf{f}_{\mathrm{ac}-\mathrm{dc}}}{\partial \mathbf{x}_{\mathrm{ac} 2}} & \frac{\partial \mathbf{f}_{\mathrm{ac}-\mathrm{dc}}}{\partial \mathbf{x}_{\mathrm{ac}-\mathrm{dc}}} & \frac{\partial \mathbf{f}_{\mathrm{ac}-\mathrm{dc}}}{\partial \mathbf{x}_{\mathrm{dc}}} \\
\frac{\partial \mathbf{f}_{\mathrm{dc}}}{\partial \mathbf{x}_{\mathrm{ac} 1}} & \frac{\partial \mathbf{f}_{\mathrm{dc}}}{\partial \mathbf{x}_{\mathrm{ac} 2}} & \frac{\partial \mathbf{f}_{\mathrm{dc}}}{\partial \mathbf{x}_{\mathrm{ac}-\mathrm{dc}}} & \frac{\partial \mathbf{f}_{\mathrm{dc}}}{\partial \mathbf{x}_{\mathrm{dc}}}
\end{array}\right] \\
& \quad=\left[\begin{array}{cccc}
\mathbf{J}_{\mathrm{a}-\mathrm{a} 1} & \mathbf{J}_{\mathrm{a}-\mathrm{a} 2} & \mathbf{0} & \mathbf{0} \\
\mathbf{J}_{\mathrm{ad}-\mathrm{a} 1} & \mathbf{J}_{\mathrm{ad}-\mathrm{a} 2} & \mathbf{J}_{\mathrm{ad}-\mathrm{ad}} & \mathbf{0} \\
\mathbf{J}_{\mathrm{d}-\mathrm{a} 1} & \mathbf{J}_{\mathrm{d}-\mathrm{a} 2} & \mathbf{J}_{\mathrm{d}-\mathrm{ad}} & \mathbf{J}_{\mathrm{d}-\mathrm{d}}
\end{array}\right] . \tag{9}
\end{align*}
$$

The power flow equation of VSC-HVDC is given as follows:

$$
\left[\begin{array}{l}
\mathbf{f}_{1}  \tag{10}\\
\mathbf{f}_{2}
\end{array}\right]=-\left[\begin{array}{ll}
\mathbf{J}_{11} & \mathbf{J}_{12} \\
\mathbf{J}_{21} & \mathbf{J}_{22}
\end{array}\right]\left[\begin{array}{l}
\Delta \mathbf{x}_{1} \\
\Delta \mathbf{x}_{2}
\end{array}\right]
$$

where $\mathbf{f}_{1}=\mathbf{f}_{\mathrm{ac}}, \mathbf{f}_{2}=\left[\mathbf{f}_{\mathrm{ac}-\mathrm{dc}}^{\mathrm{T}}, \mathbf{f}_{\mathrm{dc}}^{\mathrm{T}}\right]^{\mathrm{T}}, \Delta \mathbf{x}_{1}=\Delta \mathbf{x}_{\mathrm{ac} 1}, \Delta \mathbf{x}_{2}=$ $\left[\Delta \mathbf{x}_{\mathrm{ac} 2}^{\mathrm{T}}, \Delta \mathbf{x}_{\mathrm{ac}-\mathrm{dc}}^{\mathrm{T}}, \Delta \mathbf{x}_{\mathrm{dc}}^{\mathrm{T}}\right]^{\mathrm{T}}, \mathbf{J}_{11}=\mathbf{J}_{\mathrm{a}-\mathrm{a} 1}, \mathbf{J}_{21}=\left[\begin{array}{c}\mathbf{J}_{\mathrm{ad}-a 1} \\ \mathbf{J}_{\mathrm{d}-\mathrm{al}}\end{array}\right], \mathbf{J}_{12}=$ $\left[\begin{array}{lll}\mathbf{J}_{\mathrm{a}-\mathrm{a} 2} & , \mathbf{0}, \mathbf{0}\end{array}\right], \mathbf{J}_{22}=\left[\begin{array}{ccc}\mathbf{J}_{\text {ad-a2 }} & \mathbf{J}_{\mathrm{ad}-\mathrm{ad}} & \mathbf{0} \\ \mathrm{J}_{\mathrm{d}-\mathrm{az}} & \mathbf{J}_{\mathrm{d}-\mathrm{ad}} & \mathbf{J}_{\mathrm{d}-\mathrm{d}}\end{array}\right]$.

The number of the power flow equations for (10) is $2(n-$ $1)+4 n_{\text {VSC }}$, and the variables number is $2(n-1)+6 n_{\mathrm{VSC}}$. The $2 n_{\text {VSC }}$ variables can be eliminated by control modes of VSC-HVDC. So, (10) has solutions. The dimensions of $\mathbf{f}_{1}$ are $2\left(n_{\mathrm{AC}}-1\right)$. The dimensions of $\Delta \mathbf{x}_{1}$ are $2\left(n_{\mathrm{AC}}-1\right)$. And the inverse matrix of $\mathbf{J}_{11}$ is exists. The dimensions of $\mathbf{f}_{2}$ are $6 n_{\text {VSC }}$. The dimensions of $\Delta \mathbf{x}_{2}$ are $8 n_{\mathrm{VSC}}$, and the $2 n_{\mathrm{VSC}}$ dimensions
of $\Delta \mathbf{x}_{2}$ can be eliminated by control modes of VSC-HVDC, and so the inverse matrix of $\mathbf{J}_{22}$ exists.

To expand (10),

$$
\begin{align*}
& -\left(\mathbf{J}_{11} \Delta \mathbf{x}_{1}+\mathbf{J}_{12} \Delta \mathbf{x}_{2}\right)=\mathbf{f}_{1} \\
& -\left(\mathbf{J}_{21} \Delta \mathbf{x}_{1}+\mathbf{J}_{22} \Delta \mathbf{x}_{2}\right)=\mathbf{f}_{2} \tag{11}
\end{align*}
$$

Then the new modified sequential iterative form is given by

$$
\begin{align*}
& \Delta \mathbf{x}_{1}=-\left[\mathbf{J}_{11}-\mathbf{J}_{12}\left(\mathbf{J}_{22}\right)^{-1} \mathbf{J}_{21}\right]^{-1}\left[\mathbf{f}_{1}-\mathbf{J}_{12}\left(\mathbf{J}_{22}\right)^{-1} \mathbf{f}_{2}\right] \\
& \Delta \mathbf{x}_{2}=-\left[\mathbf{J}_{22}-\mathbf{J}_{21}\left(\mathbf{J}_{11}\right)^{-1} \mathbf{J}_{12}\right]^{-1}\left[\mathbf{f}_{2}-\mathbf{J}_{21}\left(\mathbf{J}_{11}\right)^{-1} \mathbf{f}_{1}\right] \tag{12}
\end{align*}
$$

It can be seen from the previous derivation process that the modified algorithm does not make any hypothesis. The mutual influence between the AC system and DC system is fully considered in the iteration solution procedure. By means of the previous method, the problem of AC variables coupling DC variables is solved strictly in the mathematics expression. The matrix $\mathbf{J}_{11}$ and $\mathbf{J}_{12}$ can be obtained by the program of pure AC system.

### 3.2. Parameter-Dependent Power Flow and Converter Equa-

 tions of VSC-HVDC. According to connected or not connected with a converter transformer, the buses of AC/DC systems with VSC-HVDC are divided into two kinds, DC bus and pure AC bus [23]. The bus connected to primary side of a converter transformer is considered as a DC bus. The bus not connected to a converter transformer is considered as a pure AC bus. $n$ is the total bus number of the system. $n_{\text {VSC }}$ is the number of VSC converters and also is the number of DC buses. So, the number of pure AC buses is $n_{\mathrm{AC}}=n-n_{\mathrm{VSC}}$.Considering the load changes in several areas or a particular area (at a bus and/or at a group of buses) of the AC/DC systems with VSC-HVDC, the power flow equations for a pure AC bus are given by

$$
\begin{align*}
\Delta P_{\mathrm{a} i}= & P_{\mathrm{a} i}-U_{\mathrm{a} i} \sum_{j \in i} U_{j}\left(G_{i j} \cos \theta_{i j}+B_{i j} \sin \theta_{i j}\right) \\
& +\left(P_{\mathrm{G} i}-P_{\mathrm{L} i}\right) \lambda=0,  \tag{13}\\
\Delta Q_{\mathrm{a} i}= & Q_{\mathrm{a} i}-U_{\mathrm{a} i} \sum_{j \in i} U_{j}\left(G_{i j} \sin \theta_{i j}-B_{i j} \cos \theta_{i j}\right) \\
& +\left(Q_{\mathrm{G} i}-\mathrm{Q}_{\mathrm{L} i}\right) \lambda=0,
\end{align*}
$$

where the subscript " a " identifies that the bus is a pure AC bus, $\mathrm{a}=1,2, \ldots, n_{\mathrm{AC}}$. The subscript " $i$ " is the number of the buses, $i=1,2, \ldots, n$. The subscript " $j$ " identifies that all the buses connected to the bus " $i$ " (expressed in the terms of $j \in i) . U$ and $\theta$ are the bus voltage amplitude and phase angle, respectively. $G$ and $B$ are the real part and imaginary part of nodal admittance matrix, respectively. $P_{\mathrm{G} i}$ and $Q_{\mathrm{G} i}$ are the power of generator. $P_{\mathrm{L} i}$ and $\mathrm{Q}_{\mathrm{L} i}$ are the loads at the bus $i$. $\lambda \in R$ are the parameters such real/reactive power demand at the buses and transmission line parameters.

For a DC bus, the power flow equations are given by

$$
\begin{align*}
\Delta P_{\mathrm{t} i}= & P_{\mathrm{t} i}-U_{\mathrm{t} i} \sum_{j \in i} U_{j}\left(G_{i j} \cos \theta_{i j}+B_{i j} \sin \theta_{i j}\right) \pm P_{\mathrm{t} i} \\
& +\left(P_{\mathrm{G} i}-P_{\mathrm{L} i}\right) \lambda=0  \tag{14}\\
\Delta Q_{\mathrm{t} i}= & Q_{\mathrm{t} i}-U_{\mathrm{t} i} \sum_{j \in i} U_{j}\left(G_{i j} \sin \theta_{i j}-B_{i j} \cos \theta_{i j}\right) \pm Q_{\mathrm{t} i} \\
& +\left(Q_{\mathrm{G} i}-Q_{\mathrm{L} i}\right) \lambda=0,
\end{align*}
$$

where the subscript " t " identifies the bus as a DC bus. " $\pm$ " signs correspond to the rectifiers and inverters of VSCHVDC, respectively.

The basic power flow equations for VSC-HVDC converters are given as follows:

$$
\begin{align*}
\Delta d_{k 1}= & P_{\mathrm{t} k}+\frac{\sqrt{6}}{4} M_{k} U_{\mathrm{t} k} U_{\mathrm{d} k}|Y| \cos \left(\delta_{k}+\alpha_{k}\right) \\
& -U_{\mathrm{t} k}^{2}|Y| \cos \alpha_{k}=0 \\
\Delta d_{k 2}= & Q_{\mathrm{t} k}+\frac{\sqrt{6}}{4} M_{k} U_{\mathrm{t} k} U_{\mathrm{d} k}|Y| \sin \left(\delta_{k}+\alpha_{k}\right)-U_{\mathrm{t} k}^{2}|Y| \sin \alpha_{k} \\
& -\frac{U_{\mathrm{t} k}^{2}}{X_{\mathrm{f} k}}=0 \\
\Delta d_{k 3}= & U_{\mathrm{t} k} I_{\mathrm{d} k}-\frac{\sqrt{6}}{4} M_{k} U_{\mathrm{t} k} U_{\mathrm{d} k}|Y| \cos \left(\delta_{k}-\alpha_{k}\right) \\
& +\frac{3}{8}\left(M_{k} U_{\mathrm{d} k}\right)^{2}|Y| \cos \alpha_{k}=0 \tag{15}
\end{align*}
$$

where the subscript "d" identifies that the variable as the DC side of VSC. " $k$ " is the $k$ th of VSC connected to DC network, $k=1,2, \ldots, n_{\mathrm{VSC}}$.

For the $i$ th VSC converter, there are four unknown variables in (15); that is, $\mathbf{x}_{\mathrm{d} i}=\left[U_{\mathrm{d} i}, I_{\mathrm{d} i}, \delta_{i}, M_{i}\right]^{\mathrm{T}}$, and one more equation is needed to solve (15); that is,

$$
\begin{equation*}
\mathbf{I}_{\mathrm{d}}=\mathbf{G}_{\mathrm{d}} \mathbf{U}_{\mathrm{d}} \tag{16}
\end{equation*}
$$

where $\mathbf{G}_{\mathrm{d}}$ is the nodal admittance matrix of DC network.
Moreover, the DC network equation is given by

$$
\begin{equation*}
\Delta d_{k 4}= \pm I_{\mathrm{d} k}-\sum_{s=1}^{n_{\mathrm{AC}}} g_{\mathrm{d} k s} U_{\mathrm{d} s}=0 \tag{17}
\end{equation*}
$$

where $g_{\mathrm{d} k}$ is the matrix element of nodal admittance matrix of DC network, $s=1,2, \ldots, n_{\mathrm{AC}}$.

In this paper, the concerned parameters are the real and reactive loads changes at the buses that can vary according to the following equations:

$$
\begin{align*}
P_{i} & =P_{i 0}(1+\lambda) \\
Q_{i} & =Q_{i 0}(1+\lambda) \tag{18}
\end{align*}
$$

where $P_{i 0}$ and $Q_{i 0}$ are the initial active and reactive loads, respectively. $P_{i}$ and $Q_{i}$ are the active and reactive loads at bus $i$, respectively.

For the previously AC/DC systems with VSC-HVDC, the simplified power flow equation with parameter $\lambda$-dependent is given by

$$
\begin{equation*}
\mathbf{f}(\mathbf{x}, \lambda)=0 \tag{19}
\end{equation*}
$$

where $\mathbf{f} \in R^{2(n-1)+4 n_{\mathrm{VSC}}+1}, \mathbf{x} \in R^{2(n-1)+4 n_{\mathrm{VSC}}+1} . \mathbf{f}$ is the balance equation of power flow. $\mathbf{x}$ is the system state variable such bus voltage magnitude and phase angles, DC system variables. $n_{1}$ and $n_{2}$ are the number of PQ and PV AC buses. The number of power flow equations for (19) is $2(n-1)+4 n_{\text {VSC }}+1=$ $2 n_{1}+n_{2}+4 n_{\mathrm{VSC}}+1$.

### 3.3. Continuation Power Flow Algorithm Based on Sequential

 Iteration Method. CPF is a powerful tool to numerically generate P-V curve to trace power system stationary behavior due to a set of power injection variations [25, 26]. It uses predictor-corrector scheme to find a solution path of a set of power flow equations that have been reformulated to include a load parameter. $\left(x_{l}, \lambda_{l}\right)^{\mathrm{T}}$ is the initial state of the power flow solution curve, where the subscript " $l$ " is the iteration number of predictor-corrector scheme based on NewtonRaphson algorithm in CPF.3.3.1. The Predictor Step. The predictor step is a stage in firstorder differential form. Once a base solution has been found ( $\lambda=0$ ), a prediction of the next solution can be made by taking an appropriately sized step in a direction tangent to the solution path. Taking the derivatives of (19) will result in the following total differential form equation:

$$
\left[\begin{array}{ll}
f_{x}^{\prime} & f_{\lambda}^{\prime}
\end{array}\right]\left[\begin{array}{l}
d_{x}  \tag{20}\\
d_{\lambda}
\end{array}\right]=0
$$

where $f_{x}^{\prime}=\partial f / \partial x$ is the Jacobian matrix of power flow equation with $x . f_{\lambda}^{\prime}=\partial f / \partial \lambda$ is the partial derivative of power flow equation with $\lambda .\left[d_{x} d_{\lambda}\right]^{\mathrm{T}}$ is the tangent vector, which is the computation target of the predictor step.

Since the insertion of $\lambda$ in the power flow equation added an unknown variable, one more equation is needed to solve the previously equation. This is satisfied by setting one of the components of the tangent vector to +1 or -1 . This component is referred to as the continuation parameter. Equation (20) now becomes [23]

$$
\left[\begin{array}{cc}
f_{x}^{\prime} & f_{\lambda}^{\prime}  \tag{21}\\
e_{K} &
\end{array}\right]\left[\begin{array}{l}
d_{x} \\
d_{\lambda}
\end{array}\right]=\left[\begin{array}{c}
0 \\
\pm 1
\end{array}\right]
$$

where $e_{K}$ is a row vector with all elements equal to zero except for the $K$ th element being equal 1 . The dimensions of $e_{K}$ are $1 \times\left[2(n-1)+6 n_{\mathrm{VSC}}+1\right]$. The introduction of the additional equation makes the Jacobian matrix nonsingular at the critical operation point. The modified equation of Newton-Raphson algorithm for (21) is

$$
\begin{equation*}
\Delta \mathbf{f}=-\mathbf{J} \Delta \mathbf{x} \tag{22}
\end{equation*}
$$

where $\Delta \mathbf{f}=\left[\mathbf{f}_{\mathrm{ac}}^{\mathrm{T}}, \mathbf{f}_{\mathrm{ac}-\mathrm{d} c}^{\mathrm{T}}, \mathbf{f}_{\mathrm{dc}}^{\mathrm{T}}, \mathbf{f}_{\lambda}^{\mathrm{T}}\right]^{\mathrm{T}}, \mathbf{f}_{\lambda}= \pm 1, \Delta \mathbf{x}=\left[\Delta x_{\mathrm{ac1}}^{\mathrm{T}}, \Delta x_{\mathrm{ac} 2}^{\mathrm{T}}\right.$, $\left.\Delta x_{\mathrm{ac}-\mathrm{dc}}^{\mathrm{T}}, \Delta x_{\mathrm{dc}}^{\mathrm{T}}, \Delta x_{\lambda}^{\mathrm{T}}\right]^{\mathrm{T}}, \mathbf{J}$ is referred to as the Jocobian matrix

Table 1: System parameters of operation and control modes of VSC-HVDC.

| Operation modes | Control parameters of VSC converters (p.u.) |  |
| :---: | :---: | :---: |
|  | $\mathrm{VSC}_{1}$ | $\mathrm{VSC}_{2}$ |
| 1 | Constant DC voltage $U_{\text {d1 }}^{\text {ref }}=2.0000$ | Constant AC active power $P_{\mathrm{s} 2}^{\text {ref }}=-0.8993$ |
|  | Constant AC reactive power $Q_{s l}^{\text {ref }}=0.1220$ | Constant AC reactive power $Q_{s 2}^{\text {ref }}=0.1734$ |
| 2 | Constant DC voltage $U_{\text {d1 }}^{\text {ref }}=2.0000$ | Constant AC active power $P_{s 2}^{\text {ref }}=-0.8993$ |
|  | Constant AC reactive power $Q_{s 1}^{\text {ref }}=0.1220$ | Constant AC voltage $U_{s 2}^{\text {ref }}=1.0186$ |
| 3 | Constant DC voltage $U_{\mathrm{d} 2}^{\text {ref }}=1.9863$ | Constant AC active power $P_{\mathrm{sl}}^{\text {ref }}=0.9194$ |
|  | Constant AC voltage $U_{s 2}^{\text {ref }}=1.0186$ | Constant AC reactive power $Q_{s l}^{\text {ref }}=0.1220$ |
| 4 | Constant DC voltage $U_{\mathrm{d} 2}^{\text {ref }}=1.9863$ | Constant AC active power $P_{\mathrm{sl}}^{\text {ref }}=0.9194$ |
|  | Constant AC voltage $U_{\mathrm{s} 2}^{\text {ref }}=1.0186$ | Constant AC voltage $U_{s 1}^{\text {ref }}=1.0203$ |

Table 2: Initial DC variable parameters (p.u.) of VSC-HVDC system.

| $N_{\text {bus }}$ | $R$ | $X_{1}$ | $P_{\mathrm{L}}$ | $Q_{\mathrm{L}}$ | $R_{\mathrm{d}}$ | $U_{\mathrm{s}}$ | $\theta_{\mathrm{s}}$ | $P_{\mathrm{s}}$ | $Q_{\mathrm{s}}$ | $U_{\mathrm{d}}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 13 | 0.0060 | 0.1500 | 0.1350 | 0.0580 | 0.0300 | 1.0000 | 0.0000 | 0.9190 | 0.1220 | 2.0000 |
| 14 | 0.0060 | 0.1500 | 0.1490 | 0.0500 | 0.0300 | 1.0000 | 0.0000 | -0.8990 | 0.1730 | 2.0000 |

of (22), and the dimensions of $\mathbf{J}$ are $\left[2(n-1)+4 n_{\mathrm{VSC}}+1\right] \times$ $\left[2(n-1)+6 n_{\mathrm{VSC}}+1\right]$.

The initial values of the variables of VSC-HVDC system for power flow program iteration are given by

$$
\begin{gather*}
U_{\mathrm{d} k}^{(0)}=U_{\mathrm{d} k}^{\mathrm{ref}}, \quad(k \in C V) \\
U_{\mathrm{d} k}^{(0)}=U_{\mathrm{d} k}^{\mathrm{N}}, \quad(k \notin C V), \\
I_{\mathrm{d} k}^{(0)}=\frac{P_{\mathrm{t} k}}{U_{\mathrm{d} k}^{(0)}},  \tag{23}\\
\delta_{k}^{(0)}=\arctan \left(\frac{P_{\mathrm{t} k}}{\left(U_{\mathrm{t} k}^{2} / X_{\mathrm{L} k}\right)+\left(U_{\mathrm{t} k}^{2} / X_{\mathrm{fk}}\right)-\mathrm{Q}_{\mathrm{t} k}}\right), \\
M_{k}^{(0)}=\frac{2 \sqrt{6}}{3} \frac{P_{\mathrm{t} k} X_{\mathrm{L} k}}{U_{\mathrm{t} k} U_{\mathrm{d} k}^{(0)} \sin \delta_{k}^{(0)}},
\end{gather*}
$$

where $k \in C V$ identifies that the $k$ th VSC is constant DC voltage control mode. $k \notin C V$ identifies that the $k$ th is not constant DC voltage control mode. Superscript " 0 " identifies the initial value of the 0th iteration. Superscript "ref" identifies that the variable value is reference value. Superscript "N" identifies rated value.

The $P_{\mathrm{t} k}$ is given in estimation by

$$
\begin{equation*}
P_{\mathrm{t} k}=-\sum_{s=1, s \notin C V}^{n_{\mathrm{VSC}}} P_{\mathrm{ts}}^{\mathrm{ref}} . \tag{24}
\end{equation*}
$$

Based on the previous analysis, the tangent vector $\left[d_{x} d_{\lambda}\right]^{\mathrm{T}}$ is obtained. The prediction value of the next solution is given by

$$
\left[\begin{array}{l}
x_{l+1}^{\prime}  \tag{25}\\
\lambda_{l+1}^{\prime}
\end{array}\right]=\left[\begin{array}{l}
x_{l} \\
\lambda_{l}
\end{array}\right]+h\left[\begin{array}{l}
d_{x} \\
d_{\lambda}
\end{array}\right]
$$

where $\left[\begin{array}{ll}x_{l+1}^{\prime} & \lambda_{l+1}^{\prime}\end{array}\right]^{\mathrm{T}}$ is prediction value, which is an approximate solution. $h$ is the step size of the prediction.
3.3.2. The Corrector Step. In the corrector step, the prediction value of $\left[\begin{array}{ll}x_{l+1}^{\prime} & \lambda_{l+1}^{\prime}\end{array}\right]^{\mathrm{T}}$ is substituted into (19), and its iteration form is reformulated as

$$
\left[\begin{array}{cc}
f_{x}^{\prime} & f_{\lambda}^{\prime}  \tag{26}\\
0 & 1
\end{array}\right]\left[\begin{array}{c}
\Delta x \\
\Delta \lambda
\end{array}\right]=-\left[\begin{array}{c}
f(x, \lambda) \\
0
\end{array}\right]
$$

The iteration form is now reformulated as

$$
\left[\begin{array}{cc}
f_{x}^{\prime} & f_{\lambda}^{\prime}  \tag{27}\\
e_{K} & 0
\end{array}\right]\left[\begin{array}{c}
\Delta x \\
\Delta \lambda
\end{array}\right]=-\left[\begin{array}{c}
f(x, \lambda) \\
0
\end{array}\right]
$$

## 4. Case Studies and Validations

Two cases are considered and compared: (1) the system with existing AC transmission line and (2) the system with a new dc transmission line based on VSC-HVDC.

The system parameters of the four different control modes and the VSC converters adopted in the case studies are prespecified as listed in Table 1. The initial DC variable parameters of VSC-HVDC system are shown in Table 2. In Table 2, the $N_{\text {bus }}$ identifies bus number of VSC-HVDC link connected to AC systems. The $P_{\mathrm{L}}$ and $\mathrm{Q}_{\mathrm{L}}$ are the load power of the bus VSC connected to. The $R_{\mathrm{d}}$ is the resistance of DC network. $R, X_{l}, P_{\mathrm{L}}$, and $\mathrm{Q}_{\mathrm{L}}$ are given parameters. $P_{\mathrm{s}}$ and $\mathrm{Q}_{\mathrm{s}}$ are calculated by power flow calculation of the original AC system, which is equal to the branch power of the original AC system.
4.1. Modified IEEE 14-Bus Text System. First, the proposed method has been applied to the modified IEEE 14-bus system shown in Figure 2. The AC line parameters of the system are the same as the IEEE 14-bus system. The difference is that a two-terminal VSC-HVDC transmission line is placed at bus 13 and bus 14 to replace the AC transmission line 13-14; that is, the $\mathrm{VSC}_{1}$ and $\mathrm{VSC}_{2}$ are connected to AC line of bus 13 and bus 14, respectively.

This paper chooses the commutation bus of the buses 9 , 12,13 , and 14 as the research objects. According to the different


Figure 2: The modified IEEE 14-bus system with VSC-HVDC.


Figure 3: PV curves for IEEE-14 bus system.
control modes of VSC-HVDC showed in Table 1, the detailed analysis of the voltage stability bifurcation for the system can be divided into three cases.
4.1.1. The Operation Mode of " 1 ". Figure 3 shows the P-V curves and load margins of partial buses (buses 9, 12, 13, and 14) of the original pure IEEE-14 system ("AC" for short). Figure 4 shows the P-V curves and load margins of partial buses (buses $9,12,13$, and 14) of the modified IEEE-14 system ("AC/DC" for short), in which the AC/DC system operates under the control mode " 1 " in Table 1 as the load increases. And Table 3 shows the power flow data of the VSC-HVDC operating in mode " 1 " at initial state and the maximum load state of the AC/DC system. Here, the maximum load state is corresponding to the saddle node bifurcation point of the system.

In Figure 3, when $\lambda_{\mathrm{AC}}=5.2120$ p.u., the saddle node bifurcation point is acquired, and the system is in maximum loading state. In Table 3, the " AC bus" of the lower half part

Table 3: Power flow data at initial and maximum load states for mode "1."

| Variable | $U_{\mathrm{d}}$ | $I_{\mathrm{d}}$ | $\delta$ | $M$ | $P_{\mathrm{d}}$ | $Q_{\mathrm{d}}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| VSC $_{1}$ | 2.0000 | 0.4551 | 0.1370 | 0.8108 | 0.9153 | 0.1220 |
| parameters | 2.0000 | 0.4584 | 0.1947 | 0.6840 | 0.9240 | 0.1220 |
|  |  |  |  |  |  |  |
| VSC $_{2}$ | 1.9863 | -0.4551 | -0.1292 | 0.9387 | -0.8993 | 0.1734 |
| parameters | 1.9862 | -0.4584 | -0.3065 | 0.5505 | -0.8993 | 0.1734 |
| Variable | $\mathrm{Q}_{\mathrm{s} 1}$ | $\mathrm{Q}_{\mathrm{s} 2}$ | $\mathrm{Q}_{\mathrm{s} 3}$ | $\mathrm{Q}_{\mathrm{s} 6}$ | $\mathrm{Q}_{\mathrm{s} 8}$ | - |
| AC bus | -0.6999 | 0.1404 | 0.2347 | 0.0010 | 0.1045 | - |
| parameters | -0.0438 | 7.6425 | 1.5550 | 0.0014 | 0.5365 | - |



Figure 4: PV curves for modified IEEE-14 bus system in operating mode "1."
includes the four PV buses (bus 2, bus 3, bus 6, and bus 8) and a slack bus (bus 1) of the modified IEEE-14 test system. The upper row of the twin-row is the initial state data, and the lower row of the twin-row is the maximum load state (corresponding with the $\lambda_{\mathrm{AC} / \mathrm{DC}}=5.6358$ p.u. in Figure 4) data of power flow.

It can be seen in Figure 4 and Table 3, in operation mode " 1 ," that the voltage magnitude decreases as the load increases, and the bus 14 (AC/DC) has the weakest voltage profile; so, it is the critical voltage bus needing reactive power support. When the load margin at $\lambda_{\mathrm{AC} / \mathrm{DC}}=5.6358$ p.u., the modified $\mathrm{AC} / \mathrm{DC}$ systems present a collapse or saddle node bifurcation point, where the system Jacobian matrix becomes singular. And now the voltage drop of bus 14 ( $\mathrm{AC} / \mathrm{DC}$ ) is the most obvious: $U_{14}=0.6451<U_{13}=0.8381<U_{12}=0.8816<$ $U_{9}=0.9083$ (the parameters are all the AC/DC systems).
4.1.2. The Operation Modes of " 2 " and " 3 ". Figures 5 and 6 show the P-V curves and load margins of partial buses (buses $9,12,13$, and 14 ) of the AC/DC system under the control mode of " 2 " and " 3 ".

As shown in Figures 5 and 6, in operating mode " 2 " and mode " 3 ," the voltage magnitudes of bus 14 (AC/DC) and bus 13 (AC/DC) remain almost constant as the load increases, respectively. This is because the $\mathrm{VSC}_{2}$ in mode " 2 " and $\mathrm{VSC}_{1}$ in mode " 3 " both are in constant AC voltage control mode.


Figure 5: PV curves for modified IEEE-14 bus system in operating mode "2."

Besides, the load margins obtained in mode " 2 " $\left(\lambda_{\mathrm{AC} / \mathrm{DC}}=\right.$ 6.9192 p.u.) and mode " 3 " ( $\lambda_{\mathrm{AC} / \mathrm{DC}}=5.8980$ p.u. $)$ are bigger than those in mode " 1 ," which demonstrated that the VSCHVDC system supplies voltage support to the AC bus voltage, thanks to the benefits of the fast and independent reactive power output of VSC-HVDC.
4.1.3. The Operation Mode of " 4 ". Figure 7 shows the P-V curves and load margins of partial buses (buses 9, 12, 13, and 14) of the AC/DC system under the control mode of " 4 " in Table 1 with the load varies.

As shown in Figure 7, the voltage magnitude of bus 13 (AC/DC) and bus 14 (AC/DC) remains almost constant as the load increases. This because to the control modes of $\mathrm{VSC}_{1}$ and $\mathrm{VSC}_{2}$ are constant AC voltage. The load margin for this operation mode is $\lambda_{\mathrm{AC} / \mathrm{DC}}=7.7192$ p.u., which is bigger than those in other modes. Therefore, in the case of the VSCHVDC operation in mode " 4 ," the AC/DC system has better voltage stabilization. But as has been pointed out in [27], when the AC/DC system is disturbed (such as kinds of faults), in order to maintain AC bus voltage, the VSC converter in mode " 4 " has to provide large amount of reactive power to the AC/DC system. Consequently, the overload degree of VSC converter is more severe under this operation mode than overload degree of VSC converter under other control modes.

