Journal of Function Spaces

Developments in Geometric Function Theory

Lead Guest Editor: Gangadharan Murugusundaramoorthy
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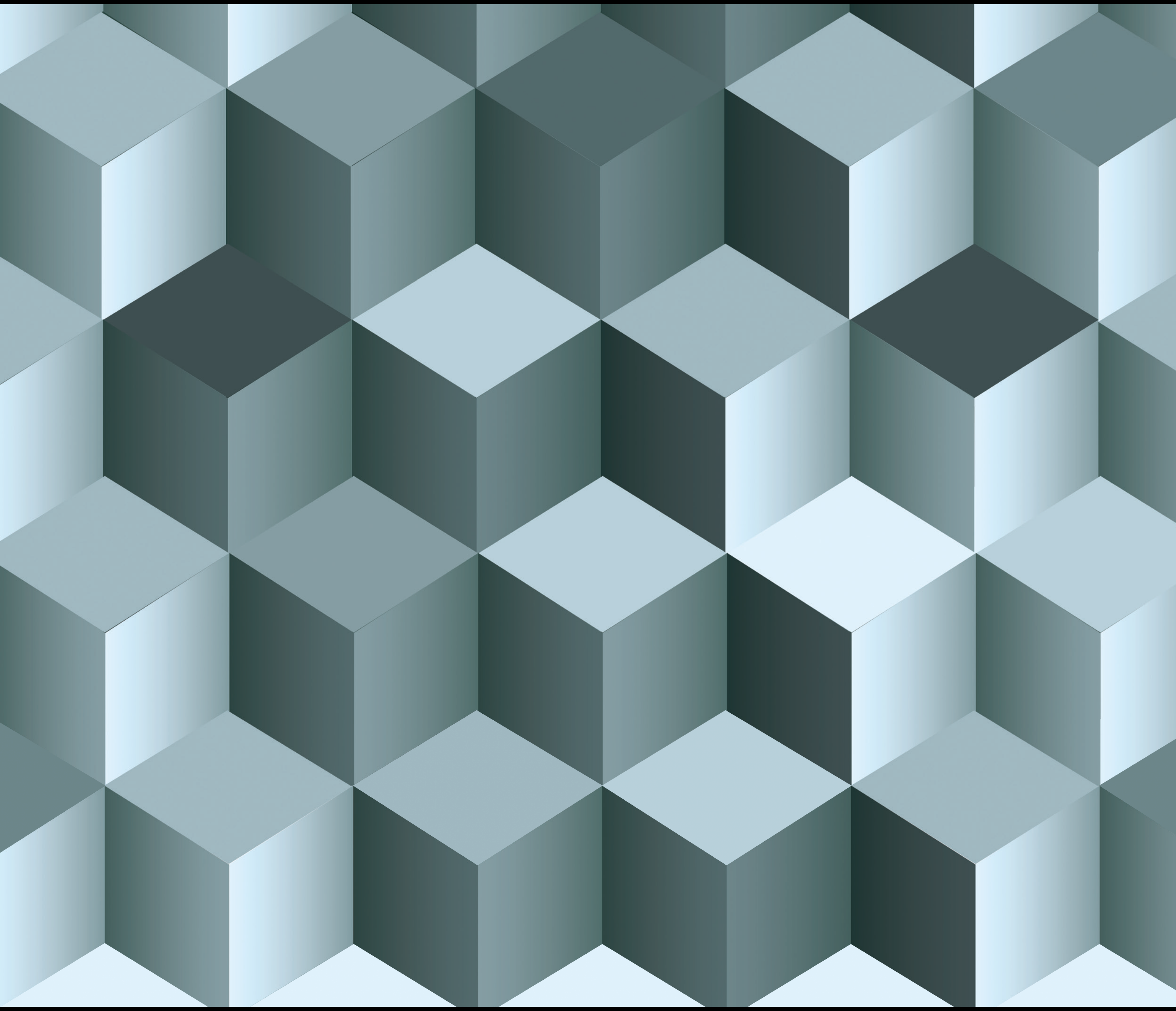

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Contents

Contents

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Research Article

Univalent Functions by Means of Chebyshev Polynomials

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The primary motivation of the paper is to define a new class $Ch_\delta(\alpha, \beta, \gamma)$ which consists of univalent functions associated with Chebyshev polynomials. For this class, we determine the coefficient bound and convolution preserving property. Furthermore, by using subordination structure, two new subclasses of $Ch_\delta(\alpha, \beta, \gamma)$ are introduced and denoted by $M(\lambda_1, \lambda_2, s)$ and $N(\lambda_1, \lambda_2, s)$, respectively. For these subclasses, we obtain coefficient estimate, extreme points, integral representation, convexity, geometric interpretation, and inclusion results. Moreover, we prove that, under some restrictions on parameters, $Ch_\delta(\alpha, \beta, \gamma) = N(\lambda_1, \lambda_2, s)$.

1. Introduction

Let $\mathscr{A}$ be the class of analytic univalent functions in the open unit disk:

$$\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}, \tag{1}$$

with Taylor expansion series of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

Also, denote by $\mathscr{S}$ the class of univalent functions which are normalized by $f(0) = f'(0) - 1 = 0$, see [1, 2]. Furthermore, suppose that $\mathscr{N}$ be the subclass of $\mathscr{A}$ consisting of functions with negative coefficients of the type:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{3}$$

The significance of Chebyshev polynomials in numerical analysis is very important in both practical and theoretical points of view. There are four kinds of such polynomials. Many researchers consider orthogonal polynomials of Chebyshev and obtain many interest results.

The Chebyshev polynomials of the first and second kinds are well known and introduced by

$$T_n(t) = \cos\theta \,\text{and}\, U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, (-1 < t < 1), \tag{4}$$

where $t = \cos\theta$ and n is the degree of polynomial. For more details, one may refer to [1–6]. The polynomials in (4) are connected by the following relations:

$$\begin{aligned} \frac{dT_n(t)}{dt} &= nU_{n-1}(t), \\ T_n(t) &= U_n(t) - tU_{n-1}(t), \end{aligned} \tag{5}$$

$$2T_n(t) = U_n(t) - U_{n-2}(t). \tag{6}$$

We note that if $t = \cos\theta$, where $\theta \in (-\pi/3, \pi/3)$, then

$$\begin{aligned} H(z,t) &= \frac{1}{1 - 2\cos\theta z + z^2} \\ &= 1 + \sum_{k=2}^{\infty} \frac{\sin(k+1)\theta}{\sin\theta} z^k, \quad (z \in \mathbb{D}). \end{aligned} \tag{7}$$

Also, we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots, \quad (z \in \mathbb{D}, \ -1 < t < 1), \tag{8}$$

where

$$U_{n-1}(t) = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}}, \quad (n \in \mathbb{N}), \tag{9}$$

are the Chebyshev polynomials of the second kind, see [7–9].

The Hadamard product (convolution) for functions

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{10}$$

is denoted by $f * g$ and defined as follows:

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \quad (z \in \mathbb{D}). \tag{11}$$

The generating function of the first kind of Chebyshev polynomial $T_n(t)$, $t \in [-1,1]$ is given by

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2}, \quad (z \in \mathbb{D}), \tag{12}$$

see [10].

Now, we consider the functions:

$$H_1(z) = 1 + (2\cos\theta + 1)z - H(z,t), \quad H_2(z) = 1 + (1+\cos\theta)z - \frac{1-tz}{1-2tz+z^2}, \tag{13}$$

$$V(z) = (H_1 * H_1) * (H_2 * H_2) * f(z), \tag{14}$$

where $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{N}$ and " $*$ " denotes the convolution. By a simple calculation, we conclude that $V(z) \in \mathcal{N}$ and in the form

$$V(z) = z - \sum_{k=2}^{\infty} T_k^2(t) \frac{\sin^2(k+1)\theta}{\sin^2\theta} a_k z^k, \tag{15}$$

where $\theta \in (-\pi/3, \pi/3)$ and $t = \cos\theta$.

Let $Ch_\delta(\alpha,\beta,\gamma)$ denote the subclass of $\mathcal{N}$ consisting of functions of form (15) satisfying the condition:

$$\left| \frac{V'(z) + zV''(z) - 1}{2\gamma V'(z) - \alpha(1+\beta)\gamma} \right| < \delta, \tag{16}$$

where $0 < \alpha \le 1$, $0 < \beta \le 1$, $0 \le \gamma \le 1$, $0 \le \delta < 1$, and $V(z)$ is given by (15), see [11].

2. Main Results

In this section, we introduce a sharp coefficient bound for $V(z) \in Ch_\delta(\alpha,\beta,\gamma)$. Also, convolution preserving property under parameters α and β is proved.

Theorem 1. *$V(z) \in Ch_\delta(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{k(k+2\gamma\delta) T_k^2(t) \sin^2(k+1)\theta}{\sin^2\theta} a_k \le \delta\gamma(2 - \alpha(1+\beta)). \tag{17}$$

Proof. Let inequality (17) hold true, and suppose $z \in \partial\mathbb{D} = \{z \in \mathbb{D}: |z| = 1\}$. Then, we obtain

$$\begin{aligned} &\left|V'(z) + zV''(z) - 1\right| - \delta\left|2\gamma V'(z) - \alpha(1+\beta)\gamma\right| \\ &= \left| -\sum_{k=2}^{\infty} \left[\frac{kT_k(t)(\sin(k+1)\theta)}{\sin\theta}\right]^2 a_k z^k \right| \\ &\quad - \delta\left| 2\gamma - \sum_{k=2}^{\infty} \frac{2\gamma k T_k^2(t)\sin^2(k+1)\theta}{\sin^2\theta} a_k z^{k-1} - \alpha(1+\beta)\gamma \right| \\ &= \sum_{k=2}^{\infty} \frac{k(k+2\gamma\delta)T_k^2(t)\sin^2(k+1)\theta}{\sin^2\theta} - \delta\gamma(2-\alpha(1+\beta)) \le 0. \end{aligned} \tag{18}$$

Hence, by maximum modulus theorem, we conclude that $V(z) \in Ch_\delta(\alpha,\beta,\gamma)$.

Conversely, let $V(z)$, defined by (15), be in the class $Ch_\delta(\alpha,\beta,\gamma)$, so condition (16) yields

$$\left| \frac{V'(z) + 2V''(z) - 1}{2\gamma V'(z) - \alpha(1+\beta)\gamma} \right| = \left| \frac{\sum_{k=2}^{\infty} k^2 T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta a_k z^{k-1}}{2\gamma - \sum_{k=2}^{\infty} 2\gamma k T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta a_k z^{k-1} - \alpha(1+\beta)\gamma} \right| < \delta. \tag{19}$$

Since, for any z, $|\mathrm{Re} z| < |z|$, then

$$\mathrm{Re}\left\{ \frac{\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta\left[k^2\right] a_k z^{k-1}}{\gamma(2-\alpha(1+\beta)) - \sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta[2\gamma k] a_k z^{k-1}} \right\} < \delta. \tag{20}$$

By letting $z \longrightarrow 1$ through real values, we obtain

$$\sum_{k=2}^{\infty}\frac{T_k^2(t)\sin^2(k+1)\theta}{\sin^2\theta}\left[k^2\right]a_k \le \delta\gamma(2-\alpha(1+\beta)) - \sum_{k=2}^{\infty}\frac{T_k^2(t)\sin^2(k+1)\theta}{\sin^2\theta}[\alpha\gamma\delta k]a_k, \quad (21)$$

and this completes the proof. □

Remark 1. We note that the function,

$$W(z) = z - \frac{\left(\sin^2\theta\right)\gamma\delta(2-\alpha(1+\beta))}{4(1+\gamma\delta)\cos^2 2\theta\,\sin^2 3\theta}z^2, \quad (22)$$

shows that inequality (17) is sharp.

Theorem 2. *If $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ are in the class $Ch_\delta(\alpha,\beta,\gamma)$, then*

(i) $(f * g)(z)$ belongs to $Ch_\delta(\alpha,\beta,\gamma)[\alpha^*]$, where

$$\alpha^* \le \frac{2}{1+\beta} - \left(\frac{\sin\theta(2-\alpha(1+\beta))}{T_k(t)\sin(k+1)\theta}\right)^2 \frac{\gamma\delta}{k(k+2\gamma\delta)(1+\beta)}. \quad (23)$$

(ii) $(f * g)(z)$ belongs to $Ch_\delta(\alpha,\beta^*,\gamma)$, where

$$\beta^* \le \frac{2}{\alpha} - \left(1 + \left(\frac{\sin\theta(2-\alpha(1+\beta))}{T_k(t)\sin(k+1)\theta}\right)^2 \frac{\gamma\delta}{k(k+2\gamma\delta)(1+\beta)}\right). \quad (24)$$

Proof. (i) It is sufficient to show that

$$\sum_{k=2}^{\infty}\frac{k(k+2\gamma\delta)T_k^2(t)\sin^2(k+1)\theta}{\sin^2\theta\delta\gamma(2-\alpha(1+\beta))[\alpha^*]}a_k b_k \le 1. \quad (25)$$

By using the Cauchy–Schwarz inequality from (31), we obtain

$$\sum_{k=2}^{\infty}\frac{k(k+2\gamma\delta)\sin^2(k+1)\theta}{\sin^2\theta\delta\gamma(2-\alpha(1+\beta))[\alpha^*]}\sqrt{a_k b_k} \le 1. \quad (26)$$

Hence, we find the largest α^* such that

$$\sum_{k=2}^{\infty}\frac{k(k+2\gamma\delta)\sin^2(k+1)\theta}{\sin^2\theta\delta\gamma(2-\alpha^*(1+\beta))}a_k b_k \le \sum_{k=2}^{\infty}\frac{k(k+2\gamma\delta)\sin^2(k+1)\theta}{\sin^2\theta\delta\gamma(2-\alpha(1+\beta))}\sqrt{a_k b_k} \le 1. \quad (27)$$

This inequality holds if

$$\frac{\sin^2\theta\delta\gamma(2-\alpha(1+\beta))}{k(k+2\gamma\delta)T_k^2(t)\sin^2(k+1)\theta} \le \frac{2-\alpha^*(1+\beta)}{2-\alpha(1+\beta)}, \quad (28)$$

or equivalently

$$\alpha^* \le \frac{2}{1+\beta} + \left(\frac{\sin\theta(2-\alpha(1+\beta))}{T_k(t)\sin(k+1)\theta}\right)^2 \frac{\gamma\delta}{k(k+2\gamma\delta)(1+\beta)}. \quad (29)$$

(ii) By using the same techniques as in the part (i), we can easily prove the part (ii), so the proof is complete. □

3. Subclass of $Ch_\delta(\alpha,\beta,\gamma)$ and Their Geometric Properties

In this section, we introduce two new subclasses of $Ch_\delta(\alpha,\beta,\gamma)$ and conclude their geometric properties.

For analytic functions $f(z)$ and $F(z)$ in $\mathbb{D}$, we say f is subordinate to F, written $f \prec F$, if there exists a function w analytic in $\mathbb{D}$, with $w(0)=0$ and $|w(z)|<1$ such that $f(z) = F(w(z))$, see [3, 12]. If F is univalent, then

$$f \prec F \Leftrightarrow f(0) = F(0), \quad f(\mathbb{D}) \subset F(\mathbb{D}). \quad (30)$$

Let $M(\lambda_1,\lambda_2,s)$ consist of all analytic functions $g(z) \in \mathbb{D}$ for which $g(0)=1$ and

$$g(z) \prec \frac{1+(\lambda_2+(\lambda_1-\lambda_2)(1-s))z}{1+\lambda_2 z}, \quad (31)$$

where $-1 \le \lambda_1 < \lambda_2 \le 1$, $0 < \lambda_2 \le 1$, and $0 \le s < 1$. Let $N(\lambda_1,\lambda_2,s)$ denote the class of all functions $V(z) \in Ch_\delta(\alpha,\beta,\gamma)$ for which

$$\frac{zV'(z)}{V(z)} \in M(\lambda_1,\lambda_2,s), \quad (32)$$

where $V(z)$ is given by (15).

Theorem 3. *$V(z) \in N(\lambda_1,\lambda_2,s)$ if and only if*

$$\sum_{k=2}^{\infty}\left(1 + \frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_k < 1. \quad (33)$$

Proof. Let $V(z) \in N(\lambda_1,\lambda_2,s)$; then, by (16), (31), and (32), we have

$$\left|\frac{\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta[(k-1)]a_k z^k}{(\lambda_2-\lambda_1)(1-s)-\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta[\lambda_2(k-1)+(\lambda_2-\lambda_1)(1-s)]a_k z^k}\right|<1, \tag{34}$$

which implies that

$$\operatorname{Re}\left\{\frac{\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta a_k z^k}{(\lambda_2-\lambda_1)(1-s)-\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta[\lambda_2(k-1)+(\lambda_2-\lambda_1)(1-s)]a_k z^k}\right\}<1. \tag{35}$$

Now, we choose the values of z on the real axis, and letting $z \longrightarrow 1^-$, we get the required result.

Conversely, assume that condition (33) holds true. We must show that $V(z) \in N(\lambda_1, \lambda_2, s)$ or equivalently

$$|Y| = \left|\frac{V(z)-zV'(z)}{\lambda_2 zV'(z)-(\lambda_2+(\lambda_2-\lambda_1)(1-s))V(s)}\right|<1.$$

However, we have (36)

$$|Y| = \left|\frac{\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta a_k}{(\lambda_2-\lambda_1)(1-s)-\sum_{k=2}^{\infty} T_k^2(t)\sin^2(k+1)\theta/\sin^2\theta[\lambda_2(k-1)+(\lambda_2-\lambda_1)(1-s)]}\right|.$$

By using (33), we get $|Y|<1$, so the proof is complete. □

Corollary 1. *Let $V \in N(\lambda_1, \lambda_2, s)$; then,*

$$a_k < \frac{(\lambda_2-\lambda_1)(1-s)(T_k(t)\sin(k+1)\theta/\sin\theta)^2}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}. \tag{37}$$

Theorem 4. *Let $\lambda_2 \neq 1$, $V(z) \in N(\lambda_1, \lambda_2, s)$, and*

$$\frac{zV'(z)}{V(z)} = a+ib = X. \tag{38}$$

Then, the values of X lie in the circle.

Proof. By (31) and (32), we have

$$X = a+ib = \frac{1+(\lambda_2+(\lambda_2-\lambda_1)(1-s))W(z)}{1+\lambda_2 W(z)}, \quad (|W(z)|<1). \tag{39}$$

Then,

$$\begin{aligned}(a+ib)(1+\lambda_2 W(z)) &= 1+(\lambda_2+(\lambda_2-\lambda_1)(1-s))W(z)\\ \text{or } a-1+ib &= [\lambda_2+(\lambda_2-\lambda_1)(1-s)-a\lambda_2-ib\lambda_2]W(z).\end{aligned} \tag{40}$$

After a simple calculation, we obtain

$$\left[a-\frac{1-\lambda_2(\lambda_2+(\lambda_2-\lambda_1)(1-s))}{1-\lambda_2^2}\right]^2+b^2<\left[\frac{(\lambda_2-\lambda_1)(1-s)}{1-\lambda_2^2}\right]^2. \tag{41}$$

Hence, the value of X lies in the circle with center at

$$\left(\frac{1-\lambda_2(\lambda_2+(\lambda_2-\lambda_1)(1-s))}{1-\lambda_2^2}, 0\right) \tag{42}$$

and radius $(\lambda_2-\lambda_1)(1-s)/1-\lambda_2^2$. □

Theorem 5. *If*

$$\frac{\lambda_2+1}{(\lambda_2-\lambda_1)(1-s)} \leq \frac{k^2+2\gamma\delta(k-1)+\gamma\alpha\delta(1+\beta)}{\delta\gamma(2-\alpha(1+\beta))}, \tag{43}$$

then $Ch_\delta(\alpha,\beta,\gamma) = N(\lambda_1, \lambda_2, s)$.

Proof. By equation (32), we have

$$N(\lambda_1, \lambda_2, s) \subseteq Ch_\delta(\alpha,\beta,\gamma). \tag{44}$$

Now, assume that $V \in Ch_\delta(\alpha,\beta,\gamma)$; then, by Theorem 1, we have

$$\sum_{k=2}^{\infty} k(k+2\gamma\delta)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_k \le \delta\gamma(2-\alpha(1+\beta)). \tag{45}$$

By Theorem 3, it is enough to show that (33) holds true, which is possible when

$$\left[1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)}\right] \le \frac{k(k+2\gamma\delta)}{\delta\gamma(2-\alpha(1+\beta))}, \tag{46}$$

or equivalently

$$\frac{(k-1))(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)} \le \frac{k^2+2\gamma\delta(k-1)+\delta\gamma\alpha(1+\beta)}{\delta\gamma(2-\alpha(1+\beta))}. \tag{47}$$

Since k starts from 2, then $k-1\ge 1$, and hence, from the last inequality, we obtain the required result. □

In the next theorems, we prove the inclusion property and convex combination concept. Also, extreme points and integral representation are introduced.

Theorem 6. *Let $0\le s_2<s_1<1$; then,*

$$N(\lambda_1,\lambda_2,s)[s_1] \subset N(\lambda_1,\lambda_2,s)[s_2]. \tag{48}$$

Proof. Suppose $f\in N(\lambda_1,\lambda_2,s)[s_1]$; then, by Theorem 3, we have

$$\sum_{k=2}^{\infty}\left(1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)[s_1]}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_k \le \delta\gamma(2-\alpha(1+\beta)). \tag{49}$$

We have to prove

$$\sum_{k=2}^{\infty}\left(1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s))}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_k \le \delta\gamma(2-\alpha(1+\beta)). \tag{50}$$

However, the last inequality holds true if

$$1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)} \le 1+\frac{(k-1)(\lambda_2+1)}{(\lambda_2-\lambda_1)(1-s)}, \tag{51}$$

and this inequality by hypothesis $(s_2<s_1)$ definitely holds true, so the proof is complete. □

Theorem 7. *$N(\lambda_1,\lambda_2,s)$ is a convex set.*

Proof. We have to prove that if

$$V_j(z) = z-\sum_{k=2}^{\infty}\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_{k,j}z^k, \quad (j=1,2,\ldots,m), \tag{52}$$

is in the class $N(\lambda_1,\lambda_2,s)$, then the function,

$$L(z) = \sum_{j=1}^{m} d_j V_j(z), \tag{53}$$

is also in $N(\lambda_1,\lambda_2,s)$, where $\sum_{j=1}^{m} d_j = 1$. However, we have

$$L(z) = z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 d_j a_{k,j}\right)z^k. \tag{54}$$

We have to prove that if $f_j(z)$ $(j=1,2,\ldots,m)$ is in the class $N(\lambda_1,\lambda_2,s)$, then the function $L(z)=\sum_{j=1}^{m} d_j f_j(z)$ is also in $N(\lambda_1,\lambda_2,s)$, where $\sum_{j=1}^{m} d_j = 1$. We have

$$L(z) = z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} d_j a_{k,j}\right)z^k. \tag{55}$$

Since, by Theorem 3,

$$\begin{aligned}&\sum_{k=2}^{\infty}\left(1+\frac{(\lambda_2+1)(k-1)}{ls}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2\left(\sum_{j=1}^{m} d_j a_{k,j}\right)\\ &\quad=\sum_{j=1}^{m}\left[\sum_{k=2}^{\infty}\left(1+\frac{(\lambda_2+1)(k-1)}{(\lambda_2-\lambda_1)(1-s)}\right)\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 a_{k,j}\right]d_j\\ &\quad<\sum_{j=1}^{m} d_j = 1\end{aligned} \tag{56}$$

so $L\in N(\lambda_1,\lambda_2,s)$ and the proof is complete. □

Theorem 8. *The function $V_1(z)=z$ and*

$$V_k(z) = z - \left(\frac{(\lambda_2-\lambda_1)(1-s)}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}\right)\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 a_k z^k, \quad (k\geq 2), \tag{57}$$

are the extreme points of $N(\lambda_1,\lambda_2,s)$.

Proof. We have to prove that $L \in N(\lambda_1,\lambda_2,s)$ if and only if

$$L(z) = \sum_{k=2}^{\infty}[1]d_k V_k(z), \tag{58}$$

where $d_k \geq 0$ $(k\geq 1)$ and $\sum_{k=2}^{\infty}[1]d_k = 1$. □

Proof. Let $L \in N(\lambda_1,\lambda_2,s)$. If we set

$$d_k = \frac{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}{(\lambda_2-\lambda_1)(1-s)}\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 a_k, \quad (k\geq 2), \tag{59}$$

we get $d_k \geq 0$, and if we put $d_1 = 1-\sum_{k=2}^{\infty}d_k$, then we obtain

$$\begin{aligned} L(z) &= z - \sum_{k=2}^{\infty}\frac{(\lambda_2-\lambda_1)(1-s)}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 d_k z^k \\ &= z - \sum_{k=2}^{\infty}d_k(z-V_k(z)) \\ &= \sum_{k=2}^{\infty}d_k V_k(z). \end{aligned} \tag{60}$$

Conversely, suppose

$$L(z) = \sum_{k=2}^{\infty}[1]d_k V_k(z). \tag{61}$$

Then, we have

$$\begin{aligned} L(z) &= d_1V_1(z) + \sum_{k=2}^{\infty}d_kV_k(z) \\ &d_1 z + \sum_{k=2}^{\infty}\left[z - \frac{(\lambda_2-\lambda_1)(1-s)}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 d_k z^k\right] \\ &= z - \sum_{k=2}^{\infty}\frac{(\lambda_2-\lambda_1)(1-s)}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2 d_k z^k. \end{aligned} \tag{62}$$

Since

$$\sum_{k=2}^{\infty} d_k \left(1+\frac{(\lambda_2+1)(k-1)}{(\lambda_2-\lambda_1)(1-s)}\right)\left(\frac{(\lambda_2-\lambda_1)(1-s)}{(\lambda_2-\lambda_1)(1-s)+(\lambda_2+1)(k-1)}\right)\times$$
$$\times\left(\frac{\sin\theta}{T_k(t)\sin(k+1)\theta}\right)^2\left(\frac{T_k(t)\sin(k+1)\theta}{\sin\theta}\right)^2 \tag{63}$$
$$=\sum_{k=2}^{\infty} d_k = 1-d_1<1,$$

therefore, by Theorem 3, we conclude the result. □

Theorem 9. *Let $f\in N(\lambda_1,\lambda_2,s)$; then,*

$$V(z)=\exp\left(\int_0^z \frac{1-(\lambda_2+(\lambda_2-\lambda_1)(1-s))W(t)}{t(1-\lambda_2 W(t))}\mathrm{d}t\right), \tag{64}$$

where $|W(z)|<1$.

Proof. By letting $U(z)=zV'/V$, since $f\in N(\lambda_1,\lambda_2,s)$, so

$$U(z)\prec\frac{1+(\lambda_2+(\lambda_1-\lambda_2)(1-s))z}{1+\lambda_2 z}, \tag{65}$$

or equivalently

$$\left|\frac{U(z)-1}{U(z)\lambda_2-(\lambda_2+(\lambda_1-\lambda_2)(1-s))}\right|<1. \tag{66}$$

Therefore,

$$\frac{U(z)-1}{U(z)\lambda_2-(\lambda_2+(\lambda_1-\lambda_2)(1-s))}=W(z), \quad (|w(z)|<1). \tag{67}$$

Hence, we can write

$$\frac{V'(z)}{V(z)}=\frac{1-(\lambda_2+(\lambda_1-\lambda_2)(1-s))W(z)}{z(1-\lambda_2 W(z))}. \tag{68}$$

After integration, we get the required result. □

4. Conclusion

Univalent functions have always been the main interests of many researchers in geometric function theory. Many studies recently related to Chebyshev polynomials revolved around classes of analytic normalized univalent functions. In this particular work, the geometric properties are obtained for functions in more general class denoted by $Ch_\delta(\alpha,\beta,\gamma)$ using the Chebyshev polynomials associated with the convolution structure. Some other geometric results are introduced for the subclasses of $Ch_\delta(\alpha,\beta,\gamma)$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Radius Problems for Starlike Functions Associated with the Tan Hyperbolic Function

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The aim of this particular article is at studying a holomorphic function f defined on the open-unit disc $\mathfrak{D} = \{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| < 1\}$ for which the below subordination relation holds $\mathfrak{z}f'(\mathfrak{z})/f(\mathfrak{z}) \prec q_0(\mathfrak{z}) = 1 + \tan h(\mathfrak{z})$. The class of such functions is denoted by $\mathfrak{S}^*_{\tan h}$. The radius constants of such functions are estimated to conform to the classes of starlike and convex functions of order β and Janowski starlike functions, as well as the classes of starlike functions associated with some familiar functions.

1. Introduction

To completely comprehend the mathematical concepts used throughout our key observations, some of the essential literature of the geometric function theory must be described and analyzed here. Let us begin with the symbol $\mathfrak{A}_n$ which describes the family of holomorphic (or analytic) functions in a subset $\mathfrak{D}$ of the complex plan $\mathbb{C}$ having the following series expansion

$$f(\mathfrak{z}) = \mathfrak{z} + a_{n+1}\mathfrak{z}^{n+1} + a_{n+2}\mathfrak{z}^{n+2} + \cdots. \quad (1)$$

Also, let the family of all univalent functions be denoted by $\mathfrak{S}$ and is a subset of the class $\mathfrak{A}_1 = \mathfrak{A}$. Next, we define that the subordination between the function belongs to the class $\mathfrak{A}$. Let $g_1, g_2 \in \mathfrak{A}$. Then, $g_1 \prec g_2$ or $g_1(\mathfrak{z}) \prec g_2(\mathfrak{z})$, the mathematical form of the subordination between g_1 and g_2, if a holomorphic function w occurs in $\mathfrak{D}$ with the restriction $w(0) = 0$ and $|w(\mathfrak{z})| < 1$ in such a way that $f(\mathfrak{z}) = g(w(\mathfrak{z}))$ hold. Further, if $g_2 \in \mathfrak{S}$ in $\mathfrak{D}$, then, the following relation holds:

$$g_1(\mathfrak{z}) \prec g_2(\mathfrak{z}), \Leftrightarrow g_1(0) = g_2(0) \text{ and } g_1(\mathfrak{D}) \subset g_2(\mathfrak{D}). \quad (2)$$

Three significant subfamilies of $\mathfrak{S}$, which are well studied and have nice geometric interpretations, are the families of starlike $\mathfrak{S}^*(\xi)$, convex $\mathcal{K}(\xi)$, and strongly starlike $\mathfrak{S}\mathfrak{S}^*(\zeta)$ functions of order $\xi(0 \le \xi < 1)$ and $\zeta(0 < \zeta \le 1)$, respectively. These families are defined as follows:

$$\begin{aligned}
\mathfrak{S}\mathfrak{S}^*(\zeta) &:= \left\{ f \in \mathfrak{S} : \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} \prec \left(\frac{1+\mathfrak{z}}{1-\mathfrak{z}}\right)^{\zeta}, (\mathfrak{z} \in \mathfrak{D}) \right\}, \\
\mathfrak{S}^*(\xi) &:= \left\{ f \in \mathfrak{S} : \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} \prec \frac{1+(1-2\xi)\mathfrak{z}}{1-\mathfrak{z}}, (\mathfrak{z} \in \mathfrak{D}) \right\}, \\
\mathcal{K}(\xi) &:= \left\{ f \in \mathfrak{S} : \frac{\left(\mathfrak{z}f'(\mathfrak{z})\right)'}{f'(\mathfrak{z})} \prec \frac{1+(1-2\xi)\mathfrak{z}}{1-\mathfrak{z}}, (\mathfrak{z} \in \mathfrak{D}) \right\}.
\end{aligned} \quad (3)$$

Particularly, the notations $\mathfrak{S}\mathfrak{S}^*(1) = \mathfrak{S}^*(0) = \mathfrak{S}^*$ and $\mathscr{K}(0) = \mathscr{K}$ represent familiar families of starlike and convex functions, respectively. These subfamilies of $\mathfrak{S}$ satisfy the following relationship

$$\mathscr{K} \subset \mathfrak{S}^* \subset \mathfrak{S}. \tag{4}$$

The reverse of the above relation hold only under certain restriction of the domain. That is; if $f \in \mathfrak{S}$ in $\mathfrak{D}$, then, it was given in [1], Corollary, p. 98, that f maps the disc $|\mathfrak{z}| < r$ onto a region which is star shaped about the origin for every $r \le r_0 = \tan h(\pi/4)$. The constant r_0 is known as the radius of starlikness for the family $\mathfrak{S}$. Also, given in [1], Corollary, p. 44, the radius of convexity for the families $\mathfrak{S}^*$ and $\mathfrak{S}$ is $2 - \sqrt{3}$.

To make a radius statement for other things than starlikeness and convexity, we choose two subfamilies $\mathscr{G}$ and $\mathscr{H}$ of the set $\mathfrak{A}$. The $\mathscr{G}$ radius for the family $\mathscr{H}$, represented by $R_{\mathscr{G}}(\mathscr{H})$, is the largest number R such that $r^{-1}f(r\mathfrak{z}) \in \mathscr{G}$ for every $0 < r \le R$ and $f \in \mathscr{H}$. Consequently, an alternative formulation of the radius of starlikeness for $\mathfrak{S}$ is that the $\mathfrak{S}^*$ radius for the family $\mathfrak{S}$ is $R_{\mathfrak{S}^*}(\mathfrak{S}) = \tan h(\pi/4)$.

In 1992, Ma and Minda [2] considered the general form of the families as

$$\begin{aligned} \mathfrak{S}^*(\varphi) &:= \left\{ f \in \mathfrak{A} : \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} \prec \varphi(\mathfrak{z}) \right\}, \\ \mathscr{K}(\varphi) &:= \left\{ f \in \mathfrak{A} : 1 + \frac{\mathfrak{z}f''(\mathfrak{z})}{f(\mathfrak{z})} \prec \varphi(\mathfrak{z}) \right\}, \end{aligned} \tag{5}$$

where φ is a holomorphic function with $\varphi'(0) > 0$ and has positive real part. Also, the function φ maps $\mathfrak{D}$ onto a star-shaped region with respect to $\varphi(0) = 1$ and is symmetric about the real axis. They addressed some specific results such as distortion, growth, and covering theorems. In recent years, several subfamilies of the set $\mathfrak{A}$ were studied as a special case of the class $\mathfrak{S}^*(\varphi)$. For example,

(i) If we take $\varphi(\mathfrak{z}) = (1 + L\mathfrak{z})/(1 + M\mathfrak{z})$ with $-1 \le M < L \le 1$, then, we achieved the class $\mathfrak{S}^*[L, M] \equiv \mathfrak{S}^*((1 + L\mathfrak{z})/(1 + M\mathfrak{z}))$ which is described by the functions of the Janowski starlike class investigated in [3]. Furthermore, $\mathfrak{S}^*(\xi) := \mathfrak{S}^*[1 - 2\xi, -1]$ is the familiar starlike function family of order ξ with $0 \le \xi < 1$

(ii) The family $\mathfrak{S}^*_{\mathscr{L}} := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ with $\varphi(\mathfrak{z}) = \sqrt{1 + \mathfrak{z}}$ was developed in [4] by Sokol and Stankiewicz. The function $\varphi(\mathfrak{z}) = \sqrt{1 + \mathfrak{z}}$ maps the region $\mathfrak{D}$ onto the the image domain which is bounded by $|w^2 - 1| < 1$

(iii) The class $\mathfrak{S}^*_{\text{car}} := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ with $\varphi(\mathfrak{z}) = 1 + (4/3)\mathfrak{z} + (2/3)\mathfrak{z}^2$ was examined by Sharma and his coauthors [5] which consists of function $f \in \mathfrak{A}$ in such a manner that $(\mathfrak{z}f'(\mathfrak{z}))/(f(\mathfrak{z}))$ is located in the region bounded by the cardioid given by

$$\left(9x^2 + 9y^2 - 18x + 5\right)^2 - 16\left(9x^2 + 9y^2 - 6x + 1\right) = 0 \tag{6}$$

(iv) The family $\mathfrak{S}^*_R := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ with $\varphi(\mathfrak{z}) = 1 + (\mathfrak{z}/J)(J + \mathfrak{z}/J - \mathfrak{z})$, $J = \sqrt{2} + 1$ is studied in [6] while $\mathfrak{S}^*_{\cos} := \mathfrak{S}^*(\cos(\mathfrak{z}))$ and $\mathfrak{S}^*_{\cos h} := \mathfrak{S}^*(\cosh(\mathfrak{z}))$ were contributed by Raza and Bano [7] and Alotaibi et.al [8], respectively

(v) By choosing $\varphi(\mathfrak{z}) = 1 + \sin\mathfrak{z}$, we obtain the class $\mathfrak{S}^*_{\sin} := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ which was established in [9]. The authors determined radius problems in this article for the defined class $\mathfrak{S}^*_{\sin}$

(vi) The class $\mathfrak{S}^*_e := \mathfrak{S}^*(e^{\mathfrak{z}})$ was explored recently in [10]. For such a class $\mathfrak{S}^*_e$, the authors calculated Hankel determinant bounds of order three in [11]. Also, the class $\mathfrak{S}^*_{\mathscr{RL}} := \mathfrak{S}^*(h_{\mathscr{RL}}(\mathfrak{z}))$ with

$$h_{\mathscr{RL}}(\mathfrak{z}) = \sqrt{2} - \left(\sqrt{2} - 1\right)\sqrt{\frac{1 - \mathfrak{z}}{1 + 2\left(\sqrt{2} - 1\right)\mathfrak{z}}} \tag{7}$$

was contributed by Mendiratta et al. [12] in which they investigated the radius problems

(vii) The family $\mathfrak{S}^*_{\mathscr{C}} := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ with $\varphi(\mathfrak{z}) = \mathfrak{z} + \sqrt{1 + \mathfrak{z}^2}$ was introduced and studied by Raina and Sokół [13]

(viii) By considering the function $\varphi(\mathfrak{z}) = 1 + \sin h^{-1}\mathfrak{z}$, we get the recently examined family $\mathfrak{S}^*_\rho := \mathfrak{S}^*(1 + \sin h^{-1}\mathfrak{z})$ introduced by Kumar and Arora [14]. They discussed relationships of this class with the already known classes. For more particular classes, see the articles [15–20]

In the present paper, we consider a trigonometric function $\varphi_1(\mathfrak{z}) = 1 + \tan h\mathfrak{z}$ with $\varphi_1(0) = 1$. Also, one can easily obtain that $\mathfrak{Re}\varphi_1(\mathfrak{z}) > 0$. By using this function, we define the below family of functions as

$$\mathfrak{S}^*_{\tan h} := \left\{ f \in \mathfrak{S} : \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} \prec 1 + \tan h\mathfrak{z}, (\mathfrak{z} \in \mathfrak{D}) \right\}. \tag{8}$$

In other words, a function $f \in \mathfrak{S}^*_{\tan h}$ if and only if there exists a holomorphic function q, fulfilling $q(\mathfrak{z}) \prec q_0(\mathfrak{z}) =: 1 + \tan h\mathfrak{z}$, such that

$$f(\mathfrak{z}) = \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{q(t) - 1}{t} dt\right). \tag{9}$$

Now, we construct some examples of our newly described family $\mathfrak{S}^*_{\tan h}$. For this, consider the following functions

$$\begin{aligned} q_1(\mathfrak{z}) &= 1 + \frac{\mathfrak{z}}{3}, \\ q_2(\mathfrak{z}) &= \frac{4+2\mathfrak{z}}{4+\mathfrak{z}}, \\ q_3(\mathfrak{z}) &= \frac{7+3\mathfrak{z}}{7+\mathfrak{z}}, \\ q_4(\mathfrak{z}) &= 1 + (\tan h1)\mathfrak{z}. \end{aligned} \tag{10}$$

Since $q_0(\mathfrak{z}) = 1 + \tanh \mathfrak{z}$ is univalent in $\mathfrak{D}$, $q_i(0) = 1 = q_0(0),(i = 1, 2, 3, 4)$, and $q_i(\mathfrak{D}) \subseteq q_0(\mathfrak{D})$, this implies that for each $i = 1, 2, 3, 4$, the relation $q_i \prec q_0$ holds. Thus, from (8), the functions

$$\begin{aligned} f_1(\mathfrak{z}) &= \mathfrak{z}e^{\mathfrak{z}/3}, \\ f_2(\mathfrak{z}) &= \mathfrak{z} + \frac{\mathfrak{z}^2}{4}, \\ f_3(\mathfrak{z}) &= \mathfrak{z}\left(1 + \frac{\mathfrak{z}}{7}\right)^2, \\ f_4(z) &= \mathfrak{z}e^{(\tan h1)\mathfrak{z}}, \end{aligned} \tag{11}$$

corresponding to the functions $q_1(\mathfrak{z})$,$q_2(\mathfrak{z})$,$q_3(\mathfrak{z})$, and $q_4(\mathfrak{z})$, respectively, belong to the family $\mathfrak{S}^*_{\tanh}$. By taking $q(\mathfrak{z}) = q_0(\mathfrak{z}) = 1 + \tan h\mathfrak{z}$ in (8), we get the below function that plays a role of the extremal in many problems of the class $\mathfrak{S}^*_{\tan h}$.

$$\begin{aligned} f_0(\mathfrak{z}) &= \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{\tan ht}{t} dt\right) \\ &= \mathfrak{z} + \mathfrak{z}^2 + \frac{1}{2} + \frac{1}{18}\mathfrak{z}^4 - \frac{5}{72}\mathfrak{z}^5 + \cdots. \end{aligned} \tag{12}$$

In this article, we work on determining the radius of starlikeness and convexity and $\mathfrak{S}^*_{\tanh}$ radius for some subfamilies of starlike functions, mentioned above in which mostly have simple geometric interpretation. Besides these subfamilies, we also discuss the $\mathfrak{S}^*_{\tanh}$ radius for some families of $\mathfrak{A}$, whose functions have been expressed as a ratio between two functions.

2. Radii of Starlikeness and Convexity

In this portion, we examined the radius of starlikeness and convexity for the family $\mathfrak{S}^*_{\tanh}$.

Theorem 1. *The $\mathfrak{S}^*(\xi)$ radius for the family $\mathfrak{S}^*_{\tan h}$ is $r_0 = \tan^{-1}(1 - \xi)$ with $0 \le \xi < 1$.*

Proof. If $f \in \mathfrak{S}^*_{\tan h}$, then, by virtue of (7), a Schwarz function w exists with $|w(\mathfrak{z})| \le |\mathfrak{z}|$ such as

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} = 1 + \tan h(w(\mathfrak{z})). \tag{13}$$

Now, let $w(\mathfrak{z}) = Re^{iv}$ with $R \le |\mathfrak{z}| = r, -\pi \le v \le \pi$. After easy computation, we get

$$\left|\tan h\left(Re^{iv}\right)\right|^2 = \frac{\cos h^2(R \cos v) - \cos^2(R \sin v)}{\cos h^2(R \cos v) + \cos^2(R \sin v) - 1} = \Psi(v). \tag{14}$$

The equation $\Psi'(v) = 0$ has five roots in $[-\pi, \pi]$, namely, $0, \pm\pi$ and $\pm(\pi/2)$. Since $\Psi(v) = \Psi(-v)$, it is sufficient to show that $v \in [0, \pi]$. Furthermore, we can see that $\Psi(0) = \tan h^2R = \Psi(\pi)$, $\Psi(\pi/2) = \tan^2 R$, and

$$\max\left\{\Psi(0), \Psi(\pi), \Psi\left(\frac{\pi}{2}\right)\right\} = \Psi\left(\frac{\pi}{2}\right) = \tan^2 R \le \tan^2 r. \tag{15}$$

Thus, we have

$$\mathfrak{Re}\left(\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}\right) \ge 1 - |\tanh(w(\mathfrak{z}))|, \ge 1 - \tan r \ge \xi, \tag{16}$$

whenever $1 - \tan r - \xi \ge 0$. The radius of starlikeness of order ξ, for the family $\mathfrak{S}^*_{\tan h}$, is the smallest positive root $r_0 \in (0, 1)$ of $1 - \tan r - \xi = 0$. □

Taking $\xi = 0$ in the above Theorem 1, we obtain the following corollary.

Corollary 1. *The $\mathfrak{S}^*$ radius, for the family $\mathfrak{S}^*_{\tan h}$, is $r_0 = \tan^{-1}(1) \approx 0.78$.*

Theorem 2. *The $\mathcal{K}(\xi)$ radius r_0, for the family $f \in \mathfrak{S}^*_{\tan h}$, is $r_0 = \min\{r_1, r_2\}$, where r_1 is the smallest root of the equation*

$$\left(1 - \xi - \tan hr \sec^2 r\right)\left(1 - r^2\right)(1 - \tan r) - r \sec^2 r = 0, \tag{17}$$

and r_2 is such that $1 - \tan r_2 > 0$.

Proof. If $f \in \mathfrak{S}^*_{\tanh}$, then, a holomorphic function w occurs with $w(0) = 0$ and $|w(\mathfrak{z})| \le |\mathfrak{z}|$, such that

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} = 1 + \tan hw(\mathfrak{z}). \tag{18}$$

By simple computation, it gives

$$1 + \frac{\mathfrak{z}f''(\mathfrak{z})}{f'(\mathfrak{z})} = 1 + \tan hw(\mathfrak{z}) + \frac{\mathfrak{z}w'(\mathfrak{z}) \sec h^2 w(\mathfrak{z})}{1 + \tan hw(\mathfrak{z})}. \tag{19}$$

From (18), we get

$$\begin{aligned} \mathfrak{Re}\left(1 + \frac{\mathfrak{z}f''(\mathfrak{z})}{f'(\mathfrak{z})}\right) &\ge 1 + \mathfrak{Re}(\tan hw(\mathfrak{z})) \\ &\quad - \frac{|\mathfrak{z}||w'(\mathfrak{z})||\sec h^2 w(\mathfrak{z})|}{1 - |\tan h(w(\mathfrak{z}))|}. \end{aligned} \tag{20}$$

Assume that $w(z) = Re^{iv}$, with $R \leq |z| = r, -\pi \leq v \leq \pi$ for calculating the minumum value of the right side of the last inequality. A simple calculation reveals that

$$\mathfrak{Re}\tan h\left(Re^{iv}\right) = \frac{\tan h(Rx)\sec^2(Ry)}{1+\tan h^2(Rx)\tan^2(Ry)}, \quad (21)$$

where $y = \sin v, x = \cos v$, and $x, y \in [-1, 1]$. It is easy to observe that

$$\begin{aligned} \tan h(Rx) &\geq -\tan hR \geq -\tan hr, \\ 1 &\leq \sec^2(Ry) \leq \sec^2 R \leq \sec^2 r. \end{aligned} \quad (22)$$

Consequently, we have

$$\mathfrak{Re}(1+\tan hw(z)) \geq 1 - \tan hR\sec^2 R \geq 1 - \tan hr\sec^2 r. \quad (23)$$

Now, consider that

$$\left|\sec h^2\left(e^{iv}R\right)\right|^2 = \frac{1}{\left(\cos h^4(R\cos v) + 2\cos h^2(R\cos v)\cos^2(R\sin v) + \cos^4(R\sin v) - 2\cosh^2(R\cos v) - 2\cos^2(R\sin v) + 1\right)} =: \phi(v). \quad (24)$$

The equation $\phi'(v) = 0$ attained has five roots in $[-\pi, \pi]$, namely, 0, $\pm\pi$ and $\pm(\pi/2)$. Also, $\phi(v) = \phi(-v)$; it is enough to consider only those roots which lie in $[0, \pi]$. Furthermore, we seen that $\phi(0) = \sec h^4 R = \phi(\pi)$, and $\phi(\pi/2) = \sec^4 R$; therefore

$$\max\left\{\phi(0), \phi(\pi), \phi\left(\frac{\pi}{2}\right)\right\} = \phi\left(\frac{\pi}{2}\right) = \sec^4 R \leq \sec^4 r. \quad (25)$$

Hence,

$$\left|\sec h^2\left(e^{iv}R\right)\right| \leq \sec^2 R \leq \sec^2 r. \quad (26)$$

Also,

$$|\tan hw(z)| \leq \tan R \leq \tan r. \quad (27)$$

Using the above facts along with the well-known inequality of Schwarz functions w(seeradii19), we have

$$\left|w'(z)\right| \leq \frac{1-|w(z)|^2}{1-|z|^2} = \frac{1-R^2}{1-|z|^2} \leq \frac{1}{1-|z|^2} \leq \frac{1}{1-r^2}. \quad (28)$$

Using (19), we obtain

$$\begin{aligned} \mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &\geq 1 + \mathfrak{Re}(\tan hw(z)) \\ &\quad - \frac{\left|\sec h^2 w(z)\right|\left|zw'(z)\right|}{1-|\tan hw(z)|}, \\ &\geq 1 - \tan hr\sec^2 r \\ &\quad - \frac{\left(\sec^2 r\right)r}{(1-\tan r)(1-r^2)} \geq \xi. \end{aligned} \quad (29)$$

The last inequality is true if $(1-\xi-\tan hr\sec^2 r)(1-r^2)(1-\tan r) - r\sec^2 r \geq 0$ with $\tan r < 1$ holds.

Hence, $\mathcal{K}(\xi)$ radius r_0 for the family $\mathfrak{S}^*_{\tan h}$ is the minumum of r_1 and r_2, where r_1 is the smallest positive root of the equation

$$\left(1-\xi-\tan hr\sec^2 r\right)\left(1-r^2\right)(1-\tan r) - r\sec^2 r = 0, \quad (30)$$

and r_2 is such that $\tan r_2 < 1$. □

Corollary 2. *The $\mathcal{K}$ radius, for the family $\mathfrak{S}^*_{\tan h}$, is $r_0 \approx 0.33286$.*

Remark 1. The result in the last Theorem is not the best one. Considering the function f_0 described by (11) provides a sharp result. For the function f_0, we have

$$\phi(z) = \mathfrak{Re}\frac{\left(zf_0'(z)\right)'}{f_0'(z)} = \mathfrak{Re}\left(1+\tan hz + \frac{z\sec h^2 z}{1+\tan hz}\right), \quad (31)$$

and $\phi(r) = 0$.

3. Radius Problems

To address our main results in this portion, first, we consider a few well-known families as follows.

$$\mathcal{P}_n[L, M] := \left\{p(z) = 1 + \sum_{k=n}^{\infty} c_n z^n : p(z) \prec \frac{1+Lz}{1+Mz}, -1 \leq M < L \leq 1\right\}. \quad (32)$$

Also, for $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{P}_n(\xi) &:= \mathcal{P}_n[1-2\xi, -1], \\ \mathcal{P}_n &:= \mathcal{P}_n(0). \end{aligned} \quad (33)$$

If we put $p(\mathfrak{z}) =: (\mathfrak{z}f'(\mathfrak{z}))/(f(\mathfrak{z}))$, for $f \in \mathfrak{A}_n$, then, the family $\mathscr{P}_n[L, M]$ is reduced to $\mathfrak{S}_n^*[L, M]$ and $\mathfrak{S}_n^*(\xi) =: \mathfrak{S}_n^*[1-2\xi, -1]$. Let the family $\mathscr{M}(\beta)$ contains the functions $f \in \mathfrak{A}_n$ satisfying that $\operatorname{Re}((\mathfrak{z}f'(\mathfrak{z}))/(f(\mathfrak{z}))) < \beta$, for $\beta > 1$. Furthermore, let

$$\begin{aligned}\mathfrak{S}^*_{\tan h,n} &:= \mathfrak{A}_n \cap \mathfrak{S}^*_{\tan h}, \mathfrak{S}^*_n(\xi) = \mathfrak{A}_n \cap \mathfrak{S}^*(\xi), \mathfrak{S}^*_{\mathscr{L},n} \\ &= \mathfrak{A}_n \cap \mathfrak{S}^*_{\mathscr{L}}, \\ \mathscr{M}_n(\beta) &= \mathfrak{A}_n \cap \mathscr{M}(\beta).\end{aligned} \tag{34}$$

Ali et al. [21] recently studied the below families

$$\begin{aligned}\mathfrak{S}_n &:= \left\{ f \in \mathfrak{A}_n : \frac{f(\mathfrak{z})}{\mathfrak{z}} \in \mathscr{P}_n \right\}, \\ \mathfrak{C}\mathfrak{S}_n(\xi) &:= \left\{ f \in \mathfrak{A}_n : \frac{f(\mathfrak{z})}{g(\mathfrak{z})} \in \mathscr{P}_n, g \in \mathfrak{S}^*_n(\xi) \right\}\end{aligned} \tag{35}$$

and calculated $\mathfrak{S}^*_{\mathscr{L},n}$ radii for certain families. Further, they achieved the conditions on L and M such that $\mathfrak{S}^*_n[L, M] \subset \mathfrak{S}^*_{\mathscr{L},n}$. In this portion, $\mathfrak{S}^*_{\tan h,n}$ radii for the family of Janowski starlike function and some other geometrically described families are explored. To get our results, we employ the following lemmas.

Lemma 1 [22]. *If $p \in \mathscr{P}_n(\xi)$, then, for $|\mathfrak{z}| = r$,*

$$\left| \frac{\mathfrak{z}p'(\mathfrak{z})}{p(\mathfrak{z})} \right| \le \frac{2nr^n(1-\xi)}{(1+(1-2\xi)r^n)(1-r^n)}. \tag{36}$$

Lemma 2 [23]. *If $p \in \mathscr{P}_n[L, M]$, then, for $|\mathfrak{z}| = r$,*

$$\left| p(\mathfrak{z}) - \frac{1 - LMr^{2n}}{1 - M^2r^{2n}} \right| \le \frac{(L-M)r^n}{1 - M^2r^{2n}}. \tag{37}$$

In particular, if $p \in \mathscr{P}_n(\xi)$, then, for $|\mathfrak{z}| = r$,

$$\left| p(\mathfrak{z}) - \frac{(1 + (1-2\xi)r^{2n})}{1 - r^{2n}} \right| \le \frac{2r^n(1-\xi)}{1 - r^{2n}}. \tag{38}$$

The aim of the following lemma is at finding the largest and the smallest disks centered at $(a, 0)$ and (1,0), respectively, such that the domain $\Omega_{\tan h} := q_0(\mathfrak{D})$, where $q_0(\mathfrak{z}) := 1 + \tan h\mathfrak{z}$, is contained in the smallest disk and contains the largest disk.

Lemma 3. *Let*

$$1 - \tan h1 \le a \le 1 + \tan h1. \tag{39}$$

And $r_a = \tan h1 - |a - 1|$. Then, the following inclusions holds

$$\{w \in \mathbb{C} : |w - a| < r_a\} \subset \Omega_{\tan h} \subset \{w \in \mathbb{C} : |w - 1| < \tan 1\}. \tag{40}$$

Proof. Since $w(\mathfrak{z}) = Re^{iv}$ with $R \le |\mathfrak{z}| = r$, we have

$$1 + \tan hw(\mathfrak{z}) = \sigma(v) - i\rho(v), \tag{41}$$

with

$$\begin{aligned}\sigma(v) &= 1 + \frac{\tan h(R\cos(v))\sec^2(R\sin(v))}{1 + \tan h^2(R\cos(v))\tan^2(R\sin(v))}, \\ \rho(v) &= \frac{\tan h(R\sin(v))\sec h^2(R\cos(v))}{1 + \tan h^2(R\cos(v))\tan^2(R\sin(v))}.\end{aligned} \tag{42}$$

First, we consider the square of the distance from $(a, 0)$ to a point on the boundary of $\Omega_{\tan h}$, which is given by

$$\begin{aligned}h(v) = d^2(v) &= \left[a - 1 - \frac{\tan h(\cos(v))\sec^2(\sin(v))}{1 + \tan h^2(\cos(v))\tan^2(\sin(v))} \right]^2 \\ &+ \left[\frac{\tan h(\sin(v))\sec h^2(\cos(v))}{1 + \tan h^2(\cos(v))\tan^2(\sin(v))} \right]^2.\end{aligned} \tag{43}$$

To show that $|w - a| < r_a$ is the largest disk contained in $\Omega_{\tan h}$, it is sufficient to show that $\min_{-\pi \le v \le \pi} d(v) = r_a$. But since $h(v) = h(-v)$, therefore, we consider the range $0 \le v \le \pi$. Now, it can easily be obtained that $h'(v) = 0$ has three roots $0, \pi$, and $v_0 \in (0, \pi)$. The root v_0 is dependent on a. The graph of $h(v)$ shows that it is decreasing in $[v_0, \pi]$ and increasing in the interval $[0, v_0]$. Hence, the minimum of $h(v)$ is calculated on either π or 0. A computation provides

$$\begin{aligned}h(\pi) &:= (a - 1 + \tan h1)^2, \\ h(0) &:= (a - 1 - \tan h1)^2.\end{aligned} \tag{44}$$

Thus, we get

$$\begin{aligned}\min_{-\pi \le v \le \lambda} h(v) &= \min(h(\pi), h(0)) \\ &= \begin{cases} h(\pi), & \text{if } -\tan h1 \le a - 1 \le 0, \\ h(0), & \text{if } 0 \le a - 1 \le \tan h1. \end{cases}\end{aligned} \tag{45}$$

Therefore, we can write that

$$\min_{-\pi \le v \le \lambda} d(v) = \begin{cases} \tan h1 + (a-1), & \text{if } -\tanh 1 \le a - 1 \le 0, \\ \tan h1 - (a-1), & \text{if } 0 \le a - 1 \le \tanh 1, \end{cases} \tag{46}$$

or equivalently

$$\min_{-\pi\le v\le\lambda} d(v) = \tan h1 - |a-1|. \tag{47}$$

For the circle of the minimum radius centered at $(1,0)$, which contains $f(\mathfrak{D}) = 1 + \tanh \mathfrak{z}$, we find the maximum distance from $(1,0)$ to a point on the boundary of $f(\mathfrak{D}) = \Omega_{\tan h}$ and the square of this distance function is given by

$$\phi(v) = \frac{\cosh^2(\cos(v) - \cos^2(\sin(v)))}{\cosh^2(\cos(v)) + \cos^2(\cos(v)) - 1}. \tag{48}$$

It is easy to verify that $\phi(v)$ achieves its maximum value at $\pi/2$, which is $\phi(\pi/2) = \tan^2 1$. Hence, the radius of the smallest disk which contains $\Omega_{\tan h}$ is $\tan 1$.□ □

In the following examples, we apply Lemma 3 , to find the necessary and sufficient conditions for two specific functions that belong to the family $\mathfrak{S}^*_{\tan h}$.

Example 1.

(a) The function

$$f(\mathfrak{z}) = \mathfrak{z} + d_2\mathfrak{z}^2 \in \mathfrak{S}^*_{\tan h}, \tag{49}$$

if and only if

$$|d_2| \le \frac{\tan h1}{1+\tan h1} \approx 0.43233 \tag{50}$$

(b) The function

$$f(\mathfrak{z}) = \frac{\mathfrak{z}}{(1-b\mathfrak{z})^2} \in \mathfrak{S}^*_{\tan h}, \tag{51}$$

if and only if

$$|b| \le \frac{\tan h1}{2+\tan h1} \approx 0.27578 \tag{52}$$

Proof.

(a) We know that $f(\mathfrak{z}) = \mathfrak{z} + d_2\mathfrak{z}^2 \in \mathfrak{S}^*$, if and only if $|d_2| \le (1/2)$. Since $\mathfrak{S}^*_{\tan h} \subset \mathfrak{S}^*$, we get $|d_2| \le (1/2)$, whenever $f \in \mathfrak{S}^*_{\tan h}$. The function

$$w(\mathfrak{z}) = \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} = \frac{1+2d_2\mathfrak{z}}{1+d_2\mathfrak{z}}, \tag{53}$$

maps $\mathfrak{D}$ onto the disk

$$\left| w - \frac{1-2|d_2|^2}{1-|d_2|^2} \right| < \frac{|d_2|}{1-|d_2|^2}. \tag{54}$$

Since

$$\frac{1-2|d_2|^2}{1-|d_2|^2} \le 1, \tag{55}$$

then, from Lemma 3 , the above disk will be contained in $\Omega_{\tan h}$ if

$$\begin{aligned} 1 - \tan h1 &\le \frac{1-2|d_2|^2}{1-|d_2|^2}, \\ \frac{|d_2|}{1-|d_2|^2} &\le \frac{1-2|d_2|^2}{1-|d_2|^2} - 1 + \tan h1. \end{aligned} \tag{56}$$

The above two inequalities give

$$\begin{aligned} |d_2| &\le \sqrt{\frac{\tan h1}{1+\tan h1}}, \\ |d_2| &\le \frac{\tan h1}{1+\tan h1}, \end{aligned} \tag{57}$$

respectively. Thus, we have

$$|d_2| \le \min\left\{ \frac{\tan h1}{1+\tan h1}; \sqrt{\frac{\tan h1}{1+\tan h1}} \right\} = \frac{\tan h1}{1+\tan h1} \tag{58}$$

(b) Logarithmic differentiation of the function

$$f(\mathfrak{z}) = \frac{\mathfrak{z}}{(1-b\mathfrak{z})^2}, \tag{59}$$

yields that

$$w(\mathfrak{z}) = \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} = \frac{(1+b\mathfrak{z})}{(1-b\mathfrak{z})}, \tag{60}$$

maps $\mathfrak{D}$ onto the disk

$$\left| w(\mathfrak{z}) - \frac{1+|b|^2}{1-|b|^2} \right| \le \frac{2|b|}{1-|b|^2}, \tag{61}$$

since

$$1 \le \frac{1+|b|^2}{1-|b|^2}. \tag{62}$$

The disk above is contained in $\Omega_{\tan h}$, in Lemma 3, whenever

$$\frac{1+|b|^2}{1-|b|^2} \leq 1+\tan 1, \qquad \frac{2|b|}{1-|b|^2} \leq 1+\tan 1 - \frac{1+|b|^2}{1-|b|^2}. \tag{63}$$

The above two inequalities give

$$|b| \leq \sqrt{\frac{\tan h1}{2+\tan h1}}, \qquad |b| \leq \frac{\tan h1}{2+\tan h1}, \tag{64}$$

respectively. Thus, we have

$$|b| \leq \min\left\{\sqrt{\frac{\tan h1}{2+\tan h1}}; \frac{\tan h1}{2+\tan h1}\right\} = \frac{\tan h1}{2+\tan h1}. \tag{65}$$

This completes the required proof. □

Theorem 3. *The sharp $\mathfrak{S}^*_{\tan h,n}$ radius for the family $\mathfrak{S}_n$ is given by*

$$\mathscr{R}_{\mathfrak{S}_{\tan h,n}}(\mathfrak{S}_n) = \left(\frac{\tan h1}{\sqrt{n^2+\tan h^2 1}+n}\right)^{1/n}. \tag{66}$$

Proof. Suppose that $f \in \mathfrak{S}_n$. Consider the function $h : \mathfrak{D} \longrightarrow \mathbb{C}$ described by

$$h(\mathfrak{z}) = \frac{f(\mathfrak{z})}{\mathfrak{z}}. \tag{67}$$

Using logarithmic differentiation, we get

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1 = \frac{\mathfrak{z}h'(\mathfrak{z})}{h(\mathfrak{z})}. \tag{68}$$

Implementing Lemma 1 , we have

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| = \left|\frac{\mathfrak{z}h'(\mathfrak{z})}{h(\mathfrak{z})}\right| \leq \frac{2nr^n}{1-r^{2n}}. \tag{69}$$

According to Lemma 3 , if the following inequality holds, the image of $|\mathfrak{z}| \leq r$ under the function $(\mathfrak{z}f'(\mathfrak{z}))/(f(\mathfrak{z}))$ lies on disk $\Omega_{\tan h}$:

$$\frac{2nr^n}{1-r^{2n}} \leq \tan h1, \tag{70}$$

or equivalently

$$(\tan h1)r^{2n} + 2nr^n - \tan h1 \leq 0. \tag{71}$$

Thus, $\mathfrak{S}^*_{\tan h,n}$ radius of $\mathfrak{S}_n$ is the smallest positive root of

$$(\tan h1)r^{2n} + 2nr^n - \tan h1 = 0, \tag{72}$$

in (0,1). Assume the function

$$f_0(\mathfrak{z}) = \frac{\mathfrak{z}(1+\mathfrak{z}^n)}{(1-\mathfrak{z}^n)}. \tag{73}$$

Then, it is clear to see that $\mathfrak{Re}((f_0(\mathfrak{z}))/\mathfrak{z}) > 0$ in the unit disk $\mathfrak{D}$. Hence, $f_0 \in \mathfrak{S}_n$ and

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} = 1 + \frac{2n\mathfrak{z}^n}{1-\mathfrak{z}^{2n}}. \tag{74}$$

Further, f_0 assures the sharpness of the results since at $\mathfrak{z} = \mathscr{R}_{\mathfrak{S}^*_{\tan h,n}}(\mathfrak{S}_n)$, we obtain

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \frac{2n\mathfrak{z}^n}{1-\mathfrak{z}^{2n}} = \tan h1. \tag{75}$$

This completes the proof. □

Theorem 4. *The sharp $\mathfrak{S}^*_{\tan h,n}$ radius for the family $\mathfrak{CS}_n(\xi)$ is given by*

$$\mathscr{R}_{\mathfrak{S}^*_{\tan h,n}}(\mathfrak{CS}_n(\xi)) = \left(\frac{\tan h1}{(n-\xi+1)+\sqrt{[\tan h1 + 2(1-\xi)]\tan h1 + (1+n-\xi)^2}}\right)^{1/n}. \tag{76}$$

Proof. Let $f \in \mathfrak{CS}_n(\xi)$ and describe a function

$$h(\mathfrak{z}) = \frac{f(\mathfrak{z})}{g(\mathfrak{z})}, \tag{77}$$

where $g \in \mathfrak{S}^*_n(\xi)$. Then, $h \in \mathscr{P}_n$. According to the definition of h, we get

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} = \frac{\mathfrak{z}h'(\mathfrak{z})}{h(\mathfrak{z})} + \frac{\mathfrak{z}g'(\mathfrak{z})}{g(\mathfrak{z})}. \tag{78}$$

Utilizing Lemma 1 and Lemma 2, we conclude that

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}-\frac{1+(1-2\xi)r^{2n}}{1-r^{2n}}\right|\leq\frac{2(1+n-\xi)r^{n}}{1-r^{2n}}. \tag{79}$$

Since

$$\frac{1+(1-2\xi)r^{2n}}{1-r^{2n}}\geq 1, \tag{80}$$

it follows from Lemma 3 and (78) that the function $f \in \mathfrak{S}^{*}_{\tan h,n}$ if the following holds:

$$\frac{1+(1-2\xi)r^{2n}+2(1+n-\xi)r^{n}}{1-r^{2n}}\leq 1+\tan h1, \tag{81}$$

or equivalently, the inequality

$$(2-2\xi+\tan h1)r^{2n}+2(1+n-\xi)r^{n}-\tan h1\leq 0 \tag{82}$$

holds. Thus, the $\mathfrak{S}^{*}_{\tan h,n}$ radius for the class $\mathfrak{CS}_{n}(\xi)$ is the smallest positive root of

$$2(1+n-\xi)r^{n}+(2-2\xi+\tan h1)r^{2n}-\tan h1=0. \tag{83}$$

Now, assume the functions described by

$$\begin{aligned} f_0(\mathfrak{z})&=\frac{\mathfrak{z}(1+\mathfrak{z}^{n})}{(1-\mathfrak{z}^{n})^{(n+2-2\xi)/n}},\\ g_0(\mathfrak{z})&=\frac{\mathfrak{z}}{(1-\mathfrak{z}^{n})^{(2(1-\xi))/n}}. \end{aligned} \tag{84}$$

Then, we get

$$\frac{f_0(\mathfrak{z})}{g_0(\mathfrak{z})}=\frac{(1+\mathfrak{z}^{n})}{(1-\mathfrak{z}^{n})}\ \text{and}\ \frac{zg_0'(\mathfrak{z})}{g_0(\mathfrak{z})}=\frac{(1+(1-2\xi)z^{n})}{(1-\mathfrak{z}^{n})}. \tag{85}$$

Furthermore, it is obvious that

$$\begin{aligned} \mathfrak{Re}\left(\frac{f_0(\mathfrak{z})}{g_0(\mathfrak{z})}\right)&>0,\\ \mathfrak{Re}\left(\frac{\mathfrak{z}g_0'(\mathfrak{z})}{g_0(\mathfrak{z})}\right)&>\xi, \end{aligned} \tag{86}$$

in the unit disk $\mathfrak{D}$. Therefore, $f_0 \in \mathfrak{CS}_{n}(\xi)$. The function f_0 described in (83), at $\mathfrak{z}=\mathscr{R}_{\mathfrak{S}^{*}_{\tan h,n}}(\mathfrak{CS}_{n}(\xi))$ satisfies that

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})}=\frac{1+(1-2\xi)\mathfrak{z}^{2n}+2(1+n-\xi)\mathfrak{z}^{n}}{1-\mathfrak{z}^{2n}}=1+\tan h(1). \tag{87}$$

Hence, the verified result is sharp. □

Theorem 5. *The $\mathfrak{S}^{*}_{\tan h,n}$ radius for the family $\mathfrak{S}^{*}_{n}[L,M]$ is given by*

$$\mathscr{R}_{\mathfrak{S}^{*}_{\tan h,n}}(\mathfrak{S}^{*}_{n}[L,M])=\left\{\begin{array}{c}\min(1:r_1),-1\leq M\leq 0<L\leq 1\\ \min(1:r_2),0<M<L\leq 1\end{array}\right\}, \tag{88}$$

where

$$\begin{aligned} r_1&=\left(\frac{2\tan h1}{(L-M)+\sqrt{(L-M)^2+4[M^2(1+\tan h1)-LM]\tan h1}}\right)^{1/n},\\ r_2&=\left(\frac{2\tanh 1}{(L-M)+\sqrt{(L-M)^2+4[M^2(\tanh 1-1)+LM]\tanh 1}}\right)^{1/n}. \end{aligned} \tag{89}$$

Proof. Let $f \in \mathfrak{S}^{*}_{n}[L,M]$. Then, by Lemma 2 , we get

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}-b\right|\leq\frac{(L-M)r^{n}}{1-M^2r^{2n}}, \tag{90}$$

where center of the disk is $b=(1-LMr^{2n})/(1-M^2r^{2n})$, $|z|=r$. Applying Lemma 3 , it is easy to see that $b\geq 1$ for $M<0$ and we achieved

$$\frac{1-LMr^{2n}+(L-M)r^{n}}{1-M^2r^{2n}}\leq 1+\tan h1. \tag{91}$$

After some simple calculation, we have

$$r\leq\left(\frac{2\tan h1}{(L-M)+\sqrt{(L-M)^2+4[M^2(1+\tan h1)-LM]\tan h1}}\right)^{1/n}=r_1. \tag{92}$$

In addition, if $M=0$ for $b=1$ and from (89), we get

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}-1\right|\leq Lr^{n}\ (0<L\leq 1). \tag{93}$$

Implementing Lemma 3 with $a=1$ leads to $f \in \mathfrak{S}^{*}_{\tan h,n}$, if

$$r\leq\left(\frac{\tanh 1}{L}\right)^{1/n}. \tag{94}$$

For $0<M<L\leq 1$, we get $b<1$. Thus, from (89) and Lemma 3 , we see that $f \in \mathfrak{S}^{*}_{\tan h,n}$, if the following holds:

$$\frac{(L-M)r^{n}+LMr^{2n}-1}{1-M^2r^{2n}}\leq b+\tan h1-1, \tag{95}$$

or equivalently, if

$$r \le \left(\frac{2 \tan h1}{(L-M) + \sqrt{(L-M)^2 + 4[M^2(\tan h1 - 1) + LM] \tan h1}} \right)^{1/n} = r_2. \quad (96)$$

This completes the proof.□ □

Theorem 6. *Let $-1 < M < L \le 1$. If either*

(a) $L - 1 \le (1 - M)(\tanh 1 - 1)$ *and* $(1 - \tanh 1)(1 - M^2) \le L - M \le 1 - M^2$ *or*

(b) $L + 1 \le (1 + M)(\tanh 1 + 1)$ *and* $1 - M^2 \le L - M \le 1 + \tanh 1$

holds, then, $\mathfrak{S}_n^*[L, M] \subset \mathfrak{S}^*_{\tan h,n}$.

Proof. Let $p(z) = (zf'(z))/(f(z))$. Since $f \in \mathfrak{S}_n^*[L, M]$, using Lemma 2, we get

$$\left| p(z) - \frac{1 - LM}{1 - M^2} \right| \le \frac{L - M}{1 - M^2}. \quad (97)$$

Therefore, either $1 - LM/1 - M^2 \le 1$ or $1 - LM/1 - M^2 \ge 1$.

For $(1 - LM)/(1 - M^2) \le 1$, using Lemma 3 , we see that $f \in \mathfrak{S}^*_{\tan h,n}$, if the following holds:

$$\begin{aligned} \frac{L - M}{1 - M^2} &\le \frac{1 - LM}{1 - M^2} - (1 - \tan h1), \\ 1 - \tan h1 &\le \frac{L - M}{1 - M^2} \le 1, \end{aligned} \quad (98)$$

which, upon simplification, reduces to the condition stated in (a).

For $(1 - LM)/(1 - M^2) \ge 1$, again, applying Lemma 3 , we see that $f \in \mathfrak{S}^*_{\tan h,n}$, if the following holds:

$$\begin{aligned} \frac{L - M}{1 - M^2} &\le (1 + \tan h1) - \frac{1 - LM}{1 - M^2}, \\ 1 &\le \frac{L - \mathrm{M}}{1 - M^2} \le (1 + \tan h1), \end{aligned} \quad (99)$$

which, upon simplification, reduces to the condition stated in (b).□ □

Theorem 7. *The sharp $\mathfrak{S}^*_{\tan h}$ radii for the families $\mathfrak{S}^*_{\mathscr{L}}$, $\mathfrak{S}^*_{\mathscr{RL}}$, $\mathfrak{S}^*_{car}$,$\mathfrak{S}^*_{\mathscr{C}}$,$\mathscr{B}\mathfrak{S}^*(\xi)$,$\mathscr{M}(\beta)$,$\mathfrak{S}^*_{\mathscr{CT}}$,$\mathfrak{S}\mathfrak{S}^*(\xi)\mathfrak{S}^*_{\mathfrak{B}}$, and $\mathfrak{S}^*_{\mathscr{T}}(u)$ are*

$$\begin{aligned}
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{L}}) &= (2 - \tan h1) \tan h1 \approx 0.944, \\
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{RL}}) &= \frac{\left(2 + \left(1 + \sqrt{2}\right) \tan h1\right) \tan h1}{6 - 3\sqrt{2} + 4\left(\sqrt{2} - 1\right) \tan h1 - \sec h^2 2} \\
&\approx 0.992, \\
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{car}) &= \frac{1}{2}\left(\sqrt{2(2 + 3 \tan h1)} - 2\right) \approx 0.463, \\
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{C}}) &= \frac{\tan h1(2 + \tan h1)}{2(1 + \tan h1)} \approx 0.217, \\
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathscr{B}\mathfrak{S}^*(\xi)) &= \frac{2 \tan h1}{1 + \sqrt{1 + 4\xi \tan h^2 1}}, \quad \text{for } 0 \le \xi < 1, \\
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathscr{M}(\beta)) &= \frac{\tan h1}{2(\beta - 1) + \tan h1}, \quad \beta > 1, \\
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{CT}}) &= \sqrt{1 + 2 \tan h1} - 1 \approx 0.589, \\
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}\mathfrak{S}^*(\xi)) &= \frac{(1 + \tan h1)^{1/\xi} - 1}{(1 + \tan h1)^{1/\xi} + 1}, \quad (0 \le \xi \le 1), \\
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathfrak{B}}) &= \ln\left[1 + \ln\left(1 + \tan h1\right)\right] \approx 0.449, \\
\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{T}}(u)) &= \frac{\sqrt{1 + 2u \ln\left(1 + \tan h1\right)} - 1}{u}.
\end{aligned} \quad (100)$$

Proof.

(1) Let $f \in \mathfrak{S}^*_{\mathscr{L}}$, then,

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1 + z}. \quad (101)$$

Thus, for $|z| = r$, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le 1 - \sqrt{1 - r} \le \tan h1. \quad (102)$$

For

$$r \le (2 - \tan h1) \tan h1 = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{L}}). \quad (103)$$

For checking the sharpness of the result, we assume the function f_0 described by

$$f_0(z) = \frac{4z \exp\left\{2\left(\sqrt{z + 1} - 1\right)\right\}}{\left(1 + \sqrt{1 + z}\right)^2}. \quad (104)$$

Since

$$\frac{zf_0'(z)}{f_0(z)} = \sqrt{1 + z}, \quad (105)$$

it follows that $f_0 \in \mathfrak{S}^*_{\mathscr{L}}$ and at $z = -\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{L}})$ and we see that

$$\frac{z f'_0(z)}{f_0(z)} - 1 = -\tan h1, \tag{106}$$

and hence, the result is sharp

(2) For function $f \in \mathfrak{S}^*_{\mathscr{RL}}$, then,

$$\frac{z f'(z)}{f(z)} \prec \sqrt{2} - \left(\sqrt{2} - 1\right)\sqrt{\frac{1 - z}{1 + 2\left(\sqrt{2} - 1\right)z}}. \tag{107}$$

Thus, for $|z| = r$, we get

$$\begin{aligned} \left|\frac{z f'(z)}{f(z)} - 1\right| &\le \left|\sqrt{2} - \left(\sqrt{2} - 1\right)\sqrt{\frac{1 - z}{1 + 2\left(\sqrt{2} - 1\right)z}} - 1\right| \\ &\le 1 - \sqrt{2} - \left(\sqrt{2} - 1\right)\sqrt{\frac{1 + r}{1 - 2\left(\sqrt{2} - 1\right)r}} \\ &\le \tan h1. \end{aligned} \tag{108}$$

For

$$r \le \frac{\left(2 + \left(1 + \sqrt{2}\right)\tan h1\right)\tan h1}{6 - 3\sqrt{2} + 4\left(\sqrt{2} - 1\right)\tan h1 - \sec h^2 2} = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}\left(\mathfrak{S}^*_{\mathscr{RL}}\right). \tag{109}$$

For checking the sharpness, assume the function f_0 described by

$$f_0(z) = z \exp\left(\int_0^z \frac{q_0(t) - 1}{t}\,dt\right), \tag{110}$$

where

$$q_0(z) = \sqrt{2} - \left(\sqrt{2} - 1\right)\sqrt{\frac{1 - z}{\left(1 + 2\left(\sqrt{2} - 1\right)z\right)}}. \tag{111}$$

Since $q_0(z) = (z f'_0(z))/(f_0(z))$ and from the definition of f_0 at $z = -\mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{RL}})$, we have

$$\frac{z f'_0(z)}{f_0(z)} = \sqrt{2} - \left(\sqrt{2} - 1\right)\sqrt{\frac{1 - z}{\left(1 + 2\left(\sqrt{2} - 1\right)z\right)}} = -\tan h1, \tag{112}$$

and hence, the sharpness of the result is verified

(3) Let $f \in \mathfrak{S}^*_{car}$, then,

$$\frac{z f'(z)}{f(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}. \tag{113}$$

Therefore, for $|z| = r$, we get

$$\left|\frac{z f'(z)}{f(z)} - 1\right| = \left|1 + \frac{4z}{3} + \frac{2z^2}{3} - 1\right| \le \frac{4r}{3} + \frac{2r^2}{3} \le \tan h1. \tag{114}$$

For

$$r \le \frac{\sqrt{2(2 + 3\tan h1)} - 2}{2}. \tag{115}$$

For checking the sharpness, assume the function f_0 described by

$$f_0(z) = z \exp\left\{\frac{4z + z^2}{3}\right\}. \tag{116}$$

Since

$$\frac{z f'_0(z)}{f_0(z)} = 1 + \frac{4z}{3} + \frac{2z^2}{3}, \tag{117}$$

it follows that $f_0 \in \mathfrak{S}^*_{car}$ and at $z = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{car})$ and we get

$$\frac{z f'_0(z)}{f_0(z)} - 1 = \tan h1. \tag{118}$$

Hence, the result is sharp

(4) Let $f \in \mathfrak{S}^*_{\mathscr{C}}$. Then,

$$\frac{z f'(z)}{f(z)} \prec z + \sqrt{1 + z^2}. \tag{119}$$

Thus, for $|z| = r$, we get

$$\left|\frac{z f'(z)}{f(z)} - 1\right| \le r + \sqrt{1 + r^2} - 1 \le \tan h1, \tag{120}$$

for

$$r \le \frac{\tan h1(2 + \tan h1)}{2(1 + \tan h1)}. \tag{121}$$

For checking the sharpness of the result, consider the function f_0 defined by

$$f_0(\mathfrak{z}) = \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{q_0(t)-1}{t}\, dt\right). \tag{122}$$

Since

$$q(z_0) = \frac{\mathfrak{z} f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} = \mathfrak{z} + \sqrt{1+\mathfrak{z}^2}, \tag{123}$$

it follows that $f_0 \in \mathfrak{S}^*_{\mathscr{C}}$ and at $\mathfrak{z} = \mathscr{R}_{\mathfrak{S}^*_{\tanh}}(\mathfrak{S}^*_{\mathscr{C}})$, we have

$$\frac{\mathfrak{z} f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tan h1. \tag{124}$$

Hence, the result is sharp

(5) For function $f \in \mathscr{B}\mathfrak{S}^*(\xi)$, we have

$$\frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} \prec 1 + \frac{\mathfrak{z}}{(1-\xi\mathfrak{z}^2)}, (0 \le \xi < 1). \tag{125}$$

Therefore, for $|\mathfrak{z}| = r$, we have

$$\left|\frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \le \frac{r}{(1-\xi r^2)} \le \tan h1. \tag{126}$$

For $0 \le \xi < 1$, we obtain

$$r \le \frac{2 \tan h1}{1 + \sqrt{1 + 4\xi \tan h^2 1}} = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathscr{B}\mathfrak{S}^*(\xi)). \tag{127}$$

For checking the sharpness of the result, we assume the function

$$f_0(\mathfrak{z}) = \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{q_0(t)-1}{t}\, dt\right), \tag{128}$$

where

$$q_0(\mathfrak{z}) = 1 + \frac{\mathfrak{z}}{(1-\xi\mathfrak{z}^2)}. \tag{129}$$

Since $q_0(\mathfrak{z}) = (\mathfrak{z} f_0'(\mathfrak{z}))/(f_0(\mathfrak{z}))$, it follows that $f_0 \in \mathscr{B}\mathfrak{S}^*(\xi)$ and at $\mathfrak{z} = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathscr{B}\mathfrak{S}^*(\xi))$ and we have

$$\frac{\mathfrak{z} f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tan h1. \tag{130}$$

Hence, the verified result is sharp

(6) Let $f \in \mathscr{M}(\beta)$. Then, by Lemma 2 , for $n = 1$, we have

$$\left|\frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} - \frac{1+(1-2\beta)^2}{1-r^2}\right| \le \frac{2(\beta-1)r}{1-r^2}. \tag{131}$$

Obviously,

$$\left|\frac{1+(1-2\beta)r^2}{1-r^2}\right| \le 1. \tag{132}$$

Hence, by Lemma 3 , the above disk contains in $\Omega_{\tan h}$, so

$$\begin{aligned} 1 - \tan h1 &\le \frac{1+(1-2\beta)r^2}{1-r^2}, \\ \frac{2(\beta-1)r}{1-r^2} &\le \frac{1+(1-2\beta)r^2}{1-r^2} - 1 + \tan h1. \end{aligned} \tag{133}$$

Simple calculation gives

$$r \le \frac{\tan h1}{2(\beta-1) + \tan h1}. \tag{134}$$

For checking the sharpness, assume the function defined as

$$f_0(\mathfrak{z}) = \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{q_0(t)-1}{t}\, dt\right). \tag{135}$$

Since

$$\frac{\mathfrak{z} f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} = 1 + 2(\beta-1)\mathfrak{z} + [2(\beta-1) + \tanh 1]\mathfrak{z}^2, \tag{136}$$

it follows that $f_0 \in \mathscr{M}(\beta)$, at $\mathfrak{z} = \mathscr{R}_{\mathfrak{S}^*_{\tanh}}(\mathscr{M}(\beta))$, we get

$$\frac{\mathfrak{z} f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tanh 1. \tag{137}$$

Hence, this verified that the result is sharp.

(7) Let $f \in \mathfrak{S}^*_{\mathscr{C}\mathscr{T}}$. Then

$$\frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} \prec 1 + \mathfrak{z} + \frac{\mathfrak{z}^2}{2}. \tag{138}$$

Therefore, for $|\mathfrak{z}| = r$, it gives

$$\left|\frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \le r + \frac{r^2}{2} \le \tanh 1. \tag{139}$$

For

$$r \le \sqrt{1 + 2 \tan h1} - 1 = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{C}\mathscr{T}}). \tag{140}$$

For checking the sharpness, assume the function defined as

$$f_0(\mathfrak{z}) = \mathfrak{z}e^{\left(\mathfrak{z}+\left(\mathfrak{z}^2/4\right)\right)}. \tag{141}$$

Since

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} = 1 + \mathfrak{z} + \frac{\mathfrak{z}^2}{2}, \tag{142}$$

it follows that $f_0 \in \mathfrak{S}^*_{\mathscr{C}\mathscr{T}}$ and at $\mathfrak{z} = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{C}\mathscr{T}})$ and we have

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tanh 1. \tag{143}$$

This shows that the result is sharp

(8) Supposing that $f \in \mathfrak{S}\mathfrak{S}^*(\zeta)$, we have

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} \prec \left(\frac{(1+\mathfrak{z})}{(1-\mathfrak{z})}\right)^{\zeta}, \quad (0 < \zeta \leq 1). \tag{144}$$

Thus, for $|\mathfrak{z}| = r$, we get

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \leq \left(\frac{(1+r)}{(1-r)}\right)^{\zeta} - 1 \leq \tan h1. \tag{145}$$

For

$$r \leq \frac{(1+\tan h1)^{1/\zeta} - 1}{(1+\tan h1)^{1/\zeta} + 1} = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathcal{S}\mathfrak{S}^*(\zeta)). \tag{146}$$

For checking the sharpness, assume the function described as

$$f_0(\mathfrak{z}) \coloneqq \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{(q_0(t)-1)}{t} dt\right), \tag{147}$$

where

$$q_0(\mathfrak{z}) = \left(\frac{(1+\mathfrak{z})}{(1-\mathfrak{z})}\right)^{\zeta}. \tag{148}$$

Since $q_0(\mathfrak{z}) = (\mathfrak{z}f_0'(\mathfrak{z}))/(f_0(\mathfrak{z}))$, it follows that $f_0 \in \mathfrak{S}\mathfrak{S}^*(\zeta)$ and at $\mathfrak{z} = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}\mathfrak{S}^*(\zeta))$ and we have

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tan h1. \tag{149}$$

Hence, this showed that the result is sharp

(9) Supposing that $f \in \mathfrak{S}^*_{\mathfrak{B}}$, then,

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} \prec e^{e^{\mathfrak{z}}-1}. \tag{150}$$

Thus, for $|\mathfrak{z}| = r$, we have

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \leq e^{e^r-1} - 1 \leq \tan h1. \tag{151}$$

For

$$r \leq \ln\left[1 + \ln\left(1 + \tan h1\right)\right]. \tag{152}$$

To show the sharpness of the result, we assume the function described by

$$f_0(\mathfrak{z}) = \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{q_0(t)-1}{t} dt\right), \tag{153}$$

where

$$q_0(\mathfrak{z}) = e^{e^{\mathfrak{z}}-1} = \frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})}. \tag{154}$$

Since $(\mathfrak{z}f_0'(\mathfrak{z}))/(f_0(\mathfrak{z})) = q_0(\mathfrak{z})$, it follows that $f_0 \in \mathfrak{S}^*_{\mathfrak{B}}$ and $\mathfrak{z} = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathfrak{B}})$ and we have

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tan h1, \tag{155}$$

and hence, the sharpness of the result is verified.

(10) Let $f \in \mathfrak{S}^*_{\mathscr{T}}(u)$. Then,

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} \prec e^{\mathfrak{z}+u\mathfrak{z}^2/2}, \quad (u \geq 1). \tag{156}$$

Thus, for $|\mathfrak{z}| = r$, we easily get

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \leq e^{r+ur^2/2} \leq \tan h1. \tag{157}$$

For

$$r \leq \frac{\sqrt{1+2u\ln\left(1+\tan h1\right)} - 1}{u} = \mathscr{R}_{\mathfrak{S}^*_{\tan h}}(\mathfrak{S}^*_{\mathscr{T}}(u)). \tag{158}$$

Now, we choose the following function to confirm its sharpness

$$f_0(\mathfrak{z}) = \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{q_0(t)-1}{t} dt\right). \tag{159}$$

Since

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} = q_0(\mathfrak{z}) = e^{\mathfrak{z}+u\mathfrak{z}^2/2}, \tag{160}$$

it follows that $f_0 \in \mathfrak{S}^*_{\mathcal{T}}(u)$ and at $\mathfrak{z} = \mathcal{R}_{\mathfrak{S}^*_{\tan\, h}}(\mathfrak{S}^*_{\mathcal{T}}(u))$ and we have

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tan h1. \tag{161}$$

This result is sharp. □

4. Functions Defined in terms of the Ratio of Functions

Now, for the following families, we will talk about the radius problem. For brevity, we shall denote them by

$$\begin{aligned}
\mathcal{F}_1 &= \left\{ f \in \mathfrak{A}_n : \mathfrak{Re}\left(\frac{f(\mathfrak{z})}{g(\mathfrak{z})}\right) > 0 \text{ and } \mathfrak{Re}\left(\frac{g(\mathfrak{z})}{\mathfrak{z}}\right) > 0, g \in \mathfrak{A}_n \right\}, \\
\mathcal{F}_2 &= \left\{ f \in \mathfrak{A}_n : \mathfrak{Re}\left(\frac{f(\mathfrak{z})}{g(\mathfrak{z})}\right) > 0 \text{ and } \mathfrak{Re}\left(\frac{g(\mathfrak{z})}{\mathfrak{z}}\right) > \frac{1}{2}, g \in \mathfrak{A}_n \right\}, \\
\mathcal{F}_3 &= \left\{ f \in \mathfrak{A}_n : \left|\frac{f(\mathfrak{z})}{g(\mathfrak{z})} - 1\right| < 1 \text{ and } \mathfrak{Re}\left(\frac{g(\mathfrak{z})}{\mathfrak{z}}\right) > 0, g \in \mathfrak{A}_n \right\}.
\end{aligned} \tag{162}$$

Theorem 8. *The sharp $\mathfrak{S}^*_{\tan\, h,n}$ radii for function in the families $\mathcal{F}_1$,$\mathcal{F}_2$,$\mathcal{F}_3$, respectively, are*

$$\begin{aligned}
\mathcal{R}_{\mathfrak{S}^*_{\tan\, h,n}}(\mathcal{F}_1) &= \left(\sqrt{1 + 4n^2 \cot h^2 1} - 2n \cot h1\right)^{1/n}, \\
\mathcal{R}_{\mathfrak{S}^*_{\tan\, h,n}}(\mathcal{F}_2) &= \left(\frac{\sqrt{9n^2 + 4n \tan h1 + 4 \tan h^2 1} - 3n}{2(n + \tan h1)}\right)^{1/n}, \\
\mathcal{R}_{\mathfrak{S}^*_{\tanh,n}}(\mathcal{F}_3) &= \left(\frac{\sqrt{9n^2 + 4n \tan h1 + 4 \tan h^2 1} - 3n}{2(n + \tan h1)}\right)^{1/n}.
\end{aligned} \tag{163}$$

Proof.

(1) Let $f \in \mathcal{F}_1$ and describe the function $p,h : \mathfrak{D} \longrightarrow \mathbb{C}$ by

$$\begin{aligned}
p(\mathfrak{z}) &= \frac{g(\mathfrak{z})}{\mathfrak{z}}, \\
h(\mathfrak{z}) &= \frac{f(\mathfrak{z})}{g(\mathfrak{z})}.
\end{aligned} \tag{164}$$

Then, obviously $p,h \in \mathcal{P}_n$. Since

$$f(\mathfrak{z}) = \mathfrak{z}p(\mathfrak{z})h(\mathfrak{z}), \tag{165}$$

it follows from Lemma 1 that

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \le \frac{4nr^n}{1 - r^{2n}} \le \tan h1, \tag{166}$$

for

$$r \le \left(\sqrt{4n^2 \cot h^2 1 + 1} - 2n \cot h1\right)^{1/n} = \mathcal{R}_{\mathfrak{S}^*_{\tan\, h,n}}(\mathcal{F}_1). \tag{167}$$

For checking the sharpness of the result, we assume the functions

$$\begin{aligned}
f_0(\mathfrak{z}) &= \mathfrak{z}\left(\frac{1 + \mathfrak{z}^n}{1 - \mathfrak{z}^n}\right)^2, \\
g_0(\mathfrak{z}) &= \mathfrak{z}\left(\frac{1 + \mathfrak{z}^n}{1 - \mathfrak{z}^n}\right).
\end{aligned} \tag{168}$$

Thus, obviously,

$$\begin{aligned}
\mathfrak{Re}\left(\frac{f_0(\mathfrak{z})}{g_0(\mathfrak{z})}\right) &> 0, \\
\mathfrak{Re}\left(\frac{g_0(\mathfrak{z})}{\mathfrak{z}}\right) &> 0,
\end{aligned} \tag{169}$$

and hence, $f \in \mathcal{F}_1$. A computation shows that at $\mathfrak{z} = \mathcal{R}_{\mathfrak{S}^*_{\tan\, h,n}}(\mathcal{F}_1)e^{i(\pi/n)}$

$$\frac{\mathfrak{z}f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} = 1 + \frac{4n\mathfrak{z}^n}{1 - \mathfrak{z}^{2n}} = 1 - \tan h1. \tag{170}$$

Hence, the result is sharp

(2) Let $f \in \mathcal{F}_2$. Describe the function $p,h : \mathfrak{D} \longrightarrow \mathbb{C}$ by

$$\begin{aligned}
p(\mathfrak{z}) &= \frac{g(\mathfrak{z})}{\mathfrak{z}}, \\
h(\mathfrak{z}) &= \frac{f(\mathfrak{z})}{g(\mathfrak{z})}.
\end{aligned} \tag{171}$$

Then, $p \in \mathcal{P}_n$ and $h \in \mathcal{P}_n(1/2)$. Since

$$f(\mathfrak{z}) = \mathfrak{z}p(\mathfrak{z})h(\mathfrak{z}), \tag{172}$$

it follows from Lemma 1 that

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \le \frac{nr^n}{1 - r^2} + \frac{4nr^n}{1 - r^{2n}} = \frac{nr^{2n} + 3nr^n}{1 - r^{2n}} \le \tan h1. \tag{173}$$

For

$$r \le \left(\frac{\sqrt{9n^2 + 4n \tan h1 + 4 \tan h^2 1} - 3n}{2(n + \tan h1)} \right)^{1/n} = \mathscr{R}_{\mathfrak{S}^*_{\tan\ h,n}}(\mathscr{F}_2). \tag{174}$$

Thus, $f \in \mathfrak{S}^*_{\tan\ h,n}$ for $r \le \mathscr{R}_{\mathfrak{S}^*_{\tan\ h,n}}(\mathscr{F}_2)$. For checking the sharpness of the result, assume the functions

$$\begin{aligned} f_0(\mathfrak{z}) &= \frac{\mathfrak{z}(1+\mathfrak{z}^n)}{(1-\mathfrak{z}^n)^2}, \\ g_0(\mathfrak{z}) &= \frac{\mathfrak{z}}{1-\mathfrak{z}^n}. \end{aligned} \tag{175}$$

Then, obviously,

$$\begin{aligned} \mathfrak{Re}\left(\frac{f_0(\mathfrak{z})}{g_0(\mathfrak{z})}\right) &\ge 0, \\ \mathfrak{Re}\left(\frac{g_0(\mathfrak{z})}{\mathfrak{z}}\right) &\ge \frac{1}{2}, \end{aligned} \tag{176}$$

and hence, $f \in \mathscr{F}_2$. The sharpness is obvious, since at $\mathfrak{z} = \mathscr{R}_{\mathfrak{S}^*_{\tan\ h,n}}(\mathscr{F}_2)$, we get

$$\frac{\mathfrak{z} f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \frac{3n\mathfrak{z}^n + n\mathfrak{z}^{2n}}{1-\mathfrak{z}^{2n}} = \tan h1 \tag{177}$$

(3) Let $f \in \mathscr{F}_3$. Describe the functions $p,h : \mathfrak{D} \longrightarrow \mathbb{C}$ by

$$\begin{aligned} p(\mathfrak{z}) &= \frac{g(\mathfrak{z})}{\mathfrak{z}}, \\ h(\mathfrak{z}) &= \frac{g(\mathfrak{z})}{f(\mathfrak{z})}. \end{aligned} \tag{178}$$

Then, $p \in \mathscr{P}_n$. We know that $|(1/h(\mathfrak{z})) - 1| < 1$ if and only if Re $(h(\mathfrak{z})) > (1/2)$, and therefore, $h \in \mathscr{P}_n(1/2)$. Using Lemma 1 , we have

$$\left| \frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})} - 1 \right| \le \frac{nr^{2n} + 3nr^n}{1 - r^{2n}}. \tag{179}$$

Applying Lemma 3, we obtain

$$\frac{nr^{2n} + 3nr^n}{1 - r^{2n}} \le \tan h1. \tag{180}$$

For checking the sharpness, consider the functions

$$\begin{aligned} f_0(\mathfrak{z}) &= \frac{(1+\mathfrak{z}^n)^2 \mathfrak{z}}{(1-\mathfrak{z}^n)}, \\ g_0(\mathfrak{z}) &= \frac{(1+\mathfrak{z}^n)\mathfrak{z}}{1-\mathfrak{z}^n}. \end{aligned} \tag{181}$$

From the definition of f_0 and g_0, we get

$$\begin{aligned} \mathfrak{Re}\left(\frac{g_0(\mathfrak{z})}{f_0(\mathfrak{z})}\right) &= \mathfrak{Re}\left(\frac{1}{1+\mathfrak{z}^n}\right) > \frac{1}{2}, \\ \mathfrak{Re}\left(\frac{g_0(\mathfrak{z})}{\mathfrak{z}}\right) &= \mathfrak{Re}\left(\frac{1+\mathfrak{z}^n}{1-\mathfrak{z}^n}\right) > 0. \end{aligned} \tag{182}$$

Hence, $f_0 \in \mathscr{F}_3$. Now, at $\mathfrak{z} = \mathscr{R}_{\mathfrak{S}^*_{\tan\ h,n}}(\mathscr{F}_3)e^{i\pi/n}$, we get

$$\frac{\mathfrak{z} f_0'(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \frac{3n\mathfrak{z}^n - n\mathfrak{z}^{2n}}{1-\mathfrak{z}^{2n}} = -\tan h1. \tag{183}$$

This result is sharp. □

Data Availability

No data are used.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors have equally contributed to complete this manuscript.

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Research Article

Existence, Decay, and Blow-Up of Solutions for a Higher-Order Kirchhoff-Type Equation with Delay Term

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This article deals with the study of the higher-order Kirchhoff-type equation with delay term in a bounded domain with initial boundary conditions, where firstly, we prove the global existence result of the solution. Then, we discuss the decay of solutions by using Nakao's technique and denote polynomially and exponentially. Furthermore, the blow-up result is established for negative initial energy under appropriate conditions.

1. Introduction

In this paper, we establish the higher-order Kirchhoff-type equation with delay term as follows:

$$\begin{cases} u_{tt} + \left(\int_\Omega \left|A^{m/2}u\right|^2 dx\right)^q A^m u + \mu_1 |\mu_t(x,t)|^{r-1}\mu_t(x,t) & \\ +\mu_2 |\mu_t(x,t-\tau)|^{r-1}u_t(x,t-\tau) = |u|^{p-1}u, & (x,t)\in\Omega\times(0,T), \\ u(x,0)=u_0(x),\, u_t(x,0)=u_1(x) & x\in\Omega \\ u_t(x,t-\tau) = f_0(x,t-\tau) & x\in\Omega,\, t\in(0,\tau), \\ \dfrac{\partial^i u}{\partial \nu^i} = 0,\ i=0,\,1,\cdots,m-1 & x\in\partial\Omega, \end{cases} \tag{1}$$

where $A=-\Delta$, $m\geq 1$is a natural number, $q, r\geq 0$ are real numbers, $p>1$ is a real number and is a bounded domain with smooth boundary $\partial\Omega$ in R^n, $n=1\,;2\,;3$; ν the outer normal. $\tau>0$ denotes time delay, and $_1$ and $_2$ are positive real numbers. The functions (u_0, u_1, f_0) are the initial data belong to a suitable space.

The problem (1) is a general form of a model introduced by Kirchhoff [1]. To be more precise, Kirchhoff recommended a model denoted by the equation for $f=g=0$,

$$ph\frac{\partial^2 u}{\partial t^2} + \delta\frac{\partial u}{\partial t} + g\left(\frac{\partial u}{\partial t}\right) = \left\{\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} + f(u), \tag{2}$$

for $0<x<L, t\geq 0$, where $u(x,t)$ is the lateral displacement, ρ is the mass density, h is the cross-section area, E is the Young modulus, L is the length, ρ_0 is the initial axial tension, and f, g are the external forces. Furthermore, (2) is called a degenerate equation when $\rho_0=0$ and nondegenerate one when $\rho_0>0$.

Time delays often appear in many various problems, such as thermal, economic phenomena, biological, chemical, and physical. Recently, the partial di/erential equations with time delay have become an active area, (see [2, 3] and references therein). Datko et al. [4] indicated that a small delay in a boundary control is a source of instability. An arbitrarily small delay may destabilize a system which is uniformly asymptotically stable without delay unless additional conditions or control terms have been used in many cases [5]. Additional control terms will be necessary to stabilize hyperbolic systems including delay terms, (see [6–8] and references therein). In [6], Nicaise and Pignotti studied the equation as follows:

$$u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a u_t(x,t-\tau) = 0, \tag{3}$$

where a_0 and a are positive real parameters. The authors obtained that, under the condition $0 \le \alpha \le a_0$, the system is exponentially stable. In the case $\alpha \ge \alpha_0$, they obtained a sequence of delays that shows the solution is instable. In [8], Xu et al. obtained the same result similar to the [6] for the one space dimension by adopting the spectral analysis approach. In [9], Nicaise et al. studied the wave equation in one space dimension in the case of time-varying delay. In that work, the authors showed that an exponential stability result under the condition

$$\alpha \le \sqrt{1 - d\alpha_0}, \tag{4}$$

where d is a constant such that

$$\tau'(t) \le d < 1,\ \forall t > 0. \tag{5}$$

In recent years, some other authors investigate hyperbolic type equation with delay term (see [10–16]).

Without delay term $(\mu_2 |u_t(x,t\text{-}\tau)|^{r\text{-}1} u_t(x,t-\tau))$, in 2004, Li [17] studied the higher-order Kirchho/-type equation as follows:

$$\mu_{tt} + \left(\int_\Omega \left|D^{m/2}u\right|^2 dx\right)^q (-\Delta)^m u + u_t|u_t|^r = |u|^p u, \tag{6}$$

where $m > 1$ is a positive integer, and $q, p, r > 0$ is a positive constant. The author obtained that the solution exists globally if $p \le r$, while if $p > \max\{r, 2q\}$. He also established the blow-up result for $E(0) < 0$. Later, in 2007, Messaoudi and Houari [18] obtained the blow-up of solutions with $E(0) > 0$ of the equation (6). Then, Piskin and Polat [19] considered global existence and decay estimates utilizing Nakao's inequality of the equation (6).

Without delay term, when $m = 1$ and $q = 0$, equation (1) takes the form of a semilinear hyperbolic equation as follows:

$$u_{tt} - \Delta u + u_t|u_t|^{r-1} = |u|^{p-1}u. \tag{7}$$

Georgiev and Todorova [20] obtained the blow-up of solutions for $E(0) < 0$ if $1 < r < p$ $(1 < p < n/(n-2)$ for $n \ge 3$, $p > 1$ for $n < 3)$ of the equation (7). Under the condition of positive upper bounded initial energy, Vitillaro [21] proved the same results of equation (7). Also, Ohta [22, 23] studied related problems for the blow-up results of the equation (7).

Messaoudi [24] studied the following equation

$$u_{tt} + \Delta^2 u + |u_t|^{r2} u_t = |u|^{p2} u \tag{8}$$

and obtained an existence result for the equation (8) and proved that the solution continues to exists globally if $r \ge p$; however, if $r < p$ and the initial energy is negative, the solution blows up in finite time. Chen [25] established that the solution of (8) blows up with $E(0) > 0$. In the presence of strong damping term $(-\Delta u_t)$, Piskin and Polat [26] obtained the decay estimates by using Nakao's inequality of equation (8).

When $m = 1$ and without delay term, equation (1) takes the form the following Kirchhoff-type equation:

$$u_{tt}\text{-}\left(\int_\Omega |D_u|^2 dx\right)^\gamma \Delta_u + u_t|u_t|^r = |u|^p u. \tag{9}$$

Many authors had studied existence and blow-up results at night time for equation (9) (see [27–30]). Ono [30] proved the blow-up results if $p > \max\{r, 2\gamma\}$ $(p < 2/(n-4)$ for $n \ge 5$, $p > 0$ for $n \le 4)$ and $E(0) < 0$ for equation (9). Later, Benaissa and Messaoudi [31] obtained the similar result for the generalized Kirchhoff-type equation as follows:

$$u_{tt} - M\left(\int_\Omega e^{\phi(x)}|\nabla u|^2 dx\right) e^{-\phi(x)} \operatorname{div}\left(e^{\phi(x)}\nabla u\right) + \alpha|u_t|^{r-2}u_t = b|u|^{p-2}u, \tag{10}$$

where $M : R^+ \longrightarrow R^+$ and $\phi(x)$ are bounded functions. Then, Wu [32], verified the same result of the general Kirchhoff-type equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + |u_t|^{r-2}u_t = |u|^{p-2}u, \tag{11}$$

with the positive upper bounded initial energy. In 2013, Ye [33] considered the global existence results by constructing a stable set in $H_0^1(\Omega)$ and showed the decay by using a lemma of Komornik for the nonlinear Kirchhoff-type equation (11) with dissipative term. Moreover, Ye [34] obtained the global existence results by constructing a stable set in $H_0^m(\Omega)$ and showed the energy decay by using a lemma of V. Komornik for a nonlinear higher-order Kirchhoff-type equation with dissipative term is as follows:

$$u_{tt} + \left\|A^{1/2}u\right\|^{2p} Au + a|u_t|^{r-2}u_t = b|u|^{p-2}u, \tag{12}$$

where $A = (-\Delta)^m$, $m > 1$ is a positive integer.

