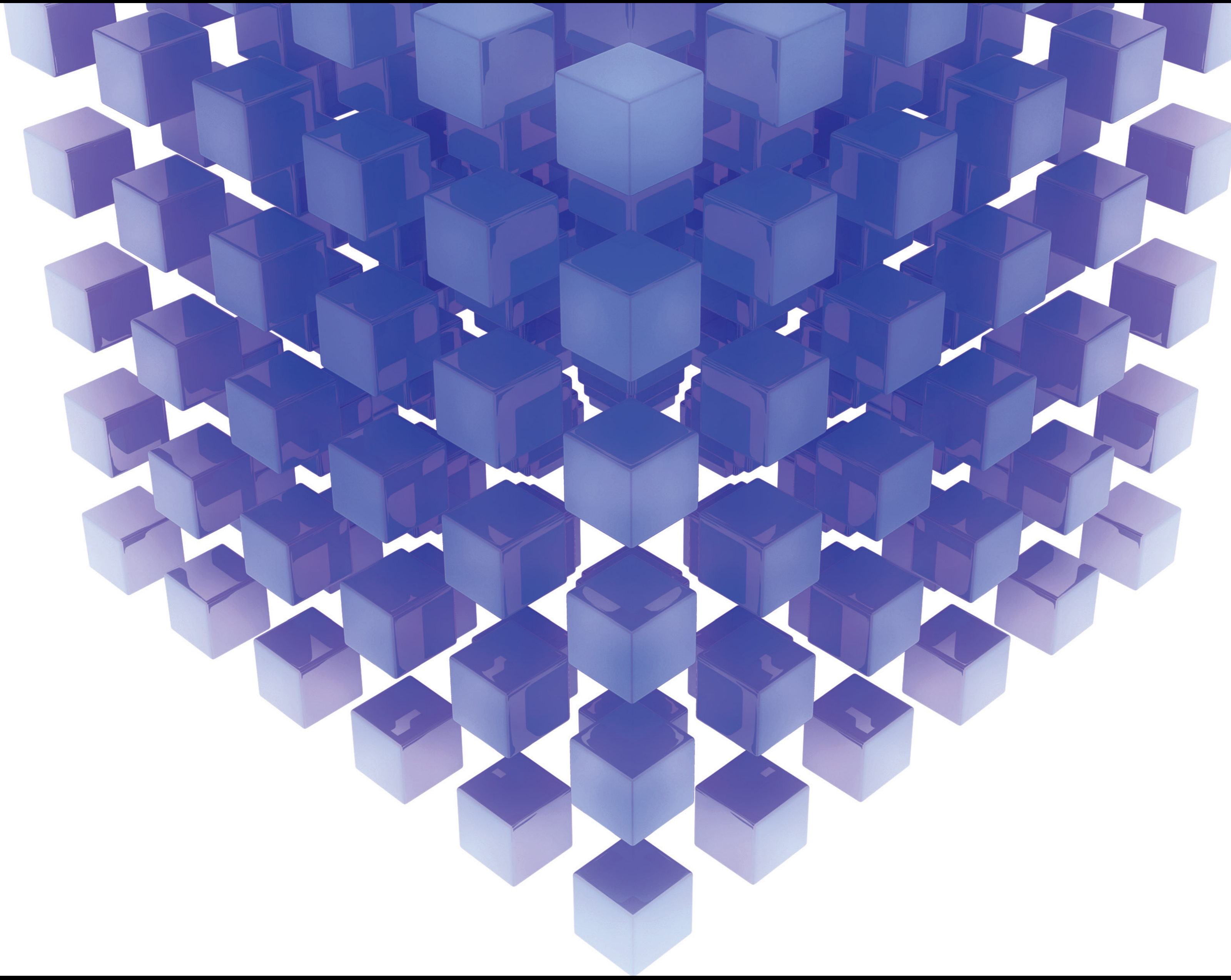
Mathematical Problems in Engineering

Recent Trends in Special Functions and Analysis of Differential Equations

Lead Guest Editor: Praveen Agarwal
Guest Editors: Michel Waldschmidt, Shigeru Kanemitsu, and Shilpi Jain



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Research Article

A Novel Method for Solution of Fractional Order Two-Dimensional Nonlocal Heat Conduction Phenomena

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In this paper, we have extended the operational matrix method for approximating the solution of the fractional-order two-dimensional elliptic partial differential equations (FPDEs) under nonlocal boundary conditions. We use a general Legendre polynomials basis and construct some new operational matrices of fractional order operations. These matrices are used to convert a sample nonlocal heat conduction phenomenon of fractional order to a structure of easily solvable algebraic equations. The solution of the algebraic structure is then used to approximate a solution of the heat conduction phenomena. The proposed method is applied to some test problems. The obtained results are compared with the available data in the literature and are found in good agreement.

Dedicated to my father Mr. Sher Mumtaz, (1955-2021), who gave me the basic knowledege of mathematics.

1. Introduction

The first approach to study a physical experiment is to derive a mathematical expression, which formulates the dynamics of the experiment under certain assumption. Most of the physical phenomena are formulated in terms of ordinary differential equations. Some problems in which the quantity of interest also changes with respect to both space and time result in partial differential equations. A wide range of scientists devoted their very precise time to investigate various important aspects of partial differential equations. In [1], thermoelastic damping of in-plane vibration of a functionally graded material has been studied based on the Eringen nonlocal theory. In [2], a fractional sideways heat flow problem is investigated, in which the interior measurements at two interior points are given by continuous data with deterministic noises. The work in [3] deals with the exothermic reactions model having a constant heat source in the porous media with strong memory effects. This article explains the behavior of heat profile under the effect of different definitions of the derivative. In [4], the Newtonian liquid flow porous stretching/shrinking sheet utilizing a Brinkman mode is investigated.

Nonlocal partial differential equations (PDEs) arise in the mathematical modeling of various problems in physics, engineering, ecology, and biological sciences [5–7]. The term nonlocal problems means that the solution of PDEs on the boundary is connected with the solution on some interior points of the domain. The case arises when the solution at the boundary is not known. Such formulation is placed in a separate class known as nonlocal boundary value problems. Some of the numerical investigations regarding PDEs with nonlocal boundary conditions reported in the literature can be found in [8–15]. Among others, some of the well-known methods that can be effectively applied to BVPs are finite difference methods, mesh-free methods, finite element methods, etc. For instance, the one-dimensional heat equation with nonlocal boundary conditions has been studied in [12, 16–18]. Two-dimensional diffusion problems with nonlocal boundary conditions have been discussed in [13, 19]. The numerical solution of the Laplace equation with integral boundary condition is explored in [8]. Similarly, the numerical solution of multidimensional linear elliptic equations with integral boundary conditions is explored in [20].

To be specific, the fundamental problem of interest in this article is to find an approximate solution of the fractional-order two-dimensional Poisson equation, given as follows:

$$\frac{\partial^\sigma u}{\partial x^\sigma} + \frac{\partial^\sigma u}{\partial y^\sigma} = -f(x, y). \tag{1}$$

The above model is subject to two-point nonlocal boundary conditions:

$$\begin{aligned} &u(x,0) = \mu_3(x), \\ &u(x,1) = \mu_4(x), \quad 0 \le x \le 1, \\ &u(0,y) = \mu_1(y), \\ &u(1,y) = \gamma u(\xi, y) + \mu_2(y), \quad 0 \le y \le 1, 0 \le \xi < 1. \end{aligned} \tag{2}$$

The functions $f(x, y), \mu_1(y), \mu_2(y), \mu_3(x)$ and $\mu_4(x)$ are given smooth functions. The parameters ξ and γ are two positive constants. The parameter σ $(1 < \sigma \le 2)$ represents the order of derivative defined in Caputo sense.

Recently, many authors devoted their studies to the approximate solution of integer order version of the above problem. In [21], Yang et al. approximated the solution of fractional order partial differential subjected to the simple initial condition of the form:

$$u(x,0) = g(x)u(0,t) = h(t). \tag{3}$$

The approach presented in [21], is interesting. However, it can not be utilized directly to the approximate solution of fractional order PDEs subject to nonlocal boundary condition (2). Islam et al. [22] presented a comprehensive text on the solution to the above problem. They implemented two different methods for the solution of integer order counterpart of (1). Their first approach is based on Haar wavelets. In the same paper, they also implemented a modified form of the mesh-less method to solve the integer order problem. Sajavicius [23] implemented the radial base function approach to integer order problems and studied some computational aspects of the proposed approach. The results presented in [22, 23] are the motivating factor of our interest to study the approximate solution of the fractional-order Poisson equation subject to nonlocal boundary conditions.

Our approach is based on shifted Legendre polynomials and their operational matrices. We derived some new operational matrices to handle the problem. The new operational matrices can handle the nonlocal boundary conditions. The interesting readers may find useful results and some new strategies of this method in [24–27]. Application of orthogonal polynomials combined with Tau and Collection method can be found in [28–31].

The rest of the paper is organized as follows: in Section 2, we recall some primary results from the fractional calculus and approximation theory. In Section 3, we recall some previously derived operational matrices and develop some new operational matrices. In Section 4, a theoretical base is developed for the conversion of the nonlocal FPDEs to the matrix equation. Convergence analysis and error estimation are also developed in the same section. In Section 5, the proposed method is applied to some benchmark problems. In the same section, the obtained results are demonstrated and compared with other methods in the literature. Section 6 is devoted to the conclusions.

2. Preliminaries

In this section, we present some useful results and notations which are of primary importance in our further investigation.

Definition 1 (see [32–34]). Given an interval $[a, b] \subset \mathbb{R}$, the Riemann–Liouville fractional order integral of a function $\phi \in (L^1[a, b], \mathbb{R})$ of order $\alpha \in \mathbb{R}_+$ is defined by the following:

$$\mathcal{I}^\alpha_{a+}\phi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\phi(s)\mathrm{d}s, \tag{4}$$

provided that the integral on the right-hand side exists.

Definition 2 (Caputo derivative). For a given function $\phi(x) \in C^n[a, b]$, the Caputo fractional order derivative is defined as follows:

$$D^\alpha \phi(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\phi^{(n)}(t)}{(x-t)^{\alpha+(1-n)}}\mathrm{d}t, \quad n-1 \le \alpha < n, n \in N, \tag{5}$$

where $n = [\alpha] + 1$.

Hence, it follows that

$$\begin{aligned} D^\alpha x^k &= \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} x^{k-\alpha}, \\ I^\alpha x^k &= \frac{\Gamma(1+k)}{\Gamma(1+k+\alpha)} x^{k+\alpha}, \\ D^\alpha C &= 0, \text{ for a constant } C. \end{aligned} \tag{6}$$

2.1. The Shifted Legendre Polynomials. The shifted Legendre polynomials [35] defined on $[0, 1]$ are given by the following relation:

$$P_i(x) = \sum_{k=0}^{i} \Delta_{(i,k)} x^k, \quad \text{where } \Delta_{(i,k)} = (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k!)^2}. \tag{7}$$

These polynomials are bounded by 1, and we have the following relation:

$$\max_{x\in[0,1]}\left|P_i(x)\right| = 1. \tag{8}$$

These polynomials are orthogonal on the domain $[0,1]$, and the orthogonality condition is given as follows:

$$\int_0^1 P_i(x)P_j(x)\mathrm{d}x = \begin{cases} \dfrac{1}{2i+1}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad \frac{\mathrm{d}y}{\mathrm{d}x} \tag{9}$$

which implies that any $f(x) \in C[0,1]$ can be approximated by Legendre polynomials as follows:

$$\begin{aligned} f(x) &\approx \sum_{a=0}^{m} C_a P_a(x), \quad \text{where } C_a \\ &= (2a+1)\int_0^1 f(x)P_a(x)\mathrm{d}x. \end{aligned} \tag{10}$$

In vector notation, we write the following:

$$f(x) \approx K_M^T \widehat{P}_M(x), \tag{11}$$

where $M = m+1$ is the scale level of approximation. K is the coefficient vector and $\widehat{P}_M(x)$ is M terms function vector. These notations can be easily extended to two-dimensional space [35] and two-dimensional Legendre polynomials of the order M are defined as a product function of two Legendre polynomials

$$\begin{aligned} &P_n(x,y) = P_a(x)P_b(y), \\ &\qquad n = Ma+b+1, \quad a = 0,1,2,\ldots,m, \quad b = 0,1,2,\ldots,m. \end{aligned} \tag{12}$$

The orthogonality condition of $P_n(x,y)$ is as follows:

$$\begin{aligned} &\int_0^1\int_0^1 P_a(x)P_b(y)P_c(x)P_d(y)\mathrm{d}x\mathrm{d}y \\ &\quad = \begin{cases} \dfrac{1}{(2a+1)(2b+1)}, & \text{if } a = c, b = d; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{13}$$

Any $f(x,y) \in C([0,1]\times[0,1])$ can be approximated by the polynomials $P_n(x,y)$ as follows:

$$\begin{aligned} f(x,y) &\approx \sum_{a=0}^{m}\sum_{b=0}^{m} C_{ab}P_a(x)P_b(y), \\ &\qquad \text{where } C_{ab} = (2a+1)(2b+1) \\ &\qquad \cdot \int_0^1\int_0^1 f(x,y)P_a(x)P_b(y)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{14}$$

For simplicity, use the notation $C_n = C_{ab}$ where $n = Ma+b+1$ and rewrite (14) as follows:

$$f(x,y) \approx \sum_{n=1}^{M^2} C_n P_n(x,y) = \mathbf{K}_{M^2}\mathbf{\Psi}(x,y), \tag{15}$$

where $\mathbf{K}_{M^2}$ is $1\times M^2$ coefficient row vector and is $M^2\times 1$ column vector of functions defined by the following:$\Psi(x,y)$

$$\mathbf{\Psi}(x,y) = \left[\psi_{11}(x,y),\ldots,\psi_{1M}(x,y)\psi_{21}(x,y),\ldots,\psi_{2M}(x,y),\ldots,\psi_{MM}(x,y)\right]^T, \tag{16}$$

where $\psi_{i+1,j+1}(x,y) = P_i(x)P_j(y)$.

3. New Operational Matrices

The operational matrices of the fractional derivatives and integrals play a vital role in converting the FPDEs to the system of algebraic equations. The operational matrices of all derivatives are explicitly derived in our previous report [35]. We will need operational matrices in integration. The operational matrices of integration w.r.t x or y is not a difficult task and can be easily derived using the same procedure as in [35]. To make this study a self-contained material and for the ease and interest of our readers, we have provided detailed proof of deriving operational matrices of integration.

Lemma 1. *Let $\Psi(x,y)$ be the function vector as defined in (16), then the fractional integral of order σ of $\Psi(x,y)$ w.r.t y is given by the following:*

$$I_y^{\sigma}(\Psi(x,y)) \simeq \mathbf{G}^{\sigma,y}_{M^2\times M^2}\mathbf{\Psi}(x,y), \tag{17}$$

where $\mathbf{G}^{\sigma,y}_{M^2\times M^2}$ is the operational matrix of the fractional integration of order σ and is defined as follows:

$$\mathbf{G}^{\sigma,y}_{M^2\times M^2} = \left[\Delta'_{q,r}\right], \tag{18}$$

where $q = Mi+j+1$, $r = Ma+b+1$, for $i,j,a,b = 0,1,2,\ldots,m$ and

$$\begin{aligned} \Delta'_{q,r} &= C_{i,j,b,a,k} = \sum_{k=0}^{a}\delta_{i,a}(2j+1) \\ &\quad \cdot \sum_{l=0}^{j}\frac{(-1)^{j+l+b+k}(j+l)!(b+k)!}{(j-l)!(l!)^2(k+l+\sigma+1)(b-k)!k!\Gamma(k+\sigma+1)}. \end{aligned} \tag{19}$$

Proof. Taking the element $P_n(x,y)$ defined by (12), then the fractional-order integration of $P_n(x,y)$ w.r.t y follows:

$$I_y^\sigma\left(P_n(x,y)\right)=P_a(x)\sum_{k=0}^{b}(-1)^{b+k}\frac{(b+k)!}{(b-k)!(k!)^2}I_y^\sigma y^k$$
$$=\sum_{k=0}^{b}(-1)^{b+k}\frac{(b+k)!}{(b-k)!(k!)^2}P_a(x)I_y^\sigma y^k. \quad (20)$$

Using the definition of fractional integration, we obtain the following:

$$I_y^\sigma P_a(x)P_b(y)=\sum_{k=0}^{b}(-1)^{b+k}\frac{(b+k)!}{(b-k)!(k!)\Gamma(k+\sigma+1)}\cdot P_a(x)y^{k+\sigma},\quad b=0,1,\ldots,m, \quad (21)$$

Approximating $P_a(x)y^{k+\sigma}$ by M terms of Legendre polynomials in two variables yields

$$P_a(x)y^{k+\sigma}\approx\sum_{i=0}^{m}\sum_{j=0}^{m}C_{ij}P_i(x)P_j(y), \quad (22)$$

where $C_{ij}=(2i+1)(2j+1)\int_0^1\int_0^1 P_a(x)y^{k+\sigma}P_i(x)P_j(y)\,\mathrm{d}x\mathrm{d}y$, which in view of the orthogonality conditions implies that

$$C_{ij,a}=\delta_{i,a}(2j+1)\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!(l!)^2(k+l+\sigma+1)}, \quad (23)$$

where

$$\delta_{ia}=\begin{cases}1, & \text{if } i=a,\\ 0, & \text{if } i\neq a,\end{cases} \quad (24)$$

and hence, it follows that

$$\begin{aligned}I_y^\sigma P_a(x)P_b(y)&\approx\sum_{k=0}^{b}(-1)^{b+k}\frac{(b+k)!}{(b-k)!(k!)\Gamma(k+\sigma+1)}\sum_{i=0}^{m}\sum_{j=0}^{m}C_{ij,a}P_i(x)P_j(y)\\&\approx\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{b}(-1)^{b+k}\frac{(b+k)!}{(b-k)!(k!)\Gamma(k+\sigma+1)}C_{ij,a}P_i(x)P_j(y)\\&\approx\sum_{i=0}^{m}\sum_{j=0}^{m}C_{ij,a,b,k}P_i(x)P_j(y),\quad b=0,1,\ldots,M,\end{aligned} \quad (25)$$

where

$$C_{ij,a,b,k}=\sum_{k=0}^{b}\delta_{i,a}(2j+1)\sum_{l=0}^{j}\frac{(-1)^{j+l+b+k}(j+l)!(b+k)!}{(j-l)!(l!)^2(k+l+\sigma+1)(b-k)!k!\Gamma(k+\sigma+1)}. \quad (26)$$

Using the notations, $q=Mi+j+1$, $r=Ma+b+1$ and $\Delta'_{q,r}=C_{i,j,b,a,k}$ for $i,j,a,b=0,1,2,3,\ldots,m$, we get the desired result. □

Lemma 2. *Let $\Psi(x,y)$ be as defined in (16), then the integration of order σ of $\Psi(x,y)$ w.r.t x is given by the following:*

$$I_x^\sigma(\mathbf{\Psi}(x,y))\simeq\mathbf{G}_{M^2\times M^2}^{\sigma,x}\mathbf{\Psi}(x,y), \quad (27)$$

where $\mathbf{G}_{M^2\times M^2}^{\sigma,x}$ is the operational matrix of derivative of order σ and is defined as follows:

$$\mathbf{G}_{M^2\times M^2}^{\sigma,x}=\left[\Theta'_{q,r}\right], \quad (28)$$

and $r=Mi+j+1$, $q=Ma+b+1$, $\Theta'_{q,r}=S_{i,j,b,a,k}$ for $i,j,a,b=0,1,2,\ldots,m$ and

$$S_{i,j,b,a,k}=\sum_{k=0}^{a}\delta_{j,b}(2i+1)\sum_{l=0}^{i}\frac{(-1)^{i+l+a+k}(i+l)!(a+k)!}{(i-l)!(l!)^2(k+l+\sigma+1)(a-k)!k!\Gamma(k+\sigma+1)}. \quad (29)$$

Proof. The proof of this lemma is similar to the above lemma. □

The operational matrices derived in the above two lemmas are essential for our further analysis. However, these matrices are not enough to fulfill our requirements. In our analysis, we will face terms like $cx^n(1/\Gamma(\sigma))\int_0^\xi(\xi-s)^{\sigma-1}u(s,y)ds$ and $cy^n(1/\Gamma(\sigma))\int_0^\xi(\xi-s)^{\sigma-1}u(x,s)ds$, where c and n are some positive constants and $0<\xi\leq 1$. Therefore to replace such terms with their equivalent matrix form, we need to derive two more operational matrices. The operational matrices used to replace such term by their equivalent matrix form are derived in the following lemmas.

Lemma 3. *Let $u(x,y)=\mathbf{K}_{M^2}\Psi(x,y)$, then for some constants c and n, the following relation holds:*

$$cy^n\frac{1}{\Gamma(\sigma)}\int_0^\xi(\xi-s)^{\sigma-1}u(x,s)ds=\mathbf{K}_{M^2}\mathbf{P}^{(c,n,\sigma,\xi,y)}_{M^2\times M^2}\mathbf{\Psi}(x,y), \tag{30}$$

where

$$\mathbf{P}^{(c,n,\sigma,\xi,y)}_{M^2\times M^2}=\begin{bmatrix}\Delta_{1,1} & \Delta_{1,2} & \cdots & \Delta_{1,M^2}\\ \Delta_{1,2} & \Delta_{2,2} & \cdots & \Delta_{2,M^2}\\ \vdots & \vdots & \ddots & \vdots\\ \Delta_{M^2,1} & \Delta_{M^2,2} & \cdots & \Delta_{M^2,M^2}\end{bmatrix}, \tag{31}$$

$\Delta_{r,q}=d^{c,n,\sigma,\xi}_{a,b,i,j}$, *where $r=Mi+j+1$, $q=Ma+b+1$ and*

$$d^{c,n,\sigma,\xi}_{a,b,i,j}=\delta_{a,i}c(2b+1)\sum_{k=0}^{j}\sum_{l=0}^{b}\Delta_{(j,k)}\cdot\frac{(-1)^{b+l}(b+l)!\Gamma(k+1)\xi^{\sigma+k}}{(b-l)!(l!)^2(n+l+1)\Gamma(\sigma+k)}. \tag{32}$$

Proof. Let

$$u(x,y)=\mathbf{K}_{M^2}\mathbf{\Psi}(x,y). \tag{33}$$

For some constant c and n, we can write the following expression:

$$\begin{aligned}&cy^n\frac{1}{\Gamma(\sigma)}\int_0^\xi(\xi-s)^{\sigma-1}u(x,s)ds\\&=\mathbf{K}_{M^2}cy^n\frac{1}{\Gamma(\sigma)}\int_0^\xi(\xi-s)^{\sigma-1}\mathbf{\Psi}(x,s)ds.\end{aligned} \tag{34}$$

Now, considering the general term of $\Psi(x,s)$, we can write the following:

$$\begin{aligned}&cy^n\frac{1}{\Gamma(\sigma)}\int_0^\xi(\xi-s)^{\sigma-1}P_r(x,s)ds\\&=cy^nP_i(x)\frac{1}{\Gamma(\sigma)}\int_0^\xi(\xi-s)^{\sigma-1}P_j(s)ds,\end{aligned} \tag{35}$$

where we use the notation $r=Mi+j+1$. By using the definition of Legendre polynomials, we can write the following:

$$\begin{aligned}&cy^n\frac{1}{\Gamma(\sigma)}\int_0^\xi(\xi-s)^{\sigma-1}P_r(x,s)ds\\&=\sum_{k=0}^{j}c\Delta_{(j,k)}\frac{\Gamma(k+1)\xi^{\sigma+k}}{\Gamma(\sigma+k)}y^nP_i(x).\end{aligned} \tag{36}$$

Approximating $P_i(x)y^n$ by Legendre polynomials in two variables yields

$$P_i(x)y^n\approx\sum_{a=0}^{m}\sum_{b=0}^{m}d_{ab}P_a(x)P_b(y), \tag{37}$$

where $d_{ab}=(2a+1)(2b+1)\int_0^1\int_0^1P_i(x)y^nP_a(x)P_b(y)\,dxdy$, which in view of the orthogonality conditions implies that

$$d_{ab}=\delta_{a,i}(2b+1)\sum_{l=0}^{b}\frac{(-1)^{b+l}(b+l)!}{(b-l)!(l!)^2(n+l+1)}, \tag{38}$$

where

$$\delta_{a,i}=\begin{cases}1, & \text{if } i=a,\\ 0, & \text{if } i\neq a.\end{cases} \tag{39}$$

Hence, it follows that

$$\begin{aligned}&cy^n\frac{1}{\Gamma(\sigma)}\int_0^\xi(\xi-s)^{\sigma-1}P_r(x,s)ds\\&\approx\sum_{k=0}^{j}c\Delta_{(j,k)}\frac{\Gamma(k+1)\xi^{\sigma+k}}{\Gamma(\sigma+k)}\sum_{a=0}^{m}\sum_{b=0}^{m}d_{ab}P_a(x)P_b(y)\\&\approx\sum_{a=0}^{m}\sum_{b=0}^{m}\sum_{k=0}^{j}c\Delta_{(j,k)}\frac{\Gamma(k+1)\xi^{\sigma+k}}{\Gamma(\sigma+k)}d_{ab}P_a(x)P_b(y)\\&\approx\sum_{a=0}^{m}\sum_{b=0}^{m}d^{c,n,\sigma,\xi}_{a,b,i,j,k}P_a(x)P_b(y),\end{aligned} \tag{40}$$

where

$$d^{c,n,\sigma,\xi}_{a,b,i,j}=\delta_{a,i}c(2b+1)\sum_{k=0}^{j}\sum_{l=0}^{b}\Delta_{(j,k)}\cdot\frac{(-1)^{b+l}(b+l)!\Gamma(k+1)\xi^{\sigma+k}}{(b-l)!(l!)^2(n+l+1)\Gamma(\sigma+k)}. \tag{41}$$

Using the notations, $r=Mi+j+1$, $q=Ma+b+1$ and $\Delta_{r,q}=d^{c,n,\sigma,\xi}_{a,b,i,j}$ for $i,j,a,b=0,1,2,3,\ldots,m$, we get the desired result. □

Lemma 4. *Let $u(x,y)=\mathbf{K}_{M^2}\Psi(x,y)$, then for some constants c and n, the following relation holds:*

$$cx^n \frac{1}{\Gamma(\sigma)} \int_0^{\xi} (\xi - s)^{\sigma-1} u(s, y) \mathrm{d}s = \mathbf{K}_{M^2} \mathbf{P}^{(c,n,\sigma,\xi,x)}_{M^2\times M^2} \mathbf{\Psi}(x, y), \tag{42}$$

where

$$\mathbf{P}^{(c,n,\sigma,\xi,x)}_{M^2\times M^2} = \begin{bmatrix} \Delta_{1,1} & \Delta_{1,2} & \cdots & \Delta_{1,M^2} \\ \Delta_{1,2} & \Delta_{2,2} & \cdots & \Delta_{2,M^2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{M^2,1} & \Delta_{M^2,2} & \cdots & \Delta_{M^2,M^2} \end{bmatrix}, \tag{43}$$

$\Delta_{r,q} = d^{c,n,\sigma,\xi}_{a,b,i,j}$, *where* $q = Mi + j + 1$, $r = Ma + b + 1$ *and*

$$d^{c,n,\sigma,\xi}_{a,b,i,j} = \delta_{b,j} c(2b+1) \sum_{k=0}^{i} \sum_{l=0}^{a} \Delta_{(i,k)} \frac{(-1)^{a+l}(a+l)!\Gamma(k+1)\xi^{\sigma+k}}{(a-l)!(l!)^2(n+l+1)\Gamma(\sigma+k)}. \tag{44}$$

Proof. This lemma can be easily proved by following similar steps as in the previous lemma. This is left as an exercise for interested readers. □

4. Main Result: Application of Operational Matrices

The operational matrices developed in the previous section have a wide range of applications. As an application of the above matrices, we solve the following fractional order Poisson equation:

$$\frac{\partial^{\sigma} u}{\partial x^{\sigma}} + \frac{\partial^{\sigma} u}{\partial y^{\sigma}} = -f(x, y), \tag{45}$$

subject to nonlocal two-point boundary conditions

$$\begin{aligned} &u(x,0) = \mu_3(x), u(x,1) = \mu_4(x), \quad 0 \le x \le 1, \\ &u(0,y) = \mu_1(y), u(1,y) = \gamma u(\xi, y) + \mu_2(y), \\ &\qquad 0 \le y \le 1, 0 \le \xi < 1. \end{aligned} \tag{46}$$

Readers may see how simple steps lead us to the approximate solution of such complicated problems with high accuracy. As usual, we seek the solution to the problem in terms of shifted Legendre polynomials given by the following:

$$\frac{\partial^{\sigma} u}{\partial y^{\sigma}} = \mathbf{K}_{M^2} \mathbf{\Psi}(x, y). \tag{47}$$

On application of the fractional integral of order σ and making use of Lemma 2, we get the following relation:

$$u(x, y) = \mathbf{K}_{M^2} \mathbf{G}^{\sigma,y}_{M^2\times M^2} \mathbf{\Psi}(x, y) + c_0 + c_1 y. \tag{48}$$

Using the conditions at $y = 0$ and $y = 1$, we get the following relation:

$$\begin{aligned} c_0 &= \mu_3(x), \\ c_1 &= \mu_4(x) - \mathbf{K}_{M^2} \mathbf{G}^{\sigma,y}_{M^2\times M^2} \mathbf{\Psi}(x, 1) + \mu_3(x). \end{aligned} \tag{49}$$

Using the values of c_0 and c_1 in (48), we get the following:

$$\begin{aligned} u(x, y) = {} & \mathbf{K}_{M^2} \mathbf{G}^{\sigma,y}_{M^2\times M^2} \mathbf{\Psi}(x, y) + \mu_3(x)(1-y) \\ & + \mu_4(x) y - y \mathbf{K}_{M^2} \mathbf{G}^{\sigma,y}_{M^2\times M^2} \mathbf{\Psi}(x, 1). \end{aligned} \tag{50}$$

which in view of Lemma 3 can be written as follows:

$$u(x, y) = \mathbf{K}_{M^2}\left(\mathbf{G}^{\sigma,y}_{M^2\times M^2} + \mathbf{P}^{(-1,1,\sigma,1,y)}\right)\mathbf{\Psi}(x, y) + \mathbf{F}_{1M^2} \mathbf{\Psi}(x, y), \tag{51}$$

where $\mathbf{F}_{1M^2}\Psi(x, y) = \mu_3(x)(1-y) + \mu_4(x)y$.

Now approximating the source term $f(x, y) = \mathbf{F}_{M^2}\Psi(x, y)$, and using (47) in (45), we can write the following:

$$\frac{\partial^{\sigma} u}{\partial x^{\sigma}} = -\mathbf{K}_{M^2} \mathbf{\Psi}(x, y) - \mathbf{F}_{M^2} \mathbf{\Psi}(x, y). \tag{52}$$

On application of fractional integral of order σ w.r.t. x, we can write the following:

$$\begin{aligned} u(x, y) = {} & -\mathbf{K}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(x, y) - \mathbf{F}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(x, y) \\ & + \mathrm{d}_0 + \mathrm{d}_1 x. \end{aligned} \tag{53}$$

Using the initial conditions at $x = 0$ we can easily $d_0 = \mu_1(y)$; however, the second constant is not known. We can use the two-point boundary conditions:

$$\begin{aligned} u(1, y) = {} & -\mathbf{K}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(1, y) - \mathbf{F}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(1, y) \\ & + \mu_1(y) + \mathrm{d}_1, \\ u(\xi, y) = {} & -\mathbf{K}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(\xi, y) - \mathbf{F}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(\xi, y) \\ & + \mu_1(y) + \mathrm{d}_1 \xi. \end{aligned} \tag{54}$$

Using the equality $u(1, y) = \gamma u(\xi, y) + \mu_2(y)$, we get the following:

$$\begin{aligned} & -\mathbf{K}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(1, y) - \mathbf{F}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(1, y) + \mu_1(y) + \mathrm{d}_1 \\ & \quad = -\gamma \mathbf{K}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(\xi, y) \\ & - \gamma \mathbf{F}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(\xi, y) + \gamma \mu_1(y) + \mathrm{d}_1 \gamma \xi + \mu_2(y). \end{aligned} \tag{55}$$

From which we can calculate the value of d_1 as follows:

$$\begin{aligned} d_1 = {} & r_1 \mathbf{K}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(1, y) - r_1 \gamma \mathbf{K}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(\xi, y) \\ & + r_1 \mathbf{F}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(1, y) - r_1 \gamma \mathbf{F}_{M^2} \mathbf{G}^{\sigma,x}_{M^2\times M^2} \mathbf{\Psi}(\xi, y) \\ & + (\gamma - 1) r_1 \mu_1(y) + r_1 \mu_2(y), \end{aligned} \tag{56}$$

where $r_1 = (1/1 - \gamma\xi) \neq 0$. Using the value of d_0 and d_1 in (53), we can write the following:

$$\begin{aligned} u(x,y) = &-\mathbf{K}_{M^2}\mathbf{G}^{\sigma,x}_{M^2\times M^2}\boldsymbol{\Psi}(x,y) - \mathbf{F}_{M^2}\mathbf{G}^{\sigma,x}_{M^2\times M^2}\boldsymbol{\Psi}(x,y)\\ &+\mu_1(y) + r_1 x\mathbf{K}_{M^2}\mathbf{G}^{\sigma,x}_{M^2\times M^2}\boldsymbol{\Psi}(1,y) -\\ &r_1 x\gamma\mathbf{K}_{M^2}\mathbf{G}^{\sigma,x}_{M^2\times M^2}\boldsymbol{\Psi}(\xi,y) + r_1 x\mathbf{F}_{M^2}\mathbf{G}^{\sigma,x}_{M^2\times M^2}\boldsymbol{\Psi}(1,y)\\ &- r_1 x\gamma\mathbf{F}_{M^2}\mathbf{G}^{\sigma,x}_{M^2\times M^2}\boldsymbol{\Psi}(\xi,y) +\\ &((\gamma-1)r_1\mu_1(y) + r_1\mu_2(y))x, \end{aligned} \tag{57}$$

which in view of Lemma 4, can be written as follows:

$$\begin{aligned} (x,y) = &-\mathbf{K}_{M^2}\mathbf{G}^{\sigma,x}_{M^2\times M^2}\boldsymbol{\Psi}(x,y) - \mathbf{F}_{M^2}\mathbf{G}^{\sigma,x}_{M^2\times M^2}\boldsymbol{\Psi}(x,y)\\ &+\mathbf{K}_{M^2}\mathbf{P}^{(r_1,1,\sigma,1,x)}_{M^2\times M^2}\boldsymbol{\Psi}(,y) -\\ &\mathbf{K}_{M^2}\mathbf{P}^{(r_1\gamma,1,\sigma,\xi,x)}_{M^2\times M^2}\boldsymbol{\Psi}(x,y) + F_{M^2}P^{(r_1,1,\sigma,1,x)}_{M^2\times M^2}\Psi(x,y)\\ &- F_{M^2}P^{(r_1\gamma,1,\sigma,\xi,x)}_{M^2\times M^2}\Psi(x,y) + F_{2M^2}\Psi(x,y), \end{aligned} \tag{58}$$

where $\mathbf{F}_{2M^2}\Psi(x,y) = \mu_1(y) + (\gamma-1)r_1\mu_1(y) + r_1\mu_2(y))x$. In simplified notation, we can write the following:

$$\begin{aligned} u(x,y) = &\mathbf{K}_{M^2}\left(\mathbf{P}^{(r_1,1,\sigma,1,x)}_{M^2\times M^2} - \mathbf{G}^{\sigma,x}_{M^2\times M^2} - \mathbf{P}^{(r_1\gamma,1,\sigma,\xi,x)}_{M^2\times M^2} r_1\gamma, 1, \sigma, \xi, x\right)\boldsymbol{\Psi}(x,y) +\\ &\mathbf{F}_{M^2}\left(\mathbf{P}^{(r_1,1,\sigma,1,x)}_{M^2\times M^2} - \mathbf{G}^{\sigma,x}_{M^2\times M^2} - \mathbf{P}^{(r_1\gamma,1,\sigma,\xi,x)}_{M^2\times M^2}\right)\boldsymbol{\Psi}(x,y) + \mathbf{F}_{2M^2}\boldsymbol{\Psi}(x,y). \end{aligned} \tag{59}$$

On comparing equation (51) and (59), we can write the following:

$$\begin{aligned} \mathbf{K}_{M^2}\left(\mathbf{G}^{\sigma,y}_{M^2\times M^2} + \mathbf{P}^{(-1,1,\sigma,1,y)}_{M^2\times M^2}\right)\boldsymbol{\Psi}(x,y) = &\mathbf{K}_{M^2}\left(\mathbf{P}^{(r_1,1,\sigma,1,x)}_{M^2\times M^2} - \mathbf{G}^{\sigma,x}_{M^2\times M^2} - \mathbf{P}^{(r_1\gamma,1,\sigma,\xi,x)}_{M^2\times M^2}\right)\boldsymbol{\Psi}(x,y) +\\ &\mathbf{F}_{M^2}\left(\mathbf{P}^{(r_1,1,\sigma,1,x)}_{M^2\times M^2} - \mathbf{G}^{\sigma,x}_{M^2\times M^2} - \mathbf{P}^{(r_1\gamma,1,\sigma,\xi,x)}_{M^2\times M^2}\right)\boldsymbol{\Psi}(x,y) + \mathbf{F}_{2M^2}\boldsymbol{\Psi}(x,y) - \mathbf{F}_{1M^2}\boldsymbol{\Psi}(x,y). \end{aligned} \tag{60}$$

Canceling out the common term, we can write the following:

$$\begin{aligned} \mathbf{K}_{M^2}\left(\mathbf{G}^{\sigma,y}_{M^2\times M^2} + \mathbf{P}^{(-1,1,\sigma,1,y)}_{M^2\times M^2}\right) = &\mathbf{K}_{M^2}\left(\mathbf{P}^{(r_1,1,\sigma,1,x)}_{M^2\times M^2} - \mathbf{G}^{\sigma,x}_{M^2\times M^2} - \mathbf{P}^{(r_1\gamma,1,\sigma,\xi,x)}_{M^2\times M^2}\right) +\\ &\mathbf{F}_{M^2}\left(\mathbf{P}^{(r_1,1,\sigma,1,x)}_{M^2\times M^2} - G^{\sigma,x}_{M^2\times M^2} - P^{(r_1\gamma,1,\sigma,\xi,x)}_{M^2\times M^2}\right) + \mathbf{F}_{2M^2} - \mathbf{F}_{1M^2}. \end{aligned} \tag{61}$$

which is a linear system of equations and can be easily solved for the unknown vector $\mathbf{K}_{M^2}$, which can be used in (51) or (59) to get an approximate solution to the problem.

4.1. Error Bound. Considering a sufficiently smooth function $g(x,y)$ on $\Delta = [0,\eta]\times[0,\eta]$, let $\prod_{M,M}(x,y)$ is the space span by m term Legendre polynomials. We assume that $g_{(M,M)}(x,y)$ is its best approximation in $\prod_{(M,M)}(x,y)$. For this purpose, consider a polynomial $\widehat{P}_{(M,M)}(x,y)$ is any polynomial of degree $\leq M$ in variable x and y, respectively. Then, from the definition of best approximation,

$$\left\|g(x,y) - g_{(M,M)}(x,y)\right\|_2 \leq \left\|g(x,y) - P_{(M,M)}(x,y)\right\|_2. \tag{62}$$

The inequality in (62) also holds if $P_{(M,M)}(x,y)$ is interpolating polynomial at point (x_i, y_j); then by the similar arguments as in [36], the error of the approximation is given by the following:

$$\left\|g(x,y) - P_{(M,M)}(x,y)\right\|_2 \leq \left(C_1 + C_2 + C_3\frac{1}{M^{M+1}}\right)\frac{1}{M^{M+1}}, \tag{63}$$

where

$$C_1 = \frac{1}{4}\max_{(x,y)\in[0,1]\times[0,1]}\left|\frac{\partial^{M+1}}{\partial x^{M+1}} g(x,y)\right|,$$
$$C_2 = \frac{1}{4}\max_{(x,y)\in[0,1]\times[0,1]}\left|\frac{\partial^{M+1}}{\partial y^{M+1}} g(x,y)\right|, \quad (64)$$
$$C_3 = \frac{1}{16}\max_{(x,y)\in[0,1]\times[0,1]}\left|\frac{\partial^{2M+2}}{\partial x^{M+1}\partial y^{M+1}} g(x,y)\right|.$$

We refer the reader to [37] for the proof of the above result. From the above result, it is clear that the error of approximation of a function decreases with the increase of M.

5. Test Problems

We solve the fractional order generalization of some bench mark problems from [22, 23].

Test Problem 1 (see [22, 23]). Consider (45) with the following functions:

$$\begin{aligned} f(x,y) &= -2e^{(x+y)}, \\ \mu_1(y) &= e^{y}, \\ \mu_2(y) &= e^{(1+y)} - \gamma e^{(\xi+y)}, \\ \mu_3(x) &= e^{x}, \\ \mu_4(x) &= e^{(x+1)}. \end{aligned} \quad (65)$$

The exact solution of the problem for fix $\sigma = 2$ is $e^{(x+y)}$.

Test Problem 2 (see [22, 23]). Consider (45) with the following functions:

$$\begin{aligned} f(x,y) &= -2\pi^2 \sin(\pi x)\sin(\pi y), \\ \mu_1(y) &= 0, \\ \mu_2(y) &= -\sin(\pi x)\sin(\pi y), \\ \mu_3(x) &= 0, \\ \mu_4(x) &= 0. \end{aligned} \quad (66)$$

The exact solution of the problem for the fix $\sigma = 2$ is $\sin(\pi x)\sin(\pi y)$.

Test Problem 3 (see [22, 23]). Consider (45) with the following functions.

$$\begin{aligned} f(x,y) = &-\left(\frac{\Gamma(5)}{\Gamma(5-\sigma)}\left(x^{(4-\sigma)} y - y^{(4-\sigma)}\right)\right. \\ &- \frac{\Gamma(4)}{\Gamma(4-\sigma)}\left(x^{(3-\sigma)}(3y-1) - 4y^{(3-\sigma)}\right) \\ &\left.+3\frac{\Gamma(3)}{\Gamma(3-\sigma)}\left(x^{(2-\sigma)} y - y^{(2-\sigma)}\right)\right), \\ \mu_1(y) = &\ y^3 - y(y-1)^3, \\ \mu_2(y) = &\ \frac{y}{16} + \frac{7}{8}, \\ \mu_3(x) = &\ x^3, \\ \mu_4(x) = &\ x(x-1)^3 + x^3 + 1. \end{aligned} \quad (67)$$

The exact solution of the problem for $1 < \sigma \le 2$ is $x^3 - y(y-1)^3 + y^3 + xy(x-1)^3$.

6. Results and Discussion

We solve the above problems with the proposed method. The first two problems are selected from [22, 23], while the third problem is a constructed problem. In [22, 23], these problems are studied and solved with two different methods, Haar wavelets and a family of the mesh-less method based on the radial base functions. We solved these problems with the operational matrices and compared our results with the results reported in these references. To measure accuracy, we calculate the following parameters:

$$L_\infty = \max_{j=1,2,\ldots,M}\left|u\left(x_j,y_j\right) - \overline{u}\left(x_j,y_j\right)\right|, \frac{\Delta y}{\Delta x}$$
$$L_{\text{rms}} = \sqrt{\sum_{j=0}^{M}\frac{\left(u\left(x_j,y_j\right) - \overline{u}\left(x_j,y_j\right)\right)^2}{M}}. \quad (68)$$

If $\sigma \ne 2$, then the exact solution of the first two problems is not known. We use the residual error norms to measure the accuracy of the proposed method for the fractional values of σ. These residual norms are defined as follows:

$$R_\infty = \max_{j=1,2,\ldots,M}\left|r\left(x_j,y_j\right)\right|,$$
$$R_{\text{rms}} = \sqrt{\sum_{j=0}^{M}\frac{r\left(x_j,y_j\right)}{M}}, \quad (69)$$

where $r(x,y)$ is defined as follows:

Table 1: L_{rms} of Test Problem 1 obtained with the proposed method (PM) at different values of M and its comparison with HWCM [22], MCTMQ [22], and MCTSMQ [22].

HWCM [22]			MCTMQ [22]	MCTSMQ [22]	Proposed method	
M (n)	L_{rms}	N	L_{rms}	L_{rms}	M (n)	L_{rms}
1 (12)	4.0860×10^{-3}	4	1.7629×10^{-4}	1.2407×10^{-4}	5 (25)	1.0404×10^{-4}
2 (32)	1.7549×10^{-3}	8	3.4559×10^{-6}	5.5859×10^{-6}	6 (36)	5.0660×10^{-6}
4 (98)	6.4795×10^{-4}	16	2.5542×10^{-5}	3.6689×10^{-6}	7 (49)	1.6107×10^{-7}
8 (320)	2.3142×10^{-4}	32	3.6732×10^{-4}	6.6522×10^{-7}	8 (64)	4.3049×10^{-9}
16 (1152)	8.2022×10^{-3}	64	4.1256×10^{-4}	3.2737×10^{-7}	9 (81)	1.3381×10^{-10}
32 (4325)	2.9017×10^{-3}		Not available		10 (100)	3.9221×10^{-12}

Here, we fix $\xi = 0.5$ and $\gamma = 1$.

Table 2: L_∞ of Example 5.1 obtained with proposed method (PM) at different values of M and its comparison with HWCM [22], MCTMQ [22], and MCTSMQ [22].

HWCM [22]			MCTMQ [22]	MCTSMQ [22]	Proposed method	
M(n)	L_∞	N	L_∞	L_∞	M(n)	L_∞
1 (12)	3.8454×10^{-3}	4	4.9876×10^{-4}	2.8688×10^{-4}	5 (25)	2.9475×10^{-4}
2 (32)	1.4767×10^{-3}	8	1.1276×10^{-5}	1.5987×10^{-5}	6 (36)	1.4687×10^{-5}
4 (98)	3.9916×10^{-4}	16	7.7560×10^{-5}	1.7883×10^{-5}	7 (49)	9.7986×10^{-7}
8 (320)	1.0055×10^{-4}	32	1.0744×10^{-3}	3.4368×10^{-6}	8 (64)	9.9163×10^{-9}
16 (1152)	2.5230×10^{-4}	64	1.4665×10^{-3}	2.4120×10^{-6}	9 (81)	3.8661×10^{-10}
32 (4325)	6.3147×10^{-4}		Not available		10 (100)	1.2467×10^{-11}

Here, we fix $\xi = 0.5$ and $\gamma = 1$.

$$r(x, y) = \frac{\partial^\sigma \overline{u}}{\partial x^\sigma} + \frac{\partial^\sigma \overline{u}}{\partial y^\sigma} + f(x, y). \tag{70}$$

The first problem is solved using Haar wavelets (HWCM), mesh-less method without splitting (MCTMQ), and with splitting (MCTSMQ). We fixed $\sigma = 2$ and obtained an approximate solution of Test Problem 1 for different values of scale level M. We compared our results with HWCM, MCTMQ, and MCTSMQ. The comparison of L_{rms} of Test Problem 1 obtained with the proposed method and HWCM, MCTMQ, and MCTSMQ are shown in Table 1. We can easily note that L_{rms} obtained with HWCM at $M = 16$ is 8.2022×10^{-3}; note that at this level, HWCM converts the problem to a system of algebraic equations of 1152 unknowns. While the proposed method yields $L_{rms} = 1.3381 \times 10^{-10}$, while converting the problem to a system of algebraic equations of 81 unknowns. L_{rms} of this problem obtained with MCTMQ and MCTSMQ at $N = 64$ is 4.1256×10^{-4} and 3.2737×10^{-7}, respectively. This shows the superiority of the proposed method over HWCM and meshless methods.

The parameters L_∞ for Test Problem 1 obtained with the proposed method are also compared with HWCM, MCTMQ, and MCTSMQ. The results are displayed in Table 2. It is observed that L_∞ for this problem obtained with the proposed method at $M = 8$ is 3.8661×10^{-10} while HWCM yields 2.5230×10^{-4} at $M = 16$, and meshless method MCTMQ and MCTSMQ yield 1.4665×10^{-3} and 2.4120×10^{-6}.

The proposed method along with HWCM and the meshless method convert the problem to the system of linear algebraic equations. The computational cost and stability of the resulting algebraic equations are different for different methods. Often some method yields a very approximate solution, but the computational cost is much higher. We compared the condition number κ and CPUtime of the proposed method with MCTMQ and MCTSMQ. It is observed that the proposed method is more robust than these methods. At $M = 64$, MCTMQ solves the algebraic equations in 53.78 seconds, and MCTSMQ takes 51.72 seconds to solve the system, while the proposed method solves the system in 0.09516 seconds. The condition number of the proposed method is much less than MCTMQ and MCTSMQ. It means that the proposed method converts the problem to the system of algebraic equations, which is more stable as compared to MCTMQ and MCTSMQ. The comparison of CPU time and conditions number of the proposed method with MCTQM and MCTSMQ at different scale levels is shown in Table 3.

The accuracy of the present method is analyzed at different values of γ. We chose $\gamma = -8, 2, 8$ and calculate L_∞ and L_{rms} for scale levels. We observed that the accuracy of the proposed method does not depend on the values of γ. The values of L_∞ and L_{rms} obtained with the proposed method at different values of γ using scale level $M = 10$ are compared with the L_∞ and L_{rms} obtained with the HWCM at $M = 32$ in Table 4. One can see that the accuracy obtained with the proposed method is very high as compared to HWCM. The error norms at different values of γ are also compared with MCTSMQ and the results are displayed in Figure 1. One can note that the accuracy remains the same for all values of γ, also at $M = 8$ and $M = 9$, the error norms are much less than the error norms obtained using MCTSMQ at $N = 64$. The CPU time at different values of γ and $M = 10$ are shown in Figure 2(a). While the condition number of the resulting matrix equation at different values of γ is shown

Table 3: CPU time and condition number κ of the proposed method and its comparison with MCTMQ [22] and MCTSMQ [22].

MCTMQ [22]			MCTSMQ [22]		Proposed method		
N	κ	CPU time	κ	CPU time	M	κ	CPU time
4	2.8541×10^{10}	0.00	3.0574×10^{12}	0.00	6	1.3472×10^{17}	0.00255
8	9.3912×10^{18}	0.00	4.7522×10^{16}	0.00	7	2.6632×10^{16}	0.00257
16	2.2347×10^{19}	0.04	4.3691×10^{18}	0.02	8	4.4841×10^{16}	0.00371
32	5.4623×10^{21}	0.86	5.7284×10^{19}	0.78	9	6.8584×10^{16}	0.01689
64	4.4325×10^{21}	53.78	3.2699×10^{20}	51.72	10	1.3208×10^{17}	0.09516

Table 4: L_∞ and $L_{\rm rms}$ of Example 5.1 for different values of γ at $M = 10$ and its comparison with HWCM [22] ($M = 32$).

γ	Haar wavelets [22]		Proposed method	
	L_∞	$L_{\rm rms}$	L_∞	$L_{\rm rms}$
−8	6.1545×10^{-6}	1.5072×10^{-5}	3.5506×10^{-11}	8.5152×10^{-12}
−6	5.8644×10^{-6}	1.4942×10^{-5}	3.5660×10^{-11}	3.4538×10^{-12}
−4	5.2613×10^{-6}	1.4837×10^{-5}	3.1595×10^{-11}	6.7567×10^{-12}
−2	3.8832×10^{-5}	1.5076×10^{-5}	1.4811×10^{-11}	3.6505×10^{-12}
0	4.3925×10^{-5}	1.9345×10^{-5}	7.4749×10^{-12}	2.9916×10^{-12}
2	3.5678×10^{-5}	9.5986×10^{-5}	2.5216×10^{-11}	8.7175×10^{-12}
4	3.3974×10^{-5}	6.2947×10^{-5}	3.3240×10^{-11}	5.7501×10^{-12}
6	4.7081×10^{-5}	6.8228×10^{-5}	1.0725×10^{-10}	1.6889×10^{-11}
8	3.4825×10^{-5}	4.7649×10^{-5}	1.4925×10^{-10}	1.8421×10^{-11}

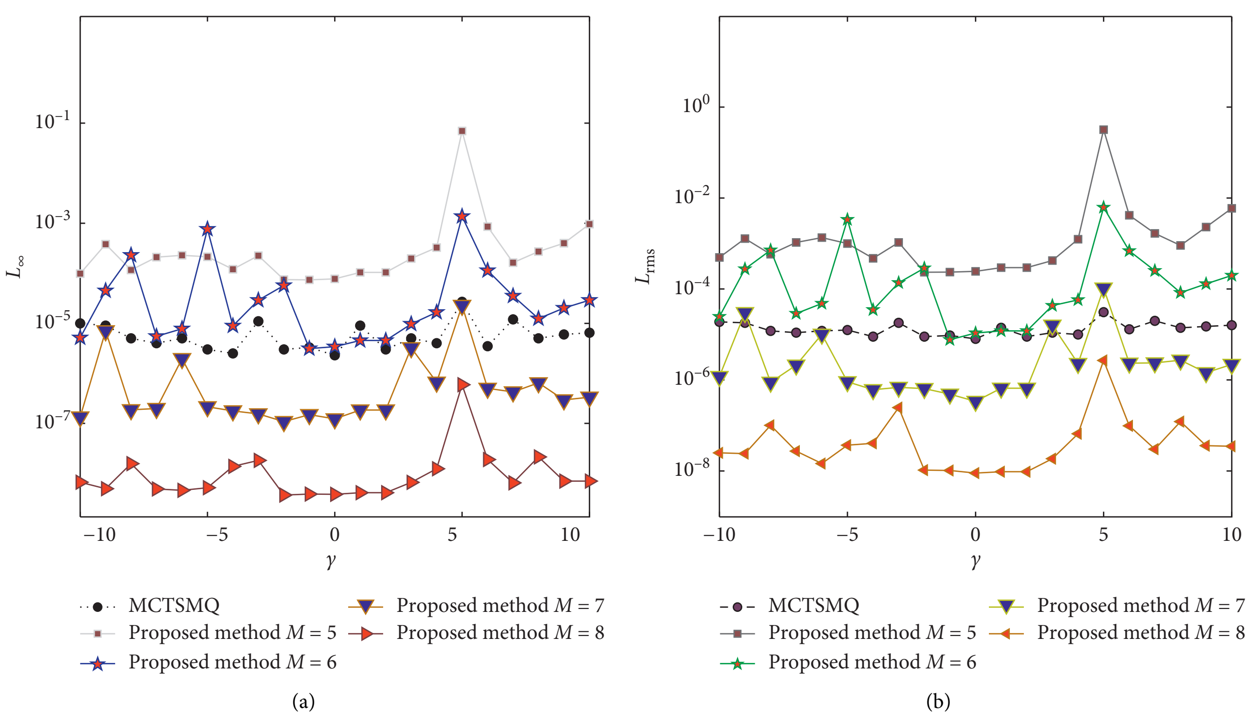
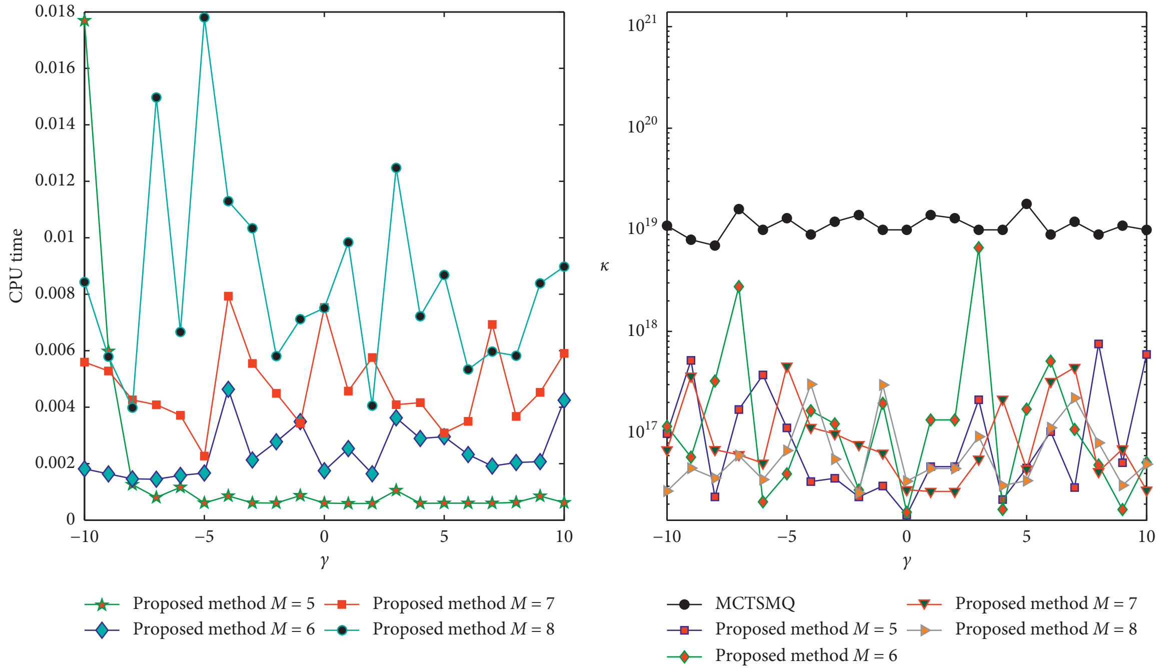


Figure 1: (a) L_∞ of Test Problem 1 obtained with the proposed method at different values of γ and its comparison with MCTSMQ (black dots). (b) $L_{\rm rms}$ of Test Problem 1 obtained with the proposed method at different values of γ and its comparison with MCTSMQ (black dots). Here we fix $\xi = 0.5$ and obtain results at different scale levels.

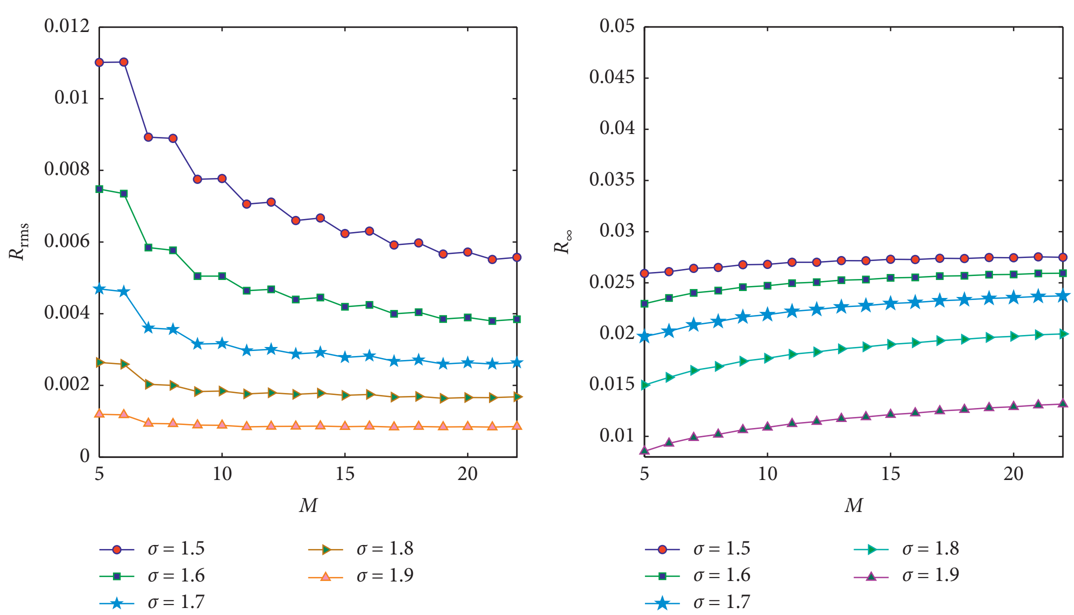
in Figure 2(b), also in the same figure, we plot the condition number obtained with MCTSMQ. We see that the condition number for MCTSMQ is approximately equal to 10^{19}, while the condition number of the proposed method is approximately equal to 10^{17}, which guarantees the robustness and stability of the proposed method as compared to MCTSMQ. From the above observations, we see that the proposed method provides a very accurate estimate of the solution of the problem.

HWCM, MCTMQ, and MCTSMQ can only handle integer order Poisson equations. Besides high accuracy, one



(a) (b)

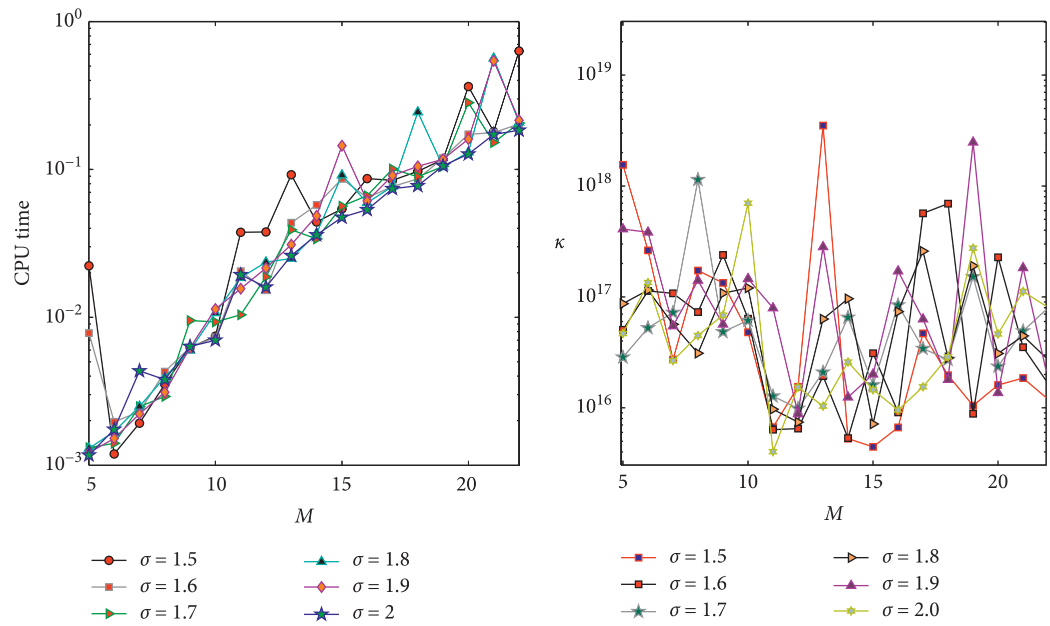
FIGURE 2: (a) CPU time of Test Problem 1 consumed by proposed method at different values of γ and different scale levels. (b) Condition number of proposed method at different scale level and different values of γ, and its comparison with condition number obtained with MCTSMQ(black dots).



(a) (b)

FIGURE 3: (a) Residual norm R_{rms} of Test Problem 1 at different scale levels for different values of σ . (b) Residual norm R_{∞} of Test Problem 1 at different scale levels for different values of σ.

of the significant advantages of the proposed method is that it can also solve the fractional order Poisson equation (the case when $1 < \sigma \leq 2$). Note that if $\sigma \neq 2$ then the exact solution of the first two problems is not known. Therefore, to check the accuracy of the approximate solution, we use two parameters R_{rms} and R_{∞}. We approximate solution for some fractional values of σ; i.e., $\sigma = 1.5, 0.1, 2$, for each value of σ we calculate the residual norms R_{rms} and R_{∞} at scale level



(a) (b)

Figure 4: (a) CPU time of Test Problem 1 at different scale levels for different values of σ . (b) Condition number κ of Problem 5.1 at different scale levels for different values of σ.

Table 5: L_∞ of Test Problem 2 obtained with the proposed method and its comparison with HWCM, MCTMQ, and MCTSMQ [22].

HWCM		MCTMQ		MCTSMQ	Proposed method	
M (n)	L_∞	N	L_∞	L_∞	M (n)	L_∞
1 (12)	8.4747×10^{-2}	4	5.3642×10^{-3}	5.3050×10^{-3}	5 (25)	2.7635×10^{-3}
2 (32)	2.9680×10^{-2}	8	3.1607×10^{-5}	1.0564×10^{-4}	6 (36)	3.2254×10^{-3}
4 (98)	8.0890×10^{-3}	16	1.1891×10^{-4}	6.2451×10^{-6}	7 (49)	7.6056×10^{-5}
8 (320)	2.0735×10^{-3}	32	1.4152×10^{-2}	3.5764×10^{-6}	8 (64)	1.1291×10^{-4}
16 (1152)	5.2168×10^{-4}	64	5.8808×10^{-3}	1.1688×10^{-6}	9 (81)	2.26363×10^{-7}
32 (4325)	1.3062×10^{-4}		Not available		10 (100)	1.9237×10^{-7}

These results are obtained using $\xi = 0.5$ and $\gamma = 1$.

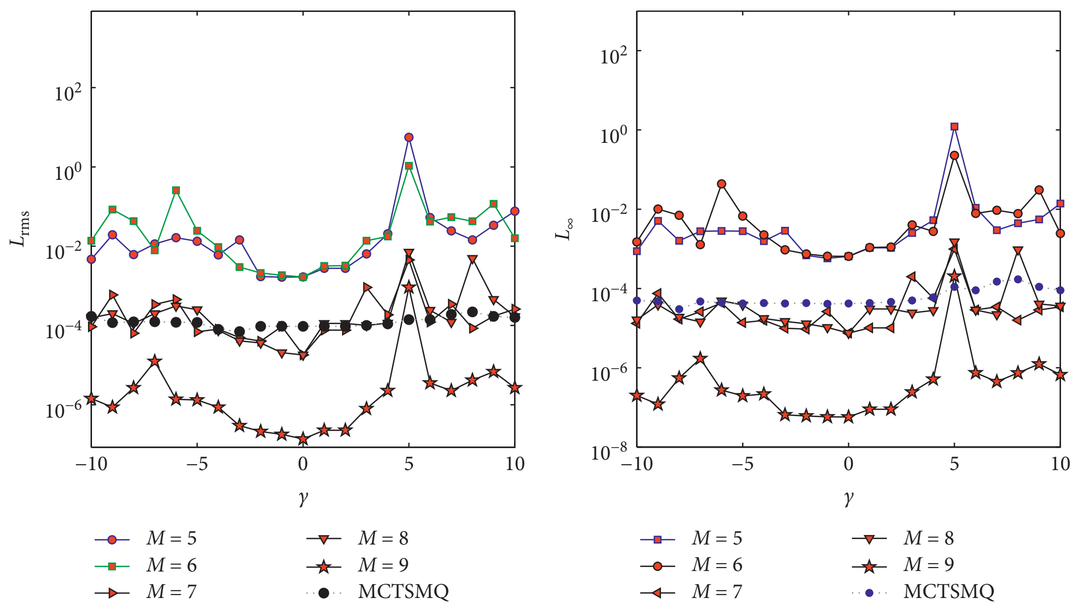
Table 6: L_{rms} of Test Problem 2 obtained with the proposed method and its comparison with HWCM, MCTMQ, and MCTSMQ [22].

HWCM		MCTMQ		MCTSMQ	Proposed method	
M (n)	L_{rms}	N	L_{rms}	L_{rms}	M (n)	L_{rms}
1 (12)	9.5549×10^{-2}	4	2.8660×10^{-3}	2.8212×10^{-3}	5 (25)	1.0591×10^{-3}
2 (32)	3.7649×10^{-2}	8	1.6169×10^{-5}	5.5406×10^{-5}	6 (36)	1.1128×10^{-3}
4 (98)	1.3624×10^{-2}	16	6.2722×10^{-5}	3.3011×10^{-6}	7 (49)	1.0155×10^{-5}
8 (320)	4.8439×10^{-3}	32	3.6982×10^{-3}	1.6629×10^{-6}	8 (64)	3.0552×10^{-5}
16 (1152)	1.7150×10^{-3}	64	1.7118×10^{-3}	5.2915×10^{-7}	9 (81)	9.0136×10^{-8}
32 (4325)	6.0655×10^{-4}		Not available		10 (100)	8.5282×10^{-8}

These results are obtained using $\xi = 0.5$ and $\gamma = 1$.

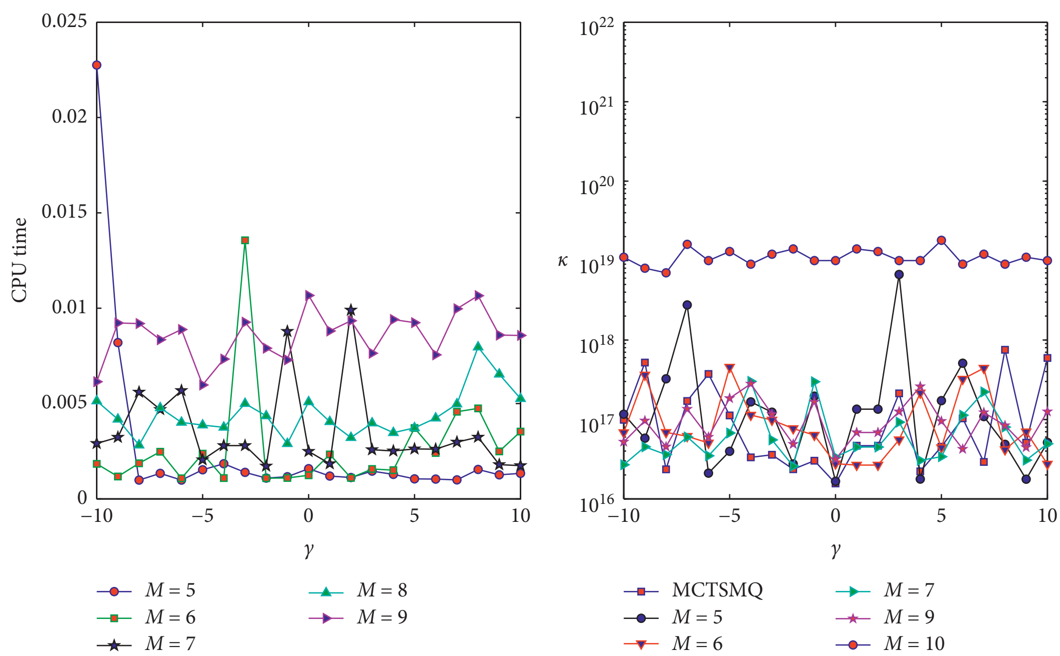
Table 7: L_∞ and L_{rms} of Test Problem 3 for different values of γ at $M = 10$ and its comparison with HWCM [22] ($M = 32$).

	HWCM [22]		Proposed method	
γ	L_∞	L_{rms}	L_∞	L_{rms}
−8	1.5165×10^{-4}	3.5243×10^{-4}	2.6752×10^{-6}	5.5132×10^{-7}
−6	1.4002×10^{-4}	3.4396×10^{-4}	1.3673×10^{-6}	2.7300×10^{-7}
−4	1.2134×10^{-4}	3.3437×10^{-4}	8.6599×10^{-7}	2.1811×10^{-8}
−2	8.6389×10^{-5}	3.3085×10^{-4}	2.1177×10^{-7}	6.0997×10^{-8}
0	1.0031×10^{-4}	4.0150×10^{-4}	1.3658×10^{-7}	5.7502×10^{-8}
2	6.8998×10^{-4}	1.8333×10^{-3}	2.2636×10^{-7}	9.0136×10^{-7}
4	5.7799×10^{-4}	1.1488×10^{-3}	2.2760×10^{-6}	5.1555×10^{-7}
6	3.5700×10^{-4}	6.7976×10^{-4}	3.5376×10^{-6}	7.4677×10^{-7}
8	2.9952×10^{-4}	5.6825×10^{-4}	4.1572×10^{-6}	7.4056×10^{-7}



(a) (b)

FIGURE 5: (a) L_∞ of Test Problem 2 obtained with the proposed method at different values of γ and its comparison with MCTSMQ (black dots). (b) L_{rms} of Test Problem 2 obtained with the proposed method at different values of γ and its comparison with MCTSMQ (blue dots). Here we fix $\xi = 0.5$ and obtain results at different scale levels.



(a) (b)

FIGURE 6: (a) CPU time of Test Problem 2 consumed by the proposed method at different values of γ and different scale levels. (b) Condition number of the proposed method at different scale level and different values of γ, and its comparison with condition number obtained with MCTSMQ (red dots).

ranges from 5 to 25. We observe that the solution is not too accurate for low values of σ. As the value of σ increases, the solution becomes more and more accurate. Also, as the value of M increases, the residual norms decreases, which guarantees the convergence of approximate solution for the fractional values of σ. The R_{rms} and R_∞ for Test problem 1 are shown in Figure 3. From the simulation of fractional values of σ, we observe that the CPU time and condition number κ remain the same as for $\sigma = 2$. These results, for some typical values of σ, are shown in Figure 4.

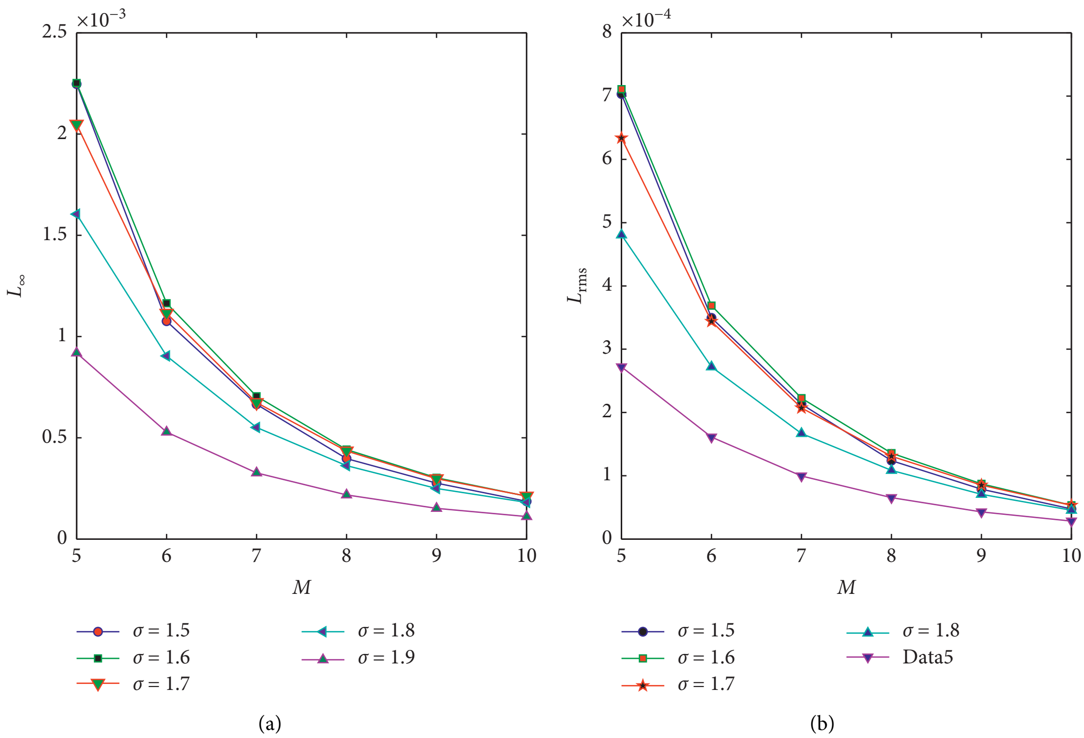


FIGURE 7: (a) L_∞ of Test Problem 3 obtained with the proposed method at different values of σ and using different scale levels. (b) L_{rms} of Test Problem 3 obtained with the proposed method at different values of σ and using different scale levels. Here, we fix $\xi = 0.5$ and obtain results at different scale levels.

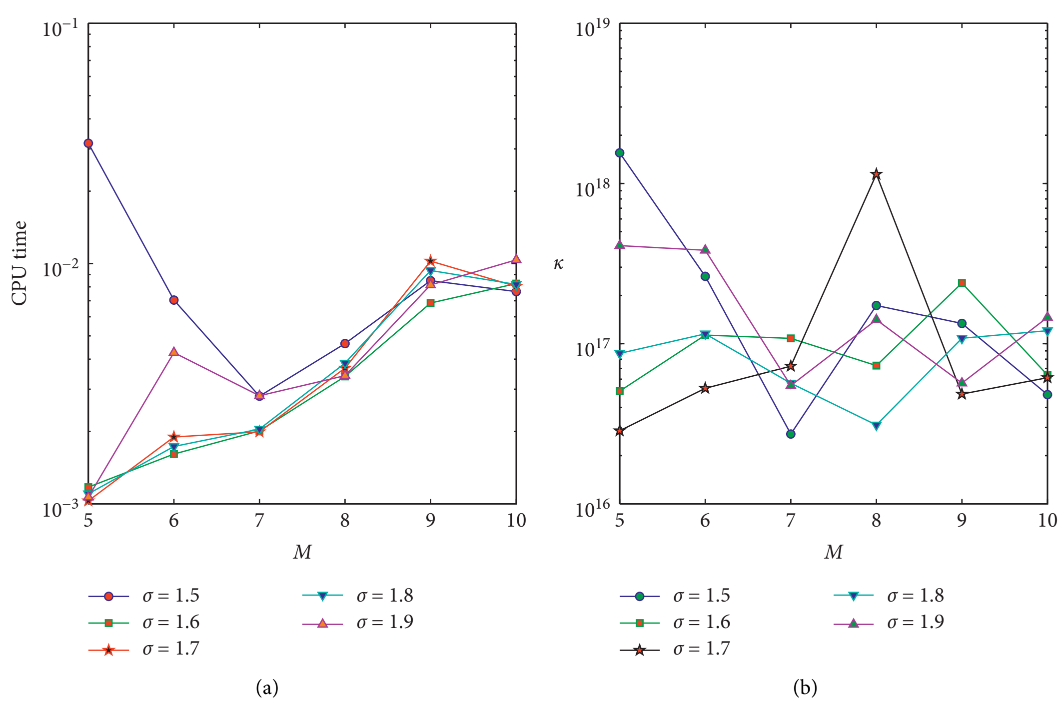
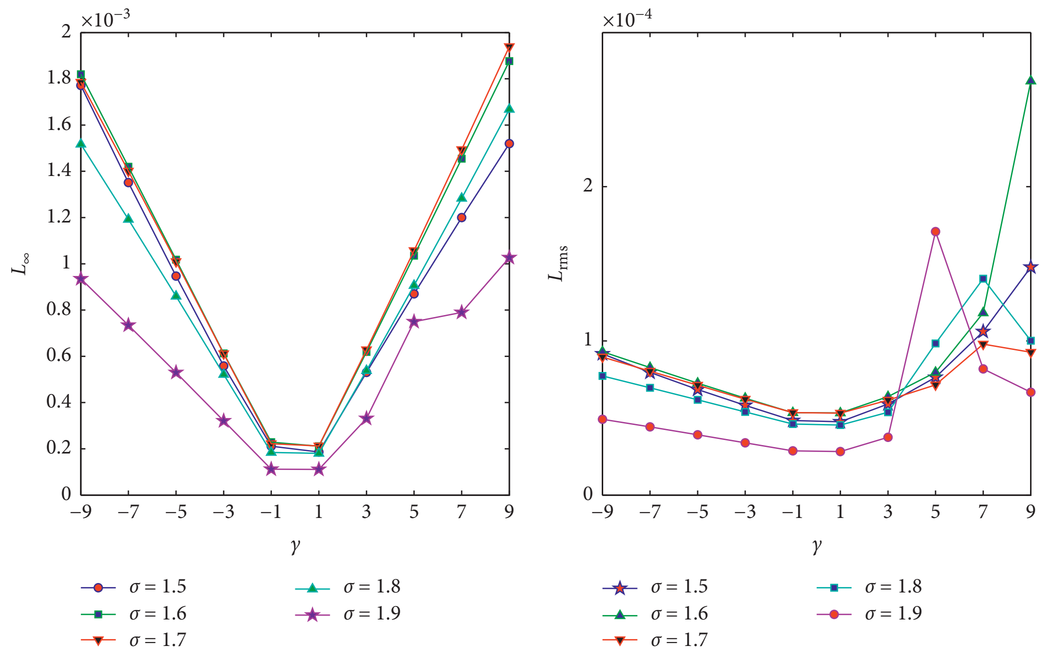
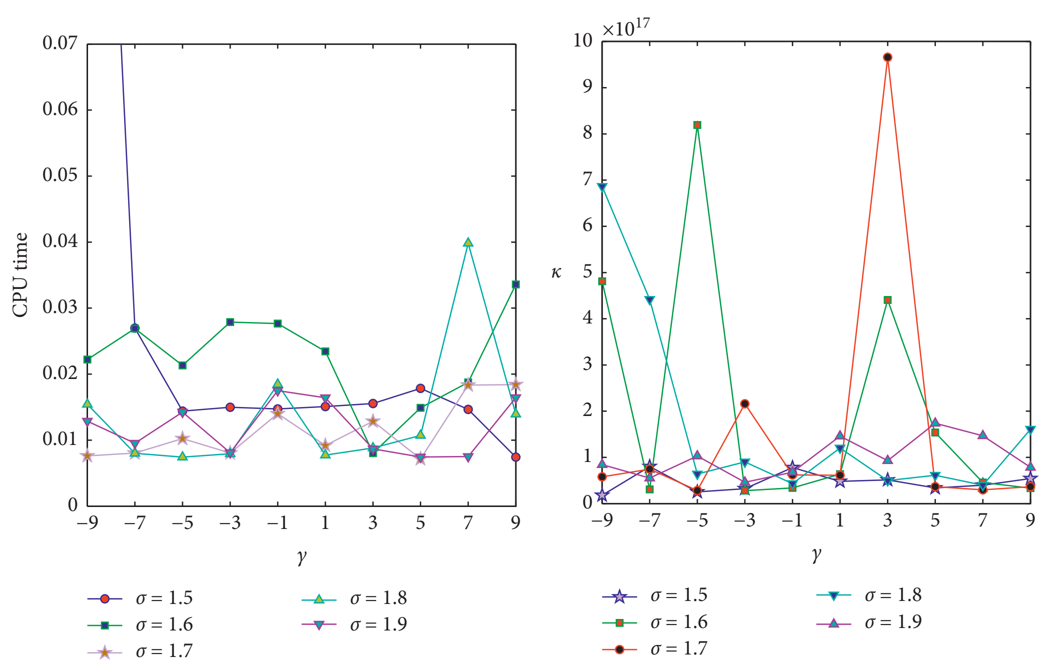


FIGURE 8: (a) CPU time of Test Problem 3 obtained with the proposed method at different values of σ and using different scale levels. (b) Condition number κ of Test Problem 3 obtained with the proposed method at different values of σ and using different scale levels. Here, we fix $\xi = 0.5$ and obtain results at different scale levels.



(a) (b)

FIGURE 9: (a) L_∞ of Test Problem 3 obtained with the proposed method at different values of γ and the fractional values of σ using scale level $M = 10$. (b) L_{rms} of Problem 5.3 obtained with the proposed method at different values of γ and the fractional values of σ using scale level $M = 10$. Here, we fix $\xi = 0.5$ and obtain results at different scale levels.



(a) (b)

FIGURE 10: (a) CPU time of Test Problem 3 obtained with the proposed method at different values of γ and the fractional values of σ using scale level $M = 10$. (b) Condition number κ of Test Problem 3 obtained with the proposed method at different values of γ and the fractional values of σ using scale level $M = 10$. Here, we fix $\xi = 0.5$ and obtain results at different scale levels.

Test Problem 2 is also analyzed, and the same conclusion is made. We fix $\xi = 0.5$ and $\sigma = 2$ and solve Test Problem 2 with the proposed method using different values of M. We observe that the proposed method yields a more accurate solution as compared to HWCM, MCTMQ, and MCTSMQ. The error norm L_∞ of the proposed method obtained using

$M = 9$ is 2.26363×10^{-7}, while L_∞ for HWCM is 5.2168×10^{-4}, and for MCTMQ and MCTSMQ, the error norm is found to be 5.8808×10^{-3} and 1.1688×10^{-6}, respectively. The detailed results of the comparison of L_∞ of the proposed method and other methods are displayed in Table 5. One can easily see that the proposed method yields better results than other methods.

Similarly, the error norm L_{rms} is also compared with HWCM, MCTMQ, and MCTSMQ. The proposed method yields $L_{\text{rms}} = 9.0136 \times 10^{-8}$ at $M = 9$, while the value of L_{rms} obtained with HWCM is 1.7150×10^{-3} and that for MCTMQ and MCTSMQ is 1.7118×10^{-3} and 5.2915×10^{-7}, respectively. It is a clear indication of the superiority of the proposed method over these methods. The detailed results of the comparison of L_{rms} are displayed in Table 6.

We fixed $\sigma = 2$ and $M = 10$, and simulated the algorithm using different choices of parameter γ ranges from -8 to 8. For every value of γ, we record the value of L_∞ and L_{rms}. It is found that maximum values of L_∞ and L_{rms} are 4.1572×10^{-6} and 7.4056×10^{-7}, which are obtained at $\gamma = 8$. We compared L_∞ and L_{rms}, for every value of γ, with HWCM, and it is observed that for every value of γ, the proposed method yields a correct solution. A detailed comparison is presented in Table 7. L_∞ and L_{rms} for different values of γ of the proposed method are also compared with MCTSMQ. The results are displayed in Figure 5. The CPU time and condition number κ of the proposed method for this problem are shown in Figure 6. In the same figure, the condition number is compared with the condition number obtained using MCTSMQ, and it is shown that the proposed method is more stable for the current problem.

Test Problem 3 is analyzed using the proposed method. We simulate the algorithm using different scale levels and record the value of L_∞ and L_{rms} at fractional values of σ ranges from 1.5 to 1.9. The results are displayed in Figure 7. It can be easily seen that the error norm decreases with the increase of scale level M and the rate of convergence is approximately the same for all values of σ. It is also observed that the error norm at low values of σ is relatively high as compared to the high value of σ. The CPU time and condition number κ for the fractional values of σ are shown in Figure 8. It can be easily noted that the condition number is approximately equal to 10^{17}. Also, an increase of CPU time with an increase of scale level is observed.

The error norms are also calculated at different values of the parameter γ and the fractional value of σ. We observed that the error norm for this problem is low at a low absolute value of γ; as the absolute value of γ increases, the error norms increases. These results are displayed in Figure 9. However, the CPU time and condition number do not show any considerable change with changing values of γ. These results are displayed in Figure 10.

7. Conclusion and Future Work

The main advantage of the proposed method is its applicability to the fractional order Poisson equations. The method can easily handle fractional order problems with two-point boundary conditions. The method converts the heat flow phenomena to an algebraic structure, whose condition number is independent of the order of derivative. The proposed method yields a very accurate approximation when applied to fractional order Poisson equations. The comparison of results of the proposed method with some recent methods, such as, HWCM, MCTMQ, and MCTSMQ, shows that the proposed method is more appropriate for integer order problems. One of the significant advantages of this method is the computational cost. The computational time is compared to the other mentioned methods, and it is observed that the proposed method solves the problem in a very short time. By measuring the condition number of the algebraic system, the proposed method shows that the condition number of the structure is very small compared to the other mentioned methods. The proposed method also solves the fractional order partial differential equations with two-point nonlocal boundary conditions. The convergence of the proposed method is shown with test problems. One of the main targets of our plan is to study the convergence and stability of the proposed method. The extension of this method to other applied problems also lies in the domain of our future work.

Data Availability

All the data are available in the manuscript.

Conflicts of Interest

All the authors declare that there are no conflicts of interest.

Authors' Contributions

All the authors have equally contributed to the preparation of the manuscript. All authors read and approved the final manuscript.

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Research Article

Asymptotic Behavior of Solutions of Even-Order Advanced Differential Equations

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In this paper, we establish the qualitative behavior of the even-order advanced differential equation $(a(v)(y^{(\kappa-1)}(v))^{\beta})' + \sum_{i=1}^{j} q_i(v)g(y(\eta_i(v))) = 0$, $v \geq v_0$. The results obtained are based on the Riccati transformation and the theory of comparison with first- and second-order equations. This new theorem complements and improves a number of results reported in the literature. Two examples are presented to demonstrate the main results.

1. Introduction

Advanced differential equations are of practical importance, which model a phenomenon in which the rate of change of a quantity depends on present and future values of the quantity. Myschkis was the first, who discussed such equations in 1955 [1] and after him Cooke and Bellman worked further on it in 1963 [2]. These types of equations have been used in modeling of various physical and engineering phenomena. For example, population genetics [3], the study of wavelets [4], population growth [5], the field of time symmetric electrodynamics [6], neural networks [7], optimal control problems with delay [8], economics [8], dynamical systems, mathematics of networks, optimization, electrical power systems, materials, energy $j \geq 1$, etc. [9] have been studied using advanced differential equations and many approaches discussed in [10–22] can be presented for solution of such equations.

In 1980, Shah et al. [23] discussed the uniqueness and existence of the solution to nonlinear and linear such types equations, while the oscillation properties of the solution were investigated by Ladas and Stavroulakis [24], and after that, particularly in the last decade, Further refinements and improvements in the theory of advanced differential equations have been made by different researchers and it is still an active of research in engineering and applied sciences. The present paper deals with the investigation of the qualitative behavior of even-order advanced differential equation:

$$\left(a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}\right)' + \sum_{i=1}^{j} q_i(v)g\left(y\left(\eta_i(v)\right)\right) = 0, \quad v \geq v_0, \tag{1}$$

where $j \geq 1$ and β are a quotient of odd positive integers. Throughout this work, we suppose that

C1: $a \in C^1([v_0,\infty),\mathbb{R})$, $a(v) > 0$, $a'(v) \geq 0$

C2: $q_i, \eta_i \in C([v_0,\infty),\mathbb{R})$, $q_i(v) \geq 0$, $\eta_i(v) \geq v$, $\lim_{v\to\infty}\eta_i(v) = \infty$, $i = 1, 2, \ldots, j$

C3: $g \in C(\mathbb{R},\mathbb{R})$ such that $g(x)/x^{\beta} \geq k > 0$, for $x \neq 0$ and under the condition

$$\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} ds = \infty. \tag{2}$$

By a solution of (1), we mean a function $y \in C^{\kappa-1}[v_y,\infty), v_y \geq v_0$, which has the property $a(v)(y^{(\kappa-1)}(v))^{\beta} \in C^1[v_y,\infty)$, and satisfies (1) on $[v_y,\infty)$. We consider only those solutions y of (1) which satisfy $\sup\{|y(v)|: v \geq v_y\} > 0$. A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[v_y,\infty)$; otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

2. The Motivation of Studying this Paper

During this decade, several works have been accomplished in the development of the oscillation theory of higher-order advanced equations by using the Riccati transformation and the theory of comparison between first- and second-order delay equations [25–39]. Further, the oscillation theory of fourth- and second-order equations has been studied and developed by using integral averaging technique and the Riccati transformation [40–45].

The study of oscillation has been carried to fractional equations in the setting of fractional operators with singular and nonsingular kernels as well (see [46, 47] and the references therein).

The main aim of this paper is to complement and improve the results of [48–49]. For this purpose we discuss these results.

Moaaz et al. [26] considered the fourth-order differential equation:

$$\left(a(v)(y'''(v))^{\beta}\right)' + q(v)y^{\alpha}(\eta(v)) = 0, \tag{3}$$

where β, α are quotients of odd positive integers.

Grace et al. [27] considered the equation

$$\left(a(v)(y''(v))^{\beta}\right)'' + q(v)g(y(\eta(v))) = 0, \tag{4}$$

where $\eta(v) \leq v$, β is a quotient of odd positive integers.

In particular, by using the comparison technique, the equation

$$\left(\left(y^{(\kappa-1)}(v)\right)^{\beta}\right)' + q(v)y^{\beta}(\eta(v)) = 0, \tag{5}$$

has been studied by Agarwal and Grace [48], and they proved it oscillatory if

$$\liminf_{v\longrightarrow\infty} \int_{v}^{\eta(v)} (\eta(s) - s)^{\kappa-2}\left(\int_{s}^{\infty} q(v)\mathrm{d}v\right)^{1/\beta}\mathrm{d}s > \frac{(\kappa-2)!}{e}. \tag{6}$$

Agarwal and Grace [48] extended the Riccati transformation to obtain new oscillatory criteria for (5) as condition

$$\limsup_{v\longrightarrow\infty} v^{\beta(\kappa-1)} \int_{v}^{\infty} q(s)\mathrm{d}s > ((\kappa-1)!)^{\beta}. \tag{7}$$

Authors in [50] studied oscillatory behavior of (5) where $\beta = 1$ and if there exists a function $\tau \in C^1([v_0,\infty),(0,\infty))$, also, they proved it oscillatory by using the Riccati transformation if

$$\int_{v_0}^{\infty}\left(\tau(s)q(s) - \frac{(\kappa-2)!(\tau'(s))^2}{2^{3-2\kappa}s^{\kappa-2}\tau(s)}\right)\mathrm{d}s = \infty. \tag{8}$$

To prove this, we apply the previous results to the equation

$$y^{(4)}(v) + \frac{q_0}{v^4}y(2v) = 0, \quad v \geq 1. \tag{9}$$

(1) By applying condition (6) in [48], we get

$$q_0 > 25.5. \tag{10}$$

(2) By applying condition (7) in [49], we get

$$q_0 > 18. \tag{11}$$

(3) By applying condition (8) in [50], we get

$$q_0 > 1728. \tag{12}$$

From the above, we find the results in [49] improve results [50]. Moreover, the results in [48] improve results [49, 50].

Thus, the motivation in studying this paper is complement and improve results [48–50].

We shall employ the following lemmas.

Lemma 1 (see [44]). *If $y^{(i)}(v) > 0$, $i = 0, 1, \ldots, \kappa$, and $y^{(\kappa+1)}(v) < 0$, then*

$$\frac{y(v)}{v^{\kappa}/\kappa!} \geq \frac{y'(v)}{v^{\kappa-1}/(\kappa-1)!}. \tag{13}$$

Lemma 2 (see [44]). *Suppose that $y \in C^{\kappa}([v_0,\infty),(0,\infty))$, $y^{(\kappa)}$ is of a fixed sign on $[v_0,\infty)$, $y^{(\kappa)}$ not identically zero and there exists a $v_1 \geq v_0$ such that*

$$y^{(\kappa-1)}(v)y^{(\kappa)}(v) \leq 0, \tag{14}$$

for all $v \geq v_1$. If we have $\lim_{v\longrightarrow\infty} y(v) \neq 0$, then there exists $v_\theta \geq v_1$ such that

$$y(v) \geq \frac{\theta}{(\kappa-1)!}v^{\kappa-1}\left|y^{(\kappa-1)}(v)\right|, \tag{15}$$

for every $\theta \in (0,1)$ and $v \geq v_\theta$.

Lemma 3 (see [34]). *Let β be a ratio of two odd numbers, $V > 0$ and U are constants. Then*

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\frac{U^{\beta+1}}{V^{\beta}}, \quad V > 0. \tag{16}$$

Lemma 4 (see [29]). *Suppose that y is an eventually positive solution of (1). Then, there exist two possible cases:*

$(\mathbf{S}_1)$ $y(v)>0, y'(v)>0,$ $y''(v)>0, y^{(\kappa-1)}(v)>0,$ $y^{(\kappa)}(v)<0$

$(\mathbf{S}_2)$ $y(v)>0,$ $y^{(r)}(v)>0,$ $y^{(r+1)}(v)<0$ *for all odd integer,* $r\in\{1,3,\ldots,\kappa-3\}$, $y^{(\kappa-1)}(v)>0,$ $y^{(\kappa)}(v)<0$

For $v\geq v_1$, where $v_1\geq v_0$ is sufficiently large.

3. Comparison Theorems with Second/First-Order Equations

Theorem 1. *Assume that (2) holds. If the differential equations*

$$\left(\frac{(\kappa-2)!a^{1/\beta}(v)}{(\theta v^{\kappa-2})^{\beta}}(y'(v))^{\beta}\right)'+k\sum_{i=1}^{j}q_i(v)y^{\beta}(v)=0, \quad (17)$$

$$y''(v)+y(v)\frac{1}{(\kappa-4)!}\int_v^{\infty}(\varsigma-v)^{\kappa-4}\cdot\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j}q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma=0. \quad (18)$$

are oscillatory. Then every solution of (1) is oscillatory.

Proof. Assume the contrary that y is a positive solution of (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive for all $v\geq v_1$ sufficiently large. From Lemma 4, we have two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

Let case $(\mathbf{S}_1)$ holds. Using Lemma 2, we find

$$y'(v)\geq\frac{\theta}{2}v^{\kappa-2}y^{(\kappa-1)}(v), \quad (19)$$

for every $\theta\in(0,1)$ and for all large v.

Define

$$\varphi(v):=\tau(v)\left(\frac{a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}}{y^{\beta}(v)}\right), \quad (20)$$

we see that $\varphi(v)>0$ for $v\geq v_1$, where $\tau\in C^1([v_0,\infty),(0,\infty))$ and

$$\varphi'(v)=\tau'(v)\frac{a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}}{y^{\beta}(v)}+\tau(v)\frac{\left(a\left(y^{(\kappa-1)}\right)^{\beta}\right)'(v)}{y^{\beta}(v)}-\beta\tau(v)\frac{y^{\beta-1}(v)y'(v)a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}}{y^{2\beta}(v)}. \quad (21)$$

Using (19) and (20), we obtain

$$\begin{aligned}\varphi'(v)&\leq\frac{\tau_+'(v)}{\tau(v)}\varphi(v)+\tau(v)\frac{\left(a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}\right)'}{y^{\beta}(v)}-\beta\tau(v)\frac{\theta}{(\kappa-2)!}v^{\kappa-2}\frac{a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta+1}}{y^{\beta+1}(v)}\\&\leq\frac{\tau'(v)}{\tau(v)}\varphi(v)+\tau(v)\frac{\left(a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}\right)'}{y^{\beta}(v)}-\frac{\beta\theta v^{\kappa-2}}{(\kappa-2)!(\tau(v)a(v))^{1/\beta}}\varphi(v)^{\beta+1/\beta}.\end{aligned} \quad (22)$$

From (1) and (22), we obtain

$$\varphi'(v)\leq\frac{\tau'(v)}{\tau(v)}\varphi(v)-k\tau(v)\frac{\sum_{i=1}^{j}q_i(v)y^{\beta}(\eta_i(v))}{y^{\beta}(v)}-\frac{\beta\theta v^{\kappa-2}}{(\kappa-2)!(\tau(v)a(v))^{1/\beta}}\varphi(v)^{\beta+1/\beta}. \quad (23)$$

Note that $y'(v)>0$ and $\eta_i(v)\geq v$; thus, we find

$$\varphi'(v)\leq\frac{\tau'(v)}{\tau(v)}\varphi(v)-k\tau(v)\sum_{i=1}^{j}q_i(v)-\frac{\beta\theta v^{\kappa-2}}{(\kappa-2)!(\tau(v)a(v))^{1/\beta}}\varphi(v)^{\beta+1/\beta}. \quad (24)$$

If we set $\tau(v)=k=1$ in (24), then we find

$$\varphi'(v)+\frac{\beta\theta v^{\kappa-2}}{(\kappa-2)!a^{1/\beta}(v)}\varphi(v)^{\beta+1/\beta}+\sum_{i=1}^{j}q_i(v)\leq 0. \quad (25)$$

From [25], we can see that equation (17) is non-oscillatory, which is a contradiction.

Let case $(\mathbf{S}_2)$ holds. Define

$$\psi(v):=\vartheta(v)\frac{y'(v)}{y(v)}, \quad (26)$$

we see that $\psi(v)>0$ for $v\geq v_1$, where $\vartheta\in C^1([v_0,\infty),(0,\infty))$. By differentiating $\psi(v)$, we find

$$\psi'(v)=\frac{\vartheta'(v)}{\vartheta(v)}\psi(v)+\vartheta(v)\frac{y''(v)}{y(v)}-\frac{1}{\vartheta(v)}\psi(v)^2. \quad (27)$$

Now, integrating (1) from v to m and using $y'(v)>0$, we find

$$a(m)\left(y^{(\kappa-1)}(m)\right)^{\beta}-a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}=-\int_{v}^{m}\sum_{i=1}^{j}q_i(s)g(y(\eta_i(s)))\mathrm{d}s. \tag{28}$$

By virtue of $y'(v)>0$ and $\eta_i(v)\geq v$, we get

$$\begin{aligned}&a(m)\left(y^{(\kappa-1)}(m)\right)^{\beta}-a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}\\&\leq -ky^{\beta}(v)\int_{v}^{u}\sum_{i=1}^{j}q_i(s)\mathrm{d}s.\end{aligned} \tag{29}$$

Letting $m\longrightarrow\infty$, we see that

$$a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}\geq ky^{\beta}(v)\int_{v}^{\infty}\sum_{i=1}^{j}q_i(s)\mathrm{d}s, \tag{30}$$

and so

$$y^{(\kappa-1)}(v)\geq y(v)\left(\frac{k}{a(v)}\int_{v}^{\infty}\sum_{i=1}^{j}q_i(s)\mathrm{d}s\right)^{1/\beta}. \tag{31}$$

Integrating again from v to ∞ for a total of $(\kappa-4)$ times, we get

$$\begin{aligned}&y''(v)+\frac{y(v)}{(\kappa-4)!}\int_{v}^{\infty}(\varsigma-v)^{\kappa-4}\\&\cdot\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j}q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma\leq 0.\end{aligned} \tag{32}$$

From (27) and (32), we obtain

$$\begin{aligned}\psi'(v)\leq&\frac{\vartheta'(v)}{\vartheta(v)}\psi(v)-\frac{\vartheta(v)}{(\kappa-4)!}\int_{v}^{\infty}(\varsigma-v)^{\kappa-4}\\&\cdot\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j}q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma-\frac{1}{\vartheta(v)}\psi(v)^2.\end{aligned} \tag{33}$$

If we now set $\vartheta(v)=k=1$ in (33), then we obtain

$$\begin{aligned}&\psi'(v)+\psi^2(v)+\frac{1}{(\kappa-4)!}\int_{v}^{\infty}(\varsigma-v)^{\kappa-4}\\&\cdot\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j}q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma\leq 0.\end{aligned} \tag{34}$$

From [25], we see equation (18) is nonoscillatory, which is a contradiction. Theorem 1 is proved. □

Remark 1. It is well known (see [42]) that if

$$\begin{aligned}&\int_{v_0}^{\infty}\frac{1}{a(v)}\mathrm{d}v=\infty,\\&\liminf_{v\longrightarrow\infty}\left(\int_{v_0}^{v}\frac{1}{a(s)}\mathrm{d}s\right)\int_{v}^{\infty}q(s)\mathrm{d}s>\frac{1}{4},\end{aligned} \tag{35}$$

then equation

$$\left[a(v)\left(y'(v)\right)^{\beta}\right]'+q(v)y^{\beta}(g(v))=0,\quad v\geq v_0, \tag{36}$$

where $\beta=1$ is oscillatory.

Based on the above results and Theorem 1, we can easily obtain the following Hille and Nehari type oscillation criteria for (1) with $\beta=1$.

Theorem 2. *Let $\beta=k=1$. Assume that (2) holds. If*

$$\int_{v_0}^{\infty}\frac{\theta v^{\kappa-2}}{(\kappa-2)!a(v)}\mathrm{d}v=\infty, \tag{37}$$

$$\liminf_{v\longrightarrow\infty}\left(\int_{v_0}^{v}\frac{\theta s^{\kappa-2}}{(\kappa-2)!a(s)}\mathrm{d}s\right)\int_{v}^{\infty}\sum_{i=1}^{j}q_i(s)\mathrm{d}s>\frac{1}{4}, \tag{38}$$

also, if

$$\begin{aligned}&\liminf_{v\longrightarrow\infty}v\int_{v_0}^{v}\frac{1}{(\kappa-4)!}\int_{v}^{\infty}(\varsigma-v)^{\kappa-4}\\&\cdot\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j}q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma\mathrm{d}v>\frac{1}{4},\end{aligned} \tag{39}$$

for some constant $\theta\in(0,1)$. Then all solution of (1) is oscillatory.

In the theorem, we compare the oscillatory behavior of (1) with the first-order differential equations:

Theorem 3. *Assume that (2) holds. If the differential equations*

$$x'(v)+k\sum_{i=1}^{j}q_i(v)\left(\frac{\theta v^{\kappa-2}}{(\kappa-2)!a^{1/\beta}(v)}\right)^{\beta}x(\eta(v))=0, \tag{40}$$

$$\begin{aligned}&z'(v)+z(v)\frac{v}{(\kappa-4)!}\int_{v}^{\infty}(\varsigma-v)^{\kappa-4}\\&\cdot\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j}q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma=0,\end{aligned} \tag{41}$$

are oscillatory, then every solution of (1) is oscillatory.

Proof. Assume the contrary that y is a positive solution of (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

In the case where $(\mathbf{S}_1)$ holds, from Lemma 2, we see

$$y(v) \geq \frac{\theta v^{\kappa-2}}{(\kappa-2)! a^{1/\beta}(v)} \left(a^{1/\beta}(v) y^{(\kappa-1)}(v)\right), \tag{42}$$

for every $\theta \in (0,1)$ and for all large v. Thus, if we set

$$x(v) = a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta} > 0, \tag{43}$$

then we see that ψ is a positive solution of the inequality.

$$x'(v) + k\sum_{i=1}^{j} q_i(v)\left(\frac{\theta v^{\kappa-2}}{(\kappa-2)! a^{1/\beta}(v)}\right)^{\beta} x(\eta(v)) \leq 0. \tag{44}$$

From [?, Theorem 1], we see that the equation (40) also has a positive solution, which is a contradiction.

In the case where $(\mathbf{S}_2)$ holds, from Lemma 1, we get

$$y(v) \geq v y'(v). \tag{45}$$

From (32) and (45), we get

$$y''(v) + y'(v)\frac{v}{(\kappa-4)!}\int_v^{\infty}(\varsigma - v)^{\kappa-4} \cdot \left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j} q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma \leq 0. \tag{46}$$

Now, we set

$$z(v) = y'(v). \tag{47}$$

Thus, we find ψ is a positive solution of the inequality

$$z'(v) + z(v)\frac{v}{(\kappa-4)!}\int_v^{\infty}(\varsigma - v)^{\kappa-4} \cdot \left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j} q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma \leq 0. \tag{48}$$

It is well known (see [?, Theorem 1]) that the equation (41) also has a positive solution, which is a contradiction. The proof is complete. □

Corollary 1. *Let (2) holds. If*

$$\begin{aligned}
&\liminf_{v\longrightarrow\infty}\int_{\eta_i(v)}^{v}\sum_{i=1}^{j} q_i(s)\left(\frac{\theta v^{\kappa-2}}{(\kappa-2)! a^{1/\beta}(v)}\right)^{\beta}\mathrm{d}s > \frac{((\kappa-1)!)^{\beta}}{\mathrm{e}} \\
&\liminf_{v\longrightarrow\infty}\int_{\eta_i(v)}^{v}\frac{s}{(\kappa-4)!}\int_v^{\infty}(\varsigma - v)^{\kappa-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j} q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma\,\mathrm{d}s > \frac{1}{\mathrm{e}},
\end{aligned} \tag{49}$$

then every solution of (1) is oscillatory.

Example 1. Let the equation

$$y^{(4)}(v) + \frac{q_0}{v^4} y(3v) = 0, \quad v \geq 1, \tag{50}$$

where $q_0 > 0$ is a constant. Note that $\beta = 1, \kappa = 4, a(v) = 1, q(v) = q_0/v^4$, and$\eta(v) = 3v$. If we set $k = 1$, then condition (38) becomes

$$\begin{aligned}
&\liminf_{v\longrightarrow\infty}\left(\int_{v_0}^{v}\frac{\theta s^{\kappa-2}}{(\kappa-2)! a(s)}\mathrm{d}s\right)\int_v^{\infty}\sum_{i=1}^{j} q_i(s)\mathrm{d}s \\
&= \liminf_{v\longrightarrow\infty}\left(\frac{v^3}{3}\right)\int_v^{\infty}\frac{q_0}{s^4}\mathrm{d}s = \frac{q_0}{9} > \frac{1}{4},
\end{aligned} \tag{51}$$

and condition (39) becomes

$$\begin{aligned}
&\liminf_{v\longrightarrow\infty} v\int_{v_0}^{v}\frac{1}{(\kappa-4)!}\int_v^{\infty}(\varsigma - v)^{\kappa-4} \cdot \left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j} q_i(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma\,\mathrm{d}v \\
&= \liminf_{v\longrightarrow\infty} v\left(\frac{q_0}{6v}\right) = \frac{q_0}{6} > \frac{1}{4}.
\end{aligned} \tag{52}$$

Therefore, from Theorem 2, all solution equation (51) is oscillatory if $q_0 > 2.25$.

Remark 2. We compare our result with the known related criteria for oscillation of this equation as follows (Table 1).

Therefore, our result improves results [48–50].

Example 2. Consider a differential equation (9) where $q_0 > 0$ is a constant. Note that $\beta = 1, \kappa = 4, a(v) = 1, q(v) = q_0/v^4$, and $\eta(v) = 2v$. If we set $k = 1$, then condition (38) becomes

Table 1: Comparision of results.

The condition	(6)	(7)	(8)
The criterion	$q_0 > 13.6$	$q_0 > 18$	$q_0 > 576$

$$\frac{q_0}{9} > \frac{1}{4}. \quad (53)$$

Therefore, from Theorem 2, all solution equation (9) is oscillatory if $q_0 > 2.25$.

Remark 3. Our result improves results [48–50].

4. Conclusion

In this article, we study the oscillatory behavior of a class of nonlinear even-order differential equations and establish sufficient conditions for oscillation of an even-order differential equation by using the theory of comparison with first- and second-order delay equations and Riccati substitution technique.

For researchers interested in this field, and as part of our future research, there is a nice open problem which is finding new results in the following case:

$$\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} \mathrm{d}s < \infty. \quad (54)$$

For all this, there is some research in progress.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest associated with this publication.

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Research Article

A Highly Efficient and Accurate Finite Iterative Method for Solving Linear Two-Dimensional Fredholm Fuzzy Integral Equations of the Second Kind Using Triangular Functions

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This work introduces a computational method for solving the linear two-dimensional fuzzy Fredholm integral equation of the second form (2D-FFIE-2) based on triangular basis functions. We have used the parametric form of fuzzy functions and transformed a 2D-FFIE-2 with three variables in crisp case to a linear Fredholm integral equation of the second kind. First, a method based on the use of two m-sets of orthogonal functions of triangular form is implemented on the integral equation under study to be changed to coupled algebraic equation system. In order to solve these two schemes, a finite iterative algorithm is then applied to evaluate the coefficients that provided the approximate solution of the integral problems. Three examples are given to clarify the efficiency and accuracy of the method. The obtained numerical results are compared with other direct and exact solutions.

1. Introduction

Several methods have been developed to estimate the solution of integral equation systems [1–3]. Many simple functions are used to approximate the solution of integral equations, such as orthogonal bases dependent on wavelets [4]. In addition, Maleknejad and Mirzaee developed the rationalized Haar functions [5] to approximate the solutions of the Fredholm linear integral equation method. In addition, second-type Fredholm integral equations are solved using direct triangular functions method as seen in [6] and using iterative algorithm-hybrid triangular functions method presented by Ramadan and Ali [7] where this hybrid method treats Fredholm integral equation of one dimension. More recently, Ramadan et al. [8] implemented such hybrid method to tackle system of two linear Fredholm integral equations of one dimension.

Furthermore, Maleknejad et al. [9] suggested by block pulse functions a numerical solution of the integral second-type equation.

It is explained using a series of orthogonal triangular functions, derived from the series of block pulses. Nevertheless, the fuzzy integral equations (FIEs) are required to solve and research a wide number of problems in various applied mathematics subjects, such as connection to physics, spatial, medical, and biology. FIEs therefore require approximate numerical solutions, as they are typically difficult to analytically solve. This thesis introduces a methodology used by the triangular functions (TFs) to solve the fuzzy linear FIE method of the second kind. In various implementation problems, certain parameters are typically represented by a fuzzy number rather than a crisp state, which involves the creation of mathematical models and computational algorithms to handle and solve the general fuzzy integral equations. A general method for solving the fuzzy Fredholm second-type integral equation is proposed in [10]. Recently, numerical methods have been developed to solve linear fuzzy Fredholm integral equation of the second kind in one-dimensional space (1D-FFIE-2) and two-dimensional space (2D-FFIE-2). Also, Fredholm fuzzy integral equations of the second kind are solved using the triangular functions [11], and numerical solution of linear Fredholm fuzzy equation of the second kind by block pulse functions is considered in [12]. Barkhordary et al. and Ramadan et al. [13, 14] presented a numerical technique for solving the fuzzy Fredholm integral equation of second kind. Numerical solution of two-dimensional fuzzy Fredholm integral equations of the second kind is presented via direct method using triangular functions [15]. Nouriani et al. [16] proposed a quadrature iterative method for solving the two-dimensional fuzzy Fredholm integral equations. Ezzati and Ziari [17], Hengamian Asl and Saberi-Nadjafi [18], and Bica and Popescu [19] illustrated a solution of the two-dimensional fuzzy Fredholm integral equations. A modified homotopy perturbation method for solving the two-dimensional fuzzy Fredholm integral equation is detailed in [20]. A two-dimensional nonlinear Volterra–Fredholm fuzzy integral equation is solved by using the Adomian decomposition method [21] and fuzzy bivariate triangular functions [22].

The aim of paper is to generalize the work proposed in [7] and [8] of these basis orthogonal triangular functions on (0, 1) to solve two-dimensional fuzzy Fredholm integral equations.

$$\tilde{u}(x,y) = \tilde{f}(x,y) + \lambda \int_a^b \int_c^d k(x,y,s,t)\,\tilde{u}(s,t)\,\mathrm{d}s\,\mathrm{d}t. \quad (1)$$

Section 2 presents some definitions and properties of the orthogonal triangular functions (TFs) (1D-TFs and 2D-TFs). Also, it expands functions by TFs. In Section 3, the definitions and properties of fuzzy function are given while a finite iterative algorithm is presented to solve coupled system of matrix equations in Section 4. The two-dimensional fuzzy integral equation is demonstrated and explained in Section 5 while the suggested method and the proposed iterative algorithm are detailed in Section 6. The illustrative examples and numerical results obtained are presented and discussed in Section 8.