By contrasting the four control modes of VSC converters, the results show that the requirements of essential reactive power for AC system can be supplied by VSC-HVDC system, and the certain voltage support capability to AC bus by VSCHVDC link is validated. But it should be pointed out that the appropriate control pattern is the basis to exploit the reactive power compensation property of the VSC-HVDC system.
4.2. Modified IEEE 118-Bus Text System. The modified IEEE 118 -bus system is analyzed in this section [28]. The relevant part of the network is shown in Figure 8, which shows the locations of the VSC-HVDC link. The VSC-HVDC replaced an existing AC transmission line (75-118), as shown in


Figure 6: PV curves for modified IEEE-14 bus system in operating mode "3."


Figure 7: PV curves for modified IEEE-14 bus system in operating mode "4."

Table 4: Performance comparison of the modified IEEE-118 test system for four different operating modes.

| The two cases | Operation <br> modes | Load margin <br> $\lambda$ (p.u.) | Iteration <br> numbers | CPU time <br> in seconds |
| :--- | :---: | :---: | :---: | :---: |
| AC/DC system | 1 | 4.6188 | 15 | 29.7642 |
| AC/DC system | 2 | 4.8750 | 15 | 32.0300 |
| AC/DC system | 3 | 4.8625 | 15 | 31.2754 |
| AC/DC system | 4 | 6.2750 | 15 | 76.4324 |

Figure 8, and the $\mathrm{VSC}_{1}$ and $\mathrm{VSC}_{2}$ are connected to AC line of bus 75 and bus 118, respectively.

The performance comparison of the modified IEEE-118 test system for four different operating modes is shown in Table 4. In Table 4, the CPU time for operation mode of " 4 " is longer than other operation modes. The reason is that the operation mode of " 4 " is in constant AC voltage control mode. As the load increases, the VSC-HVDC system supplies voltage support to the AC bus voltage, and as a result, the load margin of the AC/DC system in mode " 4 " is bigger


Figure 8: Relevant part of the modified IEEE 118-bus system withVSC-HVDC.
than other operation modes, and the calculation program for the load margin of the mode " 4 " needs more iteration; so, the CPU time is longer. Table 4 shows that the model and algorithm presented in this paper have certain flexibility with the increase of the network scale. However, it is to be remarked that the CPU time is long. The reason is that with the embedded VSC-HVDC transmission line, the types and the number of the system variables have greatly increased, that is, $P_{\mathrm{ci}}, Q_{\mathrm{c} i}, U_{\mathrm{d} i}, I_{\mathrm{d} i}, \delta_{i}, M_{i}, U_{\mathrm{c} i}, \theta_{\mathrm{c} i}$, and so forth, and the dimensions of the system equations and the Jacobian matrix of the $\mathrm{AC} / \mathrm{DC}$ system have a higher order than pure AC systems.

## 5. Conclusions

In this paper, a new method has been developed to analyze voltage stability for AC/DC systems with VSC-HVDC. The impacts of load variations and different VSC-HVDC control patterns on P-V curves and saddle node bifurcation point of the system were numerically analyzed. The simulation results indicate that the constant AC voltage control of VSC converter is superior to other control modes in voltage stability and also show that VSC-HVDC significantly improved the stability of the system compared to a pure AC line. At last, the importance of suitable control mode for the operating of VSC-HVDC was discussed, and some numerical examples have been included to demonstrate the validity of the obtained results.

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# Measuring and Forecasting Volatility in Chinese Stock Market Using HAR-CJ-M Model 

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#### Abstract

Basing on the Heterogeneous Autoregressive with Continuous volatility and Jumps model (HAR-CJ), converting the realized Volatility (RV) into the adjusted realized volatility (ARV), and making use of the influence of momentum effect on the volatility, a new model called HAR-CJ-M is developed in this paper. At the same time, we also address, in great detail, another two models (HAR-ARV, HAR-CJ). The applications of these models to Chinese stock market show that each of the continuous sample path variation, momentum effect, and ARV has a good forecasting performance on the future ARV, while the discontinuous jump variation has a poor forecasting performance. Moreover, the HAR-CJ-M model shows obviously better forecasting performance than the other two models in forecasting the future volatility in Chinese stock market.


## 1. Introduction

Persistent volatility in financial markets is one of the most ubiquitous forms by which economic phenomena may be observed. Thus, it does not come as a surprise that a principal aim of the scholars in the fields of financial practices, ranging from the financial risk measuring to asset pricing, and to financial derivatives pricing, is the search for mechanisms to measure and forecast the volatility.

To measuring and forecasting the volatility, Engle [1], Bollerslev [2], and Taylor [3] proposed the ARCH model, GARCH model, and SV model, respectively. Hereafter, these models have been extended continuously and formed into the GARCH-type and SV-type models. Although the GARCHtype and SV-type models have made certain progress in measuring and forecasting the volatility of financial markets, they cannot describe the whole-day volatility information well enough as they are set up in low-frequency time sequences. Therefore, there exist some flaws in these models. With the great development in computer technology in recent years, the cost of recording and saving financial high-frequency
data has been greatly reduced; thus, the financial highfrequency data has increasingly made an important means of studying the volatility of financial markets. Andersen and Bollerslev [4] first used the high-frequency data to propose a new method of measuring volatility, that is, the realized volatility (RV). Compared with the historical GARCH and SV model, RV carries superiority with it that it has no model, provides convenience for calculation, and is more accurate in measuring the volatility of financial markets. Thus, its appearance has greatly promoted the development of volatility models. Meanwhile, it can be widely applied to the fields of financial theory study and investment.

Since Andersen and Bollerslev [4] proposed RV, volatility models that take the high-frequency data as sample have developed rapidly and made great success in measuring and forecasting the volatility in financial markets. Andersen et al. [5] gave the theoretical explanation to RV and found that RV had obvious a long memory character by studying American exchange or stock markets. Koopman et al. [6] added RV to the SV and ARFIMA model to set up the SV-RV and ARFIMA-RV model, respectively, and found that new models
with RV added had obviously better volatility forecasting performance than the old ones. Wei and Yu [7] and Wei [8] assessed many volatility models of their forecasting accuracy in future volatility on Shanghai composite index and Hushen 300 index in China, finding that the ARFIMA-lnRV and SVRV model had better forecasting performance which were obviously better than volatility models like the GARCH model, whose conclusion was similar to that of Koopman et al. [6].

Furthermore, Corsi [9] proposed a Heterogeneous Autoregressive with Realized Volatility (HAR-RV) model in accordance with the Heterogeneous Market Hypothesis proposed by Müller et al. [10] and the long memory character of RV. The result showed that the HAR-RV model had good forecasting performance on future volatility which was obviously better than models like the GARCH and ARFIMA-RV model. In China, Zhang et al. [11] also found the HAR-RV model showed much better out-ofsample forecasting performance than the ARFIMA model. Andersen et al. [12] and Wang et al. [13] decomposed RV into the continuous sample path variation and discontinuous jump variation on the basis of the HAR-RV model, and set up a Heterogeneous Auto-Regressive with Continuous volatility and Jumps (HAR-CJ) model, which greatly improved the accuracy of forecasting future volatility. Andersen et al. [14] found that the overnight return variance played an important role in the daily asset volatility, so they added the overnight return variance to the HAR-CJ model and set up an HAR-CJN model. With comparative analysis on model's forecasting performance, they found that the HAR-CJN model performed better than the GARCH and HAR-RV model in forecasting the future volatility at 1 day, 1 week, and 1 month.

From the above-mentioned studies, we can find that the RV-type models (especially the HAR-RV and HAR-CJ model) always have better forecasting performance on the future volatility than the GARCH and SV model, and the HAR-CJ model has the best forecasting performance in these models. Although the HAR-CJ model has good forecasting performance for the forecasting of future volatility, higher accuracy is more favorable to the analysis of practical financial problems such as financial risk measuring, asset pricing, and financial derivatives pricing. Therefore, it is necessary to further improve the forecasting performance of model. So as to improve the forecasting accuracy of models, scholars used to add some variables to existed models according to financial theories and market operational mechanism, such as the SV-RV model based on SV model set up by Koopman et al. [6] and Wei [8], the HAR-RV-J model based on HARRV model set up by Zhang et al. [11], the HAR-L-M model based on HAR-RV model set up by Zhang and Tian [15] and so on, which all have better forecasting accuracies than their base models. Grounded on this, we attempt to add the irrational factors of investors to the HAR-CJ model for improving its forecasting performance on the volatility of Chinese stock market. Many researches show that investors' irrational behaviors produce great influences on the volatility of financial markets. Jegadeesh and Titman [16] brought forward the momentum effect, and they pointed out that the
return of stock had a trend of lasting the previous direction of moving. Researches of Grinblatt and Han [17] and Frazzini [18] also showed that the momentum effect made it a positive correlation between the previous gains and losses of financial asset and the current ones, respectively. It can be concluded that the momentum effect can help with the rise and fall of the market, increasing the volatility of market. Thus, we propose in the perspective of Behavioral Finance Theory, add the momentum effect factor (the capital gain overhang) to the HAR-CJ model, consider the overnight return variance at the same time, convert RV into adjusted realized volatility (ARV), and set up the HAR-CJ-M model. Afterwards, we proceed to use the HAR-CJ-M, HAR-ARV, and HAR-CJ model to study the volatility in Chinese stock market. On one hand, we are to test the influence of momentum effect in Chinese stock market volatility; on the other hand, with the comparison of this new model with the HAR-ARV and HAR-CJ model on their volatility forecasting performance in Chinese stock market, it can help us find better models to measuring and forecasting volatility in Chinese stock market.

The remainder of this paper is organized as follows. In Section 2, the theories about the HAR-CJ-M model are introduced. In Section 3, the HAR-ARV, HAR-CJ and HAR-CJ-M model are established. In Section 4, the comparative analyses of the model's volatility measuring and forecasting performance in Chinese stock market are given. We also conclude this paper in Section 5.

## 2. Preliminaries and Theories

2.1. Adjusted Realized Volatility. According to the calculation method of RV by Andersen and Bollerslev [4], we suppose a trading day $t$, divide the total day trading into $N$ parts, and $P_{t, i}$ is the $i$ th $(i=1, \ldots, N)$ closing price of the trading day $t$. What is more, we suppose $r_{t, i}$ is the return of the $i$ th on trading day $t$, namely, $r_{t, i}=100\left(\ln P_{t, i}-\ln P_{t, i-1}\right)$. Therefore the RV on trading day $t\left(\mathrm{RV}_{t}\right)$ can be written as

$$
\begin{equation*}
\mathrm{RV}_{t}=\sum_{i=1}^{N} r_{t, i}^{2} \tag{1}
\end{equation*}
$$

Hansen and Lunde [19] pointed out that Andersen and Bollerslev [4] researched RV on exchange market. But trade was not made continuously in 24 hours on stock market like that on exchange market, so RV calculated with expression (1) could only reflect the market volatility for trading periods but not for the market volatility information in periods which no trading was made (namely, the market volatility aroused by overnight information-the overnight return variance from the closing of the previous day to the opening of that day). In addition, Hansen and Lunde found that only when the overnight return variance and RV were combined could they become more approximate to the consistency estimation of integrated volatility. Research of Andersen et al. [14] also showed that the overnight return variance $r_{t, n}^{2}$ in SP and US markets made up $16.0 \%$ and $16.5 \%$ of the total return volatility, respectively, namely, $r_{t, n}^{2} /\left(\mathrm{RV}_{t}+r_{t, n}^{2}\right)$ equaled 0.160 and 0.165 , respectively. Consequently, the overnight return variance played a quite important part in calculating the total
daily return volatility, while most literatures on RV at present (such as Wang et al. [13] and Corsi [9]) have not taken it into consideration. According to researches of Martens [20] and Koopman et al. [6], considering the overnight return variance, we adjust RV as

$$
\begin{equation*}
\mathrm{ARV}_{t}=\mathrm{RV}_{t}+r_{t, n}^{2}=\sum_{j=1}^{M} r_{t, j}^{2}, \tag{2}
\end{equation*}
$$

where $r_{t, 1}$ and $r_{t, n}$ stand for the overnight return, $r_{t, 1}=$ $r_{t, n}=100\left(\ln P_{t, o}-\ln P_{t-1, c}\right), P_{t, o}$ represents the opening price of phase $t$, and $P_{t-1, c}$ denotes the closing price of phase $t-1 ; r_{t, 2}$ is the 1st return after the opening of phase $t$, $r_{t, 2}=100\left(\ln P_{t, 1}-\ln P_{t, o}\right), P_{t, 1}$ is the first closing price after the opening of phase $t ; r_{t, 3}$ shows the second return after the opening of phase $t, r_{t, 3}=100\left(\ln P_{t, 2}-\ln P_{t, 1}\right) ; \ldots ; r_{t, M}$ means the $(M-1)$ th return after the opening of phase $t$, and $r_{t, M}=100\left(\ln P_{t, M-1}-\ln P_{t, M-2}\right)$.
2.2. Decomposition of $A R V$. In the practical financial markets, the price volatility of financial asset is not continuous but containing jumps because of the influence aroused by information shock on the market and the investors' irrational behavior. To separate the discontinuous jump variation out, Barndorff-Nielsen and Shephard [21,22] proposed the realized bipower variation (RBV), that is,

$$
\begin{equation*}
\operatorname{RBV}_{t}=z_{1}^{-2}\left(\frac{M}{M-2}\right) \sum_{j=3}^{M}\left|r_{t, j-2}\right|\left|r_{t, j}\right| \tag{3}
\end{equation*}
$$

where $z_{1}=E\left(Z_{t}\right)=\sqrt{\pi / 2}, Z_{t}$ is a random variable which is in standardized normal distribution, and $M /(M-$ 2) is the amendment to sample capacity. According to the research of Barndorff-Nielsen and Shephard, the difference value between $A R V_{t}$ and $R B V_{t}$ is just the consistent estimate of the discontinuous jump variation when $M \rightarrow \infty$, that is,

$$
\begin{equation*}
\mathrm{ARV}_{t}-\mathrm{RBV}_{t} \xrightarrow{M \rightarrow \infty} J_{t} . \tag{4}
\end{equation*}
$$

In limited sample capacity, the discontinuous jump variation calculated with the above expression cannot be all nonnegative numbers. Hence, to guarantee the nonnegative character of the discontinuous jump variation, we define the discontinuous jump variation $J_{t}$ as

$$
\begin{equation*}
J_{t}=\max \left[\mathrm{ARV}_{t}-\mathrm{RBV}_{t}, 0\right] \tag{5}
\end{equation*}
$$

In the process of calculating the discontinuous jump variation, if the daily frequency of extracting sample data is different, it may lead to different calculation errors. To improve the accuracy of calculating the discontinuous jump variation, it is necessary for us to introduce some statistics to test the significance on the discontinuous jump variation.

We adopt the statistics $Z_{t}$ which is extracted by BarndorffNielsen and Shephard [21,22] on the basis of bipower variation theory to distinguish the discontinuous jump variation. The expression of statistics $Z_{t}$ is defined by

$$
\begin{array}{r}
Z_{t}=\frac{\left(\mathrm{ARV}_{t}-\mathrm{RBV}_{t}\right) \mathrm{ARV}_{t}^{-1}}{\sqrt{\left((\pi / 2)^{2}+\pi-5\right)(1 / M) \max \left(1, \mathrm{RTQ}_{t} / \mathrm{RBV}_{t}^{2}\right)}} \\
\longrightarrow N(0,1) \tag{6}
\end{array}
$$

where $\quad \mathrm{RTQ}_{t} \quad=\quad M \mu_{4 / 3}^{-3}(M /(M)-$ 4)) $\sum_{j=4}^{M}\left|r_{t, j-4}\right|^{4 / 3}\left|r_{t, j-2}\right|^{4 / 3}\left|r_{t, j}\right|^{4 / 3}\left(\mu_{4 / 3}\right.$ $\left.E\left(\left|Z_{T}\right|^{4 / 3}\right)=2^{2 / 3} \Gamma(7 / 6) \Gamma(1 / 2)^{-1}\right)$.

The calculation of traditional RBV is greatly correlated with the sampling frequency. Therefore, with the increase of sampling frequency, the estimate value of RBV cannot converge to integrated volatility because of the influence of factors like microstructure of the market. Thus, adopting RBV as the robust estimator to test the discontinuous jump variation contains errors in itself. We thus adopt a brand-new estimator MedRV ${ }_{t}$ which is proposed by Andersen et al. [23] instead of $\mathrm{RBV}_{t}$. MedRV ${ }_{t}$ is defined by

$$
\begin{align*}
\operatorname{MedRV}_{t}= & \frac{\pi}{6-4 \sqrt{3}+\pi}\left(\frac{M}{M-2}\right) \\
& \times \sum_{j=2}^{M-1} \operatorname{Med}\left(\left|r_{t, j-1}\right|\left|r_{t, j}\right|\left|r_{t, j+1}\right|\right)^{2} \tag{7}
\end{align*}
$$

Accordingly, $\mathrm{RTQ}_{1, t}$ of statistics $Z_{t}$ in expression (6) is also replaced by $\mathrm{MedRTQ}_{t}$, which is proposed by Andersen et al. [23] and can be defined by

$$
\begin{align*}
\operatorname{MedRTQ}_{t}= & \frac{3 \pi M}{9 \pi}+72+-52 \sqrt{3} \\
& \left(\frac{M}{M-2}\right)  \tag{8}\\
& \times \sum_{j=2}^{M-1} \operatorname{Med}\left(\left|r_{t, j-1}\right|,\left|r_{t, j}\right|,\left|r_{t, j+1}\right|\right)^{4}
\end{align*}
$$

By calculating the statistics $Z_{t}$ after replacing $\mathrm{RBV}_{t}$ with MedRV $_{t}$, and RTQ ${ }_{t}$ with MedRTQ ${ }_{t}$ in expression (6), when the significance level is $1-\alpha$, we get the estimate value of discontinuous jump variation as

$$
\begin{equation*}
J_{t}=I\left(Z_{t}>\phi_{\alpha}\right)\left(\mathrm{ARV}_{t}-\mathrm{MedRV}_{t}\right) \tag{9}
\end{equation*}
$$

The estimator of continuous sample path variation is

$$
\begin{equation*}
C_{t}=I\left(Z_{t} \leq \phi_{\alpha}\right) \mathrm{ARV}_{t}+I\left(Z_{t}>\phi_{\alpha}\right) \operatorname{MedRV}_{t} \tag{10}
\end{equation*}
$$

We need to choose appropriate confidence level $\alpha$ in the calculating process. In this paper, we choose the confidence level $\alpha$ at 0.99 according to previous studies. In addition, with the above test of the statistics $Z_{t}$ and bipower variation theory, we can get the estimator of both the continuous sample path variation $C_{t}$ and discontinuous jump variation $J_{t}$ of the return volatility in financial markets. Based on this, we can establish models to make empirical researches on both $C_{t}$ and $J_{t}$ in the return volatility to forecast the future volatility in financial markets.
2.3. Momentum Effect. Jegadeesh and Titman [16] first proposed the momentum effect, and then many scholars made studies on it from different perspectives, in which the research of Grinblatt and Han [17] is a representative. Grinblatt and Han proposed the capital gain overhang when studying the momentum effect, which can be used to study the influence of gains or losses in previous phases on the return and volatility in current phase or future market. Grinblatt and Han defined the capital gain overhang $g_{t}$ as: $g_{t}=\left(P_{t-1}-\mathrm{RP}_{t}\right) / P_{t-1}$ (where $P_{t-1}$ is the closing price in phase $t-1 ; \mathrm{RP}_{t}$ is investor's reference price in phase $t$ ). However, most of literature (like Frazzini [18]) afterwards usually defined $g_{t}$ as $g_{t}=\left(P_{t}-\right.$ $\left.\mathrm{RP}_{t}\right) / P_{t}$; thus this paper also defines $g_{t}$ as $g_{t}=\left(P_{t}-\mathrm{RP}_{t}\right) / P_{t}$.

The choice of reference price $\mathrm{RP}_{t}$ is very crucial when using the capital gain overhang to study the momentum effect. When Grinblatt and Han [17] proposed the capital gain overhang, they used the weighting average value of the stock in the past 260 weeks as reference price. In this paper, as the influence of three kinds (short term, medium term, long term) of investors on the volatility of Chinese stock market is to be considered, and each kind of investors chooses different reference prices. Therefore, that we choose the weighting average value of the stock in the past 260 weeks as a reference price does not fit our study. In stock market, there are different investors buy and sell stocks in every phase, and there is a great deal of information arriving at the market which will certainly affect investors' behaviors and decisions in every phase, so the reference price for each kind of investors should be changeable in every phase, that is, a dynamic price. Besides, the choice of reference price should consider not only the theoretical rationality, but also sufficient practical operations of investors in their investing processes. Therefore, we propose a series of new reference prices according to the expression of 5-day, 5-week (25 days), and 5-month (110 days) moving average, this is,

$$
\begin{equation*}
\mathrm{RP}_{t}=\frac{P_{t}+P_{t-1}+\cdots+P_{t-n+1}}{n} \tag{11}
\end{equation*}
$$

The expression is a 5 -day moving average when $n=5$, which shows the reference price for short-term investors. When $n=$ 25 , it is a 5 -week ( 25 days) moving average, representing the reference price for medium-term investors; when $n=110$, it is a 5-month (110 days) moving average which shows the reference price for long-term investors. The moving average is an important trend indicator in security technical analysis. In stock investing, investors will make analyses on these trend curves and decide whether to buy or sell their stocks. In trend analysis, investors usually focus on the corresponding reference prices of moving average, among which those of the 5-day, 5 -week ( 25 days), and 5-month (110 days) moving average are relatively more concerned. These three reference prices are closely related with investors' investment and are updated every phase; thus using them as reference prices for the short-term, medium-term, and long-term investors on the whole stock market is reasonable.

## 3. Characterization of the Models

### 3.1. Introduction to the HAR-ARV and HAR-CJ Models

3.1.1. The HAR-ARV Model. According to the Heterogeneous Market Hypothesis proposed by Müller et al. [10], Corsi [9] pointed out that the different participants are likely to settle for different prices and decide to execute their transactions in different market situations; hence they create volatility. He categorized the market volatility into the short-term, medium-term, and long-term ones, in which the shortterm volatility referred to volatility brought about by the short-term investors' daily or more frequent trading; the medium-term volatility referred to volatility aroused by the medium-term investors' weekly trading; the long-term volatility referred to volatility brought about by the long-term investors' monthly trading or trading every several months. Based on this, Corsi [9] set up a volatility forecasting model according to the long memory character of market volatility, that is, the HAR-RV model. It was defined as

$$
\begin{equation*}
\mathrm{RV}_{t+H}^{d}=\alpha_{0}+\alpha_{d} \mathrm{RV}_{t}^{d}+\alpha_{w} \mathrm{RV}_{t}^{w}+\alpha_{m} \mathrm{RV}_{t}^{m}+\varepsilon_{t+H} \tag{12}
\end{equation*}
$$

We substitute ARV for RV and get the HAR-ARV model:

$$
\begin{equation*}
\mathrm{ARV}_{t+H}^{d}=\alpha_{0}+\alpha_{d} \mathrm{ARV}_{t}^{d}+\alpha_{w} \mathrm{ARV}_{t}^{w}+\alpha_{m} \mathrm{ARV}_{t}^{m}+\varepsilon_{t+H} \tag{13}
\end{equation*}
$$

where $H=1,2, \ldots, \mathrm{ARV}_{t+H}^{d}=\left(\mathrm{ARV}_{t+1}^{d}+\mathrm{ARV}_{t+2}^{d}+\cdots+\right.$ $\left.\mathrm{ARV}_{t+H}^{d}\right) / H$, it represents ARV in the future $H$ days; $\mathrm{ARV}_{t}^{d}$ is the daily ARV in phase $t ; \mathrm{ARV}_{t}^{w}=\left(\mathrm{ARV}_{t}^{d}+\mathrm{ARV}_{t-1}^{d}+\right.$ $\left.\cdots+\mathrm{ARV}_{t-4}^{d}\right) / 5$ means the weekly ARV in phase $t ; \mathrm{ARV}_{t}^{m}=$ $\left(\mathrm{ARV}_{t}^{d}+\mathrm{ARV}_{t-1}^{d}+\cdots+\mathrm{ARV}_{t-21}^{d}\right) / 22$ shows the monthly ARV in phase $t$. The model mainly reflects that the market volatility is a complexly mixed volatility mingled by different volatility, which is the combined result of short-term, medium-term and long-term, investors' trading behaviors.

Corsi [9] found that the logarithm of ARV sequence is more approximate to normal distribution than the original ARV sequence. Thus, we start from the robustness and volatility forecasting accuracy of the model and change model (13) into logarithm form, that is,

$$
\begin{align*}
\ln \left(\mathrm{ARV}_{t+H}^{d}\right)= & \alpha_{0}+\alpha_{d} \ln \left(\mathrm{ARV}_{t}^{d}\right)+\alpha_{w} \ln \left(\mathrm{ARV}_{t}^{w}\right)  \tag{14}\\
& +\alpha_{m} \ln \left(\mathrm{ARV}_{t}^{m}\right)+\varepsilon_{t+H}
\end{align*}
$$

3.1.2. The HAR-CJ Model. Andersen et al. [12] separated ARV into the continuous sample path variation (C) and discontinuous jump variation ( $J$ ) and set up the HAR-CJ model on the basis of HAR-RV model to test the different functions of the different components of volatility in forecasting the future ARV. We still use ARV instead of RV and decompose ARV into $C$ and $J$ with the method mentioned in Section 2.2, and we get the HAR-CJ model, that is,

$$
\begin{align*}
\mathrm{ARV}_{t+H}^{d}= & \beta_{0}+\beta_{c d} C_{t}^{d}+\beta_{c w} C_{t}^{w}+\beta_{c m} C_{t}^{m} \\
& +\beta_{j d} J_{t}^{d}+\beta_{j w} J_{t}^{w}+\beta_{j m} J_{t}^{m}+\varepsilon_{t+H} \tag{15}
\end{align*}
$$

where $C_{t}^{d}$ is the daily continuous sample path variation in phase $t ; C_{t}^{w}=\left(C_{t}^{d}+C_{t-1}^{d}+\cdots+C_{t-4}^{d}\right) / 5$ means the weekly continuous sample path variation in phase $t ; C_{t}^{m}=\left(C_{t}^{d}+\right.$ $\left.C_{t-1}^{d}+\cdots+C_{t-21}^{d}\right) / 22$ means the monthly continuous sample path variation in phase $t . J_{t}^{d}$ is the daily discontinuous jump variation in phase $t ; J_{t}^{w}=\left(J_{t}^{d}+J_{t-1}^{d}+\cdots+J_{t-4}^{d}\right) / 5$ shows the weekly discontinuous jump variation in phase $t$; $J_{t}^{m}=$ $\left(J_{t}^{d}+J_{t-1}^{d}+\cdots+J_{t-21}^{d}\right) / 22$ represents the monthly discontinuous jump variation in phase $t$.

According to the research of Andersen et al. [12], we transfer model (15) to logarithm form, that is,

$$
\begin{align*}
\ln \left(\operatorname{ARV}_{t+H}^{d}\right)= & \beta_{0}+\beta_{c d} \ln \left(C_{t}^{d}\right)+\beta_{c w} \ln \left(C_{t}^{w}\right) \\
& +\beta_{c m} \ln \left(C_{t}^{m}\right)+\beta_{j d} \ln \left(J_{t}^{d}+1\right) \\
& +\beta_{j w} \ln \left(J_{t}^{w}+1\right)+\beta_{j m} \ln \left(J_{t}^{m}+1\right)+\varepsilon_{t+H} \tag{16}
\end{align*}
$$

3.2. Construction of the HAR-CJ-M Model. The basis of constructing HAR-ARV model is the Heterogeneous Market Hypothesis. The Heterogeneous Market Hypothesis is also a key hypothesis in Behavioral Finance Theory. According to Behavioral Finance Theory, we can know that financial markets are not always effective, and the investors' irrational behaviors produce certain influence on the volatility of financial markets. Therefore, when studying the volatility of financial markets, it is necessary to consider the influence of investors' irrational behaviors on volatility. Grinblatt and Han [17] and Frazzini [18] found that the disposition effect made stock price inadequate in reflecting information, and the momentum effect emerged. Accordingly, the previous gains and losses became positively correlated with the current gains and losses, respectively. Therefore, the momentum effect plays a part in the rise and fall of the market, thus increasing the volatility of stock markets. In accordance with Grinblatt and Han's research, we adopt the capital gain overhang $g_{t}$ to measure the return and loss in, previous market in this paper. Meanwhile, considering the difference in previous gains and losses for the short-term, medium-term, and longterm investors, we divide $g_{t}$ into three kinds (daily, weekly, and monthly) in accordance with the constructing thought of HAR-ARV model. Moreover, as the ARV sequence is a positive sequence, and there are positive and negative values for the $g_{t}$ sequence, to consider different influence of the previous gains and losses on the current or future volatility, we divide the $g_{t}$ sequence into a nonnegative sequence and a negative sequence.