Gao et al. [35] considered the Kirchhoff-type equation without delay term as follows:

$$u_{tt} + M\left(\|D^m u\|_2^2\right)(-\Delta)^m u + |u_t|^{r-2}u_t = |u|^{p-2}u. \tag{13}$$

The authors obtained the blow-up of solutions for $E(0) > 0$ under appropriate conditions for equation (13).

In [36–40], some authors studied abstract evolution equations as follows:

$$[P(u_t)]_t A(t,u) + Q(t,u_t) = F(u) \tag{14}$$

on suitable Banach space, and they proved some global nonexistence of solutions. Some other authors studied related problems (see [41–45]).

Motivated by the above works, we deal with the existence, decay, and blow-up results for the higher-order Kirchhoff type equation (1) with delay term and source term. There is no research, to our best knowledge, related to the higher-order Kirchhoff-type $((\int_\Omega |A^{m/2}u|^2 dx)^q A^m u)$ equation (1) with delay $(u_2|u_t(x,t-\tau)|^{r-1}u_t(x,t-\tau))$ and source $(|u|^{p-1}u)$ terms; hence, our work is the generalization of the above studies.

This work consists of five sections in addition to the introduction: Firstly, in Sect. 2, we recall some lemmas and assumptions. Then, in Section 3, we get the global existence of solutions. Moreover, in Section 4, we establish the decay results by using Nakao's tecnique. Finally, in Section 5, we obtain the blow-up of solutions for negative initial energy.

2. Preliminaries

In this part, we present some lemmas and assumptions for the proof of our result. Let $H^m(\Omega)$ denote the Sobolev space with the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}., \tag{15}$$

$H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm, $\|\cdot\|$ denotes $L^2(\Omega)$ norm, and we write equivalent norm $\|\nabla\cdot\|$instead of $H_0^1(\Omega)$ norm $\|\cdot\|H_0^1(\Omega)$. We denote by $C_i (i=1,2,\cdots,n)$ various positive constants which may be different at different occurrences.

Lemma 1 (see [46, 47] Sobolev-Poincarė inequality). *If $2\le p\le (2n/[n-2m]^+)(2\le p<\infty$ if $n=2m)$, then for some C_*,*
$\|u\|_p \le C_*\|(-\Delta)^{m/2}u\|$ for $u\in H_0^m(\Omega)$,
where we put $[\alpha]^+ = \max\{0,\alpha\}$, $1/[\alpha]^+ = \infty$ if $[\alpha]^+ = 0$.

Lemma 2 (see [48]). *Let $\phi(t)$ be nonincreasing and nonnegative function defined on $[0,T]$, $T>1$ and satisfies*

$$\phi^{1+\alpha}(t) \le w_0(\phi(t)-\phi(t+1)),\ t\in[0,T] \tag{16}$$

for w_0 is a positive constant, and α is a nonnegative constant. Then, we have for each $t\in[0,T]$,

$$\begin{cases} \phi(t)\le \phi(0)e^{-w_1[t-1]^+}, & \alpha=0,\\ \phi(t)\le \left(\phi(0)^{-\alpha}+w_0^{-1}\alpha[t-1]^+\right)^{-1/\alpha}, & \alpha=0, \end{cases} \tag{17}$$

where $[t-1]^+ = \max\{t-1,0\}$, and $w_1 = \ln(w_0/w_0-1)$.

We make the assumptions on parameters r, p, and m as follows:

(A1)

$$\begin{cases} 1<p<\infty, & n\le 2m,\\ 1<p<\dfrac{n}{n-2m}, & n>2m, \end{cases} \tag{18}$$

(A2)

$$\begin{cases} 1<r<\infty, & n\le 2m,\\ 1<r<\dfrac{n+2m}{n-2m}, & n>2m. \end{cases} \tag{19}$$

3. Global Existence

In this part, we consider the global existence results of the problem (1). Firstly, we introduce the new function z similar to the [7],

$$z(x,k,t) = u_t(x,t-\tau k), x\in\Omega, k\in(0,1). \tag{20}$$

Thus, we have

$$\tau z_t(x,k,t) + z_k(x,k,t) = 0, \text{in}\Omega\times(0,1)\times(0,\infty). \tag{21}$$

Hence, problem (1) can be transformed as follows:

$$\begin{cases} u_{tt} + \left(\int_\Omega |A^{m/2}u|^2 dx\right)^q A^m u + \mu_1|u_t(x,t)|^{r-1}u_t(x,t) + \mu_2|z(x,1,t)|^{r-1}z(x,1,t) = |u|^{p-1}u & (x,t)\in\Omega\times(0,T),\\ \tau x_t(x,k,t) + z_k(x,k,t) = 0 & \text{in }\Omega\times(0,1)\times(0,\infty),\\ z(x,k,0) = f_0(x,-\tau k) & x\in\Omega,\\ u(x,0)=u_0(x), u_t(x,0)=u_1(x) & x\in\Omega,\\ \dfrac{\partial^i u}{\partial v^i} = 0, i=0,1,\cdots,m-1 & x\in\partial\Omega. \end{cases} \tag{22}$$

We define the energy functional for any regular solution of (22) as follows:

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2(q+1)}\left\|A^{m/2}u\right\|^{2(q+1)} - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{\varsigma}{r+1}\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx, \tag{23}$$

such that

$$\tau r|\mu_2| < \varsigma < \tau((r+1)\mu_1 - |\mu_2|). \tag{24}$$

Also, have

$$J(t) = J(u(t)) = \frac{1}{2(q+1)}\left\|A^{m/2}u\right\|^{2(q+1)} - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{\varsigma}{r+1}\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx, \tag{25}$$

$$I(t) = I(u(t)) = \left\|A^{m/2}u\right\|^{2(q+1)} - \|u\|_{p+1}^{p+1} + \varsigma\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx. \tag{26}$$

We easily see that

$$E(t) = J(t) + \frac{1}{2}\|u_t\|^2. \tag{27}$$

Furthermore, we define

$$\mathscr{W} = \left\{u : u \in H_0^m(\Omega)\cap H^{2m}(\Omega), I(u) > 0\right\}\cup\{0\}. \tag{28}$$

Next, lemma gives that the energy functional $E(t)$ is a nonincreasing.

Lemma 3. *Assume that (u, z) is the solution of (22), then for $t \geq 0$,*

$$E'(t) = -\left(\mu_1 - \frac{\varsigma}{\tau(r-1)} - \frac{\mu_2}{r+1}\right)\|u_t(t)\|_{r+1}^{r+1} - \left(\frac{\varsigma}{\tau(r+1)} - \frac{\mu_2 r}{r+1}\right)\int_\Omega z^{r+1}(x,1,t)dx \leq 0. \tag{29}$$

Proof. We multiply the first equation in (22) by u_t, integrate over, and use integration by parts, and we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2(q+1)}\left\|A^{m/2}u\right\|^{2(q+1)} - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\right] + \mu_1\|u_t(t)\|_{r+1}^{r+1}\int_\Omega \mu_2|z(x,1,t)|^{r-1}z(x,1,t)u_t(x,t)dx = 0. \tag{30}$$

Integrating (30) over $(0, t)$, we get

$$\left[\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2(q+1)}\left\|A^{m/2}u\right\|^{2(q+1)} - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\right] + \int_0^t \mu_1\|u_s(s)\|_{r+1}^{r+1}ds + \mu_2\int_0^t\int_\Omega |z(x,1,s)|^{r-1}z(x,1,s)u_s(x,s)dxds = \frac{1}{2}\|u_1\|^2. \tag{31}$$

We multiply the second equation in (22) by $\varsigma|z|^{r-1}z$ and integrate the result over $\Omega\times(0,1)\times(0,t)$, and we get

$$\begin{aligned}\frac{\varsigma}{r+1}\frac{d}{dt}\int_0^t\int_\Omega\int_0^1 |z(x,k,t)|^{r-1}(x,k,t)z_t(x,k,t)dkdxds &= -\frac{\varsigma}{\tau(r+1)}\int_0^t\int_\Omega\int_0^1 \frac{\partial}{\partial k}|z(x,k,t)|^{r+1}dkdxds\\ &= -\frac{\varsigma}{\tau(r+1)}\int_0^t\int_\Omega\left[|z(x,1,t)|^{r+1} - |z(x,0,t)|^{r+1}\right]dxds\\ &= -\frac{\varsigma}{\tau(r+1)}\int_0^t\int_\Omega |z(x,1,t)|^{r+1}dxds + \frac{\varsigma}{\tau(r+1)}\int_0^t\|u_t(t)\|_{r+1}^{r+1}ds.\end{aligned} \tag{32}$$

By combining (31) and (32), we arrive at

$$E(t) + \left(\mu_1 - \frac{\varsigma}{\tau(r+1)}\right)\int_0^t\|u_s(s)\|_{r+1}^{r+1}ds + \frac{\varsigma}{\tau(r+1)}\int_0^t\int_\Omega |z(x,1,s)|^{r+1}dxds + \mu_2\int_0^t\int_\Omega |z(x,1,s)|^{r-1}z(x,1,s)u_s(x,s)dxds = E(0). \tag{33}$$

Utilizing the Young inequality on the fourth term of the left hand side of (33), we conclude that

$$E(t) + \left(\mu_1 - \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2}{r+1}\right)\int_0^t\|u_s(s)\|_{r+1}^{r+1}ds + \left(\frac{\varsigma}{\tau(r+1)} - \frac{\mu_2 r}{r+1}\right)\int_0^t\int_\Omega |z(x,1,s)|^{r+1}dxds = E(0). \tag{34}$$

Deriving the (34), we have the desired result. Hence, the proof is completed. □

Remark 4. From the condition (24), we obtain

$$c_1 = \left(\mu_1 - \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2}{r+1}\right) > 0, c_2 = \left(\frac{\varsigma}{\tau(r+1)} - \frac{\mu_2 r}{r+1}\right) > 0. \tag{35}$$

Lemma 5. *Assume that (19) and $p > 2q + 1$ hold. Let $u_0 \in \mathscr{W}$ and $u_1 \in H_0^m(\Omega)$, such that*

$$\beta = C_* \left(\frac{2(q+1)(p+1)}{p-2q-1} E(0) \right)^{p-2q-1/2(q+1)} < 1, \tag{36}$$

then $u \in \mathscr{W}$ for each $t \geq 0$.

Proof. It follows the continuity of $u(t)$, since $I(0) > 0$, such that

$$I(t) > 0, \tag{37}$$

for some interval near $t = 0$. Assume that $T_m > 0$ is a maximal time, when (26) holds on $[0, T_m]$.

By (25) and (26), we obtain

$$\begin{aligned} J(t) &= \frac{1}{p+1} I(t) + \frac{p-2q-1}{2(q+1)(p+1)} \left\| A^{m/2} u \right\|^{2(q+1)} \\ &\quad + \frac{(p-r)}{(r+1)(p+1)} \left(\varsigma \int_\Omega \int_0^1 z^{r+1}(x,k,t)\, dk dx \right) \\ &\geq \frac{p-2q-1}{2(q+1)(p+1)} \left\| A^{m/2} u \right\|^{2(q+1)}. \end{aligned} \tag{38}$$

From (23), (38), and Lemma 3, we have

$$\begin{aligned} \left\| A^{m/2} u \right\|^{2(q+1)} &\leq \frac{2(q+1)(p+1)}{p-2q-1} J(t) \leq \frac{2(q+1)(p+1)}{p-2q-1} E(t) \\ &\leq \frac{2(q+1)(p+1)}{p-2q-1} E(0). \end{aligned} \tag{39}$$

Using Lemma 1 and (39), we get

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\leq C_* \left\| A^{m/2} u \right\|^{p+1} = C_* \left\| A^{m/2} u \right\|^{p-2q-1} \left\| A^{m/2} u \right\|^{2(q+1)} \\ &\leq C_* \left(\frac{2(q+1)(p+1)}{p-2q-1} E(0) \right)^{p-2q-1/2(q+1)} \left\| A^{m/2} u \right\|^{2(q+1)} \\ &= \beta \left\| A^{m/2} u \right\|^{2(q+1)} < \left\| A^{m/2} u \right\|^{2(q+1)} \text{ on } t \in [0, T_m]. \end{aligned} \tag{40}$$

Thus, from (26), we arrive at $I(t) > 0$ for all $t \in [0, T_m]$. T_m is extended to T, by repeating the procedure. Hence, the proof is completed. □

Lemma 6. *Suppose that the assumptions of Lemma 5 hold. Then, there exists $\eta_1 = 1 - \beta$, such that*

$$\|u\|_{p+1}^{p+1} \leq (1 - \eta_1) \left\| A^{m/2} u \right\|^{2(q+1)}. \tag{41}$$

Proof. By (40), we obtain

$$\|u\|_{p+1}^{p+1} \leq \beta \left\| A^{m/2} u \right\|^{2(q+1)}. \tag{42}$$

Let $\eta_1 = 1 - \beta$; therefore, we obtain the result. □

Remark 7. By Lemma 6, we conclude that

$$\left\| A^{m/2} u \right\|^{2(q+1)} \leq \frac{1}{\eta_1} I(t). \tag{43}$$

Theorem 8. *Assume that the assumptions (A2), $\mu_2 < \mu_1$, and $p > 2q + 1$ hold. Let $u_0 \in \mathscr{W}$ satisfying (36) and $f_0 \in L^2(\Omega \times (0, 1))$ be given. Then, the solution of problem (22) is global.*

Proof. It is sufficient to show that $\|u_t\|^2 + \|A^{m/2}u\|^{2(q+1)}$ is bounded independently of t. To obtain this, by using (23) and (26), we have

$$\begin{aligned} E(0) \geq E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2(q+1)} \left\| A^{m/2} u \right\|^{2(q+1)} - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \\ &\quad + \frac{\varsigma}{r+1} \int_\Omega \int_0^1 z^{r+1}(x,k,x)\, dk dx = \frac{1}{2}\|u_t\|^2 \\ &\quad + \frac{p-2q-1}{2(q+1)(p+1)} \left\| A^{m/2} u \right\|^{2(q+1)} \\ &\quad + \frac{(p-r)}{(r+1)(p+1)} \left(\varsigma \int_\Omega \int_0^1 z^{r+1}(x,k,t)\, dk dx \right) \\ &\quad + \frac{1}{p+1} I(t) \geq \frac{1}{2}\|u_t\|^2 + \frac{p-2q-1}{2(q+1)(p+1)} \left\| A^{m/2} u \right\|^{2(q+1)}, \end{aligned} \tag{44}$$

since $I(t) \geq 0$. Thus,

$$\|u_t\|^2 + \left\| A^{m/2} u \right\|^{2(q+1)} \leq C E(0), \tag{45}$$

where $C = \max\{2, (2(q+1)(p+1)/p - 2q - 1)\}$. Therefore, we obtain the global existence of solutions. Therefore, we completed the proof. □

4. Decay of Solution

In this part, we obtain the decay of solutions of the problem (22) by using Nakao's technique.

Theorem 9. *Assume that the assumption (A2) and (36) hold. Let $u_0 \in \mathscr{W}$, $f_0 \in L^2(\Omega \times (0, 1))$, be given. Hence, we have following decay estimates:*

$$E(t) \leq \begin{cases} E(0) e^{-w_1 [t-1]^+}, & \text{if } r = 1, \\ \left(E(0)^{-\alpha} + C_7^{-1} \alpha [t-1]^+ \right)^{-1/\alpha}, & \text{if } r > 1, \end{cases} \tag{46}$$

where w_1, α and C_7 are positive constants which will be defined later.

Proof. We integrate (29) over $[t, t+1]$, $t > 0$, to get

$$E(t) - E(t+1) = [D(t)]^{r+1}, \tag{47}$$

where

$$[D(t)]^{r+1} = c_1\int_t^{t+1} \|u_t\|_{r+1}^{r+1} ds + c_2\int_t^{t+1}\int_\Omega z^{r+1}(x,1,s)dxds. \tag{48}$$

From (48) and Hölder inequality, we see that

$$\int_t^{t+1}\int_\Omega |u_t|^2 dxdt + \int_t^{t+1}\int_\Omega |z(x,1,s)|^2 dxds \le x(\Omega)[D(t)]^2, \tag{49}$$

where $c(\Omega) = vol(\Omega)$.Therefore, by (49), there exists $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$, so that

$$\|u_t(t_i)\|^2 + \|z(x,1,t_i)\|^2 \le c(\Omega)[D(t)]^2, i = 1, 2. \tag{50}$$

We multiply the first equation in (22) by u and integrate over $\Omega \times [t_1, t_2]$. Use integration by parts, Hölder's inequality, adding, and subtracting the term $\int_{t_1}^{t_2}\int_\Omega\int_0^1 \varsigma z^{r+1}(x,k,t)dk\, dxdt$, we have

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt \le{}& \|u_t(t_1)\|_2\|u(t_1)\|_2 + \|u_t(t_2)\|_2\|u(t_2)\|_2 \\ &+ \int_{t_1}^{t_2}\|u_t\|^2dt + \int_{t_1}^{t_2}\int_0^1\varsigma\int_\Omega z^{r+1}(x,k,t)dxdkdt \\ &- \mu_1\int_{t_1}^{t_2}\int_\Omega |u_t|^{r-1}|u_t|udxdt \\ &- \mu_2\int_{t_1}^{t_2}\int_\Omega |z(x,1,t)|^{r-1}z(x,1,t)udxdt.\end{aligned} \tag{51}$$

Now, we estimate the right hand side for (51).
From (39), (50), and Lemma 1, we obtain

$$\|u_t(t_i)\|_2\|u(t_i)\|_2 \le C_1 D(t)\sup_{t_1\le s\le t_2} E^{1/2}(s), \tag{52}$$

where $C_1 = 2C_*((2(q+1)(p+1)/p-2q-1)E(0))^{1/2(q+1)}$.
By using (32) that

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^1\int_\Omega z^{r+1}(x,k,t)dxdkdt &\le \frac{1}{2\tau}\int_{t_1}^{t_2}\int_{t_1}^{t_2}\|u_t(s)\|_{r+1}^{r+1}dsdv \\ &\le \left(\int_{t_1}^{t_2}dv\right)\left(\int_{t_1}^{t_2}\|u_t(s)\|_{r+1}^{r+1}ds\right) \le (t_2-t_1)[D(t)]^{r+1}.\end{aligned} \tag{53}$$

Utilizing Hölder inequality, we get

$$\int_{t_1}^{t_2}\int_\Omega |u_t|^{r-1}|u_t|udxdt \le \int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r\|u(t)\|_{r+1}. \tag{54}$$

Utilizing the Sobolev-Poincare inequality and (39), we have

$$\begin{aligned}\left|\int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r\|u(t)\|_{r+1}dt\right| &\le C_*\int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r\left\|A^{m/2}u\right\|dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r E^{1/2}(s)dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1,\le s\le t_2}E^{1/2}(s)\int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1,\le s\le t_2}E^{1/2}(s)[D(t)]^r.\end{aligned} \tag{55}$$

Also, we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |z(x,1,t)|^{r-1}z(x,1,t)udxdt &\le C_*\int_{t_1}^{t_2}\|z(x,1,t)\|_{r+1}^r\|u\|_{r+1}dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\int_{t_1}^{t_2}\|z(x,1,t)\|_{r+1}^r E^{1/2}(s)dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1\le s\le t_2}E^{1/2}(s)\int_{t_1}^{t_2}\|z(x,1,t)\|_{r+1}^r dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1\le s\le t_2}E^{1/2}(s)[D(t)]^r.\end{aligned} \tag{56}$$

Then, from (51)-(56), we get

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt \le C_2\Bigg[&\sup_{t_1\le s\le t_2}E^{1/2}(s)D(t) + [D(t)]^2 + (t_2-t_1)[D(t)]^{r+1} \\ &+ 2C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1\le s\le t_2}E^{1/2}(s)[D(t)]^r\Bigg].\end{aligned} \tag{57}$$

Moreover, by (23), (26), and Remark 7, we have

$$E(t) \le \frac{1}{2}\|u_t\|^2 + C_3 I(t), \tag{58}$$

where $C_3 = (1/\eta_1)(p-2q-1/2(q+1)(p+1)) + (1/p+1)$

.

Integrating (58) over $[t_1, t_2]$, we get

$$\int_{t_1}^{t_2} E(t)dt \le \frac{1}{2}\int_{t_1}^{t_2}\|u_t\|^2 dt + C_3\int_{t_1}^{t_2} I(t)dt. \tag{59}$$

Hence, from (57) and (59), we obtain

$$\int_{t_1}^{t_2} E(t)dt \le \frac{1}{2}C[D(t)]^2 + C_3C_2\left[\sup_{t_1\le s\le t_2} E^{1/2}(s)D(t) + [D(t)]^2 + (t_2-t_1)[D(t)]^{r+1} + 2C_* \cdot \left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)} \sup_{t_1\le s\le t_2} E^{1/2}(s)[D(t)]^r\right]. \quad (60)$$

Integrating $(d/dt)E(t)$ over $[t, t2]$, we conclude that

$$E(t) = E(t_2) + \int_t^{t_2}\left(\mu_1 - \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2}{r+1}\right)\|u_s(s)\|_{r+1}^{r+1}ds + \int_t^{t_2}\left(\frac{\varsigma}{\tau(r+1)} - \frac{\mu_2 r}{r+1}\right)\int_\Omega |z(x,1,s)|^{r+1}dxds. \quad (61)$$

Thus, since $t_2 - t_1 \ge 1/2$, we arrive at

$$\int_{t_1}^{t_2} E(t)dt \ge (t_2-t_1)E(t_2) \ge \frac{1}{2}E(t_2). \quad (62)$$

Hence,

$$E(t_2) \le 2\int_{t_1}^{t_2} E(t)dt. \quad (63)$$

As a result, from (47), (60), (61), (63), and since $t_1, t_2 \in [t, t+1]$, we get

$$E(t) \le 2\int_{t_1}^{t_2} E(t)dt + \int_t^{t+1}\left(\mu_1 - \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2}{r+1}\right)\|u_s(s)\|_{r+1}^{r+1}ds + \int_t^{t+1}\left(\frac{\varsigma}{\tau(r+1)} - \frac{\mu_2 r}{r+1}\right)\int_\Omega |z(x,1,s)|^{r+1}dxds = 2\int_{t_1}^{t_2} E(t)dt + [D(t)]^{r+1}. \quad (64)$$

Then, from (60), we have

$$E(t) \le \left(\frac{1}{2}C + C_3C_2\right)[D(t)]^2 + C_3C_2[D(t)]^{r+1} + C_4[D(t) + [D(t)]^r]E^{1/2}(t). \quad (65)$$

Thus, utilizing Young inequality, we have

$$E(t) \le C_5\left[[D(t)]^2 + [D(t)]^{r+1} + [d(t)]^{2r}\right]. \quad (66)$$

□

Hence, we have the decay estimates as follows:

Case 1. If $r = 1$, by (66), we obtain

$$E(t) \le 3C_5[D(t)]^2 = 3C_5[E(t) - E(t-1)]. \quad (67)$$

Utilizing Lemma 2, we have

$$E(t) \le E(0)e^{-w_1[t-1]^+}, \quad (68)$$

where $w_1 = \ln(3C_5/3C_5 - 1)$.

Case 2. If $r > 1$, by (66), we have

$$E(t) \le C_5[D(t)]^2\left(1 + [D(t)]^{r-1} + [D(t)]^{2r-2}\right). \quad (69)$$

Then, by (47), since $E(t) \le E(0), \forall t \ge 0$, we see that

$$E(t) \le C_5\left(1 + E^{r-1/r+1}(0) + E^{2(r-1)/r+1}(0)\right)[D(t)]^2 \le C_6[D(t)]^2, t \ge 0. \quad (70)$$

Then, we get

$$E(t)^{r+1/2} \le C_7[D(t)]^{r+1} \le C_7(E(t) - E(t+1)). \quad (71)$$

Therefore, by (71) and Lemma 2, we obtain

$$E(t) \le \left(E(0)^{-\alpha} + C_7^{-1}\alpha[t-1]^+\right)^{-1/\alpha}. \quad (72)$$

Thus, we completed the proof of Theorem 9.

5. Blow-Up of Solution

In this part, we get the blow-up of solutions for negative initial energy, in the case $r > 1$.

Theorem 10. *Let* $(u_0, u_1) \in (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ *and* $f_0 \in L^2(\Omega \times (0,1))$ *be given. Assume that* $p > \max\{2, r, 2q+1\}$ *and the assumptions (A1)-(A2) hold. Then, the solution of (22) blows up in a finite time with* $E(0) < 0$.

Proof. Setting

$$H(t) = -E(t), \quad (73)$$

from Lemma 3, we obtain

$$H'(t) \ge -\left(\mu_1 - \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2}{r+1}\right)\|u_t(t)\|_{r+1}^{r+1} - \left(\frac{\varsigma}{\tau(r+1)} - \frac{\mu_2 r}{r+1}\right)\int_\Omega z^{r+1}(x,1,t)dx. \quad (74)$$

Thus,

$$0 < H(0) \le H(t) \le \frac{1}{p+1}\|u\|_{p+1}^{p+1}, t > 0. \quad (75)$$

Let

$$M(t) = \|u\|_2^2. \tag{76}$$

Differentiating (76) twice, we get

$$\begin{aligned} M'(t) &= 2\int_\Omega u_t u dx, \\ M''(t) &= 2\|u_t\|^2 + 2\int_\Omega u_{tt} u dx. \end{aligned} \tag{77}$$

Using the first equation in (22), to have

$$\begin{aligned} M''(t) = 2\|u_t\|^2 - 2\left\|A^{\frac{m}{2}}u\right\|_2^{2(q+1)} - 2\mu_1 \int_\Omega |u_t(x,t)|^{r-1} u u_t(x,t) dx \\ - 2\mu_2 \int_\Omega |z(x,1,t)|^{r-1} u z(x,1,t) dx + 2\|u\|_{p+1}^{p+1}, \end{aligned} \tag{78}$$

we add and subtract the term $2(p+1)H(t)$, and then (78) becomes the form

$$\begin{aligned} M''(t) \geq{}& (p+3)\|u_t\|^2 + 2(p+1)H(t) \\ &+ \left(\frac{p+1}{q+1} - 2\right)\left\|A^{\frac{m}{2}}u\right\|_2^{2(q+1)} \\ &- 2\mu_1 \int_\Omega |u_t(x,t)|^{r-1} u u_t(x,t) dx \\ &- 2\mu_2 \int_\Omega |z(x,1,t)|^{r-1} u z(x,1,t) dx \\ &+ \frac{2\varsigma(p+1)}{r+1} \int_\Omega \int_0^1 z^{r+1}(x,k,s) dk dx. \end{aligned} \tag{79}$$

Now, we define

$$L(t) = H(t)^{1-\kappa} + 2\epsilon M'(t). \tag{80}$$

Differentiating (80), we obtain

$$L'(t) = (1-\kappa)H(t)^{-\kappa}H'(t) + 2\varepsilon M''(t). \tag{81}$$

Replacing (79) in (81), we arrive at

$$\begin{aligned} L'(t) \geq{}& (1-\kappa)H(t)^{-\kappa}H'(t) 2\varepsilon(p+3)\|\mu_t\|^2 \\ &+ 4\varepsilon(p+1)H(t) + 2\varepsilon\left(\frac{p+1}{q+1} - 2\right)\left\|A^{\frac{m}{2}}\mu\right\|_2^{2(q+1)} \\ &- 4\varepsilon\mu_1 \int_\Omega |\mu_t(x,t)|^{r-1} u u_t(x,t) dx \\ &- 4\varepsilon\mu_2 \int_\Omega |z(x,1,t)|^{r-1} u z(x,1,t) dx \\ &+ \frac{4\varepsilon\varsigma(p+1)}{r+1} \int_\Omega \int_0^1 z^{r+1}(x,k,s) dk dx. \end{aligned} \tag{82}$$

From (75) and utilizing Hölder inequality, we get

$$\begin{aligned} \left|\int_\Omega |u_t)x,t|^{r-1} u u_t(x,t) dx\right| &\leq \|u_t\|_{r+1}^r \|u\|_{r+1} \\ &\leq c_1 \|u\|_{r+1}^{r+1/p+1} \|u\|_{r+1}^{1-r+1/p+1} \|u\|_{r+1}^r \\ &\leq c_2 \|u\|_{r+1}^{r+1/p+1} H(t)^{\frac{1}{p+1} - \frac{r+1}{(p+1)^2}} \|u_t\|_{r+1}^r. \end{aligned} \tag{83}$$

From Young's inequality and (74), we have

$$\begin{aligned} &\left|\int_\Omega |u_t(x,t)|^{r-1} u u_t(x,t) dx\right| \\ &\quad \leq c_3\left(\rho^{1/1+p}\|\mu\|_{r+1}^{r+1} H(0)^{-\bar{k}} + \rho^{-r'} H(0)^{k-\bar{k}} H'(t) H(t)^{-k}\right), \end{aligned} \tag{84}$$

where $\bar{k} = 1/p + 1 - r + 1/(p+1)^2 > 0, \rho > 0, r' = r + 1/r,$ letting $0 < k < \bar{k}$. In a similar way, we obtain

$$\begin{aligned} &\left|\int_\Omega u|z(x,1,t)|^{r-1} z(x,1,t) dx\right| \\ &\quad \leq c_3\left(\rho^{\frac{1}{1+p}}\|\mu\|_{r+1}^{r+1} H(0)^{-\bar{k}} + \rho^{-r'} H(0)^{k-\bar{k}} H'(t) H(t)^{-k}\right). \end{aligned} \tag{85}$$

By using (82), (84), and (85), to have

$$\begin{aligned} L'(t) \geq{}& \left[(1-\kappa) - 4\varepsilon(\mu_1+\mu_2)H^{k-\bar{k}}(0)\rho^{-r'}\right] H(t)^{-\kappa}H'(t) \\ &- 4\varepsilon(\mu_1+\mu_2)H(0)^{-\bar{k}}\rho^{1/1+p}\|u\|_{r+1}^{r+1} + 2\varepsilon(p+3)\|u_t\|^2 \\ &+ 4\varepsilon(p+1)H(t) + 2\varepsilon\left(\frac{p+1}{q+1} - 2\right)\left\|A^{m/2}\mu\right\|_2^{2(q+1)} \\ &+ \frac{4\varepsilon\varsigma(p+1)}{r+1} \int_\Omega \int_0^1 z^{r+1}(x,k,s) dk dx, \end{aligned} \tag{86}$$

for ε sufficiently small, we obtain

$$\left[(1-\kappa) - 4\varepsilon(\mu_1+\mu_2)H^{k-\bar{k}}(0)\rho^{-r'}\right] \geq 0. \tag{87}$$

Setting $s = r + 1 \leq p + 1$ such that

$$\|\mu\|_{r+1}^s \leq c_1\left(\left\|A^{m/2}\mu\right\| + \|\mu\|_{p+1}^{p+1}\right), \tag{88}$$

where $c = 4(\mu_1+\mu_2)H(0)^{-\bar{k}}\rho^{1/p+1}c_1$ and taking $(p + 1/q + 1 - 2) > c$. Thus, we have

$$\begin{aligned} L'(t) \geq{}& 2\varepsilon\left(\frac{p+1}{q+1} - 2 - c\right)\left\|A^{m/2}u\right\|_2^{2(q+1)} \\ &- \varepsilon c\|\mu\|_{p+1}^{p+1} + 4\varepsilon(p+1)H(t) + 2\varepsilon(p+3)\|\mu\|^2 \\ &+ 4\varepsilon\frac{\zeta(p+1)}{r+1} \int_\Omega \int_0^1 z^{r+1}(x,k,s) dk dx. \end{aligned} \tag{89}$$

By using the notations $a_1 = 2(p + 1/q + 1 - 2 - c)$, $a_2 = c$, $a_3 = 4(p + 1)$, and $a_4 = 2(p + 3)$, (89) takes the form

$$L'(t) \geq a_1 \varepsilon \left\| A^{m/2} \mu \right\|_2^{2(q+1)} - \varepsilon a_2 \|\mu\|_{p+1}^{p+1} + \varepsilon a_3 H(t) + \varepsilon a_4 \|\mu_t\|^2. \tag{90}$$

Similarly to the approach of Messaoudi [49], we assume that $p = 2a_5 + (p - 2a_5)$, where $a_5 < \min\ (a_1, a_2, a_3, a_4)$, and then (90) becomes the form

$$\begin{aligned} L'(t) \geq{} & (a_1 - a_5)\varepsilon \left\| A^{m/2} \mu \right\|_2^{2(q+1)} + \varepsilon (a_5 - a_2) \|\mu\|_{p+1}^{p+1} \\ & + \varepsilon (a_3 - a_5) H(t) + \varepsilon (a_4 - a_5) \|\mu_t\|^2. \end{aligned} \tag{91}$$

Then,

$$L'(t) \geq \delta \varepsilon \left[\left\| A^{\frac{m}{2}} u \right\|_2^{2(q+1)} + \|\mu_t\|_{p+1}^{p+1} + H(t) + \|u_t\|^2 \right]. \tag{92}$$

We conclude that

$$L'(t) \geq \delta \varepsilon \left[\|u\|_{p+1}^{p+1} + H(t) + \|u_t\|^2 \right], \tag{93}$$

where $\delta > 0$ is the minimum of the coefficients of $\|u\|_{p+1}^{p+1}$, $H(t)$, $\|u_t\|^2$. Pick out ε such that

$$L(0) = H^{1-\kappa}(0) + 2\varepsilon \int_\Omega u_1 u_0 dx > 0. \tag{94}$$

As a result, we getsetting $\varpi = 1/1 - k$, and since $k < \bar{k} < 1$, we see that $1 < \varpi < 1/1 - \bar{k}$. Set

$$L(t) = H(t)^{1-\kappa} + 2\varepsilon \int_\Omega u u_t dx. \tag{95}$$

Then,

$$\begin{aligned} L(t) = H(t)^{1/\varpi} + 2\varepsilon \int_\Omega u u_t dx \leq H(t)^{1/\varpi} \\ + 2\varepsilon \int_\Omega u u_t dx + 2E_1 \left(\|u\|_{p+1} \right)^{p+1/\varpi}. \end{aligned} \tag{96}$$

Utilizing Young, Hölder's inequalities, and (96), we conclude that

$$\begin{aligned} L(t)^\varpi & \leq \left[H(t)^{1/\varpi} + 2\varepsilon \int_\Omega u u_t dx + 2E_1 \left(\|u\|_{p+1} \right)^{p+1/\varpi} \right]^\varpi \\ & \leq 2^{\varpi-1} \left[H(t) + \left(2\varepsilon \int_\Omega u u_t dx + 2E_1 \left(\|u\|_{p+1} \right)^{p+1/\varpi} \right)^\varpi \right] \\ & \leq 2^{\varpi-1} \left[H(t) + 2^{\varpi-1} \left(\left(2\varepsilon \int_\Omega u u_t dx \right)^\varpi + 2E_1 \|u\|_{p+1}^{p+1} \right) \right] \\ & \leq 2^{\varpi-1} \left[H(t) + 2^{\varpi-1} \left(\beta^\varpi \|u_t\|_2^\varpi \|u\|_2^\varpi + 2E_1 \|u\|_{p+1}^{p+1} \right) \right] \\ & \leq 2^{\varpi-1} \left[H(t) + 2^{\varpi-1} \left(\beta^\varpi \|u_t\|_2^\varpi \|u\|_2^\varpi \right) + 2^{\varpi-1} \left(2E_1 \|u\|_{p+1}^{p+1} \right) \right] \\ & \leq c_2 [H(t) +] \|u_t\|_2^\varpi \|u\|_2^\varpi + \|u\|_{p+1}^{p+1}, \end{aligned} \tag{97}$$

where $c_2 = \max\ \{2^{\varpi-1}, \beta^\varpi\}$. Furthermore, for $p > 1$, utilizing Hölder and Young inequalities, we obtain

$$\begin{aligned} \|u_t\|_2^\varpi \|u\|_2^\varpi \leq c_3 \|u_t\|_2^\varpi \|u\|_{p+1}^\varpi \leq c_4 \left(\|u_t\|_2^2 + \|u\|_{p+1}^{2(1-k)/1-2k} \right), \\ \|u\|_{p+1}^{2(1-k)/1-2k} = \|u\|_{p+1}^{p+1} \|u\|_{p+1}^{2(1-k)/1-2k-(p+1)} \\ \leq c_5 H(0)^{2(1-k)/(1-2k)(p+1)-1} \|u\|_{p+1}^{p+1}. \end{aligned} \tag{98}$$

Then, (97) becomes the form

$$L(t)^\varpi \leq c_6 \left(H(t) + \|u_t\|^2 + \|u\|_{p+1}^{p+1} \right). \tag{99}$$

By combining (93) and (99), we conclude that

$$L'(t) \leq c_7 L(t)^\varpi, c_7 > 0, \varpi > 1. \tag{100}$$

Therefore, a simple integration over $(0, t)$, we have the desired result. Hence, we completed the proof. □

6. Conclusions

Time delays often appear in many various problems, such as, thermal, economic phenomena, biological, chemical, and physical. Recently, the partial differential equations with time delay have become an active area (see [2, 3] and references therein). In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there were no global existence, decay, and blow-up results for the higher-order Kirchhoff-type equation with delay term. Firstly, we have been obtained the global existence result. Later, we have been established the decay results by using Nakao's technique. Finally, we have proved the blow-up of solutions with negative initial energy for the problem (1) under the sufficient conditions in a bounded domain. In the next work, we will extend our current study to more general case of the problem (1).

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they do not have any conflicts of interest.

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Review Article

Fourth Toeplitz Determinants for Starlike Functions Defined by Using the Sine Function

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In this article, we aim to study the upper bounds of the fourth Toeplitz determinant $T_4(2)$ for the function class $\mathcal{S}_s^*$, which are connected with the sine function.

1. Introduction

Suppose that $\mathcal{A}$ represents the class of analytic functions f which in the open unit disk $\mathbb{D}=\{z:|z|<1\}$ of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots(z\in\mathbb{D}), \tag{1}$$

and suppose that $\mathcal{S}$ is the subclass of $\mathcal{A}$ consisting of univalent functions.

Let $\mathcal{P}$ denotes the class of analytic functions p normalized by

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots, \tag{2}$$

and meeting the condition $\Re(p(z))>0(z\in\mathbb{D})$. Let f and g be analytic functions in $\mathbb{D}$. Then, we say that the function g is subordinate to the function f, and we write

$$g(z)\prec f(z)(z\in\mathbb{D}), \tag{3}$$

if there exists a Schwarz function $\omega(z)$ with $\omega(0)=0$ and $|\omega(z)|<1$, such that (see [1])

$$g(z)=f(\omega(z))(z\in\mathbb{D}). \tag{4}$$

In 2018, Cho et al. [2] introduced the following function class S_s^*:

$$S_s^*:=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec(1+\sin z)(z\in\mathbb{D})\right\}, \tag{5}$$

which means that the quantity $zf'(z)/f(z)$ lies in an eight-shaped region in the right-half plane.

Thomas and Halim [3] defined the symmetric Toeplitz determinant $T_q(n)$ as follows:

$$T_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-1} \\ \vdots & & \vdots & \vdots \\ a_{n+q-1} & a_{n+q+2} & \cdots & a_n \end{vmatrix}(n\geq1,q\geq1). \tag{6}$$

As a special case, we have

$$T_4(2)=\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_2 & a_3 & a_4 \\ a_4 & a_3 & a_2 & a_3 \\ a_5 & a_4 & a_3 & a_2 \end{vmatrix}(n=2,q=4). \tag{7}$$

That is,

$$T_4(2) = \left(a_2^2 - a_3^2\right)^2 + 2\left(a_3^2 - a_2a_4\right)\left(a_2a_4 - a_3a_5\right) - \left(a_2a_3 - a_3a_4\right)^2 + \left(a_4^2 - a_3a_5\right)^2 - \left(a_3a_4 - a_2a_5\right)^2. \quad (8)$$

Many and many researchers have studied several Hankel and Toeplitz determinants for various classes of functions. For example, Janteng et al. [4, 5] investigated second Hankel determinant for a function with a positive real part derivative and starlike and convex functions, respectively; Bansal [6] and Lee et al. [7] discussed the second Hankel determinant for certain analytic functions; Bansal et al. [8], Zaprawa [9], Zhang et al. [10] and Babalola [11] derived third-order Hankel determinant for certain different univalent functions; Raza et al. [12] and Shi et al. [13, 14] studied upper bounds of the third Hankel determinant for some classes of analytic functions related to lemniscate of Bernoulli, cardioid domain and exponential function; Mahmood et al. [15] found third Hankel determinant for a subclass of q-starlike functions. Following the above work, Zhang et al. [16] recently considered fourth-order Hankel determinants of starlike functions related to the sine function. On the other hand, Thomas et al. [3] and Ali et al. [17] studied Toeplitz matrices whose elements are the coefficients of starlike, close-to-convex, and univalent functions. Besides, Tang et al. [18] studied third-order Hankel and Toeplitz determinant for a subclass of multivalent q-starlike functions of order α; Zhang et al. [19] considered third-order Hankel and Toeplitz determinants of starlike functions, which are defined by using the sine function; Ramachandran et al. [20] derived an estimation for the Hankel and Topelitz determinant with domains bounded by conical sections involving Ruscheweygh derivative; Srivastava et al. [21] found the Hankel determinant and the Toeplitz matrices for this newly-defined class of analytic q-starlike functions. Based on the work of Shi et al. [14], Zhang and Tang [16], Thomas and Halim [3], and Ali et al. [17], in the present paper, we aim to investigate the fourth-order Toeplitz determinant $T_4(2)$ for this function class $\mathcal{S}_s^*$ associated with sine function and obtain the upper bounds for the determinants $T_4(2)$.

2. Main Results

Due to prove our desired results, we require the following lemmas.

Lemma 1 (see [22]). *If $p(z) \in \mathcal{P}$, then exists some x, z with $|x| \le 1, |z| \le 1$, such that*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \quad (9)$$

$$4c_3 = c_1^3 + 2c_1x\left(4 - c_1^2\right) - \left(4 - c_1^2\right)c_1x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z. \quad (10)$$

Lemma 2 (see [23]). *Let $p(z) \in \mathcal{P}$, then*

$$\left|c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4\right| \le 2\,; \quad (11)$$

$$\left|c_1^5 + 3c_1c_2^2 + 3c_1^2c_3 - 4c_1^3c_2 - 2c_1c_4 - 2c_2c_3 + c_5\right| \le 2\,; \quad (12)$$

$$\left|c_1^6 + 6c_1^2c_2^2 + 4c_1^3c_3 + 2c_1c_5 + 2c_2c_4 + c_3^2 - c_2^3 - 5c_1^4c_2 - 3c_1^2c_4 - 6c_1c_2c_3 - c_6\right| \le 2\,; \quad (13)$$

$$|c_n| \le 2, n = 1, 2, \cdots. \quad (14)$$

Lemma 3 (see [24]). *Let $p(z) \in \mathcal{P}$, then, we have*

$$\left|c_2 - \frac{c_1^2}{2}\right| \le 2 - \frac{|c_1|^2}{2}\,; \quad (15)$$

$$|c_{n+k} - \mu c_n c_k| < 2, 0 \le \mu \le 1\,; \quad (16)$$

$$|c_{n+2k} - \mu c_n c_k^2| \le 2(1 + 2\mu). \quad (17)$$

The following are the main conclusions of this paper and related proof.

Theorem 1. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then*

$$|a_2| \le 1, |a_3| \le \frac{1}{2}, |a_4| \le 0.344, |a_5| \le \frac{3}{8}, |a_6| \le \frac{67}{120}, |a_7| \le \frac{5587}{10800}. \quad (18)$$

Proof. Because $f(z) \in \mathcal{S}_s^*$, by the definition of subordination, so there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$\frac{zf'(z)}{f(z)} = 1 + \sin(\omega(z)). \quad (19)$$

□

Now

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z + \sum_{n=2}^{\infty} na_nz^n}{z + \sum_{n=2}^{\infty} a_nz^n} = \left(1 + \sum_{n=2}^{\infty} na_nz^{n-1}\right) \\ &\cdot \left[1 - a_2z + \left(a_2^2 - a_3\right)z^2 - \left(a_2^3 - 2a_2a_3 + a_4\right)z^3 + \left(a_2^4 - 3a_2^2a_3 + 2a_2a_4 - a_5\right)z^4 + \cdots\right] \\ &= 1 + a_2z + \left(2a_3 - a_2^2\right)z^2 + \left(a_2^3 - 3a_2a_3 + 3a_4\right)z^3 \\ &+ \left(4a_5 - a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2\right)z^4 \\ &+ \left(5a_6 - 5a_2a_5 + a_2^5 - 5a_3a_4 - 5a_2^3a_3 + 5a_2^2a_4 + 5a_2a_3^2\right)z^5 \\ &+ \left(6a_7 - 6a_2a_6 + 6a_2^2a_5 - 6a_3a_5 + 12a_2a_3a_4 - a_2^6 - 6a_2^3a_4 - 3a_4^2 + 2a_3^3 - 9a_2^2a_3^2 + 6a_2^4a_3\right)z^6 + \cdots.\end{aligned} \quad (20)$$

Define a function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1z + c_2z^2 + \cdots. \quad (21)$$

Apparently so, $p(z) \in \mathscr{P}$ and

$$\omega(z) = \frac{p(z)-1}{1+p(z)} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}. \quad (22)$$

On the other hand,

$$\begin{aligned} 1 + \sin(\omega(z)) = {} & 1 + \frac{1}{2} c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{5c_1^3}{48} + \frac{c_3 - c_1 c_2}{2}\right) z^3 \\ & + \left(\frac{c_4 - c_1 c_3}{2} + \frac{5c_1^2 c_2}{16} - \frac{c_2^2}{4} - \frac{c_1^4}{32}\right) z^4 \\ & + \left(\frac{c_5 - c_1 c_4 - c_2 c_3}{2} + \frac{5c_1^2 c_3 + c_1 c_2^2}{16} - \frac{c_1^3 c_2}{8} + \frac{c_1^5}{3840}\right) z^5 \\ & + \left(\frac{c_6 - c_1 c_5 - c_2 c_4}{2} + \frac{5c_1 c_2 c_3}{8} + \frac{5c_2^3}{48} - \frac{c_3^2}{4} \right. \\ & \left. + \frac{5c_1^6}{512} + \frac{c_1^4 c_2}{768} - \frac{3c_1^2 c_2^2}{16} + \frac{5c_1^2 c_4}{16} - \frac{c_1^3 c_3}{8}\right) z^6 + \cdots. \end{aligned} \quad (23)$$

Comparing the coefficients of $z, z^2, z^3, z^4, z^5, z^6$ between the equations (20) and (23), we obtain

$$\begin{aligned} a_2 &= \frac{c_1}{2}, a_3 = \frac{c_2}{4}, a_4 = \frac{c_3}{6} - \frac{c_1 c_2}{24} - \frac{c_1^3}{144}, a_5 \\ &= \frac{c_4}{8} - \frac{c_1 c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2 c_2}{192} - \frac{c_2^2}{32}, \end{aligned} \quad (24)$$

$$a_6 = \frac{-3c_1 c_4}{80} - \frac{7c_2 c_3}{120} - \frac{11c_1^5}{4800} - \frac{43c_1 c_2^2}{960} + \frac{71c_1^3 c_2}{5760} + \frac{c_5}{10}, \quad (25)$$

$$\begin{aligned} a_7 = {} & \frac{c_1^2 c_4}{480} + \frac{c_1 c_2 c_3}{480} + \frac{833c_1^6}{691200} - \frac{41c_1^2 c_2^2}{3840} - \frac{109c_1^4 c_2}{11520} \\ & - \frac{c_1 c_5}{30} - \frac{5c_2 c_4}{96} + \frac{5c_2^3}{1152} + \frac{c_6}{12} + \frac{c_1^3 c_3}{144}. \end{aligned} \quad (26)$$

By virtue of Lemma 2, we can obtain

$$|a_2| \le 1, |a_3| \le \frac{1}{2}, \quad (27)$$

$$|a_4| = \left|\frac{c_3}{6} - \frac{c_1 c_2}{24} - \frac{c_1^3}{144}\right| = \left|\frac{1}{6}\left[c_3 - \frac{c_1 c_2}{3}\right] + \frac{c_1}{72}\left[c_2 - \frac{c_1^2}{2}\right]\right|. \quad (28)$$

Let $c_1 = c, c \in [0, 2]$ and using Lemma 3, we get

$$|a_4| = \left|\frac{1}{6}\left[c_3 - \frac{c_1 c_2}{3}\right] + \frac{c_1}{72}\left[c_2 - \frac{c_1^2}{2}\right]\right| \le \frac{1}{3} + \frac{c(2 - c^2/2)}{72}, \quad (29)$$

setting

$$F(c) = \frac{1}{3} + \frac{c(2 - c^2/2)}{72}, \quad (30)$$

It can be easily verified that $F(c)$ takes its maximum value at $c = 2\sqrt{3}/3$, that is

$$|a_4| \le F\left(\frac{2\sqrt{3}}{3}\right) = \frac{1}{3} + \frac{\sqrt{3}}{162} \approx 0.344, \quad (31)$$

$$\begin{aligned} |a_5| &= \left|\frac{c_4}{8} - \frac{c_1 c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2 c_2}{192} - \frac{c_2^2}{32}\right| \\ &= \left|\frac{1}{8}\left[c_4 - \frac{c_1 c_3}{3}\right] - \frac{c_1^2}{576}\left[c_2 - \frac{c_1^2}{2}\right] - \frac{c_2}{32}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{7c_1^2 c_2}{576}\right|. \end{aligned} \quad (32)$$

Let $c_1 = c, c \in [0, 2]$ from Lemma 3, we obtain

$$|a_5| \le \frac{1}{4} + \frac{5c^2(2 - c^2/2)}{576} + \frac{1}{16}\left(2 - \frac{c^2}{2}\right) + \frac{7c^2}{288}, \quad (33)$$

taking

$$F(c) = \frac{1}{4} + \frac{5c^2(2 - c^2/2)}{576} + \frac{1}{16}\left(2 - \frac{c^2}{2}\right) + \frac{7c^2}{288}, \quad (34)$$

It can be easily verified that maximum of $F(c)$ occurs at $c = 0$, that is,

$$|a_5| \le F(0) = \frac{3}{8}, \quad (35)$$

$$\begin{aligned} |a_6| &= \left|\frac{-3c_1 c_4}{80} - \frac{7c_2 c_3}{120} - \frac{11c_1^5}{4800} - \frac{43c_1 c_2^2}{960} + \frac{71c_1^3 c_2}{5760} + \frac{c_5}{10}\right| \\ &= \left|\frac{1}{24}\left[c_5 - \frac{9c_1 c_4}{10}\right] + \frac{7}{120}[c_5 - c_2 c_3] + \frac{11c_1^3}{2400}\left[c_2 - \frac{c_1^2}{2}\right] \right. \\ & \quad \left. - \frac{43c_1 c_2}{960}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{211c_1^3 c_2}{14400}\right|. \end{aligned} \quad (36)$$

Assume $c_1 = c, c \in [0, 2]$, by Lemma 3, we get

$$|a_6| \le \frac{7}{60} + \frac{1}{12} + \frac{11c^3(2 - c^2/2)}{2400} + \frac{43}{240}\left(2 - \frac{c^2}{2}\right) + \frac{211c^3}{7200}, \quad (37)$$

putting

$$F(c) = \frac{7}{60} + \frac{1}{12} + \frac{11c^3(2 - c^2/2)}{2400} + \frac{43}{240}\left(2 - \frac{c^2}{2}\right) + \frac{211c^3}{7200}, \quad (38)$$

it is demonstrable that maximum of $F(c)$ occurs at $c = 0$, that is,

$$|a_6| \le F(0) = \frac{67}{120}. \quad (39)$$

$$\begin{aligned}|a_7| &= \left|\frac{c_1^2c_4}{480} + \frac{c_1c_2c_3}{480} + \frac{833c_1^6}{691200} - \frac{41c_1^2c_2^2}{3840} - \frac{109c_1^4c_2}{11520}\right.\\ &\quad\left. - \frac{c_1c_5}{30} - \frac{5c_2c_4}{96} + \frac{5c_2^3}{1152} + \frac{c_6}{12} + \frac{c_1^3c_3}{144}\right|\\ &= \left|\frac{-37c_1^6}{691200} - \frac{25c_1^2c_2^2}{5760} - \frac{c_1c_5}{30} + \frac{c_1^2\left[c_4 - c_2^2\right]}{480} + \frac{c_1c_2[c_3 - c_1c_2]}{480}\right.\\ &\quad + \frac{c_1^3[c_3 - c_1c_2]}{144} - \frac{29c_1^4\left[c_2 - c_1^2/2\right]}{11520}\\ &\quad\left. + \frac{5c_2^2\left[c_2 - c_1^2/2\right]}{1152} + \frac{\left[c_6 - 5/8c_2c_4\right]}{12}\right|.\end{aligned} \tag{40}$$

Let $c_1 = c, c \in [0, 2]$ and applying Lemma 3, we get

$$\begin{aligned}|a_7| &\le \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{29c^4\left(2 - c^2/2\right)}{11520} + \frac{37c^6}{691200}\\ &\quad + \frac{c^3}{72} + \frac{25c^2}{1440} + \frac{5\left(2 - c^2/2\right)}{288},\end{aligned} \tag{41}$$

showing

$$\begin{aligned}F(c) &= \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{29c^4\left(2 - c^2/2\right)}{11520} + \frac{37c^6}{691200}\\ &\quad + \frac{c^3}{72} + \frac{25c^2}{1440} + \frac{5\left(2 - c^2/2\right)}{288},\end{aligned} \tag{42}$$

further, we get

$$F'(c) \ge 0. \tag{43}$$

So, the function $F(c)$ takes its maximum value at $c = 2$, that is,

$$|a_7| \le F(2) = \frac{5587}{10800}. \tag{44}$$

Theorem 2. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we get*

$$|a_3^2 - a_2^2| \le \frac{5}{4}. \tag{45}$$

Proof. According to equation (26), we have

$$|a_3^2 - a_2^2| = \left|\frac{c_2^2}{16} - \frac{c_1^2}{4}\right|. \tag{46}$$

By applying Lemma 1, we get

$$|a_3^2 - a_2^2| = \left|\frac{c_1^4}{64} + \frac{x^2\left(4 - c_1^2\right)^2}{64} + \frac{c_1^2x\left(4 - c_1^2\right)}{32} - \frac{c_1^2}{4}\right|. \tag{47}$$

Let $|x| = t, t \in [0, 1], c_1 = c, c \in [0, 2]$. Then, by the triangle inequality, we obtain

$$|a_3^2 - a_2^2| \le \frac{c^2t\left(4 - c^2\right)}{32} + \frac{t^2\left(4 - c^2\right)^2}{64} + \frac{c^4}{64} + \frac{c^2}{4}. \tag{48}$$

Suppose that

$$F(c, t) = \frac{c^2t\left(4 - c^2\right)}{32} + \frac{t^2\left(4 - c^2\right)^2}{64} + \frac{c^4}{64} + \frac{c^2}{4}, \tag{49}$$

then $\forall t \in [0, 1], \forall c \in [0, 2]$, the upper bound of $F(c, t)$ corresponds to $t = 1, c = 2$. Hence,

$$|a_3^2 - a_2^2| \le F(1, 2) = \frac{5}{4}. \tag{50}$$

□

Theorem 3. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we have*

$$|a_2a_3 - a_3a_4| \le \frac{25}{36}. \tag{51}$$

Proof. From (26), we have

$$\begin{aligned}|a_2a_3 - a_3a_4| &= \left|\frac{c_1c_2}{8} + \frac{c_1^3c_2}{576} - \frac{c_2c_3}{24} + \frac{c_1c_2^2}{96}\right|\\ &= \left|\frac{c_2}{4}\left[\frac{c_1}{2} - \frac{(c_3 - c_1c_2/4)}{6} + \frac{c_1^3}{144}\right]\right|.\end{aligned} \tag{52}$$

If we insert $c_1 = c, c \in [0, 2]$ and according to Lemma 3, we get

$$|a_2a_3 - a_3a_4| \le \frac{1}{2}\left[\frac{c}{2} + \frac{1}{3} + \frac{c^3}{144}\right]. \tag{53}$$

Assume that

$$F(c) = \frac{1}{2}\left[\frac{c}{2} + \frac{1}{3} + \frac{c^3}{144}\right]. \tag{54}$$

Therefore, we have $\forall c \in (0, 2)$

$$F'(c) = \frac{1}{4} + \frac{c^2}{96} > 0, \tag{55}$$

namely, the maximum value of $F(c)$ can be obtained at $c = 2$, that is,

$$|a_2a_3 - a_3a_4| \le F(2) = \frac{25}{36}. \tag{56}$$

□

Theorem 4. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we get*

$$|a_2a_4 - a_3^2| \leq \frac{1}{4}. \tag{57}$$

Proof. Suppose that $f(z) \in \mathcal{S}_s^*$, then, through equation (26), we get

$$|a_2a_4 - a_3^2| = \left|\frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{c_2^2}{16}\right|. \tag{58}$$

□

Now, according to Lemma 1, we obtain

$$\begin{aligned}|a_2a_4 - a_3^2| &= \left|\frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{c_2^2}{16}\right| \\ &= \left|-\frac{5c_1^4}{576} - \frac{x^2c_1^2(4-c_1^2)}{48} - \frac{x^2(4-c_1^2)^2}{64} + \frac{c_1(4-c_1^2)(1-|x|^2)z}{24}\right|.\end{aligned} \tag{59}$$

If we insert $c_1 = c, c \in [0, 2], |x| = t, t \in [0, 1]$. Then, by the triangle inequality, we get

$$|a_2a_4 - a_3^2| \leq \frac{t^2c^2(4-c^2)}{48} + \frac{(1-t^2)c(4-c^2)}{24} + \frac{t^2(4-c^2)^2}{64} + \frac{5c^4}{576}. \tag{60}$$

Putting

$$F(c,t) = \frac{t^2c^2(4-c^2)}{48} + \frac{(1-t^2)c(4-c^2)}{24} + \frac{t^2(4-c^2)^2}{64} + \frac{5c^4}{576}, \tag{61}$$

then, $\forall t \in [0, 1], \forall c \in [0, 2]$, the upper bound of $F(c, t)$ corresponds to $t = 1, c = 0$. Hence,

$$|a_2a_4 - a_3^2| \leq F(0,1) = \frac{1}{4}. \tag{62}$$

Theorem 5. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we get*

$$|a_2a_5 - a_3a_4| \leq \frac{11}{36}. \tag{63}$$

Proof. Assume that $f(z) \in \mathcal{S}_s^*$, then, on the basis of equation (26), we obtain

$$\begin{aligned}|a_2a_5 - a_3a_4| &= \left|\frac{5c_1^5}{2304} + \frac{c_1c_4}{16} - \frac{c_1c_2^2}{192} - \frac{c_1^2c_3}{48} - \frac{c_1^3c_2}{1152} - \frac{c_2c_3}{24}\right| \\ &= \left|-\frac{c_1^3[c_2 - c_1^2/2]}{1152} - \frac{c_3[c_2 - c_1^2/2]}{24} + \frac{c_1[c_4 - c_1c_3]}{24} + \frac{c_1^5}{576} + \frac{c_1[c_4 - 1/4c_2^2]}{48}\right|.\end{aligned} \tag{64}$$

□

If we insert $c_1 = c, c \in [0, 2]$, from Lemma 3, we obtain

$$|a_2a_5 - a_3a_4| \leq \frac{c^3[2 - c^2/2]}{1152} + \frac{[2 - c^2/2]}{12} + \frac{c}{8} + \frac{c^5}{576}. \tag{65}$$

Taking

$$F(c) = \frac{c^3[2 - c^2/2]}{1152} + \frac{[2 - c^2/2]}{12} + \frac{c}{8} + \frac{c^5}{576}. \tag{66}$$

Then, easy to show that maximum of $F(c)$ occurs at $c = 2, \forall c \in [0, 2]$, also which is

$$|a_2a_5 - a_3a_4| \leq F(2) = \frac{11}{36}. \tag{67}$$

Theorem 6. *Suppose that $f(z) \in \mathcal{S}_s^*$ and in the form (1), then, we get*

$$|a_3a_5 - a_2a_4| \leq \frac{9}{16}. \tag{68}$$

Proof. Assume that $f(z) \in \mathcal{S}_s^*$, then, according to equation (26), we get

$$\begin{aligned}|a_3a_5 - a_2a_4| &= \left|\frac{c_2^3}{128} + \frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{5c_1^4c_2}{4608} - \frac{c_2c_4}{32} + \frac{c_1c_2c_3}{96} + \frac{c_1^2c_2^2}{768}\right| \\ &= \left|\frac{[c_1[c_3 - c_1c_2/4]}{12} + \frac{5c_1^2c_2[c_2 - c_1^2/2]}{2304} - \frac{c_2[c_4 - 1/3c_1c_3]}{32} + \frac{c_2^2[c_2 - c_1^2/2]}{128} + \frac{7c_1^2c_2^2}{2304} - \frac{c_1^4}{288}\right|.\end{aligned} \tag{69}$$

□

If we insert $c_1 = c, c \in [0, 2]$ and in view of Lemma 3, we have

$$|a_3a_5 - a_2a_4| \leq \frac{c}{6} + \frac{1}{8} + \frac{5c^2[2 - c^2/2]}{1152} + \frac{[2 - c^2/2]}{32} + \frac{7c^2}{576} + \frac{c^4}{288}. \tag{70}$$

Taking

$$F(c) = \frac{c}{6} + \frac{1}{8} + \frac{5c^2\left[2 - c^2/2\right]}{1152} + \frac{\left[2 - c^2/2\right]}{32} + \frac{7c^2}{576} + \frac{c^4}{288}. \tag{71}$$

Then, $\forall c \in [0, 2]$, the demonstrable function $F(c)$ obtains the maximum value at $c = 2$, that is,

$$|a_3 a_5 - a_2 a_4| \le F(2) = \frac{9}{16}. \tag{72}$$

Theorem 7. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we get*

$$|a_5 a_3 - a_4^2| \le \frac{97}{324}. \tag{73}$$

Proof. Suppose that $f(z) \in \mathcal{S}_s^*$, then, by the equation (26), we obtain

$$\begin{aligned} |a_5 a_3 - a_4^2| &= \left|\frac{7c_1^4 c_2}{13824} + \frac{c_2 c_4}{32} + \frac{c_1 c_2 c_3}{288} - \frac{c_2^3}{128} + \frac{c_1^3 c_3}{432} - \frac{7c_1^2 c_2^2}{2304} - \frac{c_3^2}{36} - \frac{c_1^6}{20736}\right| \\ &= \left|\frac{c_2[c_4 - c_1 c_3/9]}{32} - \frac{c_3[c_3 - c_1 c_2/4]}{36} - \frac{c_2^2\left[c_2 - c_1^2/2\right]}{128}\right. \\ &\quad \left. - \frac{c_1^2 c_2\left[c_2 - c_1^2/2\right]}{144} + \frac{c_1^3[c_3 - 31/32 c_1 c_2]}{432} - \frac{5c_1^4 c_2}{6912} - \frac{c_1^6}{20736}\right|. \end{aligned} \tag{74}$$

□

If we insert $c_1 = c, c \in [0, 2]$ and by Lemma 3, we obtain

$$|a_5 a_3 - a_4^2| \le \frac{1}{8} + \frac{1}{9} + \frac{\left[2 - c^2/2\right]}{32} + \frac{c^2\left[2 - c^2/2\right]}{72} + \frac{c^3}{216} + \frac{5c^4}{3456} + \frac{c^6}{20736}. \tag{75}$$

Putting

$$F(c) = \frac{1}{8} + \frac{1}{9} + \frac{\left[2 - c^2/2\right]}{32} + \frac{c^2\left[2 - c^2/2\right]}{72} + \frac{c^3}{216} + \frac{5c^4}{3456} + \frac{c^6}{20736}. \tag{76}$$

$\forall c \in (0, 2), F'(c) > 0$, Then, maximum of $F(c)$ occurs at $c = 2$, that is

$$|a_5 a_3 - a_4^2| \le F(2) = \frac{97}{324}. \tag{77}$$

Theorem 8. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we get*

$$|T_4(2)| \le \frac{263384.5}{104976} \approx 2.51. \tag{78}$$

Proof. Since

$$\begin{aligned} T_4(2) &= \left(a_2^2 - a_3^2\right)^2 + 2\left(a_3^2 - a_2 a_4\right)(a_2 a_4 - a_3 a_5) \\ &\quad - (a_2 a_3 - a_3 a_4)^2 + \left(a_4^2 - a_3 a_5\right)^2 - (a_3 a_4 - a_2 a_5)^2, \end{aligned} \tag{79}$$

then, by applying the triangle inequality, we get

$$\begin{aligned} |T_4(2)| &\le \left|a_2^2 - a_3^2\right|^2 + 2\left|a_3^2 - a_2 a_4\right||a_2 a_4 - a_3 a_5| \\ &\quad + |a_2 a_3 - a_3 a_4|^2 + \left|a_4^2 - a_3 a_5\right|^2 + |a_3 a_4 - a_2 a_5|^2. \end{aligned} \tag{80}$$

Now, substituting (18), (45)–(73) into (80), we easily obtain the desired assertion (78). □

3. Conclusion

In this paper, based on the paper [15], we continuously discuss the problem of the fourth-order Toeplitz determinant of starlike functions, which are connected with the sine function and get the upper bounds of the determinant. In the next step, we can consider the fourth-order Toeplitz determinant of other function classes defined by various linear or nonlinear operators and also make the related discussion on the fifth-order Toeplitz determinant for certain function classes.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Linearly Independent Solutions and Integral Representations for Certain Quadruple Hypergeometric Function

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Recently, hypergeometric functions of four variables are investigated by Bin-Saad and Younis. In this manuscript, our goal is to initiate a new quadruple hypergeometric function denoted by $X_{84}^{(4)}$, and then, we ensure the existence of solutions of systems of partial differential equations for this function. We also establish some integral representations involving the quadruple hypergeometric function $X_{84}^{(4)}$.

1. Introduction

Special functions, in recent years, are a piece of research that turned out to be very attractive to many scholars, hunting generalisations which are almost always evoked by applications. Hypergeometric functions of several variables have many applications in mathematical physics, statistical sciences, physics, dynamics, quantum mechanics, chemistry, and engineering (see, e.g., [1–7]). Multivariable hypergeometric functions occur in diverse areas of mathematics such as approximation theory, partition theory, representation theory, group theory, mirror symmetry, and algebraic geometry. They possess important properties such as recurrence and explicit relations, summation formulas, symmetric and convolution identities, and algebraic properties. Furthermore, multidimensional hypergeometric functions are used to solve boundary value problems (Dirichlet problem, Neumann problem, and Holmgren problem) for multidimensional degenerate differential equations (see [8–12]).

In [13], Bin-Saad and Younis introduced several integral representations of Euler type and Laplace type for new hypergeometric functions in four variables. The authors, in [14], defined four new quadruple hypergeometric functions, namely, $X_{80}^{(4)}, X_{81}^{(4)}, X_{82}^{(4)}$, and $X_{83}^{(4)}$, and they obtained fractional derivative formulas, integral representations, and operational formulas for these quadruple hypergeometric functions. More recently, Younis et al. [15] introduced and studied further quadruple hypergeometric functions denoted by $X_{85}^{(4)}, X_{86}^{(4)}, \cdots, X_{90}^{(4)}$. Each quadruple function in [13–15] can be expressed as

$$X^{(4)}(\cdot) = \sum_{m,n,p,q=0}^{\infty} \Theta(m,n,p,q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1}$$

where $\Theta(m,n,p,q)$ is a sequence of complex parameters and there are twelve parameters in every series $X^{(4)}(\cdot)$

(eight a's and four c's). The 1st, 2nd, 3rd, and 4th parameters in $X^{(4)}(\cdot)$ are connected with the integers m, n, p, and q, respectively. Every repeated parameter in the series $X^{(4)}(\cdot)$ points out a term with double parameters in $\Theta(m, n, p, q)$. Hence, it is possible to form various combinations of indices. It seems that there is no way to establish independently the number of distinct Gaussian hypergeometric series for each arbitrary integer $n \geq 2$ without giving explicitly all such series. Hence, in each situation with $n = 4$, one ought to start with actually building the set like the case $n = 3$ (refer to [16]).

Motivated by the works [13–15], we define here the following quadruple hypergeometric function:

$$\begin{aligned} &X_{84}^{(4)}(\ell_1, \ell_2, \ell_3, \ell_4; j_1, j_2, j_3; x, y, z, t) \\ &\quad = \sum_{m,n,p,q=0}^{\infty} \frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_{p+q}(\ell_4)_q}{(j_1)_{m+p}(j_2)_n(j_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!} \\ &\qquad \cdot \left(|x| < \frac{1}{4}, |y| < 1 < |z| < 1 < |t| < 1\right), \end{aligned} \tag{2}$$

where $(\ell)_n$ is the well-known Pochhammer symbol given as

$$(\ell)_n := \begin{cases} 1, & n = 0, \\ \ell(\ell+1)\cdots(\ell+n-1), & n \in \mathbb{N} := \{1, 2, \cdots\}. \end{cases} \tag{3}$$

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$, and $\mathbb{C}$ denote the sets of positive integers, negative integers, and complex numbers, respectively. Also,

$$\begin{aligned} \mathbb{N}_0 &:= \mathbb{N} \cup \{0\}, \\ \mathbb{Z}_0^- &:= \mathbb{Z}^- \cup \{0\}. \end{aligned} \tag{4}$$

Recently, various interesting hypergeometric functions in several variables have been investigated by many authors (see, e.g., [17–24]). In Section 2, we show how to find the linearly independent solutions of partial differential equations satisfied by the function $X_{84}^{(4)}$. Section 3 is aimed at presenting some integral representations of Euler type for our quadruple function.

2. Solving Systems of Partial Differential Equations

Following the theory of multiple hypergeometric functions [25], the system of partial differential equations for the quadruple hypergeometric function $X_{84}^{(4)}$ is given as follows:

$$\begin{cases} \left(j_1 + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial x} + 1\right)x^{-1}u - \left(\ell_1 + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 1\right)\left(\ell_1 + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}\right)u = 0, \\ \left(j_2 + y\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial y} + 1\right)y^{-1}u - \left(\ell_1 + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}\right)\left(\ell_2 + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u = 0, \\ \left(j_1 + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial z} + 1\right)z^{-1}u - \left(\ell_2 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(\ell_3 + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right)u = 0, \\ \left(j_3 + t\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t} + 1\right)t^{-1}u - \left(\ell_3 + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right)\left(\ell_4 + t\frac{\partial}{\partial t}\right)u = 0, \end{cases} \tag{5}$$

where $u = X_{84}^{(4)}(\ell_1, \ell_2, \ell_3, \ell_4; j_1, j_2, j_3; x, y, z, t)$.

Starting from (5) and by making use of some elementary calculations, we define the system of second-order partial differential equations:

$$\begin{cases} x(1-4x)u_{xx} - 4xyu_{xy} + zu_{xz} - y^2u_{yy} + [j_1 - 2(2\ell_1 + 3)x]u_x - 2(\ell_1 + 1)yu_y - \ell_1(\ell_1 + 1)u = 0, \\ y(1-y)u_{yy} - 2xyu_{xy} - 2xzu_{xz} + yzu_{xz} - 2\ell_2xu_x + [j_2 - (\ell_1 + \ell_2 + 1)y]u_y - \ell_1zu_z - \ell_1\ell_2u = 0, \\ z(1-z)u_{zz} - xu_{xz} - yzu_{yz} + ytu_{yt} - ztu_{zt} - \ell_3yu_y + [j_1 - (\ell_3 + \ell_2 + 1)z]u_z - \ell_2tu_t - \ell_2\ell_3u = 0, \\ t(1-t)u_{tt} - ztu_{zt} - \ell_4zu_z + [j_3 - (\ell_3 + \ell_4 + 1)t]u_t - \ell_3\ell_4u = 0. \end{cases} \tag{6}$$

It is noted that four equations of the system (6) are simultaneous. In fact, the hypergeometric function $X_{84}^{(4)}$ verifies the system. To find the linearly independent solutions of system (6), we will search the solutions in the form

$$u = x^{\alpha} y^{\beta} z^{\gamma} t^{\delta} w, \tag{7}$$

where w is an unknown function and α, β, γ, and δ are constants, which are to be determined. So, substituting $u = x^{\alpha} y^{\beta} z^{\gamma} t^{\delta} w$ into the system (6), we get

$$\begin{cases} x(1-4x)w_{xx} - 4xyw_{xy} + zw_{xz} - y^2 w_{yy} + \{j_1 + \gamma + 2\alpha - 2[2(l_1 + 2\alpha + \beta) + 3]x\}w_x - 2(\ell_1 + 2\alpha + \beta + 1)yw_y + \alpha x^{-1} z w_z - \left[-\alpha(j_1 + \alpha + \gamma - 1)x^{-1} + (\ell_1 + 2\alpha + \beta)(\ell_1 + 2\alpha + \beta + 1)\right] w = 0, \\ y(1-y)w_{yy} - 2xyw_{xy} - 2xzw_{xz} - yzw_{yz} - 2(\ell_2 + \alpha + \beta)xw_x + \{j_2 + 2\beta - [(\ell_1 + 2\alpha + \beta) + (\ell_2 + \beta + \gamma) + 1]y\}w_y - (\ell_1 + 2\alpha + \beta)zw_z - \left[-\beta(j_2 + \beta - 1)y^{-1} + (\ell_1 + 2\alpha + \beta)(\ell_2 + \beta + \gamma)\right] w = 0, \\ z(1-z)w_{zz} + xw_{xz} - yzw_{yz} - ytw_{yt} - ztw_{zt} + \gamma x z^{-1} w_x - (\ell_3 + \gamma + \delta)yw_y + \{j_1 + \alpha + 2\gamma - [(\ell_2 + \beta + \gamma) + (\ell_3 + \gamma + \delta) + 1]z\}w_z - (\ell_2 + \beta + \gamma)tw_t - \left\{-\gamma(j_1 + \alpha + \gamma - 1)z^{-1} + (\ell_2 + \beta + \gamma)(\ell_3 + \gamma + \delta)\right\} w = 0, \\ t(1-t)w_{tt} - ztw_{zt} - (\ell_4 + \delta)zw_z + \{j_3 + 2\delta - [(\ell_3 + \gamma + \delta) + (\ell_4 + \delta) + 1]t\}w_t - \left[-\delta(j_3 + \delta - 1)t^1 + (\ell_3 + \gamma + \delta)(\ell_4 + \delta)\right] w = 0. \end{cases} \tag{8}$$

Systems (8) and (6) have the same structure and can therefore be approached with similar techniques. System (8) implies

$$\begin{cases} \alpha = 0, \\ \beta(\beta + j_2 - 1) = 0, \\ \gamma = 0, \\ \delta(\delta + j_3 - 1) = 0. \end{cases} \tag{9}$$

Therefore, system (9) has the following solutions:

$$\begin{array}{lcccc} & 1 & 2 & 3 & 4 \\ \alpha := & 0 & 0 & 0 & 0 \\ \beta := & 0 & 1 - j^2 & 0 & 1 - j^2 \\ \gamma := & 0 & 0 & 0 & 0 \\ \delta := & 0 & 0 & 1 - j^3 & 1 - j^3 \end{array}. \tag{10}$$

Finally, substituting two solutions of the system (10) into (8), we find the following linearly independent solutions of the system (6) at the origin:

$$\begin{aligned} u_1(x,y,z,t) &= X_{84}^{(4)}\left(\ell_1, \ell_2, \ell_3, \ell_4 ; j_1, j_2, j_3 ; x, y, z, t\right), \\ u_2(x,y,z,t) &= y^{1-j_2} X_{84}^{(4)}\left(\ell_1 + 1 - j_2, \ell_2 + 1 - j_2, \ell_3, \ell_4 ; j_1, 2 - j_2, j_3 ; x, y, z, t\right), \\ u_3(x,y,z,t) &= t^{1-j_3} X_{84}^{(4)}\left(\ell_1, \ell_2, \ell_3 + 1 - j_3, \ell_4 + 1 - j_3 ; j_1, j_2, 2 - j_3 ; x, y, z, t\right), \\ u_4(x,y,z,t) &= y^{1-j_2} t^{1-j_3} X_{84}^{(4)}\left(\ell_1 + 1 - j_2, \ell_2 + 1 - j_2, \ell_3 + 1 - j_3, \ell_4 + 1 - j_3 ; j_1, 2 - j_2, 2 - j_3 ; x, y, z, t\right). \end{aligned} \tag{11}$$

3. Integral Representations of Euler Type

Here, we give eight integral representations of Euler type for $X_{84}^{(4)}$ whose kernel contains the Gaussian hypergeometric function ${}_2F_1$ (see [16]), Appell function F_3 (see for details [16, 25]), the Exton triple functions X_{16}, X_{17}, X_{19} [26], Lauricella's function of three variables F_N [16], and the quadruple functions $X_4^{(4)}, X_{24}^{(4)}$ (see [20, 21]):

$$\begin{aligned} & X_{84}^{(4)}\left(\ell_1, \ell_2, \ell_3, \ell_4 ; j_1, j_2, j_3 ; x, y, z, t\right) \\ &= \frac{\Gamma(j_2)\Gamma(j_3)}{\Gamma(\ell_2)\Gamma(\ell_4)\Gamma(j_2 - \ell_2)\Gamma(j_3 - \ell_4)} \\ &\quad \cdot \int_0^{\infty}\int_0^{\infty} \left(e^{-\alpha}\right)^{\ell_2}\left(1 - e^{-\alpha}\right)^{j_2 - \ell_2 - 1}\left(e^{-\beta}\right)^{\ell_4} \\ &\quad \times \left(1 - e^{-\beta}\right)^{j_3 - \ell_4 - 1}\left(1 - ye^{-\alpha}\right)^{-\ell_1}\left(1 - te^{-\beta}\right)^{-\ell_3} \\ &\quad \times F_3\left(\frac{\ell_1}{2}, 1 + \ell_2 - j_2, \frac{\ell_1 + 1}{2}, \ell_3 ; j_1 ; \right. \\ &\quad \left. \frac{4x}{\left(1 - ye^{-\alpha}\right)^2}, \frac{-ze^{-\alpha}}{\left(1 - e^{-\alpha}\right)\left(1 - te^{-\beta}\right)}\right) d\alpha d\beta \left(\Re(\ell_2) \right. \\ &\quad \left. > 0, \Re(\ell_4) > 0, \Re(j_2 - \ell_2) > 0, \Re(j_3 - \ell_4) > 0\right), \end{aligned} \tag{12}$$

$$\begin{aligned} & X_{84}^{(4)}\left(\ell_1, \ell_2, \ell_3, \ell_4 ; j_1, j_2, j_3 ; x, y, z, t\right) \\ &= \frac{2M^{\ell_2}\Gamma(j_2)}{\Gamma(\ell_2)\Gamma(j_2 - \ell_2)} \int_0^{\infty} \frac{\cosh\alpha\left(\sinh^2\alpha\right)^{\ell_2 - 1/2}}{\left(1 + M\sinh^2\alpha\right)^{j_2 - \ell_1}} \\ &\quad \times \left[\left(1 + M\sinh^2\alpha\right) - My\sinh^2\alpha\right]^{-\ell_1} F_N \\ &\quad \cdot \left(\ell_4, \frac{\ell_1}{2}, 1 + \ell_2 - j_2, \ell_3, \frac{\ell_1 + 1}{2}, \ell_3 ; j_3, j_1, j_1 ; \right. \\ &\quad \left. \cdot\, t, \frac{4x\left(1 + M\sinh^2\alpha\right)^2}{\left[\left(1 + M\sinh^2\alpha\right) - My\sinh^2\alpha\right]^2}, -zM\sinh^2\alpha\right) \\ &\quad \cdot\, d\alpha\left(\Re(\ell_2) > 0, \Re(j_2 - \ell_2) > 0, M > 0\right), \end{aligned} \tag{13}$$

$$X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t) = \frac{2(1+M)^{\ell_1}\Gamma(j_2)}{\Gamma(\ell_1)\Gamma(j_2-\ell_1)}\int_0^{\pi/2}\frac{(\sin^2\alpha)^{\ell_1-1/2}(\cos^2\alpha)^{j_2-\ell_1-1/2}}{(1+M\sin^2\alpha)^{j_2-\ell_2}} \times\left[(1+M\sin^2\alpha)-(1+M)y\sin^2\alpha\right]^{-\ell_2}F_N \cdot\left(\ell_4,\frac{1+\ell_1-j_2}{2},\ell_2,\ell_3,\frac{\ell_1-j_2}{2}+1,\ell_3;j_3,j_1,j_1;t,4x(1+M)^2\tan^4\alpha, \cdot\frac{z(1+M\sin^2\alpha)}{\left[(1+M\sin^2\alpha)-(1+M)y\sin^2\alpha\right]}\right) \cdot d\alpha\,(\Re(\ell_1)>0,\Re(j_2-\ell_1)>0,M>-1), \tag{14}$$

$$X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t) = \frac{2M^{j_3-\ell_3}\Gamma(j_3)}{\Gamma(\ell_3)\Gamma(j_3-\ell_3)}\int_0^{\pi/2}\frac{(\sin^2\alpha)^{j_3-\ell_3-1/2}(\cos^2\alpha)^{\ell_3-1/2}}{(\cos^2\alpha+M\sin^2\alpha)^{j_3-\ell_4}} \times\left[(\cos^2\alpha+M\sin^2\alpha)-t\cos^2\alpha\right]^{-\ell_4}X_{16} \cdot\left(\ell_1,\ell_2,1+\ell_3-j_3;j_1,j_2;x,y,-\frac{z\cot^2\alpha}{M}\right)d\alpha \cdot(\Re(\ell_3)>0,\Re(j_3-\ell_3)>0,M>-1), \tag{15}$$

$$X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t) = \frac{\Gamma(\ell_3+\ell_4)\Gamma(j_2)}{\Gamma(\ell_1)\Gamma(\ell_3)\Gamma(\ell_4)\Gamma(j_2-\ell_1)(W_1-V_1)^{j_2-\ell_2-1}(W_2-V_2)^{\ell_3+\ell_4-1}} \cdot\int_{V_1}^{W_1}\int_{V_2}^{W_2}\times(\alpha-V_1)^{\ell_1-1}(W_1-\alpha)^{j_2-\ell_1-1}(\beta-V_2)^{\ell_3-1} \cdot(W_2-\beta)^{\ell_4-1}\times[(W_1-V_1)-(\alpha-V_1)y]^{-\ell_2}X_{19} \cdot\left(\ell_3+\ell_4,\ell_2,\frac{1+\ell_1-j_2}{2},\frac{\ell_1-j_2}{2}+1;j_3,j_1; \cdot\frac{(\beta-V_2)(W_2-\beta)t}{(W_2-V_2)^2},\frac{(\beta-V_2)z}{(W_2-V_2)},4\left(\frac{\alpha-V_1}{W_1-\alpha}\right)^2x\right) \cdot d\alpha d\beta\,(\Re(\ell_1)>0,\Re(\ell_3)>0,\Re(\ell_4)>0,\Re(j_2-\ell_1)>0,V_1<W_1,V_2<W_2), \tag{16}$$

$$X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t) = \frac{\Gamma(\ell_3+b)\Gamma(j_1)}{2^{\ell_3+b+j_1-2}\Gamma(\ell_3)\Gamma(a)\Gamma(b)\Gamma(j_1-a)}\int_{-1}^{1}\int_{-1}^{1}(1+\alpha)^{\ell_3-1} \cdot(1-\alpha)^{b-1}\times(1+\beta)^{a-1}(1-\beta)^{j_1-a-1}X_{17}\left(\ell_1,\ell_2,\ell_3+b;a,j_2,j_1-a;\frac{(1+\beta)x}{2},y,\frac{(1+\alpha)(1-\beta)z}{4}\right)_2 \cdot F_1\left(\ell_4,1-b;j_3,\left(\frac{1+\alpha}{\alpha-1}\right)t\right)d\alpha d\beta\,(\Re(\ell_3)>0,\Re(a)>0,\Re(b)>0,\Re(j_1-a)>0), \tag{17}$$

$$X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t) = \frac{4\Gamma(\ell_1+\ell_4)\Gamma(\ell_2+\ell_3)}{\Gamma(\ell_1)\Gamma(\ell_2)\Gamma(\ell_3)\Gamma(\ell_4)}\int_0^{\pi/2}\int_0^{\pi/2}(\sin^2\alpha)^{\ell_1-1/2} \cdot(\cos^2\alpha)^{\ell_4-1/2}(\sin^2\beta)^{\ell_2-1/2}\times(\cos^2\beta)^{\ell_3-1/2}X_4^{(4)} \cdot\left(\ell_1,\ell_1,\ell_2,\ell_1,\ell_1,\ell_2,\ell_2,\ell_2;j_1,j_2,j_1,j_3; \cdot x\sin^4\alpha,y\sin^2\alpha\sin^2\beta,\frac{z\sin^2 2\beta}{4},t\cos^2\alpha\cos^2\beta\right) \cdot d\alpha d\beta\,(\Re(\ell_i)>0\,(i=1,2,3,4)), \tag{18}$$

$$X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t) = \frac{(1+M)^{\ell_1}\Gamma(\ell_1+\ell_4)}{\Gamma(\ell_1)\Gamma(\ell_4)}\int_0^1\frac{\alpha^{\ell_1-1}(1-\alpha)^{\ell_4-1}}{(1+M\alpha)^{\ell_1+\ell_4}}\times X_{24}^{(4)} \cdot\left(\ell_1,\ell_1,\ell_2,\ell_1,\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_1,j_3; \cdot\frac{(1+M)^2\alpha^2x}{(1+M\alpha)^2},\frac{(1+M)\alpha y}{(1+M\alpha)},z,\frac{(1-\alpha)t}{(1+M\alpha)}\right) \cdot d\alpha\,(\Re(\ell_1)>0,\Re(\ell_4)>0,M>-1), \tag{19}$$

where the Gaussian hypergeometric function ${}_2F_1$, Appell function F_3, Lauricella triple hypergeometric function F_N, Exton hypergeometric functions X_{16}, X_{17}, X_{19}, and the quadruple functions $X_4^{(4)}, X_{24}^{(4)}$ are defined, respectively, by

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\;(|x|<1), \tag{20}$$

$$F_3(a,b,c,d;e;x,y) = \sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\;(\max\{|x|,|y|\}<1), \tag{21}$$

$$F_N(\ell_1,\ell_2,\ell_3,b_1,b_2,b_1;j_1,j_2,j_2;x,y,z) = \sum_{m,n,p=0}^{\infty}\frac{(\ell_1)_m(\ell_2)_n(\ell_3)_p(b_1)_{m+p}(b_2)_n}{(j_1)_m(j_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p} \cdot(r+s<1\wedge v<1,|x|\le r,|y|\le s,|z|\le v), \tag{22}$$

$$X_{16}(\ell_1,\ell_2,\ell_3;j_1,j_2;x,y,z) = \sum_{m,n,p=0}^{\infty}\frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_p}{(j_1)_{m+p}(j_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \cdot\left(s<1\wedge v\le T^v(s)\wedge r<\frac{1}{4}(1-s)^2\right) \cup\left(s<1\wedge T^v(s)<v<1-s\wedge r<\frac{v}{1-v}\ell_2\left(\frac{s}{1-v}\right)\right), \tag{23}$$

$$T^{v}(s)=\begin{cases}\left(\dfrac{1-s}{1-1/2s}\right)^{2}, & 0<s<\dfrac{2}{3},\\ \dfrac{1-s}{2s}, & \dfrac{2}{3}\leq s<1\end{cases}\quad (|x|\leq r,|y|\leq s,|z|\leq v), \tag{24}$$

$$\begin{aligned}&X_{17}(\ell_1,\ell_2,\ell_3;j_1,j_2,j_3;x,y,z)\\&\quad=\sum_{m,n,p=0}^{\infty}\frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_p}{(j_1)_m(j_2)_n(j_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\\&\qquad\cdot\left(r<\frac{1}{4}\wedge v<1\wedge s<\left(1-2\sqrt{r}\right)(1-v),|x|\leq r,|y|\leq s,|z|\leq v\right),\end{aligned} \tag{25}$$

$$\begin{aligned}&X_{19}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2;x,y,z)\\&\quad=\sum_{m,n,p=0}^{\infty}\frac{(\ell_1)_{2m+n}(\ell_2)_n(\ell_3)_p(\ell_4)_p}{(j_1)_m(j_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\\&\qquad\cdot\left(s+2\sqrt{r}<1\wedge v<1,|x|\leq r,|y|\leq s,|z|\leq v\right),\end{aligned} \tag{26}$$

$$\begin{aligned}&X_4^{(4)}(\ell_1,\ell_1,\ell_2,\ell_1,\ell_1,\ell_2,\ell_2,\ell_2;j_1,j_2,j_1,j_3;x,y,z,t)\\&\quad=\sum_{m,n,p,q=0}^{\infty}\frac{(\ell_1)_{2m+n+q}(\ell_2)_{q+n+2p}}{(j_1)_{m+p}(j_2)_n(j_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}\\&\qquad\cdot\left(|x|<\frac{1}{4},|y|<1,|z|<\frac{1}{4},|t|<1\right),\end{aligned} \tag{27}$$

$$\begin{aligned}&X_{24}^{(4)}(\ell_1,\ell_1,\ell_2,\ell_1,\ell_1,\ell_2,\ell_3,\ell_3;j_1,j_2,j_1,j_3;x,y,z,t)\\&\quad=\sum_{m,n,p,q=0}^{\infty}\frac{(\ell_1)_{2m+n+q}(\ell_2)_{n+p}(\ell_3)_{p+q}}{(j_1)_{m+p}(j_2)_n(j_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}\\&\qquad\cdot\left(|x|<\frac{1}{4},|y|<1,|z|<1,|t|<1\right).\end{aligned} \tag{28}$$

Proof. We begin by recalling the following integral representations of the beta function (see, for example, [27, 28]):

$$B(a,b)=\begin{cases}\displaystyle\int_0^1\alpha^{\alpha-1}(1-\alpha)^{b-1}d\alpha & (\Re(a)>0,\Re(b)>0),\\ \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} & (a,b\mathbb{C}/Z_0^-),\end{cases} \tag{29}$$

$$\begin{aligned}B(a,b)&=\int_0^1\alpha^{a-1}(1-\alpha)^{b-1}d\alpha\\&=(W-V)^{1-a-b}\int_V^W(\alpha-V)^{a-1}(W-\alpha)^{b-1}d\alpha\\&\quad\cdot(\Re(a)>0,\Re(b)>0,V<W),\end{aligned} \tag{30}$$

$$\begin{aligned}B(a,b)&=2\int_0^{\pi/2}(\sin\alpha)^{2a-1}(\cos\alpha)^{2b-1}d\alpha\\&=\int_0^{\infty}(e^{-\alpha})^a(1-e^{-\alpha})^{b-1}d\alpha\,(\Re(a)>0,\Re(b)>0),\end{aligned} \tag{31}$$

$$\begin{aligned}B(a,b)&=2^{1-a-b}\int_{-1}^1(1+\alpha)^{a-1}(1-\alpha)^{b-1}d\alpha\\&=2M^a\int_0^{\infty}\frac{\cosh\alpha(\sinh\alpha)^{2a-1}}{\left(1+M\sinh^2\alpha\right)^{a+b}}d\alpha\,(\Re(a)\\&\quad>0,\Re(b)>0,M>0).\end{aligned} \tag{32}$$

For convenience, let $\mho$ denote the right-hand side of relation (12). Then, by substituting the expression of F_3 from definition (21) into the right-hand side of (12) and using (31), we have

$$\begin{aligned}\mho&=\sum_{m,n,p,q=0}^{\infty}\frac{(\ell_1)_{2m}(\ell_1+2m)_n(1+\ell_2-j_2)_p(\ell_3)_p(\ell_3+p)_q(-1)^p}{(j_1)_{m+p}}\\&\quad\times\frac{\Gamma(j_2)}{\Gamma(\ell_2)\Gamma(j_2-\ell_2)}\int_0^{\infty}(e^{-\alpha})^{\ell_2+n+p}(1-e^{-\alpha})^{j_2-\ell_2-p-1}d\alpha\\&\quad\times\frac{\Gamma(j_3)}{\Gamma(\ell_4)\Gamma(j_3-\ell_4)}\int_0^{\infty}\left(e^{-\beta}\right)^{\ell_4+q}\left(1-e^{-\beta}\right)^{j_3-\ell_4-1}d\beta\\&\quad\times\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}=\sum_{m,n,p,q=0}^{\infty}\frac{(1+\ell_2-j_2)_p\Gamma(j_2-\ell_2-p)(-1)^p}{\Gamma(j_2-\ell_2)}\\&\quad\times\frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_{p+q}(\ell_4)_q}{(j_1)_{m+p}(j_2)_n(j_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}\\&=\sum_{m,n,p,q=0}^{\infty}\frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_{p+q}(\ell_4)_q}{(j_1)_{m+p}(j_2)_n(j_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}\\&=X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t);\end{aligned} \tag{33}$$

we are led to the desired result. A similar argument in the proof of relation (12) will be able to establish the results (13)–(19). So, details of the proof are omitted. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Applications of Mittag-Leffer Type Poisson Distribution to a Subclass of Analytic Functions Involving Conic-Type Regions

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In this article, we introduce a new subclass of analytic functions utilizing the idea of Mittag-Leffler type Poisson distribution associated with the Janowski functions. Further, we discuss some important geometric properties like necessary and sufficient condition, convex combination, growth and distortion bounds, Fekete-Szegö inequality, and partial sums for this newly defined class.

1. Introduction, Definitions, and Motivation

Let $\mathscr{A}$ represent the collections of holomorphic (analytic) functions f defined in the open unit disc:

$$\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}, \tag{1}$$

such that the Taylor series expansion of f is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \, (z \in \mathbb{D}). \tag{2}$$

By convention, $\mathscr{S}$ stands for a subclass of class $\mathscr{A}$ comprising of univalent functions of the form (2) in the open unit disc $\mathbb{D}$. Let $\mathscr{P}$ represent the class of all functions p that are holomorphic in $\mathbb{D}$ with the condition

$$\Re(p(z)) > 0, \tag{3}$$

and has the series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \, (z \in \mathbb{D}). \tag{4}$$

Next, we recall the definition of subordination, for two functions $h_1, h_2 \in \mathscr{A}$, we say h_1 is subordinated to h_2 and is symbolically written as

$$h_1 \prec h_2, \tag{5}$$

if there exists an analytic function $w(z)$ with the properties

$$|w(z)| \leq |z|, w(0) = 0, \tag{6}$$

such that

$$h_1(z) = h_2(w(z)). \tag{7}$$

Further if $h_2 \in \mathcal{S}$, then the above condition becomes

$$\begin{gathered} h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0), \\ h_1(\mathbb{D}) \leq h_2(\mathbb{D}). \end{gathered} \tag{8}$$

Now, recall the definition of convolution, let $f \in \mathcal{A}$ given by (2) and $h(z)$ given by

$$h(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{9}$$

then their convolution denoted by $(f * h)(z)$ is given by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \ (z \in \mathbb{D}). \tag{10}$$

The most important and well-known family of analytic functions is the class of starlike functions denoted by $\mathcal{S}^*$ and is defined as

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \ (\forall z \in \mathbb{D}) \right\}. \tag{11}$$

Next, for $-1 \leq B < A \leq 1$, Janowski [1] generalized the class $\mathcal{S}^*$ as follows.

Definition 1. A function h with property that $h(0) = 1$ is placed in the class $\mathcal{P}[A, B]$ if and only if

$$h(z) \prec \frac{1 + Az}{1 + Bz} \ (-1 \leq B < A \leq 1). \tag{12}$$

Janowski also proved that for a function $p \in \mathcal{P}$, a function $h(z)$ belongs to $\mathcal{P}[A, B]$ if the following relation holds

$$h(z) = \frac{(A+1)p(z) - (A-1)}{(B+1)p(z) - (B-1)}. \tag{13}$$

Also, function f of form (2) belongs to the class $\mathcal{S}^*[A, B]$ if

$$\frac{zf'(z)}{f(z)} = \frac{(A+1)p(z) - (A-1)}{(B+1)p(z) - (B-1)} (-1 \leq B < A \leq 1). \tag{14}$$

Kanas et al. (see [2, 3]; see also [4, 5]) were the first to define the conic domain $\Omega_k (k \geqq 0)$ as follows:

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{15}$$

Moreover, for fixed k, Ω_k represents the conic region bounded successively by the imaginary axis $(k = 0)$. For $k = 1$, it is a parabola, and for $0 < k < 1$, it is the right-hand branch of the hyperbola, and for $k > 1$, it represents an ellipse.

For these conic regions, the following functions play the role of extremal functions:

$$p_k(z) = \begin{cases} \chi_1(k, z), & (k = 0), \\ \chi_2(k, z), & (k = 1), \\ \chi_3(k, z), & (0 \leqq k < 1), \\ \chi_4(k, z), & (k > 1), \end{cases} \tag{16}$$

where

$$\chi_1(k, z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots,$$

$$\chi_2(k, z) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,$$

$$\chi_3(k, z) = 1 + \frac{2}{1-k^2} \sinh^2\left\{\left(\frac{2}{\pi} \arccos k\right) \operatorname{arctanh}\left(\sqrt{z}\right)\right\},$$

$$\chi_4(k, z) = 1 + \frac{1}{k^2-1} \sin\left(\frac{\pi}{2K(\kappa)} \int_0^{u(z)/\sqrt{\kappa}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right) + \frac{1}{k^2-1},$$

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa z}} (\forall z \in \mathbb{D}), \tag{17}$$

and $\kappa \in (0, 1)$ is chosen such that $\lambda = \cosh(\pi K'(\kappa)/(4K(\kappa)))$. Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1-\kappa^2})$, that is, $K'(\kappa)$ is the complementary integral of $K(\kappa)$. Assume that

$$p_k(z) = 1 + P_1 z + P_2 z^2 + \cdots (\forall z \in \mathbb{D}). \tag{18}$$

Then, in [6], it has been shown that, for (16), one can have

$$P_1 = \begin{cases} \dfrac{2N^2}{1-k^2}, & (0 \leqq k < 1), \\ \dfrac{8}{\pi^2}, & (k = 1), \\ \dfrac{\pi^2}{4k^2(\kappa)^2(1+\kappa)\sqrt{\kappa}}, & (k > 1), \end{cases} \tag{19}$$

$$P_2 = D(k) P_1, \tag{20}$$

where

$$D(k)=\begin{cases}\dfrac{N^2+2}{3}, & (0\leqq k<1),\\ \dfrac{2}{3}, & (k=1),\\ \dfrac{[4K(\kappa)]^2\left(\kappa^2+6\kappa+1\right)-\pi^2}{24[K(\kappa)]^2(1+\kappa)\sqrt{\kappa}}, & (k>1),\end{cases} \tag{21}$$

with

$$N=\frac{2}{\pi}\arccos k. \tag{22}$$

Definition 2. A function f of the form (2) is said to be in class $k-\mathcal{ST}$, if and only if

$$l\frac{zf'(z)}{f(z)}\prec p_k(z), k\geq 0. \tag{23}$$

Noor and Malik [7] combined the concepts of the Janowski functions and the conic regions and gave the following definition.

Definition 3. A function $h\in\mathcal{P}$ is said to be in the class $k-\mathcal{P}[A,B]$ if and only if

$$h(z)\prec\frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, -1\leq B<A\leq 1, k\geq 0. \tag{24}$$

Geometrically, $h(z)\in k-\mathcal{P}[A,B]$ takes all values in domain $\Delta_k[A,B]$, which is defined as follows

$$\Delta_k[A,B]=\left\{w:\Re\left(\frac{(B-1)w-(A-1)}{(B+1)w-(A+1)}\right)>k\left|\frac{(B-1)w-(A-1)}{(B+1)w-(A+1)}-1\right|\right\}, \tag{25}$$

the domain $\Delta_k[A,B]$ represents conic-type regions, which was introduced and studied by Noor and Malik [7] and is further generalized by the many authors, see for example [8] and the references cited therein.

Definition 4 [7]. A function $f\in\mathcal{A}$ is said to be in the class $k-\mathcal{S}^*[A,B]$ if and only if

$$\frac{zf'(z)}{f(z)}\prec\frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}. \tag{26}$$

The generalized exponential series:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)}, \alpha, z\in\mathbb{C} \text{ and } \Re(\alpha)>0, \tag{27}$$

is one special-type function with single parameter α, was introduced by Mittag-Leffer (see [9]), and is therefore known as the Mittag-Leffler function. Another function $E_{\alpha,\beta}(z)$ with two parameters α and β having similar properties to those of Mittag-Leffler function is given by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)}(\alpha,\beta,z\in\mathbb{C}), \tag{28}$$

and was introduced by Wiman [10, 11] Agrawal [12], and by the many other (see for example [13–16]). It can be seen that the series $E_{\alpha;\beta}(z)$ converges for all finite values of z if

$$\Re(\alpha)>0, \Re(\beta)>0. \tag{29}$$

During the last years, the interest in Mittag-Leffler type functions has considerably increased due to their vast potential of applications in applied problems such as fluid flow, electric networks, probability, and statistical distribution theory. For a detailed account of properties, generalizations and applications of functions (27) and (28), one may refer to [17–19] and [20].

Geometric properties including starlikeness, convexity, and close-to-convexity for the Mittag-Leffler function $E_{\alpha,\beta}(z)$ were recently investigated by Bansal and Prajapat in [21]. Differential subordination results associated with generalized Mittag-Leffler function were also obtained in [22].

A variable $\mathcal{N}$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \cdots$ with probabilities $e^{-\psi}, \psi e^{-\psi}/1!, \psi^2 e^{-\psi}/2!, \psi^3 e^{-\psi}/3!, \cdots$ respectively, where ψ is called the parameter. Thus,

$$P_n(\mathcal{N}=n)=\frac{\psi^n e^{-\psi}}{n!}\ (n=0,1,2,3,\cdots). \tag{30}$$

It is easy to see that (30) is a mass probability function because

$$\begin{aligned}&P(\psi,\alpha,\beta;n)(z)\geq 0,\\&\sum_{n=0}^{\infty}P(\psi,\alpha,\beta;n)(z)=1.\end{aligned} \tag{31}$$

The power series $Y(\psi,z)$ given by

$$Y(\psi,z)=z+\sum_{n=2}^{\infty}\frac{\psi^{n-1}e^{-\psi}}{(n-1)!}z^n\ (\forall z\in\mathbb{D} \text{ and } \psi>0), \tag{32}$$

which coefficients are probabilities of Poisson distribution is introduced by Porwal [23]. We can see that by ratio test the radius of convergence of $Y(\psi,z)$ is infinity. Porwal [23] also defined and introduced the following series:

$$\mathcal{G}(\psi,z)=2z-Y(\psi,z)=z-\sum_{n=2}^{\infty}\frac{\psi^{n-1}e^{-\psi}}{(n-1)!}z^n\ (\forall z\in\mathbb{D} \text{ and } \psi>0). \tag{33}$$

The works of Porwal [23] motivate researchers to introduced a new probability distribution if it assumes the positive values and its mass function is given by (30) (see for example [24–26]).

It was Porwal and Dixit [24] who studied and connected the Poisson distribution and the well-known Mittag-Leffer function systematically. They called it the Mittag-Leffer type Poisson distribution and prevailed moments. The Mittag-Leffer type Poisson distribution is given by (see [24])

$$Y(\psi, \alpha, \beta)(z) = z + \sum_{n=2}^{\infty} \frac{\psi^{n-1}}{\Gamma(\alpha(n-1)+\beta) E_{\alpha;\beta}(\psi)} z^n, \quad (34)$$

where $Y(\psi, \alpha, \beta)(z)$ is a normalized function of class $\mathcal{S}$, since

$$Y(\psi, \alpha, \beta)(0) = 0 = Y'(\psi, \alpha, \beta)(0) - 1. \quad (35)$$

The probability mass function for the Mittag-Leffer type Poisson distribution series is given by

$$P(\psi, \alpha, \beta ; n)(z) = \frac{\psi^n}{E_{\alpha;\beta}(\psi) \Gamma(\alpha n + \beta)} \ (n = 0, 1, 2, 3, \cdots), \quad (36)$$

where $E_{\alpha;\beta}(\psi)$ is given by (28). It is worthy to note that the Mittag-Leffer type Poisson distribution is a generalization of Poisson distribution. Furtheremore, Bajpai [27] also studied and obtain some necessary and sufficient conditions for this distribution series.

Very recently, using the Mittag-Leffer type Poisson distribution series, Alessa et al. [28] defined the convolution operator as

$$\Omega(\psi, \alpha, \beta) f(z) = Y(\psi, \alpha, \beta) * f(z) = z + \sum_{n=2}^{\infty} \varphi_{\psi}^{n}(\alpha, \beta) a_n z^n, \quad (37)$$

where

$$\varphi_{\psi}^{n}(\alpha, \beta) = \frac{\psi^{n-1}}{\Gamma(\alpha(n-1)+\beta) E_{\alpha;\beta}(\psi)}. \quad (38)$$

Using this convolution operator, they defined and studied a new subclass of analytic function systematically. They obtained certain coefficient estimates, neighborhood results, partial sums, and convexity and compactness properties for their defined functions class.

In recent years, binomial distribution series, Pascal distribution series, Poisson distribution series, and Mittag-Leffer type Poisson distribution series play important role in the geometric function theory of complex analysis. The sufficient ways were innovated for certain subclasses of starlike and convex functions involving these special functions (see for example [26, 29–32]). Motivated by the abovementioned works and from the work of Alessa et al. [28], in this article, by mean of certain convolution operator for Mittag-Leffer type Poisson distribution, we shall define a new subclass of starlike functions involving both the conic-type regions and the Janowski functions. We then obtain some interesting properties for this newly defined function class including for example necessary and sufficient condition, convex combination, growth and distortion bounds, Fekete-Szegö inequality, and partial sums. We now define a subclass of Janowski-type starlike functions involving the conic domains by mean of certain convolution operator for Mittag-Leffer type Poisson distribution as follows.