2. Review of Triangular Functions (TFs)

2.1. Triangular Functions (TFs) of One Dimension

Definition 1. Two m-sets of triangular functions (TFs) are defined over the interval $[0, T)$ [5]:

$$T1_i(t) = \begin{cases} 1 - \dfrac{t - ih}{h}, & ih \le t < (1+i)h, \\ 0, & \text{o.w}, \end{cases}$$
$$T2_i(t) = \begin{cases} \dfrac{t - ih}{h}, & ih \le t < (1+i)h, \\ 0, & \text{o.w}, \end{cases} \quad (2)$$

where $i = 0, 1, \ldots, m-1$; m has a positive integer value; $h = (T/m)$; $T1_i$ is the ith left-handed triangular function; and $T2_i$ is the ith right-handed triangular function. Assuming $T = 1$, the TFs are defined over $[0, 1)$ and $h = (1/m)$. Based on this definition, it is clear that TFs are disjoint, orthogonal, and complete [5]. Therefore, one may write

$$\int_0^1 T1_i(t) T1_j(t)\mathrm{d}t = \int_0^1 T2_i(t) T2_j(t)\mathrm{d}t = \begin{cases} \dfrac{h}{3}, & i = j, \\ 0, & i \ne j, \end{cases} \quad (3)$$

$$\int_0^1 T1_i(t) T2_j(t)\mathrm{d}t = \int_0^1 T2_i(t) T1_j(t)\mathrm{d}t = \begin{cases} \dfrac{h}{6}, & i = j, \\ 0, & i \ne j. \end{cases} \quad (4)$$

The first m terms in the left-hand triangular functions and in the right-hand triangular functions can be written concisely in m-vectors format as

$$\begin{aligned} T1(t) &= [T1_0(t), T1_1(t), \ldots, T1_{m-1}(t)]^T, \\ T2(t) &= [T2_0(t), T2_1(t), \ldots, T2_{m-1}(t)]^T, \end{aligned} \quad (5)$$

where $T1(t)$ and $T2(t)$ are called left-handed triangular function (LHTF) vector and right-handed triangular function (RHTF) vector, respectively. The product of two TF vectors yields the following properties:

$$T1(t)T1^T(t) \cong \begin{pmatrix} T1_0(t) & 0 & \cdots & 0 \\ 0 & T1_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T1_{m-1}(t) \end{pmatrix},$$
$$T2(t)T2^T(t) \cong \begin{pmatrix} T2_0(t) & 0 & \cdots & 0 \\ 0 & T2_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T2_{m-1}(t) \end{pmatrix}, \quad (6)$$
$$T1(t)T2^T(t) \cong 0,$$
$$T2(t)T1^T(t) \cong 0,$$

where 0 is the zero $m \times m$ matrix. Also,

$$\begin{aligned} \int_0^1 T1(t)T1^T(t)\mathrm{d}t &= \int_0^1 T2(t)T2^T(t)\mathrm{d}t \cong \frac{h}{3}I, \\ \int_0^1 T1(t)T2^T(t)\mathrm{d}t &= \int_0^1 T2(t)T1^T(t)\mathrm{d}t \cong \frac{h}{6}I, \end{aligned} \quad (7)$$

in which I is an $m \times m$ identity matrix.

2.2. Two-Dimensional Triangular Functions and Their Properties [15]. An $(m_1 \times m_2)$-set of 2D-TFs on the region $(\Omega = [0,1] \times [0.1])$ is defined by

$$T_{i,j}^{1,1}(s,t)=\begin{cases}\left(1-\frac{s-ih_1}{h_1}\right)\left(1-\frac{t-jh_2}{h_2}\right), & ih_1\le s\le (i+1)h_1,\\ & jh_2\le t\le (j+1)h_2,\\ 0, & \text{otherwise},\end{cases}$$

$$T_{i,j}^{1,2}(s,t)=\begin{cases}\left(1-\frac{s-ih_1}{h_1}\right)\left(\frac{t-jh_2}{h_2}\right), & ih_1\le s\le (i+1)h_1,\\ & jh_2\le t\le (j+1)h_2,\\ 0, & \text{otherwise},\end{cases}$$

$$T_{i,j}^{2,1}(s,t)=\begin{cases}\left(\frac{s-ih_1}{h_1}\right)\left(1-\frac{t-jh_2}{h_2}\right), & ih_1\le s\le (i+1)h_1,\\ & jh_2\le t\le (j+1)h_2,\\ 0, & \text{otherwise},\end{cases}$$

$$T_{i,j}^{2,2}(s,t)=\begin{cases}\left(\frac{s-ih_1}{h_1}\right)\left(\frac{t-jh_2}{h_2}\right), & ih_1\le s\le (i+1)h_1,\\ & jh_2\le t\le (j+1)h_2,\\ 0, & \text{otherwise},\end{cases}\tag{8}$$

where: $i=0,1,2,\ldots;m_1-1;j=0,1,2,\ldots,m_2-1$; $h_1=(1/m_1)$; $h_2=(1/m_2)$; and m_1 and m_2 are arbitrary positive integers. Therefore,

$$\begin{cases}T_{i,j}^{1,1}(s,t)=T1_i(s)\cdot T1_j(t),\\ T_{i,j}^{1,2}(s,t)=T1_i(s)\cdot T2_j(t),\\ T_{i,j}^{2,1}(s,t)=T2_i(s)\cdot T1_j(t),\\ T_{i,j}^{2,2}(s,t)=T2_i(s)\cdot T2_j(t).\end{cases}\tag{9}$$

Furthermore,

$$T_{i,j}^{1,1}(s,t)+T_{i,j}^{1,2}(s,t)+T_{i,j}^{2,1}(s,t)+T_{i,j}^{2,2}(s,t)=\phi_{i,j}(s,t),\tag{10}$$

where $\varphi_{i,j}(s,t)$ is the $\{im_2+j+1\}$th block pulse function defined on $ih_1\le s\le (i+1)h_1$ and $jh_2\le t\le (j+1)h_2$ as

$$\varphi_{i,j}(s,t)=\begin{cases}1, & ih_1\le s\le (i+1)h_1,\\ & jh_2\le t\le (j+1)h_2,\\ 0, & \text{otherwise}.\end{cases}\tag{11}$$

Each of the sets $\{T_{i,j}^{1,1}(s,t),T_{i,j}^{1,2}(s,t),T_{i,j}^{2,1}(s,t)\}$ and $\{T_{i,j}^{2,2}(s,t)\}$ is obviously disjoint:

$$T_{i_1j_1}^{p_1,q_1}(s,t)\cdot T_{i_2j_2}^{p_2,q_2}(s,t)\approx\begin{cases}T_{i_1j_1}^{p_1,q_1}(s,t), & p_1=p_2,q_1=q_2,\\ & i_1=i_2,j_1=j_2,\\ 0, & \text{otherwise}.\end{cases}\tag{12}$$

For $p,q\in\{1,2\}$, $i_1,i_2=0,1,2,\ldots,m_1-1$ and $j_1,j_2=0,1,2,\ldots,m_2-1$.

Also, the 2D-TFs are orthogonal, that is,

$$\int_0^1\int_0^1 T_{i_1j_1}^{p_1,q_1}(s,t)\cdot T_{i_2j_2}^{p_2,q_2}(s,t)\,ds\,dt=\Delta_{p_1,p_2}\delta_{i_1,i_2}\cdot\Delta_{q_1,q_2}\delta_{j_1,j_2},\tag{13}$$

where δ denotes the Kronecker delta function and

$$\Delta_{\alpha,\beta}=\begin{cases}\frac{h}{3}, & \alpha=\beta\in\{1,2\},\\ \frac{h}{6}, & \alpha\ne\beta.\end{cases}\tag{14}$$

On the other hand, if

$$\begin{aligned}T11(s,t)&=\left[T_{0,0}^{1,1}(s,t),\ldots,T_{i,m_2-1}^{1,1}(s,t),T_{1,0}^{1,1}(s,t),\ldots,T_{m_1-1,m_2-1}^{1,1}(s,t)\right]^T,\\ T12(s,t)&=\left[T_{0,0}^{1,2}(s,t),\ldots,T_{i,m_2-1}^{1,2}(s,t),T_{1,0}^{1,2}(s,t),\ldots,T_{m_1-1,m_2-1}^{1,2}(s,t)\right]^T,\\ T21(s,t)&=\left[T_{0,0}^{2,1}(s,t),\ldots,T_{i,m_2-1}^{2,1}(s,t),T_{1,0}^{2,1}(s,t),\ldots,T_{m_1-1,m_2-1}^{2,1}(s,t)\right]^T,\\ T22(s,t)&=\left[T_{0,0}^{2,2}(s,t),\ldots,T_{i,m_2-1}^{2,2}(s,t),T_{1,0}^{2,2}(s,t),\ldots,T_{m_1-1,m_2-1}^{2,2}(s,t)\right]^T,\end{aligned}\tag{15}$$

then $T(s,t)$, the 2D-TF vector, can be defined as

$$T(s,t)=\begin{bmatrix} T11(s,t) \\ T12(s,t) \\ T21(s,t) \\ T22(s,t) \end{bmatrix}_{4m_1m_2\times 1},$$

$$T11\cdot T11^T \simeq \begin{bmatrix} T_{0,0}^{1,1} & 0 & \cdots & 0 \\ 0 & T_{0,1}^{1,1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & T_{m_1-1,m_2-1}^{1,1} \end{bmatrix} = \operatorname{diag}(T11),$$

$$\begin{aligned} T11\cdot T12^T &\simeq 0_{m_1m_2\times m_1m_2}, \\ T11\cdot T21^T &\simeq 0_{m_1m_2\times m_1m_2}, \\ T11\cdot T22^T &\simeq 0_{m_1m_2\times m_1m_2}. \end{aligned} \tag{16}$$

These relations are also satisfied for $T12(s,t), T21(s,t), T22(s,t)$, similarly. Hence,

$$T\cdot T^T \simeq \begin{bmatrix} \operatorname{diag}(T11) & 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} \\ 0_{m_1m_2\times m_1m_2} & \operatorname{diag}(T12) & 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} \\ 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} & \operatorname{diag}(T21) & 0_{m_1m_2\times m_1m_2} \\ 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} & \operatorname{diag}(T22) \end{bmatrix}, \tag{17}$$

$$\text{or } T(s,t)\cdot T^T(s,t) \simeq \operatorname{diag}(T(s,t)).$$

Finally by the orthogonality of $T11$, we have

$$\begin{aligned} \int_0^1\int_0^1 T11(s,t)^T T11(s,t)\,ds\,dt &= \frac{h_1}{3}I_{m_1\times m_1}\otimes\frac{h_2}{3}I_{m_2\times m_2}, \\ \int_0^1\int_0^1 T11(s,t)^T T12(s,t)\,ds\,dt &= \frac{h_1}{3}I_{m_1\times m_1}\otimes\frac{h_2}{6}I_{m_2\times m_2}, \\ \int_0^1\int_0^1 T11(s,t)^T T21(s,t)\,ds\,dt &= \frac{h_1}{6}I_{m_1\times m_1}\otimes\frac{h_2}{3}I_{m_2\times m_2}, \\ \int_0^1\int_0^1 T11(s,t)^T T22(s,t)\,ds\,dt &= \frac{h_1}{6}I_{m_1\times m_1}\otimes\frac{h_2}{6}I_{m_2\times m_2}, \end{aligned} \tag{18}$$

where $\otimes$ denotes the Kronecker product defined for two arbitrary matrices P and Q as

$$P\otimes Q = P_{i,j}Q. \tag{19}$$

The same equations are implied for $T12(s,t)$, $T21(s,t)$, and $T22(s,t)$, by similar computations. Hence, we can carry out double integration of $T(s,t)$:

$$\int_0^1\int_0^1 T(s,t)^T T(s,t)\,ds\,dt = D, \tag{20}$$

where D is $4m_1m_2\times 4m_1m_2$ matrix as follows:

$$D=\begin{bmatrix} \frac{h_1}{3}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{6}I_2 \\ \frac{h_1}{3}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{3}I_2 \\ \frac{h_1}{6}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{6}I_2 \\ \frac{h_1}{6}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{3}I_2 \end{bmatrix}, \tag{21}$$

where $I_1 = I_{m_1\times m_1}$ and $I_2 = I_{m_2\times m_2}$.

2.3. Function Expansion with 1D-TFs and 2D-TFs. The expansion of functions using triangular functions occurs in four situations.

(1) The expansion of function $f(t)$ over $[0,1)$ with respect to 1D-TFs is compactly written as

$$f(t) \cong \sum_{i=0}^{m-1} c_i T1_i(t) + \sum_{i=0}^{m-1} d_i T2_i(t) = c^T T1(t) + d^T T2(t), \tag{22}$$

where we may put $c_i = f(ih)$ and $d_i = f((i+1)h)$ for $i = 0, 1, \ldots, m-1$.

(2) The expansion of the function $f(s,t)$ defined over Ω $([0,1)\times[0,1))$ by 2D-TFs is as follows:

$$f(s,t)=\sum_{0}^{m_1-1}\sum_{0}^{m_2-1}c_{i,j}T_{i,j}^{1,1}(s,t)+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}d_{i,j}T_{i,j}^{1,2}(s,t)+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}e_{i,j}T_{i,j}^{2,1}(s,t)+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}l_{i,j}T_{i,j}^{2,2}(s,t)$$
$$f(s,t)=C^TT11(s,t)+D^TT12(s,t)+E^TT21(s,t)+L^TT22(s,t), \quad (23)$$

where F is a $4m_1m_2$ vector given by

$$F=\begin{bmatrix}C^T & D^T & E^T & L^T\end{bmatrix}^T, \quad (24)$$

and $T(s,t)$ is defined in equation (21). The 2D-TF coefficients in $C, D, E,$ and L can be computed by sampling the function $f(s,t)$ at grid points s_i and t_j such that $s_i=ih_1$ and $t_j=jh_2$, for various i and j. So, we have

$$\begin{aligned}
C_k&=c_{i,j}=f\left(s_i,t_j\right),\\
D_k&=d_{i,j}=f\left(s_i,t_{j+1}\right),\\
E_k&=e_{i,j}=f\left(s_{i+1},t_j\right),\\
L_k&=l_{i,j}=f\left(s_{i+1},t_{j+1}\right),
\end{aligned} \quad (25)$$

where $k=im_2+j$ and $i=0,1,2,\ldots,m_1-1$, $j=0,1,2,\ldots,m_2-1$. The $4m_1m_2$ vector F is called the 2D-TF coefficient vector.

(3) The expansion of the function $f(s,t,r)$ of three variables on $(\Omega\times[0,1])$ with respect to 2D-TFs and 1D-TFs is as follows:

$$f(s,t,r)=T^T(s,t)\cdot F\cdot T(r), \quad (26)$$

where $T(s,t)$ and $T(r)$ are 2D-TF vector and 1D-TF vector of dimension $4m_1m_2$ and $2m_3$, respectively, and F is a $(4m_1m_2\times 2m_3)$ 2D-TF coefficient matrix. This matrix can be represented as

$$F=\begin{bmatrix}F11 & F12\\ F21 & F22\\ F31 & F32\\ F41 & F42\end{bmatrix}, \quad (27)$$

where each block of F is an $(m_1m_2\times m_3)$-matrix that can be computed by sampling the function $f(s,t,r)$ at grid points (s_i,t_j,r_k) such that

$$\begin{aligned}
s_i&=ih_1, \quad i=0,1,\ldots,m_1-1,\ h_1=\frac{1}{m_1},\\
t_j&=jh_2, \quad j=0,1,\ldots,m_2-1,\ h_2=\frac{1}{m_2},\\
r_k&=kh_3, \quad k=0,1,\ldots,m_3-1,\ h_3=\frac{1}{m_3}.
\end{aligned} \quad (28)$$

Let $l=im_2+j$; then,

$$\begin{aligned}
F11_{l,k}&=f\left(s_i,t_j,r_k\right),\\
F12_{l,k}&=f\left(s_i,t_j,r_{k+1}\right),\\
F21_{l,k}&=f\left(s_i,t_{j+1},r_k\right),\\
F22_{l,k}&=f\left(s_i,t_{j+1},r_{k+1}\right),\\
F21_{l,k}&=f\left(s_i,t_{j+1},r_k\right),\\
F22_{l,k}&=f\left(s_i,t_{j+1},r_{k+1}\right),\\
F31_{l,k}&=f\left(s_{i+1},t_j,r_k\right),\\
F32_{l,k}&=f\left(s_{i+1},t_j,r_{k+1}\right),\\
F31_{l,k}&=f\left(s_{i+1},t_j,r_k\right),\\
F32_{l,k}&=f\left(s_{i+1},t_j,r_{k+1}\right),\\
F41_{l,k}&=f\left(s_{i+1},t_{j+1},r_k\right),\\
F42_{l,k}&=f\left(s_{i+1},t_{j+1},r_{k+1}\right).
\end{aligned} \quad (29)$$

(4) The expansion of the function $k(s,t,x,y)$ of four variables on $(\Omega\times\Omega)$ with respect to 2D-TFs is as follows:

$$k(s,t,x,y)=T^T(s,t)\cdot K\cdot T(x,y), \quad (30)$$

where $T(S,T)$ and $T(x,y)$ are 2D-TF vectors of dimension $4m_1m_2$ and $4m_3m_4$, respectively, and K is a $(4m_1m_2\times 4m_3m_4)$ 2D-TF coefficient matrix. This matrix can be represented as

$$K = \begin{bmatrix} K11 & K12 & K13 & K14 \\ K21 & K22 & K23 & K24 \\ K31 & K32 & K33 & K34 \\ K41 & K42 & K43 & K44 \end{bmatrix}, \tag{31}$$

where each block of K is an $(m_1 m_2 \times m_3 m_4)$ matrix that can be computed by sampling the function $k(s,t,x,y)$ at grid points $(s_{i_1}, t_{j_1}, x_{i_2}, y_{j_2})$ such that

$$\begin{aligned} s_{i_1} &= i_1 h_1, \quad i_1 = 0,1,\ldots,m_1-1,\ h_1 = \frac{1}{m_1}, \\ t_{j_1} &= j_1 h_2, \quad j_1 = 0,1,\ldots,m_2-1,\ h_2 = \frac{1}{m_2}, \\ x_{i_2} &= i_2 h_3, \quad i_2 = 0,1,\ldots,m_3-1,\ h_3 = \frac{1}{m_3}, \\ y_{j_2} &= j_2 h_3, \quad j_2 = 0,1,\ldots,m_4-1,\ h_4 = \frac{1}{m_4}. \end{aligned} \tag{32}$$

Let $p = i_1 m_2 + j_1$ and $q = i_2 m_4 + j_2$; then,

$$\begin{aligned} K11_{p,q} &= k\left(s_{i_1}, t_{j_1}, x_{i_2}, y_{j_2}\right), \\ K12_{p,q} &= k\left(s_{i_1}, t_{j_1}, x_{i_2}, y_{j_2+1}\right), \\ K13_{p,q} &= k\left(s_{i_1}, t_{j_1}, x_{i_2+1}, y_{j_2}\right), \\ K14_{p,q} &= k\left(s_{i_1}, t_{j_1}, x_{i_2+1}, y_{j_2+1}\right), \\ K21_{p,q} &= k\left(s_{i_1}, t_{j_1+1}, x_{i_2}, y_{j_2}\right), \\ K22_{p,q} &= k\left(s_{i_1}, t_{j_1+1}, x_{i_2}, y_{j_2+1}\right), \\ K23_{p,q} &= k\left(s_{i_1}, t_{j_1+1}, x_{i_2+1}, y_{j_2}\right), \\ K24_{p,q} &= k\left(s_{i_1}, t_{j_1+1}, x_{i_2+1}, y_{j_2+1}\right), \\ K31_{p,q} &= k\left(s_{i_1+1}, t_{j_1}, x_{i_2}, y_{j_2}\right), \\ K32_{p,q} &= k\left(s_{i_1+1}, t_{j_1}, x_{i_2}, y_{j_2+1}\right), \\ K33_{p,q} &= k\left(s_{i_1+1}, t_{j_1}, x_{i_2+1}, y_{j_2}\right), \\ K34_{p,q} &= k\left(s_{i_1+1}, t_{j_1}, x_{i_2+1}, y_{j_2+1}\right), \\ K41_{p,q} &= k\left(s_{i_1+1}, t_{j_1+1}, x_{i_2}, y_{j_2}\right), \\ K42_{p,q} &= k\left(s_{i_1+1}, t_{j_1+1}, x_{i_2}, y_{j_2+1}\right), \\ K43_{p,q} &= k\left(s_{i_1+1}, t_{j_1+1}, x_{i_2+1}, y_{j_2}\right), \\ K44_{p,q} &= k\left(s_{i_1+1}, t_{j_1+1}, x_{i_2+1}, y_{j_2+1}\right). \end{aligned} \tag{33}$$

In this paper, we suppose that $m_1 = m_2 = m_3 = m_4 = M$ for convergence.

3. Fuzzy Functions

We now remember through the paper some definitions that are required.

Definition 2. A fuzzy number is a fuzzy set u: $R^1 \longrightarrow [0, 1]$ that conforms to the following condition [23]:

(a) u is upper semicontinuous

(b) $u(x) = 0$ outside some interval $[c, d]$

(c) There are real numbers a and b, $c \le a \le b \le d$, for which

  (i) $u(x)$ is increasing in monotonic manner on $[c, a]$

  (ii) $u(x)$ is decreasing in monotonic manner on $[b, d]$

  (iii) $u(x) = 1$ for $a \le x \le b$

Definition 3. A fuzzy number u is a pair $(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, satisfying the following requirement [5]:

(a) $\underline{u}(r)$ is bounded monotonic increasing left continuous function

(b) $\overline{u}(r)$ is bounded monotonic decreasing left continuous function

(c) $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$

For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$, and $k > 0$, we define addition $(u + v)$ and multiplication by k as

$$\begin{aligned} \left(\underline{u+v}\right)(r) &= \underline{u}(r) + \underline{v}(r) \\ \left(\overline{u+v}\right)(r) &= \overline{u}(r) + \overline{v}(r), \\ \left(\underline{ku}\right)(r) &= k\,\underline{u}(r), \\ \left(\overline{ku}\right)(r) &= k\overline{u}(r). \end{aligned} \tag{34}$$

4. Solving Coupled System of Matrix Equations Using Finite Iterative Algorithm [5]

Matrix equations can be solved using various forms of the finite iterative algorithms example [1–3, 5]. We consider iterative solutions to coupled system similar to the forms of Sylvester matrix equations [5].

$$AV + BW = C, \tag{35}$$

and second algorithm to solve coupled system of Sylvester matrix equations:

$$\begin{aligned} A_1 V + B_1 W &= C_1, \\ A_2 V + B_2 W &= C_2. \end{aligned} \tag{36}$$

Algorithm 1 (see [5]). A finite iterative algorithm is developed to solve equation (35) as follows:

(1) Input A, B, C.

(2) Pick arbitrary matrices $V \in \mathfrak{R}^{n\times p}$ and $W_1 \in \mathfrak{R}^{r\times p}$.

(3) Set

$$\begin{aligned} R_1 &= C - AV_1 - BW_1, \\ P_1 &= A^T R_1, \\ Q_1 &= B^T R_1, \\ K &= 1. \end{aligned} \tag{37}$$

(4) If $R_K = 0$, then stop and V_K and W_K are the final solutions; else, let $K = K + 1$ and go to step 5.

(5) Calculate

$$\begin{aligned} V_{K+1} &= V_K + \frac{\|R_K\|^2}{\|P_K\|^2 + \|Q_K\|^2} P_K, \\ W_{K+1} &= W_K + \frac{\|R_K\|^2}{\|P_K\|^2 + \|Q_K\|^2} Q_K, \\ R_{K+1} &= C - AV_{K+1} - BW_{K+1} \\ &= R_K - \frac{\|R_K\|^2}{\|P_K\|^2 + \|Q_K\|^2} |AP_K + BQ_K|, \\ P_{K+1} &= A^T R_{K+1} + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} P_K, \\ Q_{K+1} &= B^T R_{K+1} + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} Q_K. \end{aligned} \tag{38}$$

Algorithm 2 (see [5]). The following finite iterative algorithm is proposed to solve coupled system of Sylvester matrix equation (36):

(1) Input matrices: A_1; B_1; A_2; B_2; C_1; C_2.

(2) Pick arbitrary matrices $Y_{1_1} \in C^{n\times p}$ and $Y_{2_1} \in C^{r\times p}$.

(3) Set

$$\begin{aligned} R_1 &= \operatorname{diag}\left(C_1 - f\left(Y_{1_1}, Y_{2_1}\right), C_2 - g\left(Y_{1_1}, Y_{2_1}\right)\right), \\ S_1 &= A_1^T\left(C_1 - f\left(Y_{1_1}, Y_{2_1}\right)\right) + A_2^T\left(C_2 - g\left(Y_{1_1}, Y_{2_1}\right)\right), \\ T_1 &= B_1^T\left(C_1 - f\left(Y_{1_1}, Y_{2_1}\right)\right) + B_2^T\left(C_2 - g\left(Y_{1_1}, Y_{2_1}\right)\right). \end{aligned} \tag{39}$$

(4) If $R_K = 0$, then stop and Y_{1_K} and Y_{2_K} are the solutions; else set $K = K + 1$ and then go to step 5.

(5) Calculate

$$\begin{aligned} Y_{1_{K+1}} &= Y_{1_K} + \frac{\|R_K\|^2}{\|S_K\|^2 + \|T_K\|^2} S_K, \\ Y_{2_{K+1}} &= Y_{2_K} + \frac{\|R_K\|^2}{\|S_K\|^2 + \|T_K\|^2} T_K, \\ R_{K+1} &= \operatorname{diag}\left(C_1 - f\left(Y_{1_{K+1}}, Y_{2_{K+1}}\right), C_2 - g\left(Y_{1_{K+1}}, Y_{2_{K+1}}\right)\right) \\ &= R_K - \frac{\|R_K\|^2}{\|S_K\|^2 + \|T_K\|^2} \operatorname{diag}\left(f\left(S_K, T_K\right), g\left(S_K, T_K\right)\right) \\ S_{K+1} &= A_1^T\left(C_1 - f\left(Y_{1_{K+1}}, Y_{2_{K+1}}\right)\right) + A_2^T\left(C_2 - g\left(Y_{1_{K+1}}, Y_{2_{K+1}}\right)\right) + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} S_K, \\ T_{K+1} &= B_1^T\left(C_1 - f\left(Y_{1_{K+1}}, Y_{2_{K+1}}\right)\right) + B_2^T\left(C_2 - g\left(Y_{1_{K+1}}, Y_{2_{K+1}}\right)\right) + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} T_K. \end{aligned} \tag{40}$$

5. Two-Dimensional Fuzzy Fredholm Integral Equation

Two-dimensional FIE of the second kind is defined as follows [24]:

$$u(x, y) = f(x, y) + \lambda \int_a^b \int_c^d k(x, y, s, t)\, u(s, t) \mathrm{d}s\, \mathrm{d}t. \tag{41}$$

The linear (2D-FFIE-2) is defined as

$$\widetilde{u}(x, y) = \widetilde{f}(x, y) + \lambda \int_a^b \int_c^d (x, y, s, t) \widetilde{u}(s, t) \mathrm{d}s\, \mathrm{d}t, \tag{42}$$

where $\widetilde{u}(x, y)$ and $\widetilde{f}(x, y)$ are fuzzy real functions on $S = [a, b] \times [c, d]$, $k(x, y, s, t)$ is an arbitrary kernel function over $V = [a, b] \times [c, d] \times [a, b] \times [c, d]$, and $\widetilde{u}(x, y)$ is unknown on S.

Throughout this paper, we consider 2D-FFIE-2 with $a = c = 0$, $b = d = 1$, and $\lambda = 1$.

Now, introduce parametric form of a 2D-FFIE-2 with respect to Definition 3. Let $(\underline{f}(x,y,r), \overline{f}(x,y,r))$ and $(\underline{u}(x,y,r), \overline{u}(x,y,r))$, $0 \le r \le 1$, $(x,y) \in S$, be parametric form of $\widetilde{f}(x,y)$ and $\widetilde{u}(x,y)$, respectively. Then parametric form of 2D-FFIE-2 is as follows:

$$\underline{u}(x,y,r) = \underline{f}(x,y,r) + \int\int_0^1 v_1(x,y,s,t,\underline{u}(x,y,r),\overline{u}(x,y,r))\mathrm{d}s\,\mathrm{d}t, \tag{43}$$

$$\overline{u}(x,y,r) = \overline{f}(x,y,r) + \int\int_0^1 v_2(x,y,s,t,\underline{u}(x,y,r),\overline{u}(x,y,r))\mathrm{d}s\,\mathrm{d}t, \tag{44}$$

$$\begin{aligned} v_1(x,y,s,t,\underline{u}(x,y,r),\overline{u}(x,y,r)) &= \begin{cases} k(x,y,s,t)\,\underline{u}(s,t,r), & k(x,y,s,t) \ge 0, \\ k(x,y,s,t)\overline{u}(s,t,r), & k(x,y,s,t) < 0, \end{cases} \\ v_1(x,y,s,t,\underline{u}(x,y,r),\overline{u}(x,y,r)) &= \begin{cases} k(x,y,s,t)\overline{u}\,\underline{u}(s,t,r), & k(x,y,s,t) \ge 0, \\ k(x,y,s,t)\,\underline{u}(s,t,r), & k(x,y,s,t) < 0, \end{cases} \end{aligned} \tag{45}$$

for each $0 \le x, y \le 1$ and $0 \le r \le 1$. We can see that equations (43) and (44) are system of Fredholm integral equation of the second kind with three variables in crisp case.

6. Proposed Hybrid Iterative Technique

6.1. Converting Linear Two-Dimensional FIEs of Second Kind to Two Crisp Coupled Systems. This section presents an efficient method for soling a 2D-FFIE-2 by using 2D-TFs.

First, consider the following equation:

$$\widetilde{u}(x,y) = \widetilde{f}(x,y) + \lambda\int_a^b\int_c^d k(x,y,s,t)\widetilde{u}(s,t)\mathrm{d}s\,\mathrm{d}t. \tag{46}$$

Now, the problem is to find the TF coefficients of $\widetilde{u}(x,y)$ from the known functions $\widetilde{f}(x,y)$ and kernel $k(x,y,s,t)$. 2D-TFs are applied for equations

$$\underline{u}(x,y,r) = \underline{f}(x,y,r) + \int\int_0^1 k(x,y,s,t)\,\underline{u}(s,t,r)\mathrm{d}s\,\mathrm{d}t, \tag{47}$$

$$\overline{u}(x,y,r) = \overline{f}(x,y,r) + \int\int_0^1 k(x,y,s,t)\,\overline{u}(s,t,r)\mathrm{d}s\,\mathrm{d}t. \tag{48}$$

To describe the approach of equation (47), first expand $\underline{u}(x,y,r)$, $\underline{f}(x,y,r)$, and $k(x,y,s,t)$ by 2D-TFs as follows:

$$\underline{u}(x,y,r) \simeq T^T(x,y)UT(r), \tag{49}$$

$$\begin{aligned} \underline{u}(x,y,r) \simeq{}& T11^T(x,y)U11T1(r) + T12^T(x,y)U21T1(r) + T21^T(x,y)U31T1(r) \\ &+ T22^T(x,y)U41T1(r) + T11^T(x,y)U12T2(r) + T12^T(x,y)U22T2(r) \\ &+ T21^T(x,y)U32T2(r) + T22^T(x,y)U42T2(r), \end{aligned} \tag{50}$$

$$\underline{f}(x,y,r) \simeq T^T(x,y)FT(r), \tag{51}$$

$$\begin{aligned} \underline{f}(x,y,r) \simeq{}& T11^T(x,y)F11T1(r) + T12^T(x,y)F21T1(r) \\ &+ T21^T(x,y)F31T1(r) + T22^T(x,y)F41T1(r) + T11^T(x,y)F12T2(r) \\ &+ T12^T(x,y)F22T2(r) + T21^T(x,y)F32T2(r) + T22^T(x,y)F42T2(r), \end{aligned} \tag{52}$$

$$k(x,y,s,t) \simeq T^T(x,y)KT(s,t), \tag{53}$$

where $T(x,y)$ and $T(r)$ are defined in equations (3) and (21), respectively, U and F are $(4M^2 \times 2M)$ matrix of 2D-TF coefficients of $\underline{u}(x,y,r)$ and $\underline{f}(x,y,r)$, respectively, and K is $(4M^2 \times 4M^2)$-matrix 2D-TF coefficients of (x,y,s,t).

To obtain the solution of equation (47) from equations (49), (51), and (53), we have

$$\begin{aligned}
&T^T(x,y)UT(r) = T^T(x,y)FT(r) + \int\int_0^1 T^T(x,y)KT(s,t)T^T(x,y)UT(r)\,\mathrm{d}s\,\mathrm{d}t,\\
&T^T(x,y)UT(r) = T^T(x,y)FT(r) + T^T(x,y)K\left(\int\int_0^1 T^T(x,y)KT(s,t)T^T(x,y)\,\mathrm{d}s\,\mathrm{d}t\right)UT(r).
\end{aligned} \tag{54}$$

Using equation (22), we have

$$T^T(x,y)UT(r) = T^T(x,y)FT(r) + T^T(x,y)KDUT(r), \tag{55}$$

and then

$$U = F + KDU, \tag{56}$$

where U and F are $4M^2 \times 2M$-matrix and KD is $4M^2 \times 4M^2$- matrix, so KDU is $4M^2 \times 2M$-matrix, where U is unknown.

Then, we have

$$(I - KD)U = F, \tag{57}$$

with

$$\begin{aligned}
&U = \begin{bmatrix} U11 & U12 \\ U21 & U22 \\ U31 & U32 \\ U41 & U42 \end{bmatrix},\\
&F = \begin{bmatrix} F11 & F12 \\ F21 & F22 \\ F31 & F32 \\ F41 & F42 \end{bmatrix},\\
&K = \begin{bmatrix} K11 & K12 & K13 & K14 \\ K21 & K22 & K23 & K24 \\ K31 & K32 & K33 & K34 \\ K41 & K42 & K43 & K44 \end{bmatrix},\\
&D = \begin{bmatrix}
\frac{h_1}{3}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{6}I_2 \\
\frac{h_1}{3}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{3}I_2 \\
\frac{h_1}{6}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{6}I_2 \\
\frac{h_1}{6}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{6}I_1\otimes\frac{h_2}{3}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{6}I_2 & \frac{h_1}{3}I_1\otimes\frac{h_2}{3}I_2
\end{bmatrix},\\
&I_{4M^2\times 4M^2} = \begin{bmatrix} I_{M^2} & 0 & 0 & 0 \\ 0 & I_{M^2} & 0 & 0 \\ 0 & 0 & I_{M^2} & 0 \\ 0 & 0 & 0 & I_{M^2} \end{bmatrix},\\
&KD = \begin{bmatrix}
K11D11 + K12D21 + K13D31 + K14D41 & K11D12 + K12D22 + K13D32 + K14D42 & K11D13 + K12D23 + K13D33 + K14D43 & K11D14 + K12D24 + K13D34 + K14D44 \\
K21D11 + K22D21 + K23D31 + K24D41 & K21D12 + K22D22 + K23D32 + K24D42 & K21D13 + K22D23 + K23D33 + K24D43 & K21D14 + K22D24 + K23D34 + K24D44 \\
K31D11 + K32D21 + K33D31 + K34D41 & K31D12 + K32D22 + K33D32 + K34D42 & K31D13 + K32D23 + K33D33 + K34D43 & K31D14 + K32D24 + K33D34 + K34D44 \\
K41D11 + K42D21 + K43D31 + K44D41 & K41D12 + K42D22 + K43D32 + K44D42 & K41D13 + K42D23 + K43D33 + K44D43 & K41D14 + K42D24 + K43D34 + K44D44
\end{bmatrix},\\
&(I - KD) = \begin{bmatrix}
I_{M^2} - (K11D11 + K12D21 + K13D31 + K14D41) & -(K11D12 + K12D22 + K13D32 + K14D42) & -(K11D13 + K12D23 + K13D33 + K14D43) & -(K11D14 + K12D24 + K13D34 + K14D44) \\
-(K21D11 + K22D21 + K23D31 + K24D41) & I_{M^2} - (K21D12 + K22D22 + K23D32 + K24D42) & -(K21D13 + K22D23 + K23D33 + K24D43) & -(K21D14 + K22D24 + K23D34 + K24D44) \\
-(K31D11 + K32D21 + K33D31 + K34D41) & -(K31D12 + K32D22 + K33D32 + K34D42) & I_{M^2} - (K31D13 + K32D23 + K33D33 + K34D43) & -(K31D14 + K32D24 + K33D34 + K34D44) \\
-(K41D11 + K42D21 + K43D31 + K44D41) & -(K41D12 + K42D22 + K43D32 + K44D42) & -(K41D13 + K42D23 + K43D33 + K44D43) & I_{M^2} - (K41D14 + K42D24 + K43D34 + K44D44)
\end{bmatrix},\\
&(I - KD)U = F\\
&\quad \left(I_{M^2} - (K11D11 + K12D21 + K13D31 + K14D41)\right)U11 - (K11D12 + K12D22 + K13D32 + K14D42)U21 - (K11D13 + K12D23 + K13D33 + K14D43)U31 - (K11D14 + K12D24 + K13D34 + K14D44)U41 = F11,\\
&\quad -(K21D11 + K22D21 + K23D31 + K24D41)U11 + I_{M^2} - (K21D12 + K22D22 + K23D32 + K24D42)U21 - (K21D13 + K22D23 + K23D33 + K24D43)U31 - (K21D14 + K22D24 + K23D34 + K24D44)U41 = F21,\\
&\quad -(K31D11 + K32D21 + K33D31 + K34D41)U11 - (K31D12 + K32D22 + K33D32 + K34D42)U21 + \left(I_{M^2} - (K31D13 + K32D23 + K33D33 + K34D43)\right)U31 - (K31D14 + K32D24 + K33D34 + K34D44)U41 = F31,\\
&\quad -(K41D11 + K42D21 + K43D31 + K44D41)U11 - (K41D12 + K42D22 + K43D32 + K44D42)U21 - (K41D13 + K42D23 + K43D33 + K44D43)U31 + \left(I_{M^2} - (K41D14 + K42D24 + K43D34 + K44D44)\right)U41 = F41.
\end{aligned} \tag{58}$$

Set

$$\begin{aligned}
A_1 &= I_{M^2} - (K11D11 + K12D21 + K13D31 + K14D41),\\
A_2 &= -(K11D12 + K12D22 + K13D32 + K14D42),\\
A_3 &= -(K11D13 + K12D23 + K13D33 + K14D43),\\
A_4 &= -(K11D14 + K12D24 + K13D34 + K14D44),\\
B_1 &= -(K21D11 + K22D21 + K23D31 + K24D41),\\
B_2 &= I_{M^2} - (K21D12 + K22D22 + K23D32 + K24D42),\\
B_3 &= -(K21D13 + K22D23 + K23D33 + K24D43),\\
B_4 &= -(K21D14 + K22D24 + K23D34 + K24D44),\\
C_1 &= -(K31D11 + K32D21 + K33D31 + K34D41),\\
C_2 &= -(K31D12 + K32D22 + K33D32 + K34D42),\\
D_1 &= -(K41D11 + K42D21 + K43D31 + K44D41),\\
D_2 &= -(K41D12 + K42D22 + K43D32 + K44D42),\\
D_3 &= -(K41D13 + K42D23 + K43D33 + K44D43),\\
D_4 &= I_{M^2} - (K41D14 + K42D24 + K43D34 + K44D44),
\end{aligned} \tag{59}$$

which lead to the following two crisp linear systems:

$$A_1U11 + A_2U21 + A_3U31 + A_4U41 = F11, \tag{60}$$

$$B_1U11 + B_2U21 + B_3U31 + B_4U41 = F21, \tag{61}$$

$$C_1U11 + C_2U21 + C_3U31 + C_4U41 = F31, \tag{62}$$

$$D_1U11 + D_2U21 + D_3U31 + D_4U41 = F41, \tag{63}$$

and

$$A_1U12 + A_2U22 + A_3U32 + A_4U42 = F12, \tag{64}$$

$$B_1U12 + B_2U22 + B_3U32 + B_4U42 = F22, \tag{65}$$

$$C_1U12 + C_2U22 + C_3U32 + C_4U42 = F32, \tag{66}$$

$$D_1U12 + D_2U22 + D_3U32 + D_4U42 = F42. \tag{67}$$

Similarly, we expand $\overline{u}(x, y, r)$ and $\overline{f}(x, y, r)$ by 2D-TFs, and by substituting them into equation (42), two coupled crisp linear systems, similar to (61) and (65), are obtained. It is clear that all matrices in the two coupled crisp linear systems (61) and (65) are square matrices of dimensions $M \times M$. Thus, we need to obtain the coefficient matrices $U11, U12, U21, U22, U31, U32, U41$, and $U42$ in order to get the approximate numerical solution of the form:

$$\underline{u}(x, y, r) = T^T(x, y)UT(r). \tag{68}$$

6.2. Proposed Iterative Algorithm for Solving Coupled Systems (61) and (65). An iterative algorithm is proposed here to solve the two coupled systems (61) and (65) as a generalization of Algorithm 2.

Algorithm 3. Algorithm 2 is modified and generalized to work out for systems (61) and (65) as follows.

First, for coupled system (61):

(1) Input $A_1; A_2; A_3;\ A_4; B_1; B_2; B_3; B_4; C_1; C_2; C_3; C_4; D_1; D_2; D_3; D_4; F11; F21; F31; F41$.

(2) Choose arbitrary matrices $U11, U21, U31, U41$.

(3) For $k = 1$, set

$$\begin{aligned}
R_k &= \operatorname{diag}(F11 - f1(U11, U21, U31, U41), F21 - f2(U11, U21, U31, U41), F31 - f3(U11, U21, U31, U41), F41\\
&\quad - f4(U11, U21, U31, U41))\\
S1_k &= A_1^T(F11 - f1(U11, U21, U31, U41)) + B_1^T(F21 - f2(U11, U21, U31, U41)) + C_1^T(F31 - f3(U11, U21, U31, U41))\\
&\quad + D_1^T(F41 - f4(U11, U21, U31, U41)),\\
S2_k &= A_2^T(F11 - f1(U11, U21, U31, U41)) + B_2^T(F21 - f2(U11, U21, U31, U41)) + C_2^T(F31 - f3(U11, U21, U31, U41))\\
&\quad + D_2^T(F41 - f4(U11, U21, U31, U41)),\\
S3_k &= A_3^T(F11 - f1(U11, U21, U31, U41)) + B_3^T(F21 - f2(U11, U21, U31, U41)) + C_3^T(F31 - f3(U11, U21, U31, U41))\\
&\quad + D_3^T(F41 - f4(U11, U21, U31, U41)),\\
S4_k &= A_4^T(F11 - f1(U11, U21, U31, U41)) + B_4^T(F21 - f2(U11, U21, U31, U41)) + C_4^T(F31 - f3(U11, U21, U31, U41))\\
&\quad + D_4^T(F41 - f4(U11, U21, U31, U41)).
\end{aligned} \tag{69}$$

(4) If $R_K = 0$, then stop and $U11, U21, U31, U41$ are the solutions; else, let $K = K + 1$ and go to step 5.

(5) Compute

$$
\begin{aligned}
U11 &= U11 + \frac{\left\|R_K\right\|^2}{\left\|S1_k\right\|^2 + \left\|S2_k\right\|^2 + \left\|S3_k\right\|^2 + \left\|S4_k\right\|^2} S1_k, \\
U21 &= U21 + \frac{\left\|R_K\right\|^2}{\left\|S1_k\right\|^2 + \left\|S2_k\right\|^2 + \left\|S3_k\right\|^2 + \left\|S4_k\right\|^2} S2_k, \\
U31 &= U31 + \frac{\left\|R_K\right\|^2}{\left\|S1_k\right\|^2 + \left\|S2_k\right\|^2 + \left\|S3_k\right\|^2 + \left\|S4_k\right\|^2} S3_k, \\
U41 &= U41 + \frac{\left\|R_K\right\|^2}{\left\|S1_k\right\|^2 + \left\|S2_k\right\|^2 + \left\|S3_k\right\|^2 + \left\|S4_k\right\|^2} S4_k, \\
R_{k+1} &= \operatorname{diag}(F11 - f1(U11, U21, U31, U41), F21 - f2(U11, U21, U31, U41), F31 - f3(U11, U21, U31, U41), \\
&\quad \cdot F41 - f4(U11, U21, U31, U41)), \\
S1_{k+1} &= A_1^T(F11 - f1(U11, U21, U31, U41)) + B_1^T(F21 - f2(U11, U21, U31, U41)) + C_1^T(F31 - f3(U11, U21, U31, U41)) \\
&\quad + D_1^T(F41 - f4(U11, U21, U31, U41)) + \frac{\left\|R_{K+1}\right\|^2}{\left\|R_K\right\|^2} S1_K, \\
S2_{k+1} &= A_2^T(F11 - f1(U11, U21, U31, U41)) + B_2^T(F21 - f2(U11, U21, U31, U41)) + C_2^T(F31 - f3(U11, U21, U31, U41)) \\
&\quad + D_2^T(F41 - f4(U11, U21, U31, U41)) + \frac{\left\|R_{K+1}\right\|^2}{\left\|R_K\right\|^2} S2_K, \\
S3_{k+1} &= A_3^T(F11 - f1(U11, U21, U31, U41)) + B_3^T(F21 - f2(U11, U21, U31, U41)) + C_3^T(F31 - f3(U11, U21, U31, U41)) \\
&\quad + D_3^T(F41 - f4(U11, U21, U31, U41)) + \frac{\left\|R_{K+1}\right\|^2}{\left\|R_K\right\|^2} S3_K, \\
S4_{k+1} &= A_4^T(F11 - f1(U11, U21, U31, U41)) + B_4^T(F21 - f2(U11, U21, U31, U41)) + C_4^T(F31 - f3(U11, U21, U31, U41)) \\
&\quad + D_4^T(F41 - f4(U11, U21, U31, U41)) + \frac{\left\|R_{K+1}\right\|^2}{\left\|R_K\right\|^2} S4_K.
\end{aligned} \tag{70}
$$

For coupled system (65), the algorithm is repeated with replacing $U11, U21, U31, U41$ by $U12, U22, U32, U42$ and $F11, F21, F31, F41$ by $F12, F22, F32, F42$ where the $2M \times 2M$ block U matrix is computed.

The approximate crisp numerical solution for equation (51) of the form $\underline{u}(x, r)_{\text{approx.}} = T^T(x)UT(r)$ is then obtained.

In a similar manner, the crisp numerical solution for equation (53) of the form $\overline{u}(x, r)_{\text{approx.}} = T^T(x)UT(r)$ can be obtained by carrying out the above-proposed algorithm for the other coupled crisp systems similar to (61) and (65). Finally, the solution for linear 2D-FFIE-2 by 2D-TFs is then given as

$$u(x,y)=\left(\underline{u}_{\text{approx}}(x,y,r),\overline{u}_{\text{approx}}(x,y,r)\right),\quad 0\le r\le 1,\ x,y\in[0,1). \tag{71}$$

7. Convergence Analysis of the Proposed Method

In this section, we obtain error estimate for the numerical method proposed in previous section.

Theorem 1. *The solution of the two-dimensional Fredholm fuzzy integral equations given by equation (1) by using 2D-TFs converges to exact solution if*

$$S=\max_{0\le x,y,s,t\le 1}|k(x,y,s,t)|<1. \tag{72}$$

Proof. Assume $\tilde{u}_{\text{exact}}(x,y)$ and $\tilde{u}_{\text{approx}}(x,y)$ represent the exact and approximate solutions of equation (1), respectively. Therefore,

$$\begin{aligned}\tilde{u}_{\text{approx}}(x,y)={}&\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}c_{i,j}T_{i,j}^{1,1}(s,t)+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}d_{i,j}T_{i,j}^{1,2}(s,t)\\&+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}e_{i,j}T_{i,j}^{2,1}(s,t)+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}l_{i,j}T_{i,j}^{2,2}(s,t).\end{aligned} \tag{73}$$

By using equation (1), we can write

$$\begin{aligned}e(x,y)&=\left\|\tilde{u}_{\text{exact}}(x,y)-\tilde{u}_{\text{approx}}(x,y)\right\|=\max_{(x,y)\in[0,1]}\left|\tilde{u}_{\text{exact}}(x,y)-\tilde{u}_{\text{approx}}(x,y)\right|\\&=\max_{(x,y)\in[0,1]}\Bigg|\Bigg(\int_0^1\int_0^1 k(x,y,s,t)\tilde{u}_{\text{exact}}(s,t)\,\mathrm{d}s\,\mathrm{d}t-\int_0^1\int_0^1 k(x,y,s,t)\Bigg(\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}c_{i,j}T_{i,j}^{1,1}(s,t)+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}d_{i,j}T_{i,j}^{1,2}(s,t)\\&\quad+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}e_{i,j}T_{i,j}^{2,1}(s,t)+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}l_{i,j}T_{i,j}^{2,2}(s,t)\Bigg)\mathrm{d}s\,\mathrm{d}t\Bigg)\Bigg|\\&\le S\left(\int_0^1\int_0^1\max_{(x,y)\in[0,1]}\left|\tilde{u}_{\text{exact}}(s,t)-\tilde{u}_{\text{approx}}(s,t)\right|\mathrm{d}s\,\mathrm{d}t\right)=S\left(\int_0^1\int_0^1\left\|\tilde{u}_{\text{exact}}(s,t)-\tilde{u}_{\text{approx}}(s,t)\right\|\mathrm{d}s\,\mathrm{d}t\right),\end{aligned} \tag{74}$$

where

$$S=\max_{0\le x,y,s,t\le 1}|k(x,y,s,t)|<\infty. \tag{75}$$

Also, we have $\lim_{M\longrightarrow\infty}\tilde{u}_{\text{approx}}(x,y)=\tilde{u}_{\text{exact}}(x,y)$, so $\|\tilde{u}_{\text{exact}}(x,y)-\tilde{u}_{\text{approx}}(x,y)\|\longrightarrow 0$ as $M\longrightarrow\infty$ and since S is bounded.

Thus,

$$\lim_{M\longrightarrow\infty}\left\|\tilde{u}_{\text{exact}}(x,y)-\tilde{u}_{\text{approx}}(x,y)\right\|\longrightarrow 0, \tag{76}$$

so the proof of the theorem is completed.

Remark 1. In our theoretical investigation for the proposed method, we take $m1=m2=M$.

8. Numerical Results and Discussion

This section demonstrates the effectiveness and the accuracy of our proposed hybrid method, 2D-TFs and an iterative algorithm, on some examples. The solution of each example is obtained for different values of x,y,r, and M and is compared with the exact solution, the direct method, and the presented method when the tolerance criteria residual is $>e^{-4}$ and $>e^{-8}$.

Example 1. The 2D-FFIE-2 given in Mirzaee et al. [15] is considered as

$$\begin{aligned}\underline{f}(x,y,r)&=r\left(xy+\frac{1}{676}\left(x^2+y^2-2\right)\right),\\\overline{f}(x,y,r)&=(2-r)\left(xy+\frac{1}{676}\left(x^2+y^2-1\right)\right),\\k(x,y,s,t)&=\frac{1}{169}\left(x^2+y^2-2\right)\left(s^2+t^2-2\right),\quad 0\le x,\ t\le 1\ \text{and}\ \lambda=1.\end{aligned} \tag{77}$$

In this case, the exact solution is given by

$$\tilde{u}(x,y,r)=\left(\underline{u}(x,y,r),\overline{u}(x,y,r)\right)=(rxy,(2-r)xy). \tag{78}$$

The number of iterations for solving the two coupled matrix equations to obtain the coefficient matrices taken by our proposed iterative algorithm is $k=3$ when the tolerance criteria residual is $>e^{-4}$ which indicates that the hybrid

Table 1: Numerical results with TF system for Example 1 for $x = 0.1$, $y = 0.4$, and $M = 4$.

r	Exact solution $\underline{u}(x, y, r)$	Direct method [15]	Absolute error	Presented method	Absolute error
0	0.00000000	0.00000000	$8.4015e-006$	0.00000002	$1.89187044e-008$
0.1	0.00400000	0.00398678	$1.3220e-005$	0.00399416	$5.83560842e-006$
0.2	0.00800000	0.00797356	$2.6440e-005$	0.00798831	$1.16901355e-005$
0.3	0.01200000	0.01196034	$3.9660e-005$	0.01198246	$1.75446627e-005$
0.4	0.01600000	0.01594712	$5.2880e-005$	0.01597660	$2.33991898e-005$
0.5	0.02000000	0.01993390	$6.6100e-005$	0.01997075	$2.92537169e-005$
0.6	0.02400000	0.02392068	$7.9320e-005$	0.02396489	$3.51082441e-005$
0.7	0.02800000	0.02700746	$9.9254e-004$	0.02796099	$4.09627712e-005$
0.8	0.03200000	0.03189424	$1.0576e-004$	0.03195318	$4.68172983e-005$
0.9	0.03600000	0.03588102	$1.1898e-004$	0.03594733	$5.26718254e-005$

Table 2: Numerical results with TF system for Example 1 for $x = 0.1$, $y = 0.4$, and $M = 4$.

r	Exact solution $\overline{u}(x, y, r)$	Direct method [15]	Absolute error	Presented method	Absolute error
0	0.08000000	0.07973560	$2.6440e-004$	0.07988291	$1.17085832e-004$
0.1	0.07600000	0.07574882	$2.5118e-004$	0.07588878	$1.11222414e-004$
0.2	0.07200000	0.07176204	$2.3796e-004$	0.07189464	$1.05358995e-004$
0.3	0.06800000	0.06777526	$2.2474e-004$	0.06790050	$9.94955768e-005$
0.4	0.06400000	0.06378849	$2.1151e-004$	0.06390637	$9.36321584e-005$
0.5	0.06000000	0.05980170	$1.9830e-004$	0.05991223	$8.77687400e-005$
0.6	0.05600000	0.05581492	$1.8508e-004$	0.05592806	$8.19053216e-005$
0.7	0.05200000	0.05182814	$1.7186e-004$	0.05192396	$7.60419032e-005$
0.8	0.04800000	0.04784136	$1.5864e-004$	0.04792982	$7.01784848e-005$
0.9	0.04400000	0.04385458	$1.4542e-004$	0.04393568	$6.43150664e-005$

proposed method is quite efficient and has good accuracy as seen from Tables 1 and 2.

Remark 2. The numerical results for the approximate solution using the direct method in Tables 1 and 2 are taken from Table 3 in [15], while the numerical results using the direct and the proposed iterative methods in Tables 3 and 4 are obtained using our own program written using MATLAB R2018b. Also, the number of iterations for solving the two coupled matrix equations to obtain the coefficient matrices taken by our proposed iterative algorithm is $k = 3$ when the tolerance criteria residual is $>e^{-4}$ which indicates that the hybrid proposed method is quite efficient and has good accuracy as seen from Tables 3 and 4.

Example 2. Consider the following made up linear two-dimensional fuzzy Fredholm integral equations (2D-FFIE-2):

$$\underline{f}(x, y, r) = \frac{5}{12} r(x + y),$$
$$\overline{f}(x, y, r) = \frac{5}{12}(2 - r)(x + y), \tag{79}$$
$$k(x, y, s, t) = xs + yt, \quad 0 \le x, t \le 1 \text{ and } \lambda = 1.$$

In this case, the exact solution is given by

$$\tilde{u}(x, y, r) = (\underline{u}(x, y, r), \overline{u}(x, y, r)) = (r(x + y), (2 - r)(x + y)). \tag{80}$$

Remark 3. The number of iterations for solving the two coupled matrix equations to obtain the coefficient matrices taken by our proposed iterative algorithm is $k = 3$ when the tolerance criteria residual is $>e^{-4}$ which indicates that the hybrid proposed method is quite efficient and has good accuracy as seen from Tables 5 and 6.

Example 3. Consider the following 2D-FFIE-2:

$$\underline{f}(x, y, r) = \frac{11}{36} r x^2 y^2,$$
$$\overline{f}(x, r) = \frac{11}{36}(2 - r) x^2 y^2,$$
$$k(x, y, s, t) = x^2 y^2 (1 + 2t)(1 + 2s), \quad 0 \le x, t \le 1 \text{ and } \lambda = 1. \tag{81}$$

In this case, the exact solution is given by

$$\begin{aligned} \underline{u}(x, y, r) &= r x^2 y^2, \\ \overline{u}(x, r) &= (2 - r) x^2 y^2. \end{aligned} \tag{82}$$

TABLE 3: Numerical results with TF system for Example 1 for $x = 0.1$, $y = 0.4$, and $M = 6$.

r	Exact solution $\underline{u}(x, y, r)$	Direct method	Absolute error	Presented method	Absolute error
0	0.00000000	0.00000239	$2.39402667e-006$	0.00000010	$9.77129847e-008$
0.1	0.00400000	0.00399511	$4.89026667e-006$	0.00399758	$2.42072449e-006$
0.2	0.00800000	0.00799261	$7.38650667e-006$	0.00799506	$4.93916197e-006$
0.3	0.01200000	0.01199012	$9.88274667e-006$	0.01199254	$7.45759945e-006$
0.4	0.01600000	0.01598762	$1.23789867e-005$	0.01599002	$9.97603693e-006$
0.5	0.02000000	0.01998512	$1.48752267e-005$	0.01998751	$1.24944744e-005$
0.6	0.02400000	0.02398263	$1.73714667e-005$	0.02398499	$1.50129119e-005$
0.7	0.02800000	0.02798013	$1.98677067e-005$	0.02798247	$1.75313494e-005$
0.8	0.03200000	0.03197764	$2.23639467e-005$	0.03197995	$2.00497868e-005$
0.9	0.03600000	0.03597514	$2.48601867e-005$	0.03597743	$2.25682243e-005$

TABLE 4: Numerical results with TF system for Example 1 for $x = 0.1$, $y = 0.4$, and $M = 6$.

r	Exact solution $\overline{u}(x, y, r)$	Direct method	Absolute error	Presented method	Absolute error
0	0.08000000	0.07994709	$5.29077333e-005$	0.07994927	$5.07341947e-005$
0.1	0.07600000	0.07594863	$5.13736533e-005$	0.07595198	$4.80210603e-005$
0.2	0.07200000	0.07195016	$4.98395733e-005$	0.07195469	$4.53079259e-005$
0.3	0.06800000	0.06795169	$4.83054933e-005$	0.06795741	$4.25947915e-005$
0.4	0.06400000	0.06395323	$4.67714133e-005$	0.06396012	$3.98816571e-005$
0.5	0.06000000	0.05995476	$4.52373333e-005$	0.05996283	$3.71685227e-005$
0.6	0.05600000	0.05595630	$4.37032533e-005$	0.05596554	$3.44553883e-005$
0.7	0.05200000	0.05195783	$4.21691733e-005$	0.05196826	$3.17422539e-005$
0.8	0.04800000	0.04795936	$4.06350933e-005$	0.04797097	$2.90291195e-005$
0.9	0.04400000	0.04396090	$3.91010133e-005$	0.04397368	$2.63159851e-005$

TABLE 5: Numerical results with TF system for Example 2 for $x = 0.1$, $y = 0.1$, and $M = 4$.

r	Exact solution $\underline{u}(x, y, r)$	Presented method	Absolute error	Exact solution $\overline{u}(x, y, r)$	Presented method	Absolute error
0	0.00000000	0.00000000	$1.734723476e-17$	0.40000000	0.40000000	$0.000000000e+00$
0.1	0.02000000	0.02000000	$1.387778781e-17$	0.38000000	0.38000000	$0.000000000e+00$
0.2	0.04000000	0.04000000	$1.387778781e-17$	0.36000000	0.36000000	$5.551115123e-17$
0.3	0.06000000	0.06000000	$0.000000000e+00$	0.34000000	0.34000000	$0.000000000e+00$
0.4	0.08000000	0.08000000	$0.000000000e+00$	0.32000000	0.32000000	$0.000000000e+00$
0.5	0.10000000	0.10000000	$0.000000000e+00$	0.30000000	0.30000000	$0.000000000e+00$
0.6	0.12000000	0.12000000	$2.775557562e-17$	0.28000000	0.28000000	$5.551115123e-17$
0.7	0.14000000	0.14000000	$2.775557562e-17$	0.26000000	0.26000000	$5.551115123e-17$
0.8	0.16000000	0.16000000	$0.000000000e+00$	0.24000000	0.24000000	$2.775557562e-17$
0.9	0.18000000	0.18000000	$1.734723476e-17$	0.22000000	0.22000000	$5.551115123e-17$

TABLE 6: Numerical results with TF system for Example 2 for $x = 0.1$, $y = 0.1$, and $M = 6$.

r	Exact solution $\underline{u}(x, y, r)$	Presented method	Absolute error	Exact solution $\overline{u}(x, y, r)$	Presented method	Absolute error
0	0.00000000	0.00000000	$0.000000000e+00$	0.40000000	0.40000000	$0.000000000e+00$
0.1	0.02000000	0.02000000	$0.000000000e+00$	0.38000000	0.38000000	$0.000000000e+00$
0.2	0.04000000	0.04000000	$0.000000000e+00$	0.36000000	0.36000000	$0.000000000e+00$
0.3	0.06000000	0.06000000	$0.000000000e+00$	0.34000000	0.34000000	$0.000000000e+00$
0.4	0.08000000	0.08000000	$0.000000000e+00$	0.32000000	0.32000000	$0.000000000e+00$
0.5	0.10000000	0.10000000	$0.000000000e+00$	0.30000000	0.30000000	$0.000000000e+00$
0.6	0.12000000	0.12000000	$0.000000000e+00$	0.28000000	0.28000000	$0.000000000e+00$
0.7	0.14000000	0.14000000	$0.000000000e+00$	0.26000000	0.26000000	$0.000000000e+00$
0.8	0.16000000	0.16000000	$0.000000000e+00$	0.24000000	0.24000000	$0.000000000e+00$
0.9	0.18000000	0.18000000	$0.000000000e+00$	0.22000000	0.22000000	$0.000000000e+00$

TABLE 7: Numerical results with TF system for Example 3 for $x = 0.25$, $y = 0.25$, and $M = 6$.

r	Exact solution $\underline{u}(x, y, r)$	Presented method	Absolute error	Exact solution $\overline{u}(x, y, r)$	Presented method	Absolute error
0	0.00000000	0.00000232	$2.32197716e-006$	0.00781250	0.00609664	$1.71585866e-003$
0.1	0.00039063	0.00030680	$8.38225958e-005$	0.00742187	0.00579187	$1.63000792e-003$
0.2	0.00078125	0.00061128	$1.69967169e-004$	0.00703125	0.00548709	$1.54415718e-003$
0.3	0.00117188	0.00091576	$2.56111742e-004$	0.00664062	0.00518232	$1.45830644e-003$
0.4	0.00156250	0.00122024	$3.42256315e-004$	0.00625000	0.00487754	$1.37245569e-003$
0.5	0.00195313	0.00152472	$4.28400888e-004$	0.00585938	0.00457277	$1.28660495e-003$
0.6	0.00234375	0.00182920	$5.14545461e-004$	0.00546875	0.00426800	$1.20075421e-003$
0.7	0.00273437	0.00213368	$6.00690034e-004$	0.00507813	0.00396322	$1.11490347e-003$
0.8	0.00312500	0.00243817	$6.86834607e-004$	0.00468750	0.00365845	$1.02905273e-003$
0.9	0.00351562	0.00274265	$7.72979179e-004$	0.00429688	0.00335367	$9.43201983e-004$

Remark 4. In Table 7, the number of iterations for this example by our proposed iterative algorithm is $k = 3$ when the tolerance criteria residual is $>e^{-4}$ which indicates that the hybrid proposed method is quite efficient. Moreover, we can see that our method has good accuracy which can be further improved by increasing the residual.

9. Conclusion

Fuzzy control applications and a large proportion of applied mathematical topics require the solution of the fuzzy integral equations. The paper introduced the 2D-TFs method for approximating the solution of linear 2D-FFIE-2, which is a hybrid of triangular functions and an iterative algorithm. The method is simple, efficient, and accurate and is based on converting the original equation into two crisp systems (2D-FFIE-2). The efficiency and simplicity of the proposed method are demonstrated via numerical examples with known exact solutions, and the results are given. Furthermore, the exceptional value of the proposed method is low cost of the equation setting with no need for any projection method or integration.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the manuscript.

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Research Article

Design and Numerical Solutions of a Novel Third-Order Nonlinear Emden–Fowler Delay Differential Model

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In this study, the design of a novel model based on nonlinear third-order Emden–Fowler delay differential (EF-DD) equations is presented along with two types using the sense of delay differential and standard form of the second-order EF equation. The singularity at $\xi = 0$ at single or multiple points of each type of the designed EF-DD model are discussed. The detail of shape factors and delayed points is provided for both types of the designed third-order EF-DD model. For the verification and validation of the model, two numerical examples are presented of each case and numerical results have been performed using the artificial neural network along with the hybrid of global and local capabilities. The comparison of the obtained numerical results with the exact solutions shows the perfection and correctness of the designed third-order EF-DD model.

1. Introduction

The delay differential (DD) equation is known as one of the historical and important equations. Recently, DD equation has attained much attention of the researcher's community due to its vast applications in many biological models, as well as scientific phenomena such as communication system model, dynamical population model, economical systems, engineering system, and transport and propagation model [1–5]. It is always interested to find the solution of DD equations and many researchers have applied different numerical/analytical techniques. Brunner et al. [6] solved DD equation by applying a discontinuous Galerkin numerical scheme. Hsiao and Wu [7] applied Haar wavelet to solve DD equations, while Wang [8] presented the solution of DD equations using Legendre wavelet. Adomian and Rach [9] solved DD equation using the Adomian decomposition scheme. Shakeri and Dehghan [10] found the solutions of DD initial value problems using the homotopy perturbation scheme. Erdogan et al. [11] implemented finite difference approach on layer-adapted mesh using the singularly perturbed DD equations. The general form of the DD model is written as [12, 13]

$$\begin{cases} \dfrac{d^3u}{d\xi^3} = h\left(\xi, u(\xi-\tau), \dfrac{du(\xi-\tau)}{d\xi}, \dfrac{d^2u(\xi-\tau)}{d\xi^2}\right), \\ u(0) = A, \dfrac{du(0)}{d\xi} = B, \dfrac{d^2u(0)}{d\xi^2} = C, \end{cases} \tag{1}$$

where h shows the linear/nonlinear function and τ is the delayed term, whereas A, B, and C are the constants.

The singular study has become very significant in the modern era due to the variety of applications in technology, engineering, and biological and physical sciences. The singular nature models are always difficult, grim, and challengeable to solve for the research community. One of the important, famous, historical, and singular models is

Emden–Fowler (EF) model that shows the singularity at the origin. Since its invention, this model has been solved by various analytical and numerical schemes, and it has a number of applications in the study of relativistic mechanics, fluid dynamics, population growth model, pattern creation, and the study of chemical reactor models. The literature form of the EF model is written as [14–16]

$$\begin{cases} \dfrac{d^2u}{d\xi^2} + \dfrac{\kappa}{\xi}\dfrac{du}{d\xi} + g(\xi)h(u) = 0, \\ \\ u(0) = A_1, \dfrac{du(0)}{d\xi} = A_2, \end{cases} \tag{2}$$

where $\kappa \geq 1$ is the shape vector. The EF model (1) becomes the Lane–Emden model by taking $h(u) = 1$ and is written as follows:

$$\begin{cases} \dfrac{d^2u}{d\xi^2} + \dfrac{\kappa}{\xi}\dfrac{du}{d\xi} + h(u) = 0, \\ \\ u(0) = A_1, \dfrac{du(0)}{d\xi} = A_2. \end{cases} \tag{3}$$

The above singular models have been achieved from the work of Homer Lane and Robert Emden. These models designate inner construction of polytropic stars, gas cloud model, cluster galaxies, and radiative cooling. Due to the worth of these models, no one can deny the value and importance of such models, which has vast applications in the physical science field [17], isotropic continuous media [18], density of gaseous star [19], morphogenesis [20], dusty fluid models [21], stellar structure models [22], reactions based on catalytic diffusion [23], oscillating magnetic systems [24], isothermal gas sphere models [25], mathematical physics [26], catalytic diffusion reactions [23], classical/quantum mechanics [27], and electromagnetic theory [28].

Due to the fame of these models, the researcher's community is interested to solve these models and only a few methods are available in the literature that has been investigated. One of the well-known methods used to solve these models is the Adomian decomposition method, which is proposed by Shawagfeh and Wazwaz [29, 30]. Parand and Razzaghi [31] implemented a famous numerical scheme to solve singular equations. Liao [32] applied an analytic technique to avoid the difficulty of singular points. Bender et al. [33] proposed a perturbative scheme to solve the singular models. Nouh [34] presented two techniques' power series and Pade approximation to solve the singular models.

The aim of this study is to design a novel third-order Emden–Fowler delay differential (EF-DD) model along with two types. Two examples of the designed third-order EF-DD model have been presented for both of the types. For the correctness of the model, the numerical investigations have been performed by using an artificial neural network along with its global/local competences. The singular ordinary differential equations are much important and have many applications in engineering as well as scientific applications, e.g., optimization and control theory, reactant application in the area of chemical reactor, theory of boundary layer, and biological sciences.

The structure of remaining paper is summarized as follows. Section 2 defines the construction of the third-order EF-DD model along with two types. Methodology and the detail of the results for solving the third-order EF-DD equations are provided in the Section 3. The conclusions along with future research directions are drawn in the Section 4.

2. Construction of Third-Order EF-DD Model

In this section, two different types are presented based on the third-order EF-DD model. The construction of the third-order EF-DD model along with the singular points, delayed points, and shape factors for both of the types is discussed. The initial conditions of the designed third-order EF-DD model are achieved using the standard form of the Lane–Emden. To derive the third-order EF-DD model system of Emden–Fowler equations, the mathematical form is used as follows:

$$\xi^{-k}\frac{d^p}{d\xi^p}\left(\xi^k\frac{d^q}{d\xi^q}\right)u(\xi-\tau) + g(\xi)h(u) = 0, \tag{4}$$

where k is real positive number. To determine the third-order DD-EF model, the values of p and q should be designated as follows:

$$p + q = 3, \quad p, q \geq 1. \tag{5}$$

The following two possibilities satisfy equation (5) as follows:

$$\begin{aligned} p &= 2, \\ q &= 1, \end{aligned} \tag{6}$$

$$\begin{aligned} p &= 1, \\ q &= 2. \end{aligned} \tag{7}$$

2.1. Type 1. Using equations (6), the updated form of equation (4) is

$$\xi^{-k}\frac{d^2}{d\xi^2}\left(\xi^k\frac{d}{d\xi}\right)y(\xi-\tau) + g(\xi)h(u) = 0. \tag{8}$$

The derivative part of the above equation is obtained as follows:

$$\begin{aligned} \frac{d^2}{d\xi^2}\left(\xi^k\frac{d}{d\xi}\right)u(\xi-\tau) &= \xi^k\frac{d^3}{d\xi^3}u(\xi-\tau) + 2k\xi^{k-1} \\ &\frac{d^2}{d\xi^2}u(\xi-\tau) + k(k-1)u(\xi-\tau)\xi^{k-2}\frac{d}{d\xi}u(\xi-\tau). \end{aligned} \tag{9}$$

Using the above expression in equation (8), the third-order EF-DD equation becomes

$$\begin{cases} \dfrac{d^3}{d\xi^3}u(\xi-\tau)+\dfrac{2k}{\xi}\dfrac{d^2}{d\xi^2}u(\xi-\tau)+\dfrac{k(k-1)}{\xi^2}\dfrac{d}{d\xi}u(\xi-\tau)+g(\xi)h(u)=0, \\ u(0)=\alpha, \dfrac{du(0)}{d\xi}=0, \dfrac{d^2u(0)}{d\xi^2}=0, \end{cases} \quad (10)$$

where the singular point at $\xi = 0$ appears two times as $\xi = 0$ and $\xi^2 = 0$. The shape factors expressed in equation (10) are $2k$ and $k(k-1)$, respectively. The multiple delays have been noticed in the first, second, and third term of equation (10). Moreover, the third expression vanishes for $k = 1$ and the shape factor reduces to 2.

2.2. Type 2. Equation (4) by putting $p = 1$ and $q = 2$ takes the form as follows:

$$\xi^{-k}\frac{d}{d\xi}\left(\xi^k\frac{d^2}{d\xi^2}\right)u(\xi-\tau)+g(\xi)h(u)=0. \quad (11)$$

The derivative part of the above equation is obtained as follows:

$$\frac{d}{d\xi}\left(\xi^k\frac{d^2}{d\xi^2}\right)u(\xi-\tau)=\xi^k\frac{d^3}{d\xi^3}u(\xi-\tau)+k\xi^{k-1}\frac{d^2}{d\xi^2}u(\xi-\tau). \quad (12)$$

Using the above value in equation (11), the third-order EF-DD model becomes as follows:

$$\begin{cases} \dfrac{d^3}{d\xi^3}u(\xi-\tau)+\dfrac{k}{\xi}\dfrac{d^2}{d\xi^2}u(\xi-\tau)+g(\xi)h(u)=0, \\ u(0)=\alpha, \dfrac{du(0)}{d\xi}=\beta, \dfrac{d^2u(0)}{d\xi^2}=0. \end{cases} \quad (13)$$

The single singularity at $\xi = 0$ has been noticed in the above equation (13). The shape factor is k and delayed expression appears twice in the above equation.

Some prime features of the designed model are presented as follows:

- The design of third-order Emden–Fowler delay differential model is presented by using the sense of standard Emden–Fowler equation and delay-differential equation
- Two types of the designed model are presented and two numerical nonlinear examples of each type are designed based on the designed model
- The shape factors, delay expressions, and singularities are discussed in both of the types
- The artificial neural network is used to check the perfection and correctness of the designed third-order Emden–Fowler model

3. Methodology and Numerical Examples

Two numerical examples based on the EF-DD novel model are presented in this section. The numerical investigations of the examples are performed using the artificial neural network. The error function is provided by using the sense of the differential equations and initial conditions. The optimization of the error function is performed using the hybrid of global and local search captaincies, which are genetic algorithm (GA) and active-set method (ASM). The artificial neural network is famous and widely applied in many well-known recent applications, see [35–41]. To approximate the results, feedforward ANN system along with its respective derivatives is used as follows:

$$\hat{u}=\sum_{i=1}^{m} l_i P(\alpha_i\xi+b_i), \quad (14)$$

$$\hat{u}^{(n)}=\sum_{i=1}^{m} l_i P^{(n)}(\alpha_i\xi+b_i), \quad (15)$$

where l_i, m_i, and n_i are the ith components of l, α, and b vectors, while n is the order of derivative. An activation log-sigmoid function, i.e., $P(\xi) = (1 + \ⅇ^{-\xi})^{-1}$ along with its third derivative is used as follows:

$$\hat{u}=\sum_{i=1}^{m} l_i\left(1+e^{-(\alpha_i\xi+b_i)}\right)^{-1}, \quad (16)$$

$$\hat{u}^{(n)}=\sum_{i=1}^{n} l_i\frac{d^n}{d\xi^n}\left(\left(1+e^{-(\alpha_i\xi+b_i)}\right)^{-1}\right). \quad (17)$$

The third-order derivative is provided as follows:

$$\hat{u}'''(\xi)=\sum_{i=1}^{m} l_i\xi_i^3\left(\frac{6e^{-3(\alpha_i\xi+b_i)}}{\left(1+e^{-(\alpha_i\xi+b_i)}\right)^4}-\frac{6e^{-2(\alpha_i\xi+b_i)}}{\left(1+e^{-(\alpha_i\xi+b_i)}\right)^3}+\frac{e^{-(\alpha_i\xi+b_i)}}{\left(1+e^{-(\alpha_i\xi+b_i)}\right)^2}\right). \quad (18)$$

The fitness function is given as follows:

$$E=E_1+E_2, \quad (19)$$

where E_1 and E_2 are the respective error functions related to differential equation and initial conditions.

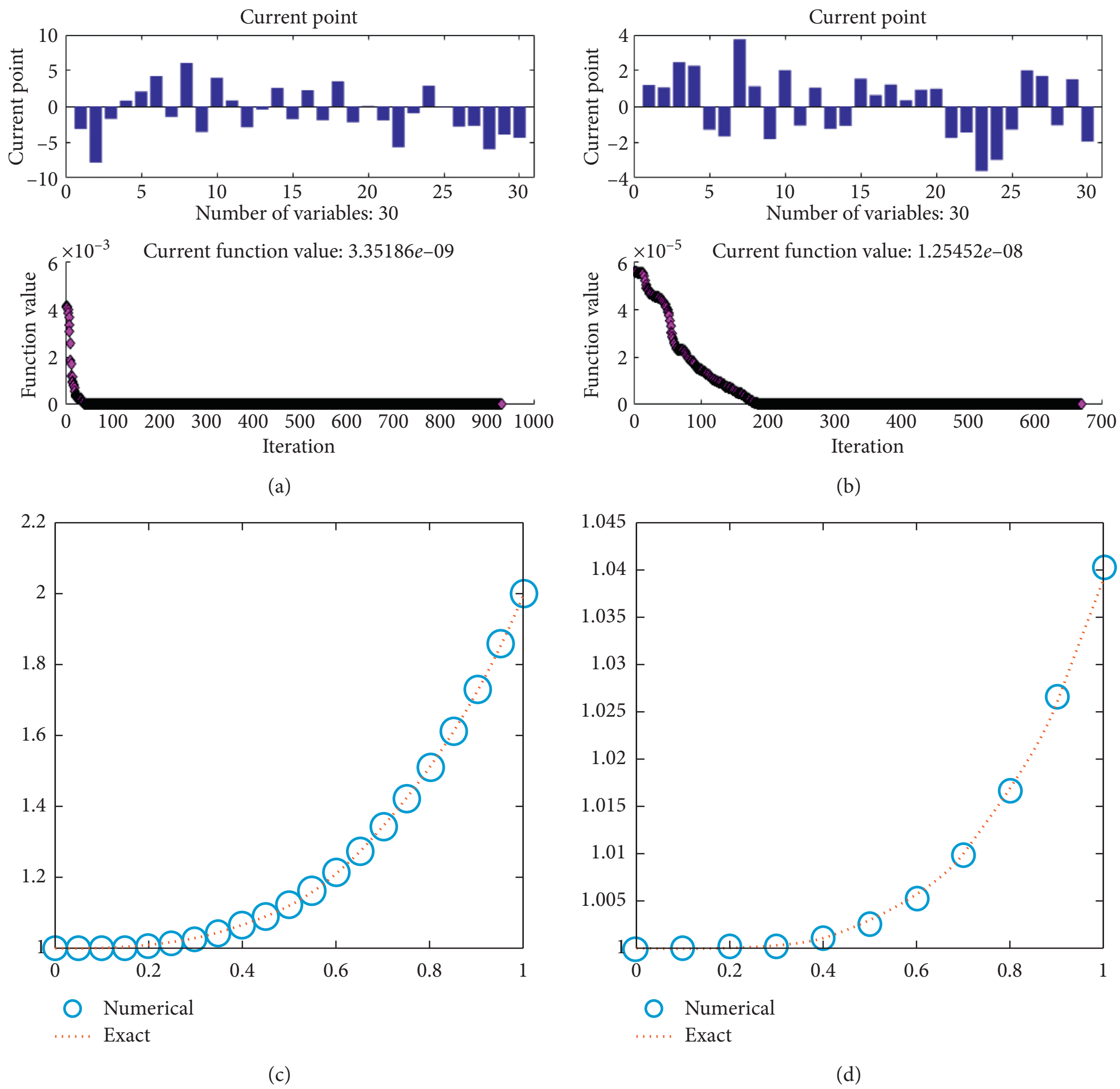

FIGURE 1: Optimization variables, learning curves, and comparison of results of the GA-AS scheme for nonlinear EF-DD equations (1) and (2) of type 1. (a) Set of best weights and current function values for 10 neurons based on third-order nonlinear EF-DD equation (1). (b) Set of best weights and current function values for 10 neurons based on third-order nonlinear EF-DD equation (2). (c) Comparison of the numerical and exact solutions of third-order nonlinear EF-DD equation (1). (d) Comparison of the numerical and exact solutions of third-order nonlinear EF-DD equation (2).

3.1. EF-DD Equation of Type 1. In this type, two different third-order EF-DD-based equations will be discussed. The updated form of equation (10) using $k = 2$ is given as follows.

Example 1. Consider the nonlinear third-order EF-DD equation having multiple singularities is shown as follows:

$$\begin{cases} \dfrac{d^3}{d\xi^3}u(\xi-1) + \dfrac{4}{\xi}\dfrac{d^2}{d\xi^2}u(\xi-1) + \dfrac{2}{\xi^2}\dfrac{d}{d\xi}u(\xi-1) + \xi u^2 = \xi^7 + 2\xi^4 + \xi + 30 - \dfrac{36}{\xi} + \dfrac{6}{\xi^2}, \\ u(0) = 1, \quad \dfrac{du(0)}{d\xi} = 0, \quad \dfrac{d^2u(0)}{d\xi^2} = 0. \end{cases} \tag{20}$$

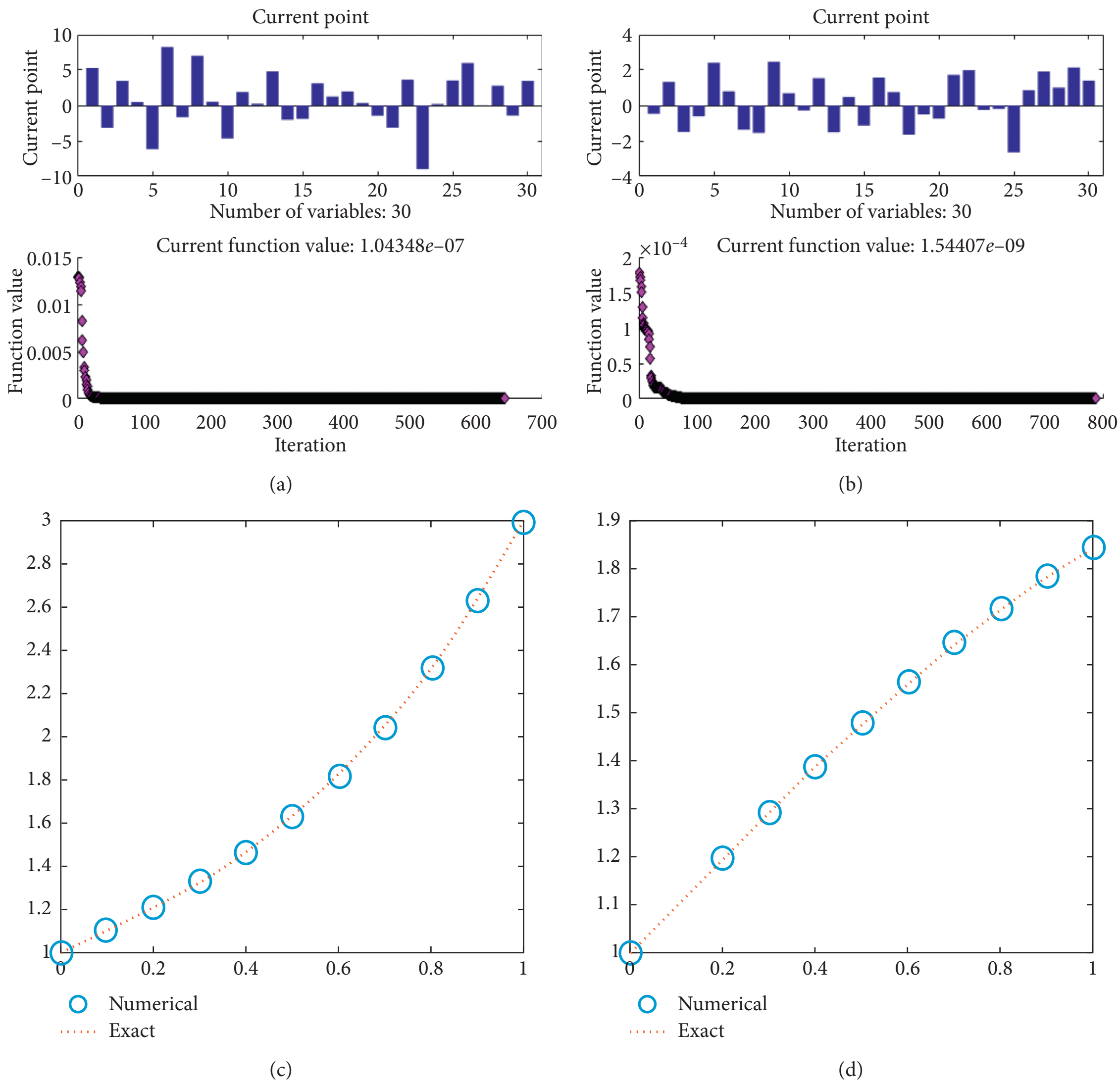

Figure 2: Optimization variables, learning curves, and comparison of results of the GA-AS scheme for nonlinear EF-DD equations (1) and (2) of type 2. (a) Set of best weights and current function values for 10 neurons based on third-order nonlinear EF-DD equation (1) of type 2. (b) Set of best weights and current function values for 10 neurons based on third-order nonlinear EF-DD equation (2) of type 2. (c) Comparison of the numerical and exact solutions of third-order nonlinear EF-DD equation (1) of type 2. (d) Comparison of the numerical and exact solutions of third-order nonlinear EF-DD equation (2) of type 2.

The exact solution of equation (20) is $1 + \xi^3$.

Example 2. Consider the nonlinear third-order EF-DD equation having multiple singularities and trigonometric functions is written as follows:

$$\begin{cases} \dfrac{d^3}{d\xi^3}u(\xi-1) + \dfrac{4}{\xi}\dfrac{d^2}{d\xi^2}u(\xi-1) + \dfrac{2}{\xi^2}\dfrac{d}{d\xi}u(\xi-1) + \xi u^2 = \dfrac{\xi^5}{4} - \dfrac{2}{\xi^2} + \dfrac{6}{\xi} + \\ \dfrac{\xi^2-2}{\xi^2}\sin(\xi-1) - \dfrac{4}{\xi}\cos(\xi-1) + \xi^3\cos\xi + \xi\cos^2\xi, \\ u(0) = 1, \dfrac{du(0)}{d\xi} = 0, \dfrac{d^2u(0)}{d\xi^2} = 0. \end{cases} \tag{21}$$

The exact solution of equation (21) is $\cos\xi + (1/2)\xi^2$.

3.2. EF-DD Equation of Type 2.

In this type, two different third-order EF-DD-based equations will be discussed. The updated form of equation (13) using $k = 1$ is given in the form of two examples.

Example 3. Consider the nonlinear third-order EF-DD equation having exponential function is given as follows:

$$\begin{cases} \dfrac{d^3}{d\xi^3}u(\xi-1) + \dfrac{1}{\xi}\dfrac{d^2}{d\xi^2}u(\xi-1) + \xi e^u = 12 - \dfrac{6}{\xi} + \xi e^{1+\xi+\xi^3}, \\ u(0) = 1, \; \dfrac{du(0)}{d\xi} = 1, \; \dfrac{d^2u(0)}{d\xi^2} = 0. \end{cases} \tag{22}$$

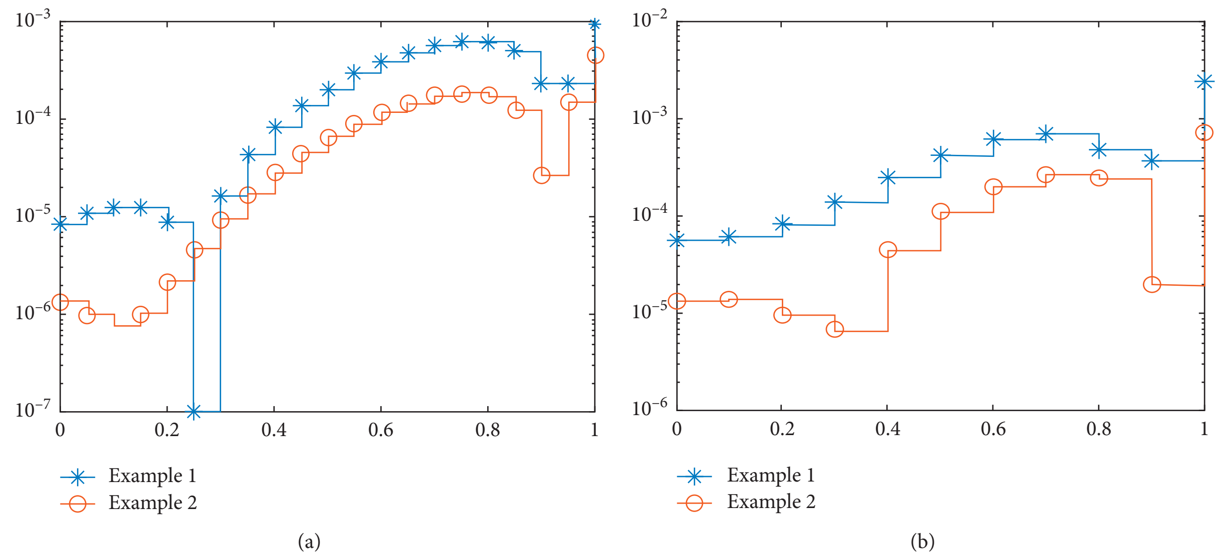

FIGURE 3: Absolute error based on the nonlinear EF-DD equations (1) and (2) of types 1 and 2. (a) AE for examples 1 and 2 of type 1. (b) AE for examples 1 and 2 of type 2.

The exact solution of equation (22) is $1 + \xi + \xi^3$.

Example 4. Consider the nonlinear third-order EF-DD equation having multi trigonometric function is given as follows:

$$\begin{cases} \dfrac{d^3}{d\xi^3}u(\xi-1) + \dfrac{1}{\xi}\dfrac{d^2}{d\xi^2}u(\xi-1) + \xi u^2 = \xi\sin^2\xi + 2\xi\sin\ \xi + \xi - \cos(\xi-1) - \dfrac{1}{\xi}\sin(\xi-1), \\ u(0) = 1,\ \dfrac{du(0)}{d\xi} = 1,\ \dfrac{d^2u(0)}{d\xi^2} = 0. \end{cases} \tag{23}$$

The exact solution of equation (23) is $1 + \sin\ \ \xi$.

Figures 1 and 2 represent the current point and function values using 10 neurons based on the hybrid combination of GA-AS scheme for both of the examples of types 1 and 2. The current function values (CFVs) are 10^{-09} and 10^{-08} for both of the examples of type 1 and 10^{-07} and 10^{-09} for both of the examples of 2 using 10 numbers of neurons. The comparison of results is presented in the rest of the figures for both examples of types 1 and 2. The overlapping of the exact and obtained results shows the correctness and the perfection of the novel third-order nonlinear EF-DD model.

The plots of the absolute error (AE) for both types of examples 1 and 2 based on the third-order nonlinear EF-DD model are provided in Figure 3. It is clear that most of the values lie around 10^{-04} to 10^{-05} for both types of examples 1 and 2, which indicates the exactness of the designed model. These witnesses prove the correctness of the designed third-order nonlinear EF-DD model. Comparison of the obtained results from GA-ASM for solving the nonlinear EF-DD model based on both problems of both types is tabulated in Tables 1 and 2. The exact solution, proposed results from GA-ASM, and the AE are provided in these tables. One can conclude on the behalf of AE the exactness and accurateness of the proposed model, as well as designed scheme.

4. Conclusion

In the present study, a novel design of third-order Emden–Fowler delay differential model is presented. The designed model is obtained by using the sense of fundamental Emden–Fowler model. The details of singular points, delay expressions, and the shape factors are also provided of the modeled equations of each type. The singularity at $\xi = 0$ appears twice in the first type, while single singularity is noticed in the second type. Similarly, the shape factor is unique in the standard form of the Emden–Fowler model, while the occurrence of shape factor is noticed twice in the type 1; however, single shape factor is noticed in type 2. For the perfection of the designed model, two nonlinear examples are presented of each type and numerical

Table 1: Comparison of the obtained results from GA-ASM for solving the nonlinear EF-DD model based on both problems of type 1.

ξ	Problem I			Problem 2		
	Exact	GA-ASM	AE	Exact	GA-ASM	AE
0	1.00000000	0.99998788	$1.212101E-05$	1.00000000	0.99999570	$4.2979068E-06$
0.05	1.00012500	1.00010387	$2.112830E-05$	1.00000026	0.99999586	$4.4025337E-06$
0.1	1.00100000	1.00097307	$2.693406E-05$	1.00000417	0.99999959	$4.5773499E-06$
0.15	1.00337500	1.00335045	$2.454778E-05$	1.00002108	1.00001622	$4.8547562E-06$
0.2	1.00800000	1.00799333	$6.670848E-06$	1.00006658	1.00006131	$5.2678390E-06$
0.25	1.01562500	1.01566134	$3.633505E-05$	1.00016242	1.00015658	$5.8417298E-06$
0.3	1.02700000	1.02711600	$1.160012E-04$	1.00033649	1.00032991	$6.5812201E-06$
0.35	1.04287500	1.04311966	$2.446627E-04$	1.00062271	1.00061526	$7.4548770E-06$
0.4	1.06400000	1.06443360	$4.336027E-04$	1.00106099	1.00105262	$8.3764421E-06$
0.45	1.09112500	1.09181538	$6.903829E-04$	1.00169710	1.00168792	$9.1849453E-06$
0.5	1.12500000	1.12601542	$1.015419E-03$	1.00258256	1.00257294	$9.6257224E-06$
0.55	1.16637500	1.16777299	$1.397990E-03$	1.00377452	1.00376519	$9.3353752E-06$
0.6	1.21600000	1.21781202	$1.812016E-03$	1.00533561	1.00532778	$7.8346627E-06$
0.65	1.27462500	1.27683704	$2.212044E-03$	1.00733380	1.00732926	$4.5343334E-06$
0.7	1.34300000	1.34552994	$2.529940E-03$	1.00984219	1.00984343	$1.2400050E-06$
0.75	1.42187500	1.42454775	$2.672747E-03$	1.01293887	1.01294907	$1.0196762E-05$
0.8	1.51200000	1.51452198	$2.521980E-03$	1.01670671	1.01672970	$2.2992646E-05$
0.85	1.61412500	1.61605929	$1.934293E-03$	1.02123315	1.02127325	$4.0102911E-05$
0.9	1.72900000	1.72974297	$7.429666E-04$	1.02660997	1.02667161	$6.1637149E-05$
0.95	1.85737500	1.85613407	$1.240925E-03$	1.03293309	1.03302018	$8.7089331E-05$
1	2.00000000	1.99577063	$4.229373E-03$	1.04030231	1.04041732	$1.1501021E-04$

Table 2: Comparison of the obtained results from GA-ASM for solving the nonlinear EF-DD model based on both problems of type 2.

ξ	Problem I			Problem 2		
	Exact	GA-ASM	AE	Exact	GA-ASM	AE
0	1.00000000	1.00022916	$2.291613E-04$	1.00000000	0.99963919	$3.60806E-04$
0.05	1.05012500	1.05035972	$2.347211E-04$	1.04997917	1.04960463	$3.74535E-04$
0.1	1.10100000	1.10123006	$2.300602E-04$	1.09983342	1.09944214	$3.91281E-04$
0.15	1.15337500	1.15357541	$2.004064E-04$	1.14943813	1.14902494	$4.13190E-04$
0.2	1.20800000	1.20812735	$1.273485E-4$	1.19866933	1.19822683	$4.42503E-04$
0.25	1.26562500	1.26561619	$8.813729E-06$	1.24740396	1.24692273	$4.81224E-04$
0.3	1.32700000	1.32677477	$2.252260E-04$	1.29552021	1.29498952	$5.30689E-04$
0.35	1.39287500	1.39234339	$5.316061E-04$	1.34289781	1.34230679	$5.91015E-04$
0.4	1.46400000	1.46307475	$9.252507E-04$	1.38941834	1.38875787	$6.60468E-04$
0.45	1.54112500	1.53973795	$1.387048E-03$	1.43496553	1.43423077	$7.34760E-04$
0.5	1.62500000	1.62312048	$1.879521E-03$	1.47942554	1.47861923	$8.06304E-04$
0.55	1.71637500	1.71402744	$2.347561E-03$	1.52268723	1.52182376	$8.63464E-04$
0.6	1.81600000	1.81327792	$2.722078E-03$	1.56464247	1.56375263	$8.89845E-04$
0.65	1.92462500	1.92169852	$2.926483E-03$	1.60518641	1.60432276	$8.63645E-04$
0.7	2.04300000	2.04011427	$2.885735E-03$	1.64421769	1.64346056	$7.57124E-04$
0.75	2.17187500	2.16933723	$2.537774E-03$	1.68163876	1.68110255	$5.36205E-04$
0.8	2.31200000	2.31015280	$1.847204E-03$	1.71735609	1.71719584	$1.60246E-04$
0.85	2.46412500	2.46330377	$8.212322E-04$	1.75128041	1.75169842	$4.18013E-04$
0.9	2.62900000	2.62947213	$4.721265E-04$	1.78332691	1.78457923	$1.25232E-03$
0.95	2.80737500	2.80925864	$1.883638E-03$	1.81341550	1.81581807	$2.40257E-03$
1	3.00000000	3.00316042	$3.160417E-03$	1.84147098	1.84540533	$3.93434E-03$

investigations have been performed using the powerful artificial neural networks. The comparison of the results is also plotted and overlapping of the proposed and exact solution enhanced more satisfaction of the model. The graphs of absolute error show that most of the values are found in good ranges for all examples of both types, which shows the exactness, worth, and the precision of the designed third-order Emden–Fowler delay differential model.

In the future, the proposed scheme ANN-GA-ASM can be applied as an accurate and efficient stochastic numerical solver for nonlinear singular models [42–44], computational models of fluid dynamics [45–48], fractional models [49–52], and biological models [53–57].

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Certain Generating Relations Involving the Generalized Multi-Index Bessel–Maitland Function

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Generating relations involving the special functions have already proved their important role in mathematics and other fields of sciences. In this paper, we aim to provide some presumably new generating relations in connection with the generalized multi-index Bessel–Maitland function $J_{(\nu_j)_m,q}^{(\lambda_j)_m,\gamma}(.)$. The main results presented here, being very general, can yield a number of particular or equivalent identities, some of which are explicitly demonstrated.

1. Introduction and Preliminaries

Here and elsewhere, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers, positive integers, and nonpositive integers, respectively.

The Bessel–Maitland function $J_\nu^\lambda(z)$ is defined as (see Marichev [1])

$$J_\nu^\lambda(z) = \sum_{r=0}^{\infty} \frac{(-z)^r}{\Gamma(\lambda r + \nu + 1) r!}, \quad \lambda \in \mathbb{R}^+, z \in \mathbb{C}. \tag{1}$$

Pathak [2] gave the following more generalized form of generalized Bessel–Maitland function (1):

$$J_{\nu,q}^{\lambda,\gamma}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\Gamma(\lambda r + \nu + 1)} \frac{(-z)^r}{r!}, \tag{2}$$

$$(\lambda, \nu, \gamma \in \mathbb{C}, \Re(\lambda) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, q \in (0,1) \cup \mathbb{N}). \tag{3}$$

Remark 1. Even though Pathak excluded $q = 0$ in (2), the case $q = 0$ yields (1).

If $q = 1$, $\gamma = 1$, ν is replaced by $\nu - 1$, and z is replaced by $-z$ in (2), then generalized Bessel–Maitland function reduces to the Mittag–Leffler function which was studied by Wiman [3] as follows:

$$J_{\nu-1,1}^{\lambda,1}(-z) = E_{\lambda,\nu}(z), \quad \Re(\lambda) > 0, \Re(\nu) > 0. \tag{4}$$

If ν is replaced by $\nu - 1$ and z is replaced by $-z$ in (2), then the generalized Bessel–Maitland function reduces to the well-known generalized Mittag–Leffler function $E_{\lambda,\nu}^{\gamma,q}(z)$ which was introduced by Shukla and Prajapati [4] as follows:

$$J_{\nu-1,q}^{\lambda,\gamma}(-z) = E_{\lambda,\nu}^{\gamma,q}(z), \tag{5}$$

$$(\Re(\lambda) > 0, \Re(\nu) > 0, \Re(\gamma) > 0; q \in (0,1) \cup \mathbb{N}). \tag{6}$$

Jain and Agarwal [5] generalized Bessel–Maitland function $J_\nu^\lambda(z)$ (1) as follows:

$$J_{\nu,\mu}^\lambda(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{\nu+2\mu+2r}}{\Gamma(\lambda r + \nu + \mu + 1)\Gamma(\mu + r + 1)}, \tag{7}$$

$$(\lambda \in \mathbb{R}^+, \nu, \mu \in \mathbb{C}, z \in \mathbb{C} \setminus (-\infty, 0]). \tag{8}$$

Choi and Agarwal [6] investigated the following generalized multi-index Bessel function:

$$J_{(\nu_j)_{m,q}}^{(\lambda_j)_{m,\gamma}}(z)=\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma\left(\lambda_j r+\nu_j+1\right)}\frac{(-z)^r}{r!}, \tag{9}$$

where $m\in\mathbb{N}$ and λ_j, ν_j, γ, q, and $z\in\mathbb{C}$ $(j=1,\ldots,m)$ such that

$$\sum_{j=1}^{m}\Re\left(\lambda_j\right)>\max\{0,\Re(q)-1\},\quad \Re\left(\nu_j\right)>-1,\ \Re(\gamma)>0,\ q\in(0,1)\cup\mathbb{N}. \tag{10}$$

Remark 2. It is easily found that generalized multi-index Bessel–Maitland function (9) is equivalent to the generalized multi-index Mittag–Leffler function defined and studied by Saxena and Nishimoto [7] (see also [8]).

Pohlen [9] introduced the Hadamard product (or the convolution) $f*g$ of two analytic functions f and g as follows:

$$(f*g)(z):=\sum_{n=0}^{\infty}a_n b_n z^n=(g*f)(z),\quad (|z|<R), \tag{11}$$

where $R\geq R_f\cdot R_g$. Here, $f(z)$ and $g(z)$ are analytic at $z=0$ whose Maclaurin series with their respective radii of convergence R_f and R_g are

$$\begin{aligned} f(z)&=\sum_{n=0}^{\infty}a_n z^n,\quad \left(|z|<R_f\right),\\ g(z)&=\sum_{n=0}^{\infty}b_n z^n,\quad \left(|z|<R_g\right). \end{aligned} \tag{12}$$

The concept of the Hadamard product has turned out to be useful, particularly, in factorizing a newborn function, which is usually expressed as a Maclaurin series, into two known functions (see, e.g., [10–13]).

The k-th derivative of the function $f(p)=p^{-\lambda-n\xi}$ $(\lambda,\xi\in\mathbb{C},n\in\mathbb{N})$ is easily found to be given in terms of gamma function as follows:

$$f^{(k)}(p)=(-1)^k p^{-\lambda-n\xi-k}\frac{\Gamma(\lambda+n\xi+k)}{\Gamma(\lambda+n\xi)},\quad \left(k\in\mathbb{N}_0\right). \tag{13}$$

Generating functions have been widely used in exploring certain properties and formulas involving sequences and polynomials in a wide range of research subjects. Many researchers have developed a remarkably large number of generating functions associated with a variety of special functions. For some works on this subject, one may refer, for example, to an extensive monograph [14–25] and the literature cited therein. In this search, we aim to provide some presumably new generating relations in connection with generalized multi-index Bessel–Maitland function (9). The main results developed here, being very general, can be reduced to produce a large number of presumably new and potentially useful generating relations for other known functions, some of which are demonstrated.

2. Generating Relations

We give two generating relations involving generalized multi-index Bessel–Maitland function (9) asserted by the following theorems.

Theorem 1. *Let $m\in\mathbb{N}$ and λ_j, ν_j, γ, q, and $z\in\mathbb{C}$ $(j=1,\ldots,m)$ such that*

$$\sum_{j=1}^{m}\Re\left(\lambda_j\right)>\max\{0,\Re(q)-1\},\quad \Re\left(\nu_j\right)>-1,\ \Re(\gamma)>0,\ q\in(0,1)\cup\mathbb{N}. \tag{14}$$

Also, let $|t|<1$. Then,

$$\begin{aligned}&(1+t)^{-\sigma}J_{(\nu_j)_m,q}^{(\lambda_j)_m,\gamma}\left(\frac{z}{1+t}\right)\\ &\quad=\sum_{k=0}^{\infty}(-1)^k(\sigma)_k J_{(\nu_j)_m,q}^{(\lambda_j)_m,\gamma}(z)*{}_1F_1(\sigma+k;\sigma;-z)\frac{t^k}{k!}.\end{aligned} \tag{15}$$

Proof. We replace $1+t$ by s in the left-hand side of (15) and denote the resulting expression by $g(s)$. Then, using form (9), on expanding the function in series, gives

$$g(s)=s^{-\sigma}J_{(\nu_j)_{m,q}}^{(\lambda_j)_{m,\gamma}}\left(\frac{z}{s}\right)=\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma\left(\lambda_j r+\nu_j+1\right)}\frac{(-z)^r}{r!}s^{-\sigma-r}. \tag{16}$$

Differentiating k times both sides of (16) with respect to s with the aid of (13) (term-by-term differentiation can be verified under the given conditions), we find

$$g^{(k)}(s)=(-1)^k s^{-\sigma-k}\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma\left(\lambda_j r+\nu_j+1\right)}\frac{\Gamma(\sigma+r+k)}{\Gamma(\sigma+r)}\left(\frac{-z}{s}\right)^r\frac{1}{r!}, \tag{17}$$

which is simplified to yield

$$g^{(k)}(s) = (-1)^k s^{-\sigma-k}(\sigma)_k \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\lambda_j r + \nu_j + 1)} \frac{(\sigma+k)_r}{(\sigma)_r}\left(\frac{-z}{s}\right)^r \frac{1}{r!}. \tag{18}$$

Decomposing series (18) into Hadamard product (11), we obtain

$$g^{(k)}(s) = (-1)^k s^{-\sigma-k}(\sigma)_k J_{(\nu_j)_m, q}^{(\lambda_j)_m, \gamma}\left(\frac{z}{s}\right) * {}_1F_1\left(\sigma+k; \sigma; -\frac{z}{s}\right). \tag{19}$$

Expanding $g(s+t)$ as the Taylor series gives

$$g(s+t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} g^{(k)}(s). \tag{20}$$

Combining (16), (19), and (20), we obtain

$$(s+t)^{-\sigma} J_{(\nu_j)_m, q}^{(\lambda_j)_m, \gamma}\left(\frac{z}{s+t}\right) = \sum_{k=0}^{\infty} \frac{(-t)^k s^{-\sigma-k}}{k!}(\sigma)_k J_{(\nu_j)_m, q}^{(\lambda_j)_m, \gamma}\left(\frac{z}{s}\right) * {}_1F_1\left(\sigma+k; \sigma; -\frac{z}{s}\right). \tag{21}$$

Finally, setting $s = 1$ yields desired result (15). □

Theorem 2. *Let $m \in \mathbb{N}$ and λ_j, ν_j, γ, q, and $z \in \mathbb{C}$ $(j = 1, \ldots, m)$ such that*

$$\sum_{j=1}^{m} \Re(\lambda_j) > \max\{0, \Re(q) - 1\}, \quad \Re(\nu_j) > -1, \Re(\gamma) > 0, q \in (0,1) \cup \mathbb{N}. \tag{22}$$

Also, let $|t| < 1$. Then,

$$\sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} J_{(\nu_j)_m, q}^{(\lambda_j)_m, \gamma+k}(z) t^k = (1-t)^{-\gamma} J_{(\nu_j)_m, q}^{(\lambda_j)_m, \gamma}\left(\frac{z}{(1-t)^q}\right). \tag{23}$$

Proof. Let J be the left-hand side of (23). Using (9), on expanding the function in series, gives

$$\mathrm{J} = \sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} \left\{ \sum_{r=0}^{\infty} \frac{(\gamma+k)_{qr}}{\prod_{j=1}^{m}\Gamma(\lambda_j r + \nu_j + 1)} \frac{(-z)^r}{r!} \right\} t^k. \tag{24}$$

Interchanging the order of summations in (24) and using the known identity (see, e.g., [26, p. 5])

$$\binom{\gamma}{k} = \frac{\Gamma(\gamma+1)}{k!\Gamma(\gamma-k+1)}, \quad k \in \mathbb{N}_0, \gamma \in \mathbb{C}, \tag{25}$$

we have

$$\mathrm{J} = \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\lambda_j r + \nu_j + 1)} \left\{ \sum_{k=0}^{\infty} \binom{\gamma+qr+k-1}{k} t^k \right\} \frac{(-z)^r}{r!}. \tag{26}$$

Using the generalized binomial expansion, we find that the inner sum in (26) gives

$$\sum_{k=0}^{\infty} \binom{\gamma+qr+k-1}{k} t^k = (1-t)^{-(\gamma+qr)}, \quad |t| < 1. \tag{27}$$

Finally, interpreting (26) with the help of (27) yields desired result (23). □

3. Further Remarks

Here, we choose to give some equivalent identities and particular cases of the results in Theorems 1 and 2. As noted in Remark 2, setting ν_j by $\nu_j - 1$ and z by $-z$ in (15) and (23) gives two corresponding generating relations involving the generalized multi-index Mittag–Leffler function $E_{(\lambda_j,\nu_j)_m}^{\gamma,q}(z)$, which are asserted, respectively, in Corollaries 1 and 2.

Corollary 1. *Let $m \in \mathbb{N}$ and λ_j, ν_j, γ, q, and $z \in \mathbb{C}$ $(j = 1, \ldots, m)$ such that*

$$\sum_{j=1}^{m} \Re(\lambda_j) > \max\{0, \Re(q) - 1\}, \quad \Re(\nu_j) > 0, \Re(\gamma) > 0, q \in (0,1) \cup \mathbb{N}. \tag{28}$$

Also, let $|t| < 1$. Then,

$$(1+t)^{-\sigma} E_{(\lambda_j,\nu_j)_m}^{\gamma,q}\left(\frac{z}{1+t}\right) = \sum_{k=0}^{\infty} (-1)^k (\sigma)_k E_{(\lambda_j,\nu_j)_m}^{\gamma,q}(z) * {}_1F_1(\sigma+k; \sigma; -z)\frac{t^k}{k!}. \tag{29}$$

Corollary 2. *Let $m \in \mathbb{N}$ and λ_j, ν_j, γ, q, and $z \in \mathbb{C}$ $(j = 1, \ldots, m)$ such that*

$$\sum_{j=1}^{m} \Re(\lambda_j) > \max\{0, \Re(q) - 1\}, \quad \Re(\nu_j) > 0, \Re(\gamma) > 0, q \in (0,1) \cup \mathbb{N}. \tag{30}$$

Also, let $|t| < 1$. Then,

$$\sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} E_{(\lambda_j,\nu_j)_m}^{\gamma+k,q}(z) t^k = (1-t)^{-\gamma} E_{(\lambda_j,\nu_j)_m}^{\gamma,q}\left(\frac{z}{(1-t)^q}\right). \tag{31}$$

The particular cases of (15), (23), (29), and (31) when $m = 1$ give the following generating relations, stated, respectively, in Corollaries 3–6.

Corollary 3. *Let σ, λ, ν, γ, and $z \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, and $q \in (0,1) \cup \mathbb{N}$. Also, let $|t| < 1$. Then,*

$$(1+t)^{-\sigma} J_{\nu,q}^{\lambda,\gamma}\left(\frac{z}{1+t}\right) = \sum_{k=0}^{\infty} (-1)^k (\sigma)_k J_{\nu,q}^{\lambda,\gamma}(z) * {}_1F_1(\sigma+k;\sigma;-z)\frac{t^k}{k!}. \tag{32}$$

Corollary 4. *Let σ, λ, ν, γ, and $z \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, and $q \in (0,1) \cup \mathbb{N}$. Also, let $|t| < 1$. Then,*

$$\sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} J_{\nu,q}^{\lambda,\gamma+k}(z) t^k = (1-t)^{-\gamma} J_{\nu,q}^{\lambda,\gamma}\left(\frac{z}{(1-t)^q}\right). \tag{33}$$

Corollary 5. *Let σ, λ, ν, γ, and $z \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\nu) \geq 0$, $\Re(\gamma) > 0$, and $q \in (0,1) \cup \mathbb{N}$. Also, let $|t| < 1$. Then,*

$$(1+t)^{-\sigma} E_{\lambda,\nu}^{\gamma,q}\left(\frac{z}{1+t}\right) = \sum_{k=0}^{\infty} (-1)^k (\sigma)_k E_{\lambda,\nu}^{\gamma,q}(z) * {}_1F_1(\sigma+k;\sigma;-z)\frac{t^k}{k!}. \tag{34}$$

Corollary 6. *Let σ, λ, ν, γ, and $z \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\nu) \geq 0$, $\Re(\gamma) > 0$, and $q \in (0,1) \cup \mathbb{N}$. Also, let $|t| < 1$. Then,*

$$\sum_{k=0}^{\infty} \binom{\gamma+k-1}{k} E_{\lambda,\nu}^{\gamma+k,q}(z) t^k = (1-t)^{-\gamma} E_{\lambda,\nu}^{\gamma,q}\left(\frac{z}{(1-t)^q}\right). \tag{35}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Study on Fuzzy Neural Sliding Mode Guidance Law with Terminal Angle Constraint for Maneuvering Target

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Aiming at the requirement that the guidance law should meet the minimum miss distance and the desired terminal angle at the same time, a sliding mode variable structure control method is introduced. In order to improve the fuzzy variable structure guidance law for maneuvering target attack effect, a neural network to the optimization design is carried out on the guidance law. The neural network is trained by the samples, which is under the condition of different error coefficient of angle, the coefficient of reaching law, and the coefficient of on-off item about target. Fuzzy neural sliding mode guidance law with terminal angle constraint can increase the performance of the large maneuvering target. In addition, on the basis of the traditional PC platform visual simulation system, a new guidance law simulation platform based on embedded system and virtual reality technology is formed. The platform can verify the validity of the guidance law.

1. Introduction

An air-to-surface missile or guided bombs are precision weapons to attack ground targets launched from the aircraft, where the precision strike is concerned with many other factors, for example, the guidance system of the terminal guidance law design is critical, and it directly affects the final precision strike weapon capacity.