According to the way of deducing the HAR-RV model by Corsi [9], we suppose short-term investors are influenced by the long-term volatility while long-term investors are not influenced by the short-term volatility. We define a partial volatility $\widetilde{\sigma}_{t}$, where $\widetilde{\sigma}_{t}^{d}$ means the short-term (1-day)
volatility component, $\tilde{\sigma}_{t}^{w}$ represents the medium-term (1week) volatility component, and $\widetilde{\sigma}_{t}^{m}$ is the long-term (1month) volatility component. $\widetilde{\sigma}_{t}^{d}, \widetilde{\sigma}_{t}^{w}$, and $\widetilde{\sigma}_{t}^{m}$ can be written, respectively, as

$$
\begin{gather*}
\tilde{\sigma}_{t+1 m}^{m}=c_{m}+\phi_{m} \mathrm{RV}_{t}^{m}+\widetilde{\varepsilon}_{t+1 m}^{m}  \tag{17a}\\
\widetilde{\sigma}_{t+1 w}^{w}=c_{w}+\phi_{w} \mathrm{RV}_{t}^{w}+\gamma_{w} E\left(\widetilde{\sigma}_{t+1 m}^{m}\right)+\widetilde{\varepsilon}_{t+1 w}^{w}  \tag{17b}\\
\widetilde{\sigma}_{t+1 d}^{d}=c_{d}+\phi_{d} \mathrm{RV}_{t}^{d}+\gamma_{d} E\left(\widetilde{\sigma}_{t+1 w}^{w}\right)+\widetilde{\varepsilon}_{t+1 d}^{d} \tag{17c}
\end{gather*}
$$

Here, we still substitute ARV for RV and divide ARV into $C$ and $J$, then introduce the three $g_{t}$ to the above three models, then we get three new models, that is,

$$
\begin{align*}
\widetilde{\sigma}_{t+1 m}^{m}= & c_{m}+\phi_{c m} C_{t}^{m}+\phi_{j m} J_{t}^{m}+\phi_{g p m} g_{t}^{m} d p_{t}^{m} \\
& +\phi_{g n m} g_{t}^{m} d n_{t}^{m}+\widetilde{\varepsilon}_{t+1 m}^{m}  \tag{18a}\\
\widetilde{\sigma}_{t+1 w}^{w}= & c_{w}+\phi_{c w} C_{t}^{w}+\phi_{j w} J_{t}^{w}+\phi_{g p w} g_{t}^{w} d p_{t}^{w} \\
& +\phi_{g n w} g_{t}^{w} d n_{t}^{w}+\gamma_{w} E\left(\widetilde{\sigma}_{t+1 m}^{m}\right)+\widetilde{\varepsilon}_{t+1 w}^{w}  \tag{18b}\\
\widetilde{\sigma}_{t+1 d}^{d}= & c_{d}+\phi_{c d} C_{t}^{d}+\phi_{j d} J_{t}^{d}+\phi_{g p d} g_{t}^{d} d p_{t}^{d} \\
& +\phi_{g n d} g_{t}^{d} d n_{t}^{d}+\gamma_{d} E\left(\widetilde{\sigma}_{t+1 w}^{w}\right)+\widetilde{\varepsilon}_{t+1 d}^{d} \tag{18c}
\end{align*}
$$

where $g_{t}^{m}=\left(P_{t}-\mathrm{RP}_{t}^{m}\right) / P_{t}$ (where $\mathrm{RP}_{t}^{m}=\left(P_{t}+P_{t-1} \cdots+\right.$ $\left.\left.P_{t-109}\right) / 110\right), g_{t}^{m}$ denotes the monthly capital gain overhang in phase $t$, which can affect the trading decisions of longterm investors and can produce certain momentum effect, thus affecting the long-term market volatility; $g_{t}^{w}=\left(P_{t}-\right.$ $\left.\mathrm{RP}_{t}^{w}\right) / P_{t}\left(\right.$ where $\left.\mathrm{RP}_{t}^{w}=\left(P_{t}+P_{t-1} \cdots+P_{t-21}\right) / 25\right), g_{t}^{w}$ represents the weekly capital gain overhang in phase $t$, which can affect the trading decisions of medium-term investors and can similarly produce certain momentum effect, thus affecting the medium-term market volatility; $g_{t}^{d}=\left(P_{t}-\right.$ $\left.\mathrm{RP}_{t}^{d}\right) / P_{t}\left(\right.$ where $\left.\mathrm{RP}_{t}^{d}=\left(P_{t}+P_{t-1} \cdots+P_{t-4}\right) / 5\right), g_{t}^{d}$ is the daily capital gain overhang in phase $t$, which can affect the trading decisions of short-term investors and can also produce certain momentum effect, thus affecting the shortterm market volatility. Therefore, the above three kinds of capital gain overhang $g_{t}$ can all produce the momentum effect and affect the volatility of the whole market. $d p_{t}^{m}, d n_{t}^{m}, d p_{t}^{w}$, $d n_{t}^{w}, d p_{t}^{d}$, and $d n_{t}^{d}$ are defined by

$$
\begin{align*}
& d p_{t}^{m}=\left\{\begin{array}{ll}
1, & g_{t}^{m} \geq 0, \\
0, & g_{t}^{m}<0,
\end{array} \quad d n_{t}^{m}=1-d p_{t}^{m}\right. \\
& d p_{t}^{w}=\left\{\begin{array}{ll}
1, & g_{t}^{w} \geq 0, \\
0, & g_{t}^{w}<0,
\end{array} \quad d n_{t}^{w}=1-d p_{t}^{w}\right.  \tag{19}\\
& d p_{t}^{d}=\left\{\begin{array}{ll}
1, & g_{t}^{d} \geq 0, \\
0, & g_{t}^{d}<0,
\end{array} \quad d n_{t}^{d}=1-d p_{t}^{d}\right.
\end{align*}
$$

The volatility innovations $\widetilde{\varepsilon}_{t+1 m}^{m}, \widetilde{\varepsilon}_{t+1 w}^{w}$, and $\widetilde{\varepsilon}_{t+1 d}^{d}$ are all contemporaneously and serially independent zero-mean nuisance variables.

According to Corsi's research [9], the composite model (18a), (18b), and (18c), $\widetilde{\sigma}_{t+1 d}^{d}$ can be defined by

$$
\begin{aligned}
\widetilde{\sigma}_{t+1 d}^{d}= & \gamma_{0}+\gamma_{c d} C_{t}^{d}+\gamma_{c w} C_{t}^{w}+\gamma_{c m} C_{t}^{m}+\gamma_{j d} J_{t}^{d}+\gamma_{j w} J_{t}^{w} \\
& +\gamma_{j m} J_{t}^{m}+\gamma_{g p d} g_{t}^{d} d p_{t}^{d}+\gamma_{g n d} g_{t}^{d} d n_{t}^{d}+\gamma_{g p w} g_{t}^{w} d p_{t}^{w} \\
& +\gamma_{g n w} g_{t}^{w} d n_{t}^{w}+\gamma_{g p m} g_{t}^{m} d p_{t}^{m}+\gamma_{g n m} g_{t}^{m} d n_{t}^{m}+\widetilde{\varepsilon}_{t+1 d}^{d}
\end{aligned}
$$

As $\widetilde{\sigma}_{t+1 d}^{d}$ can also be written as $\widetilde{\sigma}_{t+1 d}^{d}=\mathrm{ARV}_{t+1 d}^{d}+$ $\varepsilon_{t+1 d}^{d}$, we can get an ARV forecasting model, namely, the Heterogeneous Autoregressive with Continuous volatility, Jumps and Momentum (HAR-CJ-M) model. The HAR-CJ-M model can be written as

$$
\begin{align*}
\mathrm{ARV}_{t+1 d}^{d}= & \gamma_{0}+\gamma_{c d} C_{t}^{d}+\gamma_{c w} C_{t}^{w}+\gamma_{c m} C_{t}^{m}+\gamma_{j d} J_{t}^{d} \\
& +\gamma_{j w} J_{t}^{w}+\gamma_{j m} J_{t}^{m}+\gamma_{g p d} g_{t}^{d} d p_{t}^{d}+\gamma_{g n d} g_{t}^{d} d n_{t}^{d} \\
& +\gamma_{g p w} g_{t}^{w} d p_{t}^{w}+\gamma_{g n w} g_{t}^{w} d n_{t}^{w}+\gamma_{g p m} g_{t}^{m} d p_{t}^{m} \\
& +\gamma_{g n m} g_{t}^{m} d n_{t}^{m}+\varepsilon_{t+1 d} \tag{21}
\end{align*}
$$

with $\varepsilon_{t+1 d}=\widetilde{\varepsilon}_{t+1 d}^{d}-\varepsilon_{t+1 d}^{d}$.
According to Andersen et al. [12], we adopt similar method of their disposal in changing $J_{t}$ into logarithm form for those independent variables with $g_{t}$ in model (21), that is, to change the nonnegative parts into logarithm form $\ln \left(g_{t} d p_{t}+1\right)$ and the negative parts into logarithm form $\ln \left(-g_{t} d \mathrm{n}_{t}+1\right)$. Consequently, with model (21) being changed into logarithm form and forecast period being extended to $H$ phase, we can get the logarithm form of HAR-CJ-M model, that is,

$$
\begin{align*}
\ln \left(\mathrm{ARV}_{t+H}^{d}\right)= & \gamma_{0}+\gamma_{c d} \ln \left(C_{t}^{d}\right)+\gamma_{c w} \ln \left(C_{t}^{w}\right)+\gamma_{c m} \ln \left(C_{t}^{m}\right) \\
& +\gamma_{j d} \ln \left(J_{t}^{d}+1\right)+\gamma_{j w} \ln \left(J_{t}^{w}+1\right) \\
& +\gamma_{j m} \ln \left(J_{t}^{m}+1\right)+\gamma_{g p d} \ln \left(g_{t}^{d} d p_{t}^{d}+1\right) \\
& +\gamma_{g n d} \ln \left(-g_{t}^{d} d n_{t}^{d}+1\right) \\
& +\gamma_{g p w} \ln \left(g_{t}^{w} d p_{t}^{w}+1\right) \\
& +\gamma_{g n w} \ln \left(-g_{t}^{w} d n_{t}^{w}+1\right) \\
& +\gamma_{g p m} \ln \left(g_{t}^{m} d p_{t}^{m}+1\right) \\
& +\gamma_{g n m} \ln \left(-g_{t}^{m} d n_{t}^{m}+1\right) \\
& +\varepsilon_{t+H} . \tag{22}
\end{align*}
$$

## 4. Empirical Evidence

4.1. Data and Summary Statistics. CSI 300 is the component stock index which is made from 300 samples that are well chosen from Shanghai and Shenzhen stock markets. It covers about $60 \%$ stock values of Shanghai and Shenzhen stock markets, and its daily correlation coefficient to Shanghai and Shenzhen stock indexes reaches $98.4 \%$ and $97.6 \%$, respectively. So it can well represent the operation state of Chinese stock market. In addition, the daily sample data extracting frequency also greatly affects the result of the study. On one hand, low frequency of extracting cannot reflect well the volatility information of that day. On the other hand, high frequency may lead to micronoise and affect the result. As a result, we take both the influences into consideration, refer to previous studies of different scholars, and use CSI 300 with 5-minute high-frequency data as samples to study the volatility in Chinese stock market, the data comes from the WIND financial database. The sample period begins on April 20, 2007, and ends on April 20, 2012. There are 1199 trading days and 58751 effective data altogether. The variables needed in this paper like ARV $_{t}$ and $C_{t}$ are all disposed by Matlab 7.0 or Excel 2003. By dealing with and calculating the above-mentioned 58751 data, we find that the overnight return variance $r_{t, n}^{2}$ in Chinese stock market makes up 26.4\% of the whole market volatility, namely, $r_{t, n}^{2} /\left(\mathrm{RV}_{t}+r_{t, n}^{2}\right)$ equals 0.264 . Upon that, the overnight return variance should be considered in calculating RV of Chinese stock market. So the adjustment of RV in the paper is necessary.

Table 1 is the descriptive statistical results of the daily adjusted realized volatility $\mathrm{ARV}_{t}$, the daily continuous sample path variation $C_{t}$, the daily discontinuous jump variation $J_{t}$, the nonnegative part of daily capital gain overhang $g_{t}^{d} d p_{t}^{d}$, the negative part of daily capital gain overhang $g_{t}^{d} d n_{t}^{d}$, the nonnegative part of weekly capital gain overhang $g_{t}^{w} d p_{t}^{w}$, the negative part of weekly capital gain overhang $g_{t}^{w} d n_{t}^{w}$, the nonnegative part of monthly capital gain overhang $g_{t}^{m} d p_{t}^{m}$, and the negative part of monthly capital gain overhang $g_{t}^{m} d n_{t}^{m}$ in Chinese stock market. We can see from Table 1 that the $\mathrm{ARV}_{t}$ sequence shows an obvious sharp peak and fat tail which is not normally distributed, which shows the extent of volatility in Chinese stock market is great. Besides, the ADF test shows that every sequence refuses obviously the hypothesis of existence the unit root at confidence intervals of $90 \%$, so it can be concluded that every sequence is steady. Thus further modeling analysis can be made.

In Figure 1, ARV, C, $J, g d p, g d n, g w p, g w n, g m p$, and $g m n$, respectively, represents $\mathrm{ARV}_{t}, C_{t}, J_{t}, g_{t}^{d} d p_{t}^{d}, g_{t}^{d} d n_{t}^{d}, g_{t}^{w} d p_{t}^{w}$, $g_{t}^{w} d n_{t}^{w}, g_{t}^{m} d p_{t}^{m}$, and $g_{t}^{m} d n_{t}^{m}$ in Chinese stock market. Figure 1 shows, for the CSI 300 series studied in this paper, the lagged correlation function between the estimated daily integrated variance $\mathrm{ARV}_{t+h}$ with $X_{t}$ as a function of $h$, with $X_{t}$ being $\mathrm{ARV}_{t}$ itself, $C_{t}, J_{t}, g_{t}^{d} d p_{t}^{d}, g_{t}^{d} d n_{t}^{d}, g_{t}^{w} d p_{t}^{w}, g_{t}^{w} d n_{t}^{w}, g_{t}^{m} d p_{t}^{m}$, and $g_{t}^{m} d n_{t}^{m}$. Seeing from the correlation function between $\mathrm{ARV}_{t}$ and $\mathrm{ARV}_{t+h}$ (namely, the autocorrelation function of $\mathrm{ARV}_{t}$ ), we can find that $\mathrm{ARV}_{t}$ in Chinese stock market has obvious long memory character. Thus, the past $A R V_{t}$ has certain forecast effect on future $\mathrm{ARV}_{t}$, which is in line with the

Table 1: Descriptive statistics for CSI 300.

|  | Mean | Std. dev. | Skewness | Kurtosis | Jarque-Bera | ADF- $t$ statistic |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| ARV $_{t}$ | 4.3471 | 6.8081 | 5.9174 | 49.075 | $113054^{* * *}$ | $-10.710^{* * *}$ |
| $C_{t}$ | 3.3412 | 4.3445 | 5.9456 | 65.248 | $200640^{* * *}$ | $-7.7154^{* * *}$ |
| $J_{t}$ | 1.0058 | 4.8285 | 9.5878 | 109.80 | $588176^{* * *}$ | $-16.145^{* * *}$ |
| $g_{t}^{d} d p_{t}^{d}$ | 0.8629 | 1.2545 | 1.8612 | 7.5625 | $1726.4^{* * *}$ | $-16.555^{* * *}$ |
| $g_{t}^{d} d n_{t}^{d}$ | -0.9552 | 1.5382 | -2.1191 | 7.9438 | $2111.3^{* * *}$ | $-17.202^{* * *}$ |
| $g_{t}^{w} d p_{t}^{w}$ | 2.3478 | 3.3012 | 1.4196 | 4.1784 | $472.08^{* * *}$ | $-6.4500^{* * *}$ |
| $g_{t}^{w} d d_{t}^{w}$ | -2.8430 | 4.3645 | -1.9078 | 6.5785 | $1367.1^{* * *}$ | $-7.0652^{*^{* * *}}$ |
| $g_{t}^{m} d p_{t}^{m}$ | 5.7086 | 8.7686 | 1.4304 | 3.7846 | $439.66^{* * *}$ | $-3.0898^{* *}$ |
| $g_{t}^{m} d n_{t}^{m}$ | -7.9652 | 12.647 | -1.8924 | 5.9452 | $1149.0^{* * *}$ | $-2.8197^{*}$ |

${ }^{* * *},{ }^{* *}$, and ${ }^{*}$ in the table mean obvious at significance level of $1 \%, 5 \%$, and $10 \%$, respectively, same for the following table.


Figure 1: Lagged correlation function between $\mathrm{ARV}_{t+h}$ and $X_{t}$.
conclusions of previous studies. In addition, from correlation functions between $\mathrm{ARV}_{t+h}$ and other 8 variables, we can find that all function values in future 25 phases are greater than 0 , so all the past values of these variables contain some forecast information towards the future $\mathrm{ARV}_{t}$ in Chinese stock market. However, the correlation function value of $J_{t}$ and $g_{t}^{d} d p_{t}^{d}$ to $\mathrm{ARV}_{t+h}$ is very small, which shows that these two variables have relatively weaker forecasting performance on the future $\mathrm{ARV}_{t}$ in Chinese stock market. Based on the above analyses, it can be seen that the capital gain overhang $g_{t}$ in Chinese stock market carries with it provides more information of forecasting the future $\mathrm{ARV}_{t}$. Therefore, we can roughly judge that introducing the momentum effect (capital gain overhang) in the HAR-ARV-CJ model can improve the model's forecasting performance of the future $\mathrm{ARV}_{t}$ in Chinese stock market.
4.2. Parameter Estimation. To show the superiority of measuring volatility in Chinese stock market of the new model (HAR-CJ-M model) in this paper, we first estimate the parameters in the HAR-CJ-M model, and also to that of HAR-ARV and HAR-CJ model for comparisons (the HAR-ARV-CJ-M, HAR-ARV, and HAR-CJ models mentioned here and that followed are all logarithm forms, that is, model (22), model (14), and model (16).) As the HAR-type models
mainly focus on different market participations of different frequency in daily, weekly, and monthly markets when considering the heterogeneous character of the market, this paper chooses three values for $H$ (1,5 and 22), namely, $\mathrm{ARV}_{t+1}^{d}, \mathrm{ARV}_{t+5}^{d}$, and $\mathrm{ARV}_{t+22}^{d}$ represent, respectively, the ARV of future 1-day, 1-week, and 1-month in Chinese stock market. Standard OLS regression is consistent and normally distributed, but when multistep ahead forecast is considered, the presence of regressors, which overlap, makes the usual inference no longer appropriate. Therefore, we estimate above models by OLS with Newey-West covariance correction.

The estimation results of the HAR-CJ-M model are shown in Table 2. When forecasting future 1-day, 1-week, and 1-month ARV in Chinese stock market, coefficients of the daily continuous sample path variation $\ln \left(C_{t}^{d}\right)$, weekly continuous sample path variation $\ln \left(C_{t}^{w}\right)$, and monthly continuous sample path variation $\ln \left(C_{t}^{w}\right)$ in phase $t$ are all obviously positive at significance level of $1 \%$. It shows that the past continuous sample path variation in Chinese stock market contains forecasting information on the future ARV. However, the coefficient of the daily discontinuous jump variation $\ln \left(J_{t}^{d}\right)$ in phase $t$ is only significant when forecasting the future 1-day ARV, while neither the coefficient of the weekly discontinuous jump variation $\ln \left(J_{t}^{w}\right)$ nor that of the monthly discontinuous jump variation $\ln \left(J_{t}^{m}\right)$ is significant. Therefore, the discontinuous jump variation in Chinese stock market is weak in forecasting the future ARV. For the newly added the momentum effect factor (capital gain overhang $g_{t}$ ) in the HAR-CJ model, except that the coefficient of the nonnegative part of daily capital gain overhang $g_{t}^{d} d p_{t}^{d}$ is not significant when forecasting the future 1-week and 1-month ARV, the rest of coefficients of $g_{t}$ are all obviously positive at significance level of $10 \%$. This shows that the information contained in the capital gain overhang $g_{t}$ in Chinese stock market has good forecasting performance on the future ARV. In this paper, we consider CSI 300 as a stock portfolio, and then we can use the momentum effect to explain part of the estimation results of the HAR-CJ-M model. We know from Grinblatt and Han's research that the momentum effect leads to the positive correlation between the previous gains and losses (which is expressed by the capital gain overhang $g_{t}$ ) of CSI 300 and current gains and losses, respectively; hence the momentum effect helps in the rise and fall of CSI 300

Table 2: Results of parameter estimation for HAR-CJ-M model.

|  | $H=1$ (1 day) |  | $H=5$ (1 week) |  | $H=22$ (1 month) |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Coefficient | Std. error | Coefficient | Std. error | Coefficient | Std. error |
| $\gamma_{0}$ | $-0.2543^{* * *}$ | 0.0556 | -0.1941*** | 0.0751 | -0.0818 | 0.1006 |
| $\gamma_{c d}$ | $0.1749^{* * *}$ | 0.0408 | $0.1458^{* * *}$ | 0.0325 | 0.0861*** | 0.0293 |
| $\gamma_{c w}$ | $0.3592^{* * *}$ | 0.0627 | $0.2585 * * *$ | 0.0712 | 0.0931 | 0.0735 |
| $\gamma_{c m}$ | $0.1773^{* * *}$ | 0.0543 | $0.2125^{* * *}$ | 0.0764 | $0.2995^{* * *}$ | 0.0852 |
| $\gamma_{j d}$ | 0.0649** | 0.0385 | 0.0125 | 0.0268 | 0.0098 | 0.0182 |
| $\gamma_{j w}$ | -0.0160 | 0.0418 | 0.0122 | 0.0504 | 0.0741 | 0.0459 |
| $\gamma_{j m}$ | 0.0676 | 0.0534 | 0.0704 | 0.0750 | -0.0065 | 0.0814 |
| $\gamma_{\text {gpd }}$ | 0.0812* | 0.0448 | 0.0291 | 0.0443 | 0.0557 | 0.0399 |
| $\gamma_{\text {gnd }}$ | $0.3335^{* * *}$ | 0.0472 | $0.1836^{* * *}$ | 0.0467 | 0.0820* | 0.0447 |
| $\gamma_{\text {gpw }}$ | $0.0878^{* * *}$ | 0.0306 | $0.1053 * * *$ | 0.0392 | $0.1512^{* * *}$ | 0.0512 |
| $\gamma_{\text {gnu }}$ | 0.0512* | 0.0312 | 0.0774* | 0.0398 | $0.1350^{* * *}$ | 0.0447 |
| $\gamma_{\text {gpm }}$ | $0.1128^{* * *}$ | 0.0248 | $0.2023^{* * *}$ | 0.0355 | $0.2062^{* * *}$ | 0.0473 |
| $\gamma_{\text {gnm }}$ | $0.1108^{* * *}$ | 0.0227 | $0.1868^{* * *}$ | 0.0344 | 0.2070*** | 0.0455 |
| Adj- $R^{2}$ | 0.6224 |  | 0.6807 |  | 0.6270 |  |

and adds to its volatility. Therefore, the nonnegative part of past capital gain overhang in Chinese stock market is positive correlation with the future ARV, and negative correlation with the negative part, and can help with the forecasting on the future ARV to some extent. We make further analysis on the capital gain overhang of different phases (daily, weekly, and monthly), the daily capital gain overhang $g_{t}^{d}$ can represent the behaving characters of short-term investors in phase $t$ in Chinese stock market, and the reference price of shortterm investors is the 5-day moving average $\mathrm{RP}_{t}^{d}$. When the price in phase $t$ is higher than $R P_{t}^{d}$ (namely, $g_{t}^{d}>0$ ), the disposition effect suppresses further rise of the stock price; when the price in phase $t$ is lower than $\operatorname{RP}_{t}^{d}$ (namely, $g_{t}^{d}<0$ ), the disposition effect suppresses further fall of the stock price, thereupon the stock price reflects insufficient information of phase $t$; thus the momentum effect emerges. After phase $t$, the market gradually begins to reflect the previous information, so the momentum effect helps in the rise and fall of the market and increases the market volatility. Hence, the nonnegative part of the daily capital gain overhang $g_{t}^{d} d p_{t}^{d}$ is positive correlation with the future ARV, and the negative part of capital gain overhang $g_{t}^{d} d n_{t}^{d}$ is negative correlation with the future ARV. We can see from Table 2 that the value of $\gamma_{g n d}$ is obviously greater than that of $\gamma_{g p d}$, and $\gamma_{g p d}$ is not significant when forecasting the future 1 -week and 1 -month volatility. It means that short-term investors in Chinese stock market hold different attitudes towards the same amount of gains and losses in previous phases. The influence of previous losses on short-term investors is obviously greater than that of gains, which may be caused by the loss aversion of short-term investors. Similarly, the momentum effect can be adopted to explain the forecasting performance of the weekly capital gain overhang $g_{t}^{w}$ and monthly capital gain overhang $g_{t}^{m}$ on the future ARV in Chinese stock market. Different from the daily capital gain overhang $g_{t}^{d}$, coefficients of the nonnegative part and negative part of both the weekly capital gain overhang
$g_{t}^{w}$ and monthly capital gain overhang $g_{t}^{m}$ are, approximately, showing that the medium-term and long-term investors in Chinese stock market are basically the same in their attitudes towards the same amount of gains and losses in previous phases, and their loss aversion is not obvious. This also reflects that medium-term and long-term investors are more rational than short-term ones.

The estimation results of the HAR-ARV and HAR-CJ models are shown in Tables 3 and 4, respectively. With analysis of the estimation results in Table 3, we find that coefficients of the daily ARV $\left(\ln \left(\mathrm{ARV}_{t}^{d}\right)\right)$, the weekly ARV $\left(\ln \left(\operatorname{ARV}_{t}^{w}\right)\right)$, and monthly ARV $\left(\ln \left(\mathrm{ARV}_{t}^{m}\right)\right)$ in phase $t$ are all positive at significance level of $1 \%$ when the model forecast the future 1-day, 1-week or 1-month ARV in Chinese stock market. This shows that ARV in Chinese stock market has strong long memory character, and the past volatility contains forecasting information of future volatility. Meanwhile, it also shows that the volatility in Chinese stock market is affected by the past different volatility components. Different volatility components are produced by investor behaviors with different holding terms (short-term, medium-term, and longterm). This result also proves the existence of heterogeneous investors in Chinese stock market, which is in line with the Heterogeneous Market Hypothesis. With analysis of the estimation results in Table 4, when forecasting the future 1day, 1-week, and 1-month ARV in Chinese stock market, it can be seen from the significance level of coefficients of $\ln \left(C_{t}^{d}\right)$, $\ln \left(C_{t}^{w}\right), \ln \left(C_{t}^{m}\right), \ln \left(J_{t}^{d}\right), \ln \left(J_{t}^{w}\right)$ and $\ln \left(J_{t}^{m}\right)$ that the continuous sample path variation has good forecasting performance on the future ARV, while the discontinuous jump variation component has weak forecasting performance on the future ARV. It is in line with the analysis conclusion from the HAR-CJ-M model.

Comparing the adjusted coefficient of determination $A d j-R^{2}$ of the HAR-CJ-M, HAR-ARV, and HAR-CJ models, we find that $A \mathrm{dj}-R^{2}$ of the HAR-CJ-M model is obviously

TABLE 3: Estimation results of the HAR-ARV model.

|  | $H=1$ (1 day) |  | $H=5$ (1 week) |  | $H=22(1 \mathrm{month})$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Coefficient | Std. error | Coefficient | Std. error | Coefficient | Std. error |
| $\alpha_{0}$ | 0.0021 | 0.0317 | $0.1475^{* * *}$ | 0.0532 | $0.3474^{* * *}$ | 0.0708 |
| $\alpha_{d}$ | $0.3658^{* * *}$ | 0.0436 | $0.2469^{* * *}$ | 0.0423 | $0.1420^{* * *}$ | 0.0267 |
| $\alpha_{w}$ | $0.2082^{* * *}$ |  | 0.0626 | $0.2151^{* * *}$ | 0.0734 | $0.1813^{* * *}$ |
| $\alpha_{m}$ | $0.3196^{* * *}$ |  | 0.0522 | $0.3830^{* * *}$ | 0.0693 | $0.3889^{* * *}$ |
| Adj- $R^{2}$ |  | 0.5642 |  |  | 0.6088 |  |

Table 4: Estimation results of the HAR-CJ model.

|  | $H=1$ (lday) |  | $H=5$ (lweek) |  | $H=22$ (lmonth) |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Coefficient | Std. error | Coefficient | Std. error | Coefficient | Std. error |
| $\beta_{0}$ | $0.1540^{* * *}$ | 0.0303 | $0.2986^{* * *}$ | 0.0490 | $0.4776^{* * *}$ | 0.0624 |
| $\beta_{c d}$ | $0.3172^{* * *}$ | 0.0381 | $0.2438^{* * *}$ | 0.0345 | $0.1393^{* * *}$ | 0.0298 |
| $\beta_{c w}$ | $0.3975^{* * *}$ | 0.0617 | $0.3352^{* * *}$ | 0.0683 | $0.2096^{* * *}$ | 0.0778 |
| $\beta_{c m}$ | $0.1405^{* *}$ | 0.0558 | $0.2079^{* * *}$ | 0.0784 | $0.3037^{* * *}$ | 0.0903 |
| $\beta_{j d}$ | $0.1115^{* * *}$ | 0.0411 | 0.0416 | 0.0305 | 0.0256 | 0.0211 |
| $\beta_{j w}$ | -0.0446 | 0.0390 | -0.0097 | 0.0570 | 0.0725 | 0.0545 |
| $\beta_{j m}$ | $0.1402^{* * *}$ | 0.0529 | $0.1767^{* *}$ | 0.0762 | 0.0994 | 0.0813 |
| Adj-R |  | 0.5868 |  | 0.6297 |  | 0.5495 |

greater than that of the HAR-CJ and HAR-ARV models. When the three models measure ARV at future 1-day, 1-week, and 1-month, $A \mathrm{dj}-R^{2}$ of the HAR-CJ-M model is 0.0356 , 0.0510 , and 0.0775 higher than that of the HAR-CJ model, respectively, and $0.0582,0.0719$, and 0.0825 higher than that of HAR-ARV model respectively. This shows that the past capital gain overhang in Chinese stock market contains much information of forecasting the future ARV.
4.3. Robustness to Models. This paper adopts the method of Grinblatt and Han [17] to give explanation to the momentum effect, in this way, the choice of reference price in the capital gain overhang can make great influence on the study of the momentum effect. So the choice of reference price is crucial in this paper. In the empirical evidence above, we take the 5-day, 5 -week ( 25 days), and 5-month (110 days) moving average as the reference price for those short-term, medium-term, and long-term investors in Chinese stock market, respectively. Here we will adopt the 10 -day, 10 -week ( 50 days), and 10 month ( 220 days) moving average of CSI 300 in Chinese stock market as the reference price to do the robustness tests to the result in Section 4.2. The evaluation result of the HAR-CJ-M model is shown in Table 5, most of the coefficients of the capital gain overhang $g_{t}^{d}$ are significant, showing that the past capital gain overhang in Chinese stock market is helpful in forecasting the future ARV to some extent. Moreover, $A d j-R^{2}$ of the HAR-CJ-M model which takes the 10-day, 10week ( 50 days), and 10 -month ( 220 days) moving average of CSI 300 in Chinese stock market as the reference price is obviously greater than that of the HAR-CJ and HARARV models, which accords with the result in Section 4.2.

However, its $A \mathrm{dj}-R^{2}$ is smaller than that of the HAR-CJM model which takes the 5-day, 5-week (25 days), and 5month (110 days) moving average as the reference price. This shows that the 5 -day, 5 -week ( 25 days), and 5-month (110 days) moving average affects more of the decision-making behaviors of those short-term, medium-term, and long-term investors in Chinese stock market. Therefore, adopting the 5day, 5-week ( 25 days), and 5-month (110 days) moving average as the reference price to forecast the future ARV in Chinese stock market is more suitable.