Definition 5. For $-1 \le B < A \le 1$, a function $f \in \mathcal{A}$ is in class $k - \Omega\mathcal{S}^*(\alpha, \beta, A, B)$ if

$$\begin{aligned} &\Re\left(\frac{(B-1)\vartheta(f ; \psi, \alpha, \beta) - (A-1)}{(B+1)\vartheta(f ; \psi, \alpha, \beta) - (A+1)} - 1\right) \\ &\quad \ge k\left|\frac{(B-1)\vartheta(f ; \psi, \alpha, \beta) - (A-1)}{(B+1)\vartheta(f ; \psi, \alpha, \beta) - (A+1)} - 1\right|, \end{aligned} \quad (39)$$

where

$$\vartheta(f ; \psi, \alpha, \beta) = \frac{z(\Omega(\psi, \alpha, \beta) f(z))'}{\Omega(\psi, \alpha, \beta) f(z)}. \quad (40)$$

For the proofs of our key findings, we need the following lemma.

Lemma 6 [33]. *Let $p \in \mathcal{P}$ have the series expansion of form (4), then*

$$\left|a_3 - \zeta a_2^2\right| \le 2 \max\{1, |2\zeta - 1|\}, \text{where } \zeta \in \mathbb{C}. \quad (41)$$

2. Main Results

Theorem 7. *Let $f \in k - \Omega\mathcal{S}^*(\alpha, \beta, A, B)$ and is of the form (2), then*

$$\sum_{n=2}^{\infty} [2(k+1)|1-n| + |n(1+B) - (1+A)|] \varphi_{\psi}^{n}(\alpha, \beta) |a_n| < |B - A|. \quad (42)$$

The result is sharp for the function given in (51).

Proof. Suppose that inequality (42) holds true, then it is enough to show that

$$k\left|\frac{(B-1)\vartheta(f ; \psi, \alpha, \beta) - (A-1)}{(B+1)\vartheta(f ; \psi, \alpha, \beta) - (A+1)} - 1\right| - \Re\left(\frac{(B-1)\vartheta(f ; \psi, \alpha, \beta) - (A-1)}{(B+1)\vartheta(f ; \psi, \alpha, \beta) - (A+1)} - 1\right) < 1. \quad (43)$$

For this, consider

$$k\left|\frac{(B-1)\vartheta(f ; \psi, \alpha, \beta) - (A-1)}{(B+1)\vartheta(f ; \psi, \alpha, \beta) - (A+1)} - 1\right| - \Re\left(\frac{(B-1)\vartheta(f ; \psi, \alpha, \beta) - (A-1)}{(B+1)\vartheta(f ; \psi, \alpha, \beta) - (A+1)} - 1\right). \quad (44)$$

As we have set

$$\vartheta(f;\psi,\alpha,\beta)=\frac{z(\Omega(\psi,\alpha,\beta)f(z))'}{\Omega(\psi,\alpha,\beta)f(z)}, \tag{45}$$

therefore, after some straightforward simplifications, we have

$$\begin{aligned}
&k\left|\frac{(B-1)\vartheta(f;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f;\psi,\alpha,\beta)-(A+1)}-1\right|-\Re\left(\frac{(B-1)\vartheta(f;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f;\psi,\alpha,\beta)-(A+1)}-1\right),\\
&\le(k+1)\left|\frac{(B-1)z(\Omega(\psi,\alpha,\beta)f(z))'-(A-1)\Omega(\psi,\alpha,\beta)f(z)}{(B+1)z(\Omega(\psi,\alpha,\beta)f(z))'-(A+1)\Omega(\psi,\alpha,\beta)f(z)}-1\right|,\\
&=2(k+1)\left|\frac{\Omega(\psi,\alpha,\beta)f(z)-z(\Omega(\psi,\alpha,\beta)f(z))'}{(B+1)z(\Omega(\psi,\alpha,\beta)f(z))'-(A+1)\Omega(\psi,\alpha,\beta)f(z)}\right|,\\
&=2(k+1)\left|\frac{\sum_{n=2}^{\infty}(1-n)\varphi_{\psi}^{n}(\alpha,\beta)a_nz^n}{(B-A)z+\sum_{n=2}^{\infty}[n(1+B)-(1+A)]\varphi_{\psi}^{n}(\alpha,\beta)a_nz^n}\right|,\\
&\le\frac{2(k+1)\sum_{n=2}^{\infty}|1-n|\varphi_{\psi}^{n}(\alpha,\beta)|a_n|}{|B-A|-\sum_{n=2}^{\infty}|n(1+B)-(1+A)|\varphi_{\psi}^{n}(\alpha,\beta)|a_n|}.
\end{aligned} \tag{46}$$

By using (42), the above inequality is bounded above by 1, and hence, the proof is completed. □

Example 8. For the function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{|B-A|}{[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)}x_nz^n\ (z\in\mathbb{D}), \tag{47}$$

such that

$$\sum_{n=2}^{\infty}|x_n|=1, \tag{48}$$

we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)|a_n|\\
&\quad=\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)\\
&\qquad\cdot\frac{|B-A|}{[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)}|x_n|\\
&\quad=|B-A|\sum_{n=2}^{\infty}|x_n|=|B-A|.
\end{aligned} \tag{49}$$

Hence, $f\in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$ and the result is sharp.

Corollary 9. *Let the function f of the form (2) be in the class $k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$. Then,*

$$|a_n|\le\frac{|B-A|}{[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)}. \tag{50}$$

The result is sharp for the function $f_t(z)$ given by

$$f_t(z)=z+\frac{|B-A|}{[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)}z^n. \tag{51}$$

Theorem 10. *The class $k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$ is closed under convex combination.*

Proof. Let $f_k(z)\in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$ such that

$$lf_k(z)=z+\sum_{n=2}^{\infty}a_{n,k}z^n,\ k\in\{1,2\}. \tag{52}$$

It is enough to show that

$$tf_1(z)+(1-t)f_2(z)\in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)\ (t\in[0,1]). \tag{53}$$

As

$$ltf_1(z)+(1-t)f_2(z)\quad=z+\sum_{n=2}^{\infty}[ta_{n,1}+(1-t)a_{n,2}]z^n. \tag{54}$$

Now, by Theorem 7, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\\
&\quad\cdot\varphi_{\psi}^{n}(\alpha,\beta)\left|ta_{n,1}+(1-t)a_{n,2}\right|\\
&\le\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\\
&\quad\cdot\varphi_{\psi}^{n}(\alpha,\beta)\left[t\left|a_{n,1}\right|+(1-t)\left|a_{n,2}\right|\right]\\
&\le t\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\\
&\quad\cdot\varphi_{\psi}^{n}(\alpha,\beta)\left|a_{n,1}\right|+(1-t)\\
&\quad\cdot\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)\left|a_{n,2}\right|\\
&<t|B-A|+(1-t)|B-A|=|B-A|.
\end{aligned} \tag{55}$$

Hence,

$$tf_1(z)+(1-t)f_2(z)\in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B), \tag{56}$$

which completes the proof. □

Theorem 11. *Let $f \in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$, then for $|z|=r$*

$$\begin{aligned} r-\frac{|B-A|}{2(k+1)+|2B-A+1|\varphi_\psi^2(\alpha,\beta)}r^2 &\le |f(z)| \\ &\le r+\frac{|B-A|}{2(k+1)+|2B-A+1|\varphi_\psi^2(\alpha,\beta)}r^2. \end{aligned} \tag{57}$$

The result is sharp for the function given in (51) for $n=2$.

Proof. Let $f \in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$. Using Theorem 7, we can deduce the following inequity:

$$\begin{aligned} |f(z)| &\le |z|+\sum_{n=2}^{\infty}|a_n||z^n| \le |z|+|z|^2\sum_{n=2}^{\infty}|a_n| \\ &\le r+\frac{|B-A|}{2(k+1)+|2B-A+1|\varphi_\psi^2(\alpha,\beta)}r^2. \end{aligned} \tag{58}$$

Similarly,

$$\begin{aligned} |f(z)| &\ge |z|-\sum_{n=2}^{\infty}|a_n||z^n| \ge |z|-|z|^2\sum_{n=2}^{\infty}|a_n| \\ &\ge r-\frac{|B-A|}{2(k+1)+|2B-A+1|\varphi_\psi^2(\alpha,\beta)}r^2. \end{aligned} \tag{59}$$

□

Theorem 12. *Let $f \in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$, then for $|z|=r$*

$$\begin{aligned} |f'(z)| &\le 1+\frac{2|B-A|}{2(k+1)+|2B-A+1|\varphi_\psi^2(\alpha,\beta)}r, \\ |f'(z)| &\ge 1-\frac{2|B-A|}{2(k+1)+|2B-A+1|\varphi_\psi^2(\alpha,\beta)}r. \end{aligned} \tag{60}$$

The result is sharp for the function given in (51) for $n=2$.

Proof. The proof is quite similar to Theorem 11, so left for reader. □

Now, we evaluate a kind of Hankel determinant problem, which is also known as the Fekete-Szegö functional.

Theorem 13. *If f is of the form (2) and belongs to $k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$, then*

$$\left|a_3-\xi a_2^2\right| \le \frac{P_1(A-B)}{4\varphi_\psi^3(\alpha,\beta)}\max\left\{1,\left|\frac{(B-2P_2+1)\varphi_\psi(\alpha,\beta)-2\xi(A-B)P_1^2}{2P_1\varphi_\psi(\alpha,\beta)}\right|\right\}, \tag{61}$$

where P_1 and P_2 are defined by (19) and (20), respectively.

Proof. To prove inequality (61), we let

$$\vartheta(f;\psi,\alpha,\beta)=\frac{z(\Omega(\psi,\alpha,\beta)f(z))'}{\Omega(\psi,\alpha,\beta)f(z)}, \tag{62}$$

then from (26), we have

$$\vartheta(f;\psi,\alpha,\beta) \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}=\Phi(z)\ (\text{say}). \tag{63}$$

Thus, if

$$p_k(z)=1+P_1z+P_2z^2+\cdots, \tag{64}$$

then by simple computation, we get

$$\Phi(z)=1+\frac{1}{2}P_1(A-B)z+\frac{1}{4}(A-B)\left(2P_2-(1+B)P_1^2\right)z^2+\cdots. \tag{65}$$

Now, from (63), there exists an analytic function $h(z)$ such that

$$h(z)=\frac{1+\Phi^{-1}(\vartheta(f;\psi,\alpha,\beta))}{1-\Phi^{-1}(\vartheta(f;\psi,\alpha,\beta))}=1+c_1z+c_2z^2+\cdots, \tag{66}$$

is analytic and

$$\Re(h(z))>0, \tag{67}$$

in open unit disc $\mathbb{D}$. Also, we have

$$\vartheta(f;\psi,\alpha,\beta)=\Phi\left(\frac{h(z)-1}{h(z)+1}\right), \tag{68}$$

where

$$\frac{z(\vartheta(f;\psi,\alpha,\beta))'}{\vartheta(f;\psi,\alpha,\beta)}=1+\varphi_\psi^2(\alpha,\beta)a_2z+\left(2\varphi_\psi^3(\alpha,\beta)a_3-\varphi_\psi^4(\alpha,\beta)a_2^2\right)z^2+\cdots. \tag{69}$$

$$\begin{aligned} \Phi\left(\frac{h(z)-1}{h(z)+1}\right) &= 1+\frac{1}{4}(A-B)P_1c_1z+\frac{1}{4}(A-B) \\ &\quad\cdot\left[P_1c_2+\left(\frac{P_2}{2}-\frac{1+B}{4}-\frac{P_1}{2}\right)c_1^2\right]z^2+\cdots. \end{aligned} \tag{70}$$

After comparing the (69) and (70), we get

$$a_2=\frac{1}{4\varphi_\psi^2(\alpha,\beta)}(A-B)P_1c_1, \tag{71}$$

$$a_3=\frac{1}{8\varphi_\psi^3(\alpha,\beta)}(A-B)\left(P_1c_2+\left(\frac{P_2}{2}-\frac{1+B}{4}-\frac{P_1}{2}\right)c_1^2\right). \tag{72}$$

Now, by making use of (71) and (72), in conjunction with

Lemma, we have

$$\left|a_3-\xi a_2^2\right| \le \frac{P_1(A-B)}{4\varphi_\psi^3(\alpha,\beta)} \max\left\{1, \left|\frac{(B-2P_2+1)\varphi_\psi(\alpha,\beta)-2\xi(A-B)P_1^2}{2P_1\varphi_\psi(\alpha,\beta)}\right|\right\}, \tag{73}$$

which is the required result. ◻

3. Partial Sums

In this section, we will examine the ratio of a function of form (2) to its sequence of partial sums

$$f_j(z) = z + \sum_{n=2}^{j} a_n z^n, \tag{74}$$

when the coefficients of f are sufficiently small to satisfy condition (42). We will determine sharp lower bounds for

$$\begin{gathered} \Re\left(\frac{f(z)}{f_j(z)}\right), \\ \Re\left(\frac{f_j(z)}{f(z)}\right), \\ \Re\left(\frac{f'(z)}{f_j'(z)}\right), \\ \Re\left(\frac{f_j'(z)}{f'(z)}\right). \end{gathered} \tag{75}$$

Theorem 14. *If f of form (2) satisfies condition (42), then*

$$\Re\left(\frac{f(z)}{f_j(z)}\right) \ge 1 - \frac{1}{\rho_{j+1}} \quad (\forall z \in \mathbb{D}), \tag{76}$$

$$\Re\left(\frac{f_j(z)}{f(z)}\right) \ge \frac{\rho_{j+1}}{1+\rho_{j+1}} \quad (\forall z \in \mathbb{D}), \tag{77}$$

where

$$\rho_j = \frac{\left[2(k+1)|1-n| + |n(1+B)-(1+A)|\right]\varphi_\psi^n(\alpha,\beta)}{|A-B|}. \tag{78}$$

The result is sharp for the function given in (51).

Proof. It is easy to verify that

$$\rho_{n+1} > \rho_n > 1 \text{ for } n > 2. \tag{79}$$

Thus, in order to prove the inequality (76), we set

$$\begin{aligned} \rho_{j+1}\left[\frac{f(z)}{f_j(z)} - \left(1 - \frac{1}{\rho_{j+1}}\right)\right] &= \frac{1+\sum_{n=2}^{j} a_n z^{n-1} + \rho_{j+1}\sum_{n=j+1}^{\infty} a_n z^{n-1}}{1+\sum_{n=2}^{j} a_n z^{n-1}} \\ &= \frac{1+h_1(z)}{1+h_2(z)}. \end{aligned} \tag{80}$$

We now set

$$\frac{1+h_1(z)}{1+h_2(z)} = \frac{1+w(z)}{1-w(z)}. \tag{81}$$

Then, we find after some suitable simplification that

$$w(z) = \frac{h_1(z)-h_2(z)}{2+h_1(z)+h_2(z)}. \tag{82}$$

Thus, clearly, we find that

$$w(z) = \frac{\rho_{j+1}\sum_{n=j+1}^{\infty} a_n z^{n-1}}{2+2\sum_{n=2}^{j} a_n z^{n-1} + \rho_{j+1}\sum_{n=j+1}^{\infty} a_n z^{n-1}}. \tag{83}$$

By applying the trigonometric inequalities with $|z|<1$, we arrived at the following inequality:

$$|w(z)| \le \frac{\rho_{j+1}\sum_{n=j+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{j}|a_n| - \rho_{j+1}\sum_{n=j+1}^{\infty}|a_n|}. \tag{84}$$

We can now see that

$$|w(z)| \le 1, \tag{85}$$

if and only if

$$2\rho_{j+1}\sum_{n=j+1}^{\infty}|a_n| \le 2 - 2\sum_{n=2}^{j}|a_n|, \tag{86}$$

which implies that

$$\sum_{n=2}^{j}|a_n| + \rho_{j+1}\sum_{n=j+1}^{\infty}|a_n| \le 1. \tag{87}$$

Finally, to prove the inequality in (76), it suffices to show that the left-hand side of (87) is bounded above by the following sum:

$$\sum_{n=2}^{\infty}\rho_n|a_n|, \tag{88}$$

which is equivalent to

$$\sum_{n=2}^{j}(\rho_n - 1)|a_n| + \sum_{n=j+1}^{\infty}\left(\rho_n - \rho_{j+1}\right)|a_n| \geq 0. \tag{89}$$

In virtue of (89), the proof of inequality in (76) is now completed.

Next, in order to prove the inequality (77), we set

$$\begin{aligned}&\left(1+\rho_{j+1}\right)\left(\frac{f_j(z)}{f(z)} - \frac{\rho_{j+1}}{1+\rho_{j+1}}\right)\\ &\quad = \frac{1+\sum_{n=2}^{j}a_n z^{n-1} - \rho_{j+1}\sum_{n=j+1}^{\infty}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}a_n z^{n-1}}, = \frac{1+w(z)}{1-w(z)},\end{aligned} \tag{90}$$

where

$$|w(z)| \leq \frac{\left(1+\rho_{j+1}\right)\sum_{n=j+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{j}|a_n| - \left(\rho_{j+1}-1\right)\sum_{n=j+1}^{\infty}|a_n|} \leq 1. \tag{91}$$

This last inequality in (91) is equivalent to

$$\sum_{n=2}^{j}|a_n| + \rho_{j+1}\sum_{n=j+1}^{\infty}|a_n| \leq 1. \tag{92}$$

Finally, we can see that the left-hand side of the inequality in (92) is bounded above by the following sum:

$$\sum_{n=2}^{\infty}\rho_n|a_n|, \tag{93}$$

so we have completed the proof of the assertion (77). ☐

We next turn to ratios involving derivatives.

Theorem 15. *If f of the form (2) satisfies condition (42), then*

$$\begin{aligned}&\Re\left(\frac{f'(z)}{f_j'(z)}\right) \geq 1 - \frac{j+1}{\rho_{j+1}} \quad (\forall z \in \mathbb{D}),\\ &\Re\left(\frac{f_j'(z)}{f'(z)}\right) \geq \frac{\rho_{j+1}}{\rho_{j+1}+j+1} \quad (\forall z \in \mathbb{D}),\end{aligned} \tag{94}$$

where ρ_j is given by (78). The result is sharp for the function given in (51).

Proof. The proof of Theorem 15 is similar to that of Theorem 14; we here choose to omit the analogous details. ☐

4. Concluding Remarks and Observations

In our present work, by making use of the idea of Mittag-Leffler type Poisson distribution, we have defined and studied certain new subclasses of starlike functions involving the Janowski functions. Further, we have discussed some important geometric properties like necessary and sufficient condition, convex combination, growth and distortion bounds, Fekete-Szegö inequality, and partial sums for this newly defined functions class.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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Research Article

Discrete Fractional Inequalities Pertaining a Fractional Sum Operator with Some Applications on Time Scales

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This content replicates some discrete nonlinear fractional inequalities by virtue of the fractional sum operator $\bar{\Psi}$ on time scales. Through the recognition of the principle of discrete fractional calculus, we are able to recover the precise estimates for unknown functions of inequalities of the Gronwall type. The resultant inequalities are of unique structure comparative with the latest reviewing disclosures and can be described as a complementary tool for numerically testing the solutions of discrete partial differential equations. The foremost consequences are probably confirmed via handling of assessment procedure and technique of mean value speculation. We display few examples of the proposed inequalities to represent the incentives of our effort.

1. Introduction and Essentials

Fractional calculus and its conceptual applications have acquired a huge amount of potential in terms of the reality that fractional operators are becoming a valuable asset with more specificity and success in demonstrating a few complicated discoveries in numerous seemingly diverse and wide fields of science and in many areas, such as fluid flow, physics, chaos, image analysis, virology, and financial economy [1, 2]. A few years earlier, fractional differential equations and dynamic systems have been validated as being significant gadgets in showing a few marvels in different parts of applied and pure sciences. They draw enormous importance in research-oriented fields (see the basic monograph and the interesting paper [3, 4]. The set of implications that encouraged the formation of a discrete fractional theory is established [5].

The aim of the paper is to impose discrete fractional sum equations in order to build up a procedure to comprehend certain equations and to extract corresponding Gronwall sort of inequality. Especially, Gronwall's inequality is illustrated to be among the primary inequalities for the foundation of differential equations. From now on and into the near future, various assumptions and development of such inequalities have ended up being a major component. Discrete Gronwall inequality was suggested by Sugiyama [6] in 1969. He carried out the related framework of reliable and discrete type of Gronwall inequality:

Theorem 1. *Let $\mathscr{L}(\bar{\Omega})$ and $\mathfrak{k}(\bar{\Omega})$ be real-valued functions defined for $\bar{\Omega} \in \mathbb{N}_0$ and $\mathfrak{k}(\bar{\Omega}) > 0$ for every $\bar{\Omega} \in \mathbb{N}_0$. If*

$$\mathscr{L}(\bar{\Omega}) \leq \widehat{\omega}_0 + \sum_{\bar{\rho}=0}^{\bar{\Omega}-1} \mathfrak{k}(\bar{\rho})\mathscr{L}(\bar{\rho}), \bar{\Omega} \in \mathbb{N}_0, \quad (1)$$

where $\widehat{\omega}_0$ is a nonnegative constant, and then

$$\mathscr{L}(\bar{\Omega}) \leq \widehat{\omega}_0 \prod_{\bar{\rho}=0}^{\bar{\Omega}-1} [1 + \mathfrak{k}(\bar{\rho})]. \quad (2)$$

Theorem 1 is often used differential and integral equations that possess the unification of discrete factor models.

It is interesting that discretization cycle is among the most demanded tools for researchers who are captivated in multiplication and computational assessment. Keep in mind that not all discrete operators have identical characteristics to continuous ones, and the formation of discrete fractional calculus is becoming an essential prerequisite. Several other authors have dedicated their resources to the quest for arbitrary new operators. Definitely, the range of such methods provides analysts with more chances to adapt them in multiple models.

Fractional calculus that consists of derivative and integral of noninteger order is normal augmentations of the standard integer order calculus. Fractional calculus is by all accounts universal in light of its fascinating applications with regards to different aspects of science, for example, viscoelastic materials, dispersion, central nervous biology, regulation hypothesis, and statistical data [7–9].

Despite the existence of a rigorous scientific standard for the continuity of fractional calculus, the possibility of improving a discrete fractional calculus has been insufficient. Although we all realize, discovering fractional difference equations requires a thorough understanding of system identification. Recently, surprising achievement has been produced as a result of arduous attempts in fractional difference structures by Du and Jia [10]. The existence and uniqueness of solutions are the foundation for examining the stability problem that has been exploited using fractional Gronwall and Bihari inequality, for example [11, 12].

The essentially identical to discrete hypothesis by a fractional sum of order $\bar{\Psi} > 0$ was identified due to Miller and Ross [13] through solution based on linear differential equation, and many key aspects of proposed operator were tested. Moreover, Atici and Eloe [14] implemented a discrete method for Laplace transformation containing a fractional class of finite difference equations. Atici and Eloe [15] identified the causes of the initial value in the discrete fractional analysis. Atici and Eloe [16] investigated the structure of a discrete fractional calculus with the nabla operator. They created exponential laws and the item rule to the forward fractional calculus. Atici and Sengul [17] built up the Leibniz rule and summation by part equation in discrete fractional theory. Bastos and Torres [18] introduced the more broad discrete fractional operator which was specified by delta and nabla fractional sums. Holm [19] presented operators with fractional sums and applied one such hypothesis to tackle fractional initial value problems. Anastassiou [20] determined the privilege discrete nabla fractional of Taylor equation. The innovation that made look like a consequence of this depiction was charming to several readers and now subjected to outrageous review, in numerous approaches: discrete nature and precision of fractional equations, tumor formation simulating [21], consistency of tumor cell solutions related order of Legendre's derivative $\bar{\Psi}$ [22], and Euler-Lagrange equation and Legendre's optimality condition for the calculus of variations problems [23]. The idea of a discrete version of fractional calculus is adopted just as late, usually because of the impact of exploration in fractional analysis (see [24–26]).

Inequalities of finite difference that demonstrate distinct bounds of undefined functions suggest a highly useful and beneficial way to enhance understanding of finite difference equations. As a consequence, difference equations tend to be a realistic instrument that correctly represents real-life scenarios like question queueing, power systems, and financial measurements, and to attempt such kind of mechanism, this protection is mandated. Probably the least impossible enormously difference equations right now have begun to achieve the attention [27].

In the stage when we have to examine many features of a differential equation, there are multiple interpretations for certain categories of inequalities. Essentially, based on capability of the aforementioned inquiry, we formulate in this material some generalizations of discrete fractional nonlinear inequalities linked to the fractional sum operator $\bar{\Psi}$ that assemble to describe fractional inequalities and incorporate some proven publication tests. To reflect theoretical hypotheses, it was shown that the transmitted inequalities may be used to evaluate certain classifications of discrete fractional equations. In order to explore the usefulness and drawbacks of the usage of fractional sum difference equations, the completion of this paper secures a few instances.

Definitive portions of the document are classified as such. We address relevant actual considerations and basic assumptions in Section 2. Section 3 is committed to the theoretical experiences of nonlinear discrete fractional inequalities with some remarks. The remaining section is considered in accomplishing the theoretical examination specifications.

2. Preliminaries

And with that initiative, without the absence of a specific argument, let $\widehat{M}$ be a constant, $\mathbb{N}_{\bar{\rho}} = \{\widehat{\rho}, \widehat{\rho}+1, \widehat{\rho}+2, \cdots\}$, $H_{\bar{\rho}} = [\bar{\rho}, \widehat{M}] \cap \mathbb{N}_{\bar{\rho}}$, where $\widehat{M}, \bar{\rho} \in \mathbb{N}_{\bar{\rho}}$, $\sum_{\bar{\rho}=\hat{c}}^{\widehat{\vartheta}} \mathfrak{k}(\bar{\rho}) = 0$, $\mathbb{R}_+ = [0, \infty)$, and difference operator of a be assigned as $\Delta a(\widehat{\vartheta}) = a(\widehat{\vartheta}) - a(\widehat{\vartheta}-1)$, $\widehat{\vartheta} \in \mathbb{N}_{\bar{\rho}}$.

A part of primitive specifications and theorems of discrete fractional measurement is represented as

Definition 2 (see [17]). Let $\bar{\Psi}$ be any positive real number, $\mathfrak{g}$ be any real number and $\sigma(\bar{\rho}) = \bar{\rho}+1$, and then $\bar{\Psi} - th$ fractional sum of $\mathfrak{k}$ is defined for $\bar{\Omega} = \mathfrak{g}$ (mod 1) by

$$\Delta_{\mathfrak{g}}^{-\bar{\Psi}} \mathfrak{k}(\bar{\rho}) = \frac{1}{\Gamma(\bar{\Psi})} \sum_{\widehat{\Omega}=\mathfrak{g}}^{\bar{\rho}-\bar{\Psi}} \left(\bar{\rho} - \sigma(\bar{\Omega})\right)^{\bar{\Psi}-1} \mathfrak{k}(\bar{\Omega}), \tag{3}$$

such that $(\bar{\rho})^{\bar{\Psi}} = \Gamma(\bar{\rho}+1)/\Gamma(\bar{\rho}-\bar{\Psi}+1)$, and $\Delta_{\mathfrak{g}}^{-\bar{\Psi}} \mathfrak{k}$ is defined for $\bar{\rho} = \mathfrak{g} + \bar{\Psi}$ (mod1) and $\Delta_{\mathfrak{g}}^{-\bar{\Psi}} : \mathbb{N}_{\mathfrak{g}} \longrightarrow \mathbb{N}_{\mathfrak{g}+\bar{\Psi}}$.

Definition 3 (see [17]). Let $\widehat{\nu} > 0$,and $\widehat{\Lambda} - 1 < \widehat{\nu} < \widehat{\Lambda}$. Then, $\widehat{\nu}$ – th fractional difference of $\mathfrak{k}$ is classified as

$$\Delta^{\nu\wedge} \mathfrak{k}(\bar{\rho}) = \Delta^{\Lambda\wedge - \bar{\Psi}} \mathfrak{k}(\bar{\rho}) = \Delta^{\Lambda\wedge} \left(\Delta^{-\bar{\Psi}} \mathfrak{k}(\bar{\rho})\right), \tag{4}$$

where $\widehat{\Lambda}$ is a positive integer and $-\bar{\Psi} = \widehat{\nu} - \widehat{\Lambda}$.

Theorem 4 (see [15]). *If a real-valued function $\mathfrak{k}$ be prescribed on $\mathbb{N}_{\mathfrak{g}}$, such that $\widehat{\nu}, \bar{\Psi} > 0$, so*

$$\Delta^{-\bar{\Psi}}\left(\Delta^{-\nu\wedge}\mathfrak{k}(\bar{\rho})\right) = \Delta^{-\left(\bar{\Psi}+\nu\wedge\right)}\mathfrak{k}(\bar{\rho}) = \Delta^{-\nu\wedge}\left(\Delta^{-\bar{\Psi}}\mathfrak{k}(\bar{\rho})\right). \quad (5)$$

Theorem 5 (see [15]). *Let $\bar{\Psi} > 0$ and $\mathfrak{k}$ be a function which is real valued on $\mathbb{N}_{\mathfrak{g}}$, and then*

$$\Delta^{-\bar{\Psi}}\Delta\,\mathfrak{k}(\bar{\rho}) = \Delta\Delta^{-\bar{\Psi}}\,\mathfrak{k}(\bar{\rho}) = \frac{(\bar{\rho}-1)^{\bar{\Psi}-1}}{\Gamma(\bar{\Psi})}\mathfrak{k}(\mathfrak{g}). \quad (6)$$

The reader may bring attention to [15, 17] for further desirable characteristics on a discrete fractional proposition.

3. Result Declaration

Presently, we will adjust the basic tests.

Theorem 6. *Suppose that $\mathscr{L} \in \mathbb{N}_{\bar{\Psi}-1} \longrightarrow \mathbb{R}_+$, $\mathfrak{J}, \mathfrak{U} : \mathbb{N}_{\bar{\Psi}} \longrightarrow \mathbb{R}_+$ are functions, $0 < \bar{\Psi} \le 1$ and $\widehat{\omega}_0 \ge 0$ are constants, and $\widehat{\Pi} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nondecreasing continuous function with $\widehat{\Pi}(\mathfrak{t}) > 0$ for $\mathfrak{t} > 0$. If*

$$\begin{aligned}\mathscr{L}(\bar{\rho}) \le{}& \widehat{\omega}_0 + \Delta_0^{-\bar{\Psi}}\left[\mathfrak{J}(\bar{\rho})\widehat{\Pi}\left(\mathscr{L}\left(\bar{\rho}+\bar{\Psi}-1\right)\right)\right] \\ &+ \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}}\left(\bar{\rho}-\bar{\Omega}-1\right)^{\bar{\Psi}-1}\mathfrak{U}\left(\bar{\Omega}\right)\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\widehat{\Pi} \\ &\cdot\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right), \bar{\rho} \in H_{\bar{\Psi}-1},\end{aligned} \quad (7)$$

is satisfied, then

$$\begin{aligned}\mathscr{L}(\bar{\rho}) \le{}& u\wedge^{-1}\left[\aleph\wedge^{-1}\left\{\widehat{\aleph}\left(\widehat{u}\left(\widehat{\omega}_0\right) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}}\mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right)\right)\right.\right. \\ &\left.\left.+ \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}}\mathfrak{u}\left(\bar{\Omega}-\bar{\Psi}\right)\right\}\right], \bar{\rho} \in H_{\bar{\Psi}-1},\end{aligned} \quad (8)$$

for $0 \le \bar{\rho} \le \bar{\rho}_1$, provided with

$$\widehat{u}(\mathfrak{n}) = \int_{\mathfrak{n}_0}^{\mathfrak{n}}\frac{1}{\widehat{\Pi}(\mathfrak{a})}d\mathfrak{a}, \mathfrak{n} \ge \mathfrak{n}_0 > 0, \lim_{\mathfrak{n}\longrightarrow\infty}\widehat{u}(\mathfrak{n}) = \infty, \quad (9)$$

$$\widehat{\aleph}(\mathfrak{l}) = \int_{\mathfrak{l}_0}^{\mathfrak{l}}\frac{1}{u\wedge^{-1}(\mathfrak{q})}d\mathfrak{q}, \mathfrak{l} \ge \mathfrak{l}_0 > 0, \lim_{\mathfrak{l}\longrightarrow\infty}\widehat{\aleph}(\mathfrak{l}) = \infty. \quad (10)$$

$u\wedge^{-1}$, $\aleph\wedge^{-1}$ is the inverses of $\widehat{u}$, $\widehat{\aleph}$, and $\bar{\rho}_1 \in H_{\bar{\Psi}-1}$ is chosen so that

$$\widehat{\aleph}\left(\widehat{u}\left(\widehat{\omega}_0\right) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}}\mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right)\right) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}}\mathfrak{u}\left(\bar{\Omega}-\bar{\Psi}\right) \in Dom\left(\aleph\wedge^{-1}\right), \quad (11)$$

$$\mathfrak{J}\left(\bar{\Omega},\bar{\rho}\right) = \frac{1}{\Gamma(\bar{\Psi})}\left(\bar{\rho}-\bar{\Omega}-1\right)^{\bar{\Psi}-1}\mathfrak{J}\left(\bar{\Omega}\right), \quad (12)$$

$$\mathfrak{U}\left(\bar{\Omega},\bar{\rho}\right) = \frac{1}{\Gamma(\bar{\Psi})}\left(\bar{\rho}-\bar{\Omega}-1\right)^{\bar{\Psi}-1}\mathfrak{U}\left(\bar{\Omega}\right). \quad (13)$$

Proof. Let $\widehat{\omega}_0 > 0$. Defining

$$\begin{aligned}\mathfrak{y}(\bar{\rho}) ={}& \widehat{\omega}_0 + \Delta_0^{-\bar{\Psi}}\left[\mathfrak{J}(\bar{\rho})\widehat{\Pi}\left(\mathscr{L}\left(\bar{\rho}+\bar{\Psi}-1\right)\right)\right] \\ &+ \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}}\left(\bar{\rho}-\bar{\Omega}-1\right)^{\bar{\Psi}-1}\mathfrak{U}\left(\bar{\Omega}\right)\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\widehat{\Pi} \\ &\cdot\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right),\end{aligned} \quad (14)$$

then one has

$$\mathscr{L}(\bar{\rho}) \le \mathfrak{y}(\bar{\rho}), \bar{\rho} \in H_{\bar{\Psi}-1}. \quad (15)$$

From Definition 2 and (14), we have

$$\begin{aligned}\mathfrak{y}(\bar{\rho}) ={}& \widehat{\omega}_0 + \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}}\left(\bar{\rho}-\bar{\Omega}-1\right)^{\bar{\Psi}-1}\mathfrak{J}\left(\bar{\Omega}\right)\widehat{\Pi}\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right) \\ &+ \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}}\left(\bar{\rho}-\bar{\Omega}-1\right)^{\bar{\Psi}-1}\mathfrak{U}\left(\bar{\Omega}\right)\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\widehat{\Pi} \\ &\cdot\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right), \bar{\rho} \in H_{\bar{\Psi}-1} = \widehat{\omega}_0 + \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}}\mathfrak{J}\left(\bar{\Omega},\bar{\rho}\right)\widehat{\Pi} \\ &\cdot\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right) + \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}}\mathfrak{U}\left(\bar{\Omega},\bar{\rho}\right)\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\widehat{\Pi} \\ &\cdot\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right),\end{aligned} \quad (16)$$

where $\mathfrak{J}(\bar{\Omega},\bar{\rho})$ and $\mathfrak{U}(\bar{\Omega},\bar{\rho})$ are defined as in (11) and (13). Hence, $\mathfrak{y}(\bar{\rho}) \ge 0$ is nondecreasing. Now, $\mathfrak{J}(\bar{\Omega},\bar{\rho})$, $\mathfrak{U}(\bar{\Omega},\bar{\rho})$, and $(\bar{\rho})^{\bar{\Psi}}$ by their definition and $\mathfrak{J}(\bar{\Omega},\bar{\rho})$, $\mathfrak{U}(\bar{\Omega},\bar{\rho})$ is decreasing in $\bar{\rho}$ for each $\bar{\Omega} \in \mathbb{N}_0$. In the equation (14) using straightforward computation for $\bar{\rho} \in H_{\bar{\Psi}}$ and (8), we get

$$\begin{aligned}\mathfrak{y}(\bar{\rho}) - \mathfrak{y}(\bar{\rho}-1) ={}& \mathfrak{J}\left(\bar{\rho}-\bar{\Psi},\bar{\rho}\right)\widehat{\Pi}(\mathscr{L}(\bar{\rho}-1)) + \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}-1}\left[\mathfrak{J}\left(\bar{\Omega},\bar{\rho}\right)\right. \\ &\left.- \mathfrak{J}\left(\bar{\Omega},\bar{\rho}-1\right)\right]\widehat{\Pi}\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right) \\ &+ \mathfrak{U}\left(\bar{\rho}-\bar{\Psi},\bar{\rho}\right)\mathscr{L}(\bar{\rho}-1)\widehat{\Pi}(\mathscr{L}(\bar{\rho}-1)) \\ &+ \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}-1}\left[\mathfrak{U}\left(\bar{\Omega},\bar{\rho}\right) - \mathfrak{U}\left(\bar{\Omega},\bar{\rho}-1\right)\right] \\ &\times\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\widehat{\Pi}\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right) \\ ={}& \mathfrak{j}\left(\bar{\rho}-\bar{\Psi}\right)\widehat{\Pi}(\mathfrak{y}(\bar{\rho}-1)) + \mathfrak{u}\left(\bar{\rho}-\bar{\Psi}\right)\mathfrak{y}(\bar{\rho}-1)\widehat{\Pi} \\ &\cdot(\mathfrak{y}(\bar{\rho}-1)),\end{aligned} \quad (17)$$

monotonicity of $\widehat{\Pi}$, and $\mathfrak{y}$ produces

$$\widehat{\Pi}(\mathfrak{y}(\bar{\rho}-1)) > \widehat{\Pi}\left(\mathfrak{y}\left(\bar{\Psi}-1\right)\right) = \widehat{\Pi}(\widehat{\omega}_0) > 0, \bar{\rho} \in H_{\bar{\Psi}}, \quad (18)$$

from (17) and (18), and one has

$$\frac{\Delta\mathfrak{y}(\bar{\rho}-1)}{\widehat{\Pi}(\mathfrak{y}(\bar{\rho}-1))} \le \mathfrak{j}\left(\bar{\rho}-\bar{\Psi}\right) + \mathfrak{u}\left(\bar{\rho}-\bar{\Psi}\right)\mathfrak{y}(\bar{\rho}-1), \bar{\rho} \in H_{\bar{\Psi}}. \quad (19)$$

By mean value theorem, it can be seen that

$$\begin{aligned}\Delta\widehat{\mathrm{u}}(\mathfrak{y}(\bar{\rho}-1)) &= \widehat{\mathrm{u}}(\mathfrak{y}(\bar{\rho})) - \widehat{\mathrm{u}}(\mathfrak{y}(\bar{\rho}-1)) = \mathrm{u}\wedge'(\widehat{\rho})\Delta\mathfrak{y}(\bar{\rho}-1) \\ &= \frac{\Delta\mathfrak{y}(\bar{\rho}-1)}{\widehat{\Pi}(\widehat{\rho})} \le \frac{\Delta\mathfrak{y}(\bar{\rho}-1)}{\widehat{\Pi}(\mathfrak{y}(\bar{\rho}-1))}; \widehat{\rho} \in [\mathfrak{y}(\bar{\rho}-1), \mathfrak{y}(\bar{\rho})],\end{aligned} \quad (20)$$

relations (19) and (20) summarize into

$$\Delta\widehat{\mathrm{u}}(\mathfrak{y}(\bar{\rho}-1)) \le \mathfrak{j}\left(\bar{\rho}-\bar{\Psi}\right) + \mathfrak{u}\left(\bar{\rho}-\bar{\Psi}\right)\mathfrak{y}(\bar{\rho}-1), \bar{\rho} \in H_{\bar{\Psi}}, \quad (21)$$

summation (21) from $\bar{\Psi}$ to $\bar{\rho}-1$ and from (9), we obtain

$$\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \Delta\widehat{\mathrm{u}}\left(\mathfrak{y}\left(\bar{\Omega}-1\right)\right) \le \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{u}\left(\bar{\Omega}-\bar{\Psi}\right)\mathfrak{y}\left(\bar{\Omega}-1\right), \quad (22)$$

particularly

$$\widehat{\mathrm{u}}(\mathfrak{y}(\bar{\rho}-1)) \le \widehat{\mathrm{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{u}\left(\bar{\Omega}-\bar{\Psi}\right)\mathfrak{y}\left(\bar{\Omega}-1\right), \quad (23)$$

$$\mathfrak{y}(\bar{\rho}-1) \le \mathrm{u}\wedge^{-1}(\mathfrak{v}(\bar{\rho})), \quad (24)$$

where

$$\mathfrak{v}(\bar{\rho}) = \widehat{\mathrm{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{u}\left(\bar{\Omega}-\bar{\Psi}\right)\mathfrak{y}\left(\bar{\Omega}-1\right), \quad (25)$$

for $0 \le \bar{\rho} \le \bar{\rho}_1$, and we get

$$\mathfrak{v}\left(\bar{\Psi}-1\right) = \widehat{\mathrm{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}_1-1} \mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right), \quad (26)$$

and furthermore with (24) and (25), we proceed to

$$\mathfrak{v}(\bar{\rho}-1) - \mathfrak{v}(\bar{\rho}) = \mathfrak{u}\left(\bar{\rho}-\bar{\Psi}-1\right)\mathfrak{y}(\bar{\rho}-2) \le \mathfrak{u}\left(\bar{\rho}-\bar{\Psi}-1\right)\mathrm{u}\wedge^{-1}(\mathfrak{v}(\bar{\rho}-1)), \quad (27)$$

related moves from (18)-(20) with acceptable improvements to the above inequality yields

$$\frac{\Delta\mathfrak{v}(\bar{\rho}-1)}{\mathrm{u}\wedge^{-1}(\mathfrak{v}(\bar{\rho}-1))} \le \mathfrak{u}\left(\bar{\rho}-\bar{\Psi}-1\right), \quad (28)$$

sum over $[\bar{\Psi}, \bar{\rho}-1]$ in (28) and from (10), (26) with $\bar{\rho}_1$ is chosen arbitrary, and we acquire

$$\begin{aligned}\widehat{\aleph}(\mathfrak{v}(\bar{\rho}-1)) \le{}& \widehat{\aleph}\left(\widehat{\mathrm{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right)\right) \\ &+ \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{u}\left(\bar{\Omega}-\bar{\Psi}\right), \bar{\rho} \in H_{\bar{\Psi}},\end{aligned} \quad (29)$$

and thus

$$\begin{aligned}\widehat{\aleph}(\mathfrak{v}(\bar{\rho})) \le{}& \widehat{\aleph}\left(\widehat{\mathrm{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right)\right) \\ &+ \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{u}\left(\bar{\Omega}-\bar{\Psi}\right), \bar{\rho} \in H_{\bar{\Psi}-1},\end{aligned} \quad (30)$$

and the conclusion of (8) can be followed by substituting (30) in (15) and (24) for $\bar{\rho} \in H_{\bar{\Psi}-1}$. □

Remark 7. By inserting $\bar{\Psi} = 1$, $\Gamma(1) = 1$ (gamma function property), $\widehat{\Pi}(\mathscr{L}) = \mathscr{L}$, and $\mathfrak{U} = 0$ in (7), hence Theorem 6 shifts to Theorem 1 [6].

Remark 8. Theorem 6 converted into Theorem 7 by Du and Jia [24] if $\mathfrak{J} = 1$ and $\mathfrak{U} = 0$ in (7).

Theorem 9. *Under the same suppositions of $\mathscr{L}$, $\mathfrak{J}$, $\widehat{\Pi}$, $\bar{\Psi}$, $\widehat{\omega}_0$, and $\mathfrak{J}(\bar{\Omega}, \bar{\rho})$ of Theorem 6, if the inequality*

$$\mathscr{L}(\bar{\rho}) \le \widehat{\omega}_0 + \Delta_0^{-\bar{\Psi}}\left[\mathfrak{J}(\bar{\rho})\mathscr{L}^{\mathfrak{r}}\left(\bar{\rho}+\bar{\Psi}-1\right)\widehat{\Pi}\left(\mathscr{L}\left(\bar{\rho}+\bar{\Psi}-1\right)\right)\right], \bar{\rho} \in H_{\bar{\Psi}-1}, \quad (31)$$

satisfies for $\mathfrak{r} > 0$, $\mathfrak{r} \ne 1$, $\mathfrak{r}$ is a constant, then

$$\mathscr{L}(\bar{\rho}) \le \left\{\aleph\wedge_1^{-1}\left[\aleph\wedge_1\left(\omega\wedge_0^{\frac{1}{1-\mathfrak{r}}}\right) + (1-\mathfrak{r})\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right)\right]\right\}^{\frac{1}{1-\mathfrak{r}}}, \bar{\rho} \in H_{\bar{\Psi}-1}, \quad (32)$$

where

$$\widehat{\aleph}_1(\mathfrak{f}) = \int_{\mathfrak{f}_0}^{\mathfrak{f}} \frac{1}{\widehat{\Pi}\left(\mathfrak{q}^{1/\mathfrak{r}}\right)} d\mathfrak{q}, \mathfrak{f} \ge \mathfrak{f}_0 > 0, \lim_{\mathfrak{f}\longrightarrow\infty} \widehat{\aleph}_1(\mathfrak{f}) = \infty. \quad (33)$$

$\widehat{\aleph}_1^{-1}$ is the inverse of $\widehat{\aleph}_1$, so that

$$\widehat{\aleph}_1(\widehat{\omega}_0) + (1-\mathfrak{r})\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right) \in Dom\left(\widehat{\aleph}_1^{-1}\right). \quad (34)$$

Proof. Infer $\widehat{\omega}_0 > 0$ and denoting

$$\mathfrak{p}_1(\bar{\rho}) = \widehat{\omega}_0 + \Delta_0^{-\bar{\Psi}}\left[\mathfrak{J}(\bar{\rho})\mathscr{L}^{\mathfrak{r}}(\bar{\rho}+\bar{\Psi}-1)\widehat{\Pi}\left(\mathscr{L}(\bar{\rho}+\bar{\Psi}-1)\right)\right], \tag{35}$$

therefore, one has

$$\mathscr{L}(\bar{\rho}) \le \mathfrak{p}_1(\bar{\rho}), \tag{36}$$

and employing Definition 2 to (35), we deduce

$$\begin{aligned}\mathfrak{p}_1(\bar{\rho}) = \widehat{\omega}_0 + \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}}\left(\bar{\rho}-\bar{\Omega}-1\right)^{\bar{\Psi}-1}\mathfrak{J}(\bar{\Omega})\mathscr{L}^{\mathfrak{r}} \\ \cdot\left(\bar{\Omega}+\bar{\Psi}-1\right)\widehat{\Pi}\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right), \bar{\rho}\in H_{\bar{\Psi}-1},\end{aligned} \tag{37}$$

where $J(\bar{\Omega},\bar{\rho})$ is defined in (11), and $\mathfrak{p}_1(\bar{\rho}) \ge 0$ is nondecreasing. With the assistance of direct calculation for $\bar{\rho}\in H_{\bar{\Psi}}$, decreasing property of $\mathfrak{J}(\bar{\Omega},\bar{\rho})$ for $\bar{\Omega}\in\mathbb{N}_0$, the definition of $\mathfrak{J}(\bar{\Omega},\bar{\rho})$, $(\bar{\rho})^{\bar{\Psi}}$, (35), and (36), we conclude

$$\begin{aligned}\mathfrak{p}_1(\bar{\rho}) - \mathfrak{p}_1(\bar{\rho}-1) &= \mathfrak{J}\left(\bar{\rho}-\bar{\Psi},\bar{\rho}\right)\mathscr{L}^{\mathfrak{r}}(\bar{\rho}-1)\widehat{\Pi}(\mathscr{L}(\bar{\rho}-1)) \\ &\quad + \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}-1}\left[\mathfrak{J}\left(\bar{\Omega},\bar{\rho}\right)-\mathfrak{J}\left(\bar{\Omega},\bar{\rho}-1\right)\right]\mathscr{L}^{\mathfrak{r}} \\ &\quad \cdot\left(\bar{\Omega}+\bar{\Psi}-1\right)\widehat{\Pi}\left(\mathscr{L}\left(\bar{\Omega}+\bar{\Psi}-1\right)\right) \\ &\le \mathfrak{j}\left(\bar{\rho}-\bar{\Psi}\right)\mathfrak{p}_1^{\mathfrak{r}}(\bar{\rho}-1)\widehat{\Pi}(\mathfrak{p}_1(\bar{\rho}-1)).\end{aligned} \tag{38}$$

By mean value axiom, we accomplish

$$\frac{1}{1-\mathfrak{r}}\left[\mathfrak{p}_1^{1-\mathfrak{r}}(\bar{\rho}) - \mathfrak{p}_1^{1-\mathfrak{r}}(\bar{\rho}-1)\right] = \frac{\mathfrak{p}_1^{1-\mathfrak{r}}(\bar{\rho}) - \mathfrak{p}_1^{1-\mathfrak{r}}(\bar{\rho}-1)}{[\rho\wedge(\bar{\rho})]^{\mathfrak{r}}}, \tag{39}$$

for some $\widehat{\rho}(\bar{\rho})\in[\mathfrak{p}_1(\bar{\rho}-1),\mathfrak{p}_1(\bar{\rho})]$. Therefore,

$$\begin{aligned}\mathfrak{p}_1^{1-\mathfrak{r}}(\bar{\rho}) - \mathfrak{p}_1^{1-\mathfrak{r}}(\bar{\rho}-1) &\le (1-\mathfrak{r})\left[\frac{\mathfrak{p}_1(\bar{\rho})-\mathfrak{p}_1(\bar{\rho}-1)}{[\mathfrak{p}_1(\bar{\rho}-1)]^{\mathfrak{r}}}\right] \\ &\le (1-\mathfrak{r})\mathfrak{j}\left(\bar{\rho}-\bar{\Psi}\right)\widehat{\Pi}(\mathfrak{p}_1(\bar{\rho}-1)), \bar{\rho}\in H_{\bar{\Psi}},\end{aligned} \tag{40}$$

summing prior inequality from $\bar{\Psi}$ to $\bar{\rho}-1$ and taking into account $\mathfrak{p}_1(\bar{\Psi}-1) = \widehat{\omega}_0$ implies

$$\mathfrak{p}_1^{1-\mathfrak{r}}(\bar{\rho}-1) \le \bar{\omega}_0^{1-\mathfrak{r}} + (1-\mathfrak{r})\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1}\mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right)\widehat{\Pi}\left(\mathfrak{p}_1\left(\bar{\Omega}-1\right)\right). \tag{41}$$

Let

$$\mathfrak{p}_2(\bar{\rho}) = \bar{\omega}_0^{1-\mathfrak{r}} + (1-\mathfrak{r})\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1}\mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right)\widehat{\Pi}\left(\mathfrak{p}_1\left(\bar{\Omega}-1\right)\right), \tag{42}$$

from (41) and (42), and we have

$$\mathfrak{p}_1^{1-\mathfrak{r}}(\bar{\rho}-1) \le \mathfrak{p}_2(\bar{\rho}) \Rightarrow \mathfrak{p}_1(\bar{\rho}-1) \le \mathfrak{p}_2^{\frac{1}{1-\mathfrak{r}}}(\bar{\rho}), \tag{43}$$

in addition from (42); we see

$$\begin{aligned}\mathfrak{p}_2(\bar{\rho}) - \mathfrak{p}_2(\bar{\rho}-1) &= (1-\mathfrak{r})\mathfrak{j}\left(\bar{\rho}-\bar{\Psi}-1\right)\widehat{\Pi}(\mathfrak{p}_1(\bar{\rho}-2)) \\ &\le (1-\mathfrak{r})\mathfrak{j}\left(\bar{\rho}-\bar{\Psi}-1\right)\widehat{\Pi}\left(\mathfrak{p}_2^{\frac{1}{1-\mathfrak{r}}}(\bar{\rho}-1)\right),\end{aligned} \tag{44}$$

monotonicity of $\widehat{\Pi}$, and $\mathfrak{p}_2$ gives

$$\widehat{\Pi}\left(\mathfrak{p}_2^{\frac{1}{1-\mathfrak{r}}}(\bar{\rho}-1)\right) > \widehat{\Pi}\left(\mathfrak{p}_2^{\frac{1}{1-\mathfrak{r}}}(\bar{\Psi}-1)\right) = \widehat{\Pi}\left(\widehat{\omega}_0^{\frac{1}{1-\mathfrak{r}}}\right) > 0, \bar{\rho}\in H_{\bar{\Psi}}, \tag{45}$$

equation (44) with inequality (45) becomes

$$\frac{\Delta\mathfrak{p}_2(\bar{\rho}-1)}{\widehat{\Pi}\left(\mathfrak{p}_2^{1/1-\mathfrak{r}}(\bar{\rho}-1)\right)} \le (1-\mathfrak{r})\mathfrak{j}\left(\bar{\rho}-\bar{\Psi}-1\right), \bar{\rho}\in H_{\bar{\Psi}}. \tag{46}$$

analysis of mean value theorem approaches to

$$\begin{aligned}\Delta\widehat{\aleph}_1(\mathfrak{p}_2(\bar{\rho}-1)) &= \widehat{\aleph}_1(\mathfrak{p}_2(\bar{\rho})) - \widehat{\aleph}_1(\mathfrak{p}_2(\bar{\rho}-1)) \\ &= \widehat{\aleph}_1'(\widehat{\rho})\Delta\mathfrak{p}_2(\bar{\rho}-1) = \frac{\Delta\mathfrak{p}_2(\bar{\rho}-1)}{\widehat{\Pi}(\rho\wedge)^{1/1-\mathfrak{r}}} \\ &\le \frac{\Delta\mathfrak{p}_2(\bar{\rho}-1)}{\widehat{\Pi}\left(\mathfrak{p}_2^{1/1-\mathfrak{r}}(\bar{\rho}-1)\right)}\ ; \widehat{\rho}\in[\mathfrak{p}_2(\bar{\rho}-1),\mathfrak{p}_2(\bar{\rho})],\end{aligned} \tag{47}$$

inequalities (46) and (47) that offer

$$\Delta\widehat{\aleph}_1(\mathfrak{p}_2(\bar{\rho}-1)) \le (1-\mathfrak{r})\mathfrak{j}\left(\bar{\rho}-\bar{\Psi}-1\right), \bar{\rho}\in H_{\bar{\Psi}}, \tag{48}$$

inequality (48) by summing from $\bar{\Psi}$ to $\bar{\rho}-1$ and utilizing $\mathfrak{p}_2(\bar{\Psi}-1) = \bar{\omega}_0^{1/1-\mathfrak{r}}$ equals to

$$\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1}\Delta\widehat{\aleph}_1\left(\mathfrak{p}_2\left(\bar{\Omega}-1\right)\right) \le (1-\mathfrak{r})\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1}\mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right), \tag{49}$$

that is,

$$\widehat{\aleph}_1(\mathfrak{p}_2(\bar{\rho}-1)) \le \widehat{\aleph}_1\left(\widehat{\omega}_0^{\frac{1}{1-\mathfrak{r}}}\right) + (1-\mathfrak{r})\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1}\mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right), \bar{\rho}\in H_{\bar{\Psi}}, \tag{50}$$

$$\mathfrak{p}_2(\bar{\rho}) \le \widehat{\aleph}_1^{-1}\left[\widehat{\aleph}_1\left(\widehat{\omega}_0^{\frac{1}{1-\mathfrak{r}}}\right) + (1-\mathfrak{r})\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}}\mathfrak{j}\left(\bar{\Omega}-\bar{\Psi}\right)\right], \bar{\rho}\in H_{\bar{\Psi}-1}, \tag{51}$$

substitute the resulting inequality in (43) and (36) to get the acquired bound in (32) with $\bar{\rho}\in H_{\bar{\Psi}-1}$. □

Remark 10. By taking Theorem 9 alters into [28], Lemma 5 (β_1), by letting $\mathfrak{r} = 1$, $\bar{\Psi} = 1$, $\Gamma(1) = 1$, $\widehat{\omega}_0 = \widehat{\omega}(\bar{\rho})$, and $\widehat{\Pi}(\mathscr{L}(\bar{\rho})) = 1$.

Remark 11. Theorem 9 changes to Theorem 7 by taking $\mathfrak{r} = 0$ and $\mathfrak{J} = 1$ due to Du and Jia [24].

4. Boundedness and Uniqueness

This segment is related to a valid procedure of Theorem 6 to determine boundedness and uniqueness of discrete fractional inequalities. Consider the following pattern of fractional difference equation:

$$\mathscr{L}(\bar{\rho}) = \widehat{\omega}_0 + \Delta_0^{-\bar{\Psi}}\left[\mathfrak{S}\left(\bar{\rho}, \mathscr{L}\left(\bar{\rho} + \bar{\Psi} - 1\right)\right)\right], \bar{\rho} \in H_{\bar{\Psi}-1}, \quad (52)$$

where $\mathfrak{S} : \mathbb{N}_0 \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\bar{\rho}$, $\bar{\Psi}$, $\widehat{\omega}_0$, $\mathscr{L}$, and $\widehat{\Pi}$ be the same as in Theorem 6.

The accompanying example can describe the boundedness on the solutions of (52).

Example 12. Suppose that

$$|\mathfrak{S}(\bar{\rho}, \mathscr{L})| \leq \mathfrak{J}(\bar{\rho})\left|\widehat{\Pi}(\mathscr{L})\right| + \mathfrak{U}(\bar{\rho})|\mathscr{L}|\left|\widehat{\Pi}(\mathscr{L})\right|, \quad (53)$$

for $\bar{\rho} \in \mathbb{N}_0$, $\mathscr{L} \in \mathbb{R}$. If $\mathscr{L}(\bar{\rho})$ is a solution of (52), then

$$\begin{aligned}|\mathscr{L}(\bar{\rho})| \leq \mathrm{u}\wedge^{-1}\Bigg[\aleph\wedge^{-1}\Bigg\{\widehat{\aleph}\Bigg(\widehat{\mathrm{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}(\bar{\Omega} - \bar{\Psi})\Bigg) \\ + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{u}(\bar{\Omega} - \bar{\Psi})\Bigg\}\Bigg], \bar{\rho} \in H_{\bar{\Psi}-1},\end{aligned} \quad (54)$$

Proof. Equation (52) with the blend of Definition 2 is encoded into

$$\mathscr{L}(\bar{\rho}) = \widehat{\omega}_0 + \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{S}(\bar{\Omega}, \mathscr{L}(\bar{\Omega} + \bar{\Psi} - 1)). \quad (55)$$

Evidently, equation (55) with the utilization of (53) takes the form

$$\begin{aligned}|\mathscr{L}(\bar{\rho})| &= \left|\widehat{\omega}_0 + \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{S}(\bar{\Omega}, \mathscr{L}(\bar{\Omega} + \bar{\Psi} - 1))\right| \\ &\leq |\widehat{\omega}_0| + \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \times \left[\mathfrak{J}(\bar{\Omega})\left|\widehat{\Pi}(\mathscr{L}(\bar{\Omega} + \bar{\Psi} - 1))\right| \right. \\ &\quad \left. + \mathfrak{U}(\bar{\Omega})|\mathscr{L}(\bar{\Omega} + \bar{\Psi} - 1)|\left|\widehat{\Pi}(\mathscr{L}(\bar{\Omega} + \bar{\Psi} - 1))\right|\right] \\ &\leq |\widehat{\omega}_0| + \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{J}(\bar{\Omega})\left|\widehat{\Pi}(\mathscr{L}(\bar{\Omega} + \bar{\Psi} - 1))\right| \\ &\quad + \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{U}(\bar{\Omega})|\mathscr{L}(\bar{\Omega} + \bar{\Psi} - 1)|\left|\widehat{\Pi}\right. \\ &\quad \left. \cdot (\mathscr{L}(\bar{\Omega} + \bar{\Psi} - 1))\right|.\end{aligned} \quad (56)$$

The rest of the calculations can be performed by assuming the right composition of Theorem 6 in order to gather the necessary inequality (54). □

The uniqueness of (52) solutions can be defined from an illustration below.

Example 13. Let

$$\begin{aligned}|\mathfrak{S}(\bar{\rho}, \mathscr{L}_1) - \mathfrak{S}(\bar{\rho}, \mathscr{L}_2)| &\leq \mathfrak{J}(\bar{\rho})\left|\widehat{\Pi}(\mathscr{L}_1) - \widehat{\Pi}(\mathscr{L}_2)\right| \\ &\quad + \mathfrak{U}(\bar{\rho})|\mathscr{L}_1 - \mathscr{L}_2|\left|\widehat{\Pi}(\mathscr{L}_1) - \widehat{\Pi}(\mathscr{L}_2)\right|,\end{aligned} \quad (57)$$

and then (52) has at most one solution.

Proof. Equation (52) with solutions $\mathscr{L}_1(\bar{\rho})$ and $\mathscr{L}_2(\bar{\rho})$ can be represented as

$$\begin{aligned}\mathscr{L}_1(\bar{\rho}) - \mathscr{L}_2(\bar{\rho}) &= \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{S}(\bar{\Omega}, \mathscr{L}_1(\bar{\Omega} + \bar{\Psi} - 1)) \\ &\quad - \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{S}(\bar{\Omega}, \mathscr{L}_2(\bar{\Omega} + \bar{\Psi} - 1)).\end{aligned} \quad (58)$$

Assertion (57) with the prior inequality generates

$$\begin{aligned}|\mathscr{L}_1(\bar{\rho}) - \mathscr{L}_2(\bar{\rho})| &= \left|\frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{S}(\bar{\Omega}, \mathscr{L}_1(\bar{\Omega} + \bar{\Psi} - 1)) \right. \\ &\quad \left. - \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{S}(\bar{\Omega}, \mathscr{L}_2(\bar{\Omega} + \bar{\Psi} - 1))\right| \\ &\leq \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{J}(\bar{\Omega})\left|\widehat{\Pi}(\mathscr{L}_1(\bar{\Omega} + \bar{\Psi} - 1)) \right. \\ &\quad \left. - \widehat{\Pi}(\mathscr{L}_2(\bar{\Omega} + \bar{\Psi} - 1))\right| \\ &\quad + \frac{1}{\Gamma(\bar{\Psi})}\sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\overline{\bar{\Psi}-1}} \mathfrak{U}(\bar{\Omega})|\mathscr{L}_1(\bar{\Omega} + \bar{\Psi} - 1) \\ &\quad - \mathscr{L}_2(\bar{\Omega} + \bar{\Psi} - 1)| \times \left|\widehat{\Pi}(\mathscr{L}_1(\bar{\Omega} + \bar{\Psi} - 1)) \right. \\ &\quad \left. - \widehat{\Pi}(\mathscr{L}_2(\bar{\Omega} + \bar{\Psi} - 1))\right|.\end{aligned} \quad (59)$$

The previous inequality by having a few amendments to $|\mathscr{L}_1(\bar{\rho}) - \mathscr{L}_2(\bar{\rho})|$ in the process of Theorem 6 introduces

$$|\mathscr{L}_1(\bar{\rho}) - \mathscr{L}_2(\bar{\rho})| \leq 0. \quad (60)$$

Subsequently, $\mathscr{L}_1(\bar{\rho}) = \mathscr{L}_2(\bar{\rho})$ and at least one solution of fractional difference equation (52) exist. □

5. Concluding Remarks

Discrete fractional calculus has made great progress of real-world phenomena, like fractional chaotic maps, image coding, and more discrete time modeling. One of the preeminent crucial issues in investigation of difference equations is to explore the subjective attributes of solutions of these previously mentioned fields. Discrete fractional variants are notable pathways that speed disabling. In this article, fixed on the framework of discrete fractional analytics and with the aid of fractional sum inequalities, we proposed new kinds of discrete fractional Gronwall inequalities. We also extracted the expansion of the decreasing feature sequences in the time-scale domain frame. Such inequalities can be shown not only to recall explicit estimates for solutions of fractional difference equations of a discrete form but also to the uniqueness and continuous dependency on initial value of the solutions in the literature.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there are no competing interests.

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Research Article

Approximation of Fixed Points for Mean Nonexpansive Mappings in Banach Spaces

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In this paper, we establish weak and strong convergence theorems for mean nonexpansive maps in Banach spaces under the Picard–Mann hybrid iteration process. We also construct an example of mean nonexpansive mappings and show that it exceeds the class of nonexpansive mappings. To show the numerical accuracy of our main outcome, we show that Picard–Mann hybrid iteration process of this example is more effective than all of the Picard, Mann, and Ishikawa iterative processes.

1. Introduction and Preliminaries

Suppose $\mathscr{Y}$ is a Banach space and $\emptyset \neq \mathscr{W} \subseteq \mathscr{Y}$. Consider a selfmap $S : \mathscr{W} \longrightarrow \mathscr{W}$. If an element $e_0 \in \mathscr{W}$ exists such that $e_0 = Se_0$, then we say that e_0 is a fixed point for S. In this manuscript, we essentially represent the set $\{e_0 \in \mathscr{W} : e_0 = Se_0\}$ by F_S. The selfmap $S : \mathscr{W} \longrightarrow \mathscr{W}$ is called contraction [1] if

$$\|Sp - Sq\| \leq \alpha \|p - q\|, \text{ forall } p, q \in \mathscr{W} \text{ and } \alpha \in [0, 1). \tag{1}$$

The selfmap $S : \mathscr{W} \longrightarrow \mathscr{W}$ is called nonexpansive if (1) holds for the value $\alpha = 1$. In 1965, Browder [2] and Gohde [3] proved a fixed point theorem for a nonexpansive map $S : \mathscr{W} \longrightarrow \mathscr{W}$ under the restriction that $\mathscr{Y}$ is a uniformly convex Banach space (UCBS) and $\emptyset \neq \mathscr{W} \subseteq \mathscr{Y}$ is bounded as well as closed and convex.

In [4], Zhang provided the following class of mappings.

Definition 1. Let $\mathscr{W} \neq \emptyset$ be a subset of a Banach space. A selfmap $S : \mathscr{W} \longrightarrow \mathscr{W}$ is called mean nonexpansive if for all p, $q \in \mathscr{W}$ there are non-negative real numbers a, b such that $a + b \leq 1$, we have

$$\|Sp - Sq\| \leq a\|p - q\| + b\|p - Sq\|. \tag{2}$$

Zhang [4] provided an existence of fixed point result for mean nonexpansive mappings in Banach space setting under the normal structure assumption. After this, Wu and Zhang [5] and Zuo [6] investigated some other elementary properties and fixed point results for these mappings. In [7], Zhou and Cui used Ishikawa [8] iteration for approximating fixed points for these maps. The main aim here is to suggest some weak and strong convergence theorems for these mappings under the Picard-Mann hybrid [9] iteration and to show by a new example of mean nonexpansive maps that it converges better than the Ishikawa [8] and Mann [10] iteration processes.

Remark 2. It is easy to observe that each nonexpansive mapping is mean nonexpansive. Once again, in this research, we shall provide a new example to show that the converse is not true in general, that is, the class of mean nonexpansive maps properly includes the class of nonexpansive maps.

In the following example, S is mean nonexpansive but not nonexpansive.

Example 3 (see [6]). Consider $\mathscr{W} = [0, 1]$ and set $S : \mathscr{W} \longrightarrow \mathscr{W}$ by

$$Sp = \begin{cases} \dfrac{p}{5} & \text{if} \quad p < \dfrac{1}{2} \\ \dfrac{p}{6} & \text{if} \quad p \geq \dfrac{1}{2}. \end{cases} \tag{3}$$

The Banach [1] celebrated fixed point theorem suggests the existence and uniqueness of a fixed point for a self contraction $S : \mathscr{W} \longrightarrow \mathscr{W}$ under the restriction that $\mathscr{Y}$ is complete metric space and $\emptyset \neq \mathscr{W} \subseteq \mathscr{Y}$ is closed. Also this theorem essentially uses the Picard iteration [11] for finding this unique fixed point. Nevertheless, in the case of nonexpansive maps and hence for generalized nonexpansive maps, the Picard iteration fails to converge in the associated fixed point set. For some more literature on iterative schemes, please cite the work in [12–14]. Assume that $\mathscr{W}$ is any nonempty subset of a Banach space and $S : \mathscr{W} \longrightarrow \mathscr{W}$.

The Picard [11] iterative process is stated as:

$$\begin{cases} p_1 \in \mathscr{W}, \\ p_{m+1} = Sp_m, m \geq 1, \end{cases} \tag{4}$$

Mann [10] iterative process is stated as:

$$\begin{cases} p_1 \in \mathscr{W}, \\ p_{m+1} = (1 - \alpha_m)p_m + \alpha_m Sp_m, m \geq 1, \end{cases} \tag{5}$$

where $\alpha_m \in (0, 1)$.

Ishikawa [8] iteration process may be viewed as a two-step Mann iteration, stated as follows:

$$\begin{cases} p_1 \in \mathscr{W}, \\ q_m = (1 - \beta_m)p_m + \beta_m Sp_m, \\ p_{m+1} = (1 - \alpha_m)p_m + \alpha_m Sq_m, m \geq 1, \end{cases} \tag{6}$$

where $\alpha_m, \beta_m \in (0, 1)$.

Khan [9] introduced the Picard–Mann hybrid iteration as follows:

$$\begin{cases} p_1 \in \mathscr{W}, \\ q_m = (1 - \alpha_m)p_m + \alpha_m Sp_m, \\ p_{m+1} = Sq_m, m \geq 1, \end{cases} \tag{7}$$

where $\alpha_m \in (0, 1)$.

Khan [9] provided the weak and strong convergence of the scheme (7) for the class of nonexpansive operators. Furthermore, he proved that the Picard–Mann hybrid iteration process is more effective than the Picard (4), Mann (5) and Ishikawa (6) iteration processes in the setting of nonexpansive maps. In this paper, we connect this scheme with the class of mean nonexpansive mappings, and in this way, we extend his results in the more general setting of mean nonexpansive mappings.

Now we provide some elementary definitions and results, which will be used in sequel.

Definition 4 [15]. A Banach space $\mathscr{Y}$ is called UCBS if and only if for every choice of $r \in (0, 2]$, one has a $s > 0$ such that for every p, q in $\mathscr{Y}$, $1/2\|p + q\| \leq (1 - s)$ whenever $\|p\| \leq 1$, $\|q\| \leq 1$ and $\|p - q\| > r$.

Definition 5 [16]. A Banach space $\mathscr{Y}$ is said to satisfy the Opial's property if any weakly convergent sequence $\{p_m\}$ in $\mathscr{Y}$ which admits a weak limit $c \in \mathscr{Y}$, one has

$$\limsup_{m \longrightarrow \infty} \|p_m - c\| < \limsup_{m \longrightarrow \infty} \|p_m - c'\|, \text{ foreach } c' \in \mathscr{Y} - \{c\}. \tag{8}$$

Definition 6 [17]. Suppose $\mathscr{W} \neq \emptyset$ is any subset of a Banach space $\mathscr{Y}$. A selfmap $S : \mathscr{W} \longrightarrow \mathscr{W}$ is said to be endowed with the condition (I) if and only if a function $L : [0,\infty) \longrightarrow [0,\infty)$ exists sucht that $L(0) = 0$ and $L(r) > 0$ for all $r > 0$ and $\|p - Sp\| \geq L(d(p, F_S))$ for each $p \in \mathscr{W}$.

The following lemma holds, which suggests many examples of mean nonexpansive mappings.

Lemma 7. *If S is a selfmap and nonexpansive on a subset $\mathscr{W} \neq \emptyset$ of a Banach space. Then S is mean nonexpansive.*

From the definition of mean nonexpansive maps, we have the following facts.

Lemma 8. *If S is a selfmap and mean nonexpansive on a subset $\mathscr{W} \neq \emptyset$ of a Banach space $\mathscr{Y}$. Then F_S is closed. Moreover, if $\mathscr{Y}$ is strictly convex and $\mathscr{W}$ is convex, then F_S is also convex.*

Theorem 9 [6]. *Let $\mathscr{W} \neq \emptyset$ be a subset of a reflexive Banach space (RBS) $\mathscr{Y}$ having Opial property. Let $S : \mathscr{W} \longrightarrow \mathscr{W}$ be a mean nonexpansive mapping. If $\{p_m\} \subseteq \mathscr{W}$ be such that.*

(a_0) $\{p_m\}$ converges weakly to e_0,

(b_0) $\lim_{m \longrightarrow \infty} \|Sp_m - p_m\| = 0$,

then $e_0 = Se_0$.

Any UCBS can be characterized by the following way.

Lemma 10 [18]. *If $\mathscr{Y}$ is a UCBS and If $\{s_m\}$ and $\{w_m\}$ are two sequences in $\mathscr{Y}$ such that $\limsup_{m \longrightarrow \infty} \|s_m\| \leq l$, $\limsup_{m \longrightarrow \infty} \|w_m\| \leq l$ and $\lim_{m \longrightarrow \infty} \|\delta_m s_m + (1 - \delta_m)w_m\| = l$ for some $l \geq 0$ and $0 < q \leq \delta_m \leq p < 1$. Then, $\lim_{m \longrightarrow \infty} \|s_m - w_m\| = 0$.*

2. Main Results

The following results are the main outcome of this section. Notice that all these results improve and extend some main results of Khan [9] from the case of nonexpansive maps to case of mean nonexpansive maps.

Lemma 11. *Let $\mathscr{W} \neq \emptyset$ be a convex closed subset of a UCBS $\mathscr{Y}$ and $S : \mathscr{W} \longrightarrow \mathscr{W}$ be a mean nonexpansive mapping and $F_S \neq \emptyset$. Assume that $\{p_m\}$ is a sequence of Picard–Mann hybrid* *iterative process (7). Consequently* $\lim_{m\longrightarrow\infty}\|p_m - e_0\|$ *exists for each* $e_0 \in F_S$.

Proof. Let $e_0 \in F_S$. Then using (7), we have

$$\begin{aligned}\|q_m - e_0\| &\le (1-\alpha_m)\|p_m - e_0\| + \alpha_m\|p_m - e_0\| \\ &\le (1-\alpha_m)\|p_m - e_0\| + \alpha_m\|Sp_m - e_0\| \\ &= (1-\alpha_m)\|p_m - e_0\| + \alpha_m\|Sp_m - Se_0\| \\ &\le (1-\alpha_m)\|p_m - e_0\| + \alpha_m(a\|p_m - e_0\| + b\|p_m - Se_0\|) \\ &= (1-\alpha_m)\|p_m - e_0\| + \alpha_m(a\|p_m - e_0\| + b\|p_m - e_0\|) \\ &= (1-\alpha_m)\|p_m - e_0\| + \alpha_m((a+b)\|p_m - e_0\|) \\ &= (1-\alpha_m)\|p_m - e_0\| + \alpha_m\|p_m - e_0\| \\ &\le \|p_m - e_0\|. \end{aligned} \tag{9}$$

This implies that

$$\begin{aligned}\|p_{m+1} - e_0\| &= \|Sq_m - e_0\| \\ &= \|Sq_m - Se_0\| \\ &\le a\|q_m - e_0\| + b\|q_m - Se_0\| \\ &= a\|q_m - e_0\| + b\|q_m - e_0\| \\ &= (a+b)\|q_m - e_0\| \\ &\le \|q_m - e_0\| \\ &\le \|p_m - e_0\|. \end{aligned} \tag{10}$$

We have showed that $\|p_{m+1} - e_0\| \le \|p_m - e_0\|$. It follows that $\{\|p_m - e_0\|\}$ is nonincreasing and bounded. Thus $\lim_{m\longrightarrow\infty}\|p_m - e_0\|$ exists for each $e_0 \in F_S$. □

Theorem 12. *Let* $\mathscr{W} \ne \emptyset$ *be a convex closed subset of a UCBS* $\mathscr{Y}$ *and* $S : \mathscr{W} \longrightarrow \mathscr{W}$ *be a mean nonexpansive mapping and* $F_S \ne \emptyset$. *Assume that* $\{p_m\}$ *is a sequence of Picard–Mann hybrid iterative process (7). Consequently,* $\{p_m\}$ *is bounded in* $\mathscr{Y}$ *with the property* $\lim_{m\longrightarrow\infty}\|Sp_m - p_m\| = 0$.

Proof. Since the set F_S is nonempty so we may choose any $e_0 \in F_S$. By Lemma 11, $\lim_{m\longrightarrow\infty}\|p_m - e_0\|$ exists and $\{p_m\}$ is bounded. Suppose that

$$\lim_{m\longrightarrow\infty}\|p_m - e_0\| = s. \tag{11}$$

By looking in the proof of Lemma 11, one see

$$\|q_m - e_0\| \le \|p_m - e_0\|, \Rightarrow \limsup_{m\longrightarrow\infty}\|q_m - e_0\| \le \limsup_{m\longrightarrow\infty}\|p_m - e_0\| = s. \tag{12}$$

Now

$$\begin{aligned}\|Sp_m - e_0\| &= \|Sp_m - Se_0\| \\ &\le a\|p_m - e_0\| + b\|p_m - Se_0\| \\ &= (a+b)\|p_m - e_0\| \\ &\le \|p_m - e_0\|. \end{aligned} \tag{13}$$

It follows that

$$\limsup_{m\longrightarrow\infty}\|Sp_m - e_0\| \le \limsup_{m\longrightarrow\infty}\|p_m - e_0\| = s. \tag{14}$$

Again by looking in the proof of Lemma 11, one see

$$\|p_{m+1} - q\| \le \|q_m - q\|. \tag{15}$$

It follows that

$$s \le \liminf_{m\longrightarrow\infty}\|q_m - e_0\|. \tag{16}$$

From (12) and (16), we obtain

$$s = \lim_{m\longrightarrow\infty}\|q_m - e_0\|. \tag{17}$$

From (17), we have

$$\begin{aligned} s &= \lim_{m\longrightarrow\infty}\|q_m - q\| \\ &= \lim_{m\longrightarrow\infty}\|(1-\alpha_m)p_m + \alpha_m Sp_m - e_0\| \\ &= \lim_{m\longrightarrow\infty}\|(1-\alpha_m)(p_m - e_0) + \alpha_m(Sp_m - e_0)\|. \end{aligned} \tag{18}$$

Hence,

$$s = \lim_{m\longrightarrow\infty}\|(1-\alpha_m)(p_m - e_0) + \alpha_m(Sp_m - e_0)\|. \tag{19}$$

Now from (11), (14) and (19) together with Lemma 10, we obtain

$$\lim_{m\longrightarrow\infty}\|Sp_m - p_m\| = 0. \tag{20}$$

□

We now provide a weak convergence theorem under the assumption of the Opial's condition.

Theorem 13. *Let* $\mathscr{W} \ne \emptyset$ *be a convex closed subset of a UCBS* $\mathscr{Y}$ *and* $S : \mathscr{W} \longrightarrow \mathscr{W}$ *be a mean nonexpansive mapping and* $F_S \ne \emptyset$. *Assume that* $\{p_m\}$ *is a sequence of Picard–Mann hybrid iterative process (7). If* $\mathscr{Y}$ *has the Opial property, then* $\{p_m\}$ *converges weakly to a point of of* F_S.

Proof. By Theorem 12, the sequence $\{p_m\}$ is bounded and $\lim_{n\longrightarrow\infty}\|Sp_m - p_m\| = 0$. Since $\mathscr{Y}$ is UCBS, it follows that $\mathscr{Y}$ is RBS. Thus one has a weakly convergent subsequence $\{p_{m_i}\}$ of $\{p_m\}$ exists with some weak limit $q_1 \in \mathscr{W}$. By Theorem 9, $q_1 \in F_S$. Next we show that $\{p_m\}$ is weakly convergent to q_1. We may suppose that $\{p_m\}$ is not weakly convergent to q_1, that is, one has a weakly convergent subsequence $\{p_{m_j}\}$ of $\{p_m\}$ with a weak limit $q_2 \in \mathscr{W}$ and $q_2 \ne q_1$. Again applying Theorem 9, $q_2 \in F_S$. By applying Opial's condition and keeping Lemma 11 in mind, it follow that

$$\begin{aligned}\lim_{m\longrightarrow\infty}\|p_m-q_1\| &= \lim_{i\longrightarrow\infty}\left\|p_{m_i}-q_1\right\| \\ &< \lim_{i\longrightarrow\infty}\left\|p_{m_i}-q_2\right\| \\ &= \lim_{m\longrightarrow\infty}\|p_m-q_2\| \\ &= \lim_{j\longrightarrow\infty}\left\|p_{m_j}-q_2\right\| \\ &< \lim_{j\longrightarrow\infty}\left\|p_{m_j}-q_1\right\| \\ &= \lim_{m\longrightarrow\infty}\|p_m-q_1\|.\end{aligned} \tag{21}$$

Hence we have seen a contradiction. Accordingly, we have $q_1 = q_2$. Thus, $\{p_m\}$ converges weakly to $q_1 \in F_S$. □.

The strong convergence theorem under the assumption of compactness is established as follows.

Theorem 14. *Let $\mathscr{W} \neq \emptyset$ be a convex compact subset of a UCBS $\mathscr{Y}$ and $S : \mathscr{W} \longrightarrow \mathscr{W}$ be a mean nonexpansive mapping and $F_S \neq \emptyset$. Assume that $\{p_m\}$ is a sequence of Picard–Mann hybrid iterative process (7). If $\mathscr{W}$ is compact, then $\{p_m\}$ converges strongly to an element of F_S.*

Proof. Since $\mathscr{W}$ is compact, and $\{p_m\} \subseteq \mathscr{W}$. One can choose a strongly convergent subsequence $\{p_{m_k}\}$ of $\{p_m\}$ such that $p_{m_k} \longrightarrow u$. Now we show that $Su = u$. For this

$$\begin{aligned}\|u-Su\| &\leq \left\|u-p_{m_k}\right\| + \left\|p_{m_k}-Sp_{m_k}\right\| + \left\|Sp_{m_k}-Su\right\| \\ &\leq \left\|u-p_{m_k}\right\| + \left\|p_{m_k}-Sp_{m_k}\right\| \\ &\quad + \left(a\left\|p_{m_k}-u\right\| + b\left\|u-Sp_{m_k}\right\|\right) \\ &\leq \left\|u-p_{m_k}\right\| + \left\|p_{m_k}-Sp_{m_k}\right\| \\ &\quad + \left(a\left\|p_{m_k}-u\right\| + b\left\|u-p_{m_k}\right\| + b\left\|p_{m_k}-Sp_{m_k}\right\|\right) \\ &= (a+b+1)\left\|u-p_{m_k}\right\| + (b+1)\left\|p_{m_k}-Sp_{m_k}\right\|.\end{aligned} \tag{22}$$

□

Consequently, we obtained

$$\|u-Su\| \leq (a+b+1)\left\|u-p_{m_k}\right\| + (b+1)\left\|p_{m_k}-Sp_{m_k}\right\|. \tag{23}$$

According to Theorem 12, we have $\lim_{k\longrightarrow\infty}\|p_{m_k} - Sp_{m_k}\| = 0$, so applying $k \longrightarrow \infty$, we obtain $Su = u$. This shows that $u \in F_S$. By Lemma 11, $\lim_{m\longrightarrow\infty}\|p_m - u\|$ exists. Consequently, u is the strong limit of $\{p_m\}$ and element of F_S.

The strong convergence theorem without the compactness assumption is established as follows.

Theorem 15. *Let $\mathscr{W} \neq \emptyset$ be a convex closed subset of a UCBS $\mathscr{Y}$ and $S : \mathscr{W} \longrightarrow \mathscr{W}$ be a mean nonexpansive mapping and $F_S \neq \emptyset$. Assume that $\{p_m\}$ is a sequence of Picard–Mann hybrid iterative process (7). Then $\{p_m\}$ converges strongly to an element of F_S if and only if $\liminf_{m\longrightarrow\infty} d(p_m, F_S) = 0$.*

Proof. The necessity is obvious.

Conversely, suppose that $\liminf_{m\longrightarrow\infty} d(p_m, F_S) = 0$ and $e_0 \in F_S$. From the Lemma 11, $\lim_{m\longrightarrow\infty}\|p_m - e_0\|$ exists. Therefore $\lim_{m\longrightarrow\infty} d(p_m, F_S) = 0$, by assumption. We prove that $\{p_m\}$ is a Cauchy sequence in $\mathscr{W}$. As $\lim_{m\longrightarrow\infty} d(p_m, F_S) = 0$, for a given $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for each $m \geq r_0$,

$$d(p_m, F_S) < \frac{\varepsilon}{2}. \Rightarrow \ \inf\{\|p_m - e_0\| : e_0 \in F_S\} < \frac{\varepsilon}{2}. \tag{24}$$

In particular $\inf\{\|p_{r_0} - e_0\| : e_0 \in F_S\} < \varepsilon/2$. Therefore there exists $e_0 \in F_S$ such that

$$\left\|p_{r_0} - e_0\right\| < \frac{\varepsilon}{2}. \tag{25}$$

Now for $r, m \geq k_0$,

$$\begin{aligned}\|p_{m+r} - p_m\| &\leq \|p_{m+r} - e_0\| + \|p_m - e_0\| \\ &\leq \left\|p_{r_0} - e_0\right\| + \left\|p_{r_0} - e_0\right\| \\ &= 2\left\|p_{r_0} - e_0\right\| < \varepsilon.\end{aligned} \tag{26}$$

This shows that $\{p_m\}$ is a Cauchy sequence in $\mathscr{W}$. As $\mathscr{W}$ is closed subset of a Banach space $\mathscr{Y}$, so there exists a point $e_0 \in \mathscr{W}$ such that $\lim_{m\longrightarrow\infty} p_m = e_0$. Now $\lim_{m\longrightarrow\infty} d(p_m, F_S) = 0$ gives that $d(e_0, F_S) = 0$. Since from Lemma 8, we have the set F_S a closed set in $\mathscr{W}$. Hence $e_0 \in F_S$. □.

The below facts are essentially due to Sentor and Dotson [17].

Definition 16. Let $\mathscr{W} \neq \emptyset$ be a subset of a Banach space $\mathscr{Y}$. A selfmap $S : \mathscr{W} \longrightarrow \mathscr{W}$ is said to be endowed with the condition (I) if and only if a function $L : [0,\infty) \longrightarrow [0,\infty)$ exists such that $L(0) = 0$ and $L(r) > 0$ for all $r > 0$ and $\|p - Sp\| \geq L(d(p, F_S))$ for each $p \in \mathscr{W}$.

The strong convergence theorem under the assumption of condition (I) is established as follows.

Theorem 17. *Let $\mathscr{W} \neq \emptyset$ be a convex closed subset of a UCBS $\mathscr{Y}$ and $S : \mathscr{W} \longrightarrow \mathscr{W}$ be a mean nonexpansive mapping and $F_S \neq \emptyset$. Assume that $\{p_m\}$ is a sequence of Picard–Mann hybrid iterative process (7). If S is endowed with condition (I), then $\{p_m\}$ converges strongly to an element of F_S.*

Proof. From Theorem 12, we have

$$\lim_{m\longrightarrow\infty}\|Sp_m - p_m\| = 0. \tag{27}$$

Table 1: Strong convergence of Picard–Mann hybrid, Picard, Ishikawa and Mann iterations to the fixed point $e_0 = 0$ of S in Example 19.

m	Picard-Mann hybrid (7)	Picard (4)	Ishikawa (6)	Mann (5)
1	0.90000000000000	0.90000000000000	0.90000000000000	0.90000000000000
2	0.07920000000000	0.18000000000000	0.33048000000000	0.39600000000000
3	0.00696960000000	0.03600000000000	0.12135225000000	0.17424000000000
4	0.00061332480000	0.00720000000000	0.04456054840320	0.07666560000000
5	0	0.00144000000000	0.01636263337365	0.03373286400000
6	0	0	0.00600835897480	0.01484246010000
7	0	0	0.00220626941554	0.00653068247000
8	0	0	0.00066188082466	0.00287350028697
9	0	0	0.00019856424739	0.00126434012626
10	0	0	0.00005956927421	0.00037930203788
11	0	0	0.00001787078226	0.00011379061136

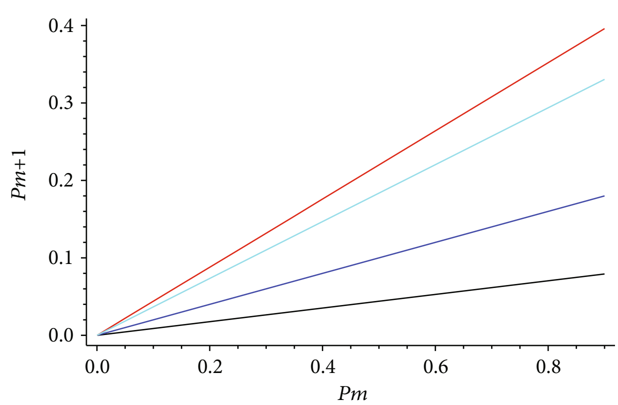

Figure 1: Behaviors of Picard-Mann hybrid (black), Picard (blue), Ishikawa (cyan) and Mann (red) iterates to the unique fixed point $e_0 = 0$ of the selfmap S.

Condition (I) of S provides

$$\lim_{m\longrightarrow\infty} d(p_m, F_S) = 0. \tag{28}$$

Now all the requirements of the Theorem 15 are available, so we conclude that $\{p_m\}$ converges strongly to an element of F_S. □

Remark 18. In the view of Lemma 7, our results contains the case of nonexpansive mappings.

3. Example

Now we want to provide a new example of mean nonexpansive maps.

Example 19. Let $S : [0, 1] \longrightarrow [0, 1]$ be defined by $Sp = 0$ if $0 \le p < 1/515$ and $Sp = p/5$ for $1/515 \le p \le 1$. Suppose $a = 1/4 = b$. Then it is easy to show $a + b \le 1$.

Now we may consider the following cases.

(i) Suppose that $0 \le p, q < 1/515$. Then

$$|Sp - Sq| = 0 \le a|p - q| + b|p - Sq|. \tag{29}$$

(ii) Suppose that $1/515 \le p, q \le 1$. Then

$$\begin{aligned} a|p-q| + b|p-Sq| &= \frac{1}{4}|p-q| + b\left|p - \frac{q}{5}\right| \\ &\ge \frac{1}{4}|p-q| \\ &\ge \frac{1}{5}|p-q| \\ &= |Sp - Sq| \end{aligned} \tag{30}$$

(iii) Suppose that $0 \le q < 1/515$ and $1/515 \le p \le 1$

$$\begin{aligned} a|p-q| + b|p-Sq| &= a|p-q| + \frac{1}{4}|p-0| \\ &\ge \frac{1}{4}|p| \\ &\ge \frac{1}{5}|p| \\ &= |Sp - Sq|. \end{aligned} \tag{31}$$

(iv) Suppose that $0 \le p < 1/515$ and $1/515 \le q \le 1$

$$\begin{aligned} a|p-q| + b|p-Sq| &= \frac{1}{4}|p-q| + \frac{1}{4}\left|p - \frac{q}{5}\right| \\ &\ge \frac{1}{4}\left|(p-q) - \left(p - \frac{q}{5}\right)\right| \\ &= \frac{1}{4}\left|\frac{4q}{5}\right| = \frac{1}{5}|q| = |Sp - Sq|. \end{aligned} \tag{32}$$

Thus we conclude that S is mean nonexpansive. Notice that, when $p = 1/600$ and $q = 1/515$, then $|Sp - Sq| > |p - q|$ and so S is not nonexpansive. Choose $\alpha_m = 0.70$ and $\beta_m = 0.65$, then the strong convergence and effectiveness of the Picard-Mann hybrid iteration can be seen in Table 1 and Figure 1.

4. Conclusions

We begun the finding of fixed points for mean nonexpansive operators in Banach spaces under the Picard–Mann hybrid iterative process. Some convergence results are established under different assumptions. It has been showen by an example that the notion of mean nonexpansive maps is properly more general than the notion of nonexpansive maps. Also, the Picard-Mann hybrid iterates of this example converge faster than the Picard, Ishikawa and Mann iterates. In particular, our results essentially improve and extended the results Khan [9] from the setting of nonexpansive operators to the larger frame work of mean nonexpansive operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

We have no conflict of interest.

Authors' Contributions

Every author listed in this manuscript has contributed equally to each part.

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Research Article

Logarithmic Coefficient Bounds and Coefficient Conjectures for Classes Associated with Convex Functions

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It is well-known that the logarithmic coefficients play an important role in the development of the theory of univalent functions. If $\mathcal{S}$ denotes the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ analytic and univalent in the open unit disk $\mathbb{U}$, then the logarithmic coefficients $\gamma_n(f)$ of the function $f\in\mathcal{S}$ are defined by $\log(f(z)/z)=2\sum_{n=1}^{\infty}\gamma_n(f)z^n$. In the current paper, the bounds for the logarithmic coefficients γ_n for some well-known classes like $\mathcal{C}(1+\alpha z)$ for $\alpha\in(0,1]$ and $\mathcal{CV}_{\text{hpl}}(1/2)$ were estimated. Further, conjectures for the logarithmic coefficients γ_n for functions f belonging to these classes are stated. For example, it is forecasted that if the function $f\in\mathcal{C}(1+\alpha z)$, then the logarithmic coefficients of f satisfy the inequalities $|\gamma_n|\le\alpha/(2n(n+1))$, $n\in\mathbb{N}$. Equality is attained for the function $L_{\alpha,n}$, that is, $\log(L_{\alpha,n}(z)/z)=2\sum_{n=1}^{\infty}\gamma_n(L_{\alpha,n})z^n=(\alpha/n(n+1))z^n+\cdots,z\in\mathbb{U}$.

Dedicated to the memory of Professor Gabriela Kohr (1967-2020)

1. Introduction

Let $\mathbb{U}:=\{z\in\mathbb{C}:|z|<1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ be the category of analytic functions f in $\mathbb{U}$ for which f has the following representation:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n,\quad z\in\mathbb{U}. \tag{1}$$

Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$. Then, the *logarithmic coefficients* γ_n of the function $f\in\mathcal{S}$ are defined with the aid of the following series expansion:

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\gamma_n(f)z^n,\quad z\in\mathbb{U}. \tag{2}$$

These coefficients play an important role for different estimates in the theory of univalent functions, and note that we use γ_n instead of $\gamma_n(f)$. Kayumov [1] solved Brennan's conjecture for conformal mappings with the help of studying the logarithmic coefficients. The significance of the logarithmic coefficients follows from Lebedev-Milin inequalities ([2], chapter 2; see also [3, 4]), where estimates of the logarithmic coefficients were applied to obtain bounds on the coefficients of f. Milin [2] conjectured the inequality

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2-\frac{1}{k}\right)\le 0,\quad n=1,2,3,\cdots, \tag{3}$$

that implies Robertson's conjecture [5] and hence Bieberbach's conjecture [6], which was the well-known coefficient problem in the theory of univalent functions. De Branges

[7] proved Bieberbach's conjecture by establishing Milin's conjecture.

Recall that we can rewrite (2) in the power series form as follows:

$$2\sum_{n=1}^{\infty}\gamma_n z^n = a_2 z + a_3 z^2 + a_4 z^3 + \cdots - \frac{1}{2}\left(a_2 z + a_3 z^2 + a_4 z^3 + \cdots\right)^2 + \frac{1}{3}\left(a_2 z + a_3 z^2 + a_4 z^3 + \cdots\right)^3 + \cdots, \quad z \in \mathbb{U}, \tag{4}$$

and equating the coefficients of z^n for $n = 1, 2, 3$, it follows that

$$\begin{cases} 2\gamma_1 = a_2, \\ 2\gamma_2 = a_3 - \dfrac{1}{2}a_2^2, \\ 2\gamma_3 = a_4 - a_2 a_3 + \dfrac{1}{3}a_2^3. \end{cases} \tag{5}$$

If the functions f and g are analytic in $\mathbb{U}$, the function f is called to be *subordinate* to the function g, written $f(z) \prec g(z)$, if there exists a function w analytic in $\mathbb{U}$ with $|w(z)| < 1$, $z \in \mathbb{U}$, and $w(0) = 0$, such that $f = g \prec w$. In particular, if g is univalent in $\mathbb{U}$, then the following equivalence relationship holds true:

$$\begin{gathered} f(z) \prec g(z) \Leftrightarrow f(0) = g(0), \\ f(\mathbb{U}) \subset g(\mathbb{U}). \end{gathered} \tag{6}$$

Using the principle of subordination, Ma and Minda [8] introduced the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$, where we make here the weaker assumptions that the function φ is analytic in the open unit disk $\mathbb{U}$ and satisfies $\varphi(0) = 1$, such that it has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, z \in \mathbb{U}, \quad \text{with } B_1 \neq 0. \tag{7}$$

They considered the abovementioned classes as follows:

$$\begin{aligned} \mathcal{S}^*(\varphi) &:= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \\ \mathcal{C}(\varphi) &:= \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}. \end{aligned} \tag{8}$$

Some special subclasses of the class $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ play a significant role in the *Geometric Function Theory* because of their geometric properties.

For example, taking $\varphi(z) = (1 + Az)/(1 + Bz)$ where $A \in \mathbb{C}$, $-1 \leq B \leq 0$, and $A \neq B$, we get the classes $\mathcal{S}^*[A, B]$ and $\mathcal{C}[A, B]$, respectively (see also [9, 10]). The mentioned classes with the restriction $-1 \leq B < A \leq 1$ reduce to the popular *Janowski starlike* and *Janowski convex functions*, respectively. By replacing $A = 1 - 2\alpha$ and $B = -1$, where $0 \leq \alpha < 1$, we obtain the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ of the *starlike functions of order* α and *convex functions of order* α, respectively. In particular, $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the class of starlike functions and of convex functions in the open unit disk $\mathbb{U}$, respectively. Further, by altering $A = \alpha$ and $B = 0$, where $0 \leq \alpha < 1$, we get the classes $\mathcal{S}^*(1 + \alpha z)$ and $\mathcal{C}(1 + \alpha z)$, which are the extensions of the classes $\mathcal{S}^*(1 + z)$ and $\mathcal{C}(1 + z)$, respectively (see [11]), that is,

$$\begin{aligned} \mathcal{S}^*(1 + \alpha z) &:= \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha \right\}, \\ \mathcal{C}(1 + \alpha z) &:= \left\{ f \in \mathcal{A} : \left| \frac{zf''(z)}{f'(z)} \right| < \alpha \right\}, \end{aligned} \tag{9}$$

where $0 < \alpha \leq 1$.

Supposing that $\Psi_{\alpha,n} \in \mathcal{S}^*(1 + \alpha z)$ is such that

$$\frac{z\Psi'_{\alpha,n}(z)}{\Psi_{\alpha,n}(z)} = 1 + \alpha z^n, \quad n \in \mathbb{N}, \tag{10}$$

each function $\Psi_{\alpha,n}$ is of the form

$$\Psi_{\alpha,n}(z) = z \exp\left(\int_0^z \frac{1 + \alpha t^n - 1}{t} \, \mathrm{d}t \right) = z + \frac{\alpha}{n} z^{n+1} + \cdots, \quad z \in \mathbb{U}, \tag{11}$$

and is the extremal function for various problems in $\mathcal{S}^*(1 + \alpha z)$. Also, suppose that $L_{\alpha,n} \in \mathcal{C}(1 + \alpha z)$ is such that

$$1 + \frac{zL''_{\alpha,n}(z)}{L'_{\alpha,n}(z)} = 1 + \alpha z^n, \quad n \in \mathbb{N}. \tag{12}$$

Then, each function $L_{\alpha,n}$ is of the form

$$\begin{aligned} L_{\alpha,n}(z) &= \int_0^z \exp\left(\int_0^x \frac{1 + \alpha t^n - 1}{t} \, \mathrm{d}t \right) \mathrm{d}x \\ &= z + \frac{\alpha}{n(n+1)} z^{n+1} + \cdots, \quad z \in \mathbb{U}, \end{aligned} \tag{13}$$

and plays as extremal function for some extremal problems in the set $\mathcal{C}(1 + \alpha z)$.

Lately, Kanas et al. [12] introduced the categories $\mathcal{ST}_{\mathrm{hpl}}(s)$ and $\mathcal{CV}_{\mathrm{hpl}}(s)$ by

$$\begin{aligned} \mathcal{ST}_{\mathrm{hpl}}(s) &=: \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \mathfrak{q}_s(z) := \frac{1}{(1 - z)^s}, 0 < s \leq 1 \right\}, \\ \mathcal{CV}_{\mathrm{hpl}}(s) &:= \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \mathfrak{q}_s(z) := \frac{1}{(1 - z)^s}, 0 < s \leq 1 \right\}, \end{aligned} \tag{14}$$

and obtained some geometric properties in these categories. Further, the functions

$$\Phi_{s,n}(z) = z \exp\left(\int_0^z \frac{\mathfrak{q}_s(t^n) - 1}{t} \, dt\right) = z + \frac{s}{n} z^{n+1} + \cdots, \quad z \in \mathbb{U}, n \in \mathbb{N},$$

$$K_{s,n}(z) = \int_0^z \exp\left(\int_0^x \frac{\mathfrak{q}_s(t^n) - 1}{t} \, dt\right) dx = z + \frac{s}{n(n+1)} z^{n+1} + \cdots, \quad z \in \mathbb{U}, n \in \mathbb{N}, \tag{15}$$

play as extremal functions for some issues of the families $\mathcal{ST}_{\text{hpl}}(s)$ and $\mathcal{CV}_{\text{hpl}}(s)$, respectively.

Lately, several researchers have subsequently investigated same problems regarding the logarithmic coefficients and the coefficient problems [9, 13–23], to mention a few of them. For instance, the rotation of the *Koebe function* $k(z) = z(1 - e^{i\theta} z)^{-2}$ for each $\theta \in \mathbb{R}$ has the logarithmic coefficients $\gamma_n = e^{i\theta n}/n$, $n \geq 1$. If $f \in \mathcal{S}$, then by applying the Bieberbach inequality for the first relation of (5), it follows that $|\gamma_1| \leq 1$, and using the *Fekete-Szegö inequality* for the second relation of (5) (see [24], Theorem 3.8) leads to

$$|\gamma_2| = \frac{1}{2}\left|a_3 - \frac{1}{2}a_2^2\right| \leq \frac{1}{2}\left(1 + 2e^{-2}\right) \simeq 0.635\cdots. \tag{16}$$

It was established in ([25], Theorem 4) that the logarithmic coefficients γ_n of $f \in \mathcal{S}$ satisfy the inequality

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6}, \tag{17}$$

and the equality is obtained for the Koebe function. For $f \in \mathcal{S}^*$, the inequality $|\gamma_n| \leq 1/n$ holds but is not true for the full class $\mathcal{S}$, even in order of magnitude (see [24], Theorem 8.4). In 2018, some first logarithmic coefficients γ_n were estimated for special subclasses of *close-to-convex functions* in [15, 20]. However, the problem of the best upper bounds for the logarithmic coefficients of univalent functions for $n \geq 3$ is presumably still a concern. In [13], the authors obtained the bounds of logarithmic coefficients γ_n, $n \in \mathbb{N}$, for the general class $\mathcal{S}^*(\varphi)$, and the bounds of the logarithmic coefficients γ_n when $n = 1, 2, 3$ for the class $\mathcal{K}(\varphi)$, while the estimated bounds would generalize many of the previous outcomes.

In the present study, which is motivated essentially by the recent works [13, 16], the bounds for the logarithmic coefficients γ_n, $n \in \mathbb{N}$, of the class $\mathcal{C}(1 + \alpha z)$ for $\alpha \in (0, 1]$ and $\mathcal{CV}_{\text{hpl}}(1/2)$ were estimated. Further, conjectures for the logarithmic coefficients γ_n for f belonging to these classes are stated.

2. Main Results

First, we will obtain the bounds for γ_n of the classes $\mathcal{S}^*(1 + \alpha z)$ and $\mathcal{C}(1 + \alpha z)$ for $\alpha \in (0, 1]$. In this regard, the following outcomes will be employed in the key results.

Lemma 1 (see [13], Theorem 1). *Let $f \in \mathcal{S}^*(\varphi)$. If φ is convex univalent, then the logarithmic coefficients of f satisfy the following inequalities:*

$$|\gamma_n| \leq \frac{|B_1|}{2n}, \quad n \in \mathbb{N}, \tag{18}$$

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} \frac{|B_n|^2}{n^2}. \tag{19}$$

The inequalities in (18) and (19) are sharp, such that for any $n \in \mathbb{N}$, there exist the function f_n given by $zf_n'(z)/f_n(z) = \varphi(z^n)$ and the function f given by $zf'(z)/f(z) = \varphi(z)$, respectively, for those equalities we obtain.

Lemma 2 (see [13], Theorem 2). *Let $f \in \mathcal{C}(\varphi)$. Then, the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_1| \leq \frac{|B_1|}{4}, \tag{20}$$

$$|\gamma_2| \leq \begin{cases} \dfrac{|B_1|}{12}, & \text{if} \quad \left|4B_2 + B_1^2\right| \leq 4|B_1|, \\[2ex] \dfrac{\left|4B_2 + B_1^2\right|}{48}, & \text{if} \quad \left|4B_2 + B_1^2\right| > 4|B_1|, \end{cases} \tag{21}$$

and if B_1, B_2, and B_3 are real values, then

$$|\gamma_3| \leq \frac{|B_1|}{24} H(q_1; q_2), \tag{22}$$

where $H(q_1; q_2)$ is given in ([26], Lemma 2) (or [9], Lemma 5), *$q_1 = (B_1 + (4B_2/B_1))/2$, and $q_2 = (B_2 + (2B_3/B_1))/2$. The* bounds (20) and (21) are sharp.

Lemma 3 (see [18], Theorem 30). *If $f \in \mathcal{CV}_{hpl}(1/2)$, then*

$$|\gamma_1| \leq \frac{1}{8}, |\gamma_2| \leq \frac{1}{24}, |\gamma_3| \leq \frac{1}{48}. \tag{23}$$

The first two bounds are sharp for $f = K_{1/2,1}$ and $f = K_{1/2,2}$, respectively.

If we consider Lemma 1 with the function $\varphi(z) =: 1 + \alpha z$, then we immediately get the next result:

Theorem 4. *If $f \in \mathcal{S}^*(1 + \alpha z)$, then*

$$|\gamma_n| \leq \frac{\alpha}{2n}, \quad n \in \mathbb{N}, \qquad \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\alpha}{4}. \tag{24}$$

These inequalities are sharp for $f = \Psi_{\alpha,n}$ and $f = \Psi_{\alpha,1}$, respectively.

Corollary 5. *Let $f \in \mathscr{C}(1+\alpha z)$. Then, the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_1| \le \frac{\alpha}{4},\quad |\gamma_2| \le \frac{\alpha}{12},\quad |\gamma_3| \le \frac{\alpha}{24}. \tag{25}$$

Equalities in these inequalities are attained for the functions $L_{\alpha,n}$ for $n = 1, 2, 3$, respectively.