The performance of the guidance system directly affects the missile's precise guidance capability. The entire guidance process of the missile will be divided into 3 stages: the first stage guidance, the middle stage guidance, and the last stage guidance, and the performance of the last stage guidance will directly determine whether the missile can effectively strike the target, so the study of the final guidance law is to improve the overall missile. The guidance ability of the system is of great help, and it is in this context that the research work on the terminal guidance law of the missile is carried out. The guidance law is to control the missile to intercept the target according to a certain trajectory according to the relative motion information of the missile and the target. Therefore, the problem solved by the guidance law is the flight trajectory of the missile intercepting the target [1–4].

For precision-guided weapons, the main task of the guidance system is to output appropriate commands, which ultimately makes the missile's end miss distance as small as possible. However, under certain special circumstances, while requiring the missile to accurately hit the target, it also requires the missile to have an optimal attitude when hitting the target. It is necessary to study the guidance law with the angle-of-restriction in depth and design a guidance law that can meet the requirements of miss distance and angle-of-fall constraint at the same time.

The current guidance laws in engineering practice are mostly the classic guidance laws formed in the 1960s to the 1970s, or improved versions based on these classic guidance laws. The typical guidance law representative is proportional guidance because it has the most improved versions. Proportional guidance was initially designed only for the target to be stationary, that is, the target is not maneuvering, and under the condition that the control energy is not constrained, then proportional guidance is the optimal guidance law for zero miss. However, when targeting a maneuverable target, the proportional guidance law has a relatively large off-target volume, which simply cannot meet the accuracy index required by the missile. Therefore, it is necessary to expand the proportional guidance method for development requirements. The modern guidance law has been developed with the progress of modern control theory and gradually applied to engineering. Typical representatives include optimal guidance law, variable structure guidance law, neural network guidance law, and fuzzy logic guidance law.

Our main contribution in the present paper is that we simulate and analyze the guidance law through MATLAB software. Then BP neural network fuzzy guidance law has been optimized. Therefore, a new type of fuzzy neural network variable structure terminal guidance law is obtained. Meanwhile, in this paper, a new guidance law simulation platform based on embedded platform and PC platforms using virtual reality technology is achieved, compared with the traditional MATLAB software simulation platform, the new platform is close to the underlying algorithm engineering practice, and the effect is closer to the actual battlefield display, making it easier to verify the excellent characteristics of guidance law.

The rest of this paper is organized as follows. In Section 1, the missile-target mathematical model is established. In Section 2, the terminal angle constraint in terminal guidance is analyzed. And in Section 3, variable structure terminal guidance law with terminal angular constraint is derived. We formalize fuzzy variable structure terminal guidance law with terminal angle constraint in Section 4, and we discuss some numerical results in Sections 5 and 6. In Section 7, we design a guidance law simulation platform based on virtual reality technology. In Section 8, the conclusion is given.

2. Establishment of Missile-Target Mathematical Model

Both missiles and targets can be seen as two different particles in space, missile and target coordinate systems are simplified into the same coordinate system, and the coordinate system is established with the distance between the missile and the target as the X-axis and the space height between the missile and the target as the Y-axis [5].

The relative motion relationship between the missile and the target is shown in Figure 1. The horizontal line parallel to the X-axis in the figure is used as the reference line, and r is the relative distance between the missile and the target. q is the angle of sight from missile to target, V_M is the velocity of the missile, V_T is the velocity of the target, σ_M is the ballistic inclination of the missile, and σ_T is the movement inclination of the target.

Equation of relative motion for both missile and target is as follows:

$$\dot{r} = V_T \cos(q - \sigma_T) - V_M \cos(q - \sigma_M), \tag{1}$$

$$r\dot{q} = -V_T \sin(q - \sigma_T) + V_M \sin(q - \sigma_M). \tag{2}$$

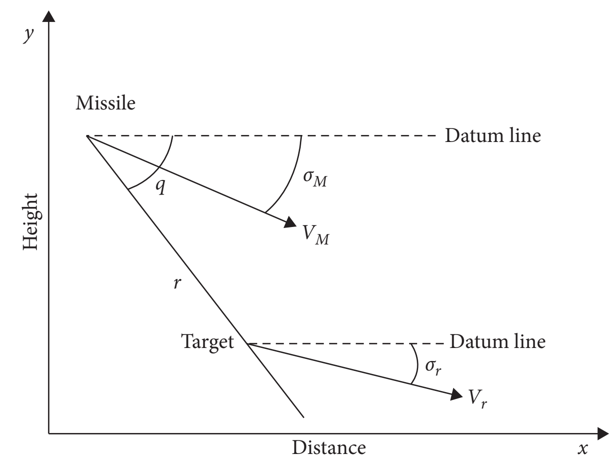

Figure 1: The relative two-dimensional relation of the terminal missile.

where q is the angle of sight between the missile and the target; $r$is the relative distance between missile and target; V_M is the missile speed; σ_M is the ballistic inclination of the missile; V_T is the target speed; and σ_T is the angle of inclination of the target as it moves.

Derivation of time on both sides of equation (1(can be obtained as follows:

$$\begin{aligned}\ddot{r} &= \left[\dot{V}_T \cos(q - \sigma_T) + V_T \sin(q - \sigma_T)\dot{\sigma}_T\right] \\ &\quad - \left[\dot{V}_M \cos(q - \sigma_M) + V_M \sin(q - \sigma_M)\dot{\sigma}_M\right] \\ &\quad - \left[V_T \sin(q - \sigma_T)\dot{q} + V_M \sin(q - \sigma_M)\dot{q}\right].\end{aligned} \tag{3}$$

For equation (3), make $\omega_R = \dot{V}_M \cos(q - \sigma_M) + V_M \sin(q - \sigma_M)\dot{\sigma}_M$, where ω_R represents the component of the target acceleration in the line of sight. Make $u_R = \dot{V}_M \cos(q - \sigma_M) + V_M \sin(q - \sigma_M)\dot{\sigma}_M$, where u_R represents the component of the missile acceleration in the line of sight.

Putting equation (1) into equation (3), we get

$$\ddot{r} = \frac{(r\dot{q})^2}{r} + \omega_R + u_R. \tag{4}$$

Derivation of time on both sides of equation (2) can be obtained as follows:

$$\begin{aligned}\dot{r}\dot{q} + r\ddot{q} &= \left[-V_T \cos(q - \sigma_T) + V_M \cos(q - \sigma_M)\right]\dot{q} \\ &\quad + \left[V_T \cos(q - \sigma_T)\dot{\sigma}_T - \dot{V}_T \sin(q - \sigma_T)\right] \\ &\quad - \left[V_M \cos(q - \sigma_M)\dot{\sigma}_M - \dot{V}_M \sin(q - \sigma_M)\right].\end{aligned} \tag{5}$$

In equation (5), in order to facilitate analysis and calculation, simplify the complex formula to the following:

Let $\omega_Q = \dot{V}_T \cos(q - \sigma_T)\dot{\sigma}_T - \dot{V}_T \sin(q - \sigma_T)$, where ω_Q represents the component of the target acceleration in the normal line of sight. Make $u_R = \dot{V}_M \cos(q - \sigma_M)$ $\dot{\sigma}_M + \dot{V}_M \sin(q - \sigma_M)$, where u_Q represents the component of the missile acceleration in the normal line of sight.

Substituting equation (2) into equation (5), we get

$$\dot{r}\dot{q} + r\ddot{q} = -\frac{\dot{r}r\dot{q}}{r} + \omega_Q - u_Q. \quad (6)$$

From equations (1) to (6), the following equations are derived simultaneously:

$$\begin{cases} \ddot{r} = \left((r\dot{q})^2/r \right) + \omega_R + u_R, \\ \dot{r}\dot{q} + rq = -(\dot{r}r\dot{q}/r) + \omega_Q - u_Q. \end{cases} \quad (7)$$

The equation is integrated and simplified according to formula (7) as follows:

$$\ddot{q} = -\frac{2\dot{r}}{r}\dot{q} + \frac{1}{r}\omega_Q - \frac{1}{r}u_Q. \quad (8)$$

Equation (8) contains multiple parameters. From these parameters, it can be seen that the rate of line-of-sight (LOS) angle q and u_Q show a nonlinear relationship. Among them, there is a certain proportional relationship with ω_Q and the distance r between the missile-target. Therefore, the key to designing the terminal guidance law is how to control the change in the line-of-sight angular rate $\dot{q}$ between the missile and the target through u_Q and make $\dot{q}$ gradually approach 0, so that the missile can approach the target in parallel and achieve the goal of maximizing destruction.

3. Terminal Angle Constraint in Terminal Guidance

In the usual missile terminal guidance, the designers hope that the guided weapon can strike ground targets at high impact angle or even vertical angles, it is necessary to ensure that the miss distance is the smallest, and the large terminal angle control of the hit target is required, and this puts forward higher requirements for the missile's terminal guidance. Therefore, in the design of the guidance law, it is necessary to consider the issue of miss distance and the control of the missile's terminal angle [6–8].

According to the terminal angle requirement in the terminal guidance, the related relational expression in equation (8) cannot describe the issue of missile terminal angle control; therefore, by introducing a terminal angle parameter, the designers hope that the terminal angle control can be realized in the guidance law. In this way, two control variables appear in the guidance law. Both the missile's precise target hit at the end and the terminal angle control when it hits the target must be met; therefore, a state-space design method is introduced here for the sake of design convenience.

Let the expected terminal angle of the end of the missile be q_d, and let there be two state variables x_1 and x_2 in the guidance of the missile end, the state variable x_1 indicates a state where the end has a terminal angle control, and the state variable x_2 indicates the state of the missile hitting the target:

$$\begin{cases} x_1 = q - q_d, \\ x_2 = \dot{q}. \end{cases} \quad (9)$$

It can be seen from equation (9) that if the state variable x_1 approaches zero, then the missile can approach the target at the expected desired attack angle and then the target can be destroyed; if the state variable x_2 approaches zero, the guidance law can meet the requirements for the missile to successfully hit the target. This article is to design such a guidance law that can meet these two requirements at the same time, that is, the state variables x_1 and x_2 are both reach zero.

Derivation of time for each variable in equation (9) can be obtained as follows:

$$\begin{cases} \dot{x}_1 = \dot{q}, \\ \dot{x}_2 = \ddot{q}. \end{cases} \quad (10)$$

Substitute equation (8) into equation (10) and simplify it to get

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{2\dot{r}}{r}\dot{q} + \frac{1}{r}\omega_Q - \frac{1}{r}u_Q. \end{cases} \quad (11)$$

The joint expressions (10) and (11) can further simplify expression (11) to a spatial state expression including the state variable x_1 and the state variable x_2:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\left(\frac{2\dot{r}}{r}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{1}{r}\right) \end{bmatrix} \omega_Q - \begin{bmatrix} 0 \\ \left(\frac{1}{r}\right) \end{bmatrix} u_Q. \quad (12)$$

In the equation of state of equation (12), u_Q is regarded as the control variable and ω_Q is regarded as the interference value.

4. Variable Structure Terminal Guidance Law with Terminal Angular Constraint

Equation (12) is a typical system of nonlinear equations, so solving this system of equations is a typical nonlinear problem. There are many methods for solving nonlinear equations, but in order to satisfy these two states at the same time, the idea of sliding mode variable structure control is used here to introduce a suitable sliding surface. Through continuous switching of this sliding surface, make two state variables meet at the same time [9–12].

4.1. Design Sliding Mode Reaching Law. For the problem with the terminal angular constraint, the purpose of the guidance law design is to obtain zero miss distance and the expected terminal angle at the same time, that is, the outputs x_1 and x_2 in (12) approach 0 in a limited time. $\dot{q} = 0$ represents the ideal state, and the missile can finally hit the target; if the requirement of the end attack angle constraint is to be achieved, $q - q_d = 0$ should be set; therefore, the sliding surface of the design should have at least two state variables; they are $x_1 = q - q_d$ and $x_2 = \dot{q}$.

The switching function of the sliding surface is as follows:

$$S = \lambda \frac{\dot{r}}{r} x_1 + x_2, \quad (13)$$

where λ is the normal number, which represents the angular error coefficient.

The physical meaning of this formula is as follows: when the relative distance r between the missile and the target is large, the second term $\dot{q}$ of the switching surface plays a major role, which is to guide the missile fly to the target; when the relative distance r is very small and almost approaches 0, the first term of the sliding surface plays the main role, that is, the guidance law is expected to hit the target at the desired attack angle, so that the original design requirements can be met.

Substitute equation (9) into equation (13), and then, equation (13) can be further simplified as follows:

$$S = \lambda \frac{\dot{r}}{r} \left(q - q_d\right) + \dot{q}. \quad (14)$$

In order to ensure that the state of the system can reach the sliding mode and have excellent dynamic characteristics in the process of reaching the sliding mode, the reaching law can be used to derive the controller.

The general exponential reaching law and constant velocity reaching law can only be applied to linear time-invariant systems, and the system state equation (14) is a linear time-varying system, so it is necessary to construct a sliding mode approximation with an adaptive time-varying parameter law to ensure that the sliding mode meets the conditions and good dynamic characteristics [13–15].

The general expression of the sliding mode reaching law for a linear time-invariant system is given by

$$\begin{cases} \dot{S} = F(S) - \varepsilon \operatorname{sgn} S, \\ F(0) = 0, \\ SF(s) > 0, \end{cases} \quad (15)$$

where $F(s)$ is a function about S.

The general expression of the adaptive sliding mode reaching law is as follows:

$$\begin{cases} \dot{S} = F(S, p) - \varepsilon(p) \operatorname{sgn} S, \\ F(0, p) = 0, \\ SF(S, p) > 0, \end{cases} \quad (16)$$

$$\dot{S} = K \frac{\dot{r}}{r} S - \frac{\varepsilon}{r} \operatorname{sgn} S. \quad (17)$$

In equation (17), K represents the reaching law coefficient and ε represents the gain coefficient of the switching term.

The physical meaning of equation (17) is as follows: when the relative distance r between the missile and the target is relatively large, the sliding mode approach rate can be adjusted slowly; as r approaches 0, the sliding mode's approaching rate will increase rapidly. This will ensure that $\dot{q}$ does not divergence, so that the accuracy of the missile will be very high. The adaptive adjustment approach law can reduce the sliding mode jitter.

Differentiating (14) gives the following equation:

$$\dot{S} = \dot{x}_2 - \frac{\lambda \ddot{r} x_1 - \lambda \dot{r} \dot{x}_1}{r^2}, \quad (18)$$

$$\dot{S} = \dot{x}_2 - \frac{\lambda \ddot{r}}{r^2} x_1 + \lambda \frac{\dot{r}}{r^2} \dot{x}_1. \quad (19)$$

Substituting (17) into (19), we get

$$K \frac{\dot{r}}{r} S - \frac{\varepsilon}{r} \operatorname{sgn} S = \dot{x}_2 - \frac{\lambda \ddot{r}}{r^2} x_1 + \lambda \frac{\dot{r}}{r^2} \dot{x}_1. \quad (20)$$

Bringing (13) into (20) gives

$$k \frac{\dot{r}}{r} \left(-\lambda \frac{\dot{r}}{r} x_1 + x_2\right) - \frac{\varepsilon}{r} \operatorname{sgn} S = -\frac{2\dot{r}}{r} x_2 + \frac{1}{r} \omega_Q + \frac{1}{r} u_Q - \frac{\lambda \ddot{r}}{r^2} x_1 + \lambda \frac{\dot{r}}{r^2} x_2. \quad (21)$$

Equation (21) can be simplified as follows:

$$u_Q = \left(K\dot{r} + 2\dot{r} - \lambda \frac{\dot{r}}{r}\right) x_2 - \left(\lambda \frac{(\dot{r})^2}{r} + \frac{\lambda \ddot{r}}{r}\right) x_1 - \omega_Q - \varepsilon \operatorname{sgn} S. \quad (22)$$

The adaptive sliding mode guidance law has relatively strong robustness to changes in system parameters, and the speed change during the missile's terminal guidance process is not very large, so it can be made equivalent processing, which is $\dot{r} \approx V_M$ and $\ddot{r} \approx 0$.

So the law of guidance is obtained as follows:

$$u_Q = \left(K + 2 - \lambda \frac{1}{r}\right) V_M x_2 - K\lambda \frac{1}{r} \left(V_M\right)^2 x_1 - \omega_Q - \varepsilon \operatorname{sgn} S. \quad (23)$$

Bring equation (9) into equation (23) to get the mathematical relationship between the final command acceleration and the terminal angle:

$$u_Q = \left(K + 2 - \lambda \frac{1}{r}\right) V_M \dot{q} - K\lambda \frac{1}{r} \left(V_M\right)^2 \left(q - q_d\right) - \omega_Q - \varepsilon \operatorname{sgn} S, \quad (24)$$

where u_Q is the final output command acceleration; ω_Q is the component of the target acceleration in the line of sight; S is the sliding surface switching function; r is the missile-target relative distance; q is the missile-target line of sight; $\dot{q}$ is the missile-target line-of-sight angular rate; V_M is the missile speed; q_d is the end restraint angle; K is the reaching law coefficient; λ is the angular error coefficient; and ε is the switch gain coefficient.

In formula (22), u_Q represents the component of the missile acceleration in the normal line of sight, which is the guidance law of the final output. ω_Q is the component of the target acceleration in the normal line of sight. The sliding mode switching function S contains three important parameters: the reaching law coefficient K, the angular error coefficient λ, and the gain coefficient ε.

5. Fuzzy Variable Structure Terminal Guidance Law with Terminal Angle Constraint

The reasoning process of the fuzzy system is as follows: first, compare the differences between the input variables and membership functions to obtain the membership of each language; then the inference engine finds the corresponding rules in the knowledge base through inference operations; finally, all the results are superimposed for fuzzy output.

To perform fuzzy processing on $\varepsilon \operatorname{sgn} S$, first construct a two-dimensional fuzzy controller. The command switching function S is used as an input to the fuzzy controller, and the change rate $\dot{S}$ of the switching function is used as the other input of the fuzzy controller. Nonlinear control quantity u is used as output. Then, the output of u is the fuzzy output of $\varepsilon \operatorname{sgn} S$, which is $u \approx \varepsilon \operatorname{sgn} S$. The final guidance law is written as follows:

$$u_Q = \left(K + 2 - \lambda \frac{1}{r}\right) V_M \dot{q} - K\lambda \frac{1}{r} (V_M)^2 (q - q_d) - \omega_Q - u. \tag{25}$$

When the system is running, S and $\dot{S}$ are calculated, and then, the quantization factors K_s and $K_{\dot{S}}$ are calculated according to the universe. The two input variables are quantized into fuzzy language variables S and SC, and then the fuzzy variable U is obtained according to fuzzy control rules and fuzzy logic reasoning. Finally, the fuzzy variable U is multiplied by the output scale factor K_u to obtain the precise control amount u.

Define the fuzzy language words set of input variables and output variables as follows: {negative large, negative middle, negative small, zero, positive small, positive middle, positive large}, which is expressed as characters: {NB , NM , NS , O , PS , PM , PB}.

The fuzzy universes of the input variables S and $\dot{S}$ are as follows: $[-6, +6]$, which is expressed as $\{-6, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, +6\}$; Figure 2 shows the membership function of input variable S, and Figure 3 shows the membership function of input variable SC.

The fuzzy set universe of output variable U is $[-7, +7]$, which is expressed as $\{-7, -6, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, +6, +7\}$. Figure 4 shows the membership function of the output variable.

In order to ensure that each fuzzy language variable can cover the entire universe better, here each fuzzy language word set uses 7 variables, and each fuzzy set universe contains 15 quantization levels, so that the universe elements is twice the number of elements in the fuzzy language words set, to achieve full coverage of the universes.

The fuzzy control rules are shown in Table 1. Fuzzy reasoning uses the maximum-minimum method to generate the most likely solution. This reasoning method is very simple and efficient and suitable for real-time control applications. Figure 5 shows a simulation diagram of the output fuzzy surface.

6. Simulation and Analysis

Simulation was done in MATLAB software. Simulation conditions are as follows: an air-to-ground missile attacks an object on the ground, let the initial position of the missile be (0, 2000), missile speed $V_M = 300\,\text{m/s}$, the initial missile ballistic inclination $\sigma_M = 0°$, the location of the target is (1000, 0), and the simulation step size is 0.01 s.

6.1. Terminal Angular Constraint and Target Speed Change

(1) When the terminal angular constraint is $-90°$, the speeds of the targets are $V_T = 15\,\text{m/s}$, $10\,\text{m/s}$, $5\,\text{m/s}$, and $0\,\text{m/s}$. For several situations, the constant parameter takes $k = 1$ and $\lambda = 1$, the trajectory obtained by simulation is shown in Figure 6, and the data analysis is shown in Table 2.

It can be seen from Figure 6 that when the target is stationary, the missile's trajectory is the smoothest. As the target speed increases, the ballistic curve of the missile will fluctuate, and the amplitude of the fluctuation will increase as the speed increases. It can be seen from Table 2 that the stationary target is the easiest to attack, the miss distance and terminal angle deviation are also very small, and the flight time is the least, which is also completely in line with the actual missile attack target situation.

(2) When the target speed is $V_T = 15\,\text{m/s}$, the missile's terminal angle constraint is divided into $-90°$, $-80°$, $-70°$, and $-60°$. For several situations, the constant value selection takes $k = 1$ and $\lambda = 1$. The ballistic trajectory obtained by simulation is shown in Figure 7, and the data analysis is shown in Table 3.

It can be seen from Figure 7 that when the terminal angle is constrained to 90 degrees, that is, when the target is hit vertically, the ballistic trajectory will have a large reverse turn to adjust the vertical strike angle. As the angle decreases, the trajectory is relatively smooth.

From Table 3, it can be seen that the maximum miss distance when hitting the target is at 90 degrees, and at 80 degrees, the deviation between the miss distance and the terminal angle is very small. This also shows that the effectiveness of the warhead can be fully exerted when striking the target vertically, and at the same time, the accuracy of the missile is also affected; therefore, a balance must be made between the terminal angle of restraint and the effectiveness of the warhead.

6.2. Reaching Law Coefficient. Parameter values are as follows: target speed = 15 m/s, terminal angle constraint = $-80°$, and $\lambda = 1$.

Figures 8–10 show the ballistic trajectory, trajectory inclination angle, and normal acceleration at different k

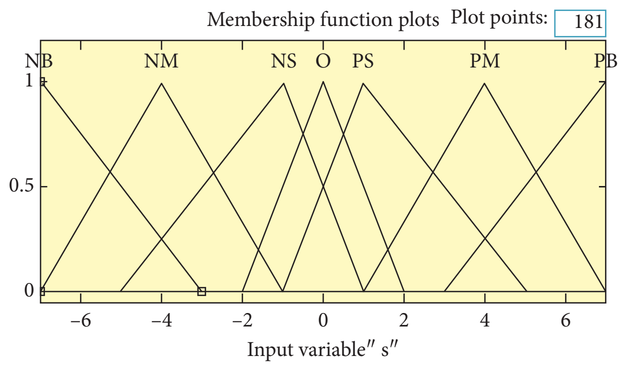

Figure 2: Membership functions of input variable S.

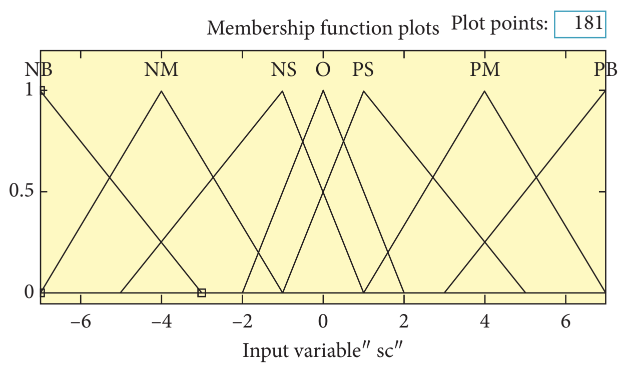

Figure 3: Membership functions of input variable SC.

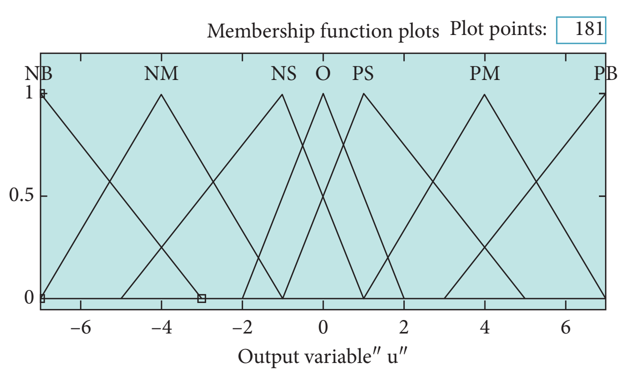

Figure 4: Membership functions of output variable u.

Table 1: Fuzzy logic control rule table.

	NB	NM	NS	O	PS	PM	PB
NB	PB	PB	PM	PM	PS	O	O
NM	PB	PB	PM	PS	PS	O	O
NS	PM	PM	PM	PS	O	NS	NS
O	PM	PM	PS	O	NS	NM	NM
PS	PS	PS	O	NS	NS	NM	NM
PM	PB	O	NS	NM	NM	NM	NB
PB	O	O	NM	NM	NM	NB	NB

values. Table 4 shows the guidance effect data under different k values.

It can be seen from the simulation diagram that when $k = 0.1$, the missile is fully capable of a 90° angle dive attack

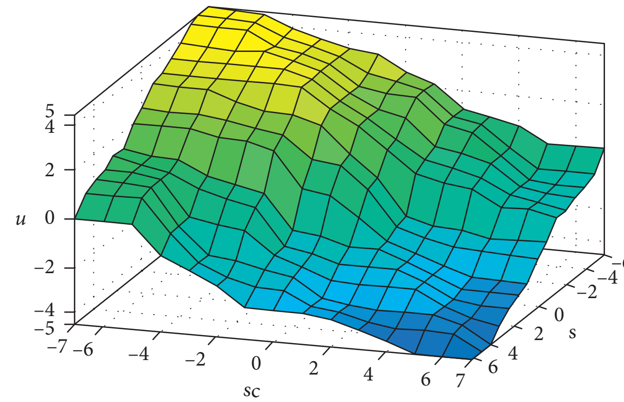

Figure 5: The output of the fuzzy surface.

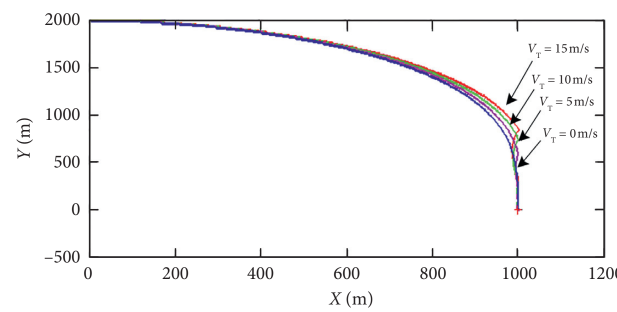

Figure 6: Trajectories of different velocities.

target, but in order to achieve the initial terminal angle constraint, it can be seen that the missile has a turning curve that intentionally maintains the landing angle. As the value of k increases, the ballistic trajectory gradually becomes flat; the inclination of the missile has a zigzag wave in the middle stage when $k = 0.1$, but as the value of k increases, the ballistic inclination is gradually smooth. The normal acceleration of the missile does not change much with the value of k; considering the miss distance and the falling angle deviation, the smaller the reaching law coefficient, the better the effect. In summary, it is better to choose k value between 1 and 10.

6.2.1. Angular Error Coefficients. Parameter values are as follows: target speed = 15 m/s, terminal angle constraint = −80°, and $k = 1$.

Figures 11–13 show the ballistic trajectory ballistic inclination, and normal acceleration under different angular error coefficients. It can be seen from the figure that when the error coefficient $\lambda = 0.1$, the missile did not hit the target at 1000 meters, and when $\lambda = 0.5$, the missile successfully hit the target. This means that the value of λ cannot be too small to hit the target; when $\lambda = 0.1$, the ballistic inclination obviously fluctuates too much, and the normal acceleration is also obviously not normal, so this also verifies that λ cannot be smaller.

As can be seen from Table 5, $\lambda = 0.5$ is a more appropriate value. The miss distance and the falling angle deviation are also very small.

Table 2: Guidance effect under different velocities.

Speed (m/s)	Miss distance (m)	Falling angle deviation (°)	Time of flight (s)
15	3.3107	47.8968	8.3200
10	15.6580	10.1036	8.2500
5	2.0579	33.4270	8.2700
0	0.9258	0.0012	8.2300

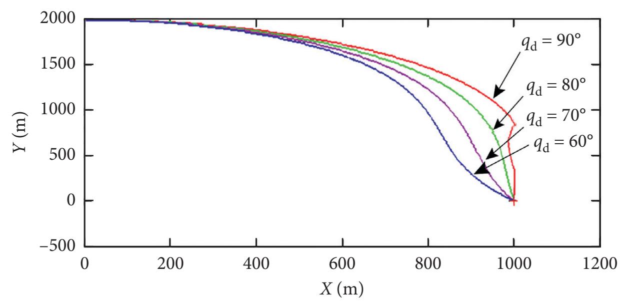

Figure 7: Trajectories of different terminal angles.

Table 3: Guidance effect under different angles.

Terminal angle constraint	Miss distance (m)	Falling angle deviation (°)	Time of flight (s)
−90	3.3107	47.8968	8.3200
−80	1.2544	5.7171	8.1500
−70	1.6046	15.840	8.0300
−60	2.2710	26.0932	7.9600

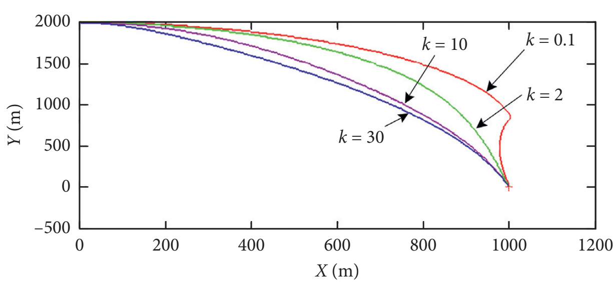

Figure 8: Ballistic trajectory at different values of k.

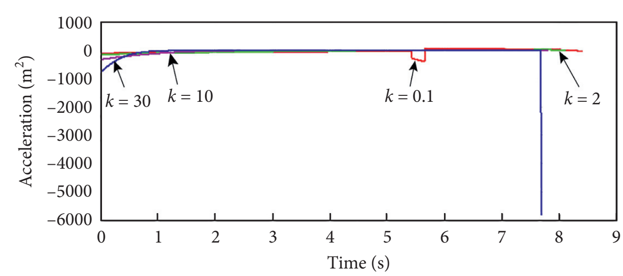

Figure 10: Missile normal acceleration at different values of k.

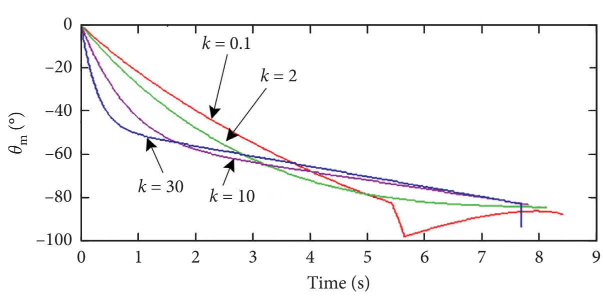

Figure 9: Trajectory inclination angle at different values of k.

By setting different reaching law coefficients and angular error coefficients, different ballistic trajectories, trajectory inclination, and missile normal accelerations are obtained. Through the analysis of miss distance data and falling angle error data, it can be seen that the two problems of end-guided miss distance and terminal angle constraint are solved simultaneously. Through these characteristic curves, comparing the parameters such as miss distance and time of flight, different parameter value ranges are obtained under the conditions of terminal angle constraints and precise guidance.

7. Optimization of Fuzzy Variable Structure Terminal Guidance Law Based on Neural Network

The missile uses fuzzy variable structure terminal guidance law at the end, which can better hit low-speed ground targets, such as tanks and armored vehicles with a speed of 15 m/s.

But for high-speed targets, the effect is not ideal. The main reason is to achieve the constraint of the terminal angle, and the missile needs to track the target trajectory in time by increasing the overload. But in the process of terminal guidance, the time is very short, and it is very difficult

Table 4: Guidance effect under different values of reaching law coefficient k.

Reaching law coefficient k	Miss distance (m)	Falling angle deviation (°)	Time of flight (s)
0.1	0.3023	2.0896	8.4000
2	2.2023	5.2486	8.1200
10	0.3531	6.6584	7.8100
30	2.3912	173.0585	7.6900

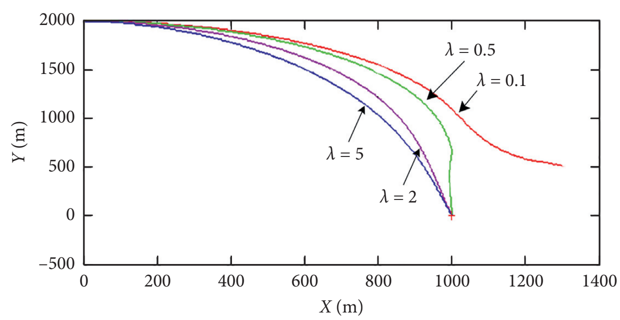

Figure 11: Ballistic trajectory at different angular error coefficients.

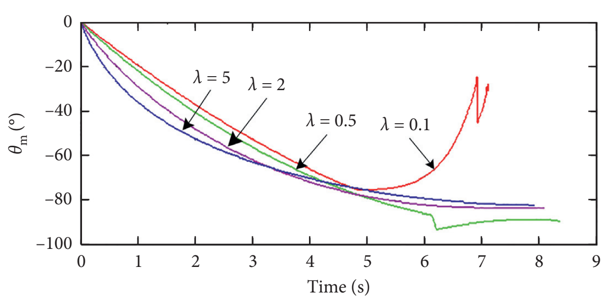

Figure 12: Trajectory inclination at different angular error coefficients.

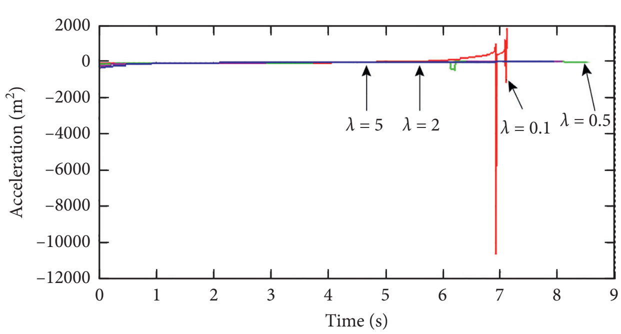

Figure 13: Missile normal acceleration at different angular error coefficients.

Table 5: Guidance effect at different values of angular error coefficient λ.

Reaching law coefficient	Miss distance (m)	Falling angle deviation (°)	Time of flight (s)
0.1	592.8644	149.5891	7.1100
0.5	0.8330	0.1252	8.3600
2	1.4469	6.4344	8.0900
5	2.6349	7.7193	7.9300

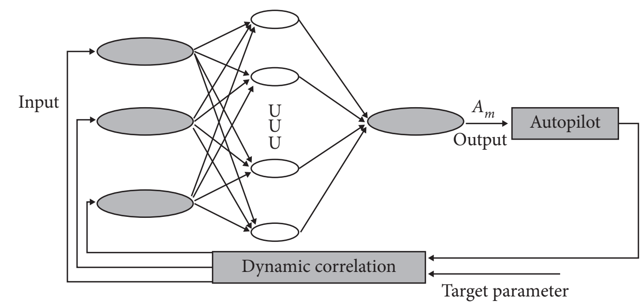

Figure 14: Neural network guidance loop.

for the missile to provide a large overload in a short time. So for missiles, the guidance law is always expected to provide a straight or smooth trajectory [16–18].

Different reaching law coefficients and angular error coefficients in variable structures have an impact on the final guidance law; when tracking a large maneuvering target, these two coefficients have an optimal range, so when the target is a large maneuver, they can automatically learn and adjust these two coefficients. The neural network system formed by this training should be able to track well big maneuvering target.

In the network structure design, a single hidden layer BP neural network is used. The theory proves that a feedforward network with a single hidden layer can map all continuous functions, and only two hidden layers are needed when learning discontinuous functions. Therefore, a single hidden layer can be used to map the fuzzy guidance law. The input layer has three input variables, which are the input line-of-sight angular rate $\dot{q}$, relative speed $\dot{r}$, and line-of-sight angle q; the output layer has only one variable, which is the command acceleration A_m.

Figure 14 shows a neural network guidance circuit. The command acceleration A_m is generated under the excitation of the input line-of-sight angular rate $\dot{q}$, relative speed $\dot{r}$, and line-of-sight angle q. It can be expressed as follows:

$$A_m = \xi(\dot{q}, \dot{r}, q). \tag{26}$$

In the above formula, ξ represents a nonlinear function, which is used to realize the nonlinear mapping of (3-21). Finally, the commanded acceleration A_m generated is sent to the autopilot, and the missile guidance can be completed.

Considering the number of neurons in the input and output layers and the number of training samples, the number of hidden neurons was finally determined to be 20 after several simulations.

The transfer function of the hidden neuron is a nonlinear transfer function $O_i = 1/(1 + \exp(-x_i))$, where O_i is the

Table 6: Fuzzy rule table.

	NB	NM	NS	O	PS	PM	PB
NB	PB	PM	PM	PM	PS	O	O
NM	PS	PB	PM	PS	PS	O	O
NS	PM	PM	PM	PS	O	NS	NS
O	PB	PM	PS	O	NS	NM	NS
PS	PS	PS	O	NS	NS	NB	NM
PM	PB	O	NS	NM	NM	NM	NB
PB	O	O	NM	PS	NM	NB	NB

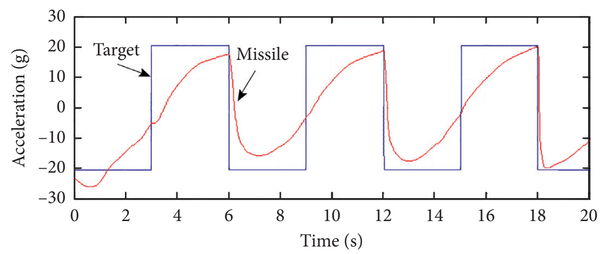

Figure 15: Fuzzy neural network training results.

output of the i-th hidden layer neuron and $x_i = \sum_{j=1}^{2} W_{ij} I_j$ is the input of the i-th hidden layer neuron, in which I_j is the j-th input of the input layer and W_{ij} is the connection weight between I_j and x_i; the transfer function of the output layer neuron is a linear transfer function $f = x$, where f is the output of the output layer neuron and $x = \sum_{i=1}^{20} W_{2i} O_i$ is the input, in which W_{2i} is the connection weight of the a hidden layer neuron and the output neuron and Q_i is the output of the i-th hidden layer neuron. When determining the training samples of the BP neural network, in order to track large maneuvering targets, step maneuvers are selected as the typical maneuvering modes of the target, and the best reaching law coefficient and angular error coefficient are selected at the same time. For both cases, the line-of-sight angular rate, relative angular velocity, and line-of-sight angle obtained by the angularly constrained fuzzy variable structure guidance law are used as the input set of the training sample space, and the corresponding command acceleration is used as the output set.

Based on the fuzzy logic controller, this group of samples is obtained by adjusting the sizes of k and λ under different conditions and the normal overload of the target maneuver.

Select several typical situations are as follows:

(1) $k = 0.5$, $\lambda = 1$, $q = 0°$, $\sigma_M = 30°$, $a_t = 200\,\mathrm{m/s^2}$, and $r = 2000\,\mathrm{m}$

(2) $k = 1$, $\lambda = 2$, $q = 0°$, $\sigma_M = 30°$, $a_t = 200\,\mathrm{m/s^2}$, and $r = 2000\,\mathrm{m}$

(3) $k = 0.5$, $\lambda = 1$, $q = 30°$, $\sigma_M = 30°$, $a_t = 100\,\mathrm{m/s^2}$,and $r = 2000\,\mathrm{m}$

(4) $k = 1$, $\lambda = 2$, $q = 30°$, $\sigma_M = 30°$, $a_t = 100\,\mathrm{m/s^2}$, and $r = 2000\,\mathrm{m}$

(5) $k = 2$, $\lambda = 5$, $q = 60°$, $\sigma_M = 30°$, $a_t = 50\,\mathrm{m/s^2}$, and $r = 2000\,\mathrm{m}$

(6) $k = 2$, $\lambda = 5$, $q = 60°$, $\sigma_M = 30°$, $a_t = 50\,\mathrm{m/s^2}$, and $r = 2000\,\mathrm{m}$

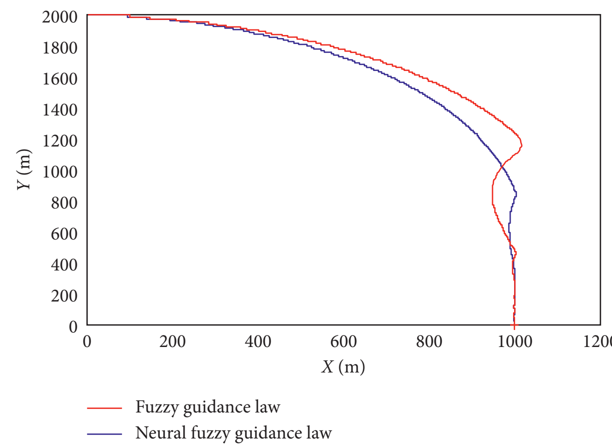

Figure 16: Ballistic trajectory of two algorithms.

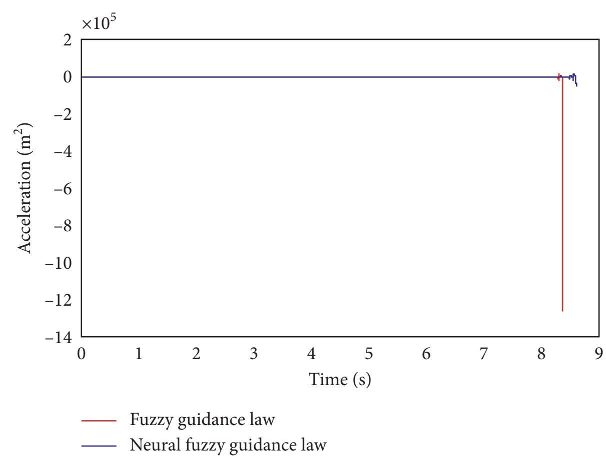

Figure 17: Normal acceleration of the two algorithms.

In a neural network, to determine the weight matrix from the hidden layer to the output layer, it represents 49 regular outputs. Use the rule table in Table 6 to assign values to it. The process of neural network training is to continuously adjust and refresh this weight matrix. Through continuous training of samples, an optimized fuzzy rule table can be obtained, as shown in Table 6. The results after training are shown in Figure 15.

An air-to-surface missile attacks an object on the ground, let the initial position of the missile be (0, 2000), the speed of the missile is $V_M = 300\,\mathrm{m/s}$, the initial missile ballistic inclination is $\sigma_M = 0°$, the location of the target is

Table 7: Guidance effect comparison.

	Miss distance (m)	Falling angle deviation (°)	Time of flight (s)
Fuzzy guidance law	0.8238	7.2829	8.1100
Neural fuzzy guidance law	0.4029	1.3893	8.1500

(1000, 0), the target speed is $V_T = 300$ m/s, let the simulation step be 0.01 s, and terminal angle constraint is 80 degrees, and the comparison of the curves when $k = 1, \lambda = 1$, and $\varepsilon = 1$ is shown in Figures 16 and 17.

From the ballistic trajectory in Figure 16, it can be seen that when the target is maneuvering, the fuzzy variable structure guidance law can still successfully hit the target, but the ballistic curve is too curved. In the actual missile guidance process, the missile cannot produce such a large amount of maneuver. The ballistic trajectory produced by the neural fuzzy guidance law optimized by the neural network has improved significantly, is smoother, and has less jitter.

Figure 17 shows a comparison of the normal acceleration of the missile. It can be seen from the figure that the missile using fuzzy guidance law has a large normal acceleration before hitting the target, and the normal acceleration of the neural fuzzy guidance law has been relatively stable. Therefore, it can be seen that the fuzzy guidance law optimized by the neural network has greatly improved its performance.

Table 7 shows the comparison of various parameters of the guidance effect. It can be seen that compared with the fuzzy guidance law, the neural fuzzy guidance law has significantly improved the guidance accuracy and other aspects.

8. Design of Guidance Law Simulation Platform Based on Virtual Reality Technology

Virtual reality simulation first solves the mathematical model of the simulation system by a numerical analysis method and then displayed on the screen by the display technology of the computer system. This will give people an intuitive and realistic experience. The actual missile guidance law is solved by a missile-borne computer, which is a typical embedded system, which is quite different from a PC platform. If the designed guidance law can be calculated in an embedded system, the performance of the guidance law will be better verified. The above-mentioned fuzzy variable structure terminal guidance law with terminal angle constraint is combined with two technologies of embedded system and virtual reality to design a new guidance law simulation platform to intuitively show the research results of guidance law [19–22].

Figure 18 shows a schematic diagram of the framework structure of the entire simulation platform. On the left is the embedded platform. The hardware uses the MPC8247 processor and runs the VxWorks operating system. This part is mainly responsible for the calculation of the guidance law algorithm and finally outputs the command acceleration to the PC platform through the serial port. The PC platform

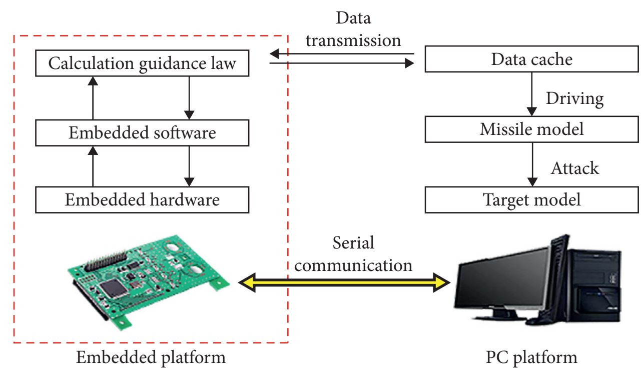

Figure 18: New style simulation platform framework.

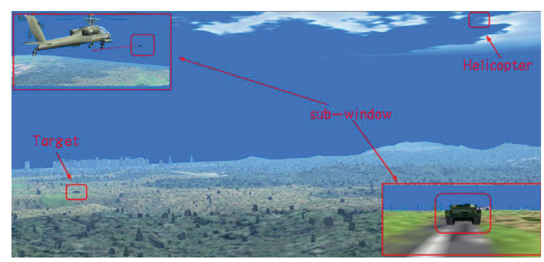

Figure 19: Virtual helicopter launching missile graph.

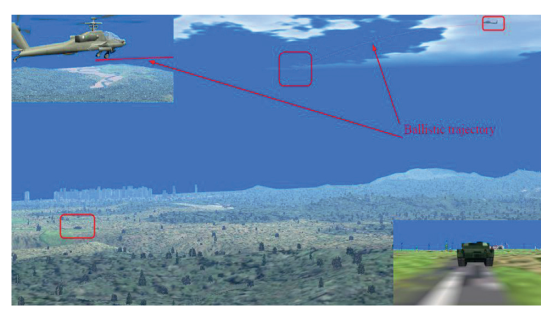

Figure 20: Virtual missile flying graph in the air.

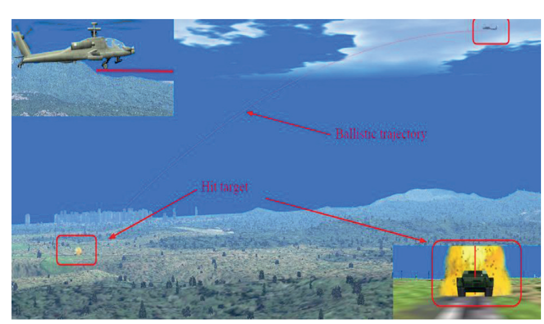

Figure 21: Virtual missile accurate hit target graph.

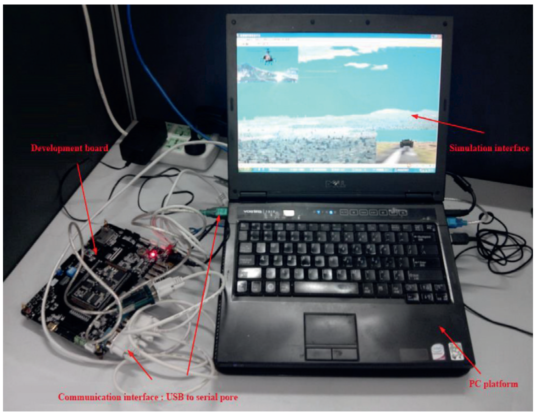

Figure 22: Physical map of the simulation platform.

uses creator and Vega Prime to develop and design a three-dimensional visual environment, and this part mainly completes the three-dimensional display of the battlefield environment and finishes receiving the guidance law calculated under the embedded platform, transmitting the guidance law parameters to the missile model, and then driving the missile model to approach the target along a certain trajectory, and then destroys the target.

Simulation parameters are as follows: the armed helicopter found a ground armored target and was ready to launch an air-to-ground missile to destroy it. Let the initial position of the missile be 2000 m above the ground, the speed of the missile is $V_M = 300\,\mathrm{m/s}$, the target speed is $V_T = 300\,\mathrm{m/s}$, the target fixed position is 5000 m from the helicopter horizontal distance of 5000 m, and the terminal angular constraint is 70 degrees.

In Figure 19, the helicopter found a ground target and launched an air-to-ground missile. The missile's ballistic trajectory can be seen in the upper subwindow. In Figure 20, the missile flies in the air according to a predetermined trajectory, and the trajectory can be clearly seen. In Figure 21, the missile accurately hits the target, and it can be clearly seen in the lower subwindow that, from the perspective of the entire trajectory, the terminal angular constraint is basically satisfied. It can be seen that the new guidance law simulation platform combining embedded systems and virtual reality technology has well demonstrated the entire process of accurate terminal guidance of air-to-ground missiles.

Figure 22 shows the real graph of the simulation platform. The left side is the embedded system platform MPC8247 development board, the red light on the development board indicates that the development board is in normal working state. The two platforms communicate through a serial port protocol; use USB to serial port to complete data transmission.

9. Conclusion

The missile's terminal guidance not only needs to meet the miss distance but also requires that the terminal angle of the attack be restricted. In order to meet this requirement, this paper proposes a terminal guidance law based on sliding mode variable structure, and blurs the jitter problem in the guidance law by the fuzzy logic method. By setting different approach law coefficients and angular error coefficients, different ballistic trajectories, trajectory inclination, and missile normal accelerations are obtained. Through the analysis of miss distance data and terminal angular error data, it can be seen that the two problems of terminal guidance miss distance and terminal angular constraint are solved simultaneously. At the same time, these characteristic curves, by comparing graphics, missed targets, flight time, etc., summarized the different parameter value ranges that meet both the terminal angle constraint and the precise guidance conditions.

The BP neural network is used to optimize the fuzzy variable structure terminal guidance law with terminal angle constraint, which effectively solves the problem of fuzzy variable structure terminal guidance law for large maneuvering targets. By the neural network self-learning and adaptive capabilities using large maneuvering targets as sample inputs, use the best reaching law coefficient and angle error coefficient. The simulation results show that the fuzzy variable structure terminal guidance law optimized by the neural network has improved the guidance accuracy and other aspects significantly.

Based on this, a new guidance law simulation platform based on the combination of embedded system and virtual reality technology is designed. Simulation experiments verify the correctness of the guidance law and the display effect is better.

Data Availability

All data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Study of Fractional Differential Equation with a Positive Constant Coefficient via Hilfer Fractional Derivative

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The aim of the paper is to consider the existence and uniqueness of solution of the fractional differential equation with a positive constant coefficient under Hilfer fractional derivative by using the fixed-point theorem. We also prove the bounded and continuous dependence on the initial conditions of solution. Besides, Hyers–Ulam stability and Hyers–Ulam–Rassias stability are discussed. Finally, we provide an example to demonstrate our main results.

1. Introduction

In recent years, the study of the fractional differential and integral equation (FDE and IDE for short) has become the topic of the applied mathematics. FDE and IDE have been used as a tool mathematical to the modeling of many phenomena in various fields for example, in theory of signal processing, physics, economics, and chaotic dynamics. The reader can refer to the books (see Varsha [1], Kilbas et al. [2], Miller et al. [3], Abbas et al. [4], and Zhou et al. [5]) or the papers (see Zin et al. [6], Ma et al. [7], P. Agarwal et al. [8, 9], Rameshet al. [10], Vivek et al. [11], O'Regan et al. [12], and Duc et al. [13]).

Tate and Dinde [14] proved the existence of solution of the problem:

$$ {}^{c}\mathfrak{D}_{0^+}^{\alpha}\xi(t) = \lambda\xi(t) + \zeta(t,\xi(t)), \quad \xi(0) = x_0 \in \mathfrak{R}, \lambda > 0, \tag{1} $$

where the symbol ${}^{C}\mathfrak{D}_{0^+}^{\alpha}$ is Caputo fractional derivative and $\zeta \in C([0,a],\mathfrak{R})$. Besides, the authors also considered the properties of solutions of this problem such as the boundedness of solution and the continuous dependence of solutions on the initial conditions. In [15], Tate et al. also performed the same study as [14] for the class of fractional integro-differential equations with positive constant coefficient.

In 2018, Sousa and Oliveira [16] have introduced a new fractional derivative with respect to another function the so-called ψ-Hilfer fractional derivative, and the properties of this concept were also presented. Besides, the authors considered the relationship between the ψ−Hilfer fractional derivative and the other fractional derivative such as Riemann–Liouville fractional derivative, Caputo fractional derivative, Hadamard fractional derivative, Katugampola fractional derivative, and Chen fractional derivative. By using the ψ−Hilfer fractional derivative, Sousa and Oliveira [17] studied the existence and uniqueness of solution of the initial valued problem for FDEs. The continuous dependence of solution on the initial condition was also considered. The stability theory for FDIs and IDEs via ψ−Hilfer fractional derivative have also been discussed (see [18–22]). In [23], by using Gronwall inequality and Picard operator theory, Kharade and Kucche proved the existence and uniqueness of solutions for impulsive implicit delay

ψ–Hilfer fractional differential equations. The Ulam–Hyers–Mittag–Leffler stability was also considered.

Motivated by Tate el al. [14, 15], Sousa et al. [16], and Kharade et al. [23], in this paper, we investigate the existence and uniqueness of solutions and some properties of solutions of the following fractional differential equation with the constant coefficient $\lambda > 0$:

$$ {}^{\mathbb{H}}_{0^+}\mathscr{D}^{h,v;g}\xi(t) = \lambda\xi(t) + \zeta(t,\xi(t)), \tag{2}$$

with the initial condition

$$\mathfrak{I}_{0^+}^{1-\gamma,g}\xi(0) = A, \tag{3}$$

where the symbol ${}^{\mathbb{H}}_{0^+}\mathscr{D}^{h,v;g}\xi(t)$ is the g–Hilfer fractional derivative of ξ with $0<h<1, 0\le v\le 1$ and ξ: $[0,a] \longrightarrow \mathfrak{R}$ is a continuous function, ζ is a continuous function with respect to t and ξ on $[0,a]\times\mathfrak{R}$, $\mathfrak{I}_{0^+}^{1-\gamma,g}(\cdot)$ is g–Riemann–Liouville fractional integral with $0\le\gamma = h + v(1-h) < 1$, and A is a given constant.

2. Preliminaries

In this section, we introduce some notations and some concepts which are used throughout this paper. This result can be found in the books [3, 8] and the papers [16, 17].

Let $C([0,a],\mathfrak{R})$ be the space of all continuous functions ζ: $[0,a] \longrightarrow \mathfrak{R}$ and $C^n([0,a],\mathfrak{R})$ be the space of all n–times continuously differentiable functions on $[0,a]$. We will introduce the weighted spaces of all continuous functions

$$C_{\gamma,g}([0,a]) \coloneqq \{\zeta\colon (0,a] \longrightarrow \mathfrak{R}\colon (g(t)-g(0))^{\gamma}\zeta(t)\in C([0,a])\}, \tag{4}$$

with the norm

$$\begin{aligned}\|\zeta\|_{C_{\gamma,g}([0,a])} &= \left\|(g(t)-g(0))^{\gamma}\zeta(t)\right\|_{C([0,a])}\\ &= \max_{t\in[0,a]}\left|(g(t)-g(0))^{\gamma}\zeta(t)\right|,\\ C_{1-\gamma,g}([0,a]) &\coloneqq \left\{\zeta\colon (0,a] \longrightarrow \mathfrak{R}\colon (g(t)-g(0))^{1-\gamma}\zeta(t) \in C([0,a])\right\},\end{aligned} \tag{5}$$

with the norm

$$\begin{aligned}\|\zeta\|_{C_{1-\gamma,g}([0,a])} &= \left\|(g(t)-g(0))^{1-\gamma}\zeta(t)\right\|_{C([0,a])}\\ &= \max_{t\in[0,a]}\left|(g(t)-g(0))^{1-\gamma}\zeta(t)\right|.\end{aligned} \tag{6}$$

Let function f be integrable on $[0,a]$ and function g: $[0,a] \longrightarrow \mathfrak{R}_+$ be increasing on $[0,a]$ with $g'(t)\neq 0$ for all $t\in[0,a]$. Then, the g–Riemann–Liouville fractional integral of f with respect to g is defined by

$$\mathfrak{I}_{0^+}^{h;g}\xi(t) = \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)\mathrm{d}s, \quad h>0, \tag{7}$$

where $\mathfrak{Q}_g^h(t,s) = g'(s)(g(t)-g(s))^{h-1}$ for all $t,s\in[0,a]$.

Let $n-1<h<n, n\in\mathbb{N}$, and $f\in C^n([0,a],\mathfrak{R})$. Then, the Hilfer fractional derivative ${}^{H}\mathfrak{D}_{0^+}^{h,v;g}\xi(\cdot)$ of the function ξ of order h and $v\in[0,1]$ is given by

$$ {}^{\mathbb{H}}\mathscr{D}_{0^+}^{h,v;g}\xi(t) = \mathfrak{I}_{0^+}^{(n-h)v;g}\left(\frac{1}{g\prime(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \mathfrak{I}_{0^+}^{(n-h)(1-v);g}\xi(t), \tag{8}$$

where $n = [h]+1$.

Theorem 1 (see [16, 17]). *If $\zeta\in C^1([0,a],\mathfrak{R})$, $h>0$, and $v\in[0,1]$, then*

(i) ${}^{\mathbb{H}}\mathscr{D}_{0^+}^{h,v;g}\mathfrak{I}_{0^+}^{h;g}\xi(t) = \xi(t)$ and $\forall t\in[0,a]$

(ii) $\mathfrak{I}_{0^+}^{h;g\mathbb{H}}\mathfrak{D}_{0^+}^{h,v;g}\xi(t) = \xi(t) - \sum_{k=0}^{n}(g(t)-g(0))^{\gamma-k}/\Gamma(\gamma-k+1)\mathfrak{I}_{0^+}^{(1-h)(1-v);g}\xi(0)$ and $\gamma = h+v(1-h)$

Lemma 1 (see [17]). *Let $a(t)$ and $b(t)$ be integrable functions and $c\in C([0,a],\mathfrak{R})$. Let $g\in C^1([0,a],\mathfrak{R})$ such that $g'(t)>0$ for any $t\in[0,a]$. Assume that $a(t)$ and $b(t)$ are nonnegative and g is nonnegative and nondecreasing. If*

$$a(t)\le b(t) + c(t)\int_0^t \mathfrak{Q}_g^h(t,s)a(s)\mathrm{d}s, \tag{9}$$

then

$$a(t)\le b(t)\mathbb{E}_h\left(c(t)\Gamma(h)(g(t)-g(0))^h\right), \quad \forall t\in[0,a], \tag{10}$$

where $\mathbb{E}_h(\xi)$ is Mittag–Leffler function is defined by

$$\mathbb{E}_h(\xi) = \sum_{k=0}^{\infty}\frac{\xi^k}{\Gamma(1+kh)}, \quad h>0, k\in\mathbb{N}. \tag{11}$$

Definition 1 (see [21, 23]). Problem (2) is called Hyers–Ulam stable if there exists a positive constant C_ζ such that, for any $\varepsilon>0$ and for each $\xi\in C_{1-\gamma,g}([0,a])$ satisfying the inequality,

$$\left|{}^{\mathbb{H}}_{0^+}\mathscr{D}^{h,v;g}\xi(t) - \lambda\xi(t) - \zeta(t,\xi(t))\right| \le \varepsilon, \quad \forall t\in[0,a], \tag{12}$$

there exists a solution $\widehat{\xi}\in C_{1-\gamma,g}([0,a])$ of problem (2) satisfying

$$|\xi(t) - \widehat{\xi}(t)| \le C_\zeta\varepsilon, \quad \forall t\in[0,a]. \tag{13}$$

Definition 2 (see [21, 23]). Problem (2) is called Hyers–Ulam–Rassias stable, with respect to $\phi\in C([0,a],\mathfrak{R})$, if there exists a positive constant $C_{\zeta,\phi}$ such that, for any $\varepsilon>0$ and for each $\xi\in C_{1-\gamma,g}([0,a])$ satisfying the inequality,

$$\left|{}^{\mathbb{H}}_{0^+}\mathscr{D}^{h,v;g}\xi(t) - \lambda\xi(t) - \zeta(t,\xi(t))\right| \le \epsilon\phi(t), \quad \forall t\in[0,a], \tag{14}$$

there exists a solution $\widehat{\xi}\in C_{1-\gamma,g}([0,a])$ of problem (2) satisfying

$$|\xi(t) - \widehat{\xi}(t)| \le C_{\zeta,\phi}\epsilon\phi(t), \quad \forall t\in[0,a]. \tag{15}$$

3. Main Results

Firstly, we note that applying the fractional integral operator $\mathfrak{I}_{0^+}^{h;g}(\cdot)$ to both sides of equation (2), we obtain

$$\mathfrak{I}_{0^+}^{h;g}{}^{\mathbb{H}}_{0^+}\mathfrak{D}^{h,v;g}\xi(t) = \lambda\mathfrak{I}_{0^+}^{h;g}\xi(t) + \mathfrak{I}_{0^+}^{h;g}\zeta(t,\xi(t)), \quad \forall t \in [0,a]. \tag{16}$$

Using Theorem 1 and the initial condition (3), we have the following integral equation:

$$\begin{aligned}\xi(t) &= \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{I}_{0^+}^{h;g}\xi(0) + \lambda\mathfrak{I}_{0^+}^{h;g}\xi(t) \\ &\quad + \mathfrak{I}_{0^+}^{(1-h)(1-v);g}\zeta(t,\xi(t)) \\ &= \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A + \frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s,\end{aligned} \tag{17}$$

for any $t \in [0,a]$.

On the contrary, if ξ satisfies equation (17), then ξ satisfies equations (2 and 3. Moreover, operating the fractional derivative operator ${}^{\mathbb{H}}_{0^+}\mathfrak{D}^{h,v;g}(\cdot)$ on both sides of equation (17), we obtain

$$\begin{aligned}{}^{\mathbb{H}}_{0^+}\mathscr{D}^{h,v;g}\xi(t) &= {}^{\mathbb{H}}_{0^+}\mathfrak{D}^{h,v;g}\left(\frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{I}_{0^+}^{h;g}\xi(0)\right) \\ &\quad + {}^{\mathbb{H}}_{0^+}\mathfrak{D}^{h,v;g}\left(\lambda\mathfrak{I}_{0^+}^{h;g}\xi(t)\right) + {}^{\mathbb{H}}_{0^+}\mathfrak{D}^{h,v;g} \\ &\quad \cdot\left(\mathfrak{I}_{0^+}^{(1-h)(1-v);g}\zeta(t,\xi(t))\right), \quad \forall t \in [0,a].\end{aligned} \tag{18}$$

By Theorem 1, we have

$${}^{\mathbb{H}}_{0^+}\mathscr{D}^{h,v;g}\left(\frac{(g(t)-g(0))^{h-1}}{\Gamma(\gamma)}\mathfrak{I}_{0^+}^{h;g}\xi(0)\right) = 0, \quad 0<h<1. \tag{19}$$

Combining (18) and (19), we imply

$${}^{\mathbb{H}}_{0^+}\mathscr{D}^{h,v;g}\xi(t) = \lambda\xi(t) + \zeta(t,\xi(t)), \quad \forall t \in [0,a]. \tag{20}$$

Next, we verify that the initial condition 3 holds. Indeed, applying the Riemann–Liouville fractional integral $\mathfrak{I}_{0^+}^{1-v}(\cdot)$ on both sides of equation (17), we have

$$\begin{aligned}\mathfrak{I}_{0^+}^{1-v}\xi(t) &= \mathfrak{I}_{0^+}^{1-v}\left(\frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{I}_{0^+}^{h;g}\xi(0)\right) \\ &\quad + \mathfrak{I}_{0^+}^{1-v}\left(\lambda\mathfrak{I}_{0^+}^{h;g}\xi(t)\right) + \mathfrak{I}_{0^+}^{1-v}\left(\mathfrak{I}_{0^+}^{(1-h)(1-v);g}\zeta(t,\xi(t))\right), \quad t \in [0,a].\end{aligned} \tag{21}$$

Taking $t \longrightarrow 0$ in equation (21), we have

$$\mathfrak{I}_{0^+}^{1-v}\xi(t) = A. \tag{22}$$

In summary, we can conclude that ξ satisfies equations (2 and 3 if and only if ξ satisfies the following integral equation:

$$\begin{aligned}\xi(t) &= \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A + \frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s,\end{aligned} \tag{23}$$

for any $t \in [0,a]$.

3.1. Existence and Uniqueness Solution for Problem (2). In this section, we will prove the existence and uniqueness solution of equation (2) with the initial condition (3). Firstly, we assume that the function ζ satisfies the following assumption: there exists a constant $L>0$ such that

$$\text{(H1)}\ \left|\zeta(t,\xi_1) - \zeta(t,\xi_2)\right| \le L\left|\xi_1 - \xi_2\right|, \quad t \in [0,a]. \tag{24}$$

Theorem 2. *Assume that assumption (H1) is satisfied. Then, problems (2) and (3) have at least one solution*

Proof. We set $\tilde{\zeta} := \zeta(t,0)$ for any $t \in [0,a]$. Let us define the set

$$\mathscr{U}_r := \left\{\xi \in C_{1-\gamma;g}([0,a]) \colon \|\xi\|_{C_{1-\gamma;g}} \le r\right\}, \tag{25}$$

with

$$\begin{aligned}r &:= \frac{|A|}{\Gamma(\gamma)} + \frac{B(h,\gamma)(g(a)-g(0))^h}{\Gamma(h)}(\lambda+L)\|\xi\|_{C_{1-\gamma;g}} \\ &\quad + \frac{B(h,\gamma)(g(a)-g(0))^h}{\Gamma(h)}\|\tilde{\zeta}\|_{C_{1-\gamma;g}}, \quad \forall t \in [0,a].\end{aligned} \tag{26}$$

It is easy to see that $\mathscr{U}_r$ is a nonempty, closed, bounded, and convex subset of Banach space $C_{1-\gamma,g}([0,a])$.

Consider the operator $\mathbb{K} \colon \mathscr{U}_r \longrightarrow \mathscr{U}_r$ given by

$$\begin{aligned}(\mathbb{K}\xi)(t) &= \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(h)}A + \frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s,\end{aligned} \tag{27}$$

for any $t \in [0,a]$.

Firstly, we prove that the fixed point of the operator $\mathbb{K}$ is a solution of equations (2) and (3). For any $\xi \in C_{1-\gamma;g}([0,a])$ and for each $t \in [0,a]$, we have the following estimate:

$$\begin{aligned}
\left|(\mathbb{K}\xi)(t)(g(t)-g(0))^{1-\gamma}\right| &= \left|\frac{A}{\Gamma(\gamma)} + \frac{\lambda(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)\mathrm{d}s + \frac{(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s\right| \\
&\le \frac{|A|}{\Gamma(\gamma)} + \frac{\lambda(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s)|\mathrm{d}s + \frac{(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\zeta(s,\xi(s))|\mathrm{d}s \\
&\le \left|\frac{|A|}{\Gamma(\gamma)} + \frac{\lambda(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s)|\mathrm{d}s + \frac{(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\zeta(s,\xi(s))\right. \\
&\quad \left. -\zeta(s,0)+\zeta(s,0)\right|\mathrm{d}s \\
&\le \frac{|A|}{\Gamma(\gamma)} + \frac{\lambda(g(t)-g(0))^{1-\gamma}}{\Gamma(h)} B(h,\gamma)(g(t)-g(0))^{h+\gamma-1}\|\xi\|_{C_{1-\gamma;g}} \\
&\quad + \frac{(g(t)-g(0))^{1-\gamma}}{\Gamma(h)} B(h,\gamma)(g(t)-g(0))^{h+\gamma-1} L\|\xi\|_{C_{1-\gamma;g}} \\
&\quad + \frac{(g(t)-g(0))^{1-\gamma}}{\Gamma(h)} B(h,\gamma)(g(t)-g(0))^{h+\gamma-1}\|\widetilde{\zeta}\|_{C_{1-\gamma;g}},
\end{aligned} \tag{28}$$

for any $t \in [0,a]$.

Combining the estimation above with the definition of $\mathcal{U}_r$, we infer that

$$\begin{aligned}
\left|(\mathbb{K}\xi)(t)(g(t)-g(0))^{1-\gamma}\right| &\le \frac{|A|}{\Gamma(\gamma)} + \frac{B(h,\gamma)(g(a)-g(0))^h}{\Gamma(h)} \\
&\cdot (\lambda+L)\|\xi\|_{C_{1-\gamma;g}} + \frac{B(h,\gamma)(g(a)-g(0))^h}{\Gamma(h)}\|\widetilde{\xi}\|_{C_{1-\gamma;g}} = r.
\end{aligned} \tag{29}$$

Hence, we conclude the operator $\mathbb{K}$ maps into itself.

Secondly, $\mathbb{K}(\mathcal{U}_r)$ is uniformly bounded since $\mathbb{K}(\mathcal{U}_r) \subset \mathcal{U}_r$.

Thirdly, we will prove that the operator $\mathbb{K}$ is continuous. Let the sequence $\{\xi_n\} \in \mathcal{U}_r$ and $\xi \in \mathcal{U}_r$ such that

$$\left\|\xi_n - \xi\right\|_{C_{1-\gamma;g}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{30}$$

For any $t \in [0,a]$, we have the following estimate:

$$\begin{aligned}
\left|((\mathbb{K}\xi_n)(t) - (\mathbb{K}\xi)(t))(g(t)-g(0))^{1-\gamma}\right| &\le \frac{\lambda(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\left|\xi_n(s)-\xi(s)\right|\mathrm{d}s \\
&\quad + \frac{(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\left|\zeta(s,\xi_n(s)) - \zeta(s,\xi(s))\right|\mathrm{d}s \\
&\le \frac{B(h,\gamma)}{\Gamma(h)}(g(t)-g(0))^h \lambda\left\|\xi_n(\cdot)-\xi(\cdot)\right\|_{C_{1-\gamma;g}} \\
&\quad + \frac{B(h,\gamma)}{\Gamma(h)}(g(t)-g(0))^h\left\|\zeta(\cdot,\xi_n(\cdot)) - \zeta(\cdot,\xi(\cdot))\right\|_{C_{1-\gamma;g}}, \quad \forall t \in [0,a].
\end{aligned} \tag{31}$$

Hence, for any $t \in [0,a]$, we have

$$\begin{aligned}
\left\|\mathbb{K}\xi_n - \mathbb{K}\xi\right\|_{C_{1-\gamma;g}} &\le \frac{B(h,\gamma)}{\Gamma(h)}(g(a)-g(0))^h \lambda\left\|\xi_n(\cdot)-\xi(\cdot)\right\|_{C_{1-\gamma;g}} \\
&\quad + \frac{B(h,\gamma)}{\Gamma(h)}(g(a)-g(0))^h \\
&\quad \cdot \left\|\zeta(\cdot,\xi_n(\cdot)) - \zeta(\cdot,\xi(\cdot))\right\|_{C_{1-\gamma;g}}.
\end{aligned} \tag{32}$$

Based on the continuity of function ζ, we obtain

$$\left\|\zeta(\cdot,\xi_n(\cdot)) - \zeta(\cdot,\xi(\cdot))\right\|_{C_{1-\gamma;g}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{33}$$

So, for any $t \in [0,a]$, we obtain

$$\left\|\mathbb{K}\xi_n - \mathbb{K}\xi\right\|_{C_{1-\gamma;g}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{34}$$

which leads to operator $\mathbb{K}$ which is continuous.

Next, we will show that $\mathbb{K}(\mathscr{U}_r)$ is equicontinuous. Let $t_1, t_2 \in [0, a]$ such that $0 < t_1 < t_2 < a$ and $\xi \in \mathscr{U}_r$, and we have

$$\begin{aligned}
&\left|(\mathbb{K}\xi)(t_2)(g(t_2)-g(0))^{1-\gamma}-(\mathbb{K}\xi)(t_1)(g(t_1)-g(0))^{1-\gamma}\right| \\
&=\left|\frac{\lambda(g(t_2)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^{t_2}\mathfrak{Q}_g^h(t_2,s)\xi(s)\mathrm{d}s+\frac{(g(t_2)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^{t_2}\mathfrak{Q}_g^h(t_2,s)\zeta(s,\xi(s))\mathrm{d}s\right. \\
&\quad\left.-\frac{\lambda(g(t_1)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^{t_1}\mathfrak{Q}_g^h(t_1,s)\xi(s)\mathrm{d}s-\frac{(g(t_1)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^{t_1}\mathfrak{Q}_g^h(t_1,s)\zeta(s,\xi(s))\mathrm{d}s\right| \\
&\le\frac{1}{\Gamma(h)}B(h,\gamma)\left(\lambda\|\xi\|_{C_{1-\gamma;g}}+\|\zeta\|_{C_{1-\gamma;g}}\right)\left|(g(t_2)-g(0))^h-(g(t_1)-g(0))^h\right|.
\end{aligned}\tag{35}$$

As $t_2 \longrightarrow t_1$, we imply the right-hand side of the estimation above tend to 0, that is, $\mathbb{K}(\mathscr{U}_r)$ is equicontinuous.

Finally, we see that all conditions of Schauder fixed point theorem are satisfied. So, we can conclude that problem (2) and (3) has at least one solution. The proof is complete. □

Theorem 3. *Let $\tilde{a}\in (0, a)$. Assume that assumption (H1) is satisfied. If*

$$\frac{B(h,\gamma)(g(\tilde{a})-g(0))^h(\lambda+L)}{\Gamma(h)}<1,\tag{36}$$

then problems (2) and (3) have a unique solution on $[0,\tilde{a}]$.

Proof. To prove this theorem, divide the proof into two steps. Now, we define the following set:

$$\mathscr{U}_\rho := \left\{\xi\in C_{1-\gamma,g}([0,\tilde{a}]) \colon \|\xi\|_{C_{1-\gamma,g}([0,\tilde{a}])}\le\rho\right\},\tag{37}$$

where

$$\rho := \frac{A\Gamma(h)}{\Gamma(\gamma)\left[\Gamma(h)-(1+\lambda+L)(g(\tilde{a})-g(0))^h\right]}.\tag{38}$$

Let us define the operator $\mathscr{B}\colon C_{1-\gamma,g}([0,\tilde{a}]) \longrightarrow C_{1-\gamma,g}([0,\tilde{a}])$ as follows:

$$\begin{aligned}
(\mathscr{B}\xi)(t)&=\frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A+\frac{\lambda}{\Gamma(h)}\int_0^t\mathfrak{Q}_g^h(t,s)\xi(s)\mathrm{d}s \\
&\quad+\frac{1}{\Gamma(h)}\int_0^t\mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s,\quad \forall t\in[0,\tilde{a}].
\end{aligned}\tag{39}$$

Step 1: similarl to the proof of Theorem 2, we can infer that the functions of $\mathscr{B}\mathscr{U}_\rho$ are uniformly bounded in $C_{1-\gamma,g}([0,\tilde{a}])$.

Step 2: we will show that $\mathscr{B}$ is a contraction on $C_{1-\gamma,g}([0,\tilde{a}])$. For any $t\in[0,\tilde{a}]$ and $\xi_1,\xi_2\in C_{1-\gamma,g}([0,\tilde{a}])$, we have

$$\begin{aligned}
\left|((\mathbb{K}\xi_1)(t)-(\mathbb{K}\xi_2)(t))(g(t)-g(0))^{1-\gamma}\right|&\le\frac{\lambda(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t\mathfrak{Q}_g^h(t,s)\left|\xi_1(s)-\xi_2(s)\right|\mathrm{d}s \\
&\quad+\frac{(g(t)-g(0))^{1-\gamma}}{\Gamma(h)}\int_0^t\mathfrak{Q}_g^h(t,s)\left|\zeta(s,\xi_1(s))-\zeta(s,\xi_2(s))\right|\mathrm{d}s \\
&\le\frac{B(h,\gamma)}{\Gamma(h)}(g(t)-g(0))^h(\lambda+L)\left\|\xi_1-\xi_2\right\|_{C_{1-\gamma,g}([0,\tilde{a}])}.
\end{aligned}\tag{40}$$

Combining the estimation above with assumption (36), we obtain

$$\left\|\mathbb{K}\xi_1-\mathbb{K}\xi_2\right\|_{C_{1-\gamma,g}([0,\tilde{a}])}\le\left\|\xi_1-\xi_2\right\|_{C_{1-\gamma,g}([0,\tilde{a}])},\tag{41}$$

that is, $\mathscr{B}$ is a contraction on $C_{1-\gamma,g}([0,\tilde{a}])$.

Here, we see that all conditions in the Banach fixed point theorem are satisfied. Therefore, there exists a unique solution of problems (2) and (3). This proof is completed. □

3.2. Continuous Dependence and Boundedness of Solution of Problem (2). In this section, we will study the continuous dependence of solutions on initial conditions and the boundedness of solution of equations (2) and (3). Now, we consider the following problems:

$$\begin{aligned}
&{}^{\mathbb{H}}_{0^+}\mathfrak{D}^{h,v;g}\xi(t)=\lambda\xi(t)+\zeta(t,\xi(t)),\quad \mathfrak{I}_{0^+}^{1-\gamma}\xi(0)=A, \\
&{}^{\mathbb{H}}_{0^+}\mathfrak{D}^{h,v;g}\xi^*(t)=\lambda\xi^*(t)+\zeta(t,\xi^*(t)),\quad \mathfrak{I}_{0^+}^{1-\gamma}\xi^*(0)=A^*,
\end{aligned}\tag{42}$$

where the functions $\zeta(\cdot,\xi(\cdot)),\zeta(\cdot,\xi^*(\cdot)) \in C_{1-\gamma,g}([0,a])$ satisfy assumption (H1), for any $\xi,\xi^* \in C_{1-\gamma,g}([0,a])$.

Theorem 4. *Assume that functions ζ and ζ^* satisfy assumption (H1). Let $\xi(t)$ and $\xi^*(t)$ be the solutions of problems (42), respectively. Then, we have the following estimate:*

$$\left|\xi(t)-\xi^*(t)\right| \leq \frac{|A-A^*|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)}\mathbb{E}_h\left((\lambda+L)[g(t)-g(0)]^h\right), \tag{43}$$

for any $t \in [0,a]$.

Proof. Since $\xi(t)$ and $\xi^*(t)$ are the solutions of problems (42), respectively. For any $t \in [0,a]$, we have

$$\begin{aligned}\xi(t) &= \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(h)}A + \frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)ds\\ &\quad + \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\,ds,\\ \xi^*(t) &= \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(h)}A^* + \frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi^*(s)ds\\ &\quad + \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi^*(s))\,ds.\end{aligned} \tag{44}$$

Using assumption (H1) and for any $t \in [0,a]$, we have the following estimate:

$$\begin{aligned}\left|\xi(t)-\xi^*(t)\right| &\leq \frac{|A-A^*|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)} + \frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\left|\xi(s)-\xi^*(s)\right|ds\\ &\quad + \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\left|\zeta(s,\xi(s))-\zeta(s,\xi^*(s))\right|ds\\ &\leq \frac{|A-A^*|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)} + \frac{(\lambda+L)}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\left|\xi(s)-\xi^*(s)\right|ds.\end{aligned} \tag{45}$$

If we put

$$\begin{aligned}\tilde{v}(t) &:= \left|\xi(t)-\xi^*(t)\right|,\\ \tilde{u}(t) &:= \frac{|A-A^*|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)},\\ \tilde{g}(t) &:= \frac{(\lambda+L)}{\Gamma(h)},\end{aligned} \tag{46}$$

for any $t \in [0,a]$, then inequality (45) becomes

$$\tilde{v}(t) \leq \tilde{u}(t) + \tilde{g}(t)\int_0^t \mathfrak{Q}_g^h(t,s)\tilde{v}(s)ds. \tag{47}$$

Applying Gronwall Lemma 1 to (47), we obtain

$$\tilde{v}(t) \leq \tilde{u}(t)\mathbb{E}_h\left((\lambda+L)[g(t)-g(0)]^h\right). \tag{48}$$

This gives inequality (43). □

Theorem 5. *Assume that assumption (H1) is satisfied. If ξ is any solution of problems (2) and (3), then*

$$\begin{aligned}|\xi(t)| &\leq \left(\frac{|A|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)} + \frac{\hat{\zeta}(g(t)-g(0))^h}{\Gamma(1+h)}\right)\\ &\quad \times \mathbb{E}_h\left((\lambda+L)[g(t)-g(0)]^h\right), \quad \forall t \in [0,a],\end{aligned} \tag{49}$$

where $\hat{\zeta} = \max_{s\in[0,a]}\zeta(s,0)$.

Proof. Let ξ be any solution of problems (2) and (3). Then, we have

$$\begin{aligned}\xi(t) &= \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A + \frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)ds\\ &\quad + \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))ds, \quad \forall t \in [0,a].\end{aligned} \tag{50}$$

Using assumption (H1) and for any $t \in [0,a]$, we have

$$\begin{aligned}|\xi(t)| &\leq \frac{|A|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)}+\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s)|\mathrm{d}s \\ &\quad+\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\zeta(s,\xi(s))|\mathrm{d}s \leq \frac{|A|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)}+\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s)|\mathrm{d}s \\ &\quad+\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\zeta(s,\xi(s))-\zeta(s,0)+\zeta(s,0)|\mathrm{d}s \\ &\leq \frac{|A|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)}+\frac{(\lambda+L)}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s)|\mathrm{d}s+\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\zeta(s,0)|\mathrm{d}s \\ &\leq \frac{|A|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)}+\frac{\widehat{\zeta}(g(t)-g(0))^h}{\Gamma(1+h)}+\frac{(\lambda+L)}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s)|\mathrm{d}s.\end{aligned} \tag{51}$$

We put

$$\begin{aligned}\nu(t) &:= \frac{|A|(g(t)-g(0))^{1-\gamma}}{\Gamma(\gamma)}+\frac{\widehat{\zeta}(g(t)-g(0))^h}{\Gamma(1+h)}, \\ u(t) &:= \frac{(\lambda+L)}{\Gamma(h)},\end{aligned} \tag{52}$$

for any $t \in [0,a]$, which leads to the following estimate:

$$|\xi(t)| \leq \nu(t)+u(t)\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s)|\mathrm{d}s, \quad \forall t \in [0,a]. \tag{53}$$

Applying Gronwall Lemma 1, we obtain

$$|\xi(t)| \leq \nu(t)\mathbb{E}_h\big(u(t)\Gamma(h)[g(t)-g(0)]^h\big), \quad \forall t \in [0,a]. \tag{54}$$

The proof is completed. □

3.3. Hyers–Ulam Stability and Hyers–Ulam–Rassias Stability for Problem (2)

Theorem 6. *Assume that assumption (H1) and (36) are satisfied. Then, problem (2) is Hyers–Ulam stable.*

Proof. Let $\xi(t) \in C_{1-\gamma,g}([0,a])$ be a solution of (12) and let $\widehat{\xi}(t) \in C_{1-\gamma,g}([0,a])$ be a unique solution of (2). Then, for any, we have

$$\begin{aligned}\widehat{\xi}(t) &= \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A+\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\widehat{\xi}(s)\mathrm{d}s \\ &\quad+\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\widehat{\xi}(s))\mathrm{d}s, \quad \forall t \in [0,a].\end{aligned} \tag{55}$$

For any $t \in [0,a]$, we have

$$\begin{aligned}&|\xi(t)-\widehat{\xi}(t)| \\ &= \left|\xi(t)-\frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A-\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\widehat{\xi}(s)\mathrm{d}s-\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\widehat{\xi}(s))\mathrm{d}s\right| \\ &\leq \left|\xi(t)-\frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A-\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\widehat{\xi}(s)\mathrm{d}s-\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\widehat{\xi}(s))\mathrm{d}s\right| \\ &\quad+\left|\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\widehat{\xi}(s)\mathrm{d}s+\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s-\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\widehat{\xi}(s)\mathrm{d}s-\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s\right| \\ &\leq \left|\xi(t)-\frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A-\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\widehat{\xi}(s)\mathrm{d}s-\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\widehat{\xi}(s))\mathrm{d}s\right| \\ &\quad+\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s)-\widehat{\xi}(t)|\mathrm{d}s+\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\zeta(s,\xi(s))-\zeta(s,\widehat{\xi}(s))|\mathrm{d}s \\ &= \left|\xi(t)-\frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A-\frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\widehat{\xi}(s)\mathrm{d}s-\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\widehat{\xi}(s))\mathrm{d}s\right| \\ &\quad+\frac{(\lambda+L)}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s)-\widehat{\xi}(t)|\mathrm{d}s.\end{aligned} \tag{56}$$

Moreover, due to the function $\xi(t)$ which satisfies inequality (12), there exists a function h: $[0,a] \longrightarrow \mathfrak{R}$ such that $|h(t)| \leq \varepsilon$, for any $t \in [0,a]$ and

$$ {}_{0^+}^{\mathbb{H}}\mathfrak{D}^{h,v;g}\xi(t) - \lambda\xi(t) - \zeta(t,\xi(t)) = h(t), \quad \forall t \in [0,a]. \tag{57} $$

Applying the fractional integral $\mathfrak{J}_{0^+}^{h;g}(\cdot)$ to both sides of equation (57) and by using Theorem 1, we obtain

$$ \mathfrak{J}_{0^+}^{h;g}\left({}_{0^+}^{\mathbb{H}}\mathfrak{D}^{h,v;g}\xi(t) - \lambda\xi(t) - \zeta(t,\xi(t))\right) = \mathfrak{J}_{0^+}^{h;g}h(t), \quad \forall t \in [0,a]. \tag{58} $$

Thus, we have

$$ \begin{aligned} &\left|\xi(t) - \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A - \frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)\mathrm{d}s \right.\\ &\left. - \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s\right| \leq \mathfrak{J}_{0^+}^{h;g}|h(t)| \\ &\leq \mathfrak{J}_{0^+}^{h;g}\epsilon = \frac{(g(t)-g(0))^h}{\Gamma(h)}\epsilon, \quad \forall t \in [0,a]. \end{aligned} \tag{59} $$

Combining inequality (57) with inequality (59), we obtain

$$ \begin{aligned} |\xi(t) - \widehat{\xi}(t)| \leq{}& \frac{(g(t)-g(0))^h}{\Gamma(h)}\epsilon + \frac{(\lambda+L)}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s) \\ &- \widehat{\xi}(t)|\mathrm{d}s, \quad \forall t \in [0,a]. \end{aligned} \tag{60} $$

Applying Gronwall Lemma 1 to (60), we obtain

$$ \begin{aligned} |\xi(t) - \widehat{\xi}(t)| &\leq \frac{(g(t)-g(0))^h}{\Gamma(h)}\mathbb{E}_h\left((\lambda+L)[g(t)-g(0)]^h\right)\epsilon \\ &\leq \frac{(g(a)-g(0))^h}{\Gamma(h)}\mathbb{E}_h\left((\lambda+L)[g(t)-g(0)]^h\right)\epsilon \\ &\leq C_\zeta\epsilon, \quad \forall t \in [0,a], \end{aligned} \tag{61} $$

where $C_\zeta := (g(a)-g(0))^h/\Gamma(h)\mathbb{E}_h((\lambda+L)[g(t)-g(0)]^h)$.

Based on the inequality above and Definition 2, we infer that problem (2) is Hyers–Ulam stable. The proof is completed. □

Theorem 7. *Assume that assumption (H1) and (36) are satisfied. If there exists function $\phi \in C([0,a],\mathfrak{R})$ and the positive constant C_ϕ such that*

$$ \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\phi(s)\mathrm{d}s \leq C_\phi\phi(t), \quad \forall t \in [0,a], \tag{62} $$

then problem (2) is Hyers–Ulam–Rassias stable.

Proof. Let $\xi(t) \in C_{1-\gamma,g}([0,a])$ be a solution of (12) and let $\widehat{\xi}(t) \in C_{1-\gamma,g}([0,a])$ be a unique solution of (2). Performing the same calculations as in Theorem 3, we have the estimation as follows:

$$ \begin{aligned} |\xi(t) - \widehat{\xi}(t)| \leq{}& \xi(t) - \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A - \frac{\lambda}{\Gamma(h)} \\ &\cdot \int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)\mathrm{d}s \\ &- \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s \\ &+ \frac{(\lambda+L)}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s) - \widehat{\xi}(t)|\mathrm{d}s. \end{aligned} \tag{63} $$

On the contrary, applying the fractional integral $\mathfrak{J}_{0^+}^{h;g}(\cdot)$ to both sides of inequality (14) and by Theorem 1 and assumptions (61), we obtain

$$ \begin{aligned} &\left|\xi(t) - \frac{(g(t)-g(0))^{\gamma-1}}{\Gamma(\gamma)}A - \frac{\lambda}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\xi(s)\mathrm{d}s \right.\\ &\left. - \frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\zeta(s,\xi(s))\mathrm{d}s\right| \\ &\leq \epsilon\mathfrak{J}_{0^+}^{h;g}\phi(t) \leq C_\phi\phi(t)\epsilon, \quad \forall t \in [0,a]. \end{aligned} \tag{64} $$

From estimations (63) and (64), we imply

$$ |\xi(t) - \widehat{\xi}(t)| \leq C_\phi\phi(t)\epsilon + \frac{(\lambda+L)}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)|\xi(s) - \widehat{\xi}(t)|\,\mathrm{d}s, \quad \forall t \in [0,a]. \tag{65} $$

Using Gronwall Lemma 1 and (64), we obtain

$$ \begin{aligned} |\xi(t) - \widehat{\xi}(t)| &\leq C_\phi\mathbb{E}_h\left((\lambda+L)[g(t)-g(0)]^h\right)\phi(t)\epsilon \\ &\leq C_{\zeta,\phi}\epsilon, \quad \forall t \in [0,a], \end{aligned} \tag{66} $$

where $C_{\zeta,\phi} := C_\phi\mathbb{E}_h((\lambda+L)[g(t)-g(0)]^h)$.

Similarly, based on the inequality above and Definition 2, we infer that problem (2) is Hyers–Ulam–Rassias stable. The proof is completed. □

3.4. Example. Let us consider the following problem:

$$ {}_{0^+}^{\mathbb{H}}\mathfrak{D}^{h,v;g}\xi(t) = \frac{1}{3}\xi(t) + \frac{1+\xi(t)}{9+e^t}, \quad \forall t \in [0,1/3], \tag{67} $$

with the initial condition $\mathfrak{J}_{0^+}^{1-\gamma,g}\xi(0) = 0$.

For this example, we only consider the two situation as below. The other one is considered similarly.

Situation 1. Let $g(t) = t$, taking the limit $v \longrightarrow 1$ on both sides of (8). Then, by Sousa et al. [16], we have

$$ {}_{0^+}^{\mathbb{H}}\mathfrak{D}^{h,v;g}\xi(t) = {}_{0^+}^{C}\mathfrak{D}^h\xi(t), \quad \forall t \in [0,1]. \tag{68} $$

Combining (67) with (68), we infer that

$$ {}_{0^+}^{C}\mathfrak{D}^{h}\xi(t) = \frac{1}{3}\xi(t) + \frac{1+\xi(t)}{9+e^{t}}, \quad \forall t \in [0,1/3], \tag{69}$$

with the initial condition $\xi(0)=0$.

We consider $h=1/2$, $\lambda=1/3$, and

$$\zeta(t,\xi(t)) \coloneqq \frac{1+\xi(t)}{9+e^{t}}, \quad \forall t \in [0,1/3]. \tag{70}$$

It is easy to see that ζ is a continuous function and it satisfies assumption (H1) with Lipschitz constant $L=1/10$. Indeed, for any $\xi_1, \xi \in C([0,a],\mathfrak{R})$, we have the following estimate:

$$\begin{aligned}|\zeta(t,\xi_1(t)) - \zeta(t,\xi_2(t))| &= \left|\frac{1+\xi_1(t)}{9+e^{t}} - \frac{1+\xi_2(t)}{9+e^{t}}\right| \\ &\le \frac{1}{9+e^{t}}|\xi_1(t)-\xi_2(t)| \\ &\le \frac{1}{10}|\xi_1(t)-\xi_2(t)|, \quad \forall t \in [0,1/3].\end{aligned} \tag{71}$$

Moreover, we have

$$\begin{aligned}&\frac{B(h,\gamma)(g(\tilde{a})-g(0))^{h}(\lambda+L)}{\Gamma(h)} \\ &= \frac{B(1/2,1/2)(1/3-0)^{1/2}(1/3+1/10)}{\Gamma(1/2)} \approx 0.45 < 1.\end{aligned} \tag{72}$$

We see that all the assumptions of Theorem 2 is satisfied. So, we infer that problem (69) has a unique solution on $[0,1/3]$.

Situation 2. Let $h=2/3$, $\upsilon=1/2$, and $g(t)=t^{\rho}$, $\rho>0$, for any $t \in [0,1/3]$. Then, by Sousa et al. [16], we have

$$ {}_{0^+}^{\rho}\mathfrak{D}^{(2/3),(1/2)}\xi(t) = \frac{1}{3}\xi(t) + \frac{1+\xi(t)}{9+e^{t}}, \quad \forall t \in [0,1/3], \tag{73}$$

with the initial condition ${}^{\rho}\mathfrak{I}_{0^+}^{5/6}\xi(0)=0$.

Performing the same calculations as in Case 1, then it is also easy to check that problem (73) has a unique solution on $[0,1/3]$.

We put $\phi(t)=t^{3\rho/2-1}$ for any $t \in [0,1/3]$. Now, we will prove that problem (73) is Hyers–Ulam–Rassias stable. For any $t \in [0,1/3]$, we have

$$\begin{aligned}\frac{1}{\Gamma(h)}\int_0^t \mathfrak{Q}_g^h(t,s)\phi(s)\mathrm{d}s &= \frac{\rho^{1/3}}{\Gamma(2/3)}\int_0^t s^{-\rho/2}(t^{\rho}-s^{\rho})^{-1/3}\phi(s)\mathrm{d}s \\ &= \frac{\rho^{1/3}}{\Gamma(2/3)}\int_0^t s^{-\rho/2}(t^{\rho}-s^{\rho})^{-1/3}s^{3\rho/2-1}\mathrm{d}s \\ &= \frac{\rho^{1/3}}{\Gamma(2/3)}\int_0^t (t^{\rho}-s^{\rho})^{-1/3}s^{\rho-1}\mathrm{d}s \\ &\le \frac{3\rho^{-2/3}}{2\Gamma(2/3)}\left(\frac{1}{3}\right)^{1-5\rho/6}\phi(t), \quad \forall t \in [0,1/3].\end{aligned} \tag{74}$$

Hence, assumption (73) of Theorem 3 is satisfied by $C_{\phi} \coloneqq 3\rho^{-2/3}/2\Gamma(2/3)(1/3)^{1-5\rho/6} > 0$ with $\rho>0$. All the assumptions of Theorem 3 are satisfied. So, we imply problem (73) is Hyers–Ulam–Rassias stable.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Design of a Novel Second-Order Prediction Differential Model Solved by Using Adams and Explicit Runge–Kutta Numerical Methods

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In this study, a novel second-order prediction differential model is designed, and numerical solutions of this novel model are presented using the integrated strength of the Adams and explicit Runge–Kutta schemes. The idea of the present study comes to the mind to see the importance of delay differential equations. For verification of the novel designed model, four different examples of the designed model are numerically solved by applying the Adams and explicit Runge–Kutta schemes. These obtained numerical results have been compared with the exact solutions of each example that indicate the performance and exactness of the designed model. Moreover, the results of the designed model have been presented numerically and graphically.

1. Introduction

The historical delay differential equations (DDEs) are applied in the pioneer work of Newton and Leibnitz in the last years of the 16^{th} century. To understand the worth and importance of the DDEs, one can see their extensive and wide-ranging applications in the field of scientific wonders. Few mentioned applications are population dynamics, economical systems, engineering systems, transports, and communication models [1–4]. Many researchers worked to solve DDEs in different years, e.g., Kondorse studied DDEs in the seventh decade of the 17^{th} century, but properly used the applications of DDEs in the 19^{th} century. Kuang [2] and Hale and LaSalle [5] presented the detailed theory, solution schemes, and applications of DDEs. Perko [6] studied linear/nonlinear differential models for the dynamical system and configuration. Beretta and Kuang [7] worked on the geometric constancy of DDEs with the constraints of delay values. Frazier [8] explained the DDEs of the second kind by applying the wavelet Galerkin scheme. Rangkuti and Noorani [9] established the exact solution of DDEs by applying the iterative method named as a coupled variation scheme with the support of the Taylor series method. Chapra [10] discussed the scheme of Runge–Kutta for solving both types of differential delay and nondelay models. Adel and Sabir [11] presented the numerical solutions of a nonlinear second-order Lane–Emden pantograph delay differential model via the Bernoulli collocation method. Sabir et al. [12, 13] solved the nonlinear functional differential models of second and third order. Erdogan et al. [14] applied the finite difference method on a layer-adapted mesh for singularly perturbed DDEs. Some more details of DDEs are provided in references [15–20]. The literature form of the second-order DDE is given as follows [21]:

$$\begin{cases} \dfrac{d^2y}{dx^2} = g(x, y(x), y(x-\tau_1)), & \tau_1 > 0, a \le x \le b, \\ y(x) = \theta(x), \quad \rho \le x \le a, & 0 \le \tau_1 \le |a-\rho|, \\ \dfrac{dy(a)}{dx} = \omega, \end{cases} \tag{1}$$

where $\theta(x)$ is used for the initial condition and τ_1 is the delayed term in the abovementioned equation. It is clear in understanding that the appearance of this term $y(x-\tau)$in any differential equation shows the DDE that means to subtract some values from time. The question arises here when we add some values in time then what happens, i.e., $y(x+\tau)$. This clearly indicates the prediction and aim of the present work is related to design a new model based on the prediction differential equation. The general form of the new second-order prediction differential model along with initial conditions is presented as

$$\begin{cases} \dfrac{d^2y}{dx^2} = g(x, y(x), y(x+\tau)), & \tau > 0, a \le x \le b, \\ y(x) = \theta(x), \quad \rho \le x \le a, & 0 \le \tau \le |\rho - a|, \\ \dfrac{dy(a)}{dx} = \omega, \end{cases} \tag{2}$$

where τ is used as a prediction term in equation (2). This prediction differential model can be used to forecast the weather, transport, engineering, stock markets, technology, biological models, astrophysics, and many more. Moreover, the obtained numerical results from both of the schemes have been compared with the exact solution to verify the correctness and exactness of the designed model presented in equation (2).

Some salient features of the designed model are given as follows:

(i) The novel prediction model is successfully designed by considering the worth of the delay differential model

(ii) For verification of the designed model, the obtained numerical results have been compared with the exact solutions

(iii) Easily comprehensible procedures with effortless implementation, conserved accuracy in close locality of the input interval, broader, and extendibility applicability are other considerable advantages.

The remaining parts of the paper are organized as follows. Section 2 shows the designed methodology. Section 3 represents the detailed results. The conclusions and future research directions of the present study are provided in the last section.

2. Methodology

In the present study, the strength of predictor-corrector Adams technique [22, 23] and explicit Runge–Kutta numerical technique [24, 25] is exploited to solve the second-order prediction differential model.

2.1. Predictor-Corrector Adams Numerical Scheme. To find the numerical solutions of the novel designed prediction differential model, the predictor-corrector numerical technique is applied, which takes further two steps to complete.

Step 1: the approximate measures of prediction are accomplished

Step 2: to find if the numerical solutions of correction are capable with the similar contributions of prediction.

$$\begin{aligned} \frac{dy}{dx} &= h(x, y), \\ u(x_0) &= y_0. \end{aligned} \tag{3}$$

The generalized Adams–Bashforth two-step numerical scheme using the predictor-corrector techniques is given as

$$D_{n+1} = y_n + \frac{3}{2} gh(x_n, y_n) - \frac{1}{2} gh(x_{n-1}, y_{n-1}). \tag{4}$$

The Adams-Moulton two-step corrector scheme is shown as follows:

$$y_{n+1} = y_n + \frac{1}{2} gh((x_{n+1}, D_{n+1}) + h(x_n, y_n)). \tag{5}$$

The 4-step predictor-corrector scheme is provided as follows:

$$\begin{aligned} D_{n+1} = y_n + \frac{1}{24} g(55h(x_n, y_n) - 59g(x_{n-1}, y_{n-1}) \\ + 37g(x_{n-2}, y_{n-2}) - 9g(x_{n-3}, y_{n-3})). \end{aligned} \tag{6}$$

The Adams–Bashforth–Moulton 4-step scheme is written as follows:

$$\begin{aligned} y_{n+1} = y_n + \frac{1}{24} g(9h(x_{n+1}, D_{n+1}) + 19g(x_n, y_n) \\ - 5g(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})). \end{aligned} \tag{7}$$

2.2. Explicit Runge–Kutta Numerical Scheme. The explicit Runge–Kutta scheme is applied to solve the novel designed second-order prediction model. The general form of the explicit Runge–Kutta scheme is considered as

$$\begin{aligned} y_{n+1} &= y_n + g \sum_{j=1}^{s} b_j I_j, \\ I_1 &= h(x_n, y_n), \\ I_2 &= h(x_n + c_2 g, y_n + g(a_{21} I_1)), \\ I_3 &= h(x_n + c_3 g, u_n + g(a_{31} I_1 + a_{32} I_2)), \\ &\vdots \\ I_s &= h\left(x_n + c_s g, y_n + g\left(a_{s_1} I_1 + a_{s_2} I_2 + a_{s_3} I_3 + \ldots + a_{ss-1} I_{s-1}\right)\right). \end{aligned} \tag{8}$$

The first step is to consider the obtained initial results, and slopes for all variables are predictable. These attained numerical conclusions for slopes (theI_{1s}) at the middle point of the interval domain are taken to make the dependent variable designs, while in the second phase, the slopes of the central point (theI_{2s}) are obtained by using these accomplished values based on the middle points. The calculated numerical values for slopes are twisted back using the first point of the other set of central point values that are instigated for the new slope of predictions at the central point (theI_{3s}). These numerical calculated values are complementary functional to make the predictions to develop the slopes at the ending point of the interval domain (theI_{ss}). Similarly, all the numerical values for I_s are accomplished to make an additional set of growth functions and, finally, take the initial point to make the last prediction.

3. Simulations and Results

In this section of the study, the prediction differential model presented in equation (2) is solved by using the four numerical examples based on the predictor-corrector Adams technique and explicit Runge–Kutta method. Furthermore, the obtained numerical results using both the schemes have been compared with the exact solutions of each example.

Example 1. Consider the second-order prediction differential equation along with the initial conditions given as follows:

$$\begin{cases} 2\dfrac{d^2y}{dx^2} + y(x) - y(x+\pi) = 0, \\ y(0) = 1, \\ \dfrac{dy(0)}{dx} = 1. \end{cases} \tag{9}$$

The exact solution of equation (9) is $1 + \sin x$.

Example 2. Consider the second-order prediction differential equation along with boundary conditions given as follows:

$$\begin{cases} \dfrac{d^2y}{dx^2} - y(x) + y(x+1) - 2x = 0, \\ y(0) = 2, \\ \dfrac{dy(3)}{dx} = 3. \end{cases} \tag{10}$$

The exact solution of equation (10) is $x^2 - 3x + 2$.

Example 3. Consider the second-order prediction differential equation involving trigonometric functions along with initial conditions given as follows:

$$\begin{cases} \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}(x+1) + y(x+1) + y(x) + \cos(x+1) - \sin(x+1) = 0, \\ y(0) = 0, \\ \dfrac{dy(0)}{dx} = 1. \end{cases} \tag{11}$$

The exact solution of equation (11) is $\sin x$.

Example 4. Consider the second-order prediction differential equation along with initial conditions given as follows:

$$\begin{cases} \dfrac{d^2y}{dx^2} - \dfrac{dy(x+1)}{dx} + y(x+1) - y(x) = 0, \\ y(0) = 1, \\ \dfrac{dy(0)}{dx} = 1. \end{cases} \tag{12}$$

The exact solution of equation (12) is e^x.

It is clearly seen that the prediction term is involved in the form of $y(x+\pi)$, $y(x+1)$ in Examples 1 and 2, respectively. The prediction terms are involved four times in Example 3, i.e., $(dy/dx)(x+1)$, $y(x+1)$, $\cos(x+1)$, and $\sin(x+1)$. Moreover, the prediction terms appeared twice in Example 4, i.e., $(dy/dx)(x+1)$ and $y(x+1)$.

The graphic illustration based on the numerical results for all four examples is provided in Figure 1. The explicit Runge–Kutta scheme is used to find the graphical values for all the examples. The plots of Figures 1(a) to 1(d) are based on Examples 1 to 4. Figure 1(a) is plotted in the domain of [0, 30], and the results are found to be positive in all intervals. The plots of Figure 1(b) are plotted in the domain of [0, 3]. The results represent positive values in most of the intervals. However, negative values have been noticed in the subinterval [1, 2]. Figures 1(c) and 1(d) are plotted in the domain of $[0, \pi]$ and [0, 1], respectively. It is noticed in the table that positive results have seen in both Examples 3 and 4. For comparison of the results, the plots of exact and numerical solutions have been drawn in Figure 2. The overlapping of the results shows the exactness and accurateness of the designed model. For more clear results of all the examples, the numerical results of exact solutions and the predictor-corrector Adams numerical scheme are tabulated in Tables 1 and 2. The comparison of the Adams numerical results and exact solutions are same up to a higher level.

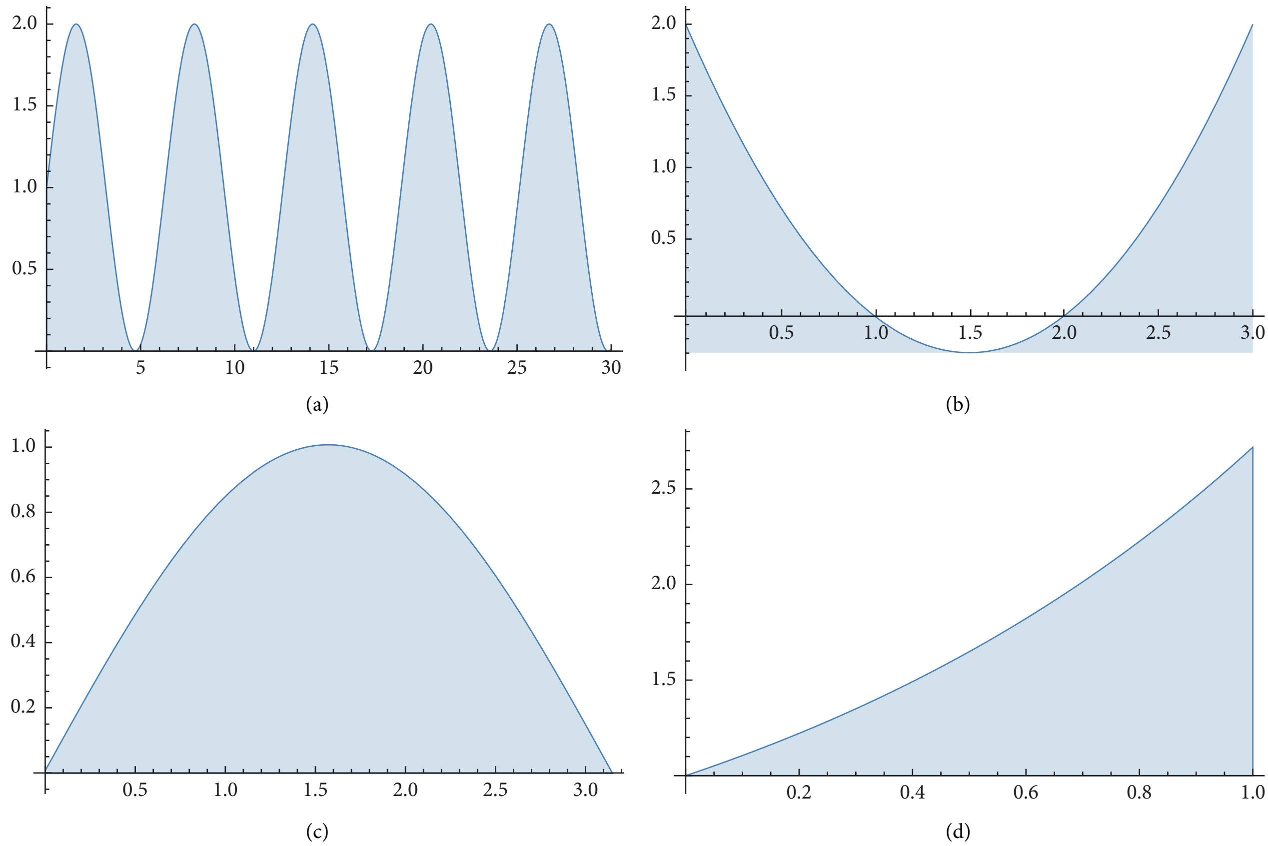


FIGURE 1: Graphical illustration of the numerical results for Examples 1, 2, 3, and 4. (a) Plot results of Example 1. (b) Plot results of Example 2. (c) Plot results of Example 3. (d) Plot results of Example 4.

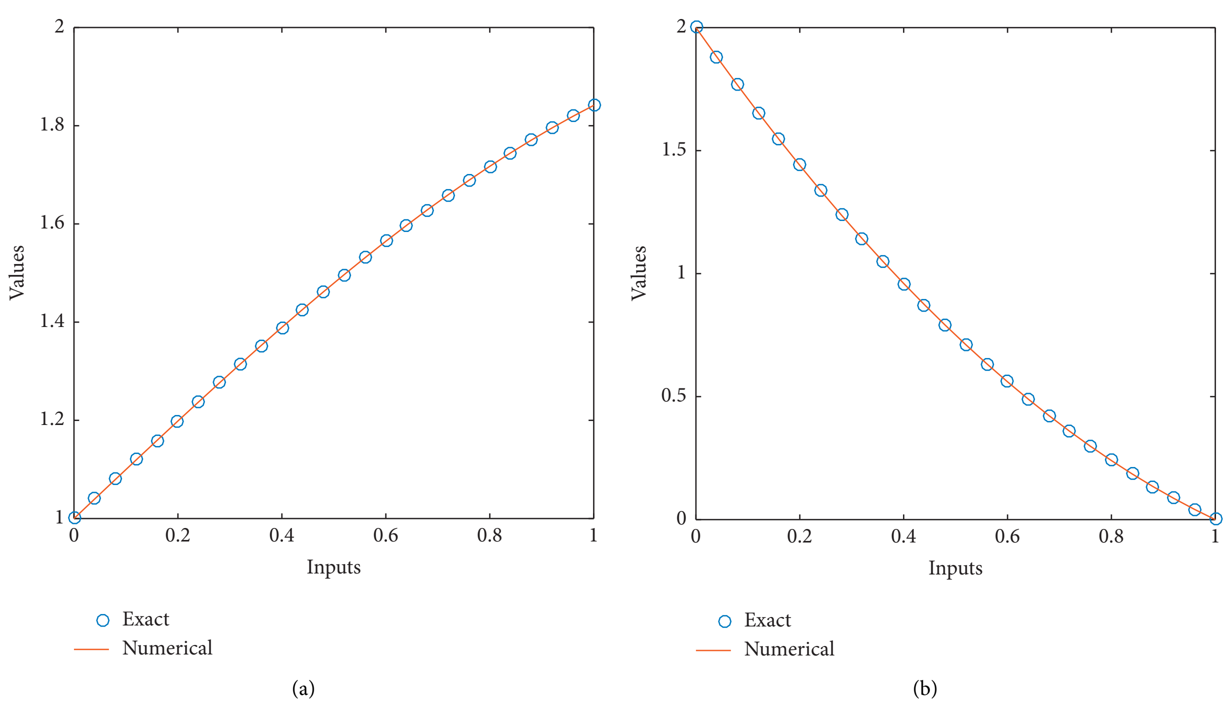
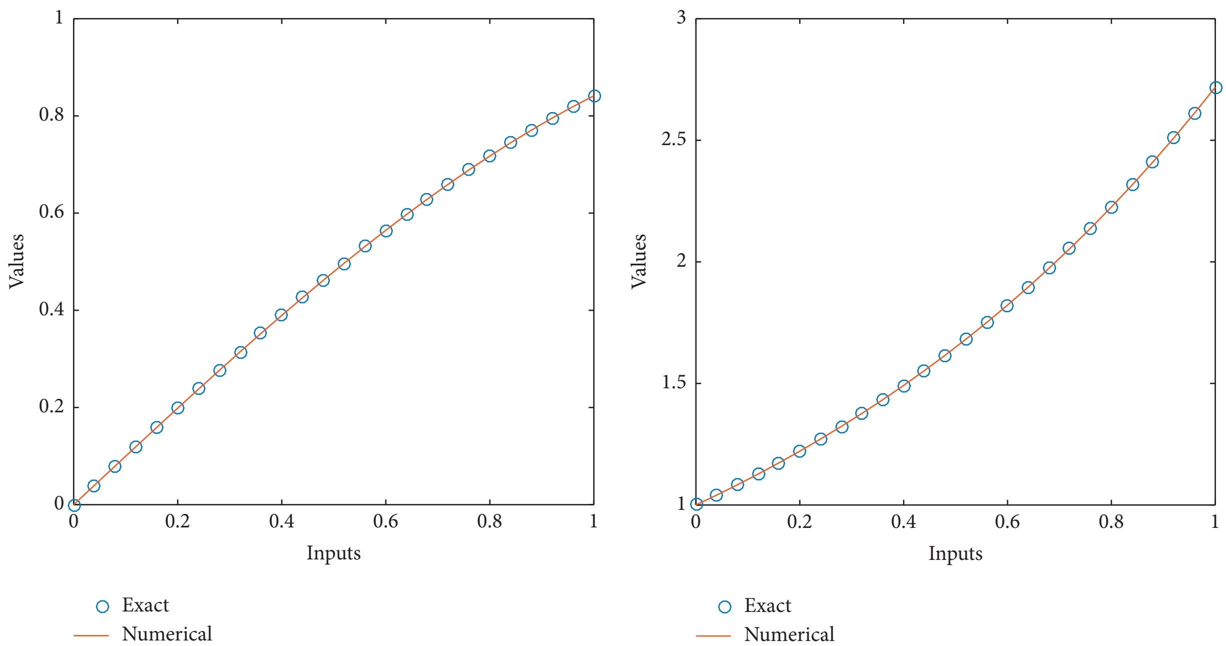


FIGURE 2: Continued.