### 4.4. Forecasts

4.4.1. In-Sample Forecasts. Figures 2(a), 2(b), and 2(c) contain three in-sample forecast volatility sequences that are obtained by the HAR-CJ-M, HAR-ARV, and HAR-CJ models and a real volatility sequence. We adopt the loss functions to evaluate the volatility forecasting performance in Chinese stock market of the HAR-CJ-M, HAR-ARV, and HAR-CJ model. We mainly choose four loss functions to evaluation. They are the mean absolute error (MAE), mean absolute percentage error (MAPE), root mean squared error (RMSE), the heteroskedastic adjusted root mean squared error (HRMSE), and Theil coefficient. The smaller the values of these four loss functions are, the better the forecasting performance of the volatility models in future Chinese stock market is. The MAE, MAPE, RMSE, HRMSE and Theil coefficient for the in-sample forecasts from each of the three different models based on the data over the full sample period are reported in Table 6. Consider

$$
\mathrm{MAE}=\frac{1}{n} \sum_{t=T+1}^{t=T+n}\left[\ln \left(\mathrm{ARV}_{t+H}\right)-\ln \left(\widehat{\left.\left.\mathrm{A} R V_{t+H}\right)\right], ~}\right.\right.
$$

Table 5: Estimation results of the HAR-CJ-M model.

|  | $H=1(1$ day $)$ |  | $H=5(1$ week $)$ |  | $H=22(1$ month $)$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Coefficient | Std. error | Coefficient | Std. error | Coefficient | Std. error |
| $\gamma_{0}$ | $-0.2092^{* * *}$ | 0.0618 | $-0.2042^{* *}$ | 0.0869 | -0.0809 | 0.1243 |
| $\gamma_{c d}$ | $0.2373^{* * *}$ | 0.0398 | $0.1743^{* * *}$ | 0.0333 | $0.0776^{* * *}$ | 0.0279 |
| $\gamma_{c w}$ | $0.3039^{* * *}$ | 0.0643 | $0.2500^{* * *}$ | 0.0727 | $0.1334^{*}$ | 0.0755 |
| $\gamma_{c m}$ | $0.2154^{* * *}$ | 0.0625 | $0.2263^{* * *}$ | 0.0854 | $0.3056^{* * *}$ | 0.0971 |
| $\gamma_{j d}$ | $0.0891^{* *}$ | 0.0394 | 0.0243 | 0.0277 | 0.0131 | 0.0178 |
| $\gamma_{j w}$ | -0.0404 | 0.0416 | -0.0148 | 0.0501 | 0.0593 | 0.0486 |
| $\gamma_{j m}$ | $0.0914^{*}$ | 0.0534 | 0.1217 | 0.0753 | 0.0248 | 0.0862 |
| $\gamma_{g p d}$ | 0.0407 | 0.0369 | 0.0626 | 0.0453 | $0.0997^{* *}$ | 0.0465 |
| $\gamma_{g n d}$ | $0.1798^{* * *}$ | 0.0386 | $0.1338^{* * *}$ | 0.0428 | $0.1100^{* *}$ | 0.0450 |
| $\gamma_{g p w}$ | $0.1304^{* * *}$ | 0.0338 | $0.1676^{* * *}$ | 0.0454 | $0.1895^{* * *}$ | 0.0446 |
| $\gamma_{g n u}$ | $0.0975^{*}$ | 0.0323 | $0.1421^{* * *}$ | 0.0433 | $0.1554^{* * *}$ | 0.0447 |
| $\gamma_{g p m}$ | 0.0317 | 0.0258 | $0.0794^{* *}$ | 0.0360 | 0.0778 | 0.0502 |
| $\gamma_{g n m}$ | $0.0520^{* *}$ | 0.0225 | $0.0929^{* * *}$ | 0.0309 | $0.1141^{* *}$ | 0.0514 |
| Adj-R2 |  | 0.6083 |  |  | 0.6677 |  |

Table 6: In-sample forecast statistics.

|  |  | MAE | MAPE | RMSE | HRMSE | Theil coefficient |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|   HAR-CJ-M$\quad 0.4352$ | 1.6175 | 0.5641 | 0.6006 | 0.2228 |  |  |
|  | HAR-ARV | 0.4674 | 1.9335 | 0.6083 | 0.6856 | 0.2425 |
|  | HAR-CJ | 0.4530 | 1.5980 | 0.5913 | 0.6731 | 0.2346 |
|  | HAR-CJ-M | 0.4437 | 4.3431 | 0.4037 | 0.4984 | 0.1469 |
|  | HAR-ARV | 0.6015 | 5.3230 | 0.4518 | 0.9004 | 0.1660 |
|  | HAR-CJ | 0.5514 | 4.4688 | 0.4368 | 0.7801 | 0.1609 |
|  | HAR-CJ-M | 0.5700 | 4.7136 | 0.7002 | 0.5000 | 0.2579 |
|  | HAR-ARV | 0.6514 | 6.5020 | 0.7896 | 0.5266 | 0.2897 |
|  | HAR-CJ | 0.6174 | 5.9764 | 0.7466 | 0.5152 | 0.2742 |

$$
\begin{gathered}
\text { MAPE }=\frac{1}{n} \sum_{t=T+1}^{t=T+n} \frac{\ln \left(\mathrm{ARV}_{t+H}\right)-\ln \left(\widehat{\operatorname{ARV}}{ }_{t+H}\right)}{\ln \left(\mathrm{ARV}_{t+H}\right)}, \\
\mathrm{RMSE}=\sqrt{\frac{1}{n} \sum_{t=T+1}^{t=T+n}\left[\ln \left(\mathrm{ARV}_{t+H}\right)-\ln \left(\widehat{\left.\left.\mathrm{A} R V_{t+H}\right)\right]^{2}},\right.\right.} \\
\mathrm{HRMSE}=\sqrt{\frac{1}{n} \sum_{t=T+1}^{t=T+n}\left[\frac{\ln \left(\mathrm{ARV}_{t+H}\right)-\ln \left(\widehat{\mathrm{A}} \mathrm{AV}_{t+H}\right)}{\ln \left(\mathrm{ARV}_{t+H}\right)}\right]^{2}},
\end{gathered}
$$

Theil cofficient

$$
\begin{align*}
= & \sqrt{\frac{1}{n} \sum_{t=T+1}^{t=T+n}\left[\ln \left(\mathrm{ARV}_{t+H}\right)-\ln \left(\widehat{\mathrm{A}} \mathrm{~V}_{t+H}\right)\right]^{2}} \\
& \times\left(\sqrt{\frac{1}{n} \sum_{t=T+1}^{t=T+n}\left[\ln \left(\widehat{\mathrm{~A}} \mathrm{RV}_{t+H}\right)\right]^{2}}\right. \\
& \left.+\sqrt{\frac{1}{n} \sum_{t=T+1}^{t=T+n}\left[\ln \left(\mathrm{ARV}_{t+H}\right)\right]^{2}}\right)^{-1} \tag{23}
\end{align*}
$$

where $n$ is the number of samples predicted, $\ln \left(\operatorname{ARV}_{t+H}\right)$ represents the true volatility, and $\ln \left(\widehat{\mathrm{A}} R V_{t+H}\right)$ represents the forecast volatility.

In Table 6, we can find except that the MAPE of the HAR-CJ-M model is greater than that of HAR-CJ model when the model forecasts the 1-day ARV, the other MAE, MAPE, RMSE, HRMSE, and Theil coefficient of the HAR-CJ-M model are all smaller than those of the HAR-CJ model, and the MAE, MAPE, RMSE, HRMSE, and Theil coefficient of HAR-CJ model are all smaller than those of the HAR-ARV model. Therefore, the in-sample forecasting performance of the HAR-CJ-M model on future volatility in Chinese stock market is better than that of the HAR-CJ model, and the HAR-ARV-CJ model is better than that of the HAR-ARV model.
4.4.2. Out-of-Sample Forecasts. Compared with the insample forecasting performance, we are more concerned with the out-of-sample forecasting performance of the model, for the out-of-sample forecasting performance is more significant to the study of volatility in Chinese stock market. In order to make effective evaluation to the out-of-sample forecasting performance of the model, we divide the whole sample interval (from April 20, 2007 to April 20, 2012) into

Table 7: Out-of-sample Forecast Statistics.

|  |  | MAE | MAPE | RMSE | HRMSE | Theil coefficient |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|   HAR-CJ-M$\quad 0.4623$ | 2.9898 | 0.6273 | 0.4201 | 0.5489 |  |  |
|  | HAR-ARV | 0.5129 | 4.9047 | 0.6826 | 0.4501 | 0.9499 |
|  | HAR-CJ | 0.4793 | 3.3916 | 0.6600 | 0.4338 | 0.5869 |
|  | HAR-CJ-M | 0.4886 | 6.5385 | 0.6361 | 0.2202 | 0.5135 |
|  | HAR-ARV | 0.5446 | 4.3915 | 0.7162 | 0.2932 | 0.6501 |
|  | HAR-CJ | 0.4966 | 4.4688 | 0.6615 | 0.2484 | 0.5522 |
|  | HAR-CJ-M | 0.5713 | 12.093 | 0.7168 | 0.4258 | 0.5137 |
|  | HAR-ARV | 0.6524 | 12.249 | 0.8089 | 0.4852 | 0.6075 |
|  | HAR-CJ | 0.6380 | 12.133 | 0.7652 | 0.4432 | 0.5245 |


(a)

(b)

(c)

Figure 2: (a) Comparison of the in-sample forecasting performance of the HAR-ARV, HAR-CJ, and HAR-CJ-M models (1 day). ARV represents the true volatility; HAR-ARV, HAR-CJ, and HAR-CJ-M represent the forecast volatility of the HAR-ARV, HAR-CJ, and HAR-CJ-M models, respectively. (b) Comparison of the in-sample forecasting performance of the HAR-ARV, HAR-CJ and HAR-CJ-M model (1 week). ARV represents the true volatility; HAR-ARV, HAR-CJ, and HAR-CJ-M represent the forecast volatility of the HAR-ARV, HAR-CJ, and HAR-CJ-M models, respectively. (c) Comparison of the in-sample forecasting performance of the HAR-ARV, HAR-CJ, and HAR-CJ-M model (1 month). In the figure, ARV represents the true volatility; HAR-ARV, HAR-CJ, and HAR-CJ-M represent the forecast volatility of the HAR-ARV, HAR-CJ, and HAR-CJ-M models, respectively.
two parts the former part (from April 20, 2007 to May 31, 2011) has 1000 samples in all as the estimation intervals of the model; the latter part (from June 1, 2011 to April 20, 2012) has 199 samples in all as the forecasting intervals of the model. Figures 3(a), 3(b), and 3(c) contain three out-of-sample forecast volatility sequences that are obtained by the HAR-CJ-M, HAR-ARV, and HAR-ARV-CJ models and a real volatility sequence. In addition, the method of analyzing is the same with that of the Section 4.4.1, that is, using
the loss functions to evaluate the out-of-sample forecasting performance of the model. The results are shown in Table 7.

In Table 7, it can be found that except that the MAPE of HAR-CJ-M model is greater than that of HAR-ARVCJ model, and that of HAR-ARV-CJ model greater than HAR-ARV model when forecasting the 1-week ARV, the rest values of MAE, MAPE, RMSE, HRMSE, and Theil coefficient of HAR-CJ-M model are all smaller than those of HAR-ARV-CJ model, and the MAE, MAPE, RMSE, HRMSE and


Figure 3: (a) Comparison of the out-of-sample forecasting performance of the HAR-ARV, HAR-CJ, and HAR-CJ-M model (1 day). ARV represents the true volatility; HAR-ARV, HAR-CJ and HAR-CJ-M represent the forecast volatility of the HAR-ARV, HAR-CJ, and HAR-CJM models, respectively. (b) Comparison of the out-of-sample forecasting performance of the HAR-ARV, HAR-CJ, and HAR-CJ-M model (1 week). ARV represents the true volatility; HAR-ARV, HAR-CJ, and HAR-CJ-M represent the forecast volatility of the HAR-ARV, HAR-CJ, and HAR-CJ-M models, respectively. (c) Comparison of the out-of-sample forecasting performance of the HAR-ARV, HAR-CJ, and HAR-CJ-M model (1 month). ARV represents the true volatility; HAR-ARV, HAR-CJ, and HAR-CJ-M represent the forecast volatility of the HAR-ARV, HAR-CJ, and HAR-CJ-M models, respectively.

Theil coefficient of HAR-ARV-CJ model are smaller than those of HAR-ARV model. Therefore, the HAR-CJ-M model has better out-of-sample forecasting performance on future performance in Chinese stock market than the HAR-ARV-CJ model, and the HAR-CJ model is better than the HAR-ARV model.

Combining the analyses in Sections 4.4.1 and 4.4.2, we can conclude that the forecasting performance of the above three volatility models of future volatility in Chinese stock market from the best to the weakest is in the following order: HAR-CJ-M model, HAR-ARV-CJ model, and then HARARV model.

## 5. Conclusion

Considering the crucial role of the overnight return variance in volatility of the whole Chinese stock market, we convert RV into ARV and set up a HAR-CJ-M model on the basis of the HAR-CJ model and momentum effect. After that, we take the 5-minute high-frequency data of CSI 300 as samples for empirical evidence and estimate parameters on the HAR-CJM, HAR-ARV, and HAR-CJ models. Then we compare these
three models of their forecasting performance of the future ARV in Chinese stock market by using the loss functions.

In the HAR-CJ-M model, most coefficients of the momentum effect (capital gain overhang) of different term limits (daily, weekly, and monthly) are significant, showing that the irrational behaviors of different kinds of investors in Chinese stock market help in forecasting the future volatility to some extent. In addition, from the estimate results of this model and the HAR-CJ model, we can see that the past continuous sample path variation in Chinese stock market can help with the forecast of future volatility, while the past discontinuous jump variation has very poor forecasting performance, which is in line with the conclusion of Wang et al. [13]. The estimate results of the HAR-ARV model show that the volatility of Chinese stock market can be influenced by the past different volatility components, and different volatility components are produced by behaviors of investors with different holding term limits (short-term, medium-term, and long-term). Thus, this result also proves the existence of the heterogeneous character of Chinese stock investors, which accords with the Heterogeneous Market Hypothesis. Besides, the comparative analysis of the above three models' forecasting performance shows that the HAR-CJ-M model
which has added the momentum effect forecasts much better than the other two models on the future volatility of Chinese stock market. Therefore, it shows that the irrational factors of investors do affect the volatility of Chinese stock market. Based on this, the volatility model which has taken the irrational factors of investors into consideration can forecast better on the volatility of Chinese stock market, and the HAR-CJ-M model is more favorable to the study of practical problems such as financial risk measuring, asset pricing, and financial derivatives pricing. Although the HAR-CJ-M model has good forecasting performance on future volatility in Chinese stock market, its $A \mathrm{dj}-R^{2}$ is all smaller than 0.7 when it forecasts the future 1-day, 1-week, and 1-month volatility in Chinese stock market. So it is necessary to further improve the accuracy of the model's forecasting volatility of Chinese stock market. Our work will be paid more consideration into irrational factors of investors on the basis of this paper so that further improve the forecasting accuracy of the model for the volatility in Chinese stock market.

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## Research Article

# Synchronization of General Complex Networks with Hybrid Couplings and Unknown Perturbations 

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#### Abstract

The issue of synchronization for a class of hybrid coupled complex networks with mixed delays (discrete delays and distributed delays) and unknown nonstochastic external perturbations is studied. The perturbations do not disappear even after all the dynamical nodes have reached synchronization. To overcome the bad effects of such perturbations, a simple but all-powerful robust adaptive controller is designed to synchronize the complex networks even without knowing a priori the functions and bounds of the perturbations. Based on Lyapunov stability theory, integral inequality Barbalat lemma, and Schur Complement lemma, rigorous proofs are given for synchronization of the complex networks. Numerical simulations verify the effectiveness of the new robust adaptive controller.


## 1. Introduction

Over the past decade, complex networks have attracted much attention from authors of many disciplines since the pioneer works of Watts and Strogatz [1, 2]. In fact, many phenomena in nature and our daily life can be explained by using complex networks, such as the Internet, World Wide Web, social networks, and neural networks. A complex network can be considered as a graph which consists of a set of nodes and edges connecting these nodes [3].

In recent years, chaos synchronization [3-7] has been intensively studied due to its important applications in many different areas, such as secure communication, biological systems, and information science [8-11]. Particularly, the synchronization of all the dynamical nodes in complex networks has become a hot research topic [3], and several results have been appeared in the literature. The authors of [12] studied the synchronization in complex networks with switching topology. In [13], Wu and Jiao investigated the synchronization in complex dynamical networks with nonsymmetric coupling. They showed that the synchronizability of a dynamical network with nonsymmetric coupling is not always characterized by its second-largest eigenvalue, even
though all the eigenvalues of the nonsymmetric coupling matrix are real. Liu and Chen [14] gave some criteria for the global synchronization of complex networks in virtual of the left eigenvector corresponding to the zero eigenvalue of the coupling matrix. For a given network with identical node dynamics, the authors of [15] showed that two key factors influencing the network synchronizability are the network inner linking matrix and the eigenvalues of the network topological matrix. Some synchronization criteria were given in [16-19] for coupled neural networks with or without delayed couplings. In [20], the robust impulsive synchronization of coupled delayed neural networks with uncertainties is considered; several new criteria are obtained to guarantee the robust synchronization via impulses.

Complex networks have the properties of robustness and fragility. A complex network can synchronize itself when parameter mismatch is within some limit. If parameter mismatch exceeds this limit, networks cannot realize synchronization themselves. Thus the controlled synchronization of coupled networks is believed to be a rather significant topic in both theoretical research and practical applications [2129]. Some effective control scheme has been proposed, for instance, state feedback control with constant control gains,
impulsive control, intermittent control, and adaptive control. Adaptive control method receives particular attention of researchers in recently rears. In [3], the authors studied synchronization in complex networks by using distributed adaptive control scheme. By designing a simple adaptive controller, authors of [23] investigated the locally and globally adaptive synchronization of an uncertain complex dynamical network. Authors in [24] investigated synchronization of neural networks with time-varying delays and distributed delays via adaptive control method. By using the adaptive feedback control scheme, Chen and Zhou [25] studied synchronization of complex nondelayed networks and Cao et al. [26] investigated the complete synchronization in an array of linearly stochastically coupled identical networks with delays. By using adaptive pinning control method, Zhou et al. [27] studied local and global synchronization of complex networks without delays, authors of $[28,29]$ considered the global synchronization of the complex networks with nondelayed and delayed couplings and the authors of [30] investigated lag synchronization of complex networks via state feedback pinning strategy. Outer synchronization of complex delayed networks with uncertain parameters was considered by using adaptive coupling in [31]. However, models in the previous references are special; that is, each of them does not consider general complex networks in which every dynamical node has mixed delays (discrete delay and distributed delay), and the complex networks have nondelayed, discrete-delayed, and distributed-delayed couplings.

Complex networks are always affected by some unknown external perturbations due to environmental causes and human causes. White noises brought by some random fluctuations in the course of transmission and other probabilities causes have received extensive attention in the literatures [21, 24, 32-35]. However, not all the external perturbations are white noise, and some of them may be nonlinear and nonstochastic perturbations. When complex networks are disturbed by nonlinear and nonstochastic perturbations, the states of the nodes will be changed dramatically, which will affect the stability and synchronization of the complex networks. Due to the fragility of complex networks, if some important nodes are perturbed by such external perturbations, whole states of the network will be affected or even the network cannot operate normally. Hence, how to realize synchronization of all nodes for complex networks with uncertain nonlinear nonstochastic external perturbations is an urgent practical problem to be solved. Obviously, the controllers for stability and synchronization of stochastic perturbations are not applicable to the case of nonlinear nonstochastic perturbations, especially when the functions and bounds of the perturbations are unknown. Therefore, to enhance antiperturbations capability and to realize synchronization of complex networks, more effective controller should be designed.

Motivated by the previous analysis, in this paper, a class of more general complex networks is proposed. The new model has nondelayed, discrete-delayed, and distributeddelayed couplings, and every dynamical node has mixed delays. Unknown nonstochastic external perturbations to the
complex networks are also considered. Then we study the global complete synchronization of the proposed model. A new simple but robust adaptive controller is designed to overcome the effects of such perturbations and synchronize the complex networks even without knowing the exact functions and bounds of the perturbations. Moreover, the adaptive controller can also synchronize coupled systems with stochastic perturbations since it includes existing adaptive controller as special case. Two cases are considered: all nodes or partial nodes are perturbed. All nodes should be controlled for the former case. Pinning control scheme can also be used for the latter case. Based on Lyapunov stability theory, integral inequality, Barbalat lemma, and Schur Complement lemma, rigorous proofs are given for synchronization of the complex networks with unknown perturbations of the previous two cases. It should be noted that our new adaptive controllers can also prevent external perturbations. Therefore, the new adaptive controllers are better than those in [23-29]. Numerical simulations verify the effectiveness of our theoretical results.

Notations. In the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions. $I_{N}$ denotes the identity matrix of $N$ dimension. The Euclidean norm in $\mathbb{R}^{n}$ is denoted as $\|\cdot\|$; accordingly, for vector $x \in \mathbb{R}^{n},\|x\|=$ $\sqrt{x^{T} x}$, where $T$ denotes transposition. $A=\left(a_{i j}\right)_{m \times m}$ denotes a matrix of $m$ dimension, $\|A\|=\sqrt{\lambda_{\max }\left(A^{T} A\right)}$, and $A^{s}=$ $(1 / 2)\left(A+A^{T}\right) . A>0$ or $A<0$ denotes that the matrix $A$ is symmetric and positive or negative definite matrix. $\lambda_{\text {min }}\left(A^{s}\right)$ is the minimum eigenvalues of the symmetric matrices $A^{s}$, and $A_{l}$ denotes the matrix of the first $l$ row-column pairs of A. $A_{l}^{c}$ denotes the minor matrix of matrix $A$ by removing all the first $l$ row-column elements of $A$.

The rest of this paper is organized as follows. In Section 2, a class of general complex networks with mixed delays and external perturbations is proposed. Some necessary assumptions and lemmas are also given in this section. In Section 3, synchronization of the complex networks with all nodes perturbed is studied. Synchronization with only partial nodes perturbed is considered in Section 4. Then, in Section 5, numerical simulations are given to show the effectiveness of our results. Finally, in Section 6, conclusions are given.

## 2. Preliminaries

The general complex networks consisting of $N$ identical nodes with external perturbations and mixed-delay couplings are described as

$$
\begin{aligned}
\dot{x}_{i}(t)= & C x_{i}(t)+A f\left(x_{i}(t)\right)+B f\left(x_{i}(t-\tau(t))\right) \\
& +D \int_{t-\theta(t)}^{t} f\left(x_{i}(s)\right) d s+I(t)+\alpha \sum_{j=1}^{N} u_{i j} \Phi x_{j}(t) \\
& +\beta \sum_{j=1}^{N} v_{i j} \curlyvee x_{j}(t-\tau(t))+\gamma \sum_{j=1}^{N} w_{i j} \Lambda \int_{t-\theta(t)}^{t} x_{j}(s) d s
\end{aligned}
$$

$$
\begin{align*}
& +\sigma_{i}\left(t, x_{i}(t), x_{i}(t-\tau(t)), \int_{t-\theta(t)}^{t} x_{i}(s) d s\right) \\
& +R_{i}, \quad i=0,1, \ldots, N \tag{1}
\end{align*}
$$

where $x_{i}(t)=\left[x_{i 1}(t), \ldots, x_{i n}(t)\right]^{T} \in \mathbb{R}^{n}$ represents the state vector of the $i$ th node of the network at time $t$, and $C, A, B, D$ are matrices with proper dimension. $f(\cdot)$ is a continuous vector function. $I(t)$ is the external input vector. $R_{i} \in \mathbb{R}^{n}$ is the control input. $\tau(t)>0, \theta(t)>0$ are time-varying discrete delay and distributed delay, respectively. Constants $\alpha>0$, $\beta>0, \gamma>0$ are coupling strengths of the whole network corresponding to nondelay, discrete delay, and distributed delay, respectively. $\Phi, \Upsilon, \Lambda \in \mathbb{R}^{n \times n}$ are inner coupling matrices of the networks, which describe the individual coupling between two subsystems. Matrices $U=\left(u_{i j}\right)_{N \times N}, V=$ $\left(v_{i j}\right)_{N \times N}, W=\left(w_{i j}\right)_{N \times N}$ are outer couplings of the whole networks satisfying the following diffusive conditions:

$$
\begin{align*}
& u_{i j} \geq 0 \quad(i \neq j), \quad u_{i i}=-\sum_{j=1, j \neq i}^{N} u_{i j} \\
& v_{i j} \geq 0 \quad(i \neq j), \quad v_{i i}=-\sum_{j=1, j \neq i}^{N} u_{i j}  \tag{2}\\
& w_{i j} \geq 0 \quad(i \neq j), \quad w_{i i}=-\sum_{j=1, j \neq i}^{N} w_{i j}
\end{align*}
$$

where $i, j=1,2, \ldots, N$. Vector $\sigma_{i}\left(t, x_{i}(t), x_{i}(t-\tau(t))\right.$, $\left.\int_{t-\theta(t)}^{t} x_{i}(s) d s\right) \in \mathbb{R}^{n}$ describes the unknown perturbation to $i$ th node of the complex networks. In this paper, we always assume that $\dot{\tau} \leq h_{\tau}<1$ and $\dot{\theta} \leq h_{\theta}<1 . \theta(t)$ is bounded and we denote $\theta_{\min }>0$ the minimum of $\theta(t)$ and $\theta_{\max }$ the maximum of $\theta(t)$.

We assume that (1) has a unique continuous solution for any initial condition in the following form:

$$
\begin{equation*}
x_{i}(s)=\varphi_{i}(s), \quad-\varrho \leq s \leq 0, \quad i=0,1,2, \ldots, N \tag{3}
\end{equation*}
$$

where $\varrho=\max \left\{\tau_{\text {max }}, \sigma_{\max }\right\}$ and $\tau_{\text {max }}$ is the maximum of $\tau(t)$.
For convenience of writing, in the sequel, we denote $\sigma_{i}\left(t, x_{i}(t), x_{i}(t-\tau(t)), \int_{t-\theta(t)}^{t} x_{j}(s) d s\right)$ with $\sigma_{i}(t)$.

The system of an isolate node without external perturbation is described as

$$
\begin{align*}
\dot{z}(t)= & C z(t)+A f(z(t))+B f(z(t-\tau(t))) \\
& +D \int_{t-\theta(t)}^{t} f(z(s)) d s+I(t), \tag{4}
\end{align*}
$$

and $z(t)$ can be any desired state: equilibrium point, a nontrivial periodic orbit, or even a chaotic orbit.

Remark 1. The nonstochastic perturbations $\sigma_{i}\left(t, x_{i}(t), x_{i}(t-\right.$ $\left.\tau(t)), \int_{t-\theta(t)}^{t} x_{i}(s) d s\right)$ are different from stochastic ones in the literature $[21,24,32-35]$. The distinct feature of the
such stochastic perturbations is that the stochastic perturbations disappear when the synchronization goal is realized. However, perturbations of this paper still exist even when complete synchronization has been achieved. Therefore, the controllers in most of existing papers including those in [21, 24, 32-35] are invalid for perturbations of this paper.

When system (4) is perturbed, then (4) turns to the following system:

$$
\begin{align*}
\dot{\bar{z}}(t)= & C \bar{z}(t)+A f(\bar{z}(t))+B f(\bar{z}(t-\tau(t))) \\
& +D \int_{t-\theta(t)}^{t} f(\bar{z}(s)) d s+I(t)  \tag{5}\\
& +\sigma_{i}\left(t, \bar{z}(t), \bar{z}(t-\tau(t)), \int_{t-\theta(t)}^{t} \bar{z}(s) d s\right) .
\end{align*}
$$

Generally, the state of a system will be changed when the system is perturbed. We assume that the state of system (5) remains to be any one of the previous three states but not necessarily the original one.

The following assumptions are needed in this paper:
$\left(\mathrm{H}_{1}\right) f(0) \equiv 0$, and there exists positive constant $h$ such that

$$
\begin{equation*}
\|f(u)-f(v)\| \leq h\|u-v\|, \quad \text { for any } u, v \in \mathbb{R}^{n} \tag{6}
\end{equation*}
$$

$\left(\mathrm{H}_{2}\right) \sigma_{i k}(t, 0,0,0) \equiv 0$, and there exist positive constants $M_{i k}$ such that $\left|\sigma_{i k}(t, u, v, w)\right| \leq M_{i k}$ for any bounded $u, v, w \in \mathbb{R}^{n}, i=1,2, \ldots, N, k=1,2, \ldots, n$.

Remark 2. Note that (4) unifies many well-known chaotic systems with or without delays, such as Chua system, Lorenz system, Rössler system, Chen system, and chaotic neural networks with mixed delays [12-29]. Hence, results of this paper are general.

Remark 3. Condition $\left(\mathrm{H}_{2}\right)$ is very mild. We do not impose the usual conditions such as Lipschitz condition, differentiability on the external perturbation functions. It can be discontinuous or even impulsive functions. If the state of (5) is a equilibrium point or a nontrivial periodic orbit, the condition $\left(\mathrm{H}_{2}\right)$ can be easily satisfied. If the state of (5) is a chaotic orbit, the condition $\left(\mathrm{H}_{2}\right)$ can also be satisfied. Since chaotic system has strange attractors, there exists a bounded region containing all attractors of it such that every orbit of the system never leaves them. Anyway, condition $\left(\mathrm{H}_{2}\right)$ can be satisfied for equilibrium point, a nontrivial periodic orbit, and a chaotic orbit. Moreover, we will subsequently prove that the complex networks (1) can be synchronized even without knowing the exact values of $h$ and $M_{i k}, i=1,2, \ldots, N, k=$ $1,2, \ldots, n$.

The aim of this paper is to synchronize all the states of complex networks (1) to the following manifold:

$$
\begin{equation*}
x_{1}(t)=x_{2}(t)=\cdots=x_{N}(t)=z(t), \tag{7}
\end{equation*}
$$

where $z(t)$ is immune to external perturbations.

Lemma 4 ((Schur Complement) see [36]). The linear matrix inequality (LMI)

$$
S=\left[\begin{array}{ll}
S_{11} & S_{12}  \tag{8}\\
S_{12}^{T} & S_{22}
\end{array}\right]<0
$$

is equivalent to any one of the following two conditions:

$$
\begin{aligned}
& \left(\mathrm{L}_{1}\right) S_{11}<0, S_{22}-S_{12}^{T} S_{11}^{-1} S_{12}<0, \\
& \left(\mathrm{~L}_{2}\right) S_{22}<0, S_{11}-S_{12} S_{22}^{-1} S_{12}^{T}<0,
\end{aligned}
$$

where $S_{11}=S_{11}^{T}, S_{22}=S_{22}^{T}$.
Lemma 5 (see [37]). For any constant matrix $D \in \mathbb{R}^{n \times n}, D^{T}=$ $D>0$, scalar $\sigma>0$, and vector function $\omega:[0, \sigma] \rightarrow \mathbb{R}^{n}$, one has

$$
\begin{equation*}
\sigma \int_{0}^{\sigma} \omega^{T}(s) D \omega(s) d s \geq\left(\int_{0}^{\sigma} \omega(s) d s\right)^{T} D \int_{0}^{\sigma} \omega(s) d s \tag{9}
\end{equation*}
$$

provided that the integrals are all well defined.
Lemma 6 ((Barbalat lemma) see [38]). If $f(t): \mathbb{R} \rightarrow \mathbb{R}^{+}$is a uniformly continuous function for $t \geq 0$ and if the limit of the integral

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \int_{0}^{t} f(s) d s \tag{10}
\end{equation*}
$$

exists and is finite, then $\lim _{t \rightarrow \infty} f(t)=0$.

## 3. Synchronization with All the Nodes Perturbed

In this section, we consider the case when all the nodes are perturbed. To realize synchronization goal (7), we have to introduce an isolate node (4).

Let $e_{i}(t)=x_{i}(t)-z(t)$. Subtracting (4) from (1), we get the following error dynamical system:

$$
\begin{align*}
\dot{e}_{i}(t)= & C e_{i}(t)+A g\left(e_{i}(t)\right)+B g\left(e_{i}(t-\tau(t))\right) \\
& +D \int_{t-\theta(t)}^{t} g\left(e_{i}(s)\right) d s+\alpha \sum_{j=1}^{N} u_{i j} \Phi e_{j}(t) \\
& +\beta \sum_{j=1}^{N} v_{i j} \gamma e_{j}(t-\tau(t))+\sigma_{i}(t)+R_{i}  \tag{11}\\
& +\gamma \sum_{j=1}^{N} w_{i j} \Lambda \int_{t-\theta(t)}^{t} e_{j}(s) d s,
\end{align*}
$$

where $g\left(e_{i}\right)=f\left(x_{i}(t)\right)-f(z(t)), i=1,2, \ldots, N$.
From $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$ we know that (11) admits a trivial solution $e_{i}(0) \equiv 0, i=1,2, \ldots, N$. Obviously, to reach the goal (7), we have only to prove that system (11) is asymptotically stable at the origin.

Theorem 7. Under the assumption conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$, the networks (1) are synchronized with the following adaptive controllers:

$$
\begin{gather*}
R_{i}=-\alpha \varepsilon_{i} e_{i}(t)-\omega \beta_{i} \operatorname{sign}\left(e_{i}(t)\right), \\
\dot{\varepsilon}_{i}=p_{i} e_{i}(t)^{T} e_{i}(t),  \tag{12}\\
\dot{\beta}_{i}=\xi_{i} \sum_{k=1}^{n}\left|e_{i k}(t)\right|
\end{gather*}
$$

where $\omega>1, p_{i}>0$, and $\xi_{i}>0$ are arbitrary constants, respectively, $i=1,2, \ldots, N$.