Proof. For $\varphi(z) = 1 + \alpha z$, where $B_1 = \alpha, B_2 = B_3 = 0$, in Theorem 6, we obtain the required result. Also, since

$$\begin{aligned}\log \frac{L_{\alpha,1}(z)}{z} &= 2\sum_{n=1}^{\infty} \gamma_n(L_{\alpha,1}) z^n = \frac{\alpha}{2} z + \cdots, \quad z \in \mathbb{U},\\ \log \frac{L_{\alpha,2}(z)}{z} &= 2\sum_{n=1}^{\infty} \gamma_n(L_{\alpha,2}) z^n = \frac{\alpha}{6} z^2 + \cdots, \quad z \in \mathbb{U},\\ \log \frac{L_{\alpha,3}(z)}{z} &= 2\sum_{n=1}^{\infty} \gamma_n(L_{\alpha,3}) z^n = \frac{\alpha}{12} z^3 + \cdots, \quad z \in \mathbb{U},\end{aligned} \tag{26}$$

it follows that these inequalities are attained for the functions $L_{\alpha,n}$ for $n = 1, 2, 3$, respectively. □

Theorem 6. *Let $f \in \mathscr{C}(1+\alpha z)$. Then, the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_n| \le \frac{\alpha}{4n}, \quad n \in \mathbb{N}. \tag{27}$$

This inequality is sharp for $|\gamma_1|$ for the function $L_{\alpha,1}$.

Proof. If $f \in \mathscr{C}(1+\alpha z)$, this is equivalent to $f \in \mathscr{A}$ and

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \alpha z =: \varphi_\alpha(z). \tag{28}$$

If we define $p(z) := zf'(z)/f(z)$, then $p(0) = 1$, and the above subordination relation can be written as

$$p(z) + \frac{zp'(z)}{p(z)} \prec \varphi_\alpha(z). \tag{29}$$

Supposing that the function ψ_α satisfies the differential equation

$$\psi_\alpha(z) + \frac{z\psi_\alpha'(z)}{\psi_\alpha(z)} = \varphi_\alpha(z), \quad \psi_\alpha(0) = 1, \tag{30}$$

we will prove that ψ_α is a convex univalent function in $\mathbb{U}$.

The function φ_α has positive real part in $\mathbb{U}$ whenever $\alpha \in (0, 1]$. Therefore, using ([27], Theorem 1) for $\beta = 1$, $\gamma = 0$, and $c = 1$, it follows that the solution ψ_α of the differential equation (30) is analytic in $\mathbb{U}$, with $\operatorname{Re} \psi_\alpha(z) > 0$ for all $z \in \mathbb{U}$, and

$$\begin{aligned}\psi_\alpha(z) &= H(z)\left(\int_0^z \frac{H(t)}{t}\,\mathrm{d}t\right)^{-1} = \frac{\alpha z \exp(\alpha z)}{\exp(\alpha z) - 1}\\ &= 1 + \frac{\alpha}{2} z + \cdots, \quad z \in \mathbb{U},\end{aligned} \tag{31}$$

where

$$H(z) = z \exp\left(\int_0^z \frac{\varphi_\alpha(t) - 1}{t}\,\mathrm{d}t\right) = z \exp(\alpha z), \tag{32}$$

and all powers are considered at the principal branch, that is, $\log 1 = 0$.

Since φ_α is convex and ψ_α is analytic with $\operatorname{Re} \psi_\alpha(z) > 0$ for all $z \in \mathbb{U}$, using [28] (Theorem 3.2i) for $n = 1$, we deduce that ψ_α is univalent in $\mathbb{U}$. Moreover, from Figure 1 made with MAPLE software, we get

$$\Psi(z) := \operatorname{Re}\left(1 + \frac{z\psi_\alpha''(z)}{\psi_\alpha'(z)}\right) > 0, \quad z \in \mathbb{U}, \tag{33}$$

and $\psi_\alpha'(0) = \alpha/2 \neq 0$, so ψ_α is a convex function. Hence, it follows that ψ_α is a convex univalent function in $\mathbb{U}$.

Therefore, according to [28] (Theorem 3.2i), the differential subordination (29) implies

$$p(z) \prec \psi_\alpha(z), \tag{34}$$

for all $0 < \alpha \le 1$, and ψ_α is the best dominant. Thus,

$$\frac{zf'(z)}{f(z)} \prec \psi_\alpha(z), \tag{35}$$

for all $0 < \alpha \le 1$. Hence,

$$\mathscr{C}(1+\alpha z) \subset \mathscr{S}^*(\psi_\alpha). \tag{36}$$

From the above relation, we get

$$\sup\{|\gamma_n(f)| : f \in \mathscr{C}(1+\alpha z)\} \le \sup\{|\gamma_n(f)| : f \in \mathscr{S}^*(\psi_\alpha)\}. \tag{37}$$

Hence, from Lemma 1, we obtain

$$\sup\{|\gamma_n(f)| : f \in \mathscr{C}(1+\alpha z)\} \le \frac{\alpha}{4n}. \tag{38}$$

Therefore, for $f \in \mathscr{C}(1+\alpha z)$ and for all $n \in \mathbb{N}$, we conclude that

$$|\gamma_n(f)| \le \frac{\alpha}{4n}. \tag{39}$$

□

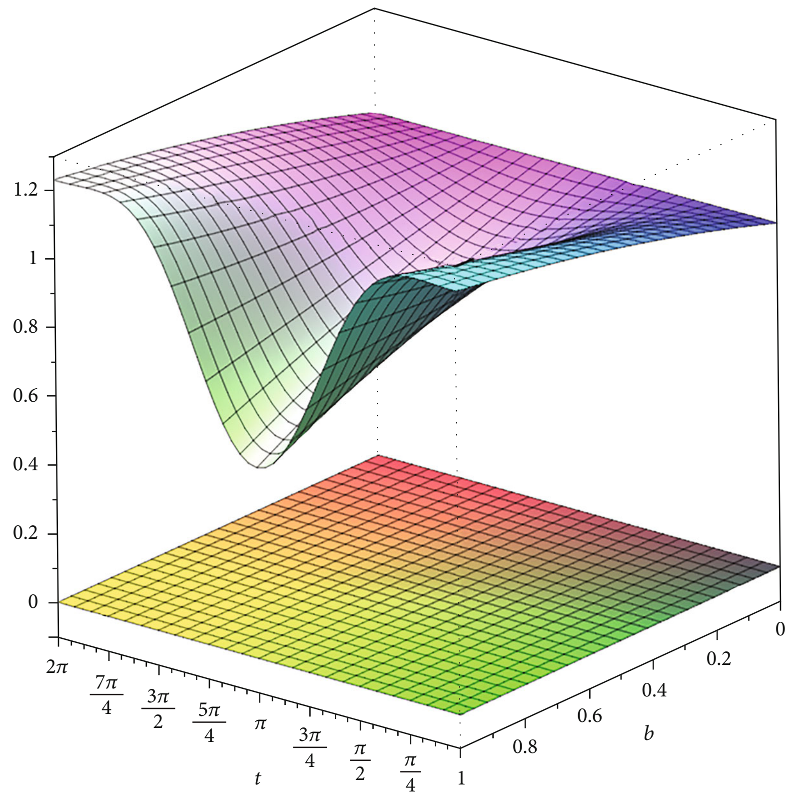

FIGURE 1: The plot of $\Psi(\mathrm{Re}^{it})$ for $t \in [0, 2\pi)$, $R = 1$, and $\alpha \in (10^{-6}, 1]$.

Remark 7. If we compare the results of Corollary 5 with those of Theorem 6, then we conclude that the results of Theorem 6 are not the best possible. We conjecture that if the function $f \in \mathscr{C}(1+\alpha z)$, then the logarithmic coefficients of f satisfy the inequalities

$$|\gamma_n| \leq \frac{\alpha}{2n(n+1)}, \quad n \in \mathbb{N}. \tag{40}$$

Equality is attained for the function $L_{\alpha,n}$, that is,

$$\log \frac{L_{\alpha,n}(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n (L_{\alpha,n}) z^n = \frac{\alpha}{n(n+1)} z^n + \cdots, \quad z \in \mathbb{U}. \tag{41}$$

Theorem 8. *Let $f \in \mathscr{CV}_{hpl}(1/2)$. Then, the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_n| \leq \frac{1}{8n}, \quad n \in \mathbb{N}. \tag{42}$$

This inequality is sharp for $|\gamma_1|$ for the function $K_{1/2,1}$.

Proof. Letting $f \in \mathscr{CV}_{\mathrm{hpl}}(1/2)$, it follows that

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1}{\sqrt{1-z}} =: \mathfrak{q}_{\frac{1}{2}}(z). \tag{43}$$

Suppose that $\mathfrak{p}$ satisfies the differential equation

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)} = \frac{1}{\sqrt{1-z}}. \tag{44}$$

If we define $p(z) := zf'(z)/f(z)$, then the subordination (43) can be rewritten as

$$p(z) + \frac{zp'(z)}{p(z)} \prec \mathfrak{q}_{\frac{1}{2}}(z). \tag{45}$$

According to the inequality (20) of [12] (Theorem 2.3), the function $\mathfrak{q}_{1/2}$ is analytic with positive real part in $\mathbb{U}$. Therefore, using [27] (Theorem 1) for $\beta = 1$, $\gamma = 0$, and $c = 1$, it follows that the solution $\mathfrak{p}$ of the differential equation (44) is analytic in $\mathbb{U}$ with $\operatorname{Re} \mathfrak{p}(z) > 0$, $z \in \mathbb{U}$, and

$$\begin{aligned}\mathfrak{p}(z) &= H(z)\left(\int_0^z \frac{H(t)}{t}\,\mathrm{d}t\right)^{-1} = \frac{4z}{\left(1+\sqrt{1-z}\right)^2} \\ &\quad\cdot \frac{1}{\left(-8/\left(1+\sqrt{1-z}\right)\right) - 8\ln\left(1+\sqrt{1-z}\right) + 4 + 8\ln 2} \\ &= 1 + \frac{1}{4}z + \cdots, \quad z \in \mathbb{U},\end{aligned} \tag{46}$$

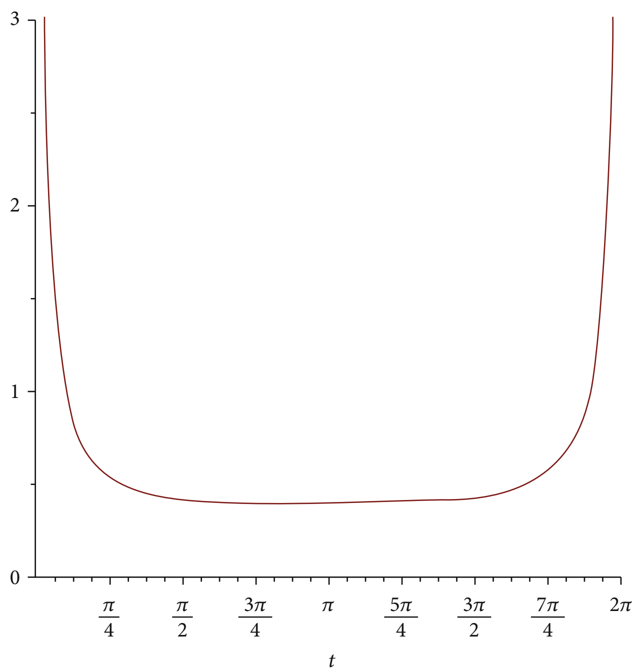

FIGURE 2: The plot of $\Psi(Re^{it})$ for $t \in [0, 2\pi)$ and $R = 1$.

where

$$H(z) = z \exp\left(\int_0^z \frac{\mathfrak{q}_{1/2}(t) - 1}{t} \mathrm{d}t\right) = \frac{4z}{\left(1 + \sqrt{1 - z}\right)^2}, \quad (47)$$

and all powers are considered at the principal branch, that is, $\log 1 = 0$. Moreover, from Figure 2 made with MAPLE software, we get

$$\Psi(z) := \mathrm{Re}\left(1 + \frac{z\mathfrak{p}''(z)}{\mathfrak{p}'(z)}\right) > 0, \quad z \in \mathbb{U}, \quad (48)$$

and $\mathfrak{p}'(0) = 1/4 \neq 0$. Hence, ψ is convex in $\mathbb{U}$.

Since p satisfies in the subordination (45), using [28] (Theorem 3.2i), we conclude that $p(z) \prec \psi(z)$, that is,

$$\frac{zf'(z)}{f(z)} \prec \mathfrak{p}(z), \quad (49)$$

and $\mathfrak{p}$ is the best dominant. Thus, $f \in \mathscr{CV}_{\mathrm{hpl}}(1/2)$ implies $f \in \mathscr{S}^*(\mathfrak{p})$, that is,

$$\mathscr{CV}_{\mathrm{hpl}}(1/2) \subset \mathscr{S}^*(\mathfrak{p}). \quad (50)$$

Therefore, since $\mathfrak{p}$ is convex univalent, from Lemma 1, it follows that

$$\sup\left\{|\gamma_n(f)|: f \in \mathscr{CV}_{\mathrm{hpl}}\left(\frac{1}{2}\right)\right\} \leq \sup\left\{|\gamma_n(f)|: f \in \mathscr{S}^*(\mathfrak{p})\right\} = \frac{1}{8n}, \quad (51)$$

and we obtain the result. This completes the proof.□

Remark 9. If we compare the results of Lemma 1 with those of Theorem 8, then we conclude that the results of Theorem 8 are not the best possible. We conjecture that if the function $f \in \mathscr{CV}_{\mathrm{hpl}}(1/2)$, then the logarithmic coefficients of f satisfy the inequalities

$$|\gamma_n| \leq \frac{1}{4n(n+1)}, \quad n \in \mathbb{N}. \quad (52)$$

Equality is attained for the function $K_{1/2,n}$, that is,

$$\log \frac{K_{1/2,n}(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(K_{1/2,n}) z^n = \frac{1}{2n(n+1)} z^n + \cdots, \quad z \in \mathbb{U}. \quad (53)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

k -Fractional Variants of Hermite-Mercer-Type Inequalities via s-Convexity with Applications

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This article is aimed at studying novel generalizations of Hermite-Mercer-type inequalities within the Riemann-Liouville k -fractional integral operators by employing s-convex functions. Two new auxiliary results are derived to govern the novel fractional variants of Hadamard-Mercer-type inequalities for differentiable mapping Ψ whose derivatives in the absolute values are convex. Moreover, the results also indicate new lemmas for Ψ', Ψ'', and Ψ''' and new bounds for the Hadamard-Mercer-type inequalities via the well-known Hölder's inequality. As an application viewpoint, certain estimates in respect of special functions and special means of real numbers are also illustrated to demonstrate the applicability and effectiveness of the suggested scheme.

1. Introduction

Recently, two fundamental notions have been introduced in pure and applied analysis having potential utilities in every field and are known as convexity and concavity. Interestingly, the convexity theory is attributed to Jensen. Several monographs and articles have played a prominent role in the developments, speculations, and modifications of convexity in different directions such as η-convexity, harmonic convexity, h-convexity, and strong convexity. Moreover, a strong connection has been developed between diverse kinds of convex functions and inequality theory. Their fertile applications in optimization theory, functional analysis, physics, and statistical theory have made it a much fascinating subject, and hence, it is assumed as an incorporative subject between combinatorics, orthogonal polynomials, hypergeometric functions, quantum theory, and linear programming. This is the major motivation behind the keen investigation and progress of the integral inequalities in the literature [1, 2].

Let $0 < \zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_n$ and let $\rho = (\rho_1, \rho_2, \cdots, \rho_n)$ be nonnegative weights such that $\sum_{j=1}^{n} \rho_j = 1$. The famous Jensen's inequality (see [1]) in the literature states that if Ψ is a convex function on the interval $[\theta_1, \theta_2]$, then

$$\Psi\left(\sum_{j=1}^{n} \rho_j \zeta_j\right) \leq \left(\sum_{j=1}^{n} \rho_j \Psi(\zeta_j)\right), \tag{1}$$

for all $\zeta_j \in [\theta_1, \theta_2]$, $\rho_j \in [0, 1]$, and $j = 1, 2, \cdots, n$. It is one of the key inequalities that helps to extract bounds for useful distances in information theory (see [3, 4]).

In 2003, a new variant of Jensen's inequality was introduced by Mercer [5].

If Ψ is a convex function on $[\theta_1, \theta_2]$, then

$$\Psi\left(\theta_1 + \theta_2 - \sum_{j=1}^{n} \rho_j \zeta_j\right) \leq \Psi(\theta_1) + \Psi(\theta_2) - \sum_{j=1}^{n} \rho_j \Psi(\zeta_j), \tag{2}$$

for all $\zeta_j \in [\theta_1, \theta_2]$, $\rho_j \in [0, 1]$, and $j = 1, 2, \cdots, n$.

Matkovic and Pećarić proposed several generalizations on Jensen-Mercer operator inequalities [6]. Later on, Niezgoda [7] has provided several extensions to higher dimensions for Mercer-type inequalities. Recently, the Jensen-Mercer-type inequality has made a significant contribution to inequality theory due to its prominent characterizations.

In the present study, we consider $s \in (0,\infty)$, the class of Breckner s-convex functions (which for $0 < s < 1$ were called in [8, 9]), s-convex in the second sense. In [10], Dragomir and Fitzpatrick introduce the concept of a real-valued Breckner s-convex function Ψ on a convex subset C of a linear space V as

$$\Psi(\rho_1\zeta_1 + \rho_2\zeta_2) \le \rho^s\psi(\zeta_1) + (1-\rho)^s\Psi(\zeta_2), \tag{3}$$

whenever $\rho_1, \rho_2 \ge 0$ with $\rho_1 + \rho_2 = 1$ and $\zeta_1, \zeta_2 \in C$. For $s = 1$, it reduces to the usual notion of convexity. As a result, he generalizes Jensen's inequality (1) as

$$\Psi\left(\sum_{j=1}^{n}\rho_j\zeta_j\right) \le \sum_{j=1}^{n}\rho_j^s\Psi(\zeta_j), \tag{4}$$

whenever $\rho_i \ge 0$, $\zeta_i \in C$, and $\sum_{i=1}^{n}\rho_i = 1$.

In [9], the class of s-convex functions in the first and second senses is introduced along with their significant properties.

Definition 1. Let $s \in (0, 1]$, a real-valued function Ψ on an interval $I = [0,\infty)$ is s-convex in the second sense provided that (3) holds for all $\zeta_1, \zeta_2 \in I$ and $\rho_1, \rho_2 \ge 0$ with $\rho_1 + \rho_2 = 1$.

They denote this class of function by $(\Psi \in K_s^2)$. Moreover, they proved that the class $(\Psi \in K_s^2)$ is stronger than convexity in the first and original sense for $0 < s < 1$. Several properties of s-convex functions in both senses are presented in a comprehensive manner with supporting examples. It is interesting to see that if $0 < s < 1$ and $\Psi \in K_s^2$, then ψ is nonnegative. This result may not hold in general when the function is convex (i.e., $s = 1$). Also, the situation is more interesting when $f(0) = 0$.

Viewing this literature, we intend to extend the Jensen-Mercer inequality for Breckner s-convex functions. For this, we use the ideology of Mercer's concept [5] and give the following important lemma.

Lemma 2. *If Ψ is a real-valued Breckner s-convex function on the interval $[\theta_1, \theta_2] \subset \mathbb{R}^+$ and $s > 0$ such that $\rho_1, \rho_2 \ge 0$, $\rho_1 + \rho_2 = 1$, and $\rho_1^s + \rho_2^s \le 1$, then for any finite positive increasing sequence $\{\zeta_n\}_{k=1}^{n} \in [\theta_1, \theta_2]$, we have*

$$\Psi(\theta_1 + \theta_2 - \zeta_k) \le \Psi(\theta_1) + \Psi(\theta_2) - \Psi(\zeta_k), \tag{5}$$

for all $1 \le k \le n$.

Proof. Let $y_k = \theta_1 + \theta_2 - \zeta_k$. Then, $\theta_1 + \theta_2 = y_k + \zeta_k$, so the pairs θ_1, θ_2 and y_k, ζ_k possess the same midpoint. Since that is the case, there exist $\rho_1, \rho_2 \in [0, 1]$, with $\rho_1 + \rho_2 = 1$ such that $\zeta_k = \rho_1\theta_1 + \rho_2\theta_2$ and $y_k = \rho_2\theta_1 + \rho_1\theta_2$. Therefore, employing (3) and the assumed condition, we get

$$\begin{aligned}\Psi(y_k) &= \Psi(\rho_2\theta_1 + \rho_1\theta_2) \le \rho_2^s\Psi(\theta_1) + \rho_1^s\Psi(\theta_2)\\ &\le (1-\rho_1^s)\Psi(\theta_1) + (1-\rho_2^s)\Psi(\theta_2) = \Psi(\theta_1)\\ &\quad + \Psi(\theta_2) - \rho_1^s\Psi(\theta_1) - \rho_2^s\Psi(\theta_2) = \Psi(\theta_1)\\ &\quad + \Psi(\theta_2) - [\rho_1^s\Psi(\theta_1) + \rho_2^s\Psi(\theta_2)] \le \Psi(\theta_1)\\ &\quad + \Psi(\theta_2) - \Psi(\rho_1\theta_1 + \rho_2\theta_2) = \Psi(\theta_1)\\ &\quad + \Psi(\theta_2) - \Psi(\zeta_k),\end{aligned} \tag{6}$$

which completes the proof.

Now, we give the result for the Jensen-Mercer inequality in the Breckner s-sense.

Theorem 3. *Let $\rho_1, \rho_2, \cdots, \rho_n$ be positive real numbers $n \ge 2$ such that $\sum_{k=1}^{n}\rho_k = 1$ and $\sum_{k=1}^{n}\rho_k^s \le 1$. If ψ is a real-valued Breckner s-convex function on $[\theta_1, \theta_2] \subset \mathbb{R}^+$, then for any finite positive increasing sequence $\{\zeta_n\}_{k=1}^{n} \in [\theta_1, \theta_2]$, we have*

$$\Psi\left(\theta_1 + \theta_2 - \sum_{k=1}^{n}\rho_k\zeta_k\right) \le \Psi(\theta_1) + \Psi(\theta_2) - \sum_{k=1}^{n}(\rho_k)^s\Psi(\zeta_k). \tag{7}$$

Proof. One can prove it by following a similar idea as in [5]; however, we need to employ Lemma 2 and generalized Jensen's inequality (4) for Breckner s-convex functions.

For further properties and applications of s-convex functions, see [9, 11] and references therein. The following lemma is of great interest for applications.

Lemma 4 (see [11]). *Let $\Psi : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ be a convex function. Then, the following results hold:*

(i) If Ψ is nonnegative, then it is s-convex for $s \in (0, 1]$

(ii) If Ψ is nonpositive, then it is s-convex for $s \in [1,\infty)$

One of the famous integral inequalities for convex functions is the Hermite-Hadamard inequality (see [2]):

$$\Psi\left(\frac{\theta_1 + \theta_2}{2}\right) \le \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2}\Psi(\lambda)d\lambda \le \frac{\Psi(\theta_1) + \Psi(\theta_2)}{2}, \tag{8}$$

provided that if a mapping $\Psi : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function on I and $\theta_1, \theta_2 \in I$.

Fractional-order calculus deals with more general behavior than integer-order calculus, and it not only provides new mathematical methods for practical systems but also has been applied into various fields due to its accurate description in many active fields, such as fraction-order memristive chaotic circuit, fractional-orderrelaxation-oscillation model, mathematical biology, and economics (see [12]).

The theory of Riemann-Liouville k-fractional integrals is a pertinent extension of Riemann-Liouville fractional integrals. It is important to note that if $k \neq 1$, the properties of Riemann-Liouville k-fractional integrals are quite dissimilar from those of general fractional integrals. For this, the Riemann-Liouville k-fractional integrals have agitated the interest of many researchers. Now, we demonstrate some essential concepts of k-fractional calculus for the investigation of our results.

Definition 5 (see [13]). Diaz and Pariguan have defined the k-gamma function $\Gamma_k(\cdot)$, a generalization of the classical gamma function, which is given by the following formula:

$$\Gamma_k(\alpha) = \lim_{m\longrightarrow\infty} \frac{m!k^m (mk)^{(\alpha/k)-1}}{(\alpha)_{m,k}}, \quad k>0, \alpha \in C \setminus kZ^-, \tag{9}$$

where $(\alpha)_{m,k}$ is the Pochhammer k-symbol given by

$$(\alpha)_{m,k} = \alpha(\alpha+k)(\alpha+2k), \cdots, (\alpha+(m-1)k). \tag{10}$$

It is shown that the Mellin transform of the exponential function $e^{-\mu^k/k}$ is the k-gamma function clearly given by

$$\Gamma_k(\alpha) = \int_0^\infty e^{-\mu^k/k}\mu^{\alpha-1} d\mu, \tag{11}$$

for Re $(\alpha) > 0$ with $\alpha\Gamma_k(\alpha) = \Gamma_k(\alpha+k)$, where $\Gamma_k(.)$ stands for the k-gamma function.

Many researchers have generalized the classical and fractional operators by introducing a parameter $k>0$ about a decade ago. Mubeen et al. [14] used special k-function theory in fractional calculus for the first time in the literature in the form of Riemann-Liouville k-fractional integrals.

Definition 6 (see [15]). Let $\Psi \in L_1[\theta_1, \theta_2]$. The Riemann-Liouville k-fractional integrals of Ψ with $k>0$, ${}^kJ^\alpha_{\theta_1+}\Psi$ and ${}^kJ^\alpha_{\theta_2-}\Psi$ of order $\alpha>0$ with $\theta_1 \ge 0$ are defined by

$$\begin{aligned} {}^kJ^\alpha_{\theta_1+}\Psi(\zeta) &= \frac{1}{k\Gamma(\alpha)}\int_{\theta_1}^{\zeta} (\zeta-\lambda)^{(\alpha/k)-1}\Psi(\lambda)d\lambda, \quad \zeta>\theta_1, \\ {}^kJ^\alpha_{\theta_2-}\Psi(\zeta) &= \frac{1}{k\Gamma(\alpha)}\int_{\zeta}^{\theta_2} (\lambda-\zeta)^{(\alpha/k)-1}\Psi(\lambda)d\lambda, \quad \zeta<\theta_2, \end{aligned} \tag{12}$$

respectively.

Remark 7. If $k=1$, then Riemann-Liouville k-fractional integrals reduce to classical Riemann-Liouville fractional integrals. And if $\alpha=1$ and $k=1$, the fractional integral reduces to the classical integral.

Recently, many researchers are presenting new fractional differential and integral operators and they generalized by using the iteration procedure and by introducing a new parameter $k>0$. They also found relationships of these generalized fractional operators with existing fractional and classical operators under the special values of the parameter k. Many k-fractional operators, their properties, related identities, and inequalities are proved during the past years. For instance, see [16, 17] and references therein.

2. Main Results

This section contains several new generalizations of Hermite-Hadamard-Mercer-type inequalities for s-convex functions in the second sense (Breckner sense) via k-fractional calculus theory.

Throughout the paper, we assumed the following assumptions:

A_1: let $\zeta_1, \zeta_2 \in [\theta_1, \theta_2] \subseteq \mathbb{R}^+$ with $\zeta_1 < \zeta_2$, $\alpha, k > 0$, and for some fixed $s \in (0,1]$, $\lambda \in [0,1]$ and $\Gamma_k(.)$ is the k-gamma function.

A_2: for $\lambda \in [0,1]$, $((1+\lambda)/2)^s + ((1-\lambda)/2)^s \le 1$, whenever we use the definition of the Jensen-Mercer inequality for the s-convex function.

Theorem 8. *Let $\Psi : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ be the s-convex function such that $\Psi \in L_1([\theta_1, \theta_2])$ along with assumptions A_1 and A_2. Then, the following Riemann-Liouville k-fractional integral inequalities hold:*

$$\begin{aligned} &\frac{1}{2^{(\alpha/k)-s}}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right) \le \frac{\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}} \\ &\times\left\{{}^kJ^\alpha_{(\theta_1+\theta_2-\zeta_1)-}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right. \\ &\left.+{}^kJ^\alpha_{(\theta_1+\theta_2-\zeta_2)+}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right\} \le \frac{\Psi(\theta_1)+\Psi(\theta_2)}{2^{(\alpha/k)-1}} \\ &-\frac{\alpha}{k2^{(\alpha/k)+s}}\left\{U(\alpha,k,s,\lambda)+B\left(\frac{\alpha}{k},s+1\right)\right\}(\Psi(\zeta_1)+\Psi(\zeta_2)), \end{aligned} \tag{13}$$

$$\begin{aligned} &\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right) \le [\Psi(\theta_1)+\Psi(\theta_2)] - \frac{2^{(\alpha/k)-s}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}} \\ &\times\left\{{}^kJ^\alpha_{\zeta_1^+}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+{}^kJ^\alpha_{\zeta_2^-}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right\} \le \Psi(\theta_1)+\Psi(\theta_2) \\ &-\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right), \end{aligned} \tag{14}$$

where $U(\alpha,k,s,\lambda) = \int_0^1 \lambda^{(\alpha/k)-1}(1+\lambda)^s d\lambda$ and $B(-,-)$ is the beta function.

Proof. By employing the definition of the s-convex function Ψ, we get

$$\begin{aligned} &\Psi\left(\theta_1+\theta_2-\frac{u+v}{2}\right) = \Psi\left(\frac{\theta_1+\theta_2-u+\theta_1+\theta_2-v}{2}\right) \\ &\le \frac{1}{2^s}(\Psi(\theta_1+\theta_2-u)+\Psi(\theta_1+\theta_2-v))(\forall u,v \in [\theta_1,\theta_2]). \end{aligned} \tag{15}$$

By change of variables $u = ((1+\lambda)/2)\zeta_1 + ((1-\lambda)/2)\zeta_2$ and $v = ((1-\lambda)/2)\zeta_1 + ((1+\lambda)/2)\zeta_2$, $\lambda \in [0,1]$, we get

$$2^s\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\le\left[\Psi\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)+\Psi\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right]. \tag{16}$$

Now, multiplying the above inequality by $\lambda^{(\alpha/k)-1}$ and then integrating w.r.t. λ over $[0, 1]$ yield

$$2^s\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\int_0^1\lambda^{(\alpha/k)-1}d\lambda\le\int_0^1\lambda^{(\alpha/k)-1}\cdot\left[\Psi\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)+\Psi\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right]d\lambda. \tag{17}$$

By change of variable, we have

$$\frac{1}{2^{(\alpha/k)-s}}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\le\frac{\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\times\left\{{}^kJ^\alpha_{(\theta_1+\theta_2-\zeta_1)-}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+{}^kJ^\alpha_{(\theta_1+\theta_2-\zeta_2)+}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right\}, \tag{18}$$

and so the first inequality of (13) is proved.

Now, for the proof of the second inequality of (13), we first note that if Ψ is an s-convex function, then for $\lambda\in[0,1]$, it gives

$$\Psi\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\le\Psi(\theta_1)+\Psi(\theta_2)-\left[\left(\frac{1+\lambda}{2}\right)^s\Psi(\zeta_1)+\left(\frac{1-\lambda}{2}\right)^s\Psi(\zeta_2)\right], \tag{19}$$

$$\Psi\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\le\Psi(\theta_1)+\Psi(\theta_2)-\left[\left(\frac{1-\lambda}{2}\right)^s\Psi(\zeta_1)+\left(\frac{1+\lambda}{2}\right)^s\Psi(\zeta_2)\right]. \tag{20}$$

By adding the inequalities of (19) and (20), we have

$$\Psi\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)+\Psi\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\le2(\Psi(\theta_1)+\Psi(\theta_2))-\left[\left(\frac{1-\lambda}{2}\right)^s+\left(\frac{1+\lambda}{2}\right)^s\right](\Psi(\zeta_1)+\Psi(\zeta_2)). \tag{21}$$

Now, multiplying the above inequality by $\lambda^{(\alpha/k)-1}$ and then integrating w.r.t. λ over $[0, 1]$, we get

$$\int_0^1\lambda^{(\alpha/k)-1}\left[\Psi\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)+\Psi\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right]d\lambda\le\frac{2}{\alpha/k}(\Psi(\theta_1)+\Psi(\theta_2))\times\frac{1}{2^s}\left\{\int_0^1\lambda^{(\alpha/k)-1}(1+\lambda)^sd\lambda+\int_0^1\lambda^{(\alpha/k)-1}(1-\lambda)^sd\lambda\right\}(\Psi(\zeta_1)+\Psi(\zeta_2)). \tag{22}$$

Consequently, we get

$$\frac{2^{\alpha/k}k\Gamma_k(\alpha)}{(\zeta_2-\zeta_1)^{\alpha/k}}\times\left\{{}^kJ^\alpha_{(\theta_1+\theta_2-\zeta_1)-}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+{}^kJ^\alpha_{(\theta_1+\theta_2-\zeta_2)+}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right\}\le\frac{2}{\alpha/k}(\Psi(\theta_1)+\Psi(\theta_2))-\frac{1}{2^s}\left\{U(\alpha,k,s,\lambda)+B\left(\frac{\alpha}{k},s+1\right)\right\}(\Psi(\zeta_1)+\Psi(\zeta_2)), \tag{23}$$

$$\frac{\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\times\left\{{}^kJ^\alpha_{(\theta_1+\theta_2-\zeta_1)-}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+{}^kJ^\alpha_{(\theta_1+\theta_2-\zeta_2)+}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right\}\le\frac{\Psi(\theta_1)+\Psi(\theta_2)}{2^{(\alpha/k)-1}}-\frac{\alpha}{k2^{(\alpha/k)+s}}\left\{U(\alpha,k,s,\lambda)+B\left(\frac{\alpha}{k},s+1\right)\right\}(\Psi(\zeta_1)+\Psi(\zeta_2)). \tag{24}$$

Combining (18) and (23), one can get (13). In order to prove (14), we employ the Jensen-Mercer inequality for the s-convex function Ψ in the second sense; then, for $\lambda\in[0,1]$, it yields

$$\Psi\left(\theta_1+\theta_2-\frac{u+v}{2}\right)\le\Psi(\theta_1)+\Psi(\theta_2)-\frac{\Psi(u)+\Psi(v)}{2^s},\quad u,v\in[\theta_1,\theta_2]. \tag{25}$$

Now, by change of variables $u=((1+\lambda)/2)\zeta_1+((1-\lambda)/2)\zeta_2$ and $v=((1-\lambda)/2)\zeta_1+((1+\lambda)/2)\zeta_2$, $\forall\zeta_1,\zeta_2\in[\theta_1,\theta_2]$ and $\lambda\in[0,1]$ in (25), we have

$$\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\le\Psi(\theta_1)+\Psi(\theta_2)-\frac{\Psi(((1+\lambda)/2)\zeta_1+((1-\lambda)/2)\zeta_2)+\Psi(((1-\lambda)/2)\zeta_1+((1+\lambda)/2)\zeta_2)}{2^s}. \tag{26}$$

Multiplying by $\lambda^{(\alpha/k)-1}$ and then integrating w.r.t. λ over $[0,1]$ give

$$\begin{aligned}\frac{2^s k}{\alpha}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right) &\le \frac{2^s k}{\alpha}[\Psi(\theta_1)+\Psi(\theta_2)] \\ &- \int_0^1 \lambda^{(\alpha/k)-1}\left[\Psi\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right. \\ &+ \left.\Psi\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right]d\lambda.\end{aligned} \tag{27}$$

It follows that

$$\begin{aligned}\frac{2^s k}{\alpha}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right) &\le \frac{2^s k}{\alpha}(\Psi(\theta_1)+\Psi(\theta_2)) \\ &- \frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\left\{{}^kJ^{\alpha}_{\zeta_1^-}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right. \\ &+ \left.{}^kJ^{\alpha}_{\zeta_2^+}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right\}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right) \\ &\le [\Psi(\theta_1)+\Psi(\theta_2)] - \frac{2^{(\alpha/k)-s}\Gamma_k(\alpha+k)}{(\theta_2-\theta_1)^{\alpha/k}} \\ &\times \left\{{}^kJ^{\alpha}_{\zeta_1^+}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+{}^kJ^{\alpha}_{\zeta_2^-}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right\},\end{aligned} \tag{28}$$

and so the first inequality of (14) is proved.

Now, for the proof of the second inequality of (14), we first note that if Ψ is an s-convex function, then for $\lambda \in [0,1]$, it gives

$$\begin{aligned}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right) &= \Psi\left(\frac{((1+\lambda)/2)\zeta_1+((1-\lambda)/2)\zeta_2+((1-\lambda)/2)\zeta_1+((1+\lambda)/2)\zeta_2}{2}\right) \\ &\le \frac{\Psi(((1+\lambda)/2)\zeta_1+((1-\lambda)/2)\zeta_2)+\Psi(((1-\lambda)/2)\zeta_1+((1+\lambda)/2)\zeta_2)}{2^s}.\end{aligned} \tag{29}$$

Multiplying by $\lambda^{(\alpha/k)-1}$ and then integrating w.r.t. λ over $[0,1]$ give

$$\begin{aligned}\frac{2^s k}{\alpha}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right) &\le \left\{\int_0^1 \lambda^{(\alpha/k)-1}\left(\Psi\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right.\right. \\ &+ \left.\left.\Psi\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)d\lambda\right\}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right) \\ &\le \frac{2^{(\alpha/k)-s}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\left\{{}^kJ^{\alpha}_{\zeta_1^+}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+{}^kJ^{\alpha}_{\zeta_2^-}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right\}.\end{aligned} \tag{30}$$

Therefore, we have

$$\begin{aligned}&-\frac{2^{(\alpha/k)-s}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha}}\left\{{}^kJ^{\alpha}_{\zeta_1^+}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right. \\ &+ \left.{}^kJ^{\alpha}_{\zeta_2^-}\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right\} \le -\Psi\left(\frac{\zeta_1+\zeta_2}{2}\right).\end{aligned} \tag{31}$$

Adding $\Psi(\theta_1)+\Psi(\theta_2)$ to both sides in (31), we get the second inequality of (14).

Remark 9. Under the assumption of Theorem 8 for inequality (13) with $s=\alpha=k=1$, one has

$$\begin{aligned}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right) &\le \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\Psi(\theta_1+\theta_2-\lambda)d\lambda \le \Psi(\theta_1) \\ &+ \Psi(\theta_2)-\left(\frac{\Psi(\zeta_1)+\Psi(\zeta_2)}{2}\right).\end{aligned} \tag{32}$$

The inequality (32) is proposed by Kian and Moslehian in [18].

Remark 10. If we choose $s=k=1$ in Theorem 8, we get Theorem 2 in [19].

3. New Identities and Related Results via Riemann-Liouville k-Fractional Integrals

Lemma 11. *Let $\Psi:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be a differentiable mapping on (θ_1,θ_2) with $\theta_1<\theta_2$ along with assumption A_1. If $\Psi' \in L[\theta_1,\theta_2]$, then the following equality for Riemann-Liouville k-fractional integrals holds:*

$$\begin{aligned}&\left(\frac{\Psi(\theta_1+\theta_2-\zeta_1)+\Psi(\theta_1+\theta_2-\zeta_2)}{2}\right)-\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}} \\ &\times\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_2)^+}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right. \\ &+\left.{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_1)^-}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right\} \\ &= \frac{\zeta_2-\zeta_1}{4}\left[\int_0^1\lambda^{\alpha/k}\left\{\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right.\right. \\ &-\left.\left.\Psi'\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right\}d\lambda\right].\end{aligned} \tag{33}$$

Proof. It suffices to write that

$$I=\frac{\zeta_2-\zeta_1}{4}\{I_1-I_2\}, \tag{34}$$

where

$$\begin{aligned}I_1 &= \int_0^1\lambda^{\alpha/k}\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)d\lambda \\ &= \frac{2\Psi(\theta_1+\theta_2-\zeta_1)}{(\zeta_2-\zeta_1)}-\frac{2^{(\alpha/k)+1}(\alpha/k)}{(\zeta_2-\zeta_1)^{(\alpha/k)+1}}\int_{\theta_1+\theta_2-((\zeta_1+\zeta_2)/2)}^{\theta_1+\theta_2-\zeta_1} \\ &\cdot\left(w-\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)^{(\alpha/k)-1}\Psi(w)dw = \frac{2\Psi(\theta_1+\theta_2-\zeta_1)}{(\zeta_2-\zeta_1)} \\ &- \frac{2^{(\alpha/k)+1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{(\alpha/k)+1}}\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_1)^-}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right\}.\end{aligned} \tag{35}$$

Analogously,

$$\begin{aligned}I_2 &= \int_0^1 \lambda^{\alpha/k}\Psi'\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)d\lambda\\ &= -\frac{2\Psi(\theta_1+\theta_2-\zeta_2)}{(\zeta_2-\zeta_1)}+\frac{2^{(\alpha/k)+1}(\alpha/k)}{(\zeta_2-\zeta_1)^{(\alpha/k)+1}}\\ &\quad\times\int_{\theta_1+\theta_2-\zeta_2}^{\theta_1+\theta_2-((\zeta_1+\zeta_2)/2)}\left(\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-w\right)^{(\alpha/k)-1}\Psi(w)dw\\ &= -\frac{2\Psi(\theta_1+\theta_2-\zeta_2)}{(\zeta_2-\zeta_1)}+\frac{2^{(\alpha/k)+1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{(\alpha/k)+1}}\\ &\quad\cdot\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_2)^+}\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right\}.\end{aligned} \tag{36}$$

Combining (35) and (36) with (34), we get (33).

Corollary 12. *For $\alpha = k = 1$ in Lemma 11, we acquire*

$$\begin{aligned}&\frac{\Psi(\theta_1+\theta_2-\zeta_1)+i(\theta_1+\theta_2-\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\theta_1+\theta_2-\zeta_2}^{\theta_1+\theta_2-\zeta_1}\Psi(u)du\\ &= \frac{\zeta_2-\zeta_1}{4}\int_0^1\lambda\left\{\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right.\\ &\quad\left.-\Psi'\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right\}d\lambda.\end{aligned} \tag{37}$$

Remark 13. Taking $\alpha = k = 1$ with $\zeta_1 = \theta_1$ and $\zeta_2 = \theta_2$ in Lemma 11, we get Lemma 2.1 in [20] and the following equality holds:

$$\begin{aligned}&\frac{\Psi(\theta_1)+\Psi(\theta_2)}{2}-\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\Psi(u)du = \frac{\theta_2-\theta_1}{4}\int_0^1\lambda\left\{\Psi'\right.\\ &\quad\cdot\left(\frac{1-\lambda}{2}\theta_1+\frac{1+\lambda}{2}\theta_2\right)-\Psi'\\ &\quad\left.\cdot\left(\frac{1+\lambda}{2}\theta_1+\frac{1-\lambda}{2}\theta_2\right)\right\}d\lambda.\end{aligned} \tag{38}$$

Theorem 14. *Suppose that $\Psi : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ is a differentiable mapping on (θ_1, θ_2) with $\theta_1 < \theta_2$ and $\Psi' \in L[\theta_1, \theta_2]$ along with assumptions A_1 and A_2. If $|\Psi'|$ is an s-convex function on $[\theta_1, \theta_2]$, then the following inequality for Riemann-Liouville k-fractional integrals holds:*

$$\begin{aligned}&\left|\frac{\Psi(\theta_1+\theta_2-\zeta_1)+\Psi(\theta_1+\theta_2-\zeta_2)}{2}-\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\right.\\ &\quad\times\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_1)^-}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right.\\ &\quad\left.\left.+{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_2)^+}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right\}\right|\\ &\le\frac{\zeta_2-\zeta_1}{4}\left[\frac{2}{((\alpha/k)+1)}\left(|\Psi'(\theta_1)|+|\Psi'(\theta_2)|\right)\right.\\ &\quad\left.-\frac{1}{2^s}\left\{U(\alpha,k,s,\lambda)+B\left(\frac{\alpha}{k},s+1\right)\right\}(\Psi(\zeta_1)+\Psi(\zeta_2))\right],\end{aligned} \tag{39}$$

where $U(\alpha, k, s, \lambda) = \int_0^1 \lambda^{\alpha/k}(1+\lambda)^s d\lambda$.

Proof. By using Lemma 11 and the Jensen-Mercer inequality and the s-convexity of $|\Psi'|$, we have

$$\begin{aligned}&\left|\left(\frac{\Psi(\theta_1+\theta_2-\zeta_1)+\Psi(\theta_1+\theta_2-\zeta_2)}{2}\right)-\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\right.\\ &\quad\times\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_2)^+}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right.\\ &\quad\left.\left.+{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_1)^-}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right\}\right|\\ &\le\frac{\zeta_2-\zeta_1}{4}\left[\int_0^1\lambda^{\alpha/k}\left\{\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right|\right.\right.\\ &\quad\left.\left.+\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right|\right\}d\lambda\right]\\ &\le\frac{\zeta_2-\zeta_1}{4}\left[\int_0^1\lambda^{\alpha/k}\left\{|\Psi'(\theta_1)|+|\Psi'(\theta_2)|\right.\right.\\ &\quad\left.-\left(\left(\frac{1+\lambda}{2}\right)^s|\Psi'(\zeta_1)|+\left(\frac{1-\lambda}{2}\right)^s|\Psi'(\zeta_2)|\right)\right\}d\lambda\\ &\quad+\int_0^1\lambda^{\alpha/k}\left\{|\Psi'(\theta_1)|+|\Psi'(\theta_2)|-\left(\left(\frac{1-\lambda}{2}\right)^s|\Psi'(\zeta_1)|\right.\right.\\ &\quad\left.\left.\left.+\left(\frac{1+\lambda}{2}\right)^s|\Psi'(\zeta_2)|\right)\right\}d\lambda\right].\end{aligned} \tag{40}$$

After simple computations, we get the required result of (39).

Corollary 15. *For $s = \alpha = k = 1$ in Theorem 14, we get*

$$\begin{aligned}&\left|\frac{\Psi(\theta_1+\theta_2-\zeta_1)+\Psi(\theta_1+\theta_2-\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\theta_1+\theta_2-\zeta_2}^{\theta_1+\theta_2-\zeta_1}\Psi(u)du\right|\\ &\le\frac{\zeta_2-\zeta_1}{4}\left\{|\Psi'(\theta_1)|+|\Psi'(\theta_2)|-\left(\frac{|\Psi'(\zeta_1)|+|\Psi'(\zeta_2)|}{2}\right)\right\}.\end{aligned} \tag{41}$$

Remark 16. Taking $s = \alpha = k = 1$ with $\zeta_1 = \theta_1$ and $\zeta_2 = \theta_2$ in Theorem 14, we recapture Theorem 2.2 in [21]:

$$\begin{aligned}&\left|\frac{\Psi(\theta_1)+\Psi(\theta_2)}{2}-\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\Psi(u)du\right|\\ &\le\frac{\theta_2-\theta_1}{4}\left(\frac{|\Psi'(\theta_1)|+|\Psi'(\theta_2)|}{2}\right).\end{aligned} \tag{42}$$

Remark 17. If we choose $s = k = 1$ in Theorem 14, we get Theorem 4 in [19].

Theorem 18. *Suppose that $\Psi : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ is a differentiable function on (θ_1, θ_2) with $\theta_1 < \theta_2$ along with assumptions A_1 and A_2. If $\Psi' \in L[\theta_1, \theta_2]$ and $|\Psi'|^q$ is an s-convex function on $[\theta_1, \theta_2]$, where $(1/r) + (1/q) = 1$, $r > 1$, with $q = r/(r-1)$, then the following inequality for Riemann-Liouville k-fractional integrals holds:*

$$\begin{aligned}&\left|\left(\frac{\Psi(\theta_1+\theta_2-\zeta_1)+\Psi(\theta_1+\theta_2-\zeta_2)}{2}\right)-\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\right.\\&\times\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_2)^+}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right.\\&\left.\left.+{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_1)^-}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right\}\right|\le\frac{\zeta_2-\zeta_1}{4}\\&\times\left[\frac{1}{((\alpha/k)r+1)^r}\left\{\left(\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q\right.\right.\right.\\&\left.-\left(\frac{2^{S+1}-1}{2^s(s+1)}\left|\Psi'(\zeta_1)\right|^q+\frac{1}{2^s(s+1)}\left|\Psi'(\zeta_2)\right|^q\right)\right)^{1/q}\\&+\left(\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q-\left(\frac{1}{2^s(s+1)}\left|\Psi'(\zeta_1)\right|^q\right.\right.\\&\left.\left.\left.\left.+\frac{2^{S+1}-1}{2^s(s+1)}\left|\Psi'(\zeta_2)\right|^q\right)\right)^{1/q}\right\}\right].\end{aligned}\tag{43}$$

Proof. Employing Lemma 11 and the Jensen-Mercer inequality with noted Hölder's inequality and utilizing the s-convexity of $|\Psi'|^q$, we have

$$\begin{aligned}&\left|\left(\frac{\Psi(\theta_1+\theta_2-\zeta_1)+\Psi(\theta_1+\theta_2-\zeta_2)}{2}\right)-\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\right.\\&\times\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_2)^+}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right.\\&\left.\left.+{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_1)^-}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right\}\right|\\&\le\frac{\zeta_2-\zeta_1}{4}\left[\int_0^1\lambda^{\alpha/k}\left\{\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right|\right.\right.\\&\left.\left.+\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right|\right\}d\lambda\right]\\&\le\frac{\zeta_2-\zeta_1}{4}\left(\int_0^1\lambda^{(\alpha/k)r}d\lambda\right)^{1/r}\\&\cdot\left[\left(\int_0^1\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)^q d\lambda\right|\right)^{1/q}\right.\\&\left.+\left(\int_0^1\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right|^q d\lambda\right)^{1/q}\right]\\&\le\frac{\zeta_2-\zeta_1}{4}\left(\int_0^1\lambda^{(\alpha/k)r}d\lambda\right)^{1/r}\times\left[\left(\int_0^1\left\{\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q\right.\right.\right.\\&\left.\left.-\left(\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'(\zeta_1)\right|^q+\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'(\zeta_2)\right|^q\right)\right\}d\lambda\right)^{1/q}\\&+\left(\int_0^1\left\{\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q-\left(\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'(\zeta_1)\right|^q\right.\right.\\&\left.\left.\left.+\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'(\zeta_2)\right|^q\right)\right\}d\lambda\right)^{1/q}\right]\le\frac{\zeta_2-\zeta_1}{4}\\&\cdot\left[\frac{1}{((\alpha/k)r+1)^{1/r}}\left\{\left(\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q-\left(\frac{2^{S+1}-1}{2^s(s+1)}\left|\Psi'(\zeta_1)\right|^q\right.\right.\right.\\&\left.\left.+\frac{1}{2^s(s+1)}\left|\Psi'(\zeta_2)\right|^q\right)\right)^{1/q}+\left(\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q\right.\\&\left.\left.\left.-\left(\frac{1}{2^s(s+1)}\left|\Psi'(\zeta_1)\right|^q+\frac{2^{S+1}-1}{2^s(s+1)}\left|\Psi'(\zeta_2)\right|^q\right)\right)^{1/q}\right\}\right].\end{aligned}\tag{44}$$

This completes the proof.

Theorem 19. *Suppose that $\Psi:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ is a differentiable function on (θ_1,θ_2) with $\theta_1<\theta_2$ along with assumptions A_1 and A_2. If $\Psi'\in L[\theta_1,\theta_2]$ and $|\Psi'|^q$ is an s-convex function on $[\theta_1,\theta_2]$, where $(1/r)+(1/q)=1$, $r\ge1$, with $q=r/(r-1)$, then the following inequality for Riemann-Liouville k-fractional integrals holds:*

$$\begin{aligned}&\left|\left(\frac{\Psi(\theta_1+\theta_2-\zeta_1)+\Psi(\theta_1+\theta_2-\zeta_2)}{2}\right)\right.\\&-\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\\&\times\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_2)^+}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right.\\&\left.\left.+{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_1)^-}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right\}\right|\\&\le\frac{\zeta_2-\zeta_1}{4}\left[\left(\frac{1}{(\alpha/k)+1}\right)^{1-(1/q)}\right.\\&\times\left\{\left(\frac{\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q}{(\alpha/k)+1}\right.\right.\\&-\left(\frac{U(\alpha,k,s,\lambda)}{2^s}\left|\Psi'(\zeta_1)\right|^q\right.\\&\left.\left.+\frac{B((\alpha/k)+1,s+1)}{2^s}\left|\Psi'(\zeta_2)\right|^q\right)\right)^{1/q}\\&+\left(\frac{\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q}{(\alpha/k)+1}\right.\\&-\left(\frac{B((\alpha/k)+1,s+1)}{2^s}\left|\Psi'(\zeta_1)\right|^q\right.\\&\left.\left.\left.\left.+\frac{U(\alpha,k,s,\lambda)}{2^s}\left|\Psi'(\zeta_2)\right|^q\right)\right)^{1/q}\right\}\right],\end{aligned}\tag{45}$$

where $U(\alpha,k,s,\lambda)=\int_0^1\lambda^{\alpha/k}(1+\lambda)^s d\lambda$.

Proof. For $r\ge1$, taking into account Lemma 11 and the Jensen-Mercer inequality with the noted power-mean inequality and utilizing the s-convexity of $|\Psi'|^q$, we have

$$\begin{aligned}&\left|\left(\frac{\Psi(\theta_1+\theta_2-\zeta_1)+\Psi(\theta_1+\theta_2-\zeta_2)}{2}\right)\right.\\&-\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\\&\times\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-\zeta_2)^+}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\right.\end{aligned}$$

$$+{}^{k}J^{\alpha}_{(\theta_1+\theta_2-\zeta_1)^-}\left(\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)\Bigg\}\Bigg|$$

$$\leq \frac{\zeta_2-\zeta_1}{4}\Bigg[\int_0^1 \lambda^{\alpha/k}$$

$$\times\left\{\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right|\right.$$

$$\left.+\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right|\right\}d\lambda\Bigg]$$

$$\leq \frac{\zeta_2-\zeta_1}{4}\left[\left(\int_0^1\lambda^{\alpha/k}d\lambda\right)^{1-(1/q)}\right.$$

$$\times\left\{\left(\int_0^1\lambda^{\alpha/k}\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right|^q d\lambda\right)^{1/q}\right.$$

$$\left.\left.+\left(\int_0^1\lambda^{\alpha/k}\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right|^q d\lambda\right)^{1/q}\right\}\right]$$

$$\leq \frac{\zeta_2-\zeta_1}{4}\left[\left(\int_0^1\lambda^{\alpha/k}d\lambda\right)^{1-(1/q)}\right.$$

$$\times\left\{\left(\int_0^1\lambda^{\alpha/k}\left\{\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q\right.\right.\right.$$

$$\left.\left.-\left(\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'(\zeta_1)\right|^q+\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'(\zeta_2)\right|^q\right)\right\}d\lambda\right)^{1/q}$$

$$+\left(\int_0^1\lambda^{\alpha/k}\left\{\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q\right.\right.$$

$$\left.\left.\left.\left.-\left(\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'(\zeta_1)\right|^q+\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'(\zeta_2)\right|^q\right)\right\}d\lambda\right)^{1/q}\right\}\right]. \tag{46}$$

Simple computations yield the desired inequality (45).

Next, we demonstrate results for twice differentiable functions Ψ''. For that, we give the following new lemma.

Lemma 20. *Let $\Psi:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be a differentiable mapping on (θ_1,θ_2) with $\theta_1<\theta_2$ along with assumption A_1. If $\Psi''\in L[\theta_1,\theta_2]$, then the following equality for Riemann-Liouville k-fractional integrals holds:*

$$\begin{aligned}
&\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\left\{{}^{k}J^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\Psi(\theta_1+\theta_2-\zeta_1)\right.\\
&\quad\left.+{}^{k}J^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\Psi(\theta_1+\theta_2-\zeta_2)\right\}-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\\
&=\frac{(\zeta_2-\zeta_1)^2}{8((\alpha/k)+1)}\left\{\int_0^1(1-\lambda)^{(\alpha/k)+1}\Psi''\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1\right.\right.\right.\\
&\quad\left.\left.\left.+\frac{1-\lambda}{2}\zeta_2\right)\right)d\lambda+\int_0^1(1-\lambda)^{(\alpha/k)+1}\Psi''\\
&\quad\left.\cdot\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)d\lambda\right\}.
\end{aligned} \tag{47}$$

Proof. It suffices to write that

$$I=\frac{(\zeta_2-\zeta_1)^2}{8((\alpha/k)+1)}\{I_1+I_2\}, \tag{48}$$

where

$$\begin{aligned}
I_1&=\int_0^1(1-\lambda)^{(\alpha/k)+1}\Psi''\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)d\lambda\\
&=-\frac{2}{(\zeta_2-\zeta_1)}\Psi'\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{2((\alpha/k)+1)}{(\theta_2-\theta_1)}\\
&\quad\times\int_0^1(1-\lambda)^{\alpha/k}\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)d\lambda\\
&=-\frac{2}{(\zeta_2-\zeta_1)}\Psi'\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{4((\alpha/k)+1)}{(\zeta_2-\zeta_1)^2}\Psi\\
&\quad\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{4((\alpha/k)+1)}{(\zeta_2-\zeta_1)^2}\int_0^1(1-\lambda)^{(\alpha/k)-1}\Psi\\
&\quad\cdot\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)d\lambda\\
&=-\frac{2}{(\zeta_2-\zeta_1)}\Psi'\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{4((\alpha/k)+1)}{(\zeta_2-\zeta_1)^2}\Psi\\
&\quad\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{2^{(\alpha/k)+2}((\alpha/k)+1)\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{(\alpha/k)+2}}\\
&\quad\cdot\left({}^{k}J^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\right)(\Psi(\theta_1+\theta_2-\zeta_1)),
\end{aligned} \tag{49}$$

and similarly, we can find

$$\begin{aligned}
I_2&=\int_0^1(1-\lambda)^{(\alpha/k)+1}\Psi''\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)d\lambda\\
&=\frac{2}{(\zeta_2-\zeta_1)}\Psi'\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{4((\alpha/k)+1)}{(\zeta_2-\zeta_1)^2}\Psi\\
&\quad\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{2^{(\alpha/k)+2}((\alpha/k)+1)\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{(\alpha/k)+2}}\\
&\quad\cdot\left({}^{k}J^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\right)(\Psi(\theta_1+\theta_2-\zeta_2)).
\end{aligned} \tag{50}$$

Combining (49) and (50) with (48), we get the identity (47).

Remark 21. In Lemma 20, taking $k=1$, with $\zeta_1=\theta_1$ and $\zeta_2=\theta_2$, recaptures Lemma 1 in [22].

Remark 22. For $\alpha=k=1$, with $\zeta_1=\theta_1$ and $\zeta_2=\theta_2$, in Lemma 20, it reduces to Lemma 2 proved in [22].

Theorem 23. *Suppose that $\Psi:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ is a differentiable mapping on (θ_1,θ_2) with $\theta_1<\theta_2$ and $\Psi\in L[\theta_1,\theta_2]$ along with assumptions A_1 and A_2. If $|\Psi''|$ is an s-convex function on $[\theta_1,\theta_2]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\Psi(\theta_1+\theta_2-\zeta_1)+{}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\Psi(\theta_1+\theta_2-\zeta_2)\right\}-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\\&\le\frac{(\zeta_2-\zeta_1)^2}{8((\alpha/k)+1)}\left[2\left(\frac{|\Psi''(\theta_1)|+|\Psi''(\theta_2)|}{(\alpha/k)+2}\right)-\left\{\left(\frac{U_1(\alpha,k,s,\lambda)}{2^s}+\frac{\Gamma((\alpha/k)+s+2)}{2^s\Gamma((\alpha/k)+s+3)}\right)\cdot\left(|\Psi''(\zeta_1)|+|\Psi''(\zeta_2)|\right)\right\}\right],\end{aligned}\tag{51}$$

where $U_1(\alpha,k,s,\lambda)=\int_0^1(1-\lambda)^{(\alpha/k)+1}(1+\lambda)^s d\lambda$.

Proof. By using Lemma 20 with the Jensen-Mercer inequality and the s-convexity of $|\Psi''|$, we have

$$\begin{aligned}&\left|\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\Psi(\theta_1+\theta_2-\zeta_1)+{}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\Psi(\theta_1+\theta_2-\zeta_2)\right\}-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\\&\le\frac{(\zeta_2-\zeta_1)^2}{8((\alpha/k)+1)}\left\{\int_0^1(1-\lambda)^{(\alpha/k)+1}\left|\Psi''\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right|d\lambda+\int_0^1(1-\lambda)^{(\alpha/k)+1}\left|\Psi''\cdot\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right|d\lambda\right\}\le\frac{(\zeta_2-\zeta_1)^2}{8((\alpha/k)+1)}\\&\times\left[\int_0^1(1-\lambda)^{(\alpha/k)+1}\left\{|\Psi''(\theta_1)|+|\Psi''(\theta_2)|-\left(\left(\frac{1+\lambda}{2}\right)^s\cdot|\Psi''(\zeta_1)|+\left(\frac{1-\lambda}{2}\right)^s|\Psi''(\zeta_2)|\right)\right\}d\lambda+\int_0^1(1-\lambda)^{(\alpha/k)+1}\right.\\&\left.\cdot\left\{|\Psi''(\theta_1)|+|\Psi''(\theta_2)|-\left(\left(\frac{1-\lambda}{2}\right)^s|\Psi''(\zeta_1)|+\left(\frac{1+\lambda}{2}\right)^s|\Psi''(\zeta_2)|\right)\right\}d\lambda\right].\end{aligned}\tag{52}$$

After simplifications, we get the required result.

Remark 24. For choosing $k=1$ with $\zeta_1=\theta_1$ and $\zeta_2=\theta_2$ in Theorem 23, we will get Theorem 2 proved in [22].

Theorem 25. *Suppose that $\Psi:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ is a differentiable function on I along with assumptions A_1 and A_2. If $\Psi''\in L[\theta_1,\theta_2]$ and $|\Psi''|^q$ is an s-convex function, where $(1/r)+(1/q)=1$, $q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\Psi(\theta_1+\theta_2-\zeta_1)+{}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\Psi(\theta_1+\theta_2-\zeta_2)\right\}-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\\&\le\frac{(\zeta_2-\zeta_1)^2}{8((\alpha/k)+1)}\left[\left(\frac{1}{((\alpha/k)+1)r+1}\right)^{1/r}\times\left\{\left(|\Psi''(\theta_1)|^q+|\Psi''(\theta_2)|^q-\left(\frac{2^{S+1}-1}{2^s(s+1)}|\Psi''(\zeta_1)|^q+\frac{1}{2^s(s+1)}|\Psi''(\zeta_2)|^q\right)\right)^{1/q}\right.\right.\\&\left.\left.+\left(|\Psi''(\theta_1)|^q+|\Psi''(\theta_2)|^q-\left(\frac{1}{2^s(s+1)}|\Psi''(\zeta_1)|^q+\frac{2^{S+1}-1}{2^s(s+1)}|\Psi''(\zeta_2)|^q\right)\right)^{1/q}\right\}\right].\end{aligned}\tag{53}$$

Proof. By using Lemma 20 and the well-known Hölder's inequality and the Jensen-Mercer inequality along with the fact that $|\Psi''|^q$ is an s-convex function, we have

$$\begin{aligned}&\left|\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\times\left\{{}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\Psi(\theta_1+\theta_2-\zeta_1)+{}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\Psi(\theta_1+\theta_2-\zeta_2)\right\}-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\\&\le\frac{(\zeta_2-\zeta_1)^2}{8((\alpha/k)+1)}\left\{\int_0^1(1-\lambda)^{(\alpha/k)+1}\times\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right|d\lambda\right.\\&\left.+\int_0^1(1-\lambda)^{(\alpha/k)+1}\times\left|\Psi'\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right|d\lambda\right\}\\&\le\frac{(\zeta_2-\zeta_1)^2}{8((\alpha/k)+1)}\times\left\{\left(\int_0^1(1-\lambda)^{r((\alpha/k)+1)}d\lambda\right)\times\left(\int_0^1\left|\Psi'(\theta_1)+\Psi'(\theta_2)-\left(\frac{1+\lambda}{2}\Psi'(\zeta_1)+\frac{1-\lambda}{2}\Psi'(\zeta_2)\right)\right|^q d\lambda\right)^{1/q}\right.\\&+\left(\int_0^1(1-\lambda)^{r((\alpha/k)+1)}d\lambda\right)^{1/r}\end{aligned}$$

$$\times\left(\int_0^1\left|\Psi'(\theta_1)+\Psi'(\theta_2)\right.\right.$$
$$\left.\left.-\left(\frac{1-\lambda}{2}\Psi'(\zeta_1)+\frac{1+\lambda}{2}\Psi'(\zeta_2)\right)\right|^q d\lambda\right)^{1/q}\Bigg\}$$
$$\le\frac{(\zeta_2-\zeta_1)^2}{8((\alpha/k)+1)}\left[\left(\left(\int_0^1(1-\lambda)^{((\alpha/k)+1)r}d\lambda\right)^{1/r}\right.\right.$$
$$\times\left\{\left(\int_0^1\left(\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q\right.\right.\right.$$
$$\left.\left.\left.-\left(\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'(\zeta_1)\right|^q+\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'(\zeta_2)\right|^q\right)\right)\right)^{1/q}$$
$$+\left(\int_0^1\left(\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'(\theta_1)\right|^q+\left|\Psi'(\theta_2)\right|^q\right.\right.$$
$$\left.\left.\left.\left.-\left(\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'(\zeta_1)\right|^q+\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'(\zeta_2)\right|^q\right)\right)\right)^{1/q}\right\}\right]. \tag{54}$$

By direct computations, we get the required result.

Remark 26. For choosing $k=1$ with $\zeta_1=\theta_1$ and $\zeta_2=\theta_2$ in Theorem 25, we will get Theorem 3 proved in [22].

Corollary 27. *For $\alpha=k=1$ in Theorem 25, we get*

$$\left|\frac{1}{\zeta_2-\zeta_1}\int_{\theta_1+\theta_2-\zeta_2}^{\theta_1+\theta_2-\zeta_1}\Psi(u)du-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\le\frac{(\zeta_2-\zeta_1)^2}{16}\left(\frac{1}{2r+1}\right)^{1/r}\times\left[\left\{\left(\left|\Psi''(\theta_1)\right|^q+\left|\Psi''(\theta_2)\right|^q\right)\right.\right.$$
$$\left.-\left(\frac{2^{s+1}-1}{2^s(s+1)}\left|\Psi''(\zeta_1)\right|^q+\frac{1}{2^s(s+1)}\left|\Psi''(\zeta_2)\right|^q\right)\right\}^{1/q}$$
$$+\left\{\left(\left|\Psi''(\theta_1)\right|^q+\left|\Psi''(\theta_2)\right|^q\right)-\left(\frac{1}{2^s(s+1)}\left|\Psi''(\zeta_1)\right|^q\right.\right.$$
$$\left.\left.\left.+\frac{2^{s+1}-1}{2^s(s+1)}\left|\Psi''(\zeta_2)\right|^q\right)\right\}^{1/q}\right]. \tag{55}$$

Finally, we state our results for third-order differentiable functions Ψ'''.

Lemma 28. *Let $\Psi:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be a differentiable mapping on (θ_1,θ_2) with $\theta_1<\theta_2$ along with assumption A_1. If $\Psi'''\in L[\theta_1,\theta_2]$, then the following equality for Riemann-Liouville k -fractional integrals holds:*

$$\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}$$
$$\times\left\{\left({}^kJ^\alpha_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\right)(\Psi(\theta_1+\theta_2-\zeta_1))+\left({}^kJ^\alpha_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\right)\right.$$
$$\left.\cdot(\Psi(\theta_1+\theta_2-\zeta_2))\right\}-\frac{(\zeta_2-\zeta_1)^2}{4((\alpha/k)+1)((\alpha/k)+2)}\Psi''$$
$$\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)$$
$$=\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}\left\{\int_0^1(1-\lambda)^{(\alpha/k)+2}\Psi'''\right.$$
$$\cdot\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)d\lambda$$
$$\left.-\int_0^1(1-\lambda)^{(\alpha/k)+2}\Psi'''\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)d\lambda\right\}. \tag{56}$$

Proof. It suffices to write that

$$I=\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}\{I_1-I_2\}, \tag{57}$$

where

$$I_1=\int_0^1(1-\lambda)^{(\alpha/k)+2}\Psi'''\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)d\lambda$$
$$=-\frac{2}{(\zeta_2-\zeta_1)}\Psi''\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{2((\alpha/k)+2)}{(\zeta_2-\zeta_1)}$$
$$\cdot\int_0^1(1-\lambda)^{(\alpha/k)+1}\Psi''\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)d\lambda$$
$$=-\frac{2}{(\zeta_2-\zeta_1)}\Psi''\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{4((\alpha/k)+2)}{(\zeta_2-\zeta_1)^2}\Psi'$$
$$\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{4((\alpha/k)+1)((\alpha/k)+2)}{(\zeta_2-\zeta_1)^2}$$
$$\cdot\int_0^1(1-\lambda)^{\alpha/k}\Psi'\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)d\lambda$$
$$=-\frac{2}{(\zeta_2-\zeta_1)}\Psi''\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{4((\alpha/k)+2)}{(\zeta_2-\zeta_1)^2}\Psi'$$
$$\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{8((\alpha/k)+2)((\alpha/k)+1)}{(\zeta_2-\zeta_1)^3}\Psi$$
$$\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{2^{(\alpha/k)+3}((\alpha/k)+1)((\alpha/k)+2)\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{(\alpha/k)+3}}$$
$$\cdot\left({}^kJ^\alpha_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\right)(\Psi(\theta_1+\theta_2-\zeta_1))$$
$$=-\frac{2}{(\zeta_2-\zeta_1)}\Psi''\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{4((\alpha/k)+2)}{(\zeta_2-\zeta_1)^2}\Psi'$$
$$\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{8((\alpha/k)+2)((\alpha/k)+1)}{(\zeta_2-\zeta_1)^3}\Psi$$
$$\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{2^{(\alpha/k)+3}\Gamma_k(\alpha+3k)}{(\zeta_2-\zeta_1)^{(\alpha/k)+3}}$$
$$\cdot\left(J^\alpha_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\right)(\Psi(\theta_1+\theta_2-\zeta_1)), \tag{58}$$

and similarly, we can find

$$\begin{aligned}I_2 &= \int_0^1 (1-\lambda)^{(\alpha/k)+2}\Psi'''\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)d\lambda\\
&= \frac{2}{(\zeta_2-\zeta_1)}\Psi''\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{4((\alpha/k)+2)}{(\zeta_2-\zeta_1)^2}\Psi'\\
&\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{8((\alpha/k)+2)((\alpha/k)+1)}{(\zeta_2-\zeta_1)^3}\Psi\\
&\cdot\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\frac{2^{(\alpha/k)+3}\Gamma_k(\alpha+3k)}{(\zeta_2-\zeta_1)^{(\alpha/k)+3}}\\
&\cdot\left(J^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\right)(\Psi(\theta_1+\theta_2-\zeta_2)).\end{aligned} \tag{59}$$

Replacing the values of the integrals I_1 and I_2 in (57), we get the identity (56).

Remark 29. In Lemma 28, choosing $k=1$ with $\zeta_1=\theta_1$ and $\zeta_2=\theta_2$, it recaptures Lemma 3.1 proved in [23].

Remark 30. For $k=\alpha=1$ with $\zeta_1=\theta_1$ and $\zeta_2=\theta_2$ in Lemma 28, it reduces to Lemma 2.1 proved in [24].

Theorem 31. *Suppose that $\Psi:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ is a three times differentiable mapping on (θ_1,θ_2) with $\theta_1<\theta_2$ and $\Psi\in L[\theta_1,\theta_2]$ along with assumptions A_1 and A_2. If $|\Psi'''|$ is an s-convex function on $[\theta_1,\theta_2]$, then the following inequality for k-fractional integrals holds:*

$$\begin{aligned}&\left|\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{(\alpha/k)}}\times\left\{\left({}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\right)(\Psi(\theta_1+\theta_2-\zeta_1))\right.\right.\\
&\left.+\left({}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\right)(\Psi(\theta_1+\theta_2-\zeta_2))\right\}\\
&-\frac{(\zeta_2-\zeta_1)^2}{4((\alpha/k)+1)((\alpha/k)+2)}\Psi''\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\\
&\left.-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\le\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}\\
&\cdot\left[2\left(\frac{|\Psi'''(\theta_1)|+|\Psi'''(\theta_2)|}{(\alpha/k)+3}\right)-\left\{\left(\frac{U_2(\alpha,k,s,\lambda)}{2^s}\right.\right.\right.\\
&\left.\left.\left.+\frac{1}{2^s((\alpha/k)+s+3)}\right)\left(|\Psi'''(\zeta_1)|+|\Psi'''(\zeta_2)|\right)\right\}\right],\end{aligned} \tag{60}$$

where $U_2(\alpha,k,s,\lambda)=\int_0^1(1-\lambda)^{(\alpha/k)+2}(1+\lambda)^s d\lambda$.

Proof. By using Lemma 28 and the Jensen-Mercer inequality and the s-convexity of $|\Psi'''|$, we have

$$\begin{aligned}&\left|\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{(\alpha/k)}}\times\left\{\left({}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\right)(\Psi(\theta_1+\theta_2-\zeta_1))\right.\right.\\
&\left.+\left({}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\right)(\Psi(\theta_1+\theta_2-\zeta_2))\right\}\\
&-\frac{(\zeta_2-\zeta_1)^2}{4((\alpha/k)+1)((\alpha/k)+2)}\Psi''\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\\
&\left.-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\le\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}\\
&\cdot\left\{\int_0^1(1-\lambda)^{(\alpha/k)+2}\left|\Psi'''\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right|d\lambda\right.\\
&\left.\cdot-\int_0^1(1-\lambda)^{(\alpha/k)+2}\left|\Psi'''\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right|d\lambda\right\}\\
&\le\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}\left[\int_0^1(1-\lambda)^{(\alpha/k)+2}\left\{|\Psi'''(\theta_1)|\right.\right.\\
&\left.+|\Psi'''(\theta_2)|-\left(\left(\frac{1+\lambda}{2}\right)^s|\Psi'''(\zeta_1)|+\left(\frac{1-\lambda}{2}\right)^s|\Psi'''(\zeta_2)|\right)\right\}d\lambda\\
&+\int_0^1(1-\lambda)^{(\alpha/k)+2}\left\{|\Psi'''(\theta_1)|+|\Psi'''(\theta_2)|-\left(\left(\frac{1-\lambda}{2}\right)^s|\Psi'''(\zeta_1)|\right.\right.\\
&\left.\left.\left.+\left(\frac{1+\lambda}{2}\right)^s|\Psi'''(\zeta_2)|\right)\right\}d\lambda\right]\le\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}\\
&\cdot\left[2\left(\frac{|\Psi'''(\theta_1)|+|\Psi'''(\theta_2)|}{(\alpha/k)+3}\right)-\left\{\left(\frac{U_2(\alpha,k,s,\lambda)}{2^s}\right.\right.\right.\\
&\left.\left.\left.+\frac{1}{2^s((\alpha/k)+s+3)}\right)\left(|\Psi'''(\zeta_1)|+|\Psi'''(\zeta_2)|\right)\right\}\right].\end{aligned} \tag{61}$$

Remark 32. Choosing $k=s=1$ with $\zeta_1=\theta_1$ and $\zeta_2=\theta_2$ in Theorem 31, we will get Theorem 18 proved in [23].

Theorem 33. *Suppose that $\Psi:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ is a three times differentiable function on I along with assumptions A_1 and A_2. If $\Psi'''\in L[\theta_1,\theta_2]$ and $|\Psi'''|^q$ is an s-convex function, where $(1/r)+(1/q)=1$, $q>1$, then the following k-fractional integral inequality holds:*

$$\begin{aligned}&\left|\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{(\alpha/k)}}\times\left\{\left({}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^+}\right)(\Psi(\theta_1+\theta_2-\zeta_1))\right.\right.\\
&\left.+\left({}^kJ^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\right)(\Psi(\theta_1+\theta_2-\zeta_2))\right\}\\
&-\frac{(\zeta_2-\zeta_1)^2}{4((\alpha/k)+1)((\alpha/k)+2)}\Psi''\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\\
&\left.-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\le\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}\\
&\cdot\left[\left(\frac{1}{((\alpha/k)+2)r+1}\right)^{1/r}\times\left\{\left(|\Psi'''(\theta_1)|^q+|\Psi'''(\theta_2)|^q\right.\right.\right.\\
&\left.-\left(\frac{2^{S+1}-1}{2^s(s+1)}|\Psi'''(\zeta_1)|^q+\frac{1}{2^s(s+1)}|\Psi'''(\zeta_2)|^q\right)\right)^{1/q}\\
&+\left(|\Psi'''(\theta_1)|^q+|\Psi'''(\theta_2)|^q-\left(\frac{1}{2^s(s+1)}|\Psi'''(\zeta_1)|^q\right.\right.\\
&\left.\left.\left.\left.+\frac{2^{S+1}-1}{2^s(s+1)}|\Psi'''(\zeta_2)|^q\right)\right)^{1/q}\right\}\right].\end{aligned} \tag{62}$$

Proof. By using Lemma 28 with the Jensen-Mercer inequality and the well-known Hölder's inequality on the fact that $|\Psi'''|^q$ is s-convexity, we have

$$\left|\frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(\zeta_2-\zeta_1)^{\alpha/k}}\right.$$

$$\times\left\{\left({}^{k}J^{\alpha}_{\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))+}\right)(\Psi(\theta_1+\theta_2-\zeta_1))\right.$$
$$\left.+\left({}^{k}J^{\alpha}_{(\theta_1+\theta_2-((\zeta_1+\zeta_2)/2))^-}\right)(\Psi(\theta_1+\theta_2-\zeta_2))\right\}$$
$$-\frac{(\zeta_2-\zeta_1)^2}{4((\alpha/k)+1)((\alpha/k)+2)}$$
$$\times\Psi''\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)-\Psi\left(\theta_1+\theta_2-\frac{\zeta_1+\zeta_2}{2}\right)\Bigg|$$
$$\leq\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}$$
$$\times\left[\left\{\int_0^1(1-\lambda)^{(\alpha/k)+2}\right\}\right.$$
$$\times\left|\Psi'''\left(\theta_1+\theta_2-\left(\frac{1+\lambda}{2}\zeta_1+\frac{1-\lambda}{2}\zeta_2\right)\right)\right|d\lambda$$
$$-\int_0^1(1-\lambda)^{(\alpha/k)+2}$$
$$\left.\times\left|\Psi'''\left(\theta_1+\theta_2-\left(\frac{1-\lambda}{2}\zeta_1+\frac{1+\lambda}{2}\zeta_2\right)\right)\right|d\lambda\right\}\Bigg]$$
$$\leq\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}$$
$$\times\left[\left(\int_0^1(1-\lambda)^{((\alpha/k)+2)r}d\lambda\right)^{1/r}\right.$$
$$\times\left\{\int_0^1\left|\Psi'''(\theta_1)+\Psi'''(\theta_2)\right.\right.$$
$$\left.\left.-\left(\frac{1+\lambda}{2}\Psi'''(\zeta_1)+\frac{1-\lambda}{2}\Psi'''(\zeta_2)\right|^q d\lambda\right)^{1/q}\right\}$$
$$+\left(\int_0^1(1-\lambda)^{((\alpha/k)+2)r}d\lambda\right)^{1/r}$$
$$\times\left\{\int_0^1\left|\Psi'''(\theta_1)+\Psi'''(\theta_2)\right.\right.$$
$$\left.\left.\left.-\left(\frac{1-\lambda}{2}\Psi'''(\zeta_1)+\frac{1+\lambda}{2}\Psi'''(\zeta_2)\right|^q d\lambda\right)^{1/q}\right\}\right]$$
$$\leq\frac{(\zeta_2-\zeta_1)^3}{16((\alpha/k)+1)((\alpha/k)+2)}$$
$$\times\left[\left(\int_0^1(1-\lambda)^{((\alpha/k)+2)r}d\lambda\right)^{1/r}\right.$$
$$\times\left\{\left(\int_0^1\left(\left|\Psi'''(\theta_1)\right|^q+\left|\Psi'''(\theta_2)\right|^q\right.\right.\right.$$
$$\left.\left.\left.-\left(\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'''(\zeta_1)\right|^q+\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'''(\zeta_2)\right|^q\right)\right)\right)^{1/q}\right.$$
$$+\left(\int_0^1\left(\left|\Psi'''(\theta_1)\right|^q+\left|\Psi'''(\theta_2)\right|^q\right.\right.$$
$$\left.\left.\left.\left.-\left(\left(\frac{1-\lambda}{2}\right)^s\left|\Psi'''(\zeta_1)\right|^q+\left(\frac{1+\lambda}{2}\right)^s\left|\Psi'''(\zeta_2)\right|^q\right)\right)\right)^{1/q}\right\}\right].\tag{63}$$

After some simplifications, we get the required results.

Remark 34. Choosing $k=s=1$ with $\zeta_1=\theta_1$ and $\zeta_2=\theta_2$ in Theorem 33, we will get Theorem 19 proved in [23].

4. Applications

4.1. Some Applications of the Means. Let us consider the following special means for different values of a_1 and a_2:

(1) The arithmetic mean:

$$A(a_1,a_2)=\frac{a_1+a_2}{2}.\tag{64}$$

(2) The geometric mean:

$$G(a_1,a_2)=(a_1a_2)^{1/2}.\tag{65}$$

(3) The harmonic mean:

$$H=\frac{2_{a_1a_2}}{a_1+a_2}.\tag{66}$$

Proposition 35. *Suppose $a_1,a_2\in\mathbb{R}, 0<a_1<a_2, 0\notin[a_1,a_2]$. Then, for all $r>1$, the following inequality holds:*

$$\left|\frac{1}{(\zeta_2-\zeta_1)(s+1)}\left[(2A(a_1,a_2)-a_1)^{s+1}-(2A(a_1,a_2)-a_2)^{s+1}\right]\right.$$
$$\left.-(2A(a_1,a_2))^s\right|\leq\frac{(\zeta_2-\zeta_1)^2}{16}s(s-1)\left(\frac{1}{2r+1}\right)^{1/r}$$
$$\times\left[\left\{\left(2A\left(a_1^{(s-2)q},a_2^{(s-2)q}\right)\right)-\left(\frac{2^{s+1}-1}{2^s(s+1)}\left|a_1^{(s-2)}\right|^q\right.\right.\right.$$
$$\left.\left.+\frac{1}{2^s(s+1)}\left|a_2^{(s-2)}\right|^q\right)\right\}^{1/q}+\left\{\left(2A\left(a_1^{(s-2)q},a_2^{(s-2)q}\right)\right)\right.$$
$$\left.\left.-\left(\frac{1}{2^s(s+1)}\left|a_1^{(s-2)}\right|^q+\frac{2^{s+1}-1}{2^s(s+1)}\left|a_2^{(s-2)}\right|^q\right)\right\}^{1/q}\right].\tag{67}$$

Proof. The proof is an immediate consequence from Corollary 27 by selecting $\Psi(x)=x^s$ and $s\in(0,1)$.

Proposition 36. *Suppose $a_1, a_2 \in \mathbb{R}, s \in (0,1), 0 < a_1 < a_2, 0 \notin [a_1, a_2]$. Then, for all $r > 1$, the following inequality holds:*

$$\left|\frac{1}{(\zeta_2-\zeta_1)(1-s)}\left[\frac{(2A(a_1,a_2)-a_2)^{s-1}-(2A(a_1,a_2)-a_1)^{s-1}}{(G^2(2A(a_1,a_2)-a_1),(2A(a_1,a_2)-a_2))^{s-1}}\right] - (H(a_1,a_2))^s\right| \le \frac{(\zeta_2-\zeta_1)^2}{16}s(s+1)\left(\frac{1}{2r+1}\right)^{1/r} \times\left[\left\{2A\left(a_1^{-(s+2)q},a_2^{-(s+2)q}\right)-\left(\frac{2^{s+1}-1}{2^s(s+1)}\left|a_1^{-(s+2)}\right|^q + \frac{1}{2^s(s+1)}\left|a_2^{-(s+2)}\right|^q\right)\right\}^{1/q} + \left\{2A\left(a_1^{-(s+2)q},a_2^{-(s+2)q}\right) - \left(\frac{1}{2^s(s+1)}\left|a_1^{-(s+2)}\right|^q + \frac{2^{s+1}-1}{2^s(s+1)}\left|a_2^{-(s+2)}\right|^q\right)\right\}^{1/q}\right]. \tag{68}$$

Proof. The proof is an immediate consequence from Corollary 27 by taking into account Lemma 4 by selecting $\Psi(x) = 1/x^s$ and $s \in (0,1)$.

4.2. **q***-Digamma Function.* Suppose $0 < \mathbf{q} < 1$; the $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$ is the $\mathbf{q}$-analogue of the digamma function φ (see [25]) given as

$$\varphi_{\mathbf{q}} = -\ln(1-\mathbf{q}) + \ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k+\zeta}}{1-\mathbf{q}^{k+\zeta}} = -\ln(1-\mathbf{q}) + \ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k\zeta}}{1-\mathbf{q}^{k\zeta}}. \tag{69}$$

For $\mathbf{q} > 1$ and $\zeta > 0$, the $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$ can be given as

$$\begin{aligned}\varphi_{\mathbf{q}} &= -\ln(\mathbf{q}-1) + \ln\mathbf{q}\left[\zeta - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{\mathbf{q}^{-(k+\zeta)}}{1-\mathbf{q}^{-(k+\zeta)}}\right]\\ &= -\ln(\mathbf{q}-1) + \ln\mathbf{q}\left[\zeta - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{\mathbf{q}^{-k\zeta}}{1-\mathbf{q}^{-k\zeta}}\right].\end{aligned} \tag{70}$$

Proposition 37. *Let a_1, a_2, q, and $\mathbf{q}$ be real numbers such that $0 < a_1 < a_2$, $q > 1$, and $0 < \mathbf{q} < 1$. Then, the following inequality holds:*

$$\left|\frac{\varphi_{\mathbf{q}}(2A(a_1,a_2)-a_1)-\varphi_{\mathbf{q}}(2A(a_1,a_2)-a_2)}{\zeta_2-\zeta_1} - \varphi'_{\mathbf{q}}(A(a_1,a_2))\right| \le \frac{(\zeta_2-\zeta_1)^2}{16}\left(\frac{1}{2r+1}\right)^{1/r}\times\left[\left\{\left(\left|\varphi^{(3)}_{\mathbf{q}}(a_1)\right|^q + \left|\varphi^{(3)}_{\mathbf{q}}(a_2)\right|^q\right) - \left(\frac{2^{s+1}-1}{2^s(s+1)}\left|\varphi^{(3)}_{\mathbf{q}}(a_1)\right|^q + \frac{1}{2^s(s+1)}\left|\varphi^{(3)}_{\mathbf{q}}(a_2)\right|^q\right)\right\}^{1/q} + \left\{\left(\left|\varphi^{(3)}_{\mathbf{q}}(a_1)\right|^q + \left|\varphi^{(3)}_{\mathbf{q}}(a_2)\right|^q\right) - \left(\frac{1}{2^s(s+1)}\left|\varphi^{(3)}_{\mathbf{q}}(a_1)\right|^q + \frac{2^{s+1}-1}{2^s(s+1)}\left|\varphi^{(3)}_{\mathbf{q}}(a_2)\right|^q\right)\right\}^{1/q}\right]. \tag{71}$$

Proof. By employing the definition of the $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}(\zeta)$, it is easy to notice that the $\mathbf{q}$-trigamma function $\zeta \longrightarrow \varphi'_{\mathbf{q}}(\zeta)$ is completely monotonic on $(0,\infty)$. This ensures that the function $\varphi^{(3)}_{\mathbf{q}}$ is again completely monotonic on $(0,\infty)$ for each $\mathbf{q} \in (0,1)$ and consequently is convex and nonnegative (see [26], p. 167). Now, by applying Corollary 27, we extract that the inequality (71) is valid for $\mathbf{q} \in (0,1)$.

As another application of inequality (71), we can deliver the following inequalities for the $\mathbf{q}$-trigamma and $\mathbf{q}$-polygamma functions and the analogue of harmonic numbers $H_{n\mathbf{q}}$ defined by

$$H_{n\mathbf{q}} = \sum\frac{\mathbf{q}^k}{1-\mathbf{q}^k}, \quad n \in N. \tag{72}$$

So, from inequality (71), we use the equation

$$\varphi_{\mathbf{q}}(n+1) = \varphi_{\mathbf{q}}(1) - \log(\mathbf{q})H_{n\mathbf{q}}, \quad n \in N. \tag{73}$$

Consequently, we obtain the following result.

Corollary 38. *Suppose $n \in N$, $q > 1$, and $0 < \mathbf{q} < 1$. Then, the following inequality holds:*

$$\left|-\frac{\log(\mathbf{q})H_{n\mathbf{q}}}{n} - \varphi'_{\mathbf{q}}(A(1,n+1))\right| \le \frac{n^2}{16}\left(\frac{1}{2r+1}\right)^{1/r}\times\left[\left\{\left(\left|\varphi^{(3)}_{\mathbf{q}}(1)\right|^q + \left|\varphi^{(3)}_{\mathbf{q}}(n+1)\right|^q\right) - \left(\frac{2^{s+1}-1}{2^s(s+1)}\left|\varphi^{(3)}_{\mathbf{q}}(1)\right|^q + \frac{1}{2^s(s+1)}\left|\varphi^{(3)}_{\mathbf{q}}(n+1)\right|^q\right)\right\}^{1/q} + \left\{\left(\left|\varphi^{(3)}_{\mathbf{q}}(1)\right|^q + \left|\varphi^{(3)}_{\mathbf{q}}(n+1)\right|^q\right) - \left(\frac{1}{2^s(s+1)}\left|\varphi^{(3)}_{\mathbf{q}}(1)\right|^q + \frac{2^{s+1}-1}{2^s(s+1)}\left|\varphi^{(3)}_{\mathbf{q}}(n+1)\right|^q\right)\right\}^{1/q}\right]. \tag{74}$$

Proposition 39. *Suppose n is an integer and $q > 1$. Then, the following inequality holds:*

$$\left|\frac{H_n}{n} - \varphi'(A(1,n+1))\right| \le \frac{n^2}{16}\left(\frac{1}{2r+1}\right)^{1/r}\times\left[\left\{\left(\left|\varphi^{(3)}(1)\right|^q + \left|\varphi^{(3)}(n+1)\right|^q\right) - \left(\frac{2^{s+1}-1}{2^s(s+1)}\left|\varphi^{(3)}(1)\right|^q + \frac{1}{2^s(s+1)}\left|\varphi^{(3)}(n+1)\right|^q\right)\right\}^{1/q} + \left\{\left(\left|\varphi^{(3)}(1)\right|^q + \left|\varphi^{(3)}(n+1)\right|^q\right) - \left(\frac{1}{2^s(s+1)}\left|\varphi^{(3)}(1)\right|^q + \frac{2^{s+1}-1}{2^s(s+1)}\left|\varphi^{(3)}(n+1)\right|^q\right)\right\}^{1/q}\right]. \tag{75}$$

Proof. From inequality (74), when $\mathbf{q} \longrightarrow 1$, we use the relation

$$\lim_{\mathbf{q}\longrightarrow 1}\log(\mathbf{q})H_{n\mathbf{q}} = \lim_{\mathbf{q}\longrightarrow 1}\left[\left(\frac{\log(\mathbf{q})}{\mathbf{q}-1}\right).(\mathbf{q}-1)H_{n\mathbf{q}}\right] = -\lim_{\mathbf{q}\longrightarrow 1}\sum_{k=1}^{n}\frac{1-\mathbf{q}}{1-\mathbf{q}^k} = -H_n. \quad (76)$$

We obtain the required result.

Remark 40. By using the equation

$$H_n = \gamma + \varphi(n+1), \quad (77)$$

where γ is the Euler-Mascheroni constant, the inequality (75) becomes

$$\begin{aligned}\left|\frac{\gamma+\varphi(n+1)}{n} - \varphi'(A(1,n+1))\right| &\le \frac{n^2}{16}\left(\frac{1}{2r+1}\right)^{1/r} \\ &\times\left[\left\{\left(\left|\varphi^{(3)}(1)\right|^q + \left|\varphi^{(3)}(n+1)\right|^q\right) - \left(\frac{2^{s+1}-1}{2^s(s+1)}\left|\varphi^{(3)}(1)\right|^q\right.\right.\right. \\ &+ \left.\left.\frac{1}{2^s(s+1)}\left|\varphi^{(3)}(n+1)\right|^q\right)\right\}^{1/q} + \left\{\left(\left|\varphi^{(3)}(1)\right|^q + \left|\varphi^{(3)}(n+1)\right|^q\right)\right. \\ &- \left.\left.\left(\frac{1}{2^s(s+1)}\left|\varphi^{(3)}(1)\right|^q + \frac{2^{s+1}-1}{2^s(s+1)}\left|\varphi^{(3)}(n+1)\right|^q\right)\right\}^{1/q}\right].\end{aligned} \quad (78)$$

5. Conclusion

In this paper, we have explored new k-fractional variants of Hermite-Mercer-type integral inequalities for s-convex functions. New results and novel connections are built for the left and right sides of Hermite-Hadamard-type inequalities for differentiable mappings whose derivatives in absolute values at certain powers are s-convex in the second sense. New integral identities for differentiable mappings are obtained, and related results are established. In the application viewpoint, our findings illustrate new generalizations with the connection of special function theory (special means of real numbers and q-digamma function) and harmonic numbers. It is quite open to think about Jensen-Hermite-Mercer variants for generalized integral operators having nonlocal and nonsingular kernels by applying generalized convexities. However, it is not easy to extend such inequalities for other existing types of convexities. The suggested scheme is viable, effective, and computationally appealing in fractional differential equations, optimization theory, and other related areas of convexity.

Data Availability

All data required for this paper is included within this paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to this paper.

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Research Article

Some Identities of the Degenerate Multi-Poly-Bernoulli Polynomials of Complex Variable

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In this paper, we introduce degenerate multi-poly-Bernoulli polynomials and derive some identities of these polynomials. We give some relationship between degenerate multi-poly-Bernoulli polynomials degenerate Whitney numbers and Stirling numbers of the first kind. Moreover, we define degenerate multi-poly-Bernoulli polynomials of complex variables, and then, we derive several properties and relations.

1. Introduction

For any $\lambda \in \mathbb{R}/\{0\}$ (or $\mathbb{C}/\{0\}$), degenerate version of the exponential function $e_\lambda^x(t)$ is defined as follows (see [1–15])

$$e_\lambda^x(t) := (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \tag{1}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$ for $n \geq 1$, (*cf.* [1–15]). It follows from (1) is $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$. Note that $e_\lambda^1(t) := e_\lambda(t)$.

Carlitz [1] introduced the degenerate Bernoulli polynomials as follows:

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_n(x;\lambda) \frac{t^n}{n!}. \tag{2}$$

Upon setting $x=0$, $\beta_n(0;\lambda) := \beta_n(\lambda)$ are called the degenerate Bernoulli numbers.

Note that

$$\lim_{\lambda\to 0} \beta_n(x;\lambda) = B_n(x), \tag{3}$$

where $B_n(x)$ are the familiar Bernoulli polynomials (*cf.* [1, 3, 4, 6, 8, 11, 12, 14, 16–22])

$$\frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad (|t| < 2\pi). \tag{4}$$

For $k \in \mathbb{Z}$, the polyexponential function $\mathrm{Ei}_k(x)$ is defined by (see [21])

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)! n^k}, (k \in \mathbb{Z}). \tag{5}$$

Setting $k=1$ in (5), we have $\mathrm{Ei}_1(x) = e^x - 1$.

The degenerate modified polyexponential function [12] is defined, for $k \in \mathbb{Z}$ and $|x| < 1$, by

$$\mathrm{Ei}_{k;\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{(n-1)!n^k} x^n. \tag{6}$$

Note that $\mathrm{Ei}_{1;\lambda}(x) = e_\lambda(x) - 1$.

Let $k \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$. The degenerate poly-Bernoulli polynomials, *cf.* [12], are defined by

$$\frac{\mathrm{Ei}_{k;\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}, \tag{7}$$

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1} (1)_{n,\frac{1}{\lambda}} \frac{t^n}{n!}, \ (\lambda \in \mathbb{R}), \tag{8}$$

where $\log_\lambda(1+t)$ are called the degenerate version of the logarithm function (*cf.* [8, 12]), which is also the inverse function of the degenerate exponential function $e_\lambda(t)$ as shown below (*cf.* [8])

$$e_\lambda(\log_\lambda(1+t)) = \log_\lambda(e_\lambda(1+t)) = 1+t. \tag{9}$$

Letting $x=0$ in (7), $B_{n,\lambda}^{(k)}(0) \coloneqq B_{n,\lambda}^{(k)}$ are called the type 2 degenerate poly-Bernoulli numbers.

The degenerate Stirling numbers of the first kind (*cf.* [8, 13]) and second kind (*cf.* [4–6, 9, 17]) are defined, respectively, by

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, (k \geq 0), \tag{10}$$

and (*cf.* [1–27])

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}, (k \geq 0). \tag{11}$$

Note that $\lim_{\lambda \to 0}$ in (10) and (1.8), we have (*cf.* [8, 13])

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!} (k \geq 0), \tag{12}$$

and (*cf.* [4–6, 9, 17, 24])

$$\frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!} (k \geq 0), \tag{13}$$

where $S_1(n,k)$ and $S_2(n,k)$ are called the Stirling numbers of the first kind and second kind.

The following paper is as follows. In Section 2, we define the degenerate multi-poly-Bernoulli polynomials and numbers by using the degenerate multiple polyexponential functions and derive some properties and relations of these polynomials. In Section 3, we consider the degenerate multi-poly-Bernoulli polynomials of a complex variable and then we derive several properties and relations. Also, we examine the results derived in this study [28, 29].

2. Degenerate Multi-Poly-Bernoulli Polynomials and Numbers

Let $k_1, k_2, \cdots, k_r \in \mathbb{Z}$. The degenerate multiple polyexponential function $\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(x)$ is defined (*cf.* [15]) by

$$\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(x) = \sum_{0<n_1<n_2<\cdots<n_r} \frac{(1)_{n_1,\lambda} \cdots (1)_{n_r,\lambda} x^{n_r}}{(n_1-1)!\cdots(n_r-1)! n_1^{k_1} \cdots n_r^{k_r}}, \tag{14}$$

where the sum is over all integers $n_1, n_2, \cdots, n_r$ satisfying $0 < n_1 < n_2 < \cdots < n_r$. Utilizing this function, Kim et al. [15] introduced and studied the degenerate multi-poly-Genocchi polynomials given by

$$\frac{2^r \mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)+1)^r} e_\lambda^x(t) = \sum_{n=0}^{\infty} g_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x) \frac{t^n}{n!}. \tag{15}$$

Inspired by the definition of degenerate multi-poly-Genocchi polynomials, using the degenerate multiple polyexponential function (14), we give the following definition.

Definition 1. Let $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$, we consider the degenerate multi-poly-Bernoulli polynomials are given by

$$\frac{r! \mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) \frac{t^n}{n!}. \tag{16}$$

Upon setting $x=0$ in (16), the degenerate multi-poly-Bernoulli polynomials reduce to the corresponding numbers, namely, the type 2 degenerate multi-poly-Bernoulli numbers $\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(0) \coloneqq \mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}$.

Remark 2. As $\lambda \to 0$, the degenerate multi-poly-Bernoulli polynomials reduce to the multi-poly-Bernoulli polynomials given by

$$\frac{r! \mathrm{Ei}_{k_1,k_2,\cdots,k_r}(\log(1+t))}{(e^t-1)^r} e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(k_1,k_2,\cdots,k_r)}(x) \frac{t^n}{n!}. \tag{17}$$

Remark 3. Upon setting $r=1$ in (16), the degenerate multi-poly-Bernoulli polynomials reduce to the degenerate poly-Bernoulli polynomials in (7).

Before going to investigate the properties of the degenerate multi-poly-Bernoulli polynomials, we first give the following result.

Proposition 4 (Derivative Property). *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$, we have*

$$\frac{d}{dx} Ei_{k_1,k_2,\cdots,k_r;\lambda}(x) = \frac{1}{x} Ei_{k_1,k_2,\cdots,k_r-1;\lambda}(x). \tag{18}$$

Proof. By (14), we see that

$$\begin{aligned}\frac{d}{dx}\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(x) &= \frac{d}{dx}\sum_{0<n_1<n_2<\cdots<n_r} \cdot \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}x^{n_r}}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}} \\ &= \frac{1}{x}\sum_{0<n_1<n_2<\cdots<n_r} \cdot \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}x^{n_r}}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r-1}} \\ &= \frac{1}{x}\mathrm{Ei}_{k_1,k_2,\cdots,k_r-1;\lambda}(x).\end{aligned} \tag{19}$$

Theorem 5. *The following relationship*

$$\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \sum_{j=0}^{n}\binom{n}{j}\mathfrak{B}_{n-j,\lambda}^{(k_1,k_2,\cdots k_r)}(x)_{j,\lambda}, \tag{20}$$

holds for $n \geq 0$.

Proof. Recall Definition 1 that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!} &= \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^x(t) \\ &= \sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}\frac{t^n}{n!}\sum_{j=0}^{\infty}(x)_{j,\lambda}\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\mathfrak{B}_{n-j,\lambda}^{(k_1,k_2,\cdots k_r)}(x)_{j,\lambda}\right)\frac{t^n}{n!},\end{aligned} \tag{21}$$

which gives the asserted result (20).

The degenerate Bernoulli polynomials of order r are given by the following series expansion:

$$\sum_{n=0}^{\infty}\beta_n^{(r)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t), \tag{22}$$

(*cf.* [3, 6, 8, 17]).

We provide the following theorem.

Theorem 6. *For $n \geq r$. Then*

$$\begin{aligned}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \sum_{m=0}^{n+r}\sum_{0<n_1<n_2<\cdots<n_r\leq m} &\cdot \binom{n+r}{m}\beta_{n+r-m}^{(r)}(x;\lambda)S_{1.\lambda}(m,n_r) \\ &\times \frac{n!r!n_r!(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}}{(n+r)!(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}}.\end{aligned} \tag{23}$$

Proof. Recall from Definition 1 and (10) that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!} &= \frac{r!e_\lambda^x(t)}{(e_\lambda(t)-1)^r}\sum_{0<n_1<n_2<\cdots<n_r} \cdot \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}(\log_\lambda(1+t))^{n_r}}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}} \\ &= \frac{r!e_\lambda^x(t)}{(e_\lambda(t)-1)^r}\sum_{0<n_1<n_2<\cdots<n_r} \cdot \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}n_r!}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}} \cdot \sum_{m=n_r}^{\infty}S_{1.\lambda}(m,n_r)\frac{t^m}{m!} \\ &= \frac{r!}{t^r}\left(\frac{t^r e_\lambda^x(t)}{(e_\lambda(t)-1)^r}\right)\sum_{m=n_r}^{\infty} \cdot \left(\sum_{0<n_1<n_2<\cdots<n_r\leq m}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1.\lambda}(m,n_r)n_r!}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}}\right) \\ &\cdot \frac{t^m}{m!} = \frac{r!}{t^r}\sum_{l=0}^{\infty}\beta_l^{(r)}(x;\lambda)\frac{t^l}{l!}\sum_{m=n_r}^{\infty} \cdot \left(\sum_{0<n_1<n_2<\cdots<n_r\leq m}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1.\lambda}(m,n_r)n_r!}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}}\right) \\ &\cdot \frac{t^m}{m!} = \sum_{n=r}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{0<n_1<n_2<\cdots<n_r\leq m} \cdot \frac{r!n_r!(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}} \\ &\cdot \beta_{n-m}^{(r)}(x;\lambda)S_{1.\lambda}(m,n_r)\frac{t^{n-r}}{n!},\end{aligned} \tag{24}$$

which means the claimed result (23).

Theorem 7. *The following formula*

$$\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}(y)_{j,\lambda}\mathfrak{B}_{n-j,\lambda}^{(k_1,k_2,\cdots,k_r)}(x), \tag{25}$$

is valid for $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ and $n \geq 0$.

Proof. In view of Definition 1, we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+y)\frac{t^n}{n!}&=\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^{x+y}(t)\\&=\sum_{i=0}^{\infty}\mathfrak{B}_{i,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^i}{i!}\sum_{j=0}^{\infty}(y)_{j,\lambda}\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(y)_{m,\lambda}\mathfrak{B}_{n-j,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\right)\frac{t^n}{n!},\end{aligned}\tag{26}$$

which implies the desired result (25).

Theorem 8. *The following relation*

$$\frac{d}{dx}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)=\mathop{}_{l=1}^{n}\binom{n}{l}\mathfrak{B}_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)(-\lambda)^{l-1}(l-1)!,\tag{27}$$

is valid for $k_1,k_2,\cdots,k_r\in\mathbb{Z}$ and $n\geq 0$.

Proof. To investigate the derivative property of $\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)$ that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}&=\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}\frac{d}{dx}e_\lambda^{x}(t)\\&=\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\frac{1}{\lambda}\ln(1+\lambda t)\\&=\left(\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\right)\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\lambda^{l-1}t^l\\&=\sum_{n=0}^{\infty}\sum_{l=1}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{(-1)^{l+1}}{l}\lambda^{l-1}\frac{t^{n+l}}{n!},\end{aligned}\tag{28}$$

which provides the asserted result (27).

We here give a relation including the degenerate multi-poly-Bernoulli polynomials with numbers and the degenerate Stirling numbers of the second kind.

Theorem 9. *The following correlation*

$$\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_{2,\lambda}(m,l)\mathfrak{B}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)},\tag{29}$$

is valid for $k_1,k_2,\cdots,k_r\in\mathbb{Z}$ and $n\geq 0$.

Proof. By means of Definition 1, we attain that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}&=\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^{x}(t)\\&=\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}(e_\lambda(t)-1+1)^x\\&=\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}\sum_{l=0}^{\infty}\binom{x}{l}(e_\lambda(t)-1)^l\\&=\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}\frac{t^n}{n!}\sum_{l=0}^{\infty}(x)_l\sum_{m=l}^{\infty}S_{2,\lambda}(m,l)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_{2,\lambda}(m,l)\mathfrak{B}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}\right)\frac{t^n}{n!},\end{aligned}\tag{30}$$

where the notation $(x)_l$ is falling factorial that is defined by $(x)_0=1$ and $(x)_n=x(x-1)\cdots(x-(n-1))$ for $n\geq 1$, (*cf.* [1, 2, 5–14, 21, 23, 24]). So, the proof is completed.