(c) (d)

Figure 2: Comparison of the exact results and numerical solutions for Examples 1, 2, 3, and 4. (a) Comparison plots of Example 1. (b) Comparison plots of Example 2. (c) Comparison plots of Example 3. (d) Comparison plots of Example 4.

Table 1: Numerical values of the Adams and explicit Runge–Kutta scheme for Examples 1 and 2.

x	Example 1		Example 2	
	Exact	Adams	Exact	Adams
0.00	1.000000	1.000000	2.000000	2.000000
0.04	1.040000	1.039989	1.881600	1.881600
0.08	1.079900	1.079915	1.766400	1.766400
0.12	1.119700	1.119712	1.654400	1.654400
0.16	1.159300	1.159318	1.545600	1.545600
0.20	1.198700	1.198669	1.440000	1.440000
0.24	1.237700	1.237703	1.337600	1.337600
0.28	1.276400	1.276356	1.238400	1.238400
0.32	1.314600	1.314567	1.142400	1.142400
0.36	1.352300	1.352274	1.049600	1.049600
0.40	1.389400	1.389418	0.960000	0.960000
0.44	1.425900	1.425939	0.873600	0.873600
0.48	1.461800	1.461779	0.790400	0.790400
0.52	1.496900	1.496880	0.710400	0.710400
0.56	1.531200	1.531186	0.633600	0.633600
0.60	1.564600	1.564642	0.560000	0.560000
0.64	1.597200	1.597195	0.489600	0.489600
0.68	1.628800	1.628793	0.422400	0.422400
0.72	1.659400	1.659385	0.358400	0.358400
0.76	1.688900	1.688921	0.297600	0.297600
0.80	1.717400	1.717356	0.240000	0.240000
0.84	1.744600	1.744643	0.185600	0.185600
0.88	1.770700	1.770739	0.134400	0.134400
0.92	1.795600	1.795602	0.086400	0.086400
0.96	1.819200	1.819192	0.041600	0.041600
1.00	1.841500	1.841471	0.000000	0.000000

Table 2: Numerical values of the Adams and explicit Runge-Kutta scheme for Examples 3 and 4.

x	Example 3		Example 4	
	Exact	Adams	Exact	Adams
0.00	0.000000	0.000000	1.000000	1.000000
0.04	0.040000	0.039989	1.040800	1.040811
0.08	0.079900	0.079915	1.083300	1.083287
0.12	0.119700	0.119712	1.127500	1.127497
0.16	0.159300	0.159318	1.173500	1.173511
0.20	0.198700	0.198669	1.221400	1.221403
0.24	0.237700	0.237703	1.271200	1.271249
0.28	0.276400	0.276356	1.323100	1.323130
0.32	0.314600	0.314567	1.377100	1.377128
0.36	0.352300	0.352274	1.433300	1.433329
0.40	0.389400	0.389418	1.491800	1.491825
0.44	0.425900	0.425939	1.552700	1.552707
0.48	0.461800	0.461779	1.616100	1.616074
0.52	0.496900	0.496880	1.682000	1.682028
0.56	0.531200	0.531186	1.750700	1.750673
0.60	0.564600	0.564642	1.822100	1.822119
0.64	0.597200	0.597195	1.896500	1.896481
0.68	0.628800	0.628793	1.973900	1.973878
0.72	0.659400	0.659385	2.054400	2.054433
0.76	0.688900	0.688921	2.138300	2.138276
0.80	0.717400	0.717356	2.225500	2.225541
0.84	0.744600	0.744643	2.316400	2.316367
0.88	0.770700	0.770739	2.410900	2.410900
0.92	0.795600	0.795602	2.509300	2.509290
0.96	0.819200	0.819192	2.611700	2.611696
1.00	0.841500	0.841471	2.718300	2.718282

4. Conclusions

The present study is carried out to design a novel second-order prediction differential model by manipulating the strength of the Adams numerical scheme and explicit Runge–Kutta scheme. The designed novel prediction differential model will be very useful and can be applied in many applications. Four different variants of the designed model have been solved by using the Adams and Runge–Kutta schemes and compared the obtained numerical results with the exact solutions. The overlapping of the exact and numerical reference solutions show the worth and accuracy of the novel designed prediction differential model. It is clear in understanding that the proposed methods are valuable and suitable for solving the second-order prediction differential model due to accurate results for all the examples of the second-order prediction differential model. For solving all four examples, the proposed Adams and explicit Runge–Kutta schemes are found to be very good in terms of accuracy and convergence. Software used for solving the prediction differential model is MATLAB R 2017(a) package and Mathematica 10.4.

In future, the nonlinear prediction Lane–Emden model and nonlinear prey-predator singular prediction model can be designed and solved via an artificial neural network [26–33].

Data Availability

Our manuscript is not data-based.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A New Faster Iterative Scheme for Numerical Fixed Points Estimation of Suzuki's Generalized Nonexpansive Mappings

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The purpose of this paper is to introduce a new four-step iteration scheme for approximation of fixed point of the nonexpansive mappings named as S^*-iteration scheme which is faster than Picard, Mann, Ishikawa, Noor, Agarwal, Abbas, Thakur, and Ullah iteration schemes. We show the stability of our proposed scheme. We present a numerical example to show that our iteration scheme is faster than the aforementioned schemes. Moreover, we present some weak and strong convergence theorems for Suzuki's generalized nonexpansive mappings in the framework of uniformly convex Banach spaces. Our results extend, improve, and unify many existing results in the literature.

1. Introduction

Most of the nonlinear equations can be transformed into a fixed point problem as follows:

$$\mathcal{F}u = u, \tag{1}$$

where $\mathcal{F}$ is a self-map on a certain distance space $\mathcal{X}$ and the solution of the aforementioned equation is considered as a fixed point of the mapping $\mathcal{F}$. Banach [1] proved that if a self-map $\mathcal{F}$ on a complete metric space is such that

$$d(\mathcal{F}u, \mathcal{F}v) \leq q\, d(u, v), \tag{2}$$

for $0 \leq q < 1$, then it possesses a unique fixed point u^*. Moreover, the iterative process

$$\mathcal{F}u_n = u_{n+1}, \tag{3}$$

called the Picard iteration process, converges to u^*. It is worth mentioning that Picard iteration process is useful for the approximation of the fixed point of the contraction mappings but the case when ones dealing with nonexpansive mappings it may fail to converge to the fixed point even if $\mathcal{F}$ has a unique fixed point. Krasnosel'skii [2] showed that Mann [3] iteration process can approximate the fixed points of a nonexpansive mapping. In this iteration scheme, the sequence (u_n) is generated by an arbitrary $u_0 \in C$ as

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n \mathcal{F}u_n, \quad \forall\, n \geq 0, \tag{4}$$

where (α_n) is in $(0, 1)$.

In 1974, Ishikawa [4] developed an iterative scheme to approximate the fixed point of nonexpansive mappings, where (u_n) is defined iteratively starting from $u_0 \in C$ by

$$\left.\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n \mathcal{F}v_n \\ v_n &= (1 - \beta_n)u_n + \beta_n \mathcal{F}u_n \end{aligned}\right\}, \tag{5}$$

for all $n \geq 0$, where (α_n) and (β_n) are in $(0, 1)$.

For the approximation of the fixed point of nonexpansive mappings, Mann and Ishikawa iterative methods have been studied by several authors (see e.g., [5–9]).

Another iteration scheme was proposed by Noor [10] in 2000, for $u_0 \in C$, the sequence (u_n) is defined by

$$\left.\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n \mathcal{F} v_n \\ v_n &= (1-\beta_n)u_n + \beta_n \mathcal{F} w_n \\ w_n &= (1-\gamma_n)u_n + \gamma_n \mathcal{F} u_n \end{aligned}\right\}, \tag{6}$$

for all $n \geq 0$, where (α_n), (β_n), and (γ_n) are in $(0,1)$.

Agarwal et al. [11], in 2007, proposed the following iterative scheme: for arbitrary $u_0 \in C$, a sequence $\{u_n\}$ is generated by

$$\left.\begin{aligned} u_{n+1} &= (1-\alpha_n)\mathcal{F} u_n + \alpha_n \mathcal{F} v_n \\ v_n &= (1-\beta_n)u_n + \beta_n \mathcal{F} u_n \end{aligned}\right\}, \tag{7}$$

for all $n \geq 0$, where (α_n) and (β_n) are in $(0,1)$. They proved that this procedure converges faster than Mann iteration for contraction mappings.

In 2014, Abbas and Nazir [12] developed an iterative scheme which is faster than Agarwal et al.'s [11] scheme, where a sequence (u_n) is formulated from arbitrary $u_0 \in C$ by

$$\left.\begin{aligned} u_{n+1} &= (1-\alpha_n)\mathcal{F} v_n + \alpha_n \mathcal{F} w_n \\ v_n &= (1-\beta_n)\mathcal{F} u_n + \beta_n \mathcal{F} w_n \\ w_n &= (1-\gamma_n)u_n + \gamma_n \mathcal{F} u_n \end{aligned}\right\}, \tag{8}$$

for all $n \geq 0$, where (α_n), (β_n), and (γ_n) are in $(0,1)$.

Later in 2016, Thakur et al. [13] developed the following iterative procedure, where a sequence (u_n) is generated iteratively by arbitrary $u_0 \in C$ and

$$\left.\begin{aligned} u_{n+1} &= (1-\alpha_n)\mathcal{F} w_n + \alpha_n \mathcal{F} v_n \\ v_n &= (1-\beta_n)w_n + \beta_n \mathcal{F} w_n \\ w_n &= (1-\gamma_n)u_n + \gamma_n \mathcal{F} u_n \end{aligned}\right\}, \tag{9}$$

for all $n \geq 0$, where (α_n), (β_n), and (γ_n) are in $(0,1)$.

Recently, in 2018, Ullah and Arshad developed a new iteration process which converges faster than all the aforementioned process, where the sequence is constructed by taking arbitrary $u_0 \in C$ and

$$\left.\begin{aligned} u_{n+1} &= \mathcal{F} v_n \\ v_n &= \mathcal{F}\left((1-\alpha_n)w_n + \alpha_n \mathcal{F} w_n\right) \\ w_n &= (1-\beta_n)u_n + \beta_n \mathcal{F} u_n \end{aligned}\right\}, \tag{10}$$

for all $n \geq 0$, where (α_n), (β_n), and (γ_n) are in $(0,1)$.

Our aim is to introduce a new faster iteration process than those mentioned above and to prove the convergence results for Suzuki's generalized nonexpansive mappings in the context of uniformly convex Banach spaces. We also show that our process is stable analytically. Numerically, we compare the rate of convergence of our iteration process with the existing iteration processes.

2. Preliminaries

Throughout this paper, $\mathcal{E}$ is a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{X}$, $\mathbb{N}$ denotes the set of all positive integers and $F(\mathcal{F})$ denotes the set of all fixed points of $\mathcal{F}$, that is,

$$F(\mathcal{F}) := \{y\colon \mathcal{F} y = y\}. \tag{11}$$

Definition 1 (see [14]). A Banach space $\mathcal{X}$ is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists a $\delta > 0$ such that for all $u, v \in \mathcal{X}$,

$$\left.\begin{aligned} &\|u\| \leq 1 \\ &\|v\| \leq 1 \\ &\|u-v\| > \epsilon \end{aligned}\right\} \quad \text{implies} \quad \left\|\frac{u+v}{2}\right\| \leq \delta. \tag{12}$$

Definition 2 (see [15]). A Banach space $\mathcal{X}$ is said to satisfy Opial property if for each sequence (u_n) in $\mathcal{X}$, converging weakly to $u \in \mathcal{X}$, we have

$$\limsup_{n\longrightarrow\infty}\left\|u_n - u\right\| < \limsup_{n\longrightarrow\infty}\left\|u_n - v\right\|, \tag{13}$$

for all $v \in \mathcal{X}$ such that $u \neq v$.

Definition 3. A mapping $\mathcal{F}\colon \mathcal{E} \longrightarrow \mathcal{E}$ is called a contraction if there exists $\alpha \in (0,1)$, such that

$$\|\mathcal{F} u - \mathcal{F} v\| \leq \alpha\|u - v\|, \quad \text{for all}\, u, v \in \mathcal{E}. \tag{14}$$

Definition 4. A mapping $\mathcal{F}\colon \mathcal{E} \longrightarrow \mathcal{E}$ quasi-nonexpansive if for all $u \in \mathcal{E}$ and $p \in F(\mathcal{F})$ and $\alpha \in (0,1)$, we have

$$\|\mathcal{F} u - p\| \leq \alpha\|u - p\|. \tag{15}$$

Definition 5 (see [16]). A mapping $\mathcal{F}\colon \mathcal{E} \longrightarrow \mathcal{E}$ is called Suzuki's generalized nonexpansive mapping if for all $u, v \in \mathcal{E}$, we have

$$\frac{1}{2}\|u - \mathcal{F} u\| \leq \alpha\|u - v\| \quad \text{implies} \quad \|\mathcal{F} u - \mathcal{F} v\| \leq \|u - v\|. \tag{16}$$

Suzuki [16] proved that the generalized nonexpansive mapping is weaker than nonexpansive mapping and stronger than quasi-nonexpansive mapping and obtained some fixed points and convergence theorems for Suzuki's generalized nonexpansive mappings. Recently, many authors have studied fixed-point theorems for Suzuki's generalized nonexpansive mapping (see, e.g., [17]).

Senter and Dotson [7] introduced a class of mappings satisfying condition (I).

Definition 6. A mapping $\mathcal{F}\colon \mathcal{E} \longrightarrow \mathcal{E}$ is said to satisfy condition (I), if there exists a nondecreasing function $f\colon [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$ and $f(\delta) > 0$ for all

$\delta > 0$ such that $\|u - \mathcal{F}u\| \geq f(d(u, F(\mathcal{F})))$, for all $u \in \mathcal{E}$, where $d(u, F(\mathcal{F})) = \inf_{p\in F(\mathcal{F})} \|u - p\|$.

Proposition 1 (see [16]). *Let $\mathcal{F}\colon \mathcal{E} \longrightarrow \mathcal{E}$ be any mapping. Then,*

(i) If $\mathcal{F}$ is nonexpansive, then $\mathcal{F}$ is a Suzuki's generalized nonexpansive mapping.

(ii) If $\mathcal{F}$ is a Suzuki's generalized nonexpansive mapping and has a fixed point, then $\mathcal{F}$ is a quasi-nonexpansive mapping.

(iii) If $\mathcal{F}$ is a Suzuki generalized nonexpansive mapping, then

$$\|u - \mathcal{F}v\| \leq 3\|\mathcal{F}u - u\| + \|u - v\|, \quad \forall u, v \in \mathcal{E}. \tag{17}$$

Lemma 1 (see [16]). *Suppose $\mathcal{F}\colon \mathcal{E} \longrightarrow \mathcal{E}$ is Suzuki's generalized nonexpansive mapping satisfying Opial property. If (u_n) converges weakly to u and $\lim_{n\longrightarrow\infty}\|\mathcal{F}u_n - u_n\| = 0$, then $\mathcal{F}u = u$.*

Lemma 2 (see [16]). *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{E}$ be a weakly convex compact subset of $\mathcal{X}$. Assume that $\mathcal{F}\colon \mathcal{E} \longrightarrow \mathcal{E}$ is Suzuki's generalized nonexpansive mapping. Then, $\mathcal{F}$ has a fixed point.*

Lemma 3 (see [18]). *Let $\mathcal{X}$ be a uniformly convex Banach space and (t_n) be any real sequence such that $0 < p \leq u_n \leq q < 1$ for all $n \geq 1$. Suppose that (u_n) and (v_n) be any two sequences of $\mathcal{X}$ such that $\limsup_{n\longrightarrow\infty}\|u_n\| \leq r$, $\limsup_{n\longrightarrow\infty}\|v_n\| \leq r$, and $\limsup_{n\longrightarrow\infty}\|t_n u_n + (1 - t_n)v_n\| = r$ hold for some $r \geq 0$. Then, $\limsup_{n\longrightarrow\infty}\|u_n - v_n\| = 0$.*

Definition 7. (see [19]). Let $\mathcal{X}$ be a Banach space and $\mathcal{E}$ be a nonempty closed convex subset of $\mathcal{X}$. Assume that (u_n) is a bounded sequence in $\mathcal{X}$. For $u \in \mathcal{X}$, we set $r(u, (u_n)) = \limsup_{n\longrightarrow\infty}\|u_n - u\|$. The asymptotic radius of (u_n) relative to $\mathcal{E}$ is the set $r(\mathcal{E}, (u_n)) = \inf\{r(u, (u_n))\colon u \in \mathcal{E}\}$ and the asymptotic center of (u_n) relative to $\mathcal{E}$ is given by the following set:

$$\mathcal{A}(\mathcal{E}, (u_n)) = \{u \in \mathcal{E}\colon r(u, (u_n)) = r(\mathcal{E}, (u_n))\}. \tag{18}$$

It is known that, in a uniformly convex Banach space, $\mathcal{A}(\mathcal{E}, (u_n))$ consists of exactly one point.

Definition 8. (see [20]). Let $\mathcal{X}$ be a Banach space and $\mathcal{F}\colon \mathcal{X} \longrightarrow \mathcal{X}$. Suppose that $u_0 \in \mathcal{X}$ and $u_{n+1} = f(\mathcal{F}, u_n)$ define an iteration procedure which gives a sequence of points (u_n) in $\mathcal{X}$. Assume that (x_n) converges to the fixed point p. Suppose (v_n) be a sequence in $\mathcal{X}$ and (ϵ_n) be a sequence in $\mathbb{R}^+ = [0, \infty)$ given by $\epsilon_n = \|v_{n+1} - f(\mathcal{F}, v_n)\|$. Then, the iteration procedure defined by $u_{n+1} = f(\mathcal{F}, u_n)$ is said to be $\mathcal{F}$-stable or stable with respect to $\mathcal{F}$ if

$$\lim_{n\longrightarrow\infty} \epsilon_n = 0 \quad \text{iff} \quad \lim_{n\longrightarrow\infty} v_n = p. \tag{19}$$

Definition 9 (see [21]). Let $\mathcal{X}$ be a Banach space and $\mathcal{F}\colon \mathcal{X} \longrightarrow \mathcal{X}$. Then, $\mathcal{F}$ is called a contractive mapping on $\mathcal{X}$ if there exist $L \geq 0$, $a \in [0, 1)$ such that for each $u, v \in \mathcal{X}$,

$$\|\mathcal{F}u - \mathcal{F}v\| \leq L\|u - \mathcal{F}u\| + a\|u - v\|. \tag{20}$$

By using (7), Osilike [21] established several stability results most of which are generalizations of the results of Rhoades [22] and Harder and Hicks [23].

Definition 10 (see [24]). Let $\mathcal{X}$ be a Banach space and $\mathcal{F}\colon \mathcal{X} \longrightarrow \mathcal{X}$. Then, $\mathcal{F}$ is called a contractive mapping on $\mathcal{X}$ if there exist $b \in [0, 1)$ and a monotone increasing function $\varphi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\varphi(0) = 0$, such that for each $u, v \in \mathcal{X}$,

$$\|\mathcal{F}u - \mathcal{F}v\| \leq \varphi(\|u - \mathcal{F}u\|) + b\|u - v\|. \tag{21}$$

Lemma 4 (see [25]). *If λ is a real number such that $0 \leq \lambda < 1$, and (ϵ_n) is the sequence of positive numbers such that*

$$\lim_{n\longrightarrow\infty} \epsilon_n = 0, \tag{22}$$

then for any sequence of positive numbers v_n satisfying

$$v_{n+1} \leq \lambda v_n + \epsilon_n, \quad \text{for } n = 1, 2, \ldots, \tag{23}$$

we have

$$\lim_{n\longrightarrow\infty} v_n = 0. \tag{24}$$

3. S^*-Iteration Process

Throughout this section, C be a nonempty set of a Banach space $\mathcal{X}$, and for all $n \geq 0$, (α_n), (β_n), (γ_n) and (ζ_n) are real sequences in the interval $(0, 1)$.

We generate the sequence (u_n) iteratively, taking arbitrary $u_0 \in C$, by

$$\left.\begin{aligned} u_{n+1} &= \mathcal{F}((1 - \alpha_n)v_n + \alpha_n\mathcal{F}v_n) \\ v_n &= \mathcal{F}((1 - \beta_n)w_n + \beta_n\mathcal{F}w_n) \\ w_n &= \mathcal{F}((1 - \gamma_n)x_n + \gamma_n\mathcal{F}x_n) \\ x_n &= \mathcal{F}((1 - \zeta_n)u_n + \zeta_n\mathcal{F}u_n) \end{aligned}\right\}. \tag{25}$$

First, we show that S^*-iteration scheme (25) converges faster than all aforementioned iteration schemes for contractive mappings due to Berinde [26] and is stable.

4. Convergence and Stability Results of S^*-Iteration Process

First, we establish convergence results for S^*-iteration process:

Theorem 1. *Let $\mathcal{X}$ be a Banach space and $\mathscr{E}$ be a nonempty closed convex subset of $\mathcal{X}$. Let $\mathscr{F}$ be a nonexpansive self mapping on $\mathscr{E}$, (u_n) be a sequence defined by (25), and $F(\mathscr{F}) \neq \phi$. Then, $\lim_{n\longrightarrow\infty}\|u_n - p\|$ exists for all $p \in F(\mathscr{F})$.*

Proof. Let $p \in F(\mathscr{F})$ for all $n \in \mathbb{N}$. From (16), we have

$$\begin{aligned}\|x_n - p\| &= \|\mathscr{F}((1-\zeta_n)u_n + \zeta_n\mathscr{F}u_n) - p\|\\ &\le \|(1-\zeta_n)u_n + \zeta_n\mathscr{F}u_n - p\|\\ &\le (1-\zeta_n)\|u_n - p\| + \zeta_n\|\mathscr{F}u_n - p\|\\ &\le (1-\zeta_n)\|u_n - p\| + \zeta_n\|u_n - p\|\\ &= \|u_n - p\|,\end{aligned} \tag{26}$$

$$\begin{aligned}\|w_n - p\| &= \|\mathscr{F}((1-\gamma_n)x_n + \gamma_n F x_n) - p\|\\ &\le \|(1-\gamma_n)x_n + \gamma_n\mathscr{F}x_n - p\|\\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|\mathscr{F}x_n - p\|\\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\|\\ &= \|x_n - p\|\\ &\le \|u_n - p\|,\end{aligned} \tag{27}$$

$$\begin{aligned}\|v_n - p\| &= \|\mathscr{F}((1-\beta_n)w_n + \beta_n\mathscr{F}w_n) - p\|\\ &\le \|(1-\beta_n)w_n + \beta_n\mathscr{F}w_n - p\|\\ &\le (1-\beta_n)\|w_n - p\| + \beta_n\|\mathscr{F}w_n - p\|\\ &\le (1-\beta_n)\|w_n - p\| + \beta_n\|w_n - p\|\\ &= \|w_n - p\|\\ &\le \|u_n - p\|.\end{aligned} \tag{28}$$

Thus,

$$\begin{aligned}\|u_{n+1} - p\| &= \|\mathscr{F}((1-\alpha_n)v_n + \alpha_n\mathscr{F}v_n) - p\|\\ &\le \|(1-\alpha_n)v_n + \alpha_n\mathscr{F}v_n - p\|\\ &\le (1-\alpha_n)\|v_n - p\| + \alpha_n\|\mathscr{F}v_n - p\|\\ &\le (1-\alpha_n)\|u_n - p\| + \alpha_n\|v_n - p\|\\ &\le (1-\alpha_n)\|u_n - p\| + \alpha_n\|u_n - p\|\\ &= \|u_n - p\|.\end{aligned} \tag{29}$$

Hence, $\lim_{n\longrightarrow\infty}\|u_n - p\|$ exists for all $p \in F(\mathscr{F})$. □

Theorem 2. *Let $\mathcal{X}$ a uniformly convex Banach space and $\mathscr{E}$ be a nonempty closed convex subset of $\mathcal{X}$. Let $\mathscr{F}: \mathscr{E} \longrightarrow \mathscr{E}$ be a nonexpansive mapping. Suppose that (u_n) is defined by the iteration process (25) and $F(\mathscr{F}) \neq \phi$. Then, the sequence $\{u_n\}$ converges to a point of $F(\mathscr{F})$ if and only if $\liminf_{n\longrightarrow\infty} d(u_n, F(\mathscr{F})) = 0$ where $d(u, F(\mathscr{F})) = \inf\{\|u - p\|: p \in F(\mathscr{F})\}$.*

Proof. Necessity is obvious. Suppose that $\liminf_{n\longrightarrow\infty} d(u_n, F(\mathscr{F})) = 0$. As proved in Theorem 1, $\lim_{n\longrightarrow\infty}\|u_n - u\|$ exists for all $u \in F(\mathscr{F})$, so $\lim_{n\longrightarrow\infty} d(u_n, F(\mathscr{F}))$ exists and $\liminf_{n\longrightarrow\infty} d(u_n, F(\mathscr{F})) = 0$ by assumption. Now, we will prove that (u_n) is a Cauchy sequence in $\mathscr{E}$. For given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$,

$$d(u_n, F(\mathscr{F})) < \frac{\epsilon}{2}. \tag{30}$$

In particular, $\inf\{\|u_N - p\|: p \in F(\mathscr{F})\} < (\epsilon/2)$. Hence, there exists $p^* \in F(\mathscr{F})$ such that $\|u_N - p^*\| < \epsilon/2$. Now for all $m, n \ge N$,

$$\|u_{m+n} - u_n\| \le \|u_{m+n} - p^*\| + \|u_n - p^*\| \le 2\|u_N - p^*\| < \epsilon, \tag{31}$$

which shows that (u_n) is a Cauchy sequence in $\mathscr{E}$. But $\mathscr{E}$ is a closed subset of $\mathcal{X}$, so there exists $p \in \mathscr{E}$ such that $\lim_{n\longrightarrow\infty} u_n = p$. Now, $\lim_{n\longrightarrow\infty} d(u_n, F(\mathscr{F})) = 0$ gives $d(p, F(\mathscr{F}))$ which implies $p \in F(\mathscr{F})$.

Next, we prove that our iteration process is $\mathscr{F}$-stable or stable with respect to $\mathscr{F}$. □

Theorem 3. *Let $\mathcal{X}$ be a Banach space and $\mathscr{F}: \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping satisfying (21). Suppose $\mathscr{F}$ has a fixed point p. Let*

(u_n) be a sequence in $\mathcal{X}$ satisfying (9). Then, S^*-iteration process (9) is $\mathcal{F}$-stable.

Proof. Let (t_n) be an arbitrary sequence in $\mathcal{X}$ and the sequence generated by (25) is $u_{n+1} = f(\mathcal{F}, u_n)$ converging to a unique fixed point p and $\epsilon_n = \|t_{n+1} - f(\mathcal{F}, t_n)\|$. We will prove that $\lim_{n\longrightarrow\infty}\epsilon_n = 0 \Longleftrightarrow \lim_{n\longrightarrow\infty}t_n = p$. Assume that $\lim_{n\longrightarrow\infty}\epsilon_n = 0$ and

$$\begin{aligned}
\|t_{n+1} - p\| &= \|t_{n+1} - f(\mathcal{F}, t_n) + f(\mathcal{F}, t_n) - p\| \\
&\le \|t_{n+1} - f(\mathcal{F}, t_n)\| + \|f(\mathcal{F}, t_n) - p\| \\
&\le \|t_{n+1} - \mathcal{F}((1-\alpha_n)s_n + \alpha_n\mathcal{F}s_n)\| + \|\mathcal{F}((1-\alpha_n)s_n + \alpha_n\mathcal{F}s_n) - p\| \\
&\le \epsilon_n + b[1 - \alpha_n(1-b)]\|s_n - p\| \\
&= \epsilon_n + b[1 - \alpha_n(1-b)]\|\mathcal{F}((1-\beta_n)r_n + \beta_n\mathcal{F}r_n) - p\| \\
&\le \epsilon_n + b^2[1 - \alpha_n(1-b)][(1 - \beta_n(1-b)]\|r_n - p\| \\
&= \epsilon_n + b^2[1 - \alpha_n(1-b)][(1 - \beta_n(1-b)]\|\mathcal{F}((1-\gamma_n)v_n + \gamma_n\mathcal{F}v_n) - p\| \\
&\le \epsilon_n + b^3[1 - \alpha_n(1-b)][(1 - \beta_n(1-b)][1 - \gamma_n(1-b)]\|v_n - p\| \\
&= \epsilon_n + b^3[1 - \alpha_n(1-b)][(1 - \beta_n(1-b)][1 - \gamma_n(1-b)]\|\mathcal{F}((1-\zeta_n)t_n + \zeta_n\mathcal{F}t_n) - p\| \\
&\le \epsilon_n + b^4[1 - \alpha_n(1-b)][(1 - \beta_n(1-b)][1 - \gamma_n(1-b)][1 - \zeta_n(1-b)\|t_n - p\|.
\end{aligned} \tag{32}$$

Since $b \in [0, 1)$ and (α_n), (β_n), (γ_n), and (ζ_n) are in $[0, 1]$,

$$\begin{aligned}
&b^4[1 - \alpha_n(1-b)][(1 - \beta_n(1-b)][1 - \gamma_n(1-b)] \\
&\quad \cdot [1 - \zeta_n(1-b)]\| < 1.
\end{aligned} \tag{33}$$

Hence by Lemma 4, we have $\lim_{n\longrightarrow\infty}\|t_n - p\| = 0$, which gives $\lim_{n\longrightarrow\infty}t_n = p$. On the other hand, suppose that $\lim_{n\longrightarrow\infty}t_n = p$. Then,

$$\begin{aligned}
\epsilon_n &= \|t_{n+1} - f(\mathcal{F}, t_n)\| \\
&= \|t_{n+1} - p + p - f(\mathcal{F}, t_n)\| \\
&\le \|t_{n+1} - p\| + b[(1-\alpha_n) + b\alpha_n]\|s_n - p\| \\
&= \|t_{n+1} - p\| + b[(1-\alpha_n) + b\alpha_n]\|\mathcal{F}((1-\beta_n)r_n + \beta_n\mathcal{F}r_n) - p\| \\
&\le \|t_{n+1} - p\| + b^2[(1-\alpha_n) + b\alpha_n][(1 - \beta_n(1-b)]\|r_n - p\| \\
&= \|t_{n+1} - p\| + b^2[(1-\alpha_n) + b\alpha_n](1-\beta_n)\|\mathcal{F}((1-\gamma_n)v_n + \gamma_n\mathcal{F}v_n) - p\| \\
&\le \|t_{n+1} - p\| + b^3[1 - \alpha_n(1-b)][(1 - \beta_n(1-b)][1 - \gamma_n(1-b)]\|v_n - p\| \\
&= \|t_{n+1} - p\| + b^3[1 - \alpha_n(1-b)][(1 - \beta_n(1-b)][1 - \gamma_n(1-b)]\|\mathcal{F}((1-\zeta_n)t_n + \zeta_n\mathcal{F}t_n) - p\| \\
&\le \|t_{n+1} - p\| + b^4[1 - \alpha_n(1-b)][(1 - \beta_n(1-b)][1 - \gamma_n(1-b)][1 - \zeta_n(1-b)]\|t_n - p\|.
\end{aligned} \tag{34}$$

Taking limit as $n \longrightarrow \infty$ in (34), we get $\lim_{n\longrightarrow\infty}\epsilon_n = 0$.

Now, we present an example to compare the rate of convergence of our iteration scheme with others. □

Example 1. Let $\mathcal{X} = \mathbb{R}$ and $C = [1, 50]$. Let $\mathcal{F}\colon C \longrightarrow C$ be a mapping defined by $\mathcal{F}u = \sqrt{u^2 - 9u + 54}$ for all $u \in C$. For $u_1 = 30$ and $\alpha_n = \beta_n = \gamma_n = 3/4$, $n = 1, 2, 3, \ldots$.From Table 1, we can see that all the iteration procedures are converging to $p^* = 6$. Clearly, our iteration process requires the least number of iteration as compared to other iteration schemes.

In Figure 1, black curve represents our iteration process. The graphical view shows that our iteration process requires less number of iterations as compared to the other iteration processes. The number of iterations in which these processes attain the fixed point is given in Table 2:

5. Some Convergence Results for Suzuki's Generalized Nonexpansive Mappings

This section contains some weak and strong convergence results for a sequence generated by S^*-iteration process for Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces.

Table 1: Comparison of the rate of convergence with various iteration schemes.

Step	Picard	Ishikawa	Noor	Agarwal	Abbas	Thakur	K. Ullah	S^*-iter.
1	30.00000000	30.00000000	30.00000000	30.00000000	30.00000000	30.00000000	30.00000000	30.00000000
2	26.15339366	25.01198240	23.48910332	24.05033082	22.61079008	21.30667585	17.14034293	7.939900241
3	22.41917610	20.25475590	17.46681907	18.43727194	15.82815627	13.58899597	7.920241534	6.000499545
4	18.83737965	15.85090878	12.32658573	13.39382036	10.25820641	8.112973955	6.038818684	6.000000071
5	15.46966242	12.01330515	8.727576617	9.372555587	7.001837925	6.225674626	6.000469229	6.000000000
6	12.41303724	9.068862033	6.958571160	6.993935718	6.119154210	6.015130221	6.000005614	6.000000000
7	9.816626625	7.282040026	6.310214626	6.186206786	6.011213258	6.000960494	6.000000067	6.000000000
8	7.875056741	6.466803146	6.097925567	6.028369366	6.001024303	6.000060749	6.000000001	6.000000000
9	6.718705828	6.160065238	6.030680843	6.004133882	6.000093304	6.000003841	6.000000000	6.000000000
10	6.218734240	6.053725040	6.009590308	6.000598188	6.000008497	6.000000242	6.000000000	6.000000000
11	6.058386534	6.017902837	6.002995608	6.000086472	6.000000774	6.000000016	6.000000000	6.000000000
12	6.014862308	6.005951431	6.000935492	6.000012498	6.000000071	6.000000001	6.000000000	6.000000000
13	6.003732823	6.001976848	6.000292122	6.000001806	6.000000005	6.000000000	6.000000000	6.000000000
14	6.00093429	6.000656462	6.000091217	6.000000261	6.000000001	6.000000000	6.000000000	6.000000000
15	6.000233641	6.000217976	6.000028483	6.000000037	6.000000000	6.000000000	6.000000000	6.000000000
16	6.000058415	6.000072376	6.000008894	6.000000005	6.000000000	6.000000000	6.000000000	6.000000000
17	6.000014603	6.000024032	6.000002778	6.000000001	6.000000000	6.000000000	6.000000000	6.000000000
18	6.000003651	6.000007979	6.000000866	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
19	6.000000912	6.000002649	6.000000270	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
20	6.000000227	6.000000880	6.000000084	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
21	6.000000057	6.000000293	6.000000026	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
22	6.000000014	6.000000097	6.000000008	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
23	6.000000003	6.000000032	6.000000003	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
24	6.000000001	6.000000010	6.000000001	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
25	6.000000000	6.000000003	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
26	6.000000000	6.000000001	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
27	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
28	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
29	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
30	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000

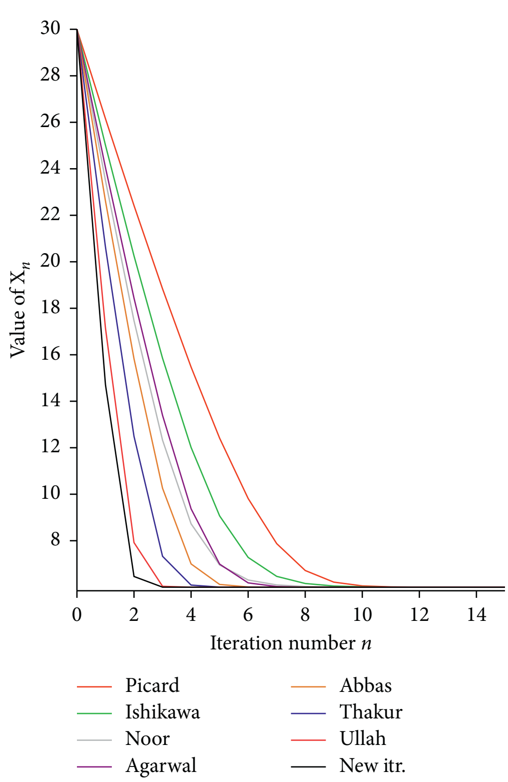

Figure 1: Graphical representation of convergence of iterative schemes.

Table 2: Number of iterations in which fixed point attains.

Iterative method	Number of iterations
Picard	25
Ishikawa	27
Noor	25
Agarwal	18
Abbas	15
Thakur	13
K. Ullah	9
S^*-iter.	5

Lemma 5. *Suppose that $\mathscr{E}$ be a nonempty closed convex subset of a Banach space $\mathscr{X}$. Let $\mathscr{F}: \mathscr{E} \longrightarrow \mathscr{E}$ be a Suzuki generalized nonexpansive mapping with $F(\mathscr{F}) \neq \phi$. For $u_0 \in \mathscr{E}$, the sequence (u_n) generated by S^*-iteration process, $\lim_{n\longrightarrow\infty}\|u_n - p\|$ exists for all $p \in F(\mathscr{F})$.*

Proof. Result follows from Proposition 1 (i) and Theorem 1. □

Lemma 6. *Suppose that $\mathscr{E}$ be a nonempty closed convex subset of a uniformly Banach space $\mathscr{X}$. Let $\mathscr{F}: \mathscr{E} \longrightarrow \mathscr{E}$ be a Suzuki's generalized nonexpansive mapping with $F(\mathscr{F}) \neq \phi$. For arbitrarily chosen $u_0 \in \mathscr{E}$, the sequence (u_n) is generated by S^*-iteration process. Then, $F(\mathscr{F}) \neq \phi$ if and only if (u_n) is bounded and $\lim_{n\longrightarrow\infty}\|\mathscr{F}u_n - u_n\| = 0$.*

Proof. Suppose that $F(\mathcal{F}) \neq \phi$ and let $p \in \mathcal{E}$. Then, by Lemma 5, $\lim_{n\longrightarrow\infty}\|u_n - p\|$ exists and (u_n) is bounded. Let

$$\lim_{n\longrightarrow\infty}\|u_n - p\| = r. \tag{35}$$

From (26) and (37), we have

$$\limsup_{n\longrightarrow\infty}\|u_n - p\| \leq \limsup_{n\longrightarrow\infty}\|u_n - p\| = r. \tag{36}$$

By the Proposition 1 (iii), we have

$$\limsup_{n\longrightarrow\infty}\|\mathcal{F}x_n - p\| \leq \limsup_{n\longrightarrow\infty}\|u_n - p\| = r, \tag{37}$$

$$\begin{aligned}
\|u_{n+1} - p\| &= \|\mathcal{F}((1-\alpha_n)v_n + \alpha_n\mathcal{F}v_n) - p\| \\
&\leq \|(1-\alpha_n)v_n + \alpha_n\mathcal{F}v_n - p\| \\
&\leq \|v_n - p\| \\
&= \|\mathcal{F}((1-\beta_n)w_n + \beta_n\mathcal{F}w_n) - p\| \\
&\leq \|w_n - p\| \\
&= \|\mathcal{F}((1-\gamma_n)x_n + \gamma_n\mathcal{F}x_n) - p\| \\
&\leq (1-\gamma_n)\|u_n - p\| + \gamma_n\|x_n - p\| \\
&= \|u_n - p\| - \gamma_n\|u_n - p\| + \gamma_n\|x_n - p\|.
\end{aligned} \tag{38}$$

This implies that

$$\begin{aligned}
&\frac{\|u_{n+1} - p\| - \|u_n - p\|}{\gamma_n} \leq \left[\|x_n - p\| - \|u_n - p\|\right], \\
&\|u_{n+1} - p\| - \|u_n - p\| \leq \frac{\|u_{n+1} - p\| - \|u_n - p\|}{\gamma_n} \\
&\qquad \leq \left[\|x_n - p\| - \|u_n - p\|\right], \\
&\|u_{n+1} - p\| \leq \|x_n - p\|,
\end{aligned} \tag{39}$$

$$r \leq \liminf_{n\longrightarrow\infty}\|x_n - p\|, \tag{40}$$

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\|x_n - p\| = r, \\
&\lim_{n\longrightarrow\infty}\|(1-\zeta_n)u_n + \zeta_n\mathcal{F}(u_n) - p\| = r, \\
&\lim_{n\longrightarrow\infty}\|(1-\zeta_n)(u_n - p) + \zeta_n(\mathcal{F}(u_n) - p)\| = r.
\end{aligned} \tag{41}$$

From equations (26) and (37) and Lemma 3, we have

$$\lim_{n\longrightarrow\infty}\|\mathcal{F}u_n - u_n\| = 0. \tag{42}$$

Conversely, assume that (u_n) is bounded and $\lim_{n\longrightarrow\infty}\|\mathcal{F}u_n - u_n\| = 0$. Suppose that $p \in \mathcal{A}(\mathcal{E}, (u_n))$. Using Proposition 1 (iii), we get

$$\begin{aligned}
r(\mathcal{F}p, (u_n)) &= \limsup_{n\longrightarrow\infty}\|u_n - \mathcal{F}p\| \\
&\leq \limsup_{n\longrightarrow\infty}\left[3\|\mathcal{F}u_n - u_n\| + \|u_n - p\|\right] \\
&\leq \limsup_{n\longrightarrow\infty}\|u_n - p\| \\
&= r(p, (u_n)).
\end{aligned} \tag{43}$$

This shows that $\mathcal{F}p \in \mathcal{A}(\mathcal{E}, (u_n))$. Since $\mathcal{X}$ is uniformly convex, $\mathcal{A}(\mathcal{E}, (u_n))$ is singleton. Thus, we have $\mathcal{F}p = p$, that is, $F(\mathcal{F}) \neq \phi$. □

Theorem 4 (weak convergence theorem). *Suppose that $\mathcal{E}$ be a nonempty closed convex subset of a uniformly Banach space $\mathcal{X}$ with the Opial property. Let $\mathcal{F}: \mathcal{E} \longrightarrow \mathcal{E}$ be Suzuki's generalized nonexpansive mapping. For arbitrarily chosen $u_0 \in \mathcal{E}$, let the sequence (u_n) be generated by S^*-iteration process with $F(\mathcal{F}) \neq \phi$. Then, (u_n) converges weakly to a fixed point of $\mathcal{F}$.*

Proof. Since $F(\mathcal{F}) \neq \phi$, by Lemma 6, the sequence (u_n) is bounded and $\lim_{n\longrightarrow\infty}\|\mathcal{F}u_n - u_n\| = 0$. Also, as $\mathcal{X}$ is uniformly convex so $\mathcal{X}$ is reflexive, thus by Eberlin's theorem, there exists a subsequence of (u_n) say (u_{n_i}) which converges weakly to some $q_1 \in \mathcal{X}$. Now, since $\mathcal{E}$ is closed and convex so by Mazur's theorem $q_1 \in \mathcal{E}$. Hence, by Lemma 1, $q_1 \in F(\mathcal{F})$. We show that (u_n) converges weakly to q_1. On contrary, suppose that it is not true. Then, there must exist a subsequence of (u_n), say (u_{n_j}), such that (u_{n_j}) converges weakly to $q_2 \in \mathcal{E}$ with $q_1 \neq q_2$. Using Lemma 1, we have $q_2 \in F(\mathcal{F})$. Now, since $\lim_{n\longrightarrow\infty}\|u_n - p\|$ exists for all $p \in F(\mathcal{F})$. Using Lemma 6 and Opial property, we have

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\|u_n - q_1\| &= \lim_{i\longrightarrow\infty}\|u_{n_i} - q_1\| \\
&< \lim_{i\longrightarrow\infty}\|u_{n_i} - q_2\| \\
&= \lim_{n\longrightarrow\infty}\|u_n - q_2\| \\
&= \lim_{j\longrightarrow\infty}\|u_{n_j} - q_2\| \\
&< \lim_{j\longrightarrow\infty}\|u_{n_j} - q_1\| \\
&= \lim_{n\longrightarrow\infty}\|u_n - q_1\|,
\end{aligned} \tag{44}$$

which is a contradiction; hence, $q_1 = q_2$. This shows that (u_n) converges weakly to a fixed point of $\mathcal{F}$. □

Theorem 5 (strong convergence theorem). *Suppose that $\mathcal{E}$ be a nonempty closed convex subset of a uniformly Banach space $\mathcal{X}$. Let $\mathcal{F}: \mathcal{E} \longrightarrow \mathcal{E}$ be a Suzuki's generalized nonexpansive mapping. For arbitrarily chosen $u_0 \in \mathcal{E}$, let the sequence (u_n) be generated by S^*-iteration process with $F(\mathcal{F}) \neq \phi$. Then, (u_n) converges strongly to a fixed point of $\mathcal{F}$.*

Proof. Using Lemma 2, we get $F(\mathcal{F}) \neq \phi$ and hence by Lemma 6, we have $\lim_{n\longrightarrow\infty}\|\mathcal{F}u_n - u_n\| = 0$. By the compactness of $\mathcal{E}$, there exists a subsequence of (u_n), say (u_{n_i}), converging strongly to p for some $p \in \mathcal{E}$. Now by using Proposition 1 (iii), we get

$$\|u_{n_i} - \mathcal{F}p\| \leq 3\|\mathcal{F}u_{n_i} - u_{n_i}\| + \|u_{n_i} - p\|. \tag{45}$$

Taking limit $i \longrightarrow \infty$, we get $\mathcal{F}p = p$, that is, $p \in F(\mathcal{F})$. By using Lemma 5, $\lim n \longrightarrow \infty\|u_n - p\|$ exists for all $p \in F(\mathcal{T})$; hence, $\{u_n\}$ converges strongly to p. □

Theorem 6. *Suppose that $\mathscr{E}$ be a nonempty closed convex subset of a uniformly Banach space $\mathscr{X}$. Let $\mathscr{F}: \mathscr{E} \longrightarrow \mathscr{E}$ be a Suzuki's generalized nonexpansive mapping. For arbitrarily chosen $u_0 \in \mathscr{E}$, the sequence (u_n) be generated by S^*-iteration process with $F(\mathscr{F}) \neq \phi$. If $\mathscr{F}$ satisfies condition (I), then (u_n) converges strongly to a fixed point of $\mathscr{F}$.*

Proof. By Lemma 5, $\lim n \longrightarrow \infty \|u_n - p\|$ exists for all $p \in F(\mathscr{F})$; hence, $\lim_{n\longrightarrow\infty} d(u_n, F(\mathscr{F}))$ exists. Let $\lim_{n\longrightarrow\infty}\|u_n - p\| = \delta$ for some $\delta \geq 0$. Now if $\delta = 0$, then there is nothing to prove. Suppose $\delta > 0$; from condition (I) and the hypothesis, we have

$$f(d(u_n, F(\mathscr{F}))) \leq \|\mathscr{F}u_n - u_n\|. \tag{46}$$

As $F(\mathscr{F}) \neq \phi$, by Lemma 5, we have $\lim_{n\longrightarrow\infty}\|\mathscr{F}u_n - u_n\| = 0$. Hence, (46) implies that

$$\lim_{n\longrightarrow\infty} f(d(u_n, F(\mathscr{F}))) = 0. \tag{47}$$

Since f is a nondecreasing function, by equation (47), we get $\lim_{n\longrightarrow\infty}(d(u_n, F(\mathscr{F})) = 0$. Thus, we have a subsequence (u_{n_i}) of (u_n) and a sequence (y_i) in $F(\mathscr{F})$ such that

$$\|u_{n_i} - y_i\| < \frac{1}{2^i}, \quad \text{for all } i \in \mathbb{N}. \tag{48}$$

From equation (48),

$$\begin{aligned} \|u_{n_{i+1}} - y_i\| &\leq \|u_{n_i} - y_i\| < \frac{1}{2^i}, \\ \|y_{i+1} - y_i\| &\leq \|y_{i+1} - u_{i+1}\| + \|u_{i+1} - y_i\| \\ &\leq \frac{1}{2^{i+1}} + \frac{1}{2^i} \\ &< \frac{1}{2^{i-1}}. \end{aligned} \tag{49}$$

Letting $i \longrightarrow \infty$, we get $1/2^{i-1} \longrightarrow 0$. Hence, $\{y_i\}$ is a Cauchy sequence in $F(\mathscr{F})$, so it converges to p. As $F(\mathscr{F})$ is closed, $p \in F(\mathscr{F})$ and then (u_{n_i}) converges strongly to p. Since $\lim_{n\longrightarrow\infty}\|u_n - p\|$ exists, we have $u_n \longrightarrow p \in F(\mathscr{F})$. This completes the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have contributed equally.

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Research Article

Simpson's Integral Inequalities for Twice Differentiable Convex Functions

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Integral inequality is an interesting mathematical model due to its wide and significant applications in mathematical analysis and fractional calculus. In the present research article, we obtain new inequalities of Simpson's integral type based on the φ-convex and φ-quasiconvex functions in the second derivative sense. In the last sections, some applications on special functions are provided and shown via two figures to demonstrate the explanation of the readers.

1. Introduction

Integral inequality is a modern model of approximation theory that describes the growth rate of competing mathematical analysis. This model is also used in various fields such as ordinary differential equations [1–5] and fractional calculus [6–17].

Among the several known inequalities, the most simple is Simpson's type, which has been successfully applied in several models of ordinary differential equations [18–29] and fractional differential equations [30–32]. Simpson's integral inequality is as follows: for any four times continuously differentiable function $\overline{F}$: $[\xi_1, \xi_2] \longrightarrow R$ on (ξ_1, ξ_2), Simpson's integral inequality is defined as follows:

$$\left|\frac{1}{3}\left[\frac{\overline{F}(\xi_1)+\overline{F}(\xi_1)}{2}+2\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right] - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right| \leq \frac{1}{2880}\left\|\overline{F}^{(4)}\right\|_{\infty}(\xi_2-\xi_1)^4, \quad (1)$$

where $\|\overline{F}^{(4)}\|_{\infty} = \sup_{x\in(\xi_1,\xi_2)}|\overline{F}^{(4)}(x)| < \infty$.

If the function $\overline{F}$ is neither four times differentiable nor is the fourth derivative $\overline{F}^{(4)}$ bounded on (ξ_1, ξ_2), then we cannot apply the classical Simpson quadrature formula.

The following literature results obtained by Alomari et al. [18] and Sarikaya et al. [23] become a special case in our findings in Sections 2 and 3.

Lemma 1 (see [18]). *Let $\overline{F}$: $\mathfrak{J} \longrightarrow R$ be twice differentiable function on $\mathfrak{J}$ with $\overline{F}'' \in L_1[\xi_1, \xi_2]$, then we have*

$$\frac{\overline{F}(\xi_1)+\overline{F}(\xi_2)}{2} - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x = \frac{(\xi_2-\xi_1)^2}{2}\int_0^1 t(1-t)\overline{F}''(t\xi_1+(1-t)\xi_2)\mathrm{d}t. \quad (2)$$

Lemma 2 (see [23]). Let $\overline{F}$: $\mathfrak{J} \longrightarrow R$ be twice differentiable function on $\mathfrak{J}$ such that $\overline{F}'' \in L_1[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in \mathfrak{J}$ with $\xi_1 < \xi_2$, then we have

$$\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x$$
$$=(\xi_2-\xi_1)^2\int_0^1 \hbar(t)\overline{F}''(t\xi_2+(1-t)\xi_1)\mathrm{d}t, \tag{3}$$

where

$$\hbar(t)=\begin{cases}\dfrac{t}{2}\left(\dfrac{1}{3}-t\right); & \text{if } t\in\left[0,\dfrac{1}{2}\right),\\ (1-t)\left(\dfrac{t}{2}-\dfrac{1}{3}\right); & \text{if } t\in\left[\dfrac{1}{2},1\right].\end{cases} \tag{4}$$

Through this paper, R represents the set of real numbers and $\mathfrak{J}$ be an interval in R and $\varphi\colon R\times R\longrightarrow R$ be a bifunction apart from some special cases.

This paper deals with the notations of φ-convex and φ-quasiconvex functions which were introduced by Gordji et al. [33] as follows.

Definition 1. A function $\overline{F}\colon \mathfrak{J}\longrightarrow R$ is called convex with respect to φ (or briefly φ-convex), if

$$\overline{F}(t\xi_1+(1-t)\xi_2)\le\overline{F}(\xi_2)+t\varphi(\overline{F}(\xi_1),\overline{F}(\xi_2)), \tag{5}$$

for all $\xi_1,\xi_2\in\mathfrak{J}$ and $t\in[0,1]$. Furthermore, $\overline{F}$ is called φ-quasiconvex, if

$$\overline{F}(t\xi_1+(1-t)\xi_2)\le\max\{\overline{F}(\xi_2),\overline{F}(\xi_2)+\varphi(\overline{F}(\xi_1),\overline{F}(\xi_2))\}, \tag{6}$$

for all $\xi_1,\xi_2\in\mathfrak{J}$ and $t\in[0,1]$.

Remark 1. (i) It is easy to see the definition that every φ-convex function is φ-quasiconvex

(ii) If we take $\varphi(\xi_1,\xi_2)=\xi_1-\xi_2$ in Definition 1, then the definitions of φ-convex and φ-quasiconvex are reduced to the definition of convex function and quasiconvex function, respectively

Next, we will give examples for the above definitions.

Example 1. Let $\overline{F}(x)=x^2$, then $\overline{F}$ is convex and φ-convex with $\varphi(\gamma_1,\gamma_2)=2\gamma_1+\gamma_2$; indeed,

$$\begin{aligned}\overline{F}(t\xi_1+(1-t)\xi_2)&=(t\xi_1+(1-t)\xi_2)^2\\&\le\xi_2^2+t\xi_1^2+2t(1-t)\xi_1\xi_2\\&\le\xi_2^2+t\xi_1^2+t(1-t)\left(\xi_1^2+\xi_2^2\right)\\&\le\xi_2^2+t\left(\xi_1^2+\xi_1^2+\xi_2^2\right)\\&=\overline{F}(\xi_2)+t\varphi(\overline{F}(\xi_1),\overline{F}(\xi_2)).\end{aligned} \tag{7}$$

Example 2. Let $\overline{F}(x)=x^3$, then $\overline{F}$ is not convex but is φ-convex with $\varphi(\gamma_1,\gamma_2)=3\gamma_2^2(\gamma_1-\gamma_2)+3\gamma_2(\gamma_1-\gamma_2)^2+(\gamma_1-\gamma_2)^3$; indeed,

$$\begin{aligned}\overline{F}(t\xi_1+(1-t)\xi_2)&=(t\xi_1+(1-t)\xi_2)^3=(\xi_2+t(\xi_1-\xi_2))^3\\&=\xi_2^3+3\xi_2^2t(\xi_1-\xi_2)+3\xi_2t^2(\xi_1-\xi_2)^2+t^3(\xi_1-\xi_2)^3\\&=\overline{F}(\xi_2)+t\left[3\xi_2^2(\xi_1-\xi_2)+3\xi_2t(\xi_1-\xi_2)^2+t^2(\xi_1-\xi_2)^3\right]\\&\le\overline{F}(\xi_2)+t\left[3\xi_2^2(\xi_1-\xi_2)+3\xi_2(\xi_1-\xi_2)^2+(\xi_1-\xi_2)^3\right]\\&=\overline{F}(\xi_2)+t\varphi(\overline{F}(\xi_1),\overline{F}(\xi_2)).\end{aligned} \tag{8}$$

Example 3. Let $\overline{F}\colon[\xi_1,\xi_2]\longrightarrow R, 0<\xi_1<\xi_2$, with $\overline{F}(x)=1/x^2$. We observe that $\overline{F}$ is convex on $[\xi_1,\xi_2]$ and therefore φ-quasiconvex with $\varphi(\gamma_1,\gamma_2)=\gamma_1-\gamma_2$.

Example 4. Let $\overline{F}\colon[\xi_1,\xi_2]\longrightarrow R, 0<\xi_1<\xi_2$, with $\overline{F}(x)=2/x^3$. We observe that $\overline{F}$ is convex on $[\xi_1,\xi_2]$ and therefore φ-quasiconvex with $\varphi(\gamma_1,\gamma_2)=\gamma_1-\gamma_2$.

Example 5. Let $\overline{F}\colon[\xi_1,\xi_2]\longrightarrow R, 0<\xi_1<\xi_2$, with $\overline{F}(x)=2$. We obviously see that $\overline{F}$ is φ-quasiconvex with $\varphi(\gamma_1,\gamma_2)=\gamma_1-\gamma_2$.

The essential object of this study is to establish new Simpson's integral inequalities for the φ-convex and φ-quasiconvex functions in the second derivative sense at certain powers.

2. Simpson's Inequality for φ-Convex

In this section, we give a new refinement of Simpson integral inequality for twice differentiable functions.

Theorem 1. *Let $\overline{F}\colon\mathfrak{J}\longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$ such that $\overline{F}''\in L_1[\xi_1,\xi_2]$, where $\xi_1,\xi_2\in\mathfrak{J}$ with $\xi_1<\xi_2$. If $|\overline{F}''|$ is φ-convex on $[\xi_1,\xi_2]$, then we have*

$$\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right|$$
$$\le\frac{(\xi_2-\xi_1)^2}{81}\left[\left|\overline{F}''(\xi_1)\right|+\frac{1}{2}\varphi\left(\left|\overline{F}''(\xi_1)\right|,\left|\overline{F}''(\xi_2)\right|\right)\right]. \tag{9}$$

Proof. By making the use of Lemma 2 and the φ-convexity of $|\overline{F}''|$, we find that

$$\begin{aligned}
&\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right| \\
&\leq (\xi_2-\xi_1)^2\int_0^1 |k(t)||\overline{F}''(t\xi_2+(1-t)\xi_1)|\mathrm{d}t \\
&\leq (\xi_2-\xi_1)^2\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(|\overline{F}''(\xi_1)|+t\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\right)\mathrm{d}t \\
&\quad+(\xi_2-\xi_1)^2\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left(|\overline{F}''(\xi_1)|+t\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\right)\mathrm{d}t \\
&:= q(\xi_2-\xi_1)^2(\top_1+\top_2),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\top_1 &:= \int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(|\overline{F}''(\xi_1)|+t\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\right)\mathrm{d}t \\
&= \int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right||\overline{F}''(\xi_1)|\mathrm{d}t+\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|t\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\mathrm{d}t \\
&= |\overline{F}''(\xi_1)|\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t+\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|t\mathrm{d}t \\
&= |\overline{F}''(\xi_1)|\int_0^{1/3}\frac{t}{2}\left(\frac{1}{3}-t\right)\mathrm{d}t-|\overline{F}''(\xi_1)|\int_{\frac{1}{3}}^{1/2}\frac{t}{2}\left(\frac{1}{3}-t\right)\mathrm{d}t \\
&\quad+\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\int_0^{1/3}\frac{t}{2}\left(\frac{1}{3}-t\right)t\mathrm{d}t-\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\int_{1/3}^{1/2}\frac{t}{2}\left(\frac{1}{3}-t\right)t\mathrm{d}t \\
&= \frac{1}{162}|\overline{F}''(\xi_1)|+\frac{59}{31104}\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right), \\
\top_2 &:= \int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left(|\overline{F}''(\xi_1)|+t\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\right)\mathrm{d}t \\
&= \int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right||\overline{F}''(\xi_1)|\mathrm{d}t+\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|t\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\mathrm{d}t \\
&= |\overline{F}''(\xi_1)|\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t+\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right)\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|t\mathrm{d}t \\
&= \frac{1}{162}|\overline{F}''(\xi_1)|+\frac{133}{31104}\varphi\left(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|\right).
\end{aligned} \tag{11}$$

A simple rearrangement gives us the proof. □

Corollary 1. *Theorem 1 with $\overline{F}(\xi_1)=\overline{F}((\xi_1+\xi_2)/2)=\overline{F}(\xi_2)$ gives the following new inequality:*

$$\left|\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)dx-\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right| \le \frac{(\xi_2-\xi_1)^2}{81}\left[\left|\overline{F}''(\xi_1)\right|+\frac{1}{2}\varphi\left(\left|\overline{F}''(\xi_1)\right|,\left|\overline{F}''(\xi_2)\right|\right)\right]. \tag{12}$$

Remark 2. Inequality (9) with $\varphi(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|)=|\overline{F}''(\xi_2)|-|\overline{F}''(\xi_1)|$ becomes

$$\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right| \le \frac{(\xi_2-\xi_1)^2}{162}\left[\left|\overline{F}''(\xi_1)\right|+\left|\overline{F}''(\xi_2)\right|\right]. \tag{13}$$

Moreover, inequality (12) with $\varphi(|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_2)|)=|\overline{F}''(\xi_2)|-|\overline{F}''(\xi_1)|$ becomes

$$\left|\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x-\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right| \le \frac{(\xi_2-\xi_1)^2}{162}\left[\left|\overline{F}''(\xi_1)\right|+\left|\overline{F}''(\xi_2)\right|\right]. \tag{14}$$

These are both obtained by Sarikaya et al. [23] in Theorem 2.2 and Corollary 2.3, respectively.

Theorem 2. *Let $\overline{F}\colon \mathfrak{J}\longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$ such that $\overline{F}''\in L_1[\xi_1,\xi_2]$, where $\xi_1,\xi_2\in\mathfrak{J}$ with $\xi_1<\xi_2$. If $|\overline{F}''|^q$ is φ-convex on $[\xi_1,\xi_2]$ and $q\ge 1$, then we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right|\\
&\le(\xi_2-\xi_1)^2\left(\frac{1}{162}\right)^{1-(1/q)}\left\{\left(\frac{1}{162}\left|\overline{F}''(\xi_1)\right|^q\right.\right.\\
&\quad\left.+\frac{59}{31104}\varphi\left(\left|\overline{F}''(\xi_1)\right|^q,\left|\overline{F}''(\xi_2)\right|^q\right)\right)^{1/q}\\
&\quad\left.+\left(\frac{1}{162}\left|\overline{F}''(\xi_1)\right|^q+\frac{133}{31104}\varphi\left(\left|\overline{F}''(\xi_1)\right|^q,\left|\overline{F}''(\xi_2)\right|^q\right)\right)^{\frac{1}{q}}\right\},\end{aligned} \tag{15}$$

where $1/p+1/q=1$.

Proof. Let $q\ge1$, then by using Lemma 2, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right|\\
&\le(\xi_2-\xi_1)^2\int_0^1|k(t)|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|\mathrm{d}t\\
&=(\xi_2-\xi_1)^2\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|\mathrm{d}t\\
&\quad+(\xi_2-\xi_1)^2\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|\mathrm{d}t.\end{aligned} \tag{16}$$

By making the use of the Hölder's inequality for the above integrals, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right|\\
&\le(\xi_2-\xi_1)^2\left(\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t\right)^{1-1/q}\left(\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|^q\mathrm{d}t\right)^{1/q}\\
&\quad+(\xi_2-\xi_1)^2\left(\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t\right)^{1-1/q}\left(\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|^q\mathrm{d}t\right)^{1/q}.\end{aligned} \tag{17}$$

By φ-convexity of $|\overline{F}''|^q$ for the last two integrals, we have

$$\begin{aligned}
&\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|\overline{F}''\left(t\xi_2+(1-t)\xi_1\right)\right|^q\mathrm{d}t\\
&\quad\le\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left[\left|\overline{F}''\left(\xi_1\right)\right|^q+t\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right)\right]\mathrm{d}t\\
&\quad=\left|\overline{F}''\left(\xi_1\right)\right|^q\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t+\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right)\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|t\mathrm{d}t\\
&\quad=\frac{1}{162}\left|\overline{F}''\left(\xi_1\right)\right|^q+\frac{59}{31104}\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right),
\end{aligned}\tag{18}$$

$$\begin{aligned}
&\int_{1/2}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|\overline{F}''\left(t\xi_2+(1-t)\xi_1\right)\right|^q\mathrm{d}t\\
&\quad\le\int_{1/2}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left[\left|\overline{F}''\left(\xi_1\right)\right|^q+t\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right)\right]\mathrm{d}t\\
&\quad=\left|\overline{F}''\left(\xi_1\right)\right|^q\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t+\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right)\int_{1/2}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|t\mathrm{d}t\\
&\quad=\frac{1}{162}\left|\overline{F}''\left(\xi_1\right)\right|^q+\frac{133}{31104}\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right).
\end{aligned}\tag{19}$$

By substituting (18) and (19) into (17), we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\overline{F}\left(\xi_1\right)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}\left(\xi_2\right)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right|\\
&\le\left(\xi_2-\xi_1\right)^2\left(\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t\right)^{1-1/q}\left(\frac{1}{162}\left|\overline{F}''\left(\xi_1\right)\right|^q+\frac{59}{31104}\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right)\right)^{1/q}\\
&\quad+\left(\xi_2-\xi_1\right)^2\left(\int_{1/2}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t\right)^{1-1/q}\left(\frac{1}{162}\left|\overline{F}''\left(\xi_1\right)\right|^q+\frac{133}{31104}\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right)\right)^{1/q}\\
&=\left(\xi_2-\xi_1\right)^2\left(\frac{1}{162}\right)^{1-1/q}\left[\left(\frac{1}{162}\left|\overline{F}''\left(\xi_1\right)\right|^q+\frac{59}{31104}\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right)\right)^{1/q}\right.\\
&\quad\left.+\left(\frac{1}{162}\left|\overline{F}''\left(\xi_1\right)\right|^q+\frac{133}{31104}\varphi\left(\left|\overline{F}''\left(\xi_1\right)\right|^q,\left|\overline{F}''\left(\xi_2\right)\right|^q\right)\right)^{1/q}\right],
\end{aligned}\tag{20}$$

where we used the identity

$$\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t=\int_{1/2}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t=\frac{1}{162}.\tag{21}$$

Thus, we are done. □

Corollary 2. *Theorem 2 with* $\overline{F}\left(\xi_1\right)=\overline{F}\left(\left(\xi_1+\xi_2\right)/2\right)=\overline{F}\left(\xi_2\right)$ *gives the following new inequality:*

$$\left|\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x-\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right|$$
$$\le(\xi_2-\xi_1)^2\left(\frac{1}{162}\right)^{1-1/q}\left[\left(\frac{1}{162}|\overline{F}''(\xi_1)|^q+\frac{59}{31104}\varphi\left(|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_2)|^q\right)\right)^{1/q}+\left(\frac{1}{162}|\overline{F}''(\xi_1)|^q+\frac{133}{31104}\varphi\left(|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_2)|^q\right)\right)^{1/q}\right]. \tag{22}$$

Remark 3. Inequality (15) with $\varphi(|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_2)|^q)=|\overline{F}''(\xi_2)|^q-|\overline{F}''(\xi_1)|^q$ becomes

$$\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right|$$
$$\le(\xi_2-\xi_1)^2\left(\frac{1}{162}\right)^{1-1/q}\left\{\left(\frac{133}{31104}|\overline{F}''(\xi_1)|^q+\frac{59}{31104}|\overline{F}''(\xi_2)|^q\right)^{1/q}+\left(\frac{59}{31104}|\overline{F}''(\xi_1)|^q+\frac{133}{31104}|\overline{F}''(\xi_2)|^q\right)^{1/q}\right\}. \tag{23}$$

Moreover, inequality (22) with $\varphi(|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_2)|^q)=|\overline{F}''(\xi_2)|^q-|\overline{F}''(\xi_1)|^q$ becomes

$$\left|\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x-\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right|$$
$$\le(\xi_2-\xi_1)^2\left(\frac{1}{162}\right)^{1-1/q}\left\{\left(\frac{133}{31104}|\overline{F}''(\xi_1)|^q+\frac{59}{31104}|\overline{F}''(\xi_2)|^q\right)^{1/q}+\left(\frac{59}{31104}|\overline{F}''(\xi_1)|^q+\frac{133}{31104}|\overline{F}''(\xi_2)|^q\right)^{1/q}\right\}. \tag{24}$$

These are both obtained by Sarikaya et al. [23] in Theorem 2.5 and Corollary 2.6, respectively.

Remark 4. Theorem 2 and Corollary 2 with $q=1$ become Theorem 1 and Corollary 1, respectively.

3. Simpson's Inequality for φ-Quasiconvex

Theorem 3. *Let $\overline{F}\colon \mathfrak{J}\longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$ provided $\overline{F}''\in L_1[\xi_1,\xi_2]$, where $\xi_1,\xi_2\in\mathfrak{J}$ with $\xi_1<\xi_2$. If $|\overline{F}''|$ is φ-quasiconvex on $[\xi_1,\xi_2]$, then we have*

$$\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right|$$
$$\le\frac{(\xi_2-\xi_1)^2}{81}\max\left\{|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_1)|+\varphi\left(|\overline{F}''(\xi_2)|,|\overline{F}''(\xi_1)|\right)\right\}. \tag{25}$$

Proof. By making use of φ-quasiconvexity of $|\overline{F}''|$ and Lemma 2, we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right|\\
&\le(\xi_2-\xi_1)^2\int_0^1|k(t)|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|\mathrm{d}t\\
&\le(\xi_2-\xi_1)^2\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\max\left\{|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_1)|+\varphi\left(|\overline{F}''(\xi_2)|,|\overline{F}''(\xi_1)|\right)\right\}\mathrm{d}t\\
&\quad+(\xi_2-\xi_1)^2\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\max\left\{|\overline{F}''(\xi_1)|,|\overline{F}''(\xi_1)|+\varphi\left(|\overline{F}''(\xi_2)|,|\overline{F}''(\xi_1)|\right)\right\}\mathrm{d}t\\
&=(\xi_2-\xi_1)^2\left(\overline{\mathsf{T}_1}+\overline{\mathsf{T}_2}\right),
\end{aligned} \tag{26}$$

where

$$
\begin{aligned}
\overline{\mathrm{T}_1} &= \int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\max\left\{|\overline{F}''(\xi_1)|, |\overline{F}''(\xi_1)| + \varphi\left(|\overline{F}''(\xi_2)|, |\overline{F}''(\xi_1)|\right)\right\}\mathrm{d}t \\
&= \max\left\{|\overline{F}''(\xi_1)|, |\overline{F}''(\xi_1)| + \varphi\left(|\overline{F}''(\xi_2)|, |\overline{F}''(\xi_1)|\right)\right\}\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t \\
&= \max\left\{|\overline{F}''(\xi_1)|, |\overline{F}''(\xi_1)| + \varphi\left(|\overline{F}''(\xi_2)|, |\overline{F}''(\xi_1)|\right)\right\}\left[\int_0^{1/3}\frac{t}{2}\left(\frac{1}{3}-t\right)\mathrm{d}t - \int_{1/3}^{1/2}\frac{t}{2}\left(\frac{1}{3}-t\right)\mathrm{d}t\right] \\
&= \frac{1}{162}\max\left\{|\overline{F}''(\xi_1)|, |\overline{F}''(\xi_1)| + \varphi\left(|\overline{F}''(\xi_2)|, |\overline{F}''(\xi_1)|\right)\right\}, \\
\overline{\mathrm{T}_2} &= \int_{1/2}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\max\left\{|\overline{F}''(\xi_1)|, |\overline{F}''(\xi_1)| + \varphi\left(|\overline{F}''(\xi_2)|, |\overline{F}''(\xi_1)|\right)\right\}\mathrm{d}t \\
&= \max\left\{|\overline{F}''(\xi_1)|, |\overline{F}''(\xi_1)| + \varphi\left(|\overline{F}''(\xi_2)|, |\overline{F}''(\xi_1)|\right)\right\}\int_{1/2}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t \\
&= \frac{1}{162}\max\left\{|\overline{F}''(\xi_1)|, |\overline{F}''(\xi_1)| + \varphi\left(|\overline{F}''(\xi_2)|, |\overline{F}''(\xi_1)|\right)\right\}.
\end{aligned}
\tag{27}
$$

A simple rearrangement completes the proof. □

Corollary 3. *Theorem 3 with $\overline{F}(\xi_1) = \overline{F}((\xi_1+\xi_2)/2) = \overline{F}(\xi_2)$ becomes*

$$
\left|\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x - \overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right| \le \frac{(\xi_2-\xi_1)^2}{81}\max\left\{|\overline{F}''(\xi_1)|, |\overline{F}''(\xi_1)| + \varphi\left(|\overline{F}''(\xi_2)|, |\overline{F}''(\xi_1)|\right)\right\}. \tag{28}
$$

Theorem 4. *Let $\overline{F}\colon \mathfrak{J} \longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$ provided $\overline{F}'' \in L_1[\xi_1,\xi_2]$, where $\xi_1, \xi_2 \in \mathfrak{J}$ with $\xi_1 < \xi_2$. If $|\overline{F}''|^q$ is φ-quasiconvex on $[\xi_1,\xi_2]$ and $q \ge 1$, then we have*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\overline{F}(\xi_1) + 4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right) + \overline{F}(\xi_2)\right] - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right| \\
&\quad\le \frac{(\xi_2-\xi_1)^2}{81}\left(\max\left\{|\overline{F}''(\xi_1)|^q, |\overline{F}''(\xi_1)|^q + \varphi\left(|\overline{F}''(\xi_2)|^q, |\overline{F}''(\xi_1)|^q\right)\right\}\right)^{1/q},
\end{aligned}
\tag{29}
$$

where $1/p + 1/q = 1$.

Proof. Let $q \ge 1$, then by using Lemma 2, we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\overline{F}(\xi_1) + 4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right) + \overline{F}(\xi_2)\right] - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right| \\
&\quad\le (\xi_2-\xi_1)^2\int_0^1|k(t)|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|\mathrm{d}t \\
&\quad= (\xi_2-\xi_1)^2\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|\mathrm{d}t \\
&\qquad+ (\xi_2-\xi_1)^2\int_{1/2}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|\mathrm{d}t.
\end{aligned}
\tag{30}
$$

By making the use of the Hölder's inequality for the above integrals, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right| \\
&\quad\leq(\xi_2-\xi_1)^2\left(\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t\right)^{1-(1/q)}\left(\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|^q\mathrm{d}t\right)^{1/q} \\
&\qquad+(\xi_2-\xi_1)^2\left(\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t\right)^{1-(1/q)}\left(\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|^q\mathrm{d}t\right)^{1/q}.
\end{aligned} \tag{31}$$

By φ-quasiconvexity of $|\overline{F}''|^q$ for the last two integrals, we have

$$\begin{aligned}
&\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|^q\mathrm{d}t \\
&\quad\leq\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}\mathrm{d}t \\
&\quad=\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t \\
&\quad=\frac{1}{162}\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|\overline{F}''(t\xi_2+(1-t)\xi_1)\right|^q\mathrm{d}t \\
&\quad\leq\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}\mathrm{d}t \\
&\quad=\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t \\
&\quad=\frac{1}{162}\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}.
\end{aligned} \tag{33}$$

By substituting (32) and (33) into (31), we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\overline{F}(\xi_1)+4\overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)+\overline{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x\right| \\
&\quad\leq(\xi_2-\xi_1)^2\left(\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t\right)^{1-(1/q)}\left(\frac{1}{162}\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}\right)^{1/q} \\
&\qquad+(\xi_2-\xi_1)^2\left(\int_{1/2}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t\right)^{1-(1/q)}\left(\frac{1}{162}\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}\right)^{1/q} \\
&\quad=2(\xi_2-\xi_1)^2\left(\frac{1}{162}\right)^{1-(1/q)}\left(\frac{1}{162}\right)^{1/q}\left(\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}\right)^{1/q} \\
&\quad=\frac{(\xi_2-\xi_1)^2}{81}\left(\max\left\{|\overline{F}''(\xi_1)|^q,|\overline{F}''(\xi_1)|^q+\varphi\left(|\overline{F}''(\xi_2)|^q,|\overline{F}''(\xi_1)|^q\right)\right\}\right)^{1/q}.
\end{aligned} \tag{34}$$

where we used the following identity

$$\int_0^{1/2}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\mathrm{d}t = \int_{1/2}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\mathrm{d}t = \frac{1}{162}. \quad (35)$$

Thus we are done. □

Corollary 4. *Theorem 4 with $\overline{F}(\xi_1) = \overline{F}((\xi_1+\xi_2)/2) = \overline{F}(\xi_2)$ becomes*

$$\begin{aligned}&\left|\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x - \overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right| \\ &\quad\le \frac{(\xi_2-\xi_1)^2}{81}\left(\max\left\{\left|\overline{F}''(\xi_1)\right|^q, \left|\overline{F}''(\xi_1)\right|^q + \varphi\left(\left|\overline{F}''(\xi_2)\right|^q, \left|\overline{F}''(\xi_1)\right|^q\right)\right\}\right)^{1/q}.\end{aligned} \quad (36)$$

Remark 5. Theorem 4 and Corollary 4 with $q = 1$ become Theorem 3 and Corollary 3, respectively.

Corollary 5. *Theorem 4 with $\overline{F}(\xi_1) = \overline{F}((\xi_1+\xi_2)/2) = \overline{F}(\xi_2)$ becomes*

$$\begin{aligned}&\left|\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\overline{F}(x)\mathrm{d}x - \overline{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right| \\ &\quad\le \frac{(\xi_2-\xi_1)^2}{81}\left(\max\left\{\left|\overline{F}''(\xi_1)\right|^2, \left|\overline{F}''(\xi_1)\right|^2 + \varphi\left(\left|\overline{F}''(\xi_2)\right|^2, \left|\overline{F}''(\xi_1)\right|^2\right)\right\}\right)^{1/2}.\end{aligned} \quad (37)$$

4. Applications

Some applications for our findings are presented.

4.1. Applications to Special Means. The special means are itemized as follows:

(i) The arithmetic mean:

$$\mathscr{A} = \mathscr{A}(\xi_1,\xi_2) = \frac{\xi_1+\xi_2}{2}, \quad \xi_1,\xi_2 \ge 0. \quad (38)$$

(ii) The harmonic mean:

$$\mathscr{H} = \mathscr{H}(\xi_1,\xi_2) = \frac{2\xi_1\xi_2}{\xi_1+\xi_2}, \quad \xi_1,\xi_2 > 0. \quad (39)$$

(iii) The logarithmic mean:

$$\mathscr{L} = \mathscr{L}(\xi_1,\xi_2) = \begin{cases}\dfrac{\xi_2-\xi_1}{\ln\xi_2-\ln\xi_1}; & \text{if } \xi_1 \ne \xi_2, \\ \xi_1; & \text{if } \xi_1 = \xi_2,\end{cases} \quad (40)$$

for $\xi_1,\xi_2 > 0$.

(iv) The p-logarithmic mean:

$$\mathscr{L}_p = \mathscr{L}_p(\xi_1,\xi_2) = \begin{cases}\left[\dfrac{\xi_2^{p+1}-\xi_1^{p+1}}{(p+1)(\xi_2-\xi_1)}\right]^{1/p}; & \text{if } \xi_1 \ne \xi_2, \\ \xi_1; & \text{if } \xi_1 = \xi_2,\end{cases} \quad (41)$$

for $p \in R\setminus\{-1,0\}$; $\xi_1,\xi_2 > 0$.

We know that $\mathscr{L}_p$ is a monotonic nondecreasing function over $p \in R$ with $\mathscr{L}_{-1} = \mathscr{L}$. In particular, we can say that $\mathscr{H} \le \mathscr{L} \le \mathscr{A}$.

Now, using our findings in Section 2, we conclude the following new inequalities.

Proposition 1. *Let $\xi_1,\xi_2 \in R$ with $0 < \xi_1 < \xi_2$. Then, we have*

$$\left|\frac{1}{3}\mathscr{A}\left(\xi_1^4,\xi_2^4\right) + \frac{2}{3}\mathscr{A}^4(\xi_1,\xi_2) - \mathscr{L}_5^5(\xi_1,\xi_2)\right| \le \frac{(\xi_2-\xi_1)^2}{27}\left[8\xi_1^2+\xi_2^2\right]. \quad (42)$$

Proof. The assertion follows from Theorem 1 with $\overline{F}(x) = x^4/12, x \in [\xi_1,\xi_2]$ and a simple computation, where $|\overline{F}''|$ is φ-convex function with $\varphi(x,y) = 2x+y$ (see Example 1). □

Proposition 2. *Let $\xi_1,\xi_2 \in R$, $0 < \xi_1 < \xi_2$. Then, we have*

$$\begin{aligned}&\left|\frac{1}{3}\mathscr{A}\left(\xi_1^5,\xi_2^5\right) + \frac{2}{3}\mathscr{A}^5(\xi_1,\xi_2) - \mathscr{L}_6^6(\xi_1,\xi_2)\right| \\ &\quad\le \frac{10(\xi_2-\xi_1)^2}{81}\left[2\xi_1^3+\xi_1^9-\xi_2^9\right].\end{aligned} \quad (43)$$

Proof. The assertion follows from Theorem 1 and a simple computation applied to $\overline{F}(x) = x^5/20, x \in [\xi_1,\xi_2]$, where $|\overline{F}''|$ is φ-convex function with $\varphi(x,y) = 3y^2(x-y) + 3y(x-y)^2 + (x-y)^3$ (see Example 2).

The following proposition is a particular case of Corollary 11 in [34] when $\lambda = 1/3$ (see Remark 12 in [34]). □

Proposition 3. *Let $\xi_1, \xi_2 \in R$, $0 < \xi_1 < \xi_2$. Then, we have*

$$\left|\frac{1}{3}\mathscr{A}\left(\xi_1^2, \xi_2^2\right) + \frac{2}{3}\mathscr{A}^2(\xi_1, \xi_2) - \mathscr{L}_3^3(\xi_1, \xi_2)\right| \leq \frac{2(\xi_2 - \xi_1)^2}{81}. \tag{44}$$

Proof. The assertion follows from Theorem 3 and a simple computation applied to $\overline{F}(x) = x^2, x \in [\xi_1, \xi_2]$, where $|\overline{F}''(x)| = 2$ is φ-quasiconvex function with $\varphi(x, y) = x - y$ (see Example 5). □

Proposition 4. *Let $\xi_1, \xi_2 \in R$, $0 < \xi_1 < \xi_2$. Then, for all $q > 1$, we have*

$$\left|\frac{1}{3}\mathscr{H}^{-1}(\xi_1, \xi_2) + \frac{2}{3}\mathscr{A}^{-1}(\xi_1, \xi_2) - \mathscr{L}^{-1}(\xi_1, \xi_2)\right| \leq \frac{(\xi_2 - \xi_1)^2}{81} \max\left\{\frac{2^q}{\xi_1^{3q}}, \frac{2^q}{\xi_2^{3q}}\right\}. \tag{45}$$

Proof. The assertion follows from Theorem 4 and a simple computation applied to $\overline{F}(x) = 1/x, x \in [\xi_1, \xi_2]$, where $|\overline{F}''(x)| = |2/x^3|$ is φ-quasiconvex function with $\varphi(x, y) = x - y$ (see Example 4). □

4.2. Applications to Simpson's Formula. Let $\mathscr{P}$ be a partition of the interval $[\xi_1, \xi_2]$; that is $\mathscr{P}$: $\xi_1 = s_0 < s_1 < \cdots < s_{n-1} < s_n = \xi_2$; $h_i = (s_{i+1} - s_i)/2$ and consider Simpson's formula:

$$S(\overline{F}, \mathscr{P}) = \sum_{i=1}^{n-1} \frac{\overline{F}(s_i) + 4\overline{F}(s_i + h_i) + \overline{F}(s_{i+1})}{6}(s_{i+1} - s_i). \tag{46}$$

We know that if $\overline{F}$: $[\xi_1, \xi_2] \longrightarrow R$ is differentiable such that $\overline{F}^{(q)}(x)$ exists on (ξ_1, ξ_2) and $K = \max_{x\in[\xi_1,\xi_2]}|\overline{F}^{(q)}(x)| < \infty$. Then, we have

$$I = \int_{\xi_1}^{\xi_2} \overline{F}(s)\mathrm{d}s = S(\overline{F}, \mathscr{P}) + E_s(\overline{F}, \mathscr{P}), \tag{47}$$

where the approximation error $E_s(\overline{F}, \mathbf{L})$ satisfies

$$\left|E_s(\overline{F}, \mathbf{L})\right| \leq \frac{K}{90} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^5. \tag{48}$$

It is clear that if the function $\overline{F}$ is not four times differentiable or $\overline{F}^{(4)}$ is not bounded on (ξ_1, ξ_2), then (47) cannot be applied.

Theorem 5. *Let $\overline{F}$: $\mathfrak{J} \subseteq [0, \infty) \longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$ such that $\overline{F}'' \in L_1[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in \mathfrak{J}$ with $\xi_1 < \xi_2$. If $|\overline{F}''|$ is φ-convex on $[\xi_1, \xi_2]$, then for every division $\mathscr{P}$ of $[\xi_1, \xi_2]$ we have*

$$\left|E_s(\overline{F}, \mathscr{P})\right| \leq \frac{1}{81} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[\overline{F}''(s_i) + \frac{1}{2}\varphi\left(\overline{F}''(s_i), \overline{F}''(s_{i+1})\right)\right]. \tag{49}$$

Proof. By applying Theorem 1 on the subintervals $[s_i, s_{i+1}]$, $(i = 0, 1, 2, \ldots, n-1)$ of the division $\mathscr{P}$ to get

$$\left|\frac{(s_{i+1} - s_i)}{6}\left[\overline{F}(s_i) + 4\overline{F}\left(\frac{s_{i+1} - s_i}{2}\right) + \overline{F}(s_{i+1})\right] - \int_{s_i}^{s_{i+1}} \overline{F}(s)\mathrm{d}s\right| \leq \frac{(s_{i+1} - s_i)^3}{6}\left\{\overline{F}''(s_i) + \frac{1}{2}\varphi\left(\overline{F}''(s_i), \overline{F}''(s_{i+1})\right)\right\}. \tag{50}$$

By summing over i from 0 to $n - 1$ and taking into account that $|\overline{F}''|$ is φ-convex to get

$$\left|S(\overline{F}, \mathscr{P}) - \int_{\xi_1}^{\xi_2} \overline{F}(s)\mathrm{d}s\right| \leq \sum_{i=0}^{n-1} \frac{(s_{i+1} - s_i)^3}{81}\left[\overline{F}''(s_i) + \frac{1}{2}\varphi\left(\overline{F}''(s_i), \overline{F}''(s_{i+1})\right)\right], \tag{51}$$

which completes our proof. □

Corollary 6. *Theorem 5 with $\varphi(x, y) = y - x$ becomes*

$$\left|E_s(\overline{F}, \mathscr{P})\right| \leq \frac{1}{162} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[\overline{F}''(s_i) + \overline{F}''(s_{i+1})\right]. \tag{52}$$

Theorem 6. *Let $\overline{F}$: $\mathfrak{J} \subseteq [0, \infty) \longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$ such that $\overline{F}'' \in L_1[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in \mathfrak{J}$ with $\xi_1 < \xi_2$. If $|\overline{F}''|$ is φ-quasiconvex on $[\xi_1, \xi_2]$, then for every division $\mathscr{P}$ of $[\xi_1, \xi_2]$ we have*

$$\left|S(\overline{F}, \mathscr{P}) - \int_{\xi_1}^{\xi_2} \overline{F}(s)\mathrm{d}s\right| \leq \frac{1}{81} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \max\left\{\left|\overline{F}''(s_i)\right|, \left|\overline{F}''(s_i)\right| + \varphi\left(\left|\overline{F}''(s_i)\right|, \left|\overline{F}''(s_{i+1})\right|\right)\right\}. \tag{53}$$

Proof. By applying Theorem 3 and by the same method used for proof of the previous theorem, we can produce the desired result. □

Proposition 5. *Let $\overline{F}$: $\mathfrak{J} \subseteq [0, \infty) \longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$ such that $\overline{F}'' \in L_1[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in \mathfrak{J}$ with $\xi_1 < \xi_2$. If $|\overline{F}''|^q q \geq 1$ is φ-convex on $[\xi_1, \xi_2]$, then we have*

$$\left|E_s(\overline{F}, \mathscr{P})\right| \leq \left(\frac{1}{162}\right)^{1-(1/q)} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[K_\varphi^q\left(\overline{F}''(s_i), \overline{F}''(s_{i+1})\right)\right], \tag{54}$$

where

$$K_{\varphi}^{q}\left(\overline{F}''(s_i),\overline{F}''(s_{i+1})\right)=\left(\frac{1}{162}\left|\overline{F}''(s_i)\right|^{q}+\frac{59}{31104}\varphi\left(\left|\overline{F}''(s_i)\right|^{q},\left|\overline{F}''(s_{i+1})\right|^{q}\right)\right)^{1/q}+\left(\frac{1}{162}\left|\overline{F}''(s_i)\right|^{q}+\frac{133}{31104}\varphi\left(\left|\overline{F}''(s_i)\right|^{q},\left|\overline{F}''(s_{i+1})\right|^{q}\right)\right)^{1/q}. \tag{55}$$

Proof. The proof follows from Theorem 2 directly. □

Proposition 6. *Let $\overline{F}$: $\mathfrak{J}\subseteq[0,\infty)\longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$ such that $\overline{F}''\in L_1[\xi_1,\xi_2]$, where $\xi_1,\xi_2\in\mathfrak{J}$ with $\xi_1<\xi_2$. If $|\overline{F}''|^q q\geq 1$ is φ-quasiconvex on $[\xi_1,\xi_2]$, then we have*

$$\left|E_s(\overline{F},\mathscr{P})\right|\leq\frac{1}{81}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\max\left\{\left|\overline{F}''(s_i)\right|^{q},\left|\overline{F}''(s_i)\right|^{q}+\varphi\left(\left|\overline{F}''(s_{i+1})\right|^{q},\left|\overline{F}''(s_i)\right|^{q}\right)\right\}. \tag{56}$$

Proof. The proof follows from Theorem 4 directly. □

4.3. Applications to the Midpoint Formula. Let $\mathscr{P}$ be a partition as before. Here we consider the midpoint formula:

$$M(\overline{F},\mathscr{P})=\sum_{i=0}^{n-1}(s_{i+1}-s_i)\overline{F}\left(\frac{s_i+s_{i+1}}{2}\right). \tag{57}$$

Suppose that the function $\overline{F}$: $[\xi_1,\xi_2]\longrightarrow R$ is differentiable with $\overline{F}''(x)$ existing on (ξ_1,ξ_2) and $K=\sup_{x\in(\xi_1,\xi_2)}|\overline{F}''(x)|<\infty$, and then, we have

$$I=\int_{\xi_1}^{\xi_2}\overline{F}(s)\mathrm{d}s=M(\overline{F},\mathscr{P})+E_M(\overline{F},\mathscr{P}), \tag{58}$$

where the approximation error $E_M(\overline{F},\mathscr{P})$ satisfies

$$\left|E_M(\overline{F},\mathscr{P})\right|\leq\frac{K}{24}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3. \tag{59}$$

Proposition 7. *Let $\overline{F}$: $\mathfrak{J}\longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$, $\xi_1,\xi_2\in\mathfrak{J}$ with $\xi_1<\xi_2$. If $|\overline{F}''|$ is φ-convex on $[\xi_1,\xi_2]$, then for any division $\mathscr{P}$ of $[\xi_1,\xi_2]$, we have*

$$\left|E_M(\overline{F},\mathscr{P})\right|\leq\frac{1}{81}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\left[\left|\overline{F}''(s_i)\right|+\frac{1}{2}\varphi\left(\left|\overline{F}''(s_i)\right|,\left|\overline{F}''(s_{i+1})\right|\right)\right]. \tag{60}$$

Proof. By applying Corollary 1 on the subintervals $[s_i,s_{i+1}]$, $(i=0,1,\ldots,n-1)$ of the division $\mathscr{P}$, to get

$$\left|(s_{i+1}-s_i)\overline{F}\left(\frac{s_{i+1}+s_i}{2}\right)-\int_{s_i}^{s_{i+1}}\overline{F}(s)\mathrm{d}s\right|\leq\frac{(s_{i+1}-s_i)^3}{81}\left[\left|\overline{F}''(s_i)\right|+\frac{1}{2}\varphi\left(\left|\overline{F}''(s_i)\right|,\left|\overline{F}''(s_{i+1})\right|\right)\right]. \tag{61}$$

By summing over i from 0 to $n-1$ to get

$$\left|E_M(\overline{F},\mathscr{P})\right|\leq\frac{1}{81}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\left[\left|\overline{F}''(s_i)\right|+\frac{1}{2}\varphi\left(\left|\overline{F}''(s_i)\right|,\left|\overline{F}''(s_{i+1})\right|\right)\right], \tag{62}$$

which completes our proof. □

Proposition 8. *Let $\overline{F}$: $\mathfrak{J}\longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$, $\xi_1,\xi_2\in\mathfrak{J}$ with $\xi_1<\xi_2$. If $|\overline{F}''|^q$ is φ-convex on $[\xi_1,\xi_2]$ and $q\geq 1$, then for any division $\mathscr{P}$ of $[\xi_1,\xi_2]$, we have*

$$\left|E_M(\overline{F},\mathscr{P})\right|\leq\left(\frac{1}{162}\right)^{1-(1/q)}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\left[K_{\varphi}^{q}\left(\overline{F}''(s_i),\overline{F}''(s_{i+1})\right)\right], \tag{63}$$

where

$$K_{\varphi}^{q}\left(\overline{F}''(s_i),\overline{F}''(s_{i+1})\right)=\left(\frac{1}{162}\left|\overline{F}''(s_i)\right|^{q}+\frac{59}{31104}\varphi\left(\left|\overline{F}''(s_i)\right|^{q},\left|\overline{F}''(s_{i+1})\right|^{q}\right)\right)^{1/q}+\left(\frac{1}{162}\left|\overline{F}''(s_i)\right|^{q}+\frac{133}{31104}\varphi\left(\left|\overline{F}''(s_i)\right|^{q},\left|\overline{F}''(s_{i+1})\right|^{q}\right)\right)^{1/q}. \tag{64}$$

Proof. By applying Corollary 2 on the subintervals $[s_i, s_{i+1}]$, $(i = 0, 1, \ldots, n-1)$ of the division $\mathscr{P}$ to get

$$\begin{aligned}&\left|(s_{i+1}-s_i)\overline{F}\left(\frac{s_{i+1}+s_i}{2}\right)-\int_{s_i}^{s_{i+1}}\overline{F}(s)\mathrm{d}s\right|\\&\quad\le\left(\frac{1}{162}\right)^{1-(1/q)}(s_{i+1}-s_i)^3\left[K_\varphi^q\left(\overline{F}''(s_i),\overline{F}''(s_{i+1})\right)\right],\end{aligned}\tag{65}$$

where

$$\begin{aligned}&K_\varphi^q\left(\overline{F}''(s_i),\overline{F}''(s_{i+1})\right)\\&\quad=\left(\frac{1}{162}\left|\overline{F}''(s_i)\right|^q+\frac{59}{31104}\varphi\left(\left|\overline{F}''(s_i)\right|^q,\left|\overline{F}''(s_{i+1})\right|^q\right)\right)^{1/q}\\&\qquad+\left(\frac{1}{162}\left|\overline{F}''(s_i)\right|^q+\frac{133}{31104}\varphi\left(\left|\overline{F}''(s_i)\right|^q,\left|\overline{F}''(s_{i+1})\right|^q\right)\right)^{1/q}.\end{aligned}\tag{66}$$

By summing over i from 0 to $n-1$ to get

$$\begin{aligned}&\left|E_M(\overline{F},\mathscr{P})\right|\\&\quad\le\left(\frac{1}{162}\right)^{1-(1/q)}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\left[K_\varphi^q\left(\overline{F}''(s_i),\overline{F}''(s_{i+1})\right)\right],\end{aligned}\tag{67}$$

which completes our proof. □

Proposition 9. *Let $\overline{F}\colon \mathfrak{J}\longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$, $\xi_1, \xi_2 \in \mathfrak{J}$ with $\xi_1 < \xi_2$. If $|\overline{F}''|$ is φ-quasiconvex on $[\xi_1, \xi_2]$, then for any division $\mathscr{P}$ of $[\xi_1, \xi_2]$, we have*

$$\left|E_M(\overline{F},\mathscr{P})\right|\le\frac{1}{81}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\max\left\{\left|\overline{F}''(s_i)\right|,\left|\overline{F}''(s_i)\right|+\varphi\left(\left|\overline{F}''(s_i)\right|,\left|\overline{F}''(s_{i+1})\right|\right)\right\}.\tag{68}$$

Proof. By applying Corollary 3 on the subintervals $[s_i, s_{i+1}]$, $(i = 0, 1, \ldots, n-1)$ of the division $\mathscr{P}$ to get

$$\begin{aligned}&\left|(s_{i+1}-s_i)\overline{F}\left(\frac{s_{i+1}+s_i}{2}\right)-\int_{s_i}^{s_{i+1}}\overline{F}(s)\mathrm{d}s\right|\\&\quad\le\frac{(s_{i+1}-s_i)^3}{81}\max\left\{\left|\overline{F}''(s_i)\right|,\left|\overline{F}''(s_i)\right|+\varphi\left(\left|\overline{F}''(s_i)\right|,\left|\overline{F}''(s_{i+1})\right|\right)\right\}.\end{aligned}\tag{69}$$

By summing over i from 0 to $n-1$ to get

$$\left|E_M(\overline{F},\mathscr{P})\right|\le\frac{1}{81}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\max\left\{\left|\overline{F}''(s_i)\right|,\left|\overline{F}''(s_i)\right|+\varphi\left(\left|\overline{F}''(s_i)\right|,\left|\overline{F}''(s_{i+1})\right|\right)\right\},\tag{70}$$

which completes our proof. □

Proposition 10. *Let $\overline{F}\colon \mathfrak{J}\longrightarrow R$ be a twice differentiable function on $\mathfrak{J}$, $\xi_1, \xi_2 \in \mathfrak{J}$ with $\xi_1 < \xi_2$. If $|\overline{F}''|^q$ is φ-quasiconvex on $[\xi_1, \xi_2]$ and $q \ge 1$, then in (30), for every division $\mathscr{P}$ of $[\xi_1, \xi_2]$, we have*

$$\left|E_M(\overline{F},\mathscr{P})\right|\le\frac{1}{81}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\max\left\{\left|\overline{F}''(s_i)\right|^q,\left|\overline{F}''(s_i)\right|^q+\varphi\left(\left|\overline{F}''(s_{i+1})\right|^q,\left|\overline{F}''(s_i)\right|^q\right)\right\}.\tag{71}$$

Proof. By applying Corollary 4 on the subintervals $[s_i, s_{i+1}]$, $(i = 0, 1, \ldots, n-1)$ of the division $\mathscr{P}$, we get

$$\begin{aligned}&\left|(s_{i+1}-s_i)\overline{F}\left(\frac{s_{i+1}+s_i}{2}\right)-\int_{s_i}^{s_{i+1}}\overline{F}(s)\mathrm{d}s\right|\\&\quad\le\frac{(s_{i+1}-s_i)^3}{81}\max\left\{\left|\overline{F}''(s_i)\right|^q,\left|\overline{F}''(s_i)\right|^q+\varphi\left(\left|\overline{F}''(s_{i+1})\right|^q,\left|\overline{F}''(s_i)\right|^q\right)\right\}.\end{aligned}\tag{72}$$

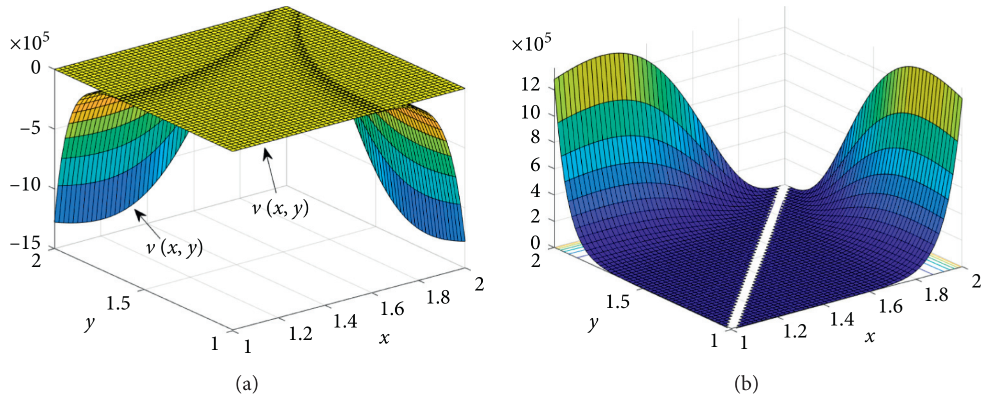

Figure 1: Plot illustrations for inequality (42). (a) For $v\,(x;\,y)$ and $V\,(x;\,y)$. (b) For $V\,(x;\,y) - v\,(x;\,y)$.

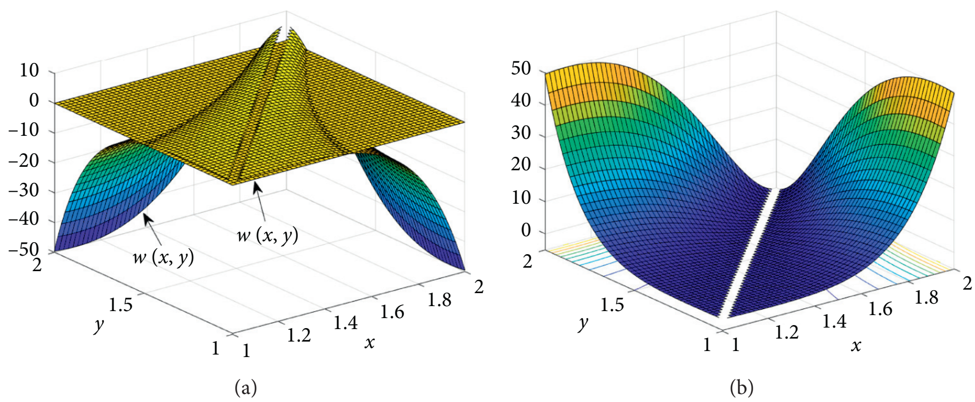

Figure 2: Plot illustrations for inequality (44). (a) For $w\,(x;\,y)$ and $W\,(x;\,y)$. (b) For $W\,(x, y) - w\,(x, y)$.

By summing over i from 0 to $n-1$ to get

$$\left|E_M\left(\overline{F}, \mathscr{P}\right)\right| \leq \frac{1}{81} \sum_{i=0}^{n-1} \left(s_{i+1} - s_i\right)^3 \max\left\{\left|\overline{F}''\left(s_i\right)\right|^q, \left|\overline{F}''\left(s_i\right)\right|^q + \varphi\left(\left|\overline{F}''\left(s_{i+1}\right)\right|^q, \left|\overline{F}''\left(s_i\right)\right|^q\right)\right\}, \tag{73}$$

which rearranges to the proof. □

5. Illustrative Plots

Finally, we present two three-dimensional plots to demonstrate the validity of the inequalities (42) and (44) in the case of φ-convex and φ-quasiconvex functions, respectively.

From inequality (42), we can define

$$\begin{aligned} v\,(x, y) &:= q\frac{1}{3}\mathscr{A}\left(x^4, y^4\right) + \frac{2}{3}\mathscr{A}^4\,(x, y) - \mathscr{L}_5^5\,(x, y) \\ V\,(x, y) &:= q\frac{(y-x)^2}{27}\left[8x^2 + y^2\right]. \end{aligned} \tag{74}$$

Thus, Figure 1 represents the plot of inequality (42) and $V\,(x, y) - v\,(x, y)$.

From inequality (44), we can define

$$\begin{aligned} w\,(x, y) &= \frac{1}{3}\mathscr{A}\left(x^2, y^2\right) + \frac{2}{3}\mathscr{A}^2\,(x, y) - \mathscr{L}_3^3\,(x, y) \\ W\,(x, y) &= \frac{2\,(y-x)^2}{81}. \end{aligned} \tag{75}$$

Thus, Figure 2 represents the plot of inequality (44) and $W\,(x, y) - w\,(x, y)$.

6. Conclusion

In this study, we have considered Simpson's type integral inequalities for the φ-convex and φ-quasiconvex functions in the second derivative sense. Some special cases of our findings are investigated to show the powerfulness of our results. Also, the proposed inequalities can be applied to other mathematical and statistical models, as we have shown in Section 4.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Exploring the Adoption of Nike+ Run Club App: An Application of the Theory of Reasoned Action

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The purpose of this study was to explore the influencing factors of users using Nike + Run Club App by Theory of Reasoned Action (TRA) as the theoretical basis and add the perceived playfulness into the research framework. This study took the users of the Nike + Run Club App as the research subject and distributed 360 questionnaires by snowballing sampling, a total of 351 valid questionnaires. All data were analyzed by descriptive statistics, confirmatory factor analysis, and structural equation models. Overall, the results reveal that extending TRA could be a well-explained users' behavior in the mobile application. The study found that the less the efforts spent in learning, the more positive the influence on attitude, thus affecting users' behavior. Therefore, this study proposed the following suggestions: People are pursuing a clearer and simple interactive function, a simplistic design; or adding instructions next to the new features will make the Nike + Run Club App more perfect. Emphasize the user's entertainment needs, develop interesting tasks, or games to make users feel interesting, and then be willing to continue to use the Nike + Run Club App.

1. Introduction

People in Taiwan use a mobile phone rather than a desktop or tablet. From the above information, we can know that, in Taiwan, the popularity of mobile devices is high, and it can be even said that they are a necessity of life. People have become accustomed to using mobile devices to do online work. The percentage of Taiwanese people using smartphones is increasing year by year; about 80% of users will carry their mobile phones with them. It can be seen that smartphones have become more and more relevant to people's lives. Smartphones have a rich user interface, functions, and complex operating platforms, and a variety of apps are available to users. With the popularity of smartphones and the maturity of mobile networks, people's lives are gradually changing, becoming more convenient.

Nowadays, people's awareness of health is improving. For example, running is the innate instinct of people, which is a leisure sport that is easy to engage where people are less burdened [1]. Chu and Cho pointed out that jogging is an economical and efficient sport that is not limited by time, space, and age [2]. Chen also believes that jogging has health benefits for people of all ages, and regular exercise promotes the quality of life and health of people [3]. In addition, the National Sports Awareness Survey found that the most popular sports in Taiwan were "jogging, brisk walking, and walking," accounting for 51.7% [4]. Obviously, jogging has gradually become the mainstream of sports and leisure, and it is not limited by time and space, making it the first choice for sports and leisure.

Along with the emergence of smartphones, many health managements related to mobile applications have been developed through mobile smart devices to download mobile applications. Sports help athletes track or record their individual sports status and promote the spirit of national sports for health or fitness purposes. App software companies also offer much different software for sports-loving users. The sports app is simple and easy to carry, and most of the sports apps have a wide range of functions and popular among sports and fitness enthusiasts. The sports app not only connects people and people, but also incorporates the concepts of smart health, medical care, and sports. It is tailor-made and records the most appropriate exercise habits and links the community to motivate each other and transform the movement into action community activities. According to foreign media reports, in recent years, the number of sports and fitness apps is growing at a rate of 150% per year. This showed that the population using mobile apps is growing and that sports apps are becoming more popular.

Since the mobile app is a medium for users to communicate with smartphone devices, it also drives the development of various types of apps, which not only creates a huge market for apps, but also enriches the additional functions of mobile devices. Many companies have also joined the ranks of creating exclusive brand apps, trying to open up more possibilities for the overall operation of the company through marketing. Based on this, Nike has created more links with consumers in the app market and has developed several apps related to its brand, which is designed for road runners. For example, Nike + Run Club App provides runners with calorie calculations, mileage, time, community links, music, GPS positioning, and more. The number of apps in the app stores exceeded 1.5 million in 2015 [5].

Sports economics has been rising rapidly globally; enterprises integrate technology to make playing sports smarter as sports applications became popular in lifestyle. Sports applications are built for people to stay exercising, but nowadays with the multiple functions of the applications, Tu et al. found that making apps fun and interesting could help consumers sustain their effort in physical activity [6]. However, there are a lot of apps on the market, and how to be popular among users is an important issue. Although there are several apps in the mobile app store, the Nike + Run Club, which is the most popular among runners, is quite in line with the needs of runners, including friends cheering, self-selected songs, and sports star voice response. Being designated, fun, and easy to use made Nike + Run Club become the preferred app for many runners. It can be seen that Nike + Run Club has a large number of users; with the accelerated of sport application, it has become the trend for global, and it can be seen that it is necessary to understand the adoption for sport application. Therefore, this study uses the Nike + Run Club App as the research topic to explore the influencing factors affecting users' use of the Nike + Run Club App and further analyze users' willingness to continue using the app.

How to effectively predict or explain whether users accept information technology is a topic of concern for enterprises and organizations. It is also one of the most mature areas of information management development today [7]. Theory of Reasoned Action (TRA) is a widely used model in the field of social psychology. The advantage is that the factors affecting action need to be determined by affecting behavioral attitudes, subjective norms, or both. TRA has been proven to be effective in predicting and interpreting practical actions in many fields, and it has been supported by many preliminary evaluation studies. Therefore, this study uses TRA as the theoretical basis to explore the behavioral intentions of Nike + Run Club App users.

With the advancement of science and technology, people's lives have begun to be closely related to these technological products. The demand for these technology products has gradually increased, and products that satisfy the user's senses and pleasures will attract consumers [8]. Lin et al. mentioned that consumers are willing to pay more attention to the function of technology products and expect products to bring out excellent experience and good quality [9]. The Nike + Run Club App is an interesting application. In the Nike + Run Club App, you can record your running time and mileage, and the data can know the pace of each minute. Besides that, you can check out friends and relatives' records to stimulate the user's active mentality and have fun in the process. In recent years, the rapid development of technology has made the "fun" element gradually attract the attention of the design field. Lin pointed out that pleasure will enhance the user's intention of IT products [10]. It can be seen that the making the app interesting has also become a new direction in mobile application design. Therefore, this study incorporates perceived playfulness into the factors of attitude influence to understand the behavior intentions of Nike + Run Club App users.

To explore the behavior intention of the Nike + Run Club App, the purpose of this study is summarized as follows: 1. Exploring the influence of perceived usefulness, perceived ease of use, and perceived playfulness on attitude. 2. Exploring the influence of Internet word-of-mouth and interpersonal influence on subjective norms. 3. Exploring the influence of attitude and subjective norms on behavioral intentions. Finally, the results of the research can be provided as a reference for the relevant mobile application developers and mobile applications and can also be used as a reference for future research.

TRA was proposed by Flanders et al. in 1975. TRA believes that the individual's behavioral intentions will be influenced by the attitude of the individual and subjective normative, and the behavior intention further influences the specific behavior being manifested. In other words, TRA assumes that "behavior occurs based on the control of the individual's will" [11]. If a person thinks that this behavior should occur, this behavior will occur, mainly used to understand and predict personal behavior. Faith represents a person's will to control their behavior. The behavior is generated by a person's belief in logical thinking, after choosing the right or not, or other decisions to take. TRA assumption is not affected by the external environment; that is, the idea of support can represent a person's behavior.

Taylor and Todd pointed out that users' attitudes toward using information technology (IT) will also be affected by perceived usefulness and perceived ease of use [12]. When the user perceives that the system is useful, the attitude toward the system will be more positive. In other words, when users feel that the function of using this innovative technology will improve the efficiency of learning or work and they need to spend less effort in learning, it will produce positive reviews. Flanders et al. argue that attitude is the result of an individual's past learning experience, producing a preference or aversion to the consistency of an object. The attitude is also an individual's overall evaluation of a particular person, thing, or idea [11]. Davis defines the attitude of use as an individual's positive or negative perception of the technology. This study divides the factors affecting attitude belief into perceived usefulness, perceived ease of use, and perceived playfulness [13].

Perceived usefulness and perceived ease of use are the two main beliefs in technology acceptance model (TAM). In 1986, Davis developed the TAM behavioral model based on TRA. It is designed for users to accept new information systems. TAM's purpose is to find an effective behavioral model to explain the behavior of users in computer technology to accept new information systems and to analyze the factors that affect user acceptance. TAM provides a theoretical basis for understanding the impact of external factors on users' beliefs, attitudes, and intentions, thus affecting the use of technology, and can be widely used to explain or predict the influencing factors of IT use.

The TAM uses perceived usefulness and perceived ease of use as independent variables, and attitudes, behavioral intentions, and usage behaviors are dependent variables. Advocating usefulness and ease of use can affect the attitude of using technology, which in turn affects specific behavioral performance. It also advocates that people's use of information technology is affected by their behavioral intentions.

Davis et al. pointed out that when the user perception system is easy to use, it will encourage users to do more work and improve job performance. Potential users subjectively believe that using a particular information system will increase their job performance or the likelihood of learning performance [14]. Davis pointed out the extent to which potential users subjectively believe that the operation of a particular information system technology is easy to use [13]. While the mind of the user causes a burden, it will produce negative emotions and then exclude the use. That is, when the user perceives the operation of the information system function of the innovative smartphone, the less the effort needed to be spent in learning, the more positive the attitude of using the system, and the perceived ease of use will also positively affect the perceived usefulness.

Perceived playfulness is defined as the degree to which a person feels euphoric when participating in an activity or adopting a system [15]. Moon and Kim applied the TAM to the World Wide Web study, citing research by Lieberman and Barnett to develop the third variable "perceived playfulness" [16–18].