Proof. Define the Lyapunov function as

$$
\begin{equation*}
V(t)=V_{1}(t)+V_{2}(t)+V_{3}(t), \tag{13}
\end{equation*}
$$

where

$$
\begin{align*}
V_{1}(t)= & \frac{1}{2} \sum_{i=1}^{N} e_{i}^{T}(t) e_{i}(t) \\
& +\frac{1}{2} \sum_{i=1}^{N} \frac{\alpha\left(\varepsilon_{i}-k_{i}\right)^{2}}{p_{i}}+\frac{1}{2} \sum_{i=1}^{N} \frac{\left(M_{i}-\beta_{i}\right)^{2}}{\xi_{i}}  \tag{14}\\
V_{2}(t)= & \int_{t-\tau(t)}^{t} \eta^{T}(s) Q \eta(s) d s \\
V_{3}(t)= & \int_{t-\theta(t)}^{t} \int_{\mu}^{t} \eta^{T}(s) G \eta(s) d s d \mu
\end{align*}
$$

$\eta(t)=\left(\left\|e_{1}(t)\right\|,\left\|e_{2}(t)\right\|, \ldots,\left\|e_{N}(t)\right\|\right)^{T}, M_{i}=\max _{1 \leq k \leq n}\left\{M_{i k}\right\}$, $k_{i}, i=1,2, \ldots, N$, are constants, $Q$ and $G$ are symmetric positive definite matrices, and $k_{i}, Q$, and $G$ are to be determined.

Differentiating $V_{1}(t)$ along the solution of (11) and from $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$, we obtain

$$
\begin{aligned}
& \dot{V}_{1}(t)=\sum_{i=1}^{N} e_{i}^{T}(t) \dot{e}_{i}(t)+\alpha \sum_{i=1}^{N}\left(\varepsilon_{i}-k_{i}\right) e_{i}^{T}(t) e_{i}(t) \\
& -\sum_{i=1}^{N}\left(M_{i}-\beta_{i}\right) \sum_{k=1}^{n}\left|e_{i k}(t)\right| \\
& \leq \sum_{i=1}^{N}\left[\left(\|C\|+\|A\| h-\alpha k_{i}\right)\left\|e_{i}(t)\right\|^{2}\right. \\
& +\|B\| h\left\|e_{i}(t)\right\|\left\|e_{i}(t-\tau(t))\right\| \\
& +\|D\| h\left\|e_{i}(t)\right\| \int_{t-\theta(t)}^{t}\left\|e_{i}(s)\right\| d s \\
& +\alpha \sum_{j=1, j \neq i}^{N} u_{i j}\|\Phi\|\left\|e_{i}(t)\right\|\left\|e_{j}(t)\right\| \\
& +\alpha \lambda_{\text {min }}\left(\Phi^{s}\right) u_{i i}\left\|e_{i}(t)\right\|^{2}
\end{aligned}
$$

$$
\begin{align*}
& +\beta \sum_{j=1}^{N}\left|v_{i j}\right|\|\Upsilon\|\left\|e_{i}(t)\right\|\left\|e_{j}(t-\tau(t))\right\| \\
& \left.\quad+\gamma \sum_{j=1}^{N}\left|w_{i j}\right|\|\Lambda\|\left\|e_{i}(t)\right\| \int_{t-\theta(t)}^{t}\left\|e_{j}(s)\right\| d s\right] \\
& =\eta^{T}(t)\left((\|C\|+\|A\| h) I_{N}+\alpha\left(\|\Phi\| \widehat{U}^{s}-K\right)\right) \eta(t) \\
& +\eta^{T}(t)\left(\|B\| h I_{N}+\beta\|\Upsilon\||V|\right) \eta(t-\tau(t)) \\
& +\eta^{T}(t)\left(\|D\| h I_{N}+\gamma\|\Lambda\||W|\right) \int_{t-\theta(t)}^{t} \eta(s) d s \\
& \leq \\
& \eta^{T}(t)\left((\|C\|+\|A\| h+1) I_{N}\right. \\
& \left.\quad+\alpha\left(\|\Phi\| \widehat{U}^{s}-K\right)\right) \eta(t) \\
& \quad+\eta^{T}(t-\tau(t)) \bar{B}^{T} \bar{B} \eta(t-\tau(t))  \tag{15}\\
& \quad+\left(\int_{t-\theta(t)}^{t} \eta(s) d s\right)^{T} \bar{D}^{T} \bar{D} \int_{t-\theta(t)}^{t} \eta(s) d s,
\end{align*}
$$

where $K=\operatorname{diag}\left(k_{1}, k_{2}, \ldots, k_{N}\right), \widehat{U}=\left(\widehat{u}_{i j}\right)_{N \times N}, \widehat{u}_{i j}=u_{i j}$, $i \neq j, \widehat{u}_{i i}=\left(\lambda_{\min }\left(\Phi^{s}\right) /\|\Phi\|\right) u_{i i}, \bar{B}=\|B\| h I_{N}+\beta\|\Upsilon\||V|, \bar{D}=$ $\|D\| h I_{N}+\gamma\|\Lambda\| \| W\left|,|V|=\left(\left|v_{i j}\right|\right)_{N \times N},|W|=\left(\left|w_{i j}\right|\right)_{N \times N}\right.$, and we have used the following deduction:

$$
\begin{align*}
& \sum_{i=1}^{N}\left[e_{i}^{T}(t)\right. \\
& \left.\sigma_{i}(t)-\omega \sum_{k=1}^{n} \beta_{i}\left|e_{i k}(t)\right|-\sum_{k=1}^{n}\left(M_{i}-\beta_{i}\right)\left|e_{i k}(t)\right|\right] \\
& \quad \leq \sum_{i=1}^{N} \sum_{k=1}^{n}\left[\left|e_{i k}(t)\right| M_{i k}-M_{i}\left|e_{i k}\right|-(\omega-1) \beta_{i}\left|e_{i k}\right|\right]  \tag{16}\\
& \\
& \leq-\sum_{i=1}^{N} \sum_{k=1}^{n}(\omega-1) \beta_{i}\left|e_{i k}\right| \leq 0
\end{align*}
$$

Differentiating $V_{2}(t)$, we get

$$
\begin{align*}
\dot{V}_{2}(t) & =\eta^{T}(t) \mathrm{Q} \eta(t)-(1-\dot{\tau}(t)) \eta^{T}(t-\tau(t)) \mathrm{Q} \eta(t-\tau(t)) \\
& \leq \eta^{T}(t) \mathrm{Q} \eta(t)-\left(1-h_{\tau}\right) \eta^{T}(t-\tau(t)) \mathrm{Q} \eta(t-\tau(t)) \tag{17}
\end{align*}
$$

Differentiating $V_{3}(t)$ from Lemma 5 we have

$$
\begin{align*}
\dot{V}_{3}(t)= & \theta(t) \eta^{T}(t) G \eta(t)-(1-\dot{\theta}(t)) \int_{t-\theta(t)}^{t} \eta^{T}(s) G \eta(s) d s \\
\leq & \theta_{\max } \eta^{T}(t) G \eta(t) \\
& -\frac{1-h_{\theta}}{\theta_{\min }}\left(\int_{t-\theta(t)}^{t} \eta(s) d s\right)^{T} G \int_{t-\theta(t)}^{t} \eta(s) d s \tag{18}
\end{align*}
$$

Take $Q=\left(1 /\left(1-h_{\tau}\right)\right) \bar{B}^{T} \bar{B}, G=\left(\theta_{\min } /\left(1-h_{\theta}\right)\right) \bar{D}^{T} \bar{D}$. From the definition of $V(t)$ we reach the following inequality:

$$
\begin{align*}
\dot{V}(t) \leq \alpha \eta^{T}(t)[ & \frac{1}{\alpha}(\|C\|+\|A\| h+1) I_{N} \\
& +\|\Phi\| \widehat{U}^{s}+\frac{1}{\alpha\left(1-h_{\tau}\right)} \bar{B}^{T} \bar{B}  \tag{19}\\
& \left.+\frac{\theta_{\min } \theta_{\max }}{\alpha\left(1-h_{\theta}\right)} \bar{D}^{T} \bar{D}-K\right] \eta(t) .
\end{align*}
$$

Let $k_{i}=\lambda_{\max }\left((1 / \alpha)(\|C\|+\|A\| h+1) I_{N}+\|\Phi\| \widehat{U}^{s}+\right.$ $\left.\left(1 / \alpha\left(1-h_{\tau}\right)\right) \bar{B}^{T} \bar{B}+\left(\theta_{\min } \theta_{\max } /\left(\alpha\left(1-h_{\theta}\right)\right)\right) \bar{D}^{T} \bar{D}\right)+1$, where $\lambda_{\max }\left((1 / \alpha)(\|C\|+\|A\| h+1) I_{N}+\|\Phi\| \widehat{U}^{s}+\left(1 / \alpha\left(1-h_{\tau}\right)\right) \bar{B}^{T} \bar{B}+\right.$ $\left.\left(\theta_{\text {min }} \theta_{\text {max }} / \alpha\left(1-h_{\theta}\right)\right) \bar{D}^{T} \bar{D}\right)$ denotes the maximum eigenvalue of $(1 / \alpha)(\|C\|+\|A\| h+1) I_{N}+\|\Phi\| \widehat{U}^{s}+\left(1 / \alpha\left(1-h_{\tau}\right)\right) \bar{B}^{T} \bar{B}+$ $\left(\theta_{\text {min }} \theta_{\text {max }} / \alpha\left(1-h_{\theta}\right)\right) \bar{D}^{T} \bar{D}$. Then, from the previous inequality, we get

$$
\begin{equation*}
\dot{V}(t) \leq-\alpha \eta^{T}(t) \eta(t) \tag{20}
\end{equation*}
$$

Integrating both sides of the previous equation from 0 to $t$ yields

$$
\begin{equation*}
V(0) \geq V(t)+\alpha \sum_{i=1}^{N} \int_{0}^{t}\left\|e_{i}(s)\right\|^{2} d s \geq \alpha \sum_{i=1}^{N} \int_{0}^{t}\left\|e_{i}(s)\right\|^{2} d s \tag{21}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\alpha \lim _{t \rightarrow \infty} \sum_{i=1}^{N} \int_{0}^{t}\left\|e_{i}(s)\right\|^{2} d s \leq V(0) \tag{22}
\end{equation*}
$$

In view of Lemma 6 and the previous inequality, one can easily get

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sum_{i=1}^{N}\left\|e_{i}(t)\right\|^{2}=0 \tag{23}
\end{equation*}
$$

which in turn means

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left\|e_{i}(t)\right\|=0, \quad i=1,2, \ldots, N \tag{24}
\end{equation*}
$$

This completes the proof.

## 4. Synchronization with Partial Nodes Perturbed

Usually, only partial nodes of complex networks are perturbed. If some important nodes are perturbed, then the entire network will not work correctly. Theoretically speaking, nodes with larger degree (undirected networks) or larger outdegree (directed networks) are more vulnerable to perturbation [39], since the states of these nodes have more effect on networks than those with smaller degree (undirected networks) or outdegree (directed networks). On the other hand, the real-world complex networks normally
have a large number of nodes; it is usually impractical and impossible to control a complex networks by adding the controllers to all nodes. Therefore, from both practical point of view and the view of reducing control cost, we can use the scheme of pinning control [27-29, 40-42] to prevent external perturbations and synchronize complex networks.

In this section, we assume that matrix $U$ is irreducible in the sense that there is no isolate cluster in the network and there are $l_{1}$ nodes affected by external perturbations.

Without loss of generality, rearrange the order of the nodes in the network, and take the first $l\left(l \geq l_{1}\right)$ nodes to be controlled. Thus, the pinning controlled network can be described as

$$
\begin{aligned}
& \dot{x}_{i}(t)=C x_{i}(t)+A f\left(x_{i}(t)\right)+B f\left(x_{i}(t-\tau(t))\right) \\
& +D \int_{t-\theta(t)}^{t} f\left(x_{i}(s)\right) d s+\alpha \sum_{j=1}^{N} u_{i j} \Phi x_{j}(t) \\
& +I(t)+\beta \sum_{j=1}^{N} v_{i j} \Upsilon x_{j}(t-\tau(t)) \\
& +\gamma \sum_{j=1}^{N} w_{i j} \Lambda \int_{t-\theta(t)}^{t} x_{j}(s) d s+\sigma_{i}(t) \\
& +R_{i}, \quad i=1,2, \ldots, l_{1}, \\
& \dot{x}_{i}(t)=C x_{i}(t)+A f\left(x_{i}(t)\right)+B f\left(x_{i}(t-\tau(t))\right) \\
& +D \int_{t-\theta(t)}^{t} f\left(x_{i}(s)\right) d s+\alpha \sum_{j=1}^{N} u_{i j} \Phi x_{j}(t) \\
& +I(t)+\beta \sum_{j=1}^{N} v_{i j} \Upsilon x_{j}(t-\tau(t)) \\
& +\gamma \sum_{j=1}^{N} w_{i j} \Lambda \int_{t-\theta(t)}^{t} x_{j}(s) d s+R_{i}, \\
& i=l_{1}+1, l_{1}+2, \ldots, l, \\
& \dot{x}_{i}(t)=C x_{i}(t)+A f\left(x_{i}(t)\right)+B f\left(x_{i}(t-\tau(t))\right) \\
& +D \int_{t-\theta(t)}^{t} f\left(x_{i}(s)\right) d s+\alpha \sum_{j=1}^{N} u_{i j} \Phi x_{j}(t) \\
& +I(t)+\beta \sum_{j=1}^{N} v_{i j} \Upsilon x_{j}(t-\tau(t)) \\
& +\gamma \sum_{j=1}^{N} w_{i j} \Lambda \int_{t-\theta(t)}^{t} x_{j}(s) d s, \\
& i=l+1, l+2, \ldots, N,
\end{aligned}
$$



Figure 1: Chaotic trajectory of (44).

Let $e_{i}(t)=x_{i}(t)-z(t)$. Subtracting (4) from (25) we obtain the following error dynamical system:

$$
\begin{align*}
& \dot{e}_{i}(t)= C e_{i}(t)+A g\left(e_{i}(t)\right)+B g\left(e_{i}(t-\tau(t))\right) \\
&+D \int_{t-\theta(t)}^{t} g\left(e_{i}(s)\right) d s+\alpha \sum_{j=1}^{N} u_{i j} \Phi e_{j}(t) \\
&+\beta \sum_{j=1}^{N} v_{i j} \curlyvee e_{j}(t-\tau(t))+\gamma \sum_{j=1}^{N} w_{i j} \Lambda \int_{t-\theta(t)}^{t} e_{j}(s) d s \\
&+\sigma_{i}(t)+R_{i}, \quad i=1,2, \ldots, l_{1}, \\
& \dot{e}_{i}(t)= C e_{i}(t)+A g\left(e_{i}(t)\right)+B g\left(e_{i}(t-\tau(t))\right) \\
&+D \int_{t-\theta(t)}^{t} g\left(e_{i}(s)\right) d s+\alpha \sum_{j=1}^{N} u_{i j} \Phi e_{j}(t) \\
&+\beta \sum_{j=1}^{N} v_{i j} \curlyvee e_{j}(t-\tau(t))+\gamma \sum_{j=1}^{N} w_{i j} \Lambda \int_{t-\theta(t)}^{t} e_{j}(s) d s \\
&+R_{i}, \quad i=l_{1}+1, l_{1}+2, \ldots, l, \\
& \dot{e}_{i}(t)= C e_{i}(t)+A g\left(e_{i}(t)\right)+B g\left(e_{i}(t-\tau(t))\right) \\
&+D \int_{t-\theta(t)}^{t} g\left(e_{i}(s)\right) d s+\alpha \sum_{j=1}^{N} u_{i j} \Phi e_{j}(t) \\
&+\beta \sum_{j=1}^{N} v_{i j} \curlyvee e_{j}(t-\tau(t))+\gamma \sum_{j=1}^{N} w_{i j} \Lambda \int_{t-\theta(t)}^{t} e_{j}(s) d s, \\
& i=l+1, l+2, \ldots, N . \tag{26}
\end{align*}
$$

Similar to Theorem 7, to reach the goal (7), we have only to prove that system (26) is asymptotically stable at the origin.


Figure 2: Chaotic trajectories of (46) with $\sigma_{1}(t)(\mathrm{a}), \sigma_{2}(t)(\mathrm{b}), \sigma_{3}(t)(\mathrm{c})$.

Theorem 8. Suppose that matrix $U$ is irreducible and the assumptions $\left(H_{1}\right)$ and $\left(H_{2}\right)$ hold. If

$$
\begin{equation*}
2 \alpha\|\Phi\|\left(\widehat{U}^{s}\right)_{l}^{c}+\Sigma I_{N-l}<0 \tag{27}
\end{equation*}
$$

then the complex networks (25) are synchronized with the adaptive pinning controllers

$$
\begin{align*}
& R_{i}=-\alpha \varepsilon_{i} e_{i}(t)-\omega \beta_{i} \operatorname{sign}\left(e_{i}(t)\right), \\
& \dot{\varepsilon}_{i}=p_{i} e_{i}(t)^{T} e_{i}(t),  \tag{28}\\
& \dot{\beta}_{i}=\xi_{i} \sum_{k=1}^{n}\left|e_{i k}(t)\right|
\end{align*}
$$

where $i=1,2, \ldots, l, \Sigma=2(\|C\|+\|A\| h)+\left(\left(\theta_{\max } \theta_{\min }+\right.\right.$ $\left.\left.1-h_{\theta}\right) /\left(1-h_{\theta}\right)\right)\left\|\|D\| h I_{N}+\gamma\right\| \Lambda\left\|\|W \mid\|+\left(\left(2-h_{\tau}\right) /(1-\right.\right.$ $\left.\left.h_{\tau}\right)\right)\left\|\|B\| h I_{N}+\beta\right\| \Upsilon\||V|\|$, and the other parameters are the same as those of Theorem 7.

Proof. We define another Lyapunov function as

$$
\begin{equation*}
\bar{V}(t)=\bar{V}_{1}(t)+V_{2}(t)+V_{3}(t), \tag{29}
\end{equation*}
$$

where

$$
\begin{align*}
\bar{V}_{1}(t)= & \frac{1}{2} \sum_{i=1}^{N} e_{i}^{T}(t) e_{i}(t) \\
& +\frac{1}{2} \sum_{i=1}^{l} \frac{\alpha\left(\varepsilon_{i}-k_{i}\right)^{2}}{p_{i}}+\frac{1}{2} \sum_{i=1}^{l_{1}} \frac{\left(M_{i}-\beta_{i}\right)^{2}}{\xi_{i}} \tag{30}
\end{align*}
$$

$k_{i}, i=1,2, \ldots, l$, are constants to be determined, and $V_{2}(t)$ and $V_{3}(t)$ are defined as those in the proof of Theorem 7.

In view of $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$, differentiating $\bar{V}_{1}(t)$ along the solution of (26) yields

$$
\begin{aligned}
\dot{\bar{V}}_{1}(t)= & \sum_{i=1}^{N} e_{i}^{T}(t) \dot{e}_{i}(t)+\alpha \sum_{i=1}^{l}\left(\varepsilon_{i}-k_{i}\right) e_{i}^{T}(t) e_{i}(t) \\
& -\sum_{i=1}^{l_{1}}\left(M_{i}-\beta_{i}\right) \sum_{k=1}^{n}\left|e_{i k}(t)\right| \\
\leq & \sum_{i=1}^{N}\left[(\|C\|+\|A\| h)\left\|e_{i}(t)\right\|^{2}\right. \\
& +\|B\| h\left\|e_{i}(t)\right\|\left\|e_{i}(t-\tau(t))\right\| \\
& \left.+\|D\| h\left\|e_{i}(t)\right\| \int_{t-\theta(t)}^{t}\left\|e_{i}(s)\right\| d s\right] \\
& -\alpha \sum_{i=1}^{l} k_{i}\left\|e_{i}(t)\right\|^{2}+\alpha \sum_{i=1}^{N} \lambda_{\min }^{\Phi^{s}} u_{i i}\left\|e_{i}(t)\right\|^{2} \\
& +\alpha \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} u_{i j}\|\Phi\|\left\|e_{i}(t)\right\|\left\|e_{j}(t)\right\| \\
& +\beta \sum_{i=1}^{N} \sum_{j=1}^{N}\left|v_{i j}\right|\|\Upsilon\|\left\|e_{i}(t)\right\|\left\|e_{j}(t-\tau(t))\right\|
\end{aligned}
$$



Figure 3: Synchronization errors of (47): (a) $e_{i 1}(t)$, (b) $e_{i 2}(t)$, (c) $e_{i 3}(t), i=1,2,3$.


Figure 4: The adaptive control gains of (47): (a) the adaptive gains $\varepsilon_{i}, 1 \leq i \leq 3$, (b) the adaptive gains $\beta_{i}, 1 \leq i \leq 3$.

$$
\begin{align*}
& +\gamma \sum_{i=1}^{N} \sum_{j=1}^{N}\left|w_{i j}\right|\|\Lambda\|\left\|e_{i}(t)\right\| \int_{t-\theta(t)}^{t}\left\|e_{j}(s)\right\| d s \\
& =\eta^{T}(t)\left((\|C\|+\|A\| h) I_{N}+\alpha\left(\|\Phi\| \widehat{U}^{s}-\bar{K}\right)\right) \eta(t) \\
& +\eta^{T}(t)\left(\|B\| h I_{N}+\beta\|\Upsilon\||V|\right) \eta(t-\tau(t)) \\
& +\eta^{T}(t)\left(\|D\| h I_{N}+\gamma\|\Lambda\||W|\right) \int_{t-\theta(t)}^{t} \eta(s) d s \tag{31}
\end{align*}
$$

where $\bar{K}=\operatorname{diag}(k_{1}, \ldots, k_{l}, \underbrace{0, \ldots, 0}_{N-l})$, and the following deduction is used:

$$
\begin{align*}
& \sum_{i=1}^{l_{1}} e_{i}^{T}(t) \sigma_{i}(t)-\omega \sum_{i=1}^{l_{1}} \sum_{k=1}^{n} \beta_{i}\left|e_{i k}(t)\right| \\
& \quad-\sum_{i=1}^{l_{1}} \sum_{k=1}^{n}\left(M_{i}-\beta_{i}\right)\left|e_{i k}(t)\right|-\omega \sum_{i=l_{1}+1}^{l} \sum_{k=1}^{n} \beta_{i}\left|e_{i k}(t)\right| \\
& \leq \sum_{i=1}^{l_{1}} \sum_{k=1}^{n}\left[\left|e_{i k}(t)\right| M_{i k}-M_{i}\left|e_{i k}\right|-(\omega-1) \beta_{i}\left|e_{i k}\right|\right]  \tag{32}\\
& \quad-\omega \sum_{i=l_{1}+1}^{l} \sum_{k=1}^{n} \beta_{i}\left|e_{i k}(t)\right| \\
& \leq-\sum_{i=1}^{l_{1}} \sum_{k=1}^{n}(\omega-1) \beta_{i}\left|e_{i k}\right|-\omega \sum_{i=l_{1}+1}^{l} \sum_{k=1}^{n} \beta_{i}\left|e_{i k}(t)\right| \leq 0
\end{align*}
$$

Combining (31) with (17) and (25), we have

$$
\begin{equation*}
\dot{\bar{V}}(t) \leq \frac{1}{2} \zeta^{T}(t) \Pi \zeta(t) \tag{33}
\end{equation*}
$$

where $\zeta(t)=\left(\eta^{T}(t), \eta^{T}(t-\tau(t)),\left(\int_{t-\theta(t)}^{t} \eta(s) d s\right)^{T}\right)^{T}$ and

$$
\boldsymbol{\Pi}=\left[\begin{array}{ccc}
\Pi_{11} & \Pi_{12} & \Pi_{13}  \tag{34}\\
\Pi_{12}^{T} & -2\left(1-h_{\tau}\right) Q & 0 \\
\Pi_{13}^{T} & 0 & -\frac{2\left(1-h_{\theta}\right)}{\theta_{\min }} G
\end{array}\right]
$$

with $\Pi_{11}=2\left((\|C\|+\|A\| h) I_{N}+\alpha\left(\|\Phi\| \widehat{U}^{s}-K\right)+Q+\theta_{\max } G\right)$, $\Pi_{12}=\|B\| h I_{N}+\beta\|\Upsilon\||V|, \Pi_{13}=\|D\| h I_{N}+\gamma\|\Lambda\| \| W \mid$.

According to Lemma $4, \Pi<0$ is equivalent to

$$
\begin{align*}
\Delta= & 2\left((\|C\|+\|A\| h) I_{N}+\alpha\left(\|\Phi\| \widehat{U}^{s}-K\right)+\right. \\
& +\frac{1}{2\left(1-h_{\tau}\right)}\left(\|B\| h I_{N}+\beta\|\Upsilon\||V|\right) Q^{-1} \\
& \times\left(\|B\| h I_{N}+\beta\|\Upsilon\||V|^{T}\right) \\
& +\frac{\theta_{\min }}{2\left(1-h_{\theta}\right)}\left(\|D\| h I_{N}+\gamma\|\Lambda\||W|\right) G^{-1}  \tag{35}\\
& \times\left(\|D\| h I_{N}+\gamma\|\Lambda\||W|^{T}\right)<0
\end{align*}
$$



Figure 5: Chaotic trajectory of (49).


Figure 6: Chaotic trajectory of (51).

Let $\mathrm{Q}=\left(1 / 2\left(1-h_{\tau}\right)\right)\| \| B\| \| I_{N}+\beta\|\Upsilon\|\|V \mid\| I_{N}, G=\left(\theta_{\text {min }} / 2(1-\right.$ $\left.\left.h_{\theta}\right)\right)\|\|D\|\| I_{N}+\gamma\|\Lambda\|\|W \mid\| I_{N}$. We have

$$
\begin{align*}
\Delta \leq & 2(\|C\|+\|A\| h) I_{N}+2 \alpha\left(\|\Phi\| \widehat{U}^{s}-K\right) \\
& +\frac{\theta_{\max } \theta_{\min }+1-h_{\theta}}{1-h_{\theta}}\| \| D\left\|h I_{N}+\gamma\right\| \Lambda\||W|\| I_{N} \\
& +\frac{2-h_{\tau}}{1-h_{\tau}}\| \| B\left\|h I_{N}+\beta\right\| \Upsilon\||V|\| I_{N-1}  \tag{36}\\
= & {\left[\begin{array}{cc}
\Delta_{11} & 2 \alpha\|\Phi\|\left(\widehat{U}^{s}\right)_{*} \\
2 \alpha\|\Phi\|\left(\widehat{U}^{s}\right)_{*}^{T} & 2 \alpha\|\Phi\|\left(\widehat{U}^{s}\right)_{l}^{c}+\Sigma I_{N-l}
\end{array}\right] }
\end{align*}
$$

where $\Delta_{11}=2 \alpha\|\Phi\|\left(\widehat{U}^{s}\right)_{l}-2 \alpha K_{l}+\Sigma I_{l}, 2 \alpha\|\Phi\|\left(\widehat{U}^{s}\right)_{*}$ is matrix with appropriate dimension.


Figure 7: WS Small-Worlds with 10 nodes. In (a) each node connects 4 nodes, and the rewire probability is 0.2 ; in (b) each node connects 2 nodes, and the rewire probability is 0.4 .


Figure 8: Synchronization errors of (52): (a) $e_{i 1}$, (b) $e_{i 2}, 1 \leq i \leq 10$.

Since $2 \alpha\|\Phi\|\left(\widehat{U}^{s}\right)_{l}^{c}+\Sigma I_{N-l}<0$ and there exist positive constants $k_{1}, k_{2}, \ldots, k_{l}$ such that

$$
\begin{align*}
& 2 \alpha\|\Phi\|\left(\widehat{U}^{s}\right)_{l}-2 \alpha K_{l}+\Sigma I_{l}-(2 \alpha\|\Phi\|)^{2}\left(\widehat{U}^{s}\right)_{*} \\
& \quad \times\left(2 \alpha\|\Phi\|\left(\widehat{U}^{s}\right)_{l}^{c}+\Sigma I_{N-l}\right)^{-1}\left(\widehat{U}^{s}\right)_{*}^{T}<0 \tag{37}
\end{align*}
$$

again, from Lemma 4 we obtain $\Delta<0$. Hence, $\Pi<0$. Denote $\lambda_{\text {min }}$ to be the minimum eigenvalue of $-\Pi$; then

$$
\begin{equation*}
\dot{\bar{V}}(t) \leq-\lambda_{\min } \sum_{i=1}^{N}\left\|e_{i}(t)\right\|^{2} \leq 0 \tag{38}
\end{equation*}
$$

Integrating both sides of the previous equation from 0 to $t$ yields

$$
\begin{align*}
\bar{V}(0) & \geq \bar{V}(t)+\lambda_{\min } \sum_{i=1}^{N} \int_{0}^{t}\left\|e_{i}(s)\right\|^{2} d s \\
& \geq \lambda_{\min } \sum_{i=1}^{N} \int_{0}^{t}\left\|e_{i}(s)\right\|^{2} d s \tag{39}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \lambda_{\min } \sum_{i=1}^{N} \int_{0}^{t}\left\|e_{i}(s)\right\|^{2} d s \leq \bar{V}(0) \tag{40}
\end{equation*}
$$

By Lemma 6 we obtain

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \lambda_{\min } \sum_{i=1}^{N}\left\|e_{i}(t)\right\|^{2}=0 \tag{41}
\end{equation*}
$$



Figure 9: The adaptive pinning control gains of (52): (a) the adaptive pinning control gains $\varepsilon_{i}$, $i=1,2$. (b) The adaptive pinning control gains $\beta_{i}, i=1,2$.
which in turn means

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left\|e_{i}(t)\right\|=0, \quad i=1,2, \ldots, N \tag{42}
\end{equation*}
$$

This completes the proof.
When there is no external perturbation, that is, $\sigma_{i}(t)=$ $0, i=1,2, \ldots, N$, one can easily get the following corollaries from Theorems 7 and 8 , respectively. We omit their proofs here.

Corollary 9. Suppose that $\sigma_{i}(t)=0, i=1,2, \ldots, N$, and the assumption condition $\left(H_{1}\right)$ holds. Then complex networks (1) are synchronized with the adaptive controllers (12). Moreover, the scalar $\omega$ can be relaxed to any positive constant.

Corollary 10. Suppose that matrix $U$ is irreducible and the assumption $\left(H_{1}\right)$ holds. The complex networks (25) are synchronized with the adaptive pinning controllers (28), if (27) holds. Moreover, the scalar $\omega$ can be relaxed to any positive constant.

Remark 11. From the inequalities (16) and (32) one can see that the designed adaptive controllers (12) and (28) are very useful. They can overcome the bad effects of the uncertain nonlinear perturbations without knowing the exact functions and bounds of the perturbations as long as the perturbed systems are chaotic. Especially, when there are only partial nodes perturbed (the first $l_{1}$ nodes in the system (25)), the designed controllers still are effective to stabilize the error system by adding them to nodes with and without such perturbations, (see the inequality (32)). Obviously, in the case of no perturbation, the parameter $\omega$ can also be taken as 0 . When $\omega=0$, the controllers (12) and (28) turn out to be the usual adaptive controller, which is extensively utilized to synchronize coupled systems with or without stochastic perturbations [8, 23-34, 40-42]. However, the controllers in [8, 23-34, 40-42] cannot synchronize coupled systems
with nonstochastic perturbations. Therefore, the designed controllers can deal with both stochastic and nonstochastic perturbations to the systems, and hence they have better robustness than usual adaptive controllers.

Remark 12. Model (1) can be extended to the following more general complex networks:

$$
\begin{array}{r}
\dot{x}_{i}(t)=C x_{i}(t)+A f\left(x_{i}(t)\right)+B f_{\tau}\left(x_{i}(t-\tau(t))\right) \\
+D \int_{t-\theta(t)}^{t} f_{\theta}\left(x_{i}(s)\right) d s+I(t) \\
+\alpha \sum_{j=1}^{N} u_{i j} \Phi x_{j}(t)+\beta \sum_{j=1}^{N} v_{i j} \Upsilon x_{j}(t-\tau(t))  \tag{43}\\
+\gamma \sum_{j=1}^{N} w_{i j} \Lambda \int_{t-\theta(t)}^{t} x_{j}(s) d s+\sigma_{i}(t)+R_{i} \\
i=0,1, \ldots, N
\end{array}
$$

Moreover, we can also consider stochastic perturbations [21] and Markovian jump [43, 44] in (43) to get more general results. For simplicity, we omit the corresponding results and only consider model (1).