Kim [5] introduced the degenerate Whitney numbers are given by

$$\frac{(e_\lambda^m(t)-1)^k}{m^k k!}e_\lambda^{\alpha}(t)=\sum_{n=k}^{\infty}W_{m,\alpha}(n,k|\lambda)\frac{t^n}{n!},\ (k\geq 0).\tag{31}$$

Kim also provided several correlations including the degenerate Stirling numbers of the second kind and the degenerate Whitney numbers (see [5]).

We now give a correlation as follows.

Theorem 10. *For $k_1,k_2,\cdots k_r\in\mathbb{Z}$ and $n\geq 0$, we have*

$$\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(xu+\alpha)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}u^l(x)_l W_{u,\alpha}(m,l|\lambda)\mathfrak{B}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}.\tag{32}$$

Proof. Using (31) and Definition 1, we acquire that

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(xu+\alpha)\frac{t^n}{n!}\\&=\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^{\alpha}(t)e_\lambda^{xu}(t)\\&=\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^{\alpha}(t)(e_\lambda^{u}(t)-1+1)^x\\&=\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^{\alpha}(t)\sum_{l=0}^{\infty}\binom{x}{l}(e_\lambda^{u}(t)-1)^l\\&=\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}\sum_{l=0}^{\infty}u^l(x)_l\frac{(e_\lambda^{u}(t)-1)^l}{l!u^l}e_\lambda^{\alpha}(t)\end{aligned}$$

$$= \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}\sum_{l=0}^{\infty}u^l(x)_l\frac{(e_\lambda^u(t)-1)^l}{l!u^l}e_\lambda^\alpha(t)$$
$$= \sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}\frac{t^n}{n!}\sum_{n=0}^{\infty}\sum_{l=0}^{n}u^l(x)_lW_{u,\alpha}(n,l|\lambda)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}u^l(x)_lW_{u,\alpha}(m,l|\lambda)\mathfrak{B}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}\right)\frac{t^n}{n!}, \tag{33}$$

which implies the asserted result (32).

3. Degenerate Multi-Poly-Bernoulli Polynomials of Complex Variable

In [25], Kim et al. defined the degenerate sine $\sin_\lambda t$ and cosine $\cos_\lambda t$ functions by

$$\sin_\lambda^{(x)}(t) = \frac{e_\lambda^{ix}(t)-e_\lambda^{-ix}(t)}{2i} \text{ and } \cos_\lambda^{(x)}(t) = \frac{e_\lambda^{ix}(t)+e_\lambda^{-ix}(t)}{2}, \tag{34}$$

where $i=\sqrt{-1}$. Note that $\lim_{\lambda\to0}\sin_\lambda^{(x)}(t)=\sin xt$ and $\lim_{\lambda\to0}\cos_\lambda^{(x)}(t)=\cos xt$. From (34), it is readily that

$$e_\lambda^{ix}(t) = \cos_\lambda^{(x)}(t) + i\sin_\lambda^{(x)}(t). \tag{35}$$

By these functions in (34), the degenerate sine-polynomials $S_{k,\lambda}(x,y)$ and degenerate cosine-polynomials $C_{k,\lambda}(x,y)$ are introduced (*cf.* [25]) by

$$\sum_{n=0}^{\infty}S_{k,\lambda}(x,y)\frac{t^n}{n!} = e_\lambda^x(t)\sin_\lambda^{(y)}(t), \tag{36}$$

$$\sum_{n=0}^{\infty}C_{k,\lambda}(x,y)\frac{t^n}{n!} = e_\lambda^x(t)\cos_\lambda^{(y)}(t). \tag{37}$$

Several properties of these polynomials in (36) and (37) are studied and investigated in [25]. Also, by means of these functions, Kim et al. [25] introduced the degenerate Euler and Bernoulli polynomials of complex variable and investigate some of their properties. Motivated and inspired by these considerations above, we define type 2 degenerate multi-poly-Bernoulli polynomials of complex variable as follows.

Definition 11. Let $k_1,k_2,\cdots,k_r\in\mathbb{Z}$. We define a new form of the degenerate multi-poly-Bernoulli polynomials of complex variable by the following generating function:

$$\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(\mathrm{e}_\lambda(t)-1)^r}e_\lambda^{x+iy}(t) = \sum_{n=0}^{\infty}\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy)\frac{t^n}{n!}. \tag{38}$$

By (34) and (38), we observe that

$$\sum_{n=0}^{\infty}\frac{\left(\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy)-\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x-iy)\right)}{2i}\frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^x(t)\sin_\lambda^{(y)}(t), \tag{39}$$

and

$$\sum_{n=0}^{\infty}\frac{\left(\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy)+\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x-iy)\right)}{2}\frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^x(t)\cos_\lambda^{(y)}(t). \tag{40}$$

In view of (39) and (40), we consider the degenerate multi-poly-sine-Bernoulli polynomials $\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x,y)$ with two parameters and the degenerate multi-poly-cosine-Bernoulli polynomials $\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y)$ with two parameters as follows:

$$\sum_{n=0}^{\infty}\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x,y)\frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^x(t)\sin_\lambda^{(y)}(t), \tag{41}$$

$$\sum_{n=0}^{\infty}\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y)\frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^x(t)\cos_\lambda^{(y)}(t). \tag{42}$$

Note that

$$\lim_{\lambda\to0}\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x,y) \coloneqq \mathrm{B}_{n}^{(k_1,k_2,\cdots,k_r;S)}(x,y) \text{ and } \lim_{\lambda\to0}\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y) \coloneqq \mathrm{B}_{n}^{(k_1,k_2,\cdots,k_r;C)}(x,y), \tag{43}$$

which are multi-poly-sine-Bernoulli polynomials $\mathrm{B}_{n}^{(k_1,k_2,\cdots,k_r;S)}(x,y)$ and multi-poly-cosine-Bernoulli polynomials $\mathrm{B}_{n}^{(k_1,k_2,\cdots,k_r;C)}(x,y)$ with two parameters.

By (39)-(42), we see that

$$\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x,y) = \frac{\left(\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy)-\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x-iy)\right)}{2i},$$
$$\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y) = \frac{\left(\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy)+\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x-iy)\right)}{2}. \tag{44}$$

We now give the two summation formulae by the following theorem.

Theorem 12. *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ and $n \geq 0$, we have*

$$B_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\lambda}^{(k_1,k_2,\cdots k_r)}(x)(iy)_{m,\lambda}, \quad (45)$$

$$B_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\lambda}^{(k_1,k_2,\cdots k_r)}(x+iy)_{m,\lambda}. \quad (46)$$

Proof. The proofs of this theorem can be done by using the same proof methods used in Theorems 5 and 7. So, we omit the proofs.

We here provide the two derivative formulae by the following theorem.

Theorem 13. *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ and $n \geq 0$, we have*

$$\frac{d}{dt} B_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy) = n B_{n-1,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy),$$

$$\frac{d}{dx} \mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy) = \sum_{l=1}^{n} \binom{n}{l} \mathfrak{B}_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+iy)(-\lambda)^{l-1}(l-1)!. \quad (47)$$

Proof. The proofs of this theorem can be done by using the same proof methods used in Theorem 8. So, we omit the proofs.

We give the following theorem.

Theorem 14. *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ and $n \geq 0$, we have*

$$\begin{aligned} B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x,y) &= \sum_{l=0}^{n} \binom{n}{l} B_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)} S_{l,\lambda}(x,y), \\ B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y) &= \sum_{l=0}^{n} \binom{n}{l} B_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)} C_{l,\lambda}(x,y). \end{aligned} \quad (48)$$

Proof. From (36), (37), and (38), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x,y) \frac{t^n}{n!} \\ &= \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r} e_\lambda^x(t) \sin_\lambda^{(y)}(t) \\ &= \sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)} \frac{t^n}{n!} \sum_{n=0}^{\infty} S_{n,\lambda}(x,y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \mathrm{B}_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)} S_{l,\lambda}(x,y) \right) \frac{t^n}{n!}, \end{aligned}$$

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y) \frac{t^n}{n!} \\ &= \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r} e_\lambda^x(t) \cos_\lambda^{(y)}(t) \\ &= \sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)} \frac{t^n}{n!} \sum_{n=0}^{\infty} C_{n,\lambda}(x,y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \mathrm{B}_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)} C_{l,\lambda}(x,y) \right) \frac{t^n}{n!}, \end{aligned} \quad (49)$$

which complete the proof of the theorem.

We note that (*cf.* [25])

$$\sin_\lambda^{(y)}(t) = \sum_{n=1}^{\infty} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \lambda^{n-2k-1}(-1)^k y^{2k+1} S_{1,\lambda}(n, 2k+1) \frac{t^n}{n!}, \quad (50)$$

$$\cos_\lambda^{(y)}(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \lambda^{n-2k}(-1)^k y^{2k} S_{1,\lambda}(n, 2k) \frac{t^n}{n!}. \quad (51)$$

We give the following theorem.

Theorem 15. *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ and $n \geq 0$, we have*

$$\begin{aligned} B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x,y) &= \sum_{l=1}^{n} \sum_{k=0}^{\left[\frac{l-1}{2}\right]} \binom{n}{l} B_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)} \\ &\quad \cdot (x) \lambda^{l-2k-1}(-1)^k y^{2k+1} S_{1,\lambda}(l, 2k+1), \\ B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y) &= \sum_{l=1}^{n} \sum_{k=0}^{\left[\frac{l}{2}\right]} \binom{n}{l} B_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)} \\ &\quad \cdot (x) \lambda^{l-2k-1}(-1)^k y^{2k+1} S_{1,\lambda}(l, 2k+1), \end{aligned} \quad (52)$$

where the notation $[\cdot]$ is Gauss' notation and represents the maximum integer which does not exceed a number in the square bracket.

Proof. By (41)–(51), we observe that

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x,y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) \frac{t^n}{n!} \sum_{n=1}^{\infty} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \lambda^{n-2k-1}(-1)^k y^{2k+1} S_{1,\lambda}(n, 2k+1) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=1}^{n} \binom{n}{l} \mathrm{B}_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) \sum_{k=0}^{\left[\frac{l-1}{2}\right]} \lambda^{l-2k-1}(-1)^k y^{2k+1} S_{1,\lambda}(l, 2k+1) \right) \frac{t^n}{n!}, \end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y)\frac{t^n}{n!} \\
&\quad = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r} e_\lambda^x(t)\cos_\lambda^{(y)}(t) \\
&\quad = \sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\,{}^{\infty}_{n=1}\,{}^{\left[\frac{n}{2}\right]}_{k=0}\lambda^{n-2k}(-1)^k y^{2k} S_{1,\lambda}(n,2k)\frac{t^n}{n!} \\
&\quad = {}^{\infty}_{n=0}\left({}^{n}_{l=1}\begin{pmatrix} n \\ l \end{pmatrix} \mathrm{B}_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\,{}^{\left[\frac{l}{2}\right]}_{k=0}\lambda^{l-2k}(-1)^k y^{2k} S_{1,\lambda}(l,2k)\right)\frac{t^n}{n!}.
\end{aligned} \tag{53}$$

So, the proof is completed.

We give the following proposition.

Proposition 16. *The following relations*

$$\begin{aligned}
B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x+u,y) &= {}^{n}_{l=0}\begin{pmatrix} n \\ l \end{pmatrix} B_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(x,y)(u)_{l,\lambda}, \\
B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x+u,y) &= {}^{n}_{l=0}\begin{pmatrix} n \\ l \end{pmatrix} B_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y)(u)_{l,\lambda}.
\end{aligned} \tag{54}$$

hold for $k_1, k_2, \cdots, k_r \in \mathbb{Z}, u \in \mathbb{C}$ *and* $n \geq 0$.

Proof. The proofs of this proposition can be done by utilizing the same proof methods used in Theorem 7. So, we omit the proofs.

Upon setting $x = 0$ in (41) and (42), we consider the degenerate multi-poly-sine-Bernoulli polynomials $\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(y)$ and the degenerate multi-poly-cosine-Bernoulli polynomials $\mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(y)$ as follows

$$\sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(y)\frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}\sin_\lambda^{(y)}(t), \tag{55}$$

$$\sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(y)\frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}\cos_\lambda^{(y)}(t). \tag{56}$$

We now provide the following theorem.

Theorem 17. *For* $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ *and* $n > 0$, *we have*

$$\begin{aligned}
\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y) = &\sum_{m=0}^{n}\sum_{l=0}^{m}\begin{pmatrix} n \\ m \end{pmatrix}(x)_l S_{2,\lambda}(m,l)\mathfrak{B}_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r;C)} \\
&\cdot (y) with \mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(x,y) = 0.
\end{aligned} \tag{57}$$

Proof. The proofs of this theorem can be done by utilizing the same proof methods in Theorem 9.

Let α be any fixed real (or complex) number. The Bernoulli polynomials of order α is defined by (*cf.* [25])

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad (|t| < 2\pi). \tag{58}$$

When $x = 0$, the Bernoulli polynomials of order α reduce to the Bernoulli numbers of order α, denoted by $B_n^{(\alpha)}$.We give the following relation.

Theorem 18. *For* $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ *and* $n \geq 0$, *we have*

$$\begin{aligned}
&B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(1,y) - B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(y) \\
&\qquad = n\sum_{l=0}^{n-1}\begin{pmatrix} n-1 \\ l \end{pmatrix} B_{n-1-l,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(y)B_l^{(-1)},
\end{aligned} \tag{59}$$

$$\begin{aligned}
&B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(1,y) - B_{n,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(y) \\
&\qquad = n\sum_{l=0}^{n-1}\begin{pmatrix} n-1 \\ l \end{pmatrix} B_{n-1-l,\lambda}^{(k_1,k_2,\cdots,k_r;C)}(y)B_l^{(-1)}.
\end{aligned} \tag{60}$$

Proof. By (55) and (56), we acquire

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(1,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(y)\frac{t^n}{n!} \\
&\quad = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r;\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}\sin_\lambda^{(y)}(t)(e_\lambda(t)-1) \\
&\quad = \sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(y)\frac{t^{n+1}}{n!}\sum_{n=0}^{\infty} B_n^{(-1)}\frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{pmatrix} n \\ l \end{pmatrix} \mathrm{B}_{n-l,\lambda}^{(k_1,k_2,\cdots,k_r;S)}(y)B_l^{(-1)}\frac{t^{n+1}}{n!}.
\end{aligned} \tag{61}$$

Thus, (59) is proved. We prove (60) in the same way.

Here is a special case of Theorem 18.

Corollary 19. *For* $k_1, k_2, \cdots, k_r \in \mathbb{Z}$ *and* $n \geq 0$, *we have*

$$B_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(1) - B_{n,\lambda}^{(k_1,k_2,\cdots,k_r)} = n\sum_{l=0}^{n-1}\begin{pmatrix} n-1 \\ l \end{pmatrix} B_{n-1-l,\lambda}^{(k_1,k_2,\cdots,k_r)}B_l^{(-1)}, \tag{62}$$

which is a relation including the degenerate multi-poly-Bernoulli polynomials.

4. Conclusions

In this paper, we defined the degenerate multi-poly-Bernoulli polynomials by employing the degenerate multiple polyexponential functions. We have established some identities and relations between degenerate Whitney numbers and degenerate Stirling numbers of the first kind. Also, we have established addition formulas and derivative formulas of degenerate multi-poly-Bernoulli polynomials. In the last section, we have defined degenerate multi-poly-Bernoulli polynomials of complex variables and then we have derived several properties and relations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Research Article

On Some Relationships of Certain K – Uniformly Analytic Functions Associated with Mittag-Leffler Function

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In this paper, we introduce and investigate several inclusion relationships of new k -uniformly classes of analytic functions defined by the Mittag-Leffler function. Also, integral-preserving properties of these classes associated with the certain integral operator are also obtained.

1. Introduction

Let $\mathscr{A}$ be the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ which in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

For $f(z)$ and $g(z) \in \mathscr{A}$, we say that the function $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \text{ or } f(z) \prec g(z), \tag{2}$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$. In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence relation (cf., e.g., [1, 2]; see also [3]):

$$f(z) \prec g\,(z) \Leftrightarrow f(0) \prec g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \tag{3}$$

Let f be as in (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then Hadamard product (or convolution) of $f(z)$ and $h(z)$ is given by

$$(f * h)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \,(z \in U). \tag{4}$$

For $\zeta, \eta \in [0, 1)$, we denote by $S^*(\zeta)$, $C(\zeta)$, $K(\zeta, \eta)$, and $K^*(\zeta, \eta)$ the subclasses of $\mathscr{A}$ consisting of all analytic functions which are, respectively, starlike of order ζ, convex of order ζ, close-to-convex of order ζ and type η, and quasiconvex of order ζ and type η in $\mathbb{U}$.

Also, let the subclasses $US(\mu, \zeta)$, $UC(\mu, \zeta)$, $USK(\mu, \zeta, \eta)$, and $UCK(\mu, \zeta, \eta)$ of $\mathscr{A}$ $(\eta \in 0, 1) < 1\,; \mu \geq 0)$ be defined as follows:

$$US(\mu, \zeta) = \left\{ f \in \mathscr{A} : \Re\left(\frac{zf'(z)}{f(z)} - \zeta\right) > \mu\left|\frac{zf'(z)}{f(z)} - 1\right| \right\},$$

$$UC(\mu, \zeta) = \left\{ f \in \mathscr{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)} - \zeta\right) > \mu\left|\frac{zf''(z)}{f'(z)}\right| \right\},$$

$$USK(\mu,\zeta,\eta)=\left\{f\in\mathscr{A}:\exists h\in US(\mu,\zeta)\right.$$
$$\left. s.t.\Re\left(\frac{zf'(z)}{h(z)}-\zeta\right)>\mu\left|\frac{zf'(z)}{h(z)}-1\right|\right\}$$
$$UCK(\mu,\zeta,\eta)=\left\{f\in\mathscr{A}:\exists h\in UC(\mu,\zeta)\right.$$
$$\left. s.t.\Re\left(\frac{\left(zf'(z)\right)'}{h'(z)}-\zeta\right)>\mu\left|\frac{\left(zf'(z)\right)'}{h'(z)}-1\right|\right\}. \tag{5}$$

We note that

$$US(0,\zeta)=S^*(\zeta),\ UC(0,\zeta)=C(\zeta),$$
$$USK(0,\zeta,\eta)=K(\zeta,\eta)\text{ and }UCK(0,\zeta,\eta)=K^*(\zeta,\eta)\ (0\le\zeta\,;\eta<1). \tag{6}$$

Moreover, let $q_{\mu,\zeta}(z)$ be an analytic function which maps $\mathbb{U}$ onto the conic domain $\Phi_{\mu,\zeta}=\{u+iv:u>k\sqrt{(u-1)^2+v^2}+\zeta\}$ such that $1\in\Phi_{\mu,\zeta}$ defined as follows:

$$q_{\mu,\zeta}(z)=\begin{cases}\dfrac{1+(1-2\zeta)z}{1-z}\ (\mu=0),\\[2mm] \dfrac{1-\zeta}{1-\mu^2}\cos\left\{\dfrac{2}{\pi}\left(\cos^{-1}\mu\right)i\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\dfrac{\mu^2-\zeta}{1-\mu^2}\ (0<\mu<1),\\[2mm] 1+\dfrac{2(1-\zeta)}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\ (\mu=1),\\[2mm] \dfrac{1-\zeta}{\mu^2-1}\sin\left\{\dfrac{\pi}{2\varsigma(\mu)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{\mu}}}\dfrac{dt}{\sqrt{1-t^2}\sqrt{1-\mu^2t^2}}\right\}+\dfrac{\mu^2-\zeta}{\mu^2-1}\ (\mu>1),\end{cases} \tag{7}$$

where $u(z)=(z-\sqrt{\mu})/(1-\sqrt{\mu}z)$ and $\varsigma(\mu)$ is such that $\mu=\cosh\left(\pi\varsigma'(z)/4\varsigma(z)\right)$. By virtue of properties of the conic domain $\Phi_{\mu,\zeta}$ (cf., e.g., [4, 5]), we have

$$\Re\left\{q_{\mu,\zeta}(z)\right\}>\frac{\mu+\zeta}{\mu+1}. \tag{8}$$

Making use of the principal of subordination and the definition of $q_{\mu,\zeta}(z)$, we may rewrite the subclasses $US(\mu,\zeta)$, $UC(\mu,\zeta)$, $USK(\mu,\zeta,\eta)$, and $UCK(\mu,\zeta,\eta)$ as follows:

$$US(\mu,\zeta)=\left\{f\in\mathscr{A}:\frac{zf'(z)}{f(z)}\prec q_{\mu,\zeta}(z)\right\},$$
$$UC(\mu,\zeta)=\left\{f\in\mathscr{A}:1+\frac{zf''(z)}{f'(z)}\prec q_{\mu,\zeta}(z)\right\},$$
$$USK(\mu,\zeta,\eta)=\left\{f\in\mathscr{A}:\exists h\in US(\mu,\eta)\ s.t.\frac{zf'(z)}{h(z)}\prec q_{\mu,\zeta}(z)\right\} \tag{9}$$

and

$$UCK(\mu,\zeta,\eta)=\left\{f\in\mathscr{A}:\exists h\in UC(\mu,\zeta)\right.$$
$$\left. s.t.\frac{\left(zf'(z)\right)'}{h'(z)}\prec q_{\mu,\zeta}(z)\right\}. \tag{10}$$

Attiya [6] introduced the operator $H^{\gamma,k}_{\alpha,\beta}(f)$, where $H^{\gamma,k}_{\alpha,\beta}(f)$: $\mathscr{A}\longrightarrow\mathscr{A}$ is defined by

$$H^{\gamma,k}_{\alpha,\beta}(f)=\mu^{\gamma,k}_{\alpha,\beta}*f(z)\ (z\in\mathbb{U}), \tag{11}$$

with $\beta,\gamma\in\mathbb{C}$, $\mathrm{Re}\,(\alpha)>\max\{0,\mathrm{Re}\,(k)-1\}$ and $\mathrm{Re}\,(k)>0$. Also, $\mathrm{Re}\,(\alpha)=0$ when $\mathrm{Re}\,(k)=1$; $\beta\neq0$. Here, $\mu^{\gamma,k}_{\alpha,\beta}$ is the generalized Mittag–Leffler function defined by [7], see also [6], and the symbol $(*)$ denotes the Hadamard product.

Due to the importance of the Mittag–Leffler function, it is involved in many problems in natural and applied science. A detailed investigation of the Mittag–Leffler function has been studied by many authors (see, e.g., [7–12]).

Attiya [6] noted that

$$H_{\alpha,\beta}^{\gamma,k}(f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\beta+\alpha n)n!} a_n z^n. \quad (12)$$

Also, Attiya [6] showed that

$$z\left(H_{\alpha,\beta}^{\gamma,k}(f(z))\right)' = \left(\frac{\gamma+k}{k}\right)\left(H_{\alpha,\beta}^{\gamma+1,k} f(z)\right) - \frac{\gamma}{k}\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right), \quad (13)$$

and

$$z\left(H_{\alpha,\beta+1}^{\gamma,k}(f(z))\right)' = \left(\frac{\alpha+\beta}{\alpha}\right)\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right) - \frac{\beta}{\alpha}\left(H_{\alpha,\beta+1}^{\gamma,k} f(z)\right). \quad (14)$$

Next, by using the operator $H_{\alpha,\beta}^{\gamma,k}(f)$, we introduce the following subclasses of analytic functions in $\mathbb{U}$

$$\begin{aligned} US_\beta^\gamma(\mu,\zeta) &= \left\{ f \in \mathscr{A} : H_{\alpha,\beta}^{\gamma,k} f(z) \in US(\mu,\zeta) \right\}, \\ UC_\beta^\gamma(\mu,\zeta) &= \left\{ f \in \mathscr{A} : H_{\alpha,\beta}^{\gamma,k} f(z) \in UC(\mu,\zeta) \right\}, \\ USK_\beta^\gamma(\mu,\zeta,\eta) &= \left\{ f \in \mathscr{A} : H_{\alpha,\beta}^{\gamma,k} f(z) \in USK(\mu,\zeta,\eta) \right\}, \\ UCK_\beta^\gamma(\mu,\zeta,\eta) &= \left\{ f \in \mathscr{A} : H_{\alpha,\beta}^{\gamma,k} f(z) \in UCK(\mu,\zeta,\eta) \right\}, \end{aligned} \quad (15)$$

where $\beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > \max\{0, \Re(k) - 1\}$ and $\Re(k) > 0$. Also, $\Re(\alpha) = 0$ when $\Re(k) = 1$; $\beta \neq 0$.

Also, we note that

$$f(z) \in UC_\beta^\gamma(\mu,\zeta) \Leftrightarrow zf'(z) \in US_\beta^\gamma(\mu,\zeta), \quad (16)$$

$$f(z) \in UCK_\beta^\gamma(\mu,\zeta,\eta) \Leftrightarrow zf'(z) \in USK_\beta^\gamma(\mu,\zeta,\eta). \quad (17)$$

In this paper, we introduce several inclusion properties of the classes $US_\beta^\gamma(\mu,\zeta)$, $UC_\beta^\gamma(\mu,\zeta)$, $USK_\beta^\gamma(\mu,\zeta,\eta)$, and $UCK_\beta^\gamma(\mu,\zeta,\eta)$. Also, integral-preserving properties of these classes associated with generalized Libera integral operator are also obtained.

2. Inclusion Properties Associated with $H_{\alpha,\beta}^{\gamma,k} f(z)$

Lemma 1 (see [13]). *If $h(z)$ is convex univalent in $\mathbb{U}$ with $h(0) = 1$ and $\Re\{\xi h(z) + \zeta\} > 0 (\zeta \in \mathbb{C})$. Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ which satisfy the following subordination relation*

$$p(z) + \frac{zp'(z)}{\xi p(z) + \zeta} \prec h(z), \quad (18)$$

then

$$p(z) \prec h(z). \quad (19)$$

Lemma 2 (see [2]). *If $h(z)$ is convex univalent in $\mathbb{U}$ and let w be analytic in $\mathbb{U}$ with $\Re\{w(z)\} \geq 0$. Let $p(z)$ be analytic in $\mathbb{U}$ and $p(0) = h(0)$ which satisfy the following subordination relation*

$$p(z) + w(z) zp'(z) \prec h(z), \quad (20)$$

then

$$p(z) \prec h(z). \quad (21)$$

Theorem 3. *If $\Re(\gamma/k) > -(\mu+\zeta)/(\mu+1)$, then $US_\beta^{\gamma+1}(\mu,\zeta) \subset US_\beta^\gamma(\mu,\zeta)$.*

Proof. Let $f(z) \in US_\beta^{\gamma+1}(\mu,\zeta)$, put

$$p(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} f(z)} \ (z \in \mathbb{U}), \quad (22)$$

we note that $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. From (13) and (22), we have

$$\frac{H_{\alpha,\beta}^{\gamma+1,k} f(z)}{H_{\alpha,\beta}^{\gamma,k} f(z)} = \frac{k}{\gamma+k}\left(p(z) + \frac{\gamma}{k}\right). \quad (23)$$

Differentiating (23) with respect to z, we obtain

$$\frac{z\left(H_{\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma+1,k} f(z)} = p(z) + \frac{zp'(z)}{p(z) + (\gamma/k)}. \quad (24)$$

From the above relation and using (7), we may write

$$p(z) + \frac{zp'(z)}{p(z) + (\gamma/k)} \prec q_{\mu,\zeta}(z) \ (z \in \mathbb{U}). \quad (25)$$

Since $\Re\{q_{\mu,\zeta}(z)\} > (\mu+\zeta)/(\mu+1)$, we see that

$$\Re\left(q_{\mu,\zeta}(z) + \frac{\gamma}{k}\right) > 0 \ (z \in \mathbb{U}). \quad (26)$$

Applying Lemma 1, it follows that $p(z) \prec q_{\mu,\zeta}(z)$, that is, $f(z) \in US_\beta^\gamma(\mu,\zeta)$.

Using the same technique in Theorem 3 with relation (14), we have the following theorem.

Theorem 4. *If $\Re(\alpha/\beta) > -(\mu+\zeta)/(\mu+1)$, then $US_\beta^\gamma(\mu,\zeta) \subset US_{\beta+1}^\gamma(\mu,\zeta)$.*

Theorem 5. *If $\Re(\gamma/k) > -(\mu+\zeta)/(\mu+1)$, then $UC_{\beta}^{\gamma+1}(\mu,\zeta) \subset UC_{\beta}^{\gamma}(\mu,\zeta)$.*

Proof. Applying Theorem 3 and relation (16), we observe that

$$\begin{aligned} f(z) \in UC_{\beta}^{\gamma+1}(\mu,\zeta) &\Leftrightarrow zf'(z) \in US_{\beta}^{\gamma+1}(\mu,\zeta) \\ &\Rightarrow zf'(z) \in US_{\beta}^{\gamma}(\mu,\zeta) \Leftrightarrow f(z) \in UC_{\beta}^{\gamma}(\mu,\zeta), \end{aligned} \tag{27}$$

which evidently proves Theorem 5.

Similarly, we can prove the following theorem.

Theorem 6. *If $\Re(\alpha/\beta) > -(\mu+\zeta)/(\mu+1)$, then $UC_{\beta}^{\gamma}(\mu,\zeta) \subset UC_{\beta+1}^{\gamma}(\mu,\zeta)$.*

Theorem 7. *If $\Re(\gamma/k) > -(\mu+\zeta)/(\mu+1)$, then $USK_{\beta}^{\gamma+1}(\mu,\zeta,\eta) \subset USK_{\beta}^{\gamma}(\mu,\zeta,\eta)$.*

Proof. Let $f(z) \in USK_{\beta}^{\gamma+1}(\mu,\zeta,\eta)$. Then, there exists a function $r(z) \in US(\mu,\zeta)$ such that

$$\frac{z\left(H_{\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{r(z)} \prec q_{\mu,\zeta}(z). \tag{28}$$

We can choose the function $h(z)$ such that $H_{\alpha,\beta}^{\gamma+1,k} h(z) = r(z)$. Then, $h(z) \in US_{\beta}^{\gamma+1}(\mu,\zeta)$ and

$$\frac{z\left(H_{\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma+1,k} h(z)} \prec q_{\mu,\zeta}(z). \tag{29}$$

Now, let

$$p(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)}, \tag{30}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=1$. Since $h(z) \in US_{\beta}^{\gamma+1}(\mu,\zeta)$, by Theorem 3, we know that $h(z) \in US_{\beta}^{\gamma}(\mu,\zeta)$. Let

$$t(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} h(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} \quad (z \in \mathbb{U}), \tag{31}$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\} > (\mu+\zeta)/(\mu+1)$. Also, from(30), we note that

$$z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)' = H_{\alpha,\beta}^{\gamma,k} zf'(z) = \left(H_{\alpha,\beta}^{\gamma,k} h(z)\right) p(z). \tag{32}$$

Differentiating both sides of (32) with respect to z, we obtain

$$\begin{aligned} \frac{z\left(H_{\alpha,\beta}^{\gamma,k} zf'(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} &= \frac{z\left(H_{\alpha,\beta}^{\gamma,k} h(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} p(z) + zp'(z) \\ &= t(z)p(z) + zp'(z). \end{aligned} \tag{33}$$

Now, using (13) and (33), we obtain

$$\begin{aligned} \frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} = \frac{H_{\alpha,\beta}^{\gamma+1,k} zf'(z)}{H_{\alpha,\beta}^{\gamma+1,k} h(z)} &= \frac{z\left(H_{\alpha,\beta}^{\gamma,k} zf'(z)\right)' + (\gamma/k) H_{\alpha,\beta}^{\gamma,k} zf'(z)}{z\left(H_{\alpha,\beta}^{\gamma,k} h(z)\right)' + (\gamma/k) H_{\alpha,\beta}^{\gamma,k} h(z)} \\ &= \frac{\left(z\left(H_{\alpha,\beta}^{\gamma,k} zf'(z)\right)'/H_{\alpha,\beta}^{\gamma,k} h(z)\right) + (\gamma/k)\left(z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'/H_{\alpha,\beta}^{\gamma,k} h(z)\right)}{\left(z\left(H_{\alpha,\beta}^{\gamma,k} h(z)\right)'/H_{\alpha,\beta}^{\gamma,k} h(z)\right) + (\gamma/k)} \\ &= \frac{t(z)p(z) + zp'(z) + (\gamma/k)p(z)}{t(z) + (\gamma/k)} = p(z) + \frac{zp'(z)}{t(z) + (\gamma/k)}. \end{aligned} \tag{34}$$

Since $\Re(\gamma/k) > -(\mu+\zeta)/(\mu+1)$, we see that

$$\Re\left\{t(z) + \frac{\gamma}{k}\right\} > 0 \; (z \in \mathbb{U}). \tag{34}$$

Hence, applying Lemma 2, we can show that $p(z) \prec q_{\mu,\zeta}(z)$, so that $f(z) \in USK_{\beta}^{\gamma}(\mu,\zeta,\eta)$. This completes the proof of Theorem 7.

Similarly, we can prove the following theorem.

Theorem 8. *If $\Re(\alpha/\beta) > -(\mu+\zeta)/(\mu+1)$, then $USK_{\beta}^{\gamma}(\mu,\zeta,\eta) \subset USK_{\beta+1}^{\gamma}(\mu,\zeta,\eta)$.*

We can also prove Theorem 9 by using Theorem 7 and relation (17).

Theorem 9. *If* $\Re(\gamma/k) > -(\mu+\zeta)/(\mu+1)$*, then* $UCK_{\beta}^{\gamma+1}(\mu,\zeta,\eta) \subset UCK_{\beta}^{\gamma}(\mu,\zeta,\eta)$.

Also, we obtain the following theorem.

Theorem 10. *If* $\Re(\alpha/\beta) > -(\mu+\zeta)/(\mu+1)$*, then* $UCK_{\beta}^{\gamma}(\mu,\zeta,\eta) \subset UCK_{\beta+1}^{\gamma}(\mu,\zeta,\eta)$.

Now, we obtain squeeze theorems for inclusion by combining the above theorems as follows:

Combining both theorems 3 and 4, we have the following corollary.

Corollary 11. *If* $(\mu+\zeta)/(\mu+1) > -\min\{\Re(\gamma/k),\Re(\alpha/\beta)\}$*, then*

$$US_{\beta}^{\gamma+1}(\mu,\zeta) \subset US_{\beta}^{\gamma}(\mu,\zeta) \subset US_{\beta+1}^{\gamma}(\mu,\zeta). \tag{36}$$

Combining both theorems 5 and 6, we have the following corollary.

Corollary 12. *If* $(\mu+\zeta)/(\mu+1) > -\min\{\Re(\gamma/k),\Re(\alpha/\beta)\}$*, then*

$$UC_{\beta}^{\gamma+1}(\mu,\zeta) \subset UC_{\beta}^{\gamma}(\mu,\zeta) \subset UC_{\beta+1}^{\gamma}(\mu,\zeta). \tag{37}$$

Combining both theorems 7 and 8, we have the following corollary.

Corollary 13. *If* $(\mu+\zeta)/(\mu+1) > -\min\{\Re(\gamma/k),\Re(\alpha/\beta)\}$*, then*

$$USK_{\beta}^{\gamma+1}(\mu,\zeta,\eta) \subset USK_{\beta}^{\gamma}(\mu,\zeta,\eta) \subset USK_{\beta+1}^{\gamma}(\mu,\zeta,\eta). \tag{38}$$

Combining both theorems 9 and 10, we have the following corollary.

Corollary 14. *If* $(\mu+\zeta)/(\mu+1) > -\min\{\Re(\gamma/k),\Re(\alpha/\beta)\}$*, then*

$$UCK_{\beta}^{\gamma+1}(\mu,\zeta,\eta) \subset UCK_{\beta}^{\gamma}(\mu,\zeta,\eta) \subset UCK_{\beta+1}^{\gamma}(\mu,\zeta,\eta). \tag{39}$$

3. Integral Preserving Properties Associated with F_δ

The generalized Libera integral operator F_δ (see [14–16], also, see related topics [17–19]) is defined by

$$F_\delta(f)(z) = \frac{\delta+1}{z^\delta}\int_0^z t^{\delta-1} f(t)\,dt, \tag{40}$$

where $f(z) \in \mathscr{A}$ and $\delta > -1$.

Theorem 15. *Let* $\delta > -(\mu+\zeta)/(\mu+1)$*. If* $f \in US_{\beta}^{\gamma}(\mu,\zeta)$*, then* $F_\delta(f) \in US_{\beta}^{\gamma}(\mu,\zeta)$.

Proof. Let $f \in US_{\beta}^{\gamma}(\mu,\zeta)$ and set

$$p(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z)} \quad (z \in \mathbb{U}), \tag{41}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=1$. From definition of $H_{\alpha,\beta}^{\gamma,k}(f)$ and (40), we have

$$z\left(H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z)\right)' = (\delta+1) H_{\alpha,\beta}^{\gamma,k} f(z) - \delta H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z). \tag{42}$$

Then, by using (41) and (42), we obtain

$$(\delta+1)\frac{H_{\alpha,\beta}^{\gamma,k} f(z)}{H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z)} = p(z) + \delta. \tag{43}$$

Taking the logarithmic differentiation on both sides of (43) and simple calculations, we have

$$p(z) + \frac{zp'(z)}{p(z)+\delta} = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} f(z)} \prec q_{\mu,\zeta}(z). \tag{44}$$

Since $\Re(q_{\mu,\zeta}+\delta) > ((\mu+\zeta)/(\mu+1)+\delta) > 0$, by virtue of Lemma 1, we conclude that $p(z) \prec q_{\mu,\zeta}(z)$ in $\mathbb{U}$, which implies that $F_\delta(f) \in US_{\beta}^{\gamma}(\mu,\zeta)$.

Theorem 16. *Let* $\delta > -(\mu+\zeta)/(\mu+1)$*. If* $f \in UC_{\beta}^{\gamma}(\mu,\zeta)$*, then* $F_\delta(f) \in UC_{\beta}^{\gamma}(\mu,\zeta)$.

Proof. By applying Theorem 15, it follows that

$$\begin{aligned} f(z) \in UC_{\beta}^{\gamma}(\mu,\zeta) &\Leftrightarrow zf'(z) \in US_{\beta}^{\gamma}(\mu,\zeta) \\ &\Rightarrow F_\delta\left(zf'\right)(z) \in US_{\beta}^{\gamma}(\mu,\zeta) \\ &\Leftrightarrow z(F_\delta(f)(z))' \in US_{\beta}^{\gamma}(\mu,\zeta) \\ &\Leftrightarrow F_\delta(f)(z) \in UC_{\beta}^{\gamma}(\mu,\zeta), \end{aligned} \tag{45}$$

which proves Theorem 16.

Theorem 17. *Let* $\delta > -(\mu+\zeta)/(\mu+1)$*. If* $f \in USK_{\beta}^{\gamma}(\mu,\zeta,\eta)$*, then* $F_\delta(f) \in USK_{\beta}^{\gamma}(\mu,\zeta,\eta)$.

Proof. Let $f(z) \in USK_{\beta}^{\gamma}(\mu, \zeta, \eta)$. Then, there exists a function $h(z) \in US_{\beta}^{\gamma}(\mu, \zeta)$ such that

$$\frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} \prec q_{\mu,\zeta}(z). \tag{46}$$

Thus, we set

$$p(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(f)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)} \ (z \in \mathbb{U}), \tag{47}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Since $h(z) \in US_{\beta}^{\gamma}(\mu, \zeta)$, we see from Theorem 15 that $F_{\delta}(h) \in US_{\beta}^{\gamma}(\mu, \zeta)$. Let

$$t(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)}, \tag{48}$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\} > (\mu + \zeta)/(\mu + 1)$. Using (47), we have

$$H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z) = \left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right) p(z). \tag{49}$$

Differentiating both sides of (49) with respect to z and simple calculations, we obtain

$$\begin{aligned}\frac{z\left(H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)} &= \frac{z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)} p(z) + z p'(z) \\ &= t(z)p(z) + z p'(z).\end{aligned} \tag{50}$$

Now, using the identity (42) and (50), we obtain

$$\begin{aligned}\frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} &= \frac{H_{\alpha,\beta}^{\gamma,k} z f'(z)}{H_{\alpha,\beta}^{\gamma,k} h(z)} = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z)\right)' + \delta H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z)}{z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)' + \delta H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)} \\ &= \frac{\left(z\left(H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z)\right)' / H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right) + \delta\left(z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(f)(z)\right)' / H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)}{\left(z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)' / H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right) + \delta} \\ &= \frac{t(z)p(z) + z p'(z) + \delta p(z)}{t(z) + \delta} = p(z) + \frac{z p'(z)}{t(z) + \delta}.\end{aligned} \tag{51}$$

Since $\delta > -(\mu + \zeta)/(\mu + 1)$ and $\Re\{t(z)\} > (\mu + \zeta)/(\mu + 1)$, we see that

$$\Re\{t(z) + \delta\} > 0 \ (z \in \mathbb{U}). \tag{52}$$

Applying Lemma 2 into relation (51), it follows that $p(z) \prec q_{\mu,\zeta}(z)$, which is $F_{\delta}(f) \in USK_{\beta}^{\gamma}(\mu, \zeta, \eta)$.

We can deduce the integral-preserving property asserted by 18 by using Theorem 17 and relation (17).

Theorem 18. *Let $\delta > (-\mu + \zeta)/(\mu + 1)$. If $f \in UCK_{\beta}^{\gamma}(\mu, \zeta, \eta)$, then $F_{\delta}(f) \in UCK_{\beta}^{\gamma}(\mu, \zeta, \eta)$.*

Data Availability

All data are available in this paper.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors contributed equally to the writing of this paper. All authors approved the final version of the manuscript.

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Research Article

Products of Composition and Differentiation between the Fractional Cauchy Spaces and the Bloch-Type Spaces

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The operators DC_Φ and $C_\Phi D$ are defined by $DC_\Phi(f) = (f \circ \Phi)'$ and $C_\Phi D(f) = f' \circ \Phi$ where Φ is an analytic self-map of the unit disc and f is analytic in the disc. A characterization is provided for boundedness and compactness of the products of composition and differentiation from the spaces of fractional Cauchy transforms F_α to the Bloch-type spaces B^β, where $\alpha > 0$ and $\beta > 0$. In the case $\beta < 2$, the operator $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact $\Leftrightarrow DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded $\Leftrightarrow \Phi' \in B^\beta, \Phi\Phi' \in B^\beta$ and $\|\Phi\|_\infty < 1$. For $\beta < 1$, $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is compact $\Leftrightarrow C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded $\Leftrightarrow \Phi \in B^\beta$ and $\|\Phi\|_\infty < 1$.

1. Introduction

Let $U = \{z \in C : |z|<1\}$ and let $H(U)$ denote the family of functions analytic on U. Let M denote the Banach space of complex Borel measures on $T = \{x \in C : |x| = 1\}$, endowed with the total variation norm. For $\alpha > 0$, the space F_α of fractional Cauchy transforms is the family of functions of the form

$$f(z) = \int_T \frac{1}{(1 - \bar{x}z)^\alpha} \, d\mu(x)(|z| < 1), \qquad (1)$$

where $\mu \in M$. The principal branch of the logarithm is used here. The space F_α is a Banach space, with norm

$$\|f\|_{F_\alpha} = \inf \|\mu\|, \qquad (2)$$

where μ varies over all measures in M for which (1) holds. The families F_α have been studied extensively [1, 2]. Interest in these spaces was first established in connection with the classical family S of normalized univalent functions. It is known that $S \subseteq F_\alpha$ for any $\alpha > 2$ [2]. The reference [2] also includes MacGregor's construction of a function $f \in S$ with $f \notin F_2$.

Let $\beta > 0$. The Bloch-type space B^β is the Banach space of functions analytic in U such that $\sup_{z\in U}(1 - |z|^2)^\beta |f'(z)| < \infty$, with norm

$$\|f\|_{B^\beta} = |f(0)| + \sup_{z\in U}\left(1 - |z|^2\right)^\beta \left|f'(z)\right|. \qquad (3)$$

The relation (1) implies that $F_\alpha \subset B^{\alpha+1}$, and there is a constant C depending only on α such that $\|f\|_{B^{\alpha+1}} \le C\|f\|_{F_\alpha}$ for all $f \in F_\alpha$.

Let Φ be an analytic self-map of U. The composition operator C_Φ is defined by $C_\Phi(f) = f \circ \Phi$ for $f \in H(U)$. The differentiation operator D is defined by $D(f) = f'$. In this paper, the products $C_\Phi D(f) = f' \circ \Phi$ and $DC_\Phi(f) = \Phi'(f' \circ \Phi)$ are studied. Conditions on Φ are given, necessary and sufficient to imply boundedness or compactness of $C_\Phi D : F_\alpha \longrightarrow B^\beta$ and $DC_\Phi : F_\alpha \longrightarrow B^\beta$.

Products of composition and differentiation on the Bloch space were studied by Ohno in [3]. In [4], Li and Stević studied $C_\Phi D$ and DC_Φ acting between the weighted Bergman spaces and the Bloch-type spaces. In [5], Hibschweiler and Portnoy studied these operators between Bergman and Hardy spaces.

2. Preliminary Results

Fix $\alpha > 0$. For fixed $z \in U$ and for $n = 0, 1, \ldots$, the relation (1) yields a constant C depending only on n such that $|f^{(n)}(z)| \le C\|f\|_{F_\alpha}/(1-|z|^2)^{\alpha+n}$ [2].

For each $w \in U$, $\|1/(1-\bar{w}z)^\alpha\|_{F_\alpha} = 1$ [2].

We follow the convention that C denotes a positive constant, the precise value of which will differ from one appearance to the next.

Lemma 1 and Lemma 2 will be used to develop test functions for F_α. Proofs appear in [6].

Lemma 1. *Fix $\alpha > 0$. For $w \in U$, define*

$$h_w(z) = \frac{1-|w|^2}{(1-\bar{w}z)^{\alpha+1}} \, (z \in U). \tag{4}$$

Then, $h_w \in F_\alpha$, and there is a constant C such that $\|h_w\|_{F_\alpha} \le C$ for all $w \in U$.

Lemma 2. *Fix $\alpha > 0$. For $w \in U$, define*

$$k_w(z) = \frac{\left(1-|w|^2\right)^2}{(1-\bar{w}z)^{\alpha+2}} \, (z \in U). \tag{5}$$

Then, $k_w \in F_\alpha$, and there is a constant C such that $\|k_w\|_{F_\alpha} \le C$ for all $w \in U$.

3. The Operator $DC_\Phi : F_\alpha \longrightarrow B^\beta$

In [7], Shapiro proved that the condition $\|\Phi\|_\infty < 1$ is necessary for $C_\Phi : X \longrightarrow X$ to be compact, for Banach spaces X obeying boundary regularity and Möbius invariance. In particular, Shapiro's result applies to the Lipschitz spaces and thus, to the space B^γ when $\gamma < 1$ [8].

Theorem 3. *Fix $\alpha > 0$ and $0 < \beta < 2$. Let Φ be an analytic self-map of U.*

$$\begin{aligned} &DC_\Phi : F_\alpha \longrightarrow B^\beta \text{ is bounded} \Leftrightarrow \\ &\Phi' \in B^\beta, \Phi\Phi' \in B^\beta \text{ and } \|\Phi\|_\infty < 1 \Leftrightarrow \\ &DC_\Phi : F_\alpha \longrightarrow B^\beta \text{ is compact}. \end{aligned} \tag{6}$$

Proof. First, assume that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded, that is, there is a constant C such that $\|DC_\Phi(f)\|_{B^\beta} \le C\|f\|_{F_\alpha}$ for all $f \in F_\alpha$. It is clear that $\Phi' = DC_\Phi(z) \in B^\beta$ and $\Phi\Phi' = DC_\Phi(z^2/2) \in B^\beta$. Thus,

$$\left(1-|z|^2\right)^\beta \left|\Phi''(z)\right| \le C, \tag{7}$$

and

$$\left(1-|z|^2\right)^\beta \left|\Phi(z)\Phi''(z) + \left(\Phi'(z)\right)^2\right| \le C, \tag{8}$$

for all $z \in U$. It follows that

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta \left|\Phi'(z)\right|^2 < \infty, \tag{9}$$

and thus, $\Phi \in B^{\beta/2}$.

Let $w \in U$ and define

$$g_w(z) = \frac{\alpha+1}{\left(1-\overline{\Phi(w)}z\right)^\alpha} - \frac{\alpha\left(1-|\Phi(w)|^2\right)}{\left(1-\overline{\Phi(w)}z\right)^{\alpha+1}} \, (z \in U). \tag{10}$$

By Lemma 1 and the preliminary results, there is a constant C independent of w such that $\|g_w\|_{F_\alpha} \le C$, and thus, $\|DC_\Phi(g_w)\|_{B^\beta} = \|(g_w' \circ \Phi)\Phi'\|_{B^\beta} \le C$. It follows that

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta \left|g_w''(\Phi(z))\left(\Phi'(z)\right)^2 + g_w'(\Phi(z))\,\Phi''(z)\right| \le C, \tag{11}$$

for all $w \in U$. Calculations yield $g_w'(\Phi(w)) = 0$ and

$$g_w''(\Phi(w)) = \frac{-\alpha(\alpha+1)\overline{\Phi(w)}^2}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}}. \tag{12}$$

The substitution $z = w$ in (11) now yields

$$\sup_{w\in U} \left(1-|w|^2\right)^\beta \frac{\alpha(\alpha+1)|\Phi(w)|^2\left|\Phi'(w)\right|^2}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} \le C, \tag{13}$$

and thus,

$$\sup_{|\Phi(w)|>1/2} \frac{\left(1-|w|^2\right)^\beta \left|\Phi'(w)\right|^2}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < \infty. \tag{14}$$

By the relation (9),

$$\sup_{|\Phi(z)|\le 1/2} \frac{\left(1-|w|^2\right)^\beta \left|\Phi'(w)\right|^2}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < \infty. \tag{15}$$

Thus,

$$C = \sup_{w\in U} \frac{\left(1-|w|^2\right)^\beta \left|\Phi'(w)\right|^2}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < \infty. \tag{16}$$

It follows that

$$\sup_{w\in U} \frac{\left(1-|w|^2\right)^{\beta/2} \left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\beta/2}} \le \sup_{w\in U} \frac{\left(1-|w|^2\right)^{\beta/2} \left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{(\alpha+2)/2}} < \infty. \tag{17}$$

By Xiao's result [9], $C_\Phi : B^{\beta/2} \longrightarrow B^{\beta/2}$ is bounded. Furthermore, (16) yields

$$\frac{\left(1-|w|^2\right)^{\beta/2}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\beta/2}} \leq C\left(1-|\Phi(w)|^2\right)^{(\alpha-\beta+2)/2} \longrightarrow 0, \quad (18)$$

as $|\Phi(w)| \longrightarrow 1$. Thus, $C_\Phi : B^{\beta/2} \longrightarrow B^{\beta/2}$ is compact [9], and it follows as in [7] that $\|\Phi\|_\infty < 1$. It has been established that the conditions $\Phi' \in B^\beta, \Phi\Phi' \in B^\beta$, and $\|\Phi\|_\infty < 1$ are necessary if $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded.

Next, assume that $\Phi' \in B^\beta$, $\Phi\Phi' \in B^\beta$, and $\|\Phi\|_\infty < 1$. To show that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact, let (f_n) be a bounded sequence in F_α with $f_n \longrightarrow 0$ uniformly on compact subsets of U as $n \longrightarrow \infty$. It is enough to prove that $\|DC_\Phi(f_n)\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$. First, note that $|f_n{}'(\Phi(0))\Phi'(0)| \longrightarrow 0$ as $n \longrightarrow \infty$. For $z \in U$, (9) yields

$$\begin{aligned}\left(1-|z|^2\right)^\beta\left|(DC_\Phi f_n)'(z)\right| &= \left(1-|z|^2\right)^\beta \Big| f_n{}''(\Phi(z))\left(\Phi'(z)\right)^2 \\ &\quad + f_n{}'(\Phi(z))\Phi''(z)\Big| \\ &\leq C \max_{|w|\leq\|\Phi\|_\infty}\left|f_n{}''(w)\right| \\ &\quad + \left\|\Phi'\right\|_{B^\beta} \max_{|w|\leq\|\Phi\|_\infty}\left|f_n{}'(w)\right|.\end{aligned} \quad (19)$$

Since $f_n{}' \longrightarrow 0$ and $f_n{}'' \longrightarrow 0$ uniformly on compact subsets as $n \longrightarrow \infty$, the argument shows that $\sup_{z\in U}(1-|z|^2)^\beta|(DC_\Phi f_n)'(z)| \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, $\|DC_\Phi(f_n)\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$, and $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact, as required.

The remaining implication is clear, and the proof is complete.

Theorem 4. *Fix $\alpha > 0$ and $\beta \geq 2$. Let Φ be an analytic self-map of U. Then,*

$$DC_\Phi : F_\alpha \longrightarrow B^\beta \text{ is bounded} \Leftrightarrow \quad (20)$$

$$\sup_{z\in U}\frac{\left(1-|z|^2\right)^\beta\left|\Phi''(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+1}} < \infty, \quad (21)$$

and

$$\sup_{z\in U}\frac{\left(1-|z|^2\right)^\beta\left|\Phi'(z)\right|^2}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} < \infty. \quad (22)$$

Proof. Fix α, β and Φ as described.

First, assume (21) and (22). Let $f \in F_\alpha$. By (21) and the introductory remarks in Section 2,

$$\begin{aligned}\left(1-|z|^2\right)^\beta\left|f'(\Phi(z))\right|\left|\Phi''(z)\right| &\leq \left(1-|z|^2\right)^\beta\left|\Phi''(z)\right|\frac{C\|f\|_{F_\alpha}}{\left(1-|\Phi(z)|^2\right)^{\alpha+1}} \\ &\leq C\|f\|_{F_\alpha}.\end{aligned} \quad (23)$$

A similar argument using (22) yields

$$\left(1-|z|^2\right)^\beta\left|f''(\Phi(z))\right|\left|\Phi'(z)\right|^2 \leq C\|f\|_{F_\alpha}, \quad (24)$$

for all $z \in U$. Thus, $\sup_{z\in U}(1-|z|^2)^\beta|(DC_\Phi f)'(z)| \leq C\|f\|_{F_\alpha}$. Since $|(DC_\Phi f)(0)| \leq C\|f\|_{F_\alpha}$, it now follows that $\|DC_\Phi(f)\|_{B^\beta} \leq C\|f\|_{F_\alpha}$, as required.

For the converse, assume that $\|DC_\Phi(f)\|_{B^\beta} \leq C\|f\|_{F_\alpha}$ for a constant C independent of $f \in F_\alpha$. In particular, $\Phi' \in B^\beta$.

The argument leading to (16) remains valid for $\beta \geq 2$. Thus, (22) holds. It remains to prove (21). First, note that

$$\sup_{|\Phi(w)|\leq 1/2}\frac{\left(1-|w|^2\right)^\beta\left|\Phi''(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+1}} \leq \left(\frac{4}{3}\right)^{\alpha+1}\left\|\Phi'\right\|_{B^\beta} < \infty. \quad (25)$$

For $w \in U$, define

$$H_w(z) = \frac{(\alpha+3)\left(1-|\Phi(w)|^2\right)}{\left(1-\bar{\Phi}(w)z\right)^{\alpha+1}} - \frac{(\alpha+1)\left(1-|\Phi(w)|^2\right)^2}{\left(1-\bar{\Phi}(w)z\right)^{\alpha+2}}, \quad (26)$$

for $z \in U$. By Lemma 1 and Lemma 2, there is a constant C independent of w such that $\|H_w\|_{F_\alpha} \leq C$. Thus, $\|DC_\Phi(H_w)\|_{B^\beta} \leq C$ for all w. It follows that

$$\sup_{z\in U}\left(1-|z|^2\right)^\beta\left|H_w{}'(\Phi(z))\Phi''(z) + \left(\Phi'(z)\right)^2 H_w{}''(\Phi(z))\right| < C, \quad (27)$$

for all $w \in U$. An argument using $H_w{}'(\Phi(w)) = (\alpha+1)\bar{\Phi}(w)/(1-|\Phi(w)|^2)^{\alpha+1}$ and $H_w{}''(\Phi(w)) = 0$ yields

$$\sup_{1/2<|\Phi(w)|}\frac{\left(1-|w|^2\right)^\beta\left|\Phi''(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+1}} < \infty. \quad (28)$$

The relations (25) and (28) establish relation (21), and the proof is complete.

Theorem 5. *Fix $\alpha > 0$ and assume $\beta \geq 2$. Let Φ be a self-map of U for which $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded.*

$$DC_\Phi : F_\alpha \longrightarrow B^\beta \text{ is compact} \Leftrightarrow \quad (29)$$

$$\lim_{|\Phi(z)|\longrightarrow 1}\frac{\left(1-|z|^2\right)^\beta\left|\Phi''(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+1}} = 0, \quad (30)$$

and

$$\lim_{|\Phi(z)|\longrightarrow 1}\frac{\left(1-|z|^2\right)^\beta\left|\Phi'(z)\right|^2}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} = 0. \quad (31)$$

Proof. First, assume that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded and relations (30) and (31) hold. Let (f_n) be a bounded sequence in F_α such that $f_n \longrightarrow 0$ uniformly on compact subsets of U. As previously noted, there is a constant C depending only on α such that

$$\left(1-|z|^2\right)^\beta \left|f_n'(\Phi(z))\right|\left|\Phi'(z)\right|^2 \le C\frac{\left(1-|z|^2\right)^\beta\left|\Phi'(z)\right|^2}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}}, \tag{32}$$

for $n = 1, 2, \ldots$ and $z \in U$. Relation (31) now implies that given $\varepsilon > 0$, there exists $r_0, 0 < r_0 < 1$, such that

$$\sup_{|\Phi(z)|>r_0} \left(1-|z|^2\right)^\beta \left|f_n''(\Phi(z))\right|\left|\Phi'(z)\right|^2 < \varepsilon, \tag{33}$$

for all n.

Since $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded, relation (9) holds, and thus,

$$\left(1-|z|^2\right)^\beta \left|f_n'(\Phi(z))\right|\left|\Phi'(z)\right|^2 < C\left|f_n''(\Phi(z))\right|, \tag{34}$$

for all $z \in U$. Since $f_n'' \longrightarrow 0$ uniformly on $\{w : |w| \le r_0\}$, there exists $N > 0$ such that

$$\sup_{|\Phi(z)|\le r_0} \left(1-|z|^2\right)^\beta \left|f_n''(\Phi(z))\right|\left|\Phi'(z)\right|^2 < \varepsilon, \tag{35}$$

for all $n > N$. The relations (33) and (35) yield

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta \left|f_n''(\Phi(z))\right|\left|\Phi'(z)\right|^2 < \varepsilon, \tag{36}$$

for $n > N$.

A similar argument using $\Phi' \in B^\beta$ and (30) yields $N_1 > 0$ such that

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta \left|f_n'(\Phi(z))\right|\left|\Phi''(z)\right| < \varepsilon, \tag{37}$$

for $n > N_1$. The relations (36) and (37) yield

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta \left|(DC_\Phi f_n)'(z)\right| \longrightarrow 0, \tag{38}$$

as $n \longrightarrow \infty$.

Since $|(DC_\Phi f_n)(0)| \longrightarrow 0$ as $n \longrightarrow \infty$, the argument yields $\|DC_\Phi(f_n)\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$ for any sequence (f_n) as described, and therefore, $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact.

For the converse, assume that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact. We may assume that $\|\Phi\|_\infty = 1$. Let (z_n) be any sequence in U with $|\Phi(z_n)| \longrightarrow 1$ as $n \longrightarrow \infty$. For $z \in U$, define

$$f_n(z) = \frac{(\alpha+3)\left(1-|\Phi(z_n)|^2\right)}{\left(1-\Phi(\bar{z}_n)z\right)^{\alpha+1}} - \frac{(\alpha+1)\left(1-|\Phi(z_n)|^2\right)^2}{\left(1-\Phi(\bar{z}_n)z\right)^{\alpha+2}}. \tag{39}$$

By the lemmas above, $\|f_n\|_{F_\alpha} \le C$. Also, $f_n \longrightarrow 0$ uniformly on compact subsets. Therefore, $\|DC_\Phi(f_n)\|_{B^\beta} \longrightarrow 0$ and

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta \left|f_n'(\Phi(z))\Phi''(z) + f_n''(\Phi(z))\left(\Phi'(z)\right)^2\right| \longrightarrow 0, \tag{40}$$

as $n \longrightarrow \infty$. Calculations yield $f_n''(\Phi(z_n)) = 0$ and

$$f_n'(\Phi(z_n)) = \frac{(\alpha+1)\Phi(\bar{z}_n)}{\left(1-|\Phi(z_n)|^2\right)^{\alpha+1}}. \tag{41}$$

Substitution into (40) yields

$$\frac{\left(1-|z_n|^2\right)^\beta |\Phi(z_n)|\left|\Phi''(z_n)\right|}{\left(1-|\Phi(z_n)|^2\right)^{\alpha+1}} \longrightarrow 0, \tag{42}$$

as $n \longrightarrow \infty$. Since (z_n) is a generic sequence with $|\Phi(z_n)| \longrightarrow 1$ as $n \longrightarrow \infty$, this yields the relation (30).

A similar argument using the functions

$$g_n(z) = \frac{(\alpha+2)\left(1-|\Phi(z_n)|^2\right)}{\left(1-\Phi(\bar{z}_n)z\right)^{\alpha+1}} - \frac{(\alpha+1)\left(1-|\Phi(z_n)|^2\right)^2}{\left(1-\Phi(\bar{z}_n)z\right)^{\alpha+2}} \tag{43}$$

yields the relation (31). The details are omitted.

Theorem 3 implies that if $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded for fixed α, β with $\beta < 2$, then $DC_\Phi : F_\gamma \longrightarrow B^\beta$ is compact for all $\gamma > 0$. The next corollary gives a related result when $\beta \ge 2$.

Corollary 6. *Fix $\alpha > 0$ and $\beta \ge 2$. Let Φ be a self-map of U and assume that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded. Then, $DC_\Phi : F_\gamma \longrightarrow B^\beta$ is compact for any $\gamma, 0 < \gamma < \alpha$.*

Proof. By assumption, there is a constant C such that $\|DC_\Phi(f)\|_{B^\beta} \le C\|f\|_{F_\alpha}$ for all $f \in F_\alpha$. Fix γ with $0 < \gamma < \alpha$ and let $f \in F_\gamma$. Then, $f \in F_\alpha$ and $\|f\|_{F_\alpha} \le \|f\|_{F_\gamma}$ [2]. Therefore, $DC_\Phi : F_\gamma \longrightarrow B^\beta$ is bounded and Theorem 5 applies.

Since $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded, (21) yields

$$\frac{\left(1-|z|^2\right)^\beta\left|\Phi''(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\gamma+1}} \le C\left(1-|\Phi(z)|^2\right)^{\alpha-\gamma}, \tag{44}$$

and therefore,

$$\lim_{|\Phi(z)|\longrightarrow 1} \frac{\left(1-|z|^2\right)^{\beta}\left|\Phi''(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\gamma+1}} = 0. \tag{45}$$

A similar argument using (22) yields

$$\lim_{|\Phi(z)|\longrightarrow 1} \frac{\left(1-|z|^2\right)^{\beta}\left|\Phi'(z)\right|^2}{\left(1-|\Phi(z)|^2\right)^{\gamma+2}} = 0. \tag{46}$$

Theorem 5 now yields $DC_{\Phi} : F_{\gamma} \longrightarrow B^{\beta}$ is compact.

4. The Operator $C_{\Phi}D$

In this section, characterizations are given for self-maps Φ for which $C_{\Phi}D : F_{\alpha} \longrightarrow B^{\beta}$ is bounded or compact. The proofs are similar to those in Section 3, so details are kept to a minimum.

Theorem 7. *Fix $\alpha > 0$ and $0 < \beta < 1$.*

$$\begin{aligned} &C_{\Phi}D : F_{\alpha} \longrightarrow B^{\beta} \text{ is bounded} \Longleftrightarrow \\ &\Phi \in B^{\beta} \text{ and } \|\Phi\|_{\infty} < 1 \Longleftrightarrow \\ &C_{\Phi}D : F_{\alpha} \longrightarrow B^{\beta} \text{ is compact}. \end{aligned} \tag{47}$$

Proof. First, assume that there is a constant C independent of $f \in F_{\alpha}$ such that $\|C_{\Phi}D(f)\|_{B^{\beta}} \le C\|f\|_{F_{\alpha}}$. In particular, $\Phi \in B^{\beta}$. For $w \in U$, define

$$g_w(z) = \frac{1}{\left(1-\overline{\Phi(w)}z\right)^{\alpha}} \quad (z \in U). \tag{48}$$

There is a constant C independent of $w \in U$ such that $\|g_w\|_{F_{\alpha}} \le C$, and it follows that

$$\sup_{z\in U}\left(1-|z|^2\right)^{\beta}\left|g_w''(\Phi(z))\Phi'(z)\right| < C, \tag{49}$$

for all $w \in U$. The substitution $z = w$ yields

$$\left(1-|w|^2\right)^{\beta}\frac{\alpha(\alpha+1)|\Phi(w)|^2\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < C, \tag{50}$$

for all $w \in U$. Therefore,

$$\sup_{1/2<|\Phi(w)|} \frac{\left(1-|w|^2\right)^{\beta}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < \infty. \tag{51}$$

Since $\Phi \in B^{\beta}$,

$$\sup_{|\Phi(w)|\le 1/2} \frac{\left(1-|w|^2\right)^{\beta}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < \infty. \tag{52}$$

It follows that

$$\sup_{w\in U} \frac{\left(1-|w|^2\right)^{\beta}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < \infty \tag{53}$$

and therefore

$$\sup_{w\in U} \frac{\left(1-|w|^2\right)^{\beta}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\beta}} < \infty. \tag{54}$$

By [9], $C_{\Phi} : B^{\beta} \longrightarrow B^{\beta}$ is bounded. A further argument as in the proof of Theorem 3 yields that $C_{\Phi} : B^{\beta} \longrightarrow B^{\beta}$ is compact. Since $\beta < 1$, Shapiro's result [7] applies and yields $\|\Phi\|_{\infty} < 1$. Thus, the conditions $\Phi \in B^{\beta}$ and $\|\Phi\|_{\infty} < 1$ are necessary in order for $C_{\Phi}D : F_{\alpha} \longrightarrow B^{\beta}$ to be bounded.

Next, assume $\Phi \in B^{\beta}$ and $\|\Phi\|_{\infty} < 1$. Let (f_n) be a bounded sequence in F_{α} with $f_n \longrightarrow 0$ uniformly on compact subsets of U. First, note that $|f_n'(\Phi(0))| \longrightarrow 0$ as $n \longrightarrow \infty$. For $z \in U$,

$$\left(1-|z|^2\right)^{\beta}\left|\left(f_n' \circ \Phi\right)'(z)\right| \le \|\Phi\|_{B^{\beta}} \max_{|w|\le\|\Phi\|_{\infty}}\left|f_n''(w)\right|. \tag{55}$$

Since $f_n'' \longrightarrow 0$ uniformly on compact subsets, the argument yields $\|C_{\Phi}D(f_n)\|_{B^{\beta}} \longrightarrow 0$ and $C_{\Phi}D : F_{\alpha} \longrightarrow B^{\beta}$ is compact.

The remaining implication is trivial, and the proof is complete.

Theorem 8. *Fix $\alpha > 0$ and $\beta \ge 1$. Let Φ be a self-map of U.*

$$\begin{aligned} &C_{\Phi}D : F_{\alpha} \longrightarrow B^{\beta} \text{ is bounded} \Longleftrightarrow \\ &\sup_{z\in U} \frac{\left(1-|z|^2\right)^{\beta}\left|\Phi'(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} < \infty. \end{aligned} \tag{56}$$

Proof. First, assume that the supremum is finite.

Let $f \in F_{\alpha}$. By previous remarks, $|f'(\Phi(0))| \le C\|f\|_{F_{\alpha}}$. By an argument as in the proof of Theorem 4,

$$\begin{aligned} \left(1-|z|^2\right)^{\beta}\left|\left(f' \circ \Phi\right)'(z)\right| &= \left(1-|z|^2\right)^{\beta}\left|f''(\Phi(z))\right|\left|\Phi'(z)\right| \\ &\le \left(1-|z|^2\right)^{\beta}\frac{C\|f\|_{F_{\alpha}}\left|\Phi'(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} \\ &\le C\|f\|_{F_{\alpha}}, \end{aligned} \tag{57}$$

and thus, $\|C_{\Phi}D(f)\|_{B^{\beta}} \le C\|f\|_{F_{\alpha}}$ as required.

To complete the proof, assume that $\|C_{\Phi}D(f)\|_{B^{\beta}} \le C \|f\|_{F_{\alpha}}$ for a constant C independent of f. The argument leading to (53) remains valid for $\beta \ge 1$. This proves the opposite implication, and the proof is complete.

Theorem 9. *Fix $\alpha > 0$ and $\beta \geq 1$. Let Φ be a self-map of U and assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded.*

$$C_\Phi D : F_\alpha \longrightarrow B^\beta \text{ is compact} \Leftrightarrow \lim_{|\Phi(z)|\longrightarrow 1} \frac{\left(1-|z|^2\right)^\beta |\Phi'(z)|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} = 0. \tag{58}$$

Proof. First, assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded and the limit condition holds. Let (f_n) be a bounded sequence in F_α with $f_n \longrightarrow 0$ uniformly on compact subsets as $n \longrightarrow \infty$. Clearly, $|f_n{}'(\Phi(0))| \longrightarrow 0$ as $n \longrightarrow \infty$. As in previous arguments,

$$\left(1-|z|^2\right)^\beta |f_n{}''(\Phi(z))||\Phi'(z)| \leq C \frac{\left(1-|z|^2\right)^\beta |\Phi'(z)|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}}, \tag{59}$$

for all $z \in U$. The hypothesis now implies that, given $\varepsilon > 0$, there exists $r_0, 0 < r_0 < 1$, such that

$$\sup_{|\Phi(z)|>r_0} \left(1-|z|^2\right)^\beta |f_n{}''(\Phi(z))||\Phi'(z)| < \varepsilon, \tag{60}$$

for all n. Since $\Phi \in B^\beta$ and since $f_n{}'' \longrightarrow 0$ uniformly on compact subsets,

$$\sup_{|\Phi(z)|\leq r_0} \left(1-|z|^2\right)^\beta |f_n{}''(\Phi(z))||\Phi'(z)| \longrightarrow 0, \tag{61}$$

as $n \longrightarrow \infty$. By (60) and (61),

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta \left|\left(f_n{}' \circ \Phi\right)'(z)\right| \longrightarrow 0, \tag{62}$$

as $n \longrightarrow \infty$. The argument yields $\|f_n{}' \circ \Phi\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$ for any sequence (f_n) as described above. Thus, $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is compact.

Now, assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is compact. We may assume that $\|\Phi\|_\infty = 1$. Let (z_n) be any sequence in U with $|\Phi(z_n)| \longrightarrow 1$ as $n \longrightarrow \infty$. For $n = 1, 2, \cdots$, define

$$f_n(z) = \frac{1-|\Phi(z_n)|^2}{\left(1-\bar{\Phi}(z_n) z\right)^{\alpha+1}}, \tag{63}$$

for $z \in U$. By Lemma 1, $\|f_n\|_{F_\alpha} \leq C$ for all n. Also, $f_n \longrightarrow 0$ uniformly on compact subsets. Therefore, $\|C_\Phi D(f_n)\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$. Given $\varepsilon > 0$, there exists $N > 0$ such that

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta |f_n{}''(\Phi(z))||\Phi'(z)| < \varepsilon, \tag{64}$$

for all $n > N$. In particular, $(1-|z_n|^2)^\beta |f_n{}''(\Phi(z_n))||\Phi'(z_n)| < \varepsilon$ for $n > N$. Calculations yield

$$\frac{\left(1-|z_n|^2\right)^\beta (\alpha+1)(\alpha+2)|\Phi(z_n)|^2|\Phi'(z_n)|}{\left(1-|\Phi(z_n)|^2\right)^{\alpha+2}} < \varepsilon, \tag{65}$$

for $n > N$. Since (z_n) is a generic sequence with $|\Phi(z_n)| \longrightarrow 1$, it follows that

$$\lim_{|\Phi(z)|\longrightarrow 1} \frac{\left(1-|z|^2\right)^\beta |\Phi'(z)|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} = 0. \tag{66}$$

The proof is complete.

Assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded for fixed $\alpha > 0$ and $\beta < 1$. By Theorem 7, $\Phi \in B^\beta$ and $\|\Phi\|_\infty < 1$. It follows that $C_\Phi D : F_\gamma \longrightarrow B^\beta$ is compact for any $\gamma > 0$. Corollary 10 gives a related result in the case $\beta \geq 1$.

Corollary 10. *Fix $\alpha > 0, \beta \geq 1$ and assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded. Then, $C_\Phi D : F_\gamma \longrightarrow B^\beta$ is compact for any $\gamma, 0 < \gamma < \alpha$.*

Proof. Fix $0 < \gamma < \alpha$ and let $f \in F_\gamma$. Then, $f \in F_\alpha$ and $\|f\|_{F_\alpha} \leq \|f\|_{F_\gamma}$ [2]. Therefore, $C_\Phi D : F_\gamma \longrightarrow B^\beta$ is bounded and Theorem 9 applies.

By Theorem 8, there is a constant C with

$$\frac{\left(1-|z|^2\right)^\beta |\Phi'(z)|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} \leq C, \tag{67}$$

for all $z \in U$. Therefore,

$$\frac{\left(1-|z|^2\right)^\beta |\Phi'(z)|}{\left(1-|\Phi(z)|^2\right)^{\gamma+2}} \leq C \left(1-|\Phi(z)|^2\right)^{\alpha-\gamma} \longrightarrow 0, \tag{68}$$

as $|\Phi(z)| \longrightarrow 1$. By Theorem 9, $C_\Phi D : F_\gamma \longrightarrow B^\beta$ is compact.

Data Availability

This manuscript does not contain any data.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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Research Article

Toeplitz-Superposition Operators on Analytic Bloch Spaces

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The important purpose of this current work is to study a new class of operators, the so-called Toeplitz-superposition operators as an expansion of the weighted known composition operators, induced by such continuous entire functions mapping on bounded specific sets. Minutely, we have deeply discussed the conditions for boundedness of this new type of operators between certain types of some holomorphic Bloch classes with some specific values of the weighted functions.

1. Introduction

Fundamentals of the needed analytic function spaces as well as the types of concerned operators are briefly introduced. The paper focuses first on the concerned setting of certain classes of function spaces and the new defined operator, which in turn is motivated essentially by some certain classical concepts of known operators such as superposition operators as well as Toeplitz operator. There is an emphasis in the concerned paper on intensive tying together the needed type of analytic function spaces and the concerned operators, to illustrate the roles of the obtained results.

All of the needed information to justify the target of this research is collected in this concerned section. Moreover, here, basic concerned concepts, the Bloch space of analytic-type, certain needed concerned lemmas, and superposition and Toeplitz operators are presented.

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ be the open unit disk in $\mathbb{C}$, and let $\mathscr{H}(\mathbb{D})$ denote the class of all analytic functions in $\mathbb{D}$. Let $dA(z)=dx\,dy$ denote the concerned Lebesgue measures on $\mathbb{D}$.

Numerous intensive studies on analytic Bloch-type spaces are researched in literature (see [1–5] and others).

Let $h\in\mathscr{H}(\mathbb{D})$ and $0<b<\infty$, the b-Bloch space $\mathscr{B}^b$ is defined by

$$\mathscr{B}^b=\left\{f\in\mathscr{H}(\mathbb{D}):\|h\|_{\mathscr{B}^b}=\sup_{\zeta\in\mathbb{D}}\left(1-|\zeta|^2\right)^b|f'(\zeta)|<\infty\right\}. \tag{1}$$

The space $\mathscr{B}^1$ is called the Bloch space and denoted by $\mathscr{B}$ (see [3]).

The following interesting needed lemma has been proved in [6].

Lemma 1. *For a given $0<a<\infty$, let the function $h\in\mathscr{B}^a$. Then, we have*

$$|h(\zeta)|\lesssim\begin{cases}\|h\|_{\mathscr{B}^a}, & \text{if } 0<a<1;\\ \|h\|_{\mathscr{B}^a}\ln\dfrac{e}{1-|\zeta|^2}, & \text{if } a=1;\\ \dfrac{\|h\|_{\mathscr{B}^a}}{\left(1-|\zeta|^2\right)^{a-1}}, & \text{if } a>1.\end{cases} \tag{2}$$

The following useful integral estimate is well known and can be found in [7].

Lemma 2. *Let $s > 0$ and $t > -1$. Then*

$$\int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^t dA(w)}{|1-\bar{z}w|^{2+t+s}} \lesssim \frac{1}{\left(1-|z|^2\right)^s}, \quad \text{for all} \quad z \in \mathbb{D}. \quad (3)$$

For $a > -1$ and $p \in (0,\infty)$, the weighted Bergman spaces $\mathscr{A}^p_a(\mathbb{D})$ is the space of all functions $h \in \mathscr{H}(\mathbb{D})$, for which

$$\|h\|^p_{\mathscr{A}^p_a} = \int_{\mathbb{D}} |h(\zeta)|^p dA_a(\zeta) < \infty, \quad \text{where} \quad dA_a(\zeta) = \left(1-|\zeta|^2\right)^a dA(\zeta). \quad (4)$$

When $a = 0$, we simply write $\mathscr{A}^p(\mathbb{D})$ for $\mathscr{A}^p_0(\mathbb{D})$, and when $p = 2$, $\mathscr{A}^2_a(\mathbb{D})$ is a Hilbert space. It is well known that the Bergman kernel $K_z(w)$ of the Hilbert space $\mathscr{A}^2_a(\mathbb{D})$ is given by $K_z(w) = (1-\bar{w}z)^{-a-2}$, where $z, w \in \mathbb{D}$. The Bergman projection P_a is the orthogonal projection from $L^2(\mathbb{D}, dA_a)$ onto Hilbert space $\mathscr{A}^2_a(\mathbb{D})$, which given as:

$$P_a h(z) = \int_{\mathbb{D}} K_z(w)h(w)dA_a(w). \quad (5)$$

For $a > -1$ and $h \in \mathscr{H}(\mathbb{D})$, the Toeplitz-type operator T^a_u with symbol $u \in H^\infty(\mathbb{D})$ is defined by

$$T^a_u h(z) = \int_{\mathbb{D}} \frac{u(w)h(w)}{(1-\bar{w}z)^{a+2}} dA_a(w). \quad (6)$$

This paper is organized as follows: during Section 2, we have defined the Toeplitz-superposition operators on the normed (metric) subspaces. Throughout Section 3, we establish the conditions for the Toeplitz-superposition operators to be bounded from a-Bloch space $\mathscr{B}^a$ into b-Bloch space $\mathscr{B}^b$, in the case $a \in (0,1)$ and $b > a$ or $b < a$. Section 4 is devoted to a study the boundedness of Toeplitz-superposition operators between weighted Bloch spaces in the case $0 < a \le b$ or $a = 0$, $b > 0$.

Remark 3. It is concerned remarkable to say that two concerned quantities N_h and N^*_h , where both depending on the concerned function $h \in \mathscr{H}(\mathbb{D})$, the expression $N_h \lesssim N^*_h$, can be satisfied when we have a concerned positive constant C_1, for which $N_h \le C_1 N^*_h$. When $N^*_h \lesssim N_h \lesssim N^*_h$, the expression $N_h \approx N^*_h$ can be written to say that there is an equivalence relation between the concerned quantities N_h and N^*_h. Furthermore, when $N_h \approx N^*_h$, we deduce that $N_h < \infty \Leftrightarrow N^*_h < \infty$.

2. Toeplitz-Superposition Operators

Let $\mathscr{E}(\mathbb{C})$ denote the set of all entire functions on the complex plane $\mathbb{C}$. For a function $\phi \in \mathscr{E}(\mathbb{C})$, the superposition operator $S_\phi : \mathscr{H}(\mathbb{D}) \to \mathscr{H}(\mathbb{D})$ is defined by $S_\phi(h) = (\phi \circ h)$. Moreover, if $u \in \mathscr{H}(\mathbb{D})$ and $\phi \in \mathscr{E}(\mathbb{C})$, the weighted superposition operator $S_{\phi,u} : \mathscr{H}(\mathbb{D}) \to \mathscr{H}(\mathbb{D})$ is defined by $S_{\phi,u}(h)(z) = u(z)\phi(h(z))$, for all $h \in \mathscr{H}(\mathbb{D})$ and $z \in \mathbb{D}$. Note that, if $u(z) = 1$, then $S_{\phi,u} = S_\phi$, for any $z \in \mathbb{D}$.

For any normed subspace $X \subset \mathscr{H}(\mathbb{D})$, we will consider the set $\mathscr{K}(X)$, defined by

$$\mathscr{K}(X) = \{h \in X : \phi \circ h \in X, \quad \text{where} \quad \phi \in \mathscr{E}(\mathbb{C})\}. \quad (7)$$

Now, we define the Toeplitz-superposition operators acting on $\mathscr{H}(\mathbb{D})$.

Definition 4. Let two functions $\phi \in \mathscr{E}(\mathbb{C})$ and $u \in \mathscr{H}(\mathbb{D})$. Then, the Toeplitz-superposition operators $T_u S_\phi$ on the normed (metric) subspace X are given by

$$T_u S_\phi(h) = T_u(\phi \circ h) = P(u \cdot (\phi \circ h)), \quad \text{for all} \quad h \in \mathscr{K}(X). \quad (8)$$

Let α, β be the scalers if ϕ is a fixed entire function and $u, v \in \mathscr{H}(\mathbb{D})$. Then, from the definition of Toeplitz-superposition operators, we have

$$\begin{aligned} T_{\alpha u+\beta v} S_\phi(h) &= T_{\alpha u+\beta v}(\phi \circ h) = P(\alpha u \cdot (\phi \circ h) + \beta v \cdot (\phi \circ h)) \\ &= \alpha P(u \cdot (\phi \circ h)) + \beta P(v \cdot (\phi \circ h)) \\ &= \alpha T_u S_\phi(h) + \beta T_v S_\phi(h), \end{aligned} \quad (9)$$

which holds for all $h \in \mathscr{K}(X)$, and hence, the Toeplitz-Superposition operators are linear on the normed subspace X.

It can be seen that whenever $u \in \mathscr{H}(\mathbb{D})$, then, the operator $T_u S_\phi$ becomes the operator $S_{\phi,u}$. So, Toeplitz-superposition operators can be taken as an extension of weighted superposition operators. The present paper is interested in answering the following interesting questions.

(i) Can we transform one holomorphic function space into another by what kinds of entire functions?

(ii) What are the holomorphic spaces that can be transformed one into another by certain weighted classes of entire functions such as specific analytic polynomials of a certain degree and certain entire-type functions of given type and order?

(iii) When does the holomorphic function φ induces a Toeplitz-superposition operators to form one holomorphic function space into another?

As a concerned result, the obtained results will introduce answers of the above mentioned questions by using the class of Toeplitz-superposition operators that are acting between different classes of Bloch functions.

Also, the answers for some of these concerned questions have been introduced by several authors; the following citations can be stated for interesting and intensive studies [8–20].

3. Boundedness in the case $a \in (0,1)$ and $b > a$ or $b < a$

Several important discussions on boundedness property of the new operator acting on the analytic Bloch spaces are presented in this concerned section. Furthermore, some essential equivalent characterizations for its boundedness are established too.

Now, we will introduce the main results of boundedness.

Theorem 5. *For $a \in (0,1)$ and $b > a$. Suppose that $u \in H^\infty(\mathbb{D})$ and let $\phi \in \mathscr{E}(\mathbb{C})$, with $\phi \neq 0$. Then, the Toeplitz-superposition operator $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^b$ is bounded.*

Proof. First, assume that $u \in H^\infty(\mathbb{D})$. Let $h \in \mathscr{B}^b$, since

$$\|h\|_{\mathscr{B}^b} \approx \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} |h(z)|, \tag{10}$$

we have

$$\begin{aligned}
\|T_u S_\phi h\|_{\mathscr{B}^b} &\approx \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} |T_u S_\phi h| \\
&= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} \left| \int_{\mathbb{D}} \frac{u(w)(\phi\circ h)(w)}{(1-\bar{w}z)^2} dA(w) \right| \\
&\lesssim \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} \int_{\mathbb{D}} \frac{|u(w)||\phi(h(w))|}{|1-\bar{w}z|^2} dA(w) \\
&\lesssim \|u\|_{H^\infty} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} \int_{\mathbb{D}} \frac{|\phi(h(w)|}{|1-\bar{w}z|^2} dA(w).
\end{aligned} \tag{11}$$

Now, let the constant $R > 0$ where $h \in \mathscr{B}^a$ such that $\|h\|_{\mathscr{B}^a} \leq R$, by Lemma 1, we have $|h(z)| \lesssim R$. Set $R_1 = \max_{|z|=R} |\phi(z)|$, then $|\phi(h(z))| \lesssim R_1$. Since $b > a$, we have the fact that $\mathscr{B}^a \subset \mathscr{B}^b$, and since $a \in (0,1)$, we have that $\mathscr{B}^a \subset H^\infty(\mathbb{D})$. Thus,

$$\begin{aligned}
\|T_u S_\phi h\|_{\mathscr{B}^b} &\lesssim R_1 \|u\|_{H^\infty} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} \int_{\mathbb{D}} \frac{dA(w)}{|1-\bar{w}z|^2} \\
&\lesssim \|u\|_{H^\infty} < \|u\|_{\mathscr{B}^a} < \infty,
\end{aligned} \tag{12}$$

where R, R_1 depended only on a, b, and ϕ. This shows that $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^b$ is bounded.

Theorem 6. *For $0 < b < a < 1$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathscr{E}(\mathbb{C})$. Then, the Toeplitz-superposition operator $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^b$ is bounded if and only if $u \in H^\infty(\mathbb{D})$ and ϕ is a constant entire function.*

Proof. It is trivial that if $u \in H^\infty(\mathbb{D})$ and ϕ is constant, then $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^b$ is bounded. If ϕ is constant, not identically 0, and $T_u S_\phi$ maps $\mathscr{B}^a$ into $\mathscr{B}^b$ then it is clear that $u \in H^\infty(\mathbb{D})$. Assume now that $u \neq 0$ and ϕ is not constant, and set $T_u S_\phi$ maps $\mathscr{B}^a$ into $\mathscr{B}^b$. Let h be the constant function defined by $h(\zeta) = \lambda$, for all $\zeta \in \mathbb{D}$, such that $\phi(\lambda) \neq 0$. Since $h \in \mathscr{B}^a$, it follows that $T_u S_\phi h(\zeta) = T_u \phi(\lambda) \in \mathscr{B}^b$. This implies that $u \in \mathscr{B}^b \subset H^\infty(\mathbb{D})$, since $0 < b < 1$. Finally, since ϕ is not constant, then there is a disk $|w - w_0| < \varepsilon$ and $\delta > 0$, on which $|\phi(w)| > \delta |w|$. Set the test function $h_0(w) = w_0 + r(1-w)^{1-a} \in \mathscr{B}^a$. Then, for all $w \in \mathbb{D}$, we have

$$\begin{aligned}
\|T_u S_\phi h_0\|_{\mathscr{B}^b} &\approx \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} |T_u S_\phi h_0| \\
&= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} \left| \int_{\mathbb{D}} \frac{u(w)(\phi\circ h_0)(w)}{(1-\bar{w}z)^2} dA(w) \right| \\
&\geq \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} \int_{\mathbb{D}} \frac{|u(w)|}{|1-w|^a |1-\bar{w}z|^2} dA(w).
\end{aligned} \tag{13}$$

But, along with the positive radius, we get $|u(w)|/(|1-w|^a |1-\bar{w}z|^2) \to \infty$, as $w \to 1$. This shows that $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^b$ is not bounded.

4. Boundedness in the case $0 < a \leq b$ or $a = 0, b > 0$

Theorem 7. *For $a > 0$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathscr{E}(\mathbb{C})$. Then, $T_u S_\phi$ is bounded on $\mathscr{B}^a$ if and only if $u \in H^\infty(\mathbb{D})$ and ϕ is an affine function (linear function plus a translation).*

Proof. First, suppose that $u \in H^\infty(\mathbb{D})$ and ϕ is an affine function. It is easy to explain $T_u S_\phi$ is bounded from $\mathscr{B}^a$ into itself.

On the other hand, assume that $u \in H^\infty(\mathbb{D})$ and $\phi \in \mathscr{E}(\mathbb{C})$ does not linear function. Then, by using the Cauchy estimates for $\phi \in \mathscr{E}(\mathbb{C})$, we can find a sequence $\{w_n\} \subset \mathbb{C}$, for each $n \in \mathbb{N}$ such that $|w_n| \to \infty$ as $n \to \infty$ and $|\phi(w_n)| = \max_{|w|=n} |\phi(w)| \gtrsim |w_n|^2$. Also, since the weight $\left(1-|\zeta|^2\right)^a$ is typical, we can find a sequence of points $\{z_n\} \subset \mathbb{D}$ such that $|z_n| \to 1^-$, with $(0.5<|z_n|<1)$ and such that $(1-|z_n|)|w_n| = 1$, for all $n \in \mathbb{N}$. Now consider the sequence of functions $\{h_n\}$ contained in $\mathscr{B}^a$ satisfies $\|h_n\|_{\mathscr{B}^a} \leq 1$ and $|h_n(z_n)| = |w_n|$. Furthermore, we can suppose that $h_n(z_n) = w_n$. Hence,

$$\begin{aligned}
\|T_u S_\phi(h_n)\|_{\mathscr{B}^b} &\approx \sup_{z_n\in\mathbb{D}} \left(1-|z_n|^2\right)^{b-1} |T_u \phi(h_n(z_n))| \\
&\gtrsim \|u\|_{H^\infty} \sup_{z\in\mathbb{D}} \left(1-|z_n|^2\right)^{b-1} \int_{\mathbb{D}} \frac{|w_n|^2}{|1-\bar{w}z_n|^2} dA(w) \\
&\gtrsim \|u\|_{H^\infty} \sup_{z\in\mathbb{D}} \left(1-|z_n|^2\right)^{b-1} \int_{\mathbb{D}} \frac{|w_n|}{|1-\bar{w}z|^2} dA(w) \\
&\to \infty, \text{as } n \to \infty.
\end{aligned} \tag{14}$$

Because $|w_n| \to \infty$ as $n \to \infty$. This shows that $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^a$ cannot be bounded if $\phi \in \mathscr{E}(\mathbb{C})$ is not a linear function.

Theorem 8. *For $0 < a \le b$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathscr{E}(\mathbb{C})$ be an increasing and continuous function. Then, $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^b$ is bounded if and only if $u \in H^\infty(\mathbb{D})$, and for each $\lambda \in (0,1)$, there is a positive constant η whenever $|w| > \eta$, such that*

$$|\phi(w)| \lesssim \phi(\lambda \mid w \mid). \tag{15}$$

Proof. First, suppose that $u \in H^\infty(\mathbb{D})$ and (15) is true. Now, consider $R_1 > 0$ and let $h \in \mathscr{B}^a$ satisfy $\|h\|_{\mathscr{B}^a} \le R_1$ and select $\lambda \in (0,1)$ such that $\lambda R_1 < 1$. Then, there is $\eta > 0$, such that $|\phi(w)| \lesssim \phi(\lambda \mid w \mid)$, whenever $|w| > \eta$. Thus, since $\bar{D}_R = \{w \in \mathbb{C} : |w| \le R\}$ is a compact set and $\phi \in \mathscr{E}(\mathbb{C})$ is a continuous function, we can assume that $|\phi(w)| \le 1$, for all $w \in \bar{D}_R$. Hence,

$$\begin{aligned}
\|T_u S_\phi h\|_{\mathscr{B}^b} &\approx \sup_{z_n \in \mathbb{D}} \left(1 - |z_n|^2\right)^{b-1} |T_u \phi(h(z))| \\
&= \sup_{\{z \in \mathbb{D}, |h(w)| \le R\}} \left(1 - |z|^2\right)^{b-1} |T_u S_\phi h| \\
&\quad + \sup_{\{z \in \mathbb{D}, |h(w)| > R\}} \left(1 - |z|^2\right)^{b-1} |T_u \phi(h(z))| \\
&\lesssim \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{b-1} \int_{\mathbb{D}} \frac{dA(w)}{|1 - \bar{w}z|^2} \\
&\quad + \|u\|_{H^\infty} \sup_{\{z \in \mathbb{D}, |h(w)| > R\}} \left(1 - |z|^2\right)^{b-1} \int_{\mathbb{D}} \phi \\
&\quad \cdot \left(\lambda \mid \frac{\|h\|_{\mathscr{B}^a}}{\left(1 - |w|^2\right)^a} \mid\right) \frac{dA(w)}{|1 - \bar{w}z|^2} \\
&\lesssim \|u\|_{H^\infty} + \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{b-1} \int_{\mathbb{D}} \phi \\
&\quad \cdot \left(\lambda \mid \frac{1}{\left(1 - |w|^2\right)^a} \mid\right) \frac{dA(w)}{|1 - \bar{w}z|^2}.
\end{aligned} \tag{16}$$

Using that the function ϕ is increasing and the fact that $\lambda < 1$, we have

$$\begin{aligned}
\|T_u S_\phi h\|_{\mathscr{B}^b} &\lesssim \|u\|_{H^\infty} + \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{b-1} \int_{\mathbb{D}} \frac{dA(w)}{|1 - \bar{w}z|^2} \\
&\lesssim \|u\|_{H^\infty}.
\end{aligned} \tag{17}$$

This shows that $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^b$ is bounded.

On the other hand, assume that $u \in H^\infty(\mathbb{D})$ and $\phi \in \mathscr{E}(\mathbb{C})$ does not satisfy (15). Then, we can find $\lambda_1 \in (0,1)$ and a sequence $\{w_n\} \subset \mathbb{C}$ such that $|w_n| \to \infty$ as $n \to \infty$ and $|\phi(w_n)| \ge \phi(\lambda_1 \mid w_n \mid)$, for all $n \in \mathbb{N}$. Since the weight $(1 - |z|^2)^a$ is typical, we can find a sequence of points $\{z_n\} \subset \mathbb{D}$ such that $|z_n| \to 1^-$ as $n \to \infty$. Thus, we can consider a sequence of functions $\{h_n\}$ contained in $\mathscr{B}^a$ satisfies $\|h_n\|_{\mathscr{B}^a} \le 1$ and $|h_n(z_n)| \le |w_n|$. Now, let $z \in \mathbb{D}$ and set the function $f_n(z) = w_n h_n(z)/h_n(z_n)$ for all $n \in \mathbb{N}$. Then, we have $f_n(z_n) = w_n$ and $\|f_n\|_{\mathscr{B}^a} \lesssim 1$. For large enough $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|T_u S_\phi f_n\|_{\mathscr{B}^b} &\approx \sup_{z_n \in \mathbb{D}} \left(1 - |z_n|^2\right)^{b-1} |T_u S_\phi f_n(z_n)| \\
&\gtrsim \sup_{z_n \in \mathbb{D}} \left(1 - |z_n|^2\right)^{b-1} \int_{\mathbb{D}} \frac{|u(w)||\phi(w_n)|}{|1 - \bar{w}z_n|^2} dA(w) \\
&\gtrsim \|u\|_{H^\infty} \sup_{z_n \in \mathbb{D}} \left(1 - |z_n|^2\right)^{b-1} \int_{\mathbb{D}} \frac{\phi(\lambda_1 |w_n|)}{|1 - \bar{w}z_n|^2} dA(w) \\
&\gtrsim \|u\|_{H^\infty} \sup_{z_n \in \mathbb{D}} \left(1 - |z_n|^2\right)^{b-1} \int_{\mathbb{D}} \phi\left(\left|\frac{1}{\left(1 - |w_n|^2\right)^a}\right|\right) \\
&\quad \cdot \frac{dA(w)}{|1 - \bar{w}z_n|^2} \to \infty, as\, n \to \infty.
\end{aligned} \tag{18}$$

Then, we conclude that $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^b$ cannot be bounded.

Theorem 9. *For $1 < a \le b$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathscr{E}(\mathbb{C})$. Then, the following are equivalent:*

(i) $T_u S_\phi$ maps $\mathscr{B}^a$ into $\mathscr{B}^b$

(ii) $u \in H^\infty(\mathbb{D})$ and ϕ is a polynomial of degree at most $b - 1/a - 1$

(iii) $T_u S_\phi : \mathscr{B}^a \to \mathscr{B}^b$ is bounded

Proof. First, suppose that (i) holds, let $\phi = z^n$ be a polynomial with $n \le (b-1)/(a-1)$, then, we have that $u \in H^\infty(\mathbb{D})$.

Now, suppose that the entire function ϕ is a polynomial of degree $m > b - 1/a - 1$. Then, for an integer n and a positive constant δ, there is a sequence $z_n \to \infty$ such that $|\phi(z_n)| \ge \delta |z_n|^m$. We may assume without losing generality that $|z_n| > 1$ and $|\arg z_n| < \min\{a\pi/4, \pi/2\}$, for an integer n. Now, we let $h_a(w) = (1 - w)^{1-a} \in \mathscr{B}^a$, then, we show that $T_u S_\phi h_a \notin \mathscr{B}^b$. The point $w_n = 1 - (z_n)^{-1/a}$ such that $|1 - w_n| < 1$ and $|\arg(1 - w_n)| < \pi/4$, and satisfies that $|1 - w_n| \lesssim (1 - w_n)$, for an integer n. Thus, we have

$$\begin{aligned}
&\left(1 - |z_n|^2\right)^{b-1} |T_u S_\phi h_a(z_n)| \\
&\quad = \left(1 - |z_n|^2\right)^{b-1} \left|\int_{\mathbb{D}} \frac{u(w)\phi(h_a(w))}{(1 - \bar{w}z_n)^2} dA(w)\right| \\
&\quad \gtrsim \left(1 - |z_n|^2\right)^{b-1} \int_{\mathbb{D}} \frac{\delta \mid u(w) \mid}{|1 - z_n|^{ma} |1 - \bar{w}z_n|^2} dA(w) \\
&\quad \gtrsim \int_{\mathbb{D}} \frac{\delta \mid u(w) \mid}{|1 - z_n|^{ma-m-b+1} |1 - w|^a |1 - \bar{w}z_n|^2} dA(w).
\end{aligned} \tag{19}$$

Since $ma - m - b + 1 > 0$, then, we obtain

$$\left(1 - |z_n|^2\right)^{b-1} |T_u S_\phi h_a(z_n)| \to \infty, as\, n \to \infty. \tag{20}$$

This implies that $T_u S_\phi h_a \notin \mathscr{B}^b$. Based on the above it is clear that (i) $\Rightarrow$ (ii).

Second, assume that $u \in H^{\infty}(\mathbb{D})$ and $\phi = z^n$ are a polynomial of degree $n \le (b-1)/(a-1)$. For all $h \in \mathscr{B}^a$ and $b-1-na-n \ge 0$, by Lemma 1 and 2, where n is bounded, we have

$$\begin{aligned}
\|T_u S_\phi h\|_{\mathscr{B}^b} &\approx \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} |T_u S_\phi h(z)| \\
&= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} \left| \int_{\mathbb{D}} \frac{u(w)(\phi\circ h)(w)}{(1-\bar{w}z)^2} dA(w) \right| \\
&\le \|u\|_{H^\infty} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} \int_{\mathbb{D}} \frac{|h(w)|^n}{|1-\bar{w}z|^2} dA(w) \\
&\le \|u\|_{H^\infty} \|h\|^n_{\mathscr{B}^a} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1-na-n} \int_{\mathbb{D}} \frac{dA(w)}{|1-\bar{w}z|^2} \\
&\le \|u\|_{H^\infty} \|h\|^n_{\mathscr{B}^a}.
\end{aligned} \tag{21}$$

This shows that (ii) $\Rightarrow$ (iii). Thus, the proof has been completed.

Theorem 10. *For $b > -1$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathscr{E}(\mathbb{C})$, with order ρ and type τ. Then, the following are equivalent:*

(i) $T_u S_\phi$ maps $\mathscr{B}$ into $\mathscr{B}^b$

(ii) $u \in H^\infty(\mathbb{D})$ and $\phi \in \mathscr{E}(\mathbb{C})$ with $\rho < 1$ or (with $\rho = 1$ and $\tau = 0$);

(iii) $T_u S_\phi : \mathscr{B} \to \mathscr{B}^b$ is bounded

Proof. First, assume that $T_u S_\phi$ maps $\mathscr{B}$ into $\mathscr{B}^b$. Now, we assume on the antithesis that (ii) does not hold. Then, the function $\phi \in \mathscr{E}(\mathbb{C})$ with $\rho > 1$ or (with $\rho = 1$ and $\tau > 0$). Thus, there is a positive constant λ and a sequence $\{w_n\}$ of complex numbers such that $|w_n| \to \infty$ and

$$|\phi(w_n)| \ge \exp(\lambda |w_n|), \quad \text{for any} \quad n \in \mathbb{N}. \tag{22}$$

Hence, as in the proof of Theorem 8, we can consider the sequence $\{w_n\} \subset \mathbb{D}$ and $\{h_n\} \subset \mathscr{B}$ satisfies $\|h_n\|_{\mathscr{B}} \le 1$ and $|h_n(z_n)| \lesssim |w_n|$. Now, let $z \in \mathbb{D}$ and set the function $f_n(z) = w_n h_n(z)/h_n(z_n)$ for all $n \in \mathbb{N}$. Then, we have $f_n(z_n) = w_n$ and $\|f_n\|_{\mathscr{B}} \lesssim 1$. For large enough, $n \in \mathbb{N}$, since $|w_n| \to \infty$, we obtain

$$\begin{aligned}
\|T_u S_\phi f_n\|_{\mathscr{B}^b} &\approx \sup_{z\in\mathbb{D}} \left(1-|z_n|^2\right)^{b-1} |T_u S_\phi f_n(z_n)| \\
&\gtrsim \sup_{z\in\mathbb{D}} \left(1-|z_n|^2\right)^{b-1} \int_{\mathbb{D}} \frac{|u(w)||\phi(w_n)|}{|1-\bar{w}z_n|^2} dA(w) \\
&\gtrsim \|u\|_{H^\infty} \sup_{z\in\mathbb{D}} \left(1-|z_n|^2\right)^{b-1} \int_{\mathbb{D}} \frac{\exp(\lambda_1 |w_n|)}{|1-\bar{w}z_n|^2} dA(w) \\
&\to \infty, as\, n \to \infty.
\end{aligned} \tag{23}$$

Then, we conclude that $T_u S_\phi : \mathscr{B} \to \mathscr{B}^b$ cannot be bounded. Based on the above results, it is clear that (i) $\Rightarrow$ (ii).

Second, set $M(t,\phi) = \max_{|w|=t} |\phi(w)|$, where $t \ge 0$, the order ρ of $\phi \in \mathscr{E}(\mathbb{C})$ is

$$\rho = \limsup_{t\to\infty} \frac{\log\log M(t,\phi)}{\log t}. \tag{24}$$

If $0 < \rho < \infty$, then the type τ of $\phi \in \mathscr{E}(\mathbb{C})$ is

$$\tau = \limsup_{t\to\infty} \frac{\log M(t,\phi)}{t^\rho}. \tag{25}$$

For given $\lambda = b/R > 0$, the condition (ii) implies that (see for example [18])

$$|\phi(w)| \lesssim \exp(\lambda |w|), \quad \text{for any} \quad w \in \mathbb{C}. \tag{26}$$

Moreover, for a function $h \in \mathscr{B}$, with $\|h\|_{\mathscr{B}} \lesssim 1$, we know that

$$|h(w)| \lesssim \left(1 + \log \frac{1}{1-|w|}\right), \quad \text{for} \quad w \in \mathbb{D}. \tag{27}$$

Then,

$$|\phi(h(w))| \lesssim \exp(\lambda |h(w)|) \le \left(\frac{e}{1-|w|}\right)^b \le (2e)^b. \tag{28}$$

Thus, we have

$$\begin{aligned}
\|T_u S_\phi h\|_{\mathscr{B}^b} &\approx \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} |T_u S_\phi h(z)| \\
&\lesssim (2e)^b \|u\|_{H^\infty} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{b-1} \int_{\mathbb{D}} \frac{dA(w)}{|1-\bar{w}z|^2} \\
&\lesssim (2e)^b \|u\|_{H^\infty} < \infty.
\end{aligned} \tag{29}$$

This shows that $T_u S_\phi : \mathscr{B} \to \mathscr{B}^b$ is bounded. So, (ii) $\Rightarrow$ (iii).

5. Conclusion and Future Study

This manuscript deals with a radical study of a concerned class of Toeplitz superposition operators acting between some certain classes of analytic function spaces of Bloch-type. Global discussions of the boundedness property of the new class of operators are presented class of the univalent Bloch functions. All concerned entire functions which transform a class of holomorphic Bloch-type spaces into another using the so-called Toeplitz superposition operators in terms of their order and type or the degree of polynomials are characterized in this paper. Moreover, all the defined Toeplitz-superposition operators induced by concerned entire functions are cleared to be bounded actually. We have cleared that for two spaces of normed-type which belonging to $\mathscr{H}(\mathbb{D})$, where $X=\mathscr{B}^a$ and $Y=\mathscr{B}^b$, we can find certain concerned functions ϕ and u, with $\phi\in\mathscr{E}(\mathbb{C})$ and $u\in\mathscr{H}(\mathbb{D})$, for which the newly Toeplitz-superposition operators T_uS_ϕ can map $\mathscr{B}^a$ into $\mathscr{B}^b$ for some specific values of a and b. Furthermore, the operator $T_uS_\phi : X\to Y$ is shown to be actually bounded.

Data Availability

The data is not applicable to this concerned article as no concerned data sets were created or used through this concerned study.

Conflicts of Interest

The authors declare that they have no competing interest.

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Research Article

A Study of Fourth-Order Hankel Determinants for Starlike Functions Connected with the Sine Function

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In this paper, upper bounds for the fourth-order Hankel determinant $H_4(1)$ for the function class $\mathcal{S}_s^*$ associated with the sine function are given.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathbb{D} = \{z : |z|<1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots (z \in \mathbb{D}), \tag{1}$$

and let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions.

Suppose that $\mathcal{P}$ is the class of analytic functions p normalized by

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{2}$$

and satisfying the condition

$$\mathfrak{R}(p(z)) > 0 (z \in \mathbb{D}). \tag{3}$$

Assume that f and g are two analytic functions in $\mathbb{D}$. Then, we say that the function g is subordinate to the function f, and we write

$$g(z) \prec f(z) (z \in \mathbb{D}), \tag{4}$$

if there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| <1$, such that (see [1])

$$g(z) = f(\omega(z)) (z \in \mathbb{D}). \tag{5}$$

In 2018, Cho et al. [2] introduced the following function class S_s^*:

$$S_s^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec (1 + \sin z) (z \in \mathbb{D}) \right\}, \tag{6}$$

which implies that the quantity $(zf'(z))/(f(z))$ lies in an eight-shaped region in the right-half plane.