Playfulness was first proposed by Lieberman, and then Barnett studied human behavior and made some observations about the meaning of playfulness: 1. He focused on the interesting features (trait of playfulness), which regards interest as a feature of motivation, mainly referred to the characteristics of individuals that are more stable and do not change with the situation. 2. He considered playfulness to be caused by the interaction between the individual and the situation. It is mainly that the individual is affected by things in the scriptures and will be affected by time, by context factors, and by interaction [15, 18].

According to Csikszentmihalyi, if a person feels pleasant when interacting with the environment, this feeling will generate perceived playfulness, and therefore people will have a positive attitude toward the environment [19]. Lu and Ma also pointed out that the main reason why the product found that it can trigger the user's "pleasure" feeling is that the product can trigger the user's "emotional" experience, and the area is no longer limited to the "function" of the product itself [8]. Lin et al. pointed out that, by enhancing the happy atmosphere, users can get a positive evaluation and thus improve their behavioral intentions [20].

According to Flanders et al., TRA refers to the social pressure exerted by others or groups (such as parents, spouses, friends, and colleagues) on an individual's specific behavior and is based on normative beliefs and motivation to comply [11]. The normative belief refers to the opinion of other people or groups on an individual's engagement in a particular behavior; motivation to comply refers to the degree of compliance of individuals with other people's opinions. Ajzen believes that individuals' behavior will be influenced by the degree of identity of others and the surrounding environment [21]. Subjective norms are individuals who are subject to the pressure of important others, groups, and society. The above individuals are obedient to the decision or behavior, and the individual will change the self-determination behavior because of external factors.

When potential adopters try to adapt, to reduce uncertainty, the experience of the previous adopter (interpersonal influence) or the experts of the mass media in the field (external influence) will be consulted in advance. The potential adopter is to form his perspective on innovation from two sources (interpersonal influence and external influence). Many studies have confirmed that reference groups have an impact on behavioral intentions [22–24]. According to Bhattacherjee, interpersonal influence is influenced by the dictation of friends, superiors, classmates, and other innovators [24].

Research on the influence of word-of-mouth has been flourished in recent years. In past research, scholars have a lot of discussions and definitions of word-of-mouth communication. Warrington defines word-of-mouth communication as to how consumers shape their attitudes and behaviors [25]. "Word-of-mouth" has always played a very important role in the communication of products and services between consumers through nonvendor marketing channels [26]. Word-of-mouth communication is often one of the main considerations when consumers are faced with purchasing decisions about goods or services [27]. Therefore, word-of-mouth is a noncommercial, two-way communication, experience-oriented, interactive, and immediate [25].

Behavior intention refers to the tendency and degree of action of an individual who wants to engage in a particular behavior, that is, the psychological strength of the individual's action in the decision-making process; in measurement, it can be transformed into whether the individual is willing to try or is willing to pay, by which variables can explain and predict the actual performance of the individual.

In summary, TRA holds that attitude and subjective norm determine the individual's behavior intention; behavior intention determines the individual behavior, and the behavior intention is determined by the attitude and subjective norms. Therefore, if an individual is a more positive attitude on a particular behavior and subjective norms support the behavior, the individual's intention to engage in the behavior will also increase.

2. Materials and Methods

2.1. Research Subject. This study took the Nike + Run Club App users in Taiwan as the research subject and used the Google form to distribute the survey by purposive and snowball sampling. In addition to posting questionnaires on the various jogging-related community website, the study also distributed questionnaires to users who use Nike + Run Club and requested them to send to each other. The questionnaires were distributed online from December 26th, 2016, to January 26th, 2017; a total of 360 questionnaires were collected, with 351 valid questionnaires; the effective recovery rate was 97.5%.

2.2. Research Tools. The scale of this study could be divided into three parts. The first part was the TAM, which was mainly referred to by Davis, Su, and Liao [13, 28, 29]. Through the Confirmatory Factor Analysis (CFA), the factor loading of attitude was between 0.85 and 0.89, Composite Reliability (CR) is 0.93, and Average Variance Extracted (AVE) is 0.76; the factor loading of perceived usefulness was between 0.88 and 0.93, CR is 0.94, and AVE is 0.80; the factor loading of perceived ease of use was between 0.84 and 0.90, CR is 0.93, and AVE is 0.75.

The second part would be the TRA, which was mainly referred to by Flanders et al., Bhattacherjee and Chen et al. [11, 24, 30], Through the CFA, the factor loading of interpersonal influence was between 0.84 and 0.90, CR is 0.90, and AVE is 0.76; the factor loading of Internet W-O-M was between 0.80 and 0.92, CR is 0.93, and AVE is 0.77; the factor loading of the subjective norm was between 0.70 and 0.91, CR is 0.88, and AVE is 0.71; the factor loading of behavior intention was between 0.64 and 0.92, CR is 0.89, and AVE is 0.68.

Finally, perceived playfulness was mainly referred to by Moon and Kim, and Cheng et al. [16, 30]. Through the CFA, the factor loading of perceived playfulness was between 0.87 and 0.90, CR is 0.93, and AVE is 0.78.

2.3. Discriminant Validity. This study used the confidence interval method to test the discriminant validity of the overall behavior model. According to Torkzadeh et al., in confidence interval method, if the confidence interval does not contain 1 representative, there is no correlation at all, indicating that the variables have discriminant validity [31]. In this study, the bootstrap method was used to estimate (2,000 times), the confidence level was below 95%, and percentile confidence intervals and bias-corrected confidence intervals were used to measure. In this study, the upper bounds of the confidence interval method are between 0.50 and 92, respectively, and the lower bounds are between 0.34 and 0.81; the confidence interval does not contain 1, indicating that each variable has discriminant validity.

2.4. Data Analysis. In this study, we used SPSS version 21.0 and AMOS version 21.0 to analyze the data and used descriptive statistics, confirmatory factor analysis, and structural equation modeling to understand the issues and purpose discussed in this study.

3. Results and Discussion

3.1. Subject Data Analysis. The subject majority in this study were females (62.7%), aged around 21–30 (55.8%); education level of the majority was university degree (79.8%); occupation of the majority was student (60.1%); average monthly income was below NTD 20,000 (60.7%).

3.2. Confirmatory Factor Analysis of the Apple Watch User Behavior Model. Data of this study were qualified with the normality test but the CR of multivariate kurtosis was 107.20, the assumption of multivariate normality distribution was not supported; therefore, this study used bootstrap proposed by Bollen and Stine to modify the overall model [32]. In the overall model fit analysis, all values have reached the standard, GFI = 0.95, RMSEA = 0.04, NFI = 0.95, RFI = 0.94, IFI = 0.98, TFI = 0.98, CFI = 0.98, PGFI = 0.87, PNFI = 0.87, CN = 0.231.88 and $\chi^2/df = 1.516$, indicating that the overall model fits well, and then continue the discussion.

For behavior intention, the attitude has a higher influence (0.73), which means that using Nike + Run Club App is meaningful and helpful to the sport, and the higher the user's attitude is, the higher the behavior intention will be (Figure 1).

For attitude, the influence of perceived ease of use is the highest (0.56), which means that, on behalf of users, the Nike + Run Club App is easy to operate and easy to use. According to Cheong and Park, the less the effort for users to learn the new information technology, the more positive the attitude on IT [33]. Followed by perceived playfulness (0.51), when using the Nike + Run Club App, on behalf of the user, it is fun to enjoy while running through the Nike + Run Club App (Figure 1). According to Csikszentmihalyi, if the user feels happy when interacting with technology, perceived playfulness will generate, and therefore the user will have a positive attitude and thus willingness to continue using Nike + Run Club App in the future [19].

For subjective norm, interpersonal influence has the higher influence (0.56), which means that users believe that the opinion and suggestion from friends and family are important for them to use the Nike + Run Club App; users

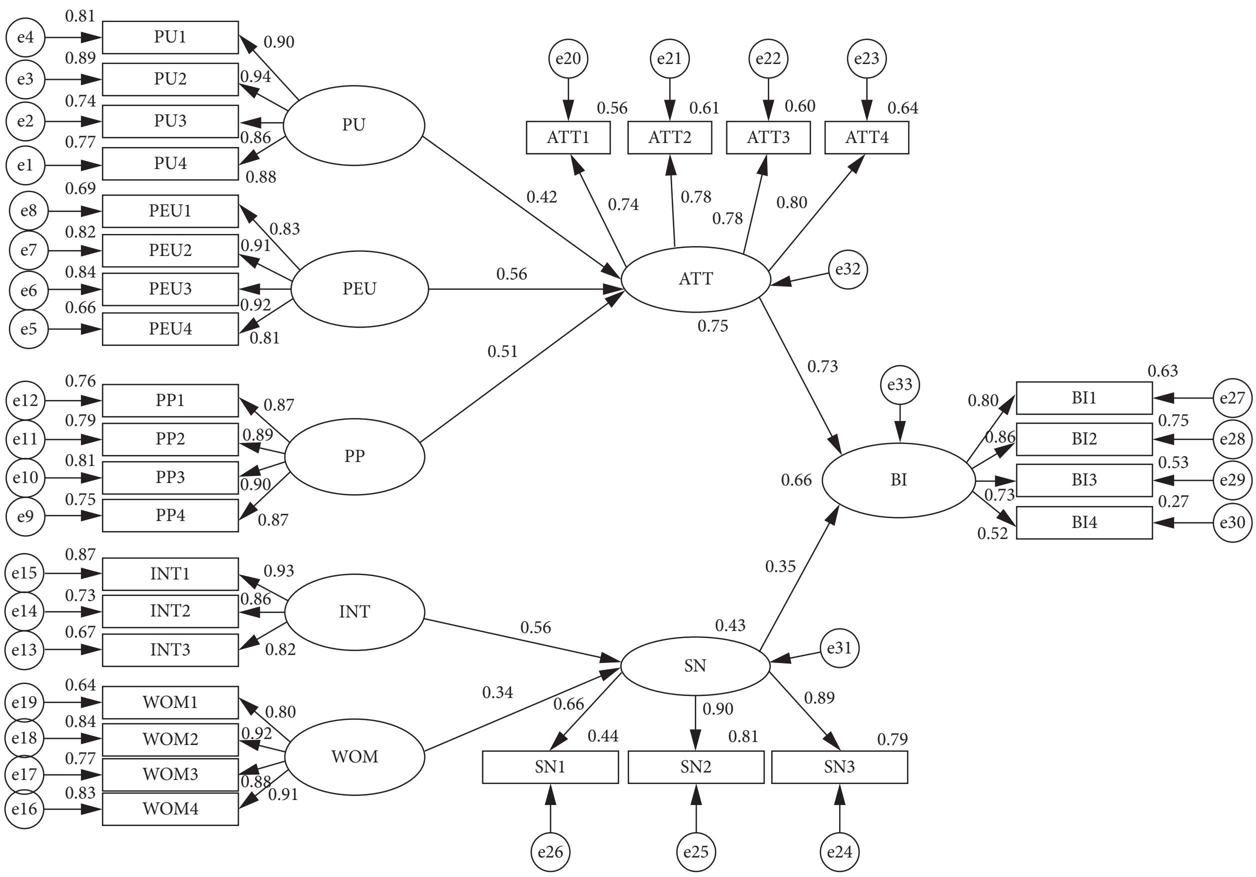

Figure 1: Path analysis of Nike Run + behavioral model. PU: perceived usefulness. PEU: perceived ease of use. PP: perceived playfulness. INT: interpersonal influence. WOM: internet word-of-mouth. ATT: altitude. SN: subjective norm. BI: behavior intention. e: error.

also hope that they can build a connection with people through the Nike + Run Club App (Figure 1). Therefore, when the interpersonal influence is more positive, the subjective norm will be higher. The results are similar to Lin's study results [34].

According to the results of this study, perceived ease of use has the greatest influence on attitudes; hence the designation of the Nike + Run Club App should be based on simple operations. The updated Nike + Run Club App is too complicated due to the new features. Although new interactive features have been added, there is no clear instruction of the operation, which leads users to update the Nike + Run Club App but cannot experience more diverse functions after the update. Therefore, this study suggests that, in a technologically advanced society, people are pursuing a clearer and simpler interactive function, so they can design in the direction of making the interactive function more simplistic or add instructions to the next function to make Nike + Run Club App More perfect.

According to the results of this study, the influence of perceived playfulness has a very high influence on attitude. When the questionnaire was further examined, it was found that users could enjoy the operation when using the Nike + Run Club App, but they could not satisfy their entertainment. The possible reason is that when jogging through the Nike + Run Club App, the user is interested because of the freshness, but if the user interacts for a long time, it lacks the incentive for the user to continue to feel the entertainment, leading to the lower entertainment demand. Therefore, this research suggests that, in addition to the user's refreshing features, it is also possible to develop interesting tasks or games into the Nike + Run Club App, emphasizing the user's entertainment needs and improving the Nike + Run Club App into a fun and enjoyable experience.

According to the results of this study, interpersonal influence has the greatest influence on the subjective norm, and the subjective norm will also influence behavior intention. It can be seen that family and friends will influence users to continue to use the Nike + Run Club App. Therefore, it is recommended that the Nike + Run Club App R&D Department can be more closely connected to the community and users so that future users can continue to use the Nike + Run Club App trend.

This study used Nike Run + App as the major study research, but there are still a lot of applications on the market. Especially in a wearable device, Rause pointed out that wearable technology will be improving the exercise habits of adults [35]. Therefore, it is suggested that future research could study the application adoption of the wearable device.

This study distributes the survey on the jogging-related community website, yet we have found that the age of the majority of subjects was around 21–30. Therefore, it is suggested that future research should distribute the survey through various paths to expand the sample to other age groups.

This study uses TRA as the theoretical basis to explore the behavioral intentions of Nike + Run Club App users. However, Lin et al. found that the adoption of application will be affected by the perceptual barrier, which has a significant negative impact on perceptual usefulness and perceived ease of use [36]. Therefore, it is suggested that future research should explore the adoption of application from the innovation resistance theory perspective.

4. Conclusions

Perceived usefulness, perceived ease of use, and perceived playfulness have positive and significant influence on attitude. Among them, perceived ease of use has the highest influence on attitude, indicating that users think that the operation of the Nike + Run Club App is the main factor affecting their attitude.

Interpersonal influence and Internet W-O-M had positive and significant influence on the subjective norm. Among them, interpersonal influence has the highest influence on the subjective norm, indicating that users believe that the recognition and suggestions of family and friends for the Nike + Run Club App are the main factors affecting subjective norm.

Attitude and subjective norm had positive and significant influence on behavior intention. Among them, the influence of attitude on behavior intention is the highest, indicating that the practicality of the Nike + Run Club App is an important factor affecting the continuous use of users.

Data Availability

The Nike Run + behavioral model data used to support the findings of this study are restricted by the Sports Research Center of Department of Leisure Services Management to protect the subject's privacy. Data are available from Dr. Chih-Wei Lin (e-mail: cwlin@cyut.edu.tw) for researchers who meet the criteria for access to confidential data.

Conflicts of Interest

The authors have no affiliations with or involvement in any organization or entity with any financial interest or nonfinancial interest in the subject matter or materials discussed in this manuscript.

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Research Article

Pathway Fractional Integral Formulas Involving $\mathcal{S}$-Function in the Kernel

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In this paper, we present several composition formulae of pathway fractional integral operators connected with $\mathcal{S}$-function. Here, we point out important links to known outcomes for some specific cases with our key results.

1. Introduction and Preliminaries

In recent years, fractional calculus has become a significant instrument for the modeling analysis and plays a significant role in different fields, for example, material science, science, mechanics, power, economy, and control theory. In addition, a number of researchers have investigated a variety of fractional calculus operators in the depth level of properties, implementation methods, and complex modifications. Other analogous topics are also very active and extensive around the world. One may refer to the research monographs in [1, 2].

$\mathcal{S}$-function. Recently, Saxena and Daiya [3] defined and studied a special function called as $\mathcal{S}$-function (also see [4]) and its relation with other special functions, which include generalized $\mathcal{K}$-function, $\mathcal{M}$-series, k-Mittag–Leffler function, Mittag–Leffler type functions, and other many special functions. These special functions have recently found essential applications in solving problems in applied sciences, biology, physics, and engineering.

The $\mathcal{S}$-function is defined for $\sigma, \eta, \varepsilon, \tau \in \mathbb{C}$, $\mathfrak{R}(\sigma) > 0$, $k \in \mathfrak{R}$, $\mathfrak{R}(\sigma) > k\mathfrak{R}(\tau)$, $l_i\,(i = 1, 2, 3, \ldots, p)$, $m_j\,(j = 1, 2, 3, \ldots, q)$, and $p < q + 1$ as

$$\mathcal{S}^{\sigma,\eta,\varepsilon,\tau,k}_{(p,q)}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta)} \frac{x^n}{n!}, \tag{1}$$

Here, k-Pochhammer symbol is as follows:

$$(\varepsilon)_{n,k} = \begin{cases} \dfrac{\Gamma_k(\varepsilon + nk)}{\Gamma_k(\varepsilon)}, & (k \in \mathfrak{R}, \varepsilon \in \mathbb{C}/\{0\}), \\ \varepsilon(\varepsilon + k)\ldots(\varepsilon + (n-1)k), & (n \in \mathbb{N}, \varepsilon \in \mathbb{C}). \end{cases} \tag{2}$$

Also, the k-gamma function is

$$\Gamma_k(\varepsilon) = k^{(\varepsilon/k)-1}\Gamma\left(\frac{\varepsilon}{k}\right), \tag{3}$$

where $\varepsilon \in \mathbb{C}, k \in \mathfrak{R}$, and $n \in \mathbb{N}$, introduced by Díaz and Pariguan [5] (see also Romero and Cerutti [6]).

Several major special cases of the $\mathcal{S}$-function are described as follows:

(i) For $p = q = 0$, the generalized k-Mittag–Leffler function from Saxena et al. [7] (see [8, 9]) is

$$\mathscr{E}^{\varepsilon,\tau}_{k,\sigma,\eta}(x) = \mathcal{S}^{\sigma,\eta,\varepsilon,\tau,k}_{(0,0)}[-;-;x] = \sum_{n=0}^{\infty}\frac{(\varepsilon)_{n\tau,k}}{\Gamma_k(n\sigma+\eta)}\frac{x^n}{n!}, \Re\left(\frac{\sigma}{k}-\tau\right) > p-q. \tag{4}$$

(ii) For $k=\tau=1$, the $\mathcal{S}$-function is the generalized $\mathscr{K}$-function, introduced by Sharma [10] (see also [11]):

$$\begin{aligned}\mathscr{K}^{\sigma,\eta,\varepsilon}_{(p,q)}\left[l_1,\ldots,l_p;m_1,\ldots,m_q;x\right] &= \mathcal{S}^{\sigma,\eta,\varepsilon,1,1}_{(p,q)}\left[l_1,\ldots,l_p;m_1,\ldots,m_q;x\right]\\ &= \sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n(\varepsilon)_n}{(m_1)_n\ldots(m_q)_n\Gamma(n\sigma+\eta)}\frac{x^n}{n!}, \quad \Re(\sigma)>p-q.\end{aligned} \tag{5}$$

(iii) For $\tau=k=\varepsilon=1$, the $\mathcal{S}$-function reduced to generalized $\mathscr{M}$-series introduced by Sharma and Jain [12](detail [13]) is

$$\begin{aligned}\mathscr{M}^{\sigma,\eta}_{(p,q)}\left[l_1,\ldots,l_p;m_1,\ldots,m_q;x\right] &= \mathcal{S}^{\sigma,\eta,1,1,1}_{(p,q)}\left[l_1,\ldots,l_p;m_1,\ldots,m_q;x\right]\\ &= \sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n x^n}{(m_1)_n\ldots(m_q)_n\Gamma(n\sigma+\eta)}, \quad \Re(\sigma)>p-q-1.\end{aligned} \tag{6}$$

Recently, an expending pathway fractional integral (PFI) operator introduced by Nair [14], which was earlier defined by Mathai [15] and Mathai and Haubold [16, 17], is defined as follows:

$$\left(\mathscr{P}^{\lambda,\varsigma}_{0+}f\right)(x) = x^{\lambda}\int_0^{[x/(a(1-\varsigma))]}\left(1-\frac{a(1-\varsigma)\xi}{x}\right)^{\lambda/(1-\varsigma)} f(\xi)\mathrm{d}\xi, \tag{7}$$

where Lebesgue measurable function $f\in\mathscr{L}(a,b)$ for real or complex term valued function, $\lambda\in\mathbb{C}$, $\Re(\lambda)>0, a>0$, and $\varsigma<1$ (ς is a pathway parameter).

The pathway model for a real scalar ς and scalar random variables is represented by the probability density function (p.d.f.) in the following manner:

$$f(x) = \frac{c}{|x|^{1-\nu}}\left[1-a(1-\varsigma)|x|^{\rho}\right]^{\lambda/(1-\varsigma)}, \tag{8}$$

where $x\in(-\infty,\infty);\lambda>0;\rho>0;\left[1-a(1-\varsigma)|x|^{\rho}\right]^{\lambda/(1-\varsigma)}>0$; $\nu>0$ and ς and c denote the pathway parameter and normalizing constant, respectively.

Additionally, for $\varsigma\in\Re$, the normalizing constants are expressed in the following way:

$$c = \begin{cases}\dfrac{1}{2}\dfrac{\rho[a(1-\varsigma)]^{\nu/\rho}\Gamma(\nu/\rho+\lambda/(1-\varsigma)+1)}{\Gamma(\nu/\rho)\Gamma(\lambda/(1-\varsigma)+1)}, & (\varsigma<1),\\[2ex] \dfrac{1}{2}\dfrac{\rho[a(1-\varsigma)]^{\nu/\rho}\Gamma(\lambda/(\varsigma-1))}{\Gamma(\nu/\rho)\Gamma(\lambda/(\varsigma-1)-\nu/\rho)}, & \left(\dfrac{1}{\varsigma-1}-\dfrac{\nu}{\rho}>0,\varsigma>1\right),\\[2ex] \dfrac{1}{2}\dfrac{[a\lambda]^{\nu/\rho}}{\Gamma(\nu/\rho)}, & (\varsigma\longrightarrow 1).\end{cases} \tag{9}$$

It is noted that if $\varsigma<1$, finite range density with $\left[1-a(1-\varsigma)|x|^{\rho}\right]^{\lambda/(1-\varsigma)}>0$ and (8) can be considered a member of the extended generalized type-1 beta family. Also, the triangular density, the uniform density, the extended type-1 beta density and various other probability density functions are precise special cases of the pathway density function defined in (8) for $\varsigma<1$.

For example, if $\varsigma>1$ and by setting $(1-\varsigma)=-(\varsigma-1)$ in (7), then we have

$$\left(\mathscr{P}_{0+}^{\lambda,\varsigma} f\right)(x) = x^{\lambda} \int_0^{[x/-a(\varsigma-1)]} \left(1 + \frac{a(\varsigma-1)\xi}{x}\right)^{\lambda/-(\varsigma-1)} f(\xi)\mathrm{d}\xi, \tag{10}$$

$$f(x) = \frac{c}{|x|^{1-\nu}}\left[1 + a(\varsigma-1)|x|^{\rho}\right]^{\lambda/-(\varsigma-1)}, \tag{11}$$

provided that $x \in (-\infty,\infty); \rho > 0; \lambda > 0;$ and $\varsigma > 1$ characterize the extended generalized type-2 beta model for real x. The specific cases of density function (11) include the type-2 beta density function, the p density function, and the Student's t density function. For $\varsigma \longrightarrow 1$, (7) diminishes to the Laplace integral transform.

In a similar way, if $\varsigma = 0$, $a = 1$, and λ takes the place of $\lambda - 1$, then (7) diminishes to the familiar Riemann–Liouville (R-L) fractional integral operator $\mathscr{I}_{0+}^{\lambda} f$ (e.g., [7]):

$$\left(\mathscr{P}_{0+}^{\lambda-1,0} f\right)(x) = \Gamma(\lambda)\left(\mathscr{I}_{0+}^{\lambda} f\right)(x), \quad (\Re(\lambda) > 1). \tag{12}$$

PFI operator (7) leads to numerous interesting illustrations such as fractional calculus associated with probability density functions and their significant in statistical theory. Nowadays, many researchers study PFI formulae associated with various special functions (see [18–27]). Motivated by these researchers, we study the $\mathcal{S}$-function, which is connected with PFI operator (7), to present their integral formulae. Suitable connections of some particular cases are also pointed out.

2. Pathway Fractional Integral Operator of $\mathcal{S}$-Function

In this section, we establish the PFI formula involving the $\mathcal{S}$-function which is stated in Theorems 1 and 2.

Theorem 1. *Suppose $w, k \in \Re, \sigma, \eta, \varepsilon, \tau \in \mathbb{C}, \Re(\sigma) > 0, \Re(\lambda) > 0, \Re(\sigma) > k\Re(\tau)$, and $p < q + 1, \Re(\lambda/(1-\varsigma)) > -1; \varsigma < 1$. Then, the following formula holds true:*

$$\begin{aligned}&\mathscr{P}_{0^+}^{\lambda,\varsigma}\left[\zeta^{(\eta/k)-1}\mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;w\zeta^{(\sigma/k)}\right]\right](x)\\&= \frac{x^{\lambda+(\eta/k)}k^{(1+(\lambda/1-\varsigma))}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{(\eta/k)}} \times \mathcal{S}_{(p,q)}^{\sigma,\eta+(1+\lambda/(1-\varsigma))k,\varepsilon,\tau,k}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;\frac{wx^{(\sigma/k)}}{(a(1-\varsigma))^{(\sigma/k)}}\right].\end{aligned} \tag{13}$$

Proof. We indicate the RHS of equation (13) by $\mathfrak{I}_1$, and invoking equations (1) and (7), we have

$$\begin{aligned}\mathfrak{I}_1 &= x^{\lambda}\int_0^{[x/a(1-\varsigma)]}\left(1 - \frac{a(1-\varsigma)\zeta}{x}\right)^{\lambda/(1-\varsigma)}\zeta^{(\eta/k)-1}\\&\times\sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n(\varepsilon)_{n\tau,k}}{(m_1)_n\ldots(m_q)_n\Gamma_k(n\sigma+\eta)}\frac{\left(w\zeta^{(\sigma/k)}\right)^n}{n!}\mathrm{d}\zeta.\end{aligned} \tag{14}$$

Now changing the order of integration and summation, we obtain

$$\begin{aligned}\mathfrak{I}_1 &= x^{\lambda}\sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n(\varepsilon)_{n\tau,k}w^n}{(m_1)_n\ldots(m_q)_n\Gamma_k(n\sigma+\eta)n!}\\&\times\int_0^{[x/a(1-\varsigma)]}\left(1-\frac{a(1-\varsigma)\zeta}{x}\right)^{\lambda/(1-\varsigma)}\zeta^{((\eta+\sigma n)/k)-1}\mathrm{d}\zeta.\end{aligned} \tag{15}$$

Using the substitution $u = a(1-\varsigma)\zeta/x$, we can change the limit of integration into the following:

$$\begin{aligned}\mathfrak{I}_1 &= x^{\lambda}\sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n(\varepsilon)_{n\tau,k}w^n}{(m_1)_n\ldots(m_q)_n\Gamma_k(n\sigma+\eta)n!}\left(\frac{x}{a(1-\varsigma)}\right)^{(\eta+\sigma n)/k}\\&\times\int_0^1(1-u)^{\lambda/(1-\varsigma)}u^{((\eta+\sigma n)/k)-1}\mathrm{d}u.\end{aligned} \tag{16}$$

Now, by calculating the inner integral and using the beta function formula, we obtain the following:

$$\begin{aligned}\mathfrak{I}_1 &= x^{\lambda}\sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n(\varepsilon)_{n\tau,k}w^n}{(m_1)_n\ldots(m_q)_n\Gamma_k(n\sigma+\eta)n!}\left(\frac{x}{a(1-\varsigma)}\right)^{(\eta+\sigma n)/k}\\&\times\frac{\Gamma(\eta/k+n\sigma/k)\Gamma(\lambda/(1-\varsigma)+1)}{\Gamma(\eta/k+(n\sigma/k)+\lambda/(1-\varsigma)+1)}.\end{aligned} \tag{17}$$

Using (3), we obtain

$$\mathfrak{J}_1 = x^{\lambda+\eta/k}\sum_{n=0}^{\infty}\frac{(l_1)_n\cdots(l_p)_n(\varepsilon)_{n\tau,k}}{(m_1)_n\cdots(m_q)_n\Gamma(n\sigma/k+\eta/k)k^{(n\sigma/k)+(\eta/k)-1}n!}$$
$$\times\frac{\Gamma(\eta/k+n\sigma/k)\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{\eta/k}\Gamma(\eta/k+(n\sigma/k)+\lambda/(1-\varsigma)+1)}$$
$$\cdot\left(w\left(\frac{x}{a(1-\varsigma)}\right)^{(\sigma)/k}\right)^{n}. \tag{18}$$

Once again, using (3), we obtain

$$\mathfrak{J}_1 = \frac{x^{\lambda+(\eta/k)}k^{(1+\lambda/(1-\varsigma))}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{(\eta/k)}}$$
$$\mathcal{S}^{\sigma,\eta+(1+\lambda/(1-\varsigma))k,\varepsilon,\tau,k}_{(p,q)}\Bigg[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots, \tag{19}$$
$$m_q;\frac{wx^{(\sigma/k)}}{(a(1-\varsigma))^{(\sigma/k)}}\Bigg],$$

which gives the required proof of Theorem 1. □

Corollary 1. *If we put $p=q=0$, then (13) leads to the subsequent result of generalized k-Mittag–Leffler function:*

$$\mathcal{P}^{\lambda,\varsigma}_{0^+}\left[\zeta^{(\eta/k)-1}\mathcal{E}^{\varepsilon,\tau}_{k,\sigma,\eta}\left(w\zeta^{(\sigma/k)}\right)\right](x)$$
$$=\frac{x^{\lambda+(\eta/k)}k^{(1+\lambda/(1-\varsigma))}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{(\eta/k)}}\mathcal{E}^{\varepsilon,\tau}_{k,\sigma,\eta+(1+\lambda/(1-\varsigma))k}(x)$$
$$\cdot\left[\frac{wx^{(\sigma/k)}}{(a(1-\varsigma))^{(\sigma/k)}}\right]. \tag{20}$$

Proof. We consider (4) and $p=q=0$ in Theorem 1, and we obtain the desired result in (13). □

Corollary 2. *If we put $k=\tau=1$, then (13) leads to the subsequent result in terms of generalized $\mathcal{K}$-function:*

$$\mathcal{P}^{\lambda,\varsigma}_{0^+}\left[\zeta^{\eta-1}\mathcal{K}^{\sigma,\eta,\varepsilon}_{(p,q)}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;w\zeta^{\sigma}\right]\right](x)$$
$$=\frac{x^{\lambda+\eta}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{\eta}}\mathcal{K}^{\sigma,\eta+(1+\lambda/(1-\varsigma)),\varepsilon}_{(p,q)}$$
$$\cdot\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;\frac{wx^{\sigma}}{(a(1-\varsigma))^{\sigma}}\right]. \tag{21}$$

Proof. If we set $k=\tau=1$ in Theorem 1 and using (5), we obtain the required result (21). □

Corollary 3. *If we put $k=\tau=1$, then (13) holds the formula in terms of generalized $\mathcal{M}$-series:*

$$\mathcal{P}^{\lambda,\varsigma}_{0^+}\left[\zeta^{\eta-1}\mathcal{M}^{\sigma,\eta}_{(p,q)}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;w\zeta^{\sigma}\right]\right](x)$$
$$=\frac{x^{\lambda+\eta}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{\eta}}\mathcal{M}^{\sigma,\eta+(1+\lambda/(1-\varsigma))}_{(p,q)}$$
$$\cdot\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;\frac{wx^{\sigma}}{(a(1-\varsigma))^{\sigma}}\right]. \tag{22}$$

Proof. If we put $\tau=k=\varepsilon=1$ in Theorem 1 and using (6), we obtain the result (22). □

Now, we use equation (10) to define the following theorem, by the case $\varsigma>1$.

Theorem 2. *Suppose $w,k\in\mathfrak{R};\sigma,\eta,\varepsilon,\tau\in\mathbb{C},\mathfrak{R}(\sigma)>0$, $\mathfrak{R}(\lambda)>0$, $\mathfrak{R}(\sigma)>k\,\mathfrak{R}(\tau)$ and $p<q+1$, and $\mathfrak{R}(1-(\lambda/\varsigma-1))>0;\varsigma>1$. Then, the following formula holds true:*

$$\mathcal{P}^{\lambda,\varsigma}_{0^+}\left[\zeta^{(\eta/k)-1}\mathcal{S}^{\sigma,\eta,\varepsilon,\tau,k}_{(p,q)}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;w\zeta^{(\sigma/k)}\right]\right]$$
$$(x)=\frac{x^{\lambda+(\eta/k)}k^{(1-\lambda/(\varsigma-1))}\Gamma(1-(\lambda/(\varsigma-1)))}{(-a(\varsigma-1))^{(\eta/k)}}$$
$$\times\mathcal{S}^{\sigma,\eta+(1-\lambda/(\varsigma-1))k,\varepsilon,\tau,k}_{(p,q)}\Bigg[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;$$
$$\frac{wx^{(\sigma/k)}}{(-a(\varsigma-1))^{(\sigma/k)}}\Bigg]. \tag{23}$$

Proof. We denote, for convenience, the RHS of equation (23) by $\mathfrak{J}_2$, and invoking equations (1) and (10), we have

$$\mathfrak{J}_2 = x^{\lambda}\int_0^{[x/-a(\varsigma-1)]}\left(1+\frac{a(\varsigma-1)\zeta}{x}\right)^{\lambda/-(\varsigma-1)}\zeta^{(\eta/k)-1}$$
$$\times\sum_{n=0}^{\infty}\frac{(l_1)_n\cdots(l_p)_n(\varepsilon)_{n\tau,k}}{(m_1)_n\cdots(m_q)_n\Gamma_k(n\sigma+\eta)}\frac{\left(w\zeta^{(\sigma/k)}\right)^n}{n!}d\zeta. \tag{24}$$

Now, changing the order of integration and summation, we obtain

$$\mathfrak{I}_2 = x^{\lambda} \sum_{n=0}^{\infty} \frac{(l_1)_n \dots (l_p)_n (\varepsilon)_{n\tau,k} w^n}{(m_2)_n \dots (m_q)_n \Gamma_k (n\sigma + \eta) n!} \times \int_0^{[x/-a(\varsigma-1)]} \left(1 + \frac{a(\varsigma-1)\zeta}{x}\right)^{\lambda/-(\varsigma-1)} \zeta^{((\eta+\sigma n)/k)-1} d\zeta. \quad (25)$$

By setting $\nu = -a(\varsigma-1)\zeta/x$, we can change the limit of integration into the following:

$$\mathfrak{I}_2 = x^{\lambda} \sum_{n=0}^{\infty} \frac{(l_1)_n \dots (l_p)_n (\varepsilon)_{n\tau,k} w^n}{(m_1)_n \dots (m_q)_n \Gamma_k (n\sigma + \eta) n!} \left(\frac{x}{-a(\varsigma-1)}\right)^{(\eta+\sigma n)/k} \times \int_0^1 (1-\nu)^{\lambda/-(\varsigma-1)} \nu^{((\eta+\sigma n)/k)-1} d\nu \times \int_0^1 (1-\nu)^{\lambda/-(\varsigma-1)} \nu^{((\eta+\sigma n)/k)-1} d\nu. \quad (26)$$

By analyzing the internal integral and using the beta function rule, we obtain

$$\mathfrak{I}_2 = x^{\lambda} \sum_{n=0}^{\infty} \frac{(l_1)_n \dots (l_p)_n (\varepsilon)_{n\tau,k} w^n}{(m_1)_n \dots (m_q)_n \Gamma_k (n\sigma + \eta) n!} \left(\frac{x}{-a(\varsigma-1)}\right)^{(\eta+\sigma n)/k} \times \frac{\Gamma(\eta/k + n\sigma/k)\Gamma(1 - \lambda/(\varsigma-1))}{\Gamma(\eta/k + (n\sigma/k) + 1 - \lambda/(\varsigma-1))}. \quad (27)$$

Using (3), we obtain

$$\mathfrak{I}_2 = x^{\lambda+\eta/k} \sum_{n=0}^{\infty} \frac{(l_1)_n \dots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \dots (m_q)_n \Gamma(n\sigma/k + \eta/k) k^{(n\sigma/k)+(\eta/k)-1} n!} \times \frac{\Gamma(\eta/k + n\sigma/k)\Gamma(1 - \lambda/(\varsigma-1))}{(-a(\varsigma-1))^{\eta/k}\Gamma(\eta/k + (n\sigma/k) + 1 - \lambda/(\varsigma-1))} \cdot \left(w\left(\frac{x}{-a(\varsigma-1)}\right)^{(\sigma)/k}\right)^n. \quad (28)$$

Once again, we arrive at the target outcome by applying (3):

$$\mathfrak{I}_2 = \frac{x^{\lambda+(\eta/k)} k^{(1-\lambda/\varsigma-1)} \Gamma(1 - \lambda/\varsigma - 1)}{(-a(\varsigma-1))^{(\eta/k)}} \mathcal{S}_{(p,q)}^{\sigma,\eta+(1-\lambda/\varsigma-1)k,\varepsilon,\tau,k} \cdot \left[l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; \frac{w x^{(\sigma/k)}}{(-a(\varsigma-1))^{(\sigma/k)}}\right]. \quad (29)$$

□

Corollary 4. *If we put $p = q = 0$, then (23) provides the result as follows:*

$$\mathscr{P}_{0^+}^{\lambda,\varsigma} \left[\zeta^{(\eta/k)-1} \mathscr{E}_{k,\sigma,\eta}^{\varepsilon,\tau}\left(w\zeta^{(\sigma/k)}\right)\right](x) = \frac{x^{\lambda+(\eta/k)} k^{(1-(\lambda/\varsigma-1))} \Gamma(1-(\lambda/\varsigma-1))}{(-a(\varsigma-1))^{(\eta/k)}} \mathscr{E}_{k,\sigma,\eta+(1-(\lambda/\varsigma-1))k}^{\varepsilon,\tau}(x) \cdot \left[\frac{w x^{(\sigma/k)}}{(-a(\varsigma-1))^{(\sigma/k)}}\right]. \quad (30)$$

Proof. We consider (4) and $p = q = 0$ in Theorem 2 and we obtain the desired result (30). □

Corollary 5. *If $k = \tau = 1$, then (23) holds the following formula:*

$$\mathscr{P}_{0^+}^{\lambda,\varsigma} \left[\zeta^{\eta-1} \mathscr{K}_{(p,q)}^{\sigma,\eta,\varepsilon}\left[l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; w\zeta^{\sigma}\right]\right](x) = \frac{x^{\lambda+\eta}\Gamma(1-(\lambda/\varsigma-1))}{(-a(\varsigma-1))^{\eta}} \mathscr{K}_{(p,q)}^{\sigma,\eta+(1-(\lambda/\varsigma-1)),\varepsilon} \cdot \left[l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; \frac{w x^{\sigma}}{(-a(\varsigma-1))^{\sigma}}\right]. \quad (31)$$

Proof. If we set $k = \tau = 1$ in Theorem 2 and using (5), we obtain the required result (31). □

Corollary 6. *If we put $k = \tau = \varepsilon = 1$, then resulting formula (23) holds true:*

$$\mathscr{P}_{0^+}^{\lambda,\varsigma} \left[\zeta^{\eta-1} \mathscr{M}_{(p,q)}^{\sigma,\eta}\left[l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; w\zeta^{\sigma}\right]\right](x) = \frac{x^{\lambda+\eta}\Gamma(1-(\lambda/\varsigma-1))}{(-a(\varsigma-1))^{\eta}} \mathscr{M}_{(p,q)}^{\sigma,\eta+(1-(\lambda/\varsigma-1))} \cdot \left[l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; \frac{w x^{\sigma}}{(-a(\varsigma-1))^{\sigma}}\right]. \quad (32)$$

Proof. If we put $\tau = k = \varepsilon = 1$ in Theorem 2 and using (6), we obtain the result (32). □

3. Concluding Remarks

In the present paper, we have established two pathway fractional integral formulae associated with the more generalized special function called as S-function. The results obtained here involve special functions such as k-Mittag–Leffler function, $\mathscr{K}$-function, and $\mathscr{M}$-series, due to their general nature and usefulness in the theory of integral operators and relevant part of computational mathematics. Also, the special functions involved here can be reduced to simpler functions, which have a number of applications in various fields of science and technology and can be found as special cases that we have not specifically stated here.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the present investigation and read and approved the final manuscript.

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Research Article

Fractional Ostrowski Type Inequalities via Generalized Mittag–Leffler Function

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If we study the theory of fractional differential equations then we notice the Mittag–Leffler function is very helpful in this theory. On the contrary, Ostrowski inequality is also very useful in numerical computations and error analysis of numerical quadrature rules. In this paper, Ostrowski inequalities with the help of generalized Mittag–Leffler function are established. In addition, bounds of fractional Hadamard inequalities are given as straightforward consequences of these inequalities.

1. Introduction

Exponential function plays a vital role in the theory of integer order differential equations. The symbol $E_\alpha(z)$ is well known as the Mittag–Leffler function and it is a generalization of exponential function. It occurs in the solutions of fractional differential equations such as exponential function which exists in the solutions of differential equations. Due to its importance, Mittag–Leffler function is generalized by many mathematicians: For example, Wiman [1], Prabhakar [2], Shukla and Prajapati [3], Salim [4], Salim and Faraj [5], and Rahman et al. [6]. Mittag–Leffler function is also used in the formation of fractional integral operators. These fractional integral operators provide generalizations of fractional differential equations and modeling of dynamic systems. Fractional integral operators also play a vital role in the advancement of classical mathematical inequalities. For example, Hadamard inequality, Ostrowski inequality, Gruss inequality, and many others have been presented for fractional integral and derivative operators, see [7–16]. The aim of this paper is to study well-known Ostrowski inequality for an integral operator which is directly associated with many fractional integral operators defined in near past.

Recently, in [7], Andrić et al. defined the extended generalized Mittag–Leffler function $E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(.;p)$ as follows.

Definition 1. Let $\mu,\alpha,l,\gamma,c\in\mathbb{C}$, $\Re(\mu),\Re(\alpha),\Re(l)>0$, and $\Re(c)>\Re(\gamma)>0$ with $p\geq 0$, $\delta>0$, and $0<k\leq\delta+\Re(\mu)$. Then, the extended generalized Mittag–Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is defined as

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)=\sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{t^n}{(l)_{n\delta}},\qquad(1)$$

where β_p is the generalized beta function defined as $\beta_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-(p/t(1-t))}dt$ and $(c)_{nk}$ is the Pochhammer symbol given by $(c)_{nk}=(\Gamma(c+nk))/\Gamma(c)$.

The corresponding left- and right-sided generalized fractional integrals $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}$ are defined as follows.

Definition 2 (see [7]). Let $\omega,\mu,\alpha,l,\gamma,c\in\mathbb{C}$, $\Re(\mu),\Re(\alpha),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$ with $p\geq 0$, $\delta>0$, and $0<k\leq\delta+\Re(\mu)$. Let $\psi_1\in L_1[a,b]$ and $x\in[a,b]$. Then, the generalized fractional integrals $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}\psi_1$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}\psi_1$ are defined as

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}\psi_1\right)(x;p)=\int_a^x (x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^\mu;p)\psi_1(t)\mathrm{d}t, \tag{2}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}\psi_1\right)(x;p)=\int_x^b (t-x)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(t-x)^\mu;p)\psi_1(t)\mathrm{d}t. \tag{3}$$

Recently, Farid defined a unified integral operator in [17] (also see [18]). This unifies several kinds of fractional and conformable integrals in a compact formula and is given as follows.

Definition 3. Let $\psi_1,\psi_2\colon [a,b]\longrightarrow\mathbb{R}$, $0<a<b$, be the functions such that ψ_1 be positive and $\psi_1\in L_1[a,b]$, and ψ_2 be differentiable and strictly increasing. Also, let ϕ/x be an increasing function on $[a,\infty)$ and $\alpha,l,\gamma,c\in\mathbb{C}$, $\mathfrak{R}(\alpha),\mathfrak{R}(l)>0$, $\mathfrak{R}(c)>\mathfrak{R}(\gamma)>0$, $p,\mu,\delta\geq 0$, and $0<k\leq\delta+\mu$. Then, for $x\in[a,b]$ the left and right integral operators are defined by

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\psi_1\right)(x,\omega;p)=\int_a^x K_x^y\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)\psi_1(y)d(\psi_2(y)), \tag{4}$$

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\psi_1\right)(x,\omega;p)=\int_x^b K_y^x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)\psi_1(y)d(\psi_2(y)), \tag{5}$$

where $K_x^y(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)=((\phi(\psi_2(x)-\psi_2(y)))/(\psi_2(x)-\psi_2(y)))E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(y))^\mu;p)$.

The following definition can be deduced from Definition 3 (see [16]).

Definition 4. Let $\psi_1,\psi_2\colon [a,b]\longrightarrow\mathbb{R}$, $0<a<b$, be the functions such that ψ_1 be positive and $\psi_1\in L_1[a,b]$, and ψ_2 be differentiable and strictly increasing and $\alpha,l,\gamma,c\in\mathbb{R}_+$, $c>\gamma$, $p,\mu,\delta\geq 0$ and $0<k\leq\delta+\mu$. Then, for $x\in[a,b]$ the left and right integral operators are defined by

$$\left({}_{\psi_2}\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}\psi_1\right)(x;p)=\int_a^x (\psi_2(x)-\psi_2(t))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(t))^\mu;p)\times\psi_1(t)\psi_2'(t)\mathrm{d}t, \tag{6}$$

$$\left({}_{\psi_2}\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}\psi_1\right)(x;p)=\int_x^b (\psi_2(t)-\psi_2(x))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(t)-\psi_2(x))^\mu;p)\times\psi_1(t)\psi_2'(t)\mathrm{d}t. \tag{7}$$

It can be noted that

$$\left({}_{\psi_2}\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}1\right)(x;p)=(\psi_2(x)-\psi_2(t))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(t))^\mu;p), \tag{8}$$

$$\left({}_{\psi_2}\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}1\right)(x;p)=(\psi_2(t)-\psi_2(x))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(t)-\psi_2(x))^\mu;p). \tag{9}$$

In the following, we state the Ostrowski inequality which is proved by Ostrowski [19] in 1938.

Theorem 1. *Let $\psi_1\colon I\longrightarrow\mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o, the interior of I and $a,b\in I^o$, $a<b$. If $|\psi_1'(t)|\leq M$ for all $t\in[a,b]$, then for $x\in[a,b]$ we have*

$$\left|\psi_1(x)-\frac{1}{b-a}\int_a^b\psi_1(t)\mathrm{d}t\right|\leq\left[\frac{1}{4}+\frac{(x-((a+b)/2))^2}{(b-a)^2}\right](b-a)M. \tag{10}$$

The Ostrowski inequality has been studied by many researchers to obtain its refinements, generalizations, and extensions. Also, their applications are analyzed for establishing the bounds of relations among special means and for estimations of numerical quadrature rules. For recent developments of Ostrowski inequality, we refer the reader to [8, 9, 11, 20–26] and references therein.

In Section 2, fractional version of Ostrowski inequalities with the help of Mittag–Leffler function has been established. The presented results may be useful in the study of fractional integral operators and their applications. Also, the error bounds of fractional Hadamard inequalities are presented in Section 3.

2. Main Results

First, we establish the following lemma for extended generalized Mittag–Leffler function.

Lemma 1. *If $\omega,\mu,\alpha,l,\gamma,c\in C$, $\mathfrak{R}(\mu),\mathfrak{R}(\alpha),\mathfrak{R}(l)>0$, $\mathfrak{R}(c)>\mathfrak{R}(\gamma)>0$ with $p\geq 0,\delta>0$ and $0<k<\delta+\mathfrak{R}(\mu)$, then*

$$\left(\frac{d}{dt}\right)\left[(\psi_2(t))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega\psi_2(t)^{\mu};p)\right] = \psi_2'(t)\psi_2(t)^{\alpha-2}E^{\gamma,\delta,k,c}_{\mu,\alpha-1,l}(\omega\psi_2(t)^{\mu};p). \tag{11}$$

Proof. We have

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[(\psi_2(t))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega\psi_2(t)^{\mu};p)\right] &= \sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{\omega^n(\mu n+\alpha-1)(\psi_2(t))^{\mu n+\alpha-2}\psi_2'(t)}{(l)_{n\delta}}\\ &= \sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha-1)}\frac{\omega^n(\psi_2(t))^{\mu n+\alpha-2}\psi_2'(t)}{(l)_{n\delta}}.\end{aligned} \tag{12}$$

After simple computation, one can obtain (11).

Next, we give the generalized fractional Ostrowski type inequality containing extended generalized Mittag–Leffler function. □

Theorem 2. *Let $\psi_1\colon I \longrightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o, the interior of I and $a,b \in I^o$, $a<b$. If ψ_1 is an integrable function, $|\psi_1'(\psi_2(t))| \leq M$ for all $t \in [a,b]$ and $\psi_2\colon [a,b] \longrightarrow \mathbb{R}$ be an increasing and positive function on $(a,b]$, having continuous derivative ψ_2' on (a,b), then for $\alpha,\beta \geq 1$, the following inequality for fractional integrals (6) and (7) holds:*

$$\begin{aligned}&\Big|\psi_1(\psi_2(x))((\psi_2(b)-\psi_2(x))^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)+(\psi_2(x)-\psi_2(a))^{\alpha-1}\\ &\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)-\left(\left({}_{\psi_2}\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}\psi_1\circ\psi_2\right)(x;p)+\left({}_{\psi_2}\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}\psi_1\circ\psi_2\right)(x;p)\right)\Big|\\ &\leq M\big((\psi_2(x)-\psi_2(a))^{\alpha}\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)+(\psi_2(b)-\psi_2(x))^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)\\ &-\left(\left({}_{\psi_2}\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}1\right)(x;p)+\left({}_{\psi_2}\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}1\right)(x;p)\right).\end{aligned} \tag{13}$$

Proof. Let $x \in [a,b]$, $t \in [a,x]$, and $\alpha \geq 1$. Then, the following inequality holds for the monotonically increasing function ψ_2 and the Mittag–Leffler function (1):

$$(\psi_2(x)-\psi_2(t))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(t))^{\mu};p)\psi_2'(t) \leq (\psi_2(x)-\psi_2(a))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)\psi_2'(t). \tag{14}$$

From (14) and given condition of boundedness of ψ_1', one can have the following integral inequalities:

$$\begin{aligned}&\int_a^x\left(M-\psi_1'(\psi_2(t))\right)(\psi_2(x)-\psi_2(t))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(t))^{\mu};p)\psi_2'(t)\mathrm{d}t\\ &\leq(\psi_2(x)-\psi_2(a))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)\int_a^x(M-\psi_1'(\psi_2(t)))\psi_2'(t)\mathrm{d}t,\end{aligned} \tag{15}$$

$$\begin{aligned}&\int_a^x(M+\psi_1'(\psi_2(t)))(\psi_2(x)-\psi_2(t))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(t))^{\mu};p)\psi_2'(t)\mathrm{d}t\\ &\leq(\psi_2(x)-\psi_2(a))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}0(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)\int_a^x(M+\psi_1'(\psi_2(t)))\psi_2'(t)\mathrm{d}t.\end{aligned} \tag{16}$$

First, we consider inequality (15) as follows:

$$
\begin{aligned}
&M\int_a^x \left(\psi_2(x)-\psi_2(t)\right)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(x)-\psi_2(t)\right)^{\mu};p\right)\psi_2'(t)\mathrm{d}t-\int_a^x \left(\psi_2(x)-\psi_2(t)\right)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(x)-\psi_2(t)\right)^{\mu};p\right)\\
&\quad\cdot\psi_1'\left(\psi_2(t)\right)\psi_2'(t)\mathrm{d}t\\
&\le\left(\psi_2(x)-\psi_2(a)\right)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)\int_a^x\left(M-\psi_1'\left(\psi_2(t)\right)\right)\psi_2'(t)\mathrm{d}t.
\end{aligned}
\tag{17}
$$

Therefore, (17) takes the following form after integrating by parts and using derivative property (11) and a simple computation:

$$
\begin{aligned}
&\left(\psi_2(x)-\psi_2(a)\right)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)\psi_1\left(\psi_2(x)\right)\\
&\quad-\left({}_{\psi_2}\Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c}\psi_1\circ\psi_2\right)(x;p)\le M\left(\left(\psi_2(x)-\psi_2(a)\right)^{\alpha}\right.\\
&E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)-\left.\left({}_{\psi_2}\Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c}1\right)(x;p)\right).
\end{aligned}
\tag{18}
$$

Similarly, adopting the same pattern from (16), one can obtain

$$
\begin{aligned}
&\left(\psi_2(x)-\psi_2(a)\right)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)\psi_1\left(\psi_2(x)\right)\\
&\quad-\left({}_{\psi_2}\Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c}\psi_1\circ\psi_2\right)(x;p)\ge -M\left(\psi_2(x)-\psi_2(a)\right)^{\alpha}\\
&E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)-\left({}_{\psi_2}\Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c}1\right)(x;p)\Big).
\end{aligned}
\tag{19}
$$

From (18) and (19), the following inequality is obtained:

$$
\begin{aligned}
&\left|\left(\psi_2(x)-\psi_2(a)\right)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)\psi_1\left(\psi_2(x)\right)-\left(\psi_2\Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c}\psi_1\circ\psi_2\right)(x;p)\right|\\
&\quad\le M\left(\left(\psi_2(x)-\psi_2(a)\right)^{\alpha}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)-\left(\psi_2\Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c}1\right)(x;p)\right).
\end{aligned}
\tag{20}
$$

Now, on the contrary, we let $x\in[a,b]$, $t\in[x,b]$, and $\beta\ge1$. Then, the following inequality holds for Mittag–Leffler function:

$$
\left(\psi_2(t)-\psi_2(x)\right)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(t)-\psi_2(x)\right)^{\mu};p\right)\psi_2'(t)\le\left(\psi_2(b)-\psi_2(x)\right)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(b)-\psi_2(x)\right)^{\mu};p\right)\psi_2'(t).
\tag{21}
$$

From (21) and the condition of boundedness of ψ_1', one can have the following integral inequalities:

$$
\begin{aligned}
&\int_x^b\left(M-\psi_1'\left(\psi_2(t)\right)\right)\left(\psi_2(t)-\psi_2(x)\right)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(t)-\psi_2(x)\right)^{\mu};p\right)\psi_2'(t)\mathrm{d}t\\
&\quad\le\left(\psi_2(b)-\psi_2(x)\right)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(b)-\psi_2(x)\right)^{\mu};p\right)\int_x^b\left(M-\psi_1'\left(\psi_2(t)\right)\right)\psi_2'(t)\mathrm{d}t,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&\int_x^b\left(M+\psi_1'\left(\psi_2(t)\right)\right)\left(\psi_2(t)-\psi_2(x)\right)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(t)-\psi_2(x)\right)^{\mu};p\right)\psi_2'(t)\mathrm{d}t\\
&\quad\le\left(\psi_2(b)-\psi_2(x)\right)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}\left(\omega\left(\psi_2(b)-\psi_2(x)\right)^{\mu};p\right)\int_x^b\left(M+\psi_1'\left(\psi_2(t)\right)\right)\psi_2'(t)\mathrm{d}t.
\end{aligned}
\tag{23}
$$

Following the same procedure as we did for (15) and (16), one can obtain from (22) and (23) the following modulus inequality:

$$\begin{aligned}&\left|(\psi_2(b)-\psi_2(x))^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)\psi_1(\psi_2(x))-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}\psi_1\circ\psi_2\right)(x;p)\right|\\&\leq M\left((\psi_2(b)-\psi_2(x))^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}1\right)(x;p)\right).\end{aligned}\tag{24}$$

Inequalities (20) and (24) give (13) which is the required inequality.

In the following, we give direct consequences of above theorem. □

Corollary 1. *If we put $\alpha=\beta$ in (13), then we get the following fractional integral inequality:*

$$\begin{aligned}&\Big|\psi_1(\psi_2(x))(\psi_2(b)-\psi_2(x))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)+(\psi_2(x)-\psi_2(a))^{\alpha-1}\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)\\&\quad-\left(\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}\psi_1\circ\psi_2\right)(x;p)+\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,b^-}\psi_1\circ\psi_2\right)(x;p)\right)\Big|\\&\leq M\Big((\psi_2(x)-\psi_2(a))^{\alpha}\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)+(\psi_2(b)-\psi_2(x))^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)\\&\quad-\left(\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}1\right)(x;p)+\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,b^-}1\right)(x;p)\right)\Big).\end{aligned}\tag{25}$$

Remark 1

(i) If we put $\psi_2(x)=x$ in (13), then we obtain Theorem 5 in [9]

(ii) If we put $\omega=p=0$ and $\psi_2(x)=x$ in (13), then we obtain Theorem 1 in [6]

(iii) If we put $\alpha=\beta=1$, $\psi_2(x)=x$, and $\omega=p=0$ in (13), then we obtain Ostrowski inequality (10)

(iv) If we put $\psi_2(x)=x$ in (25), then we obtain Corollary 1 in [9]

The next result is a general form of fractional Ostrowski inequality containing generalized Mittag–Leffler function.

Theorem 3. *Let $\psi_1\colon I\longrightarrow\mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o, the interior of I and $a,b\in I^o$, $a<b$. If ψ_1 is integrable function and $m<\psi_1'(\psi_2(t))\leq M$ for all $t\in[a,b]$ and $\psi_2\colon[a,b]\longrightarrow\mathbb{R}$ be an increasing and positive function on $(a,b]$, having continuous derivative ψ_2' on (a,b), then, for $\alpha,\beta\geq1$, the following inequalities for fractional integrals (6) and (7) hold:*

$$\begin{aligned}&\psi_1(\psi_2(x))(\psi_2(x)-\psi_2(a))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)-(\psi_2(b)-\psi_2(x))^{\beta-1}\\&\quad\times E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}\psi_1\circ\psi_2\right)(x;p)\\&\quad-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}\psi_1\circ\psi_2\right)(x;p)\Big)\leq M(\psi_2(x)-\psi_2(a))^{\alpha}\\&\quad\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}1\right)(x;p))-m\Big((\psi_2(b)-\psi_2(x))^{\beta}\\&\quad\times E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}1\right)(x;p),\end{aligned}\tag{26}$$

$$\begin{aligned}&\psi_1(\psi_2(x))(\psi_2(b)-\psi_2(x))^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)-(\psi_2(x)-\psi_2(a))^{\alpha-1}\\&E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)-\left(\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}\psi_1\circ\psi_2\right)(x;p)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}\psi_1\circ\psi_2\right)(x;p)\right)\\&\quad\leq M\left((\psi_2(b)-\psi_2(x))^{\beta}\times E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}1\right)(x;p)\right)\\&\qquad-m\left((\psi_2(x)-\psi_2(a))^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}1\right)(x;p)\right).\end{aligned}\tag{27}$$

Proof. The proof is similar to the proof of Theorem 2, just after comparing conditions on derivative of ψ_1, so we left it for the reader.

Some comments on the abovementioned result are given as follows. □

Remark 2

(i) If we put $\omega = p = 0$ and $\psi_2(x) = x$ in (26) and (27), then we obtain Theorem 1 in [6]

(ii) If we put $m = -M$ in Theorem 3, then with some rearrangements we obtain Theorem 2

(iii) If we put $\psi_2(x) = x$ in (26) and (27), then we obtain Theorem 6 in [9]

In the following, we have established a result related to fractional Ostrowski inequality containing generalized Mittag–Leffler function.

Theorem 4. *Let $\psi_1\colon I \longrightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o, the interior of I and $a, b \in I^o$, $a < b$. If ψ_1 is integrable function, $|\psi_1'(\psi_2(t))| \leq M$ for all $t \in [a,b]$and and $\psi_2\colon [a,b] \longrightarrow \mathbb{R}$ be an increasing and positive function on $(a,b]$, then, for $\alpha, \beta \geq 1$, the following inequality for fractional integrals (6) and (7) holds:*

$$\begin{aligned}
&\Big|\psi_1(\psi_2(b))(\psi_2(b)-\psi_2(x))^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)+\psi_1(\psi_2(a))\\
&\cdot(\psi_2(x)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)-\Big(\Big(\psi_2\Upsilon_{\mu,\beta-1,l,\omega,x^+}^{\gamma,\delta,k,c}\psi_1\circ\psi_2\Big)(b;p)-\Big(\psi_2\Upsilon_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c}\psi_1\circ\psi_2\Big)(a;p)\Big)\Big|\\
&\leq M\Big((\psi_2(a)-\psi_2(x))^{\alpha}\times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)+(\psi_2(b)-\psi_2(x))^{\beta}E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)\\
&-\Big(\Big(\psi_2\Upsilon_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c}1\Big)(a;p)+\Big(\psi_2\Upsilon_{\mu,\beta-1,l,\omega,x^+}^{\gamma,\delta,k,c}1\Big)(b;p)\Big)\Big).
\end{aligned}\tag{28}$$

Proof. Let $x \in [a,b]$, $t \in [a,x]$, and $\alpha \geq 1$. Then, the following inequality holds true for Mittag–Leffler function:

$$\begin{aligned}
&(\psi_2(t)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(t)-\psi_2(a))^{\mu};p)\psi_2'(t)\\
&\leq(\psi_2(x)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)\psi_2'(t).
\end{aligned}\tag{29}$$

From (29) and given condition of boundedness on ψ_1', one can have the following integral inequalities:

$$\begin{aligned}
&\int_a^x(M-\psi_1'(\psi_2(t)))(\psi_2(t)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(t)-\psi_2(a))^{\mu};p)\psi_2'(t)\mathrm{d}t\\
&\leq(\psi_2(x)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)\int_a^x(M-\psi_1'(\psi_2(t)))\psi_2'(t)\mathrm{d}t,
\end{aligned}\tag{30}$$

$$\begin{aligned}
&\int_a^x(M+\psi_1'(\psi_2(t)))(\psi_2(t)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(t)-\psi_2(t))^{\mu};p)\psi_2'(t)\mathrm{d}t\\
&\leq(\psi_2(x)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)\int_a^x(M+\psi_1'(\psi_2(t)))\psi_2'(t)\mathrm{d}t.
\end{aligned}\tag{31}$$

First, we consider inequality (30) as follows:

$$\begin{aligned}
&M\int_a^x(\psi_2(t)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(t)-\psi_2(a))^{\mu};p)\psi_2'(t)\mathrm{d}t\\
&-\int_a^x(\psi_2(t)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(t)-\psi_2(a))^{\mu};p)\psi_2'(t)\psi_1'(\psi_2(t))\mathrm{d}t\\
&\leq(\psi_2(x)-\psi_2(a))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)\int_a^x(M-\psi_1'(\psi_2(t)))\psi_2'(t)\mathrm{d}t.
\end{aligned}\tag{32}$$

Therefore, (32) takes the following form after integrating by parts and using derivative property (11) and a simple computation

$$\begin{aligned}&\left(\psi_2(x)-\psi_2(a)\right)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)\psi_1\left(\psi_2(a)\right)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}\psi_1\circ\psi_2\right)(a;p)\\&\leq M\left(\left(\psi_2(x)-\psi_2(a)\right)^{\alpha}\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}1\right)(a;p)\right).\end{aligned}\tag{33}$$

Similarly, adopting the same pattern from (31), one can obtain

$$\begin{aligned}&\left(\psi_2(x)-\psi_2(a)\right)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)\psi_1\left(\psi_2(a)\right)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}\psi_1\circ\psi_2\right)(a;p)\\&\geq -M\left(\left(\psi_2(x)-\psi_2(a)\right)^{\alpha}\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}1\right)(a;p)\right).\end{aligned}\tag{34}$$

From (33) and (34), the following inequality is obtained:

$$\begin{aligned}&\left|\left(\psi_2(x)-\psi_2(a)\right)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(\psi_2(x)-\psi_2(a)\right)^{\mu};p\right)\psi_1\left(\psi_2(a)\right)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}\psi_1\circ\psi_2\right)(a;p)\right|\\&\leq M\left(\left(\psi_2(x)-\psi_2(a)\right)^{\alpha}\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(\left(\psi_2(x)-\psi_2(a)\right)^{\alpha}\right)^{\mu};p\right)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}1\right)(a;p)\right).\end{aligned}\tag{35}$$

Now, on the contrary, we let $x\in[a,b]$, $t\in[x,b]$, and $\beta\geq 1$. Then, the following inequality holds for Mittag–Leffler function:

$$\begin{aligned}&\left(\psi_2(b)-\psi_2(t)\right)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}\left(\omega\left(\psi_2(b)-\psi_2(t)\right)^{\mu};p\right)\psi_2'(t)\leq\left(\psi_2(b)-\psi_2(x)\right)^{\beta-1}\\&\times E^{\gamma,\delta,k,c}_{\mu,\beta,l}\left(\omega\left(\psi_2(b)-\psi_2(x)\right)^{\mu};p\right)\psi_2'(t).\end{aligned}\tag{36}$$

From (36) and given condition of boundedness of ψ_1', one can have the following integral inequalities:

$$\begin{aligned}&\int_x^b\left(M-\psi_1'\left(\psi_2(t)\right)\right)\left(\psi_2(b)-\psi_2(t)\right)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}\left(\omega\left(\psi_2(b)-\psi_2(t)\right)^{\mu};p\right)\psi_2'(t)\mathrm{d}t\\&\leq\left(\psi_2(b)-\psi_2(x)\right)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}\left(\omega\left(\psi_2(b)-\psi_2(x)\right)^{\mu};p\right)\int_x^b\left(M-\psi_1'\left(\psi_2(t)\right)\right)\psi_2'(t)\mathrm{d}t,\end{aligned}\tag{37}$$

$$\begin{aligned}&\int_x^b\left(M+\psi_1'\left(\psi_2(t)\right)\right)\left(\psi_2(b)-\psi_2(t)\right)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}\left(\omega\left(\psi_2(b)-\psi_2(t)\right)^{\mu};p\right)\psi_2'(t)\mathrm{d}t\\&\leq\left(\psi_2(b)-\psi_2(x)\right)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}\left(\omega\left(\psi_2(b)-\psi_2(x)\right)^{\mu};p\right)\int_x^b\left(M+\psi_1'\left(\psi_2(t)\right)\right)\psi_2'(t)\mathrm{d}t.\end{aligned}\tag{38}$$

Following the same procedure as we did for (30) and (31), one can obtain from (37) and (38) the following modulus inequality:

$$\begin{aligned}&\left|(\psi_2(b)-\psi_2(x))^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)\psi_1(\psi_2(b))-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,x^+}\psi_1\circ\psi_2\right)(b;p)\right|\\&\quad\le M\left((\psi_2(b)-\psi_2(x))^{\beta}\times E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)-\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,x^+}1\right)(b;p)\right).\end{aligned}\tag{39}$$

Inequalities (35) and (39) give (28) which is required inequality.

Some direct consequences of the above theorem are given below. □

Corollary 2. *If we put $\alpha=\beta$ in (28), then we get the following fractional integral inequality:*

$$\begin{aligned}&\Big|\psi_1(\psi_2(b))(\psi_2(b)-\psi_2(x))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)+\psi_1(\psi_2(a))\\&\quad\cdot(\psi_2(x)-\psi_2(a))^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)-\left(\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^+}\psi_1\circ\psi_2\right)(b;p)+\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}\psi_1\circ\psi_2\right)(a;p)\right)\Big|\\&\quad\le M\Big((\psi_2(x)-\psi_2(a))^{\alpha}\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(x)-\psi_2(a))^{\mu};p)+(\psi_2(b)-\psi_2(x))^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(b)-\psi_2(x))^{\mu};p)\\&\quad-\left(\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}1\right)(a;p)+\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^+}1\right)(b;p)\right)\Big).\end{aligned}\tag{40}$$

Remark 3

(i) If we put $\psi_2(x)=x$ in (28), then we obtain Theorem 5 in [9]

(ii) If we put $\omega=p=0$ and $\psi_2(x)=x$ in (28), then we obtain Theorem 1 in [6]

(iii) If we put $\psi_2(x)=x$ in (40), then we obtain Corollary 2 in [9]

3. Applications

In this section, we just describe some applications of Theorem 4 and leave such applications of other results for the reader. By applying Theorem 4 at end points of the interval $[a,b]$ and adding the resulting inequalities, one obtains the error bounds of compact form of the fractional Hadamard inequality.

Theorem 5. *Under the assumptions of Theorem 4, the following estimation of Hadamard inequality can be obtained:*

$$\begin{aligned}&\left|\frac{(\psi_1(\psi_2(b))+\psi_1(\psi_2(b)))}{2}\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}1\right)(b;p)-\frac{1}{2}\left(\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,b^-}\psi_1\circ\psi_2\right)(a;p)+\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}\psi_1\circ\psi_2\right)(b;p)\right|\\&\quad\le\frac{M}{2}(\psi_2(b)-\psi_2(a)^{\alpha})E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(b)-\psi_2(a))^{\mu};p)-\frac{M}{2}\left(\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,b^-}1\right)(a;p)\right.\\&\quad\left.+\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}1\right)(b;p)\right).\end{aligned}\tag{41}$$

Proof. By putting $x=a$, $\alpha=\beta$, and $x=b$ in (40) then adding the resulting inequalities, we obtain

$$\begin{aligned}&\Big|(\psi_1(\psi_2(b))+\psi_1(\psi_2(b)))(\psi_2(b)-\psi_2(a))(a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(b)-\psi_2(a))^{\mu};p)\\&\quad-\left(\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,b^-}\psi_1\circ\psi_2\right)(a;p)+\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}\psi_1\circ\psi_2\right)(b;p)\right)\Big|\\&\quad\le M(\psi_2(b)-\psi_2(a))^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\psi_2(b)-\psi_2(a))^{\mu};p)-\left(\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,b^-}1\right)(a;p)+\left(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}1\right)(b;p)\right).\end{aligned}\tag{42}$$

Multiplying both sides of the above inequality by 1/2 and using (8) and (9), inequality (41) can be obtained. □

Remark 4. If in (41) α is replaced by $\alpha+1$ and ω by $\omega\prime=\omega/((g(b)-g(a))^{\mu})$, then we get an error bound of the Hadamard inequality given in Theorem 1 in [20].

Theorem 6. *Under the assumptions of Theorem 4, the following inequality can be obtained:*

$$\begin{aligned}&\Big|\psi_1(\psi_2(b))\Big(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,b^-}1\Big)\Big(\frac{a+b}{2};p\Big)+\psi_1(\psi_2(a))\Big(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}1\Big)\Big(\frac{a+b}{2};p\Big)\\&-\Big(\Big(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,(a+b/2)^+}\psi_1\circ\psi_2\Big)(b;p)+\Big(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,(a+b/2)^-}\psi_1\circ\psi_2\Big)(a;p)\Big)\Big|\\&\leq M\Bigg(\Big(\psi_2\Big(\frac{a+b}{2}\Big)-\psi_2(a)\Big)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\Big(\omega\Big(\psi_2\Big(\frac{a+b}{2}\Big)-\psi_2(a)\Big)^{\mu};p\Big)\\&+\Big(\psi_2(b)-\psi_2\Big(\frac{a+b}{2}\Big)\Big)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\Big(\omega\Big(\psi_2(b)-\psi_2\Big(\frac{a+b}{2}\Big)\Big)^{\mu};p\Big)\\&-\Big(\Big(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,(a+b/2)^-}1\Big)(a;p)+\Big(\psi_2\Upsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,(a+b/2)^+}1\Big)(b;p)\Big)\Bigg).\end{aligned}\tag{43}$$

Proof. By putting $x=(a+b)/2$ for $\alpha=\beta$ in (40), (43) can be obtained. □

4. Concluding Remarks

We have established generalized fractional integral inequalities of Ostrowski type. By applying boundedness of a differentiable function and using properties of an extended generalized Mittag–Leffler function different generalized versions of Ostrowski type inequalities are analyzed. Also, some deductions from results of this paper are connected with already published results. Furthermore, all the results can be calculated for fractional integral operators defined in [2, 3, 5, 6, 27], and we left it for the reader.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors' Contributions

All authors have equal contributions.

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Research Article

Traveling Wave Solution of the Olver–Rosenau Equation Solved by Dynamics System

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Olver–Rosenau equations presented by Olver and Rosenau can be rewritten to the dynamic system by the wave transformation. The system is a Hamiltonian system with the first integral, and its phase-space and equilibrium point analysis are given in different parameter spaces in detail. On this basis, we can derive various solutions of the original equation relating these orbits in different phase-space planes, and the theoretical basis of the numerical solution is provided for engineering application and production practice.

1. Introduction

In recent years, nonlinear partial differential equations (NLPDEs) are more and more extensively used to engineering application and production practice. Because of the difficulty of solving these equations, so many methods have been presented in the last decades. The Camassa–Holm (CH) equation including the cubic term, that is, the Fokas–Olver–Rosenau–Qiao (FORQ) equation,

$$\sigma_t + \frac{1}{2}\left[\left(v^2 \pm (v_x)^2\right)\sigma\right]_x = 0, \quad \sigma = v \pm (v_{xx}), \tag{1}$$

has been studied for a long time and achieved so many results, such as the solution of Cauchy initial value problem [1], Holder continuous [2], the algebro-geometric solutions [3], the Cauchy problem of the generalized equation [4], and the nonuniqueness for the equation [5]. A lot of solving methods for the nonlinear partial differential equation are discussed to be applied in engineering and practice areas, such as the sine-Gordon expansion method [6] and the travelling wave method and its conservation laws [7], and so many examples are in this regard. In this paper, the Olver–Rosenau equation

$$\sigma_t = bv_x + \frac{1}{2}\left[\left(v^2 \pm (v_x)^2\right)\sigma\right]_x, \quad \sigma = v \pm (v_{xx}), \tag{2}$$

was discussed by Olver and Rosenau through a reshuffling procedure of the Hamiltonian operators in 1996, and they found that the equation was changed into the Hamiltonian system by the bitransformation structure of the mKdV equation [8]. Rosenau believed the nonanalytic solitary waves of the equation in 1997 [9]. Later, the equation was derived in 2013 [10]. The resulting Hamiltonian equations are considered by the dynamical system theory and a phase-space analysis of their singular points. Those results of the study proved that the equations can support double compacton solutions. They found that the new Olver–Rosenau compactons are different from the well-known Rosenau–Hyman compacton and Cooper–Shepard–Sodano compacton.

It was recently introduced by Li in [11], but Li did not give the solution because of the complexity of the integral. We will give more detailed discussion in this paper. In the second section, we discuss bifurcations and phase portraits of the system in all parameters. In the third section, smooth (or bright) solitary wave solutions of the system and their parametric representations are obtained in detail.

Let $v = \phi(x - ct) = \phi(\zeta)$; equation (2) follows from

$$-c\left(\phi_\zeta \pm \phi_{\zeta\zeta\zeta}\right) = b\phi_\zeta + \frac{1}{2}\left[\left(\phi^2 \pm \phi_\zeta^2\right)\left(\phi \pm \phi_{\zeta\zeta}\right)\right]_\zeta. \tag{3}$$

Integrating once with respect to ζ and letting $d\phi/d\zeta = \omega$, equation (3) can be rewritten as

$$\frac{d\omega}{d\zeta} = -\frac{\phi\left(\phi^2 \pm \omega^2 + 2(b+c)\right) + k}{\omega^2 \pm \phi^2 \pm 2c}, \quad (4)$$

which has the first integral

$$h(\phi,\omega) = \frac{1}{4}\left(\phi^2 \pm \omega^2\right)^2 + (b+c)\phi^2 \pm c\omega^2 + k\phi, \quad (5)$$

where k is an integral constant.

Without loss of generality, we consider the case + instead of "±." Equation (5) is simplified as the dynamic system

$$\frac{d\phi}{d\zeta} = \omega,$$
$$\frac{d\omega}{d\zeta} = -\frac{\phi\left(\phi^2 + \omega^2 + 2(b+c)\right) + k}{\omega^2 + \phi^2 + 2c}, \quad (6)$$

with the Hamiltonian

$$h(\phi,\omega) = \frac{1}{4}\left(\phi^2 + \omega^2\right)^2 + (b+c)\phi^2 + c\omega^2 + k\phi. \quad (7)$$

Obviously, system (6) is a planar dynamical system with three parameters depended on the parameter group (b, c, and k). All possible phase portraits of (6) in the (ϕ,ω) phase are discussed under the conditions of the different parameter group (b, c, and k).

We notice that the right-hand side of the second equation in (6) is not continuous, while $\omega^2 + \phi^2 + 2c = 0$. On the circle line, $\omega^2 + \phi^2 = -2c$, in the phase plane (ϕ,ω), $\phi_{\zeta\zeta}$ is not well defined. This implies that differential system (6) could have traveling wave solutions with nonsmoothness.

2. Phase Portraits of the System

Imposing the transformation $d\zeta = (\omega^2 + \phi^2 + 2c)d\tau$, when $\omega^2 + \phi^2 + 2c \neq 0$, equation (6) leads to associated regular system

$$\begin{cases} \dfrac{d\phi}{d\tau} = \omega\left(\omega^2 + \phi^2 + 2c\right), \\ \dfrac{d\omega}{d\tau} = -\phi\left(\phi^2 + \omega^2 + 2(b+c)\right) - k. \end{cases} \quad (8)$$

This system has the same first integral as equation (6). Apparently, the singular curve $\omega^2 + \phi^2 + 2c = 0$ is related to the singular solution of equation (8). Near the circumference, the variable τ is a fast variable, while the variable ζ is a slow variable in the sense of the geometric singular perturbation theory.

In order to find the equilibrium points of system (8), let $f(\phi) = \phi(\phi^2 + 2c + 2b) + k$ and $f'(\phi) = 0$; we obtain $\phi_1 = -\sqrt{(-2/3)(b+c)}$ and $\phi_2 = -\sqrt{(-2/3)(b+c)}$ if $(b+c) < 0$. The zero points of $f(\phi)$ are estimated by the monotonicity of the function ϕ based on zero points of the derivative function $f'(\phi)$. Let $A = -6(b+c)$, $B = -9k$, and $C = (2c+2b)^2$; then, the discriminant $S = B^2 - 4AC$ of the cubic polynomial $f(\phi) = 0$ is just that $S = 81g^2 + 96(b+c)^3$. It is easy to see that, for given b and c when $k^2 < k_1^2 = (-32/27)(b+c)^3$, we have $S < 0$. It follows that there exist three simple real roots ϕ_{31}, ϕ_{32}, and ϕ_{33} of $f(\phi)$ satisfying $\phi_{31} < \phi_1 < \phi_{32} < \phi_2 < \phi_{33}$.

If $k^2 = k_1^2$, there exist two simple real roots ϕ_{21} and ϕ_{22} of $f(\phi)$ satisfying $\phi_{21} = \phi_1 < \phi_2 < \phi_{22}$

If $k^2 > k_1^2$, there exists one simple real root ϕ_{11}

If $A = B$, there exists one simple real root $\phi = 0$

On the singular circle line, there exist two equilibrium points $S_{\mp} = (-k/(2b), \mp Y_s)$ of system (8) with $Y_s = \sqrt{-k^2/(4b^2) - 2c}$ if $k^2/(4b^2) + 2c < 0$.

Let $M(\phi_j, y_j)$ be the Jacobi matrix of system (8) at an equilibrium point $E_j(\phi_j, y_j)$; we have

$$J\left(\phi_j, 0\right) = \det M\left(\phi_j, 0\right) = 3\phi^4 + 2(4c+b)\phi^2 + 4bc + 4c^2,$$
$$J\left(-\frac{k}{(2b)}, \mp Y_s\right) = \det M\left(-\frac{k}{(2b)}, \mp Y_s\right) = -\left(\frac{2k^2}{b^2} + 4b\right)Y_s^2. \quad (9)$$

By the theory of planar dynamical systems, for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$ and $(\mathrm{trace}M)^2 - 4J < 0\,(>0)$, then it is a center point (a node point); and if $J = 0$ and the Poincare index of the equilibrium point is 0, then this equilibrium point is cusped (see [12]).

Let $h_i = H(\phi_i, 0)$ and $h_s = H(-k/(2b), \mp Y_s)$, where H comes from equation (5).

(1) If $S = 0$, there exist two simple real roots, and then $k^2 = k_1^2 = (-32/27)(b+c)^3$, $\phi_{21} = (3k)/(2(b+c))$, $\phi_{22} = -(3k)/(4(b+c))$.

  (i) When $b > 0$ and $-4b < c < -b$, $(\phi_{21}, 0)$ is a saddle point. $(\phi_{22}, 0)$ is the high-order singular point, see Figure 1.

  (ii) When $c < -4b < 0$ or $b < 0$, $c < -b$, $(\phi_{21}, 0)$ is a center point. $(\phi_{22}, 0)$ is the high-order singular point, see Figures 2 and 3.

(2) If $A = B$, there exists only one simple real root $\phi = 0$, and then $k = (2/3)(b+c)$.

  (i) When $b > 0$, $-b < c < 0$ or $b < 0$, and $0 < c < -b$, $(0,0)$ is a saddle point from the high-order singular point, see Figures 4 and 5.

  (ii) When $b > 0$, $c > 0$ (Figure 6) or $b > 0$, $c < -b$ (Figure 7) or $b < 0$, $c > -b$ (Figure 8) or $b < 0$, $c < 0$ (Figure 9), $(0,0)$ is a center point from the high-order singular point.

(3) If $S < 0$, there exist three simple real roots, and then $k^2 < k_1^2 = (-32/27)(b+c)^3$, $\phi_{31} = (-2/3)\sqrt{-6(b+c)}\cos(\theta/3)$, $\phi_{32} = (1/3)\sqrt{-6(b+c)}(\cos(\theta/3) + \sqrt{3}\sin(\theta/3))$, and $\phi_{33} = (1/3)\sqrt{-6(b+c)}(\cos(\theta/3) - \sqrt{3}\sin(\theta/3))$, where $\theta = \arccos T$, $T = (27k)/(2(-6(b+c))^{3/2})$.

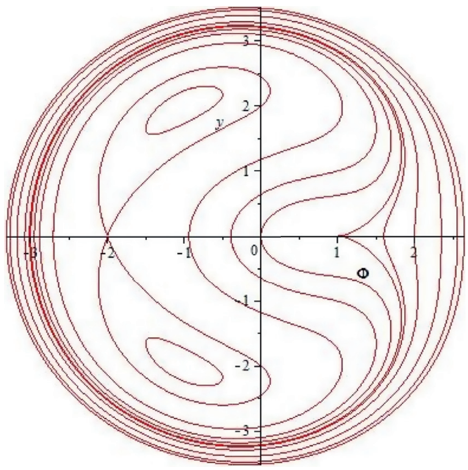

Figure 1: One is a saddle point, and another is a high-order singular point ($b > 0$ and $-4b < c < -b$).

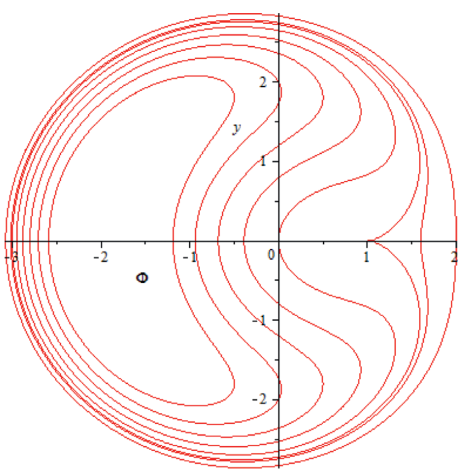

Figure 2: One is a center point, and another is a high-order singular point $(c < -4b < 0)$.

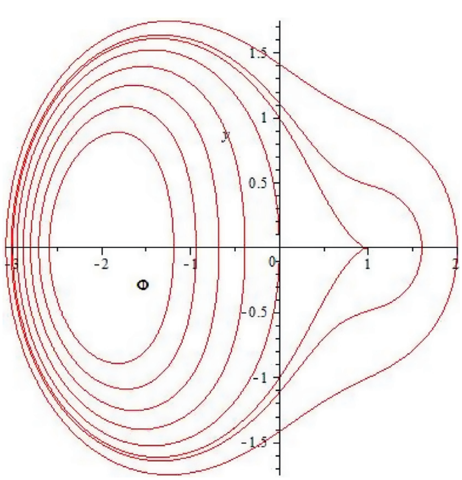

Figure 3: One is a center point, and another is a high-order singular point $(b < 0, c < -b)$.

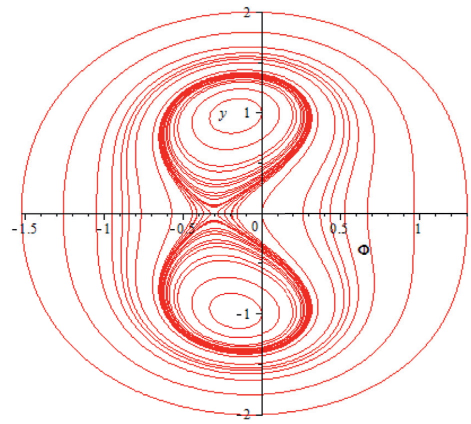

Figure 4: $(0, 0)$ is a saddle point from the high-order singular point $(b > 0, -b < c < 0)$.

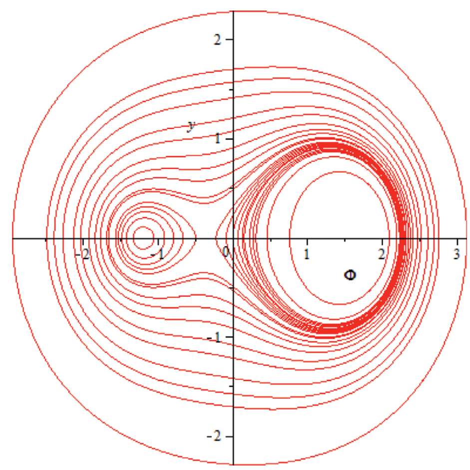

Figure 5: $(0, 0)$ is a saddle point from the high-order singular point $(b < 0, 0 < c < -b)$.

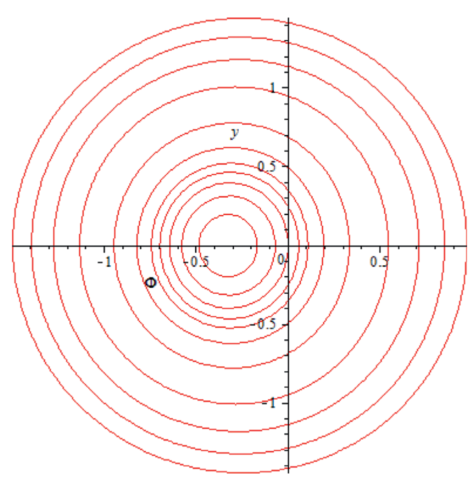

Figure 6: $(0, 0)$ is a center point from the high-order singular point $(b > 0, c > 0)$.

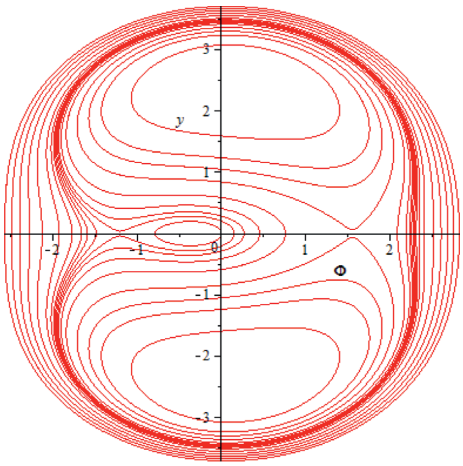

Figure 7: $(0, 0)$ is a center point from the high-order singular point $(b > 0, c < -b)$.

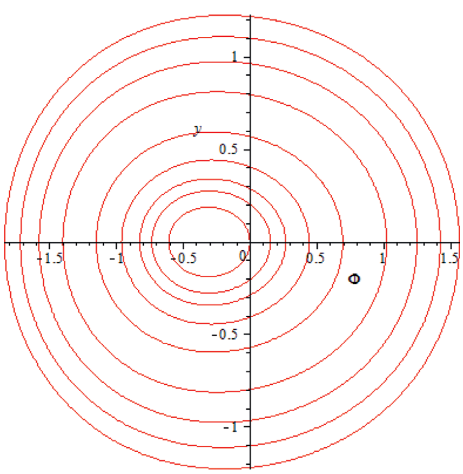

Figure 8: $(0, 0)$ is a center point from the high-order singular point $(b < 0, -b < c)$.

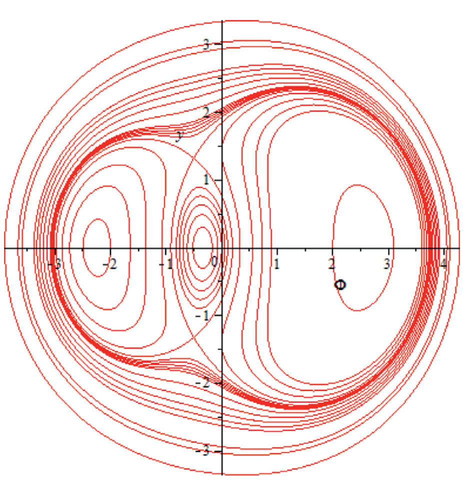

Figure 9: $(0, 0)$ is a center point from the high-order singular point $(b < 0, 0 > c)$.

(i) When $b + c < 0$, $b > 2c$, $(\phi_{31}, 0)$, $(\phi_{32}, 0)$, and $(\phi_{33}, 0)$ are centers. $(-k/(2b), \mp Y_s)$ are saddle points, see Figure 10.

(ii) When $b + c < 0$, $b < 2c$, $(\phi_{31}, 0)$, $(\phi_{32}, 0)$, and $(\phi_{33}, 0)$ are centers. $(-k/(2b), \mp Y_s)$ are saddle points, see Figure 11.

(4) If $S > 0$, there exists one simple real root, and then $k^2 > k_1^2 = (-32/27)(b + c)^3$ and $\phi_{11} = (-1/3)(Y_1^{1/3} + Y_2^{1/3})$, where $Y_1 = (3/2)(9k + \sqrt{81k^2 + 96(b + c)^3})$ and $Y_2 = 3/2(9k - \sqrt{81k^2 + 96(b + c)^3})$.

(i) When $b < 0$, $c < 0$, $(\phi_{11}, 0)$ is a center, see Figure 12.

(ii) When $b > 0$, $c > 0$, $(\phi_{11}, 0)$ is also a center, see Figure 13.

We obtain four class figures totally including 13 phase portraits under the conditions of different parameters. An orbit in a figure is related to Hamiltonian h of equation (7). An orbit derives a class solution of system (6), and at the same time, it is appropriative to the original partial differential equation. In the next section, we mainly consider how to solve system (6).

3. Smooth (Bright) Solitary Wave Solutions and Their Parametric Representations

As we know, system (6) for a fixed integral constant h has an explicit solution. However, the scope of value h is calculated by different orbits corresponding to different parameter conditions in the phase portraits from Figures 1 to 13.

$$\omega^2 = 2\sqrt{h + c^2 - b\phi^2 - k\phi} - (\phi^2 + 2c),$$
$$\frac{d\phi}{d\zeta} = \omega = \sqrt{2\sqrt{h + c^2 - b\phi^2 - k\phi} - (\phi^2 + 2c)}. \tag{10}$$

To write this equation as integral form,

$$\int \frac{d\phi}{\sqrt{2\sqrt{h + c^2 - b\phi^2 - k\phi} - (\phi^2 + 2c)}} = \int d\zeta. \tag{11}$$

As it is very difficult to solve the left integral directly, we believe that square root $F = h + c^2 - b\phi^2 - k\phi$ plays a critical role in our discussion. If F is a perfect square of the function ϕ, we can obtain analytical solution because of the simplicity of the integral. If F is a complete square, then the algebraic expression in the first root in the above formula becomes rational, and the integration is relatively easy solution.

3.1. Analytical Solution. Supporting that

$$F = h + c^2 - b\phi^2 - k\phi = -b\left(\phi + \frac{k}{2b}\right)^2, \tag{12}$$

If we want to remove the second root sign of equation (11), we must make F to be complete square, and the necessary and sufficient condition for F to be completely square is also given:

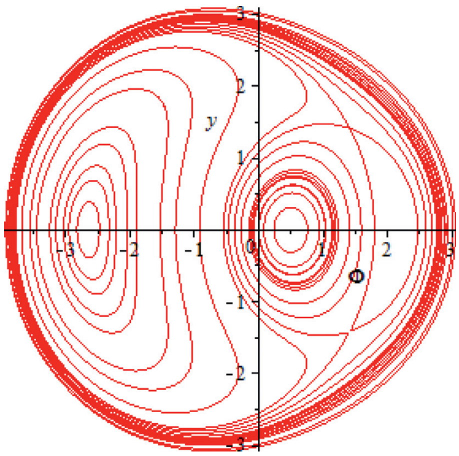

Figure 10: Three singular points are centers, and another is saddle ($b + c < 0, b > 2c$).

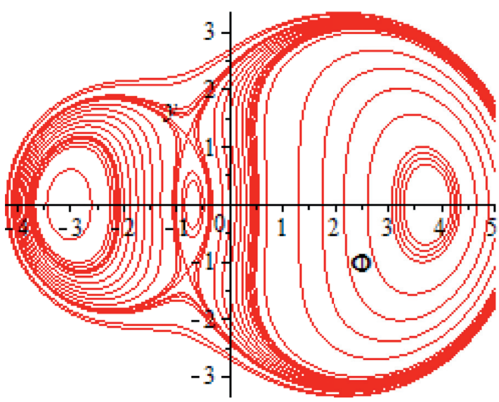

Figure 11: Three singular points are centers, and another is saddle ($b + c < 0, b < 2c$).

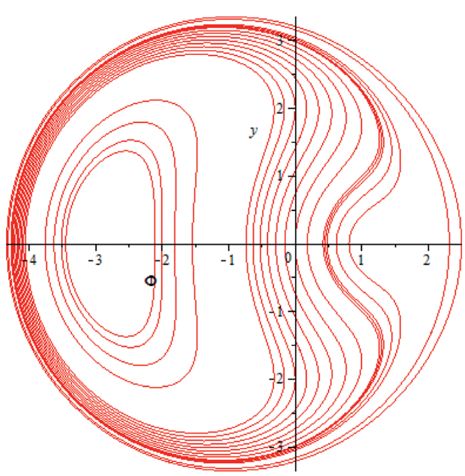

Figure 12: $b < 0$, $c < 0$, and $(\phi_{11}, 0)$ is a center.

$$\int \frac{d\phi}{\sqrt{D - (\phi - \sqrt{-b})^2}} = \int d\zeta, \tag{13}$$

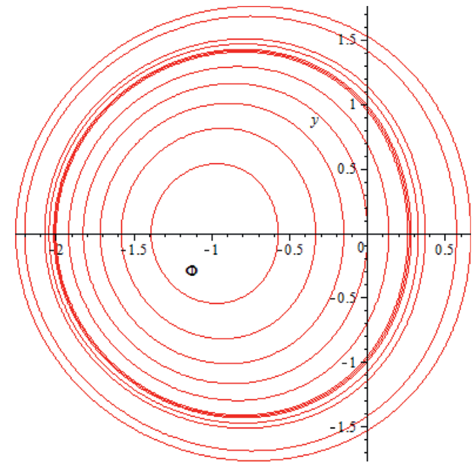

Figure 13: $b > 0$, $c > 0$, and $(\phi_{11}, 0)$ is also a center.

where $D = b^2 + 2c \pm 2\sqrt{h + c^2} \geq 0$ and $-D + \sqrt{-b} < \phi < D + \sqrt{-b}$, $b < 0$. After integrating, we obtain (C is an integral constant)

$$\frac{\phi - \sqrt{-b}}{\sqrt{D - (\phi - \sqrt{-b})^2}} = \tan(x - c * t + C). \tag{14}$$

3.2. Approximate Solution. If $F = \sqrt{h + c^2 - b\phi^2 - k\phi}$ is not a rational function, the integral of equation (11) is very difficult to be calculated directly. If F cannot be expressed as a complete square, we can only carry out Taylor expansion of F; otherwise, equation (13) cannot calculate the integral, and the solution of the original equation cannot be obtained. While we need to approximate nonlinear function, our right choice is the Taylor display:

$$\begin{aligned}(1 + x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \cdots \\ &\quad + \frac{\alpha(\alpha - 1 \cdots (\alpha - n + 1))}{n!} + R_n(x),\end{aligned} \tag{15}$$

where the remainder

$$R_n(x) = \frac{\alpha(\alpha - 1)\cdots(\alpha - n)}{(n + 1)!}(1 + \theta x)^{\alpha - n + 1}\ (0 < \theta < 1),$$

$$\begin{aligned}F &= \sqrt{h + c^2 - b\phi^2 - k\phi} = \left(h + \frac{k^2}{4b} - b\left(\phi + \frac{k}{2b}\right)^2\right)^{1/2} \\ &= \left(\frac{4b(h + c^2) + k^2}{4b} - b\left(\phi + \frac{k}{2b}\right)^2\right)^{1/2} \\ &= \frac{1}{2}\sqrt{\frac{4b(h + c^2) + k^2}{b}}\left(1 - \frac{4b^2}{4bh + 4bc^2 + k^2}\left(\phi + \frac{k}{2b}\right)^2\right)^{1/2}.\end{aligned} \tag{16}$$

Let $4b^2/(4b(h+c^2)+k^2) = A$; Taylor expansion is

$$\begin{aligned}\sqrt{h+c^2-b\phi^2-k\phi} &= \sqrt{\frac{b}{A}}\left(1-A\left(\phi+\frac{k}{2b}\right)^2\right)^{1/2} \\ &= \sqrt{\frac{b}{A}}\left[1-\frac{1}{2}A\left(\phi+\frac{k}{2b}\right)^2+R_2(x)\right],\end{aligned} \quad (17)$$

generally satisfying $|A(\phi+(k/2b))^2|<1$,

$$\begin{aligned}R_2(x) &= \frac{1/2(1/2-t1)}{2}\left(1+\theta A\left(\phi+\frac{k}{2b}\right)^2\right)^{1/2} \\ &\cdot \sqrt{h+c^2-b\phi^2-k\phi} \\ &\approx \sqrt{\frac{b}{A}}\left[1-\frac{1}{2}A\left(\phi+\frac{k}{2b}\right)^2\right].\end{aligned} \quad (18)$$

Error $|d| \leq (1/8)(1+A\theta(\phi+(k/2b)))^{1/2}$.

$$\frac{d\phi}{d\zeta} = \sqrt{2\sqrt{\frac{b}{A}}-\sqrt{bA}\left(\phi+\frac{k}{2b}\right)^2-(\phi^2+2c)}, \quad (19)$$

that is,

$$\int \frac{d\phi}{\sqrt{-\left(\sqrt{bA}\,(k^2/4b^2)+1\right)\phi^2-\sqrt{(A/b)k\phi}-\left(k^2/4b^2\right)+2\sqrt{(b/A)}+2c}} = d\zeta. \quad (20)$$

Let $A_1 = -(\sqrt{bA}\,(k^2/4b^2)+1)$, $B_1 = -\sqrt{(A/k)}k$, and $C_1 = -(k^2/4b^2)+2\sqrt{b/A}+2c$; then the above equation of the integral becomes

$$\int \frac{d\phi}{\sqrt{A_1\phi^2+B_1\phi+C_1}} = d\zeta. \quad (21)$$

Approximation solution of system (6) is

$$ln\left(\frac{1}{2}B_1+A_1\phi\right)+\sqrt{A_1\phi^2+B_1\phi+C_1} = \sqrt{A_1}*(x-c*t). \quad (22)$$

$v=\phi(x-c*t)$ is the approximation solution of the original equation.

If the order of Taylor expansion of F is higher, the accuracy of the solution will be higher.

4. Conclusion

We obtain some exact solutions and some approximation solutions of the Olver–Rosenau equation by the dynamic system method. The original equation considered by the first integral method and its phase-space analysis and equilibrium points are calculated under different parameter conditions. We can derive various solutions of the original equation relating these orbits in different phase-space planes. Nevertheless, as there are some troubles in the course of the calculation of these solutions because of the integral complexity, we need to find some numerical methods for the equation. The precision differs from the variable order of Taylor expansion.

Data Availability

This paper is mainly theoretical derivation and calculation, it does not involve data and its right to use.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Degenerate Analogues of Euler Zeta, Digamma, and Polygamma Functions

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In recent years, much attention has been paid to the role of degenerate versions of special functions and polynomials in mathematical physics and engineering. In the present paper, we introduce a degenerate Euler zeta function, a degenerate digamma function, and a degenerate polygamma function. We present several properties, recurrence relations, infinite series, and integral representations for these functions. Furthermore, we establish identities involving hypergeometric functions in terms of degenerate digamma function.

1. Introduction

The gamma, digamma, and polygamma functions have an increasing and recognized role in fractional differential equations, mathematical physics, the theory of special functions, statistics, probability theory, and the theory of infinite series. The reader may refer, for example, to [1–9]. These functions are directly connected with a variety of special functions such as zeta function, Clausen's function, and hypergeometric functions. The evaluations of series involving Riemann zeta function $\zeta(s)$ and related functions have a long history that can be traced back to Christian Goldbach (1690–1764) and Leonhard Euler (1707–1783) (see, for details, [10]). The Euler zeta function and its generalizations and extensions have been widely studied [11–15].

Later on, these functions arise in the study of matrix-valued special functions and in the theory of matrix-valued orthogonal polynomials, see e.g., [16–23] and the references therein.

Motivated by this great importance of these functions, their investigations and generalizations to the degenerate framework have been widely considered in the literature, for instance, [24–27].

In this section, we present some basic properties and well-known results on a degenerate gamma function which we need in this work. In Section 2, we introduce a degenerate Euler zeta function and discuss its region of convergence, integral representation, and infinite series representation. In Section 3, we define a degenerate digamma function along with its region of convergence and integral representation. We also give certain recurrence relations and formulae satisfied by the degenerate digamma function. In Section 4, we define a degenerate polygamma function and describe its convergence conditions. Some recurrence relations satisfied by the degenerate polygamma function are also given here. Finally, in Section 5, the hypergeometric functions are expressed in terms of the degenerate digamma function.

In [26], a degenerate gamma function, denoted Γ_λ^*, has been defined by

$$\Gamma_\lambda^*(z) = \int_0^\infty (1+\lambda)^{-t/\lambda} t^{z-1} \mathrm{d}t, \quad \lambda \in (0,1), \mathrm{Re}(z) > 0. \tag{1}$$

The basic results of this function, given in [26], can be summarized in the following lemma.

Lemma 1. *Let* $\lambda \in (0,1)$. *Then, for* $z \in \mathbb{C}$ *with* $\mathrm{Re}(z) > 0, \Gamma_\lambda^*(z)$ *satisfies*

$$\Gamma_\lambda^*(z+1) = \frac{\lambda z}{\log(1+\lambda)}\Gamma_\lambda^*(z),$$

$$\Gamma_\lambda^*(1) = \frac{\lambda}{\log(1+\lambda)}, \tag{2}$$

$$\Gamma_\lambda^*(z+1) = \frac{\lambda^{k+1} z(z-1)\cdots(z-k)}{(\log(1+\lambda))^{k+1}}\Gamma_\lambda^*(z-k), \quad k \ge 0, \tag{3}$$

$$\Gamma_\lambda^*(k+1) = \frac{\lambda^{k+1} k!}{(\log(1+\lambda))^{k+1}}, \quad k \in \mathbb{N}. \tag{4}$$

Also, we can easily show that

Corollary 1. *Let* $\lambda \in (0,1)$. *Then,* $\Gamma_\lambda^*(z)$ *satisfies*

$$\Gamma_\lambda^*(z) = \left[\frac{\lambda}{\log(1+\lambda)}\right]^z \Gamma(z), \quad z \in \mathbb{C}, \mathrm{Re}(z) > 0, \tag{5}$$

where $\Gamma(z)$ *is the gamma function. Moreover, for* $m, n \in \mathcal{N}$, *we have*

$$\Gamma_\lambda^*(m)\Gamma_\lambda^*(n) = B(m,n)\Gamma_\lambda^*(m+n), \tag{6}$$

where $B(.,.)$ *is the beta function.*

2. Degenerate Euler Zeta Function

The Euler zeta function in two complex variables s, z such that $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(z) > 0$ is defined by (see [12, 24])

$$\zeta_E(s,z) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+z)^s}. \tag{7}$$

An integral representation of $\zeta_E(s,z)$ is given as

$$\zeta_E(s,z) = \Gamma^{-1}(s)\int_0^\infty F(-t,z)t^{s-1}\mathrm{d}t, \tag{8}$$

where

$$F(t,z) = \frac{2e^{zt}}{1+e^t} = \sum_{n=0}^{\infty} E_n(z)\frac{t^n}{n!}, \tag{9}$$

where $E_n(z)$ is the Euler polynomial of degree n. When $z = 0$, $E_n = E_n(0)$ are Euler numbers (see, [12, 14]). Kim in [14] obtained that $\zeta_n(-n,z) = E_n(z), n \ge 0$.

In this section, we consider a degenerate analogue of the Euler zeta function which is given as

$$\zeta_{E_\lambda}(s,z) = \Gamma^{-1}(s)\int_0^\infty F_\lambda(-t,z)t^{s-1}\mathrm{d}t, \tag{10}$$

where $\lambda \in (0,1), s, z \in \mathbb{C}$ with $\mathrm{Re}(s) > 0, \mathrm{Re}(z) > 0$, and

$$F_\lambda(t,z) = \frac{2}{1+(1+\lambda)^{t/\lambda}}(1+\lambda)^{zt/\lambda} = \sum_{n=0}^{\infty} \mathscr{E}_n^\lambda(z)\frac{t^n}{n!}. \tag{11}$$

By (9) and (11), it follows that

$$\mathscr{E}_n^\lambda(z) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^n E_n(z), \tag{12}$$

which is the degenerate Euler polynomial of degree n.

From (10) and (11), we obtain that

$$\begin{aligned}\Gamma^{-1}(s)\int_0^\infty F_\lambda(-t,z)t^{s-1}\mathrm{d}t &= \Gamma^{-1}(s)\int_0^\infty 2\sum_{m=0}^{\infty}(-1)^m(1+\lambda)^{-(m+z)t/\lambda}t^{s-1}\mathrm{d}t \\ &= 2\Gamma^{-1}(s)\sum_{m=0}^{\infty}(-1)^m\int_0^\infty(1+\lambda)^{-\tau/\lambda}\frac{\tau^{s-1}}{(m+z)^s}\mathrm{d}\tau \\ &= 2\frac{\Gamma_\lambda^*(s)}{\Gamma(s)}\sum_{m=0}^{\infty}(-1)^m\frac{1}{(m+z)^s}.\end{aligned} \tag{13}$$

Thus, using (10) and (13), we conclude the following result.

Theorem 1. *For* $s, z \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 0, \mathrm{Re}(z) > 0$, *and* $\lambda \in (0,1)$, *the degenerate Euler zeta function* $\zeta_{E_\lambda}(s,z)$ *defined in (10) has the following infinite series representation:*

$$\zeta_{E_\lambda}(s,z) = 2\frac{\Gamma_\lambda^*(s)}{\Gamma(s)}\sum_{m=0}^{\infty}(-1)^m\frac{1}{(m+z)^s}. \tag{14}$$

Moreover, in view of (5), we have

$$\zeta_{E_\lambda}(s,z) = \zeta_E(s,z)\left(\frac{\lambda}{\ln(1+\lambda)}\right)^s, \tag{15}$$

where $\zeta_E(s,z)$ *is the Euler zeta function defined by (7).*

Furthermore, from (11), it follows

$$\Gamma^{-1}(s)\int_0^\infty F_\lambda(-t,z)t^{s-1}\mathrm{d}t = \Gamma^{-1}(s)\sum_{m=0}^{\infty}\mathscr{E}_n^\lambda(z)\frac{(-1)^m}{m!}\int_0^\infty t^{s+m-1}\mathrm{d}t. \tag{16}$$

Hence, we obtain the following results.

Theorem 2. *For $s, z \in \mathbb{C}$ with $\operatorname{Re}(s) > 0, \operatorname{Re}(z) > 0,$ and $\lambda \in (0,1)$, the degenerate Euler zeta function $\zeta_{E_\lambda}(s,z)$, defined in (10), satisfies*

$$\zeta_{E_\lambda}(s,z) = \Gamma^{-1}(s)\sum_{m=0}^{\infty}\mathscr{E}_n^{\lambda}(z)\frac{(-1)^m}{m!}\int_0^{\infty}t^{s+m-1}\mathrm{d}t. \quad (17)$$

And for $n \in \mathbb{N}\cup\{0\}$,

$$\zeta_{E_\lambda}(-n,z) = \frac{2\pi i(-1)^n}{n!\Gamma(-n)}\mathscr{E}_n^{\lambda} = \mathscr{E}_n^{\lambda}(z). \quad (18)$$

Remark 1. Note that $\zeta_{E_\lambda}(s,z)$ is an entire function in the complex s–plane.

Remark 2.

$$\lim_{\lambda\longrightarrow 0}\zeta_{E_\lambda}(-n,z) = E_n(z) = \zeta_E(-n,z). \quad (19)$$

3. Degenerate Digamma Function

The digamma function, denoted by $\psi(z)$, is the logarithmic derivative of the gamma function given by [6, 16, 28]:

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \quad (20)$$

In this section, we define a degenerate digamma function as follows:

$$\psi_\lambda^*(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma_\lambda^*(z) = \frac{\Gamma_\lambda^{*'}(z)}{\Gamma_\lambda^*(z)}, \quad (21)$$

where $\Gamma_\lambda^*(z)$ is the degenerate gamma function defined by (1). Now, we are going to obtain certain functional equations involving the degenerate digamma function $\psi_\lambda^*(z)$. Using (2) and (21), it follows that

$$\begin{aligned}\psi_\lambda^*(z+1) &= \frac{\Gamma_\lambda^{*'}(z+1)}{\Gamma_\lambda^*(z+1)} = \frac{(z\Gamma_\lambda^*(z))'}{z\Gamma_\lambda^*(z)} \\ &= \frac{\Gamma_\lambda^{*'}(z)}{\Gamma_\lambda^*(z)} + \frac{1}{z} = \psi_\lambda^*(z) + \frac{1}{z},\end{aligned} \quad (22)$$

$\operatorname{Re}(z) > 0.$

Generally, we have the following.

Theorem 3. *For $n \in \mathbb{N}$, $z \in \mathbb{C}$, and $\operatorname{Re}(z) > 0$, we have*

$$\psi_\lambda^*(z+n) = \psi_\lambda^*(z) + \sum_{m=0}^{n-1}\frac{1}{z+m}. \quad (23)$$

Furthermore, using relation (5), we find that

$$\psi_\lambda^*(z) = \psi(z) + \log\left(\frac{\lambda}{\log(1+\lambda)}\right), \quad (24)$$

where ψ is the digamma function defined by (20). According to Batir [28], we have

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty}\frac{z-1}{(n+1)(n+z)}, \quad (25)$$

where

$$\gamma = \lim_{n\longrightarrow\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = -0.577215 \quad (26)$$

is the Euler–Mascheroni constant. Hence, substituting (25) into (24), one gets the following.

Theorem 4. *For $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$, and $\lambda \in (0,1)$,*

$$\psi_\lambda^*(z) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \lim_{n\longrightarrow\infty}\left[\log n - \sum_{j=0}^{n}\frac{1}{z+j}\right], \quad (27)$$

$$\psi_\lambda^*(z) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \gamma + (z-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+z)}, \quad (28)$$

$$\psi_\lambda^*(z+1) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \gamma + z\sum_{n=1}^{\infty}\frac{1}{n(n+z)}. \quad (29)$$

Next, the degenerate digamma function $\psi_\lambda^*(z)$ defined by (21) can be expressed as a series expression in terms of Riemann's zeta function. Using

$$(n+z)^{-1} = n^{-1}\sum_{m=0}^{\infty}\left(\frac{-z}{n}\right)^m, \quad (30)$$

equation (29) can be rewritten as

$$\psi_\lambda^*(z+1) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \gamma - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}n^{-(m+1)}(-z)^m. \quad (31)$$

Thus, one gets the following.

Theorem 5. *For $z \in \mathbb{C}, \operatorname{Re}(z) > 0$, and $\lambda \in (0,1)$,*

$$\psi_\lambda^*(z+1) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) - \gamma - \sum_{m=1}^{\infty}\zeta(m+1)(-z)^m. \quad (32)$$

Note that these series converge absolutely for $|z| < 1$. Using the Legendre duplication formula [29]

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad (33)$$

and (5), one can simply find

$$\Gamma_\lambda^*\left(\frac{1}{2}\right)\Gamma_\lambda^*(2z) = 2^{2z-1}\Gamma_\lambda^*(z)\Gamma_\lambda^*\left(z+\frac{1}{2}\right), \quad (34)$$

$$\psi_\lambda^*(2z) = \log 2 + \frac{1}{2}\psi_\lambda^*(z) + \frac{1}{2}\psi_\lambda^*\left(z+\frac{1}{2}\right), \quad \operatorname{Re}(z) > 0. \quad (35)$$

Equation (35) can be extended to an arbitrary integral multiplication of z as follows.

Theorem 6. *For $z \in \mathbb{C}, \mathrm{Re}(z) > 0$, and $\lambda \in (0,1)$,*

$$\psi_\lambda^*(mz) = \log m + \frac{1}{m}\sum_{j=1}^{m}\psi_\lambda^*\left(z + \frac{j-1}{m}\right), \quad \mathrm{Re}(z) > 0. \tag{36}$$

Figures 1–3 illustrate the degenerate digamma function $\psi_\lambda^*(z)$ in (24) at different values for $\lambda \in (0,1)$.

Remark 3. Its worth to mention here that all plotted functions in the below figures were multiplied by $\sin x$, since Fourier space, for the sake of clarify the results to the reader.