## 5. Numerical Examples

In this section, we provide two examples to illustrate the general model and the advantage of the new adaptive controller.

Example 13. The Lorenz system is described as

$$
\begin{equation*}
\dot{z}(t)=C z(t)+A f(z(t)) \tag{44}
\end{equation*}
$$

where

$$
C=\left[\begin{array}{ccc}
-10 & 10 & 0  \tag{45}\\
28 & -1 & 0 \\
0 & 0 & 8 / 3
\end{array}\right], \quad A=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

$f(z(t))=\left(0,-z_{1}(t) z_{3}(t), z_{1}(t) z_{2}(t)\right)^{T}$. When initial values are taken as $z_{1}(0)=0.8, z_{2}(0)=2, z_{3}(0)=2.5$, chaotic trajectory of (44) can be seen in Figure 1.

The following three perturbed Lorenz systems are chaotic:

$$
\begin{equation*}
\dot{\bar{z}}(t)=C \bar{z}(t)+A f(\bar{z}(t))+\sigma_{i}(t), \quad i=1,2,3 \tag{46}
\end{equation*}
$$

where $\sigma_{1}(t)=\left(0.1 \bar{z}_{1}^{2}(t), 0.2 \bar{z}_{2}(t), 0.2 \bar{z}_{3}(t)\right)^{T}, \sigma_{2}(t)=$ $\left(0.1 \bar{z}_{1}(t), 0.05 \bar{z}_{2}^{2}(t), \sin \bar{z}_{3}(t)\right)^{T}, \sigma_{3}(t)=\left(0.1 \bar{z}_{1}(t), \cos \bar{z}_{2}(t)\right.$, $\left.\sin \bar{z}_{3}(t)\right)^{T}$. Chaotic trajectories of the three perturbed Lorenz systems are showed in Figure 2 with the same initial values $\bar{z}_{1}(0)=0.8, \bar{z}_{2}(0)=2, \bar{z}_{3}(0)=2.5$.

Now consider the following complex networks with each node as the previous perturbed Lorenz system:

$$
\begin{align*}
\dot{x}_{i}(t)= & C x_{i}(t)+A f\left(x_{i}(t)\right) \\
& +\alpha \sum_{j=1}^{N} u_{i j} \Phi x_{j}(t)+\sigma_{i}(t)+R_{i}, \quad i=0,1,3 \tag{47}
\end{align*}
$$

where $\alpha=0.5, \Phi=I_{3}$ and

$$
U=\left[\begin{array}{ccc}
-1 & 1 & 0  \tag{48}\\
1 & -1 & 0 \\
1 & 1 & -2
\end{array}\right]
$$

Obviously, conditions $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$ are satisfied. According to Theorem 7, the complex networks (47) can be synchronized with adaptive controllers (12).

The initial conditions of the numerical simulations are as follows: $\omega=4$, step $=0.0005, x_{1}(0)=(-2,-1,0)^{T}$, $x_{2}(0)=(1,2,3)^{T}, x_{3}(0)=(4,5,6)^{T}, \varepsilon_{i}(0)=1, \beta_{i}(0)=2, p_{i}=$ $\xi_{i}=0.5, i=1,2,3$. Figure 3 describes the synchronization errors $e_{i j}(t)=x_{i j}(t)-z_{j}(t), i, j=1,2,3$. Figure 4 shows the adaptive feedback gains. Numerical simulations verify the effectiveness of Theorem 7.

Example 14. Consider the following chaotic neural networks with mixed delays:

$$
\begin{align*}
\dot{z}(t)= & C z(t)+A f(z(t))+B f(z(t-\tau(t))) \\
& +D \int_{t-\sigma(t)}^{t} f(z(s)) d s+I(t) \tag{49}
\end{align*}
$$

where $z(t)=\left(z_{1}(t), z_{2}(t)\right)^{T}, \tau(t)=1, \sigma(t)=0.3, f(z(t))=$ $\left(\tanh \left(z_{1}(t), \tanh \left(z_{2}(t)\right)^{T}\right.\right.$,

$$
\begin{align*}
& C=\left[\begin{array}{cc}
-1.2 & 0 \\
0 & -1
\end{array}\right], \quad A=\left[\begin{array}{cc}
3 & -0.3 \\
8 & 5
\end{array}\right], \\
& I=\left[\begin{array}{l}
0 \\
2
\end{array}\right], \quad B=\left[\begin{array}{cc}
-1.4 & 0.1 \\
0.3 & -8
\end{array}\right], \quad D=\left[\begin{array}{cc}
-1.2 & 0.1 \\
-2.8 & -1
\end{array}\right] . \tag{50}
\end{align*}
$$

In the case that the initial condition is chosen as $z_{1}(t)=0.4$, $z_{2}(t)=0.6$, for all $t \in[-1,0]$, the chaotic attractor can be seen in Figure 5.

The perturbed system of (49) is

$$
\begin{align*}
\dot{\bar{z}}(t)= & C \bar{z}(t)+A f(\bar{z}(t))+B f(\bar{z}(t-\tau(t))) \\
& +D \int_{t-\sigma(t)}^{t} f(\bar{z}(s)) d s+I(t)+\sigma(t) \tag{51}
\end{align*}
$$

where $\sigma(t)=\left(0.2 \bar{z}_{1}(t-1), 0.2 \int_{t-0.3}^{t} \bar{z}_{2}(s) d s\right)^{T}$. The chaotic attractor of (51) can be seen in Figure 6 with $\bar{z}_{1}(t)=0.4$, $\bar{z}_{2}(t)=0.6$, for all $t \in[-1,0]$.

Now consider the following complex networks with each node as the previous neural networks with mixed delays (49), while the second node is disturbed with the previous $\sigma(t)$.

$$
\begin{align*}
\dot{x}_{i}(t)= & C x_{i}(t)+A f\left(x_{i}(t)\right)+B f\left(x_{i}(t-\tau(t))\right) \\
& +D \int_{t-\theta(t)}^{t} f\left(x_{i}(s)\right) d s+I(t)+\alpha \sum_{j=1}^{10} u_{i j} \Phi x_{j}(t) \\
& +\beta \sum_{j=1}^{10} v_{i j} \curlyvee x_{j}(t-\tau(t))+\gamma \sum_{j=1}^{10} w_{i j} \Lambda \int_{t-\theta(t)}^{t} x_{j}(s) d s \\
& +\sigma_{i}(t)+R_{i}, \quad i=0,1, \ldots, 10, \tag{52}
\end{align*}
$$

where $\alpha=3, \beta=\gamma=1, \sigma_{2}(t)=\sigma(t)$, else $\sigma_{i}(t)=0$.
Figure 7 depicts the WS Small-World networks [2] corresponding to nondelay (a), discrete delay, and distributed delay (b). The corresponding Laplacian matrices are shown as following:

$$
\begin{align*}
& U=\left[\begin{array}{cccccccccc}
-4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & -5 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3
\end{array}\right], \quad(5  \tag{53}\\
& V=W=\left[\begin{array}{cccccccccc}
-2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -3 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -3 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2
\end{array}\right] . \tag{54}
\end{align*}
$$

Take the first two nodes (corresponding to matrix $U$ ) to be controlled. According Theorem 8, the complex networks (52) can be synchronized with adaptive controllers (27).

The initial conditions of the numerical simulations are as follows: $\omega=2$, step $=0.0005, x_{i}(0)=(-11+2 i,-10+$ $2 i)^{T}, i=1,2, \ldots, 10 . \varepsilon_{i}(0)=\beta_{i}(0)=p_{i}=\xi_{i}=1, i=$ 1,2 . Figure 8 describes the synchronization errors $e_{i j}(t)=$ $x_{i j}(t)-z_{j}(t), i=1,2, \ldots, 10, j=1,2$. Figure 9 depicts the adaptive feedback gains. Numerical simulations verify the effectiveness of Theorem 8.

## 6. Conclusions

External perturbations to networks are unavoidable in practice. On the other hand, many chaotic models have discrete delay and distributed delay. Therefore, in this paper, we introduced a class of hybrid coupled complex networks with mixed delays and unknown nonstochastic external perturbations. A simple robust adaptive controller is designed to synchronize the complex networks even without knowing a priori the bounds and the exact functions of the perturbations. It should be emphasized that we do not assume that the coupling matrix is symmetric or diagonal. The controller can enhance robustness and reduce fragility of complex networks; hence, it has great practical significance. Moreover, we also verify the effectiveness of the theoretical results by numerical simulations.

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## Research Article

# Limit Cycles and Integrability in a Class of Systems with High-Order Nilpotent Critical Points 

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#### Abstract

A class of polynomial differential systems with high-order nilpotent critical points are investigated in this paper. Those systems could be changed into systems with an element critical point. The center conditions and bifurcation of limit cycles could be obtained by classical methods. Finally, an example was given; with the help of computer algebra system MATHEMATICA, the first 5 Lyapunov constants are deduced. As a result, sufficient and necessary conditions in order to have a center are obtained. The fact that there exist 5 small amplitude limit cycles created from the high-order nilpotent critical point is also proved.


## 1. Introduction

In the qualitative theory of ordinary differential equations, bifurcation of limit cycles for planar polynomial differential systems which belong to second part of the Hilbert 16th problem is hot but intractable issue. Of course, this problem is far from being solved now; there are many famous works about this problem. Let $H(n)$ be the maximum possible number of limit cycles bifurcating from infinity for analytic vector fields of degree $n$. It was found that $N(3) \geq 7[1,2]$, $N(5) \geq 11[3], N(7) \geq 13[4]$.

When the critical point is a nilpotent critical point, let $N(n)$ be the maximum possible number of limit cycles bifurcating from nilpotent critical points for analytic vector fields of degree $n$. It was found that $N(3) \geq 2, N(5) \geq 5$, $N(7) \geq 9$ in [5], $N(3) \geq 3, N(5) \geq 5$ in [6], and for a family of Kukles systems with 6 parameters $N(3) \geq 3$ in [7]. Recently, li and liu other found that $N(3) \geq 8, N(5) \geq 12, N(7) \geq 13$ $[8,9]$ employing the inverse integral factor method.

But when the order of nilpotent critical point is higher than three, it is very difficult to study the limit cycle because the inverse integral factor method could not be used to compute the singular values. There are few results about the bifurcations of limit cycles at a nilpotent critical point with high order.

In this paper, we will study the bifurcation of limit cycles and conditions of centers for a class of special system

$$
\begin{align*}
& \frac{d x}{d t}=y+\sum_{k=2}^{\infty} f_{k}\left(x^{n}, y\right)  \tag{1}\\
& \frac{d y}{d t}=-x^{2 n-1}+x^{n-1} \sum_{k=2}^{\infty} g_{k}\left(x^{n}, y\right)
\end{align*}
$$

Obviously, when $n=1$, the system is

$$
\begin{align*}
& \frac{d x}{d t}=y+\sum_{k=2}^{\infty} f_{k}(x, y)  \tag{2}\\
& \frac{d y}{d t}=-x+\sum_{k=2}^{\infty} g_{k}(x, y)
\end{align*}
$$

The origin is an element critical point, but it is a high order nilpotent critical point when $n>1$.

This paper will be organized as follows. In Section 2, we state some preliminary knowledge given in [10] which is useful throughout the paper. In Section 3, we make some transformations to change system (2) into a system with an element singular. As an example, a special system is investigated. Using the linear recursive formulae in [10] to do
direct computation, we obtain the first 5 Lyapunov constants and the sufficient and necessary conditions of center and the result that there exist 5 limit cycles in the neighborhood of the high-order nilpotent critical point are proved.

## 2. Preliminary Knowledge

The ideas of this section come from [10], where the center focus problem of critical points in the planar dynamical systems are studied. We first recall the related notions and results. For more details, please refer to [10].

In [10], the authors defined complex center and complex isochronous center for the following complex system:

$$
\begin{align*}
& \frac{d z}{d T}=z+\sum_{k=2}^{\infty} \sum_{\alpha+\beta=k} a_{\alpha \beta} z^{\alpha} w^{\beta}=Z(z, w),  \tag{3}\\
& \frac{d w}{d T}=-w-\sum_{k=2}^{\infty} \sum_{\alpha+\beta=k} b_{\alpha \beta} w^{\alpha} z^{\beta}=-W(z, w),
\end{align*}
$$

and gave two recursive algorithms to determine necessary conditions for a center and an isochronous center. We now restate the definitions and algorithms.

Lemma 1. For system (3), we can derive uniquely the following formal series:

$$
\begin{align*}
& \xi=z+\sum_{k+j=2}^{\infty} c_{k j} z^{k} w^{j} \\
& \eta=w+\sum_{k+j=2}^{\infty} d_{k j} w^{k} z^{j} \tag{4}
\end{align*}
$$

where $c_{k+1, k}=d_{k+1, k}=0, k=1,2, \ldots$, such that

$$
\begin{align*}
& \frac{d \xi}{d T}=\xi+\sum_{j=1}^{\infty} p_{j} \xi^{j+1} \eta^{j} \\
& \frac{d \eta}{d T}=-\eta-\sum_{j=1}^{\infty} q_{j} \eta^{j+1} \xi^{j} \tag{5}
\end{align*}
$$

Definition 2. Let $\mu_{0}=0, \mu_{k}=p_{k}-q_{k}, \tau_{k}=p_{k}+q_{k}, k=1,2, \ldots$. $\mu_{k}$ is called the $k_{\mathrm{th}}$ singular point quantity of the origin of system (3) and $\tau_{k}$ is called the $k_{\text {th }}$ period constant of the origin of system (3).

Theorem 3. For system (3), the origin is a complex center if and only if $\mu_{k}=0, k=1,2, \ldots$. The origin is a complex isochronous center if and only if $\mu_{k}=\tau_{k}=0, k=1,2, \ldots$.

Theorem 4. For system (3), we can derive successively the terms of the following formal series:

$$
\begin{equation*}
M(z, w)=\sum_{\alpha+\beta=0}^{\infty} c_{\alpha \beta} z^{\alpha} w^{\beta}, \tag{6}
\end{equation*}
$$

such that

$$
\begin{equation*}
\frac{\partial(M Z)}{\partial z}-\frac{\partial(M W)}{\partial w}=\sum_{m=1}^{\infty}(m+1) \mu_{m}(z w)^{m} \tag{7}
\end{equation*}
$$

where $c_{00}=1$, for all $c_{k k} \in R, k=1,2, \ldots$, and for any integer $m, \mu_{m}$ is determined by the following recursive formulae:

$$
\begin{align*}
& c_{\alpha \beta}=\frac{1}{\beta-\alpha} \sum_{k+j=3}^{\alpha+\beta+2}\left[(\alpha+1) a_{k, j-1}\right. \\
& \left.-(\beta+1) b_{j, k-1}\right] c_{\alpha-k+1, \beta-j+1},  \tag{8}\\
& \mu_{m}=\sum_{k+j=3}^{2 m+2}\left(a_{k, j-1}-b_{j, k-1}\right) c_{m-k+1, m-j+1} .
\end{align*}
$$

Theorem 5 (the constructive theorem of singular point quantities in [10]). A $k$-order singular point quantity of system (3) at the origin can be represented as a linear combination ofkorder monomial Lie invariants and their antisymmetry forms, that is,

$$
\begin{equation*}
\mu_{k}=\sum_{j=1}^{N} \gamma_{k j}\left(g_{k j}-g_{k j}^{*}\right), \quad k=1,2, \ldots, \tag{9}
\end{equation*}
$$

where $N$ is a positive integer and $\gamma_{k j}$ is a rational number, $g_{k j}$ and $g_{k j}^{*}$ are $k$-order monomial Lie invariants of system (3).

Theorem 6 (the extended symmetric principle in [10]). Let $g$ denote an elementary Lie invariant of system (3). If for all $g$ the symmetric condition $g=g^{*}$ is satisfied, then the origin of system (3) is a complex center. Namely, all singular point quantities of the origin are zero.

## 3. Simplification of System

In fact, system (1) is symmetric with axis when $n$ is even, so the origin is a center. Now we will consider system (1) when $n=2 k+1$ is odd.

By transformations

$$
\begin{equation*}
u=x^{2 k+1}, \quad v=y, \quad d \tau=u^{2 k /(2 k+1)} d t \tag{10}
\end{equation*}
$$

system (1) is changed into

$$
\begin{align*}
& \frac{d u}{d \tau}=(2 k+1) v+(2 k+1) \sum_{k=2}^{\infty} f_{k}(u, v)  \tag{11}\\
& \frac{d v}{d \tau}=-u+\sum_{k=2}^{\infty} g_{k}(u, v)
\end{align*}
$$

then by transformations

$$
\begin{equation*}
x=u, \quad y=\sqrt{2 k+1} v, \quad d \tau=\sqrt{2 k+1} d t \tag{12}
\end{equation*}
$$

system (11) could be changed into

$$
\begin{align*}
& \frac{d x}{d \tau}=-y-\sqrt{2 k+1} \sum_{k=2}^{\infty} f_{k}\left(x, \frac{1}{\sqrt{2 k+1}} y\right), \\
& \frac{d v}{d \tau}=x-\sum_{k=2}^{\infty} g_{k}\left(x, \frac{1}{\sqrt{2 k+1}} y\right) . \tag{13}
\end{align*}
$$

Thus, the origin of system (13) is an element critical point. It could be investigated using the classical integral factor method.

Now, we consider the following system:

$$
\begin{align*}
& \frac{d x}{d t}=y+A_{30} x^{3 n}+A_{21} x^{2 n} y+A_{12} x^{n} y^{2}+A_{03} y^{3} \\
& \frac{d y}{d t}=-x^{2 n-1}+x^{n-1}\left(B_{30} x^{3 n}+B_{21} x^{2 n} y+B_{12} x^{n} y^{2}+B_{03} y^{3}\right) \tag{14}
\end{align*}
$$

When $n=2 k+1$, by those transformations, system (14) is changed into

$$
\begin{align*}
& \frac{d x}{d t}=-y-\sqrt{2 k+1}\left(A_{30} x^{3}+\frac{A_{21}}{\sqrt{2 k+1}} x^{2} y\right. \\
& \left.+\frac{A_{12}}{2 k+1} x y^{2}+\frac{A_{03}}{(2 k+1)^{3 / 2}}\right) \\
& \frac{d y}{d t}=x-\left(B_{30} x^{3}+\frac{B_{21}}{\sqrt{2 k+1}} x^{2} y+\frac{B_{12}}{2 k+1} x y^{2}\right.  \tag{15}\\
& \left.+\frac{B_{03}}{(2 k+1)^{3 / 2}} y^{3}\right)
\end{align*}
$$

where

$$
\begin{gather*}
A_{30}=\frac{-2 A_{2}-2 A_{3}-3 A_{4}+2 A_{2} k+A_{4} k}{16(1+2 k)^{3}}, \\
A_{21}=\frac{-9 B_{1}+B_{2}+B_{3}+3 B_{1} k}{8(1+2 k)^{5 / 2}}, \\
A_{12}=\frac{-2 A_{2}-2 A_{3}+9 A_{4}+2 A_{2} k-3 A_{4} k}{16(1+2 k)^{2}}, \\
A_{03}=\frac{3 B_{1}+B_{2}+B_{3}-B_{1} k}{8(1+2 k)^{3 / 2}},  \tag{16}\\
B_{21}=\frac{-2 A_{2}+2 A_{3}-45 A_{4}+2 A_{2} k+15 A_{4} k}{16(1+2 k)^{2}}, \\
B_{12}=\frac{-9 B_{1}-B_{2}+B_{3}+3 B_{1} k}{8(1+2 k)^{3 / 2}}, \\
B_{03}= \\
\frac{-2 A_{2}+2 A_{3}+15 A_{4}+2 A_{2} k-5 A_{4} k}{16(1+2 k)} .
\end{gather*}
$$

By transformation

$$
\begin{equation*}
z=x+i y, \quad w=x-i y, \quad T=i t \tag{17}
\end{equation*}
$$

system (15) is changed into

$$
\begin{aligned}
& \frac{d z}{d T}=z+a_{30} z^{3}+a_{21} z^{2} w+a_{12} z w^{2}+a_{03} w^{3} \\
& \frac{d w}{d T}=-w-b_{30} w^{3}-b_{21} w^{2} z-b_{12} w z^{2}-b_{03} z^{3}
\end{aligned}
$$

where

$$
\begin{gather*}
a_{30}=\frac{\left(3 i A_{4}+2 B_{1}\right)(k-3)}{16(1+2 k)^{5 / 2}}, \\
a_{21}=\frac{-i A_{2}+B_{2}+i A_{2} k}{8(1+2 k)^{5 / 2}}, \\
a_{12}=-\frac{i\left(A_{3}-i B_{3}\right)}{8(1+2 k)^{5 / 2}},  \tag{19}\\
a_{03}=-\frac{i A_{4}(k-3)}{8(1+2 k)^{5 / 2}}, \quad b_{i j}=\overline{a_{i j}} .
\end{gather*}
$$

After careful computation by using formula in (4), we have the following.

Theorem 7. For system (18), the first 5 Lyapunov constants at the origin are given by

$$
\begin{align*}
& \lambda_{1}=\frac{i A_{2}(k-1)}{4(1+2 k)^{5 / 2}} \\
& \lambda_{2}=\frac{i\left(2 A_{3} B_{1}+3 A_{4} B_{3}\right)(k-3)}{64(1+2 k)^{5}} \tag{20}
\end{align*}
$$

When $A_{4} B_{1} \neq 0$

$$
\begin{align*}
& \lambda_{3}=\frac{i A_{4}\left(3 A_{4}-2 B_{1}\right)\left(3 A_{4}+2 B_{1}\right)(k-3)(-9+3 k-2 p)(-3+k+6 p)}{8192(1+2 k)^{15 / 2}}, \\
& \lambda_{4}=\frac{i A_{4} B_{2}\left(3 A_{4}-2 B_{1}\right)\left(3 A_{4}+2 B_{1}\right)(k-3)^{2}(-9+3 k-2 p)}{49152(1+2 k)^{10}}, \\
& \lambda_{5}=\frac{i A_{4} B_{1}^{2}\left(3 A_{4}-2 B_{1}\right)\left(3 A_{4}+2 B_{1}\right)(k-3)^{4}(-9+3 k-2 p)}{2654208(1+2 k)^{25 / 2}} . \tag{21}
\end{align*}
$$

When $A_{4}=0, B_{1} \neq 0$

$$
\begin{equation*}
\lambda_{2}=\frac{i A_{3} B_{1}(k-3)}{32(1+2 k)^{5}} \tag{22}
\end{equation*}
$$

When $A_{4} \neq 0, B_{1}=0$

$$
\begin{align*}
& \lambda_{2}=\frac{3 i A_{4} B_{3}(k-3)}{64(1+2 k)^{5}}, \\
& \lambda_{3}=\frac{3 i A_{4}\left(2 A_{3}-3 A_{4}+A_{4} k\right)\left(-2 A_{3}-27 A_{4}+9 A_{4} k\right)(k-3)}{8192(1+2 k)^{15 / 2}}, \\
& \lambda_{4}=-\frac{i A_{4} B_{2}\left(-2 A_{3}-27 A_{4}+9 A_{4} k\right)(k-3)^{2}}{49152(1+2 k)^{10}} . \tag{23}
\end{align*}
$$

When $A_{4}=B_{1}=0$

$$
\begin{equation*}
\lambda_{2}=\lambda_{3}=\lambda_{4}=\cdots=0 \tag{24}
\end{equation*}
$$

In the above expression of $\lambda_{k}$, one has already let $\lambda_{1}=\lambda_{2}=$ $\lambda_{3}=\lambda_{4}=0$.

From Theorem 7, we obtain the following assertion.
Proposition 8. The first 5 Lyapunov constants at the origin of system (18) are zero if and only if one of the following conditions is satisfied:

$$
\begin{align*}
& k=3, \quad A_{2}=0, \\
& A_{2}=0, \quad A_{3}=\frac{3(3 k-9)}{2} A_{4}, \quad B_{3}=-2(3 k-9) B_{1},  \tag{26}\\
& A_{2}=0, \quad 2 A_{3} B_{1}=-3 A_{4} B_{3}, \quad B_{1}=-\frac{3}{2} A_{4},  \tag{27}\\
& A_{2}=0, \quad 2 A_{3} B_{1}=-3 A_{4} B_{3}, \quad B_{1}=\frac{3}{2} A_{4},  \tag{28}\\
& A_{2}=A_{3}=A_{4}=0,  \tag{29}\\
& A_{2}=B_{1}=B_{2}=B_{3}=0, \quad A_{3}=-\frac{k-3}{2} A_{4},  \tag{30}\\
& k=1, \quad A_{3}=-9 A_{4}, \quad B_{3}=6 B_{1},  \tag{31}\\
& k=1, \quad 2 A_{3} B_{1}=-3 A_{4} B_{3}, \quad B_{1}=-\frac{3}{2} A_{4},  \tag{32}\\
& k=1, \quad 2 A_{3} B_{1}=-3 A_{4} B_{3}, \quad B_{1}=\frac{3}{2} A_{4},  \tag{33}\\
& k=1, \quad A_{3}=A_{4}=0,  \tag{34}\\
& k=1, \quad B_{1}=B_{2}=B_{3}=0, \quad A_{3}=A_{4} . \tag{35}
\end{align*}
$$

Furthermore, we have the following.
Theorem 9. The origin of system (18) is a center if and only if the first 5 Lyapunov constants are zero; that is, one of the conditions in Proposition 8 is satisfied.

Proof. When one of conditions (25), (27), (28), (29), (30), (32), (33), and (34) holds, according to Theorems 6, we get all $\mu_{k}=0, k=1,2 \ldots$

When condition (26) holds, system (18) could be written as

$$
\begin{aligned}
\frac{d x}{d t}= & -y+\frac{(k-3) A_{4}}{2(1+2 k)^{5 / 2}} x^{3}-\frac{B_{2}}{8(1+2 k)^{5 / 2}} x^{2} y \\
& +\frac{3(k-3) A_{4}}{4(1+2 k)^{5 / 2}} x y^{2}+\frac{4 B_{1} k-B_{2}-12 B_{1}}{8(1+2 k)^{5 / 2}} y^{3}, \\
\frac{d y}{d t}= & x+\frac{4 B_{1} k+B_{2}-12 B_{1}}{8(1+2 k)^{5 / 2}} x^{3}-\frac{3(k-3) A_{4}}{2(1+2 k)^{5 / 2}} x^{2} y \\
& +\frac{B_{2}}{8(1+2 k)^{5 / 2}} x y^{2}-\frac{(n-3) A_{4}}{4(1+2 k)^{5 / 2}} y^{3} .
\end{aligned}
$$

System (36) has an analytic first integral

$$
\begin{align*}
H(x, y)= & \frac{1}{2} x^{2}+\frac{1}{2} y^{2}+\frac{4 B_{1} n+B_{2}-12 B_{1}}{32(1+2 k)^{5 / 2}} x^{4} \\
& -\frac{(k-3) A_{4}}{2(1+2 k)^{5 / 2}} x^{3} y+\frac{B_{2}}{16(1+2 k)^{5 / 2}} x^{2} y^{2} \\
& -\frac{k-3}{4(1+2 k)^{5 / 2}} A_{4} x y^{3}-\frac{4 B_{1} n-B_{2}-12 B_{1}}{32(1+2 k)^{5 / 2}} y^{4} . \tag{37}
\end{align*}
$$

When condition (31) holds, system (18) could be written as

$$
\begin{align*}
\frac{d x}{d t}= & -y+\sqrt{3}\left(\frac{A_{4}}{27} x^{3}-\frac{B_{2}}{216} x^{2} y\right. \\
& \left.\quad+\frac{A_{4}}{18} x y^{2}-\frac{-8 B_{1}+B_{2}}{216} y^{3}\right) \\
\frac{d y}{d t}= & x+\frac{-8 B_{1}+B_{2}}{72 \sqrt{3}} x^{3}+\frac{A_{4}}{3 \sqrt{3}} x^{2} y  \tag{38}\\
& +\frac{B_{2}}{72 \sqrt{3}} x y^{2}+\frac{A_{4}}{18 \sqrt{3}} y^{3}
\end{align*}
$$

System (38) has an analytic first integral

$$
\begin{align*}
H(x, y)= & \frac{1}{2} x^{2}+\frac{1}{2} y^{2}-\frac{-8 B_{1}-B_{2}}{864} y^{4}+\frac{\sqrt{3}\left(-8 B_{1}+B_{2}\right)}{864} x^{4} \\
& +\frac{\sqrt{3}}{432} B_{2} x^{2} y^{2}+\frac{\sqrt{3} A_{4}}{9} x^{3} y . \tag{39}
\end{align*}
$$

When condition (35) holds, system (18) could be written as

$$
\begin{align*}
& \frac{d x}{d t}=-y+\frac{\sqrt{3} A_{4}}{108} x^{3}-\frac{\sqrt{3} A_{4}}{108} x y^{2} \\
& \frac{d y}{d t}=x+\frac{7 \sqrt{3} A_{4}}{108} x^{2} y-\frac{\sqrt{3} A_{4}}{36} y^{3} \tag{40}
\end{align*}
$$

System (40) has an analytic first integral

$$
\begin{equation*}
H(x, y)=\frac{1}{2} x^{2}+\frac{1}{2} y^{2}-\frac{\sqrt{3} A_{4}}{108} x^{3} y-\frac{\sqrt{3} A_{4}}{36} x y^{3} . \tag{41}
\end{equation*}
$$

Next, we will prove that when the critical point $O(0,0)$ is a 5 -order weak focus, the perturbed system of (15) can generate 5 limit cycles enclosing the origin of perturbation system (15).

Using the fact

$$
\begin{equation*}
\lambda_{1}=\lambda_{2}=\lambda_{3}=\lambda_{4}=0, \quad \lambda_{5} \neq 0 \tag{42}
\end{equation*}
$$

we obtain the following.

Theorem 10. The origin of system (18) is a 5-order weak focus if and only if one of the following conditions is satisfied:

$$
\begin{gather*}
A_{2}=0, \quad A_{3}=\frac{3-k}{2} A_{4}, \quad B_{3}=-\frac{3-k}{3} B_{1}, \quad B_{2}=0, \\
A_{4} B_{1}^{2}\left(3 A_{4}-2 B_{1}\right)\left(3 A_{4}+2 B_{1}\right)(k-3) \neq 0,  \tag{43}\\
k=1, \quad A_{3}=A_{4}, \quad B_{3}=-\frac{2}{3} B_{1}, \quad B_{2}=0,  \tag{44}\\
A_{4} B_{1}\left(3 A_{4}-2 B_{1}\right)\left(3 A_{4}+2 B_{1}\right) \neq 0 .
\end{gather*}
$$

We next study the perturbed system of (15) as follows:

$$
\begin{gather*}
\frac{d x}{d t}=\delta x-y-\sqrt{2 k+1}\left(A_{30} x^{3}+\frac{A_{21}}{\sqrt{2 k+1}} x^{2} y\right. \\
\left.+\frac{A_{12}}{2 k+1} x y^{2}+\frac{A_{03}}{(2 k+1)^{3 / 2}}\right) \\
\frac{d y}{d t}=\delta y+x-\left(B_{30} x^{3}+\frac{B_{21}}{\sqrt{2 k+1}} x^{2} y\right. \\
\left.+\frac{B_{12}}{2 k+1} x y^{2}+\frac{B_{03}}{(2 k+1)^{3 / 2}}\right) \tag{45}
\end{gather*}
$$

Theorem 11. If the origin of system (15) is a 5 -order weak focus, for $0<\delta \ll 1$, making a small perturbation to the coefficients of system (15), then, for system (45), in a small neighborhood of the origin, there exist exactly 5 small amplitude limit cycles enclosing the origin $O(0,0)$.