In 1976, Noonan and Thomas [3] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ of functions f as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix} (a_1 = 1). \tag{7}$$

In particular, we have

$$H_2(1)=\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}=a_3-a_2^2\,(a_1=1, n=1, q=2),$$

$$H_2(2)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2\,(n=2, q=2),$$

$$H_3(1)=\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}(n=1, q=3), \tag{8}$$

$$H_4(1)=\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_4 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix}(n=1, q=4).$$

Since $f\in\mathcal{S}$, $a_1=1$, thus

$$\begin{aligned} H_4(1)=a_7&\left\{a_3\left(a_2a_4-a_3^2\right)-a_4(a_4-a_2a_3)+a_5\left(a_3-a_2^2\right)\right\}\\ &-a_6\left\{a_3(a_2a_5-a_3a_4)-a_4(a_5-a_2a_4)+a_6\left(a_3-a_2^2\right)\right\}\\ &+a_5\left\{a_3\left(a_3a_5-a_4^2\right)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3)\right\}\\ &+a_5\left\{a_3\left(a_3a_5-a_4^2\right)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3)\right\}\\ &-a_4\left\{a_4\left(a_3a_5-a_4^2\right)-a_5(a_2a_5-a_3a_4)+a_6(a_4-a_2a_3)\right\}. \end{aligned} \tag{9}$$

We note that $H_2(1)$ is the well-known Fekete-Szegö functional (see [4–6]).

In recent years, many papers have been devoted to finding upper bounds for the second-order Hankel determinant $H_2(2)$ and the third-order Hankel determinant $H_3(1)$, whose elements are various classes of analytic functions; it is worth mentioning that [7–20]. For instance, Murugusundaramoorthy and Bulboacă [21] defined a new subclass of analytic functions $M\mathfrak{L}_c^a(\lambda,\phi)$ and got upper bounds for the Fekete-Szegö functional and the Hankel determinant of order two for $f\in M\mathfrak{L}_c^a(\lambda,\phi)$. Islam et al. [22] examined the q-analog of starlike functions connected with a trigonometric sine function and discussed some interesting geometric properties, such as the well-known problems of Fekete-Szegö, the necessary and sufficient condition, the growth and distortion bound, closure theorem, and convolution results with partial sums for this class. Zaprawa et al. [23] obtained the bound of the third Hankel determinant for the univalent starlike functions. Very recently, Arif et al. [24] studied the problem of fourth Hankel determinant $H_4(1)$ for the first time for the class of bounded turning functions and successfully obtained the bound of $H_4(1)$. Recently, Khan et al. [25] discussed some classes of functions with bounded turning which are connected to the sine functions and obtained upper bounds for the third- and fourth-order Hankel determinants related to such classes. Inspired by the aforementioned works, in this paper, we mainly investigate upper bounds for the fourth-order Hankel determinant $H_4(1)$ for the function class $\mathcal{S}_s^*$ associated with the sine function.

2. Main Results

By proving our desired results, we need the following lemmas.

Lemma 1 (see [26]). *If $p(z)\in\mathcal{P}$, then exists some x, z with $|x|\le 1, |z|\le 1$, such that*

$$\begin{aligned} 2c_2&=c_1^2+x\left(4-c_1^2\right),\\ 4c_3&=c_1^3+2c_1x\left(4-c_1^2\right)-\left(4-c_1^2\right)c_1x^2+2\left(4-c_1^2\right)\left(1-|x|^2\right)z. \end{aligned} \tag{10}$$

Lemma 2 (see [27]). *Let $p(z)\in\mathcal{P}$, then*

$$\begin{aligned} &\left|c_1^4+c_2^2+2c_1c_3-3c_1^2c_2-c_4\right|\le 2,\\ &\left|c_1^5+3c_1c_2^2+3c_1^2c_3-4c_1^3c_2-2c_1c_4-2c_2c_3+c_5\right|\le 2,\\ &|c_1^6+6c_1^2c_2^2+4c_1^3c_3+2c_1c_5+2c_2c_4+c_3^2-c_2^3\\ &\quad-5c_1^4c_2-3c_1^2c_4-6c_1c_2c_3-c_6|\le 2,\\ &|c_n|\le 2, n=1,2,\cdots. \end{aligned} \tag{11}$$

Lemma 3 (see [28]). *Let $p(z)\in\mathcal{P}$, then we have*

$$\begin{aligned} \left|c_2-\frac{c_1^2}{2}\right|&\le 2-\frac{|c_1|^2}{2},\\ |c_{n+k}-\mu c_nc_k|&<2,\quad 0\le\mu\le 1,\\ |c_{n+2k}-\mu c_nc_k^2|&\le 2(1+2\mu). \end{aligned} \tag{12}$$

We now state and prove the main results of our present investigation.

Theorem 4. *If the function $f(z)\in\mathcal{S}_s^*$ and of the form ((1)), then*

$$\begin{aligned} |a_2|&\le 1,\\ |a_3|&\le\frac{1}{2},\\ |a_4|&\le 0.344,\\ |a_5|&\le\frac{3}{8},\\ |a_6|&\le\frac{67}{120},\\ |a_7|&\le\frac{5587}{10800}. \end{aligned} \tag{13}$$

Proof. Since $f(z)\in\mathcal{S}_s^*$, according to subordination relationship, thus there exists a Schwarz function $\omega(z)$ with $\omega(0)=0$ and $|\omega(z)|<1$, satisfying

$$\frac{zf'(z)}{f(z)} = 1 + \sin(\omega(z)). \quad (14)$$

Here,

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z+\sum_{n=2}^{\infty}na_nz^n}{z+\sum_{n=2}^{\infty}a_nz^n} = \left(1+\sum_{n=2}^{\infty}na_nz^{n-1}\right)\left[1-a_2z+\left(a_2^2-a_3\right)z^2\right.\\ &\quad \left.-\left(a_2^3-2a_2a_3+a_4\right)z^3+\left(a_2^4-3a_2^2a_3+2a_2a_4-a_5\right)z^4+\cdots\right]\\ &= 1+a_2z+\left(2a_3-a_2^2\right)z^2+\left(a_2^3-3a_2a_3+3a_4\right)z^3\\ &\quad +\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)z^4\\ &\quad +\left(5a_6-5a_2a_5+a_2^5-5a_3a_4-5a_2^3a_3+5a_2^2a_4+5a_2a_3^2\right)z^5\\ &\quad +\left(6a_7-6a_2a_6+6a_2^2a_5-6a_3a_5+12a_2a_3a_4-a_2^6\right.\\ &\quad \left.-6a_2^3a_4-3a_4^2+2a_3^3-9a_2^2a_3^2+6a_2^4a_3\right)z^6+.\end{aligned} \quad (15)$$

Now, we define a function

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1z + c_2z^2 + \cdots. \quad (16)$$

It is easy to see that $p(z) \in \mathscr{P}$ and

$$\omega(z) = \frac{p(z)-1}{1+p(z)} = \frac{c_1z+c_2z^2+c_3z^3+\cdots}{2+c_1z+c_2z^2+c_3z^3+\cdots}. \quad (17)$$

On the other hand,

$$\begin{aligned}1+\sin(\omega(z)) &= 1+\frac{1}{2}c_1z+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)z^2+\left(\frac{5c_1^3}{48}+\frac{c_3-c_1c_2}{2}\right)z^3\\ &\quad +\left(\frac{c_4-c_1c_3}{2}+\frac{5c_1^2c_2}{16}-\frac{c_2^2}{4}-\frac{c_1^4}{32}\right)z^4\\ &\quad +\left(\frac{c_5-c_1c_4-c_2c_3}{2}+\frac{5c_1^2c_3+c_1c_2^2}{16}-\frac{c_1^3c_2}{8}+\frac{c_1^5}{3840}\right)z^5\\ &\quad +\left(\frac{c_6-c_1c_5-c_2c_4}{2}+\frac{5c_1c_2c_3}{8}+\frac{5c_2^3}{48}-\frac{c_3^2}{4}+\frac{5c_1^6}{512}\right.\\ &\quad \left.+\frac{c_1^4c_2}{768}-\frac{3c_1^2c_2^2}{16}+\frac{5c_1^2c_4}{16}-\frac{c_1^3c_3}{8}\right)z^6+\cdots.\end{aligned} \quad (18)$$

Comparing the coefficients of $z, z^2, z^3, z^4, z^5, z^6$ between equations (15) and (18), we obtain

$$\begin{aligned}a_2 &= \frac{c_1}{2},\\ a_3 &= \frac{c_2}{4},\\ a_4 &= \frac{c_3}{6}-\frac{c_1c_2}{24}-\frac{c_1^3}{144},\\ a_5 &= \frac{c_4}{8}-\frac{c_1c_3}{24}+\frac{5c_1^4}{1152}-\frac{c_1^2c_2}{192}-\frac{c_2^2}{32},\end{aligned} \quad (19)$$

$$a_6 = \frac{-3c_1c_4}{80}-\frac{7c_2c_3}{120}-\frac{11c_1^5}{4800}-\frac{43c_1c_2^2}{960}+\frac{71c_1^3c_2}{5760}+\frac{c_5}{10}, \quad (20)$$

$$\begin{aligned}a_7 &= \frac{c_1^2c_4}{480}+\frac{c_1c_2c_3}{480}+\frac{833c_1^6}{691200}-\frac{41c_1^2c_2^2}{3840}-\frac{109c_1^4c_2}{11520}-\frac{c_1c_5}{30}\\ &\quad -\frac{5c_2c_4}{96}+\frac{5c_2^3}{1152}+\frac{c_6}{12}+\frac{c_1^3c_3}{144}.\end{aligned} \quad (21)$$

Applying Lemma 2, we easily get

$$\begin{aligned}|a_2| &\le 1,\\ |a_3| &\le \frac{1}{2},\\ |a_4| &= \left|\frac{c_3}{6}-\frac{c_1c_2}{24}-\frac{c_1^3}{144}\right| = \left|\frac{1}{6}\left[c_3-\frac{c_1c_2}{3}\right]+\frac{c_1}{72}\left[c_2-\frac{c_1^2}{2}\right]\right|.\end{aligned} \quad (22)$$

Let $c_1 = c$, $c \in [0,2]$; by using Lemma 3, we show

$$|a_4| = \left|\frac{1}{6}\left[c_3-\frac{c_1c_2}{3}\right]+\frac{c_1}{72}\left[c_2-\frac{c_1^2}{2}\right]\right| \le \frac{1}{3}+\frac{c\left(2-c^2/2\right)}{72}; \quad (23)$$

also, let

$$F(c) = \frac{1}{3}+\frac{c\left(2-c^2/2\right)}{72}; \quad (24)$$

obviously, we find

$$F'(c) = \frac{1}{36}-\frac{c^2}{48}. \quad (25)$$

Setting $F'(c) = 0$, we have $c = 2\sqrt{3}/3$, and so, $F(c)$ has a maximum value attained at $c = 2\sqrt{3}/3$, also which is

$$\begin{aligned}|a_4| &\le F\left(\frac{2\sqrt{3}}{3}\right) = \frac{1}{3}+\frac{\sqrt{3}}{162} \approx 0.344,\\ |a_5| &= \left|\frac{c_4}{8}-\frac{c_1c_3}{24}+\frac{5c_1^4}{1152}-\frac{c_1^2c_2}{192}-\frac{c_2^2}{32}\right|\\ &= \left|\frac{1}{8}\left[c_4-\frac{c_1c_3}{3}\right]-\frac{c_1^2}{576}\left[c_2-\frac{c_1^2}{2}\right]-\frac{c_2}{32}\left(c_2-\frac{c_1^2}{2}\right)-\frac{7c_1^2c_2}{576}\right|.\end{aligned} \quad (26)$$

Let $c_1 = c$, $c \in [0,2]$, according to Lemma 3, we obtain

$$|a_5| \le \frac{1}{4}+\frac{5c^2\left(2-c^2/2\right)}{576}+\frac{1}{16}\left(2-\frac{c^2}{2}\right)+\frac{7c^2}{288}. \quad (27)$$

Putting

$$F(c) = \frac{1}{4}+\frac{5c^2\left(2-c^2/2\right)}{576}+\frac{1}{16}\left(2-\frac{c^2}{2}\right)+\frac{7c^2}{288}, \quad (28)$$

we get

$$F'(c) = -\frac{7c}{144} - \frac{5c^3}{288} \le 0. \tag{29}$$

Therefore, the function $F(c)$ has a maximum value attained at $c = 0$, also which is

$$|a_5| \le F(0) = \frac{3}{8},$$

$$\begin{aligned}|a_6| &= \left|\frac{-3c_1c_4}{80} - \frac{7c_2c_3}{120} - \frac{11c_1^5}{4800} - \frac{43c_1c_2^2}{960} + \frac{71c_1^3c_2}{5760} + \frac{c_5}{10}\right| \\ &= \left|\frac{1}{24}\left[c_5 - \frac{9c_1c_4}{10}\right] + \frac{7}{120}[c_5 - c_2c_3] + \frac{11c_1^3}{2400}\left[c_2 - \frac{c_1^2}{2}\right]\right. \\ &\quad \left. - \frac{43c_1c_2}{960}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{211c_1^3c_2}{14400}\right|.\end{aligned} \tag{30}$$

Let $c_1 = c$, $c \in [0, 2]$, in view of Lemma 3, we have that

$$|a_6| \le \frac{7}{60} + \frac{1}{12} + \frac{11c^3(2 - c^2/2)}{2400} + \frac{43}{240}\left(2 - \frac{c^2}{2}\right) + \frac{211c^3}{7200}. \tag{31}$$

Taking

$$F(c) = \frac{7}{60} + \frac{1}{12} + \frac{11c^3(2 - c^2/2)}{2400} + \frac{43}{240}\left(2 - \frac{c^2}{2}\right) + \frac{211c^3}{7200}, \tag{32}$$

we obtain

$$F'(c) = \frac{277c^2}{2400} - \frac{55c^4}{4800} - \frac{c}{240}. \tag{33}$$

Thus, $c = 0$ is the root of the function $F'(c) = 0$ and $F''(0) < 0$; we are easy to see that the function $F(c)$ has a maximum value attained at $c = 0$, also which is

$$|a_6| \le F(0) = \frac{67}{120},$$

$$\begin{aligned}|a_7| &= \left|\frac{c_1^2c_4}{480} + \frac{c_1c_2c_3}{480} + \frac{833c_1^6}{691200} - \frac{41c_1^2c_2^2}{3840} - \frac{109c_1^4c_2}{11520}\right. \\ &\quad \left. - \frac{c_1c_5}{30} - \frac{5c_2c_4}{96} + \frac{5c_2^3}{1152} + \frac{c_6}{12} + \frac{c_1^3c_3}{144}\right| \\ &= \left|\frac{-37c_1^6}{691200} - \frac{25c_1^2c_2^2}{5760} - \frac{c_1c_5}{30} + \frac{c_1^2[c_4 - c_2^2]}{480} + \frac{c_1c_2[c_3 - c_1c_2]}{480}\right. \\ &\quad + \frac{c_1^3[c_3 - c_1c_2]}{144} - \frac{29c_1^4[c_2 - c_1^2/2]}{11520} + \frac{5c_2^2[c_2 - c_1^2/2]}{1152} \\ &\quad \left. + \frac{[c_6 - 5/8c_2c_4]}{12}\right|.\end{aligned} \tag{34}$$

Let $c_1 = c$, $c \in [0, 2]$, by virtue of Lemma 3, we have that

$$\begin{aligned}|a_7| \le{}& \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{29c^4(2 - c^2/2)}{11520} + \frac{37c^6}{691200} + \frac{c^3}{72} \\ &+ \frac{25c^2}{1440} + \frac{5(2 - c^2/2)}{288}.\end{aligned} \tag{35}$$

Letting

$$\begin{aligned}F(c) ={}& \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{29c^4(2 - c^2/2)}{11520} + \frac{37c^6}{691200} + \frac{c^3}{72} \\ &+ \frac{25c^2}{1440} + \frac{5(2 - c^2/2)}{288},\end{aligned} \tag{36}$$

so we get

$$F'(c) \ge 0. \tag{37}$$

Thus, the function $F(c)$ has a maximum value attained at $c = 2$, also which is

$$|a_7| \le F(2) = \frac{5587}{10800}. \tag{38}$$

Hence, the proof is complete.

Theorem 5. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_3 - a_2^2| \le \frac{1}{2}. \tag{39}$$

Proof. Applying equation (21), we have

$$|a_3 - a_2^2| = \left|\frac{c_2}{4} - \frac{c_1^2}{4}\right|. \tag{40}$$

Then, by applying Lemma 1, we get

$$|a_3 - a_2^2| = \left|\frac{x(4 - c_1^2)}{8} - \frac{c_1^2}{8}\right|. \tag{41}$$

Suppose that $|x| = t$, $t \in [0, 1]$, $c_1 = c$, $c \in [0, 2]$. Then, using the triangle inequality, we obtain

$$|a_3 - a_2^2| \le \frac{t(4 - c^2)}{8} + \frac{c^2}{8}. \tag{42}$$

Suppose

$$F(c, t) = \frac{t(4 - c^2)}{8} + \frac{c^2}{8}, \tag{43}$$

then for any $t \in (0, 1)$ and $c \in (0, 2)$, we get

$$\frac{\partial F}{\partial t} = \frac{4-c^2}{8} > 0, \tag{44}$$

which means that $F(c,t)$ is an increasing function on the closed interval [0,1] about t. Therefore, the function $F(c,t)$ can get the maximum value at $t=1$, that is,

$$\max F(c,t) = F(c,1) = \frac{(4-c^2)}{8} + \frac{c^2}{8} = \frac{1}{2}. \tag{45}$$

So, obviously,

$$|a_3 - a_2^2| \le \frac{1}{2}. \tag{46}$$

Hence, the proof is complete.

Theorem 6. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_2a_3 - a_4| \le \frac{1}{3}. \tag{47}$$

Proof. From (21), we have

$$|a_2a_3 - a_4| = \left|\frac{c_1c_2}{8} + \frac{c_1^3}{144} - \frac{c_3}{6} + \frac{c_1c_2}{24}\right| = \left|\frac{c_1c_2}{6} - \frac{c_3}{6} + \frac{c_1^3}{144}\right|. \tag{48}$$

Now, in view of Lemma 1, we get

$$|a_2a_3 - a_4| = \left|\frac{7c_1^3}{144} + \frac{(4-c_1^2)c_1x^2}{24} - \frac{(4-c_1^2)(1-|x|^2)z}{12}\right|. \tag{49}$$

Let $|x| = t$, $t \in [0,1]$, $c_1 = c$, $c \in [0,2]$. Then, using the triangle inequality, we deduce that

$$|a_2a_3 - a_4| \le \frac{7c^3}{144} + \frac{(4-c^2)ct^2}{24} + \frac{(4-c^2)(1-t^2)}{12}. \tag{50}$$

Assume that

$$F(c,t) = \frac{7c^3}{144} + \frac{(4-c^2)ct^2}{24} + \frac{(4-c^2)(1-t^2)}{12}. \tag{51}$$

Therefore, for any $t \in (0,1)$ and $c \in (0,2)$, we have

$$\frac{\partial F}{\partial t} = \frac{(4-c^2)t(c-2)}{12} < 0, \tag{52}$$

that is, $F(c,t)$ is an decreasing function on the closed interval [0,1] about t. This implies that the maximum value of $F(c,t)$ occurs at $t=0$, which is

$$\max F(c,t) = F(c,0) = \frac{7c^3}{144} + \frac{(4-c^2)}{12}. \tag{53}$$

Define

$$G(c) = \frac{(4-c^2)}{12} + \frac{7c^3}{144}; \tag{54}$$

we clearly see that the function $G(c)$ has a maximum value attained at $c=0$, also which is

$$|a_2a_3 - a_4| \le G(0) = \frac{1}{3}. \tag{55}$$

Hence, the proof is complete.

Theorem 7. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_2a_4 - a_3^2| \le \frac{1}{4}. \tag{56}$$

Proof. Let $f(z) \in \mathcal{S}_s^*$, then by equation (21), we get

$$|a_2a_4 - a_3^2| = \left|\frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{c_2^2}{16}\right|. \tag{57}$$

Now, in terms of Lemma 1, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left|\frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{c_2^2}{16}\right| \\ &= \left|-\frac{5c_1^4}{576} - \frac{x^2c_1^2(4-c_1^2)}{48} - \frac{x^2(4-c_1^2)^2}{64}\right. \\ &\quad \left. + \frac{c_1(4-c_1^2)(1-|x|^2)z}{24}\right|. \end{aligned} \tag{58}$$

Let $|x| = t$, $t \in [0,1]$, $c_1 = c$, $c \in [0,2]$. Then, using the triangle inequality, we get

$$|a_2a_4 - a_3^2| \le \frac{t^2c^2(4-c^2)}{48} + \frac{(1-t^2)c(4-c^2)}{24} + \frac{t^2(4-c^2)^2}{64} + \frac{5c^4}{576}. \tag{59}$$

Setting

$$F(c,t) = \frac{t^2c^2(4-c^2)}{48} + \frac{(1-t^2)c(4-c^2)}{24} + \frac{t^2(4-c^2)^2}{64} + \frac{5c^4}{576}, \tag{60}$$

then, for any $t \in (0,1)$ and $c \in (0,2)$, we have

$$\frac{\partial F}{\partial t} = \frac{t(c^2 - 8c + 12)(4-c^2)}{96} > 0, \tag{61}$$

which implies that $F(c,t)$ increases on the closed interval [0,1] about t. That is, that $F(c,t)$ has a maximum value at $t=1$, which is

$$\max F(c,t)=F(c,1)=\frac{c^2\left(4-c^2\right)}{48}+\frac{\left(4-c^2\right)^2}{64}+\frac{5c^4}{576}. \quad (62)$$

Putting

$$G(c)=\frac{c^2\left(4-c^2\right)}{48}+\frac{\left(4-c^2\right)^2}{64}+\frac{5c^4}{576}, \quad (63)$$

then we have

$$G'(c)=\frac{c\left(4-c^2\right)}{24}-\frac{c^3}{24}-\frac{c\left(4-c^2\right)}{16}+\frac{5c^3}{144}. \quad (64)$$

If $G'(c)=0$, then the root is $c=0$. Also, since $G''(0)=-1/12<0$, so the function $G(c)$ can take the maximum value at $c=0$, which is

$$|a_2a_4-a_3^2|\leq G(0)=\frac{1}{4}. \quad (65)$$

Hence, the proof is complete.

Theorem 8. *If the function $f(z)\in\mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_2a_5-a_3a_4|\leq\frac{11}{36}. \quad (66)$$

Proof. Let $f(z)\in\mathcal{S}_s^*$, then by using (21), we have

$$\begin{aligned}|a_2a_5-a_3a_4|&=\left|\frac{5c_1^5}{2304}+\frac{c_1c_4}{16}-\frac{c_1c_2^2}{192}-\frac{c_1^2c_3}{48}-\frac{c_1^3c_2}{1152}-\frac{c_2c_3}{24}\right|\\&=\left|-\frac{c_1^3\left[c_2-c_1^2/2\right]}{1152}-\frac{c_3\left[c_2-c_1^2/2\right]}{24}+\frac{c_1\left[c_4-c_1c_3\right]}{24}\right.\\&\quad\left.+\frac{c_1^5}{576}+\frac{c_1\left[c_4-1/4c_2^2\right]}{48}\right|.\end{aligned} \quad (67)$$

Let $c_1=c, c\in[0,2]$, according to Lemma 3, we obtain

$$|a_5|\leq\frac{1}{4}+\frac{5c^2\left(2-c^2/2\right)}{576}+\frac{1}{16}\left(2-\frac{c^2}{2}\right)+\frac{7c^2}{288}. \quad (68)$$

Taking

$$F(c)=\frac{c^3\left[2-c^2/2\right]}{1152}+\frac{\left[2-c^2/2\right]}{12}+\frac{c}{8}+\frac{c^5}{576}. \quad (69)$$

Then, $\forall c\in(0,2)$, we have

$$F'(c)=\frac{c^2}{192}+\frac{c^4}{128}-\frac{c}{12}+\frac{1}{8}>0, \quad (70)$$

which implies that $F(c)$ increases on the closed interval [0,2] about c. Namely, the maximum value of $F(c)$ attains at $c=2$, also which is

$$|a_2a_5-a_3a_4|\leq F(2)=\frac{11}{36}. \quad (71)$$

The proof of Theorem 8 is completed.

Theorem 9. *If the function $f(z)\in\mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_5-a_2a_4|\leq\frac{13}{32}. \quad (72)$$

Proof. Assume that $f(z)\in\mathcal{S}_s^*$, then from (21), we obtain

$$\begin{aligned}|a_5-a_2a_4|&=\left|\frac{c_1^4}{128}-\frac{c_1c_3}{8}+\frac{c_1^2c_2}{64}-\frac{c_2^2}{32}+\frac{c_4}{8}\right|\\&=\left|\frac{\left[c_1^4+c_2^2+2c_1c_3-3c_1^2c_2-c_4\right]}{32}-\frac{5c_1^2\left[c_2-c_1^2/2\right]}{64}\right.\\&\quad\left.-\frac{3\left[c_4-2/3c_1c_3\right]}{32}\right|.\end{aligned} \quad (73)$$

Next, by virtue of Lemma 3, we obtain

$$|a_5-a_2a_4|\leq\frac{1}{4}+\frac{5c^2\left[2-c^2/2\right]}{64}. \quad (74)$$

Setting

$$F(c)=\frac{1}{4}+\frac{5c^2\left[2-c^2/2\right]}{64}. \quad (75)$$

Then, we have

$$F'(c)=\frac{5c}{16}-\frac{5c^3}{32}. \quad (76)$$

Let $F'(c)=0$, we get $c=0$ or $c=\sqrt{2}$ and $F'(\sqrt{2})<0$, which implies that the maximum value of $F(c)$ attains at $c=\sqrt{2}$, also which is

$$|a_5-a_2a_4|\leq F\left(\sqrt{2}\right)=\frac{13}{32}. \quad (77)$$

Hence, the proof is complete.

Theorem 10. *If the function $f(z)\in\mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_5a_3-a_4^2|\leq\frac{97}{324}. \quad (78)$$

Proof. Assume that $f(z) \in \mathcal{S}_s^*$, then from (21), we obtain

$$\begin{aligned} |a_5a_3 - a_4^2| &= \left| \frac{7c_1^4c_2}{13824} + \frac{c_2c_4}{32} + \frac{c_1c_2c_3}{288} - \frac{c_2^3}{128} + \frac{c_1^3c_3}{432} \right. \\ &\quad \left. - \frac{7c_1^2c_2^2}{2304} - \frac{c_3^2}{36} - \frac{c_1^6}{20736} \right| \\ &= \left| \frac{c_2[c_4 - c_1c_3/9]}{32} - \frac{c_3[c_3 - c_1c_2/4]}{36} - \frac{c_2^2[c_2 - c_1^2/2]}{128} \right. \\ &\quad - \frac{c_1^2c_2[c_2 - c_1^2/2]}{144} + \frac{c_1^3[c_3 - 31/32c_1c_2]}{432} \\ &\quad \left. - \frac{5c_1^4c_2}{6912} - \frac{c_1^6}{20736} \right|. \end{aligned} \tag{79}$$

Next, in terms of Lemma 3, we obtain

$$\begin{aligned} |a_5a_3 - a_4^2| \le \frac{1}{8} + \frac{1}{9} + \frac{[2 - c^2/2]}{32} + \frac{c^2[2 - c^2/2]}{72} + \frac{c^3}{216} \\ + \frac{5c^4}{3456} + \frac{c^6}{20736}. \end{aligned} \tag{80}$$

Putting

$$\begin{aligned} F(c) = \frac{1}{8} + \frac{1}{9} + \frac{[2 - c^2/2]}{32} + \frac{c^2[2 - c^2/2]}{72} + \frac{c^3}{216} \\ + \frac{5c^4}{3456} + \frac{c^6}{20736}. \end{aligned} \tag{81}$$

Then, for any $c \in (0, 2)$, we have $F'(c) > 0$, which means that the maximum value of $F(c)$ arrives at $t = 2$, also which is

$$|a_5a_3 - a_4^2| \le F(2) = \frac{97}{324}. \tag{82}$$

Hence, the proof is complete.

Theorem 11. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|H_4(1)| \le 0.81945. \tag{83}$$

Proof. Because of

$$\begin{aligned} H_4(1) = a_7\{a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)\} \\ - a_6\{a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - a_2^2)\} \\ - a_6\{a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - a_2^2)\} \\ + a_5\{a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3)\} \\ - a_4\{a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_4 - a_2a_3)\}, \end{aligned} \tag{84}$$

then, by applying the triangle inequality, we get

$$\begin{aligned} |H_4(1)| = |a_7||a_3||\,|a_2a_4 - a_3^2| + |a_7||a_4||a_4 - a_2a_3| \\ + |a_7||a_5||a_3 - a_2^2| + |a_6||a_3||a_2a_5 - a_3a_4| \\ + |a_6||a_4||a_5 - a_2a_4| + |a_6|^2|a_3 - a_2^2| \\ + |a_5||a_3||a_3a_5 - a_4^2| + |a_5|^2|a_5 - a_2a_4| \\ + |a_5||a_6||a_4 - a_2a_3| + |a_4|^2|a_3a_5 - a_4^2| \\ + |a_4||a_5||a_2a_5 - a_3a_4| + |a_4||a_6||a_4 - a_2a_3|. \end{aligned} \tag{85}$$

Next, substituting (13) and (39)–(78) into (85), we easily obtain the desired assertion (83).

3. Conclusion

In the present paper, we mainly get upper bounds of the fourth-order Hankel determinant $H_4(1)$ of starlike functions connected with the sine function. However, the results obtained in this paper are not sharp. In the future, we will consider the sharpness of the results. Also, we can discuss the related research of the fifth-order Hankel determinant and fifth-order Toeplitz determinant for this function class.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Sufficiency Criteria for q-Starlike Functions Associated with Cardioid

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This article deals with the q-differential subordinations for starlike functions associated with the lemniscate of Bernoulli and cardioid domain. The primary goal of this work is to find the conditions on γ for $1+(\gamma z\partial_q h(z))/(h^n(z)) \prec \sqrt{1+z}$, where $h(z)$ is analytic function and is subordinated by the function which is producing cardioid domain as its image domain while mapping the open unit disk. Along with this, certain sufficient conditions for q-starlikeness of analytic functions are determined.

1. Introduction

Consider the class A of analytic functions defined in open unit disk F with normalization condition $f(0)=0$ and $f'(0)=1$ which provides the Taylor series expansion of the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n, \quad z \in F. \tag{1}$$

The class S consists of functions from A which are univalent functions in F, and the class P contains the analytic functions whose codomains are bounded by the open right half plane. For more details, see [1, 2].

The concept of differential subordination plays a vital role in the study of geometric properties of analytic functions. It was first introduced by Lindelof, but Littlewood [3] did the remarkable work in this field. Many researchers contributed in the study of differential subordinations. History and the development of works in the field related to differential subordination are briefly described and included in the book by Miller and Mocanu [4]. The major development in the field of differential subordination started in 1974 by Miller et al. [5].

An analytic function f is considered to be subordinated by analytic function g, denoted as $f \prec g$, if there exists another analytic function w with the property that $w(0)=0$ and $|w(z)|<|z|$ such that $f(z)=g(w(z))$. Moreover, in case of univalent functions in F, we can have

$$\begin{aligned} f \prec g \Leftrightarrow f(0)=g(0), \\ f(F) \subset g(F). \end{aligned} \tag{2}$$

Recently, many mathematicians have used this concept of differential subordinations to prove many helpful results. Familiar Jack's lemma [6] has produced several advancements for the generalization of differential subordinations and found many applications in this field. The work of Ma and Minda [7] in this field is not negligible as they studied the function Φ which is analytic, and condition of normalization given for prescribed function is defined as $\Phi(0)=1$ and $\Phi'(0)>0$ with a positive real part. With the help of the function Φ, they introduced the following subclasses for starlike and convex functions.

$$\begin{aligned} S^*(\Phi) &= \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \Phi(z)\,;\, z \in F \right\}, \\ C(\Phi) &= \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z)\,;\, z \in F \right\}. \end{aligned} \tag{3}$$

These subclasses helped many researchers for further studies in the field of differential subordination. Ali et al. [8] used the concept of differential subordination to prove analytic functions to be Janowski starlike. Ali et al. [9] also evaluated several differential subordinations: $1+\gamma z(p'(z)/p^n(z))$ and found the γ for $p(z) \prec \sqrt{1+z}$. Raina and Sokol [10] used subordinations for coefficient estimation of starlike functions. Similar kinds of works have also been done by Sharma et al. [11] by using starlikeness for cardioid function, and Yunus et al. [12] studied for limacon.

Quantum calculus is the new branch of mathematics and is equally important for its applications both in physics and in mathematics as well. Jackson [13, 14] presented the functions of q-derivatives and q-integrals and highlighted their definitions for the first time. He also holds the credit for the systematic initiation of q-calculus. Ismail et al. [15] were the pioneers to contribute in the application of q-calculus in geometric function theory. The new form of the subclass of starlike functions $S^*(\Phi)$ with the involvement of q-derivative was introduced by Seoudy and Aouf [16]. By choosing different image domains instead of $\Phi(z)$, so many attractive subclasses of starlike functions are obtained. Mahmood et al. [17] have dealt with the class of q-starlike functions by relating them with conic domains. The most recent work related to q-starlikeness of functions is done by Srivastava et al. [18]. The contributions of Haq et al. [19] are remarkable. They proved differential subordinations with q-analogue for cardioid and limacon domain with the involvement of Janowski function and found the sufficient conditions for q-starlike functions. The q version of Jack's lemma which is the soul of our work was given by Çetinkaya and Polatoglu [20]. These recent efforts of mathematicians discussed above motivated us and provide strength to contribute in the field of differential subordinations with the involvement of its q-analogue, which is the main idea of this article. The foundation of all this work in q-analogue is the q-derivative which is defined below.

The q-derivative of a complex-valued function f, defined in the domain F, is given as follows:

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{4}$$

where $0<q<1$. This implies the following:

$$\lim_{q \longrightarrow 1^-} \left(D_q f\right)(z) = \lim_{q \longrightarrow 1^-} \frac{f(z)-f(qz)}{(1-q)z} = f'(z), \tag{5}$$

provided the function f is differentiable in domain F. The function $D_q f$ has Maclaurin's series representation

$$\left(D_q f\right)(z) = \sum_{n=0}^{\infty} [n]_q a_n z^{n-1}, \tag{6}$$

where

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ \displaystyle\sum_{k=0}^{n-1} q^k = 1+q+q^2+\cdots+q^{n-1}, & n \in \mathbb{N}. \end{cases} \tag{7}$$

For more details about q-derivatives and recent work on it, we refer the reader to [21–25].

Definition 1. The function $f(z) \in A$ is said to be in the class $S^*_{q,c}$, if

$$\frac{z\partial_q f(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad z \in F. \tag{8}$$

Lemma 2 (q-Jack's lemma, [20]). *Consider an analytic function w in F with $w(0)=0$. For a maximum value of w on the circle $|z|=1$ at $z_0 = ae^{i\theta}$, where $\theta \in [-\pi,\pi]$, and $0<q<1$, then, we have*

$$z_0 \partial q w(z_0) = m w(z_0). \tag{9}$$

Here, m is real and $m \geq 1$.

By using the above lemma, we have proved our main results.

2. Main Results

Theorem 3. *Assume that*

$$\gamma \geq \frac{3\left(\sqrt{2}+1\right)}{2(1-q)}, \tag{10}$$

and we define an analytic function h on F with $h(0)=1$ which satisfies

$$1+\gamma z\partial_q h(z) \prec \sqrt{1+z}. \tag{11}$$

In addition, we suppose that

$$1+\gamma z\partial_q h(z) = \sqrt{1+w(z)}, \tag{12}$$

where w is analytic in F with $w(0)=0$. Then,

$$h(z) \prec 1+\frac{4}{3}z+\frac{2}{3}z^2. \tag{13}$$

Proof. Consider the function

$$p(z)=1+\gamma z\partial_q h(z), \tag{14}$$

which is analytic in F with the condition $p(0)=1$ and the function

$$h(z)=1+\frac{4}{3}w(z)+\frac{2}{3}w^2(z), \tag{15}$$

where w is an analytic function in F with $w(0)=0$. To prove the result, it would be sufficient to show that $|w(z)|\le 1$ for

$$w(z)=p^2(z)-1. \tag{16}$$

From (14) and (15), we deduce the following:

$$p(z)=1+\frac{\gamma}{3}z\partial_q w(z)\left\{4(1+w(z))-2(1-q)z\partial_q w(z)\right\}, \tag{17}$$

and with this, one can have

$$\begin{aligned} w(z)=p^2(z)-1&=\left[\frac{\gamma}{3}z\partial_q w(z)\left\{4(1+w(z))-2(1-q)z\partial_q w(z)\right\}\right]^2\\ &\quad+2\left[\frac{\gamma}{3}z\partial_q w(z)\left\{4(1+w(z))-2(1-q)z\partial_q w(z)\right\}\right]. \end{aligned} \tag{18}$$

This implies that

$$\begin{aligned} \left|p^2(z)-1\right|&=\left|\left[\frac{\gamma}{3}z\partial_q w(z)\left\{4(1+w(z))-2(1-q)z\partial_q w(z)\right\}\right]^2\right.\\ &\quad\left.+2\left[\frac{\gamma}{3}z\partial_q w(z)\left\{4(1+w(z))-2(1-q)z\partial_q w(z)\right\}\right]\right|\\ &=\left|2+\frac{\gamma}{3}z\partial_q w(z)\left\{4(1+w(z))-2(1-q)z\partial_q w(z)\right\}\right|\\ &\quad\times\left|\frac{\gamma}{3}z\partial_q w(z)\left\{4(1+w(z))-2(1-q)z\partial_q w(z)\right\}\right|. \end{aligned} \tag{19}$$

Now, considering the existence of a point $z_0\in F$ such that

$$\max_{|z|\le|z|}|w(z)|=|w(z_0)|=1. \tag{20}$$

Now, we use the q-Jack's lemma which implies that there exist a number $m\ge 1$ such that $z_0\partial q w(z_0)=mw(z_0)$. This, with the consideration that $w(z_0)=e^{i\theta}, \theta\in[-\pi,\pi]$ for $z_0\in F$, we have

$$\begin{aligned} \left|p^2(z_0)-1\right|&=\left|2+\frac{\gamma}{3}z_0\partial_q w(z_0)\left\{4(1+w(z_0))-2(1-q)z_0\partial_q w(z_0)\right\}\right|.\left|\frac{\gamma}{3}z_0\partial_q w(z_0)\left\{4(1+w(z_0)-2(1-q)z_0\partial_q w(z_0)\right\}\right|\\ &=\left|2+\frac{\gamma}{3}me^{i\theta}\left\{4\left(1+e^{i\theta}\right)-2m(1-q)e^{i\theta}\right\}\right|.\left|\frac{\gamma}{3}me^{i\theta}\left\{4\left(1+e^{i\theta}\right)-2m(1-q)e^{i\theta}\right\}\right|\\ &=\sqrt{4+\frac{4}{9}\gamma^2m^2\left(4+(2-m(1-q))^2\right)+\frac{16}{3}\gamma m\left\{1+\frac{\gamma m}{3}(2-m(1-q))\right\}\cos\theta+\frac{8}{3}\gamma m(2-m(1-q))\cos 2\theta}\\ &\quad.\frac{m\gamma}{3}\sqrt{16+(4-2m(1-q))^2+8(4-2m(1-q))\cos\theta}. \end{aligned} \tag{21}$$

The function

$$\begin{aligned} G(\theta)=&\sqrt{4+\frac{4}{9}\gamma^2m^2\left(4+(2-m(1-q))^2\right)+\frac{16}{3}\gamma m\left\{1+\frac{\gamma m}{3}(2-m(1-q))\right\}\cos\theta+\frac{8}{3}\gamma m(2-m(1-q))\cos 2\theta}\\ &.\frac{m\gamma}{3}\sqrt{16+(4-2m(1-q))^2+8(4-2m(1-q))\cos\theta} \end{aligned} \tag{22}$$

is clearly an even function. So, in order to find the maximum value of G, we will consider the interval $[0, \pi]$. Thus,

$$G'(\theta) = \frac{-8\gamma m\sqrt{9+4\gamma^2m^2+\gamma^2m^2(2-m(1-q))^2+12\gamma m\cos\theta+4\gamma^2m^2(2-m(1-q))\cos\theta+6\gamma m(2-m(1-q))\cos 2\theta}\,(4-2m(1-q))\sin\theta}{9\sqrt{16+(4-2m(1-q))^2+8(4-2m(1-q))\cos\theta}} + \frac{1}{9}\frac{\gamma m\sqrt{16+(4-2m(1-q))^2+8(4-2m(1-q))\cos\theta}\,\left(-12\gamma m\sin\theta-4\gamma^2m^2(2-m(1-q))\sin\theta-12\gamma m(2-m(1-q))\sin 2\theta\right)}{\sqrt{9+4\gamma^2m^2+\gamma^2m^2(2-m(1-q))^2+12\gamma m\cos\theta+4\gamma^2m^2(2-m(1-q))\cos\theta+6\gamma m(2-m(1-q))\cos 2\theta}} \tag{23}$$

gives $G'(\theta)=0$ for $\theta=0$ and π. Also, we can see that $G''(\pi)>0$ for $1<m<2.5$, which results that $G(\theta)\geq G(\pi)$. Now, consider the function

$$\begin{aligned}\Theta(m) &= \sqrt{4+\frac{4}{9}\gamma^2m^2\left(4+(2-m(1-q))^2\right)-\frac{16}{3}\gamma m\left\{1+\frac{\gamma m}{3}(2-m(1-q))\right\}+\frac{8}{3}\gamma m(2-m(1-q))}\,\frac{m\gamma}{3}\sqrt{16+(4-2m(1-q))^2-8(4-2m(1-q))}\\ &= \frac{4}{9}\gamma^2m^4(1-q)^2-\frac{4}{3}\gamma m^2(1-q).\end{aligned} \tag{24}$$

So we have

$$\Theta'(m) = \frac{16}{3}\gamma^2m^3(1-q)^2-\frac{8}{3}\gamma m(1-q)>0. \tag{25}$$

Thus, $\Theta(m)$ is an increasing function which gives a minimum value for $m=1$. Then, we have

$$\left|p^2(z_0)-1\right| \geq \frac{4}{9}\gamma^2(1-q)^2-\frac{4}{3}\gamma(1-q). \tag{26}$$

From (10), we conclude that

$$\left|p^2(z_0)-1\right| \geq 1, \tag{27}$$

but this result contradicts (11). Hence, $|w(z)|<1$ and this leads us to the desired result.

By taking $h(z)=z\partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 4. *Let $\gamma \geq 3(\sqrt{2}+1)/2(1-q)$ and $f\in A$ satisfy the subordination*

$$1+\gamma z\partial_q\left(\frac{z\partial_q f(z)}{f(z)}\right) \prec \sqrt{1+z}. \tag{28}$$

Then, $f(z)\in S^*_{q,c}$.

Theorem 5. *Assume that*

$$\gamma \geq \frac{\sqrt{2}+1}{2(1-q)}, \tag{29}$$

and we define an analytic function h on F with $h(0)=1$ which satisfies

$$1+\frac{\gamma z\partial_q h(z)}{h(z)} \prec \sqrt{1+z}. \tag{30}$$

In addition, we suppose that

$$1+\frac{\gamma z\partial_q h(z)}{h(z)} = \sqrt{1+w(z)}, \tag{31}$$

where w is analytic in F with $w(0)=0$. Then,

$$h(z) \prec 1+\frac{4}{3}z+\frac{2}{3}z^2. \tag{32}$$

Proof. Consider the function

$$p(z) = 1+\gamma\frac{z\partial_q h(z)}{h(z)}, \tag{33}$$

which is analytic in F with the condition $p(0)=1$ and the function

$$h(z) = 1+\frac{4}{3}w(z)+\frac{2}{3}w^2(z), \tag{34}$$

where w is an analytic function in F with $w(0)=0$. Using (33) and (34), we obtain

$$p(z)=1+\frac{(\gamma/3)z\partial_q w(z)\{4(1+w(z))-2(1-q)z\partial_q w(z)\}}{1+(4/3)w(z)+(2/3)w^2(z)}. \tag{35}$$

Proving the fact that $|w(z)|\le 1$ will be sufficient to prove our assertion. For this, consider

$$\begin{aligned}&\left|p^2(z)-1\right|\\&=\left|\frac{2(3+4w(z)+2w^2(z))+\gamma z\partial_q w(z)\{4(1+w(z))-2(1-q)z\partial_q w(z)\}}{(3+4w(z)+2w^2(z))}\right|\\&\quad\cdot\left|\frac{\gamma z\partial_q w(z)\{4(1+w(z))-2(1-q)z\partial_q w(z)\}}{(3+4w(z)+2w^2(z))}\right|.\end{aligned} \tag{36}$$

Considering the existence of a point $z_0\in F$ such that

$$\max_{|z|\le|z|}|w(z)|=|w(z_0)|=1, \tag{37}$$

we can make use of q-Jack's lemma which implies that there exists a number $m\ge 1$ such that $z_0\partial q w(z_0)=mw(z_0)$. Now, consider that $w(z_0)=e^{i\theta},\theta\in[-\pi,\pi]$, then for $z_0\in F$, we have

$$\begin{aligned}&\left|p^2(z_0)-1\right|\\&=\left|\frac{2\left(3+4e^{i\theta}+2e^{2i\theta}\right)+\gamma m e^{i\theta}\left\{4\left(1+e^{i\theta}\right)-2m(1-q)e^{i\theta}\right\}}{\left(3+4e^{i\theta}+2e^{2i\theta}\right)}\right|\\&\quad\times\left|\frac{\gamma m e^{i\theta}\left\{4\left(1+e^{i\theta}\right)-2m(1-q)e^{i\theta}\right\}}{\left(3+4e^{i\theta}+2e^{2i\theta}\right)}\right|.\end{aligned} \tag{38}$$

Now, one can easily see that the function

$$\begin{aligned}G(\theta)&=\frac{\sqrt{\Psi_1}}{\sqrt{29+40\cos\theta+29\cos 2\theta}}\\&\quad\cdot\frac{m\gamma\sqrt{16+(4-2m(1-q))^2+8(4-2m(1-q))\cos\theta}}{\sqrt{29+40\cos\theta+29\cos 2\theta}}\end{aligned} \tag{39}$$

with

$$\begin{aligned}\Psi_1&=68+48\gamma m+8\gamma m^2+4\gamma^2m^4+32\gamma^2m^2-16\gamma^2m^3\\&\quad-8\gamma m^2q+16\gamma^2m^3q-8\gamma^2m^4q+4\gamma^2m^4q^2+160\cos\theta\\&\quad+48\gamma m^2q\cos 2\theta+32\cos\theta\gamma m^2q+16\cos\theta\gamma^2m^3q\\&\quad+96\cos 2\theta-48\gamma m^2\cos 2\theta+96\gamma m\cos 2\theta\\&\quad+144\cos\theta\gamma m-16\cos\theta\gamma^2m^3+32\cos\theta\gamma^2m^2\\&\quad-32\cos\theta\gamma m^2\end{aligned} \tag{40}$$

is clearly an even function. So, in order to find the maximum value of G, we will consider the interval $[0,\pi]$. Now, we have

$$G'(\theta)=0 \tag{41}$$

for $\theta=0$ and π. Also, we can see that $G''(\pi)>0$ for $m\ge1$, thus we conclude that $G(\theta)\ge G(\pi)$. So we have the function

$$\begin{aligned}\Theta(m)&=\sqrt{4-8\gamma m^2+8\gamma m^2q+4\gamma^2m^4-8^2m^4q+4\gamma^2m^4q^2}\\&\quad\cdot m\gamma(2m(1-q))=\left[2\gamma m^2(1-q)-2\right]\left[2\gamma m^2(1-q)\right].\end{aligned} \tag{42}$$

This gives

$$\Theta'(m)=16\gamma^2m^3(1-q)^2-8\gamma m(1-q)>0. \tag{43}$$

Thus, $\Theta(m)$ is an increasing function which gives a minimum value for $m=1$. Then, we have

$$\left|p^2(z_0)-1\right|\ge 4\gamma^2(1-q)^2-4\gamma(1-q). \tag{44}$$

From (29), we conclude that

$$\left|p^2(z_0)-1\right|\ge 1, \tag{45}$$

but this result contradicts (30). Hence, $|w(z)|<1$ which provides the required result.

By taking $h(z)=z\partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 6. *Let $\gamma\ge\sqrt{2}+1/2(1-q)$ and $f\in A$ satisfy the subordination*

$$1+\gamma z\left(\frac{f(z)}{z\partial_q f(z)}\right)\partial_q\left(\frac{z\partial_q f(z)}{f(z)}\right)\prec\sqrt{1+z}. \tag{46}$$

Then, $f(z)\in S^*_{q,c}$.

Theorem 7. *Assume that*

$$\gamma\ge\frac{\sqrt{2}+1}{2.3(1-q)}, \tag{47}$$

and we define an analytic function h on F with $h(0)=1$ which satisfies

$$1+\frac{\gamma z\partial_q h(z)}{h^2(z)}\prec\sqrt{1+z}. \tag{48}$$

In addition, we suppose that

$$1+\frac{\gamma z\partial_q h(z)}{h^2(z)}=\sqrt{1+w(z)}, \tag{49}$$

where w is analytic in F with $w(0)=0$. Then,

$$h(z) \prec 1+\frac{4}{3}z+\frac{2}{3}z^2. \tag{50}$$

Proof. Let us define the function

$$p(z)=1+\gamma\frac{z\partial_q h(z)}{h^2(z)}, \tag{51}$$

which is analytic in F with the condition $p(0)=1$ and the function

$$h(z)=1+\frac{4}{3}w(z)+\frac{2}{3}w^2(z), \tag{52}$$

where w is an analytic function in F with $w(0)=0$. Using (51) and (52), we get

$$p(z)=1+\frac{(\gamma/3)z\partial_q w(z)\{4(1+w(z))-2(1-q)z\partial_q w(z)\}}{(1+(4/3)w(z)+(2/3)w^2(z))^2}. \tag{53}$$

To prove the assertion, it would be enough to show that $|w(z)|\le 1$. Therefore,

$$\begin{aligned}\left|p^2(z)-1\right| &= \left|2+\frac{(\gamma/3)z\partial_q w(z)\{4(1+w(z))-2(1-q)z\partial_q w(z)\}}{(1+(4/3)w(z)+(2/3)w^2(z))^2}\right| \\ &\quad\cdot\left|\frac{(\gamma/3)z\partial_q w(z)\{4(1+w(z))-2(1-q)z\partial_q w(z)\}}{(1+(4/3)w(z)+(2/3)w^2(z))^2}\right|,\end{aligned} \tag{54}$$

which after using (9) gives

$$\begin{aligned}\left|p^2(z_0)-1\right| &= \left|\frac{2\left(3+4w(z_0)+2w^2(z_0)\right)^2+3\gamma z_0\partial_q w(z_0)\{4(1+w(z_0))-2(1-q)z_0\partial_q w(z_0)\}}{(3+4w(z_0)+2w^2(z_0))^2}\right| \\ &\quad\times\left|\frac{3\gamma z_0\partial_q w(z_0)\{4(1+w(z_0))-2(1-q)z_0\partial_q w(z_0)\}}{(3+4w(z_0)+2w^2(z_0))^2}\right|.\end{aligned} \tag{55}$$

Now, we consider the function

$$\begin{aligned}G(\theta) &= \frac{\sqrt{\Psi_2}}{(29+40\cos\theta+12\cos 2\theta)} \\ &\quad\cdot\frac{3\gamma m\sqrt{16+(4-2m(1-q))^2-8(4-2m(1-q))\cos\theta}}{(29+40\cos\theta+12\cos 2\theta)},\end{aligned} \tag{56}$$

where

$$\begin{aligned}\Psi_2 &= 1156+1104\gamma m-360\gamma m^2-960\gamma m^2\cos\theta+360\gamma m^2 q \\ &\quad+144\gamma^2 m^3 q-72\gamma^2 m^4 q+36\gamma^2 m^4 q^2+288\gamma^2 m^2 \\ &\quad-144\gamma^2 m^3+36\gamma^2 m^4+9664\cos 2\theta+768\gamma m\cos 3\theta \\ &\quad+2784\gamma m\cos 2\theta-624\gamma m^2\cos\theta+3120\gamma m\cos\theta \\ &\quad+288\gamma^2 m^2\cos\theta-144\gamma^2 m^3\cos\theta+2304\cos 4\theta \\ &\quad+7680\cos 3\theta+5440\cos\theta+624\gamma m^2 q\cos 2\theta \\ &\quad+960\gamma m^2 q\cos\theta+144\gamma^2 m^3 q\cos\theta.\end{aligned} \tag{57}$$

As we see that $G(\theta)$ is an even function, so $G'(\theta)=0$ at $\theta=0,\pi$ and also we see that $G''(\pi)>0$ for $m\ge 1$. Thus, we conclude that $G(\theta)\ge G(\pi)$ and we get a new function

$$\begin{aligned}\Theta(m) &= \sqrt{4-24\gamma m^2+24\gamma m^2 q-72\gamma^2 m^4 q+36\gamma^2 m^4 q^2+36\gamma^2 m^4} \\ &\quad\cdot 3\gamma m(2m(1-q)),\end{aligned} \tag{58}$$

and we have

$$\Theta'(m)=72\gamma^2 m^3(1-q)^2+12\left(6\gamma m^2(1-q)-2\right)\gamma m(1-q)>0. \tag{59}$$

So $\Theta(m)$ is an increasing function, and it has its minimum value at $m=1$. Then, we have

$$\left|p^2(z_0)-1\right|\ge(6\gamma(1-q)-2)(6\gamma(1-q)). \tag{60}$$

Using (47), we get

$$\left|p^2(z_0)-1\right|\ge 1, \tag{61}$$

but this result contradicts (48). Hence, $|w(z)|<1$ which proves the required result.

By taking $h(z)=z\partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 8. *Let $\gamma \geq \sqrt{2} + 1/2.3(1-q)$ and $f \in A$ satisfy the subordination*

$$1 + \gamma z \left(\frac{f(z)}{z\partial_q f(z)} \right)^2 \partial_q \left(\frac{z\partial_q f(z)}{f(z)} \right) \prec \sqrt{1+z}. \quad (62)$$

Then, $f(z) \in S^*_{q,c}$.

Theorem 9. *Assume that*

$$\gamma \geq \frac{\sqrt{2}+1}{2.3^2(1-q)}, \quad (63)$$

and we define an analytic function h on F with $h(0) = 1$ which satisfies

$$1 + \frac{\gamma z \partial_q h(z)}{h^3(z)} \prec \sqrt{1+z}. \quad (64)$$

In addition, we suppose that

$$1 + \frac{\gamma z \partial_q h(z)}{h^3(z)} = \sqrt{1+w(z)}, \quad (65)$$

where w is analytic in F with $w(0) = 0$. Then,

$$h(z) \prec 1 + \frac{4}{3} z + \frac{2}{3} z^2. \quad (66)$$

Proof. Let us define the function

$$p(z) = 1 + \gamma \frac{z\partial_q h(z)}{h^3(z)}, \quad (67)$$

which is analytic in F with the condition $p(0) = 1$ and the function

$$h(z) = 1 + \frac{4}{3} w(z) + \frac{2}{3} w^2(z), \quad (68)$$

where w is an analytic function in F with $w(0) = 0$. Using (67) and (68), we obtain

$$p(z) = 1 + \frac{(\gamma/3) z\partial_q w(z)\{4(1+w(z)) - 2(1-q) z\partial_q w(z)\}}{(1 + (4/3)w(z) + (2/3)w^2(z))^3}. \quad (69)$$

To prove the result, we have to show that $|w(z)| \leq 1$. Therefore,

$$|p^2(z) - 1| = \left| 2 + \frac{(\gamma/3) z\partial_q w(z)\{4(1+w(z)) - 2(1-q) z\partial_q w(z)\}}{(1 + (4/3)w(z) + (2/3)w^2(z))^3} \right| \cdot \left| \frac{(\gamma/3) z\partial_q w(z)\{4(1+w(z)) - 2(1-q) z\partial_q w(z)\}}{(1 + (4/3)w(z) + (2/3)w^2(z))^3} \right|. \quad (70)$$

Hence, by applying (9), we obtain

$$\begin{aligned} &|p^2(z_0) - 1| \\ &= \left| \frac{2\left(3 + 4w(z_0) + 2w^2(z_0)\right)^3 + 9\gamma z_0 \partial_q w(z_0)\{4(1+w(z_0)) - 2(1-q) z_0 \partial_q w(z_0)\}}{(3 + 4w(z_0) + 2w^2(z_0))^3} \right| \\ &\times \left| \frac{9\gamma z_0 \partial_q w(z_0)\{4(1+w(z_0)) - 2(1-q) z_0 \partial_q w(z_0)\}}{(3 + 4w(z_0) + 2w^2(z_0))^3} \right|. \end{aligned} \quad (71)$$

Now, consider the function

$$G(\theta) = \frac{\sqrt{\Psi_3}}{(29 + 40\cos\theta + 12\cos 2\theta)^{3/2}} \cdot \frac{9\gamma m \sqrt{16 + (4 - 2m(1-q))^2 + 8(4 - 2m(1-q))\cos\theta}}{(29 + 40\cos\theta + 12\cos 2\theta)^{3/2}}, \quad (72)$$

where

$$\begin{aligned} \Psi_3 = {} & 19652 + 18288\gamma m^2 q\cos 2\theta + 13824\gamma m^2 q\cos 3\theta \\ & + 4608\cos 4\theta\gamma m^2 q + 12384\cos\theta\gamma m^2 q + 1296\cos\theta\gamma^2 m^3 q \\ & + 18432\cos 5\theta\gamma m - 720\gamma m - 3384\gamma m^2 + 138720\cos\theta \\ & + 41184\gamma m\cos 2\theta + 80640\gamma m\cos 3\theta + 64512\cos 4\theta\gamma m \\ & + 3384\gamma m^2 q - 18288\gamma m^2\cos 2\theta + 2592\cos\theta\gamma^2 m^2 \\ & + 5904\cos\theta\gamma m - 1296\cos\theta\gamma^2 m^3 - 12384\cos\theta\gamma m^2 \\ & - 13824\gamma m^2\cos 3\theta - 4608\cos 4\theta\gamma m^2 + 1296\gamma^2 m^3 q \\ & - 648\gamma^2 m^4 q + 324\gamma^2 m^4 q^2 + 409632\cos 2\theta + 647680\cos 3\theta \\ & + 578304\cos 4\theta + 276480\cos 5\theta + 55296\cos 6\theta + 2592\gamma^2 m^2 \\ & - 1296\gamma^2 m^3 + 324\gamma^2 m^4. \end{aligned} \quad (73)$$

The above function is clearly an even function. So in order to find its maximum value, we will consider the interval $[0, \pi]$. Now, we have $G'(\theta) = 0$ for $\theta = 0$ and π. Clearly, $G''(\pi) > 0$, and hence, we have obtained the minimum value of G at $\theta = \pi$, and thus, we conclude that $G(\theta) \geq G(\pi)$. So now consider the function

$$\Theta(m) = \sqrt{4 - 72\gamma m^2 + 72\gamma m^2 q - 648\gamma^2 m^4 q + 324\gamma^2 m^4 q^2 + 324\gamma^2 m^4} \cdot 9\gamma m \sqrt{16 + (4 - 2m(1-q))^2 - 8(4 - 2m(1-q))}, \tag{74}$$

which gives

$$\Theta'(m) = 648\gamma^2 m^2 (1-q)^3 > 0. \tag{75}$$

Thus, $\Theta(m)$ is an increasing function. So for $m = 1$, it gives a minimum value. Then, we have

$$\left|p^2(z_0) - 1\right| \geq [18\gamma(1-q) - 2][18\gamma(1-q)]. \tag{76}$$

Using (63), we get

$$\left|p^2(z_0) - 1\right| \geq 1, \tag{77}$$

but this result contradicts (64). Hence $|w(z)| < 1$ which proves the required result.

By taking $h(z) = z\partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 10. *Let $\gamma \geq \sqrt{2} + 1/2.3^2(1-q)$ and $f \in A$ satisfy the subordination*

$$1 + \gamma z \left(\frac{f(z)}{z\partial_q f(z)}\right)^3 \partial_q \left(\frac{z\partial_q f(z)}{f(z)}\right) \prec \sqrt{1+z}. \tag{78}$$

Then, $f(z) \in S^*_{q,c}$.

Theorem 11. *Assume that*

$$\gamma \geq \frac{\sqrt{2} + 1}{2.3^{n-1}(1-q)}, \tag{79}$$

and we define an analytic function h on F with $h(0) = 1$ which satisfies

$$1 + \frac{\gamma z \partial_q h(z)}{h^n(z)} \prec \sqrt{1+z}. \tag{80}$$

In addition, we suppose that

$$1 + \frac{\gamma z \partial_q h(z)}{h^n(z)} = \sqrt{1 + w(z)}, \tag{81}$$

where w is analytic in F with $w(0) = 0$. Then,

$$h(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2. \tag{82}$$

We omit the proof of this result as it can be done by using a similar technique as applied in the above results.

By taking $h(z) = z\partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 12. *Let $\gamma \geq \sqrt{2} + 1/2.3^{n-1}(1-q)$ and $f \in A$ satisfy the subordination*

$$1 + \gamma z \left(\frac{f(z)}{z\partial_q f(z)}\right)^n \partial_q \left(\frac{z\partial_q f(z)}{f(z)}\right) \prec \sqrt{1+z}. \tag{83}$$

Then, $f(z) \in S^*_{q,c}$.

3. Conclusion

In this article, we have worked on q-differential subordinations associated with lemniscate of Bernoulli and defined sufficient conditions for q-starlikeness related to cardioid domain. We have also determined the conditions on γ to prove the starlikeness of prescribed function such as

$$1 + \frac{\gamma z \partial_q h(z)}{(h(z))^n} \prec \sqrt{1+z} \text{ for } n = 0, 1, 2, 3, \tag{84}$$

then

$$h(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2. \tag{85}$$

We can use these results to study the sufficiency criteria of other analytic functions.

Data Availability

All data generated or analyzed during this study are included within this article.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Research Article

Applications of a New q-Difference Operator in Janowski-Type Meromorphic Convex Functions

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The main aim of the present article is the introduction of a new differential operator in q-analogue for meromorphic multivalent functions which are analytic in punctured open unit disc. A subclass of meromorphic multivalent convex functions is defined using this new differential operator in q-analogue. Furthermore, we discuss a number of useful geometric properties for the functions belonging to this class such as sufficiency criteria, coefficient estimates, distortion theorem, growth theorem, radius of starlikeness, and radius of convexity. Also, algebraic property of closure is discussed of functions belonging to this class. Integral representation problem is also proved for these functions.

1. Introduction and Definitions

Let $\mathfrak{A}_p$ denote the family of all meromorphic p-valent functions f that are analytic in the punctured disc $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and obeying the normalization

$$f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Also, let $\mathscr{MC}_p(\alpha)$ denote the well-known family of meromorphic p-valent convex functions of order $\alpha (0 \le \alpha < p)$ and defined as

$$f(z) \in \mathscr{MC}_p(\alpha) \Leftrightarrow \operatorname{Re} \frac{\left(zf'(z)\right)'}{f'(z)} < -\alpha. \tag{2}$$

For $0 < q < 1$, the q-difference operator or q-derivative of a function f is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad z \neq 0, q \neq 1. \tag{3}$$

It can easily be seen that for $n \in \mathbb{N}$, where $\mathbb{N}$ stands for the set of natural numbers and $z \in \mathbb{D}$,

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] a_n z^{n-1}, \tag{4}$$

where

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^{n} q^l, \qquad [0, q] = 0. \tag{5}$$

For any nonnegative integer n, the q-number shift factorial is defined by

$$[n, q]! = \begin{cases} 1, & n = 0, \\ [1, q][2, q][3, q] \cdots [n, q], & n \in \mathbb{N}. \end{cases} \quad (6)$$

Also, the q-generalized Pochhammer symbol for $x \in \mathbb{R}$ is given by

$$[x, q]_n = \begin{cases} 1, & n = 0, \\ [x, q][x+1, q] \cdots [x+n-1, q], & n \in \mathbb{N}. \end{cases} \quad (7)$$

In (3), if $q \to 1^-$, then this operator becomes the conventional derivative in the classical calculus, so the limits can be generalized by introducing the parameter q, with condition $0 < q < 1$, and all such concepts, which have been developed thus, are known as quantum calculus (q-calculus). Many physical phenomena are better explained using this generalized operator, and as a result, this field attracted a lot of the researchers due to its various applications in the branches of mathematics and physics (see details in [1, 2]). Jackson [3, 4] was the pioneer of this field, who gave some applications of q-calculus and introduced the q-analogues of derivative and integral. Aral and Gupta [1, 2, 5] defined an operator, which is known as q-Baskakov Durrmeyer operator by using q-beta functions. The generalization of complex operators known as q-Picard and q-Gauss-Weierstrass singular integral operators was discussed by Aral and Anastassiu in [6–8]. Later, Kanas and Răducanu [9] introduced the q-analogue of a Ruscheweyh differential operator and studied its various properties. More applications of this operator can be seen in the paper [10]. Huda and Darus [11] utilized the q-analogue of a Liu-Srivastava operator and defined an integral operator. In somewhat similar way, Mohammed and Darus [12] introduced a generalized operator along with investigating a class of functions relating to q-hypergeometric functions. Later, Seoudy [13] estimated coefficient bounds for some q-starlike and q-convex functions of complex order. Recently, Arif and Ahmad defined a new q-differential operator for meromorphic multivalent functions and investigated classes related to q-meromorphic starlike and convex functions in their articles [14, 15].

In this article, we introduce a new q-differential operator for meromorphic functions and use this operator to define and study some properties of a new family of meromorphic multivalent functions associated with circular domain.

We now define the differential operator $\mathscr{D}_{\mu,q} : \mathfrak{A}_p \to \mathfrak{A}_p$ by

$$\mathscr{D}_{\mu,q} f(z) = (1 + [p, q]\mu) f(z) + \mu q^p z \partial_q f(z), \quad (8)$$

where $\mu \geq 0$.

Using (1), we can easily obtain

$$\mathscr{D}_{\mu,q} f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} (1 + [p, q]\mu + \mu q^p [n, q]) a_n z^n. \quad (9)$$

We take

$$\begin{aligned} \mathscr{D}^0_{\mu,q} f(z) &= f(z), \\ \mathscr{D}^2_{\mu,q} f(z) &= \mathscr{D}_{\mu,q}\left(\mathscr{D}_{\mu,q} f(z)\right) = \frac{1}{z^p} \\ &\quad + \sum_{n=p+1}^{\infty} (1 + [p, q]\mu + \mu q^p [n, q])^2 a_n z^n. \end{aligned} \quad (10)$$

In a similar way, for $m \in \mathbb{N} \cup \{0\}$, we get

$$\mathscr{D}^m_{\mu,q} f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} (1 + [p, q]\mu + \mu q^p [n, q])^m a_n z^n. \quad (11)$$

From (8) and (11), we get the following identity:

$$\mathscr{D}^{m+1}_{\mu,q} f(z) = \mu q^p z \partial_q \mathscr{D}^m_{\mu,q} f(z) + (1 + [p, q]\mu) \mathscr{D}^m_{\mu,q} f(z). \quad (12)$$

We now define a subfamily $\mathscr{MC}_{\mu,q}(p, m, A, B)$ of $\mathfrak{A}_p$ by using the operator $\mathscr{D}^m_{\mu,q}$ as follows.

Definition 1. For $-1 \leq B < A \leq 1$ and $0 < q < 1$, we define $f \in \mathfrak{A}_p$ to be in the class $\mathscr{MC}_{\mu,q}(p, m, A, B)$, if it satisfies

$$\frac{-q^p \partial_q \left(z \partial_q \mathscr{D}^m_{\mu,q} f(z)\right)}{[p, q] \partial_q \mathscr{D}^m_{\mu,q} f(z)} \prec \frac{1 + Az}{1 + Bz}. \quad (13)$$

Here, the relation symbol "$\prec$" is used for the subordinations.

We see that for particular values of p, m, A, B, μ, and q, we get some of the well-known classes few of which are listed below:

(1) For $m = 0$ and $q \to 1^-$, we get the class of meromorphic multivalent convex functions associated with Janowski functions denoted by $\mathscr{MC}^*_p[A, B]$

(2) For $A = 1, B = -1$, and $m = 0$, we get $\mathscr{MC}^*_{p,q}$, the class of meromorphic multivalent convex functions in q-analogue

(3) For $A = 1, B = -1, m = 0$, and $q \to 1^-$, we get the class of meromorphic multivalent convex functions denoted by $\mathscr{MC}^*_p$

(4) For $A = 1, B = -1, m = 0, p = 1$, and $q \to 1^-$, we get $\mathscr{MC}^*$, the class of meromorphic convex functions

It can easily be verified that a function $f \in \mathfrak{A}_p$ will be in the class $\mathcal{MC}_{\mu,q}(p,m,A,B)$, if and only if

$$\left|\frac{\left(q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)\right)/\left([p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)+1}{A+B\left(\left(q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)\right)q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)/\left([p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)[p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)}\right|<1. \quad (14)$$

The following lemma is used in our main results.

Lemma 2 (see [16]). *Let $h(z)$ be analytic in $\mathbb{D}$ and have the form*

$$h(z)=1+\sum_{n=1}^{\infty}d_nz^n, \quad (15)$$

and $k(z)$ is analytic and convex in $\mathbb{D}$ with series representation

$$k(z)=1+\sum_{n=1}^{\infty}k_nz^n. \quad (16)$$

So if $h(z)\prec k(z)$, then $|d_n|\le|k_1|$, for $n\in\mathbb{N}=\{1,2,\cdots\}$.

2. Main Results and Their Consequences

In this section, we start with sufficiency criteria for this newly defined class and then, we give the coefficient estimates for the functions belonging to this class. The following lemma is proved which will be used in this section.

Lemma 3. *Suppose that the sequence $\{A_{p+n}\}_{n=1}^{\infty}$ is defined by*

$$\begin{cases} A_{p+n}=\dfrac{[p,q](A-B)\psi(0)}{\phi(1)\psi(1)} & (n=1),\\ A_{p+n}=\dfrac{[p,q](A-B)}{\phi(n)\psi(n)}\left(\displaystyle\sum_{k=0}^{n-1}\psi(k)A_{p+k}\right) & (n\ge 2).\end{cases} \quad (17)$$

Then,

$$A_{p+n}=\frac{\psi(0)}{\psi(n)}\cdot\frac{[p,q](A-B)}{\phi(1)}\prod_{k=1}^{n-1}\frac{[p,q](A-B)+\phi(k)}{\phi(k+1)} \quad (n\ge 2). \quad (18)$$

Proof. From (17), we have

$$\phi(n)\psi(n)A_{p+n}=[p,q](A-B)\left(\sum_{k=0}^{n-1}\psi(k)A_k\right). \quad (19)$$

Thus, we obtain that

$$\begin{aligned}\phi(n+1)\psi(n+1)A_{p+n+1}&=[p,q](A-B)\left(\sum_{k=0}^{n}\psi(k)A_k\right)\\&=[p,q](A-B)\psi(n)A_{p+n}+[p,q](A-B)\sum_{k=0}^{n-1}\psi(k)A_k\\&=[p,q](A-B)\psi(n)A_{p+n}+\phi(n)\psi(n)A_{p+n}\\&=([p,q](A-B)+\phi(n))\psi(n)A_{p+n}.\end{aligned} \quad (20)$$

From (20), we find that

$$\frac{A_{p+n+1}}{A_{p+n}}=\frac{([p,q](A-B)+\phi(n))\psi(n)}{\phi(n+1)\psi(n+1)} \quad (n\ge 1). \quad (21)$$

Thus,

$$\begin{aligned}A_{p+n}&=\frac{A_{p+n}}{A_{p+n-1}}\cdot\frac{A_{p+n-1}}{A_{p+n-2}}\cdots\frac{A_{p+2}}{A_{p+1}}\\&\quad\cdot A_{p+1}=\frac{([p,q](A-B)+\phi(n-1))\psi(n-1)}{\phi(n)\psi(n)}\\&\quad\cdots\frac{([p,q](A-B)+\phi(1))\psi(1)}{\phi(2)\psi(2)}\cdot\frac{p,q](A-B)\psi(0)}{\phi(1)\psi(1)}\\&=\frac{\psi(0)}{\psi(n)}\cdot\frac{p,q](A-B)}{\phi(1)}\prod_{k=1}^{n-1}\frac{[p,q](A-B)+\phi(k)}{\phi(k+1)} \quad (n\ge 2).\end{aligned} \quad (22)$$

In conjunction with (17), we complete the proof of Lemma 3.

Theorem 4. *If $f\in\mathfrak{A}_p$ is of the form (1), then it will be in the class $\mathcal{MC}_{\mu,q}(p,m,A,B)$ if and only if the inequality*

$$\begin{aligned}&\sum_{n=p+1}^{\infty}\left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right)\\&\cdot\left(1+[p,q]\mu+\mu q^p[n,q]^m|a_n|\right)\le\frac{[p,q]^2(A-B)}{q^p}\end{aligned} \quad (23)$$

is satisfied.

Proof. For $f\in\mathcal{MC}_{\mu,q}(p,m,A,B)$, we need to prove the inequality (14). For this, consider

$$\begin{aligned}H &:= \left|\frac{\left(q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)\right)/\left([p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)+1}{A+B\left(\left(q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)\right)q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)/\left([p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)[p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)}\right| \\ &= \left|\frac{q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)+[p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)}{A[p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)+Bq^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)}\right|.\end{aligned} \tag{24}$$

By using (8) and with the help of (3) and (11),

$$\begin{aligned}H &:= \left|\frac{\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(q^p[n,q]^2+[p,q][n,q]\right)a_nz^{n-1}}{-\left(\left((A-B)[p,q]^2\right)(A-B)[p,q]^2/q^pz^{p+1}\right)+\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(A[p,q][n,q]+Bq^p[n,q]^2\right)a_nz^{n-1}}\right| \\ &= \left|\frac{\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(q^p[n,q]^2+[p,q][n,q]\right)a_nz^{n+p}}{-\left(\left((A-B)[p,q]^2\right)(A-B)[p,q]^2/q^p\right)+\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(A[p,q][n,q]+Bq^p[n,q]^2\right)a_nz^{n+p}}\right| \\ &\le \frac{\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(q^p[n,q]^2+[p,q][n,q]\right)|a_n|}{\left(\left((A-B)[p,q]^2\right)(A-B)[p,q]^2/q^p\right)-\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(A[p,q][n,q]+Bq^p[n,q]^2\right)|a_n|}.\end{aligned} \tag{25}$$

Now, if we use the inequality (23), then

$$H<1, \tag{26}$$

and this completes the direct part of the proof.

Conversely, let $f\in\mathcal{MC}_{\mu,q}(p,m,A,B)$ and be of the form (1); then, from (14), we have for $z\in\mathbb{D}$,

$$\begin{aligned}&\left|\frac{\left(q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)\right)/\left([p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)+1}{A+B\left(\left(q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)\right)q^p\partial_q\left(z\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)/\left([p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)[p,q]\partial_q\mathcal{D}^m_{\mu,q}f(z)\right)}\right| \\ &= \left|\frac{\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(q^p[n,q]^2+[p,q][n,q]\right)a_nz^{n+p}}{-\left(\left((A-B)[p,q]^2\right)(A-B)[p,q]^2/q^p\right)+\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(A[p,q][n,q]+Bq^p[n,q]^2\right)a_nz^{n+p}}\right|.\end{aligned} \tag{27}$$

Since $|\mathfrak{Re}z|\le|z|$, we have

$$\mathfrak{Re}\left\{\frac{\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(q^p[n,q]^2+[p,q][n,q]\right)a_nz^{n+p}}{\left(\left((A-B)[p,q]^2\right)(A-B)[p,q]^2/q^p\right)+\sum_{n=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[n,q])^m\left(A[p,q][n,q]+Bq^p[n,q]^2\right)a_nz^{n+p}}\right\}<1. \tag{28}$$

Now if the values of z are chosen on the real axis, then $(q^p\partial_q(z\partial_q\mathscr{D}^m_{\mu,q}f(z)))/([p,q]\partial_q\mathscr{D}^m_{\mu,q}f(z))$ is real. Using some calculations in the inequality (28) and letting $z\to 1^-$ through real values, we finally get (23).

Theorem 5. *If $f\in\mathscr{MC}_{\mu,q}(p,m,A,B)$ and is of the form (1), then*

$$\left|a_{p+1}\right|\le\frac{p,q](A-B)\psi(0)}{\phi(1)\psi(1)},\tag{29}$$

$$\left|a_{p+n}\right|\le\frac{\psi(0)}{\psi(n)}\cdot\frac{p,q](A-B)}{\phi(1)}\prod_{k=1}^{n-1}\frac{[p,q](A-B)+\phi(k)}{\phi(k+1)}\quad(n\ge 2),\tag{30}$$

where

$$\phi(n):=q^p[p+n,q]^2+[p,q][p+n,q],\tag{31}$$

$$\psi(n):=\left(1+[p,q]\mu+\mu q^p[p,q][p+n,q]\right)^m.\tag{32}$$

Proof. If $f\in\mathfrak{A}_p$ is in the class $\mathscr{MC}_{\mu,q}(p,m,A,B)$, then it satisfies

$$\frac{-q^p\partial_q\left(z\partial_q\mathscr{D}^m_{\mu,q}f(z)\right)}{[p,q]\partial_q\mathscr{D}^m_{\mu,q}f(z)}\prec\frac{1+Az}{1+Bz}.\tag{33}$$

Now, let

$$h(z)=\frac{-q^p\partial_q\left(z\partial_q\mathscr{D}^m_{\mu,q}f(z)\right)}{[p,q]\partial_q\mathscr{D}^m_{\mu,q}f(z)}.\tag{34}$$

Since

$$\operatorname{Re}h(z)>0,\tag{35}$$

so $h(z)$ is in the class P with its representation which is given by

$$h(z)=1+\sum_{n=1}^{\infty}d_nz^n.\tag{36}$$

Now,

$$h(z)\prec\frac{1+Az}{1+Bz}.\tag{37}$$

But

$$\frac{1+Az}{1+Bz}=1+(A-B)z+\cdots.\tag{38}$$

Now, using Lemma 2, we get

$$|d_n|\le(A-B),\tag{39}$$

now putting the series expansions of $h(z)$ and $f(z)$ in (34), simplifying and comparing the coefficients of z^{n+p} on both sides

$$\begin{aligned}-q^p&\left(1+[p,q]\mu+\mu q^p[n+p,q]\right)^m[p+n,q]^2a_{p+n}\\&=[p,q][p+n,q]\left(1+[p,q]\mu+\mu q^p[n+p,q]\right)^m a_{p+n}\\&\quad+[p,q]\sum_{i=0}^{n-1}\left(1+[p,q]\mu+\mu q^p[p+i,q]\right)^m[p+i,q]a_{p+i}d_{n-i},\end{aligned}\tag{40}$$

which implies that

$$\begin{aligned}-&\left(1+[p,q]\mu+\mu q^p[n+p,q]\right)^m\\&\cdot\left(q^p[p+n,q]^2+[p,q][p+n,q]\right)a_{p+n}\\&=[p,q]\sum_{i=0}^{n-1}\left(1+[p,q]\mu+\mu q^p[p+i,q]\right)^m[p+n,q]a_{p+i}d_{n-i}.\end{aligned}\tag{41}$$

Now, by taking absolute on both sides with using the triangle inequality and using (39), we obtain

$$\begin{aligned}&\left(1+[p,q]\mu+\mu q^p[p+n,q]\right)^m\\&\quad\cdot\left(q^p[p+n,q]^2+[p,q][p+n,q]\right)\left|a_{p+n}\right|\\&\le[p,q](A-B)\sum_{i=0}^{n-1}\left(1+[p,q]\mu+\mu q^p[p+i,q]\right)^m\left|a_{p+i}\right|.\end{aligned}\tag{42}$$

Using the notation (31) and (32) implies that

$$\left|a_{p+1}\right|\le\frac{p,q](A-B)\psi(0)}{\phi(1)\psi(1)},\tag{43}$$

$$\left|a_{p+n}\right|\le\frac{p,q](A-B)}{\phi(n)\psi(n)}\left(\sum_{k=0}^{n-1}\psi(k)\left|a_{p+k}\right|\right)\quad(n\ge 2).\tag{44}$$

Now, we define the sequence $\{A_{p+n}\}_{n=1}^{\infty}$ as follows:

$$\begin{cases}A_{p+1}=\dfrac{[p,q](A-B)\psi(0)}{\phi(1)\psi(1)} & (n=1),\\[2ex] A_{p+n}=\dfrac{[p,q](A-B)}{\phi(n)\psi(n)}\left(\displaystyle\sum_{k=0}^{n-1}\psi(k)A_{p+k}\right) & (n\ge 2).\end{cases}\tag{45}$$

In order to prove that

$$\left|a_{p+n}\right|\le A_{p+n}\quad(n\ge 2),\tag{46}$$

we use the principle of mathematical induction. It is easy to verify that

$$\left|a_{p+1}\right|\le A_{p+1}=\frac{[p,q](A-B)\psi(0)}{\phi(1)\psi(1)}.\tag{47}$$

Thus, assuming that

$$\left|a_{p+j}\right| \leq A_{p+j} \quad (j = 2, 3, \cdots, n), \tag{48}$$

we find from (44) and (48) that

$$\begin{aligned}\left|a_{p+n+1}\right| &\leq \frac{[p, q](A - B)}{\phi(n)\psi(n)}\left(\sum_{k=0}^{n}\psi(k)\left|a_{p+k}\right|\right)\\ &\leq \frac{[p, q](A - B)}{\phi(n)\psi(n)}\left(\sum_{k=0}^{n}\psi(k)A_{p+k}\right) = A_{p+n+1}.\end{aligned} \tag{49}$$

Therefore, by the principle of mathematical induction, we have

$$\left|a_{p+n}\right| \leq A_{p+n} \quad (n \geq 2). \tag{50}$$

By means of Lemma 3 and (45), we know that

$$A_{p+n} = \frac{\psi(0)}{\psi(n)} \cdot \frac{p, q](A - B)}{\phi(1)}\prod_{k=1}^{n-1}\frac{[p, q](A - B) + \phi(k)}{\phi(k + 1)} \quad (n \geq 2). \tag{51}$$

Combining (50) and (51), we readily get the coefficient estimates (30).

3. Closure Theorems

Let the functions $f_k(z), (k = 1, 2, 3, \cdots, l)$ be defined by

$$f_k(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,k}z^n \; (z \in \mathbb{D}, a_{n,k} \geq 0). \tag{52}$$

Theorem 6. *Let the functions $f_k(z)(k = 1, 2, 3, \cdots, l)$ defined by (52) be in the class $\mathscr{MC}_{\mu,q}(p, m, A, B)$. Then, the function $F \in \mathscr{MC}_{\mu,q}(p, m, A, B)$, where*

$$F(z) = \sum_{k=1}^{l}\lambda_k f_k(z)\left(\lambda_k \geq 0, \sum_{k=1}^{l}\lambda_k = 1\right). \tag{53}$$

Proof. From (53), we have

$$F(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty}\left(\sum_{k=1}^{l}\lambda_k a_{n,k}\right)z^n. \tag{54}$$

By Theorem 4, we have

$$\begin{aligned}&\sum_{n=p+1}^{\infty}\left(q^p[n, q]^2(1 + B) + (1 + A)[n, q][p, q]\right)\\ &\quad\cdot (1 + [p, q]\mu \;\; +\mu q^p[n, q])^m\left(\sum_{k=1}^{l}\lambda_k a_{n,k}\right)\\ &= \sum_{k=1}^{l}\lambda_k\left(\sum_{n=p+1}^{\infty}\left(q^p[n, q]^2(1 + B) + (1 + A)[n, q][p, q]\right)\right.\\ &\quad\left.\cdot (1 + [p, q]\mu{+}\mu q^p[n, q])^m\right)a_{n,k}\\ &\leq \sum_{k=1}^{l}\lambda_k\left(\frac{[p, q]^2(A - B)}{q^p}\right) = \frac{[p, q]^2(A - B)}{q^p}.\end{aligned} \tag{55}$$

Hence, by Theorem 4, $F \in \mathscr{MC}_{\mu,q}(p, m, A, B)$.

Theorem 7. *The class $\mathscr{MC}_{\mu,q}(p, m, A, B)$ is closed under convex combination.*

Proof. Let the function $f_k(z)(k = 1, 2)$ given by (52) be in the class $\mathscr{MC}_{\mu,q}(p, m, A, B)$. It is enough to show that

$$h(z) = \alpha f_1(z) + (1 - \alpha)f_2(z), 0 \leq \alpha \leq 1, \tag{56}$$

is in the class $\mathscr{MC}_{\mu,q}(p, m, A, B)$. Since for $0 \leq \alpha \leq 1$,

$$h(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty}[\alpha a_{n,1} + (1 - \alpha)a_{n,2}]z^n, \tag{57}$$

by Theorem 4, we have

$$\begin{aligned}&\sum_{n=p+1}^{\infty}\left(q^p[n, q]^2(1 + B) + (1 + A)[n, q][p, q]\right)\\ &\quad\cdot (1 + [p, q]\mu\alpha a_{n,1} + (1 - \alpha)a_{n,2}\\ &= \alpha_{n=p+1}^{\infty}\left(q^p[n, q]^2(1 + B) + (1 + A)[n, q][p, q]\right)\\ &\quad\cdot (1 + [p, q]\mu + \mu q^p[n, q])^m a_{n,1} + (1 - \alpha)_{n=p+1}^{\infty}\\ &\quad\cdot \left(q^p[n, q]^2(1 + B) + (1 + A)[n, q][p, q]\right)\\ &\quad\cdot (1 + [p, q]\mu + \mu q^p[n, q])^m a_{n,2}\\ &\leq \alpha\frac{[p, q]^2(A - B)}{q^p} + (1 - \alpha)\frac{[p, q]^2(A - B)}{q^p} = \frac{[p, q]^2(A - B)}{q^p}.\end{aligned} \tag{58}$$

Hence, by Theorem 4, $h(z) \in \mathscr{MC}_{\mu,q}(p, m, A, B)$.

Theorem 8. *Let the function $f_k(z)(k = 1, 2)$ given by (52) belong to $\mathscr{MC}_{\mu,q}(p, m, A, B)$; then, their weighted mean $h_j(z$ $)$ is also in the class $\mathscr{MC}_{\mu,q}(p, m, A, B)$, where $h_j(z)$ is defined by*

$$h_j(z) = \left\{\frac{(1 - j)f_1(z) + (1 + j)f_2(z)}{2}\right\}. \tag{59}$$

Proof. From (59), one can easily write

$$h_j(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} \left[\frac{(1-j)a_{n,1} + (1+j)a_{n,2}}{2}\right] z^n. \tag{60}$$

To prove $h_j(z) \in \mathcal{MC}_{\mu,q}(p, m, A, B)$, we consider

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} \left(q^p[n,q]^2(1+B) + (1+A)(n,q)[p,q]\right) \\
&\quad \cdot (1+[p,q]\mu+\mu q^p[n,q])^m \left[\frac{(1-j)a_{n,1} + (1+j)a_{n,2}}{2}\right] \\
&= \frac{(1-j)}{2} \sum_{n=p+1}^{\infty} \left(q^p[n,q]^2(1+B) + (1+A)[n,q][p,q]\right) \\
&\quad \cdot (1+[p,q]\mu+\mu q^p[n,q])^m a_{n,1} \\
&\quad + \frac{(1+j)}{2} \sum_{n=p+1}^{\infty} \left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right) \\
&\quad \cdot (1+[p,q]\mu + \mu q^p[n,q])^m a_{n,2} \le \frac{(1-j)}{2} \frac{[p,q]^2(A-B)}{q^p} \\
&\quad + \frac{(1+j)}{2} \frac{[p,q]^2(A-B)}{q^p} = \frac{[p,q]^2(A-B)}{q^p}.
\end{aligned} \tag{61}$$

Hence, by Theorem 4, $h_j(z) \in \mathcal{MC}_{\mu,q}(p, m, A, B)$.

4. Distortion Theorem

In the next two results, we shall discuss the growth and distortion theorems for our newly defined class of functions.

Theorem 9. *If f is in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$ and has the form (1), then for $|z| = r$, we have*

$$\frac{1}{r^p} - \vartheta_1 r^p \le |f(z)| \le \frac{1}{r^p} + \vartheta_1 r^p, \tag{62}$$

where

$$\vartheta_1 = \frac{[p,q]^2(A-B)}{q^p\left(q^p[p+1,q]^2(1+B) + (1+A)[p+1,q][p,q]\right)(1+[p,q]\mu + \mu q^p[p+1,q])^m}. \tag{63}$$

Proof. As

$$|f(z)| = \left|\frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_n z^n\right| \le \frac{1}{|z^p|} + \sum_{n=p+1}^{\infty} |a_n||z|^n, \tag{64}$$

for $|z| = r < 1$, we have $r^n < r^p$ for $n \ge p+1$ and

$$|f(z)| \le \frac{1}{r^p} + r^p \sum_{n=p+1}^{\infty} |a_n|. \tag{65}$$

Similarly,

$$|f(z)| \ge \frac{1}{r^p} - r^p \sum_{n=p+1}^{\infty} |a_n|. \tag{66}$$

Now, if $f \in \mathcal{MC}_{\mu,q}(p, m, A, B)$, then by (23),

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} \left(q^p[n,q]^2(1+B) + (1+A)[n,q][p,q]\right) \\
&\quad \cdot (1+[p,q]\mu + \mu q^p[n,q])^m |a_n| \le \frac{[p,q]^2(A-B)}{q^p}.
\end{aligned} \tag{67}$$

But we know that

$$\begin{aligned}
&\left(q^p[p+1,q]^2(1+B) + (1+A)[p+1,q][p,q]\right) \\
&\quad \cdot (1+[p,q]\mu + \mu q^p[p+1,q])^m \sum_{n=p+1}^{\infty} |a_n| \\
&\le \sum_{n=p+1}^{\infty} \left(q^p[n,q]^2(1+B) + (1+A)[n,q][p,q]\right) \\
&\quad \cdot (1+[p,q]\mu + \mu q^p[n,q])^m |a_n|.
\end{aligned} \tag{68}$$

Hence,

$$\begin{aligned}
&\left(q^p[p+1,q]^2(1+B) + (1+A)[p+1,q][p,q]\right) \\
&\quad \cdot (1+[p,q]\mu + \mu q^p[p+1,q])^m \sum_{n=p+1}^{\infty} |a_n| \le \frac{[p,q]^2(A-B)}{q^p},
\end{aligned} \tag{69}$$

which implies that

$$\sum_{n=p+1}^{\infty} |a_{n+p}| \le \frac{[p,q]^2(A-B)}{q^p\left(q^p[p+1,q]^2(1+B) + (1+A)[p+1,q][p,q]\right)(1+[p,q]\mu + \mu q^p[p+1,q])^m}. \tag{70}$$

Now, by putting this value in (65) and (66), we get the required proof.

Theorem 10. *Let $f \in \mathcal{MC}_{\mu,q}(p, m, A, B)$ and have the form (1). Then, for $|z| = r$,*

$$\frac{p,q]_m}{q^{mp+\zeta} r^{m+p}} - \vartheta_2 r^p \le \left|\partial_q^m f(z)\right| \le \frac{p,q]_m}{q^{mp+\zeta} r^{m+p}} + \vartheta_2 r^p, \tag{71}$$

where

$$\vartheta_2 = \frac{[p,q]^2(A-B)}{q^p(q^p[p+1,q](1+B) + (1+A)[p,q])(1+[p,q]\mu + \mu q^p[p+1,q])^m},$$
$$\zeta = {}_{n=1}^{m} n. \tag{72}$$

Proof. From the help of (3) and (4), we can write

$$\partial_q^m f(z) = \frac{(-1)^m [p,q]_m}{q^{mp+\zeta} z^{p+m}} + \sum_{n=p+1}^{\infty} [n-(m-1),q]_{m+1} a_n z^{n-m}. \tag{73}$$

Since $|z| = r < 1$ implies that $r^{n-m} \le r^p$ for $m \le n$ and $n \ge p+1$, hence

$$\left|\partial_q^m f(z)\right| \le \frac{p,q]_m}{q^{mp+\zeta} r^{m+p}} + r^p \sum_{n=p+1}^{\infty} [n-(m-1),q]_{m+1} |a_n|. \tag{74}$$

Similarly,

$$\left|\partial_q^m f(z)\right| \ge \frac{p,q]_m}{q^{mp+\zeta} r^{m+p}} - r^p \sum_{n=p+1}^{\infty} [n-(m-1),q]_{m+1} |a_n|. \tag{75}$$

Since f is in the class $\mathscr{MC}_{\mu,q}(p,m,A,B)$, so by (23), we have the inequality

$$\begin{aligned}&(q^p[p+1,q](1+B)+(1+A)[p,q])\\ &\quad\cdot(1+[p,q]\mu+\mu q^p[p+1,q])^m\\ &\quad\cdot\sum_{n=p+1}^{\infty}[n,q]|a_n| \le \frac{[p,q]^2(A-B)}{q^p},\end{aligned} \tag{76}$$

from which it can be deduced that

$$\begin{aligned}&\sum_{n=p+1}^{\infty}[n,q]|a_n|\\ &\quad\le \frac{[p,q]^2(A-B)}{q^p(q^p[p+1,q](1+B)+(1+A)[p,q])(1+[p,q]\mu+\mu q^p[p+1,q])^m},\end{aligned} \tag{77}$$

but it can easily be seen that

$$\sum_{n=p+1}^{\infty}[n-(m-1),q]\left|a_{p+n}\right| \le \sum_{n=p+1}^{\infty}[n,q]|a_n|, \tag{78}$$

which implies

$$\begin{aligned}&\sum_{n=p+1}^{\infty}[n-(m-1),q]\left|a_{p+n}\right|\\ &\quad\le \frac{[p,q]^2(A-B)}{q^p(q^p[p+1,q](1+B)+(1+A)[p,q])(1+[p,q]\mu+\mu q^p[p+1,q])^m}.\end{aligned} \tag{79}$$

Now, using this inequality in (74) and (75), we obtain the required proof.

5. Integral Representation

Theorem 11. *Let the function f given by (1) be in the class $\mathscr{MC}_{\mu,q}(p,m,A,B)$. Then, the function $G(z)$ represented by*

$$G(z) = (1+\gamma)\frac{1}{z^p} + \gamma p\int_0^z \frac{f(t)}{t}dt \, (\gamma \ge 0, z \in \mathbb{D}), \tag{80}$$

is in the class $\mathscr{MC}_{\mu,q}(p,m,A,B)$.

Proof. From (1),

$$f(z) = z^{-p} + \sum_{n=p+1}^{\infty} a_n z^n. \tag{81}$$

Then,

$$\begin{aligned}G(z) &= (1+\gamma)\frac{1}{z^p} + \gamma p\int_0^z \frac{t^{-p}+\sum_{n=p+1}^{\infty}a_n t^n}{t}dt\\ &= (1+\gamma)\frac{1}{z^p} + \gamma p\left(\frac{-1}{p}\frac{1}{z^p} + \sum_{n=p+1}^{\infty}\frac{a_n}{n}z^n\right)\\ &= z^{-p} + \sum_{n=p+1}^{\infty}\frac{a_n \gamma p}{n}z^n.\end{aligned} \tag{82}$$

Consider

$$\begin{aligned}&\sum_{n=p+1}^{\infty}\left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right)\\ &\quad\cdot(1+[p,q]\mu+\mu q^p[n,q])^m\left|\frac{a_n\gamma p}{n}\right|\\ &\le \sum_{n=p+1}^{\infty}\left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right)\\ &\quad\cdot(1+[p,q]\mu+\mu q^p[n,q])^m\gamma p|a_n| \le \frac{[p,q]^2(A-B)}{q^p},\end{aligned} \tag{83}$$

since $\gamma p \le 1$.

Therefore, by Theorem 4, $G(z) \in \mathscr{MC}_{\mu,q}(p,m,A,B)$.

6. Radius Problems

The following results are about the radii of convexity and starlikeness for the functions of the class $\mathscr{MC}_{\mu,q}(p,m,A,B)$.

Theorem 12. *If $f \in \mathscr{MC}_{\mu,q}(p,m,A,B)$, then $f \in \mathscr{MC}_p(\alpha)$ for $|z| < r_1$, where*

$$r_1 = \left(\frac{q^p p(p-\alpha)\left(q^p[n+p,q]^2(1+B)+(1+A)[n+p,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n+p,q])^m}{(p+n)(n+p+\alpha)[p,q]^2(A-B)}\right)^{1/n+2p}. \tag{84}$$

Proof. Let $f \in \mathscr{MC}_{\mu,q}(p,m,A,B)$. To prove $f \in \mathscr{MC}_p(\alpha)$, we only need to show

$$\left|\frac{zf''(z)+(p+1)f'(z)}{zf''(z)+(1+2\alpha-p)f'(z)}\right|<1. \quad (85)$$

Using (1) along with some simple computation yields

$$\sum_{n=1}^{\infty}\frac{(p+n)(p+n+\alpha)}{p(p-\alpha)}\left|a_{n+p}\right||z|^{n+2p}<1. \quad (86)$$

As f is in the class $\mathscr{MC}_{\mu,q}(p,m,A,B)$, so we have from (23)

$$\begin{aligned}&\sum_{n=p+1}^{\infty}\left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right)\\&\cdot(1+[p,q]\mu+\mu q^p[n,q])^m|a_n|\\&\le\frac{[p,q]^2(A-B)}{q^p}\Rightarrow\sum_{n=p+1}^{\infty}q^p\left(q^p[n,q]^2(1+B)\right.\\&\left.+(1+A)[n,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n,q])^m|a_n|<1.\end{aligned} \quad (87)$$

Equivalently,

$$\begin{aligned}&\sum_{n=1}^{\infty}q^p\left(q^p[n+p,q]^2(1+B)+(1+A)[n+p,q][p,q]\right)\\&\cdot(1+[p,q]\mu+\mu q^p[n+p,q])^m\left|a_{n+p}\right|<1.\end{aligned} \quad (88)$$

Now, inequality (86) will be true, if the following holds:

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(p+1)(n+p+\alpha)}{p(p-\alpha)}\left|a_{n+p}\right||z|^{n+2p}\\&<\sum_{n=1}^{\infty}q^p\frac{\left(q^p[n+p,q]^2(1+B)+(1+A)[n+p,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n+p,q])^m}{[p,q]^2(1-A)}\left|a_{n+p}\right|,\end{aligned} \quad (89)$$

which implies that

$$|z|^{n+2p}<\frac{q^pp(p-\alpha)\left(q^p[n+p,q]^2(1+B)+(1+A)[n+p,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n+p,q])^m}{(p+n)(n+p+\alpha)[p,q]^2(A-B)}, \quad (90)$$

and so

$$|z|<\left(\frac{q^pp(p-\alpha)\left(q^p[n+p,q]^2(1+B)+(1+A)[n+p,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n+p,q])^m}{(p+n)(n+p+\alpha)[p,q]^2(A-B)}\right)^{1/(n+2p)}=r_1. \quad (91)$$

We get the required condition.

Theorem 13. *Let $f\in\mathscr{MC}_{\mu,q}(p,m,A,B)$. Then, $f\in\mathscr{MS}_p^*(\alpha)$ for $|z|<r_2$, where*

$$r_2=\left(\frac{q^p(p-\alpha)\left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n,q])^m}{(n+p+\alpha)[p,q]^2(A-B)}\right)^{1/(n+2p)}. \quad (92)$$

Proof. We know that $f\in\mathscr{MS}_p^*(\alpha)$, if and only if

$$\left|\frac{zf'(z)+pf(z)}{zf'(z)-(p-2\alpha)f(z)}\right|<1. \quad (93)$$

Using (1) and with some simplification, we get

$$\sum_{n=1}^{\infty}\left(\frac{n+p+\alpha}{p-\alpha}\right)\left|a_{n+p}\right||z|^{n+2p}<1. \quad (94)$$

Now, from (23), we can easily obtain that

$$\sum_{n=p+1}^{\infty}\frac{q^p\left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n,q])^m}{[p,q]^2(A-B)}|a_n|<1. \quad (95)$$

For inequality (94) to hold, it will be enough if

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\frac{n+p+\alpha}{p-\alpha}\right)\left|a_{n+p}\right||z|^{n+2p}\\&<\sum_{n=1}^{\infty}\frac{q^p\left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n,q])^m}{[p,q]^2(A-B)},\end{aligned} \quad (96)$$

which implies that

$$|z|^{n+2p}<\frac{q^p(p-\alpha)\left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n,q])^m}{(n+p+\alpha)[p,q]^2(A-B)}, \quad (97)$$

and hence,

$$|z|<\left(\frac{q^p(p-\alpha)\left(q^p[n,q]^2(1+B)+(1+A)[n,q][p,q]\right)(1+[p,q]\mu+\mu q^p[n,q])^m}{(n+p+\alpha)[p,q]^2(A-B)}\right)^{1/(n+2p)}=r_2. \quad (98)$$

Thus, we obtain the required result.

7. Conclusion

The applications of q-calculus have been the focus point in the recent times in various branches of mathematics. This article introduces a new operator in q-analogue for meromorphic multivalent functions. Then, a new subclass of multivalent convex functions is defined and studied for some of its geometric properties like sufficient conditions, coefficient estimates, and distortion. Also, problems of closure and integral representation are discussed in detail. Many other classes can be defined using this operator which will open a lot of new opportunities for research in this and related fields.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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Research Article

Majorization Results for Certain Subfamilies of Analytic Functions

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Let $h_1(z)$ and $h_2(z)$ be two nonvanishing holomorphic functions in the open unit disc with $h_1(0) = h_2(0) = 1$. For some holomorphic function $q(z)$, we consider the class consisting of normalized holomorphic functions f whose ratios $f(z)/zq(z)$ and $q(z)$ are subordinate to $h_1(z)$ and $h_2(z)$, respectively. The majorization results are obtained for this class when $h_1(z)$ is chosen either $h_1(z) = \cos z$ or $h_1(z) = 1 + \sin z$ or $h_1(z) = \sqrt{1+z}$ and $h_2(z) = 1 + \sin z$.

1. Introduction

In order to better explain the terminology included in our key observations, some of the essential relevant literature on geometric function theory needs to be provided and discussed here. We start with symbol $\mathscr{A}$ which represents the class of holomorphic functions in the region of open unit disc $\mathbb{U}_d = \{z \in \mathbb{C} : |z| < 1\}$, and if $f(z)$ is in $\mathscr{A}$, then it satisfies the relationship $f(0) = f'(0) - 1 = 0$. Also, the family $\mathscr{S} \subset \mathscr{A}$ contains all univalent functions. Though function theory was started in 1851, in 1916, due to coefficient conjecture provided by Bieberbach [1], this field emerged as a good area of new research. This conjecture was proved by De-Branges [2] in 1985. Many good scholars of the period attempted to prove or disprove this conjecture between the years 1916 and 1985. As a result, they identified several subfamilies of a class $\mathscr{S}$ of univalent functions linked to various image domains. The families of star-like $\mathscr{S}^*$ and convex $\mathscr{K}$ functions are the most basic, mostly studied, and beautiful geometric representations of these subfamilies, which are described as

$$\mathscr{S}^* =: \left\{ f \in \mathscr{S} : \Re \frac{zf'(z)}{f(z)} > 0, (z \in \mathbb{U}_d) \right\},$$
$$\mathscr{K} =: \left\{ f \in \mathscr{S} : \Re \frac{\left(zf'(z)\right)'}{f'(z)} > 0, (z \in \mathbb{U}_d) \right\}. \tag{1}$$

In 1970, Roberston [3] established the idea of quasisubordination among holomorphic functions. Two functions $\mathscr{F}_1(z), \mathscr{F}_2(z) \in \mathscr{A}$ are related to the relationship of quasi-subordination, denoted mathematically by $\mathscr{F}_1(z) \prec_q \mathscr{F}_2(z)$, if there exist functions $\varphi(z), u(z) \in \mathscr{A}$ such that $zf'(z)/\varphi(z)$ is holomorphic in $\mathbb{U}_d$ with $|\varphi(z)| \leqq 1, u(0) = 0$, and $|u(z)| \leqq |z|$ satisfying the relationship

$$\mathscr{F}_1(z) = \varphi(z)\mathscr{F}_2(u(z)), z \in \mathbb{U}_d. \tag{2}$$

Also, by choosing $u(z) = z$ and $\varphi(z) \equiv 1$, we obtain the most useful concepts of geometric function theory known as subordination between analytic functions. In fact, if

$\mathcal{F}_2(z) \in \mathcal{S}$, then, for $\mathcal{F}_1(z), \mathcal{F}_2(z) \in \mathcal{A}$, the subordination relationship has

$$\mathcal{F}_1(z) \prec \mathcal{F}_2(z) \Leftrightarrow [\mathcal{F}_1(\mathbb{U}_d) \subset \mathcal{F}_2(\mathbb{U}_d) \text{ with } \mathcal{F}_1(0) = \mathcal{F}_2(0)]. \tag{3}$$

By taking $u(z) = z$, the above definition of quasisubordination becomes the majorization between holomorphic functions and is written mathematically by $\mathcal{F}_1(z) \ll \mathcal{F}_2(z)$, for $\mathcal{F}_1(z), \mathcal{F}_2(z) \in \mathcal{A}$. That is; $\mathcal{F}_1(z) \ll \mathcal{F}_2(z)$, if a function $\varphi(z) \in \mathcal{A}$ exists with $|\varphi(z)| \leqq 1$ in such a way that

$$\mathcal{F}_1(z) = \varphi(z)\mathcal{F}_2(z), \quad z \in \mathbb{U}_d. \tag{4}$$

This idea was introduced by MacGregor [4] in 1967. Numerous articles have been published in which this idea was used. The work of Altintas and Srivastava [5], Cho et al. [6], Goswami and Aouf [7], Goyal and Goswami [8, 9], Li et al. [10], Panigraht and El-Ashwah [11], Prajapat and Aouf [12], and the authors [13, 14] are worth mentioning on this topic.