Now, we are going to find the integral representations for the degenerate digamma function $\psi_\lambda^*(z)$, defined by (21), as follows. Note that

$$\int_0^1 \left(1 - t^{z-1}\right)(1-t)^{-1}\mathrm{d}t = \sum_{n=0}^{\infty}\int_0^1 \left(1 - t^{z-1}\right)t^n \mathrm{d}t$$
$$= (z-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+z)}. \tag{37}$$

Hence, using (28) and (37), it can be shown that

$$\psi_\lambda^*(z) = -\gamma + \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \int_0^1 \left(1 - t^{z-1}\right)(1-t)^{-1}\mathrm{d}t. \tag{38}$$

Now, substituting $t = (1+\lambda)^{-s/\lambda}$ in (37) gives

$$\psi_\lambda^*(z) = -\gamma + \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \frac{\log(1+\lambda)}{\lambda} \times \int_0^\infty \left[(1+\lambda)^{-t/\lambda} - (1+\lambda)^{-zt/\lambda}\right]\left[1 - (1+\lambda)^{-t/\lambda}\right]^{-1}\mathrm{d}t. \tag{39}$$

Since

$$z^{-1} = \frac{\log(1+\lambda)}{\lambda}\int_0^\infty (1+\lambda)^{-zt/\lambda}\mathrm{d}t \tag{40}$$

and by integrating from 1 to n, it follows that

$$\begin{aligned}\log n &= \int_0^\infty \int_1^n (1+\lambda)^{-zt/\lambda} \cdot \log(1+\lambda)^{1/\lambda}\mathrm{d}z\mathrm{d}t \\ &= \int_0^\infty \int_1^n \frac{1}{t}\mathrm{d}_z (1+\lambda)^{-zt/\lambda}\mathrm{d}t \\ &= \int_0^\infty \frac{1}{t}\left[(1+\lambda)^{-t/\lambda} - (1+\lambda)^{-nt/\lambda}\right]\mathrm{d}t.\end{aligned} \tag{41}$$

Inserting (41) and

$$(z+j)^{-1} = \frac{\log(1+\lambda)}{\lambda}\int_0^\infty (1+\lambda)^{-(z+j)t/\lambda}\mathrm{d}t \tag{42}$$

in (27), we get

$$\begin{aligned}\psi_\lambda^*(z) &= \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \lim_{n\to\infty}\int_0^\infty \left[\left((1+\lambda)^{-t/\lambda} - (1+\lambda)^{-nt/\lambda}\right)\frac{1}{t} - \sum_{j=0}^{n}\frac{\log(1+\lambda)}{\lambda}(1+\lambda)^{-(z+j)t/\lambda}\right]\mathrm{d}t \\ &= \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \lim_{n\to\infty}\int_0^\infty \left\{(1+\lambda)^{-\frac{t}{\lambda}}t^{-1} - \frac{\log(1+\lambda)}{\lambda}(1+\lambda)^{-\frac{zt}{\lambda}}\left[1 - (1+\lambda)^{-\frac{t}{\lambda}}\right]^{-1}\right\}\mathrm{d}t \\ &\quad - \lim_{n\to\infty}\int_0^\infty (1+\lambda)^{-\frac{nt}{\lambda}}\left\{t^{-1} - \frac{\log(1+\lambda)}{\lambda}(1+\lambda)^{-\frac{zt}{\lambda}}\left[1 - (1+\lambda)^{-\frac{t}{\lambda}}\right]^{-1}\right\}\mathrm{d}t.\end{aligned} \tag{43}$$

Since the last limit equals to zero, it follows

$$\psi_\lambda^*(z) = \log\left(\frac{\lambda}{\log(1+\lambda)}\right) + \int_0^\infty \left[\frac{1}{t}(1+\lambda)^{-t/\lambda} - \frac{\log(1+\lambda)}{\lambda}\left(1 - (1+\lambda)^{-t/\lambda}\right)^{-1}(1+\lambda)^{-zt/\lambda}\right]\mathrm{d}t. \tag{44}$$

The following theorem summarizes the above results.

Theorem 7. *For $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, and $\lambda \in (0,1)$, the degenerate digamma function $\psi_\lambda^*(z)$, defined by (10), can be expressed as (38), (39) as well as (44).*

4. Degenerate Polygamma Function

The polygamma function of order m is obtained by taking the $(m+1)$th derivative of the logarithm of gamma function (cf. [28]). Thus,

$$\psi^{(m)}(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\psi(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\log\Gamma(z), \quad \mathrm{Re}(z) > 0. \tag{45}$$

In this section, we define the degenerate polygamma function of order m as

$$\psi_\lambda^{*\,(m)}(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\psi_\lambda^*(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\log\Gamma_\lambda^*(z), \quad \mathrm{Re}(z) > 0, \tag{46}$$

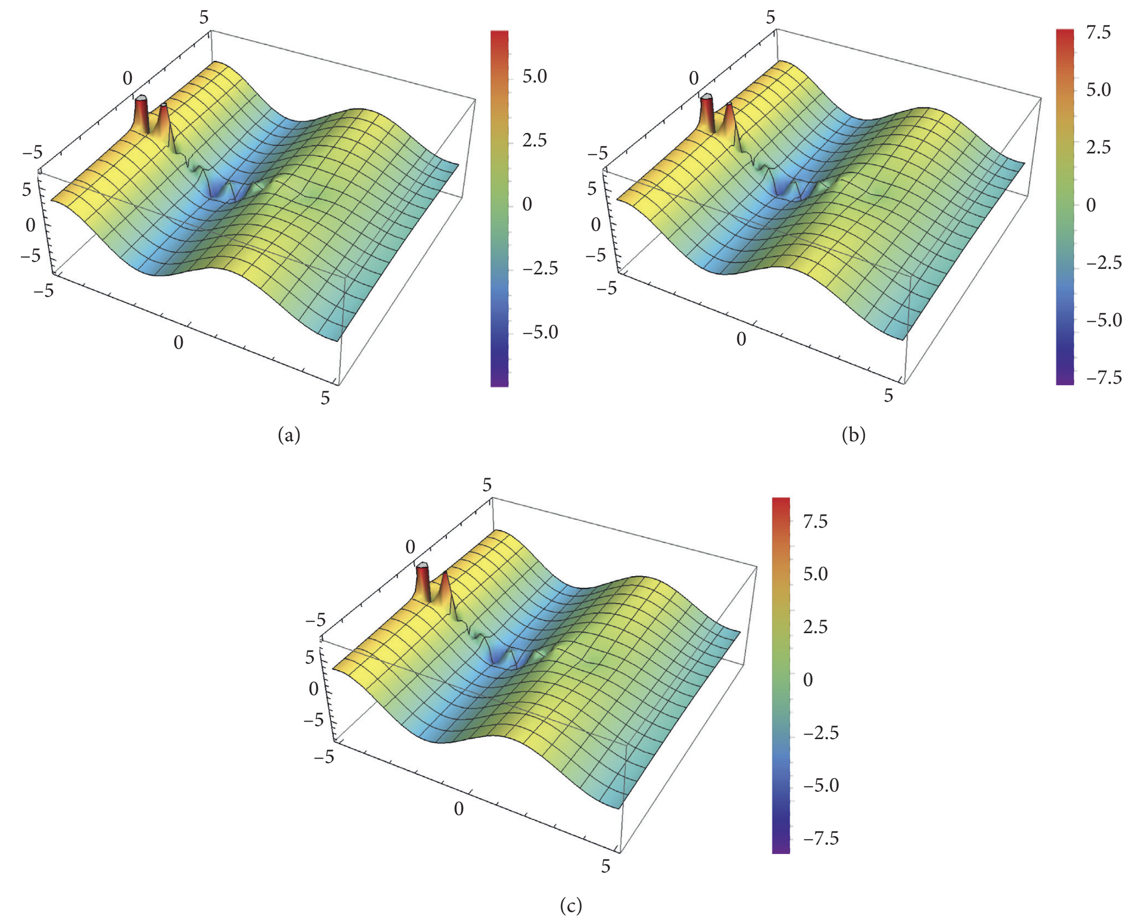


FIGURE 1: Absolute plots of the degenerate digamma function. (a) $\lambda = 0.1$. (b) $\lambda = 0.5$. (c) $\lambda = 1.0$.

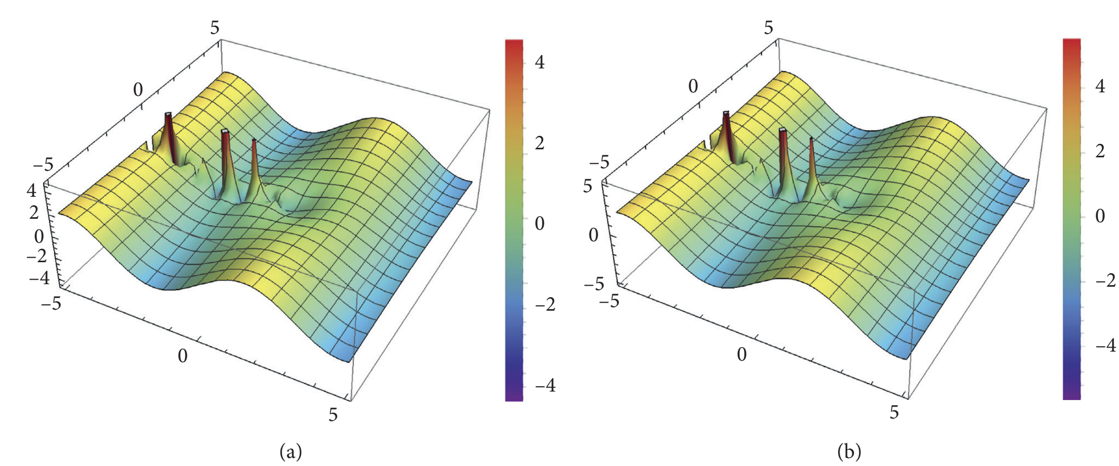
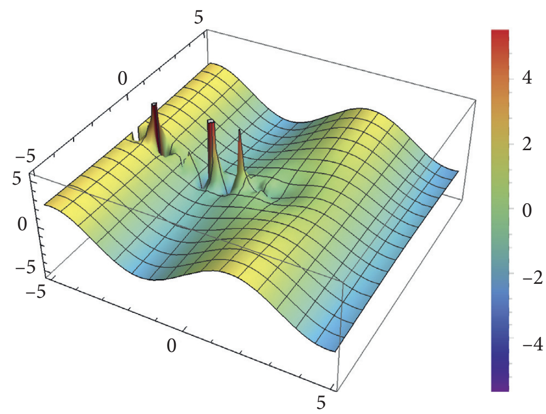
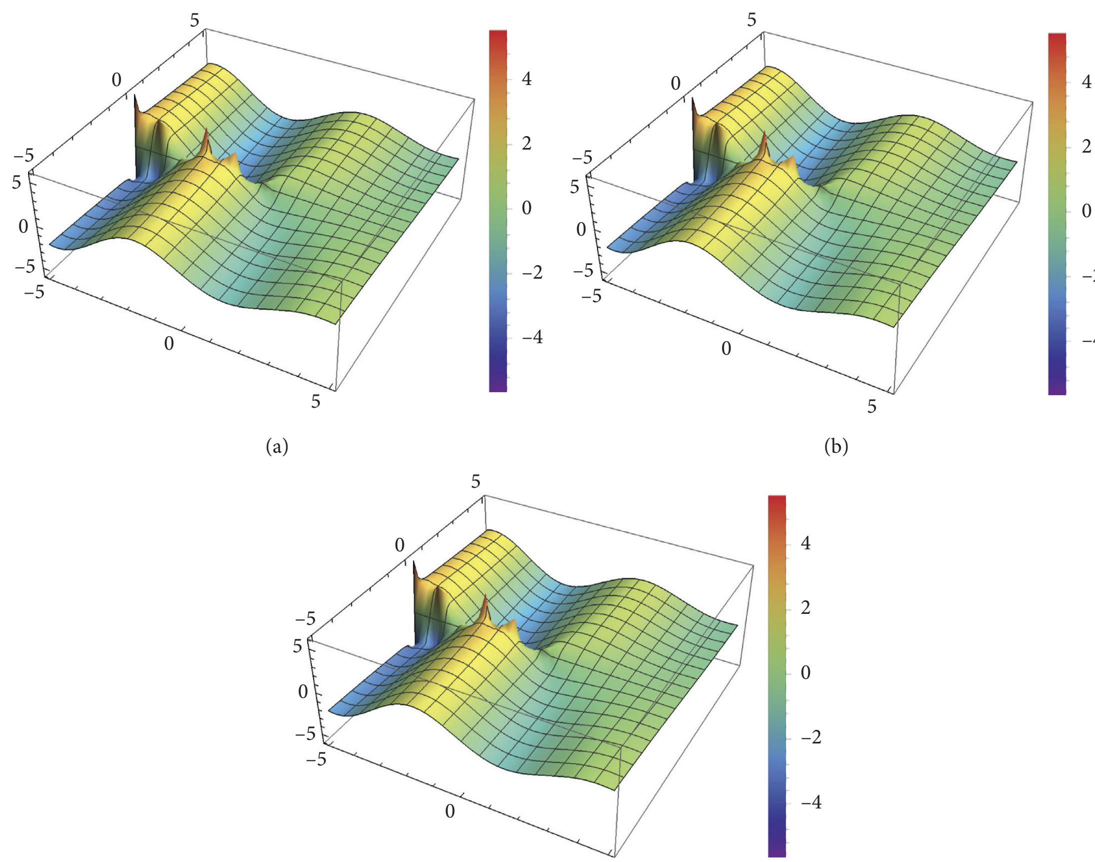


FIGURE 2: Continued.



(c)

Figure 2: Real-part plots of the degenerate digamma function. (a) $\lambda = 0.1$. (b) $\lambda = 0.5$. (c) $\lambda = 1.0$.



(a) (b)

(c)

Figure 3: Imagery-part plots of the degenerate digamma function. (a) $\lambda = 0.1$. (b) $\lambda = 0.5$. (c) $\lambda = 1.0$.

where $\Gamma_\lambda^*(z)$ is the degenerate gamma function defined by (1) and $\psi_\lambda^*(z)$ is the degenerate digamma function defined by (21).

By (24), it follows that

$$\psi_\lambda^{*\,(m)}(z) = \psi^{(m)}(z), \quad \operatorname{Re}(z) > 0. \tag{47}$$

Using (44), an integral representation for $\psi_\lambda^{*\,(m)}(z)$, given in the next theorem, can be obtained.

Theorem 8. *Let $\lambda \in (0,1)$ and $m \in \mathbb{N}$. Then, for $z \in \mathbb{C}$ with* $\operatorname{Re}(z) > 0$, *the degenerate polygamma function $\psi_\lambda^{*\,(m)}(z)$, defined by (46), can be expressed as*

$$\psi_\lambda^{*\,(m)}(z) = (-1)^m \left(\frac{\log(1+\lambda)}{\lambda}\right)^{m+1} \times \int_0^\infty t^m \left[1-(1+\lambda)^{-t/\lambda}\right]^{-1} (1+\lambda)^{-zt/\lambda} \mathrm{d}t. \tag{48}$$

The following recurrence relations for the degenerate polygamma function $\psi_\lambda^{*\,(m)}(z)$ defined by (47) can be obtained from (22)–(24), (35), and (36) as the following.

Theorem 9. *For $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, $\lambda \in (0,1)$, and $m \in \mathbb{N}$, the recurrence relations hold true:*

$$\begin{aligned}
\psi_\lambda^{*\,(m)}(z+1) &= \psi_\lambda^{*\,(m)}(z) + \frac{(-1)^m \Gamma(m+1)}{z^{m+1}},\\
\psi_\lambda^{*\,(m)}(1-z) &= (-1)^m \psi_\lambda^{*\,(m)}(z) + (-1)^m \pi \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \cot(\pi z),\\
\psi_\lambda^{*\,(m)}(z+n) &= \psi_\lambda^{*\,(m)}(z) + \sum_{k=0}^{n-1} \frac{(-1)^m \Gamma(m+1)}{(z+k)^{m+1}},\\
\psi_\lambda^{*\,(m)}(2z) &= \frac{1}{4}\psi_\lambda^{*\,(m)}(z) + \frac{1}{4}\psi_\lambda^{*\,(m)}\left(z+\frac{1}{2}\right),\\
\psi_\lambda^{*\,(m)}(nz) &= \frac{1}{n^{m+1}} \sum_{k=1}^{n} \psi_\lambda^{*\,(m)}\left(z+\frac{k-1}{n}\right), \quad \mathrm{Re}(z) > 0.
\end{aligned} \tag{49}$$

From (25), a series representation of the degenerate polygamma function $\psi_\lambda^{*\,(m)}(z)$ is given in the following result.

Theorem 10. *For $z \in \mathbb{C}, \mathrm{Re}(z) > 0, \lambda \in (0,1)$, and $m \in \mathbb{N}$, we have*

$$\psi_\lambda^{*\,(m)}(z) = (-1)^{m+1} \Gamma(m+1) \sum_{n=0}^{\infty} \frac{1}{(z+n)^{m+1}}. \tag{50}$$

Remark 4. The degenerate polygamma function $\psi_\lambda^{*\,(m)}(z)$ can be expressed in terms of the generalized zeta function

$$\zeta(m,z) = \sum_{n=0}^{\infty} (z+n)^{-m} \tag{51}$$

as

$$\psi_\lambda^{*\,(m)}(z) = (-1)^m \Gamma(m+1) \zeta(m+1,z). \tag{52}$$

Finally, using (32), a series representation in terms of the Riemann zeta function can be obtained, see the following result.

Theorem 11. *For $z \in \mathbb{C}, \mathrm{Re}(z) > 0, \lambda \in (0,1)$, and $m \in \mathbb{N}$, we have*

$$\psi_\lambda^{*\,(m)}(z+1) = \sum_{n=0}^{\infty} (-1)^{m+n+1} \Gamma(m+n+1) \zeta(m+n+1) \frac{z^n}{n!}, \quad m,n \in \mathbb{N}. \tag{53}$$

5. Applications

Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $n \in \mathbb{N}$. Then, it can be verified that

$${}_3F_2\left[\begin{matrix}(-n+2), z+1, 1\\ z+(n+1), 2\end{matrix}; 1\right] = \frac{z+n}{z(-n+1)} \times \left({}_2F_1\left[\begin{matrix}(-n+1), z\\ z+1\end{matrix}; 1\right] - 1\right). \tag{54}$$

Now, we can directly use the integral transform of Gauss hypergeometric function (see [29]) and the formulae:

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \quad \mathrm{Re}(z) > 0, \tag{55}$$

$$2F_1\left[\begin{matrix}(-n+2), z\\ z+1\end{matrix}; 1\right] = 2^{-z} \frac{\Gamma(z+n)\Gamma(n-(1/2))}{\Gamma((z/2)+n)\Gamma((z/2)+(n-(1/2)))}, \quad \mathrm{Re}(z) > 0. \tag{56}$$

Using (54) in (56) and L'Hôpital rule for complex numbers with applying equation (24) yields the following identity in terms of the degenerate digamma function:

$${}_3F_2\left[\begin{matrix}(-n+2), z+1, 1\\ z+(n+1), 2\end{matrix}; 1\right] = \frac{z+1}{z} \times \left[\psi_\lambda^*\left(\frac{1}{2}\right) + \psi_\lambda^*(z+1) - \psi_\lambda^*\left(\frac{1}{2}(z+1)\right) - \psi_\lambda^*\left(\frac{1}{2}z+1\right)\right], \quad \mathrm{Re}(z) > 0. \tag{57}$$

Similarly, we can present another identity involving hypergeometric function in terms of our degenerate digamma function in the following form:

$$
\begin{aligned}
4F_3\left[\begin{matrix}1,1,1,-n\\ 2,2,z+1\end{matrix};1\right]\times\Bigg[&\left(\psi_\lambda^*(n+2)-\log\left(\frac{\lambda}{\log(1+\lambda)}\right)\right)\left(\psi_\lambda^*(z+n+1)-\psi_\lambda^*(z)\right)\\
&-\sum_{s=1}^{n}\frac{\psi_\lambda^*(s+1)-\log(\lambda/\log(1+\lambda))}{z+1}\Bigg],\quad \mathrm{Re}(z)>0.
\end{aligned}
\tag{58}
$$

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

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Research Article

Effects of Homogeneous and Heterogeneous Chemical Features on Oldroyd-B Fluid Flow between Stretching Disks with Velocity and Temperature Boundary Assumptions

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This research endeavors the rheological features of Oldroyd-B fluid configured by infinite stretching disks in presence of velocity and thermal slip features. Additionally, the effects of homogeneous and heterogeneous chemical features are also considered. The transmuted flow equations are analytically solved with help of the homotopy analysis method (HAM). It is observed that the homogeneous chemical reaction parameter enhances the concentration distribution, while the heterogeneous reaction reduces the concentration profile. With implementations of temperature jump conditions, the heat transfer from the surfaces of both disks can be effectively controlled. The impacts of various dimensionless parameters are elaborated through graphs and tables.

1. Introduction

The fluid flow between stretching disks is the main motivation of investigators in recent years due to its leading applications in turbine engines, compression, mechanical components transient loading, semiconductor manufacturing, rotating wafers, injection modeling, power transmission, viscometer, lubrications, radial diffusers, geophysics, biomechanics, geothermal, oceanography, thrust bearings, etc. The usage of microdevices has many practical applications in different scientific areas such as surgery, biotechnology, electronic cooling, microchannels, heat pipes, and pumps. The heat and fluid flow characteristics are different for both microdevices and macroscale counterparts. This difference is constituted by velocity slip and temperature jump. The velocity slip is an important feature to analyze the behavior of microflows because no-slip boundary conditions are not applicable to the fluid flow in microelectro-mechanical-systems (MEMS). Also, no-slip boundary conditions show the impractical behavior for the cases such as corner flow, spreading of liquid on a solid substrate, and extrusion of polymer melts from a capillary tube. Therefore, no-slip boundary condition is replaced by slip boundary condition. Further, in the slip flow regime, temperature jump is significantly used to determine the heat transfer. Because of such applications of slip flow, many interesting contributions have been made by investigators in recent years. For instance, Zheng et al. [1] investigated the stretched flow of viscous fluid in presence of velocity as well as thermal slip features. The peristaltic transport of Carreau fluid through a channel with various flow features with application of velocity slip, temperature, and concentration jump has been inspected by Vajravelu et al. [2]. Khan et al. [3] discussed the double diffusion slip flow of viscous fluid over a vertical plate. Xiao et al. [4] presented a mathematical model for fully developed slip flow in a microtube gas problem. This interesting continuation contains the velocity slip of order two and the assumptions of temperature jump constraints. They claimed an effective change up to 15% in the local Nusselt number at room temperature. Similar slip effects have been performed by Rooholghdos and Roohi [5] for a nanoscale flat plate and a microscale cylinder. Another useful contribution regarding the gas flow associated with thermal slip conditions was examined by Le and Roohi [6]. The peristaltic transport of viscous fluid in an asymmetric channel in presence of velocity and temperature boundary conditions has been discussed by Sinha et al. [7]. El-Aziz and Afify [8] examined the heat transfer characteristics for slip flow of Casson fluid subjected to the induced magnetic field. Khan et al. [9] determined the analytical solution based on the Galerkin technique for an upper convected flow of Maxwell fluid in presence of slip features. Muhammad et al. [10] examined the entropy generation aspects in the flow of nanofluid under the action of the second-order slip. The investigation for fractional Maxwell fluid in presence of slip effects and porous medium was performed by Aman et al. [11].

The fluid flow encountered the heat transportation process conveying a diverse engineering and industrial significance in the metal cooling, petroleum engineering, chemical processing, food industries, thermophysical systems, fiber spinning, manufacturing of metallic sheets, and various nuclear processes. Besides this, the thermal performance of disc-shaped bodies had engaged many scholars because of its practical applications in the era of aeronautical sciences. Many engineering and mechanical processes like thermal power generation and heat transfer to automatic control systems encountered the applications of these phenomena. Due to such recurrent applications, several researchers investigate the flow over or flow between two disks. The initial contribution on this topic was led by Kármán and Uber [12] by considering viscous fluid flow between two infinite disks. This study was further extended by many researchers with different flow features. Hayat et al. [13] studied the heat transfer characteristics based on the Fourier law of conduction in third-grade liquid configured by two porous disks. Turkyilmazoglu [14] simulated the numerical solution of hydromagnetic fluid flow near the stagnation point subject to disk rotation. Heat transfer analysis in the hydromagnetic fluid flow caused by a rotating shrinking disk was also performed numerically by Turkyilmazoglu [15]. Soid et al. [16] applied the numerical technique to observe heat transfer phenomenon in viscous fluid for a radially stretching disk. Yin et al. [17] examined the flow thermal characteristics of nanofluid flow due to a rotating disk. Turkyilmazoglu [18] numerically examined the flow of Newtonian fluid through a vertically moving disk. Hashmi et al. [19] analytically explored the mixed convection flow of Oldroyd-B fluid placed between isothermal stretching disks. The idea of flow over stretching surfaces is extremely useful and involved a large number of practical applications in manufacturing processes [20–23]. The spontaneous idea of flow due to a moving surface was originally advised by Sakiadis [24, 25] which encouraged the investigators to pay attention in this direction. The exact solution for a stretching flow problem was successfully provided by Wang [26]. Another investigation in this direction has been suggested by Fang [27] which conferred the viscous fluid flow induced due to a stretched disk. In another attempt, Fang and Zhang [28] derived an exact solution based on the mathematical formulation of Navier Stokes equations modeled in cylindrical coordinates. In fact, such type of flow between two infinite stretching disks arises due to accelerated stretching velocity. Gorder et al. [29] discussed the axisymmetric flow between two infinite stretching disks. Mohyud-Din and Khan [30] implemented effects of nonlinear thermal radiation in flow of Casson fluid concedes between two stretching disks. Slip flow in presence of thermo-diffusion effects in flow of viscous fluid between stretching disks was suggested by Rashidi and his coworkers [31]. Analytical solution based on the homotopy analysis method for flow of viscous fluid through a stretchable disk has been depicted by Khan et al. [32]. In another investigation, Khan et al. [33] examined the viscous dissipation and joule heating effects on the axisymmetric flow of viscous fluid between stretching disks. Khan et al. [34] studied the entropy generation effects on flow of carbon nanotubes between two rotating and stretching disks. The heat transfer analysis based on Cattaneo–Christov heat flux expressions for the flow of micropolar fluid induced by a nonlinear stretching disk was focused by Doh et al. [35]. Renuka et al. [36] computed an analysis solution for the flow of nanofluid, additionally featuring entropy generation features induced by a stretchable spinning disk.

In the recent decade, the study of combined heat and mass transportation has inspired the scientists to examine various aspects of the simultaneous phenomenon due to its arising applications in the real-world problems like reacting systems, cooling towers, marine engineering, distillation columns, hydrometallurgical industry, crop damage via freezing, and copse of trees. The collaboration amongst homogeneous and heterogeneous responses happening on some catalytic surfaces is correlated with the production and employment of chemical species at diverse rates within the fluid and on the catalytic surfaces. Merkin [37] developed a very useful mathematical model to explore the relationship between a surface-based reaction and homogeneous and heterogeneous reactions. Another useful contribution is from Kameswaran et al. [38] where flow of nanoparticles is immersed in a porous medium with additional features of binary chemical reactions. Rashidi et al. [31] address the effects of homogeneous/heterogeneous on a peristaltic transport in a channel. Hayat et al. [39] implemented the effects of second-order velocity slip to examine the flow of chemical reactive viscous nanofluid induced by a permeable stretching surface.

In this present analysis, our focus is to evaluate the driven transport of Oldroyd-B fluid considered within two infinite stretching disks in presence of homogeneous and heterogeneous reactions. Unlike typical studies, here the idea of second-order velocity slip and temperature jump boundary conditions has been implemented. According to the literature survey, no attempt has been made by researchers for such analysis and is presented for the first time. The present flow problem is utilized in presence of applied magnetic field effects which are useful in the industry of metal-working, chemical reactors, plasma materials, modern metallurgical, oil exploration, and extraction of geothermal energy. The analytical solutions of such transmuted flow equations are determined by employing the homotopy analysis method [40–45]. The accuracy of this method is successfully obtained and expressed in a tabular form. Finally, the important feature effective parameters are graphically underlined and discussed for some velocity, temperature, and concentration profiles with technical relevance.

2. Mathematical Modeling

We consider a two-dimensional flow of Oldroyd-B due to infinite stretching disks. Let flow be axisymmetric and considered fluid be incompressible. The velocity slip and temperature jump are also considered at the walls of stretchable disks. A magnetic field with strength B_0 is imposed in z-direction. The effects of electric and induced magnetic fields are neglected. It is assumed that both lower and upper disks are maintained at temperature T_1 and T_2, respectively. Following Merkin and Chaudhary [46], the mathematical expressions repressing the homogeneous-heterogeneous reactions are expressed as

$$A + 2B \longrightarrow 3B, \quad \text{rate} = k_c \alpha \beta^2. \tag{1}$$

The isothermal, first-order reaction associated with a catalyst surface is represented as

$$A \longrightarrow B, \quad \text{rate} = k_s \alpha, \tag{2}$$

where α and β stand for concentrations of chemical species and A, B, k_c, and k_s denote the rate constants. In the present analysis, both reactions are treated as processes which are isothermal. The analysis is performed by opting a cylindrical coordinate (r, θ, z). All the involved expressions are independent of θ due to axisymmetry. The constitutive partial differential equations for Oldroyd-B fluid in presence of chemical reactions are expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{3}$$

$$\begin{aligned}
u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial \rho}{\partial r} + \nu\left(2\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 w}{\partial r\,\partial z} + \frac{\partial^2 u}{\partial z^2} + \frac{2}{r}\frac{\partial u}{\partial r} - 2\frac{u}{r^2}\right) - \lambda_1'\left(w^2\frac{\partial^2 u}{\partial z^2} + 2uw\frac{\partial^2 u}{\partial r\,\partial z} + \frac{\partial^2 u}{\partial r^2}\right) \\
&\quad + \nu\lambda_2'\left(\frac{4u^2}{r^3} - \frac{2w}{r^2}\frac{\partial u}{\partial z} - \frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2 - 2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} + w\frac{\partial^3 u}{\partial z^3} - \frac{2u}{r^2}\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial z^2}\frac{\partial u}{\partial r} - 2\left(\frac{\partial u}{\partial r}\right)^2 - \frac{1}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + \frac{2w}{r}\frac{\partial^2 u}{\partial r\,\partial z} - \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\,\partial z}\right. \\
&\quad \left. - \frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\,\partial z} + u\frac{\partial^3 u}{\partial r\,\partial z^2} + w\frac{\partial^3 w}{\partial r\,\partial z^2} + \frac{2u}{r}\frac{\partial^2 u}{\partial r^2} - 2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2} - \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2} + 2w\frac{\partial^3 u}{\partial r^2 \partial z} + u\frac{\partial^3 w}{\partial r^2 \partial z} + 2u\frac{\partial^3 u}{\partial r^3}\right) + \frac{\sigma B_0^2}{\rho}\left(-u - \lambda_1' w\frac{\partial u}{\partial z}\right),
\end{aligned} \tag{4}$$

$$\begin{aligned}
u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial \rho}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 u}{\partial r\,\partial z} + 2\frac{\partial^2 w}{\partial z^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \lambda_1'\left(w^2\frac{\partial^2 u}{\partial z^2} + 2uw\frac{\partial^2 u}{\partial r\,\partial z} + u^2\frac{\partial^2 u}{\partial r^2}\right) \\
&\quad + \nu\lambda_2'\left(-\frac{u}{r^2}\frac{\partial u}{\partial z} - \frac{1}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} + \frac{w}{r}\frac{\partial^2 u}{\partial z^2} - 2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} + 2w\frac{\partial^3 w}{\partial z^3} + \frac{u}{r^2}\frac{\partial w}{\partial r} - \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial u}{\partial z} - \frac{\partial^2 u}{\partial z^2}\frac{\partial w}{\partial r} - \frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r} + \frac{u}{r}\frac{\partial^2 u}{\partial r\,\partial z} - \frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\,\partial z}\right. \\
&\quad + \frac{w}{r}\frac{\partial^2 w}{\partial r\,\partial z} - \frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\,\partial z} + w\frac{\partial^3 w}{\partial r\,\partial z^2} + 2u\frac{\partial^2 w}{\partial r\,\partial z^2} - 2\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{u}{r}\frac{\partial^2 w}{\partial r^2} - \frac{\partial w}{\partial z}\frac{\partial^3 w}{\partial r^2} + u\frac{\partial^3 u}{\partial r^2 \partial z} + w\frac{\partial^3 w}{\partial r^2 \partial z} + u\frac{\partial^3 w}{\partial r^3},
\end{aligned} \tag{5}$$

$$u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = K\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right), \tag{6}$$

$$u\frac{\partial \alpha}{\partial r} + w\frac{\partial \alpha}{\partial z} = D_A\left(\frac{\partial^2 \alpha}{\partial r^2} + \frac{1}{r}\frac{\partial \alpha}{\partial r} + \frac{\partial^2 \alpha}{\partial z^2}\right) - k_c \alpha \beta^2, \tag{7}$$

$$u\frac{\partial \beta}{\partial r} + w\frac{\partial \beta}{\partial z} = D_B\left(\frac{\partial^2 \beta}{\partial r^2} + \frac{1}{r}\frac{\partial \beta}{\partial r} + \frac{\partial^2 \beta}{\partial z^2}\right) + k_c \alpha \beta^2, \tag{8}$$

where u and w are the radial and axial components of velocities, respectively, p is the pressure, ρ is the fluid density, μ stands for dynamic viscosity of fluid, $\nu = (\mu/\rho)$ represents the kinematic viscosity, a and c are the stretching constants, λ_1' is the constant of relaxation, λ_2' is the retardation time, T is the temperature, K is the thermal diffusivity, and D_A, D_B are the diffusion species coefficient of A and B.

2.1. Slip Boundary Conditions. As it has been mentioned earlier that the present flow problem is assisted with slip boundary conditions. For the velocity profile, the derivation of second-order velocity slip is based on the expansion of Taylor series from the first-order Maxwell conditions which are generally expressed as

$$u = u_w + \left(\frac{2-\sigma_u}{\sigma_u}\right)\frac{\partial u}{\partial n}\tau_1 + \left(\frac{2-\sigma_u}{\sigma_u}\right)\tau_1^2\frac{\partial^2 u}{\partial n^2}. \tag{9}$$

For the present analysis, we propose the following second-order boundary conditions:

$$\begin{aligned} u &= ar + \left(\frac{2-\sigma_u}{\sigma_u}\right)\tau_1\frac{\partial u}{\partial z} + \left(\frac{2-\sigma_u}{\sigma_u}\right)\tau_1^2\frac{\partial^2 u}{\partial z^2}, \quad w = 0,\ p = \frac{a\mu\beta_1 r^2}{4d^2},\ \text{at } z = 0, \\ u &= cr - \left(\frac{2-\sigma_u}{\sigma_u}\right)\tau_1\frac{\partial u}{\partial z} - \left(\frac{2-\sigma_u}{\sigma_u}\right)\tau_1^2\frac{\partial^2 u}{\partial z^2}, \quad w = 0,\ p = 0,\ \text{at } z = d, \end{aligned} \tag{10}$$

where a and b represent the stretching rates, σ_u is the tangential momentum accommodation coefficient, and τ_1 denotes the molecular mean-free path. It is a well-established fact that the molecular mean-free path is assumed positive, i.e., $\epsilon_1 > 0$ and. $\epsilon_2 < 0$.

2.2. Temperature Jump Boundary Conditions. By using Taylor series second-order expansion for K_n from the first order, Smoluchowski jump condition second-order jump conditions are proposed in [6] as follows:

$$T = T_w + \left(\frac{2-\sigma_T}{\sigma_T}\right)\left(\frac{2\xi}{\xi+1}\right)\frac{1}{\mathrm{Pr}}\tau_2\frac{\partial T}{\partial n} + \left(\frac{2-\sigma_T}{\sigma_T}\right)\left(\frac{2\xi}{\xi+1}\right)\frac{1}{\mathrm{Pr}}\frac{\tau_2^2}{2}\frac{\partial^2 T}{\partial n^2}. \tag{11}$$

The second-order temperature jump boundary conditions associated with the governing equations are

$$\begin{aligned} T &= T_1 + \left(\frac{2-\sigma_T}{\sigma_T}\right)\left(\frac{2\xi}{\xi+1}\right)\frac{1}{\mathrm{Pr}}\tau_2\frac{\partial T}{\partial z} + \left(\frac{2-\sigma_T}{\sigma_T}\right)\left(\frac{2\xi}{\xi+1}\right)\frac{1}{\mathrm{Pr}}\frac{\tau_2^2}{2}\frac{\partial^2 T}{\partial z^2}, \quad \text{at } z = 0, \\ T &= T_2 - \left(\frac{2-\sigma_T}{\sigma_T}\right)\left(\frac{2\xi}{\xi+1}\right)\frac{1}{\mathrm{Pr}}\tau_2\frac{\partial T}{\partial z} + \left(\frac{2-\sigma_T}{\sigma_T}\right)\left(\frac{2\xi}{\xi+1}\right)\frac{1}{\mathrm{Pr}}\frac{\tau_2^2}{2}\frac{\partial^2 T}{\partial z^2}, \quad \text{at } z = d, \end{aligned} \tag{12}$$

where σ_T is the thermal accommodation coefficient and ξ is the specific heat ratio. The other boundary conditions for the flow problem are prescribed by

$$\begin{aligned} &\alpha = \alpha_0 \text{ at } z = 0, \quad D_A\frac{\partial \alpha}{\partial z} = k_s\alpha \text{ at } \quad z = d, \\ &\beta = 0 \text{ at } z = 0, \quad D_B\frac{\partial \beta}{\partial z} = -k_s\alpha \text{ at } \quad z = d. \end{aligned} \tag{13}$$

Introducing the similarity variables,

$$\begin{aligned} u &= -\frac{ar}{2}H'(\eta), \\ w &= a\,dH(\eta), \\ p &= a\mu\left(P(\eta) + \frac{\beta_1 r^2}{4d^2}\right), \\ \eta &= \frac{z}{d}, \\ T &= T_1 + (T_2 - T_1)\theta(\eta), \\ \alpha &= \alpha_0\varphi(\eta), \\ \beta &= \alpha_0 g(\eta), \\ T &= T_1 + (T_2 - T_1)\theta(\eta), \\ \alpha &= \alpha_0\varphi(\eta), \\ \beta &= \alpha_0 g(\eta). \end{aligned} \tag{14}$$

In view of the above similarity variables, equations (4)–(10) yield

$$\frac{R}{2}\left(H'^2 - 2HH''\right) = -\Big[\beta_1 + H''' + \lambda_1' aR\Big(HH'H'' - H^2H'''\Big) + \lambda_2' a\Big(HH^{(iv)} - H''^2\Big)\Big] + RM\left(H' + \lambda_1' aHH''\right), \tag{15}$$

$$\theta'' - RPrH\theta' = 0, \tag{16}$$

$$\varphi'' - RSc\left(K_1\varphi g^2 + H\varphi'\right) = 0, \tag{17}$$

$$\delta g'' + RSc\left(K_1\varphi g^2 + Hg'\right) = 0, \tag{18}$$

$$P\prime = \frac{3H''}{2} - RHH' - \lambda_1 RH^2H'', \tag{19}$$

$$\begin{aligned} &H(0) = 0,\\ &H(1) = 0,\\ &H'(0) = -2 + \Big(\epsilon_1 H''(0) + \epsilon_2 H'''(0)\Big), \end{aligned} \tag{20}$$

$$H'(1) = -2\gamma - \Big(\epsilon_1 H''(0) + \epsilon_2 H'''(0)\Big), \quad P(0) = 0, \tag{21}$$

$$\begin{aligned} &\theta(0) = \epsilon_3\theta'(0) + \epsilon_4\theta''(0),\\ &\theta(1) = 1 - \left(\epsilon_3\theta'(1) + \epsilon_4\theta''(1)\right), \end{aligned} \tag{22}$$

$$\begin{aligned} &\varphi(0) = 1,\\ &\varphi'(1) = K_2\varphi(1),\\ &g(0) = 0,\\ &\delta g'(1) = -K_2 g(1), \end{aligned} \tag{23}$$

where γ is the wall stretching parameter, R stands for the Reynolds number, Pr is the Prandtl number, ϵ_1 is the first-order velocity slip parameter, M is the Hartmann number, ϵ_2 is the second-order velocity slip parameter, ϵ_3 is the first-order temperature jump parameter, ϵ_4 stands for temperature jump parameter of the second order, Sc represents the Schmidt number, δ is the ratio of the diffusion coefficient, Kn denotes the Knudsen number, K_1 is the strength of the homogeneous reaction, and K_2 is the strength of the heterogeneous reaction and is defined as

$$\begin{aligned} \gamma &= \frac{c}{a},\\ R &= \frac{ad^2}{\nu},\\ \mathrm{Pr} &= \frac{\nu}{\alpha},\\ Kn &= \frac{\tau_{1,2}}{d},\\ \varepsilon_1 &= Kn\left(\frac{2-\sigma_u}{\sigma_u}\right),\\ \varepsilon_2 &= \frac{Kn^2}{2}\left(\frac{2-\sigma_u}{\sigma_u}\right),\\ \varepsilon_3 &= Kn\left(\frac{2-\sigma_T}{\sigma_T}\right)\left(\frac{2\xi}{\xi+1}\right)\frac{1}{\mathrm{Pr}},\\ \varepsilon_4 &= \frac{Kn^2}{2}\left(\frac{2-\sigma_T}{\sigma_T}\right)\left(\frac{2\xi}{\xi+1}\right)\frac{1}{\mathrm{Pr}},\\ \delta &= \frac{D_B}{D_A}. \end{aligned} \tag{24}$$

The constant β_1 has been eliminated from equation (15) as the following procedure:

$$\begin{aligned} H^{(iv)} = {}& RHH''' - \lambda_1 R\Big(-HH'H''' - H^2H^{(iv)} + HH''^2 + H'^2H''\Big)\\ &- MR\Big[H'' + \lambda_1\Big(H'H'' + HH'''\Big)\Big]\\ &- \lambda_2\Big(-2H''H''' + H'H^{(iv)} + HH^{(v)}\Big), \end{aligned} \tag{25}$$

in which $\lambda_1 = \lambda_1' a$ is the Deborah number for relaxation time and $\lambda_2 = \lambda_2' a$ for the retardation time. It is pointed out here that the diffusion coefficients of chemical species A and B are not equal in general. So, we remarked that constants A and B are of comparable size as a special case and subsequently D_A and D_B are equal, i.e., $\delta = 1$. Equations (16) and (17) lead to the following relation:

$$\begin{aligned} &\varphi + g = 1,\\ &\varphi'' - RSc\left(K_1\varphi(1-\varphi)^2 + H\varphi'\right) = 0,\\ &\varphi(0) = 1,\\ &\varphi'(1) = K_2\varphi(1). \end{aligned} \tag{26}$$

Following mathematical expressions are suggested for the wall skin friction coefficient, local Nusselt number, and local Sherwood number at both surfaces of disks:

$$C_{1f,2f}=\frac{\tau_{rz}|_{\eta=0,1}}{(1/2)\rho(\delta r)^2}=2R^{-1}H''(\eta)|_{\eta=0,1},$$

$$N_{1u,2u}=-\frac{dk_T(\partial T/\partial z)|_{\eta=0,1}}{k_T(T_2-T_1)}=-\theta'(\eta)|_{\eta=0,1}, \tag{27}$$

$$Sh=\frac{-(D(\partial C/\partial z))|_{\eta=0,1}}{D(C_1-C_2)}=-\varphi'(\eta)|_{\eta=0,1}.$$

3. Solution Methodology

To start our simulations, first we introduce the following initial guesses for velocity, temperature, and concentration profiles:

$$\begin{aligned}H_0(\eta)&=\frac{1}{8\epsilon_1+12\epsilon_1^2}(-2\eta(1+\eta)(-1+\eta(1+\gamma)\\&\quad+2\epsilon_1(-2+\eta+\gamma(1+\eta))-6\epsilon_2(1+\eta))),\\\theta_0(\eta)&=\frac{\eta+\epsilon_3}{1+2\epsilon_3},\\\varphi_0(\eta)&=\frac{-1+K_2(1-\eta)}{-1+K_2},\end{aligned} \tag{28}$$

with auxiliary linear operators:

$$L_H=\frac{d^4}{d\eta^4},\quad L_\theta=\frac{d^2}{d\eta^2},\quad L_\varphi=\frac{d^2}{d\eta^2}. \tag{29}$$

The mathematical expressions associated with the zeroth-order deformation problem are defined as

$$\begin{aligned}(1-q)L_H[H(\eta;q)-H_0(\eta)]&=q\hbar_H N_H[H(\eta;q)],\\(1-q)L_\theta[\theta(\eta;q)-\theta_0(\eta)]&=q\hbar_\theta N_\theta[\theta(\eta;q)],\\(1-q)L_\varphi[\varphi(\eta;q)-\varphi_0(\eta)]&=q\hbar_\varphi N_\varphi[\varphi(\eta;q)],\end{aligned} \tag{30}$$

where $\hbar_H$, $\hbar_\theta$, and $\hbar_\varphi$ denote the auxiliary parameters and $q\ \epsilon\ [0,1]$ represents the embedding parameter. And,

$$\begin{aligned}N_H[H(\eta;q)]&=\frac{\partial^4H(\eta,q)}{\partial\eta^4}-RH(\eta,q)\left(\frac{\partial^3H(\eta,q)}{\partial\eta^3}\right)-R\lambda_1\left(H(\eta,q)\left(\frac{\partial H(\eta,q)}{\partial\eta}\right)\left(\frac{\partial^3H(\eta,q)}{\partial\eta^3}\right)\right.\\&\quad\left.+H^2(\eta,q)\left(\frac{\partial^4H(\eta,q)}{\partial\eta^4}\right)-\left(\frac{\partial H(\eta,q)}{\partial\eta}\right)^2\left(\frac{\partial^2H(\eta,q)}{\partial\eta^2}\right)-H(\eta,q)\left(\frac{\partial^2H(\eta,q)}{\partial\eta^2}\right)^2\right)\\&\quad+\lambda_2\left(-2\left(\frac{\partial^2H(\eta,q)}{\partial\eta^2}\right)\left(\frac{\partial^3H(\eta,q)}{\partial\eta^3}\right)+\left(\frac{\partial H(\eta,q)}{\partial\eta}\right)\left(\frac{\partial^4H(\eta,q)}{\partial\eta^4}\right)+H(\eta,q)\left(\frac{\partial^5H(\eta,q)}{\partial\eta^5}\right)\right)\\&\quad+MR\left[\frac{\partial^2H(\eta,q)}{\partial\eta^2}+\lambda_1\left(\left(\frac{\partial H(\eta,q)}{\partial\eta}\right)\left(\frac{\partial^2H(\eta,q)}{\partial\eta^2}\right)+H(\eta,q)\left(\frac{\partial^3H(\eta,q)}{\partial\eta^3}\right)\right)\right],\\N_\theta[\theta(\eta;q)]&=\frac{\partial^2\theta(\eta,q)}{\partial\eta^2}-R\mathrm{Pr}H(\eta,q)\frac{\partial\theta(\eta,q)}{\partial\eta},\\N_\varphi[\varphi(\eta;q)]&=\frac{\partial^2\varphi(\eta,q)}{\partial\eta^2}-RSc\left(K_1\varphi(\eta,q)(1-\varphi(\eta,q))^2+H(\eta,q)\frac{\partial\varphi(\eta,q)}{\partial\eta}\right).\end{aligned} \tag{31}$$

The equations for the m-th deformations of the problem are

$$
\begin{aligned}
L_H\left[H_m(\eta)-\chi_m H_{m-1}(\eta)\right] &= \hbar_H R_{1m}(\eta),\\
L_\theta\left[\theta_m(\eta)-\chi_m \theta_{m-1}(\eta)\right] &= \hbar_\theta R_{2m}(\eta),\\
L_\varphi\left[\varphi_m(\eta)-\chi_m \varphi_{m-1}(\eta)\right] &= \hbar_\varphi R_{3m}(\eta),\\
H_m(0) &= H_m(1)=0,\\
H_m'(0) &= \left(\epsilon_1 H_m''(0)+\epsilon_2 H_m'''(0)\right),\ H_m'(1)=-\left(\epsilon_1 H_m''(1)+\epsilon_2 H_m'''(1)\right),\\
\theta_m(0) &= \left(\epsilon_3\theta_m'(0)+\epsilon_4\theta_m''(0)\right),\\
\theta(1) &= -\left(\epsilon_3\theta_m'(1)+\epsilon_4\theta_m''(1)\right),\\
\varphi_m(0) &= 0,\\
\varphi_m'(1) &= K_2\varphi_m(1),\\
\chi_m &= \begin{cases} 0, & m\le 1,\\ 1, & m>0.\end{cases}
\end{aligned}
\tag{32}
$$

The series solution is computed iteratively for $m=1,2,3,\ldots$ using MATHEMATICA software.

4. Convergence of Solution

In order to obtain the comfortable accuracy of the homotopic solution, the significance of auxiliary parameters cannot be denied. This task has been completed by preparing three h-curves, organized for velocity, temperature, and concentration profiles for some dignified values of emerging parameters. The admissible values of such parameter guaranteed the convergence of the solution. The convergence of the derived series solution is controlled by auxiliary parameters $\hbar_H$, $\hbar_\theta$, and $\hbar_\varphi$. Therefore, we have sketched the $\hbar$-curves in Figure 1 to determine the admissible values of $\hbar_H$, $\hbar_\theta$, and $\hbar_\varphi$. These figures reveal that the convergence region lies within the domain $-0.8\le\hbar_H\le-0.2, -1.5\le\hbar_\theta\le-0.4$, and $-1.4\le\hbar_\varphi\le-0.7$.

In Table 1, the computations have been performed to illustrate the convergence of the obtained solution for $H''(0)$, $\theta'(0)$, and $\varphi'(0)$ at various approximations. Close observations to the table suggest that accuracy of the solution has been obtained at the 15^{th} order of approximations.

5. Physical Interpretations of Results

In this section, the effects of various arising parameters on radial and vertical velocity components, pressure, temperature, and concentration fields are discussed with relevant physical significances.

5.1. Dimensionless Velocity and Pressure Profiles. Figure 2(a) shows the impact of the Hartmann number M on the velocity vertical component by keeping other parameters fixed. The interface of stronger magnetic force is more valuable to decay the motion of fluid particles. A small increment in velocity was observed first which decreases up to a certain height. Physically, as M increases, the Lorentz force boosts up which resists the flow of liquid due to which velocity decay occurs. Therefore, the presence of magnetic field combats the transport phenomena and subsequently diminishes the vertical velocity. The effects of wall stretching parameter γ on the velocity profile are shown in Figure 2(b). The vertical velocity component rises up with a variation of γ. However, a change in the radial component is not similar to vertical components. Here, velocity increases at a specific range and then gradually decreases. Figure 2(c) delineates the significance of the Deborah number in terms of relaxation time λ_1 on vertical and radial component of velocities. A rise in the vertical component of velocity is observed for larger values of the Deborah number; however, the radial component of velocity decreases smoothly after a small increment. The variation of material parameter λ_2 on both vertical and radial velocity components is illustrated in Figure 2(d). The reverse trend is observed as compared to λ_1 for both components. We observe from Figures 2(e) and 2(f) that when we increase of first- and second-order velocity slip constants (ϵ_1,ϵ_2), the vertical velocity component also increases. Physically, with increase of velocity slip parameters, the stretching velocity affects the movement of fluid so velocity profiles get maximum values. Moreover, the amplitude of radial velocity increases up to a specific range due to the difference of the stretching rate. Figures 2(g) and 2(h) show that the skin friction coefficient increases with increase of both slip parameters. It is scrutinized from Figure 2(i) that pressure decreases in the whole domain by increasing values of the Hartmann number M. It is found from Figure 2(j) that decay in pressure is observed by increasing the velocity slip parameter.

5.2. Dimensionless Temperature Profile. In Figures 3(a) and 3(b), the dimensionless temperature $\theta(\eta)$ is plotted to study the impact of the velocity slip parameter. The temperature decreases by increasing both velocity slip parameters. It is elucidated from Figures 3(c) and 3(d) that the distribution of temperature θ boosts up due to alteration of the first- and

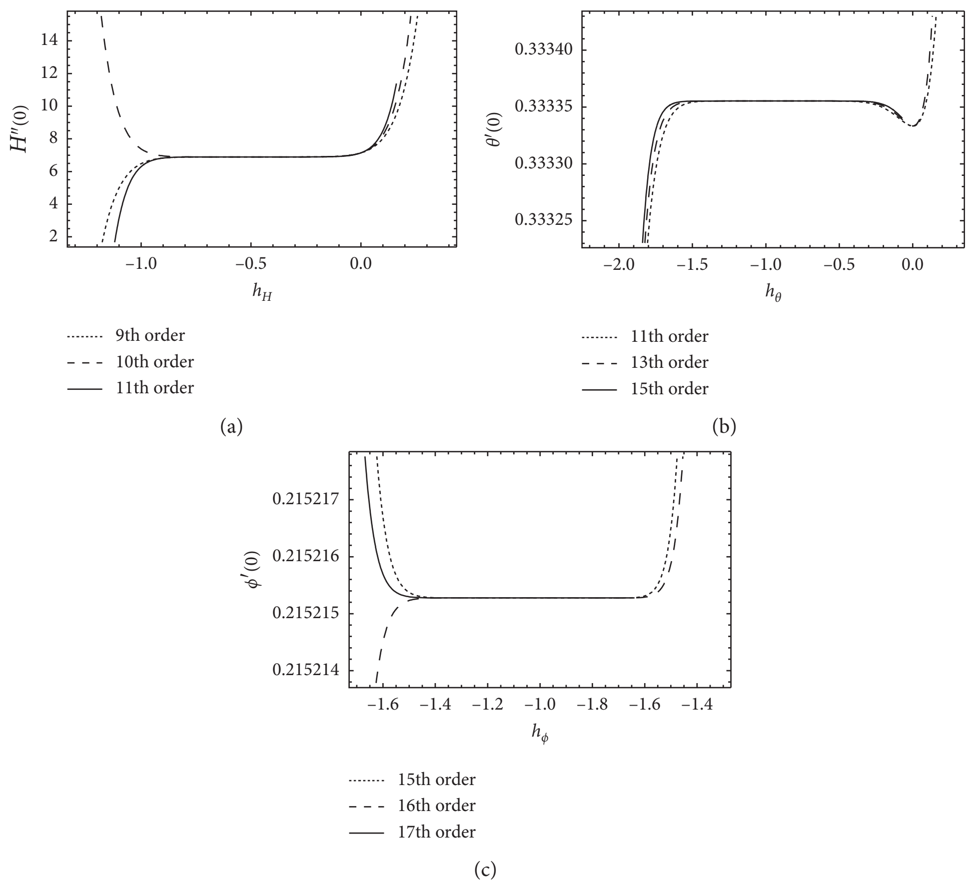


Figure 1: $\hbar$-curves for (a) $\hbar_H$, (b) $\hbar_\theta$, and (c) $\hbar_\varphi$ with $R = 2, \gamma = 0.5, M = 0.3, K_2 = 0.2, \mathrm{Pr} = 0.5, \lambda_1 = \lambda_2 = 0.2, K_1 = 0.5, Sc = 0.2, \epsilon_1 = 0.2, \epsilon_2 = 0.3, \epsilon_3 = 0.3$, and $\epsilon_4 = 0.5$..

Table 1: The HAM convergence at different order of approximations.

Approximation	$H''(0)$	$\theta'(0)$	$\varphi'(0)$
07	6.89132	0.499908	0.214794
10	6.89123	0.499907	0.215223
13	6.89122	0.499906	0.215228
14	6.89121	0.499905	0.215215
15	6.89120	0.499905	0.215215

second-order temperature jump parameters. Physically, due to slip effect, more flow penetrates through the thermal boundary with an increase in temperature jump parameters. Figure 3(e) accomplishes the significance of the Prandtl number Pr on the temperature profile. The impression Pr declined the temperature of the fluid effectively. The dimensionless number Pr depends upon thermal diffusivity which decreases by increasing Pr. Therefore, a decline in the temperature field is observed. Thus, higher values of Pr correspond to lower thermal diffusivity and subsequently declining temperature distribution. Figure 3(f) exhibits the dominant effect of the Hartmann number M on the temperature profile. As expected, the temperature of fluid increases by increasing M. Physically, the applied magnetic field produces the Lorentz force, which creates a drag force which has a tendency to enhance the temperature of the fluid between both disks.

5.3. Dimensionless Concentration Profile. Taking into account of the concentration profile φ, the effects for various parameters are encountered. First, we consider the variation of the homogeneous reaction K_1 on φ. An increase in K_1 results in diminishing of the concentration profile (Figure 4(a)). Figure 4(b) shows the consequence of heterogeneous reaction parameter K_2 on the concentration profile. The rate of mass transfer is enhanced by increasing K_2. Figure 4(c) shows that the rate of mass transfer solely decreases by varying Schmidt number Sc. Sc has an inverse relation with molecular diffusivity which decreases by increasing Sc. The variation of different values of the strengths of the homogeneous parameter K_1 and heterogeneous reaction parameter K_2 on wall concentration on both disks is shown in Figures 5 and 6, respectively. These figures indicate that values of $\varphi'(0)$ and $\varphi'(1)$ increase by increasing K_1 while contradictory behavior is noted for K_2.

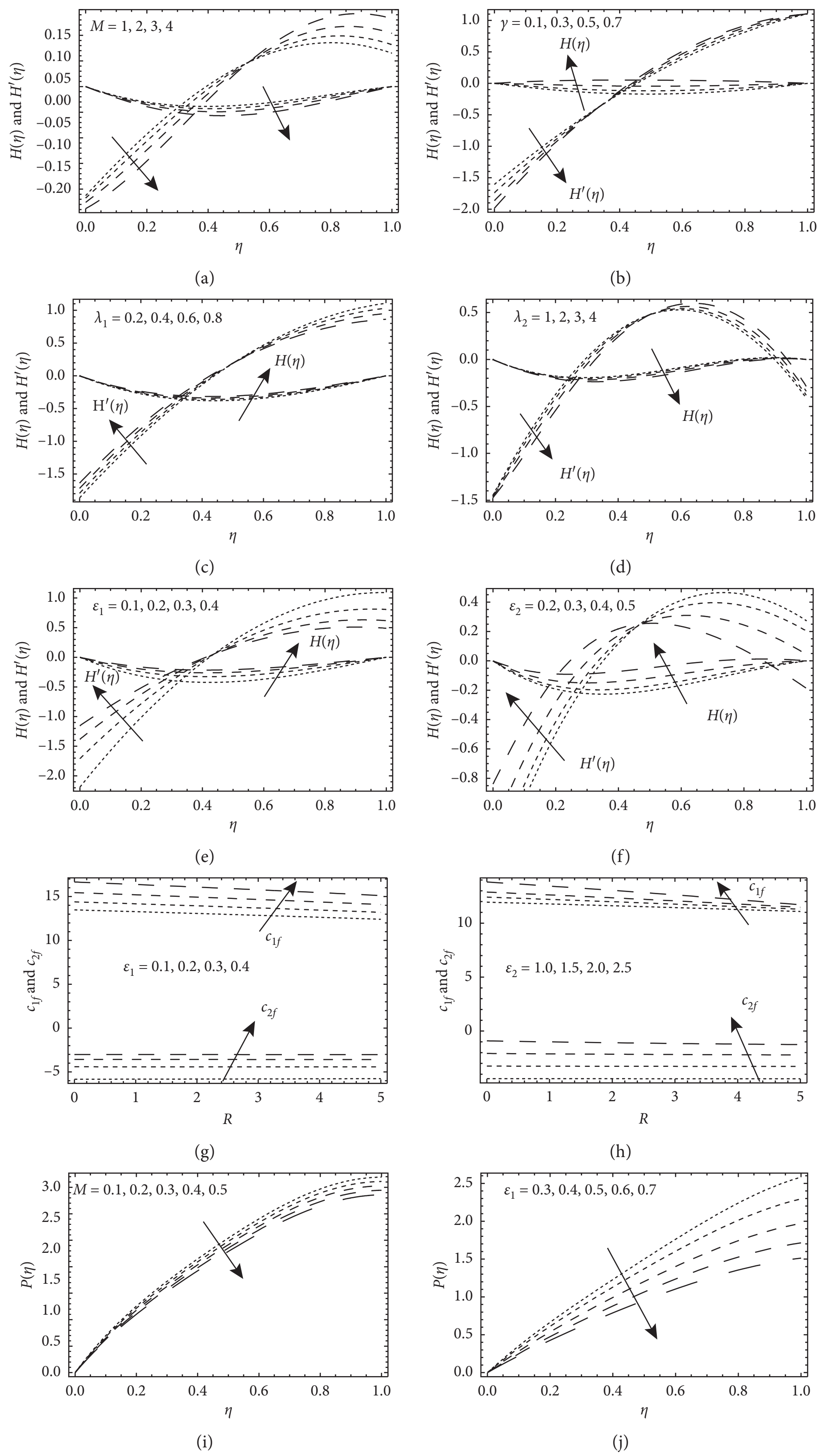


Figure 2: (a)–(f) Graphs of vertical and radial components of velocity, (g)-(h) graphs of for skin friction, and (i)-(j) graphs of pressure for different values of $\gamma = 0.5, \hbar_H = -0.5, \hbar_\theta = -1.2, \hbar_\varphi = -1.0, M = 0.3, R = 5, \epsilon_1 = 0.5, \epsilon_2 = 0.2, \lambda_1 = 0.2$, and $\lambda_2 = 0.5$. (a) Effects of the Hartmann number, (b) effects of the stretching parameter, (c) effects of the Deborah number of relaxation, (d) effects of the Deborah number of retardation, (e) effects of the first-order velocity slip parameter, (f) effects of the second-order velocity slip parameter, (g) influence of the first-order velocity slip parameter on the skin friction coefficient, (h) influence of the second-order velocity slip parameter on the skin friction coefficient, (i) Influence of Hartmann number on pressure (j) Influence of first-order velocity slip parameter on pressure.

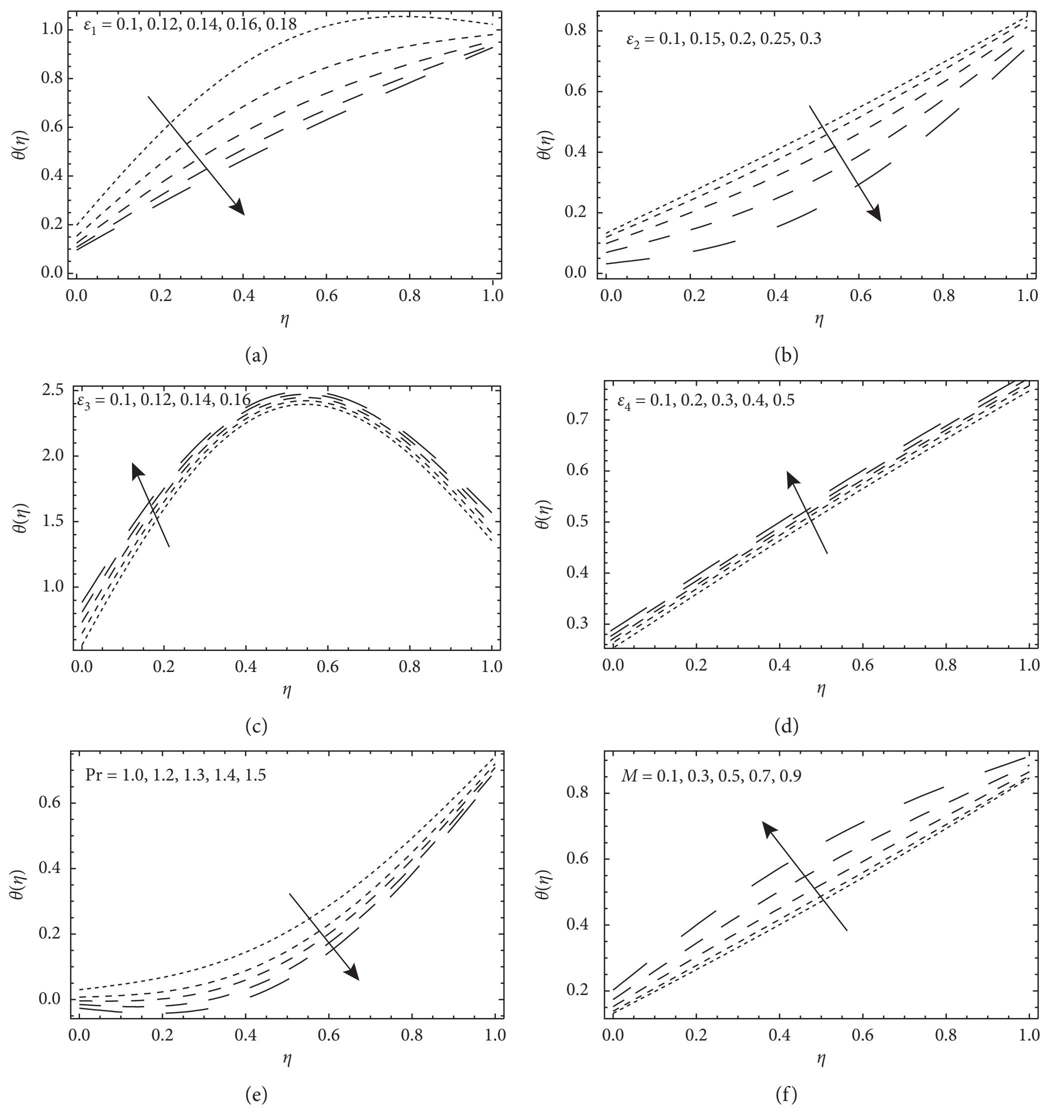


Figure 3: Temperature profile with $\gamma = 0.5, \hbar_H = -0.5, \hbar_\theta = -1.2, \hbar_\varphi = -1.0, M = 0.3, R = 5, \epsilon_1 = \epsilon_2 = 0.2, \epsilon_3 = 0.3, \epsilon_4 = 0.5, \lambda_1 = 0.2$, and $\lambda_2 = 0.5$. (a) Influence of the first-order velocity slip parameter, (b) influence of the second-order velocity slip parameter, (c) variation of the first-order temperature jump parameter, (d) variation of the second-order temperature jump parameter, (e) variation of the Hartmann number, and (f) variation of the Prandtl number.

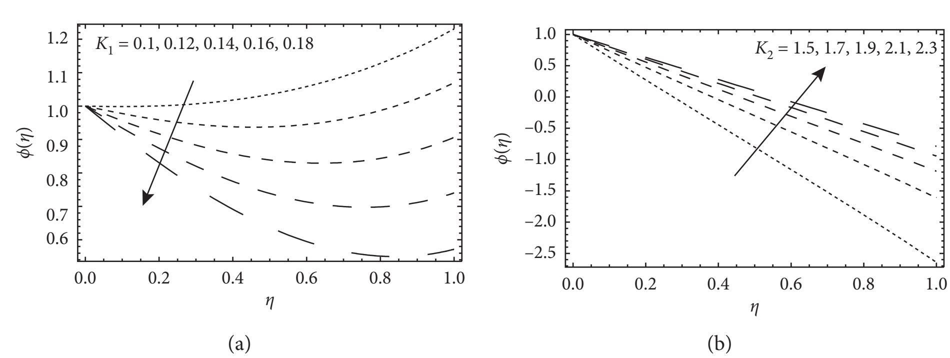
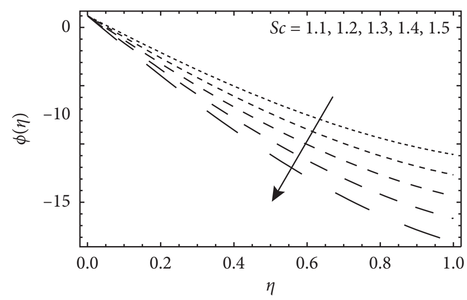
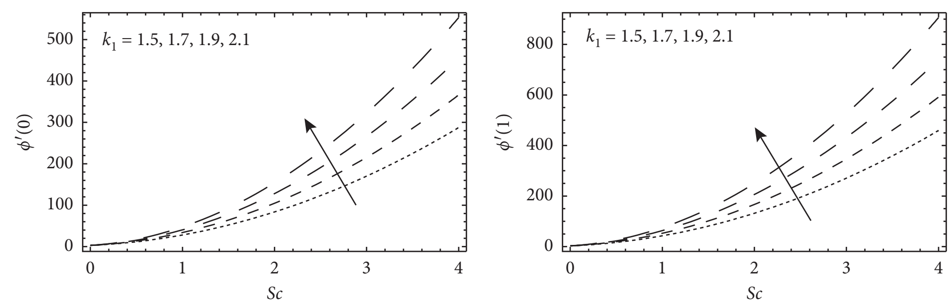


Figure 4: Continued.



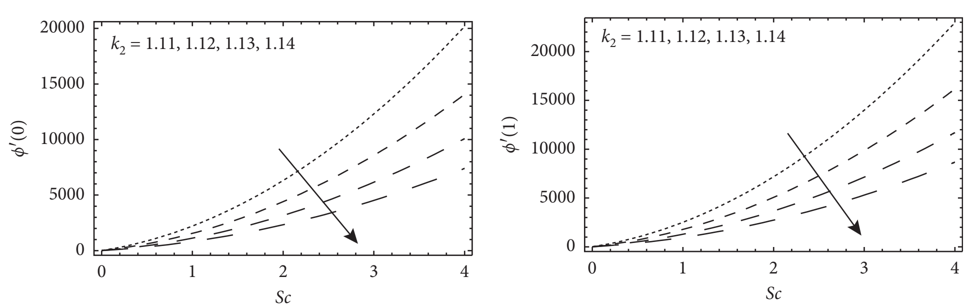
(c)

FIGURE 4: Concentration distribution for $\gamma = 0.5, \hbar_H = -0.5, \hbar_\theta = -1.2, \hbar_\varphi = -1.0, M = 0.3, R = 2, \epsilon_1 = \epsilon_2 = 0.2, \epsilon_3 = 0.3, \epsilon_4 = 0.5, \lambda_1 = 0.2$, and $\lambda_2 = 0.2$. (a) Influence of strength of the homogeneous reaction, (b) influence of strength of the heterogeneous reaction, and (c) influence of strength of the Schmidt number.



(a) (b)

FIGURE 5: Influence of strength of the homogeneous reaction.



(a) (b)

FIGURE 6: Influence of strength of the heterogeneous reaction.

5.4. Local Nusselt Number. Table 2 aims to elaborate the iterative numerical variation in the local Nusselt number against involved fluid parameters. We found that with the increase in the velocity slip parameter, the temperature profile at the lower disk increases. The heat transfer rate decreases by increasing the Hartmann number M at the lower disk. However, opposite values for M are observed for the upper disk. Such observations are made as both disks are stretched with different velocities.

6. Conclusions

In this work, a chemically reactive flow of Oldroyd-B fluid subject to stretchable disks is considered in presence of homogeneous and heterogeneous chemical reactions. The homogeneous-heterogeneous reactions are considered in the concentration equation. The physical features are visualized for various involved parameters graphically. The important observations are summarized as follows:

Table 2: Variation in the local Nusselt number at lower and upper disks.

ϵ_1	ϵ_2	ϵ_3	ϵ_4	Pr	M	Lower disk	Upper disk
0.1	0.5	0.5	0.2	1.0	0.3	−0.66799	−0.75961
0.2	0.5	0.5	0.2	1.0	0.3	−0.65734	−0.73076
0.3	0.5	0.5	0.2	1.0	0.3	−0.64662	−0.72170
0.2	0.1	0.5	0.2	1.0	0.3	−0.66799	−0.75961
0.2	0.2	0.5	0.2	1.0	0.3	−0.50683	−0.61961
0.2	0.3	0.5	0.2	1.0	0.3	−0.17056	−0.25760
0.2	0.5	0.1	0.2	1.0	0.3	−0.70753	−0.66151
0.2	0.5	0.2	0.2	1.0	0.3	−0.89202	−0.73829
0.2	0.5	0.3	0.2	1.0	0.3	−0.90542	−0.84591
0.2	0.5	0.5	1.0	1.0	0.3	−0.52624	−0.46733
0.2	0.5	0.5	2.0	1.0	0.3	−0.52626	−0.46736
0.2	0.5	0.5	3.0	1.0	0.3	−0.52628	−0.46739
0.2	0.5	0.5	0.2	0.1	0.3	−0.49842	−0.50166
0.2	0.5	0.5	0.2	0.2	0.3	−0.49684	−0.50332
0.2	0.5	0.5	0.2	0.3	0.3	−0.49527	−0.50498
0.2	0.5	0.5	0.2	1.0	0.1	−0.46169	−0.53829
0.2	0.5	0.5	0.2	1.0	0.2	−0.46974	−0.53052
0.2	0.5	0.5	0.2	1.0	0.3	−0.48423	−0.51662

(i) The velocity distribution increases with variation of slip parameters while it decreases with the Deborah number for retardation time.

(ii) The concentration distribution declines with increment of the Schmidt number and the homogeneous reaction while effects of the heterogeneous reaction parameter are quite reverse.

(iii) The temperature distribution increases by increasing the Hartmann number while lower temperature distribution is observed for larger values of the Prandtl number.

(iv) The presence of first- and second-order velocity slip results in an increment in the wall shear stress.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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Research Article

A Generalization of the Secant Zeta Function as a Lambert Series

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Recently, Lalín, Rodrigue, and Rogers have studied the secant zeta function and its convergence. They found many interesting values of the secant zeta function at some particular quadratic irrational numbers. They also gave modular transformation properties of the secant zeta function. In this paper, we generalized secant zeta function as a Lambert series and proved a result for the Lambert series, from which the main result of Lalín et al. follows as a corollary, using the theory of generalized Dedekind eta-function, developed by Lewittes, Berndt, and Arakawa.

1. Introduction

The Dedekind eta-function and its limiting values have been considered by several authors starting from Riemann's posthumous fragment [1] and Wintner [2] and later by Reyna [3] and Wang [4]. There are many generalizations of the Dedekind eta-function as a Lambert series including those of Lewittes [5], Berndt [6], and Arakawa [7, 8]. In particular cases, they reduce to the cotangent or the cosecant zeta function. Lerch [9] in 1904 introduced the cotangent zeta function for an algebraic irrational number z and an odd positive integer s as

$$\xi(z, s) := \sum_{n=1}^{\infty} \frac{\cot(n\pi z)}{n^s}. \tag{1}$$

He stated the following functional equation for the cotangent zeta function, but without proof.

Theorem 1 (see [9]). *For any algebraic irrational number z and sufficiently large positive integer $k = k(z)$, we have*

$$\xi(z, 2k+1) + z^{2k}\xi\left(\frac{1}{z}, 2k+1\right) = (2\pi)^{2k+1}\phi(z, 2k+1), \tag{2}$$

where

$$\phi(z, n) := \sum_{j=0}^{n+1} \frac{B_j B_{n+1-j}}{j!\,(n+1-j)!} z^{j-1}, \tag{3}$$

where B_i is the i-th Bernoulli number.

Berndt [10], in 1973, focused on the cotangent zeta function for general $s \in \mathbb{C}$ and proved Lerch's functional equation for cotangent zeta function. He found many interesting explicit formulae for $\xi(z, s)$ when z is a quadratic irrational and $s \geq 3$ is an odd integer. One such pleasing formula is

$$\xi\left(\frac{1+\sqrt{5}}{2}, 3\right) = -\frac{\pi^3}{45\sqrt{5}}. \tag{4}$$

In fact, Berndt's work implies that $\sqrt{j}\,\xi(\sqrt{j}, s)\pi^{-s} \in \mathbb{Q}$, where j is any positive integer and $s \geq 3$ is an odd integer.

2. Secant Zeta Function

Recently, Lalín et al. [11] considered the secant zeta function

$$\psi(z, s) := \sum_{n=1}^{\infty} \frac{\sec(n\pi z)}{n^s} \tag{5}$$

and found its special values at some particular quadratic irrational arguments. They proved the following results.

Theorem 2 (see [11], Theorem 1). *The series (5) is absolutely convergent in the following cases:*

(1) When $z = p/q$ is a rational number with q odd and $s > 1$.

(2) When z is an algebraic irrational number and $s \geq 2$.

To prove this theorem, they have used the celebrated Thue–Siegel–Roth theorem.

Theorem 3 (see [11], Theorem 3). *Let E_m denote the Euler numbers and let B_m denote the Bernoulli numbers. Suppose that l is an even positive integer. Then, for appropriate values of α,*

$$(\alpha+1)^{l-1}\psi\left(\frac{\alpha}{\alpha+1}, l\right) - (-\alpha+1)^{l-1}\psi\left(\frac{\alpha}{-\alpha+1}, l\right) = \frac{(\pi i)^l}{l!}\sum_{n=0}^{l}\left(2^{n-1}-1\right)B_n E_{l-n}\binom{l}{n}\left[(1+\alpha)^{n-1} - (1-\alpha)^{n-1}\right]. \tag{6}$$

They found the values of the secant zeta function at some quadratic irrational numbers. For $j \in \mathbb{Z}$,

$$\psi\left(\sqrt{2j(2j+1)}, 2\right) = (3j+1)\frac{\pi^2}{6},$$
$$\psi\left(\sqrt{8j(2j+1)}, 2\right) = \frac{\pi^2}{6}, \tag{7}$$
$$\psi\left(\sqrt{2j(2j+1)}, 4\right) = \frac{75j^2+46j+6}{8j+3}\frac{\pi^4}{180}.$$

After observing these values, they conjectured the following.

Conjecture 1 (see [11], Conjecture 1). *If j is any positive integer and s is an even positive integer, then*

$$\psi(\sqrt{j}, s)\pi^{-s} \in \mathbb{Q}. \tag{8}$$

By a clever use of residue theorem, Berndt and Straub [12] proved the above functional equation (6), and from it they derived

$$\psi(\sqrt{r}, s)\pi^{-s} \in \mathbb{Q}, \quad r \in \mathbb{Q}^+, s \in 2\mathbb{N}. \tag{9}$$

Furthermore, they connected the secant Dirichlet series with Eichler integrals of Eisenstein series and checked unimodularity of period polynomials. On the contrary, Charollais and Greenberg [13] related the secant Dirichlet series $\psi(\alpha, s)$ to the generalized eta-function which was studied by Arakawa [7]. They proved that for $s \in 2\mathbb{N}$,

$$\psi(\alpha, s)\pi^{-s} \in \mathbb{Q}(\alpha), \tag{10}$$

for all real quadratic irrationals α. They used Arakawa's result to give an explicit formula for $\psi(\alpha, s)$ for real quadratic irrational numbers α.

We will introduce a generalization of the secant zeta function as a Lambert series. Using the theory of generalized Dedekind eta-function due to Lewittes [5], Berndt [6], and Arakawa [7], we shall give a generalization of Theorem 3.

We begin by briefly describing the theory of generalized Dedekind eta-function, developed by Lewittes [5], Berndt [6], and Arakawa [7], which is a main tool in our study.

3. Work of Lewittes and Berndt

Lewittes and Berndt treat the case of the upper half-plane $\mathbb{H}$ while Arakawa treats the case of upper half plane limiting to an algebraic irrational number. Hereafter, we use the following notations:

$$e[w] := \exp(2\pi i w), \quad w \in \mathbb{C},$$
$$\langle x\rangle \in \mathbb{R}, \quad 0 < \langle x\rangle \leqq 1, \quad x - \langle x\rangle \in \mathbb{Z}, \tag{11}$$
$$\{x\} \in \mathbb{R}, \quad 0 \leqq \{x\} < 1, \quad x - \{x\} \in \mathbb{Z}.$$

Lewittes [5] defined the generalization of the Dedekind eta-function as a Lambert series. For a pair (r_1, r_2) of real numbers, $z \in \mathbb{H}$ and arbitrary $s \in \mathbb{C}$, he considered the series

$$A(z, s, r_1, r_2) := \sum_{m > -r_1}\sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(m+r_1)z], \tag{12}$$

where the first summation is over all integers m with $m > -r_1$. He also introduced its associate as

$$H(z, s, r_1, r_2) := A(z, s, r_1, r_2) + e\left[\frac{s}{2}\right]A(z, s, -r_1, -r_2). \tag{13}$$

Let $s = r_1 = r_2 = 0$. Put $A(z, 0, 0, 0) = A(z)$, then $H(z, 0, 0, 0) = 2A(z)$. Using the product definition of Dedekind eta-function $\eta(z)$, it is easy to show that

$$\log(\eta(z)) = \frac{\pi i}{12} - A(z). \tag{14}$$

Let us see a couple of examples.

Example 1. For special choices of parameters r_1 and r_2, the A- and H-functions reduce to the cosecant and cotangent zeta functions:

$$\begin{aligned}\frac{1}{(1+e[s/2])}H\left(z, s, \left(\frac{1}{2}, 0\right)\right) &= A\left(z, s, \frac{1}{2}, 0\right) \\ &= \sum_{m > -(1/2)}\sum_{k=1}^{\infty} k^{s-1} e\left[k\left(m+\frac{1}{2}\right)z\right] \\ &= \sum_{k=1}^{\infty} k^{s-1}\frac{e[(1/2)kz]}{1-e[kz]} \\ &= \frac{i}{2}\sum_{k=1}^{\infty}\frac{\operatorname{cosec}(\pi kz)}{k^{1-s}}.\end{aligned} \tag{15}$$

Also,

$$\frac{1}{(1+e[s/2])}H(z,s,(1,0)) = A(z,s,1,0)$$
$$= \sum_{m>-1}\sum_{k=1}^{\infty} k^{s-1}e[k(m+1)z]$$
$$= \sum_{k=1}^{\infty} k^{s-1}\frac{e[kz]}{1-e[kz]}$$
$$= \frac{1}{2}\sum_{k=1}^{\infty} k^{s-1}\left\{\frac{1+e[kz]}{1-e[kz]}-1\right\}$$
$$= \frac{i}{2}\sum_{k=1}^{\infty}\frac{\cot(\pi kz)}{k^{1-s}} - \frac{1}{2}\zeta(1-s). \tag{16}$$

Some more definitions will be required.

Definition 1 (Hurwitz zeta function). For a positive number a, the Hurwitz zeta function

$$\zeta(s,a) := \sum_{n=0}^{\infty}(n+a)^{-s}, \quad \Re(s) > 1. \tag{17}$$

Definition 2. Let Ω denote the characteristic function of integers, i.e.,

$$\Omega(a) := \begin{cases} 1, & a\in\mathbb{Z}, \\ 0, & a\notin\mathbb{Z}. \end{cases} \tag{18}$$

For any positive number λ, let $I(\lambda,\infty)$ denote the integration path consisting of the oriented line segment $(+\infty,\lambda)$, the positively oriented circle of radius λ with center at the origin, and the oriented line segment $(\lambda,+\infty)$.

Let

$$G_2(z,(\omega_1,\omega_2);t) := \frac{\exp(-zt)}{(1-\exp(-\omega_1 t))(1-\exp(-\omega_2 t))}, \tag{19}$$

for any pair (ω_1,ω_2) of positive numbers and for $z,t\in\mathbb{C}$.

Berndt [6] proved the following transformation formula.

Theorem 4 (see [6], Theorem 2). *Let $V=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$ with $c>0$. For any pair (r_1,r_2) of real numbers, set $R_1=r_1a+r_2c$, $R_2=r_1b+r_2d$, $\rho=\{R_2\}c-\{R_1\}d$. For $z\in\mathbb{H}$ with $c\Re(z)+d>0$, let $\beta=cz+d$. Then, for arbitrary $s\in\mathbb{C}$, we have*

$$\begin{aligned}\beta^{-s}H(Vz,s,r_1,r_2) - H(z,s,R_1,R_2) \\ = -\Omega(r_1)(2\pi)^{-s}e\left[\frac{s}{4}\right]\beta^{-s}\Gamma(s)\left(\zeta(s,\langle r_2\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -r_2\rangle)\right) \\ +\Omega(R_1)(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta(s,\langle -R_2\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle R_2\rangle)\right) \\ +(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(z,s,R_1,R_2,c,d),\end{aligned} \tag{20}$$

where

$$L(z,s,R_1,R_2,c,d) = -\sum_{j=1}^{c}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1-\{(jd+\varrho)/c\}+((cz+d)(j-\{R_1\}))/c)t)}{(1-\exp(-t))(1-\exp(-(cz+d)t))}dt, \quad 0<\lambda<2\pi,\frac{2\pi}{|\beta|}. \tag{21}$$

Here, $\log t$ is understood to be real-valued on the upper segment $(+\infty,\lambda)$ of $I(\lambda,\infty)$.

4. Work of Arakawa

Arakawa studied certain Lambert series associated to a complex variable s and an irrational real algebraic number α. Those Lambert series are defined as limiting (boundary) values of the generalized Dedekind eta-functions studied by Berndt [6]. Arakawa obtained transformation formulae under the action of $SL(2,\mathbb{Z})$ on those α.

For an irrational real algebraic number α and a pair (p,q) of real numbers, Arakawa [7] introduced a generalized eta-function defined as

$$\eta(\alpha,s,p,q) := \sum_{n=1}^{\infty} n^{s-1}\frac{e[n(p\alpha+q)]}{1-e[n\alpha]}, \quad s\in\mathbb{C}, \tag{22}$$

and its associate by

$$H(\alpha,s,(p,q)) := \eta(\alpha,s,\langle p\rangle,q)+e\left[\frac{s}{2}\right]\eta(\alpha,s,\langle -p\rangle,-q). \tag{23}$$

Example 2. Again, if we consider $(p,q)=(1/2,0)$ and $(p,q)=(1,0)$, then also we will get the cosecant and cotangent zeta function:

$$\frac{1}{(1+e[s/2])}H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)=\eta\left(\alpha,s,\frac{1}{2},0\right)$$
$$=\sum_{k=1}^{\infty}k^{s-1}\frac{e[(1/2)k\alpha]}{1-e[k\alpha]} \quad (24)$$
$$=\frac{i}{2}\sum_{k=1}^{\infty}\frac{\operatorname{cosec}(\pi k\alpha)}{k^{1-s}},$$

$$\frac{1}{(1+e[s/2])}H(\alpha,s,(1,0))=\eta(\alpha,s,1,0)$$
$$=\sum_{k=1}^{\infty}k^{s-1}\frac{e[kz]}{1-e[kz]}$$
$$=\frac{i}{2}\sum_{k=1}^{\infty}\frac{\cot(\pi k\alpha)}{k^{1-s}}-\frac{1}{2}\zeta(1-s), \quad (25)$$

where $s\in\mathbb{C}$ with $\mathfrak{R}(s)<0$.

Theorem 5 (see [7], Lemma 1 and Theorem 2). *Suppose $\alpha\in\mathbb{R}\cap\overline{\mathbb{Q}}$ and $\alpha\notin\mathbb{Q}$. Then, the infinite series $\eta(\alpha,s,p,q)$ is absolutely convergent if $\mathfrak{R}(s)<0$. If, in addition, $[\mathbb{Q}(\alpha):\mathbb{Q}]=2$ and $(p,q)\in\mathbb{Q}^2$, then $H(\alpha,s,p,q)$ has analytic continuation to $\mathbb{C}-\{0\}$, and the singularity at $s=0$ is at worst a simple pole.*

Arakawa proved the absolute convergence of $\eta(\alpha,s,p,q)$ for $\mathfrak{R}(s)<0$, by using the Thue–Siegel–Roth theorem.

Consider the generalized eta-function

$$\eta(z,s,p,q)=\sum_{n=1}^{\infty}n^{s-1}\frac{e[n(pz+q)]}{1-e[nz]},\quad s\in\mathbb{C} \quad (26)$$

corresponding to (22), for $z\in\mathbb{H}$ and a pair $(p,q)\in\mathbb{R}^2$ with $p>0$. Then, one can see that this series is absolutely convergent for arbitrary $s\in\mathbb{C}$. It can be easily checked that there is a link between the infinite series $A(z,s,r_1,r_2)$ and $\eta(z,s,r_1,r_2)$.

Lemma 1. *For any pair $(r_1,r_2)\in\mathbb{R}^2$ and $z\in\mathbb{H}$, we have*

$$A(z,s,r_1,r_2)=\eta(z,s,\langle r_1\rangle,r_2),\quad s\in\mathbb{C}. \quad (27)$$

Now, from the definition of H-function (13), we have

$$H(z,s,r_1,r_2)=A(z,s,r_1,r_2)+e\left[\frac{s}{2}\right]A(z,s,-r_1,-r_2). \quad (28)$$

Hence, using Lemma 1, we get

$$H(z,s,r_1,r_2)=\eta(z,s,\langle r_1\rangle,r_2)+e\left[\frac{s}{2}\right]\eta(z,s,\langle -r_1\rangle,-r_2). \quad (29)$$

Similarly, we have

Lemma 2. *For any algebraic irrational number α and a pair $(p,q)\in\mathbb{R}^2$,*

$$A(\alpha,s,p,q)=\eta(\alpha,s,\langle p\rangle,q),\quad \mathfrak{R}(s)<0. \quad (30)$$

Again by the definition of H-function (23)(due to Arakawa), we have

$$H(\alpha,s,p,q)=\eta(\alpha,s,\langle p\rangle,q)+e\left[\frac{s}{2}\right]\eta(\alpha,s,\langle -p\rangle,q). \quad (31)$$

Therefore, by Lemma 2, we get

$$H(\alpha,s,p,q)=A(\alpha,s,p,q)+e\left[\frac{s}{2}\right]A(\alpha,s,-p,q). \quad (32)$$

Proposition 1 (see [7], Proposition 1). *Let $V=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})$, α be an irrational real algebraic number, and $(p,q)\in\mathbb{R}^2$ with $p>0$. Let $z=\alpha+iy$ with $y>0$. Set $z^*=Vz$ and $\beta=V\alpha=(a\alpha+b)(c\alpha+d)^{-1}$. If $\mathfrak{R}(s)<-3$, then*

$$\lim_{y\longrightarrow 0^+}\eta(z^*,s,p,q)=\eta(\beta,s,p,q). \quad (33)$$

Arakawa obtained the following transformation formulae for $H(\alpha,s,(p,q))$, by virtue of Theorem 4 of Berndt and Proposition 1.

Theorem 6 (see [7], Theorem 1). *Let α be any real algebraic irrational, and let $V=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})$ with $c>0$ such that $\beta:=c\alpha+d>0$. For any pair (p,q) of real numbers, set $p'=pa+qc$, $q'=pb+qd$, and $\rho=\{q'\}c-\{p'\}d$. Then, for $\mathfrak{R}(s)<0$,*

$$\begin{aligned}D_1(V,\alpha,s,(p,q)) &:= \beta^{-s}H(V\alpha,s,(p,q))-H(\alpha,s,(p,q)V)\\
&=\beta^{-s}H(V\alpha,s,(p,q))-H(\alpha,s,(p',q'))\\
&=-\Omega(p)(2\pi)^{-s}e\left[\frac{s}{4}\right]\beta^{-s}\Gamma(s)\\
&\quad\cdot\left(\zeta(s,\langle q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -q\rangle)\right)\\
&\quad+\Omega(p')(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\\
&\quad\cdot\left(\zeta(s,\langle -q'\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle q'\rangle)\right)\\
&\quad+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(p',q'),c,d),\end{aligned} \quad (34)$$

where

$$L(\alpha,s,p',q',c,d)=-\sum_{j=1}^{c}\int_{I(\lambda,\infty)}t^{s-1}G_2\left(1-\left\{\frac{(jd+\rho)}{c}\right\}+\frac{(j-\{p'\})\beta}{c},(1,\beta);t\right)dt,\quad 0<\lambda<2\pi,\frac{2\pi}{\beta}. \tag{35}$$

Berndt [6] (p. 499) found the special values of $L(\alpha,s,(p',q'),c,d)$ at nonnegative integral arguments $s=-m$:

$$\begin{aligned}L(\alpha,-m,(p',q'),c,d)=&\frac{2\pi i}{(m+2)!}\sum_{j=1}^{c}\sum_{k=0}^{m+2}\binom{m+2}{k}\\&\cdot B_k\left(\frac{j-\{p'\}}{c}\right)\overline{B}_{m+2-k}\left(\frac{jd+\rho}{c}\right)\\&\cdot(-\beta)^{k-1},\end{aligned} \tag{36}$$

where $B_n(x)$ denotes the nth Bernoulli polynomial and $\overline{B}_n(x)=B_n(\{x\})$.

Lemma 3 (see [7], Lemma 4). *Let α be an irrational number in a real quadratic field $\mathbb{Q}(\Delta)$ and let (p,q) be a pair of rational numbers. Then, there exist a totally positive unit β of $\mathbb{Q}(\Delta)$ and an element $V=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ of $SL(2,\mathbb{Z})$which satisfy the conditions:*

(i) $c>0$

(ii) $(p,q)V\equiv(p,q)\bmod 1$

(iii) $\beta\begin{pmatrix}\alpha\\1\end{pmatrix}=V\begin{pmatrix}\alpha\\1\end{pmatrix}$

We choose such $\beta\in\mathbb{Q}(\Delta)$ and $V\in SL(2,\mathbb{Z})$, i.e., which satisfy the conditions of Lemma 3. Then, using condition (ii), we have

$$H(\alpha,s,(p,q))=H(\alpha,s,(p,q)V). \tag{37}$$

Since $V\alpha=\alpha$ and $c>0$, we can see easily from Theorem 6 that

$$\begin{aligned}H(\alpha,s,(p,q))=&-\Omega(p)(2\pi)^{-s}e\left[\frac{s}{4}\right]\Gamma(s)\zeta(s,\langle q\rangle)\\&+\Omega(p)(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\zeta(s,\langle q\rangle)\\&\cdot\frac{1-e[s]\beta^{-s}}{\beta^{-s}-1}+\frac{(2\pi)^{-s}e[-(s/4)]}{\beta^{-s}-1}\\&\cdot L(\alpha,s,(p,q),c,d).\end{aligned} \tag{38}$$

Example 3. Let α,β, and V as in Lemma 3 and with $(p,q)=(1,0)$ and $(p,q)=(1/2,0)$. Then,

$$\begin{aligned}H(\alpha,s,(1,0))=&(2\pi)^{-s}\left(-e\left[\frac{s}{4}\right]+e\left[-\frac{s}{4}\right]\frac{1-e[s]\beta^{-s}}{\beta^{-s}-1}\right)\Gamma(s)\zeta(s)\\&+\frac{(2\pi)^{-s}e[-(s/4)]}{\beta^{-s}-1}L(\alpha,s,(1,0),c,d),\\H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)=&\frac{(2\pi)^{-s}e[-(s/4)]}{\beta^{-s}-1}L\left(\alpha,s,\left(\frac{1}{2},0\right),c,d\right).\end{aligned} \tag{39}$$

Values at some particular matrices. Let

$$\begin{aligned}V_0&=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\\V_1&=\begin{pmatrix}1&0\\1&1\end{pmatrix},\\V_2&=V_0^2V_1^{-1}=\begin{pmatrix}-1&0\\1&-1\end{pmatrix}.\end{aligned} \tag{40}$$

Example 4. Theorem 6 gives the following:

$$\begin{aligned}D_1(V_0,\alpha,s,(p,q))=&\alpha^{-s}H\left(\frac{-1}{\alpha},s,(p,q)\right)-H(\alpha,s,(q,-p))\\=&-\Omega(p)(2\pi)^{-s}e\left[\frac{s}{4}\right]\alpha^{-s}\Gamma(s)\\&\cdot\left(\zeta(s,\langle q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -q\rangle)\right)\\&+\Omega(q)(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\\&\cdot\left(\zeta(s,\langle p\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -p\rangle)\right)\\&+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(q,-p),1,0),\end{aligned} \tag{41}$$

$$\begin{aligned}D_1(V_1,\alpha,s,(p,q))=&(\alpha+1)^{-s}H\left(\frac{\alpha}{\alpha+1},s,(p,q)\right)\\&-H(\alpha,s,(p+q,q))\\=&-\Omega(p)(2\pi)^{-s}e\left[\frac{s}{4}\right](\alpha+1)^{-s}\Gamma(s)\\&\cdot\left(\zeta(s,\langle q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -q\rangle)\right)\\&+\Omega(p+q)(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\\&\cdot\left(\zeta(s,\langle -q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle q\rangle)\right)\\&+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(p+q,q),1,1),\end{aligned} \tag{42}$$

$$\begin{aligned}D_1(V_2,\alpha,s,(p,q))=&(\alpha-1)^{-s}H\left(\frac{-\alpha}{\alpha-1},s,(p,q)\right)\\&-H(\alpha,s,(-p+q,-q))\\=&-\Omega(p)(2\pi)^{-s}e\left[\frac{s}{4}\right](\alpha-1)^{-s}\Gamma(s)\\&\cdot\left(\zeta(s,\langle q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -q\rangle)\right)\\&+\Omega(-p+q)(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\\&\cdot\left(\zeta(s,\langle q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -q\rangle)\right)\\&+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(-p+q,-q),1,-1).\end{aligned} \tag{43}$$

In particular, when $(p,q) = (1,0)$, we have

$$\begin{aligned}
D_1(V_0,\alpha,s,(1,0)) &= (2\pi)^{-s}e\left[\frac{s}{4}\right]\left\{e\left[-\frac{s}{2}\right]-\alpha^{-s}\right\}\Gamma(s)\left(1+e\left[\frac{s}{2}\right]\right)\zeta(s)+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(0,-1),1,0),\\
D_1(V_1,\alpha,s,(1,0)) &= (2\pi)^{-s}e\left[\frac{s}{4}\right]\left\{e\left[-\frac{s}{2}\right]-(\alpha+1)^{-s}\right\}\Gamma(s)\left(1+e\left[\frac{s}{2}\right]\right)\zeta(s)+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(1,0),1,1),\\
D_1(V_2,\alpha,s,(1,0)) &= (2\pi)^{-s}e\left[\frac{s}{4}\right]\left\{e\left[-\frac{s}{2}\right]-(\alpha-1)^{-s}\right\}\Gamma(s)\left(1+e\left[\frac{s}{2}\right]\right)\zeta(s)+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(-1,0),1,-1).
\end{aligned}\tag{44}$$

If we choose $(p,q) = (1/2,0)$, we get

$$\begin{aligned}
D_1\left(V_0,\alpha,s,\left(\frac{1}{2},0\right)\right) &= \alpha^{-s}H\left(\frac{-1}{\alpha},s,\left(\frac{1}{2},0\right)\right)-H\left(\alpha,s,\left(0.-\frac{1}{2}\right)\right)\\
&= (2\pi)^{-s}\left(e\left[\frac{s}{4}\right]+e\left[-\frac{s}{4}\right]\right)\Gamma(s)\zeta\left(s,\frac{1}{2}\right)+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(0,-\frac{1}{2}\right),1,0\right),\\
D_1\left(V_1,\alpha,s,\left(\frac{1}{2},0\right)\right) &= (\alpha+1)^{-s}H\left(\frac{\alpha}{\alpha+1},s,\left(\frac{1}{2},0\right)\right)-H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\\
&= (2\pi)^{-s}e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(\frac{1}{2},0\right),1,1\right),\\
D_1\left(V_2,\alpha,s,\left(\frac{1}{2},0\right)\right) &= (\alpha-1)^{-s}H\left(\frac{-\alpha}{\alpha-1},s,\left(\frac{1}{2},0\right)\right)-H\left(\alpha,s,\left(-\frac{1}{2},0\right)\right)\\
&= (2\pi)^{-s}e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(-\frac{1}{2},0\right),1,-1\right).
\end{aligned}\tag{45}$$

Note that for nonnegative integers m, we have the following explicit formulae for V_j, where $j = 0,1,2$:

$$\begin{aligned}
L\left(\alpha,-m,(1,0)V_j,c,d\right) &= \frac{2\pi i}{(m+2)!}\sum_{k=0}^{m+2}\binom{m+2}{k}\\
&\quad\cdot B_k(1)\overline{B}_{m+2-k}(1)(-\beta)^{k-1},\\
L\left(\alpha,-m,\left(\frac{1}{2},0\right)V_j,c,d\right) &= \frac{2\pi i}{(m+2)!}\sum_{k=0}^{m+2}\binom{m+2}{k}\\
&\quad\cdot B_k\left(\frac{1}{2}\right)\overline{B}_{m+2-k}\left(\frac{1}{2}\right)(-\beta)^{k-1}.
\end{aligned}\tag{46}$$

5. Generalization of the Secant Zeta Function

We introduce two Lambert series corresponding to (22) and (12). These include the generalizations of secant and tangent zeta functions as shown in Example 5. Let α be any algebraic irrational number and (p,q) a pair of real numbers. Then, we define the series η^* by

$$\eta^*(\alpha,s,p,q) := \sum_{n=1}^{\infty} n^{s-1}\frac{e[n(p\alpha+q)]}{1+e[n\alpha]}, \quad \Re(s)<0 \tag{47}$$

and another infinite series A^* by

$$A^*(z,s,r_1,r_2) := \sum_{m>-r_1}(-1)^m\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(m+r_1)z], \tag{48}$$

for a pair $(r_1,r_2)\in\mathbb{R}^2$, $z\in\mathbb{H}$, and $s\in\mathbb{C}$.

Example 5. If we take $(r_1,r_2) = (1,0)$, and $(1/2,0)$, then (48) becomes

$$\begin{aligned}
A^*(\alpha,s,1,0) &= \eta^*(\alpha,s,1,0)\\
&= \sum_{k=1}^{\infty}k^{s-1}\frac{e[k\alpha]}{1+e[k\alpha]}\\
&= \frac{1}{2}\sum_{k=1}^{\infty}k^{s-1}\left(\frac{e[k\alpha]-1}{1+e[k\alpha]}+1\right)\\
&= \frac{i}{2}\sum_{k=1}^{\infty}k^{s-1}\tan(\pi k\alpha)+\frac{1}{2}\zeta(1-s),
\end{aligned}\tag{49}$$

$$A^*\left(\alpha, s, \frac{1}{2}, 0\right) = \eta\left(\alpha, s, \frac{1}{2}, 0\right) = \frac{1}{2}\sum_{k=1}^{\infty} k^{s-1}\frac{1}{\cos(\pi k\alpha)} = \frac{1}{2}\psi(\alpha, 1-s), \tag{50}$$

respectively.

By virtue of the results of Lewittes, Berndt, and Arakawa, we have the following results.

Lemma 4. *Let α be an algebraic irrational number and (p, q) be a pair of real numbers. The series $\eta^*(\alpha, s, p, q)$ is absolutely convergent, if $s \in \mathbb{C}$ with $\Re(s) < 0$.*

Proof. One can prove this result applying the Thue–Siegel–Roth theorem, in a similar manner to Arakawa's procedure for proving the absolute convergence of the series $\eta(\alpha, s, p, q)$. □

Lemma 5. *If $z \in \mathbb{H}$ and a pair $(p, q) \in \mathbb{R}^2$ with $p > 0$, then the series $\eta^*(z, s, p, q)$ is absolutely convergent for any $s \in \mathbb{C}$.*

Proof. Since $z \in \mathbb{H}$, assume $z = x + iy$ with $y > 0$. We have

$$\left|\eta^*(z, s, p, q)\right| = \left|\sum_{n=1}^{\infty} n^{s-1}\frac{e[n(pz+q)]}{1+e[nz]}\right| \leqq \sum_{n=1}^{\infty} n^{\sigma-1}\frac{\exp(-2\pi npy)}{1-\exp(-2\pi ny)}, \tag{51}$$

for $\Re(s) = \sigma$. $1 - \exp(-2\pi ny) \geqq 1 - \exp(-2\pi y)$. And we can choose a large enough positive integer K such that for $n > K$

$$n^{\sigma-1}\exp(-2\pi npy) = \exp((\sigma-1)\log n - 2\pi npy) \leqq \exp(-\pi npy). \tag{52}$$

Thus,

$$\begin{aligned}
\left|\eta^*(z,s,p,q)\right| &\leq \sum_{n=1}^{\infty} n^{\sigma-1}\frac{\exp(-2\pi npy)}{1-\exp(-2\pi ny)} \\
&\leq \sum_{n=1}^{K} n^{\sigma-1}\frac{\exp(-2\pi npy)}{1-\exp(-2\pi ny)} + \sum_{n=K+1}^{\infty} n^{\sigma-1}\frac{\exp(-2\pi npy)}{1-\exp(-2\pi y)} \\
&\leq \sum_{n=1}^{K} n^{\sigma-1}\frac{\exp(-2\pi npy)}{1-\exp(-2\pi ny)} + \sum_{n=K+1}^{\infty}\frac{\exp(-\pi npy)}{1-\exp(-2\pi y)} \\
&\leq \sum_{n=1}^{K} n^{\sigma-1}\frac{\exp(-2\pi npy)}{1-\exp(-2\pi ny)} \\
&\quad + \frac{\exp(-\pi(K+1)py)}{1-\exp(-2\pi py)}\frac{1}{1-\exp(-2\pi y)} < \infty.
\end{aligned} \tag{53}$$

□

Lemma 6. *Let $z \in \mathbb{H}$ and α be an irrational algebraic number. Then, for any pair of real numbers (r_1, r_2), we have*

$$\begin{aligned}
A^*(z, s, r_1, r_2) &= (-1)^{-r_1+\langle r_1\rangle}\eta^*(z, s, \langle r_1\rangle, r_2), \quad s \in \mathbb{C}, \\
A^*(\alpha, s, r_1, r_2) &= (-1)^{-r_1+\langle r_1\rangle}\eta^*(\alpha, s, \langle r_1\rangle, r_2), \quad \Re(s) < 0.
\end{aligned} \tag{54}$$

Proof. If $r_1 \in \mathbb{Z}$, then $m > -r_1$ implies $m = -r_1 + r$ for $r = 1, 2, 3, \ldots,$. By the definition of $A^*(z, s, r_1, r_2)$, we know

$$\begin{aligned}
A^*(z,s,r_1,r_2) &= \sum_{m>-r_1}(-1)^m\sum_{k=1}^{\infty} k^{s-1}e[kr_2 + k(m+r_1)z] \\
&= \sum_{k=1}^{\infty} k^{s-1}\sum_{r=1}^{\infty}(-1)^{-r_1+r}e[kr_2 + krz] \\
&= (-1)^{-r_1+1}\sum_{k=1}^{\infty} k^{s-1}e[kr_2 + kz]\sum_{r=0}^{\infty}(-1)^r e[krz] \\
&= (-1)^{-r_1+1}\sum_{k=1}^{\infty} k^{s-1}\frac{e[kr_2 + kz]}{1+e[kz]} \\
&= (-1)^{-r_1+\langle r_1\rangle}\eta^*(z,s,\langle r_1\rangle,r_2),
\end{aligned}$$

since $\langle r_1\rangle = 1$.

(55)

Again, if $r_1 \notin \mathbb{Z}$, $m > -r_1$ implies $m = -\lfloor r_1\rfloor + r$ for $r = 0, 1, 2, \ldots,$. So, we will have

$$\begin{aligned}
A^*(z,s,r_1,r_2) &= \sum_{m>-r_1}(-1)^m\sum_{k=1}^{\infty} k^{s-1}e[kr_2 + k(m+r_1)z] \\
&= \sum_{k=1}^{\infty} k^{s-1}\sum_{r=0}^{\infty}(-1)^{-\lfloor r_1\rfloor+r}e[kr_2 + k(\langle r_1\rangle + r)z] \\
&= (-1)^{-\lfloor r_1\rfloor}\sum_{k=1}^{\infty} k^{s-1}e[kr_2 + k\langle r_1\rangle z]\sum_{r=0}^{\infty}(-1)^r e[krz] \\
&= (-1)^{-\lfloor r_1\rfloor}\sum_{k=1}^{\infty} k^{s-1}\frac{e[kr_2 + k\langle r_1\rangle z]}{1+e[kz]} \\
&= (-1)^{-r_1+\langle r_1\rangle}\eta^*(z,s,\langle r_1\rangle,r_2).
\end{aligned} \tag{56}$$

Similarly, we can see that

$$A^*(\alpha,s,r_1,r_2) = (-1)^{-r_1+\langle r_1\rangle}\eta^*(\alpha,s,\langle r_1\rangle,r_2), \quad \text{for } \Re(s) < 0. \tag{57}$$

□

Lemma 7. *If $z \in \mathbb{H}$, $A^*(z, s, r_1, r_2)$ is absolutely convergent for any $s \in \mathbb{C}$.*

Proof. Using Lemmas 5 and 6, we can show that $A^*(z, s, r_1, r_2)$ is absolutely convergent for $s \in \mathbb{C}$. □

6. Main Results

Consider the difference

$$D^*(V) := D^*\left(V, \alpha, s, \frac{1}{2}, 0\right) := \beta^{-s} A^*\left(V\alpha, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right), \quad (58)$$

for each V from (40). Now, the second term in the above expression is the secant zeta function in view of (50). This difference is quite natural in the sense that it expresses the surplus after the modular transformation is applied.

We interpret the main result of Lalín et al. Theorem 3 in this setting as a special case of

$$(\alpha+1)^{-s} A^*\left(V_1\alpha, s, \frac{1}{2}, 0\right) + (\alpha-1)^{-s} A^*\left(V_2\alpha, s, \frac{1}{2}, 0\right), \quad (59)$$

for $\mathfrak{R}(s) < 0$, and locate it in a natural way as we will see in Corollary 1. Our main theorem is the following.

Theorem 7. *For a real algebraic irrational α and a complex variable s with $\mathfrak{R}(s) < 0$, we have*

$$\begin{aligned} D^*(V_0) &= \alpha^{-s} A^*\left(\frac{-1}{\alpha}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \\ &= 2^{1-2s}\pi^{-s} e\left[-\frac{s}{4}\right]\left(\Phi_0 + \Gamma(s)\Omega_0\right) + 2^{1-s}\Psi_0 \\ &= -\frac{(2\pi)^{-s} e[-s/4]}{1-e[s/2]} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-(1/2+\alpha)t)}{(1+\exp(-t))(1-\exp(-\alpha t))} dt + 2^{-2s}\pi^{-s} e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right)\right) \\ &\quad - 2^{1-s}\sum_{n=1}^{\infty} n^{s-1} \frac{e[n(\alpha/2+1/4)](e[n\alpha/2]+1)}{1-e[n\alpha]} + 2^{2-s}\sum_{n=1}^{\infty} (2n)^{s-1} \frac{e[3\pi\alpha/2]}{1-e[2n\alpha]}, \end{aligned} \quad (60)$$

$$\begin{aligned} D^*(V_1) &= (\alpha+1)^{-s} A^*\left(\frac{\alpha}{\alpha+1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \\ &= 2^{1-2s}\pi^{-s} e\left[-\frac{s}{4}\right]\Phi_1 + 2^{1-s}\Psi_1 \\ &= -\frac{(2\pi)^{-s} e[-s/4]}{1-e[s/2]} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-t/2)}{(1+\exp(-t))} \frac{\exp(-(\alpha+1)t/2)}{(1-\exp(-(\alpha+1)t))} dt + 2^{-s}\sum_{n=1}^{\infty} n^{s-1} \frac{(-1)^{n-1}}{\cos(\pi n\alpha/2)}. \end{aligned} \quad (61)$$

Also,

$$\begin{aligned} D^*(V_2) &= (\alpha-1)^{-s} A^*\left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \\ &= 2^{1-2s}\pi^{-s} e\left[-\frac{s}{4}\right]\Phi_2 + 2^{1-s}\Psi_2 \\ &= -\frac{(2\pi)^{-s} e[-s/4]}{1-e[s/2]} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-t/2)}{(1+\exp(-t))} \frac{\exp(-(\alpha-1)t/2)}{(1-\exp(-(\alpha-1)t))} dt - 2^{-s}\sum_{n=1}^{\infty} n^{s-1} \frac{1}{\cos(\pi n\alpha/2)}, \end{aligned} \quad (62)$$

where Φ_k and Ψ_k and $(k = 0, 1, 2)$ are defined later. They indicate the block of L-integrals and the block of H-functions, corresponding to the matrix V_k, respectively. Also, Ω_0 is defined in (90).

We recover the main result of Lalín et al. ([11], Theorem 3), i.e., Theorem 3 by adding the equations (61) and (62). We note it as a corollary.

Corollary 1.

$$(\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1},s,\frac{1}{2},0\right)+(\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1},s,\frac{1}{2},0\right) = -\frac{(2\pi)^{-s}e[-s/4]}{1-e[s/2]}\int_{I(\lambda,\infty)}t^{s-1}\sum_{m=0}^{\infty}2^{-m-1}E_m\frac{t^m}{m!}\sum_{n=0}^{\infty}\left(2^{1-n}-1\right)B_n \times\frac{\left\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\right\}t^{n-1}}{n!}dt. \tag{63}$$

The genesis of the transformation formula of Lalín et al. ([11], Theorem 3) for the secant zeta function is given by the sum of $D^*(V_1)$ and $D^*(V_2)$, which we have seen in Corollary 1. We will see in the proof of Corollary 1 that the term $2A^*(\alpha,s,1/2,0)$ on the left side and the secant zeta function on the right hand side naturally cancel each other. As this occurs only in such a pairing, this elucidates the hidden structure of the paired transformation formula from a more general standpoint.

Deduction of the Main Theorem of Lalín et al. Firstly, we deduce Theorem 3 from Corollary 1. To do that, let $l=2k$ be an even positive integer and $s=1-l$. Then, (63) amounts to

$$\begin{aligned}
&(\alpha+1)^{2k-1}A^*\left(\frac{\alpha}{\alpha+1},-2k+1,\frac{1}{2},0\right)+(\alpha-1)^{2k-1}A^*\left(\frac{-\alpha}{\alpha-1},-2k+1,\frac{1}{2},0\right)\\
&=-\frac{(2\pi)^{2k-1}e[-((-2k+1)/4)]}{1-e[(-2k+1)/2]}\int_{I(\lambda,\infty)}t^{-2k}\sum_{m=0}^{\infty}2^{-m-1}E_m\frac{t^m}{m!}\sum_{n=0}^{\infty}\left(2^{1-n}-1\right)B_n\frac{\left\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\right\}t^{n-1}}{n!}dt\\
&=-\frac{2^{2k-1}\pi^{2k}(-1)^k}{2\pi i}\int_{I(\lambda,\infty)}t^{-2k}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{-m-1}\left(2^{1-n}-1\right)E_mB_n\left\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\right\}\frac{t^{m+n-1}}{m!n!}dt\\
&=-2^{2k-1}\pi^{2k}(-1)^k\sum_{n=0}^{2k}\frac{1}{(2k-n)!n!}2^{-2k+n-1}\left(2^{1-n}-1\right)E_{2k-n}B_n\left\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\right\}\\
&=\frac{1}{2}\pi^{2k}(-1)^k\sum_{n=0}^{2k}\frac{1}{(2k-n)!n!}\left(2^{n-1}-1\right)E_{2k-n}B_n\left\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\right\}.
\end{aligned}\tag{64}$$

This proves Theorem 3.