Proof. It is easy to check that when condition (43) or (44) holds,

$$
\begin{equation*}
\frac{\partial\left(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right)}{\partial\left(A_{2}, A_{3}, B_{3}, B_{2}\right)} \neq 0 \tag{46}
\end{equation*}
$$

From the statement mentioned above, according to the classical theory of Bautin, there exist 5 limit cycles in a small enough neighborhood of the origin.

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## Research Article

# Random Dynamics of the Stochastic Boussinesq Equations Driven by Lévy Noises 

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#### Abstract

This paper is devoted to the investigation of random dynamics of the stochastic Boussinesq equations driven by Lévy noise. Some fundamental properties of a subordinator Lévy process and the stochastic integral with respect to a Lévy process are discussed, and then the existence, uniqueness, regularity, and the random dynamical system generated by the stochastic Boussinesq equations are established. Finally, some discussions on the global weak solution of the stochastic Boussinesq equations driven by general Lévy noise are also presented.


## 1. Introduction

Dynamical systems driven by non-Gaussian processes, such as Lévy processes, have attracted a lot of attention recently. Ordinary differential equations driven by Lévy processes have been summarized in [1]. Peszat and Zabczyk [2] have presented a basic framework for partial differential equations driven by Lévy processes.

The Navier-Stokes fluid equations are often coupled with other equations, especially, with the scalar transport equations for fluid density, salinity, or temperature. These coupled equations model a variety of phenomena arising in environmental, geophysical, and climate systems. The related Boussinesq fluid equations [3-5] under Gaussian fluctuations have been recently studied, for example, existence and uniqueness of solutions [6], stochastic flow, dynamical impact under random dynamical boundary conditions [7, 8], and large deviation principles $[9,10]$, among others.

Motivated by a recent work on a simple stochastic partial differential equation with Lévy noise [11], we study the stochastic Boussinesq equations driven by some special Lévy noises, and we consider the random dynamics of this stochastic system. Specifically, for a given bounded $C^{1}$ smooth domain $D \subset \mathbb{R}^{2}$ with sufficient smooth boundary,
we consider the following stochastic Boussinesq equations driven by subordinator Lévy noise:

$$
\begin{gather*}
\frac{d u}{d t}=\left(\frac{1}{\operatorname{Re}} \Delta u-\nabla p-u \cdot \nabla u-\frac{1}{\mathrm{Fr}^{2}} \theta e_{2}\right)+d Y_{1}(t) \\
\quad \text { on } D \times \mathbb{R}_{+}, \\
\frac{d \theta}{d t}=\left(\frac{1}{\operatorname{RePr}} \Delta \theta-u \cdot \nabla \theta\right)+d Y_{2}(t), \quad \text { on } D \times \mathbb{R}_{+},  \tag{1}\\
\operatorname{div} u=0, \quad \text { on } D \times \mathbb{R}_{+}, \\
u(0)=u_{0}, \quad \theta(0)=\theta_{0},
\end{gather*}
$$

where $u=u(x, t)=\left(u^{1}, u^{2}\right) \in \mathbb{R}^{2}$ is the velocity vector, $\theta=\theta(t, x) \in \mathbb{R}$ is salinity, $p(t, x) \in \mathbb{R}$ is the pressure term, $x=(\xi, \eta) \in D \subset \mathbb{R}^{2}, \Delta$ denotes the Laplacian operator, and $\nabla$ denotes the gradient operator. Moreover, Fr is the Froude number, Re is the Reynolds number, Pr is the Prandtl number, and $e_{2} \in \mathbb{R}^{2}$ is a unit vector in the upward vertical direction. The initial data $u_{0}, \theta_{0}$ are given. Both $Y_{1}(t)$ and $Y_{2}(t)$ are subordinator Lévy processes on Hilbert spaces $H_{1}$ and $H_{2}$, which will be specified later. The present paper is
devoted to the existence, uniqueness, regularity, and the cocycle property of solution for stochastic Boussinesq equations (1).

This paper is organized as follows. In Section 2, we first present some properties of the subordinator Lévy process $Y(t)$, then review some fundamental properties of the stochastic integral with respect to Lévy process $Y(t)$. Section 3 is devoted to the existence, uniqueness, regularity, and the cocycle property of the stochastic Boussinesq equations. Finally, some discussions on the global weak solution of stochastic Boussinesq equations driven by general Lévy noise are also presented in Section 4.

## 2. Preliminaries

In this section, we introduce some operators and fraction spaces and then present some properties of the subordinator Lévy process $Y(t)$ and the stochastic integral with respect to Lévy process $Y(t)$.

In order to reformulate the stochastic Boussinesq equations (1) as an abstract stochastic evolution, we introduce the following function spaces.

Denote $L^{2}(D)$ to be the space of functions defined on $D$, which are $L^{2}$-integrable with respect to the Lebesgue measure $d x=d x_{1} d x_{2}$, endowed with the usual scalar product and norm, that is, for $u, v \in L^{2}(D)$,

$$
\begin{equation*}
(u, v)=\int_{D} u(x) v(x) d x, \quad|u|=\{(u, u)\}^{1 / 2} . \tag{2}
\end{equation*}
$$

For $m \in \mathbb{Z}^{+} \cup\{0\}$ and $q \in(1, \infty)$, define

$$
\begin{align*}
H^{m, q} & (D) \\
\quad & =\left\{u \in L^{q}(D): D^{\alpha} u \in L^{q}(D), \alpha \in \mathbb{N}^{2}, 0 \leq|\alpha| \leq m\right\} \tag{3}
\end{align*}
$$

as the usual Soblev space with scalar product

$$
\begin{equation*}
(u, v)_{m}=\sum_{0 \leq|\alpha| \leq m}\left(D^{\alpha} u, D^{\alpha} v\right)_{L^{q}(D)} \tag{4}
\end{equation*}
$$

and the induced norm

$$
\begin{equation*}
|u|_{m}=\|u\|_{H^{m}(D)}=\left(\sum_{0 \leq|\alpha| \leq m}\left|D^{\alpha} u\right|_{L^{q}(D)}^{q}\right)^{1 / 2} \tag{5}
\end{equation*}
$$

where $D^{\alpha} u$ is the $\alpha$ th order weak derivative of $u$.
For $s \in \mathbb{R}$, let $H^{s, q}(D)$ be defined by the complex interpolation method [12] as follows.

$$
\begin{equation*}
H^{\beta, q}(D)=\left[H^{k, q}(D), H^{m, q}(D)\right]_{\theta}, \tag{6}
\end{equation*}
$$

where $k, m \in \mathbb{N}, \theta \in(0,1)$, and $k<m$ are chosen to satisfy

$$
\begin{equation*}
\beta=(1-\theta) k+\theta m . \tag{7}
\end{equation*}
$$

The closure of $C_{0}^{\infty}(D)$ in the Banach space $H^{s, q}(D), s \geq 0$, $q \in(1, \infty)$, will be denoted by $H_{0}^{s, q}(D)$.

The following product spaces are needed:

$$
\begin{align*}
& \mathscr{V}=\left\{u=\left(u^{1}, u^{2}\right) \in\left(C^{\infty}(D)\right)^{2} \times C^{\infty}(D), \nabla \cdot u=0\right\}, \\
& \mathbb{L}^{q}(D)=\left(L^{q}(D)\right)^{2} \times L^{q}(D), \\
& \mathbb{H}^{s, q}(D)=\left(H^{s, q}(D)\right)^{2} \times H^{s, q}(D) \\
&=\left\{u=\left(u^{1}, u^{2}\right) \in\left(H^{s, q}(D)\right)^{2} \times H^{s, q}(D),\right. \\
&\nabla \cdot u=0\}, \\
& \mathbb{V}^{s, q}(D)=\left\{u=\left(u^{1}, u^{2}\right) \in \mathbb{H}_{0}^{s, q}(D), \nabla \cdot u=0\right\} . \tag{8}
\end{align*}
$$

Let $\mathbf{H}^{s, q}(D)$ denote the closure of $\mathscr{V}$ with respect to the $\mathbb{H}^{s, q}$-norm, $V^{s, q}(D)$ denote the closure of $\mathscr{V}$ with respect to the $\mathbb{V}^{s, q}$-norm, and $V^{\prime}$ be the dual space of $V^{s, q}(D)$. In particular, we denote by $\mathbf{H}^{1,2}$ and $V^{1,2} H$ and $V$, respectively.

Denote

$$
\begin{gather*}
A_{1} u=\Delta u(t), \quad A_{2} \theta=\Delta \theta(t), \\
B_{1}\left(u_{1}, u_{2}\right)=\left(u_{1} \cdot \nabla\right) u_{2}, \quad B_{2}\left(u_{1}, \theta_{2}\right)=\left(u_{1} \cdot \nabla\right) \theta_{2}, \\
U_{0}=\binom{u_{0}}{\theta_{0}} \in H, \quad U(t)=\binom{u(t)}{\theta(t)} \in V, \\
R(U)=\binom{-\frac{1}{\mathrm{Fr}^{2}} \theta e_{2},}{0},  \tag{9}\\
Y(t)=\binom{Y_{1}(t)}{Y_{2}(t)} \in H=H_{1} \times H_{2},
\end{gather*}
$$

where $v=1 / \operatorname{Re}$ and $k=1 / \operatorname{Re} \operatorname{Pr}$.
Now, we define the following two operators:

$$
\begin{gather*}
A: V \longrightarrow V^{\prime}: V \ni U=(u, \theta) \longmapsto A U=\binom{v A_{1} u}{k A_{2} \theta}, \\
B: V \times V \longrightarrow V^{\prime}: V \times V \ni\left(U_{1}, U_{2}\right) \tag{10}
\end{gather*}
$$

$$
\longmapsto B\left(U_{1}, U_{2}\right)=\binom{B_{1}\left(u_{1}, u_{2}\right)}{B_{2}\left(u_{1}, \theta_{2}\right)} .
$$

Then, the stochastic Boussinesq system (1) can be rewritten as the following abstract stochastic evolution equation:

$$
\begin{gather*}
d U(t)+[A U(t)+B(U(t), U(t))+R(U(t))] d t=d Y(t), \\
U(0)=U_{0} . \tag{11}
\end{gather*}
$$

In order to apply the technique of the reproducing Kernel Hilbert space, it is better to introduce the definition $\gamma$ radonifying.

Definition 1 (see [13]). Let $K$ and $X$ be Banach spaces, a bounded linear operator $L: K \rightarrow X$ is called $\gamma$-radonifying
if and only if $L\left(\gamma_{K}\right)$ is $\sigma$-additive, where $\gamma_{K}$ is the canonical cylindrical finitely additive set-valued function (also called a Gaussian distribution) on $K$.

The following is our standing assumption:
Assumption 1. Space $E \subset H \cap \mathbb{L}^{4}$ is a Hilbert space such that for some $\delta \in(0,1 / 2)$,

$$
\begin{equation*}
A^{-\delta}: E \longrightarrow H \cap \mathbb{Q}^{4} \text { is } \gamma \text {-radonifying. } \tag{12}
\end{equation*}
$$

Remark 2. Under the above assumption, we have the facts that $E \subset H$ and the Banach space $U$ is taken as $H \cap \mathbb{L}^{4}$ (see [11, 14, 15] for more details and related results). In fact, space $E$ is the reproducing kernel Hilbert space of noise $W(t)$ on $H \cap \mathbb{L}^{4}$.

It is well known that subordinators form the subclass of increasing Lévy processes, which can be thought of as a random model of time evolution (see [16]). We will present some properties of the subordinator Lévy process $Y(t), t \geq 0$, then review briefly the stochastic integral with respect to Lévy process $Y(t)$.

Definition 3 (see $[1,2,11]$ ). Let $E$ be a Banach space, and let $Y=(Y(t), t \geq 0)$ be an $E$-valued stochastic process defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Stochastic process $Y$ is called a Lévy process if
(L1) $Y(0)=0$, a.s.;
(L2) process $Y$ has independent and stationary increments; and
(L3) process $Y$ is stochastically continuous, that is, for all $\delta>0$ and for all $s \geq 0$,

$$
\begin{equation*}
\lim _{t \rightarrow s} \mathbb{P}(|Y(t)-Y(s)|>\delta)=0 \tag{13}
\end{equation*}
$$

A subordinator Lévy process is an increasing onedimensional Lévy process.

For $p>0, \operatorname{Sub}(p)$ denotes the set of all subordinator Lévy processes $Z$, whose intensity measure $\rho$ satisfies the condition $\int_{0}^{1} \eta^{p / 2} \rho(d \eta)<\infty$.

In the most interesting cases, the space $E$ is a subspace of $H$, that is, $E \subset H$, and

$$
\begin{equation*}
Y(t)=W(Z(t)), \quad t \geq 0, \tag{14}
\end{equation*}
$$

where $Z=(Z(t))_{t \geq 0}$ is an independent subordinator process belonging to class $\operatorname{Sub}(p), p \in(1,2], W=(W(t))_{t \geq 0}$ is an $H$ valued cylindrical Wiener process defined on some Banach space $U$.

We decompose the $H$-valued Lévy process $Y(t)$ into two parts $N_{1}(t)$ and $N_{2}(t)$, the first one with small jumps and the second one with (relatively) large jumps, that is,

$$
\begin{equation*}
Y(t)=N_{1}(t)+N_{2}(t), \quad t \geq 0 \tag{15}
\end{equation*}
$$

with $v$ being the intensity measure of Lévy process $Y, N_{1}$ being the Lévy process with the intensity measure:

$$
\begin{array}{r}
v_{1}(\Gamma)=v\left(\Gamma \cap B_{U}(0,1)\right), \quad \Gamma \in \mathscr{B}(U), \\
B_{U}(0,1) \text { denotes the unit ball in } U, \tag{16}
\end{array}
$$

and $N_{2}$ be the Lévy process with the intensity measure $\nu_{2}=$ $v-v_{1}$. Then $N_{2}$ can be defined as a compound Poisson process with the intensity measure $v_{2}$, and $N_{1}, N_{2}$ can be defined by the Poisson random measure $\pi$ which is defined as follows:

$$
\begin{equation*}
\pi([0,1] \times \Gamma)=\Sigma_{s \leq t} 1_{\Gamma} \Delta Y(s), \quad \Gamma \in \mathscr{B}(U) \tag{17}
\end{equation*}
$$

where $\Delta Y(s)=Y\left(s^{+}\right)-Y\left(s^{-}\right), s \geq 0$. Here, the symbol $\Delta$ denotes the increment of $Y$.

We assume that the process $Y$ is right-continuous with left-hand side limits. Thus

$$
\begin{equation*}
\Delta Y(s)=Y(s)-Y\left(s^{-}\right), \quad s \geq 0 \tag{18}
\end{equation*}
$$

Notice that as $\pi$ is a time homogenous Poisson random measure, $Y$ can be expressed as

$$
\begin{equation*}
Y(t)=\Sigma_{s \leq t} \Delta Y(s)=\int_{0}^{t} \int_{U} u \pi(d y, d s), \quad t \geq 0 \tag{19}
\end{equation*}
$$

Hence,

$$
\begin{align*}
& N_{1}(t)=\sum_{s \leq t} 1_{|\Delta Y(s)|<1} \Delta Y(s)=\int_{0}^{t} \int_{|u|<1} u \pi(d y, d s), \\
& N_{2}(t)=\Sigma_{s \leq t} 1_{|\Delta Y(s)| \geq 1} \Delta Y(s)=\int_{0}^{t} \int_{|u| \geq 1} u \pi(d y, d s) . \tag{20}
\end{align*}
$$

Assume that the operator $\Psi(t), t \in[0, T]$, is a strongly measurable function taking values in the space of all bounded linear operator from $U$ to $E$. Let $0<\tau_{1}<\tau_{2}<\tau_{3}<\cdots \rightarrow \infty$ be the jump times for $N_{2}$ and $\Delta N_{2}\left(\tau_{k}\right)=\Delta Y\left(\tau_{k}\right)=Y\left(\tau_{k}\right)-$ $Y\left(\tau_{k}-\right), k=1,2, \ldots$. Then, the stochastic integral of $\Psi(t)$ with respect to jump process $N_{2}(t), t \geq 0$, can be defined as

$$
\begin{equation*}
\int_{0}^{t} \Psi(s) d N_{2}(s)=\Sigma_{\tau_{k} \leq t} \Psi\left(\tau_{k}\right) \Delta N_{2}\left(\tau_{k}\right) \tag{21}
\end{equation*}
$$

Since the operator $\Psi$ is taking values in $E$, it follows from the decomposition of $Y$ that the sum of sequences is finite. Hence the stochastic integral of the operator $\Psi$ with respect to $N_{2}$ is taking values in $E$. Moreover, the stochastic integral of the operator $\Psi(t), t \in[0, T]$, with respect to Lévy process $Y$ can be defined by

$$
\begin{equation*}
\int_{0}^{t} \Psi(s) d Y(s)=\int_{0}^{t} \Psi(s) d N_{1}(s)+\int_{0}^{t} \Psi(s) d N_{2}(s) \tag{22}
\end{equation*}
$$

and takes values in $E$ as well (see [11] for more details).
Next, we recall some basic definitions and properties for general random dynamical systems, which are taken from [7]. Let $(H, d)$ be a complete separable metric space and $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition 4. $\left(\Omega, \mathscr{F}, \mathbb{P},(\theta)_{t \in \mathbb{R}}\right)$ is called a metric dynamical system if the mapping $\theta: \mathbb{R} \times \Omega \rightarrow \Omega$ is $(\mathscr{B}(\mathbb{R}) \times \mathscr{F}, \mathscr{F})$ measurable, $\theta_{0}=I, \theta_{s+t}=\theta_{s} \circ \theta_{t}$ for all $t, s \in R$, and $\theta_{t} \mathbb{P}=\mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 5. A random dynamical system (RDS) with time $T$ on $(H, d)$ over $\left\{\theta_{t}\right\}$ on $\left(\Omega, \mathscr{F}, \mathbb{P},\left(\theta_{t}\right)_{t \in R}\right)$ is a $\left(\mathscr{B}\left(R^{+}\right) \times \mathfrak{F} \times\right.$ $\mathscr{B}(H), B(H))$-measurable map:

$$
\begin{equation*}
\Pi: T \times H \times \Omega \longrightarrow H \times \Omega, \quad \Pi(t, s, \omega)=\left(S(t, \omega) x, \theta_{t} \omega\right) \tag{23}
\end{equation*}
$$

such that
(i) $S(0, \omega)=I d$ (identity on $H$ ) for any $\omega \in \Omega$,
(ii) (Cocycle property) $S(t+s, \omega)=S\left(t, \theta_{s} \omega\right) \circ S(s, \omega)$ for all $s, t \in T$ and $\omega \in \Omega$.

An RDS is said to be continuous or differentiable if for all $t \in T$, and an arbitrary outside outside $\mathbb{P}$-nullset $B \subset \Omega, w \in$ $B$ the map $S(t, \omega): H \rightarrow H$ is continuous or differentiable, respectively.

Assume that the bounded linear operator $A$ generates a $C_{0}$-semigroup $S=\left(e^{t A}\right)_{t \geq 0}$ on a Hilbert space $E$ and $Y$ defined on a filtered probability space $\left(\Omega, \mathscr{F},(\mathscr{F})_{t \geq 0}, \mathbb{P}\right)$ is a subordinator Lévy process taking values in a Hilbert space $U$.

Consider the following stochastic Langevin equation:

$$
\begin{gather*}
d X(t)=A X(t) d t+d Y(t), \quad t \geq t_{0} \\
X\left(t_{0}\right)=x \in E \tag{24}
\end{gather*}
$$

Definition 6. Let $x \in E$ be a square integrable $\mathscr{F}_{t_{0}}$-measurable random variable in $E$. A predicable process $X:\left[t_{0}, \infty\right) \times$ $\Omega \rightarrow E$ is called a mild solution of the Langevin equation (24) with initial data $\left(t_{0}, x\right)$ if it is an adapted $E$-valued stochastic process and satisfies

$$
\begin{equation*}
X(t)=S\left(t-t_{0}\right) x+\int_{t_{0}}^{t} S(t-s) d Y(s), \quad t \geq t_{0} \tag{25}
\end{equation*}
$$

It is well known that the Ornstein-Uhlenbeck process $X(t), t \geq 0$, has some important integrability. Here we need the Banach space to be of type $p$, for $p \in(1,2]$. First we recall the definition briefly (see [14] for more details).

Definition 7 (see [14]). For $p \in(1,2]$, the Banach space $E$ is called as type $p$, if and only if there exists a constant $K_{p}(E)>0$ such that for any finite sequence of symmetric independent identically distribution random variables $\varepsilon_{1}, \ldots, \varepsilon_{n}: \Omega \rightarrow$ $[-1,1], n \in \mathbb{N}$, and any finite sequence $x_{1}, \ldots, x_{n}$ from $E$, satisfying

$$
\begin{equation*}
\mathbb{E}\left|\sum_{i=1}^{n} \varepsilon_{i} x_{i}\right|^{p} \leq K_{p}(E) \sum_{i=1}^{n}\left|x_{i}\right|^{p} \tag{26}
\end{equation*}
$$

Moreover, if there exists a constant $L_{p}(E)>0$ such that for every $E$-valued martingale $\left\{M_{n}\right\}_{n=0}^{N}, N \in \mathbb{N}$, satisfying

$$
\begin{equation*}
\sup _{n} \mathbb{E}\left|M_{n}\right|^{p} \leq L_{p}(E) \sum_{n=0}^{N} \mathbb{E}\left|M_{n}-M_{n-1}\right|^{p}, \quad M_{-1}=0 \tag{27}
\end{equation*}
$$

the separable Banach space $E$ is called a separable martingale type $p$-Banach space.

Lemma 8 (see [11, Corollary 8.1, Proposition 8.4]). Assume that $p \in(1,2], Z$ is a subordinator Lévy process from the class $\operatorname{Sub}(p), E$ is a separable type $p$-Banach, $U$ is a separable Hilbert space $U, E \subset U$, and $W=(W(t))_{t \geq 0}$ is a $U$-valued Wiener process.

Define the $U$-valued Lévy process as

$$
\begin{equation*}
Y(t)=W(Z(t)), \quad t \geq 0 \tag{28}
\end{equation*}
$$

and define the process as

$$
\begin{equation*}
X(t)=\int_{0}^{t} e^{(t-s) A} d Y(s) \tag{29}
\end{equation*}
$$

Then, with probability 1, for all $T>0$,

$$
\begin{align*}
& \int_{0}^{T}|X(t)|_{E}^{p} d t<\infty \\
& \int_{0}^{T}|X(t)|_{L^{4}}^{4} d t<\infty . \tag{30}
\end{align*}
$$

We have the following existence and regularity results, which have been studied in $[2,11]$.

Theorem 9. Assume that $E=U, S=S(t), t \geq 0$ is the $C_{0}$ semigroup generated by the bounded linear operator $A$ in the space E. Then, if one of the following conditions is satisfied:
(i) $p \in(0,1]$ or
(ii) $p \in(1,2]$ and the Banach space $E$ is of separable martingale type $p$-Banach space,
the Langevin equation (24) admits one mild solution $X(t) \in E$, $t>0$. Moreover, if $p \in(1,2], S=S(t), t \geq 0$, is a $C_{0}$-group in the separable martingale type $p$-Banach space $E$, then the mild solution $X$ of the Langevin equation is a cádlág (rightcontinuous with left-hand side limits) process.

Proof. As $S=S(t), t \geq 0$, is a $C_{0}$-group in the separable martingale type $p$-Banach space $E$, the Hilbert space $H$ is the reproducing kernel Hilbert space of $W(1)$, and the embedding operator $i: H \hookrightarrow E$ satisfies the $\gamma$-radonifying property. The proof of Theorem 9 is just a simple application of Theorems 4.1 and 4.4 in [11].

## 3. Cocycle Property of the Stochastic Boussinesq Equations

In this section, we will show the existence, uniqueness, regularity, and the cocycle property of the stochastic Boussinesq equations (11).

It is well known that both $A_{1}$ and $A_{2}$ are positive definite, self-adjoint operators, and denote $D\left(A_{1}\right)$ and $D\left(A_{2}\right)$ to be the domains of $A_{1}$ and $A_{2}$, respectively. Hence, the domain of the operator $A$ can be represented as $D(A)=D\left(A_{1}\right) \times D\left(A_{2}\right)$.

It follows from Lemma 2.2 in [7] that there exists positive numbers $\mu_{1}, \mu_{2}$, such that

$$
\begin{gather*}
\left(A_{1} u, u\right) \geq \mu_{1}\|u\|_{\left(L^{2}\right)^{2}}^{2},  \tag{31}\\
\left(A_{2}(u, \theta),(u, \theta)\right) \geq \mu_{2}\|(u, \theta)\|^{2} .
\end{gather*}
$$

Let $\lambda=\min \left(\mu_{1}, \mu_{2}\right)$. Then

$$
\begin{equation*}
(A U, U) \geq \lambda\|U\|^{2} \tag{32}
\end{equation*}
$$

For any arbitrary $U, V, W \in \mathbb{V}$, we can define the following trilinear form $b: U \times V \times W \rightarrow \mathbb{R}$ by

$$
\begin{gather*}
b(u, v, w)=\langle B(u, v), w\rangle, \\
b(U, V, W)=b_{1}(u, v, w)+b_{2}(u, \widetilde{v}, \widetilde{w}), \\
b_{1}(u, v, w)=\int_{D} \sigma_{i, j}^{2} u_{i} \frac{\partial v_{j}}{\partial x_{i}} w_{j} d x,  \tag{33}\\
b_{2}(u, \widetilde{v}, \widetilde{w})=\int_{D} \sigma_{i}^{2} u_{i} \frac{\partial \widetilde{v}_{j}}{\partial x_{i}} \widetilde{w}_{j} d x .
\end{gather*}
$$

We have the following results.
Lemma 10 (see [7, Lemma 2.3]). If $U, V, W \in \mathbb{V}$, then

$$
\begin{align*}
b(U, V, W) & =-b(U, W, V) \\
(B(V, U), U) & =b(V, U, U)=0 \tag{34}
\end{align*}
$$

Lemma 11 (see [7, Lemma 2.4]). There exists a constant $c_{B}>$ 0 such that if $u \in V_{1}, \theta, \eta \in V_{2}, \phi=(u, \theta)$, then
(1) $\left|b_{1}(u, v, w)\right| \leq c_{B}\|u\|_{H^{1}}\|v\|_{H^{2}}\|w\|, u \in V, v \in D(A)$, $w \in H$,
(2) $\left|b_{1}(u, v, w)\right| \leq c_{B}\|u\|_{L^{2}}^{1 / 2}\|u\|_{H^{1}}^{1 / 2}\|v\|_{H^{2}}\|w\|^{1 / 2}\|w\|_{H^{1}}^{1 / 2}, u \in$ $V, v \in D(A), w \in V$,
(3) $\left|b_{1}(u, v, u)\right| \leq c_{B}\|u\|_{H^{1}}\|v\|_{H^{1}}\|v\|, u \in V, v \in V$,
(4) $\left|b_{2}(u, \theta, w)\right| \leq c_{B}\|u\|^{1 / 2}\|u\|_{H^{1}}^{1 / 2}\|\theta\|_{H^{1}}\|w\|^{1 / 2}\|w\|_{H^{1}}^{1 / 2}, u \in$ $V, \theta \in V, w \in V$,
(5) $\left|b_{2}(u, \theta, w)\right| \leq c_{B}\|u\|\|\theta\|_{H^{2}}\|w\|_{H^{1}}, u \in H, \theta \in D(A)$, $w \in V$.

Definition 12. An $H$-valued $\left(\mathscr{F}_{t}\right)_{t \geq 0}$ adapted and $\mathbf{H}^{4,2}(D)$ valued cádlág process $u(t)(t \geq 0)$ is considered as a solution to (11), if for each $T>0$,

$$
\begin{equation*}
\sup _{0 \leq t \leq T}|U(t)|_{H}^{2}+\int_{0}^{T}|U(t)|_{\mathbb{L}^{4}(D)}^{4} d t<\infty, \quad \text { a.s. } \tag{35}
\end{equation*}
$$

and for any $\psi \in V \cap \mathbf{H}^{2,2}(D)$, and for any $t>0$, $\mathbb{P}$-a.s.,

$$
\begin{align*}
& (U(t), \psi)-\left(U_{0}, \psi\right)-\int_{0}^{t}(U(s), \Delta \psi) d s \\
& \quad+\int_{0}^{t}(B(U, U), \psi(s)) d s+\int_{0}^{t}(R(U), \psi) d s=(\psi, Y(t)) \tag{36}
\end{align*}
$$

Denote
$\mathscr{H}^{1,2}(0, T)=\left\{\right.$ the space of all functions $v \in L^{2}(0, T ; V)$

$$
\begin{equation*}
\left.\cap \mathbb{H}^{2,2}(D) \text { satisfying } v^{\prime} \in L^{2}\left(0, T ; V^{\prime}\right)\right\} \tag{37}
\end{equation*}
$$

Lemma 13. Assume that $z \in L^{4}\left(0, T ; \mathbb{L}^{4}(D)\right), g \in L^{2}\left(0, T, V^{\prime}\right)$, and $v_{0} \in H$. Then there exists a unique $v \in \mathscr{H}^{1,2}(0, T)$ such that

$$
\begin{gather*}
\frac{d v}{d t}+A v+B(v, z)+B(z, v)+B(v, v)=g, \quad t \geq 0  \tag{38}\\
v(0)=v_{0}
\end{gather*}
$$

Moreover,

$$
\begin{gather*}
\sup _{t \in[0, T]}|v(t)|^{2} \leq K^{2} L^{2}, \quad \int_{0}^{T}|\nabla v(t)|^{2} d t \leq M^{2} \\
\int_{0}^{T}\left|v^{\prime}(t)\right|_{V^{\prime}}^{2} d t \leq N^{2}, \quad \int_{0}^{T}|v(t)|_{\mathbb{L}^{4}(D)}^{4} d t \leq 2 T^{1 / 2} K^{3} L^{3} M \tag{39}
\end{gather*}
$$

where

$$
\begin{gather*}
K^{2}=e^{2 \int_{0}^{T}|z(s)|_{⿺^{4}}^{4}} d s, \quad L^{2}=\left|v_{0}\right|^{2}+2 \int_{0}^{T}|g(s)|_{V^{\prime}}^{2} d s, \\
M^{2}=\left|v_{0}\right|^{2}+9 K L \int_{0}^{T}|z(t)|_{L^{4}\left(0, T, \mathbb{L}^{4}(D)\right)}^{2}+\frac{T^{1 / 4}}{\sqrt{2}} K^{3 / 2} L^{1 / 2}, \tag{40}
\end{gather*}
$$

and the mapping $L^{2}\left(0, T, V^{\prime}\right) \times H \ni\left(g_{0}, v_{0}\right) \mapsto v \in \mathscr{H}^{1,2}(0, T)$ is analytic.

Proof. It can be shown by the same approach as the one in Proposition 8.7 in [11].

Lemma 14 (see [2, Proposition 10.1]). Let $u:[0, T] \rightarrow B$ be a continuous function whose left derivative

$$
\begin{equation*}
\frac{d^{-} u}{d t}\left(t_{0}\right)=\lim _{\epsilon \rightarrow 0, \epsilon<0} \frac{u\left(t_{0}+\epsilon\right)-u\left(t_{0}\right)}{\epsilon} \tag{41}
\end{equation*}
$$

exists at $t_{0} \in[0, T]$. Then the function $\gamma(t)=|u(t)|_{B}, t \in[0, T]$, is left differentiable at $t_{0}$ and

$$
\begin{equation*}
\frac{d^{-} \gamma}{d t}\left(t_{0}\right)=\min \left\{\left\langle x^{*}, \frac{d^{-} u}{d t}\left(t_{0}\right)\right\rangle: x^{*} \in \partial\left|u\left(t_{0}\right)\right|_{B}\right\} . \tag{42}
\end{equation*}
$$

In order to apply the Yosida approximation for the solution of (11), we need to introduce some definitions of dissipative mapping (operator) (see [17] for details).