The general form of the class $\mathcal{S}^*$ was studied in 1992 by Ma and Minda [15] and was given by

$$\mathcal{S}^*(\Lambda) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \Lambda(z) \, (z \in \mathbb{U}_d) \right\}, \tag{5}$$

where $\Lambda(z)$ is a regular function with positive real part and $\Lambda'(0) > 0$. Also, the function $\Lambda(z)$ maps $\mathbb{U}_d$ onto a star-shaped region with respect to $\Lambda(0) = 1$ and is symmetric about the real axis. They addressed some specific results such as distortion, growth, and covering theorems. In recent years, several subfamilies of the set $\mathcal{A}$ were studied as a special case of the class $\mathcal{S}^*(\Lambda)$. For example,

(i) if we take $\Lambda(z) = 1 + Mz/1 + Nz$ with $-1 \leq N < M \leq 1$, then the deduced family $\mathcal{S}^*[M, N] \equiv \mathcal{S}^*(1 + M z/1 + Nz)$ is described by the functions of the Janowski star-like family established in [16] and later studied in different directions in [17, 18]

(ii) the family $\mathcal{S}_L^* \equiv \mathcal{S}^*(\Lambda(z))$ with $\Lambda(z) = \sqrt{1+z}$ was developed in [19] by Sokól and Stankiewicz. The image of the function $\Lambda(z) = \sqrt{1+z}$ demonstrates that the image domain is bounded by the Bernoullis lemniscate right-half plan specified by $|w^2(z) - 1| < 1$

(iii) by selecting $\Lambda(z) = 1 + \sin z$, the class $\mathcal{S}^*(\Lambda(z))$ leads to the family $\mathcal{S}_{\sin}^*$ which was explored in [20] while $\mathcal{S}_e^* \equiv \mathcal{S}^*(e^z)$ has been produced in the article [21] and later studied in [22]

(iv) the family $\mathcal{S}_R^* \equiv \mathcal{S}^*(\Lambda(z))$ with $\Lambda(z) = 1 + z/J(J + z)/(J - z), J = \sqrt{2} + 1$ is studied in [23] while $\mathcal{S}_{\cos}^* \coloneqq \mathcal{S}^*(\cos(z))$ and $\mathcal{S}_{\cosh}^* \coloneqq \mathcal{S}^*(\cosh(z))$ were recently examined by Raza and Bano [24], and Abdullah et.al [25], respectively

Now, let us take the nonvanishing analytic functions $h_1(z)$ and $h_2(z)$ in $\mathbb{U}_d$ with $h_1(0) = h_2(0) = 1$. Then, the families defined in this article consist of functions $f(z) \in \mathcal{A}$ whose ratios $f(z)/zq(z)$ and $q(z)$ are subordinated to $h_1(z)$ and $h_2(z)$, respectively, for some analytic function $q(z)$ with $q(0) = 1$ as

$$\begin{aligned} \frac{f(z)}{zq(z)} &\prec h_1(z), \\ q(z) &\prec h_2(z). \end{aligned} \tag{6}$$

We are now going to choose some particular functions instead of $h_1(z)$ and $h_2(z)$. These choices are

$$\begin{aligned} h_1(z) &= 1 + \sin z \\ \text{or } h_1(z) &= \cos z \\ \text{or } h_1(z) &= \sqrt{1+z}, \\ h_2(z) &= 1 + \sin z, \end{aligned} \tag{7}$$

and by applying the above-mentioned concepts, we now consider the following classes:

$$\mathcal{F}_{\cos} = \left\{ f(z) \in \mathcal{A} : \frac{f(z)}{zq(z)} \prec \cos z \& q(z) \prec h_2(z), z \in \mathbb{U}_d \right\}, \tag{8}$$

$$\mathcal{F}_{\mathcal{SL}} = \left\{ f(z) \in \mathcal{A} : \frac{f(z)}{zq(z)} \prec \sqrt{1+z} \& q(z) \prec h_2(z), z \in \mathbb{U}_d \right\}, \tag{9}$$

$$\mathcal{F}_{\sin} = \left\{ f(z) \in \mathcal{A} : \frac{f(z)}{zq(z)} \prec 1 + \sin z \& q(z) \prec h_2(z), z \in \mathbb{U}_d \right\}. \tag{10}$$

In the present article, we discuss majorization problems for each of the above-defined classes $\mathcal{F}_{\cos}, \mathcal{F}_{\mathcal{SL}}$, and $\mathcal{F}_{\sin}$.

2. Main Results

To prove majorization results for the classes $\mathcal{F}_{\cos}, \mathcal{F}_{\mathcal{SL}}$, and $\mathcal{F}_{\sin}$, we need the following lemma.

Lemma 1. *Let $q(z) \prec 1 + \sin z$ and for $|z| \leq r$. Then, $q(z)$ satisfies the following inequalities:*

$$1 - \sin r \cosh r \leq |q(z)| \leq 1 + \sin r \cosh r, \tag{11}$$

$$\left| \frac{zq'(z)}{q(z)} \right| \leq \frac{r \cosh r}{(1 - r^2)(1 - \sinh r)}. \tag{12}$$

Proof. If $q(z) \prec 1 + \sin z$, then

$$q(z) = 1 + \sin w(z), \tag{13}$$

for some Schwartz function $w(z)$. Now, after some easy calculations, we have

$$\frac{zq'(z)}{q(z)} = \frac{zw'(z)\cos w(z)}{1 + \sin w(z)}. \tag{14}$$

Let $w(z) = Re^{i\theta}$ with $|z| = r \leq R, -\pi \leq \theta \leq \pi$.A calculation shows that

$$\mathfrak{R}\left(\cos\left(Re^{i\theta}\right)\right) = \cos(Rx)\cosh(Ry), \tag{15}$$

where

$$\begin{gathered} x = \cos\theta, \\ y = \sin\theta, \\ \text{for } x, y \in [-1, 1]. \end{gathered} \tag{16}$$

Now, we can write

$$\begin{aligned} &\cos(Rx) \geq \cos R \geq \cos r, \\ &\quad 1 \leq \cosh(Ry) \leq \cosh R \leq \cosh r. \end{aligned} \tag{17}$$

Thus, we have

$$\mathfrak{R}\cos w(z) \geq \cos r. \tag{18}$$

Now, consider

$$\begin{aligned} \left|\sin\left(Re^{i\theta}\right)\right|^2 &= \cos(R\cos\theta)^2\sinh(R\sin\theta)^2 \\ &\quad + \sin(R\cos\theta)^2\cosh(R\sin\theta)^2 \\ &= \Psi(\theta). \end{aligned} \tag{19}$$

A calculation shows that the numbers, $0, \pm\pi, \pm\pi/2$, are the roots of equation (19) in $[-\pi, \pi]$. Since $\Psi(\theta)$ is an even function, it is sufficient to consider $\theta \in [0, \pi]$. We observe that $\Psi(\pi/2) = \sinh^2(R)$ and $\Psi(0) = \sin^2(R)$. Now, we can write

$$\max\left\{\Psi(0), \Psi\left(\frac{\pi}{2}\right), \Psi(\pi)\right\} = \sin \mathrm{h}^2(R). \tag{20}$$

Hence,

$$\left|\sin R\left(e^{i\theta}\right)\right| \leq \sinh(R) \leq \sinh r. \tag{21}$$

Similarly, one can easily show that

$$\cos r \leq |\cos w(z)| \leq \cosh r. \tag{22}$$

Now, from well-known inequality for Schwartz function $w(z)$, we obtain

$$\left|w'(z)\right| \leq \frac{1 - |w(z)|^2}{1 - |z|^2} = \frac{1 - R^2}{1 - |z|^2} \leq \frac{1}{1 - r^2}. \tag{23}$$

Now, by applying (21), (22), and (23) in (14), we get (12).

Theorem 2. *Let the functions $f(z) \in \mathscr{A}, g(z) \in \mathscr{F}_{\cos}$ and also suppose that $f(z) \ll g(z)$ in $\mathbb{U}_d$. Then, for $|z| \leq r_1$,*

$$\left|f'(z)\right| \leqq \left|g'(z)\right|, \tag{24}$$

where r_1 is the smallest positive root of the equation

$$\begin{aligned} &\left(\left(\left(1 - r^2\right) - r(1 + \rho)\right)\cos r - r\sinh r\right) \\ &\quad \cdot (1 - \sinh r) - r\cos r\cosh r = 0. \end{aligned} \tag{25}$$

Proof. If $g(z) \in \mathscr{F}_{\cos}$, then by the subordination relationship, we have

$$\frac{g(z)}{zq(z)} = \cos w(z). \tag{26}$$

Now, after simple calculations, we have

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zq'(z)}{q(z)} - \frac{zw'(z)\sin w(z)}{\cos w(z)}. \tag{27}$$

Now, by using (21), (22), and (23) along with Lemma 1, we obtain

$$\begin{aligned} \left|\frac{g(z)}{g'(z)}\right| &= \frac{|z|}{\left|1 + zq'(z)/q(z) - zw'(z)\sin w(z)/\cos w(z)\right|} \\ &\leqq \frac{|z|}{1 - \left|zq'(z)/q(z)\right| - \left|zw'(z)\sin w(z)/\cos w(z)\right|} \\ &\leqq \frac{r\left(1 - r^2\right)(1 - \sinh r)\cos r}{\left(1 - r^2\right)(1 - \sinh r)\cos r - r\cos r\cosh r - r\sinh r(1 - \sinh r)}. \end{aligned} \tag{28}$$

From (4), we can write

$$f(z) = \varphi(z)g(z). \tag{29}$$

Differentiating the above equality on both sides, we get

$$\begin{aligned} f'(z) &= \varphi'(z)g(z) + \varphi(z)g'(z) \\ &= g'(z)\left(\varphi(z) + \varphi'(z)\frac{zg'(z)}{g(z)}\right). \end{aligned} \tag{30}$$

Also, the Schwartz function $\varphi(z)$ fulfils the below inequality:

$$\left|\varphi'(z)\right| \leqq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} = \frac{1 - |\varphi(z)|^2}{1 - r^2}\ (z \in \mathbb{U}_d). \tag{31}$$

Now, applying (28) and (31) in (30), we have

$$|f'(z)| \leqq \left[\varphi(z) + \frac{r(1-|\varphi(z)|^2)(1-\sinh r)\cos r}{(1-r^2)(1-\sinh r)\cos r - r\cos r\cosh r - r\sinh r(1-\sinh r)}\right]|g'(z)|, \tag{32}$$

which by putting

$$|\varphi'(z)| = \rho\ (0 \leqq \rho \leqq 1) \tag{33}$$

becomes the inequality

$$|f'(z)| \leqq \Phi_1(r,\rho)|g'(z)|, \tag{34}$$

where

$$\Phi_1(r,\rho) = \varphi(z) + \frac{r(1-|\varphi(z)|^2)(1-\sinh r)\cos r}{((1-r^2)\cos r - r\sinh r)(1-\sinh r) - r\cos r\cosh r}. \tag{35}$$

To determine r_1, it is sufficient to choose

$$r_1 = \max\left(r \in [0,1): \Phi_1(r,\rho) \leqq 1, \forall \rho \in [0,1]\right), \tag{36}$$

or, equivalently,

$$r_1 = \max\left(r \in [0,1): \Psi_1(r,\rho) \geq 0, \forall \rho \in [0,1]\right), \tag{37}$$

where

$$\begin{aligned}\Psi_1(r,\rho) = {} & \left(\left(1-r^2-r(1+\rho)\right)\cos r - r\sinh r\right) \\ & \cdot (1-\sinh r) - \cos r\cosh r.\end{aligned} \tag{38}$$

Clearly, when $\rho = 1$, the function $\Psi_1(r,\rho)$ assumes its minimum value, namely,

$$\min\left(\Psi_1(r,\rho), \rho \in [0,1]\right) = \Psi_1(r,1) = \Psi_1(r), \tag{39}$$

where

$$\begin{aligned}\Psi_1(r) = {} & \left(\left(1-r^2-2r\right)\cos r - r\sinh r\right) \\ & \cdot (1-\sinh r) - r\cos r\cosh r.\end{aligned} \tag{40}$$

Next, we have the following inequalities:

$$\begin{aligned}&\Psi_1(0) = 1 > 0, \\ &\Psi_1(1) = -0.438\,51 < 0.\end{aligned} \tag{41}$$

There exists r_1 such that $\Psi_1(r) \geq 0$ for all $r \in [0, r_1]$, where r_1 is the smallest positive root of equation (25). Thus, the proof is completed.

Theorem 3. *Let $f(z) \in \mathscr{A}, g(z) \in \mathscr{F}_{\sin}$ and also suppose that $f(z) \ll g(z)$ in $\mathbb{U}_d$. Then, for $|z| \leq r_2$,*

$$|f'(z)| \leqq |g'(z)|, \tag{42}$$

where r_2 is the root (smallest positive) of the equation:

$$\left(1-r^2\right)(1-\sinh r) - 2r(\cosh r + 1 - \sinh r) = 0. \tag{43}$$

Proof. If $g(z) \in \mathscr{F}_{\sin}$, then by using (10) along with the subordination relationship, a holomorphic function $w(z)$ in $\mathbb{U}_d$ occurs with $w(0) = 0$ and $|w(z)| \leq |z|$ in such a way that

$$\frac{g(z)}{zq(z)} = 1 + \sin w(z) \tag{44}$$

hold. Now, after simple calculations, we have

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zq'(z)}{q(z)} + \frac{zw'(z)\cos w(z)}{1+\sin w(z)}. \tag{45}$$

Using (21), (22), and (23) along with Lemma 1, we obtain

$$\begin{aligned}\left|\frac{g(z)}{g'(z)}\right| &= \frac{|z|}{\left|1 + zp'(z)/p(z) + zw'(z)\cos w(z)/1+\sin w(z)\right|} \\ &\leqq \frac{r\left(1-r^2\right)(1-\sinh r)}{(1-r^2)(1-\sinh r) - 2r\cosh r}.\end{aligned} \tag{46}$$

Also, with the use of (31) and (46) in (30), we easily get

$$|f'(z)| \leqq \left[\varphi(z) + \frac{r\left(1-|\varphi(z)|^2\right)(1-\sinh r)}{(1-r^2)(1-\sinh r) - 2r\cosh r}\right]|g'(z)|. \tag{47}$$

Now, by the similar lines of Theorem 2 along with the virtue of (33), we easily obtain the required result.

Theorem 4. *Let $f(z) \in \mathscr{A}, g(z) \in \mathscr{F}_{\mathscr{SL}}$ and also suppose that $f(z)$ is majorized by $g(z)$ in $\mathbb{U}_d$. Then, for $|z| \leq r_4$,*

$$|f'(z)| \leqq |g'(z)|, \tag{48}$$

where r_4 is the positive smallest root of the equation

$$\left(1-2r^2-5r\right)(1-\sinh r) - 2r\cosh r = 0. \tag{49}$$

Proof. Let $g(z) \in \mathscr{F}_{\mathscr{SL}}$. Then, a holomorphic function $w(z)$ in $\mathbb{U}_d$ occurs with $w(0) = 0$ and $|w(z)| \leq |z|$ so that

$$\frac{g(z)}{zq(z)} = \sqrt{1+w(z)}. \tag{50}$$

Now, after simple calculations, we have

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zq'(z)}{q(z)} + \frac{zw'(z)}{2(1+w(z))}. \tag{51}$$

Using (23), we obtain

$$\frac{|z||w'(z)|}{2(1-|w(z)|)} \leq \frac{|z|(1+|w(z)|)}{2(1-|z|^2)} \leq \frac{|z|(1+|z|)}{2(1-|z|^2)} = \frac{|z|}{2(1-|z|)} \leq \frac{r}{2(1-r)}. \tag{52}$$

By virtue of (23) and Lemma 1, we obtain

$$\left|\frac{g(z)}{g'(z)}\right| \leqq \frac{|z|}{1-|zp'(z)/p(z)|-|zw'(z)/2(1+w(z))|} \leqq \frac{2r(1-r^2)(1-\sinh r)}{2(1-r^2)(1-\sinh r)-2r\cosh r-r(1+r)(1-\sinh r)}. \tag{53}$$

Now, using (31) and (53) in (30), we get

$$|f'(z)| \leqq \left[\varphi(z)+\frac{2r(1-|\varphi(z)|^2)(1-\sinh r)}{2(1-r^2)(1-\sinh r)-2r\cosh r-r(1+r)(1-\sinh r)}\right]|g'(z)|. \tag{54}$$

The required result follows directly using similar calculations as Theorem 2 along with the use of (33).

3. Conclusion

For some particular subfamilies of holomorphic functions which are connected with different shapes of image domains, we studied the problems of majorization. These problems can be examined for some other families such as for the families of meromorphic functions as well as for the families of harmonic functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this research paper.

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Hindawi
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Research Article

On Decomposition Formulas Related to the Gaussian Hypergeometric Functions in Three Variables

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In this paper, by using certain inverse pairs of symbolic operators introduced by Choi and Hasanov in 2011, we establish several decomposition formulas associated with the Gaussian triple hypergeometric functions. Some transformation formulas for these functions have also been obtained.

1. Introduction

The use of many mathematical operations goes beyond the class of elementary functions. The calculation of integrals, the summation of series, and solution of algebraic, transcendental, difference, and differential equations and their systems require expanding the class of studied functions. The development of the concept of a function, going in parallel with the development of the concepts of number and space, led to the emergence of new hypergeometric functions of many complex variables.

The great success of the theory of hypergeometric functions in a single variable has stimulated the development of the theory of hypergeometric functions in several variables by the fact that the solutions of partial differential equations arising in many applied problems of mathematical physics are given in terms of such hypergeometric functions (see, e.g., [1–6]). Multiple hypergeometric functions occur in numerous problems in hydrodynamics, control theory, electrical current, heat conduction, and classical and quantum mechanics (see, for details, [7–10], and the references cited therein). In view of theory and applications, a large number of hypergeometric functions have been developed; for example, as many as 205 hypergeometric functions are recorded in the monograph [11]. In particular, we recall the Gaussian functions $F_{11a}, F_{11c}, F_{15a}, F_{17a}, F_{17b}, F_{17c}, F_{18a}, F_{19a}, F_{20a}$, and F_{23c} in three variables defined by (see [11])

$$F_{11a}(a_1, a_2, a_3, a_4 ; c_1, c_2 ; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_m(a_4)_p}{(c_1)_{m+n}(c_2)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{1}$$

$$F_{11c}(a_1, a_2, a_3, b ; c ; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_p(b)_{m-p}}{(c_1)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2}$$

$$F_{15a}(a_1, a_2, a_3 ; c_1, c_2 ; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_{p+m}}{(c_1)_{m+n}(c_2)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{3}$$

$$F_{17a}(a_1, a_2, a_3, a_4 ; c_1, c_2, c_3 ; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_m(a_3)_n(a_4)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{4}$$

$$F_{17b}(a_1, a_2, a_3, b; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_n(a_3)_p(b)_{m-n}}{(c_1)_m(c_2)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \quad (5)$$

$$F_{17c}(a_1, a_2, b_1, b_2; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_p(b_1)_{m-n}(b_2)_{n-m}}{(c)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \quad (6)$$

$$F_{18a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_m(a_3)_n(a_4)_p}{(c_1)_{m+n}(c_2)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \quad (7)$$

$$F_{19a}(a_1, a_2, a_3, a_4; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_m(a_3)_n(a_4)_p}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \quad (8)$$

$$F_{20a}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_{m+n}(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \quad (9)$$

$$F_{23c}(a_1, a_2, a_3, b_1, b_2; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_m(a_3)_n(b_1)_{n-p}(b_2)_{2p-m}}{(c)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}. \quad (10)$$

Here, $(a)_m$ denotes the Pochhammer symbol given as

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+m-1) \; (m \in \mathbb{N} := \{1, 2, \cdots\}), (a)_0 = 1. \quad (11)$$

Burchnall and Chaundy presented the inverse pairs of symbolic operators ∇ and Δ [12, 13] (also see [14]) by means of which they established several decomposition formulas for Appell's double hypergeometric functions in terms of the Gaussian hypergeometric functions in one variable. Recently, Hasanov and Srivastava [15, 16] introduced multivariable analogues of Burchnall-Chaundy's symbolic operators, and with the help of these operators, the authors obtained a number of decomposition formulas associated with multiple Lauricella hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$, and $F_D^{(r)}$. In [17, 18], the authors gave the following multivariable symbolic operators:

$$H_{x_1,\cdots,x_r}(\alpha, \beta) = \frac{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)}{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)} = \sum_{k_1,\cdots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(\beta)_{k_1+\cdots+k_r} k_1!\cdots k_r!}, \quad (12)$$

$$\bar{H}_{x_1,\cdots,x_r}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)}{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)} = \sum_{k_1,\cdots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(1-\alpha-\delta_1-\cdots-\delta_r)_{k_1+\cdots+k_r} k_1!\cdots k_r!} \quad (13)$$

$$\left(\delta_j := x_j \frac{\partial}{\partial x_j}, j = 1, \cdots, r; r \in \mathbb{N} := \{1, 2, 3, \cdots\}\right). \quad (14)$$

Based on the operators (12) and (13), we aim in this work to establish certain decomposition formulas for second-order Gaussian hypergeometric functions in three variables (1), (2), (3), (4), (5), (6), (7), (8), (9), and (10), which are used to derive some interesting transformation formulas for these functions.

2. Symbolic Form

Applying the symbolic operators in (12) and (13), we construct the following set of operator identities involving the classical Gauss hypergeometric function ${}_2F_1$ [19], the Appell functions F_1, F_2, F_4 [20], the Horn functions G_1, H_1, H_7 [21], and the Gaussian triple hypergeometric functions defined in (1), (2), (3), (4), (5), (6), (7), (8), (9), and (10):

$$F_{11a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = H_{x,y}(a_1, c_1)(1-x)^{-a_3}(1-y)^{-a_2}\, {}_2F_1\left(a_2, a_4; c_2; \frac{z}{1-y}\right), \quad (15)$$

$$(1-x)^{-a_3}(1-y)^{-a_2}\, {}_2F_1\left(a_2, a_4; c_2; \frac{z}{1-y}\right) = \bar{H}_{x,y}(a_1, c_1) F_{11a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z), \quad (16)$$

$$F_{11c}(a_1, a_2, a_3, b; c; x, y, z) = H_{x,y}(a_1, c)(1-x)^{-b}(1-y)^{-a_2}\, {}_2F_1\left(a_2, a_3; 1-b; -\frac{(1-x)z}{1-y}\right), \quad (17)$$

$$(1-x)^{-b}(1-y)^{-a_2}\, {}_2F_1\left(a_2, a_3; 1-b; -\frac{(1-x)z}{1-y}\right) = \bar{H}_{x,y}(a_1, c) F_{11c}(a_1, a_2, a_3, b; c; x, y, z), \quad (18)$$

$$F_{15a}(a_1, a_2, a_3; c_1, c_2; x, y, z) = H_{x,y}(a_1, c_1)(1-x)^{-a_3}(1-y)^{-a_2}\, {}_2F_1\left(a_2, a_3; c_2; \frac{z}{(1-x)(1-y)}\right), \quad (19)$$

$$(1-x)^{-a_3}(1-y)^{-a_2}\, {}_2F_1\left(a_2, a_3; c_2; \frac{z}{(1-x)(1-y)}\right) = \bar{H}_{x,y}(a_1, c_1) F_{15a}(a_1, a_2, a_3; c_1, c_2; x, y, z), \quad (20)$$

$$F_{17a}(a_1,a_2,a_3,a_4;c_1,c_2,c_3;x,y,z) = H_z(a_4,c_3)(1-z)^{-a_1}F_2\left(a_1,a_2,a_3;c_1,c_2;\frac{x}{1-z},\frac{y}{1-z}\right), \quad (21)$$

$$(1-z)^{-a_1}F_2\left(a_1,a_2,a_3;c_1,c_2;\frac{x}{1-z},\frac{y}{1-z}\right) = \bar{H}_z(a_4,c_3)F_{17a}(a_1,a_2,a_3,a_4;c_1,c_2,c_3;x,y,z), \quad (22)$$

$$F_{17b}(a_1,a_2,a_3,b;c_1,c_2;x,y,z) = H_z(a_3,c_2)(1-z)^{-a_1}H_1\left(b,a_1,a_2;c_1;\frac{x}{1-z},\frac{y}{1-z}\right), \quad (23)$$

$$(1-z)^{-a_1}H_1\left(b,a_1,a_2;c_1;\frac{x}{1-z},\frac{y}{1-z}\right) = \bar{H}_z(a_3,c_2)F_{17b}(a_1,a_2,a_3,b;c_1,c_2;x,y,z), \quad (24)$$

$$F_{17c}(a_1,a_2,b_1,b_2;c;x,y,z) = H_z(a_2,c)(1-z)^{-a_1}G_1\left(a_1,b_2,b_1;\frac{x}{1-z},\frac{y}{1-z}\right), \quad (25)$$

$$(1-z)^{-a_1}G_1\left(a_1,b_2,b_1;\frac{x}{1-z},\frac{y}{1-z}\right) = \bar{H}_z(a_2,c)F_{17c}(a_1,a_2,b_1,b_2;c;x,y,z), \quad (26)$$

$$F_{18a}(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z) = H_z(a_4,c_2)(1-z)^{-a_1}F_1\left(a_1,a_2,a_3;c_1;\frac{x}{1-z},\frac{y}{1-z}\right), \quad (27)$$

$$(1-z)^{-a_1}F_1\left(a_1,a_2,a_3;c_1;\frac{x}{1-z},\frac{y}{1-z}\right) = \bar{H}_z(a_4,c_2)F_{18a}(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z), \quad (28)$$

$$F_{19a}(a_1,a_2,a_3,a_4;c;x,y,z) = H_{x,y,z}(a_1,c)(1-x)^{-a_2}(1-y)^{-a_3}(1-z)^{-a_4}, \quad (29)$$

$$(1-x)^{-a_2}(1-y)^{-a_3}(1-z)^{-a_4} = \bar{H}_{x,y,z}(a_1,c)F_{19a}(a_1,a_2,a_3,a_4;c;x,y,z), \quad (30)$$

$$F_{20a}(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z) = H_z(a_3,c_3)(1-z)^{-a_1}F_4\left(a_1,a_2;c_1,c_2;\frac{x}{1-z},\frac{y}{1-z}\right), \quad (31)$$

$$(1-z)^{-a_1}F_4\left(a_1,a_2;c_1,c_2;\frac{x}{1-z},\frac{y}{1-z}\right) = \bar{H}_z(a_3,c_3)F_{20a}(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z), \quad (32)$$

$$F_{23c}(a_1,a_2,a_3,b_1,b_2;c;x,y,z) = H_y(a_3,c)(1-y)^{-b_1}H_7(b_2,a_1,a_2;1-b_1;-(1-y)z,x), \quad (33)$$

$$(1-y)^{-b_1}H_7(b_2,a_1,a_2;1-b_1;-(1-y)z,x) = \bar{H}_y(a_3,c)F_{23c}(a_1,a_2,a_3,b_1,b_2;c;x,y,z). \quad (34)$$

Each of the operator identities (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30), (31), (32), (33), and (34) can be proved by means of Mellin and inverse Mellin transformation (see, for example, [11, 20, 22]). The proofs of the operator identities are omitted here.

3. Decomposition Formulas

In [23] (p. 93), it is proved that, for every analytic function $f(\xi)$, the following formulas hold true:

$$(-\delta)_n\{f(\xi)\} = (-1)^n\xi^n\frac{d^n}{d\xi^n}\{f(\xi)\}, \quad (35)$$

$$(\alpha+\delta)_n\{f(\xi)\} = \xi^{1-\alpha}\frac{d^n}{d\xi^n}\left\{\xi^{\alpha+n-1}f(\xi)\right\}, \quad (36)$$

where

$$\delta := \xi\frac{d}{d\xi};\ \alpha\in\mathbb{C};\ n\in\mathbb{N}_0 := \{0,1,2,\cdots\}. \quad (37)$$

In view of formulas (35) and (36), and taking into account the differentiation formula for hypergeometric functions, from operator identities (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30), (31), (32), (33), and (34), we have

$$F_{11a}(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z) = (1-x)^{-a_3}(1-y)^{-a_2}\times\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a_3)_i(a_2)_j(c_1-a_1)_{i+j}}{(c_1)_{i+j}i!j!} \cdot\left(\frac{x}{1-x}\right)^i\left(\frac{y}{1-y}\right)^j\times{}_2F_1\left(a_2+j,a_4;c_2;\frac{z}{1-y}\right), \quad (38)$$

$$(1-x)^{-a_3}(1-y)^{-a_2}{}_2F_1\left(a_2,a_4;c_2;\frac{z}{1-y}\right) = \sum_{i,j=0}^{\infty}\frac{(a_3)_i(a_2)_j(c_1-a_1)_{i+j}}{(c_1)_{i+j}i!j!} \times x^iy^jF_{11a}(a_1,a_2+j,a_3+i,a_4;c_1+i+j,c_2;x,y,z), \quad (39)$$

$$F_{11c}(a_1,a_2,a_3,b;c;x,y,z) = (1-x)^{-b}(1-y)^{-a_2}\times \cdot\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(b)_i(a_2)_j(c-a_1)_{i+j}}{(c)_{i+j}i!j!}\left(\frac{x}{1-x}\right)^i\left(\frac{y}{1-y}\right)^j\times{}_2F_1 \cdot\left(a_2+j,a_3;1-b-i;-\frac{(1-x)z}{1-y}\right), \quad (40)$$

$$(1-x)^{-b}(1-y)^{-a_2}{}_2F_1\left(a_2,a_3;1-b;-\frac{(1-x)z}{1-y}\right) = \sum_{i,j=0}^{\infty}\frac{(b)_i(a_2)_j(c-a_1)_{i+j}}{(c)_{i+j}i!j!}x^iy^j \times F_{11c}(a_1,a_2+j,a_3,b+i;c+i+j;x,y,z), \tag{41}$$

$$F_{15a}(a_1,a_2,a_3;c_1,c_2;x,y,z) = (1-x)^{-a_3}(1-y)^{-a_2} \times \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a_3)_i(a_2)_j(c_1-a_1)_{i+j}}{(c_1)_{i+j}i!j!}\left(\frac{x}{1-x}\right)^i\left(\frac{y}{1-y}\right)^j \times {}_2F_1\left(a_2+j,a_3+i;c_2;\frac{z}{(1-x)(1-y)}\right), \tag{42}$$

$$(1-x)^{-a_3}(1-y)^{-a_2}{}_2F_1\left(a_2,a_3;c_2;\frac{z}{(1-x)(1-y)}\right) = \sum_{i,j=0}^{\infty}\frac{(a_3)_i(a_2)_j(c_1-a_1)_{i+j}}{(c_1)_{i+j}i!j!}x^iy^j \times F_{15a}(a_1,a_2+j,a_3+i;c_1+i+j,c_2;x,y,z), \tag{43}$$

$$F_{17a}(a_1,a_2,a_3,a_4;c_1,c_2,c_3;x,y,z) = (1-z)^{-a_1} \times \sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c_3-a_4)_i}{(c_3)_ii!}\left(\frac{z}{1-z}\right)^iF_2 \cdot\left(a_1+i,a_2,a_3;c_1,c_2;\frac{x}{1-z},\frac{y}{1-z}\right), \tag{44}$$

$$(1-z)^{-a_1}F_2\left(a_1,a_2,a_3;c_1,c_2;\frac{x}{1-z},\frac{y}{1-z}\right) = \sum_{i=0}^{\infty}\frac{(a_1)_i(c_3-a_4)_i}{(c_3)_ii!}z^iF_{17a} \cdot(a_1+i,a_2,a_3,a_4;c_1,c_2,c_3+i;x,y,z), \tag{45}$$

$$F_{17b}(a_1,a_2,a_3,b;c_1,c_2;x,y,z) = (1-z)^{-a_1}\times\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c_2-a_3)_i}{(c_2)_ii!}\left(\frac{z}{1-z}\right)^iH_1 \cdot\left(b,a_1+i,a_2;c_1;\frac{x}{1-z},\frac{y}{1-z}\right), \tag{46}$$

$$(1-z)^{-a_1}H_1\left(b,a_1,a_2;c_1;\frac{x}{1-z},\frac{y}{1-z}\right) = \sum_{i=0}^{\infty}\frac{(a_1)_i(c_2-a_3)_i}{(c_2)_ii!}z^iF_{17b}(a_1+i,a_2,a_3,b;c_1,c_2+i;x,y,z), \tag{47}$$

$$F_{17c}(a_1,a_2,b_1,b_2;c;x,y,z) = (1-z)^{-a_1} \times\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c-a_2)_i}{(c)_ii!}\left(\frac{z}{1-z}\right)^iG_1 \cdot\left(a_1+i,b_2,b_1;\frac{x}{1-z},\frac{y}{1-z}\right), \tag{48}$$

$$(1-z)^{-a_1}G_1\left(a_1,b_2,b_1;\frac{x}{1-z},\frac{y}{1-z}\right) = \sum_{i=0}^{\infty}\frac{(a_1)_i(c-a_2)_i}{(c)_ii!}z^iF_{17c}(a_1+i,a_2,b_1,b_2;c+i;x,y,z), \tag{49}$$

$$F_{18a}(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z) = (1-z)^{-a_1} \times\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c_2-a_4)_i}{(c_2)_ii!}\left(\frac{z}{1-z}\right)^iF_1 \cdot\left(a_1+i,a_2,a_3;c_1;\frac{x}{1-z},\frac{y}{1-z}\right), \tag{50}$$

$$(1-z)^{-a_1}F_1\left(a_1,a_2,a_3;c_1;\frac{x}{1-z},\frac{y}{1-z}\right) = \sum_{i=0}^{\infty}\frac{(a_1)_i(c_2-a_4)_i}{(c_2)_ii!}z^iF_{18a}(a_1+i,a_2,a_3,a_4;c_1,c_2+i;x,y,z), \tag{51}$$

$$F_{19a}(a_1,a_2,a_3,a_4;c;x,y,z) = (1-x)^{-a_2}(1-y)^{-a_3}(1-z)^{-a_4} \times F_D^{(3)}\left(c-a_1,a_2,a_3,a_4;c;\frac{x}{x-1},\frac{y}{y-1},\frac{z}{z-1}\right), \tag{52}$$

$$(1-x)^{-a_2}(1-y)^{-a_3}(1-z)^{-a_4} = \sum_{i,j,k=0}^{\infty}\frac{(a_2)_i(a_3)_j(a_4)_k(c-a_1)_{i+j+k}}{(c)_{i+j+k}i!j!k!}x^iy^jz^k \times F_{19a}(a_1,a_2+i,a_3+j,a_4+k;c+i+j+k;x,y,z), \tag{53}$$

$$F_{20a}(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z) = (1-z)^{-a_1}\times\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c_3-a_3)_i}{(c_3)_ii!}\left(\frac{z}{1-z}\right)^iF_4 \cdot\left(a_1+i,a_2;c_1,c_2;\frac{x}{1-z},\frac{y}{1-z}\right), \tag{54}$$

$$(1-z)^{-a_1}F_4\left(a_1,a_2;c_1,c_2;\frac{x}{1-z},\frac{y}{1-z}\right) = \sum_{i=0}^{\infty}\frac{(a_1)_i(c_3-a_3)_i}{(c_3)_ii!}z^iF_{20a}(a_1+i,a_2,a_3;c_1,c_2,c_3+i;x,y,z), \tag{55}$$

$$F_{23c}(a_1,a_2,a_3,b_1,b_2;c;x,y,z) = (1-y)^{-b_1}\times\sum_{i=0}^{\infty}\frac{(-1)^i(b_1)_i(c-a_3)_i}{(c)_ii!} \cdot\left(\frac{y}{1-y}\right)^iH_7(b_2,a_1,a_2;1-b_1-i;-(1-y)z,x), \tag{56}$$

$$(1-y)^{-b_1} H_7(b_2, a_1, a_2; 1-b_1; -(1-y)z, x) = \sum_{i=0}^{\infty} \frac{(b_1)_i (c-a_3)_i}{(c)_i i!} y^i F_{23c}(a_1, a_2, a_3, b_1+i, b_2; c+i; x, y, z). \quad (57)$$

4. Transformation Formulas

The transformation formulas defined below follow from the expansion formulas (38), (39), (40), (41), (42), (43), (44), (45), (46), (47), (48), (49), (50), (51), (52), (53), (54), (55), (56), and (57):

$$F_{11a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = (1-x)^{-a_3}(1-y)^{-a_2} F_{11a} \cdot \left(c_1 - a_1, a_2, a_3, a_4; c_1, c_2; \frac{x}{x-1}, \frac{y}{y-1}, \frac{z}{1-y}\right),$$

$$F_{11c}(a_1, a_2, a_3, b; c; x, y, z) = (1-x)^{-b}(1-y)^{-a_2} F_{11c} \cdot \left(c - a_1, a_2, a_3, b; c; \frac{x}{x-1}, \frac{y}{y-1}, \frac{(1-x)z}{1-y}\right),$$

$$F_{15a}(a_1, a_2, a_3; c_1, c_2; x, y, z) = (1-x)^{-a_3}(1-y)^{-a_2} F_{15a} \cdot \left(c_1 - a_1, a_2, a_3; c_1, c_2; \frac{x}{x-1}, \frac{y}{y-1}, \frac{z}{(1-x)(1-y)}\right),$$

$$F_{17a}(a_1, a_2, a_3, a_4; c_1, c_2, c_3; x, y, z) = (1-z)^{-a_1} F_{17a} \cdot \left(a_1, a_2, a_3, c_3 - a_4; c_1, c_2, c_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1}\right),$$

$$F_{17b}(a_1, a_2, a_3, b; c_1, c_2; x, y, z) = (1-z)^{-a_1} F_{17b} \cdot \left(a_1, a_2, c_2 - a_3, b; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1}\right),$$

$$F_{17c}(a_1, a_2, b_1, b_2; c; x, y, z) = (1-z)^{-a_1} F_{17c} \cdot \left(a_1, c - a_2, b_1, b_2; c; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1}\right),$$

$$F_{18a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = (1-z)^{-a_1} F_{18a} \cdot \left(a_1, a_2, a_3, c_2 - a_4; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1}\right),$$

$$F_{19a}(a_1, a_2, a_3, a_4; c; x, y, z) = (1-x)^{-a_2}(1-y)^{-a_3}(1-z)^{-a_4} \times F_D^{(3)}\left(c - a_1, a_2, a_3, a_4; c; \frac{x}{x-1}, \frac{y}{y-1}, \frac{z}{z-1}\right),$$

$$F_{20a}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = (1-z)^{-a_1} F_{20a} \cdot \left(a_1, a_2, c_3 - a_3; c_1, c_2, c_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1}\right),$$

$$F_{23c}(a_1, a_2, a_3, b_1, b_2; c; x, y, z) = (1-y)^{-b_1} F_{23c} \cdot \left(a_1, a_2, c - a_3, b_1, b_2; c; x, \frac{y}{y-1}, (1-y)z\right). \quad (58)$$

5. Concluding Remarks

In this present paper, with the help of the inverse pairs of symbolic operators, we established a number of decomposition formulas for some Gaussian triple hypergeometric functions. Also, we investigated certain transformation formulas for these functions. We conclude that mutually inverse operators (12) and (13) can be applied to other multiple hypergeometric functions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Fekete-Szegö Inequality for Analytic and Biunivalent Functions Subordinate to Gegenbauer Polynomials

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In the present paper, a subclass of analytic and biunivalent functions by means of Gegenbauer polynomials is introduced. Certain coefficients bound for functions belonging to this subclass are obtained. Furthermore, the Fekete-Szegö problem for this subclass is solved. A number of known or new results are shown to follow upon specializing the parameters involved in our main results.

1. Introduction

Orthogonal polynomials have been studied extensively as early as they were discovered by Legendre in 1784 [1]. In mathematical treatment of model problems, orthogonal polynomials arise often to find solutions of ordinary differential equations under certain conditions imposed by the model.

The importance of the orthogonal polynomials for contemporary mathematics, as well as for a wide range of their applications in physics and engineering, is beyond any doubt. It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equations as well as in mathematical statistics. Their applications in quantum mechanics, scattering theory, automatic control, signal analysis, and axially symmetric potential theory are also known [2, 3].

Formally speaking, polynomials P_r and P_m of order r and m are orthogonal if

$$\int_a^b \Phi(x) P_r(x) P_m(x) dx = 0. \text{for } r \neq m, \tag{1}$$

where $\Phi(x)$ is nonnegative function in the interval (a, b); therefore, the integral is well-defined for all finite order polynomials $P_r(x)$.

A special case of orthogonal polynomials is Gegenbauer polynomials. They are representatively related with typically real functions T_R as discovered in [4], where the integral representation of typically real functions and generating function of Gegenbauer polynomials are using common algebraic expressions. Undoubtedly, this led to several useful inequalities appear from the Gegenbauer polynomial realm.

Typically, real functions play an important role in the geometric function theory because of the relation $T_R = \bar{co} S_R$ and its role of estimating coefficient bounds, where S_R denotes the class of univalent functions in the unit disk with real coefficients and $\bar{co} S_R$ denotes the closed convex hull of S_R.

This paper associates certain biunivalent functions with Gegenbauer polynomials and then explores some properties of the class in hand. Paving the way for mathematical notations and definitions, we provide the following section.

2. Definitions and Preliminaries

Let $\mathscr{A}$ denotes the class of all analytic functions f defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus, each $f \in \mathscr{A}$ has a Taylor-Maclaurin series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in \mathbb{U}). \tag{2}$$

Further, let $\mathscr{S}$ denotes the class of all functions $f \in \mathscr{A}$ which are univalent in $\mathbb{U}$ (for details, see [5]).

A subordination between two analytic functions f and g is written as $f \prec g$. Conceptually, the analytic function f is subordinate to g if the image under g contains the image under f. Technically, the analytic function f is subordinate to g if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$; such that

$$f(z) = g(w(z)). \tag{3}$$

Besides, if the function g is univalent in $\mathbb{U}$, then the following equivalence holds:

$$\begin{gathered} f(z) \prec g(z) \text{ if and only if } f(0) = g(0) \\ f(\mathbb{U}) \subset g(\mathbb{U}). \end{gathered} \tag{4}$$

Further on the subordination principle, we refer to [6].

It is well-known that, if $f(z)$ is an univalent analytic function from a domain $\mathbb{D}_1$ onto a domain $\mathbb{D}_2$, then the inverse function $g(z)$ defined by

$$g(f(z)) = z, (z \in \mathbb{D}_1), \tag{5}$$

is an analytic and univalent mapping from $\mathbb{D}_2$ to $\mathbb{D}_1$. Moreover, by the familiar Koebe one-quarter theorem (for details, see [5]), we know that the image of $\mathbb{U}$ under every function $f \in \mathscr{S}$ contains a disk of radius 1/4.

According to this, every function $f \in \mathscr{S}$ has an inverse map f^{-1} that satisfies the following conditions:

$$\begin{aligned} &f^{-1}(f(z)) = z \,(z \in \mathbb{U}), \\ &f\left(f^{-1}(w)\right) = w \left(|w| < r_0(f)\,; r_0(f) \ge \frac{1}{4}\right). \end{aligned} \tag{6}$$

In fact, the inverse function is given by

$$\begin{aligned} g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 \\ - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \end{aligned} \tag{7}$$

A function $f \in \mathscr{A}$ is said to be biunivalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denotes the class of biunivalent functions in $\mathbb{U}$ given by (2). Examples of functions in the class Σ are

$$\frac{z}{1-z}, - \log(1-z), \frac{1}{2} \log\left(\frac{1+z}{1-z}\right), \cdots. \tag{8}$$

It is worth noting that the familiar Koebe function is not a member of Σ, since it maps the unit disk $\mathbb{U}$ univalently onto the entire complex plane except the part of the negative real axis from $-1/4$ to $-\infty$. Thus, clearly, the image of the domain does not contain the unit disk $\mathbb{U}$. For a brief history and some intriguing examples of functions and characterization of the class Σ, see [7–15].

In 1967, Lewin [16] investigated the biunivalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [17] conjectured that $|a_2| \le \sqrt{2}$. On the other hand, Netanyahu [18] showed that $\max_{f \in \Sigma} |a_2| = 4/3$. The best-known estimate for functions in Σ has been obtained in 1984 by Tan [19], that is, $|a_2| < 1.485$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n| \, (n \in \mathbb{N} \setminus \{1, 2\})$ for each $f \in \Sigma$ given by (2) is presumably still an open problem.

The most important and well-investigated subclasses of the analytic and univalent function class $\mathscr{S}$ are the class $\mathscr{S}^*(\varsigma)$ of starlike functions of order ς in $\mathbb{U}$ and the class $\mathscr{K}(\varsigma)$ of convex functions of order ς in $\mathbb{U}$. By definition, we have

$$\begin{aligned} \mathscr{S}^*(\varsigma) &:= \left\{ f : f \in \mathscr{S} \text{ and } \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \varsigma, (z \in \mathbb{U}\,; 0 \le \varsigma < 1) \right\}, \\ \mathscr{K}(\varsigma) &:= \left\{ f : f \in \mathscr{S} \text{ and } \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \varsigma, (z \in \mathbb{U}\,; 0 \le \varsigma < 1) \right\}. \end{aligned} \tag{9}$$

For $0 \le \varsigma < 1$, a function $f \in \Sigma$ is in the class $\mathscr{S}_\Sigma^*(\varsigma)$ of bistarlike function of order ς or $\mathscr{K}_\Sigma(\varsigma)$ of biconvex function of order ς if both f and f^{-1} are, respectively, starlike or convex functions of order ς.

Very recently, Amourah [20] considered the Gegenbauer polynomials $H_\alpha(x, z)$, which are given by the following recurrence relation:

For nonzero real constant α, a generating function of Gegenbauer polynomials is defined by

$$H_\alpha(x, z) = \frac{1}{(1 - 2xz + z^2)^\alpha}, \tag{10}$$

where $x \in [-1, 1]$ and $z \in \mathbb{U}$. For fixed x, the function H_α is analytic in $\mathbb{U}$, so it can be expanded in a Taylor series as

$$H_\alpha(x, z) = \sum_{n=0}^{\infty} C_n^\alpha(x) z^n, \tag{11}$$

where $C_n^\alpha(x)$ is Gegenbauer polynomial of degree n.

Obviously, H_α generates nothing when $\alpha = 0$. Therefore, the generating function of the Gegenbauer polynomial is set to be

$$H_0(x,z) = 1 - \log\left(1 - 2xz + z^2\right) = \sum_{n=0}^{\infty} C_n^0(x) z^n, \quad (12)$$

for $\alpha = 0$. Moreover, it is worth to mention that a normalization of α to be greater than $-1/2$ is desirable [3, 21]. Gegenbauer polynomials can also be defined by the following recurrence relations:

$$C_n^\alpha(x) = \frac{1}{n}\left[2x(n+\alpha-1)C_{n-1}^\alpha(x) - (n+2\alpha-2)C_{n-1}^\alpha(x)\right], \quad (13)$$

with the initial values

$$C_0^\alpha(x) = 1,\, C_1^\alpha(x) = 2\alpha x \text{ and } C_2^\alpha(x) = 2\alpha(1+\alpha)x^2 - \alpha. \quad (14)$$

First off, we present some special cases of the polynomials $C_n^\alpha(x)$:

(1) For $\alpha = 1$, we get the Chebyshev Polynomials

(2) For $\alpha = 1/2$, we get the Legendre Polynomials

Recently, many researchers have been exploring biunivalent functions associated with orthogonal polynomials, few to mention [22–28]. For Gegenbauer polynomial, as far as we know, there is little work associated with biunivalent functions in the literatures. Initiating an exploration on the properties of biunivalent functions associated with Gegenbauer polynomials is the main goal of this paper. To do so, we take into account, the following definitions.

Definition 1. Let $\lambda \geq 1$, $\mu \geq 0, x \in (1/2, 1]$ *and* α *is a nonzero real constant. A function* f $\in \Sigma$ *given by (2) is said to be in the class* $\mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$ *if the following subordinations are satisfied:*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec H_\alpha(x,z), \quad (15)$$

$$(1-\lambda)\left(\frac{f^{-1}(w)}{w}\right)^\mu + \lambda\left(f^{-1}(w)\right)'\left(\frac{f^{-1}(w)}{w}\right)^{\mu-1} \prec H_\alpha(x,w), \quad (16)$$

where the function $f^{-1}(w)$ is defined by (7) and H_α is the generating function of the Gegenbauer polynomial given by (10).

By suitably specializing the parameters μ, λ, and α, the class $\mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$ leads to the following new subclasses of biuniavlent functions:

Example 1. If $\mu = 1$ and a function f $\in \Sigma$ given by (2) is said to be in the class $\mathfrak{B}_\Sigma(\lambda, \mathrm{x}, \alpha)$ if the following subordinations are satisfied:

$$\begin{aligned} (1-\lambda)\left(\frac{f(z)}{z}\right) + \lambda f'(z) &\prec \frac{1}{\left(1-2xz+z^2\right)^\alpha}, \\ (1-\lambda)\left(\frac{f^{-1}(w)}{w}\right) + \lambda\left(f^{-1}(w)\right)' &\prec \frac{1}{\left(1-2xz+z^2\right)^\alpha}, \end{aligned} \quad (17)$$

where $x \in (1/2, 1]$ and the function $f^{-1}(w)$ is defined by (7).

Example 2. If $\lambda = 1$ and a function $f \in \Sigma$ given by (2) is said to be in the class $\mathfrak{B}_\Sigma^\mu(x, \alpha)$ if the following subordinations are satisfied:

$$\begin{aligned} f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} &\prec \frac{1}{\left(1-2xz+z^2\right)^\alpha}, \\ \left(f^{-1}(w)\right)'\left(\frac{f^{-1}(w)}{w}\right)^{\mu-1} &\prec \frac{1}{\left(1-2xz+z^2\right)^\alpha}, \end{aligned} \quad (18)$$

where $x \in (1/2, 1]$ and the function $f^{-1}(w)$ is defined by (7).

Example 3. If $\lambda = 1$, $\mu = 1$, and a function $f \in \Sigma$ given by (2) is said to be in the class $\mathfrak{B}_\Sigma(x, \alpha)$ if the following subordinations are satisfied:

$$\begin{aligned} f'(z) &\prec \frac{1}{\left(1-2xz+z^2\right)^\alpha}, \\ \left(f^{-1}(w)\right)' &\prec \frac{1}{\left(1-2xz+z^2\right)^\alpha}, \end{aligned} \quad (19)$$

where $x \in (1/2, 1]$ and the function $f^{-1}(w)$ is defined by (7).

Example 4. If $\lambda = 1$, $\mu = 0$ and a function $f \in \Sigma$ given by (2) is said to be in the class $\mathcal{S}_\Sigma^*(x, \alpha)$ if the following subordinations are satisfied:

$$\begin{aligned} \frac{zf'(z)}{f(z)} &\prec \frac{1}{\left(1-2xz+z^2\right)^\alpha}, \\ \frac{z\left(f^{-1}(w)\right)'}{f^{-1}(w)} &\prec \frac{1}{\left(1-2xz+z^2\right)^\alpha}, \end{aligned} \quad (20)$$

where $x \in (1/2, 1]$ and the function $f^{-1}(w)$ is defined by (7).

Remark 2. The subclasses $\mathfrak{B}_\Sigma^\mu(\lambda, x, 1) = \mathfrak{B}_\Sigma^\mu(\lambda, x)$ and $\mathfrak{B}_\Sigma^1(\lambda, x, 1) = \mathfrak{B}_\Sigma(\lambda, x)$ were studied by Bulut et al. [29] and Bulut et al. [30], respectively.

In this paper, motivated by recent works of Amourah et al. [20], we use Gegenbauer polynomials to obtain the estimates on the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$.

Unless otherwise mentioned, we assume in the remainder of this paper that $\lambda \geq 1, \mu \geq 0$, $x \in (1/2, 1]$, and α is a nonzero real constant.

3. Coefficient Bounds of the Class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$

In the following theorem, we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$.

Theorem 3. *Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$. Then,*

$$
\begin{aligned}
|a_2| &\leq \frac{2|\alpha| x \sqrt{2|\alpha| x}}{\sqrt{\left|\alpha(\mu+\lambda)^2 - 2\left[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)\right]x^2\right|}}, \\
|a_3| &\leq \frac{4\alpha^2 x^2}{(\mu+\lambda)^2} + \frac{2|\alpha| x}{(\mu+2\lambda)}.
\end{aligned} \tag{21}
$$

Proof. Let $f \in \mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$. From (15) and (16), we have

$$
\begin{aligned}
&(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \\
&\quad = 1 + C_1^{\alpha}(x)c_1 z + \left[C_1^{\alpha}(x)c_2 + C_2^{\alpha}(x)c_1^2\right]z^2 + \cdots,
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&(1-\lambda)\left(\frac{f^{-1}(w)}{w}\right)^{\mu} + \lambda\left(f^{-1}(w)\right)'\left(\frac{f^{-1}(w)}{w}\right)^{\mu-1} \\
&\quad = 1 + C_1^{\alpha}(x)d_1 w + \left[C_1^{\alpha}(x)d_2 + C_2^{\alpha}(x)d_1^2\right]\big)w^2 + \cdots,
\end{aligned} \tag{23}
$$

for some analytic functions

$$
\begin{aligned}
p(z) &= c_1 z + c_2 z^2 + c_3 z^3 + \cdots (z \in \mathbb{U}), \\
q(w) &= d_1 w + d_2 w^2 + d_3 w^3 + \cdots (w \in \mathbb{U}),
\end{aligned} \tag{24}
$$

such that $p(0) = q(0) = 0$ and $|p(z)| < 1, |q(w)| < 1$ for all $z, w \in \mathbb{U}$.

It is fairly well known that if $|p(z)| < 1, (z \in \mathbb{U})$ and $|q(w)| < 1, (w \in \mathbb{U})$, then

$$
|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{25}
$$

Thus, upon comparing the corresponding coefficients in (22) and (23), we have

$$
(\mu+\lambda)a_2 = C_1^{\alpha}(x)c_1, \tag{26}
$$

$$
(\mu+2\lambda)\left[\left(\frac{\mu-1}{2}\right)a_2^2 + a_3\right] = C_1^{\alpha}(x)c_2 + C_2^{\alpha}(x)c_1^2, \tag{27}
$$

$$
-(\mu+\lambda)a_2 = C_1^{\alpha}(x)d_1, \tag{28}
$$

$$
(\mu+2\lambda)\left[\frac{\mu+3}{2}a_2^2 - a_3\right] = C_1^{\alpha}(x)d_2 + C_2^{\alpha}(x)d_1^2. \tag{29}
$$

It follows from (26) and (28) that

$$
c_1 = -d_1, \tag{30}
$$

$$
2(\mu+\lambda)^2 a_2^2 = \left[C_1^{\alpha}(x)\right]^2\left(c_1^2 + d_1^2\right). \tag{31}
$$

If we add (27) and (29), we get

$$
(\mu+2\lambda)(1+\mu)a_2^2 = C_1^{\alpha}(x)(c_2 + d_2) + C_2^{\alpha}(x)\left(c_1^2 + d_1^2\right). \tag{32}
$$

Substituting the value of $\left(c_1^2 + d_1^2\right)$ from (31) in the right-hand side of (32), we deduce that

$$
\left[(\mu+2\lambda)(1+\mu) - 2(\mu+\lambda)^2\frac{C_2^{\alpha}(x)}{\left[C_1^{\alpha}(x)\right]^2}\right]a_2^2 = C_1^{\alpha}(x)(c_2 + d_2). \tag{33}
$$

Moreover, computations using (23), (25), and (33), we find that

$$
|a_2| \leq \frac{2|\alpha| x \sqrt{2|\alpha| x}}{\sqrt{\left|\alpha(\mu+\lambda)^2 - 2\left[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)\right]x^2\right|}}. \tag{34}
$$

Moreover, if we subtract (29) from (27), we obtain

$$
2(\mu+2\lambda)\left(a_3 - a_2^2\right) = C_1^{\alpha}(x)(c_2 - d_2) + C_2^{\alpha}(x)\left(c_1^2 - d_1^2\right). \tag{35}
$$

Then, in view of (14) and (31), Eq. (35) becomes

$$
a_3 = \frac{\left[C_1^{\alpha}(x)\right]^2}{2(\mu+\lambda)^2}\left(c_1^2 + d_1^2\right) + \frac{C_1^{\alpha}(x)}{2(\mu+2\lambda)}(c_2 - d_2). \tag{36}
$$

Thus, applying (14), we conclude that

$$
|a_3| \leq \frac{4\alpha^2 x^2}{(\mu+\lambda)^2} + \frac{2|\alpha x|}{(\mu+2\lambda)}. \tag{37}
$$

4. Fekete-Szegö Problem for the Function Class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$

Fekete-Szegö inequality is one of the famous problems related to coefficients of univalent analytic functions. It was first given by [31], who stated that, if $f \in \Sigma$, then

$$
\left|a_3 - \eta a_2^2\right| \leq 1 + 2e^{-2\eta/(1-\mu)}. \tag{38}
$$

This bound is sharp when η is real.

In this section, we aim to provide Fekete-Szegö inequalities for functions in the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$. These inequalities are given in the following theorem.

Theorem 4. *Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$. Then,*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{2|\alpha| x}{(\mu+2\lambda)}, & |\eta - 1| \leq \left|\dfrac{\alpha(\mu+\lambda)^2 - 2\left[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)\right]x^2}{4\alpha(\mu+2\lambda)x^2}\right| \\ \dfrac{8|\alpha x|^3 |1-\eta|}{\left|\alpha(\mu+\lambda)^2 - 2\left[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)\right]x^2\right|}, & |\eta - 1| \geq \left|\dfrac{\alpha(\mu+\lambda)^2 - 2\left[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)\right]x^2}{4\alpha(\mu+2\lambda)x^2}\right|. \end{cases} \tag{39}$$

Proof. From (33) and (35)

$$\begin{aligned} a_3 - \eta a_2^2 &= (1-\eta)\frac{\left[C_1^{\alpha}(x)\right]^3 (c_2 + d_2)}{\left[(\mu+2\lambda)[1+\mu]\left[C_1^{\alpha}(x)\right]^2 - 2(\mu+\lambda)^2 C_2^{\alpha}(x)\right]} \\ &\quad + \frac{C_1^{\alpha}(x)}{2(\mu+2\lambda)}(c_2 - d_2) \\ &= C_1^{\alpha}(x)\left[\left[h(\eta) + \frac{1}{2(\mu+2\lambda)}\right]c_2 + \left[h(\eta) - \frac{1}{2(\mu+2\lambda)}\right]d_2\right], \end{aligned} \tag{40}$$

where

$$h(\eta) = \frac{\left[C_1^{\alpha}(x)\right]^2 (1-\eta)}{\left[(\mu+2\lambda)(1+\mu)\left[C_1^{\alpha}(x)\right]^2 - 2(\mu+\lambda)^2 C_2^{\alpha}(x)\right]}, \tag{41}$$

Then, in view of (14), we conclude that

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{\left|C_1^{\alpha}(x)\right|}{(\mu+2\lambda)} & 0 \leq |h(\eta)| \leq \dfrac{1}{2(\mu+2\lambda)}, \\ 2\left|C_1^{\alpha}(x)\right||h(\eta)| & |h(\eta)| \geq \dfrac{1}{2(\mu+2\lambda)}, \end{cases} \tag{42}$$

which completes the proof of Theorem 4.

5. Corollaries and Consequences

In this section, we apply our main results in order to deduce each of the following new corollaries and consequences.

Corollary 5. *Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, 1/2)$. Then,*

$$\begin{aligned} |a_2| &\leq \frac{x\sqrt{x}}{\sqrt{\left|1/2(\mu+\lambda)^2 - \left[3/2(\mu+\lambda)^2 - 1/2(\mu+2\lambda)(1+\mu)\right]x^2\right|}}, \\ |a_3| &\leq \frac{x^2}{(\mu+\lambda)^2} + \frac{x}{(\mu+2\lambda)}, \\ \left|a_3 - \eta a_2^2\right| &\leq \begin{cases} \dfrac{x}{(\mu+2\lambda)}, & |\eta - 1| \leq \left|\dfrac{1/2(\mu+\lambda)^2 - \left[3/2(\mu+\lambda)^2 - 1/2(\mu+2\lambda)(1+\mu)\right]x^2}{2(\mu+2\lambda)x^2}\right| \\ \dfrac{x^3|1-\eta|}{\left|1/2(\mu+\lambda)^2 - \left[3/2(\mu+\lambda)^2 - 1/2(\mu+2\lambda)(1+\mu)\right]x^2\right|}, & |\eta - 1| \geq \left|\dfrac{1/2(\mu+\lambda)^2 - \left[3/2(\mu+\lambda)^2 - 1/2(\mu+2\lambda)(1+\mu)\right]x^2}{2(\mu+2\lambda)x^2}\right|. \end{cases} \end{aligned} \tag{43}$$

Corollary 6. *Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_\Sigma^1(\lambda, x, \alpha) = \mathfrak{B}_\Sigma(\lambda, x, \alpha)$. Then,*

$$
\begin{aligned}
|a_2| &\le \frac{2|\alpha|x\sqrt{2|\alpha|x}}{\sqrt{\left|\alpha(1+\lambda)^2 - 2\left[\alpha(1+\lambda)^2(1+\alpha) - 2\alpha^2(1+2\lambda)\right]x^2\right|}},\\
|a_3| &\le \frac{4\alpha^2x^2}{(1+\lambda)^2} + \frac{2|\alpha|x}{(1+2\lambda)},\\
\left|a_3 - \eta a_2^2\right| &\le \begin{cases} \dfrac{2|\alpha|x}{(1+2\lambda)}, & |\eta-1| \le \left|\dfrac{\alpha(1+\lambda)^2 - 2\left[\alpha(1+\lambda)^2(1+\alpha) - 2\alpha^2(1+2\lambda)\right]x^2}{4\alpha(1+2\lambda)x^2}\right| \\[2ex] \dfrac{8|\alpha x|^3|1-\eta|}{\left|\alpha(1+\lambda)^2 - 2\left[\alpha(1+\lambda)^2(1+\alpha) - 2\alpha^2(1+2\lambda)\right]x^2\right|}, & |\eta-1| \ge \left|\dfrac{\alpha(1+\lambda)^2 - 2\left[\alpha(1+\lambda)^2(1+\alpha) - 2\alpha^2(1+2\lambda)\right]x^2}{4\alpha(1+2\lambda)x^2}\right|. \end{cases}
\end{aligned}
\tag{44}
$$

Corollary 7. *Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_\Sigma^\mu(1, x, \alpha) = \mathfrak{B}_\Sigma^\mu(x, \alpha)$. Then,*

$$
\begin{aligned}
|a_2| &\le \frac{2|\alpha|x\sqrt{2|\alpha|x}}{\sqrt{\left|\alpha(\mu+1)^2 - 2\left[\alpha(\mu+1)^2(1+\alpha) - \alpha^2(\mu+2)(1+\mu)\right]x^2\right|}},\\
|a_3| &\le \frac{4\alpha^2x^2}{(\mu+1)^2} + \frac{2|\alpha|x}{(\mu+2)},\\
\left|a_3 - \eta a_2^2\right| &\le \begin{cases} \dfrac{2|\alpha|x}{(\mu+2)}, & |\eta-1| \le \left|\dfrac{\alpha(\mu+1)^2 - 2\left[\alpha(\mu+1)^2(1+\alpha) - \alpha^2(\mu+2)(1+\mu)\right]x^2}{4\alpha(\mu+2)x^2}\right| \\[2ex] \dfrac{8|\alpha x|^3|1-\eta|}{\left|\alpha(\mu+1)^2 - 2\left[\alpha(\mu+1)^2(1+\alpha) - \alpha^2(\mu+2)(1+\mu)\right]x^2\right|}, & |\eta-1| \ge \left|\dfrac{\alpha(\mu+1)^2 - 2\left[\alpha(\mu+1)^2(1+\alpha) - \alpha^2(\mu+2)(1+\mu)\right]x^2}{4\alpha(\mu+2)x^2}\right|. \end{cases}
\end{aligned}
\tag{45}
$$

Corollary 8. *Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_\Sigma^0(1, x, \alpha) = \mathcal{S}_\Sigma^*(x, \alpha)$. Then,*

$$
\begin{aligned}
|a_2| &\le \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|1-2(1-\alpha)x^2|}},\\
|a_3| &\le 4\alpha^2x^2 + |\alpha|x,\\
\left|a_3 - \eta a_2^2\right| &\le \begin{cases} |\alpha|x, & |\eta-1| \le \left|\dfrac{1-2(1-\alpha)x^2}{8x^2}\right| \\[2ex] \dfrac{8\alpha^2|x^3 1-\eta|}{|1-2(1-\alpha)x^2|}, & |\eta-1| \ge \left|\dfrac{1-2(1-\alpha)x^2}{8x^2}\right|. \end{cases}
\end{aligned}
\tag{46}
$$

Corollary 9. *Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_\Sigma^1(1, x, \alpha) = \mathfrak{B}_\Sigma(x, \alpha)$. Then,*

$$
\begin{aligned}
|a_2| &\le \frac{|\alpha|x\sqrt{2x}}{\sqrt{|1-(2-\alpha)x^2|}},\\
|a_3| &\le \alpha^2x^2 + \frac{2|\alpha|x}{3},\\
\left|a_3 - \eta a_2^2\right| &\le \begin{cases} \dfrac{2|\alpha|x}{3}, & |\eta-1| \le \left|\dfrac{1-(2-\alpha)x^2}{3x^2}\right| \\[2ex] \dfrac{2\alpha^2x^3|1-\eta|}{|1-(2-\alpha)x^2|}, & |\eta-1| \ge \left|\dfrac{1-(2-\alpha)x^2}{3x^2}\right|. \end{cases}
\end{aligned}
\tag{47}
$$

6. Concluding Remark

By taking $\alpha = 1$ and specializing $\mu = 1$ or $\lambda = 1$, one can deduce the above results for various subclasses of Σ studied by Bulut et al. [29] and by Altinkaya and Yalcin [32].

Data Availability

Data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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Research Article

Sharp Bounds of the Coefficient Results for the Family of Bounded Turning Functions Associated with a Petal-Shaped Domain

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The goal of this article is to determine sharp inequalities of certain coefficient-related problems for the functions of bounded turning class subordinated with a petal-shaped domain. These problems include the bounds of first three coefficients, the estimate of Fekete-Szegö inequality, and the bounds of second- and third-order Hankel determinants.

1. Preliminary Concepts

Let the family of holomorphic (or analytic) functions in the region (or domain) of unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be described by the symbol $\mathscr{H}(\mathbb{D})$ and let $\mathscr{A}$ be the subfamily of $\mathscr{H}(\mathbb{D})$ which is defined by

$$\mathscr{A} = \left\{ f \in \mathscr{H}(\mathbb{D}) : f(z) = \sum_{k=1}^{\infty} a_k z^k \, (\text{with } a_1 = 1) \right\}. \tag{1}$$

Further, the set $\mathscr{S} \subset \mathscr{A}$ contains all normalized univalent functions in $\mathbb{D}$. For two functions $F_1, F_2 \in \mathscr{H}(\mathbb{D})$, we say that F_1 is subordinate to F_2, written symbolically by $F_1 \prec F_2$, if there exists a Schwarz function v with $v(0) = 0$ and $|v(z)| < 1$ that is analytic in $\mathbb{D}$ such that $f(z) = g(v(z)), z \in \mathbb{D}$. However, if F_2 is univalent in $\mathbb{D}$, then the following relation holds:

$$F_1(z) \prec F_2(z), (z \in \mathbb{D}) \Leftrightarrow F_1(0) = F_2(0) \text{ and } F_1(\mathbb{D}) \subset F_2(\mathbb{D}). \tag{2}$$

In geometric function theory, the most basic and important subfamilies of the set $\mathscr{S}$ are the family $\mathscr{S}^*$ of starlike functions and the family $\mathscr{C}$ of convex functions which are defined as follows:

$$\mathscr{C} = \left\{ f \in \mathscr{A} : \frac{\left(zf'(z)\right)'}{f'(z)} \prec \Lambda(z) \, (z \in \mathbb{D}) \right\} := \mathscr{C}(\Lambda),$$
$$\mathscr{S}^* = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \Lambda(z) \, (z \in \mathbb{D}) \right\} := \mathscr{S}^*(\Lambda), \tag{3}$$

with

$$\Lambda(z) = 1 + 2\sum_{n=2}^{\infty} z^n := \frac{1+z}{1-z}, (z \in \mathbb{D}). \tag{4}$$

By varying the function $\Lambda(z)$ in (18), we get some subfamilies of the set $\mathscr{S}^*$ which have significant geometric sense. For example,

(i) If we take $\Lambda(z) = (1 + Mz)/(1 + Nz)$ with $-1 \le N < M \le 1$, then the deduced family

$$\mathcal{S}^*[M,N] \equiv \mathcal{S}^*\left(\frac{1+Mz}{1+Nz}\right) \tag{5}$$

is described by the functions of the Janowski starlike family established in [1] and later studied in different directions in [2, 3]

(ii) The family $\mathcal{S}_L^* \equiv \mathcal{S}^*(\Lambda(z))$ with $\Lambda(z)=\sqrt{1+z}$ was developed in [4] by Sokól and Stankiewicz. The image of the function $\Lambda(z)=\sqrt{1+z}$ demonstrates that the image domain is bounded by the Bernoulli's lemniscate right-half plan specified by $|w^2-1|<1$

(iii) By selecting $\Lambda(z)=1+\sin z$, the class $\mathcal{S}^*(\Lambda(z))$ lead to the family $\mathcal{S}_{\sin}^*$ which was explored in [5] while $\mathcal{S}_e^* \equiv \mathcal{S}^*(e^z)$ has been produced in the article [6] and later studied in [7]

(iv) The family $\mathcal{S}_c^* := \mathcal{S}^*(\Lambda(z))$ with $\Lambda(z)=1+(4/3)z+(2/3)z^2$ was contributed by Sharma and his coauthors [8] which contains function $f \in \mathcal{A}$ such that $z f'(z)/f(z)$ is located in the region bounded by the cardioid given by

$$\left(9x^2+9y^2-18x+5\right)^2-16\left(9x^2+9y^2-6x+1\right)=0 \tag{6}$$

(v) The family $\mathcal{S}_R^* \equiv \mathcal{S}^*(\Lambda(z))$ with $\Lambda(z)=1+(z/(\sqrt{2}+1))((\sqrt{2}+1+z)/(\sqrt{2}+1-z))$ is studied in [9] while $\mathcal{S}_{\cos}^* := \mathcal{S}^*(\cos(z))$ and $\mathcal{S}_{\cosh}^* := \mathcal{S}^*(\cosh(z))$ were recently examined by Bano and Raza [10] and Alotaibi et al. [11], respectively

(vi) If we consider $\Lambda(z)=1\sinh^{-1}z$, then the class $\mathcal{S}_\rho^* := \mathcal{S}^*(1+\sinh^{-1}z)$ was provided by Kumar and Arora [12] and is defined as a function $f \in \mathcal{A}$ which is in the family $\mathcal{S}_\rho^*$ if (18) holds for the function $\Lambda(z)=\rho(z)$, where

$$\rho(z)=1+\sinh^{-1}z \tag{7}$$

Clearly, the function $\rho(z)$ is a multivalued function and has the branch cuts about the line segments $(-i\infty,-i)\cup(i,i\infty)$, on the imaginary axis, and hence, it is holomorphic in $\mathbb{D}$. In a geometric point of view, the function $\rho(z)$ maps the unit disc $\mathbb{D}$ onto a petal-shaped region Ω_ρ,

$$\Omega_\rho=\{w\in\mathbb{C}:|\sinh(w-1)|<1\}. \tag{8}$$

Using this idea, we now consider a subfamily $\mathcal{BT}_s$ of analytic functions as

$$\mathcal{BT}_s=\left\{f\in\mathcal{A}:f'(z)\prec\tilde{\Lambda}(z),\text{ and }\tilde{\Lambda}(z)\text{ is given by (8)}\right\}. \tag{9}$$

If we take the function $\Lambda(z)$, given by (4), instead of $\tilde{\Lambda}(z)$ in (9), we get the familiar class $\mathcal{R}$ of bounded turning functions. From the statement of the Nashiro-Warschowski theorem, it follows that the functions in $\mathcal{R}$ are univalent in $\mathbb{D}$. The properties of this class was studied extensively by the researchers, see [13–16].

The Hankel determinant $\mathcal{HD}_{q,n}(f)$(with $q,n\in\mathbb{N}=\{1,2,\cdots\}$ and $a_1=1$) for a function $f\in\mathcal{S}$ of the series form (1) was given by Pommerenke [17, 18] as

$$\mathcal{HD}_{q,n}(f):=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{10}$$

Specifically, the first-, second-, and third-order Hankel determinants, respectively, are

$$\begin{aligned} \mathcal{HD}_{2,1}(f)&=\begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix}=a_3-a_2^2, \\ \mathcal{HD}_{2,2}(f)&=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2, \\ \mathcal{HD}_{3,1}(f)&=\begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &=a_3\left(a_2a_4-a_3^2\right)-a_4\left(a_4-a_2a_3\right)+a_5\left(a_3-a_2^2\right). \end{aligned} \tag{11}$$

In literature, there are relatively few findings in relation to the Hankel determinant for the function f belongs to the general family $\mathcal{S}$. For the function $f\in\mathcal{S}$, the best established sharp inequality is $|\mathcal{HD}_{2,n}(f)|\le\lambda\sqrt{n}$, where λ is absolute constant, which is due to Hayman [19]. Further, for the same class $\mathcal{S}$, it was obtained in [20] that

$$\begin{aligned} \left|\mathcal{HD}_{2,2}(f)\right|&\le\lambda,\text{ for }1\le\lambda\le\frac{11}{3}, \\ \left|\mathcal{HD}_{3,1}(f)\right|&\le\mu,\text{ for }\frac{4}{9}\le\mu\le\frac{32+\sqrt{285}}{15}. \end{aligned} \tag{12}$$

The growth of $|\mathcal{HD}_{q,n}(f)|$ has often been evaluated for different subfamilies of the set $\mathcal{S}$ of univalent functions. For example, the sharp bound of $|\mathcal{HD}_{2,2}(f)|$, for the subfamilies $\mathcal{C}$, $\mathcal{S}^*$, and $\mathcal{R}$ of the set $\mathcal{S}$, was measured by Janteng et al. [21, 22]. These bounds are

$$\left|\mathscr{H}\mathscr{D}_{2,2}(f)\right| \le \begin{cases} \dfrac{1}{8}, & \text{for } f \in \mathscr{C}, \\ 1, & \text{for } f \in \mathscr{S}^*, \\ \dfrac{4}{9}, & \text{for } f \in \mathscr{R}. \end{cases} \tag{13}$$

The exact bound for the collection of close-to-convex functions of such a specific determinant is still unavailable (see [23]). On the other hand, for the set of Bazilevič functions, the best estimate of $|\mathscr{H}\mathscr{D}_{2,2}(f)|$ was proved by Krishna and RamReddy [24]. For more work on $|\mathscr{H}\mathscr{D}_{2,2}(f)|$, see References [25–29].

It is very obvious from the formulae provided in (11) that the estimate of $|\mathscr{H}\mathscr{D}_{3,1}(f)|$ is far more complicated compared with finding the bound of $|\mathscr{H}\mathscr{D}_{2,2}(f)|$. In the first paper on $|\mathscr{H}\mathscr{D}_{3,1}(f)|$, published in 2010, Babalola [30] obtained the upper bound of $|\mathscr{H}\mathscr{D}_{3,1}(f)|$ for the families of $\mathscr{C}$, $\mathscr{S}^*$, and $\mathscr{R}$. He obtained the following bounds:

$$\left|\mathscr{H}\mathscr{D}_{3,1}(f)\right| \le \begin{cases} 0.714\cdots, & \text{for } f \in \mathscr{C}, \\ 16, & \text{for } f \in \mathscr{S}^*, \\ 0.742\cdots, & \text{for } f \in \mathscr{R}. \end{cases} \tag{14}$$

Later on, using the same methodology, some other authors [31–35] published their work concerning $|\mathscr{H}\mathscr{D}_{3,1}(f)|$ for different subfamilies of analytic and univalent functions. In 2017, Zaprawa [36] improved Babalola's [30] results by applying a new technique which is given as

$$\left|\mathscr{H}\mathscr{D}_{3,1}(f)\right| \le \begin{cases} \dfrac{49}{540}, & \text{for } f \in \mathscr{C}, \\ 1, & \text{for } f \in \mathscr{S}^*, \\ \dfrac{41}{60}, & \text{for } f \in \mathscr{R}. \end{cases} \tag{15}$$

He argues that such limits are indeed not the best ones. After that, in 2018, Kwon et al. [37] enhanced Zaprawa's bound for $f \in \mathscr{S}^*$ and showed that $|\mathscr{H}\mathscr{D}_{3,1}(f)| \le 8/9$, but it is still not the best possible. The firstly examined papers in which the authors obtained the sharp bounds of $|\mathscr{H}\mathscr{D}_{3,1}(f)|$ came to the reader's hands in 2018. Such papers have been written by Kowalczyk et al. [38] and Lecko et al. [39]. These results are given as

$$\left|\mathscr{H}\mathscr{D}_{3,1}(f)\right| \le \begin{cases} \dfrac{4}{135}, & \text{for } f \in \mathscr{C}, \\ \dfrac{1}{9}, & \text{for } f \in \mathscr{S}^*\left(\dfrac{1}{2}\right), \end{cases} \tag{16}$$

where $\mathscr{S}^*(1/2)$ indicates the starlike function family of order 1/2. We would also like to acknowledge the research provided by Mahmood et al. [40] in which they examined the third Hankel determinant in the q-analog for a subfamily of starlike functions and for more contribution of such type families, see [41, 42]. In the present article, our aim is to calculate the sharp bounds of some of the problems related to Hankel determinant for the class $\mathscr{B}\mathscr{T}_s$ of bounded turning functions connected with a petal-shaped domain.

2. A Set of Lemmas

Definition 1. Let $\mathscr{P}$ represent the class of all functions p that are holomorphic in $\mathbb{D}$ with $\mathfrak{Re}(p(z)) > 0$ and has the series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \; (z \in \mathbb{D}). \tag{17}$$

For the proofs of our key findings, we need the following lemma. It contains the well-known formula for c_2, see [43], the formula for c_3 due to Libera and Zlotkiewicz [44], and the formula for c_4 proved in [45].

Lemma 2. *Let $p \in \mathscr{P}$ has the series form ((17)). Then, for x, $\sigma, \rho \in \bar{\mathbb{D}} = \mathbb{D} \cup \{1\}$,*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{18}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\sigma, \tag{19}$$

$$\begin{aligned} 8c_4 = c_1^4 + \left(4 - c_1^2\right)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4\left(4 - c_1^2\right) \\ \cdot \left(1 - |x|^2\right)\left[c(x - 1)z + \bar{x}\sigma^2 - \left(1 - |\sigma|^2\right)\rho\right]. \end{aligned} \tag{20}$$

Lemma 3. *If $p \in \mathscr{P}$ and has the series form ((17)), then*

$$|c_{n+k} - \mu c_n c_k| \le 2 \max\left(1, |2\mu - 1|\right), \tag{21}$$

$$|c_n| \le 2 \text{ for } n \ge 1, \tag{22}$$

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \le 2|J| + |K - 2J| + 2|J - K + L|, \tag{23}$$

with $J, K, L, \mu \in \mathbb{C}$ and for $B \in [0, 1]$ with $B(2B - 1) \le D \le B$, we have

$$\left|c_3 - 2Bc_1c_2 + Dc_1^3\right| \le 2. \tag{24}$$

The inequalities (21), (22), (23), and (24) in the above lemma are taken from [43, 46], [47–49], and [50], respectively.

3. Coefficient Inequalities for the Class $\mathscr{B}\mathscr{T}_s$

We begin this section by finding the absolute values of the first three initial coefficients for the function of class $\mathscr{B}\mathscr{T}_s$.

Theorem 4. *If $f \in \mathcal{BT}_s$ and has the series representation ((1)), then*

$$|a_2| \le \frac{1}{2},$$
$$|a_3| \le \frac{1}{3}, \quad (25)$$
$$|a_4| \le \frac{1}{4}.$$

These bounds are sharp.

Proof. Let $f \in \mathcal{BT}_s$. Then, (9) can be written in the form of the Schwarz function as

$$f'(z) = 1 + \sinh^{-1}(w(z)), (z \in \mathbb{D}). \quad (26)$$

Now, if $p \in \mathcal{P}$, then it may be written in terms of the Schwarz function w by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad (27)$$

equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}. \quad (28)$$

From (1), we easily get

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots. \quad (29)$$

By simplification and using the series expansion (28), we obtain

$$\begin{aligned} 1 + \sinh^{-1}(w(z)) = 1 &+ \left(\frac{1}{2}c_1\right)z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 \\ &+ \left(\frac{1}{2}c_3 + \frac{5}{48}c_1^3 - \frac{1}{2}c_1 c_2\right)z^3 \\ &\cdot \left(\frac{1}{2}c_4 - \frac{1}{4}c_2^2 - \frac{1}{32}c_1^4 + \frac{5}{16}c_1^2 c_2 - \frac{1}{2}c_1 c_3\right)z^4 + \cdots. \end{aligned} \quad (30)$$

Comparing (29) and (30),we get

$$a_2 = \frac{1}{4}c_1, \quad (31)$$

$$a_3 = \frac{1}{3}\left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right), \quad (32)$$

$$a_4 = \frac{1}{4}\left(\frac{1}{2}c_3 + \frac{5}{48}c_1^3 - \frac{1}{2}c_1 c_2\right), \quad (33)$$

$$a_5 = \frac{1}{5}\left(\frac{1}{2}c_4 - \frac{1}{4}c_2^2 - \frac{1}{32}c_1^4 + \frac{5}{16}c_1^2 c_2 - \frac{1}{2}c_1 c_3\right). \quad (34)$$

For a_2, implementing (22) in (31), we obtain

$$|a_2| \le \frac{1}{2}. \quad (35)$$

For a_3, reordering (32), we get

$$a_3 = \frac{1}{6}\left(c_2 - \frac{1}{2}c_1 c_1\right), \quad (36)$$

and using (21), we have

$$|a_3| \le \frac{1}{3}. \quad (37)$$

For a_4, we can rewrite (33) as

$$|a_4| = \frac{1}{4}\left|\frac{5}{48}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3\right|. \quad (38)$$

Application of triangle inequality plus (23), we get

$$|a_4| \le \frac{1}{4}\left[2\left|\frac{5}{48}\right| + 2\left|\frac{1}{2} - 2\left(\frac{5}{48}\right)\right| + 2\left|\frac{5}{48} - \frac{1}{2} + \frac{1}{2}\right|\right]. \quad (39)$$

By simple calculations, we obtain

$$|a_4| \le \frac{1}{4}. \quad (40)$$

These outcomes are sharp. For this, we consider a function

$$f_n'(z) = 1 + \sinh^{-1}(z^n), \text{for } n = 1, 2, 3. \quad (41)$$

Thus, we have

$$\begin{aligned} f_1(z) &= \int_0^z \left(1 + \sinh^{-1}(t)\right)dt = z + \frac{1}{2}z^2 - \frac{1}{24}z^4 + \cdots, \\ f_2(z) &= \int_0^z \left(1 + \sinh^{-1}(t^2)\right)dt = z + \frac{1}{3}z^3 - \frac{1}{42}z^7 + \cdots, \\ f_3(z) &= \int_0^z \left(1 + \sinh^{-1}(t^3)\right)dt = z + \frac{1}{4}z^4 - \frac{1}{60}z^{10} + \cdots. \end{aligned} \quad (42)$$

Now, we discussed about the Hankel determinant problem, which is explicitly related to the Fekete-Szegö functional which is an extraordinary instance of the Hankel determinant.

Theorem 5. *If f of the form ((1)) belongs to $\mathcal{BT}_s$, then*

$$\left|a_3 - \gamma a_2^2\right| \le \max\left\{1, \frac{3|\gamma|}{4}\right\}, \text{for } \gamma \in \mathbb{C}. \quad (43)$$

This inequality is sharp.

Proof. Employing (31) and (32), we may write

$$\left|a_3-\gamma a_2^2\right|=\left|\frac{c_2}{6}-\frac{c_1^2}{12}-\gamma\frac{c_1^2}{16}\right|. \tag{44}$$

By rearranging, it yields

$$\left|a_3-\gamma a_2^2\right|=\frac{1}{6}\left|\left(c_2-\left(\frac{3\gamma+4}{8}\right)c_2\right)\right|. \tag{45}$$

Application of (21) leads us to

$$\left|a_3-\gamma a_2^2\right|\le\frac{1}{6}\max\left\{2,2\left|2\frac{3\gamma+4}{8}-1\right|\right\}. \tag{46}$$

After the simplification, we obtain

$$\left|a_3-\gamma a_2^2\right|\le\frac{1}{3}\max\left\{1,\frac{3|\gamma|}{4}\right\}. \tag{47}$$

The required result is sharp and is determined by

$$f_2(z)=\int_0^z\left(1+\sinh^{-1}\left(t^2\right)\right)dt=z+\frac{1}{3}z^3-\frac{1}{42}z^7+\cdots. \tag{48}$$

Theorem 6. *If f has the form ((1)) belongs to $\mathscr{BT}_s$, then*

$$\left|a_2a_3-a_4\right|\le\frac{1}{4}. \tag{49}$$

This inequality is sharp.

Proof. Using (31), (32), and (33), we have

$$\left|a_2a_3-a_4\right|=\frac{1}{8}\left|c_3-2\left(\frac{2}{3}\right)c_1c_2+\frac{7}{24}c_1^3\right|. \tag{50}$$

From (24), we have

$$0\le B=\frac{2}{3}\le 1, B=\frac{2}{3}\ge D=\frac{7}{24}, \tag{51}$$

and also satisfy

$$B(2B-1)=\frac{2}{3}\left(\frac{4}{3}-1\right)\le D=\frac{7}{24}. \tag{52}$$

Thus, by using (24), we have

$$\left|a_2a_3-a_4\right|\le\frac{1}{4}. \tag{53}$$

Equality is achieved by using

$$f_3(z)=\int_0^z\left(1+\sinh^{-1}\left(t^3\right)\right)dt=z+\frac{1}{4}z^4-\frac{1}{60}z^{10}+\cdots. \tag{54}$$

Next, we will determine the second-order Hankel determinant $\mathscr{HD}_{2,2}(f)$ for $f\in\mathscr{BT}_s$.

Theorem 7. *If f belongs to $\mathscr{BT}_s$, then the second Hankel determinant*

$$\left|\mathscr{HD}_{2,2}(f)\right|=\left|a_2a_4-a_3^2\right|\le\frac{1}{9}. \tag{55}$$

This result is the best possible.

Proof. From (31), (32), and (33), we have

$$\mathscr{HD}_{2,2}(f)=-\frac{1}{2304}c_1^4-\frac{1}{288}c_1^2c_2+\frac{1}{32}c_1c_3-\frac{1}{36}c_2^2. \tag{56}$$

Using (18) and (19) to express c_2 and c_3 in terms of c_1 and noting that without loss in generality we can write $c_1=c$, with $0\le c\le 2$, we obtain

$$\begin{aligned}\left|\mathscr{HD}_{2,2}(f)\right|=&\left|-\frac{1}{768}c^4-\frac{1}{128}c^2\left(4-c^2\right)x^2-\frac{1}{144}\left(4-c^2\right)^2x^2\right.\\&\left.+\frac{1}{64}c\left(4-c^2\right)\left(1-|x|^2\right)z\right|,\end{aligned} \tag{57}$$

with the aid of the triangle inequality and replacing $|z|\le 1$, $|x|=b$, with $b\le 1$. So,

$$\begin{aligned}\left|\mathscr{HD}_{2,2}(f)\right|\le&\frac{1}{768}c^4+\frac{1}{128}c^2\left(4-c^2\right)b^2+\frac{1}{144}\left(4-c^2\right)^2b^2\\&+\frac{1}{64}b\left(4-c^2\right)\left(1-b^2\right)\coloneqq\phi(c,b).\end{aligned} \tag{58}$$

It is a simple exercise to show that $\phi'(c,b)\ge 0$ on $[0,1]$, so that $\phi(c,b)\le\phi(c,1)$. Putting $b=1$ gives

$$\left|\mathscr{HD}_{2,2}(f)\right|\le\frac{1}{768}c^4+\frac{1}{128}c^2\left(4-c^2\right)+\frac{1}{144}\left(4-c^2\right)^2\coloneqq\phi(c,1). \tag{59}$$

Also, $\phi'(c,1)<0$, and so $\phi(c,1)$ is a decreasing function. Thus, the maximum value at $c=0$ is

$$\left|\mathscr{HD}_{2,2}(f)\right|\le\frac{16}{144}=\frac{1}{9}. \tag{60}$$

The required second Hankel determinant is sharp and is obtained by

$$f_2(z)=\int_0^z\left(1+\sinh^{-1}\left(t^2\right)\right)dt=z+\frac{1}{3}z^3-\frac{1}{42}z^7+\cdots. \tag{61}$$

4. Third-Order Hankel Determinant

We will now determine the third-order Hankel determinant $\mathscr{HD}_{3,1}(f)$ for $f\in\mathscr{BT}_s$.

Theorem 8. *If f belongs to $\mathscr{BT}_s$, then the third Hankel determinant*

$$\left|\mathscr{HD}_{3,1}(f)\right| \leq \frac{1}{16}. \tag{62}$$

This result is sharp.

Proof. The third Hankel determinant can be written as

$$\mathscr{HD}_{3,1}(f) = \left(a_2 a_4 - a_3^2\right) a_3 - \left(a_1 a_4 - a_2 a_3\right) a_4 + \left(a_1 a_3 - a_2^2\right) a_5. \tag{63}$$

After simplification of the above equation, we have

$$\mathscr{HD}_{3,1}(f) = 2a_2 a_3 a_4 - a_3^3 - a_4^2 + a_3 a_5 - a_2^2 a_5. \tag{64}$$

Let $c_1 = c$ and putting the estimations of a_i's from (31), (32), (33), and (34), we get

$$\begin{aligned}\mathscr{HD}_{3,1}(f) = \frac{1}{552960}\big(&-151c^6 + 144c^4 c_2 + 1584c^3 c_3 - 768c^2 c_2^2 \\ &- 8064c^2 c_4 + 13824 c c_2 c_3 - 7168 c_2^3 + 9216 c_2 c_4 - 8640 c_3^2\big).\end{aligned} \tag{65}$$

To simplify computation, let $t = 4 - c^2$ in (18), (19), and (20). Now, using the simplified form of these formulae, we obtain

$$\begin{aligned}
144c^4 c_2 &= 72\left(c^6 + c^4 t x\right),\\
1584c^3 c_3 &= 396c^6 + 792c^4 t x - 396c^4 t x^2 + 792c^3 t\left(1 - |x|^2\right)\sigma,\\
768c^2 c_2^2 &= 192c^6 + 384c^4 t x + 192c^2 t^2 x^2,\\
8064c^2 c_4 &= 1008c^4 t x^3 - 4032c^3 t x\left(1 - |x|^2\right)\sigma\\
&\quad - 4032c^2 t\bar{x}\left(1 - |x|^2\right)\sigma^2 - 3024c^4 t x^2\\
&\quad + 4032c^2 t\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho + 4032c^3 t\left(1 - |x|^2\right)\sigma\\
&\quad + 3024c^4 t x + 1008c^6 + 4032c^2 t x^2,\\
13824 c c_2 c_3 &= -1728c^2 t^2 x^3 - 1728c^4 t x^2 + 3456 c t^2 x\left(1 - |x|^2\right)\sigma\\
&\quad + 3456c^2 t^2 x^2 + 3456c^3 t\left(1 - |x|^2\right)\sigma + 5184c^4 t x\\
&\quad + 1728c^6,\\
7168c_2^3 &= 896t^3 x^3 + 2688c^2 t^2 x^2 + 2688c^4 t x + 896c^6,\\
9216c_2 c_4 &= 2304c^2 t x^2 + 2304t^2 x^3 + 576c^6 + 2304c^4 t x\\
&\quad + 2304c^3 t\left(1 - |x|^2\right)\sigma + 2304c^2 t\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho\\
&\quad + 1728c^2 t^2 x^2 + 2304 c t^2 x\left(1 - |x|^2\right)\sigma\\
&\quad + 2304t^2 x\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho - 1728c^4 t x^2\\
&\quad - 2304c^2 t\bar{x}\left(1 - |x|^2\right)\sigma^2 - 2304c^3 t x\left(1 - |x|^2\right)\sigma\\
&\quad - 1728c^2 t^2 x^3 - 2304t^2 x\bar{x}\left(1 - |x|^2\right)\sigma^2 + 576c^4 t x^3\\
&\quad + 576c^2 t^2 x^4 - 2304 c t^2 x^2\left(1 - |x|^2\right)\sigma,
\end{aligned}$$

$$\begin{aligned}
8640c_3^2 &= 540c^2 t^2 x^4 - 2160 c t^2 x^2\left(1 - |x|^2\right)\sigma - 2160c^2 t^2 x^3\\
&\quad - 1080c^4 t x^2 + 2160t^2\left(1 - |x|^2\right)^2\sigma^2\\
&\quad + 4320 c t^2 x\left(1 - |x|^2\right)\sigma + 2160c^2 t^2 x^2 + 2160c^3 t\left(1 - |x|^2\right)\sigma\\
&\quad + 2160c^4 t x + 540c^6.
\end{aligned} \tag{66}$$

Substituting these expressions in (65), by simple but too long computation,

$$\begin{aligned}
\mathscr{HD}_{3,1}(f) = \frac{1}{552960}\big\{&-15c^6 + 2304t^2 x^3 - 896t^3 x^3 - 1728c^2 t x^2\\
&- 432c^4 t x^3 + 252c^4 t x^2 + 96c^4 t x + 36c^2 t^2 x^4\\
&- 1296c^2 t^2 x^3 + 144c^2 t^2 x^2 - 2160t^2\left(1 - |x|^2\right)^2\sigma^2\\
&+ 360c^3 t\left(1 - |x|^2\right)\sigma + 1728c^3 t x\left(1 - |x|^2\right)\sigma\\
&+ 1728c^2 t\bar{x}\left(1 - |x|^2\right)\sigma^2 - 1728c^2 t\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho\\
&- 144 c t^2 x^2\left(1 - |x|^2\right)\sigma - 2304t^2 x\bar{x}\left(1 - |x|^2\right)\sigma^2 + 1440 c t^2 x\left(1 - |x|^2\right)\sigma\\
&+ 2304t^2 x\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho\big\}.
\end{aligned} \tag{67}$$

Since $t = 4 - c^2$,

$$\mathscr{HD}_{3,1}(f) = \frac{1}{552960}\left(v_1(c, x) + v_2(c, x)\sigma + v_3(c, x)\sigma^2 + \Psi(c, x, \sigma)\rho\right), \tag{68}$$

where $\rho, x, \sigma \in \bar{\mathbb{D}}$, and

$$\begin{aligned}
v_1(c, x) &= -15c^6 + \left(4 - c^2\right)\big[\left(4 - c^2\right)\left(-1280x^3 - 400c^2 x^3 + 36c^2 x^4\right.\\
&\qquad \left. + 144c^2 x^2\right) - 1728c^2 x^2 - 432c^4 x^3 + 252c^4 x^2 + 96c^4 x\big],\\
v_2(c, x) &= \left(4 - c^2\right)\left(1 - |x|^2\right)\big[\left(4 - c^2\right)\left(1440 c x - 144 c x^2\right)\\
&\qquad + 1728c^3 x + 360c^3\big],\\
v_3(c, x) &= \left(4 - c^2\right)\left(1 - |x|^2\right)\left[\left(4 - c^2\right)\left(-144x^2 - 2160\right) + 1728c^2\bar{x}\right],\\
\Psi(c, x, z) &= \left(4 - c^2\right)\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\left[-1728c^2 + 2304x\left(4 - c^2\right)\right].
\end{aligned} \tag{69}$$

Now, by using $|x| = x, |\sigma| = y$ and utilizing the fact $|\rho| \leq 1$, we get

$$\begin{aligned}\left|\mathscr{HD}_{3,1}(f)\right| &\leq \frac{1}{552960}\big(|v_1(c, x)| + |v_2(c, x)|y + |v_3(c, x)|y^2\\ &\quad + |\Psi(c, x, \sigma)|\big),\end{aligned} \tag{70}$$

$$\leq \frac{1}{552960} G(c, x, y), \tag{71}$$

where

$$G(c, x, y) = g_1(c, x) + g_2(c, x)y + g_3(c, x)y^2 + g_4(c, x)\left(1 - y^2\right), \tag{72}$$

with

$$g_1(c,x)=15c^6+(4-c^2)\left[(4-c^2)(1280x^3+400c^2x^3+36c^2x^4+144c^2x^2)+1728c^2x^2+432c^4x^3+252c^4x^2+96c^4x\right],$$

$$g_2(c,x)=(4-c^2)(1-x^2)\left[(4-c^2)(1440cx+144cx^2)+1728c^3x+360c^3\right],$$

$$g_3(c,x)=(4-c^2)(1-x^2)\left[(4-c^2)(144x^2+2160)+1728c^2\bar{x}\right],$$

$$g_4(c,x)=(4-c^2)(1-x^2)\left[1728c^2+2304x(4-c^2)\right]. \tag{73}$$

Clearly, in the last four functions, $g_2(c,x)$ and $g_3(c,x)$ are nonnegative in the interval $[0,2]\times[0,1]$. So from (70) along with $y=|\sigma|$ in the interval $[0,1]$, we get

$$G(c,x,y)=G(c,x,1). \tag{74}$$

Therefore,

$$G(c,x,1)=g_1(c,x)+g_2(c,x)+g_3(c,x)+g_4(c,x)=F(c,x). \tag{75}$$

Here, we shall maximize $F(c,x)$ over the interval $[0,2]\times[0,1]$. For this purpose, we consider possible cases:

(i) By taking $x=0$, we have

$$F(c,0)=15c^6-360c^5+2160c^4+144c^3-17280c^2+34560=f_1(c). \tag{76}$$

Since $f_1'(c)<0$ in $[0,2]$, so, $f_1(c)$ is decreasing over $[0,2]$. Thus, $f_1(c)$ has its maxima at $c=0$ which is equal to 34560

(ii) By taking $x=1$, we have

$$F(c,1)=-185c^6-1968c^4+5952c^2+20480=f_2(c). \tag{77}$$

As $f_2'(c)=0$ has its optimum point at $c_0=0.674788$. Therefore, $f_2(c)$ is an increasing function for $c\le c_0$ and decreasing for $c_0\le c$. Hence, $f_2(c)$ has its maxima at $c=c_0$ that is approximately equal to 22764.68167

(iii) By taking $c=0$, we have

$$F(0,x)=-2304x^4+20480x^3-32256x^2+34560=f_3(x). \tag{78}$$

As $f_3'(x)<0$ over $[0,1]$, so, $f_3(x)$ is decreasing over $[0,1]$. Thus, $f_3(x)$ has its maxima at $x=0$ which is equal to 34560. Now, by taking $c=2$, we obtain

$$F(2,x)\le 960 \tag{79}$$

(iv) When (c,x) lies in $[0,2]\times[0,1]$. Then, some simple computation shows that there exists real solution for these equations

$$\frac{\partial F(c,x)}{\partial x}=0,\qquad \frac{\partial F(c,x)}{\partial c}=0, \tag{80}$$

lies inside this region $[0,2]\times[0,1]$ at $(c,x)\approx(0,0)$. Consequently, we obtain

$$F(c,x)=34560. \tag{81}$$

Thus, from all the above cases, we conclude that

$$G(c,x,y)\le 34560 \text{ on } [0,2]\times[0,1]\times[0,1]. \tag{82}$$

From (71), we can write

$$\left|\mathscr{H}\mathscr{D}_{3,1}(f)\right|\le\frac{1}{552960}G(c,x,y)\le\frac{1}{16}\approx 0.0625. \tag{83}$$

If $f\in\mathscr{B}\mathscr{T}_s$, then the equality is obtained from the function

$$f_3(z)=\int_0^z\left(1+\sinh^{-1}(t^3)\right)dt=z+\frac{1}{4}z^4-\frac{1}{60}z^{10}+\cdots. \tag{84}$$

5. Conclusion

For the family of bounded turning functions connected with a petal-shaped domain, we studied the problems such as the bounds of the first three coefficients, the estimate of the Fekete-Szegö inequality, and the bounds of Hankel determinants of order three. All the bounds which we investigated are sharp.

Data Availability

We have not used any data.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally in this research paper.

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Research Article

Hermite–Hadamard Type Inequalities via Generalized Harmonic Exponential Convexity and Applications

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In this work, we introduce the idea and concept of m–polynomial p–harmonic exponential type convex functions. In addition, we elaborate the newly introduced idea by examples and some interesting algebraic properties. As a result, several new integral inequalities are established. Finally, we investigate some applications for means. The amazing techniques and wonderful ideas of this work may excite and motivate for further activities and research in the different areas of science.

1. Introduction

Theory of convexity present an active, fascinating, and attractive field of research and also played prominence and amazing act in different fields of science, namely, mathematical analysis, optimization, economics, finance, engineering, management science, and game theory. Many researchers endeavor, attempt, and maintain his work on the concept of convex functions and extend and generalize its variant forms in different ways using innovative ideas and fruitful techniques. Convexity theory provides us with a unified framework to develop highly efficient, interesting, and powerful numerical techniques to tackle and to solve a wide class of problems which arise in pure and applied sciences. In recent years, the concept of convexity has been improved, generalized, and extended in many directions. The concept of convex functions also played prominence and meaningful act in the advancement of the theory of inequalities. A number of studies have shown that the theory of convex functions has a close relationship with the theory of inequalities.

The integral inequalities are useful in optimization theory, functional analysis, physics, and statistical theory. In diverse and opponent research, inequalities have a lot of applications in statistical problems, probability, and numerical quadrature formulas [1–3]. So eventually due to many generalizations, variants, extensions, widespread views, and applications, convex analysis and inequalities have become an attractive, interesting, and absorbing field for the researchers and for attention; the reader can refer to [4–6]. Recently Kadakal and Iscan [7] introduced a generalized form of convexity, namely, n–polynomial convex functions.

It is well known that the harmonic mean is the special case of power mean. It is often used for the situations when the average rates is desired and have a lot of applications in different field of sciences which are statistics, computer science, trigonometry, geometry, probability, finance, and electric circuit theory. Harmonic mean is the most appropriate measure for rates and ratios because it equalizes the weights of each data point. Harmonic mean is used to define the harmonic convex set. In 2003, first time harmonic convex set was introduced by Shi [8]. Harmonic and p–harmonic convex function was for the first time introduced and discussed by Anderson et al. [9] and Noor et al. [10], respectively. Awan et al. [11] keeping his work on generalizations, introduced a new class called n–polynomial harmonically convex function. Motivated and inspired by the ongoing activities and research in the convex analysis field, we found out that there exists a special class of function known as exponential convex function, and nowadays, a lot of people working are in this field [12, 13]. Dragomir [14] introduced the class of exponential type convexity. After Dragomir, Awan et al. [15] studied and investigated a new class of exponentially convex functions. Kadakal and İşcan introduced a new definition of exponential type convexity in [16]. Recently, Geo et al. [17] introduced n–polynomial harmonic exponential type convex functions. The fruitful benefits and applications of exponential type convexity is used to manipulate for statistical learning, information sciences, data mining, stochastic optimization and sequential prediction [7, 18, 19] and the references therein. Before we start, we need the following necessary known definitions and literature.

2. Preliminaries

In this section, we recall some known concepts.

Definition 1 (see [5]). Let $\psi : I \longrightarrow \mathbb{R}$ be a real valued function. A function ψ is said to be convex, if

$$\psi(\kappa\wp_1 + (1-\kappa)\wp_2) \le \kappa\psi(\wp_1) + (1-\kappa)\psi(\wp_2), \tag{1}$$

holds for all $\wp_1, \wp_2 \in I$ and $\kappa \in [0, 1]$.

Definition 2 (see [20]). A function $\psi : I \subseteq (0,\infty) \longrightarrow \mathbb{R}$ is said to be harmonic convex, if

$$\psi\left(\frac{\wp_1\wp_2}{\kappa\wp_2 + (1-\kappa)\wp_1}\right) \le \kappa\psi(\wp_1) + (1-\kappa)\psi(\wp_2), \tag{2}$$

holds for all $\wp_1, \wp_2 \in I$ and $\kappa \in [0, 1]$.

For the harmonic convex function, İşcan [20] provided the Hermite–Hadamard type inequality.

Definition 3 (see [21]). A function $\psi : I \longrightarrow \mathbb{R}$ is said to be p–harmonic convex, if

$$\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p + (1-\kappa)\wp_1^p}\right]^{1/p}\right) \le \kappa\psi(\wp_1) + (1-\kappa)\psi(\wp_2), \tag{3}$$

holds for all $\wp_1, \wp_2 \in I$ and $\kappa \in [0, 1]$.

Note that $\kappa = 1/2$ in (3), we get the following inequality:

$$\psi\left(\left[\frac{2\wp_1^p\wp_2^p}{\wp_1^p + \wp_2^p}\right]^{1/p}\right) \le \frac{\psi(\wp_1) + \psi(\wp_2)}{2}, \tag{4}$$

holds for all $\wp_1, \wp_2 \in I$.

The function ψ is called Jensen p–harmonic convex function.

If we put $p = -1$ and $p = 1$, then p–harmonic convex sets and p–harmonic convex functions collapse to classical convex sets, harmonic convex sets, and harmonic convex functions, respectively.

Definition 4 (see [17]). A function $\psi : I \subseteq (0,+\infty) \longrightarrow [0,+\infty)$ is called m–polynomial harmonic exponential type convex function, if

$$\begin{aligned}\psi\left(\frac{\wp_1\wp_2}{\kappa\wp_2 + (1-\kappa)\wp_1}\right) \le \frac{1}{m}\sum_{j=1}^{m}\left(e^{\kappa}-1\right)^j\psi(\wp_1)\\ + \frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\psi(\wp_2),\end{aligned} \tag{5}$$

holds for every $\wp_1, \wp_2 \in I, m \in \mathbb{N}$ and $\kappa \in [0, 1]$.

Motivated by the above results, literature, and ongoing activities and research, we organise the paper in the following way. Firstly, we will give the idea and its algebraic properties of m–polynomial p–harmonic exponential type convex functions. Secondly, we will derive the new sort of (H–H) and refinements of (H–H) type inequalities by using the newly introduced idea. Finally, we will give some applications for means and conclusion.