The following conjecture seems to be plausible.

Conjecture 2. *Let* $W_1=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}$ *and* $W_2=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}$ *be two matrices in* $PSL_2(\mathbb{Z})$ *which are inverses to each other. Then, for a pair* $(p,q)\in\mathbb{R}^2$,

$$(c_1\alpha+d_1)^{-s}A^*(W_1\alpha,s,p,q)+(c_2\alpha+d_2)^{-s}A^*(W_2\alpha,s,p,q) \tag{65}$$

can be expressible in terms of special values of the zeta and L-functions as we have seen for the sum of two explicit expressions for

$$(c_1\alpha+d_1)^{-s}A^*\left(V_j\alpha,s,\left(\frac{1}{2},0\right)\right)-A^*\left(\alpha,s,\left(\frac{1}{2},0\right)\right),\quad j=1,2. \tag{66}$$

7. A^* in Terms of A- and H-Functions

Before proving our main theorem we need to express A^* in terms of A and H. We know that given a sum $S=\sum_n a_n$ with its even and odd parts S_e and S_o, where the even part is over all even integer values and odd part over odd integer values, the sum $2S_e-S$ is the alternating sum $\sum_n(-1)^n a_n$. Using this observation, we have the following result.

Lemma 8. $A^*(z,s,r_1,r_2)=2A(2z,s,r_1/2,r_2)-A(z,s,r_1,r_2)$.

Proof. By the definition of $A^*(z,s,r_1,r_2)$, we have

$$\begin{aligned}
A^*(z,s,r_1,r_2) &= \sum_{m>-r_1}(-1)^m\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(m+r_1)z]\\
&=2\sum_{\substack{m>-r_1\\ m:even}}\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(m+r_1)z]\\
&\quad-\sum_{m>-r_1}\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(m+r_1)z]\\
&=2\sum_{2m>-r_1}\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(2m+r_1)z]\\
&\quad-\sum_{m>-r_1}\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(m+r_1)z]\\
&=2A\left(2z,s,\frac{r_1}{2},r_2\right)-A(z,s,r_1,r_2).
\end{aligned}\tag{67}$$

□

There is a duplication formula for $A(z,s,r_1,r_2)$ which is as follows:

Lemma 9. $A(z,s,r_1,r_2) + A(z,s,r_1,r_2+1/2) = 2^s A(2z,s,r_1,2r_2)$.

Proof. From Definition 1 of $A(z,s,r_1,r_2)$, we have

$$\begin{aligned}
&A(z,s,r_1,r_2) + A\left(z,s,r_1,r_2+\frac{1}{2}\right)\\
&= \sum_{m>-r_1}\sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(m+r_1)z]\\
&\quad + \sum_{m>-r_1}\sum_{k=1}^{\infty} k^{s-1} e\left[k\left(r_2+\frac{1}{2}\right) + k(m+r_1)z\right]\\
&= \sum_{m>-r_1}\sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(m+r_1)z]\left(1+e\left[\frac{1}{2}k\right]\right)\\
&= 2\sum_{m>-r_1}\sum_{k=1}^{\infty} (2k)^{s-1} e[2kr_2 + 2k(m+r_1)z]\\
&= 2^s\sum_{m>-r_1}\sum_{k=1}^{\infty} k^{s-1} e[k(2r_2) + k(m+r_1)(2z)]\\
&= 2^s A(2z,s,r_1,2r_2).
\end{aligned} \tag{68}$$

□

Using the duplication formula, i.e., Lemma 9 in Lemma 8, we get

Lemma 10.

$$\begin{aligned}
A^*(z,s,r_1,r_2) &= 2^{1-s}A\left(z,s,\frac{r_1}{2},\frac{r_2}{2}\right) + 2^{1-s}A\left(z,s,\frac{r_1}{2},\frac{r_2}{2}+\frac{1}{2}\right)\\
&\quad - A(z,s,r_1,r_2).
\end{aligned} \tag{69}$$

On the other hand,

$$\begin{aligned}
H(z,s,r_1,r_2) &= A(z,s,r_1,r_2) + e\left[\frac{s}{2}\right]A(z,s,-r_1,-r_2),\\
H(z,s,-r_1,-r_2) &= A(z,s,-r_1,-r_2) + e\left[\frac{s}{2}\right]A(z,s,r_1,r_2),\\
H(z,s,-r_1,-r_2) &= A(z,s,-r_1,-r_2) + e\left[\frac{s}{2}\right]A(z,s,r_1,r_2).
\end{aligned} \tag{70}$$

Therefore,

$$\begin{aligned}
A(z,s,r_1,r_2) &= \frac{1}{1-e[s]}\{H(z,s,r_1,r_2)\\
&\quad - e\left[\frac{s}{2}\right]H(z,s,-r_1,-r_2)\Big\}.
\end{aligned} \tag{71}$$

Substituting (71) in Lemma 10, we deduce the following proposition.

Proposition 2. *For a real algebraic irrational α, a pair (p,q) of real numbers with $p>0$, and a complex variable s with $\Re(s)<0$, we have*

$$\begin{aligned}
(1-e[s])A^*(\alpha,s,p,q) &= 2^{1-s}\Big\{H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right)\\
&\quad - e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right)\Big\}\\
&\quad + 2^{1-s}\Big\{H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right)\\
&\quad - e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right)\Big\}\\
&\quad - (1-e[s])A(\alpha,s,p,q),
\end{aligned} \tag{72}$$

where

$$\begin{aligned}
(1-e[s])A(\alpha,s,p,q) &= \Big\{H(\alpha,s,(p,q))\\
&\quad - e\left[\frac{s}{2}\right]H(\alpha,s,(-p,-q))\Big\},
\end{aligned} \tag{73}$$

as in equation (71).

Example 6. If we consider $(p,q)=(1,0)$ and $(1/2,0)$, then we get

$$\begin{aligned}
A^*(\alpha,s,1,0) &= \frac{1}{1+e[s/2]}\Big\{2^{1-s}H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\\
&\quad + 2^{1-s}H\left(\alpha,s,\left(\frac{1}{2},\frac{1}{2}\right)\right) - H(\alpha,s,(1,0))\Big\},\\
A^*\left(\alpha,s,\frac{1}{2},0\right) &= \frac{2^{1-s}}{1-e[s]}H\left(\alpha,s,\left(\frac{1}{4},0\right)\right)\\
&\quad - \frac{2^{1-s}e[s/2]}{1-e[s]}H\left(\alpha,s,\left(-\frac{1}{4},0\right)\right)\\
&\quad + \frac{2^{1-s}}{1-e[s]}H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right)\\
&\quad - \frac{2^{1-s}e[s/2]}{1-e[s]}H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right)\\
&\quad - \frac{1-e[s/2]}{1-e[s]}H\left(\alpha,s,\left(\frac{1}{2},0\right)\right).
\end{aligned} \tag{74}$$

For the last term, with s an even integer, we use either

$$H\left(\alpha,s,\left(\frac{1}{2},0\right)\right) = \frac{(2\pi)^{-s}e[-s/4]}{\beta^{-s}-1}L\left(\alpha,s,\left(\frac{1}{2},0\right),c,d\right) \tag{75}$$

or

$$\frac{1}{1+e[s/2]}H\left(\alpha,s,\left(\frac{1}{2},0\right)\right) = \frac{i}{2}\sum_{k=1}^{\infty}\frac{1}{k^{1-s}}\frac{1}{\sin(\pi k\alpha)}, \tag{76}$$

which follows from Examples 1 and 3, respectively.

8. General Procedure

The general procedure is to transform

$$D^*(V) := D^*(V,\alpha,s,p,q) := \beta^{-s}A^*(V\alpha,s,p,q) - A^*(\alpha,s,p,q). \tag{77}$$

We recall the following notations

$$D_1(V,\alpha,s,(p,q)) = \beta^{-s}H(V\alpha,s,(p,q)) - H(\alpha,s,(p,q)V), \tag{78}$$

$$D_0^*(V,\alpha,s,p,q) = \beta^{-s}A(V\alpha,s,p,q) - A(\alpha,s,p,q), \tag{79}$$

$$A(z,s,r_1,r_2) = \frac{1}{1-e[s]}\left\{H(z,s,r_1,r_2) - e\left[\frac{s}{2}\right]H(z,s,-r_1,-r_2)\right\}. \tag{80}$$

Now using Proposition 2, we can write

$$\begin{aligned} D^*(V,\alpha,s,p,q) + D_0^*(V,\alpha,s,p,q) = {} & \frac{2^{1-s}}{1-e[s]}\left(\beta^{-s}H\left(V\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right) - H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right)\right) \\ & - \frac{2^{1-s}e[s/2]}{1-e[s]}\left(\beta^{-s}H\left(V\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right) - H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right)\right) \\ & + \frac{2^{1-s}}{1-e[s]}\left(\beta^{-s}H\left(V\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right) - H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right)\right) \\ & - \frac{2^{1-s}e[s/2]}{1-e[s]}\left(\beta^{-s}H\left(V\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right) - H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right)\right). \end{aligned} \tag{81}$$

For $(p,q) = (1/2,0)$, we have

$$\begin{aligned} D_0^*\left(V,\alpha,s,\frac{1}{2},0\right) = {} & \frac{1}{1+e[s/2]}\left(\beta^{-s}H\left(V\alpha,s,\left(\frac{1}{2},0\right)\right)\right. \\ & \left. - H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\right). \end{aligned} \tag{82}$$

We now transform (81) by using (78):

$$\begin{aligned} D^*(V,\alpha,s,p,q) + D_0^*(V,\alpha,s,p,q) = {} & \frac{2^{1-s}}{1-e[s]}\left(D_1\left(V,\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right) + H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)V\right) - H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right)\right) \\ & - \frac{2^{1-s}e[s/2]}{1-e[s]}\left(D_1\left(V,\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right) + H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)V\right) - H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right)\right) \\ & + \frac{2^{1-s}}{1-e[s]}\left(D_1\left(V,\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right) + H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)V\right) - H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right)\right) \\ & - \frac{2^{1-s}e[s/2]}{1-e[s]}\left(D_1\left(V,\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right) + H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)V\right) - H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right)\right), \end{aligned} \tag{83}$$

where

$$\begin{aligned} D_0^*(V,\alpha,s,p,q) = {} & \frac{1}{1-e[s]}\left(D_1(V,\alpha,s,(p,q))\right. \\ & \left. + H(\alpha,s,(p,q)V) - H(\alpha,s,(p,q))\right) \\ & - \frac{e[s/2]}{1-e[s]}\left(D_1(V,\alpha,s,(-p,-q))\right. \\ & \left. + H(\alpha,s,(-p,-q)V) - H(\alpha,s,(-p,-q))\right), \end{aligned} \tag{84}$$

in the case of (79), while

$$\begin{aligned} D_0^*\left(V,\alpha,s,\frac{1}{2},0\right) = {} & \frac{1}{1+e[s/2]}\left(D_1\left(V,\alpha,s,\left(\frac{1}{2},0\right)\right)\right. \\ & \left. + H\left(\alpha,s,\left(\frac{1}{2},0\right)V\right) - H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\right), \end{aligned} \tag{85}$$

in the case of (82). Hence,

$$
\begin{aligned}
D^{*}(V,\alpha,s,p,q)+D_{0}^{*}(V,\alpha,s,p,q)={}&\frac{2^{1-s}}{1-e[s]}D_{1}\left(V,\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right)-\frac{2^{1-s}e[s/2]}{1-e[s]}D_{1}\left(V,\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right)\\
&+\frac{2^{1-s}}{1-e[s]}D_{1}\left(V,\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right)-\frac{2^{1-s}e[s/2]}{1-e[s]}D_{1}\left(V,\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right)\\
&+\frac{2^{1-s}}{1-e[s]}\left(H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)V\right)-H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right)\right)\\
&-\frac{2^{1-s}e[s/2]}{1-e[s]}\left(H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)V\right)-H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right)\right)+\frac{2^{1-s}}{1-e[s]}\left(H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)V\right)\right.\\
&\left.-H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right)\right)\\
&-\frac{2^{1-s}e[s/2]}{1-e[s]}\left(H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)V\right)-H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right)\right),
\end{aligned}
\tag{86}
$$

where the last term is either (79) or (85).

9. Proof of Theorem 7 (60)

The three identities in Theorem 7 are proved on similar lines. We begin by using (83) and (85).

$$
\begin{aligned}
D^{*}(V_0)={}&D^{*}\left(V_0,\alpha,s,\frac{1}{2},0\right)=\alpha^{-s}A^{*}\left(\frac{-1}{\alpha},s,\frac{1}{2},0\right)-A^{*}\left(\alpha,s,\frac{1}{2},0\right)\\
={}&\frac{2^{1-s}}{1-e[s]}D_{1}\left(V_0,\alpha,s,\left(\frac{1}{4},0\right)\right)+\frac{2^{1-s}e[s/2]}{1-e[s]}D_{1}\left(V_0,\alpha,s,\left(-\frac{1}{4},0\right)\right)\\
&+\frac{2^{1-s}}{1-e[s]}D_{1}\left(V_0,\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right)-\frac{2^{1-s}e[s/2]}{1-e[s]}D_{1}\left(V_0,\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right)\\
&-\frac{1-e[s/2]}{1-e[s]}D_{1}\left(V_0,\alpha,s,\left(\frac{1}{2},0\right)\right)+\frac{2^{1-s}}{1-e[s]}\left(H\left(\alpha,s,\left(\frac{1}{4},0\right)V_0\right)-H\left(\alpha,s,\left(\frac{1}{4},0\right)\right)\right)\\
&-\frac{2^{1-s}}{1-e[s]}\left\{H\left(\alpha,s,\left(\frac{1}{4},0\right)V_0\right)-H\left(\alpha,s,\left(-\frac{1}{4},0\right)\right)\right\}\\
&+\frac{2^{1-s}}{1-e[s]}\left\{H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)V_0\right)-H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right)\right\}\\
&-\frac{2^{1-s}e[s/2]}{1-e[s]}\left\{H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)V_0\right)-H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right)\right\}\\
&-\frac{1-e[s/2]}{1-e[s]}\left\{H\left(\alpha,s,\left(\frac{1}{2},0\right)V_0\right)-H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\right\}.
\end{aligned}
\tag{87}
$$

Then, applying (41), we deduce that

$$\begin{aligned} D^*(V_0) = {} & \frac{2^{1-s}}{1-e[s]}(2\pi)^{-s}\left\{e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(0,-\frac{1}{4}\right),1,0\right)-e\left[\frac{s}{4}\right]L\left(\alpha,s,\left(0,\frac{1}{4}\right),1,0\right)\right\} \\ & +\frac{2^{1-s}}{1-e[s]}(2\pi)^{-s}\left\{e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(\frac{1}{2},-\frac{1}{4}\right),1,0\right)-e\left[\frac{s}{4}\right]L\left(\alpha,s,\left(-\frac{1}{2},\frac{1}{4}\right),1,0\right)\right\} \\ & -\frac{1-e[s/2]}{1-e[s]}(2\pi)^{-s}e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(0,-\frac{1}{2}\right),1,0\right)-\frac{2^{1-s}e[s/2]}{1-e[s]}(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta\left(s,\frac{3}{4}\right)+e\left[\frac{s}{2}\right]\zeta\left(s,\frac{1}{4}\right)\right) \\ & +\frac{2^{1-s}}{1-e[s]}(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta\left(s,\frac{1}{4}\right)+e\left[\frac{s}{2}\right]\zeta\left(s,\frac{3}{4}\right)\right)-\frac{1-e[s/2]}{1-e[s]}(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta\left(s,\frac{1}{2}\right)+e\left[\frac{s}{2}\right]\zeta\left(s,\frac{1}{2}\right)\right) \\ & -\frac{1-e[s/2]}{1-e[s]}\left\{H\left(\alpha,s,\left(0,-\frac{1}{2}\right)\right)-H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\right\}+\frac{2^{1-s}}{1-e[s]}\left\{H\left(\alpha,s,\left(0,-\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(\frac{1}{4},0\right)\right)\right\} \\ & -\frac{2^{1-s}[s/2]}{1-e[s]}\left\{H\left(\alpha,s,\left(0,\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(-\frac{1}{4},0\right)\right)\right\}+\frac{2^{1-s}}{1-e[s]}\left\{H\left(\alpha,s,\left(\frac{1}{2},-\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right)\right\} \\ & -\frac{2^{1-s}e[s/2]}{1-e[s]}\left\{H\left(\alpha,s,\left(-\frac{1}{2},\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right)\right\}. \end{aligned} \tag{88}$$

Let

$$\begin{aligned} (1-e[s])\Phi_0 = {} & L\left(\alpha,s,\left(0,-\frac{1}{4}\right),1,0\right)+L\left(\alpha,s,\left(\frac{1}{2},-\frac{1}{4}\right),1,0\right)-e\left[\frac{s}{2}\right]L\left(\alpha,s,\left(0,\frac{1}{4}\right),1,0\right)-e\left[\frac{s}{2}\right]L\left(\alpha,s,\left(-\frac{1}{2},\frac{1}{4}\right),1,0\right) \\ & -\left(1-e\left[\frac{s}{2}\right]\right)2^{s-1}L\left(\alpha,s,\left(0,-\frac{1}{2}\right),1,0\right), \end{aligned} \tag{89}$$

$$(1-e[s])\Omega_0 = \zeta\left(s,\frac{1}{4}\right)-e[s]\zeta\left(s,\frac{1}{4}\right)-2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\left(\zeta\left(s,\frac{1}{2}\right)+e\left[\frac{s}{2}\right]\zeta\left(s,\frac{1}{2}\right)\right), \tag{90}$$

$$\begin{aligned} (1-e[s])\Psi_0 = {} & H\left(\alpha,s,\left(0,-\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(\frac{1}{4},0\right)\right)-e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(0,\frac{1}{4}\right)\right) \\ & +e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{1}{4},0\right)\right)+H\left(\alpha,s,\left(\frac{1}{2},-\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right) \\ & -e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{1}{2},\frac{1}{4}\right)\right)+e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right) \\ & -2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\left\{H\left(\alpha,s,\left(0,-\frac{1}{2}\right)\right)-H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\right\}. \end{aligned} \tag{91}$$

Now, we can express the difference $D^*(V_0)$ as

$$D^*(V_0) = 2^{1-2s}\pi^{-s}e\left[-\frac{s}{4}\right](\Phi_0+\Gamma(s)\Omega_0)+2^{1-s}\Psi_0. \tag{92}$$

Using the integral representation (35) of $L(\alpha,s,(p',q'),c,d)$, we calculate Φ_0. Therefore,

$$\begin{aligned} (1-e[s])\Phi_0 = {} & -\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/4+\alpha)t)+\exp(-(1/4+\alpha/2)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\mathrm{d}t \\ & +e\left[\frac{s}{2}\right]\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(3/4+\alpha)t)+\exp(-(3/4+\alpha/2)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\mathrm{d}t \\ & +2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/2+\alpha)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\mathrm{d}t. \end{aligned} \tag{93}$$

Combining the first two integrals, we have

$$(1-e[s])\Phi_0 = I_0 + 2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-(1/2+\alpha)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\mathrm{d}t, \tag{94}$$

where

$$I_0 = \int_{I(\lambda,\infty)} t^{s-1}\frac{-\exp(-t/4)(1-\exp(\pi is - t/2))(\exp(-\alpha t)+\exp(-\alpha t/2))}{(1-\exp(-t))(1-\exp(-\alpha t))}\mathrm{d}t. \tag{95}$$

Now, making the change of variable $t\longleftrightarrow 2t$, we get

$$I_0 = 2^s\int_{I(\lambda,\infty)} t^{s-1}\frac{-\exp(-t/2)(1-\exp(\pi is - t))(\exp(-2\alpha t)+\exp(-\alpha t))}{(1+\exp(-t))(1-\exp(-t))(1+\exp(-\alpha t))(1-\exp(-\alpha t))}\mathrm{d}t. \tag{96}$$

Hence, after eliminating the common factor, we arrive at

$$\begin{aligned}(1-e[s])\Phi_0 &= 2^s\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-t/2)(-1+\exp(\pi is - t))\exp(-\alpha t)}{(1+\exp(-t))(1-\exp(-t))(1-\exp(-\alpha t))}\mathrm{d}t\\ &+ 2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/2+\alpha)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\mathrm{d}t.\end{aligned} \tag{97}$$

Therefore,

$$\Phi_0 = -\frac{2^{s-1}}{1-e[s/2]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/2+\alpha)t)}{(1+\exp(-t))(1-\exp(-\alpha t))}\mathrm{d}t. \tag{98}$$

Our next target is to calculate Ψ_0. Using (23), we have

$$\begin{aligned}\Psi_0 &= \eta\left(\alpha,s,1,-\frac{1}{4}\right) - \eta\left(\alpha,s,\frac{1}{4},0\right) + \eta\left(\alpha,s,\frac{1}{2},-\frac{1}{4}\right)\\ &- \eta\left(\alpha,s,\frac{1}{4},\frac{1}{2}\right) - 2^s\left\{\eta\left(\alpha,s,1,\frac{1}{2}\right) - \eta\left(\alpha,s,\frac{1}{2},0\right)\right\}.\end{aligned} \tag{99}$$

Now, using the definition of the η-function, we get

$$\begin{aligned}\Psi_0 &= \sum_{n=1}^{\infty} n^{s-1}\frac{e[n(\alpha-1/4)]}{1-e[n\alpha]} + \sum_{n=1}^{\infty} n^{s-1}\frac{e[n(\alpha/2-1/4)]}{1-e[n\alpha]} - \sum_{n=1}^{\infty} n^{s-1}\frac{e[n\alpha/4]}{1-e[n\alpha]} - \sum_{n=1}^{\infty} n^{s-1}\frac{e[n(\alpha/4+1/2)]}{1-e[n\alpha]}\\ &- 2^s\sum_{n=1}^{\infty} n^{s-1}\frac{e[n(\alpha+1/2)]}{1-e[n\alpha]} + 2^s\sum_{n=1}^{\infty} n^{s-1}\frac{e[n\alpha/2]}{1-e[n\alpha]}\\ &= \sum_{n=1}^{\infty} n^{s-1}\frac{e[n(\alpha/2-1/4)](e[n\alpha/2]+1)}{1-e[n\alpha]} - 2\sum_{n=1}^{\infty}(2n)^{s-1}\frac{e[2n(\alpha/2-1/4)](1+e[n\alpha])}{1-e[2n\alpha]}\\ &- \sum_{n=1}^{\infty} n^{s-1}\frac{e[n\alpha/4](1+(-1)^n)}{1-e[n\alpha]} + 2\sum_{n=1}^{\infty}(2n)^{s-1}\frac{e[2n\alpha/4](1+e[n\alpha])}{1-e[2n\alpha]}\\ &= -\sum_{n=1}^{\infty} n^{s-1}\frac{e[n(\alpha/2+1/4)](e[n\alpha/2]+1)}{1-e[n\alpha]} + 2\sum_{n=1}^{\infty}(2n)^{s-1}\frac{e[3n\alpha/2]}{1-e[2n\alpha]}.\end{aligned} \tag{100}$$

To calculate Ω_0, we use $2^s\zeta(s,(1/2)) = \zeta(s,(1/4)) + \zeta(s,(3/4))$, and we get

$$\Omega_0 = \left(\zeta\left(s,\frac{1}{4}\right) - 2^{s-1}\zeta\left(s,\frac{1}{2}\right)\right) = 2^{-1}\left(\zeta\left(s,\frac{1}{4}\right) - \zeta\left(s,\frac{3}{4}\right)\right). \tag{101}$$

Finally, combining the expressions for Φ_0, Ψ_0, and Ω_0 we deduce Theorem 7 (60).

10. Proof of Theorem 7(61)

By using Proposition 2 and from (42), we have

$$\begin{aligned} D^*(V_1) &= D^*\left(V_1,\alpha,s,\frac{1}{2},0\right) \\ &= (\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1},s,\frac{1}{2},0\right) - A^*\left(\alpha,s,\frac{1}{2},0\right) \\ &= \frac{2^{1-s}}{1-e[s]}(2\pi)^{-s}\left\{e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(\frac{1}{4},0\right),1,1\right) - e\left[\frac{s}{4}\right]L\left(\alpha,s,\left(-\frac{1}{4},0\right),1,1\right)\right\} \\ &\quad + \frac{2^{1-s}}{1-e[s]}(2\pi)^{-s}\left\{e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(\frac{3}{4},\frac{1}{2}\right),1,1\right) - e\left[\frac{s}{4}\right]L\left(\alpha,s,\left(-\frac{3}{4},-\frac{1}{2}\right),1,1\right)\right\} \\ &\quad + \frac{2^{1-s}}{1-e[s]}\left\{H\left(\alpha,s,\left(\frac{3}{4},\frac{1}{2}\right)\right) - H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right)\right\} \\ &\quad - \frac{2^{1-s}e[s/2]}{1-e[s]}\left\{H\left(\alpha,s,\left(-\frac{3}{4},-\frac{1}{2}\right)\right) - H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right)\right\} \\ &\quad - \frac{1-e[s/2]}{1-e[s]}(2\pi)^{-s}e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(\frac{1}{2},0\right),1,1\right). \end{aligned} \tag{102}$$

Let

$$\begin{aligned} (1-e[s])\Phi_1 &= L\left(\alpha,s,\left(\frac{1}{4},0\right),1,1\right) + L\left(\alpha,s,\left(\frac{3}{4},\frac{1}{2}\right),1,1\right) \\ &\quad - e\left[\frac{s}{2}\right]L\left(\alpha,s,\left(-\frac{1}{4},0\right),1,1\right) \\ &\quad - e\left[\frac{s}{2}\right]L\left(\alpha,s,\left(-\frac{3}{4},-\frac{1}{2}\right),1,1\right) \\ &\quad - \left(1-e\left[\frac{s}{2}\right]\right)2^{s-1}L\left(\alpha,s,\left(\frac{1}{2},0\right),1,1\right). \end{aligned} \tag{103}$$

$$\begin{aligned} (1-e[s])\Psi_1 &= H\left(\alpha,s,\left(\frac{3}{4},\frac{1}{2}\right)\right) - H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right) \\ &\quad - e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{3}{4},-\frac{1}{2}\right)\right) \\ &\quad + e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right). \end{aligned} \tag{104}$$

We now express (102) as

$$D^*(V_1) = 2^{1-2s}\pi^{-s}e\left[-\frac{s}{4}\right]\Phi_1 + 2^{1-s}\Psi_1. \tag{105}$$

Now, utilizing the integral representation (35) of $L(\alpha,s,(p',q'),c,d)$, we have

$$\begin{aligned}(1-e[s])\Phi_1 &= -\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/4+3(\alpha+1)/4)t)+\exp(-(1/4+(\alpha+1)/4)t)}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\mathrm{d}t \\ &\quad + e\left[\frac{s}{2}\right]\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(3/4+(\alpha+1)/4)t)+\exp(-(3/4+3(\alpha+1)/4)t)}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\mathrm{d}t \\ &\quad + 2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/2+(\alpha+1)/2)t)}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\mathrm{d}t.\end{aligned} \tag{106}$$

Again, we write the left hand side of the above equation as

$$\begin{aligned}(1-e[s])\Phi_1 &= I_1 \\ &\quad + 2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/2+(\alpha+1)/2)t)}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\mathrm{d}t,\end{aligned} \tag{107}$$

where

$$I_1 = \int_{I(\lambda,\infty)} t^{s-1}\frac{(-\exp(-t/4)+\exp(\pi i s-(-3t/4)))(\exp(-(\alpha+1)t/4)+\exp(-3(\alpha+1)t/4))}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\mathrm{d}t. \tag{108}$$

Now, by change of variable $t\longleftrightarrow 2t$ followed by the elimination of the common factor $1+\exp(-(\alpha+1)t)$, we get

$$I_1 = 2^s\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-t/2)(-1+\exp(\pi i s-t))\exp(-(\alpha+1)t/2)}{(1+\exp(-t))(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\mathrm{d}t. \tag{109}$$

Thus, substituting I_1 in (107), we see that

$$\Phi_1 = -\frac{2^{s-1}}{1-e[s/2]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/2+(\alpha+1)/2)t)}{(1+\exp(-t))(1-\exp(-(\alpha+1)t))}\mathrm{d}t. \tag{110}$$

Using the definition of H-function, from (104), we have

$$\begin{aligned}\Psi_1 &= \eta\left(\alpha, s, \frac{3}{4}, \frac{1}{2}\right) - \eta\left(\alpha, s, \frac{1}{4}, \frac{1}{2}\right) \\ &= \sum_{n=1}^{\infty} n^{s-1}\frac{e[3n\alpha/4+1/2]}{1-e[n\alpha]} \\ &\quad - \sum_{n=1}^{\infty} n^{s-1}\frac{e[n\alpha/4+1/2]}{1-e[n\alpha]}.\end{aligned} \tag{111}$$

The nth summand is

$$n^{s-1}\frac{e[n\alpha/4+1/2](e[n\alpha/2]-1)}{1-e[n\alpha]}, \tag{112}$$

from which we may eliminate the common factor $e[(1/2)n\alpha]-1$. Therefore,

$$\Psi_1 = \sum_{n=1}^{\infty} n^{s-1}\frac{(-1)^{n-1}e[n\alpha/4]}{e[n\alpha/2]+1} = \frac{1}{2}\sum_{n=1}^{\infty} n^{s-1}\frac{(-1)^{n-1}}{\cos(\pi n\alpha/2)}. \tag{113}$$

Now, we substitute (110) and (113) in (102) and finally get

$$
\begin{aligned}
&(\alpha+1)^{-s} A^*\left(\frac{\alpha}{\alpha+1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \\
&= -\frac{(2\pi)^{-s} e[-s/4]}{(1-e[s/2])} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-t/2)}{(1+\exp(-t))} \frac{\exp(-(\alpha+1)t/2)}{(1-\exp(-(\alpha+1)t))} \mathrm{d}t \\
&\quad + 2^{-s} \sum_{n=1}^{\infty} n^{s-1} \frac{(-1)^{n-1}}{\cos(\pi\alpha n/2)}.
\end{aligned} \tag{114}
$$

This completes the proof of Theorem 7 (61).

11. Proof of Theorem 7 (62)

We follow the same route: first, we use Proposition 2 and then using (43), we obtain

$$
\begin{aligned}
D^*(V_2) &= D^*\left(V_2, \alpha, s, \frac{1}{2}, 0\right) \\
&= (\alpha-1)^{-s} A^*\left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \\
&= \frac{2^{1-s}}{1-e[s]} (2\pi)^{-s} \left\{ e\left[-\frac{s}{4}\right] L\left(\alpha, s, \left(-\frac{1}{4}, 0\right), 1, -1\right) - e\left[\frac{s}{4}\right] L\left(\alpha, s, \left(\frac{1}{4}, 0\right), 1, -1\right) \right\} \\
&\quad + \frac{2^{1-s}}{1-e[s]} (2\pi)^{-s} \left\{ e\left[-\frac{s}{4}\right] L\left(\alpha, s, \left(\frac{1}{4}, -\frac{1}{2}\right), 1, -1\right) - e\left[\frac{s}{4}\right] L\left(\alpha, s, \left(-\frac{1}{4}, \frac{1}{2}\right), 1, -1\right) \right\} \\
&\quad - \frac{1-e[s/2]}{1-e[s]} (2\pi)^{-s} e\left[-\frac{s}{4}\right] L\left(\alpha, s, \left(-\frac{1}{2}, 0\right), 1, -1\right) \\
&\quad + \frac{2^{1-s}}{1-e[s]} \left\{ H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right) - H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right) \right\} \\
&\quad - \frac{2^{1-s} e[s/2]}{1-e[s]} \left\{ H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right) - H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right) \right\} \\
&= 2^{1-2s} \pi^{-s} e\left[-\frac{s}{4}\right] \Phi_2 + 2^{1-s} \Psi_2,
\end{aligned} \tag{115}
$$

where

$$
\begin{aligned}
(1-e[s])\Phi_2 &= L\left(\alpha, s, \left(-\frac{1}{4}, 0\right), 1, -1\right) - e\left[\frac{s}{2}\right] L\left(\alpha, s, \left(\frac{1}{4}, 0\right), 1, -1\right) \\
&\quad + L\left(\alpha, s, \left(\frac{1}{4}, -\frac{1}{2}\right), 1, -1\right) - e\left[\frac{s}{2}\right] L\left(\alpha, s, \left(-\frac{1}{4}, \frac{1}{2}\right), 1, -1\right) \\
&\quad - \left(1 - e\left[\frac{s}{2}\right]\right) 2^{s-1} L\left(\alpha, s, \left(-\frac{1}{2}, 0\right), 1, -1\right),
\end{aligned} \tag{116}
$$

$$
\begin{aligned}
(1-e[s])\Psi_2 &= H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right) - H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right) \\
&\quad - e\left[\frac{s}{2}\right] H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right) + e\left[\frac{s}{2}\right] H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right) \\
&= \left(1 + e\left[\frac{s}{2}\right]\right) \left(H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right) - H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right) \right).
\end{aligned} \tag{117}
$$

To simplify Φ_2, we make use of the integral representation (35) of $L(\alpha, s, (p', q'), c, d)$. So, we have

$$(1 - e[s])\Phi_2 = I_2 + 2^{s-1}\left(1 - e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/2 + (\alpha-1)/2)t)}{(1-\exp(-t))(1-\exp(-(\alpha-1)t))}\,\mathrm{d}t, \tag{118}$$

where

$$I_2 = \int_{I(\lambda,\infty)} t^{s-1}\frac{(-\exp(-t/4) + \exp(\pi is - 3t/4))(\exp-(\alpha-1)t/4 + \exp(-3(\alpha-1)t/4))}{(1-\exp(-t))(1-\exp(-(\alpha-1)t))}\,\mathrm{d}t. \tag{119}$$

As before, by eliminating the common factor $1 - \exp(-t)$, we obtain

$$I_2 = 2^s\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-t/2)\exp(-(\alpha-1)t/2)(-1 + \exp(\pi is - t))}{(1+\exp(-t))(1-\exp(-t))(1-\exp(-(\alpha-1)t))}\,\mathrm{d}t. \tag{120}$$

Whence, it follows that

$$\Phi_2 = -\frac{2^{s-1}}{1-e[s/2]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(1/2 + (\alpha-1)/2)t)}{(1+\exp(-t))(1-\exp(-(\alpha-1)t))}\,\mathrm{d}t. \tag{121}$$

While handling (117), we decompose it as

$$\Psi_2 = \eta\left(\alpha, s, \frac{3}{4}, 0\right) - \eta\left(\alpha, s, \frac{1}{4}, 0\right). \tag{122}$$

In the series expression of Ψ_2, we factor out $e[n\alpha/4]$ as before and eliminate the common factor $(e[n\alpha/2] - 1)$ to obtain

$$\Psi_2 = -\sum_{n=1}^{\infty} n^{s-1}\frac{e[n\alpha/4]}{e[n\alpha/2]+1} = -\frac{1}{2}\sum_{n=1}^{\infty} n^{s-1}\frac{1}{\cos(\pi n\alpha/2)}. \tag{123}$$

Finally, substituting the expressions for Φ_2 and Ψ_2, we have

$$\begin{aligned}(\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) &= -\frac{(2\pi)^{-s}e[-s/4]}{1-e[(s/2)]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-t/2)}{(1+\exp(-t))}\frac{\exp(-(\alpha-1)t/2)}{(1-\exp(-(\alpha-1))t)}\,\mathrm{d}t \\ &\quad - 2^{-s}\sum_{n=1}^{\infty} n^{s-1}\frac{1}{\cos(\pi n\alpha/2)}.\end{aligned} \tag{124}$$

This finishes the proof of Theorem 7 (62).

12. Proof of Corollary 1

We conclude this chapter by finally proving Corollary 1. We add (114) and (124) and derive that

$$
\begin{aligned}
&(\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1},s,\frac{1}{2},0\right)+(\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1},s,\frac{1}{2},0\right)-2A^*\left(\alpha,s,\frac{1}{2},0\right)\\
&\quad=-\frac{(2\pi)^{-s}e[-s/4]}{1-e[s/2]}\int_{I(\lambda,\infty)}t^{s-1}\frac{\exp(-t/2)}{(1+\exp(-t))}\frac{\exp(-(\alpha+1)t/2)}{(1-\exp(-(\alpha+1)t))}dt\\
&\quad\quad-\frac{(2\pi)^{-s}e[-s/4]}{1-e[-s/2]}\int_{I(\lambda,\infty)}t^{s-1}\frac{\exp(-t/2)}{(1+\exp(-t))}\frac{\exp(-(\alpha-1)t/2)}{(1-\exp(-(\alpha-1)t))}dt\\
&\quad\quad+2^{-s}\sum_{n=1}^{\infty}n^{s-1}\frac{(-1)^{n-1}}{\cos(\pi n\alpha/2)}-2^{-s}\sum_{n=1}^{\infty}n^{s-1}\frac{1}{\cos(\pi n\alpha/2)}\\
&\quad=-\frac{(2\pi)^{-s}e[-s/4]}{1-e[-s/2]}\int_{I(\lambda,\infty)}t^{s-1}\frac{\exp(-t/2)}{1+\exp(-t)}\left\{\frac{\exp(-(\alpha+1)t/2)}{1-\exp(-(\alpha+1)t)}+\frac{\exp(-(\alpha+1)t/2)}{1-\exp(-(\alpha-1)t)}\right\}dt\\
&\quad\quad-2\cdot 2^{-s}\sum_{n=1}^{\infty}(2n)^{s-1}\frac{1}{\cos(2\pi n\alpha/2)}.
\end{aligned}
\tag{125}
$$

Now, in the above expression, $2A^*(\alpha,s,1/2,0)$ on the left hand side and secant zeta function on the right hand side will cancel each other, as they are the same (from (50)). Therefore, we have

$$
\begin{aligned}
&(\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1},s,\frac{1}{2},0\right)+(\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1},s,\frac{1}{2},0\right)\\
&=-\frac{(2\pi)^{-s}e[-(s/4)]}{1-e[s/2]}\int_{I(\lambda,\infty)}t^{s-1}\frac{\exp(-(1/2)t)}{1+\exp(-t)}\\
&\quad\cdot\left\{\frac{\exp(-(1/2)(\alpha+1)t)}{1-\exp(-(\alpha+1)t)}+\frac{\exp(-(1/2)(\alpha-1)t)}{1-\exp(-(\alpha-1)t)}\right\}dt\\
&=-\frac{(2\pi)^{-s}e[-(s/4)]}{1-e[s/2]}\int_{I(\lambda,\infty)}t^{s-1}\sum_{m=0}^{\infty}E_m\left(\frac{1}{2}\right)\frac{t^m}{2m!},
\end{aligned}
\tag{126}
$$

and thus Corollary 1 follows.

13. Future Work

By the virtue of the work of Lewittes, Berndt, and Arakawa, it would be interesting to find the general modular transformation formula for $A^*(\alpha,s,p,q)$ for all $(p,q)\in\mathbb{R}^2$ and from which one would like to see the truth of our Conjecture 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Some New Refinements of Hermite–Hadamard-Type Inequalities Involving ψ_k-Riemann–Liouville Fractional Integrals and Applications

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The main objective of this article is to establish some new fractional refinements of Hermite–Hadamard-type inequalities essentially using new ψ_k-Riemann–Liouville fractional integrals, where $k > 0$. Using this new fractional integral, we also derive two new fractional integral identities. Applications of the obtained results are also discussed.

1. Introduction and Preliminaries

Let $f: I = [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function; then,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The above inequality is known as Hermite–Hadamard's inequality [1–5]. This inequality provides us a necessary and sufficient condition for a function to be convex. It can be considered as one of the most extensively studied results pertaining to convexity. Since the appearance of this result in the literature, it gained popularity, and many new generalizations for this classical result have been obtained. This can be attributed to its applications in various other fields such as in numerical analysis and in mathematical statistics. For more details on generalizations of convexity, Hermite–Hadamard-like inequalities, and its applications, see [6–14].

Fractional calculus is a calculus in which we study about the integrals and derivatives of any arbitrary real or complex order. The history of fractional calculus is not very much old, but in the short span of time, it experienced a rapid development. Recently, the generalizations [15–25], extensions [26–32], and applications [33–46] for fractional calculus have been made by many researchers. The Riemann–Liouville fractional integrals are defined as follows.

Definition 1 (see [47]). Let $f \in L_1[a, b]$. Then, Riemann–Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a, \tag{2}$$

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b, \tag{3}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1}\mathrm{d}x, \tag{4}$$

is the well-known gamma function.

Sarikaya et al. [10] elegantly utilized this concept in establishing fractional analogue of Hermite–Hadamard's inequality. This idea motivated other researchers, and consequently, many new generalizations of Hermite–Hadamard's inequality have been obtained using the concept of Riemann–Liouville fractional integrals.

Sarikaya and Karaca [12] introduced k-analogue of Riemann–Liouville fractional integrals and discussed some of its basic properties. They defined this concept in the following way: to be more precise, let f be piecewise continuous on $I^* = (0, \infty)$ and integrable on any finite subinterval of $I = [0, \infty]$. Then, for $t > 0$, we consider k-Riemann–Liouville fractional integral of f of order α as

$$ {}_kJ_a^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{(\alpha/k)-1} f(t)\mathrm{d}t, \quad x > a, k > 0. \tag{5}$$

If $k \longrightarrow 1$, then k-Riemann–Liouville fractional integrals reduce to classical the Riemann–Liouville fractional integral. It is worth to mention here that the concept of the k-Riemann–Liouville fractional integral is a significant generalization of Riemann–Liouville fractional integrals; as for $k \neq 1$, the properties of k-Riemann–Liouville fractional integrals are quite different from the classical Riemann–Liouville fractional integrals.

Another important generalization of Riemann–Liouville fractional integrals is ψ_k-Riemann–Liouville fractional integrals.

Definition 2 (see [6]). Let (a, b) be a finite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also, let $\psi(x)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b). Then, the left- and right-sided ψ-Riemann–Liouville fractional integrals of a function f with respect to another function ψ on $[a, b]$ are defined as

$$\begin{aligned} I_{a^+}^{\alpha;\psi} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} f(t)\mathrm{d}t, \\ I_{b^-}^{\alpha;\psi} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} f(t)\mathrm{d}t, \end{aligned} \tag{6}$$

respectively; $\Gamma(\cdot)$ is the gamma function.

For some recent research works, see [48].

Recently, Liu et al. [14] obtained some interesting results pertaining to Hermite–Hadamard's inequality involving ψ_k-Riemann–Liouville fractional integrals. Motivated by the research work of Liu et al. [14], we obtain some new refinements of fractional Hermite–Hadamard's inequality essentially using ψ_k-Riemann–Liouville fractional integrals. We also discuss applications of the obtained results to means. We show that our results represent significant generalization of some previous results.

2. Hermite–Hadamard's Inequality

In this section, we derive a new refinement of Hermite–Hadamard's inequality via the ψ_k-Riemann–Liouville fractional integral.

Definition 3. Let $k > 0$, (a, b) be a finite interval of the real line $\mathbb{R}$, and $\alpha > 0$. Also, let $\psi(x)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b). Then, the left- and right-sided ψ_k-Riemann–Liouville fractional integrals of a function f with respect to another function ψ on $[a, b]$ are defined as

$$\begin{aligned} {}_kI_{a^+}^{\alpha;\psi} f(x) &= \frac{1}{k\Gamma_k(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{(\alpha/k)-1} f(t)\mathrm{d}t, \\ {}_kI_{b^-}^{\alpha;\psi} f(x) &= \frac{1}{k\Gamma_k(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{(\alpha/k)-1} f(t)\mathrm{d}t, \end{aligned} \tag{7}$$

respectively;

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-(t^k/k)}\mathrm{d}t, \quad \mathfrak{R}(x) > 0, \tag{8}$$

is the k-analogue of gamma function.

The k-analogues of beta function and incomplete beta function are, respectively, defined as

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{(x/k)-1} (1-t)^{(x/k)-1}\mathrm{d}t, \tag{9}$$

$$B_k(z; x, y) = \frac{1}{k} \int_0^z t^{(x/k)-1} (1-t)^{(b/y)-1}\mathrm{d}t. \tag{10}$$

We now derive the main result of this section.

Theorem 1. *Let $0 \le e < f$ and g: $[e, f] \longrightarrow \mathbb{R}$ be a positive function and $g \in L_1[e, f]$. Also, suppose that g is a convex function on $[e, f]$, $\psi(x)$ is an increasing and positive monotone function on $(e, f]$, having a continuous derivative $\psi'(x)$ on (e, f), and $\alpha \in (0, 1)$. Then, for $k > 0$, the following k-fractional integral inequalities hold:*

$$\begin{aligned} g\left(\frac{e+f}{2}\right) &\le \frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}} \left[{}_kI_{\psi^{-1}(e)^+}^{\alpha;\psi} (g\circ\psi)\left(\psi^{-1}(f)\right) \right. \\ &\left. \quad + {}_kI_{\psi^{-1}(f)^-}^{\alpha;\psi} (g\circ\psi)\left(\psi^{-1}(e)\right) \right] \le \frac{g(e)+g(f)}{2}. \end{aligned} \tag{11}$$

Proof. Using the convexity of g, we have

$$2g\left(\frac{e+f}{2}\right) \le g(tc + (1-t)f) + g((1-t)e + td). \tag{12}$$

Multiplying both sides by $t^{(\alpha/k)-1}$ and then integrating with respect to t on $[0, 1]$, we have

$$\begin{aligned} \frac{2k}{\alpha} g\left(\frac{e+f}{2}\right) &\le \int_0^1 t^{(\alpha/k)-1} g(tc + (1-t)f)\mathrm{d}t \\ &\quad + \int_0^1 t^{(\alpha/k)-1} g((1-t)e + td)\mathrm{d}t. \end{aligned} \tag{13}$$

Now, making the substitution $t = (\psi(v) - f/e - f)$, $s = (\psi(v) - e/f - e)$, we have

$$\begin{aligned}
&\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\\
&=\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\frac{1}{k\Gamma_k(\alpha)}\left[\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\right.\\
&\left.+\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\right]\\
&=\frac{\alpha}{2k}\left[\int_0^1 t^{(\alpha/k)-1}g(tc+(1-t)f)\mathrm{d}t+\int_0^1 t^{(\alpha/k)-1}g((1-t)e+td)\mathrm{d}t\right]\\
&\geq g\left(\frac{e+f}{2}\right).
\end{aligned}\tag{14}$$

Also, using the convexity property of g, we have

$$g(tc+(1-t)f)+g((1-t)e+td)\leq g(e)+g(f).\tag{15}$$

Multiplying both sides by $t^{(\alpha/k)-1}$ and then integrating it with respect to t on $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{(\alpha/k)-1}g(tc+(1-t)f)\mathrm{d}t+\int_0^1 t^{(\alpha/k)-1}g((1-t)e+td)\mathrm{d}t\\
&\leq\frac{k}{\alpha}[g(e)+g(f)].
\end{aligned}\tag{16}$$

This implies

$$\begin{aligned}
&\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\\
&\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\leq\frac{g(e)+g(f)}{2}.
\end{aligned}\tag{17}$$

The proof is completed. □

3. Some More Fractional Inequalities of Hermite–Hadamard Type

We now derive two new fractional integral identities involving ψ_k-Riemann–Liouville fractional integrals. These results will serve as auxiliary results for obtaining our next results.

Lemma 1. *Let $e<f$ and $g\colon [e,f]\longrightarrow\mathbb{R}$ be a differentiable mapping on (e,f). Also, suppose that $g'\in L[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha\in(0,1)$. Then, for $k>0$, the following identity holds:*

$$\begin{aligned}
&\frac{g(e)+g(f)}{2}-\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\\
&\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\\
&=\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left[(\psi(v)-e)^{(\alpha/k)}\right.\\
&\left.-(f-\psi(v))^{(\alpha/k)}\right](g'\circ\psi)(v)\psi'(v)\mathrm{d}v.
\end{aligned}\tag{18}$$

Proof. Consider $J_1=(\Gamma_k(\alpha+k)/2(f-e)^{(\alpha/k)})_k I^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)(\psi^{-1}(f))$ and $J_2=(\Gamma_k(\alpha+k)/2(f-e)^{(\alpha/k)})_k I^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)(\psi^{-1}(e))$.

Now,

$$\begin{aligned}
J_1&=\frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)-1}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\\
&=-\frac{1}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(g\circ\psi)(v)\mathrm{d}(f-\psi(v))^{(\alpha/k)}\\
&=\frac{g(e)}{2}+\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)}(g'\circ\psi)(v)\psi'(v)\mathrm{d}v.
\end{aligned}\tag{19}$$

Similarly,

$$
\begin{aligned}
J_2 &= \frac{\alpha}{2k(f-e)^{(\alpha/k)}} \int_{\psi^{-1}(e)}^{\psi^{-1}(f)} (\psi(v)-e)^{(\alpha/k)-1} \\
&\quad \cdot (g\circ\psi)(v)\psi'(v)\mathrm{d}v \\
&= \frac{1}{2k(f-e)^{(\alpha/k)}} \int_{\psi^{-1}(e)}^{\psi^{-1}(f)} (g\circ\psi)(v)\mathrm{d} \\
&\quad \cdot (\psi(v)-e)^{(\alpha/k)} \\
&= \frac{g(f)}{2} - \frac{1}{2(f-e)^{(\alpha/k)}} \int_{\psi^{-1}(e)}^{\psi^{-1}(f)} \\
&\quad \cdot (\psi(v)-e)^{(\alpha/k)}(g'\circ\psi)(v)\psi'(v)\mathrm{d}v.
\end{aligned} \tag{20}
$$

It follows that

$$
\begin{aligned}
\frac{g(e)+g(f)}{2} - (J_1+J_2) &= \frac{1}{2(f-e)^{(\alpha/k)}} \int_{\psi^{-1}(e)}^{\psi^{-1}(f)} \\
&\quad \cdot \left[(\psi(v)-e)^{(\alpha/k)} - (f-\psi(v))^{(\alpha/k)}\right] \\
&\quad \cdot (g'\circ\psi)(v)\psi'(v)\mathrm{d}v.
\end{aligned} \tag{21}
$$

□

Example 1. Let $c=2, d=3, \alpha=(1/2), k=2, g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 1 are satisfied. Observe that $(g(c)+g(d)/2)=(13/2)$.

$$
\begin{aligned}
&\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}} \left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)\right. \\
&\quad \left. + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] \\
&= \frac{\Gamma_{(2)}(1/2)}{2}\left[\frac{1}{\Gamma_{(2)}(1/2)} \int_2^3 v^2(3-v)^{-(3/4)}\mathrm{d}v\right. \\
&\quad \left. + \frac{1}{\Gamma_{(2)}(1/2)} \int_2^3 v^2(v-2)^{-(3/4)}\mathrm{d}v\right] = \frac{577}{90}.
\end{aligned} \tag{22}
$$

This implies

$$
\begin{aligned}
&\frac{g(c)+g(d)}{2} - \frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}} \left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)\right. \\
&\quad \left. + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] = \frac{4}{45}.
\end{aligned} \tag{23}
$$

Also,

$$
\begin{aligned}
&\frac{1}{2(d-c)^{(\alpha/k)}} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \left[(\psi(v)-c)^{(\alpha/k)} - (d-\psi(v))^{(\alpha/k)}\right] \\
&\quad \cdot (g'\circ\psi)(v)\psi'(v)\mathrm{d}v \\
&\quad = \int_2^3 v(v-2)^{(1/4)}\mathrm{d}v - \int_2^3 v(3-v)^{(1/4)}\mathrm{d}v = \frac{4}{45}.
\end{aligned} \tag{24}
$$

Example 2. Let $c=2, d=3, \alpha=(1/2), k=(1/2), g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 1 are satisfied. Observe that $(g(c)+g(d)/2)=(13/2)$.

$$
\begin{aligned}
&\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}} \left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] \\
&= \frac{\Gamma_{(1/2)}(1/2)}{2}\left[\frac{1}{\Gamma_{(1/2)}(1/2)} \int_2^3 v^2\mathrm{d}v + \frac{1}{\Gamma_{(1/2)}(1/2)} \int_2^3 v^2\mathrm{d}v\right] = \frac{19}{3}.
\end{aligned} \tag{25}
$$

This implies

$$
\begin{aligned}
&\frac{g(c)+g(d)}{2} - \frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}} \left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)\right. \\
&\quad \left. + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] = \frac{1}{6}.
\end{aligned} \tag{26}
$$

Also,

$$
\begin{aligned}
&\frac{1}{2(d-c)^{(\alpha/k)}} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \left[(\psi(v)-c)^{(\alpha/k)} - (d-\psi(v))^{(\alpha/k)}\right] \\
&\quad \cdot (g'\circ\psi)(v)\psi'(v)\mathrm{d}v \\
&\quad = \int_2^3 v(v-2)\mathrm{d}v - \int_2^3 v(3-v)\mathrm{d}v = \frac{1}{6}.
\end{aligned} \tag{27}
$$

Lemma 2. *Let $e<f$ and $g\colon [e,f] \longrightarrow \mathbb{R}$ be a differentiable mapping on (e,f). Also, suppose that $g' \in L[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha \in (0,1)$. Then, for $k>0$, the following identity holds:*

$$\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right] - g\left(\frac{e+f}{2}\right)$$
$$= \int_{\psi^{-1}(e)}^{\psi^{-1}(f)} h(g'\circ\psi)(v)\psi'(v)\mathrm{d}v + \frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left[(\psi(v)-e)^{(\alpha/k)} - (f-\psi(v))^{(\alpha/k)}\right](g'\circ\psi)(v)\psi'(v)\mathrm{d}v, \tag{28}$$

where

$$h = \begin{cases} \frac{1}{2}, & \text{for } \psi^{-1}\left(\frac{e+f}{2}\right)\le v\le \psi^{-1}(f), \\ -\frac{1}{2}, & \text{for } \psi^{-1}(e)\le v\le \psi^{-1}\left(\frac{e+f}{2}\right). \end{cases} \tag{29}$$

Proof. Suppose

$$I_1 = -\frac{1}{2}\int_{\psi^{-1}(e)}^{\psi^{-1}(e+f/2)}(g'\circ\psi)(v)\psi'(v)\mathrm{d}v = -\frac{1}{2}g\left(\frac{e+f}{2}\right)+\frac{g(e)}{2},$$
$$I_2 = \frac{1}{2}\int_{\psi^{-1}(e+f/2)}^{\psi^{-1}(f)}(g'\circ\psi)(v)\psi'(v)\mathrm{d}v = -\frac{1}{2}g\left(\frac{e+f}{2}\right)+\frac{g(f)}{2},$$
$$\begin{aligned} I_3 &= \frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)}(g'\circ\psi)(v)\psi'(v)\mathrm{d}v \\ &= -\frac{g(e)}{2}+\frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)-1}(g\circ\psi)(v)\psi'(v)\mathrm{d}v \\ &= -\frac{g(e)}{2}+\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right), \end{aligned} \tag{30}$$

$$\begin{aligned} I_4 &= -\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)}(g'\circ\psi)(v)\psi'(v)\mathrm{d}v \\ &= -\frac{g(f)}{2}+\frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)-1}(g\circ\psi)(v)\psi'(v)\mathrm{d}v \\ &= -\frac{g(f)}{2}+\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right). \end{aligned} \tag{31}$$

Summing I_1, I_2, I_3, and I_4, we get the required result. □

Example 3. Let $c=2, d=3, \alpha=(1/2), k=2, g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 2 are satisfied. Note that $g(c+d/2)=(25/4)$.

$$\begin{aligned} &\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] \\ &= \frac{\Gamma_{(1/2)}(1/2)}{8}\left[\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2(3-v)^{-(3/4)}\mathrm{d}v + \frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2(v-2)^{-(3/4)}\mathrm{d}v\right] = \frac{577}{90}. \end{aligned} \tag{32}$$

This implies

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] - g\left(\frac{c+d}{2}\right) = \frac{29}{180}. \tag{33}$$

Also,

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} h\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \frac{1}{4}, \tag{34}$$

where h is defined in Lemma 2.

$$\frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(d-\psi(v))^{(\alpha/k)} - (\psi(v)-c)^{(\alpha/k)}\right] \cdot \left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \int_2^3 v(v-2)^{(1/4)}\mathrm{d}v - \int_2^3 v(3-v)^{(1/4)}\mathrm{d}v = -\frac{4}{45}. \tag{35}$$

This implies

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} h\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v + \frac{1}{2(d-c)^{(\alpha/k)}} \cdot \int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(d-\psi(v))^{(\alpha/k)} - (\psi(v)-c)^{(\alpha/k)}\right]\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \frac{29}{180}. \tag{36}$$

Example 4. Let $c=2, d=3, \alpha=(1/2), k=(1/2), g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 2 are satisfied. Note that $g(c+d/2)=(25/4)$.

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] = \frac{\Gamma_{(1/2)}(1/2)}{2}\left[\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2\mathrm{d}v + \frac{1}{\Gamma_{(2)}(1/2)}\int_2^3 v^2\mathrm{d}v\right] = \frac{19}{3}. \tag{37}$$

This implies

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] - g\left(\frac{c+d}{2}\right) = \frac{1}{12}. \tag{38}$$

Also,

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} h\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \frac{1}{4}, \tag{39}$$

where h is defined in Lemma 2.

$$\frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(d-\psi(v))^{(\alpha/k)} - (\psi(v)-c)^{(\alpha/k)}\right]\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \int_2^3 v(v-2)\mathrm{d}v - \int_2^3 v(3-v)\mathrm{d}v = -\frac{1}{6}. \tag{40}$$

This implies

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} h\left(g\prime\circ\psi\right)(v)\psi'(v)\mathrm{d}v + \frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \cdot\left[(d-\psi(v))^{(\alpha/k)} - (\psi(v)-c)^{(\alpha/k)}\right] \cdot \left(g\prime\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \frac{1}{12}. \tag{41}$$

Before proceeding to next results, let us recall the definition of s-convex function of Breckner type.

Definition 4 (see [49]). A function g: $[0,\infty) \longrightarrow [0,\infty)$ is said to be s-convex function of Breckner type if

$$g((1-t)x+ty) \leq (1-t)^s g(x) + t^s g(y), \quad \forall x, y \in [0,\infty), t \in [0,1], s \in (0,1]. \tag{42}$$

Theorem 2. *Let $e<f$ and g: $[e,f] \longrightarrow \mathbb{R}$ be a differentiable mapping on (e,f). Also, suppose that $|g'|$ is Breckner type of s-convex on $[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha \in (0,1)$. Then, for $k>0$, the following inequality holds:*

$$\left|\frac{g(e)+g(f)}{2} - \frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\right| \leq \frac{f-e}{2}\left[L_1\left|g'(e)\right| + L_2\left|g'(f)\right|\right], \tag{43}$$

where

$$L_1 := 2k\mathrm{B}_k\left(\frac{1}{2}; 1+s, \frac{k+\alpha}{k}\right) + \frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha} - \mathrm{B}_k\left(1+s, \frac{k+\alpha}{k}\right), \tag{44}$$

$$L_2 := \frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha} - 2k\mathrm{B}_k\left(\frac{1}{2}; \frac{k+\alpha}{k}, 1+s\right) - \mathrm{B}_k\left(\frac{k+\alpha}{k}, 1+s\right), \tag{45}$$

respectively.

Proof. Using Lemma 1 and the fact that $|g'|$ is Breckner type of s-convex function, we have

$$\begin{aligned}
&\left|\frac{g(e)+g(f)}{2}-\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\right|\\
&\leq\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left|(\psi(v)-e)^{(\alpha/k)}-(f-\psi(v))^{(\alpha/k)}\right|\left|(g\prime\circ\psi)(v)\right|\left|\psi'(v)\right|\mathrm{d}v\\
&=\frac{f-e}{2}\int_0^1\left|(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right|\left|g'(tc+(1-t)f)\right|\mathrm{d}t\\
&\leq\frac{f-e}{2}\int_0^1\left|(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right|\left[t^s\left|g'(e)\right|+(1-t)^s\left|g'(f)\right|\right]\mathrm{d}t\\
&=\frac{f-e}{2}\left[\left|g'(e)\right|\int_0^{(1/2)}t^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t+\left|g'(f)\right|\int_0^{(1/2)}(1-t)^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t\right.\\
&\left.+\left|g'(e)\right|\int_{(1/2)}^1t^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t+\left|g'(f)\right|\int_{(1/2)}^1(1-t)^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t\right]\\
&=\frac{f-e}{2}\left[L_1\left|g'(e)\right|+L_2\left|g'(f)\right|\right],
\end{aligned}\tag{46}$$

where

$$\begin{aligned}
L_1&:=H_1+H_3=\int_0^{(1/2)}t^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t+\int_{(1/2)}^1t^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t\\
&=2k\mathrm{B}_k\left(\frac{1}{2};1+s,\frac{k+\alpha}{k}\right)+\frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha}-\mathrm{B}_k\left(1+s,\frac{k+\alpha}{k}\right),
\end{aligned}\tag{47}$$

$$\begin{aligned}
L_2&:=H_2+H_4=\int_0^{(1/2)}(1-t)^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t+\int_{(1/2)}^1(1-t)^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t\\
&=\frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha}-2k\mathrm{B}_k\left(\frac{1}{2};\frac{k+\alpha}{k},1+s\right)-\mathrm{B}_k\left(\frac{k+\alpha}{k},1+s\right).
\end{aligned}\tag{48}$$

This completes the proof. □

Theorem 3. *Let $g\colon [e,f]\longrightarrow\mathbb{R}$ be a differentiable function on (e,f) with $e<f$. Also, suppose that $|g'|$ is Breckner type of s-convex function. If $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f) and $\alpha\in(0,1)$, then for $k>0$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\right.\\
&\left.\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]-g\left(\frac{e+f}{2}\right)\right|\\
&\leq\frac{|g(f)-g(e)|}{2}+\frac{f-e}{2}\left[L_1\left|g'(e)\right|+L_2\left|g'(f)\right|\right],
\end{aligned}\tag{49}$$

where L_1 and L_2 are given by (44) and (45), respectively.

Proof. Using Lemma 2, the property of modulus, and the given hypothesis of the theorem, we have

$$\begin{aligned}
&\left|\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\right.\\
&\left.\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)(\pi)\right]-g\left(\frac{e+f}{2}\right)\right|\\
&\leq\left|\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}h(g'\circ\psi)(v)\psi'(v)\mathrm{d}v\right|\\
&+\left|\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left[(f-\psi(v))^{(\alpha/k)}\right.\right.\\
&\left.\left.-(\psi(v)-e)^{(\alpha/k)}\right](g\prime\circ\psi)(v)\psi'(v)\mathrm{d}v\right|\\
&=I_1+I_2.
\end{aligned}\tag{50}$$

Using substitution $t = (\psi(v) - e/f - e)$ and the fact that $|g'|$ is Breckner type of s-convex function, we have

$$I_1 \leq \frac{f-e}{2}\left[L_1\left|g'(e)\right| + L_2\left|g'(f)\right|\right], \tag{51}$$

where L_1 and L_2 are given by (44) and (45), respectively. And

$$I_2 = \frac{|g(f) - g(e)|}{2}. \tag{52}$$

This completes the proof. □

4. Applications

In this section, we discuss some applications of Theorem 2 to means by considering a particular example of s-convexity. First of all, we recall some previously known concepts related to means [50].

For arbitrary real numbers α, β, $\alpha \neq \beta$, we define the following:

(1) Arithmetic mean:

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}; \tag{53}$$

(2) Logarithmic mean:

$$\overline{L}(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|}, \quad \alpha,\beta \in \mathbb{R}\backslash\{0\}; \tag{54}$$

(3) Generalized log-mean:

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{(1/n)}, \quad n \in \mathbb{N}, n \geq 1, \alpha,\beta \in \mathbb{R}, \alpha < \beta. \tag{55}$$

We now give the main results of this section.

Proposition 1. *Let $e, f \in \mathbb{R}^+$ with $e < f$; then,*

$$\left|A(e^s, f^s) - L_s^s(e,f)\right| \leq \frac{s(f-e)}{2}\left[W_1|e|^{s-1} + W_2|f|^{s-1}\right], \tag{56}$$

where

$$W_1 := 2\mathrm{B}\left(\frac{1}{2}; 1+s, 2\right) + \frac{1-2^{-1-s}}{2+s} - \mathrm{B}(1+s,2), \tag{57}$$

$$W_2 := \frac{1-2^{-1-s}}{2+s} - 2\mathrm{B}\left(\frac{1}{2}; 2, 1+s\right) - \mathrm{B}(2,1+s), \tag{58}$$

respectively.

Proof. Applying Theorem 2 for $g(x) = x^s$, $\psi(x) = x$, and $\alpha = 1 = k$, we obtain the required result. □

Proposition 2. *Let $e, f \in \mathbb{R}^+$ with $e < f$; then,*

$$\left|A(e^s, f^s) - L_s^s(e,f)\right| \leq \frac{\left|f^s - g^s\right|}{2} + \frac{s(f-e)}{2}\left[W_1|e|^{s-1} + W_2|f|^{s-1}\right], \tag{59}$$

where W_1 and W_2 are given by (57) and (58), respectively.

Proof. Applying Theorem 3 for $g(x) = x^s$, $\psi(x) = x$, and $\alpha = 1 = k$, we obtain the required result. □

5. Conclusion

In this article, we obtain some new fractional estimates of Hermite–Hadamard's inequality essentially using a new k-analogue of ψ_k-fractional integrals. We derive two new fractional integral identities in the setting of k-fractional calculus. In order to check the validity of these identities, we discuss some particular examples. In the final section, we have discussed applications of Theorems 2 and 3 to means.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Around the Lipschitz Summation Formula

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Boundary behavior of important functions has been an object of intensive research since the time of Riemann. Kurokawa, Kurokawa-Koyama, and Chapman studied the boundary behavior of generalized Eisenstein series which falls into this category. The underlying principle is the use of the Lipschitz summation formula. Our purpose is to show that it is a form of the functional equation for the Lipschitz–Lerch transcendent (and in the long run, it is equivalent to that for the Riemann zeta-function) and that this being indeed a boundary function of the Hurwitz–Lerch zeta-function, one can extract essential information. We also elucidate the relation between Ramanujan's formula and automorphy of Eisenstein series.

1. Introduction

Boundary behavior of core functions has always been the object of intensive research since it exhibits a peculiar phenomenon that cannot be predicted by the behavior inside the domain. There are many instances of such unexpected behavior cf. [1–3]. Kurokawa [4] and Koyama and Kurokawa [5] studied the following limiting values by the Lipschitz summation formula:

$$\lim_{\tau \longrightarrow x}\left(E_k\left(-\frac{1}{\tau}\right)-\tau^k E_k(\tau)\right), \quad \forall x \in \mathbb{R}, \tag{1}$$

where $E_k(\tau)$ is the generalized Eisenstein series defined by (25).

It has been elucidated and generalized by Chapman [6] who also used the Lipschitz summation formula for which he appealed to [7]. Knopp and Robbins in their Remarks 1 and 2 state their own views on the Lipschitz summation formula and Stark's method [8] to the effect that they are *not directly related to the functional equation* (just as, for the Riemann zeta-function, the partial fraction expansion does not seem to be related). In [9], Murty and Sinha [10] result has been elucidated as a manifestation of one of the equivalent conditions to the functional equation, the Fourier–Bessel expansion, or the perturbed Dirichlet series ([11], Chapter 4), thereby explaining the genesis of Stark's method. The Lipschitz summation formula for quadratic fields is also deduced there. We shall turn to this toward the end of Section 4.

We cite the passage from [12] "The relation between modular forms and Dirichlet series with functional equations was discovered by Hecke, whose epoch-making work during the years 1930–1940, based on that discovery and that of the 'Hecke operators', brought out completely new aspects of a theory which many mathematicians would have regarded as a closed chapter long before."

We refer to this as part of the Riemann–Hecke–Bochner correspondence (RHB correspondence) ([11], p. 4 and 22) which is coined by Knopp [13].

Our main aim in this paper is to prove the general modular relation, Theorem 4, for the Lipschitz–Lerch transcendent (57) and deduce the general Lipschitz summation formula, Corollary 4. From this, we show that, in this case again, generalized RHB correspondence or the modular relation is the key for everything.

But prior to this, in Section 2, we state the modular relation for the Lambert series generated by the product of two Riemann zeta-functions with variables different by an odd integer and prove the automorphy of the Eisenstein series by the RHB correspondence, which of course settles the even weight case of (1). For another relation, cf. Bruinier and Funke [14].

The even difference case turns out to be a reminiscent of the Wigert–Bellman divisor problem [15] as alluded to in [9]. In Section 3, we state the results in another form based on the shifted Mellin inversion.

Here, we use a method similar to the one in [16] (pp. 73–75) of using the Ewald expansion ([11], Chapter 5) as opposed to the Fourier–Bessel expansion alluded to above. Since it is equivalent to the Lerch functional equation ([17], Theorem 5.3, p. 130) which in turn is equivalent to an asymmetric form (3) of the functional equation for the Riemann zeta-function, we thereby show that, in the long run, the genesis is in the functional equation for the Riemann zeta-function.

As a necessary step, we show that the reciprocal Hurwitz formula amounts to a ramified functional equation, Lemma 1. There are many cases of such ramified functional equations (cf. [18] and references therein). We state one of the earliest occurrences.

In what follows, we always use the notation $s = \sigma + it$ as the complex variable.

2. An Example of the Riemann–Hecke–Bochner Correspondence

Throughout in what follows, we appeal to the Riemann zeta-function defined in the first instance for $\sigma = \operatorname{Re} s > 1$ by

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2}$$

This satisfies the asymmetric form of the functional equation:

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi}{2}s\right)\Gamma(s)\zeta(s), \tag{3}$$

which is a prototype of the Hurwitz formula (cf. (64) for its reciprocal).

We fix the integer α throughout.

We consider the product of two zeta-functions:

$$\varphi(s) = \varphi(s,\alpha) = \zeta(s)\zeta(s+\alpha) = \sum_{n=1}^{\infty} \frac{\sigma_{-\alpha}(n)}{n^s}, \tag{4}$$

where the series is absolutely convergent for $\sigma > \sigma_\varphi \coloneqq \max\{1, 1-\alpha\}$, and

$$\sigma_{-\alpha}(n) = \sum_{d\mid n} d^{-\alpha}, \tag{5}$$

is the sum-of-divisors function. We note that $\varphi(s-a)$ includes the case of $\zeta(s)\zeta(s-\alpha)$ as $\varphi(s-a) = \sum_{n=1}^{\infty}((\sigma_\alpha(n))/n^s)$. This will be pursued in Section 3.

The zeta-function $\varphi(s)$ satisfies the asymmetric functional equation:

$$\varphi(1-s) = 4(2\pi)^{-2s+\alpha} \cos\frac{\pi}{2}s \cos\frac{\pi}{2}(s-\alpha)\Gamma(s)\Gamma(s-\alpha)\varphi(s-\alpha). \tag{6}$$

Noting that

$$\cos\frac{\pi}{2}(s-\alpha) = \begin{cases} \sin\frac{\pi}{2}s \sin\frac{\pi}{2}\alpha, & \alpha = 2\varkappa+1, \\ \cos\frac{\pi}{2}s\cos\frac{\pi}{2}\alpha, & \alpha = 2\varkappa, \end{cases} \tag{7}$$

the product of cosines amounts to

$$\begin{cases} \frac{1}{2}(-1)^{\varkappa}\sin\pi s, & \alpha = 2\varkappa+1, \\ \frac{1}{2}(-1)^{\varkappa}(1+\cos\pi s), & \alpha = 2\varkappa. \end{cases} \tag{8}$$

(i) First we treat the case of $\alpha = 2\varkappa + 1$ an odd integer. Then by the reciprocal relation for the gamma function, we see that the functional equation (6) amounts to

$$\varphi(1-s) = 2(-1)^{\varkappa}(2\pi)^{-2s+\alpha}\frac{\pi}{\Gamma(1-s)}\Gamma(s-\alpha)\varphi(s-\alpha). \tag{9}$$

Now by the well-known procedure—Hecke gamma transform (e.g., [16]), we have for $c > \sigma_\varphi$ and $\operatorname{Re} x > 0$

$$\sum_{n=1}^{\infty}\sigma_{-\alpha}(n)e^{-nx} = \frac{1}{2\pi i}\int_{(c)}\Gamma(s)\varphi(s)x^{-s}ds, \tag{10}$$

where (c) indicates the Bromwich contour $\sigma = c, -\infty < t < \infty$.

By a standard procedure of moving the line to the left up to (d), where $d < -\alpha < 0$ ($d = -\alpha - (1/2)$, say), whereby noting that the horizontal integrals vanish in the limit as $|t| \longrightarrow \infty$, we obtain

$$\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\varphi(s)x^{-s}ds = \frac{1}{2\pi i}\int_{(d)}\Gamma(s)\varphi(s)x^{-s}ds + P(x), \tag{11}$$

where $P(x) = P_\alpha(x)$ is the residual function consisting of the sum of residues of the integrand at $-a, \dots, -1, 0, 1, a+1$. Writing $1-s$ for s in (11), we see that the right-hand side of (11) becomes $(1/2\pi i)\int_{(1-d)}\Gamma(1-s)\varphi(1-s)x^{s-1}ds + P(x)$, whence substituting (9), we conclude that

$$\sum_{n=1}^{\infty}\sigma_{-\alpha}(n)e^{-nx} = (-1)^{\varkappa}(2\pi)^{\alpha+1}\frac{1}{x}\frac{1}{2\pi i} \cdot \int_{(1-d)}\Gamma(s-\alpha)\varphi(s-\alpha)\left(\frac{4\pi^2}{x}\right)^{-s}ds + P(x). \tag{12}$$

Substituting the absolutely convergent series in (4) and factoring out $(x/4\pi^2)^a$, we deduce that

$$\sum_{n=1}^{\infty}\sigma_{-\alpha}(n)e^{-nx} = (-1)^{\varkappa}(2\pi)^{\alpha+1}\frac{1}{x}\left(\frac{x}{4\pi^2}\right)^{\alpha}\sum_{n=1}^{\infty}\sigma_{-\alpha}(n) \times \frac{1}{2\pi i}\int_{(1-d)}\Gamma(s-\alpha)\left(\frac{4\pi^2 n}{x}\right)^{-(s+\alpha)}ds + P(x). \quad (13)$$

Hence, by the Mellin inversion again, we obtain

$$\sum_{n=1}^{\infty}\sigma_{-\alpha}(n)e^{-nx} = (-1)^{(\alpha-1)/2}\left(\frac{x}{2\pi}\right)^{\alpha-1}\sum_{n=1}^{\infty}\sigma_{-\alpha}(n)e^{-\left(4\pi^2 n\right)/x} + P(x), \quad (14)$$

for $\mathrm{Re}\,x > 0$, i.e., the Bochner modular relation [19].

Here only in the case $\alpha = 1 > 0$, $s = 1$ is a double pole, others being simple poles. In the case $\alpha = -1 < 0$, there is one more term $-(1/2)x^{-1}$.

To compute the residual function, we use Table 1, taking into account the trivial zeros of the Riemann zeta-function at negative even integers.

For $1 \neq \alpha > 0$, the residual function is

$$\begin{aligned} P_{\alpha}(x) &= \sum_{k=1}^{\alpha}\frac{(-1)^k}{k!}\zeta(-k)\zeta(-k+\alpha)x^k + \zeta(1+\alpha)x^{-1} - \frac{1}{2}\zeta(\alpha) \\ &= -\sum_{j=1}^{[\alpha/2]+1}\frac{1}{(2j-1)!}\zeta(1-2j)\zeta(\alpha-2j+1)x^{2j-1} \\ &\quad + \zeta(1+\alpha)x^{-1} - \frac{1}{2}\zeta(\alpha), \end{aligned} \quad (15)$$

on writing $k = 2j-1$. In literature, this is expressed in another form based on the explicit formula for zeta-values cf., e.g., ([20], p. 71 and 91):

$$\zeta(1-2k) = -\frac{B_{2k}}{2k} \quad (k \geq 1), \quad \zeta(2k) = \frac{2^{2k-1}}{(2k)!}B_{2k}\pi^{2k}, \quad (16)$$

where the Bernoulli numbers B_{2k} are b-notation ([20], p. 90).

In particular, for $\alpha = 2\varkappa + 1 \geq 1$, (14) with x replaced by $2\pi x$ amounts to the celebrated Ramanujan formula, cf., e.g., [21] for a general account:

$$\sum_{n=1}^{\infty}\sigma_{-2\varkappa-1}(n)e^{-2\pi nx} + (-1)^{\varkappa+1}x^{2\varkappa}\sum_{n=1}^{\infty}\sigma_{-2\varkappa-1}(n)e^{-((2\pi n)/x)} = P(x), \quad (17)$$

where $P(x) = P_{2\varkappa+1}(x)$ is given by (15) whose concrete form is given in (49). Indeed, $P_{2\varkappa+1}(x)$ is a residual function given as the sum of the residues:

$$P(x) = \sum_{\xi\in R}\operatorname*{Res}_{s=\xi}(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s+2\varkappa+1)x^{-s}, \quad (18)$$

where $R = \{-2\varkappa-1, -2\varkappa, -2\varkappa+1, -2\varkappa+3, \ldots, -3, -1, 0, 1\}$, and $s = 0$ is a double pole only when $\varkappa = 0$ (others are simple poles). Hence,

$$P_{2\varkappa+1}(x) = \sum_{j=1}^{\varkappa+1}\frac{(-1)^k}{(2j+1)!}\zeta(1-2j)\zeta(2\varkappa-2j+2)x^{2j-1} + \zeta(2\varkappa+2)x^{-1} - \frac{1}{2}\zeta(2\varkappa+1). \quad (19)$$

cf. (49) below for another expression for the residual function.

If we write $-x = 2\pi i\tau$, then $\tau \in \mathscr{H}$ and the Lambert series in Liouville's form amounts to the Eisenstein series $E_k(\tau)$ in Definition 1, where $k = a+1$ is even and (14) gives

$$\sum_{n=1}^{\infty}\sigma_{-\alpha}(n)e^{2\pi in\tau} = \tau^{\alpha-1}\sum_{n=1}^{\infty}\sigma_{-\alpha}(n)e^{-((2\pi in)/\tau)} + P(x). \quad (20)$$

In case $\alpha = -2\varkappa-1$, $\varkappa \geq 1$, (20) reads

$$\begin{aligned} \sum_{n=1}^{\infty}\sigma_{2\varkappa+1}(n)e^{2\pi in\tau} &= \tau^{\alpha-1}\sum_{n=1}^{\infty}\sigma_{2\varkappa+1}(n)e^{-((2\pi in)/\tau)} + P(x), \\ P(\tau) = P_{-2\varkappa-1}(\tau) &= (2\varkappa+1)!\zeta(2\varkappa+2) \\ &\quad \cdot (-2\pi i\tau)^{-2\varkappa-2} - \frac{1}{2}\zeta(-2\varkappa-1), \end{aligned} \quad (21)$$

cf. (51) and (52).

In slightly different notation from [20] (p. 83),

$$G_{2k}(\tau) := \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}}\frac{1}{(n\tau+m)^{2k}}. \quad (22)$$

By Proposition 4 in [20] (p. 83), $G_{2k}(\tau)$ is a modular form of weight $2k$. The Laurent expansion (or q-expansion) reads ([20], p. 92)

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi i n\tau}, \quad \tau \in \mathscr{H}. \tag{23}$$

Appealing to (16), we have similarly to [20] (Corollary, p. 92)

$$\frac{1}{2\zeta(2k)}G_{2k}(\tau) = \frac{2}{\zeta(1-2k)}E_{2k}(\tau) = -\frac{2k}{B_{2k}}E_{2k}(\tau), \tag{24}$$

where $(2/(\zeta(1-2k)))E_{2k}(\tau)$ is the Eisenstein series ([20], (34), p. 92) and $E_{2k}(\tau)$ is the even suffix case of the following.

Definition 1. For any $k \in \mathbb{N}$, Kurokawa introduces the general Eisenstein series:

$$E_k(\tau) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n\tau}, \quad \tau \in \mathscr{H}, \tag{25}$$

which is not necessarily modular for k odd.

Stating (21) in explicit form

$$\sum_{k=1}^{\infty}\sigma_{2\varkappa+1}(k)e^{-2\pi k x} - \frac{B_{2\varkappa+2}}{4\varkappa+4} = (-1)^{\varkappa+1}x^{-2\varkappa-2}\left\{\sum_{k=1}^{\infty}\sigma_{2\varkappa+1}(k)e^{-((2\pi k)/x)} - \frac{B_{2\varkappa+2}}{4\varkappa+4}\right\}, \tag{26}$$

we see that it is nothing but

$$E_{2\varkappa+2}\left(-\frac{1}{\tau}\right) = \tau^{2\varkappa+2}E_{2\varkappa+2}(\tau), \tag{27}$$

i.e., the automorphy of $E_{2\varkappa+2}(z)$, cf. (25).

Thus, we have established.

Theorem 1. *The Bochner modular relation (14) entails at one end of the spectrum $\alpha = -(2\varkappa+1)$ Ramanujan's formula (17) and at the other end $\alpha = 2\varkappa+1$ the automorphy of the Eisenstein series (27), thus abridging analytic number theory and the theory of modular forms.*

(ii) Now we turn to the case of $a = 2\varkappa$. We digress from (12) which should be replaced by

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma_a(n)e^{-nx} &= 2(-1)^{\varkappa}(2\pi)^{-2s+a}\frac{\pi}{x}\frac{1}{2\pi i}\int_{(1-d)}\frac{\pi}{\sin\pi s}\Gamma \\ &\quad\cdot(s-a)\varphi(s)\left(\frac{4\pi^2 n}{x}\right)^{-s}ds \\ &\quad + 2(-1)^{\varkappa}(2\pi)^{-2s+a}\frac{\pi}{x}\frac{1}{2\pi i}\int_{(1-d)}\pi\cot\pi s\Gamma \\ &\quad\cdot(s-a)\varphi(s)\left(\frac{4\pi^2 n}{x}\right)^{-s}ds + P(x).\end{aligned} \tag{28}$$

In the same way as we have deduced (13), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma_a(n)e^{-nx} &= 2(-1)^{\varkappa}(2\pi)^{-2s+a}\frac{\pi}{x}\frac{1}{2\pi i}\int_{(1-d)}\frac{\pi}{\sin\pi s}\Gamma \\ &\quad\cdot(s-a)\varphi(s)\left(\frac{4\pi^2 n}{x}\right)^{-s}ds \\ &\quad + 2(-1)^{\varkappa}(2\pi)^{-2s+a}\frac{\pi}{x}\frac{1}{2\pi i}\int_{(1-d)}\pi\cot\pi s\Gamma \\ &\quad\cdot(s-a)\varphi(s)\left(\frac{4\pi^2 n}{x}\right)^{-s}ds + P(x).\end{aligned} \tag{29}$$

We shall stop here since it would be difficult to express the resulting integrals and Bellman's method [15] yields an asymptotic formula rather than an equality. Partial theory of modular relations for the product of zeta-functions is given in ([11], Chapter 9, pp. 241–265), which is still in progress.

Remark 1. That the odd integer difference case (i) reduces to the one-gamma factor case to the RHB correspondence is not coincidental and is expounded in [11] (pp. 81–86), where one can also find a plausible discovery of Ramanujan of the transformation formula for the Dedekind eta-function η. Weil's paper [22] is the most well-known paper that contains the proof of the latter, but prior to this, Chowla gave a proof [23] for the discriminant function, which is the 24th power of η. Ramanujan's formula is stated as I, Entry 15 of Chapter 16 [24], Entry 21 (i), Chapter 14 of Ramanujan's Notebook II [25] (which is and also as IV, Entry 20 of [26]). The most extensive account of information surrounding Ramanujan's formula is [27], while [28] is the most informative account of special values of the zeta-functions. The intersection of references in these two excellent survey papers (which have a lot in common) is a null set. In literature, Ramanujan's formula is stated for $\alpha = \pi x > 0, \beta = (\pi/x) > 0$ satisfying the following relation:

$$\alpha\beta = \pi^2, \tag{30}$$

and in terms of Lambert series.

The Lambert series $L(z)$ is defined for $|z| < 1$ by

$$L(z) = \sum_{n=1}^{\infty} a_n \frac{z^n}{1 - z^n}, \tag{31}$$

which is transformed into the Fourier series:

$$L(z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_n (z^m)^n = \sum_{\ell=1}^{\infty} b_\ell z^\ell, \tag{32}$$

the Liouville formula, where $b_\ell = \sum_{d|\ell} a_d$. The sum-of-divisors function is the case $a_n = n^\alpha$. Original Ramanujan's formula looks like having little to do with modular forms. Equation (17) being a rephrased Lambert series in Liouville's form has amenity to the q-expansion, and so to automorphy.

3. The Ramanujan–Guinand Formula and Its Consequences

In Section 2, we established that at both ends of the spectrum, the Bochner modular relation amounts to Ramanujan's formula and the automorphy of Eisenstein series, respectively. In this section, we partially follow [29], reproduced in [11] (pp. 86–92), and elucidate the mechanism hidden in the correspondence $\varphi(s) \longleftrightarrow \varphi(s-a)$ by differentiation of the Ramanujan–Guinand formula.

In [14], they mention duality between the space of weak Maass forms of (negative) weight $k \in (1/2)\mathbb{Z}$ and the space of holomorphic cusp form of (positive) weight $2-k$.

To this end, we introduce the *Mellin inversion with shifted argument*$I_a(x)$. Let $a \geq 0$ be a fixed integer to be taken as the number of times of differentiation throughout. Let $\varphi(s)$ be the zeta-function defined by (4) with $\alpha = 2\varkappa + 1$ and $\varkappa$ a nonnegative integer:

$$\varphi(s) = \zeta(s)\zeta(s + 2\varkappa + 1), \tag{33}$$

the other case being included in Theorem 2 below. The argument of its proof goes in the lines of Section 2. For $Rex > 0$ let

$$I_a(x) = I_{a,\varkappa}(x) = \frac{1}{2\pi i}\int_{(c)} (2\pi)^{-s}\Gamma(s)\varphi(s-a)x^{-s}\mathrm{d}s, \tag{34}$$

where $c > 1 + a$. The Hecke gamma transform reads

$$I_a(x) = \sum_{k=1}^{\infty} \sigma_{-2\varkappa-1}(k)k^a e^{-2\pi kx}. \tag{35}$$

The special case,

$$I(x) = I_0(x) = \frac{1}{2\pi i}\int_{(\kappa)} (2\pi)^{-s}\Gamma(s)\varphi(s)x^{-s}\mathrm{d}s = \sum_{k=1}^{\infty}\sigma_{-2\varkappa-1}(k)e^{-2\pi kx}, \tag{36}$$

is the Lambert series appearing in Ramanujan's formula (17). Differentiating $I(x)$ a-times with respect to x, whereby we perform differentiation under integral sign, we have the additional factor

$$\prod_{j=0}^{a-1}(-s-j), \tag{37}$$

which is $(-1)^a(\Gamma(s+a)/\Gamma(s))$, whence we deduce the remarkable formula:

$$\frac{\mathrm{d}^a}{\mathrm{d}x^a} I(x) = (-2\pi)^a I_a(x), \tag{38}$$

i.e., a-times differentiation of the Lambert series (36) is effected by shifting the argument of $\varphi(s)$ by a in (36) and multiplying by $(-2\pi)^a$. In view of (38), the a-times differentiated form of Ramanujan's formula (17) amounts to a counterpart of the modular relation for $I_a(x)$.

Theorem 2 *(Ramanujan–Guinand formula). For the Mellin transform $I_a(x)$ with shifted argument as defined by (34), we have the modular relation for $\varkappa \geq 0$ and $0 \leq a \leq 2\varkappa + 1$,*

$$\begin{aligned} I_a(x) = (-1)^{\varkappa+a}(2\pi)^{-a}x^{-a+2\varkappa}\sum_{k=0}^{a}\binom{a}{k}\frac{(2\varkappa-k)!}{(2\varkappa-a)!} \\ \cdot\left(\frac{2\pi}{x}\right)^k I_k\left(\frac{1}{x}\right) + P_a(x), \end{aligned} \tag{39}$$

where $P_a(x)$ is the residual function.

Proof. Proof depends on the following equation:

$$\varphi(s-a) = (-1)^n(2\pi)^{2\varkappa+2s-2a}\frac{\Gamma(a-2\varkappa-s)}{\Gamma(s-a)}\varphi(a-2\varkappa-s), \tag{40}$$

which is a variant of the functional equation (6) in the following form:

$$(2\pi)^{-s}\Gamma(s)\varphi(s) = (-1)^n(2\pi)^{2\varkappa+s}\Gamma(-2\varkappa-s)\varphi(-2\varkappa-s). \tag{41}$$

Moving the line of integration to $\sigma = -c_1\,(c_1 > 2\varkappa + 1 - a)$, we have

$$\begin{aligned} I_a(x) &= J_a(x) + P_a(x), \\ J_a(x) &= \frac{1}{2\pi i}\int_{(-c_1)} \Gamma(s)\varphi(s-a)(2\pi x)^{-s}\mathrm{d}s, \end{aligned} \tag{42}$$

where $P(x) = P_a(x)$ denotes the sum of residues of the integrand at its poles at $s = a - 2\varkappa - 1, a - 2\varkappa, a - 2\varkappa + 1, a - 2\varkappa + 3, \ldots, 0, a + 1$.

Substitute (40) and change the variable $s \longleftrightarrow a - 2\varkappa - s$ in the integral $J_a(x)$.

Then

$$J_a(x) = \frac{(-1)^\varkappa}{2\pi i}\int_{(a-2\varkappa+c_1)}\frac{\Gamma(a-2\varkappa-s)\Gamma(s)}{\Gamma(-2\varkappa-s)}\varphi(s)(2\pi)^{-s-a}x^{s-a+2\varkappa}\mathrm{d}s. \tag{43}$$

Substituting

TABLE 1: Poles and the sum of residues in the relevant domain.

Case	σ_φ	d	Poles	$P_\alpha(x)$
$\alpha > 0$	1	$d = \alpha - (1/2)$	$-\alpha, \ldots, -1, 0, 1$	(2.12)
$\alpha < 0$	$1 - \alpha$	$d = -(1/2)$	$0, 1, 1 - \alpha$	$\Gamma(1-\alpha)\zeta(1-\alpha)x^{\alpha-1} - (1/2)\zeta(\alpha)$

%

$$\begin{aligned}\frac{\Gamma(a-2\varkappa-s)}{\Gamma(-2\varkappa-s)} &= \frac{\Gamma(1+s+2\varkappa)}{\Gamma(1+s+2\varkappa-a)}\frac{\sin\pi s}{\sin\pi(s-a)}\\ &= (-1)^a\frac{\Gamma(a+2\varkappa+1)}{\Gamma(s+2\varkappa+1-a)},\end{aligned}\tag{44}$$

we find that

$$J_a(x) = \frac{(-1)^{\varkappa+a}}{2\pi i}\int_{(a-2\varkappa+c_1)}\frac{\Gamma(s+2n\varkappa+1)\Gamma(s)}{\Gamma(s+2\varkappa+1-a)}\varphi(s)(2\pi)^{-s-a}x^{s-a+2\varkappa}\mathrm{d}s.\tag{45}$$

Since the gamma factor can be computed as follows for $0 \le a \le 2\varkappa + 1$,

$$\frac{\Gamma(s)\Gamma(s+2\varkappa+1)}{\Gamma(s+2\varkappa+1-a)} = \sum_{k=0}^{a}\binom{a}{k}\frac{(2\varkappa-k)!}{(2\varkappa-a)!}\Gamma(s+k),\tag{46}$$

where for $a = 2\varkappa + 1$, the right-hand side is to mean $\Gamma(s + 2\varkappa + 1)$, we conclude from (42) and (45) that

$$\begin{aligned}I_a(x) &= (-1)^{\varkappa+a}(2\pi)^{-a}x^{-a+2\varkappa}\sum_{k=0}^{a}\binom{a}{k}\frac{(2\varkappa-k)!}{(2\varkappa-a)!}\\ &\quad\times\frac{1}{2\pi i}\int_{(a-2\varkappa+c_1)}\Gamma(s+k)\varphi(s)\left(\frac{2\pi}{x}\right)^{-s}\mathrm{d}s + P_a(x),\end{aligned}\tag{47}$$

where the sum reduces to 1 for $a - 2\varkappa + 1$. Finally, we note that the integral on the right-hand side of (47) becomes by the change of variable $s \longleftrightarrow s + k$

$$\left(\frac{2\pi}{x}\right)^k I_k\left(\frac{1}{x}\right).\tag{48}$$

Hence, (47) leads to (39), completing the proof. □

Corollary 1

(i) The case $a = 0$ is Ramanujan's formula (17) with the residual function

$$\begin{aligned}P(x) &= \frac{(2\pi)^{2\varkappa+1}}{2x}\sum_{j=0}^{\varkappa+1}(-1)^j\frac{B_{2j}}{(2j)!}\frac{B_{2\varkappa+2-2j}}{(2\varkappa+2-2j)!}x^{2\varkappa+2-2j}\\ &\quad+\begin{cases}-\dfrac{1}{2}\zeta(2\varkappa+1)\left\{1+(-1)^{\varkappa+1}x^{2\varkappa}\right\} & \text{if } \varkappa \ge 1,\\[2ex] \dfrac{1}{2}\log x & \text{if } \varkappa = 0,\end{cases}\end{aligned}\tag{49}$$

valid for $Re x > 0$.

(ii) The case $a = 2\varkappa$ is Guinand's formula, cf. ([29], Theorem 3) with

$$\begin{aligned}P_{2\varkappa}(x) &= (2\varkappa)!\zeta(2\varkappa+2)(2\pi x)^{-2\varkappa-1} + \frac{1}{2}\zeta(-2\varkappa-1)(2\pi x)\\ &\quad+\frac{(-1)^\varkappa}{2}(2\pi)^{-2\varkappa}(2\varkappa)!\zeta(2\varkappa+1), \quad \varkappa \ge 1.\end{aligned}\tag{50}$$

(iii) The case $a = 2\varkappa + 1$ reads

$$I_{2\varkappa+1}(x) = (-1)^{\varkappa+1}(2\pi)^{-2\varkappa-1}x^{-2\varkappa-2}I_{2\varkappa+1}\left(\frac{1}{x}\right) + P_{2\varkappa+1}(x),\tag{51}$$

where

$$P_{2\varkappa+1}(x) = (2\varkappa+1)!\zeta(2\varkappa+2)(2\pi x)^{-2\varkappa-2} - \frac{1}{2}\zeta(-2\varkappa-1)(\varkappa \ge 1).\tag{52}$$

These lead to (21), thence to the automorphy (27).

(iv) The special case of (50) with $\varkappa = 1$, i.e., once differentiated form of Ramanujan's formula, yields Terras' formula [30, 31]:

$$\zeta(3) = \frac{2}{45}\pi^3 - 4\sum_{k=1}^{\infty}e^{-2\pi k}\sigma_{-3}(k)\left(2\pi^2k^2 + \pi k + \frac{1}{2}\right).\tag{53}$$

Remark 2. The Mellin inversion with shifted argument (34) is an additive version of the "pseudomodular relation principle," which is a processed modular relation with the processing gamma factor $\Gamma(s + a)$ ([11], p. 50). In this special case, the Main Formula (38) is the manifestation of the statement of Razar that the differentiation of Lambert series essentially corresponds to the shift of the argument of the associated Dirichlet series.

Remark 3. It is shown ([11], pp. 82–84) that the special case $\varkappa = 0, a = 0$ of Theorem 2, the Ramanujan–Guinand formula, is nothing other than the automorphic property of the Dedekind eta-function, cf. Remark 1 above. This can be regarded as once differentiated form of Guinand's formula. We note that, in the case of $\varkappa = 0$, the twice differentiated Guinand's formula, with a suitable modification, coincides with the automorphic property of the Eisenstein series E_2:

$$E_2(\tau) + \frac{3}{\pi y} = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} = \frac{12}{\pi i} (\log \eta(\tau))'. \tag{54}$$

Equation (51), once differentiated form of Guinand's formula ($= (2n+1)$ times differentiated form of Ramanujan's formula=the negative case thereof) leads to (27), automorphy of $G_{2\varkappa+2}$.

4. Lipschitz Summation Formula

Knopp–Robbins [7] in their Remark 1 state their view on the Lipschitz summation formula to the effect that it is conceptually simpler than Riemann's original method of using the theta series. However, at least the special case of the Lipschitz summation formula (Theorem 3) which is applied by Chapman to establish the limit relation has already been used extensively and can be readily deduced from the partial fraction expansion for the cotangent function. Since it is known that the partial fraction expansion is equivalent to the functional equation for the Riemann zeta-function, we may say that Chapman's result is a consequence of the functional equation. By Corollary 4 below, we shall show that the Lipschitz summation formula itself is equivalent to the functional equation, thereby enhancing the above statement. Pasles and Pribitkin [32] extend the Lipschitz summation formula to the two-variable case to which we hope to return elsewhere.

Theorem 3 (Lipschitz summation formula). *For the complex variables $z = x + iy$, $x > 0$, $s = \sigma + it$, $\sigma > 1$ and the real parameter $0 < \alpha \leq 1$, we have the Lipschitz summation formula:*

$$\frac{(2\pi)^s}{\Gamma(s)} \sum_{m=0}^{\infty} (m+\alpha)^{s-1} e^{-2\pi z (m+\alpha)} = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n \alpha}}{(z+in)^s}. \tag{55}$$

Under the condition $0 < \alpha < 1$, this formula holds in the wider half-plane $\sigma > 0$.

In what follows, we shall generalize Theorem 3 in a wider framework, deducing Corollary 4 to Theorem 4 as a general modular relation.

Let

$$\Phi(w, s, z) = \sum_{n=0}^{\infty} \frac{w^n}{(n+z)^s}, \tag{56}$$

be the Hurwitz–Lerch zeta-function, and let

$$\phi(x, s, z) = \Phi\left(e^{2\pi i n x}, s, x\right) = \sum_{n=0}^{\infty} \frac{e^{2\pi i n x}}{(n+z)^s}, \tag{57}$$

denote the boundary function—the Lipschitz–Lerch transcendent (([9], pp. 59–62), ([2], pp. 128–131), [33]). This is in close correspondence with the case of $L_s(z)$ and $\ell_s(x)$ considered in [34]. As a consequence of the main formula in Appendix Section, we may deduce, from the reciprocal Hurwitz formula (64), a generalized Lipschitz summation formula (78) which is indeed a form of the functional equation.

Lemma 1. *The Dirichlet series*

$$\varphi(s) = \frac{1}{\pi^{s/2}} \ell_s(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi i x n}}{(\sqrt{\pi} n)^s}, \tag{58}$$

$$\begin{aligned} \psi_1(s) &= \frac{1}{2\pi^{(s/2)}} \zeta(s, x) + \frac{1}{2\pi^{(s/2)}} \zeta(s, 1-x) \\ &= \sum_{n=1}^{\infty} \left(\frac{(1/2)}{(\sqrt{\pi}\,(n+x-1))^s} + \frac{(1/2)}{(\sqrt{\pi}\,(n-x))^s} \right), \end{aligned} \tag{59}$$

$$\begin{aligned} \psi_2(s) &= -\frac{i}{2\pi^{(s/2)}} \zeta(s, x) + \frac{i}{2\pi^{(s/2)}} \zeta(s, 1-x) \\ &= \sum_{n=1}^{\infty} \left(\frac{-(i/2)}{(\sqrt{\pi}\,(n+x-1))^s} + \frac{(i/2)}{(\sqrt{\pi}\,(n-x))^s} \right), \end{aligned} \tag{60}$$

satisfy the ramified functional equation, a special case of (A.2) with $r = 1$

$$\begin{aligned} &\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\varphi(s) \\ &= \Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\psi_1(1-s) + \Gamma\left(-\frac{s}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\psi_2(1-s), \end{aligned} \tag{61}$$

or

$$\chi(s) = \begin{cases} \Gamma\left(s \,\middle|\, -\left(0,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right);;--\right)\varphi(s), \\ \Gamma\left(1-s \,\middle|\, \left(0,\frac{1}{2}\right)\left(0,\frac{1}{2}\right);;--\right)\psi_1(1-s) + \Gamma\left(1-s \,\middle|\, \left(-\frac{1}{2},\frac{1}{2}\right)\left(-\frac{1}{2},\frac{1}{2}\right);;--\right)\psi_2(1-s). \end{cases} \tag{62}$$

Proof. From Euler's formula, we have

$$\begin{cases} e^{(s/2)i(1-s)} = \sin\dfrac{s}{2}s + i\cos\dfrac{s}{2}s, \\ e^{-(s/2)i(1-s)} = \sin\dfrac{s}{2}s - i\cos\dfrac{s}{2}s. \end{cases} \tag{63}$$

We transform the reciprocal Hurwitz formula $(0 < x < 1)$:

$$\begin{aligned}\ell_s(x) &= \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{\frac{1-s}{2}\pi i}\zeta(1-s,x) + e^{-((1-s)/2)\pi i}\zeta(1-s,1-x)\right) \\ &= f_1(s)f_2(s),\end{aligned} \tag{64}$$

say, where $f_1(s) = (\Gamma(1-s)/(2\pi)^{1-s})$ is the first factor of the right-hand side member and $f_2(s)$ is the second. Substituting (63), the second factor becomes

$$\begin{aligned}f_2(s) &= \sin\frac{\pi}{2}s(\zeta(1-s,x) + \zeta(1-s,1-x)) \\ &\quad + \cos\frac{\pi}{2}s(\zeta(1-s,x) - \zeta(1-s,1-x)).\end{aligned} \tag{65}$$

In view of this, we transform the first factor accordingly

$$\begin{aligned}f_1(s) &= \frac{\pi^{s-(3/2)}}{2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\frac{\pi}{\sin(\pi/2)s} \\ \text{or } &= \frac{\pi^{s-(3/2)}}{2}\frac{\Gamma(1-(s/2))}{\Gamma((1+s)/2)}\frac{\pi}{\cos(\pi/2)s}.\end{aligned} \tag{66}$$

Substituting (65) and (66), we transform (64) into

$$\begin{aligned}\ell_s(x) &= \frac{\pi^{s-(1/2)}\Gamma((1/2)-(s/2))}{2\Gamma((s/2))}(\zeta(1-s,x) - \zeta(1-s,1-x)) \\ &\quad + i\frac{\pi^{s-(1/2)}\Gamma(1-(s/2))}{2\Gamma((s/2)+(1/2))}(\zeta(1-s,x) - \zeta(1-s,1-x)) \\ &= \frac{\pi^{-(s/2)}\Gamma((1/2)-(s/2))}{2\Gamma(s/2)}\psi_1(s) - \frac{\pi^{-(s/2)}\Gamma(1-(s/2))}{2\Gamma((s/2)+(1/2))}\psi_2(s),\end{aligned} \tag{67}$$

where ψ_i's are defined in (59) and (60). Clearing the denominators in (67),

$$\begin{aligned}\varphi(s) &= \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\frac{1}{\pi^{s/2}}l_s(x) \\ &= \Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\psi_1(s) - \Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\psi_2(s).\end{aligned} \tag{68}$$

It remains to note that the factor $-\Gamma(s/2)\Gamma(1-(s/2))$ of the second summand may be transformed into $\Gamma(-(s/2))\Gamma(1+(s/2))$, which is indeed the case in view of the reciprocity formula.

By Lemma 1, we have the ramified functional equation. The only pole of $\chi(s)$ which is neither a pole of $\Gamma((1/2)+(s/2))$ nor of $\Gamma(1+(s/2))$ is $s_1 = 0$. Thus, by main formula (A.13), we have the following modular relation. For the notation and formulas, we refer to ([11], pp. 49–59). □

Theorem 4

$$\begin{aligned}&\frac{1}{\pi^{(s/2)}}\sum_{n=1}^{\infty}\frac{e^{2\pi inx}}{n^s}H\left(z\sqrt{\pi}n\,|\,\Delta\oplus\begin{pmatrix} - & ;- \\ ((s/2),(1/2)),((1/2)+(s/2),(1/2)); - & \end{pmatrix}\right) \\ &= \frac{1}{2\pi^{(1-s/2)}}\sum_{n=1}^{\infty}\left\{\frac{1}{(n+x-1)^{1-s}}H\left(\frac{\sqrt{\pi}(n+x-1)}{z}\,|\,\Delta^*\oplus\begin{pmatrix}((1-s/2),(1/2)); - \\ ((1-s/2),(1/2)); -\end{pmatrix}\right)\right. \\ &\quad\left. + \frac{1}{(n-x)^{1-s}}H\left(\frac{\sqrt{\pi}(n-x)}{z}\,|\,\Delta^*\oplus\begin{pmatrix}((1-s/2),(1/2)); - \\ ((1-s/2),(1/2)); -\end{pmatrix}\right)\right\} \\ &\quad + \frac{1}{2\pi^{(1-s)/2}}\sum_{k=1}^{\infty}\left\{\frac{-i}{(n+x-1)^{1-s}}H\left(\frac{\sqrt{\pi}(n+x-1)}{z}\,|\,\Delta^*\oplus\begin{pmatrix}(-(s/2),(1/2)); - \\ (-(s/2),(1/2)); -\end{pmatrix}\right)\right. \\ &\quad\left. + \frac{i}{(n-x)^{1-s}}H\left(\frac{\sqrt{\pi}(n-x)}{z}\,|\,\Delta^*\oplus\begin{pmatrix}(-(s/2),(1/2)); - \\ (-(s/2),(1/2)); -\end{pmatrix}\right)\right\} + \operatorname{Res}\left(\Gamma(w-s\,|\,\Delta)\chi(w)z^{s-w}, w=0\right) \\ &= \frac{1}{2\pi^{(1-s)/2}}\sum_{n=1}^{\infty}\left\{\frac{1}{(n+x-1)^{1-s}}\left\{H\left(\frac{\sqrt{\pi}(n+x-1)}{z}\,|\,\Delta^*\oplus\begin{pmatrix}((1-s/2),(1/2)); - \\ ((1-s/2),(1/2)); -\end{pmatrix}\right)\right.\right.\end{aligned}$$

$$-iH\left(\frac{\sqrt{\pi}\,(k+a-1)}{z}\,\Big|\,\Delta^{*}\oplus\begin{pmatrix}(-(s/2),\,(1/2));-\\(-(s/2),\,(1/2));-\end{pmatrix}\right)\Bigg\}$$
$$+\frac{1}{(n-x)^{1-s}}\left\{H\left(\frac{\sqrt{\pi}\,(n-x)}{z}\,\Big|\,\Delta^{*}\oplus\begin{pmatrix}((1-s/2),\,(1/2));-\\((1-s/2),\,(1/2));-\end{pmatrix}\right)+iH\left(\frac{\sqrt{\pi}\,(n-x)}{z}\,\Big|\,\Delta^{*}\oplus\begin{pmatrix}(-(s/2),\,(1/2));-\\(-(s/2),\,(1/2));-\end{pmatrix}\right)\right\}\Bigg\}$$
$$+\operatorname{Res}\left(\Gamma(w-s\,|\,\Delta)\chi(w)z^{s-w},w=0\right). \tag{69}$$

Corollary 2. *If* $\Delta=\begin{pmatrix}(1,1)\ ;\ -\\-\ ;\ -\end{pmatrix}$, *we have* $\Delta^{*}=\begin{pmatrix}-\ ;\ -\\(0,1)\ ;\ -\end{pmatrix}$, *and so Theorem 4 amounts to*

$$\frac{1}{\pi^{(s/2)}}\sum_{n=1}^{\infty}\frac{e^{2\pi i n a}}{n^{s}}H_{2,1}^{1,2}\left(z\sqrt{\pi}n\,\middle|\,\begin{matrix}(1,1)\\\left(\frac{s}{2},\frac{1}{2}\right),\left(\frac{1}{2}+\frac{s}{2},\frac{1}{2}\right)\end{matrix}\right)$$
$$=\frac{1}{2\pi^{(1-s/2)}}\sum_{n=1}^{\infty}\left\{\frac{1}{(n+a-1)^{1-s}}\left\{H_{1,2}^{2,1}\left(\frac{\sqrt{\pi}\,(n+a-1)}{z}\,\middle|\,\begin{matrix}\left(\frac{1-s}{2},\frac{1}{2}\right)\\(0,1),\left(\frac{1-s}{2},\frac{1}{2}\right)\end{matrix}\right)\right.\right.$$
$$\left.-iH_{1,2}^{2,1}\left(\frac{\sqrt{\pi}\,(n+a-1)}{z}\,\middle|\,\begin{matrix}\left(-\frac{s}{2},\frac{1}{2}\right)\\(0,1),\left(-\frac{s}{2},\frac{1}{2}\right)\end{matrix}\right)\right\}$$
$$+\frac{1}{(n-a)^{1-s}}\left\{H_{1,2}^{2,1}\left(\frac{\sqrt{\pi}\,(n-a)}{z}\,\middle|\,\begin{matrix}\left(\frac{1-s}{2},\frac{1}{2}\right)\\(0,1),\left(\frac{1-s}{2},\frac{1}{2}\right)\end{matrix}\right)\right.$$
$$\left.\left.+iH_{1,2}^{2,1}\left(\frac{\sqrt{\pi}\,(n-a)}{z}\,\middle|\,\begin{matrix}\left(-\frac{s}{2},\frac{1}{2}\right)\\(0,1),\left(-\frac{s}{2},\frac{1}{2}\right)\end{matrix}\right)\right\}\right\}$$
$$+\operatorname{Res}\left(\Gamma(s-w)\chi(w)z^{s-w},w=0\right). \tag{70}$$

Corollary 3 (Ewald expansion).

$$\phi(x,s,z)=\sum_{n=1}^{\infty}\frac{e^{2\pi i n x}}{(n+z)^{s}}+\frac{1}{z^{s}}$$
$$=\sum_{n=1}^{\infty}\left(\frac{e^{-2\pi(n+x-1)zi}}{\left(2\pi(n+x-1)e^{-(\pi i/2)}\right)^{1-s}}\Gamma(1-s,-2\pi(n+x-1)zi)+\frac{e^{2\pi(n-x)zi}}{\left(2\pi(n-x)e^{\pi i/2}\right)^{1-s}}\Gamma(1-s,2\pi(n-x)zi)\right) \tag{71}$$
$$+\frac{1}{2z^{s}},$$

where $\Gamma(s,c)$ is the incomplete gamma function defined as follows:

$$\Gamma(s,c) = \Gamma(s) - c^s \int_0^1 e^{-cu} u^{s-1} \mathrm{d}u. \tag{72}$$

Proof. We transform H-functions in Corollary 2 in a concrete form. The procedure is the same for the three H-functions, i.e., duplication formula, $H \longrightarrow G$ formula, and the explicit formula for the G-function, and we use the known results on them freely, cf. [11, 35], etc. We have

$$H_{2,1}^{1,2}\left(z \,\middle|\, \begin{matrix} (1,1) \\ \left(\frac{s}{2},\frac{1}{2}\right),\left(\frac{1}{2}+(s/2),\frac{1}{2}\right) \end{matrix}\right) = 2^{1-s}\sqrt{\pi} G_{1,1}^{1,1}\left(2z \,\middle|\, \begin{matrix} 1 \\ s \end{matrix}\right)$$
$$= 2^{1-s}\sqrt{\pi}\Gamma(s)\frac{(2z)^s}{(1+2z)^s}, \tag{73}$$

$$H_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \left(\frac{1}{2}-\frac{s}{2},\frac{1}{2}\right) \\ (0,1),\left(\frac{1}{2}-\frac{s}{2},\frac{1}{2}\right) \end{matrix}\right)$$
$$= \frac{1}{2\sqrt{\pi}} H_{1,3}^{3,1}\left(\frac{z}{2} \,\middle|\, \begin{matrix} \left(\frac{1}{2}-\frac{s}{2},\frac{1}{2}\right) \\ \left(0,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2}-\frac{s}{2},\frac{1}{2}\right) \end{matrix}\right)$$
$$= \frac{1}{\sqrt{\pi}} G_{1,3}^{3,1}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{1-s}{2} \\ 0,\frac{1}{2},\frac{1-s}{2} \end{matrix}\right)$$
$$= \Gamma(s)\left(e^{((1-s)/2)\pi i - zi}\Gamma(1-s,-zi) + e^{-((1-s)/2)\pi i + zi}\Gamma(1-s,zi)\right), \tag{74}$$

and similarly to (74)

$$H_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \left(-\frac{s}{2},\frac{1}{2}\right) \\ (0,1),\left(-\frac{s}{2},\frac{1}{2}\right) \end{matrix}\right)$$
$$= \Gamma(s+1)\left(e^{-(s/2)\pi i - zi}\Gamma(-s,-zi) + e^{(s/2)\pi i + zi}\Gamma(-s,zi)\right)$$
$$= \Gamma(s)\left(ie^{(1-s/2)\pi i - zi}\Gamma(1-s,-zi) - ie^{-(1-s/2)\pi i + zi}\Gamma(1-s,zi) + \frac{2}{z^s}\right). \tag{75}$$

Hence, (71) amounts to

$$\frac{1}{(2\sqrt{\pi})^{s-1}}\sum_{n=1}^{\infty}\frac{e^{2\pi inx}}{(n+\xi)^s}$$
$$= \frac{1}{\pi^{((1-s)/2)}}\sum_{n=1}^{\infty}\left\{\frac{e^{((1-s)/2)\pi i}e^{-2\pi(n+x-1)\xi i}}{(n+x-1)^{1-s}}\Gamma\right.$$
$$\cdot(1-s,-2\pi(n+x-1)\xi i)$$
$$+\frac{e^{-((1-s)/2)\pi i}e^{2\pi(n-x)\xi i}}{(n-x)^{1-s}}\Gamma(1-s,2\pi(n-x)\xi i) \tag{76}$$
$$\left.-\frac{i}{(2\pi\xi)^s(n+x-1)}+\frac{i}{(2\pi\xi)^s(n-x)}\right\}$$
$$+\frac{1}{(2\sqrt{\pi})^{s-1}\xi^s}\ell_0(x),$$

where $\xi = (1/2\sqrt{\pi}z)$. Since

$$\ell_0(x) = -\frac{1}{2}+\frac{i}{2\pi}\left(\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{1}{x+n}+\frac{1}{x-n}\right)\right), \tag{77}$$

equation (76) leads to the incomplete gamma series (71) for (57).