Definition 15. Let $\left(B,|\cdot|_{B}\right)$ be a separable Banach space, $B^{*}$ be the dual space of $B$. The subdifferential $\partial|x|_{B}$ of norm $|\cdot|_{B}$ at $x \in B$ is defined by the formula

$$
\begin{equation*}
\partial|x|_{B}:=\left\{x^{*} \in B^{*}:|x+y|_{B}-|x|_{B} \geq\left(x^{*}, y\right), \forall y \in B\right\} . \tag{43}
\end{equation*}
$$

A mapping $F: D(F) \subset B \rightarrow B$ is said to be dissipative, if for any $x, y \in D(F)$, there exists $z^{*} \in \partial|x-y|_{B}$ such that

$$
\begin{equation*}
\left\langle z^{*}, F(x)-F(y)\right\rangle \leq 0 . \tag{44}
\end{equation*}
$$

A dissipative mapping $F: D(F) \subset B \rightarrow B$ is called an $m$-dissipative mapping or maximal dissipative if the image of $I-\lambda F$ is equal to the whole space $B$ for some $\lambda>0$ (and then for any $\lambda>0$ ), that is,

$$
\begin{equation*}
\text { range }(I-\lambda A)=B, \quad \text { for some } \lambda>0 \tag{45}
\end{equation*}
$$

Assume that $F$ is an $m$-dissipative mapping. Then its resolvent $J_{\alpha}$ and respectively the Yosida approximations $F_{\alpha}$, $\alpha>0$, are defined by

$$
\begin{gather*}
J_{\alpha} x=(I-\alpha F)^{-1} x \in \operatorname{dom} F, \\
F_{\alpha} x=\frac{1}{\alpha}\left(J_{\alpha} x-x\right), \quad \forall x \in \operatorname{dom} J_{\alpha}=\operatorname{range}(I-\alpha F) . \tag{46}
\end{gather*}
$$

Lemma 16 (see [2, Proposition 10.2]). Let $F: D(F) \rightarrow B$ be an $m$-dissipative mapping on $B$. Then
(1) for all $\alpha>0$ and $x, y \in B,\left|J_{\alpha}(x)-J_{\alpha}(y)\right|_{B} \leq|x-y|_{B}$;
(2) the mapping $F_{\alpha}, \alpha>0$, are dissipative and Lipschitz continuous:

$$
\begin{equation*}
\left|F_{\alpha}(x)-F_{\alpha}(y)\right|_{B} \leq \frac{2}{\alpha}|x-y|_{B}, \quad \forall x, y \in B \tag{47}
\end{equation*}
$$

Moreover, $\left|F_{\alpha}(x)\right|_{B} \leq|F(x)|_{B}$, for all $x \in D(F)$; and
(3) $\lim _{\alpha \rightarrow 0} F_{\alpha}(x)=x$, for all $x \in \overline{D(F)}$.

The following theorem is one of the main results of this paper, which will be proved by applying the well-known Yosida approach.

Theorem 17. For every $u_{0} \in H$, under Assumption 1, the stochastic Boussinesq system (11) admits a unique cádlág mild solution $u(t)$.

Proof. Denote $Z_{A}(\omega)$ to be the stationary solution of Langevin equation (24). Let $V=U-Z_{A}$. Then (11) is converted into the following evolution equation with random coefficients:

$$
\begin{gather*}
d V=\left[A V+B\left(V+Z_{A}, V+Z_{A}\right)+R\left(V+Z_{A}\right)\right] d t, \quad t \geq 0 \\
V(0)=U_{0} \tag{48}
\end{gather*}
$$

where $(A, D(A))$ generates an analytic $C_{0}$-semigroup $S$ (see Section 2.2 in [2]). It follows from the proof of Theorem 10.1 in [2] that, for $\alpha>0, \beta>0$, and sufficiently small $\eta$, the mappings $A+\eta$ and $B(\cdot, \cdot)+R(\cdot)+\eta$ are $m$-dissipative. Hence, the Yosida approximations of the $m$-dissipative mappings $A+$ $\eta$ and $B(\cdot, \cdot)+R(\cdot)+\eta$ can be respectively denoted by

$$
\begin{aligned}
(A+\eta)_{\beta} & =\frac{1}{\beta}\left((I-\beta(A+\eta))^{-1}-I\right) \\
((B+R)+\eta)_{\alpha} & =\frac{1}{\alpha}\left((I-\alpha((B+R)+\eta))^{-1}-I\right) .
\end{aligned}
$$

Now consider the following random approximate equation:

$$
\begin{gather*}
\frac{d^{-}}{d t} Y_{\alpha, \beta}(t)=(A+\eta)_{\beta} Y_{\alpha, \beta}+(B+R+\eta)_{\alpha}\left(Y_{\alpha, \beta}+Z_{A}(t-)\right) \\
-2 \eta Y_{\alpha, \beta}-\eta Z_{A}(t-) \\
Y_{\alpha, \beta}(0)=U_{0} . \tag{50}
\end{gather*}
$$

It is easy to verify that $\left((A+\eta)_{\beta}, D\left((A+\eta)_{\beta}\right)\right)$ generates an analytic $C_{0}$-semigroup $S_{\beta}$. Notice that the Yosida approximate operators are Lipschitz. Therefore the random approximation equation (50) has a unique continuous solution $Y_{\alpha, \beta}$.

Next we will show that

$$
\begin{equation*}
\lim _{\alpha \rightarrow 0}\left[\lim _{\beta \rightarrow 0} Y_{\alpha, \beta}(t)\right]=Y(t), \quad t \geq 0 \tag{51}
\end{equation*}
$$

in $H$, and this limit is actually the mild solution of stochastic Boussinesq equation (48).

For the sake of simplification, we just present the estimations when $\eta=0$, and the remaining estimates can be obtained by the similar arguments for $\eta \neq 0$.

Let $Y_{\alpha}$ be the solution of the integral equation:

$$
\begin{array}{rl}
Y_{\alpha}(t)=S(t) U_{0}+\int_{0}^{t} & S(t-s) \\
& \times\left(B\left(Y_{\alpha}(s)+Z_{A}(s-), Y_{\alpha}(s)+Z_{A}(s-)\right)\right. \\
& \left.+R\left(Y_{\alpha}(s)+Z_{A}(s-)\right)\right)_{\alpha} d s \tag{52}
\end{array}
$$

Notice that the operator $(B(\cdot, \cdot)+R(\cdot))_{\alpha}$ is Lipschitz continuous and $Z_{A}$ is cádlág. Hence, there exists a solution of random approximate equation (50), which is continuous in $H$.

For $\alpha>0$ and $\beta>0$, direct computation implies

$$
\begin{align*}
& Y_{\alpha}-Y_{\alpha, \beta}= S(t) \\
& U_{0}-S_{\beta}(t) \\
&+\int_{0}^{t}[ \left.S(t-s)-S_{\beta}(t-s)\right] \\
& \times\left[B\left(Y_{\alpha}(s)+Z_{A}(s-), Y_{\alpha}(s)+Z_{A}(s-)\right)\right. \\
&+\left.R\left(Y_{\alpha}(s)+Z_{A}(s-)\right)\right]_{\alpha} d s \\
&+\int_{0}^{t}\left[S_{\beta}(t-s)\right] \\
& \times[ {\left[B\left(Y_{\alpha}(s)+Z_{A}(s-), Y_{\alpha}(s)+Z_{A}(s-)\right)\right.} \\
&\left.+R\left(Y_{\alpha}(s)+Z_{A}(s-)\right)\right]_{\alpha} \\
&-\left[B \left(Y_{\alpha, \beta}(s)+Z_{A}(s-),\right.\right. \\
&\left.Y_{\alpha, \beta}(s)+Z_{A}(s-)\right)  \tag{53}\\
&\left.\left.+R\left(Y_{\alpha, \beta}(s)+Z_{A}(s-)\right)\right]_{\alpha}\right] d s .
\end{align*}
$$

Since both $A$ and $B+R$ are $m$-dissipative. Therefore, there exists constant $M, \omega$, and $C_{\alpha}$ such that for all $t \geq 0, V, W \in H$,

$$
\begin{gather*}
\left\|S_{\beta}(t)\right\|_{L(H, H)} \leq M e^{\omega t},  \tag{54}\\
\left|[B(V)+R(V)]_{\alpha}-[B(U)+R(U)]_{\alpha}\right| \leq C_{\alpha}|V-U|_{H} .
\end{gather*}
$$

Then

$$
\begin{align*}
& \left|Y_{\alpha}(t)-Y_{\alpha, \beta}(t)\right| \\
& \leq\left|S(t) U_{0}-S_{\beta}(t) U_{0}\right|+M C_{\alpha} \int_{0}^{t} e^{\omega(t-s)}\left|Y_{\alpha}(s)-Y_{\alpha, \beta}(s)\right| d s \\
& +\int_{0}^{t} \mid\left[S_{\beta}(t-s)-S(t-s)\right] \\
& \times\left[B\left(Y_{\alpha, \beta}(s)+Z_{A}(s-), Y_{\alpha, \beta}(s)+Z_{A}(s-)\right)\right. \\
& \left.\quad+R\left(Y_{\alpha, \beta}(s)+Z_{A}(s-)\right)\right]_{\alpha} \mid d s \tag{55}
\end{align*}
$$

By the Hille-Yosida theorem, it follows that

$$
\begin{equation*}
S_{\beta}(t) U_{0} \longrightarrow S(t) U_{0}, \quad \text { as } \beta \longrightarrow 0 \tag{56}
\end{equation*}
$$

uniformly in $t$ on compact subsets $U_{0}$ of $H$.
Hence, it follows that

$$
\begin{equation*}
\left|Y_{\alpha}(t)-Y_{\alpha, \beta}(t)\right| \leq M C_{\alpha} \int_{0}^{t}\left|Y_{\alpha}(s)-Y_{\alpha, \beta}(s)\right| d s \tag{57}
\end{equation*}
$$

uniformly on bounded intervals as $\beta \rightarrow 0$.
By Gronwall inequality, we have

$$
\begin{equation*}
\lim _{\beta \rightarrow 0} \sup _{t \leq T}\left|Y_{\alpha}(t)-Y_{\alpha, \beta}(t)\right|=0, \quad \forall T<\infty \tag{58}
\end{equation*}
$$

By Lemma 14,

$$
\begin{align*}
& \frac{d^{-}}{d t}\left|Y_{\alpha, \beta}(t)\right| \\
& =\min \left\{\left\langle x^{*}, \frac{d^{-}}{d t} Y_{\alpha, \beta}(t)\right\rangle: x^{*} \in \partial\left|Y_{\alpha, \beta}(t)\right|\right\} \\
& =\min \left\{\left\langlex^{*}, A_{\beta} Y_{\alpha, \beta}(t)\right.\right. \\
& + \\
& +B\left(Y_{\alpha, \beta}(s)+Z_{A}(s-), Y_{\alpha, \beta}(s)+Z_{A}(s-)\right) \\
& \left.\left.\quad+R\left(Y_{\alpha, \beta}(s)+Z_{A}(s-)\right)\right]_{\alpha}\right\rangle:  \tag{59}\\
& \left.x^{*} \in \partial\left|Y_{\alpha, \beta}(t)\right|\right\} .
\end{align*}
$$

Recalling that both $A_{\beta}$ and $[B(\cdot, \cdot)+R(\cdot)]$ are $m$-dissipative and $A_{\beta}$ is linear, we obtain

$$
\begin{align*}
& \frac{d^{-}}{d t}\left|Y_{\alpha, \beta}(t)\right| \leq \mid {\left[B\left(Y_{\alpha, \beta}(t)+Z_{A}(t-), Y_{\alpha, \beta}(s)+Z_{A}(t-)\right)\right.} \\
&\left.+R\left(Y_{\alpha, \beta}(t)+Z_{A}(t-)\right)\right]_{\alpha} \mid \\
& \leq \mid B\left(Y_{\alpha, \beta}(t)+Z_{A}(t-), Y_{\alpha, \beta}(s)+Z_{A}(t-)\right) \\
&+R\left(Y_{\alpha, \beta}(t)+Z_{A}(t-)\right) \mid \tag{60}
\end{align*}
$$

that is,

$$
\begin{align*}
\left|Y_{\alpha, \beta}(t)\right| \leq\left|U_{0}\right|+\int_{0}^{t} \mid B & \left(Y_{\alpha, \beta}(s)+Z_{A}(s-), Y_{\alpha, \beta}(s)+Z_{A}(s-)\right) \\
& +R\left(Y_{\alpha, \beta}(s)+Z_{A}(s-)\right) \mid d s, \quad t \geq 0 \tag{61}
\end{align*}
$$

It follows from the estimate (58) that, for any $\alpha>0$, and $t \in$ $[0, T]$,

$$
\begin{align*}
\left|Y_{\alpha}(t)\right| \leq\left|U_{0}\right|+\int_{0}^{t} \mid & {\left[B\left(Y_{\alpha, \beta}(s)+Z_{A}(s-), Y_{\alpha, \beta}(s)+Z_{A}(s-)\right)\right.} \\
+ & R\left(Y_{\alpha, \beta}(s)+Z_{A}(s-)\right) \mid d s \tag{62}
\end{align*}
$$

Similarly, by Lemma 16 , for $t \in[0, T]$,

$$
\begin{align*}
& \frac{1}{2} \frac{d^{-}}{d t}\left|Y_{\alpha, \beta}(t)-Y_{\gamma, \beta}(t)\right|^{2} \\
&=\left\langle\frac{d^{-}}{d t}\left(Y_{\alpha, \beta}(t)-Y_{\gamma, \beta}(t)\right), Y_{\alpha, \beta}(t)-Y_{\gamma, \beta}(t)\right\rangle \\
&=\left\langle\left(A_{\beta} Y_{\alpha, \beta}(t)-A_{\beta} Y_{\gamma, \beta}(t)\right)+\left[(B+R)_{\alpha}\right]\right. \\
& \times\left(Y_{\alpha, \beta}(t)+Z_{A}(\omega)(t-)\right)-\left[(B+R)_{\gamma}\right] \\
&\left.\times\left(Y_{\gamma, \beta}(t)+Z_{A}(\omega)(t-)\right), Y_{\alpha, \beta}(t)-Y_{\gamma, \beta}(t)\right\rangle \\
& \leq\langle {\left[(B+R)_{\alpha}\right]\left(Y_{\alpha, \beta}(t)+Z_{A}(\omega)(t-)\right)-\left[(B+R)_{\gamma}\right] } \\
&\left.\quad \times\left(Y_{\gamma, \beta}(t)+Z_{A}(\omega)(t-)\right), Y_{\alpha, \beta}(t)-Y_{\gamma, \beta}(t)\right\rangle \\
& \leq(\gamma+\alpha) {\left[\left|(B+R)_{\alpha}\left(Y_{\alpha, \beta}(t)+Z_{A}(\omega)(t-)\right)\right|\right.} \\
&\left.\quad+\left|(B+R)_{\gamma}\left(Y_{\gamma, \beta}(t)+Z_{A}(\omega)(t-)\right)\right|\right]^{2} \\
& \leq(\gamma+\alpha)\left[\left|(B+R)\left(Y_{\alpha, \beta}(t)+Z_{A}(\omega)(t-)\right)\right|\right. \\
&\left.\quad+\left|(B+R)\left(Y_{\gamma, \beta}(t)+Z_{A}(\omega)(t-)\right)\right|\right]^{2} \tag{63}
\end{align*}
$$

By the dissipation of the operators $A, B$, and $R$ and estimates (63), there exists a constant $C>0$ such that

$$
\begin{equation*}
\frac{1}{2} \frac{d^{-}}{d t}\left|Y_{\alpha, \beta}(t)-Y_{\gamma, \beta}(t)\right|^{2} \leq C(\alpha+\gamma), \quad t \in[0, T] . \tag{64}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\left|Y_{\alpha, \beta}(t)-Y_{\gamma, \beta}(t)\right|^{2} \leq 2 C(\alpha+\gamma) T, \quad t \in[0, T] \tag{65}
\end{equation*}
$$

By the estimate (58),

$$
\begin{equation*}
\left|Y_{\alpha}(t)-Y_{\gamma}(t)\right|^{2} \leq 2 C(\alpha+\gamma) T, \quad t \in[0, T] . \tag{66}
\end{equation*}
$$

Thus, $Y_{\alpha}(t) \rightarrow Y(t)$ in $H$ uniformly on $[0, T]$ as $\alpha \rightarrow 0$.
Next, we are going to show that the solution $Y_{\alpha}$ of the Yosida approximations equation is a mild solution:

$$
\begin{array}{r}
Y_{\alpha}(t)=S(t) U_{0}+\int_{0}^{t} S(t-s)(B+R)_{\alpha}\left(Y_{\alpha}(s)+Z_{A}(s)\right) d s \\
t \in[0, T] \tag{67}
\end{array}
$$

By the reflexivity of $H^{1}$ and the estimate $\left\|Y_{\alpha}(t)\right\|_{H^{1}} \leq$ $C_{2}\left\|U_{0}\right\|_{H^{1}}, t \in[0, T], \alpha>0$, there exists a subsequence $\left\{Y_{\alpha, n}\right\}$, which converges weakly in $H^{1}$ and weakly converges to the function $Y(t)$ in $H^{1}$. Since $\left\{Y_{\alpha, n}(t)\right\}$ is strong convergent in $L^{2}$, and

$$
\begin{equation*}
\|Y(t)\|_{H^{1}} \leq C_{2}\left\|U_{0}\right\|_{H^{1}}, \quad t \in[0, T] . \tag{68}
\end{equation*}
$$

Let $h \in L^{2}$, then

$$
\begin{align*}
\left\langle Y_{\alpha}(t)\right. & , h\rangle_{L^{2}} \\
= & \left\langle S(t) U_{0}, h\right\rangle_{L^{2}} \\
\quad & +\int_{0}^{t}\left\langle(B+R) J_{\alpha}\left(Y_{\alpha}(s)+Z_{A}(s)\right), S^{*}(t-s) h\right\rangle_{L^{2}} d s \tag{69}
\end{align*}
$$

Moreover

$$
\begin{equation*}
J_{\alpha}\left(Y_{\alpha}(s)+Z_{A}(s)\right) \longrightarrow Y(s)+Z_{A}(s), \quad \text { as } \alpha \longrightarrow 0 \tag{70}
\end{equation*}
$$

Notice that $(B+R)\left(J_{\alpha}\left(Y_{\alpha}(s)+Z_{A}(s)\right) \rightarrow(B+R)\left(Y_{\alpha}(s)+Z_{A}(s)\right)\right.$ weakly converges in $L^{2}$. So, letting $\alpha \rightarrow 0$, we obtain

$$
\begin{align*}
\langle Y(t), h\rangle_{L^{2}}= & \left\langle S(t) U_{0}, h\right\rangle_{L^{2}} \\
& +\int_{0}^{t}\left\langle\left(S(t-s)(B+R)\left(Y(s)+Z_{A}(s)\right), h\right\rangle_{L^{2}} d s\right. \tag{71}
\end{align*}
$$

It follows from the arbitrariness of $h$ that

$$
\begin{array}{r}
Y(t)=S(t) U_{0}+\int_{0}^{t} S(t-s)(B+R)\left(Y_{\alpha}(s)+Z_{A}(s)\right) d s \\
t \in[0, T] \tag{72}
\end{array}
$$

Thus, $Y(t)$ is a mild solution of random Boussinesq equation (50).

Theorem 18. For any $U_{0} \in H$, the $\operatorname{map} \varphi: \mathbb{T} \times \Omega \times H \rightarrow H$ defined by the solution of stochastic Boussinesq equation (11) as $U(t)=\Phi\left(t, \vartheta_{t}(\omega)\right) U_{0}$ has the cocycle property; that is, the solution of stochastic Boussinesq equation (11) generates a random dynamical system $\left(\Omega, \mathscr{F}, \mathbb{P},\left(\vartheta_{t}\right)_{t \geq 0}, \Phi\right)$.

Proof. From Theorem 17, stochastic Boussinesq equation (11) admits a unique solution $V(t, Z(\omega)(t), x)$. Define the map

$$
\Phi: \mathbb{R}^{+} \times \Omega \times H \longrightarrow H
$$

$$
\begin{equation*}
\Phi(t, \omega) x=V(t \cdot Z(\omega)(t))(x-Z(\omega)(0))+Z(\omega)(t+s) \tag{73}
\end{equation*}
$$

(i) By the similar argument of Theorem 17, every solution $Y_{\alpha}(t)$ of the Yosida approximation equation (50) is measurable. Notice that $Y_{\alpha}(t) \rightarrow Y(t)$ uniformly as $\alpha \rightarrow 0$. Hence, the limit function $Y(t)$ is also measurable. Thus, the mapping $\Phi$ is measurable.
(ii) Obviously, $\Phi(0, \omega)=I$.
(iii) It suffices to verify that the cocycle property holds for the mapping $\Phi$, that is,

$$
\begin{align*}
\Phi(t+s, \omega) x= & V\left(t+s, Z_{A}(\omega)(t+s)\right)\left(x-Z_{A}(\omega)(0)\right) \\
& +Z_{A}(\omega)(t+s) \tag{74}
\end{align*}
$$

In fact, recalling that $Z_{A}(\omega)(s)=Z_{A}\left(\theta_{s} \omega\right)(0)$, it follows that

$$
\begin{align*}
\Phi\left(t, \theta_{s} \omega\right) & {[\Phi(s, \omega) x] } \\
= & V\left(t, Z_{A}\left(\theta_{s} \omega\right)(t)\right)\left(\Phi(s, \omega) x-Z_{A}\left(\theta_{s} \omega\right)(0)\right) \\
& +Z_{A}\left(\theta_{s} \omega\right)(t) \\
= & V\left(t, Z_{A}\left(\theta_{s} \omega\right)(t)\right) \\
& \times\left[V\left(s, Z_{A}(\omega)(s)\right)\left(x-Z_{A}(\omega)(0)\right)+Z(\omega)(s)\right. \\
& \left.\quad-Z\left(\theta_{s} \omega\right)(0)\right]+Z\left(\theta_{s} \omega\right)(t) \\
= & V\left(t, Z_{A}\left(\theta_{s} \omega\right)(t)\right) V\left(s, Z_{A}(\omega)(s)\right) \\
& \times\left(x-Z_{A}(\omega)(0)\right)+Z_{A}\left(\theta_{s} \omega\right)(t) \\
= & V_{1}(t) . \tag{75}
\end{align*}
$$

Moreover,

$$
\begin{align*}
V & \left(t+s, Z_{A}(\omega)(t+s)\right)\left(x-Z_{A}(\omega)(0)\right) \\
& =V\left(t, Z_{A}\left(\theta_{s} \omega\right)(t)\right) V\left(s, Z_{A}(\omega)(s)\right)\left(x-Z_{A}(\omega)(0)\right) \\
& =V_{2}(t) \tag{76}
\end{align*}
$$

Since

$$
\begin{equation*}
V\left(0, Z_{A}\left(\theta_{s} \omega\right)(0)\right)\left(x-Z_{A}\left(\theta_{s} \omega\right)(0)\right)=x-Z_{A}\left(\theta_{s} \omega\right)(0) \tag{77}
\end{equation*}
$$

Thus,

$$
\begin{align*}
V_{1}(0)= & V\left(s, Z_{A}(\omega)(s)\right)\left(x-Z_{A}(\omega)(0)\right) \\
= & V\left(0, Z_{A}\left(\theta_{s} \omega\right)(0)\right) V\left(s, Z_{A}(\omega)(s)\right) \\
& \times\left(x-Z_{A}(\omega)(0)\right)=V_{2}(0),  \tag{78}\\
\frac{d V_{1}(t)}{d t} & =\frac{d V\left((t+s), Z_{A}(\omega)\right)}{d t}(t+s) .
\end{align*}
$$

Therefore, we obtain

$$
\begin{align*}
\frac{d V_{1}(t)}{d t} & +A V_{1}(t) \\
& +B\left(V_{1}(t)+Z_{A}(\omega)(t+s), V_{1}(t)+Z_{A}(\omega)(t+s)\right) \\
= & -R\left(V_{1}(t)+Z_{A}\left(\theta_{t+s} \omega\right)\right) \\
\frac{d V_{2}(t)}{d t} & +A V_{2}(t) \\
& +B\left(V_{2}(t)+Z_{A}\left(\theta_{s} \omega\right)(t), V_{2}(t)+Z_{A}\left(\theta_{s} \omega\right)(t)\right) \\
= & -R\left(V_{2}(t)+Z_{A}\left(\theta_{s} \omega\right)(t)\right) \tag{79}
\end{align*}
$$

The uniqueness of the solution implies that almost surely $V_{1}(t)=V_{2}(t)$ holds, that is,

$$
\begin{equation*}
\Phi\left(t, \theta_{s} \omega\right)[\Phi(s, \omega) x]=\Phi\left(t+s, \theta_{t+s}(\omega)\right) x \tag{80}
\end{equation*}
$$

Thus, the cocycle property for the mapping $\Phi$ holds.
By the definition of random dynamical systems [18], the solution mapping of the stochastic Boussinesq equation (11) generates a random dynamical system $\Phi$. Thus, the proof of Theorem 18 is complete.

## 4. Discussion

In Section 3, we have studied the long-time behavior of stochastic Boussinesq equations (1) driven by subordinator Lévy noise and have shown the cocycle property of random dynamical systems generated by the mild solution of stochastic Boussinesq equation (1). To prove the existence of random attractor, it suffices to show the existence of random absorbing set and the compactness of random dynamical system $\Phi$, we refer the similar argument to [13].

Here, we are also interested in the stochastic Boussinesq equations driven by Poisson noise and Wiener noise, and we are trying to show the existence of random dynamical
systems. To the end, we consider the following stochastic Boussinesq equations driven by Lévy noises followed as

$$
\begin{align*}
& \frac{\partial u}{\partial t}+(u \cdot \nabla) u-v \Delta u+\nabla p \\
& \quad=\theta e_{2}+b_{1} d t+d W^{1}(t)+\int_{X} f(x) \widetilde{N}^{1}(d t, d x) \\
& \begin{aligned}
& \frac{\partial \theta}{\partial t}+(u \cdot \nabla) \theta-k \Delta \theta \\
&=u_{2}+b_{2} d t+d W^{2}(t)+\int_{X} g(x) \widetilde{N}_{2}(d t, d x) \\
& \quad \nabla \cdot u=0,
\end{aligned}  \tag{81}\\
& \left.\quad u\right|_{\partial D}=0, \quad u(0)=u_{0}, \quad \theta(0)=\theta_{0}
\end{align*}
$$

where $W^{1}(\cdot)$ and $W^{2}(\cdot)$ are $H$-valued Brownian motion, $b_{1}$ and $b_{2}$ are constants vector in $H, f$ and $g$ are measurable mappings from some measurable space $X$ to $H$, and $\widetilde{N}_{1}$ and $\widetilde{N}_{2}$ are compensated Poisson measure on $[0, \infty) \times X$ with intensity measure $n v_{1}$ and $n v_{2}$, respectively, where $v_{1}$ and $v_{2}$ are $\sigma$-finite measure on $\mathscr{B}(X), f(x)$, and $g(x)$ satisfying

$$
\begin{gather*}
\int_{U}|f(x)|^{2} e^{\alpha|f(x)|} v(d x)<\infty  \tag{82}\\
\int_{U}|g(x)|^{2} e^{\beta|g(x)|} v(d x)<\infty, \quad \forall \alpha>0, \quad \forall \beta>0
\end{gather*}
$$

Let $D([0, T], H)$ be the space of all cádlág paths from $[0, T]$ to $H$ endowed with the uniform convergence topology. Since there are finite jumps when the character measure $\lambda(Z)<\infty$, we can rearrange the jump time of $N(d t, d x)$ as $\sigma_{1}(\omega)<\sigma_{2}(\omega)<\cdots$. Since there is no jump on the interval [ $\left.0, \sigma_{1}(\omega)\right)$, just as the approach in [19], we can apply Banach fixed point theorem to prove that there exists a unique solution $\phi(t)$ in $L^{2}\left(\left[0, \sigma_{1}(\omega)\right) ; V\right) \cap D\left(\left[0, \sigma_{1}(\omega)\right) ; H\right)$. Define

$$
\phi^{(1)}(t)= \begin{cases}\phi(t), & 0 \leq t<\sigma_{1}(\omega)  \tag{83}\\ \phi\left(\sigma_{1}-\right)+f\left(\phi\left(\sigma_{1}-\right), P_{\sigma_{1}}\right), & t=\sigma_{1}(\omega)\end{cases}
$$

On $\left[\sigma_{1}(\omega), \sigma_{2}(\omega)\right)$, define

$$
\begin{align*}
\widetilde{\phi}_{0} & =\phi^{(1)}\left(\sigma_{1}\right) 1_{\left(\sigma_{1}<\infty\right)} \\
\widetilde{\sigma}_{2} & =\left(\sigma_{2}-\sigma_{1}\right) 1_{\left(\sigma_{1}<\infty\right)}+\infty 1_{\left(\sigma_{1}=\infty\right)}  \tag{84}\\
\widetilde{F}_{t} & =\mathscr{F}_{\sigma_{1}+t}, \quad \widetilde{P}_{t}=\left(\theta_{\sigma_{1}} P\right)_{t} 1_{\left(\sigma_{1}<\infty\right)}
\end{align*}
$$

Similar to the argument in [11], since $P_{t}$ is stationary Poisson point process on $R^{+} \times Z$ with intensity measure $\lambda(d x) d t$,
then $\widetilde{P}_{t}$ is also a stable Poisson point process on $R^{+} \times Z$ with intensity measure $\lambda(d x) d t$. Define

$$
\begin{align*}
& \phi^{(2)}(t) \\
& \quad=\left\{\begin{array}{ll}
\phi^{(1)}(t), & 0 \leq t<\sigma_{1}(\omega) \\
\widetilde{\phi}^{(2)}\left(t-\sigma_{1}\right), & \sigma_{1}(\omega)<t<\sigma_{2}(\omega), \\
\widetilde{\phi}^{(2)}\left(\left(\sigma_{2}-\sigma_{1}\right)-\right) \\
+f\left(\widetilde{\phi}^{(2)}\left(\left(\sigma_{2}-\sigma_{1}\right)-\right), P_{\sigma_{2}}\right), & t=\sigma_{2}(\omega), \\
\phi^{(n)}(t) & t<\sigma_{n-1}(\omega), \\
& = \begin{cases}\phi^{(n-1)}(t), & t=\sigma_{n-1} \\
\widetilde{\phi}^{(n)}\left(t-\sigma_{n-1}\right), \\
\widetilde{\phi}^{(n)}\left(\left(\sigma_{n}-\sigma_{n-1}\right)-\right) \\
+f\left(\widetilde{\phi}^{(n)}\left(\left(\sigma_{n}-\sigma_{n-1}\right)-\right), P_{\sigma_{n}}\right),\end{cases}
\end{array} .\right.
\end{align*}
$$

Hence, $\phi^{(n)}(t)$ is cádlág on $[0, T]$ such that $B\left(\phi^{(n)}, \phi^{(n)}\right) \in H$ and $A_{p}\left(\phi^{(n)} \in H, P\right.$ a.s. for all $t \geq 0$, and

$$
\begin{align*}
& P\left(\int _ { 0 } ^ { t } \left[|\phi(s)|+\left|B\left(\phi(s)+z_{A}(s), \phi(s)+z_{A}(s)\right)\right|\right.\right. \\
& \left.\left.\quad+2 \mu_{0}\left|R\left(\phi(s)+z_{A}(s)\right)\right|\right] d s<\infty\right)=1, \quad \forall t>0 . \tag{86}
\end{align*}
$$

Therefore, $\phi^{(n)}(t)$ is a unique global weak solution of (81). We can verify the existence of random dynamical systems generated by the global weak solution of (81).

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