We are now in a position to transform (71) into the functional equation. □

Corollary 4 (General Lipschitz summation formula)

$$\phi(x,s,z)$$
$$= \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{(1-s/2)\pi i - 2\pi xzi}\phi(-z,1-s,x)\right. \tag{78}$$
$$\left.+e^{-(1-s/2)\pi i + 2\pi(1-x)zi}\phi(z,1-s,1-x)\right),$$

where $0<z,x<1$, *and* $t=-z(u-1)$.

Proof. By (72), we have the left-hand side of (71) is equal to

$$= \Gamma(1-s)\sum_{k=1}^{\infty}\left(\frac{e^{-2\pi(n+x-1)zi}}{(2\pi(n+x-1)e^{-(\pi i/2)})^{1-s}}+\frac{e^{2\pi(n-x)zi}}{(2\pi(n-x)e^{(\pi i/2)})^{1-s}}\right)$$
$$-\frac{1}{z^{s-1}}\sum_{k=1}^{\infty}\int_0^1\left(e^{2\pi(n+x-1)zi(u-1)}+e^{-2\pi(n-x)zi(u-1)}\right)u^{-s}\mathrm{d}u+\frac{1}{2z^s}$$
$$= \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{(1-s)/2\pi i - 2\pi xzi}\phi(-z,1-s,x)\right.$$
$$\left.+e^{-(1-s/2)\pi i + 2\pi(1-z)zi}\phi(z,1-s,1-x)\right)$$
$$-\frac{1}{z^s}\sum_{k\in\mathbb{Z}}\int_0^z e^{2\pi(n-x)ti}\left(1-\frac{t}{z}\right)^{-s}\mathrm{d}t+\frac{1}{2z^s}. \tag{79}$$

Equation (78) is sometimes referred to as the Lipschitz summation formula ([36–38] etc.).

The character analogue of the Lipschitz summation formula is known ([2], pp. 128–131), and so we may naturally treat a more general case which will be conducted elsewhere.

We briefly state the Lipschitz summation formula for quadratic fields which is contained in ([11], Chapter 4). This is an elucidation of Koshlyakov's results [39–42]. Let Ω be a quadratic field whose degree $\varkappa = r_1 + r_2 = 2$ with its discriminant Δ. Let

$$A = \frac{2^{r_2}\pi^{\varkappa/2}}{\sqrt{|\Delta|}}, \tag{80}$$

and let $r = r_1 + r_2 - 1$ denote the rank of the unit group. Let $\zeta_\Omega(s)$ be the Dedekind zeta-function of Ω with the coefficients $\alpha_k = \beta_k = \alpha(k)$, with $\alpha(k)$ indicating the number of ideals of norm k, where $\lambda_k = Ak$, $\mu_k = Ak$, $r = 1$, and

$$\rho = \frac{2^{r+1}\pi^{r_2}Rh}{w\sqrt{|\Delta|}} = -\frac{2^{r+1}\pi^{r_2}\zeta_\Omega^{(r)}(s)}{\sqrt{|\Delta|}}. \tag{81}$$

The functional equation reads

$$\Gamma^{r_1}\left(\frac{1}{2}s\right)\Gamma^{r_2}(s)\varphi(s) = \Gamma^{r_1}\left(\frac{1}{2}-\frac{1}{2}s\right)\Gamma^{r_2}(1-s)\varphi(1-s), \tag{82}$$

where

$$\varphi(s) = A^{-s}\zeta_\Omega(s) = \sum_{n=1}^{\infty}\frac{\alpha(n)}{(An)^s}. \tag{83}$$

In conformity with Koshlyakov ([40], p. 241) (cf. (83)), we introduce the perturbed Dedekind zeta-function:

$$\zeta_\Omega(s,z) = -\frac{2\zeta_\Omega(0)}{w} + \sum_{n=1}^{\infty}\frac{\alpha(n)}{(n+z)^s}. \tag{84}$$

Then the Fourier–Bessel expansion gives the Lipschitz summation formula for an imaginary quadratic field ([11], (4.42)):

$$\begin{aligned}\sum_{n=-\infty}^{\infty}{}' \frac{\alpha(n)}{(z+in)^s} &= e^{(\pi i/2)s}\zeta_\Omega(s,iz) + e^{-(\pi i/2s)}\zeta_\Omega(s,-iz)\\ &= A^s\frac{z^{(1/2)(1-s)}}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{\alpha(n)}{n^{(1/2)(1-s)}}\Big(\varepsilon^{s+1}K_{1-s}(2A\varepsilon\sqrt{nz})\\ &\quad + \bar{\varepsilon}^{s+1}K_{1-s}(2A\bar{\varepsilon}\sqrt{nz})\Big)\\ &\quad - \frac{\zeta_\Omega(0)}{z^s},\end{aligned} \tag{85}$$

which is the corrected form of (23.15) in [40] , where $\varepsilon = e^{(\pi/4)i}$ and the prime on the summation sign means that the term with $n = 0$ is excluded. □

Theorem 5 (Theorem 4.6 in [11]). *The generating Dirichlet series for the Wigert–Bellman divisor problem, cf. (29) above,*

$$\begin{aligned}\zeta_\Omega(s,z) &= \frac{z^{-s}}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{\alpha(n)}{n}\frac{1}{2\pi i}\\ &\quad\cdot\int_{(c)}\tan\frac{\pi}{2}w\Gamma(1-w)\Gamma(s-w)\left(\frac{1}{nz}\right)^{-w}dw,\end{aligned} \tag{86}$$

amounts to the Lipschitz summation formula

$$\begin{aligned}\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{\alpha(n)}{(z+in)^s} &= \frac{1}{2}e^{(\pi i/2)s}\zeta_\Omega(s,iz) + \frac{1}{2}e^{-(\pi i/2)s}\zeta_\Omega(s,-iz)\\ &= A^s\frac{z^{(1/2)(1-s)}}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{\alpha(n)}{n^{(1/2)(1-s)}}\left(\frac{\varepsilon^{s+1}}{i}K_{1-s}(2A\varepsilon\sqrt{nz})\right.\\ &\quad\left.-\frac{\bar{\varepsilon}^{s+1}}{i}K_{1-s}(2A\bar{\varepsilon}\sqrt{nz})\right).\end{aligned} \tag{87}$$

This is the corrected form of (23.16) in [40].

5. Ramified Functional Equations

There are some instances of the ramified functional equations in literature.

In [43–45], they are stated in the case of zeta-functions with periodic coefficients which satisfy the ramified functional equations as a result of representations in bases consisting of the Hurwirz and Lerch zeta-functions [18]: suppose $f(n)$ be a periodic function with period M,

$$L(s,f) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s}, \tag{88}$$

be the associated Dirichlet series absolutely convergent $\sigma = Res > 1$ and that

$$\begin{aligned}f_{\text{odd}} &= \frac{1}{2}(f(n \bmod M) - f(-n \bmod M)),\\ f_{\text{even}} &= \frac{1}{2}(f(n \bmod M) + f(-n \bmod M)),\end{aligned} \tag{89}$$

be odd, resp. even part of f: $f = f_{\text{even}} + f_{\text{odd}}$. Then

$$\begin{aligned}L(1-s,f) &= \left(\frac{\pi}{M}\right)^{(1/2)-s}\left(\frac{\Gamma(s/2)}{\Gamma(1-s/2)}L\left(s,\hat{f}_{\text{even}}\right)\right.\\ &\quad\left.+\frac{\Gamma((1+s)/2)}{\Gamma(1-(s/2))}L\left(s,\hat{f}_{\text{odd}}\right)\right),\end{aligned} \tag{90}$$

which amounts to (61) on clearing the denominators and multiplying by $(\pi/M)^{(s-1)/2}$. Wang and Banerjee [46] treat the product of Hurwitz zeta-functions which satisfy a ramified functional equation as a result of the Hurwitz formula:

$$\zeta(1-s,x) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-(\pi i s/2)}\ell_s(x) + e^{(\pi i s/2)}\ell_s(1-x)\right), \tag{91}$$

whose reciprocal (64) has been used and proved to be a ramified functional equation in the proof of Lemma 1 above. More general case is that of the Hurwitz–Lerch zeta-function described in [35] (pp. 27–31) and [33] (pp. 121–126, 339–341). Another class of zeta-functions that satisfy a ramified functional equations is that of Barnes multiple zeta-functions cf. [33] (pp. 77–88) and [47]. For their rich applications [48] and references therein. Hardy and Littlewood [49, 50] use the Barnes double zeta-function.

In another context, Estermann [51] and others and in Prehomogeneous Vector Space (PHV) theory [52].

Appendix

The Main Formula

In this section, we state a special case ($H = 1$) of the Main Formula in [11] (Section 4.4, pp. 115–122) first proved in [53] which was used in the proof of Theorem 4. We have two sets of Dirichlet series $\{\phi(s)\}$ and $\{\psi_i(s)\}$, $1 \le i \le I$ that satisfy the generalized functional equation (A.2) in the following sense.

With increasing sequences $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k^{(i)}\}_{k=1}^{\infty}$ $(1 \le i \le I)$ and complex sequences $\{\alpha_k\}_{k=1}^{\infty}$, $\{\beta_k^{(i)}\}_{k=1}^{\infty}$ $(1 \le i \le I)$, we form the Dirichlet series:

$$\begin{gathered}\varphi(s) = \sum_{k=1}^{\infty} \frac{\alpha_k}{\lambda_k^s}, \\ \psi_i(s) = \sum_{k=1}^{\infty} \frac{\beta_k^{(i)}}{\mu_k^{(i)s}}, \\ 1 \le i \le I,\end{gathered} \tag{A.1}$$

which we suppose have finite abscissa of absolute convergence $\sigma_\varphi, \sigma_{\psi_i}$ $(1 \le i \le I)$, respectively.

We assume the existence of the meromorphic function χ, which satisfies, for a real number r, the functional equation:

$$\chi(s) = \begin{cases} \dfrac{\prod_{j=1}^{M}\Gamma\left(d_j + D_j s\right)\prod_{j=1}^{N}\Gamma\left(c_j - C_j s\right)}{\prod_{j=N+1}^{P}\Gamma\left(c_j + C_j s\right)\prod_{j=M+1}^{Q}\Gamma\left(d_j - D_j s\right)}\varphi_h(s), \quad Re(s) > \sigma_\varphi, \\ \displaystyle\sum_{i=1}^{I}\prod_{j=1}^{\widetilde{N}^{(i)}}\Gamma\left(e_j^{(i)} + E_j^{(i)}(r-s)\right)\frac{\prod_{j=1}^{\widetilde{M}^{(i)}}\Gamma\left(f_j^{(i)} - F_j^{(i)}(r-s)\right)}{\prod_{j=\widetilde{M}^{(i)}+1}^{\widetilde{Q}^{(i)}}\Gamma\left(-b \pm \sqrt{b^2-4ac}/2a f_j^{(i)} + F_j^{(i)}(r-s)\right)\prod_{j=\widetilde{N}^{(i)}+1}^{\widetilde{P}^{(i)}}\Gamma\left(e_j^{(i)} - E_j^{(i)}(r-s)\right)}\psi_i(r-s), Re(s) < \min_{1\le i\le I}\left(r - \sigma_{\psi_i}\right), \left(C_j, D_j, E_j^{(i)}, F_j^{(i)} > 0\right). \end{cases} \tag{A.2}$$

We assume further that only finitely many of the poles s_k $(1 \le k \le L)$ of $\chi(s)$ are neither a pole of

$$\frac{\prod_{j=1}^{N}\Gamma\left(c_j - C_j s\right)}{\prod_{j=N+1}^{P}\Gamma\left(c_j + C_j s\right)\prod_{j=M+1}^{Q}\Gamma\left(d_j - D_j s\right)}, \tag{A.3}$$

nor a pole of

$$\frac{\prod_{j=1}^{\widetilde{M}^{(i)}}\Gamma\left(f_j^{(i)} - F_j^{(i)} r + F_j^{(i)} s\right)}{\prod_{j=\widetilde{N}^{(i)}+1}^{\widetilde{P}^{(i)}}\Gamma\left(e_j^{(i)} - E_j^{(i)} r + E_j^{(i)} s\right)\prod_{j=\widetilde{M}^{(i)}+1}^{\widetilde{Q}^{(i)}}\Gamma\left(f_j^{(i)} + F_j^{(i)} r - F_j^{(i)} s\right)}. \tag{A.4}$$

We introduce the processing gamma factor:

$$\Gamma(w \mid \Delta) = \frac{\prod_{j=1}^{m}\Gamma\left(b_j + B_j w\right)\prod_{j=1}^{n}\Gamma\left(a_j - A_j w\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j + A_j w\right)\prod_{j=m+1}^{q}\Gamma\left(b_j - B_j w\right)}, \quad \left(A_j, B_j > 0\right), \tag{A.5}$$

and suppose that for any real numbers u_1, u_2 $(u_1 < u_2)$,

$$\lim_{|v| \longrightarrow \infty}\Gamma(u + iv - s \mid \Delta)\chi(u + iv) = 0, \tag{A.6}$$

uniformly in $u_1 \le u \le u_2$.

We choose $L_1(s)$ so that the poles of

$$\frac{\prod_{j=1}^{n}\Gamma\left(a_j + A_j s - A_j w\right)\prod_{j=1}^{N}\Gamma\left(c_j - C_j w\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j - A_j s + A_j w\right)\prod_{j=N+1}^{P}\Gamma\left(c_j + C_j w\right)} \times \frac{1}{\prod_{j=m+1}^{q}\Gamma\left(b_j + B_j s - B_j w\right)\prod_{j=M+1}^{Q}\Gamma\left(d_j - D_j w\right)}, \tag{A.7}$$

lie on the right of $L_1(s)$, and those of

$$\frac{\prod_{j=1}^{m}\Gamma\left(b_j - B_j s + B_j w\right)\prod_{j=1}^{M}\Gamma\left(d_j + D_j w\right)}{\prod_{j=m+1}^{q}\Gamma\left(b_j + B_j s - B_j w\right)\prod_{j=M+1}^{Q}\Gamma\left(d_j - D_j w\right)} \times \frac{1}{\prod_{j=n+1}^{p}\Gamma\left(a_j - A_j s + A_j w\right)\prod_{j=N+1}^{P}\Gamma\left(c_j + C_j w\right)}, \tag{A.8}$$

lie on the left of $L_1(s)$, and choose $L_2(s)$ so that the poles of

$$\frac{\prod_{j=1}^{m}\Gamma\left(b_j - B_j s + B_j w\right)\prod_{j=1}^{\widetilde{M}^{(i)}}\Gamma\left(f_j^{(i)} - F_j^{(i)} r + F_j^{(i)} w\right)}{\prod_{j=m+1}^{q}\Gamma\left(b_j + B_j s - B_j w\right)\prod_{j=\widetilde{M}^{(i)}+1}^{\widetilde{Q}^{(i)}}\Gamma\left(f_j^{(i)} + F_j^{(i)} r - F_j^{(i)} w\right)} \times \frac{1}{\prod_{j=n+1}^{p}\Gamma\left(a_j - A_j s + A_j w\right)\prod_{j=\widetilde{N}^{(i)}+1}^{\widetilde{P}^{(i)}}\Gamma\left(e_j^{(i)} - E_j^{(i)} r + E_j^{(i)} w\right)}, \tag{A.9}$$

lie on the left of $L_2(s)$, and those of

$$\frac{\prod_{j=1}^{n}\Gamma\left(a_j + A_j s - A_j w\right)\prod_{j=1}^{\widetilde{N}^{(i)}}\Gamma\left(e_j^{(i)} + E_j^{(i)} r - E_j^{(i)} w\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j - A_j s + A_j w\right)\prod_{j=\widetilde{N}^{(i)}+1}^{\widetilde{P}^{(i)}}\Gamma\left(e_j^{(i)} - E_j^{(i)} r + E_j^{(i)} w\right)} \times \frac{1}{\prod_{j=m+1}^{q}\Gamma\left(b_j + B_j s - B_j w\right)\prod_{j=\widetilde{M}^{(i)}+1}^{\widetilde{Q}^{(i)}}\Gamma\left(f_j^{(i)} + F_j^{(i)} r - F_j^{(i)} w\right)}, \tag{A.10}$$

lie on the right of $L_2(s)$. Further, they squeeze a compact set S such that $s_k \in S (1 \le k \le L)$. Under these conditions, we define the χ-function, key-function, $X(z, s \mid \Delta)$ by

$$X(z, s \mid \Delta) = \frac{1}{2\pi i}\int_{L_1(s)} \Gamma(w - s \mid \Delta)\chi(w) z^{-w} \mathrm{d}w, \tag{A.11}$$

where $\Gamma(s \mid \Delta)$ is the processing gamma factor (A.5).

Then, we have the following modular relation, equivalent to the functional equation (A.2):

The Main Formula H:

$$X(z, s \mid \Delta) =$$
$$\begin{cases} \sum_{k=1}^{\infty} \alpha_k H_{p+P,q+Q}^{m+M,n+N}\left(z\lambda_k \,\middle|\, \begin{matrix} \left\{\left(1 - a_j - A_j s, A_j\right)\right\}_{j=1}^{n}, \left\{\left(1 - c_j, C_j\right)\right\}_{j=1}^{N}, \left\{\left(a_j - A_j s, A_j\right)\right\}_{j=n+1}^{p}, \left\{\left(c_j, C_j\right)\right\}_{j=N+1}^{P} \\ \left\{\left(b_j - B_j s, B_j\right)\right\}_{j=1}^{m}, \left\{\left(d_j, D_j\right)\right\}_{j=1}^{M}, \left\{\left(1 - b_j - B_j s, B_j\right)\right\}_{j=m+1}^{q}, \left\{\left(1 - d_j, D_j\right)\right\}_{j=M+1}^{Q} \end{matrix} \right) \\ \text{if } L_1(s) \text{ can be taken to the right of } \sigma_{\varphi} \\ z^{-r}\sum_{i=1}^{I}\sum_{k=1}^{\infty} \beta_k^{(i)} H_{q+\widetilde{Q}^{(i)},p+\widetilde{P}^{(i)}}^{n+\widetilde{N}^{(i)},m+\widetilde{M}^{(i)}}\left(\frac{\mu_k^{(i)}}{z} \,\middle|\, \begin{matrix} \left\{\left(1 - b_j - B_j(r - s), B_j\right)\right\}_{j=1}^{m}, \left\{\left(1 - f_j^{(i)}, F_j^{(i)}\right)\right\}_{j=1}^{\widetilde{M}^{(i)}}, \left\{\left(b_j - B_j(r - s), B_j\right)\right\}_{j=m+1}^{q}, \left\{\left(f_j^{(i)}, F_j^{(i)}\right)\right\}_{j=\widetilde{M}^{(i)}+1}^{\widetilde{Q}^{(i)}} \\ \left\{\left(a_j - A_j(r - s), A_j\right)\right\}_{j=1}^{n}, \left\{\left(e_j^{(i)}, E_j^{(i)}\right)\right\}_{j=1}^{\widetilde{N}^{(i)}}, \left\{\left(1 - a_j - A_j(r - s), A_j\right)\right\}_{j=n+1}^{p}, \left\{\left(1 - e_j^{(i)}, E_j^{(i)}\right)\right\}_{j=\widetilde{N}^{(i)}+1}^{\widetilde{P}^{(i)}} \end{matrix} \right) \\ + \sum_{k=1}^{L} Res\left(\Gamma(w - s \mid \Delta)\chi(w) z^{-w}, w = s_k\right) \\ \text{if } L_2(s) \text{ can be taken to the left of } \min_{1 \le i \le I}\left(r - \sigma_{\psi_i}\right), \end{cases} \tag{A.12}$$

$$z^s X(z,s\,|\,\Delta) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{\alpha_k}{\lambda_k^s} H_{p+P,q+Q}^{m+M,n+N}\left(z\lambda_k \,\middle|\, \begin{matrix} \{(1-a_j,A_j)\}_{j=1}^n, \{(1-c_j+C_js,C_j)\}_{j=1}^N, \\ \{(b_j,B_j)\}_{j=1}^m, \{(d_j+D_js,D_j)\}_{j=1}^M, \end{matrix} \right. \\ \left. \begin{matrix} \{(a_j,A_j)\}_{j=n+1}^p, \{(c_j+C_js,C_j)\})\}_{j=N+1}^P \\ \{(1-b_j,B_j)\}_{j=m+1}^q, \{(1-d_j+D_js,D_j)\}_{j=M+1}^Q \end{matrix} \right) \\ \text{if } L_1(s) \text{ can be taken to the right of } \sigma_\varphi \\ \displaystyle\sum_{i=1}^{I}\sum_{k=1}^{\infty} \frac{\beta_k^{(i)}}{\mu_k^{(i)r-s}} H_{q+\widetilde{Q}^{(i)},p+\widetilde{P}^{(i)}}^{n+\widetilde{N}^{(i)},m+\widetilde{M}^{(i)}}\left(\frac{\mu_k^{(i)}}{z} \,\middle|\, \begin{matrix} \{(1-b_j,B_j)\}_{j=1}^m, \{(1-f_j^{(i)}+F_j^{(i)}(r-s),F_j^{(i)})\}_{j=1}^{\widetilde{M}^{(i)}}, \\ \{(a_j,A_j)\}_{j=1}^n, \{(e_j^{(i)}+E_j^{(i)}(r-s),E_j^{(i)})\}_{j=1}^{\widetilde{N}^{(i)}}, \end{matrix} \right. \\ \left. \begin{matrix} \{(b_j,B_j)\}_{j=m+1}^q, \{(f_j^{(i)}+F_j^{(i)}(r-s),F_j^{(i)})\}_{j=\widetilde{M}^{(i)}+1}^{\widetilde{Q}^{(i)}} \\ \{(1-a_j,A_j)\}_{j=n+1}^p, \{(1-e_j^{(i)}+E_j^{(i)}(r-s),E_j^{(i)})\}_{j=\widetilde{N}^{(i)}+1}^{\widetilde{P}^{(i)}} \end{matrix} \right) \\ \displaystyle + \sum_{k=1}^{L} Res\left(\Gamma(w-s\,|\,\Delta)\chi(w)z^{s-w}, w=s_k\right) \\ \text{if } L_2(s) \text{ can be taken to the left of } \min\limits_{1\le i\le I}\left(r-\sigma_{\psi_i}\right). \end{cases} \tag{A.13}$$

Data Availability

No data were used to support this study.

Disclosure

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Existence of Positive Weak Solutions for Quasi-Linear Kirchhoff Elliptic Systems via Sub-Supersolutions Concept

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By using the sub- and supersolutions concept (Schmitt, 2007), we prove in this paper the existence of positive solutions of quasi-linear Kirchhoff elliptic systems in bounded smooth domains. This work is an extension of the recent work of Boulaaras et al., 2020.

1. Introduction

The scope of nonlinear partial differential equations is quite wide. One of the main advances in the development of nonlinear PDEs has been the study of wave propagation, then comes the equations related to chemical and biological phenomena, and later, the equations related to solid mechanics, fluid dynamics, acoustics, nonlinear optics, plasma physics, quantum field theory, and engineering.

Studying these equations is a daunting task because there are no general methods for solving them. Each problem requires an appropriate approach depending on the type of linearity ([1–10]).

The p-Laplacian operator is a model of quasi-linear elliptic operators which makes it possible to model physical phenomena such as the flow of non-Newtonian aids, reaction flow systems, nonlinear elasticity, the extraction of petroleum, astronomy, through porous media, and glaciology. Several authors in this field obtained many results of existence (see, for example, [1, 3, 5, 11, 12]).

In this work, we consider the following quasi-linear elliptic system:

$$\begin{cases} -A\left(\int_{\Omega}|\nabla u|^{2}\mathrm{d}x\right)\Delta u=\lambda u^{\alpha}v^{\gamma}, & \text{in } \Omega,\\ -B\left(\int_{\Omega}|\nabla v|^{2}\mathrm{d}x\right)\Delta v=\lambda u^{\delta}v^{\beta}, & \text{in } \Omega,\\ v=u=0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^{N}\,(N\geq 3)$ is a bounded domain and its boundary $\partial\Omega$. Also, A and B are two continuous functions on $\mathbb{R}^{+}$, and the parameters α, β, δ, and γ satisfy the following conditions:

$$\begin{cases} 0\leq\alpha<1,\\ 0\leq\beta<1,\\ \delta,\gamma>0,\\ \theta=(1-\alpha)(1-\beta)-\gamma\delta>0 \text{ for each } \lambda>0. \end{cases} \tag{2}$$

Within previous studies [13–15], some nonlocal elliptical problems of the Kirchhoff type of the following model were extensively studied:

$$\begin{cases} M\left(\int_{\Omega}|\nabla u|^{2}\mathrm{d}x\right)\triangle u=h(x,u), & \text{in } \Omega,\\ u=0, & \text{in } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded open domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $h(x,u)$ the right hand side is defined for some exceptional functions similar to those in [13–16]. In addition, M is a defined and continuous function on $\mathbb{R}_+$ with values in $\mathbb{R}_+^*$. In recent years, various Kirchhoff or $p(x)$-Kirchhoff-type problems have been widely studied by many authors due to their theoretical and practical importance. Such problems are often referred to as nonlocal due to the presence of a full term on Ω or in $\mathbb{R}^n$. It is well known that this problem is analogous to the stationary problem of a model introduced by Kirchhoff [17].

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2}=0. \qquad (4)$$

More specifically, Kirchhoff proposed this model as an extension of the wave equation of the Alembert classic by considering the effects of variations in the length of the strings during vibration. The parameters of the above equation have the following meanings: E is Young's modulus of the material, ρ is the mass density, L is the length of the chain, h is the section area, and P_0 is the initial tension.

In recent work in [18], we have discussed the existence of the weak positive solution for the following Kirchhoff elliptic systems:

$$\begin{cases} -A\left(\|\nabla u\|_{L^2(\Omega)}\right)\triangle u=\lambda_1 u^\alpha+\mu_1' v^\beta, & \text{in }\Omega,\\ -B\left(\|\nabla u\|_{L^2(\Omega)}\right)\triangle v=\lambda_2' u^c+\mu_2' v^d, & \text{in }\Omega,\\ u=v=0, & \text{on }\partial\Omega, \end{cases} \qquad (5)$$

where λ_1, μ_1', λ_2', and μ_2' are positive parameters, $\alpha+c<1$, and $\beta+d<1$.

Motivated by the recent work in [13, 14, 18, 19] and by using the sub- and supersolution method which is defined in [20], existence of positive solutions of quasi-linear Kirchhoff elliptic systems is shown in bounded smooth domains.

The paper outline is as follows: some assumptions and definitions related to problem (1) are given in Section 2. Finally, our main result is given in Section 3.

2. Preliminaries and Assumptions

We assume the following hypothesis:

$(H1)$: we assume that $M\colon \mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a nonincreasing and continuous function which satisfies

$$\lim_{t\longrightarrow 0^+}M(t)=m_0, \qquad (6)$$

where $m_0>0$, and there exists $a_i,b_i>0, i=1,2$ such that

$$a_1\le A(t)\le a_2,\ b_1\le B(t)\le b_2 \quad \text{for all t}\in\mathbb{R}^+. \qquad (7)$$

$(H2)$: and

$$\begin{gathered}\alpha,\beta\in C(\overline{\Omega}),\\ \alpha(x)\ge\alpha_0>0,\beta(x)\ge\beta_0>0\end{gathered} \qquad (8)$$

for all $x\in\Omega$.

$(H3)$: f, g, h, and τ are C^1 on $(0,+\infty)$ and increasing functions, where

$$\begin{cases}\lim_{t\longrightarrow+\infty}f(t)=+\infty,\lim_{t\longrightarrow+\infty}g(t)=+\infty,\\ \lim_{t\longrightarrow+\infty}h(t)=+\infty=\lim_{t\longrightarrow+\infty}\tau(t)=+\infty.\end{cases} \qquad (9)$$

$(H4)$: $\exists\gamma>0$ such that

$$\begin{gathered}\lim_{t\longrightarrow+\infty}\frac{h(t)f\left(k\left[g(t)^\gamma\right]\right)}{t}=0,\quad \text{for all } k>0,\\ \lim_{t\longrightarrow+\infty}\frac{\tau\left(kt^\gamma\right)}{t^{\gamma-1}}=0,\quad \text{for all } k>0.\end{gathered} \qquad (10)$$

Lemma 1 (see [14]). *Under assumption $(H1)$,we suppose further that function $H(t):=tM(t^2)$ is increasing on $\mathbb{R}$.*

We assume that u and v are couple nonnegative functions, where

$$\begin{cases}-M\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)\Delta u\ge -M\left(\int_\Omega|\nabla v|^2\mathrm{d}x\right)\Delta v, & \text{in }\Omega,\\ u=v=0, \quad \text{on }\partial\Omega,\end{cases} \qquad (11)$$

and then $u\ge v$ a.e. in Ω.

Lemma 2 (see [1]). *If M verifies the conditions of Lemma 1, then for each $f\in L^2(\Omega)$, there exists a unique solution $u\in H_0^1(\Omega)$ to the M-linear problem:*

$$-M\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)\Delta u=f(x)\text{ in }\Omega\text{ and }u=0\text{ in }\partial\Omega. \qquad (12)$$

Lemma 3 (see [1]). *Let w solve $\Delta w=g$ in Ω. If $g\in C(\Omega)$, then $w\in C^{1,\alpha}(\Omega)$ for any $\alpha\in(0,1)$, so particularly, w is continuous in Ω.*

Definition 1. Let $(u,v)\in(H_0^1(\Omega)\cap L^\infty(\Omega)\times H_0^1(\Omega)\cap L^\infty(\Omega))$, and (u,v) is said a weak solution of (1) if it satisfies

$$\begin{gathered}A\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)\int_\Omega\nabla u\nabla\phi\mathrm{d}x=\lambda\int_\Omega u^\alpha v^\gamma\phi\mathrm{d}x,\quad \text{in }\Omega,\\ B\left(\int_\Omega|\nabla v|^2\mathrm{d}x\right)\int_\Omega\nabla v\nabla\psi\mathrm{d}x=\lambda\int_\Omega u^\delta v^\beta\,\psi\mathrm{d}x,\quad \text{in }\Omega,\end{gathered} \qquad (13)$$

for all $(\phi,\psi)\in(H_0^1(\Omega)\times H_0^1(\Omega))$.

Definition 2. We call the following nonnegative functions $(\underline{u},\underline{v})$, respectively; $(\overline{u},\overline{v})$ in $(H_0^1(\Omega)\cap L^\infty(\Omega)\times H_0^1(\Omega)\cap L^\infty(\Omega))$ are a weak subsolution (respectively, upersolution) of (1) if they verify $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})=(0,0)$ in $\partial\Omega$:

$$\begin{aligned}
&A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_\Omega \nabla \underline{u}\,\nabla\phi dx \le \lambda \int_\Omega \underline{u}^\alpha\,\underline{v}^\gamma \phi dx \text{ in } \Omega,\\
&B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_\Omega \nabla \underline{v}\,\nabla\psi dx \le \lambda \int_\Omega \underline{u}^\delta\,\underline{v}^\beta\,\psi dx \text{ in } \Omega,\\
&A\left(\int_\Omega |\nabla \overline{u}|^2 dx\right)\int_\Omega \nabla \overline{u}\nabla\phi dx \ge \lambda \int_\Omega \overline{u}^\alpha\overline{v}^\gamma \phi dx \text{ in } \Omega,\\
&B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right)\int_\Omega \nabla \overline{v}\nabla\psi dx \ge \lambda \int_\Omega \overline{u}^\delta\overline{v}^\beta\,\psi dx \text{ in } \Omega,
\end{aligned} \tag{14}$$

for all $(\phi,\psi) \in (H_0^1(\Omega)\times H_0^1(\Omega))$.

Before proving our main result, we need to prove the existence of weak supersolution and subsolution in the following section.

3. Weak Existence Results

3.1. Existence of Weak Supersolution. The existence of a positive weak supersolution for system (1) is established such that each component belongs to $C^{0,\rho}(\overline{\Omega})$, for $\rho \in (0,1)$.

Lemma 4. *Suppose that $(H1)$ holds, $0 \le \alpha,\beta < 1$, $\delta,\gamma > 0$, and $\theta = (1-\alpha)(1-\beta) - \gamma\delta > 0$. Then, system (1) possesses a positive weak supersolution*

$$(\overline{u},\overline{v}) \in L^2\left(0,T,C^{0,\rho_1}(\overline{\Omega})\right)\times L^2\left(0,T,C^{0,\rho_2}(\overline{\Omega})\right), \tag{15}$$

for $\rho_i \in [0,1]$, $i = 1,2$ and $\lambda > 0$.

Proof. Let $e_i \in C^{0,\rho_i}(\overline{\Omega})$, for $i = 1,2$, $\rho_i > 0$, be the solution of the following problem:

$$\begin{cases} -\triangle e_i = 1, & \text{in } \Omega,\\ e_i = 0, & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Then, by the strong maximum principle, we get $e_i > 0$ in Ω, $i = 1,2$.

We define

$$(\overline{u},\overline{v}) = (C_1e_1, C_2e_2), \tag{17}$$

where C_1 and C_2 are positive constants which we will fix them later.

Let $(\phi,\psi) \subset (H_0^1(\Omega)\times H_0^1(\Omega))$, with $(\phi,\psi) \ge 0$.

Then, we obtain

$$\begin{aligned}
A\left(\int_\Omega |\nabla\overline{u}|^2 dx\right)\int_\Omega \nabla\overline{u}\nabla\phi dx &= A\left(\int_\Omega |\nabla\overline{u}|^2 dx\right)C_1\int_\Omega \nabla e_1\nabla\phi dx\\
&= A\left(\int_\Omega |\nabla\overline{u}|^2 dx\right)C_1\int_\Omega \phi dx\\
&\ge a_1C_1\int_\Omega \phi dx,
\end{aligned} \tag{18}$$

and similarly,

$$\begin{aligned}
B\left(\int_\Omega |\nabla\overline{v}|^2 dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx &= B\left(\int_\Omega |\nabla\overline{v}|^2 dx\right)C_2\int_\Omega \psi dx\\
&\ge b_1C_2\int_\Omega \psi dx.
\end{aligned} \tag{19}$$

If

$$\begin{aligned}
&l = \|e_1\|_\infty, \quad L = \|e_2\|_\infty,\\
&0 \le \alpha < 1, \quad 0 \le \beta < 1,\\
&\lambda > 0, \quad \theta > 0,
\end{aligned} \tag{20}$$

and $(H1)$ holds, it is easy to prove that there exist positive constants C_1 and C_2 such that

$$\begin{aligned}
a_1C_1^{1-\alpha} &= \lambda C_2^\gamma l^\alpha L^\gamma,\\
b_1C_2^{1-\beta} &= \lambda C_1^\gamma l^\delta L^\beta.
\end{aligned} \tag{21}$$

Thus, from (21), we obtain for all $x \in \Omega$

$$\begin{aligned}
\lambda\overline{u}^\alpha\overline{v}^\gamma &\le \lambda C_1^\alpha C_2^\gamma l^\alpha L^\gamma \le a_1C_1,\\
\lambda\overline{u}^\delta\overline{v}^\beta &\le \lambda C_1^\delta C_2^\beta l^\delta L^\beta \le b_1C_2.
\end{aligned} \tag{22}$$

Therefore, by using (18), (19), and (22), we conclude that

$$\begin{aligned}
A\left(\int_\Omega |\nabla\overline{u}|^2 dx\right)\int_\Omega \nabla\overline{u}\nabla\phi dx &\ge \lambda\int_\Omega \overline{u}^\alpha\overline{v}^\gamma\phi dx, \quad \text{in } \Omega,\\
B\left(\int_\Omega |\nabla\overline{v}|^2 dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx &\ge \lambda\int_\Omega \overline{u}^\delta\overline{v}^\beta\psi dx, \quad \text{in } \Omega.
\end{aligned} \tag{23}$$

Hence, $(\overline{u},\overline{v}) \in C^{0,\rho_1}(\overline{\Omega})\times C^{0,\rho_2}(\overline{\Omega})$ is a positive weak supersolution of system (1). □

3.2. Existence of Weak Subsolution. Existence of a positive weak subsolution for system (1) is proved such that each component belongs to $C^0(\overline{\Omega})$.

Lemma 5. *We assume that $(H1)$ holds:*

$$\begin{aligned}
&0 \le \alpha,\beta < 1, \delta,\gamma > 0,\\
&\theta = (1-\alpha)(1-\beta) - \gamma\delta > 0.
\end{aligned} \tag{24}$$

Therefore, system (1) possesses a positive weak subsolution $(\underline{u},\underline{v}) \in C^0(\overline{\Omega})\times C^0(\overline{\Omega})$, for all $\lambda > 0$.

Proof. We assume that λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet condition with ϕ_1 which is its corresponding eigenfunction and $\phi_1$belongs to $C^{0,\rho_1}(\overline{\Omega})\times C^{0,\mu_1}(\overline{\Omega})$, $\phi_1 > 0$ in Ω and $|\nabla\phi_1| \ge \sigma_1$ on $\partial\Omega$, for some positive constants σ_1,μ_1, and ρ_1.

We define

$$(\underline{u},\underline{v}) = \left(c\phi_1^2, c^k\phi_1^2\right) \tag{25}$$

which belongs to $(C^0(\overline{\Omega})\cap C^1(\overline{\Omega}))\times(C^0(\overline{\Omega})\cap C^1(\overline{\Omega}))$, with $c > 0$ to be fixed later, and

$$\frac{\delta}{1-\beta}<k<\frac{1-\alpha}{\gamma} \tag{26}$$

because $\theta>0$, $1-\alpha>0$, and $1-\beta>0$. Then, for all $(\phi,\psi)\in(H_0^1(\Omega)\times H_0^1(\Omega))$, with ϕ, $\psi\geq 0$, we have

$$\begin{aligned} A\Big(\int_\Omega|\nabla \underline{u}|^2\mathrm{d}x\Big)\int_\Omega\nabla\underline{u}\,\nabla\phi\mathrm{d}x &= 2cA\Big(\int_\Omega|\nabla \underline{u}|^2\mathrm{d}x\Big)\int_\Omega\phi_1\nabla\phi_1\nabla\phi,\\ &= 2cA\Big(\int_\Omega|\nabla \underline{u}|^2\mathrm{d}x\Big)\int_\Omega\big[\lambda_1\phi_1^2-|\nabla\phi_1|^2\big]\phi\mathrm{d}x. \end{aligned} \tag{27}$$

Similarly,

$$\begin{aligned} B\Big(\int_\Omega|\nabla \underline{v}|^2\mathrm{d}x\Big)\int_\Omega\nabla\underline{v}\,\nabla\psi\mathrm{d}x &= 2c^kB\Big(\int_\Omega|\nabla \underline{v}|^2\mathrm{d}x\Big)\int_\Omega\\ &\cdot\big[\lambda_1\phi_1^2-|\nabla\phi_1|^2\big]\psi\mathrm{d}x. \end{aligned} \tag{28}$$

Since $\phi_1=0$ and $|\nabla\phi_1|\geq\sigma_1$ on $\partial\Omega$, there exists $\eta>0$ such that, for every $x\in\Omega_\eta=\{x\in\Omega\colon d(x,\partial\Omega)\leq\eta\}$, we have

$$\begin{aligned} &\big[\lambda_1\phi_1^2-|\nabla\phi_1|^2\big]\leq 0,\\ &\big[\lambda_1\psi_1^2-|\nabla\psi_1|^2\big]\leq 0. \end{aligned} \tag{29}$$

Then, for each $\lambda>0$, we get

$$A\Big(\int_\Omega|\nabla \underline{u}|^2\mathrm{d}x\Big)\int_{\Omega_\eta}\nabla\underline{u}\,\nabla\phi\mathrm{d}x\leq 0\leq\lambda\int_{\Omega_\eta}\underline{u}^\alpha\,\underline{u}^\gamma\phi\mathrm{d}x, \tag{30}$$

for all $\phi\in H_0^1(\Omega)$, $\phi\geq 0$, and

$$B\Big(\int_\Omega|\nabla \underline{v}|^2\mathrm{d}x\Big)\int_{\Omega_\eta}\nabla\underline{v}\,\nabla\psi\mathrm{d}x\leq 0\leq\lambda\int_{\Omega_\eta}\underline{u}^\delta\,\underline{u}^\beta\psi\mathrm{d}x, \tag{31}$$

for all $\psi\in H_0^1(\Omega)$ and $\psi\geq 0$.

Now, as $\phi_1>0$ in Ω and ϕ_1 is continuous, then there exists $\mu>0$ such that $\phi_1(x)\geq\mu>0$ for all $x\in\Omega\backslash\overline{\Omega}_\eta$. Therefore, from (26), we obtain $a_0>0$ such that the following inequalities hold:

$$2b_2\lambda_1c^{k(1-\beta)-\delta}\phi_1^{2-2\beta}(x)\leq\lambda\mu^{2\delta}\leq\lambda\phi_1^{2\delta}(x),\quad\forall x\in\frac{\Omega}{\overline{\Omega}_\eta}, \tag{32}$$

$$2a_2\lambda_1c^{1-\alpha-k\gamma}\phi_1^{2-2\alpha}(x)\leq\lambda\mu^{2\gamma}\leq\lambda\phi_1^{2\gamma}(x),\quad\forall x\in\frac{\Omega}{\overline{\Omega}_\eta}, \tag{33}$$

for each $c\in(0,a_0)$.

Then,

$$\begin{aligned} &2cA\Big(\int_\Omega|\nabla \underline{u}|^2\mathrm{d}x\Big)\big[\lambda_1\phi_1^2-|\nabla\phi_1|^2\big]\phi\\ &\quad\leq 2a_2c\lambda_1\phi_1^2\\ &\quad= 2a_2\lambda_1c^{1-\alpha-k\gamma}\phi_1^{2-2\alpha}\big[c^{k\gamma}c^\alpha\phi_1^{2\alpha}\big]. \end{aligned} \tag{34}$$

By (33), we have

$$\begin{aligned} 2cA\Big(\int_\Omega|\nabla \underline{u}|^2\mathrm{d}x\Big)\big[\lambda_1\phi_1^2-|\nabla\phi_1|^2\big] &\leq\lambda\phi_1^{2\gamma}c^{k\gamma}c^\alpha\phi_1^{2\alpha}\\ &=\lambda\,\underline{u}^\alpha\,\underline{v}^\gamma. \end{aligned} \tag{35}$$

And similarly, from (32), we have

$$\begin{aligned} &2c^kB\Big(\int_\Omega|\nabla \underline{v}|^2\mathrm{d}x\Big)\big[\lambda_1\phi_1^2-|\nabla\phi_1|^2\big]\\ &\quad\leq\lambda\,\underline{u}^\delta\,\underline{v}^\beta \end{aligned} \tag{36}$$

in $\Omega/\overline{\Omega}_\eta$ and each $c\in(0,a_0)$.

Therefore,

$$A\Big(\int_\Omega|\nabla \underline{u}|^2\mathrm{d}x\Big)\int_{\Omega/\overline{\Omega}_\eta}\nabla\underline{u}\,\nabla\phi\mathrm{d}x\leq\lambda\int_{\Omega/\overline{\Omega}_\eta}\underline{u}^\alpha\,\underline{v}^\gamma\phi\mathrm{d}x, \tag{37}$$

$$B\Big(\int_\Omega|\nabla \underline{v}|^2\mathrm{d}x\Big)\int_{\Omega/\overline{\Omega}_\eta}\nabla\underline{v}\,\nabla\psi\mathrm{d}x\leq\lambda\int_{\Omega/\overline{\Omega}_\eta}\overset{\delta\,\beta}{\underline{u}\underline{v}}\psi\mathrm{d}x. \tag{38}$$

Hence, from (30), (31), (37), and (38), it follows that

$$\left\{\begin{aligned} &A\Big(\int_\Omega|\nabla \underline{u}|^2\mathrm{d}x\Big)\Big[\int_{\Omega_\eta}\nabla\underline{u}\,\nabla\phi\mathrm{d}x+\int_{\Omega/\overline{\Omega}_\eta}\nabla\underline{u}\,\nabla\phi\mathrm{d}x\Big]\\ &=A\Big(\int_\Omega|\nabla \underline{u}|^2\mathrm{d}x\Big)\int_\Omega\nabla\underline{u}\,\nabla\phi\mathrm{d}x\leq\int_\Omega\underline{u}^\alpha\underline{v}^\gamma\phi\mathrm{d}x, \end{aligned}\right. \tag{39}$$

$$\begin{aligned} &B\Big(\int_\Omega|\nabla \underline{v}|^2\mathrm{d}x\Big)\Big[\int_{\Omega_\eta}\nabla\underline{v}\,\nabla\psi\mathrm{d}x+\int_{\Omega/\overline{\Omega}_\eta}\nabla\underline{v}\,\nabla\psi\mathrm{d}x\Big]\\ &\quad=B\Big(\int_\Omega|\nabla \underline{v}|^2\mathrm{d}x\Big)\int_\Omega\nabla\underline{v}\,\nabla\psi\mathrm{d}x\leq\lambda\int_\Omega\underline{u}^\delta\,\underline{u}^\beta\psi\mathrm{d}x. \end{aligned} \tag{40}$$

Then, by (39) and (40), $(\underline{u},\underline{v})$ is a positive weak subsolution of system (1), for each $c\in(0,a_0)$. □

4. Main Result

In this section, we give the result of the existence of the positive weak solution to quasi-linear elliptic system (1) by using the sub- and supersolution method which has been already used for some classical elliptic equations by known authors (see [1, 4, 11, 19, 21]).

Theorem 1. *Suppose that $(H1)$ holds, $0\leq\alpha,\beta<1,\delta,\gamma>0$, and $\theta=(1-\alpha)(1-\beta)-\gamma\delta>0$ as well as under the results of Lemma 4 and 5. Then, system (1) possesses a weak solution $(u,v)\in(H_0^1(\Omega)\times H_0^1(\Omega))$, where each component is positive and belongs to $C^{0,\rho}(\overline{\Omega})\cap C^{1,\mu}(\Omega)$ for some $\rho\in[0,1]$, $\mu>0$, and each $\lambda>0$.*

Proof 3. In order to obtain a weak solution of problem (1), we shall use the arguments by Azzouz and Bensedik [13]. For this purpose, we define a sequence $\{(u_n,v_n)\}\subset(H_0^1(\Omega)\times H_0^1(\Omega))$ as follows: $u_0:=\overline{u}$, $v_0=\overline{v}$, and (u_n,v_n) is the unique solution of the system

$$\begin{cases} -A\left(\int_\Omega |\nabla u_n|^2 \mathrm{d}x\right)\triangle u_n = \lambda u_{n-1}^\alpha v_{n-1}^\gamma, & \text{in } \Omega, \\ -B\left(\int_\Omega |\nabla v_n|^2 \mathrm{d}x\right)\triangle v_n = \lambda u_{n-1}^\delta v_{n-1}^\beta, & \text{in } \Omega, \\ u_n = v_n = 0, & \text{on } \partial\Omega. \end{cases} \quad (41)$$

Problem (41) is (A, B)–linear in the sense that if

$$\left(u_{n-1}, v_{n-1}\right) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right) \quad (42)$$

is given, the right-hand sides of (41) are independent of u_n, v_n.

Set

$$\begin{aligned} A(t) &= tA\left(t^2\right), \\ B(t) &= tB\left(t^2\right). \end{aligned} \quad (43)$$

Then, since

$$\begin{aligned} &A(\mathbb{R}) = \mathbb{R},\ B(\mathbb{R}) = \mathbb{R}, \\ &f\left(u_{n-1}, v_{n-1}\right) = u_{n-1}^\alpha v_{n-1}^\gamma \in L^2(\Omega), \\ &g\left(u_{n-1}, v_{n-1}\right) = u_{n-1}^\delta v_{n-1}^\beta \in L^2(\Omega). \end{aligned} \quad (44)$$

According to the result in [1], we can deduce that system (41) admits a unique solution

$$\left(u_n, v_n\right) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right). \quad (45)$$

By using (41) and the fact that (u_0, v_0) is a supersolution of (1), we have

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_0|^2 \mathrm{d}x\right)\triangle u_0 &\geq \lambda u_0^\alpha v_0^\gamma = -A\left(\int_\Omega |\nabla u_1|^2 \mathrm{d}x\right)\triangle u_1, \\ -B\left(\int_\Omega |\nabla v_0|^2 \mathrm{d}x\right)\triangle v_0 &\geq \lambda u_0^\delta v_0^\beta = -B\left(\int_\Omega |\nabla v_1|^2 \mathrm{d}x\right)\triangle v_1. \end{aligned} \quad (46)$$

Also, by using Lemma 1, $u_0 \geq u_1$ and $v_0 \geq v_1$, and since $u_0 \geq \underline{u}$, $v_0 \geq \underline{v}$, and the monotonicity of $f(u, v) = u^\alpha v^\gamma$ and $g(u, v) = u^\delta v^\beta$, one has

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2 \mathrm{d}x\right)\Delta u_1 &= \lambda u_0^\alpha v_0^\gamma \geq \lambda \underline{u}\underline{v}^{\alpha\gamma} \geq -A\left(\int_\Omega |\nabla \underline{u}|^2 \mathrm{d}x\right)\Delta \underline{u}, \\ -B\left(\int_\Omega |\nabla v_1|^2 \mathrm{d}x\right)\Delta v_1 &= \lambda u_0^\delta v_0^\beta \geq \lambda \underline{u}\underline{v}^{\delta\beta} \geq -B\left(\int_\Omega |\nabla \underline{v}|^2 \mathrm{d}x\right)\Delta \underline{v}, \end{aligned} \quad (47)$$

from which, according to Lemma 1, $u_1 \geq \underline{u}$ and $v_1 \geq \underline{v}$. For u_2, v_2, we write

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2 \mathrm{d}x\right)\Delta u_1 &= \lambda u_0^\alpha v_0^\gamma \geq \lambda u_1^\alpha v_1^\gamma = -A\left(\int_\Omega |\nabla u_2|^2 \mathrm{d}x\right)\Delta u_2, \\ -B\left(\int_\Omega |\nabla v_1| \mathrm{d}x\right)\Delta v_1 &= \lambda u_0^\delta v_0^\beta \geq \lambda u_1^\delta v_1^\beta = -B\left(\int_\Omega |\nabla v_2|^2 \mathrm{d}x\right)\Delta v_2, \end{aligned} \quad (48)$$

and then $u_1 \geq u_2$ and $v_1 \geq v_2$. Similarly, $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$-A\left(\int_\Omega |\nabla u_2|^2 \mathrm{d}x\right)\Delta u_2 = \lambda u_1^\alpha v_1^\gamma \geq \lambda u_1^\alpha \underline{v}_1^\gamma \geq -A\left(\int_\Omega |\nabla \underline{u}|^2 \mathrm{d}x\right)\Delta \underline{u},$$

$$-B\left(\int_\Omega |\nabla v_2|^2 \mathrm{d}x\right)\Delta v_2 = \lambda u_1^\delta v_1^\beta \geq \lambda \underline{u}_1^\delta \underline{v}_1^\beta \geq -B\left(\int_\Omega |\nabla \underline{v}|^2 \mathrm{d}x\right)\Delta \underline{v}. \quad (49)$$

Repeating this argument, we get a bounded monotone sequence $\{(u_n, v_n)\} \subset (H_0^1(\Omega) \times H_0^1(\Omega))$ satisfying

$$\overline{u} = u_0 \geq u_1 \geq u_2 \geq \ldots \geq u_n \geq \ldots \geq \underline{u} > 0, \quad (50)$$

$$\overline{v} = v_0 \geq v_1 \geq v_2 \geq \ldots \geq v_n \geq \ldots \geq \underline{v} > 0. \quad (51)$$

Using the continuity of the functions f and g and the definition of the sequence $\{u_n\}, \{v_n\}$, there exist constants $C_i > 0$, $i = 1, \ldots 4$, independent of n such that

$$\begin{aligned} &\left|f\left(u_{n-1}, v_{n-1}\right)\right| \leq C_1, \\ &\left|g\left(u_{n-1}, v_{n-1}\right)\right| \leq C_2, \quad \text{for all } n. \end{aligned} \quad (52)$$

From (52), we multiply the first equation of (41) by u_n; in addition, by using the Holder inequality combined with Sobolev embedding, we have

$$\begin{aligned} a_1 \int_\Omega |\nabla u_n|^2 \mathrm{d}x &\leq A\left(\int_\Omega |\nabla u_n|^2 \mathrm{d}x\right)\int_\Omega |\nabla u_n|^2 \mathrm{d}x \\ &= \lambda \int_\Omega f\left(u_{n-1}, v_{n-1}\right) u_n \mathrm{d}x \\ &\leq \lambda \int_\Omega \left|f\left(u_{n-1}, v_{n-1}\right)\right| |u_n| \mathrm{d}x \\ &\leq C_1 \lambda \left(\int_\Omega |u_n|^2\right)^{(1/2)} \mathrm{d}x \\ &\leq C_3 \|u_n\|_{H_0^1(\Omega)} \\ \text{or } \|u_n\|_{H_0^1(\Omega)} &\leq C_3, \forall n, \end{aligned} \quad (53)$$

where $C_3 > 0$ is a constant independent of n. Similarly, there exists $C_2 > 0$ independent of n such that

$$\|v_n\|_{H_0^1(\Omega)} \leq C_4, \forall n. \quad (54)$$

From (53) and (54), we deduce that the couple $\{(u_n, v_n)\}$ converges weakly in $H_0^1(\Omega, \mathbb{R}^2)$ to the couple $(u, v)$with$u \geq \underline{u} > 0$ and $v \geq v > 0$.

By using a standard regularity argument, $\{(u_n, v_n)\}$ converges to (u, v). Thus, when $n \longrightarrow +\infty$ in (41), we can see that (u, v) is a positive solution of system (1).

The proof is completed. □

5. Conclusion

As a conclusion of this contribution, we have proved the existence of positive solutions of quasi-linear Kirchhoff elliptic systems in bounded smooth domains by using the sub- and super-solution method [20], which is an extension of our recent works of Boulaaras et al. in [18]. In the next work, some other methods such as variational and Galerkin methods (see, for example, [15]) will be used for this problem, and some numerical examples will also be given [9, 22].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Authors' Contributions

The authors contributed equally to this article. They have all read and approved the final manuscript.

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Research Article

A Note on the Appell Hypergeometric Matrix Function F_2

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In this article, we introduce some of the mathematical properties of the second Appell hypergeometric matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ including integral representations, transformation formulas, and series formulas.

1. Introduction

Appell defined and studied in [1–3] four kinds of double series of two variables z, w as generalizations of the hypergeometric series:

$$F(\alpha,\beta,\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n}z^n, \tag{1}$$

where z is a main variable in the unit disk $\{z\in\mathbb{C}\colon |z|<1\}$, α,β,γ are complex parameters with $\gamma\neq 0, -1, -2, -3, \ldots,$ and $(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)$ $(n\in\mathbb{N})$ and $(\alpha)_0=1$. Here and throughout, let $\mathbb{C}$ and $\mathbb{N}$ denote the sets of complex numbers and positive integers, respectively, and let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Appell hypergeometric functions F_s, $s=\{1,2,3,4\}$ play an important role in mathematical physics in which broad practical applications can be found (see, e.g. [1, 3–7]). In particular, the Appell hypergeometric series F_2 arises frequently in various physical and chemical applications ([8–11]). The exact solutions of number of problems in quantum mechanics have been given [6, 7, 9, 12] in terms of Appell's function F_2. For readers, they can find some results of the classical second Appell hypergeometric function F_2 in [13–17].

On the other hand, many authors [18–25] generalized the hypergeometric series $F(\alpha,\beta,\gamma;z)$ by extending parameters α, β, and γ to square matrices A, B, and C in the complex space $\mathbb{C}^{d\times d}$. Recently, the extension of the classical Appell hypergeometric functions F_s, $s=\{1,2,3,4\}$, to the Appell hypergeometric matrix functions has been a subject of intensive studies [26–30]. The purpose of the present work is to study the second Appell hypergeometric matrix function $F_2(A,B_1,B_2,C_1,C_2;z,w)$ on the domain $\{(z,w)\in\mathbb{C}^2\colon |z|+|w|<1\}$, with square matrix valued parameters A, B_1, B_2, C_1, and C_2 in $\mathbb{C}^{d\times d}$. We investigate some of the mathematical properties of this matrix function and introduce new integral representations, transformation formulas, and summation formulas.

2. Some Known Definitions and Results

We begin with a brief review of some definitions and notations. A matrix E is a positive stable matrix in $\mathbb{C}^{d\times d}$ if $\mathrm{Re}(\lambda)>0$ for all $\lambda\in\sigma(E)$, where $\sigma(E)$ is the set of all eigenvalues of E. I and $\mathbf{0}$ stand for the identity matrix and the null matrix in $\mathbb{C}^{d\times d}$, respectively.

If $\Phi(z)$ and $\Psi(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and E is a matrix in $\mathbb{C}^{d\times d}$ such that $\sigma(E)\subset\Omega$; then, from the properties of the matrix functional calculus [28], it follows that

$$\Phi(E)\Psi(E)=\Psi(E)\Phi(E). \tag{2}$$

Hence, if F in $\mathbb{C}^{d\times d}$ is a matrix for which $\sigma(F)\subset\Omega$ and also if EF = FE, then

$$\Phi(E)\Psi(F) = \Psi(F)\Phi(E). \quad (3)$$

By application of the matrix functional calculus, for E in $\mathbb{C}^{d\times d}$, then from [23, 31], the Pochhammer symbol or shifted factorial defined by

$$(E)_n = \begin{cases} E(E+I)\dots(E+(n-1)I) = \Gamma^{-1}(E)\Gamma(E+nI), & n \in \mathbb{N}, \\ I, & n = 0, \end{cases} \quad (4)$$

with the condition

$$E + nI \text{ is invertible for all integers } n \in \mathbb{N}_0. \quad (5)$$

From (5), it is easy to find that

$$(E)_{n-k} = (-1)^k (E)_n \left[(I - E - nI)_k\right]^{-1}; \quad 0 \le k \le n, \quad (6)$$

$$(E)_{m+n} = (E)_n (E+nI)_m \text{ or } (E)_{m+n} = (E)_m (E+mI)_n. \quad (7)$$

From [28], one obtains

$$\frac{(-1)^k}{(n-k)!} I = \frac{(-n)_k}{n!} I = \frac{(-nI)_k}{n!}; \quad 0 \le k \le n. \quad (8)$$

Definition 1 (see [31]). If E is a matrix in $\mathbb{C}^{d\times d}$, such that $\operatorname{Re}(z) > 0$ for all eigenvalues z of E, then $\Gamma(E)$ is well defined as

$$\Gamma(E) = \int_0^\infty \tau^{E-I} e^{-\tau} d\tau, \qquad \tau^{E-I} = \exp((E-I)\ln\tau). \quad (9)$$

Definition 2 (see [31]). If E and F are positive stable matrices in $\mathbb{C}^{d\times d}$ and EF = FE, then the Beta matrix function is well defined by

$$\mathcal{B}(E,F) = \int_0^1 \tau^{E-I}(1-\tau)^{F-I} d\tau = \Gamma^{-1}(E+F)\Gamma(E)\Gamma(F). \quad (10)$$

Definition 3 (see[23]). Suppose that N_1, N_2, and N_3 are matrices in $\mathbb{C}^{d\times d}$, such that N_3 satisfies condition (5). Then, the hypergeometric matrix function ${}_2F_1(N_1, N_2; N_3; z)$ is given by

$${}_2F_1(N_1, N_2; N_3; z) = \sum_{n\ge 0} \frac{(N_1)_n (N_2)_n \left[(N_3)_n\right]^{-1}}{n!} z^n. \quad (11)$$

Definition 4. If E is the positive stable matrix in $\mathbb{C}^{d\times d}$, then the Laguerre-type matrix polynomial is defined by [28]

$$L_n^E(z) = \sum_{k=0}^n \frac{(-1)^k (z)^k}{k!(n-k)!}(E+I)_n (E+I)_k^{-1} = \frac{(E+I)_n}{n!} {}_1F_1(-nI; E+I; z), \quad n \in \mathbb{N}_0, \quad (12)$$

where ${}_1F_1$ is the confluent hypergeometric matrix function (cf. [25]).

Definition 5 (see[28, 32, 33]). Let E and F be positive stable matrices in $\mathbb{C}^{d\times d}$, then the Jacobi matrix polynomial $\mathbb{P}_n^{(E,F)}(z)$ is defined by

$$\begin{aligned}\mathbb{P}_n^{(E,F)}(z) &= \sum_{k=0}^n \binom{n}{k} \Gamma^{-1}(E+F+(n+1)I)\Gamma^{-1}(F+(k+1)I) \\ &\quad \Gamma(F+(n+1)I)\Gamma(E+F+(n+k+1)I)\frac{(-1)^{n+k}(1+z)^k}{2^k n!}, \\ &= \frac{(E+I)_n}{n!} {}_2F_1\left(-nI, E+F+(n+1)I; E+I; \frac{1-z}{2}\right).\end{aligned} \quad (13)$$

Using (6) and (11), we can write the second kind of two complex variables Appell hypergeometric matrix function in the following definition (see [26, 28]).

Definition 6. Let A, B_1, B_2, C_1, and C_2 be commutative matrices in $\mathbb{C}^{d\times d}$ with $C_1 + kI$ and $C_2 + kI$ being invertible for all integers $k \in \mathbb{N}_0$. Then, the second Appell hypergeometric matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ is defined in the following form:

$$\begin{aligned}F_2(A, B_1, B_2, C_1, C_2; z, w) &= \sum_{s_1,s_2=0}^\infty (A)_{s_1+s_2}(B_1)_{s_1}(B_2)_{s_2}\left[(C_1)_{s_1}\right]^{-1}\left[(C_2)_{s_2}\right]^{-1}\frac{z^{s_1}w^{s_2}}{s_1!s_2!} \\ &= \sum_{s_1=0}^\infty (A)_{s_1}(B_1)_{s_1}\left[(C_1)_{s_1}\right]^{-1} {}_2F_1(A+s_1I, B_2; C_2; w)\frac{z^{s_1}}{s_1!}, \quad (|z|+|w|<1).\end{aligned} \quad (14)$$

3. Main Results

In this section, we investigate some of the main properties of the second Appell hypergeometric matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ such as integral representations, transformation formulas, and summation formulas

3.1. Integral Representations

Theorem 1. *Let A, C_1, and C_2 be positive stable matrices in $\mathbb{C}^{d\times d}$. Then, for $|z|+|w|<1$, then the function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ defined in (14) can be represented in the following integer forms:*

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = \Gamma^{-1}(A) \times \int_0^\infty u^{A-I} e^{-u} {}_1F_1(B_1; C_1; zu) {}_1F_1(B_2; C_2; wu) du, \tag{15}$$

$$\begin{aligned} &F_2(A, -mI, -nI, C_1 + I, C_2 + I; z, w) \\ &= m!n! [(C_1 + I)_m]^{-1} [(C_2 + I)_n]^{-1} \Gamma^{-1}(A) \\ &\times \int_0^\infty u^{A-I} e^{-u} L_m^{C_1}(zu) L_n^{C_2}(wu) du. \end{aligned} \tag{16}$$

Proof. Replacing the Pochhammer symbol $(A)_{m+n}$ in definition (14) by its integral representation which is obtained from (5) and (9), we get the desired result (15).

Using integral formula (15) and the relation given in (12), we have

$$\begin{aligned} F_2(A, -mI, -nI, C_1 + I, C_2 + I; z, w) &= \Gamma^{-1}(A) \int_0^\infty u^{A-I} e^{-u} {}_1F_1(-mI; C_1 + I; zu) {}_1F_1(-nI; C_2 + I; wu) du \\ &= m!n! [(C_1 + I)_m]^{-1} [(C_2 + I)_n]^{-1} \Gamma^{-1}(A) \times \int_0^\infty u^{A-I} e^{-u} L_m^{C_1}(zu) L_n^{C_2}(wu) du, \end{aligned} \tag{17}$$

which completes proof relation (16). □

3.2. Transformation Formulas

Theorem 2. *For the matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$, we have the following transformations:*

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = (1 - z)^{-A} F_2\left(A, C_1 - B_1, B_2, C_1, C_2; \frac{-z}{1 - z}, \frac{w}{1 - w}\right), \tag{18}$$

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = (1 - w)^{-A} F_2\left(A, B_1, C_2 - B_2, C_1, C_2; \frac{z}{1 - w}, \frac{w}{w - 1}\right), \tag{19}$$

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = (1 - z - w)^{-A} F_2\left(A, C_1 - B_1, C_2 - B_2, C_1, C_2; \frac{-z}{1 - z - w}, \frac{-w}{1 - z - w}\right), \tag{20}$$

where A, B_1, B_2, C_1, and C_2 are commutative matrices in $\mathbb{C}^{d\times d}$ with $C_1 + kI$ and $C_2 + kI$ being invertible for all integer $k \in \mathbb{N}_0$, and B_1, B_2, C_1, C_2, $C_1 - B1$, and $C_2 - B_2$ are positively stable.

Proof. We will prove only (18) since the others can be proved similarly. Using matrix Kummer's first formula (cf. [8]),

$${}_1F_1(B; C; z) = e^z {}_1F_1(C - B; C; -z), \tag{21}$$

in (15), we have

$$F_2(A, B_1, B_2, C_1, C_2; z, w) = \Gamma^{-1}(A) \times \int_0^\infty u^{A-I} e^{-(1-z)u} {}_1F_1(C_1 - B_1; C_1; zu) {}_1F_1(B_2; C_2; wu) du. \tag{22}$$

Substituting $t = (1 - z)u$ into (22), we obtain formula (18).

Now, connections with the Gauss hypergeometric matrix function is considered by the following theorem: □

Theorem 3. *Let $F_2(A, B, B', C_1, C_2; z, w)$ be given in (14). The following formulas hold true:*

$$F_2(A, B_1, B_2, C_1, C_2; 0, w) = {}_2F_1(A, B_2; C_2; w), \tag{23}$$

$$F_2(A, B_1, B_2, C_1, C_2; z, 0) = {}_2F_1(A, B_1; C_1; z), \tag{24}$$

$$F_2(A, \mathbf{0}, B_2, C_1, C_2; z, w) = {}_2F_1(A, B_2; C_2; z), \tag{25}$$

$$F_2(A, B_1, \mathbf{0}, C_1, C_2; z, w) = {}_2F_1(A, B_1; C_1; z), \tag{26}$$

$$F_2(A, \mathbf{0}, B_2, C_1, C_2; z, 0) = (1 - z)^{-A} {}_2F_1\left(A, B_2; C_2; \frac{z}{1 - z}\right), \tag{27}$$

$$F_2(A, B_1, \mathbf{0}, C_1, C_2; 0, w) = (1 - w)^{-A} {}_2F_1\left(A, B_1; C_1; \frac{w}{1 - w}\right), \tag{28}$$

where ${}_2F_1$ is the Gauss hypergeometric matrix function defined in (11).

Proof. The proof of (23)–(26) is a direct consequence of definition (27). The relation (27) is obtained setting $C_1 = B_1$ in (18) and then using (25). Similarly, the relation (28) is derived setting $C_2 = B_2$ in (19) and then using (26). □

3.3. Some Summation Formulas. We now present the summation formulas behavior of the second Appell hypergeometric matrix function $F_2(A, B_1, B_2, C_1, C_2; z, w)$ by the following results.

Theorem 4. *The following finite summation formula holds true:*

$$\sum_{n=0}^{\mu} \frac{(\mathbf{C}+I)_n}{n!} F_2(\mathbf{A}, -nI, -nI, \mathbf{C}+I, \mathbf{C}+I; z, w)$$
$$= \frac{(\mathbf{A}-I)^{-1}(\mathbf{C}+I)_{\mu+1}}{(z-w)\mu!} \{F_2(\mathbf{A}-I, -\mu I, -(\mu+1)I, \mathbf{C}$$
$$+ I, \mathbf{C}+I; z, w) + z \rightleftharpoons w\}, \tag{29}$$

where $\mathbf{A}$ and $\mathbf{C}$ are positively stable in $\mathbb{C}^{d\times d}$ and $z \rightleftharpoons w$ indicates the presence of a second term that originates from the first by interchanging z and w.

Proof. Using (16), we find that

$$\sum_{n=0}^{\mu} \frac{(\mathbf{C}+I)_n}{n!} F_2(\mathbf{A}, -nI, -nI, \mathbf{C}+I, \mathbf{C}+I; z, w)$$
$$= \Gamma^{-1}(\mathbf{A}) \sum_{n=0}^{\mu} n! \left[(\mathbf{C}+I)_n\right]^{-1} \int_0^{\infty} u^{\mathbf{A}-I} e^{-u} L_\mu^{\mathbf{C}}(zu) L_n^{\mathbf{C}}(wu) \mathrm{d}u. \tag{30}$$

By interchanging the order of summation and integration and applying the following formula [28]:

$$\sum_{n=0}^{\mu} n! \left[(\mathbf{C}+I)_n\right]^{-1} L_n^{\mathbf{C}}(z) L_n^{\mathbf{C}}(w)$$
$$= (\mu+1)! \left[(\mathbf{C}+I)_\mu\right]^{-1} (z-w)^{-1} \left\{L_\mu^{\mathbf{C}}(z) L_{\mu+1}^{\mathbf{C}}(w) - L_{\mu+1}^{\mathbf{C}}(z) L_\mu^{\mathbf{C}}(w)\right\}, \tag{31}$$

and then taking into consideration (16), we obtain formula (29).

To extend this theorem, we propose to obtain some more formulas centering around the Appell's matrix function F_2; it follows that □

Theorem 5. *Suppose that $\mathbf{A}$ and $\mathbf{B}$ are positively stable in $\mathbb{C}^{d\times d}$ such that $\mathbf{B}$ satisfies spectral condition (5), with $|t| < 1$, $|zt/((1-w)(1-t))| < 1$ and $|w/((1-w)(1-t))| < 1$. The following generating matrix function holds true:*

$$\sum_{n=0}^{\infty} \frac{(\mathbf{B}+I)_{\mu+n}}{n!} F_2(\mathbf{A}, -nI, -(n+\mu)I; \mathbf{B}+I, \mathbf{B}+I; z, w) t^n$$
$$= (\mathbf{B}+I)_\mu (1-w)^{-\mathbf{A}} (1-t)^{-(\mathbf{B}+(1+\mu)I)}$$
$$\times F_4\left(\mathbf{A}, \mathbf{B}+(1+\mu)I; \mathbf{B}+I, \mathbf{B}+I; \frac{-zt}{(1-w)(1-t)}; \frac{-w}{(1-w)(1-t)}\right). \tag{32}$$

where F_4 is the four Appell's matrix function defined in [27–29].

Proof. To prove (32), we require formula (19) and the relations (12); thus, we have

$$\sum_{n=0}^{\infty} \frac{(\mathbf{B}+I)_{\mu+n}}{n!} F_2(\mathbf{A}, -nI, -(n+\mu)I; \mathbf{B}+I, \mathbf{B}+I; z, w) t^n$$
$$= (1-w)^{-\mathbf{A}} \times \sum_{n=0}^{\infty} \frac{(\mathbf{B}+I)_{\mu+n}}{n!} F_2\left(\mathbf{A}, -nI; \mathbf{B}+(\mu+n+1)I; \mathbf{B}+I, \mathbf{B}+I; \frac{z}{(1-w)}; \frac{w}{(w-1)}\right)$$
$$= (1-w)^{-\mathbf{A}} \sum_{s,n=0}^{\infty} \sum_{r=0}^{n} (\mathbf{A})_{s+r} \left[(\mathbf{B}+I)_r\right]^{-1} \left[(\mathbf{B}+I)_s\right]^{-1} \cdot (\mathbf{B}+I)_{\mu+n+r} \times \frac{t^n}{(n-r)!s!r!} \left(\frac{-z}{(1-w)}\right)^r \left(\frac{-w}{(1-w)}\right)^s$$
$$= (1-w)^{-\mathbf{A}} \sum_{s,r=0}^{\infty} \frac{1}{s!r!} (\mathbf{A})_{s+r} \left[(\mathbf{B}+I)_r\right]^{-1} \left[(\mathbf{B}+I)_s\right]^{-1} \cdot (\mathbf{B}+I)_{\mu+s+r} \times \left(\frac{-zt}{(1-w)}\right)^r \left(\frac{-w}{(1-w)}\right)^s \sum_{n=0}^{\infty} \frac{(\mathbf{B}+(1+\mu+s+r)I)_n}{n!}$$
$$= (\mathbf{B}+I)_\mu (1-w)^{-\mathbf{A}} (1-t)^{-(\mathbf{B}+(1+\mu)I)}$$
$$\times F_4\left(\mathbf{A}, \mathbf{B}+(1+\mu)I; \mathbf{B}+I, \mathbf{B}+I; \frac{-zt}{(1-w)(1-t)}; \cdot \frac{-w}{(1-w)(1-t)}\right). \tag{33}$$

This completes the proof of Theorem 5.

Putting $\mu = 0$ and then using the following formula,

$$F_4(\mathbf{A}, \mathbf{B}; \mathbf{B}, \mathbf{B}; z, w) = (1 - z - w)^{-\mathbf{A}}{}_2F_1\left(\frac{\mathbf{A}}{2}, \frac{1}{2}(\mathbf{A} + I); \mathbf{B}; \frac{4zw}{(1 - z - w)^2}\right). \quad (34)$$

Thus, (32) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\mathbf{B} + I)_{\mu+n}}{n!} F_2(\mathbf{A}, -nI, -(n+\mu)I; \mathbf{B} + I, \mathbf{B} + I; z, w)t^n &= (1 - w)^{-\mathbf{A}} \\ \times \sum_{n=0}^{\infty} \frac{(\mathbf{B} + I)_{\mu+n}}{n!} F_2\left(\mathbf{A}, -nI; \mathbf{B} + (\mu + n + 1)I; \mathbf{B} + I, \mathbf{B} + I; \frac{z}{(1 - w)}; \frac{w}{(w - 1)}\right) \\ = (1 - t)^{\mathbf{A}-(\mathbf{B}+I)}[1 - (1 - z - w)t]^{-\mathbf{A}}{}_2F_1\left(\frac{\mathbf{A}}{2}, \frac{1}{2}(\mathbf{A} + I); \mathbf{B} + I; \frac{4zwt}{[1 - (1 - z - w)t]^2}\right).\end{aligned} \quad (35)$$

Now, we shall see that (32) and (35) enable us to obtain some new formulas. By (29), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathbf{A}^{-1} \frac{(\mathbf{B} + I)_{n+1}}{(z - w)n!} [F_2(\mathbf{A} - I, -nI, -(n+1)I; \mathbf{B} + I, \mathbf{B} + I; z, w) \\ &\quad + z \rightleftharpoons w]t^n \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(\mathbf{B} + I)_k}{k!} F_2(\mathbf{A}, -kI; -kI; \mathbf{B} + I, \mathbf{B} + I; z, w)t^n.\end{aligned} \quad (36)$$

Using (35), we arrive at

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(\mathbf{B} + I)_{n+1}}{n!} [F_2(\mathbf{A} - I, -nI, -(n+1)I; \mathbf{B} + I, \mathbf{B} + I; z, w) \\ &\quad + z \rightleftharpoons w]t^n \\ &= F_2(\mathbf{A}, -kI; -kI; \mathbf{B} + I, \mathbf{B} + I; z, w)t^n \\ &= \mathbf{A}(z - w)(1 - t)^{\mathbf{A}-(\mathbf{B}+2I)}[1 - (1 - z - w)t]^{-\mathbf{A}} \\ &\quad \times {}_2F_1\left(\frac{\mathbf{A}}{2}, \frac{1}{2}(\mathbf{A} + I); \mathbf{B} + I; \frac{4zwt}{[1 - (1 - z - w)t]^2}\right).\end{aligned} \quad (37)$$

With the help of a generating function for Jacobi matrix polynomials (see [28, 32, 33]),

$$\begin{aligned}&\sum_{n=0}^{\infty} (\mathbf{B} + \mathbf{C} + I)_n [(\mathbf{B} + I)_n]^{-1} \mathbb{P}_n^{(\mathbf{B},\mathbf{C})}(z)t^n \\ &= (1 - t)^{-(\mathbf{B}+\mathbf{C}+I)}{}_2F_1\left(\frac{1}{2}(\mathbf{B} + \mathbf{C} + I), \frac{1}{2}(\mathbf{B} + \mathbf{C} + 2I); \mathbf{B} \right. \\ &\quad \left. + I; \frac{2t(z - 1)}{(1 - t)^2}\right).\end{aligned} \quad (38)$$

We rewrite (35) as

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(\mathbf{B} + I)_{n+1}}{n!} [F_2(\mathbf{B} + \mathbf{C}, -nI, -(n+1)I; \mathbf{B} + I, \mathbf{B} + I; z, w) \\ &\quad + z \rightleftharpoons w]t^n \\ &= (\mathbf{B} + \mathbf{C} + I)(z - w)(1 - t)^{\mathbf{C}-I} \sum_{r=0}^{\infty} (\mathbf{B} + \mathbf{C} + I)_r [(\mathbf{B} + I)_r]^{-1} \\ &\quad \times \mathbb{P}_r^{(\mathbf{B},\mathbf{C})}\left(\frac{(1 - w)(1 - z) + zw}{(1 - z - w)}\right)(1 - z - w)^r t^r,\end{aligned} \quad (39)$$

which yields

$$\begin{aligned}&\mathbb{P}_n^{(\mathbf{B},\mathbf{C})}\left(\frac{(1 - w)(1 - z) + zw}{(1 - z - w)}\right) = (\mathbf{B} + I)_n (\mathbf{B} + \mathbf{C} + I) \\ &\quad \cdot [(\mathbf{B} + \mathbf{C} + I)_n]^{-1} \\ &\quad \times (z - w)^{-1}(1 - z - w)^{-n} \sum_{r=0}^{n} \frac{(\mathbf{C} - I)_r}{(n - r)!} \frac{(\mathbf{B} + I)_{r+1}}{r!} \\ &\quad \times [F_2(\mathbf{B} + \mathbf{C}, -rI, -(r+1)I; \mathbf{B} + I, \mathbf{B} + I; z, w) + z \rightleftharpoons w].\end{aligned} \quad (40)$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to writing of this article. All the authors read and approved the final manuscript.

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Research Article

On a New Model Based on Third-Order Nonlinear Multisingular Functional Differential Equations

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In this study, a novel mathematical model based on third-order nonlinear multisingular functional differential equations (MS-FDEs) is presented. The designed model is solved by using a well-known differential transformation (DT) scheme that is a very credible tool for solving the nonlinear third-order nonlinear MS-FDEs. In order to check the exactness, efficacy, and convergence of the scheme, some numerical examples are presented based on nonlinear third-order MS-FDEs and numerically solved by using DT scheme. The scheme of differential transformation allows us to find a complete solution and a closed approximate solution of the differential equation. The distinctive advantage of the computational technique is to deal with the complex and monotonous physical problems that are obtained in various branches of engineering and natural sciences. Moreover, a comparison of the obtained numerical outcomes from the exact solutions shows the correctness, accurateness, and exactness of the designed model as well as the presented scheme.

1. Introduction

The singular study along with functional differential equations (FDEs) is considered very significant for the researcher's community, and the implementations of the FDEs have been noticed in the sixth decade of the nineteen century. The FDEs have a huge variety of applications in many fields; to mention few of them are, models of population growth [1], electrodynamics [2], infection models of HIV-1 [3], models of tumor growth [4], models based on chemical kinetics [5], B-virus infection hepatitis models [6], models of the gene regulations [7], and models of viral infections [8], and many more [9–14]. The singular study based on the differential models is very interesting, complicated, experimental, and challenging for the researchers due to the singularity appearance at origin. There are many singular models in the literature; one of the famous models is Lane–Emden that represents singularity at $x = 0$ always. The model of the Lane–Emden is famous as its historic point of view and has been applied broadly due to its huge important and significant applications in the fields of science and technology. Some of the important applications of the Lane–Emden model are that it is used in various phenomena of mathematical physics structure and in the study of astrophysics, such as models of the stellar structure [15], study of thermal explosions model [16], study of the model of isothermal gas spheres [17], oscillating magnetic fields [18], and thermionic currents [19].

In recent decades, the research community is interested to solve the singular nonlinear FDEs numerically due to the singularity and functionality in differential equations. For example, to present the solutions of these nonlinear FDEs, Kadalbajoo and Sharma [20, 21] applied a numerical scheme. In order to solve differential-difference based model, Mirzaee and Hoseini [22] implemented a numerical collocation scheme. Xu and Jin [23] explained the singularly functional perturbed differential model by applying the fractional steps and boundary functions. Geng et al [24] applied a numerical approach for the delay differential equations on the basis of singularly perturbed model.

The purpose of the recent study is to present the model based on the nonlinear multisingular (MS) functional differential equations, i.e., MS-FDEs of order three. The modeled form of the third-order nonlinear MS-FDE along with its initial conditions (ICs) is written as

$$\left\{ y'''(x+\tau_1) + \frac{\alpha}{x} y''(x+\tau_2) + \frac{\beta}{x^2} y'(x+\tau_3) + x y(x+\tau_4) = f(x), \quad y(0) = a, y'(0) = b, y''(0) = c. \right. \tag{1}$$

The parameters τ_i $(i = 1, 2, 3, 4)$ and $\alpha, \beta, a, b,$ and c are the real constant values.

The idea of the above model is achieved by extending the work of Sabir et al. [25] that is used to explain the nonlinear singular FDEs of second order. For the verification and correctness of the designed MS-FDEs model, three different examples have been modeled and numerically solved by using the well-known differential transformation (DT) scheme, and the obtained numerical outcomes of DT scheme are compared with the exact solutions. The DT scheme has been applied to solve many stiff, nonstiff, singular, nonsingular, linear, and nonlinear types of problems. Zhou [26], for the first time, presented the idea of the DT scheme at the end of the 19$^{\text{th}}$ century to solve the linear/nonlinear initial value problems based on the analysis of electrical circuit. The DT scheme is basically a numerical approach, which works on the basis of the expansion of TS, which constitutes a polynomial form of the analytic results. The quality of the numerical DT scheme is to require less work and does not require linearization as well as assumptions. This numerical scheme is designed on the basis of an analytical solution by using the polynomial expressions, such as the Taylor series (TS) expansion. But its procedure is more easier than the conventional higher-order TS scheme, which achieves symbolic computation of the necessary derivatives using the data-based functions. Three explanatory and illustrative examples based on model (1) are provided to show the efficacy of the obtained results from the DT scheme. These numerical outcomes are compared with the exact solutions that indicate the proficiency of the designed model as well as the proposed scheme.

Some major key factors of the present study are summarized as follows:

The mathematical modeled form of the third-order nonlinear MS-FDEs is presented successfully by extending the work of Sabir et al. [25]

The designed nonlinear MS-FDEs based on the designed model are addressed numerically by using the famous DT scheme.

Manipulation of the present scheme is to apply the brilliance-obtained outcomes for nonlinear MS-FDEs with better precision and outstanding consistency.

The reliability and correctness of the designed model are authentic through the comparison of the numerical results obtained by the DT scheme and the exact results. The overlapping of these results indicates perfection, excellence, and faultlessness for the model.

The third-order functional differential model given in equation (1) is not easy to solve because of nonlinearity, multisingularity, functionality, and harder in nature. DT scheme is the best choice and good selection to handle these types of complicated and complex models.

The rest of the paper is described as follows: the designed detailed methodology on the basis of DT scheme is provided in Section 2. Results and discussion are provided in Section 3. Conclusion along with future research direction is provided in the last section.

2. Methodology (Differential Transform Scheme)

The mathematical definition of DT scheme using $y(\zeta)$ is given as

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(\zeta)}{d\zeta^k}\right]_{x=0}. \tag{2}$$

The original function in the above equation (2) is $y(\zeta)$, whereas the transformed function (TF) is denoted by $\mathbf{Y}(\mathbf{k})$, which is also called the T-function. The inverse of DT scheme of $\mathbf{Y}(\mathbf{k})$ is provided as

$$y(\zeta) = \sum_{k=0}^{\infty} Y(k)(\zeta-\zeta_0)^k \equiv D^{-1}Y(k). \tag{3}$$

Using the results of equations (2) and (3), the obtained function becomes

$$y(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{k!}\left[\frac{d^k y(\zeta)}{d\zeta^k}\right]_{x=0}. \tag{4}$$

Equation (4) provides the concepts of DT scheme that are derived from the TS expansion, and this scheme has not been applied for symbolical assessment of the derivatives. Moreover, comparative derivative values are achieved by using the iterative procedure, which is defined by the transformed original function. In this study, the lowercase and uppercase letters are used to show the original function and the TF, respectively. Using the nature of the above two equations, one can easily prove the TFs have the basic mathematical values provided in Table 1.

In real applications, $y(\zeta)$ is obtained by a finite series and equation (3) can be described as

Table 1: The essential operations of DT scheme.

Unique function	TF
$y(\zeta) = u(\zeta) \pm v(\zeta)$	$Y(k) = U(k) \pm V(k)$
$y(\zeta) = cu(\zeta)$	$Y(k) = cU(k)$
$y(\zeta) = d^m u(\zeta)/d\zeta^m$	$Y(k) = (k+1) + (k+2)\cdots(k+m)U(k+m)$
$y(\zeta) = u(c\zeta)$	$Y(k) = c^k U(k)$
$y(\zeta) = u(\zeta/c)$	$Y(k) = U(k)/c^k$
$y(\zeta) = d^m/d\zeta^m u(c\zeta)$	$Y(k) = (k+1) + (k+2)\cdots(k+m)c^{k+m}U(k+m)$

$$y(\zeta) = \sum_{k=0}^{m} \zeta^k Y(k). \tag{5}$$

Equation (5) shows the term $\sum_{k=m+1}^{\infty} \zeta^k Y(k)$, which is very small and can be neglected, while **m** shows the convergence of natural frequency.

For better explanation of the DT scheme, some important theorems are presented as follows:

Theorem 1. *If $f(\zeta) = m(\zeta)o(\zeta)$, then $F(K) = M(k) \otimes O(k) = \sum_{l=0}^{k} M(l)O(k-l)$ (here, $\otimes$ denotes the convolution):*

$$\begin{aligned}
f(\zeta) &= \sum_{k=0}^{\infty} M(k)(\zeta-\zeta_0)^k \times \sum_{k=0}^{\infty} O(k)(\zeta-\zeta_0)^k \\
&= \left(M(0) + M(1)(\zeta-\zeta_0) + M(2)(\zeta-\zeta_0)^2 + \cdots\right) \\
&= M(0)O(0) + M(U(0)O(1) + M(1)O(0))(\zeta-\zeta_0) \\
&\quad \times \left(O(0) + O(1)(\zeta-\zeta_0) + O(2)(\zeta-\zeta_0)^2 + \cdots\right) \\
&\quad + (M(0)O(2) + M(1)O(1) + M(2)O(0))(\zeta-\zeta_0)^2 + \cdots \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{k} M(l)O(k-l)(\zeta-\zeta_0)^k.
\end{aligned} \tag{6}$$

By using equation (3), we get

$$F(K) = \sum_{l=0}^{k} M(l)O(k-l). \tag{7}$$

Theorem 2. *If $f(\zeta) = \zeta^{\phi}$, then*

$$F(h) = \delta(h-\phi) = \begin{cases} 1, & h = \phi, \\ 0, & h \neq \phi. \end{cases} \tag{8}$$

Proof. By using equation (2), we have

$$F(h) = \frac{1}{h!}\frac{\partial\left(\zeta^{\phi}\right)}{\partial\zeta^h}\bigg|_{t=0} = \begin{cases} \dfrac{1}{h!}\dfrac{\partial^h\left(\zeta^h\right)}{\partial\zeta^h} = \dfrac{h!}{h!} = 1, & h = \phi, \\[2ex] \dfrac{1}{h!}\dfrac{\partial^h\left(\zeta^{\phi}\right)}{\partial\zeta^h} = 0, & h \neq \phi. \end{cases} \tag{9}$$

□

Theorem 3. *By taking $f(\zeta) = e^{\zeta+c}$, we have $F(k) = e^c/k!$.*

Proof. Using equation (2), we get

$$F(k) = \frac{1}{k!}\frac{\partial\left(e^{\zeta+c}\right)}{\partial\zeta^k}\bigg|_{t=0} = e^c\left(\frac{\partial e^k}{\partial\zeta^k}\right)\bigg|_{t=0} = \frac{e^c}{k!}. \tag{10}$$

□

Theorem 4. *If $f(\zeta) = q(\zeta+m)$, then [27]*

$$F(k) = \sum_{i=k}^{N} (m)^{i-k}\binom{i}{k} Q(i) \text{ for } N \longrightarrow \infty. \tag{11}$$

Proof. By using differential inverse transform of $Y(k)$, we have

$$\begin{aligned}
f(\zeta) &= \sum_{l=2}^{\infty} Q(k)(\zeta-\zeta_0+m)^k = Q(0) + Q(1)((\zeta-\zeta_0)+m) \\
&\quad + Q(2)((\zeta-\zeta_0)+m)^2 + Q(3)((\zeta-\zeta_0)+m)^3 + \cdots \\
&= Q(0) + Q(1)(\zeta-\zeta_0) + Q(1)m + Q(2)(\zeta-\zeta_0)^2 + Q(2)m^2 \\
&\quad + 2Q(2)(\zeta-\zeta_0)m + Q(3)m^3 + 3Q(3)(\zeta-\zeta_0)m^2 + 3Q(3)(\zeta-\zeta_0)^2 m + Q(3)(\zeta-\zeta_0)^3 + \cdots
\end{aligned}$$

$$\begin{aligned}
&= \left(Q(0) + Q(1)m + Q(2)m^2 + Q(3)m^3 + \cdots\right)\\
&\quad + (\zeta - \zeta_0)\left(Q(1) + 2Q(2)m + 3Q(3)m^2 + \cdots\right)\\
&\quad + (x - x_0)^2 (Q(2) + 3Q(3)m + \cdots) + (\zeta - \zeta_0)^3 (Q(3) + \cdots) + \cdots\\
&= \sum_{l=0}^{\infty} \frac{l!}{0!(l-0)!} m^{l-0} Q(l) + \sum_{l=0}^{\infty} \frac{l!}{1!(l-1)!} m^{l-1} Q(l)(\zeta - \zeta_0)\\
&\quad + \sum_{l=2}^{\infty} \frac{l!}{2!(l-2)!} m^{l-2} Q(l)(\zeta - \zeta_0)^2 + \sum_{l=0}^{\infty} \frac{l!}{3!(l-3)!} m^{l-3} Q(l)(\zeta - \zeta_0)^3\\
&\quad + \cdots + \sum_{l=0}^{\infty} \frac{l!}{k!(l-k)!} m^{l-k} Q(l)(\zeta - \zeta_0)^k\\
&= \sum_{k=0}^{\infty}\sum_{l=k}^{\infty} \frac{l!}{k!(l-k)!} m^{l-k} Q(l)(\zeta - \zeta_0)^k = \sum_{k=0}^{\infty}\sum_{l=k}^{\infty} \binom{l}{k} m^{l-k} Q(l)(\zeta - \zeta_0)^k.
\end{aligned} \tag{12}$$

□

By comparing equations (3) and (12), $Y(k)$ becomes

$$F(k) = \sum_{l=k}^{N} \binom{l}{k} m^{l-k} Q(l) \text{ for } N \longrightarrow \infty. \tag{13}$$

Theorem 5. *If $y(\zeta) = d^b u(\zeta + m)/d\zeta^b$, then*

$$Y(h) = \frac{(h+b)!}{h!} \sum_{l=h+b}^{N} (m)^{l-h-b} \binom{l}{h+b} U(l) \text{ for } N \longrightarrow \infty. \tag{14}$$

Proof. Suppose $y(\zeta) = u(\zeta + m)$, in equation (2), we have

$$Y(h) = \frac{1}{h!}\frac{\partial^h}{\partial \zeta^h}\left(\frac{d^b y(\zeta)}{d\zeta^b}\right) = \frac{(h+b)!}{h!} Y(h+b). \tag{15}$$

Using the $Y(k)$ values from equation (13), it becomes

$$\begin{aligned}
\frac{(h+b)!}{h!} Y(h+b) &= \frac{(h+b)!}{h!}\left(\frac{d^{h+b}\left(\sum_{l=h}^{N}\binom{l}{h} m^{l-k} U(l)\right)}{d\zeta^b}\right)\\
&= \frac{(h+b)!}{h!}\sum_{l=h+b}^{N}\binom{l}{h+b} m^{l-(h+b)} U(l),\\
Y(h) = \frac{1}{h!}\frac{\partial^h}{\partial\zeta^h}\left(\frac{d^b y(\zeta)}{d\zeta^b}\right) &= \frac{1}{h!}\frac{\partial^h}{\partial\zeta^h}\left(\frac{d^b u(\zeta+m)}{d\zeta^b}\right)\\
&= \frac{(h+b)!}{h!}\sum_{l=h+b}^{N}\binom{l}{h+b} m^{l-h-b} U(l),\\
N &\longrightarrow \infty.
\end{aligned} \tag{16}$$

□

3. Results and Discussion

To present the numerical solutions based on the designed third-order nonlinear MS-FDE model, the nonlinear study is very important and many investigations have been provided in references [28–33]. Three different examples have been presented, and the solutions of the examples are performed by using the DT scheme.

Example 1. Consider the nonlinear third-order MS-FDE given as

$$\begin{cases}
y''(\psi - 1) + \dfrac{1}{\psi} y''(\psi + 1) + \dfrac{2}{\psi^2} y'(\psi + 2) + \psi y(\psi) = e^{\psi - 1}\\
+ \dfrac{1}{\psi} e^{\psi+1} + \dfrac{2}{\psi^2} e^{\psi+2} + \psi e^{\psi},\\
y(0) = 1,\\
y'(0) = 1,\\
y''(0) = 1.
\end{cases} \tag{17}$$

Multiplied by ψ^2, the achieved form is given as

$$\begin{aligned}
&\psi^2 y'''(\psi - 1) + \psi y''(\psi + 1) + 2y'(\psi + 2) + \psi^3 y(\psi)\\
&\quad = \psi^2 e^{\psi - 1} + \psi e^{\psi+1} + 2e^{\psi+2} + \psi^3 e^{\psi},
\end{aligned} \tag{18}$$

The DT scheme is applied to solve the model given in equation (17). By using the definitions of one-dimensional DT and the corresponding transformation of equation (17), the obtained system becomes

$$\delta(k-2)\otimes(k+3)(k+2)(k+1)\sum_{\sigma=k+3}^{N}(-1)^{\sigma-k-3}\begin{pmatrix}\sigma\\ k+3\end{pmatrix}Y(\sigma)$$
$$+\delta(k-1)\otimes(k+2)(k+1)\sum_{\sigma=k+2}^{N}(1)^{\sigma-k-2}\begin{pmatrix}\sigma\\ k+2\end{pmatrix}Y(\sigma)$$
$$+2(k+1)\sum_{\sigma=k+1}^{N}(2)^{\sigma-k-1}\begin{pmatrix}\sigma\\ k+1\end{pmatrix}Y(\sigma)+\delta(k-3)\otimes Y(k)$$
$$=\delta(k-2)\otimes\frac{1}{k!}e^{-1}+\delta(k-1)\otimes\frac{1}{k!}e+2\frac{1}{k!}e^{2}+\delta(k-3)\otimes\frac{1}{k!}. \tag{19}$$

Theorem 1 is used in equation (19), we get

$$\sum_{\eta=0}^{k}\sum_{i=\eta+3}^{N}\delta(k-\eta-2)(\eta+3)(\eta+2)(\eta+1)(-1)^{i-\eta-3}\begin{pmatrix}\sigma\\ \eta+3\end{pmatrix}Y(\sigma)$$
$$+\sum_{\eta=0}^{k}\sum_{\sigma=v+2}^{N}\delta(k-\eta-1)(\eta+2)(\eta+1)\begin{pmatrix}\sigma\\ \eta+2\end{pmatrix}Y(\sigma)$$
$$+2(k+1)\sum_{i=k+1}^{N}(2)^{\sigma-k-1}\begin{pmatrix}\sigma\\ k+1\end{pmatrix}Y(\sigma)$$
$$+\sum_{\eta=0}^{k}\delta(\eta-3)Y(k-\eta)$$
$$=\sum_{\eta=0}^{k}\delta(k-\eta-2)\frac{1}{\eta!}e^{-1}+\sum_{\eta=0}^{k}\delta(k-\eta-1)\frac{1}{\eta!}e+2\frac{1}{k!}e^{2}$$
$$+\sum_{\eta=0}^{k}\delta(k-\eta-3)\frac{1}{\eta!}. \tag{20}$$

Using the ICs given in equation (17), we have

$$Y(0)=1,$$
$$Y(1)=1, \tag{21}$$
$$Y(2)=\frac{1}{2}.$$

Taking $N=4$ and $k=0$ and 1, by using equations (20) and (21), the obtained linear algebraic equation system is written as

$$\begin{cases}12Y(3)+32Y(4)+3=e^{2},\\ 30Y(3)+108Y(4)+3=e+2e^{2}.\end{cases} \tag{22}$$

By solving the above coupled equations given in (22), we have

$$y(3)=\frac{\left(-57-8e+11e^{2}\right)}{84},$$
$$y(4)=\frac{\left(9+2e-e^{2}\right)}{56}. \tag{23}$$

By using the values of $Y(k)$ for $k=0$ and 1 in $y(\psi)$, i.e., the inverse-reduced DT, the results are written as

$$y(\psi)=\sum_{k=0}^{\infty}Y(k)\psi^{k}=1+\psi+\frac{1}{2}\psi^{2}+\frac{\left(-57-8e+11e^{2}\right)}{84}\psi^{3}$$
$$+\frac{\left(9+2e-e^{2}\right)}{56}\psi^{4}+O\left(\psi^{5}\right). \tag{24}$$

Repeat the process by using the equations (20) and (25) for $N=6$ and $k=0,1,$ and 2. The solution of the obtained linear algebraic equations system is given as

$$y(3)=\frac{\left(-57-8e+11e^{2}\right)}{84},$$
$$y(4)=\frac{\left(9+2e-e^{2}\right)}{56},$$
$$y(5)=\frac{128e^{2}-128e-495}{17120}, \tag{25}$$
$$y(6)=\frac{-8e^{2}+8e+51}{5136}.$$

By using the inverse-reduced DT $y(k)$, the solutions will be as follows:

$$y(\psi)=\sum_{k=0}^{\infty}Y(k)\psi^{k}=1+\psi+\frac{1}{2}\psi^{2}+\frac{\left(-57-8e+11e^{2}\right)}{84}\psi^{3}$$
$$+\frac{\left(9+2e-e^{2}\right)}{56}\psi^{4}$$
$$+\frac{128e^{2}-128e-495}{17120}\psi^{5}+\frac{-8e^{2}+8e+51}{5136}\psi^{6}+O\left(\psi^{7}\right). \tag{26}$$

Table 2 shows the comparison of the present numerical results for $N=4$ and $N=6$ with the exact solutions. The $y(\psi)$ results are slightly varied by changing the N parameter values. It is clear in Table 2 that the proposed and exact solutions overlapped each other.

Example 2. Consider the third-order MS-FDEs with its ICs:

$$\begin{cases}y'''(\psi-1)+\frac{1}{\psi}y''(\psi+1)+\frac{2}{\psi^{2}}y'(\psi+2)+\psi y(\psi)=\psi^{5}\\ +45\psi+48+\frac{108}{\psi}+\frac{64}{\psi^{2}},\\ y(0)=1,\\ y'(0)=0,\\ y''(0)=0.\end{cases} \tag{27}$$

Equation (27) becomes

Table 2: Comparison of the obtained results and exact solutions for $N = 4$ and $N = 6$.

ψ	DT ($N = 4$)	DT ($N = 6$)	Exact solution
0.01	1.01005	1.01005	1.01005
0.02	1.02020	1.02020	1.02020
0.03	1.03045	1.03045	1.03045
0.04	1.04080	1.04080	1.04081
0.05	1.05125	1.05125	1.05127
0.06	1.06180	1.06180	1.06183
0.07	1.07246	1.07246	1.07250
0.08	1.08322	1.09408	1.08328
0.09	1.09408	1.09408	1.09417
0.1	1.10504	1.10504	1.10517
0.2	1.22044	1.22044	1.22140
0.3	1.34683	1.34685	1.34985
0.4	1.48515	1.48522	1.49182
0.5	1.63663	1.63686	1.64872
0.6	1.80282	1.80341	1.82211
0.7	1.98556	1.98688	2.01375
0.8	2.18698	2.18965	2.22554
0.9	2.40955	2.41451	2.45960

$$\begin{aligned}\psi^2 y'''(\psi-1) + \psi y''(\psi+1) + 2y'(\psi+2) + \psi^3 y(\psi) \\ = \psi^7 + 45\psi^3 + 48\psi^2 + 108\psi + 64.\end{aligned} \tag{28}$$

The definition of one-dimensional DT scheme is applied and taking the consistent transform of equation (27), the system is given as

$$\begin{aligned}&\delta(k-2)\otimes(k+3)(k+2)(k+1)\sum_{\sigma=k+3}^{N}(-1)^{\sigma-k-3}\binom{\sigma}{k+3}Y(\sigma) \\ &\quad + \delta(k-1)\otimes(k+2)(k+1)\sum_{\sigma=k+2}^{N}(1)^{\sigma-k-2}\binom{\sigma}{k+2}Y(\sigma) \\ &\quad + 2(k+1)\sum_{\sigma=k+1}^{N}(2)^{\sigma-k-1}\binom{\sigma}{k+1}Y(\sigma) + \delta(k-3)\otimes Y(k) \\ &= \delta(k-7) + 45\delta(k-3) + 48\delta(k-2) + 108\delta(k-1) + 64\delta(k).\end{aligned} \tag{29}$$

Appling Theorem 1, we have

$$\begin{aligned}&\sum_{\eta=0}^{k}\sum_{\sigma=\eta+3}^{N}\delta(k-\eta-2)(\eta+3)(\eta+2)(\eta+1)(-1)^{\sigma-\eta-3}\binom{\sigma}{\eta+3}Y(\sigma) \\ &\quad + \sum_{\eta=0}^{k}\sum_{\sigma=\eta+2}^{N}\delta(k-\eta-1)(\eta+2)(\eta+1)\binom{\sigma}{l+2}Y(\sigma) \\ &\quad + 2(k+1)\sum_{\sigma=k+1}^{N}(2)^{\sigma-k-1}\binom{\sigma}{k+1}Y(\sigma) + \sum_{\eta=0}^{k}\delta(\eta-3)Y(k-\eta) \\ &= \delta(k-7) + 45\delta(k-3) + 48\delta(k-2) + 108\delta(k-1) + 64\delta(k).\end{aligned} \tag{30}$$

Using the ICs given in equation (27), we have

$$\begin{aligned}Y(0) &= 1, \\ Y(1) &= 0, \\ Y(2) &= 0.\end{aligned} \tag{31}$$

Taking the values of $N = 4$ and $k = 0$ and 1 in equations (30) and (31), the obtained linear algebraic equations system is given as

$$\begin{cases} 12Y(3) + 32Y(4) = 32, \\ 30Y(3) + 108Y(4) = 108. \end{cases} \tag{32}$$

Solving the coupled equations given in system (32), we have

$$\begin{aligned}y(3) &= 0, \\ y(4) &= 1.\end{aligned} \tag{33}$$

Using the $Y(k)$ values for $k = 0$ and 1 in the inverse-reduced DT $y(\psi)$, the obtained results are given as

$$y(\psi) = \sum_{k=0}^{\infty} Y(k)\psi^k = 1 + \psi^4. \tag{34}$$

Repeat the same procedure using equations (30) and (31), for $N = 5$ and $k = 0, 1,$ and 2. The following linear algebraic system becomes

$$\begin{cases} 30Y(3) + 108Y(4) + 340Y(5) = 108, \\ 12Y(3) + 32Y(4) + 80Y(5) = 32, \\ 3Y(3) + 8Y(4) + 60Y(5) = 8. \end{cases} \tag{35}$$

By solving the equation system, we have

$$\begin{aligned}y(3) &= 0, \\ y(4) &= 1, \\ y(5) &= 0.\end{aligned} \tag{36}$$

By using the inverse-reduced DT $y\,(k)$, the solutions will be as follows:

$$y(\psi) = \sum_{k=0}^{\infty} Y(k)\psi^k = 1 + \psi^4. \tag{37}$$

This is the exact solution of the Example 2.

Example 3. Consider the third-order MS-FDEs

$$\begin{cases} y'''(\psi-1) + \dfrac{1}{\psi}y''(\psi+1) + \dfrac{2}{x^2}y'(\psi+2) + \psi y(\psi) = \psi^4 \\ + \psi + 18 + \dfrac{30}{\psi} + \dfrac{24}{\psi^2}, \\ y(0) = 1, \\ y'(0) = 0, \\ y''(0) = 0. \end{cases} \tag{38}$$

Multiplying by ψ^2, equation (38) takes the form as

$$\begin{aligned}\psi^2 y'''(\psi-1) + \psi y''(\psi+1))2y'(\psi+2) + \psi^3 y(\psi) \\ = \psi^6 + \psi^3 + 18\psi^2 + 30\psi + 24.\end{aligned} \tag{39}$$

Using the definitions of the one-dimensional DT scheme, we get

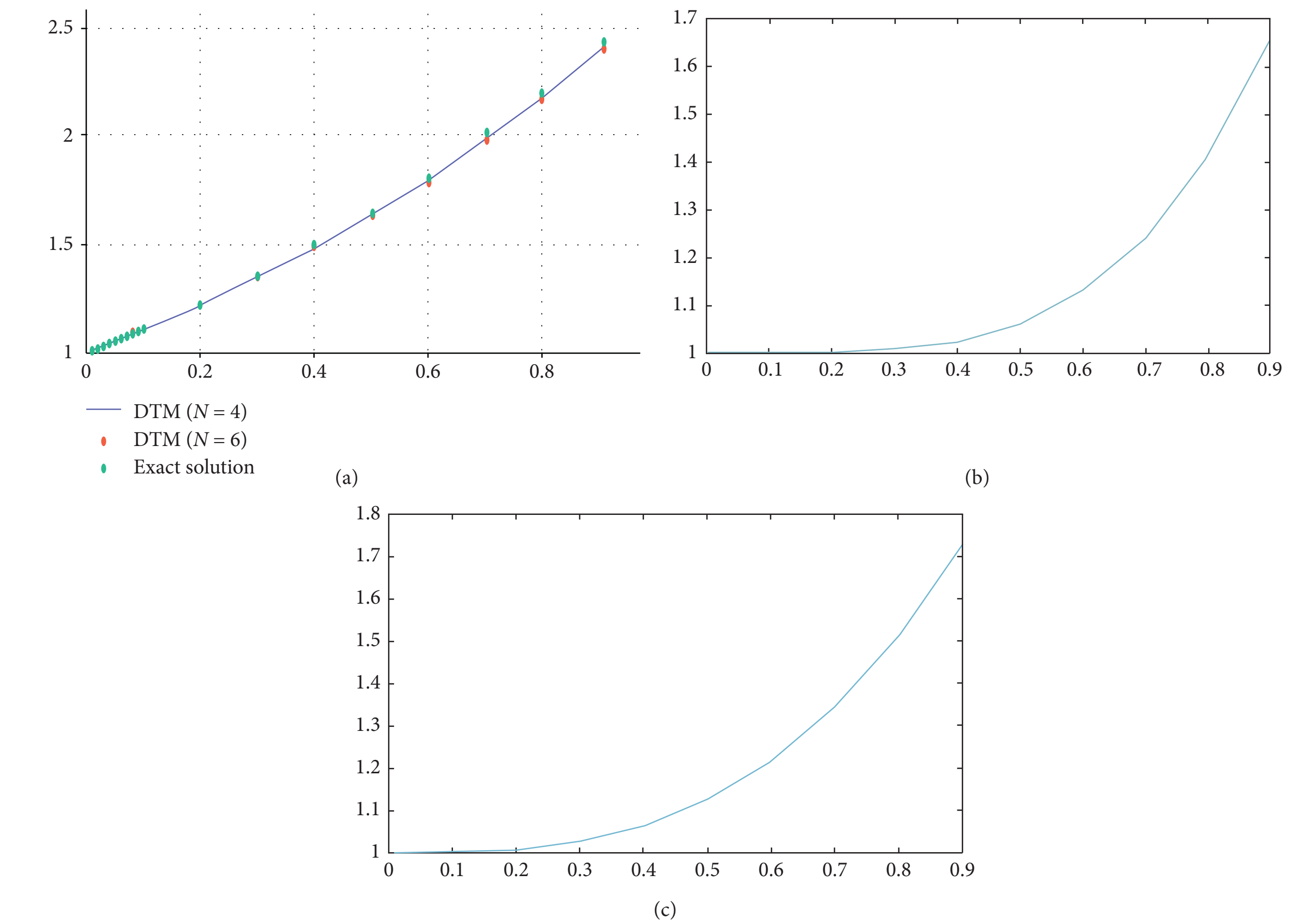

FIGURE 1: Plots of $y(\psi)$ for Examples 1–3 in the range of $0.01 < \psi < 0.9$. Comparison of numerical results of $y(\psi)$ for (a) Example 1, (b) Example 2, and (c) Example 3.

$$\begin{aligned}
&\delta(k-2)\otimes(k+3)(k+2)(k+1)\sum_{\sigma=k+3}^{N}(-1)^{\sigma-k-3}\binom{\sigma}{k+3}Y(\sigma)\\
&\quad+\delta(k-1)\otimes(k+2)(k+1)\sum_{\sigma=k+2}^{N}(1)^{\sigma-k-2}\binom{\sigma}{k+2}Y(\sigma)\\
&\quad+2(k+1)\sum_{\sigma=k+1}^{N}(2)^{\sigma-k-1}\binom{\sigma}{k+1}Y(\sigma)+\delta(k-3)\otimes Y(k)\\
&\quad=\delta(k-6)+\delta(k-3)+18\delta(k-2)+30\delta(k-1)+24\delta(k).
\end{aligned}\tag{40}$$

By using Theorem 1, we get

$$\begin{aligned}
&\sum_{\eta=0}^{k}\sum_{\sigma=\eta+3}^{N}\delta(k-\eta-2)(\eta+3)(\eta+2)(\eta+1)(-1)^{\sigma-l-3}\binom{\sigma}{\eta+3}Y(\sigma)\\
&\quad+\sum_{\eta=0}^{k}\sum_{\sigma=\eta+2}^{N}\delta(k-\eta-1)(\eta+2)(\eta+1)\binom{\sigma}{\eta+2}Y(\sigma)\\
&\quad+2(k+1)\sum_{\sigma=k+1}^{N}(2)^{\sigma-k-1}\binom{\sigma}{k+1}Y(\sigma)+\sum_{\eta=0}^{k}\delta(\eta-3)Y(k-\eta)\\
&\quad=\delta(k-6)+\delta(k-3)+18\delta(k-2)+30\delta(k-1)+24\delta(k).
\end{aligned}\tag{41}$$

Using the ICs of equation (38), we have

$$\begin{aligned}
Y(0)&=1,\\
Y(1)&=0,\\
Y(2)&=0.
\end{aligned}\tag{42}$$

Taking $N = 4$ and $k = 0$ and 1 in equations (41) and (42), the linear algebraic equations system is achieved as

$$\begin{cases}
12Y(3)+32Y(4)=12,\\
30Y(3)+108Y(4)=30.
\end{cases}\tag{43}$$

Solving the above system, we get

$$\begin{aligned}
y(3)&=1,\\
y(4)&=0.
\end{aligned}\tag{44}$$

Using the values of $Y(k)$ for $k = 0$ and 1 into the inverse-reduced DT of $y(\psi)$, the solution will be as follows:

$$y(\psi)=\sum_{k=0}^{\infty}Y(k)\psi^{k}=1+\psi^{3}.\tag{45}$$

Repeat the same process for $N = 5$ and $k = 0, 1$, and 2 by using the above equations. The obtained linear algebraic equation system is given as

$$\begin{cases} 30Y(3) + 108Y(4) + 340Y(5) = 30, \\ 12Y(3) + 32Y(4) + 80Y(5) = 12, \\ 3Y(3) + 8Y(4) + 60Y(5) = 3. \end{cases} \quad (46)$$

The solution of the obtained system of equations is

$$\begin{aligned} y(3) &= 1, \\ y(4) &= 0, \\ y(5) &= 0. \end{aligned} \quad (47)$$

By using the inverse-reduced DT of $y(k)$, the solutions becomes

$$y(\psi) = \sum_{k=0}^{\infty} Y(k)\psi^k = 1 + \psi^3. \quad (48)$$

which is the exact solution. $y(\psi)$ is calculated for different values of N and shown in Figure 1(c).

For more clear understanding, Figure 1 is plotted that has been drawn between 0.01 and 0.9. The values of N are taken as 4 and 6. One can see that the exact and present solutions for $N = 4$ and $N = 6$ are overlapped to each other in the range of 0.01 to 0.09. However, by increasing a slight value in the step size, the results are slightly different but accurate. So it can be concluded that small step size gives more accurate values as compared to large step size.

4. Conclusion

The task to model the third-order MS-FDEs is very difficult to handle as well as construct the differential equations of the designed model. The numerical differential transformation scheme is applied successfully to check the correctness and the accurateness of the designed model. The traditional/conventional techniques fail to solve such multisingular, nonlinear, functionality, and harder nature models. The numerical differential transformation scheme is a good choice to solve such types of complicated, nonlinear, and multisingular models. Consequently, the adopted scheme is effective as well as suitable too. The present study shows that the DTM is an effective and suitable technique to solve such types of equations that we have investigated here. The comparison of the exact and solutions obtained from the differential transformation scheme has also been presented in tabular form as well as graphically. The overlapping of the results shows the perfection of the designed model and establishes the worth of the designed scheme. However, it is observed that when the step size is small, the results are more accurate, but making a slight increase in the step size, the results are overlapped and the error is reduced.

In future, a system of third-order and fourth-order multisingular functional models will be modeled and it will be verified by using the differential transformation scheme as well as famous artificial neural networks [34–39].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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