3. Generalized Exponential Type Convex Functions and Its Properties

We are going to introduce a new definition called m–polynomial p–harmonic exponential type convex function and will study some of their algebraic properties. Throughout the paper, one thing gets in mind m represents finite $\mathbb{Z}^+$, m–poly p–harmonic exp convex function represents m–polynomial p–harmonic exponential type convex function and (H–H) represents Hermite–Hadamard.

Definition 5. A function $\psi : I \subseteq (0,+\infty) \longrightarrow [0,+\infty)$ is called m–poly p–harmonic exp convex, if

$$\begin{aligned}\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p + (1-\kappa)\wp_1^p}\right]^{1/p}\right) \le \frac{1}{m}\sum_{j=1}^{m}\left(e^{\kappa}-1\right)^j\psi(\wp_1)\\ + \frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\psi(\wp_2),\end{aligned} \tag{6}$$

holds for every $\wp_1, \wp_2 \in I, m \in \mathbb{N}$ and $\kappa \in [0, 1]$.

Remark 6.

(i) Taking $m = 1$ in Definition 5, we obtain the following new definition about p –harmonically exp convex function:

$$\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)\le(e^\kappa-1)\psi(\wp_1)+\left(e^{1-\kappa}-1\right)\psi(\wp_2) \tag{7}$$

(ii) Taking $m=2$ in Definition 5, we obtain the following new definition about 2–poly p–harmonically exp convex function:

$$\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)\le\left(\frac{e^{2\kappa}-e^\kappa}{2}\right)\psi(\wp_1)+\left(\frac{e^{2(1-\kappa)}-e^{1-\kappa}}{2}\right)\psi(\wp_2) \tag{8}$$

(iii) Taking $p=1$ in Definition 5, then, we get a definition, namely, m–poly harmonically exp convex function which is defined by Geo et al. [17]

(iv) Taking $p=-1$ in Definition 5, we obtain the following new definition about m–poly exp convex function:

$$\psi(\kappa\wp_1+(1-\kappa)\wp_2)\le\frac{1}{m}\sum_{j=1}^m(e^\kappa-1)^j\psi(\wp_1)+\frac{1}{m}\sum_{j=1}^m\left(e^{1-\kappa}-1\right)^j\psi(\wp_2) \tag{9}$$

(v) Taking $m=1$ and $p=1$ in Definition 5, we obtain the following new definition about harmonically exp type convex function:

$$\psi\left(\frac{\wp_1\wp_2}{\kappa\wp_2+(1-\kappa)\wp_1}\right)\le(e^\kappa-1)\psi(\wp_1)+\left(e^{1-\kappa}-1\right)\psi(\wp_2) \tag{10}$$

(vi) Taking $m=1$ and $p=-1$ in Definition 5, then, we get a definition, namely, exponential type convex function which is defined by Kadakal et al. [16]

That is the beauty of this newly introduce definition if we put the values of m and p, then, we obtain new inequalities and also found some results which connect with previous results.

Lemma 7. *The following inequalities $(1/m)\sum_{j=1}^m(e^\kappa-1)^j\ge\kappa$ and $(1/m)\sum_{j=1}^m(e^{(1-\kappa)}-1)^j\ge(1-\kappa)$ are hold. If for all $\kappa\in[0,1]$.*

Proof. The rest of the proof is clearly seen.

Proposition 8. *Every p–harmonic convex function is m–poly p–harmonic exp convex function.*

Proof. Using the definition of p–harmonic convex function and from the Lemma 7, since $\kappa\le(1/m)\sum_{j=1}^m(e^\kappa-1)^j$ and $(1-\kappa)\le(1/m)\sum_{j=1}^m(e^{1-\kappa}-1)^j$ for all $\kappa\in[0,1]$, we have

$$\begin{aligned}\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)&\le\kappa\psi(\wp_1)+(1-\kappa)\psi(\wp_2)\\&\le\frac{1}{m}\sum_{j=1}^m(e^\kappa-1)^j\psi(\wp_1)\\&\quad+\frac{1}{m}\sum_{j=1}^m\left(e^{1-\kappa}-1\right)^j\psi(\wp_2).\end{aligned} \tag{11}$$

Proposition 9. *Every m–poly p–harmonic exp convex function is p–harmonic h–convex function with $h(\kappa)=1/m\sum_{j=1}^m(e^\kappa-1)^j$.*

Proof.

$$\begin{aligned}\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)&\le\frac{1}{m}\sum_{j=1}^m(e^\kappa-1)^j\psi(\wp_1)\\&\quad+\frac{1}{m}\sum_{j=1}^m\left(e^{1-\kappa}-1\right)^j\psi(\wp_2)\\&\le h(\kappa)\psi(\wp_1)+h(1-\kappa)\psi(\wp_2).\end{aligned} \tag{12}$$

Remark 10.

(i) *If* $p=1$ in Proposition 9, then as a result, we get harmonically convex function, which is introduced by Noor et al. [22]

(ii) If $p=-1$ in Proposition 9, then as a result, we get h–convex function, which is defined by Varošanec [6]

Now, we make and investigate some examples by way of newly introduced definition.

Example 11. If $\psi(\nu)=\nu^{p+1}$, $\forall x\in(0,\infty)$ is p–harmonic convex function, then according to Proposition 8, it is an m–poly p–harmonic exp convex function.

Example 12. If $\psi(\nu)=1/\nu^{2p}$, $\forall x\in\mathbb{R}\setminus\{0\}$ is p–harmonic convex function, then according to Proposition 8, it is an m–poly p–harmonic exp convex function.

Now, we will discuss and investigate some of its algebraic properties.

Theorem 13. *Let $\psi,\varphi:[\wp_1,\wp_2]\longrightarrow\mathbb{R}$. If ψ and φ are two m–poly p–harmonic exp convex functions, then*

(i) $\psi+\varphi$ is an m–poly p–harmonic exp convex function

(ii) *For $c \in \mathbb{R}(c \geq 0)$, $c\psi$ is an m–poly p–harmonic exp convex function*

Proof.

(i) Let ψ and φ be an m–poly p–harmonic exp convex, then

$$\begin{aligned}(\psi+\varphi)\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right) &= \psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)+\varphi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)\\ &\leq \frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j\psi(\wp_1)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\psi(\wp_2)\\ &\quad+\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j\varphi(\wp_1)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\varphi(\wp_2)\\ &=\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j[\psi(\wp_1)+\varphi(\wp_1)]\\ &\quad+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j[\psi(\wp_2)+\varphi(\wp_2)]\\ &=\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j(\psi+\varphi)(\wp_1)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j(\psi+\varphi)(\wp_2)\end{aligned} \tag{13}$$

(ii) Let ψ be an m–pol p–harmonic exp convex, then

$$\begin{aligned}(c\psi)\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right) &\leq c\left[\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j\psi(\wp_1)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\psi(\wp_2)\right]\\ &=\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j c\psi(\wp_1)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j c\psi(\wp_2)\\ &=\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j(c\psi)(\wp_1)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j(c\psi)(\wp_2)\end{aligned} \tag{14}$$

which completes the proof.

Remark 14.

(i) If $m=1$ in Theorem 13, then as a result, we get the $\psi+\varphi$ and $c\psi$ are p–harmonic exp convex functions

(ii) If $p=1$ in Theorem 13, then as a result, we get Theorem 3.2 in [17]

(iii) If $m=p=1$ in Theorem 13, then as a result, we get the $\psi+\varphi$ and $c\psi$ are harmonic exp convex functions

(iv) If $p=-1$ in Theorem 13, then as a result, we get the $\psi+\varphi$ and $c\psi$ are m–poly exp convex functions

(v) If $m=1$ and $p=-1$ in Theorem 13, then as a result, we get Theorem 2.1 in [16]

Theorem 15. *Let $\psi : I=[\wp_1,\wp_2]\longrightarrow J$ be p–harmonic convex function and $\varphi : J\longrightarrow\mathbb{R}$ is nondecreasing and m–poly exp convex function. Then, the function $\varphi\circ\psi : I=[\wp_1,\wp_2]\longrightarrow\mathbb{R}$ is an m–poly p–harmonic exp convex function.*

Proof. $\forall\wp_1,\wp_2\in I$, and $\kappa\in[0,1]$, we have

$$\begin{aligned}(\varphi\circ\psi)\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right) &= \varphi\left(\psi\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)\\ &\leq\varphi(\kappa\psi(\wp_1)+(1-\kappa)\psi(\wp_2))\leq\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j\varphi(\psi(\wp_1))\\ &\quad+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\varphi(\psi(\wp_2))=\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j(\varphi\circ\psi)(\wp_1)\\ &\quad+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j(\varphi\circ\psi)(\wp_2).\end{aligned} \tag{15}$$

Remark 16.

(i) In case of $m=1$, we investigate the following new inequality:

$$\begin{aligned}(\varphi\circ\psi)\left(\left[\frac{\wp_1^p\wp_2^p}{t\wp_2^p+(1-t)\wp_1^p}\right]^{1/p}\right)&\leq(e^{\kappa}-1)(\varphi\circ\psi)(\wp_1)\\ &\quad+\left(e^{1-\kappa}-1\right)(\varphi\circ\psi)(\wp_2)\end{aligned} \tag{16}$$

(ii) In case of $p=1$, the above Theorem 15 collapses to Theorem 3.3 in [17]

(iii) In case of $m=p=1$, as a result, we obtain the following new inequality:

$$\begin{aligned}(\varphi\circ\psi)\left[\frac{\wp_1\wp_2}{\kappa\wp_2+(1-\kappa)\wp_1}\right]&\leq(e^{\kappa}-1)(\varphi\circ\psi)(\wp_1)\\ &\quad+\left(e^{1-\kappa}-1\right)(\varphi\circ\psi)(\wp_2)\end{aligned} \tag{17}$$

(iv) In case of $p=-1$, then, the above Theorem 15 collapses to the following new inequality:

$$(\varphi\circ\psi)(\kappa\wp_1+(1-\kappa)\wp_2)\le\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j(\varphi\circ\psi)(\wp_1)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j(\varphi\circ\psi)(\wp_2) \tag{18}$$

(v) In case of $m=1$ and $p=-1$, as a result, the above Theorem 15 collapses to the Theorem (2.2) in [16]

Theorem 17. *Let $0<\wp_1<\wp_2,\psi_j:[\wp_1,\wp_2]\longrightarrow[0,+\infty)$ be a class of m–poly p–harmonic exp convex functions and $\psi(u)=\sup_j\psi_j(u)$. Then, ψ is an m–poly p–harmonic exp convex function and $U=\{u\in[\wp_1,\wp_2]\colon\psi(u)<+\infty\}$ is an interval.*

Proof. Let $\wp_1,\wp_2\in U$ and $\kappa\in[0,1]$, then

$$\begin{aligned}\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)&=\sup_j\psi_j\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)\\&\le\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j\sup_j\psi_j(\wp_1)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\sup_j\psi_j(\wp_2)\\&=\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j\psi(\wp_1)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\psi(\wp_2)<+\infty,\end{aligned} \tag{19}$$

which completes the proof.

Remark 18.

(i) In case of $p=1$, as a result, we get Theorem 3.4 in [17]

(ii) In case of $m=1$ and $p=-1$ in Theorem 17, as a result, we get Theorem 2.3 in [16]

Theorem 19. *If $\psi:[\wp_1,\wp_2]\longrightarrow\mathbb{R}$ is an m–poly p–harmonic exp convex then ψ is bounded on $[\wp_1,\wp_2]$.*

Proof. Let $x\in[\wp_1,\wp_2]$ and $L=\max\{\psi(\wp_1),\psi(\wp_2)\}$, then, there exist $\exists\kappa\in[0,1]$ such that $x=[(\wp_1^p\wp_2^p)/(\kappa\wp_2^p+(1-\kappa)\wp_1^p)]^{1/p}$. Thus, since $e^{\kappa}\le e$ and $e^{1-\kappa}\le e$, we have

$$\begin{aligned}\psi(x)&=\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)\le\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j\psi(\wp_1)\\&\quad+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\psi(\wp_2)\le\frac{1}{m}\sum_{j=1}^{m}\left(e^{\kappa}+e^{1-\kappa}-2\right)^j\cdot L\\&\le\frac{2L}{m}\sum_{j=1}^{m}\left[(e-1)^j\right]=M.\end{aligned} \tag{20}$$

The above proof clearly shows that ψ is bounded above from M. For bounded below, the readers using the identical concept as in Theorem 2.4 in [16].

Remark 20.

(i) In case of $p=1$, we obtain Theorem 3.5 in [17]

(ii) In case of $m=1$ and $p=-1$, we obtain Theorem 2.4 in [16]

4. (H–H) Type Inequality via Generalized Exponential Type Convexity

The main object of this section is to investigate and prove a new version of (H–H) type inequality using m–poly p–harmonic exp convexity.

Theorem 21. *Let $\psi:[\wp_1,\wp_2]\longrightarrow[0,+\infty)$ be an m–poly p–harmonic exp convex function. If $\psi\in L[\wp_1,\wp_2]$, then*

$$\begin{aligned}\frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^j}\psi\left(\left[\frac{2\wp_1^p\wp_2^p}{\wp_1^p+\wp_2^p}\right]^{1/p}\right)&\le\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu\\&\le\left[\frac{\psi(\wp_1)+\psi(\wp_2)}{m}\right]\sum_{j=1}^{m}[e-2]^j.\end{aligned} \tag{21}$$

Proof. Since ψ is an m–poly p–harmonic exp convex function, we have

$$\psi\left(\left[\frac{x^py^p}{\kappa y^p+(1-\kappa)x^p}\right]^{1/p}\right)\le\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^j\psi(x)+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\psi(y), \tag{22}$$

which lead to

$$\psi\left(\left[\frac{2x^py^p}{x^p+y^p}\right]^{1/p}\right)\le\frac{1}{m}\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^j\psi(x)+\frac{1}{m}\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^j\psi(y). \tag{23}$$

Using the change of variables, we get

$$\begin{aligned}\psi\left(\left[\frac{2\wp_1^p\wp_2^p}{\wp_1^p+\wp_2^p}\right]^{1/p}\right)&\le\frac{1}{m}\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^j\times\left\{\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\left(\kappa\wp_2^p+(1-\kappa)\wp_1^p\right)}\right]^{1/p}\right)\right.\\&\quad\left.+\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\left(\kappa\wp_1^p+(1-\kappa)\wp_2^p\right)}\right]^{1/p}\right)\right\}.\end{aligned} \tag{24}$$

Integrating the above inequality with respect to κ on $[0,1]$, we obtain

$$\frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^{j}}\psi\left(\left[\frac{2\wp_1^p\wp_2^p}{\wp_1^p+\wp_2^p}\right]^{1/p}\right)\leq\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu, \tag{25}$$

which completes the left side inequality.

For the right side inequality, first of all, we change the variable of integration by $\nu=[(\wp_1^p\wp_2^p)/(\kappa\wp_2^p+(1-\kappa)\wp_1^p)]^{1/p}$ and using Definition 5 for the function ψ, we have

$$\begin{aligned}\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu&=\int_0^1\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)d\kappa\\&\leq\int_0^1\left[\frac{1}{m}\sum_{j=1}^{m}\left(e^{\kappa}-1\right)^j\psi(\wp_1)\right.\\&\quad\left.+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^j\psi(\wp_2)\right]d\kappa\\&=\frac{\psi(\wp_1)}{m}\sum_{j=1}^{m}\int_0^1\left(e^t-1\right)^j d\kappa\\&\quad+\frac{\psi(\wp_2)}{m}\sum_{j=1}^{m}\int_0^1\left(e^{1-\kappa}-1\right)^j d\kappa\\&=\left[\frac{\psi(\wp_1)+\psi(\wp_2)}{m}\right]\sum_{j=1}^{m}[e-2]^j,\end{aligned} \tag{26}$$

which completes the proof.

Corollary 22. *In case of m = 1 in Theorem 21, then, we get the following new (H–H) type inequality for p–harmonic exp convex functions:*

$$\begin{aligned}\frac{1}{2\left(\sqrt{e}-1\right)}\psi\left(\left[\frac{2\wp_1^p\wp_2^p}{\wp_1^p+\wp_2^p}\right]^{1/p}\right)&\leq\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu\\&\leq(e-2)[\psi(\wp_1)+\psi(\wp_2)].\end{aligned} \tag{27}$$

Corollary 23. *In case of p = −1 in Theorem 21, then as a result, we investigate the following new (H–H) type inequality for m–poly exp convex functions:*

$$\begin{aligned}\frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^{j}}\psi\left(\frac{\wp_1+\wp_2}{2}\right)&\leq\frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\psi(\nu)d\nu\\&\leq\left(\frac{\psi(\wp_1)+\psi(\wp_2)}{m}\right)\sum_{j=1}^{m}[e-2]^j.\end{aligned} \tag{28}$$

Remark 24.

(i) In case of p = 1, then as a result, we obtain Theorem 4.1 in [17]

(ii) In case of $m=1$ and $p=-1$, then as a result, we obtain Theorem 3.1 in [16]

(iii) In case of $m=1$ and $p=1$, then as a result, we obtain Corollary 1in [17]

5. Refinements of (H–H) Type Inequality via Generalized Exponential Type Convexity

In this section, in order to prove our main results regarding on some Hermite–Hadamard type inequalities for m–poly p –harmonic exp convex function, we need the following lemmas:

Lemma 25. *Let $\psi: I=[\wp_1,\wp_2]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\wp_1,\wp_2]$, then*

$$\begin{aligned}&\frac{\psi(\wp_1)+\psi(\wp_2)}{2}-\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx\\&\quad=\frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{2p}\int_0^1\frac{\mu(\kappa)}{A_\kappa^{p+1}}\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)d\kappa,\end{aligned} \tag{29}$$

where $A_\kappa=[\kappa\wp_2^p+(1-\kappa)\wp_1^p]^{1/p}$ and $\mu(\kappa)=(1-2\kappa)$.

Proof. Let

$$I=\frac{\wp_2^p-\wp_1^p}{2p\wp_1^p\wp_2^p}\int_0^1(1-2\kappa)\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1+\frac{1}{p}}\psi'\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right). \tag{30}$$

Using integration by parts

$$\begin{aligned}I&=\frac{\wp_2^p-\wp_1^p}{2p\wp_1^p\wp_2^p}\left\{\left|\frac{-p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}(1-2\kappa)\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)\right|_0^1\right.\\&\quad\left.-\frac{2p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_0^1\psi\left(\left[\frac{\wp_1^p\wp_2^p}{\kappa\wp_2^p+(1-\kappa)\wp_1^p}\right]^{1/p}\right)d\kappa\right\}\\&=\frac{\psi(\wp_1)+\psi(\wp_2)}{2}-\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx.\end{aligned} \tag{31}$$

Lemma 26 (see [23]). *Let $\psi: I=[\wp_1,\wp_2]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\wp_1,\wp_2]$, then*

$$\begin{aligned}&\frac{1}{8}\left[\psi(\wp_1)+3\psi\left(\left[\frac{3\wp_1^p\wp_2^p}{\wp_1^p+2\wp_2^p}\right]^{1/p}\right)+3\psi\left(\left[\frac{3\wp_1^p\wp_2^p}{2\wp_1^p+\wp_2^p}\right]^{1/p}\right)+\psi(\wp_2)\right]\\&\quad-\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx=\frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{p}\int_0^1\frac{\mu(\kappa)}{A_\kappa^{p+1}}\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)d\kappa,\end{aligned} \tag{32}$$

where $A_\kappa = [\kappa\wp_2^p + (1-\kappa)\wp_1^p]^{1/p}$ *and*

$$\mu(\kappa) = \begin{cases} \kappa - \dfrac{1}{8}, & \text{if } \kappa \in \left[0, \dfrac{1}{3}\right), \\ \kappa - \dfrac{1}{2}, & \text{if } \kappa \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right), \\ \kappa - \dfrac{7}{8}, & \text{if } \kappa \in \left[\dfrac{2}{3}, 1\right). \end{cases} \tag{33}$$

Theorem 27. *Let* $\psi : I = [\wp_1, \wp_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ *be differentiable function on the* I° *of I. If* $\psi' \in L[\wp_1, \wp_2]$ *and* $|\psi'|^q$ *is an m-poly p–harmonic exp convex function on I,* $q \geq 1$*, then*

$$\left|\frac{\psi(\wp_1)+\psi(\wp_2)}{2} - \frac{p\wp_1^p\wp_2^p}{\wp_2^p - \wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx\right| \leq \frac{\wp_1\wp_2(\wp_2^p - \wp_1^p)}{2p} \cdot \left\{G_1^{1-(1/q)}\left[G_2|\psi'(\wp_1)|^q + G_3|\psi'(\wp_2)|^q\right]^{1/q}\right\}, \tag{34}$$

where

$$G_1 = \int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa, \quad G_2 = \frac{1}{m}\int_0^1 \frac{|1-2\kappa|\sum_{j=1}^m (e^\kappa - 1)^j}{A_\kappa^{1+p}}d\kappa,$$
$$G_3 = \frac{1}{m}\int_0^1 \frac{|1-2\kappa|\sum_{j=1}^m (e^{1-\kappa} - 1)^j}{A_\kappa^{1+p}}d\kappa. \tag{35}$$

Proof. Using Lemma 25, properties of modulus, power mean inequality, and m–poly p–harmonic exp convexity of the $|\psi'|^q$, we have

$$\begin{aligned}
&\left|\frac{\psi(\wp_1)+\psi(\wp_2)}{2} - \frac{p\wp_1^p\wp_2^p}{\wp_2^p - \wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx\right| \leq \frac{\wp_1\wp_2(\wp_2^p - \wp_1^p)}{2p}\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}} \\
&\cdot \left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right| d\kappa \leq \frac{\wp_1\wp_2(\wp_2^p - \wp_1^p)}{2p}\left(\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa\right)^{1-(1/q)} \\
&\cdot \left(\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right|^q d\kappa\right)^{1/q} \leq \frac{\wp_1\wp_2(\wp_2^p - \wp_1^p)}{2p}\left(\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa\right)^{1-(1/q)} \\
&\times \left(\int_0^1 \frac{|1-2\kappa|\left[(1/m)\sum_{j=1}^m (e^\kappa - 1)^j|\psi'(\wp_1)|^q + (1/m)\sum_{j=1}^m (e^{1-\kappa} - 1)^j|\psi'(\wp_2)|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{1/q} \\
&\leq \frac{\wp_1\wp_2(\wp_2^p - \wp_1^p)}{2p}\left(\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa\right)^{1-(1/q)} \\
&\times \left(\frac{1}{m}\int_0^1 \frac{|1-2\kappa|\sum_{j=1}^m (e^\kappa - 1)^j}{A_\kappa^{1+p}}|\psi'(\wp_1)|^q d\kappa + \frac{1}{m}\int_0^1 \frac{|1-2\kappa|\sum_{j=1}^m (e^{1-\kappa} - 1)^j}{A_\kappa^{1+p}}|\psi'(\wp_2)|^q d\kappa\right)^{1/q} \\
&\leq \frac{\wp_1\wp_2(\wp_2^p - \wp_1^p)}{2p}\left\{G_1^{1-(1/q)}\left[G_2|\psi'(\wp_1)|^q + G_3|\psi'(\wp_2)|^q\right]^{1/q}\right\},
\end{aligned} \tag{36}$$

which completes the proof.

Corollary 28. *Under the assumptions of Theorem 27 with* $p = -1$*, we have the following new result:*

$$\left|\frac{\psi(\wp_1)+\psi(\wp_2)}{2} - \frac{1}{\wp_2 - \wp_1}\int_{\wp_1}^{\wp_2}\psi(x)dx\right| \leq \frac{(\wp_2 - \wp_1)}{2}\left(\frac{1}{2}\right)^{1-(1/q)}\frac{1}{m}\sum_{j=1}^m\left(\frac{8\sqrt{e} - 2e - 7}{2}\right)^j \cdot \left\{\left[|\psi'(\wp_1)|^q + |\psi'(\wp_2)|^q\right]^{1/q}\right\}. \tag{37}$$

Corollary 29. *Under the assumptions of Theorem 27 with* $p = 1$*, we have the following new result:*

$$\left|\frac{\psi(\wp_1)+\psi(\wp_2)}{2} - \frac{\wp_1\wp_2}{\wp_2 - \wp_1}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^2}dx\right| \leq \frac{\wp_1\wp_2(\wp_2 - \wp_1)}{2} \cdot \left\{G_1'^{1-(1/q)}\left[G_2'|\psi'(\wp_1)|^q + G_3'|\psi'(\wp_2)|^q\right]^{1/q}\right\}, \tag{38}$$

where

$$G_1' = \int_0^1 \frac{|1-2\kappa|}{A_\kappa^2}d\kappa, \quad G_2' = \frac{1}{m}\int_0^1 \frac{|1-2\kappa|\sum_{j=1}^m (e^\kappa - 1)^j}{A_\kappa^2}d\kappa,$$
$$G_3' = \frac{1}{m}\int_0^1 \frac{|1-2\kappa|\sum_{j=1}^m (e^{1-\kappa} - 1)^j}{A_\kappa^2}d\kappa. \tag{39}$$

Theorem 30. *Let* $\psi : I = [\wp_1, \wp_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ *be differentiable function on the* I° *of I. If* $\psi' \in L[\wp_1, \wp_2]$ *and* $|\psi'|^q$ *is an m–poly p–harmonic exp convex function on I,* $r, q \geq 1$, $(1/r) + (1/q) \geq 1$*, then,*

$$\left|\frac{\psi(\wp_1)+\psi(\wp_2)}{2} - \frac{p\wp_1^p\wp_2^p}{\wp_2^p - \wp_1^p}\int_a^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx\right| \leq \frac{\wp_1\wp_2(\wp_2^p - \wp_1^p)}{2p} \times \left\{G_4^{1/r}\left[G_5|\psi'(\wp_1)|^q + G_6|\psi'(\wp_2)|^q\right]^{1/q}\right\}, \tag{40}$$

where

$$G_4 = \int_0^1 |1-2\kappa|^r d\kappa, \quad G_5 = \frac{1}{m}\int_0^1 \frac{\sum_{j=1}^m (e^\kappa - 1)^j}{A_\kappa^{(1+p)q}}d\kappa,$$
$$G_6 = \frac{1}{m}\int_0^1 \frac{\sum_{j=1}^m (e^{1-\kappa} - 1)^j}{A_\kappa^{(1+p)q}}d\kappa. \tag{41}$$

Proof. Using Lemma 25, properties of modulus, Hölder's inequality, and m–poly p–harmonic exp convexity of the $|\psi'|^q$, we have

$$
\begin{aligned}
&\left|\frac{\psi(\wp_1)+\psi(\wp_2)}{2}-\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx\right| \le \frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{2p} \\
&\cdot \int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right|d\kappa \le \frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{2p} \\
&\cdot \left(\int_0^1 |1-2\kappa|^r d\kappa\right)^{1/r}\left(\int_0^1 \frac{1}{A_\kappa^{(1+p)q}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right|^q d\kappa\right)^{1/q} \\
&\le \frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{2p}\left(\int_0^1 |1-2\kappa|^r d\kappa\right)^{1/r} \\
&\times \left(\int_0^1 \frac{1}{A_\kappa^{(1+p)q}}\left[\frac{1}{m}\sum_{j=1}^m\left(e^\kappa-1\right)^j\left|\psi'(\wp_1)\right|^q+\frac{1}{m}\sum_{j=1}^m\left(e^{1-\kappa}-1\right)^j\left|\psi'(\wp_2)\right|^q\right]d\kappa\right)^{1/q} \\
&= \frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{2p}\left\{G_4^{1/r}\left[G_5\left|\psi'(\wp_1)\right|^q+G_6\left|\psi'(\wp_2)\right|^q\right]^{1/q}\right\},
\end{aligned}
\tag{42}
$$

which completes the proof.

Corollary 31. *Under the assumptions of Theorem 30 with $p=-1$, we have the following new result:*

$$
\begin{aligned}
&\left|\frac{\psi(\wp_1)+\psi(\wp_2)}{2}-\frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\psi(x)dx\right| \\
&\le \frac{(\wp_2-\wp_1)}{2}\left(\int_0^1|1-2\kappa|^r d\kappa\right)^{1/r}\frac{1}{m}\sum_{j=1}^m(e-2)^j \\
&\cdot\left\{\left[\left|\psi'(\wp_1)\right|^q+\left|\psi'(\wp_2)\right|^q\right]^{1/q}\right\}.
\end{aligned}
\tag{43}
$$

Corollary 32. *Under the assumptions of Theorem 30 with $p=1$, we have the following new result:*

$$
\begin{aligned}
&\left|\frac{\psi(\wp_1)+\psi(\wp_2)}{2}-\frac{\wp_1\wp_2}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^2}dx\right| \le \frac{\wp_1\wp_2(\wp_2-\wp_1)}{2} \\
&\cdot\left\{{G_4'}^{1/r}\left[G_5'\left|\psi'(\wp_1)\right|^q+G_6'\left|\psi'(\wp_2)\right|^q\right]^{1/q}\right\},
\end{aligned}
\tag{44}
$$

where

$$
\begin{aligned}
&G_4'=\int_0^1|1-2\kappa|^r d\kappa, G_5'=\frac{1}{m}\int_0^1\frac{\sum_{j=1}^m(e^\kappa-1)^j}{A_\kappa^{2q}}d\kappa, \\
&G_6'=\frac{1}{m}\int_0^1\frac{\sum_{j=1}^m\left(e^{1-\kappa}-1\right)^j}{A_\kappa^{2q}}d\kappa.
\end{aligned}
\tag{45}
$$

Theorem 33. *Let $\psi : I=[\wp_1,\wp_2]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\wp_1,\wp_2]$ and $|\psi'|^q$ is an m–poly p–harmonic exp convex function on I, $q\ge,1$ then*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\psi(\wp_1)+3\psi\left(\left[\frac{3\wp_1^p\wp_2^p}{\wp_1^p+2\wp_2^p}\right]^{1/p}\right)+3\psi\left(\left[\frac{3\wp_1^p\wp_2^p}{2\wp_1^p+\wp_2^p}\right]^{1/p}\right)+\psi(\wp_2)\right]\right. \\
&\left.-\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx\right| \le \frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{p} \\
&\cdot\left\{B_1^{1-(1/q)}\left[B_4\left|\psi'(\wp_1)\right|^q+B_5\left|\psi'(\wp_2)\right|^q\right]^{1/q}\right. \\
&+B_2^{1-(1/q)}\left[B_6\left|\psi'(\wp_1)\right|^q+B_7\left|\psi'(\wp_2)\right|^q\right]^{1/q} \\
&\left.+B_3^{1-(1/q)}\left[B_8\left|\psi'(\wp_1)\right|^q+B_9\left|\psi'(\wp_2)\right|^q\right]^{1/q}\right\},
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
&B_1=\int_0^{1/3}\frac{|\kappa-(1/8)|}{A_\kappa^{p+1}}d\kappa, B_2=\int_{1/3}^{2/3}\frac{|\kappa-(1/2)|}{A_\kappa^{p+1}}d\kappa, \\
&B_4=\frac{1}{m}\int_0^{1/3}\frac{|\kappa-(1/8)|\sum_{j=1}^m(e^\kappa-1)^j}{A_\kappa^{p+1}}d\kappa, \\
&B_5=\frac{1}{m}\int_0^{1/3}\frac{|\kappa-(1/8)|\sum_{j=1}^m\left(e^{1-\kappa}-1\right)^j}{A_\kappa^{p+1}}d\kappa, \\
&B_6=\frac{1}{m}\int_{1/3}^{2/3}\frac{|\kappa-(1/2)|\sum_{j=1}^m(e^\kappa-1)^j}{A_\kappa^{p+1}}d\kappa, \\
&B_7=\frac{1}{m}\int_{1/3}^{2/3}\frac{|\kappa-(1/2)|\sum_{j=1}^m\left(e^{1-\kappa}-1\right)^j}{A_\kappa^{p+1}}d\kappa, \\
&B_8=\frac{1}{m}\int_{2/3}^{1}\frac{|\kappa-(7/8)|\sum_{j=1}^m(e^\kappa-1)^j}{A_\kappa^{p+1}}d\kappa, \\
&B_9=\frac{1}{m}\int_{2/3}^{1}\frac{|\kappa-(7/8)|\sum_{j=1}^m\left(e^{1-\kappa}-1\right)^j}{A_\kappa^{p+1}}d\kappa.
\end{aligned}
\tag{47}
$$

Proof. Using Lemma 26, properties of modulus, power mean inequality, and m–poly p-harmonic exp convexity of the $|\psi'|^q$, we have

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\psi(\wp_1)+3\psi\left(\left[\frac{3\wp_1^p\wp_2^p}{\wp_1^p+2\wp_2^p}\right]^{1/p}\right)+3\psi\left(\left[\frac{3\wp_1^p\wp_2^p}{2\wp_1^p+\wp_2^p}\right]^{1/p}\right)+\psi(\wp_2)\right]-\frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx\right| \\
&\le \frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{p}\times\left[\int_0^{1/3}\frac{|\kappa-(1/8)|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right|d\kappa+\int_{1/3}^{2/3}\frac{|\kappa-(1/2)|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right|d\kappa\right. \\
&\left.+\int_{2/3}^{1}\frac{|\kappa-(7/8)|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right|d\kappa\right] \le \frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{p}\times\left[\left(\int_0^{1/3}\frac{|\kappa-(1/8)|}{A_\kappa^{1+p}}d\kappa\right)^{1-(1/q)}\right. \\
&\cdot\left(\int_0^{1/3}\frac{|\kappa-(1/8)|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right|^q d\kappa\right)^{1/q}+\left(\int_{1/3}^{2/3}\frac{|\kappa-(1/2)|}{A_\kappa^{1+p}}d\kappa\right)^{1-(1/q)} \\
&\cdot\left(\int_{1/3}^{2/3}\frac{|\kappa-(1/2)|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right|^q d\kappa\right)^{1/q}+\left(\int_{2/3}^{1}\frac{|\kappa-(7/8)|}{A_\kappa^{1+p}}d\kappa\right)^{1-(1/q)} \\
&\left.\cdot\left(\int_{2/3}^{1}\frac{|\kappa-(7/8)|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_\kappa}\right)\right|^q d\kappa\right)^{1/q}\right] \le \frac{\wp_1\wp_2\left(\wp_2^p-\wp_1^p\right)}{p}\times\left[\left(\int_0^{1/3}\frac{|\kappa-(1/8)|}{A_\kappa^{1+p}}d\kappa\right)^{1-(1/q)}\right. \\
&\times\left(\int_0^{1/3}\frac{|\kappa-(1/8)|\left[(1/m)\sum_{j=1}^m(e^\kappa-1)^j\left|\psi'(\wp_1)\right|^q+(1/m)\sum_{j=1}^m\left(e^{1-\kappa}-1\right)^j\left|\psi'(\wp_2)\right|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{1/q} \\
&+\left(\int_{1/3}^{2/3}\frac{|\kappa-(1/2)|}{A_\kappa^{1+p}}d\kappa\right)^{1-(1/q)}
\end{aligned}
$$

$$\times\left(\int_{1/3}^{2/3}\frac{|\kappa-(1/2)|\left[(1/m)\sum_{j=1}^{m}(e^{\kappa}-1)^{j}\left|\psi'(\wp_1)\right|^{q}+(1/m)\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}\left|\psi'(\wp_2)\right|^{q}\right]}{A_{\kappa}^{1+p}}d\kappa\right)^{1/q}$$
$$+\left(\int_{2/3}^{1}\frac{|\kappa-(7/8)|}{A_{\kappa}^{1+p}}d\kappa\right)^{1-(1/q)}$$
$$\left.\times\left(\int_{2/3}^{1}\frac{|\kappa-(7/8)|\left[(1/m)\sum_{j=1}^{m}(e^{\kappa}-1)^{j}\left|\psi'(\wp_1)\right|^{q}+(1/m)\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}\left|\psi'(\wp_2)\right|^{q}\right]}{A_{\kappa}^{1+p}}d\kappa\right)^{1/q}\right]$$
$$\leq\frac{\wp_1\wp_2\left(\wp_2^{p}-\wp_1^{p}\right)}{p}\times\left[\left(\int_{0}^{1/3}\frac{|\kappa-(1/8)|}{A_{\kappa}^{1+p}}d\kappa\right)^{1-(1/q)}\times\left(\frac{1}{m}\int_{0}^{1/3}\frac{|\kappa-(1/8)\mid\sum_{j=1}^{m}(e^{\kappa}-1)^{j}}{A_{\kappa}^{1+p}}\right.\right.$$
$$\left.\cdot\left|\psi'(\wp_1)\right|^{q}d\kappa+\frac{1}{m}\int_{0}^{1/3}\frac{|\kappa-(1/8)\mid\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}}{A_{\kappa}^{1+p}}\left|\psi'(\wp_2)\right|^{q}d\kappa\right)^{1/q}$$
$$+\left(\int_{1/3}^{2/3}\frac{|\kappa-(1/2)|}{A_{\kappa}^{1+p}}d\kappa\right)^{1-(1/q)}\times\left(\frac{1}{m}\int_{1/3}^{2/3}\frac{|\kappa-(1/2)\mid\sum_{j=1}^{m}(e^{\kappa}-1)^{j}}{A_{\kappa}^{1+p}}\left|\psi'(\wp_1)\right|^{q}d\kappa\right.$$
$$\left.+\frac{1}{m}\int_{1/3}^{2/3}\frac{|\kappa-(1/2)\mid\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}}{A_{\kappa}^{1+p}}\left|\psi'(\wp_2)\right|^{q}d\kappa\right)^{1/q}+\left(\frac{1}{m}\int_{2/3}^{1}\frac{|\kappa-(7/8)|}{A_{t}^{1+p}}dt\right)^{1-(1/q)}$$
$$\times\left(\frac{1}{m}\int_{2/3}^{1}\frac{|\kappa-(7/8)\mid\sum_{j=1}^{m}(e^{\kappa}-1)^{j}}{A_{\kappa}^{1+p}}\left|\psi'(\wp_1)\right|^{q}d\kappa+\frac{1}{m}\int_{2/3}^{1}\frac{|\kappa-(7/8)\mid\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}}{A_{\kappa}^{1+p}}\right.$$
$$\left.\left.\cdot\left|\psi'(\wp_2)\right|^{q}d\kappa\right)^{1/q}\right]=\frac{\wp_1\wp_2\left(\wp_2^{p}-\wp_1^{p}\right)}{p}\left\{B_1^{1-(1/q)}\left[B_4\left|\psi'(\wp_1)\right|^{q}+B_5\left|\psi'(\wp_2)\right|^{q}\right]^{1/q}\right.$$
$$\left.+B_2^{1-(1/q)}\left[B_6\left|\psi'(\wp_1)\right|^{q}+B_7\left|\psi'(\wp_2)\right|^{q}\right]^{1/q}+B_3^{1-(1/q)}\left[B_8\left|\psi'(\wp_1)\right|^{q}+B_9\left|\psi'(\wp_2)\right|^{q}\right]^{1/q}\right\},\tag{48}$$

which completes the proof.

Corollary 34. *Under the assumptions of Theorem 33 with $p=-1$ and $m=1$, we have the following new result:*

$$\left|\frac{1}{8}\left[\psi(\wp_1)+3\psi\left(\frac{2\wp_1+\wp_2}{3}\right)+3\psi\left(\frac{\wp_1+2\wp_2}{3}\right)+\psi(\wp_2)\right]\right.$$
$$\left.-\frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\psi(x)dx\right|\leq(\wp_2-\wp_1)$$
$$\cdot\left\{\left(\frac{17}{576}\right)\left[0.0069\left|\psi'(\wp_1)\right|^{q}+0.036\left|\psi'(\wp_2)\right|^{q}\right]^{1/q}\right.$$
$$+\left(\frac{0.183}{360}\right)\left[\left|\psi'(\wp_1)\right|^{q}+\left|\psi'(\wp_2)\right|^{q}\right]^{1/q}+\left(\frac{17}{576}\right)$$
$$\left.\cdot\left[0.036\left|\psi'(\wp_1)\right|^{q}+0.0069\left|\psi'(\wp_2)\right|^{q}\right]^{1/q}\right\}.\tag{49}$$

Theorem 35. *Let $\psi: I=[\wp_1,\wp_2]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\wp_1,\wp_2]$ and $|\psi'|^{q}$ is an m–poly p–harmonic exp convex function on I, $r,q\geq 1$ and $(1/r)+(1/q)\geq 1$ then*

$$\left|\frac{1}{8}\left[\psi(\wp_1)+3\psi\left(\left[\frac{3\wp_1^{p}\wp_2^{p}}{\wp_1^{p}+2\wp_2^{p}}\right]^{1/p}\right)+3\psi\left(\left[\frac{3\wp_1^{p}\wp_2^{p}}{2\wp_1^{p}+\wp_2^{p}}\right]^{1/p}\right)+\psi(\wp_2)\right]\right.$$
$$\left.-\frac{p\wp_1^{p}\wp_2^{p}}{\wp_2^{p}-\wp_1^{p}}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx\right|\leq\frac{\wp_1\wp_2\left(\wp_2^{p}-\wp_1^{p}\right)}{p}\times\left\{\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r}\right.$$
$$\cdot\left(B_{10}\left|\psi'(\wp_1)\right|^{q}+B_{11}\left|\psi'(\wp_2)\right|^{q}\right)^{1/q}+\left(\frac{2}{6^{r+1}(r+1)}\right)^{1/r}$$
$$\cdot\left(B_{12}\left|\psi'(\wp_1)\right|^{q}+B_{13}\left|\psi'(\wp_2)\right|^{q}\right)^{1/q}+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r}$$
$$\left.\cdot\left(B_{14}\left|f'(a)\right|^{q}dt+B_{15}\left|\psi'(\wp_2)\right|^{q}\right)^{1/q}\right\},\tag{50}$$

where

$$B_{10}=\frac{1}{m}\int_{0}^{1/3}\frac{\sum_{j=1}^{m}(e^{\kappa}-1)^{j}}{A_{\kappa}^{(1+p)q}}d\kappa,$$
$$B_{11}=\frac{1}{m}\int_{0}^{1/3}\frac{\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}}{A_{\kappa}^{(1+p)q}}d\kappa,$$
$$B_{12}=\frac{1}{m}\int_{1/3}^{2/3}\frac{\sum_{j=1}^{m}(e^{\kappa}-1)^{j}}{A_{\kappa}^{(1+p)q}}d\kappa,$$
$$B_{13}=\frac{1}{m}\int_{1/3}^{2/3}\frac{\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}}{A_{\kappa}^{(1+p)q}}d\kappa,\tag{51}$$
$$B_{14}=\frac{1}{m}\int_{2/3}^{1}\frac{\sum_{j=1}^{m}(e^{\kappa}-1)^{j}}{A_{\kappa}^{(1+p)q}}d\kappa,$$
$$B_{15}=\frac{1}{m}\int_{2/3}^{1}\frac{\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}}{A_{\kappa}^{(1+p)q}}d\kappa.$$

Proof. Using Lemma 26, properties of modulus, Hölder's inequality, and m–poly p–harmonic exp convexity of the $|\psi'|^{q}$, we have

$$\left|\frac{1}{8}\left[\psi(\wp_1)+3\psi\left(\left[\frac{3\wp_1^{p}\wp_2^{p}}{\wp_1^{p}+2\wp_2^{p}}\right]^{1/p}\right)+3\psi\left(\left[\frac{3\wp_1^{p}\wp_2^{p}}{2\wp_1^{p}+\wp_2^{p}}\right]^{1/p}\right)+\psi(\wp_2)\right]\right.$$
$$\left.-\frac{p\wp_1^{p}\wp_2^{p}}{\wp_2^{p}-\wp_1^{p}}\int_{\wp_1}^{\wp_2}\frac{\psi(x)}{x^{1+p}}dx\right|\leq\frac{\wp_1\wp_2\left(\wp_2^{p}-\wp_1^{p}\right)}{p}\times\left[\int_{0}^{1/3}\left|\kappa-\frac{1}{8}\right|\right.$$
$$\cdot\left|\frac{1}{A_{\kappa}^{1+p}}\psi'\left(\frac{\wp_1\wp_2}{A_{\kappa}}\right)\right|d\kappa+\int_{1/3}^{2/3}\left|\kappa-\frac{1}{2}\right|\left|\frac{1}{A_{\kappa}^{1+p}}\psi'\left(\frac{\wp_1\wp_2}{A_{\kappa}}\right)\right|d\kappa$$
$$\left.+\int_{2/3}^{1}\left|\kappa-\frac{7}{8}\right|\left|\frac{1}{A_{\kappa}^{1+p}}\psi'\left(\frac{\wp_1\wp_2}{A_{\kappa}}\right)\right|d\kappa\right]\leq\frac{\wp_1\wp_2\left(\wp_2^{p}-\wp_1^{p}\right)}{p}$$
$$\cdot\left\{\left(\int_{0}^{1/3}\left|\kappa-\frac{1}{8}\right|^{r}d\kappa\right)^{1/r}\left(\int_{0}^{1/3}\frac{1}{A_{\kappa}^{(1+p)q}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_{\kappa}}\right)\right|^{q}d\kappa\right)^{1/q}\right.$$
$$+\left(\int_{1/3}^{2/3}\left|\kappa-\frac{1}{2}\right|^{r}d\kappa\right)^{1/r}\left(\int_{1/3}^{2/3}\frac{1}{A_{\kappa}^{(1+p)q}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_{\kappa}}\right)\right|^{q}d\kappa\right)^{1/q}$$
$$\left.+\left(\int_{2/3}^{1}\left|\kappa-\frac{7}{8}\right|^{r}d\kappa\right)^{1/r}\left(\int_{2/3}^{1}\frac{1}{A_{\kappa}^{(1+p)q}}\left|\psi'\left(\frac{\wp_1\wp_2}{A_{\kappa}}\right)\right|^{q}d\kappa\right)^{1/q}\right\}$$
$$\leq\frac{\wp_1\wp_2\left(\wp_2^{p}-\wp_1^{p}\right)}{p}\times\left\{\left(\int_{0}^{1/3}\left|\kappa-\frac{1}{8}\right|^{r}d\kappa\right)^{1/r}\times\left(\int_{0}^{1/3}\frac{1}{A_{\kappa}^{(1+p)q}}\right.\right.$$
$$\left.\cdot\left[\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^{j}\left|\psi'(\wp_1)\right|^{q}+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}\left|\psi'(\wp_2)\right|^{q}\right]d\kappa\right)^{1/q}$$
$$+\left(\int_{1/3}^{2/3}\left|\kappa-\frac{1}{2}\right|^{r}d\kappa\right)^{1/r}\times\left(\int_{1/3}^{2/3}\frac{1}{A_{\kappa}^{(1+p)q}}\left[\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^{j}\left|\psi'(\wp_1)\right|^{q}\right.\right.$$
$$\left.\left.+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}\left|\psi'(\wp_2)\right|^{q}\right]d\kappa\right)^{1/q}+\left(\int_{2/3}^{1}\left|\kappa-\frac{7}{8}\right|^{r}d\kappa\right)^{1/r}$$
$$\left.\times\left(\int_{2/3}^{1}\frac{1}{A_{\kappa}^{(1+p)q}}\left[\frac{1}{m}\sum_{j=1}^{m}(e^{\kappa}-1)^{j}\left|\psi'(\wp_1)\right|^{q}+\frac{1}{m}\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}\left|\psi'(\wp_2)\right|^{q}\right]d\kappa\right)^{1/q}\right\}$$
$$=\frac{\wp_1\wp_2\left(\wp_2^{p}-\wp_1^{p}\right)}{p}\times\left\{\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r}\left(\frac{1}{m}\int_{0}^{1/3}\frac{\sum_{j=1}^{m}(e^{\kappa}-1)^{j}}{A_{\kappa}^{(1+p)q}}\left|\psi'(\wp_1)\right|^{q}d\kappa\right.\right.$$
$$\left.+\frac{1}{m}\int_{0}^{1/3}\frac{\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}}{A_{\kappa}^{(1+p)q}}\left|\psi'(\wp_2)\right|^{q}d\kappa\right)^{1/q}+\left(\frac{2}{6^{r+1}(r+1)}\right)^{1/r}$$
$$\times\left(\frac{1}{m}\int_{1/3}^{2/3}\frac{\sum_{j=1}^{m}(e^{\kappa}-1)^{j}}{A_{\kappa}^{(1+p)q}}\left|\varphi'(\wp_1)\right|^{q}d\kappa+\frac{1}{m}\int_{1/3}^{2/3}\frac{\sum_{j=1}^{m}\left(e^{1-\kappa}-1\right)^{j}}{A_{\kappa}^{(1+p)q}}\left|\psi'(\wp_2)\right|^{q}d\kappa\right)^{1/q}$$

$$
\begin{aligned}
&+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r} \\
&\times\left(\frac{1}{m}\int_{2/3}^{1}\frac{\sum_{j=1}^{m}(e^{\kappa}-1)^{j}}{A_{\kappa}^{(1+p)q}}\left|\psi'(\wp_1)\right|^{q}d\kappa+\frac{1}{m}\int_{2/3}^{1}\frac{\sum_{j=1}^{m}(e^{1-\kappa}-1)^{j}}{A_{\kappa}^{(1+p)q}}\left|\psi'(\wp_2)\right|^{q}d\kappa\right)^{1/q}\Bigg\} \\
&=\frac{\wp_1\wp_2(\wp_2^{p}-\wp_1^{p})}{p}\times\left\{\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r}\left(B_{10}\left|\psi'(\wp_1)\right|^{q}+B_{11}\left|\psi'(\wp_2)\right|^{q}\right)^{1/q}\right. \\
&\quad+\left(\frac{2}{6^{r+1}(r+1)}\right)^{1/r}\left(B_{12}\left|\psi'(\wp_1)\right|^{q}+B_{13}\left|\psi'(\wp_2)\right|^{q}\right)^{1/q} \\
&\quad\left.+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r}\left(B_{14}\left|\psi'(\wp_1)\right|^{q}d\kappa+B_{15}\left|\psi'(\wp_2)\right|^{q}\right)^{1/q}\right\},
\end{aligned}
\tag{52}
$$

which completes the proof.

Corollary 36. *Under the assumptions of Theorem 35 with $p=-1$ and $m=1$, we have the following new result:*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\psi(\wp_1)+3\psi\left(\frac{2\wp_1+\wp_2}{3}\right)+3\psi\left(\frac{\wp_1+2\wp_2}{3}\right)+\psi(\wp_2)\right]\right. \\
&\quad\left.-\frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\psi(x)dx\right|\leq(\wp_2-\wp_1)\left[\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r}\right. \\
&\quad\cdot\left(0.0623\left|\psi'(\wp_1)\right|^{q}+0.4372\left|\psi'(\wp_2)\right|^{q}\right)^{1/q} \\
&\quad+\left(\frac{1}{6^{r+1}(r+1)}\right)^{1/r}0.2188\left(\left|\psi'(\wp_1)\right|^{q}+\left|\psi'(\wp_2)\right|^{q}\right)^{1/q} \\
&\quad\left.+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r}\left(0.4372\left|\psi'(\wp_1)\right|^{q}+0.0623\left|\psi'(\wp_2)\right|^{q}\right)^{1/q}\right].
\end{aligned}
\tag{53}
$$

6. Applications

In this section, we recall the following special means of two positive number $\wp_1, \wp_2$ with $\wp_1<\wp_2$:

(1) The arithmetic mean

$$A=A(\wp_1,\wp_2)=\frac{\wp_1+\wp_2}{2} \tag{54}$$

(2) The geometric mean

$$G=G(\wp_1,\wp_2)=\sqrt{\wp_1\wp_2} \tag{55}$$

(3) The harmonic mean

$$H=H(\wp_1,\wp_2)=\frac{2\wp_1\wp_2}{\wp_1+\wp_2} \tag{56}$$

(4) The logarithmic mean

$$L=L(\wp_1,\wp_2)=\frac{\wp_2-\wp_1}{\ln\wp_2-\ln\wp_1} \tag{57}$$

These means have a lot of applications in areas and different types of numerical approximations. However, the following simple relationship are known in the literature:

$$H(\wp_1,\wp_2)\leq G(\wp_1,\wp_2)\leq L(\wp_1,\wp_2)\leq A(\wp_1,\wp_2). \tag{58}$$

Proposition 37. *Let $0<\wp_1<\wp_2$ and $p\geq 1$. Then we get the following inequality*

$$
\begin{aligned}
\frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^{j}}H_p\left(\wp_1^{p},\wp_2^{p}\right)&\leq\frac{p\wp_1^{p}\wp_2^{p}}{\wp_2^{p}-\wp_1^{p}}\left(\frac{\wp_2^{1-p}-\wp_1^{1-p}}{1-p}\right) \\
&\leq A(\wp_1,\wp_2)\frac{2}{m}\sum_{j=1}^{m}[e-2]^{j}.
\end{aligned}
\tag{59}
$$

Proof. Taking $\psi(\nu)=\nu$ for $\nu>0$ in Theorem 21, then, inequality (59) is easily captured.

Proposition 38. *Let $0<\wp_1<\wp_2$ and $p\geq 1$. Then, we get the following inequality:*

$$
\begin{aligned}
\frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^{j}}H_{2p}^{-1}\left(\wp_1^{p},\wp_2^{p}\right)&\leq\frac{p\wp_1^{p}\wp_2^{p}}{\wp_2^{p}-\wp_1^{p}}\left(\frac{\wp_2^{(1/2)-p}-\wp_1^{(1/2)-p}}{(1/2)-p}\right)^{-1} \\
&\leq A^{-1}\left(\sqrt{\wp_1},\sqrt{\wp_2}\right)\frac{2}{m}\sum_{j=1}^{m}[e-2]^{j}.
\end{aligned}
\tag{60}
$$

Proof. Taking $\psi(\nu)=1/\sqrt{\nu}$ for $\nu>0$ in Theorem 21, then, inequality (60) is easily captured.

Proposition 39. *Let $0<\wp_1<\wp_2$ and $p\geq 1$. Then, we get the following inequality:*

$$
\begin{aligned}
\frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^{j}}H\left(\wp_1^{p},\wp_2^{p}\right)&\leq\frac{p\wp_1^{p}\wp_2^{p}}{\wp_2^{p}-\wp_1^{p}}\left(\frac{\wp_2-\wp_1}{L(\wp_1,\wp_2)}\right) \\
&\leq A\left(\wp_1^{p},\wp_2^{p}\right)\frac{2}{m}\sum_{j=1}^{m}[e-2]^{j}.
\end{aligned}
\tag{61}
$$

Proof. Taking $\psi(\nu)=\nu^{p}$ for $\nu>0$ in Theorem 21, then, inequality (61) is easily captured.

Proposition 40. *Let $0<\wp_1<\wp_2$ and $p\geq 1$. Then, we get the following inequality:*

$$
\begin{aligned}
\frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^{j}}H_p^{2}\left(\wp_1^{p},\wp_2^{p}\right)&\leq\frac{p\wp_1^{p}\wp_2^{p}}{\wp_2^{p}-\wp_1^{p}}\left(\frac{\wp_2^{2-p}-\wp_1^{2-p}}{2-p}\right) \\
&\leq A\left(\wp_1^{2},\wp_2^{2}\right)\frac{2}{m}\sum_{j=1}^{m}[e-2]^{j}.
\end{aligned}
\tag{62}
$$

Proof. Taking $\psi(\nu)=\nu^{2}$ for $\nu>0$ in Theorem 21, then, inequality (62) is easily captured.

Proposition 41. *Let $0<\wp_1<\wp_2$ and $p\geq 1$. Then, we get the following inequality:*

$$\frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^{j}}\ln G(\wp_1,\wp_2)\leq \frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{-\ln x}{x^{p+1}}dx \leq \ln H_p\left(\wp_1^p,\wp_2^p\right)\frac{2}{m}\sum_{j=1}^{m}[e-2]^j. \tag{63}$$

Proof. Taking $\psi(v)=-\ln v$ for $v>0$ in Theorem 21, then, inequality (63) is easily captured.

Proposition 42. *Let $0<\wp_1<\wp_2$. Then, we get the following inequality:*

$$\frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^{j}}e^{H(\wp_1,\wp_2)}\leq \frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{e^x}{x^{p+1}}dx \leq A\left(e^{\wp_1},e^{\wp_2}\right)\frac{2}{m}\sum_{j=1}^{m}[e-2]^j. \tag{64}$$

Proof. Taking $\psi(v)=e^v$ for $v>0$ in Theorem 21, then, inequality (64) is easily captured.

Proposition 43. *Let $0<\wp_1<\wp_2$. Then, we get the following inequality:*

$$A(\sin\wp_1,\sin\wp_2)\frac{2}{m}\sum_{j=1}^{m}[e-2]^j\leq \frac{p\wp_1^p\wp_2^p}{\wp_2^p-\wp_1^p}\int_{\wp_1}^{\wp_2}\frac{\sin x}{x^{p+1}}dx \leq \frac{m}{2\sum_{j=1}^{m}\left(\sqrt{e}-1\right)^{j}}\sin H_p(\wp_1,\wp_2). \tag{65}$$

Proof. Taking $\psi(v)=\sin(-v)$ for $v\in(0,(\pi/2))$ in Theorem 21, then, inequality (65) is easily captured.

Remark 44. The above discussed means, namely, arithmetic, geometric, harmonic, and logarithmic are well known in literature because these means have remarkable applications in machine learning, probability, statistics, and numerical approximation [24]. But, in the future, we will try to find the applications of the He Chengtian mean (also called as He Chengtian average), which was introduced by the first time a famous ancient Chinese mathematician He Chengtian [25]. The He Chengtian average was extended to solve nonlinear oscillators and it is called as He's max–min approach (also called as He's max–min method), which was further developed into a frequency–amplitude formulation for nonlinear oscillators [26, 27].

7. Conclusion

We have introduced and investigated some algebraic properties of a new class of functions, namely, m–poly p–harmonic exp convex. We showed that our new introduced class of function have some nice properties. We proved that our new introduced class is very larger with respect to the known class of functions, like m–polynomial convex and m–polynomial harmonically convex. A new version of Hermite–Hadamard type inequality and an integral identity for the differentiable function are obtained. It is high time to find the applications of these inequalities along with efficient numerical methods. We believe that our new class of functions will have a very deep research in this fascinating field of inequalities and also in pure and applied sciences. The interesting techniques and wonderful ideas of this paper can be extended on the coordinates along with fractional calculus. In the future, our goal is that we will continue our research work in this direction furthermore.

Data Availability

Data will be provided on request to the first author.

Conflicts of Interest

The authors declare that there are no conflicts of interest associated with this publication.

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Research Article

A New Subclass of Analytic Functions Related to Mittag-Leffler Type Poisson Distribution Series

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The object of this work is to an innovation of a class $k-\tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ in Y with negative coefficients, further determining coefficient estimates, neighborhoods, partial sums, convexity, and compactness of this specified class.

1. Introduction

Let $Y=\{\omega : |\omega|<1\}$ be an open unit disc in $\mathscr{C}$. Consider the analytic class function A that indicates j specified on the unit disk along with normalization

$$j(0)=0, j'(0)=1 \tag{1}$$

and has the form

$$j(\omega)=\omega+\sum_{n=2}^{\infty} o_n \omega^n, \tag{2}$$

indicated by S, the subclass of A lying of functions that are univalent in Y. A function $j \in A$ is stated in $k-\mathrm{UST}(\iota)$, and $k-\mathrm{UCV}(\iota)$, "the class of k-uniformly starlike functions and convex functions of order $\iota, 0 \le \iota < 1$," if and only if

$$\begin{aligned} &\Re\left\{\frac{\nu j'(\nu)}{j(\nu)}\right\} > k\left|\frac{\nu j'(\nu)}{j(\nu)}-1\right| + \iota, (k \ge 0), \\ &\Re\left\{1+\frac{\nu j''(\nu)}{j'(\nu)}\right\} > k\left|\frac{\nu j''(\nu)}{j'(\nu)}\right| + \iota, (k \ge 0). \end{aligned} \tag{3}$$

The classes UCV and UST were introduced by Goodman [1] and studied by Ronning [2]. Due to Sakaguchi [3], the class ST_s of starlike functions w.r.t. symmetric points are defined as follows.

The function $j \in A$ is stated to be starlike w.r.t. symmetric points in Y

$$\Leftrightarrow \Re\left\{\frac{2\omega j'(\omega)}{j(\omega)-j(-\omega)}\right\} > 0, (\omega \in Y). \tag{4}$$

Owa et al. [4] defined the class $\mathrm{ST}_s(\alpha, \varsigma)$ as complies

$$\Re\left\{\frac{(1-\varsigma)\omega j'(\omega)}{j(\omega)-j(\varsigma\omega)}\right\} > \alpha, (\omega \in Y), \tag{5}$$

where $0 \le \alpha < 1$, $|\varsigma| \le 1, \varsigma \ne 1$. Here, $\mathrm{ST}_s(0,-1)=\mathrm{ST}_s$ and $\mathrm{ST}_s(\alpha,-1)=\mathrm{ST}_s(\alpha)$ is named Sakaguchi function of order α.

In recent years, binomial distribution series, Pascal distribution series, Poisson distribution series, etc., play important role in GFT. The sufficient ways were innovated for ST, UCV for some special functions in the GFT. By the motivation of the works [5–13], we develop this work.

In [14], Porwal, Poisson distribution series, gives a gracious application on analytic functions; it exposed a new way of research in GFT. Subsequently, the authors turned on the distribution series of confluent hypergeometric, hypergeometric, binomial, and Pascal and prevail necessary and sufficient stipulation for certain classes of univalent functions.

Lately, Porwal and Dixit [15] innovate Mittag-Leffler type Poisson distribution and prevailed moments, mgf, which is an abstraction of Poisson distribution using the definition of this distribution. Bajpai [16] innovated Mittag-Leffler type Poisson distribution series and discussed about necessary and sufficient conditions.

The probability mass function for this is

$$P(\hbar,\tau,\upsilon;n)(\omega)=\frac{\hbar^n}{E_{\tau,\upsilon}(\hbar)\Gamma(\tau n+\upsilon)},(n=0,1,2,\cdots),\quad(6)$$

where

$$E_{\tau,\upsilon}(\omega)=\sum_{n=0}^{\infty}\frac{\omega^n}{\Gamma(\tau n+\upsilon)},(\tau,\upsilon\in\mathscr{C},\mathfrak{R}(\tau)>0,\mathfrak{R}(\upsilon)>0).\quad(7)$$

The series (7) converges for all finite values of ω if $R(\tau)>0,\mathfrak{R}(\upsilon)>0$. This suggest that the series $E_{\tau,\upsilon}(\hbar)$ is convergent for $\tau,\upsilon,\hbar>0$. For further details of the study, see [17]. It is easy to see that the series (7) are reduced to exponential series for $\tau=\upsilon=1$.

A variable x is said to have Poisson distribution if it takes the values $0,1,2,3,\cdots$ with probabilities $e^{-\hbar},\hbar e^{-\hbar}/1!,\hbar^2e^{-\hbar}/2!,\hbar^3e^{-\hbar}/3!,\cdots$, respectively, where $\hbar$ is called the parameter. Thus,

$$P(x=n)=\frac{\hbar^n e^{-\hbar}}{n!},n=0,1,2,\cdots.\quad(8)$$

This motivates researchers (see [15, 17, 18], etc.) to introduce a new probability distribution if it assumes nonnegative values and its probability mass function is given by (6). It is easy to see that $P(\hbar,\tau,\upsilon;n)(\omega)$ given by (6) is the probability mass function because

$$P(\hbar,\tau,\upsilon;n)(\omega)\geq 0,\ \sum_{n=0}^{\infty}P(\hbar,\tau,\upsilon;n)(\omega)=1.\quad(9)$$

It is worthy to note that for $\alpha=\beta=1$, it reduces to the Poisson distribution.

Also note that

$$E_{\tau,\upsilon}(\omega)=\omega\Gamma(\upsilon)E_{\tau,\tau+\upsilon}(\omega).\quad(10)$$

In [18], Chakrabortya and Ong introduced and discussed about the Mittag-Leffler function distribution—a new generalization of hyper-Poisson distribution. The Mittag-Leffler type Poisson distribution series was innovated by Porwal and Dixit [15] and given as

$$K(\hbar,\tau,\upsilon)(\omega)=\omega+\sum_{n=2}^{\infty}\frac{\hbar^{n-1}}{\Gamma(\tau(\mathrm{n}-1)+\upsilon)E_{\tau,\upsilon}(\hbar)}\omega^n.\quad(11)$$

Equation (11) is a normalization function in S, since $K(\hbar,\tau,\upsilon)(0)=0$ and $K'(\hbar,\tau,\upsilon)(0)=1$. After that, in [19], Porwal et al. discussed about the geometric properties of (11).

For $j\in A$ given by (2) and $l(\omega)$ given by

$$l(\omega)=\omega+\sum_{n=2}^{\infty}b_n\omega^n,\quad(12)$$

their convolution, indicated by $(j*l)$, is given by

$$(j*l)(\omega)=\omega+\sum_{n=2}^{\infty}o_nb_n\omega^n=(l*j)(\omega),(\omega\in Y).\quad(13)$$

Note that $j*l\in A$.

Next, we innovate the convolution operator

$$\mathscr{J}(\hbar,\tau,\upsilon)j(\omega)=K(\hbar,\tau,\upsilon)*j(\omega)=\omega+\sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau,\upsilon)o_n\omega^n,\quad(14)$$

where $\varphi_{\hbar}^{n}(\tau,\upsilon)=\hbar^{n-1}/(\Gamma(\tau(n-1)+\upsilon)E_{\tau,\upsilon}(\hbar))$.

Then, using linear operator $\mathscr{J}(\hbar,\tau,\upsilon)$, we exemplifier a contemporary subclass of functions in A.

Definition 1. If $j\in A$ is named in the class $k-\mathrm{UST}_s(\hbar,\upsilon,\tau,\iota,\varsigma)$ if for all $\omega\in Y$

$$\mathfrak{R}\left\{\frac{(1-\varsigma)\omega(\mathscr{J}(\hbar,\tau,\upsilon)j(\omega))'}{\mathscr{J}(\hbar,\tau,\upsilon)j(\omega)-\mathscr{J}(\hbar,\tau,\upsilon)j(\varsigma\omega)}\right\}\geq k\left|\frac{(1-\varsigma)\omega(\mathscr{J}(\hbar,\tau,\upsilon)j(\omega))'}{\mathscr{J}(\hbar,\tau,\upsilon)j(\omega)-\mathscr{J}(\hbar,\tau,\upsilon)j(\varsigma\omega)}-1\right|+\iota,\quad(15)$$

for $k\geq 0,|\varsigma|\leq 1,\varsigma\neq 1,0\leq\iota<1$.

Moreover, we named that $j\in k-\mathrm{UST}_s(\hbar,\upsilon,\tau,\iota,\varsigma)$ is in the subclass $k-\tilde{\mathrm{U}}\mathrm{ST}_s(\hbar,\upsilon,\tau,\iota,\varsigma)$ if $j(\omega)$ is of the compiling form

$$j(\omega)=\omega-\sum_{n=2}^{\infty}o_n\omega^n,\quad o_n\geq 0,n\in\mathbb{N},\omega\in Y.\quad(16)$$

In this work, we analyze the bounds for coefficient, partial sums, and some neighborhood outcomes of the class $k-\tilde{\mathrm{U}}\mathrm{ST}_s(\hbar,\upsilon,\tau,\iota,\varsigma)$.

To claim our outcomes, we adopt lemmas [20].

Lemma 2. *Let w be a complex number. Then, $\alpha\leq\mathfrak{R}(w)\Leftrightarrow|w-(1+\alpha)|\leq|w+(1-\alpha)|$.*

Lemma 3. *Suppose a complex number w with real numbers α, ι. Then,*

$$\Re(w) > \alpha|w-1| + \iota \Leftrightarrow \Re\left\{w\left(1+\alpha e^{i\rho}\right) - \alpha e^{i\rho}\right\} > \iota, (-\pi < \rho \leq \pi). \tag{17}$$

2. Coefficient Bounds

Theorem 4. *A function j given by (16) is in $k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$*

$$\Leftrightarrow \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)|n(k+1) - j_n(k+\iota)|o_n \leq 1-\iota, \tag{18}$$

here $k \geq 0, |\varsigma| \leq 1, \varsigma \neq 1, 0 \leq \iota < 1$ and $j_n = 1 + \varsigma + \cdots + \varsigma^{n-1}$. The result is sharp for $j(\omega)$ is

$$j(\omega) = \omega - \frac{1-\iota}{\varphi_{\hbar}^{n}(\tau, \upsilon) \mid n(k+1) - j_n(k+\iota) \mid} \omega^n. \tag{19}$$

Proof. By Definition 1, we have

$$\Re\left\{\frac{(1-\varsigma)\omega(\mathcal{I}(\hbar, \tau, \upsilon)j(\omega))}{\mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)}\left(1+ke^{i\rho}\right) - ke^{i\rho}\right\} \geq \iota, -\pi < \rho \leq \pi. \tag{20}$$

Let $H(\omega) = (1-\varsigma)\omega(\mathcal{I}(\hbar, \tau, \upsilon)j(\omega))(1+ke^{i\rho}) - ke^{i\rho}[\mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)]$ and $K(\omega) = \mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)$.

By Lemma 2, (20) becomes

$$|H(\omega) + (1-\iota)K(\omega)| \geq |H(\omega) - (1+\iota)K(\omega)|, \text{for } 0 \leq \iota < 1. \tag{21}$$

But

$$\begin{aligned} |H(\omega) + (1-\iota)K(\omega)| &= |(1-\varsigma) \\ &\cdot \left\{(2-\iota)\omega - \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)(n + j_n(1-\iota))o_n\omega^n - ke^{i\rho}\sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)(n-j_n)o_n\omega^n\right\}| \geq |1-\varsigma| \\ &\cdot \left\{(2-\iota) \mid \omega| - \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon) \mid n + j_n(1-\iota) \mid o_n \mid \omega^n| - k\sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon) \mid n - j_n \mid o_n \mid \omega^n \mid\right\}. \end{aligned} \tag{22}$$

Also

$$\begin{aligned} |H(\omega) - (1+\iota)K(\omega)| &= |(1-\varsigma) \\ &\cdot \left\{-\iota\omega - \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)(n - j_n(1+\iota))o_n\omega^n - ke^{i\rho}\sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)(n-j_n)o_n\omega^n\right\}| \leq |1-\varsigma| \\ &\cdot \left\{\iota \mid \omega| + \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon) \mid n - j_n(1+\iota) \mid o_n \mid \omega^n| + k\sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon) \mid n - j_n \mid o_n \mid \omega^n \mid\right\}. \end{aligned} \tag{23}$$

So

$$\begin{aligned} &|H(\omega) + (1-\iota)K(\omega)| - |H(\omega) - (1+\iota)K(\omega)| \geq |1-\varsigma| \\ &\quad \times \left\{2(1-\iota) \mid \omega| - \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)[|n + j_n(1-\iota)| + |n - j_n(1+\iota)| + 2k \mid n - j_n \mid]o_n \mid \omega^n \mid\right\} \\ &\geq 2(1-\iota)|\omega| - \sum_{n=2}^{\infty} 2\varphi_{\hbar}^{n}(\tau, \upsilon)|n(k+1) - j_n(k+\iota)|o_n|\omega^n| \geq 0. \end{aligned} \tag{24}$$

Conversely, suppose (18) holds. Then, we have

$$\Re\left\{\frac{(1-\varsigma)\omega(\mathcal{I}(\hbar, \tau, \upsilon)j(\omega))\left(1+ke^{i\rho}\right) - ke^{i\rho}[\mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)]}{\mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)}\right\} \geq \iota. \tag{25}$$

Opting ω values on the +ve real axis, where $0 \leq |\omega| = r < 1$, then

$$\Re\left\{\frac{(1-\iota) - \sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)\left[n\left(1+ke^{i\rho}\right) - j_n\left(\iota + ke^{i\rho}\right)\right]o_n\omega^{n-1}}{1 - \sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)j_n o_n\omega^{n-1}}\right\} \geq 0. \tag{26}$$

Since $\Re(-e^{i\rho}) \geq -|e^{i\rho}| = -1$, then

$$\Re\left\{\frac{(1-\iota) - \sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)[n(1+k) - j_n(\iota+k)]o_n r^{n-1}}{1 - \sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)j_n o_n r^{n-1}}\right\} \geq 0. \tag{27}$$

Taking limit r tends to 1^-, we obtain our needed result.

Corollary 5. *If $j(\omega) \in k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$, then*

$$o_n \leq \frac{1}{(1-\iota)^{-1}\varphi_{\hbar}^{n}(\tau, \upsilon) \mid n(k+1) - j_n(k+\iota) \mid}, \tag{28}$$

where $k \geq 0, |\varsigma| \leq 1, \varsigma \neq 1, 0 \leq \iota < 1$ and $j_n = 1 + \varsigma + \cdots + \varsigma^{n-1}$.

3. Neighborhood Properties

The notion of β-neighbourhood was innovated and studied by Goodman [21] and Ruscheweyh [22].

Definition 6. We define the β-neighborhood of a mapping $j \in A$ and indicate by $N_\beta(j)$ lying of all mappings $g(\omega) = \omega - \sum_{n=2}^{\infty} b_n\omega^n \in S(b_n \geq 0, n \in \mathbb{N})$ satisfies the condition

$$\sum_{n=2}^{\infty} \frac{\varphi_{\hbar}^{n}(\tau, \upsilon) \mid n(k+1) - j_n(k+\iota) \mid}{1-\iota}|o_n - b_n| \leq 1-\beta, \tag{29}$$

where $k \geq 0, |\varsigma| \leq 1, \varsigma \neq 1, 0 \leq \iota < 1, \beta \geq 0$ and $j_n = 1 + \varsigma + \cdots + \varsigma^{n-1}$.

Theorem 7. *Let $j(\omega) \in k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ and every real ρ we get $\iota(e^{i\rho} - 1) - 2e^{i\rho} \neq 0$. For any $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \beta(\beta \geq 0)$, if j fulfills*

$$\frac{j(\omega) + \varepsilon\omega}{1+\varepsilon} \in k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma), \tag{30}$$

then, $N_\beta(j) \subset k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$.

Proof. It is evident that $j \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$

$$\Leftrightarrow \left|\frac{(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))\left(1+ke^{i\rho}\right) - \left(ke^{i\rho}+1+\iota\right)(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega))}{(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))(1+ke^{i\rho}) + (1-ke^{i\rho}-\iota)(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega))}\right| < 1, \quad (31)$$

where $-\pi \le \rho \le \pi$ for some $s \in \mathbb{C}$ and $|s| = 1$, we obtain

$$\frac{(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))\left(1+ke^{i\rho}\right) - \left(ke^{i\rho}+1+\iota\right)(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega))}{(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))(1+ke^{i\rho}) + (1-ke^{i\rho}-\iota)(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega))} \neq s. \quad (32)$$

In other words,

$$\begin{aligned}&(1-s)(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))\left(1+ke^{i\rho}\right) - \left(ke^{i\rho}+1+\iota+s\left(-1+ke^{i\rho}+\iota\right)\right)\\&\quad\times(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega)) \neq 0 \Rightarrow \omega\\&\quad - \sum_{n=2}^{\infty}\frac{\varphi_\hbar^n(\tau,\upsilon)\left((n-j_n)\left(1+ke^{i\rho}-ske^{i\rho}\right) - s(n+j_n) - j_n\iota(1-s)\right)}{\iota(s-1)-2s}\omega^n \neq 0.\end{aligned} \quad (33)$$

However, $j \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma) \Leftrightarrow (j * h)/\omega \neq 0, \omega \in Y - \{0\}$, where $h(\omega) = \omega - \sum_{n=2}^{\infty} c_n\omega^n$ and

$$\begin{aligned}c_n &= \frac{\varphi_\hbar^n(\tau,\upsilon)\left((n-j_n)\left(1+ke^{i\rho}-ske^{i\rho}\right) - s(n+j_n) - j_n\iota(1-s)\right)}{\iota(s-1)-2s} \Rightarrow |c_n|\\&\le \frac{\varphi_\hbar^n(\tau,\upsilon)|n(1+k) - j_n(k+\iota)|}{1-\iota},\end{aligned} \quad (34)$$

since $((j(\omega)+\varepsilon\omega)/(1+\varepsilon)) \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$; therefore, $\omega^{-1}((j(\omega)+\varepsilon\omega)/(1+\varepsilon) * h(\omega)) \neq 0$, which implies

$$\frac{(j*h)(\omega)}{(1+\varepsilon)\omega} + \frac{\varepsilon}{1+\varepsilon} \neq 0. \quad (35)$$

Now, suppose $|(j*h)(\omega)/\omega| < \beta$. Then, by (35),

$$\left|\frac{(j*h)(\omega)}{(1+\varepsilon)\omega} + \frac{\varepsilon}{1+\varepsilon}\right| > \frac{|\varepsilon|-\beta}{|1+\varepsilon|} \ge 0, \quad (36)$$

which contradicts by $|\varepsilon| < \beta$, and thus, we arrive $|(j*h)(\omega)/\omega| \ge \beta$.

If $g(\omega) = \omega - \sum_{2\le n\le\infty} b_n\omega^n \in N_\beta(j)$, then

$$\begin{aligned}\beta - \left|\frac{(g*h)(\omega)}{\omega}\right| &\le \left|\frac{((j-g)*h)(\omega)}{\omega}\right|\\&< \sum_{n=2}^{\infty}\frac{\varphi_\hbar^n(\tau,\upsilon)\,|\,n(1+k) - j_n(k+\iota)\,|}{1-\iota}|o_n - b_n| \le \beta.\end{aligned} \quad (37)$$

4. Partial Sums

Theorem 8. *If the function j is of the form (2) fulfill (18) then*

$$\Re\left\{\frac{j(\omega)}{j_m(\omega)}\right\} \ge 1 - \frac{1}{\chi_{m+1}}, \quad (38)$$

$$\chi_n = \begin{cases} 1, & \text{if } 2 \le n \le m; \\ \chi_{m+1}, & \text{if } m+1 \le n \le \infty, \end{cases} \quad (39)$$

where

$$\chi_n = \frac{\varphi_\hbar^n(\tau,\upsilon)\,|\,n(1+k) - j_n(k+\iota)\,|}{1-\iota}. \quad (40)$$

The estimate (38) is sharp, for every m, with

$$j(\omega) = \omega + \frac{\omega^{m+1}}{\chi_{m+1}}. \quad (41)$$

Proof. Now, we define $\wp$; we can define

$$\begin{aligned}\frac{1+\wp(\omega)}{1-\wp(\omega)} &= \chi_{m+1}\left\{\frac{j(\omega)}{j_m(\omega)} - \left(1 - \frac{1}{\chi_{m+1}}\right)\right\}\\&= \left[\frac{1 + \sum_{2\le n\le m} o_n\omega^{n-1} + \chi_{m+1}\sum_{n=m+1}^{\infty} o_n\omega^{n-1}}{1 + \sum_{n=2}^{m} o_n\omega^{n-1}}\right].\end{aligned} \quad (42)$$

Then, from (42), we attain

$$\begin{aligned}\wp(\omega) &= \frac{\chi_{m+1}\sum_{n=m+1}^{\infty} o_n\omega^{n-1}}{2 + 2\sum_{n=2}^{m} o_n\omega^{n-1} + \chi_{m+1}\sum_{n=m+1}^{\infty} o_n\omega^{n-1}},\\|\wp(\omega)| &\le \frac{\chi_{m+1}\sum_{n=m+1}^{\infty} o_n}{2 - 2\sum_{n=2}^{m} o_n - \chi_{m+1}\sum_{n=m+1}^{\infty} o_n}.\end{aligned} \quad (43)$$

Now, $|\wp(\omega)| \leq 1$ if

$$2\chi_{m+1}\sum_{n=m+1}^{\infty} o_n \leq 2-2\sum_{n=2}^{m} o_n \Rightarrow \sum_{n=2}^{m} o_n + \chi_{m+1}\sum_{m+1}^{\infty} o_n \leq 1. \tag{44}$$

It is enough to prove that the LHS of (44) is bounded above by $\sum_{n=2}^{\infty}\chi_n o_n$, which implies

$$\sum_{n=2}^{m}(\chi_n-1)o_n + \sum_{n=m+1}^{\infty}(\chi_n-\chi_{m+1})o_n \geq 0. \tag{45}$$

To show that the mapping disposed by (41) gives the exact result, we notice that for $\omega = re^{i\pi/n}$,

$$\frac{j(\omega)}{j_m(\omega)} = 1 + \frac{\omega^m}{\chi_{m+1}}. \tag{46}$$

Taking limit ω tends to 1^-, we have

$$\frac{j(\omega)}{j_m(\omega)} = 1 - \frac{1}{\chi_{m+1}}. \tag{47}$$

Hence, the proof is completed.

Theorem 9. *If j of the form (2) which fulfill (18) then*

$$\Re\left\{\frac{j_m(\omega)}{j(\omega)}\right\} \geq \frac{\chi_{m+1}}{1+\chi_{m+1}}. \tag{48}$$

The result is sharp with (41).

Proof. Define

$$\begin{aligned}\frac{1+\wp(\omega)}{1-\wp(\omega)} &= (1+\chi_{m+1})\left\{\frac{j_m(\omega)}{j(\omega)} - \frac{\chi_{m+1}}{1+\chi_{m+1}}\right\}\\ &= \left[\frac{1+\sum_{n=2}^{m}o_n\omega^{n-1} - \chi_{m+1}\sum_{n=m+1}^{\infty}o_n\omega^{n-1}}{1+\sum_{n=2}^{\infty}o_n\omega^{n-1}}\right],\end{aligned} \tag{49}$$

where

$$\begin{aligned}\wp(\omega) &= \frac{(1+\chi_{m+1})\sum_{n=m+1}^{\infty}o_n\omega^{n-1}}{-(2+2\sum_{n=2}^{m}o_n\omega^{n-1} - (1-\chi_{m+1})\sum_{n=m+1}^{\infty}o_n\omega^{n-1})},\\ |\wp(\omega)| &\leq \frac{(1+\chi_{m+1})\sum_{n=m+1}^{\infty}o_n}{2-2\sum_{n=2}^{m}o_n + (1-\chi_{m+1})\sum_{n=m+1}^{\infty}o_n} \leq 1.\end{aligned} \tag{50}$$

This last inequality is

$$\sum_{n=2}^{m} o_n + \chi_{m+1}\sum_{n=m+1}^{\infty} o_n \leq 1. \tag{51}$$

It is enough to prove that the LHS of (51) is bounded above by $\sum_{n=2}^{\infty}\chi_n o_n$, which implies

$$\sum_{2\leq n\leq m}(\chi_n-1)o_n + \sum_{n=m+1}^{\infty}(\chi_n-\chi_{m+1})o_n \geq 0. \tag{52}$$

This completes the proof.

Theorem 10. *If j of the form (2) fulfill (18) then*

$$\Re\left\{\frac{j'(\omega)}{j_m'(\omega)}\right\} \geq 1 - \frac{m+1}{\chi_{m+1}}, \tag{53}$$

$$\Re\left\{\frac{j_m'(\omega)}{j'(\omega)}\right\} \geq \frac{\chi_{m+1}}{1+m+\chi_{m+1}}, \tag{54}$$

where

$$\chi_n \geq \left\{\begin{array}{ll} 1, & \text{if } 1 \leq n \leq m \\ n\dfrac{\chi_{m+1}}{m+1}, & \text{if } m+1 \leq n \leq \infty \end{array}\right\}. \tag{55}$$

and χ_n is given by (40). The computation in (53) and (54) are sharp with (41).

Theorem 11. *$k-\tilde{U}ST_s(\hbar, \upsilon, \iota, \varsigma)$ is a convex and compact subset of T.*

Proof. Suppose $j_d \in k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$,

$$j_d(\omega) = \omega - \sum_{n=2}^{\infty}|a_{d,n}|\omega^n. \tag{56}$$

Then, for $0 \leq \psi < 1$, let $j_1, j_2 \in k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ be given by (56). Then,

$$\begin{aligned}\xi(\omega) &= \psi j_1(\omega) + (1-\psi)j_2(\omega) = \psi\left(\omega - \sum_{n=2}^{\infty}|a_{1,n}|\omega^n\right)\\ &\quad + (1-\psi)\left(\omega - \sum_{n=2}^{\infty}|a_{2,n}|\omega^n\right)\\ &= \omega - \sum_{n=2}^{\infty}\left(\psi|a_{1,n}| + (1-\psi)|a_{2,n}|\right)\omega^n,\end{aligned}$$

$$\begin{aligned}&\sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau,\upsilon)(n(k+1) - j_n(k+\iota))(\psi\,|\,a_{1,n}| + (1-\psi)\,|\,a_{2,n}\,|)\\ &\quad \leq \psi(1-\iota) + (1-\psi)(1-\iota) = 1-\iota.\end{aligned} \tag{57}$$

Then, $\xi(\omega) = \psi j_1(\omega) + (1-\psi)j_2(\omega) \in k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$. Therefore, $k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ is convex. Now, we have to show $k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ is compact.

For $j_d \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma), \varsigma \in \mathbb{N}$ and $|\omega| < r (0 < r < 1)$, then we arrive

$$\begin{aligned} |j_d(\omega)| &\le r + \sum_{n=2}^{\infty} |a_{d,n}| r^n \le r + \sum_{n=2}^{\infty} \varphi_{\hbar}^n(\tau, \upsilon) |n(k+1) - j_n(k+\iota)| |a_{d,n}| r^n \\ &\le r + (1+r) r^n. \end{aligned} \tag{58}$$

Therefore, $k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ is uniformly bounded.

Let $j_d(\omega) = \omega - \sum_{n=2}^{\infty} | a_{d,n} | \omega^n, \omega \in Y, d \in \mathbb{N}$.

Also, let $j(\omega) = \omega - \sum_{n=2}^{\infty} o_n \omega^n$. Then, by Theorem 4, we get

$$\sum_{n=2}^{\infty} \varphi_{\hbar}^n(\tau, \upsilon) |n(k+1) - j_n(k+\iota)| |o_n| \le 1 - \iota. \tag{59}$$

Assuming $j_d \longrightarrow j$, then we have $a_{d,n} \longrightarrow o_n$ as $n \longrightarrow \infty$, $(d \in \mathbb{N})$.

Let $\{\rho_n\}$ be the array of partial sums of the series

$$\sum_{n=2}^{\infty} \varphi_{\hbar}^n(\tau, \upsilon) |n(k+1) - j_n(k+\iota)| |o_n|. \tag{60}$$

Then, $\{\rho_n\}$ is a nondecreasing array and by (59), it is bounded above by $1 - \iota$.

Thus, it is convergent and

$$\sum_{n=2}^{\infty} \varphi_{\hbar}^n(\tau, \upsilon) |n(k+1) - j_n(k+\iota)| |a_{d,n}| = \lim_{n \to \infty} \rho_n \le 1 - \iota. \tag{61}$$

Therefore, $j \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ and the class $k - \tilde{\mathrm{U}}\mathrm{S}$ $\mathrm{T}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ is closed.

Data Availability

Our manuscript does not contain any data.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. And all the authors have read and approved the final version of the manuscript.

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Research Article

On Janowski Type Harmonic Meromorphic Functions with respect to Symmetric Point

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In this paper, we define a new class of Sakaguchi type-meromorphic harmonic functions in the Janowski domain that are starlike with respect to symmetric point. Furthermore, we investigate some important geometric properties like sufficiency criteria, distortion bound, extreme point theorem, convex combination, and weighted means.

1. Introduction and Definitions

One of the contemporary developments in Mathematics is the solicitations of harmonic analysis in other fields. Like various other fields, it has immensely influenced and nurtured the branch of geometric function theory. Jahangiri et al. [1] defined and studied a subclass of harmonic and univalent functions. Another example of such work would be an article of Porwal and Dixit [2], who used a certain convolution operator involving hypergeometric functions to define a class of univalent functions. As a consequence, many mathematicians generalized many ideas of this field and various important results with the help of some operators; the work of Porwal et al. [3], Porwal et al. [4], and Porwal and Dixit [5] are worth mentioning here. Recently, some subclasses of harmonic functions were investigated by Arif et al. [6] and Khan et al. [7]. To start with, we give preliminaries which will be useful in understanding the concepts of this research.

A real-valued function $u(x, y)$ is said to be harmonic in a domain $\mathbb{D} \subset \mathbb{C}$ if it has a continuous second partial derivative and satisfy the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a complex domain $\mathbb{U}$ if both its real and imaginary parts are real harmonic in $\mathbb{U}$. In any simply connected domain $\mathbb{U} \subset \mathbb{C}$, one can write $f = h + \bar{g}$, where h and g are analytic in $\mathbb{U}$. The class of such functions is denoted by $\mathscr{H}$. The condition $|h'(z)| > |g'(z)|$ is necessary and sufficient for f to be locally univalent and sense preserving in $\mathbb{U}$, see [8]. There are different papers on univalent harmonic functions defined in unit disc $\mathbb{D} = \{z : |z| < 1\}$, for details, see [9–14]. For $z \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}$, in the punctured open unit disc and let $\mathscr{M}_{\mathscr{H}}$ denote the class of functions

$$f(z)=h(z)+\overline{g(z)}=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n+\overline{\sum_{n=1}^{\infty}b_nz^n}, \quad (2)$$

which are harmonic in $\mathbb{D}^*$ where h is analytic in $\mathbb{D}^*$ and has a simple pole at the origin with residue 1, while g is analytic in $\mathbb{D}$. The class $\mathscr{M}_{\mathscr{H}}$ was studied in [15–17]. Furthermore, denoted by $\mathscr{M}_{\overline{\mathscr{H}}}$, a subclass of $\mathscr{M}_{\mathscr{H}}$, consisting a functions of the form

$$f(z)=h(z)+\overline{g(z)}=\frac{1}{z}+\sum_{n=1}^{\infty}|a_n|z^n-\sum_{n=1}^{\infty}|b_n|\overline{z}^n, \quad (3)$$

which are harmonic univalent in punctured unit disc $\mathbb{D}^*$.

For functions $f\in\mathscr{M}_{\mathscr{H}}$ given by (2) and $F\in\mathscr{M}_{\mathscr{H}}$ given by

$$F(z)=H(z)+\overline{G(z)}=\frac{1}{z}+\sum_{n=1}^{\infty}A_nz^n+\overline{\sum_{n=1}^{\infty}B_nz^n}, \quad (4)$$

we recall the Hadamard product (or convolution) of f and F by

$$(f*F)(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nA_nz^n+\sum_{n=1}^{\infty}\overline{b_nB_nz^n}\,(z\in\mathbb{D}^*). \quad (5)$$

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathscr{H}$ such that $(f*F)(z)$ exists for any $f\in\mathscr{H}$, and for various choices of F, we get different linear operators which have been studied in the recent past.

Recently, Khan et al. [18] introduced and studied a class of meromorphic starlike functions with respect to symmetric point in circular domain i.e.,

$$-\frac{2zf'(z)}{f(z)-f(-z)}\prec\frac{1+Az}{1+Bz}. \quad (6)$$

Motivated from the above discussion on harmonic functions and class of meromorphic starlike functions with respect to symmetric point, we introduced the class of meromorphic harmonic univalent functions as:

Let $-1\le B<A\le 1$. Then, the function $f\in\mathscr{M}_{\mathscr{H}}$ is in the class $\mathscr{M}^{**}_{\mathscr{H}}[A,B]$ if it satisfies the condition

$$-\frac{2\mathscr{D}_{\mathscr{H}}f(z)}{f(z)-f(-z)}\prec\frac{1+Az}{1+Bz},\,(z\in\mathbb{D}), \quad (7)$$

where the symbol "$\prec$" represent well-known subordination and

$$\mathscr{D}_{\mathscr{H}}f(z)=zh'(z)-\overline{zg'(z)}, \quad (8)$$

or equivalently

$$\left|\frac{\mathscr{D}_{\mathscr{H}}f(z)+(f(z)-f(-z)/2)}{B\mathscr{D}_{\mathscr{H}}f(z)+A(f(z)-f(-z)/2)}\right|<1,\,(z\in\mathbb{D}). \quad (9)$$

Furthermore, we denote $\mathscr{M}^{**}_{\overline{\mathscr{H}}}[A,B]$ subclass of $\mathscr{M}^{**}_{\mathscr{H}}[A,B]$ consisting of harmonic meromorphic functions $f=h+\overline{g}$ of the form (3).

2. Main Results

Theorem 1. *Let $f=h+\overline{g}$ be of the form (2) and satisfies the condition*

$$\sum_{n=1}^{\infty}\alpha_n|a_n|+\beta_n|b_n|\le 1, \quad (10)$$

with

$$\alpha_n=\frac{|(1+B)n+(1+A)(1-(-1)^n/2)|}{A-B}\quad and$$
$$\beta_n=\frac{|(1+B)n-(1+A)(1-(-1)^n/2)|}{A-B}, \quad (11)$$

then f is harmonic univalent sense-preserving in $\mathbb{D}^*$ and $f\in\mathscr{M}^{**}_{\mathscr{H}}[A,B]$.

Proof. For $0<|z_1|\le|z_2|<1$, we obtain

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right|&\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|\\&=1-\left|\frac{z_1z_2\sum_{n=1}^{\infty}b_n(z_1^n-z_2^n)}{(z_1-z_2)-z_1z_2\sum_{n=1}^{\infty}a_n(z_1^n-z_2^n)}\right|\\&>1-\frac{\sum_{n=1}^{\infty}n|b_n|}{1-\sum_{n=1}^{\infty}n|a_n|}>1-\frac{\sum_{n=1}^{\infty}\beta_n|b_n|}{1-\sum_{n=1}^{\infty}\alpha_n|a_n|}\ge 0,\end{aligned} \quad (12)$$

where we have used (10) and this shows that the function is univalent.

Now to show $f(z)$ is sense-preserving harmonic mapping in $\mathbb{D}^*$, consider

$$\begin{aligned}|h'(z)|&\ge\frac{1}{|z|^2}-\sum_{n=1}^{\infty}n|a_n||z|^{n-1}\ge 1-\sum_{n=1}^{\infty}\alpha_n|a_n|\ge\sum_{n=1}^{\infty}\beta_n|b_n|\\&\ge\sum_{n=1}^{\infty}n|b_n|\ge|g'(z)|.\end{aligned} \quad (13)$$

This shows that f is sense-preserving.

Now, to show that $f\in\mathscr{M}^{**}_{\mathscr{H}}[A,B]$ from (9), it is enough to show that

$$\left|\mathscr{D}_{\mathscr{H}}f(z)+\frac{f(z)-f(-z)}{2}\right|-\left|B\mathscr{D}_{\mathscr{H}}f(z)+A\frac{f(z)-f(-z)}{2}\right|<0. \quad (14)$$

For this, consider

$$\begin{aligned}
&\left|\mathscr{D}_{\mathscr{H}}f(z)+\frac{f(z)-f(-z)}{2}\right|-\left|B\mathscr{D}_{\mathscr{H}}f(z)+A\frac{f(z)-f(-z)}{2}\right|\\
&=\left|\sum_{n=1}^{\infty}\left[\left(n+\frac{1-(-1)^n}{2}\right)a_nz^n+\left(n+\frac{1-(-1)^n}{2}\right)b_n\bar{z}^n\right]\right|-\left|\frac{A-B}{z}\right.\\
&\quad\left.+\sum_{n=1}^{\infty}\left[\left(Bn+A\frac{1-(-1)^n}{2}\right)a_nz^n-\left(Bn+A\frac{1-(-1)^n}{2}\right)b_n\bar{z}^n\right]\right|\\
&\le\sum_{n=1}^{\infty}\left[\left(n+\frac{1-(-1)^n}{2}\right)|a_n||z|^n+\left(n+\frac{1-(-1)^n}{2}\right)|b_n||z|^n\right]\\
&\quad-\frac{A-B}{|z|}\sum_{n=1}^{\infty}\left[\left(Bn+A\frac{1-(-1)^n}{2}\right)|a_n||z|^n+\left(Bn+A\frac{1-(-1)^n}{2}\right)|b_n||z|^n\right]\\
&\le\sum_{n=1}^{\infty}\left\{\left|(1+B)n+(1+A)\frac{1-(-1)^n}{2}\right||a_n||z|^n+\left|(1+B)n\right.\right.\\
&\quad\left.+(1+A)\frac{1-(-1)^n}{2}\right||b_n||z|^n-\frac{A-B}{|z|}\\
&=\frac{A-B}{|z|}\sum_{n=1}^{\infty}\left[\frac{|(1+B)n+(1+A)(1-(-1)^n/2)|}{A-B}|a_n||z|^{n+1}\right.\\
&\quad\left.+\frac{|(1+B)n+(1+A)(1-(-1)^n/2)|}{A-B}|b_n||z|^{n+1}\right]-\frac{A-B}{|z|}\\
&<\frac{A-B}{|z|}\left\{\sum_{n=1}^{\infty}[\alpha_n|a_n|+\beta_n|b_n|]-1\right\}<0.
\end{aligned}\tag{15}$$

Hence, complete the proof.

Example 2. The meromorphic univalent function

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{x_n}{\alpha_n}z^n+\sum_{n=1}^{\infty}\frac{y_n}{\beta_n}\bar{z}^n,\tag{16}$$

such that $\sum_{n=1}^{\infty}(|x_n|+|y_n|)=1$, we have

$$\sum_{n=1}^{\infty}(\alpha_n|a_n|+\beta_n|b_n|)=\sum_{n=1}^{\infty}(|x_n|+|y_n|)=1.\tag{17}$$

Thus, $f\in\mathscr{M}_{\mathscr{H}}^{**}[A,B]$ and above coefficient bound given in (10) is sharp for this function.

Theorem 3. *Let $f=h+\bar{g}\in\mathscr{M}_{\mathscr{H}}$ and of the form (3), then $f\in\mathscr{M}_{\mathscr{H}}^{**}[A,B]$ if it satisfies the condition*

$$\sum_{n=1}^{\infty}\alpha_n|a_n|+\beta_n|b_n|\le 1,\tag{18}$$

with

$$\alpha_n=\frac{|(1+B)n+(1+A)(1-(-1)^n/2)|}{A-B}\quad\text{and}$$

$$\beta_n=\frac{|(1+B)n-(1+A)(1-(-1)^n/2)|}{A-B}.\tag{19}$$

Proof. The proof is similar to Theorem 1, so omitted.

Theorem 4. *Let $f=h+\bar{g}\in\mathscr{M}_{\mathscr{H}}^{**}[A,B]$ and of the form (3), $0<|z|=r<1$. Then,*

$$\frac{1}{r}-\frac{A-B}{2+A+B}r\le|f(z)|\le\frac{1}{r}+\frac{A-B}{2+A+B}r.\tag{20}$$

Proof. Consider

$$\begin{aligned}
|f(z)|&=\left|\frac{1}{z}+\sum_{n=1}^{\infty}\left(a_nz^n+b_n\bar{z}^n\right)\right|\le\frac{1}{r}+\sum_{n=1}^{\infty}(|a_n|+|b_n|)r^n\\
&\le\frac{1}{r}+r\sum_{n=1}^{\infty}(|a_n|+|b_n|)\le\frac{1}{r}+\frac{A-B}{2+A+B}r.
\end{aligned}\tag{21}$$

Similarly, proceeding as above we get

$$\begin{aligned}
|f(z)|&=\left|\frac{1}{z}+\sum_{n=1}^{\infty}\left(a_nz^n+b_n\bar{z}^n\right)\right|\ge\frac{1}{r}-r\sum_{n=1}^{\infty}(|a_n|+|b_n|)\\
&=\frac{1}{r}-\frac{A-B}{2+A+B}r.
\end{aligned}\tag{22}$$

Hence, this completes the result.

Theorem 5. *Let $f=h+\bar{g}$, and of the form (3) then $f\in\mathscr{M}_{\mathscr{H}}^{**}[A,B]$ if and only if*

$$f(z)=\sum_{n=0}^{\infty}(\gamma_nh_n+\delta_ng_n),\tag{23}$$

where

$$\begin{aligned}
h_0(z)&=\frac{1}{z},\ h_n(z)=\frac{1}{z}+\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}z^n,\ n=1,2,\cdots,\\
g_0(z)&=\frac{1}{z},\ g_n(z)=\frac{1}{z}-\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\bar{z}^n,\ n=1,2,\cdots,
\end{aligned}\tag{24}$$

where $1\ge\gamma_n\ge 0, 1\ge\delta_n\ge 0$, and $\sum_{n=0}^{\infty}(\gamma_n+\delta_n)=1$.

Proof. Let

$$\begin{aligned}
f(z)&=\gamma_0h_0+\delta_0g_0+\sum_{n=1}^{\infty}(\gamma_nh_n+\delta_ng_n)\\
&=(\gamma_0+\delta_0)\frac{1}{z}+\sum_{n=1}^{\infty}\gamma_n\left(\frac{1}{z}+\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}z^n\right)\\
&\quad+\sum_{n=1}^{\infty}\delta_n\left(\frac{1}{z}-\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\bar{z}^n\right),\\
&=\sum_{n=1}^{\infty}(\gamma_n+\delta_n)\frac{1}{z}+\sum_{n=1}^{\infty}\gamma_n\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}z^n\\
&\quad-\sum_{n=1}^{\infty}\delta_n\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\bar{z}^n.
\end{aligned}\tag{25}$$

Thus,

$$\begin{aligned}&\sum_{n=1}^{\infty}\left\{\alpha_n\left(\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\gamma_n\right)\right.\\&\quad\left.+\beta_n\left(\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\delta_n\right)\right\}\\&=\sum_{n=1}^{\infty}(\gamma_n+\delta_n)=1-\gamma_0-\delta_0\le 1,\end{aligned}\tag{26}$$

hence, $f\in\mathscr{M}^{**}_{\mathscr{H}}[A,B]$. Conversely, let $f\in\mathscr{M}^{**}_{\mathscr{H}}[A,B]$. Set

$$\begin{aligned}\gamma_n&=\frac{|(1+B)n+(1+A)(1-(-1)^n/2)|}{A-B}|a_n|,0\le\gamma_n\le 1,\delta_n\\&=\frac{|(1+B)n+(1+A)(1-(-1)^n/2)|}{A-B}|b_n|,0\le\delta_n\le 1,\gamma_0\\&=1-\sum_{n=1}^{\infty}\gamma_n-\sum_{n=1}^{\infty}\delta_n.\end{aligned}\tag{27}$$

Therefore, f can be written as

$$\begin{aligned}f(z)&=\frac{1}{z}+\sum_{n=1}^{\infty}|a_n|z^n-\sum_{n=1}^{\infty}|b_n|\bar{z}^n\\&=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\gamma_n|a_n|z^n\\&\quad-\sum_{n=1}^{\infty}\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\delta_n|b_n|\bar{z}^n\\&=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\gamma_n|a_n|z^n\\&\quad-\sum_{n=1}^{\infty}\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\delta_n|b_n|\bar{z}^n\\&=(\gamma_0+\delta_0)\frac{1}{z}+\sum_{n=1}^{\infty}\gamma_n\left(\frac{1}{z}+\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}z^n\right)\\&\quad+\sum_{n=1}^{\infty}\delta_n\left(\frac{1}{z}-\frac{A-B}{|(1+B)n+(1+A)(1-(-1)^n/2)|}\bar{z}^n\right)\\&=\sum_{n=1}^{\infty}(\gamma_n h_n+\delta_n g_n),\quad\text{hence required.}\end{aligned}\tag{28}$$

Theorem 6. *The class $\mathscr{M}^{**}_{\mathscr{H}}[A,B]$ is closed under convex combination.*

Proof. For $k\in\mathbb{N}$, let $f_k\in\mathscr{M}^{**}_{\mathscr{H}}[A,B]$, be of the form

$$f_k(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left(|a_{k,n}|z^n-|b_{k,n}|\bar{z}^n\right).\tag{29}$$

Then form (10), we get

$$\sum_{n=1}^{\infty}\left(\alpha_n|a_{k,n}|+\beta_n|b_{k,n}|\right)\le 1.\tag{30}$$

For $\sum_{k=1}^{\infty}\delta_k=1,(0\le\delta_k\le 1)$, the convex combination of f_k is

$$\sum_{k=1}^{\infty}\delta_k f_k(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\delta_k|a_{k,n}|\right)z^n-\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\delta_k|b_{k,n}|\right)\bar{z}^n.\tag{31}$$

Using (10), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\alpha_n\sum_{k=1}^{\infty}\delta_k|a_{k,n}|+\beta_n\sum_{k=1}^{\infty}\delta_k|b_{k,n}|\right)\\&=\sum_{k=1}^{\infty}\delta_k\left[\sum_{n=1}^{\infty}\left(\alpha_n|a_{k,n}|+\beta_n|b_{k,n}|\right)\right]\le\sum_{k=1}^{\infty}\delta_k=1,\end{aligned}\tag{32}$$

thus prove our desired results.

Theorem 7. *Let $f_k\in\mathscr{M}^{**}_{\mathscr{H}}[A,B]$, for $k=\{1,2\}$ be of the form (29), then, their weighted F_i mean is also in the class $\mathscr{M}^{**}_{\mathscr{H}}[A,B]$, where F_i is defined below*

$$F_i(z)=\frac{(1-i)f_1(z)+(1+i)f_2(z)}{2}.\tag{33}$$

Proof. From (33), one may easily write

$$\begin{aligned}F_i(z)&=\frac{1}{z}+\frac{(1-i)f_1(z)+(1+i)f_2(z)}{2}\\&=\frac{1}{z}+\sum_{k=1}^{\infty}\left[\frac{(1-j)|a_{n,1}|+(1+j)|a_{n,2}|}{2}z^n\right.\\&\quad\left.-\frac{(1-j)|b_{n,1}|+(1+j)|b_{n,2}|}{2}\bar{z}^n\right].\end{aligned}\tag{34}$$

To show that $F_i\in\mathscr{M}^{**}_{\mathscr{H}}[A,B]$, it is enough to show that

$$\begin{aligned}&\sum_{k=1}^{\infty}\left[\left|\frac{(1-j)|a_{n,1}|+(1+j)|a_{n,2}|}{2}\right|\alpha_n\right.\\&\quad\left.+\left|\frac{(1-j)|b_{n,1}|+(1+j)|b_{n,2}|}{2}\right|\beta_n\right]\le 1.\end{aligned}\tag{35}$$

Now consider

$$\begin{aligned}&\sum_{k=1}^{\infty}\left[\left|\frac{(1-j)|a_{n,1}|+(1+j)|a_{n,2}|}{2}\right|\alpha_n+\left|\frac{(1-j)|b_{n,1}|+(1+j)|b_{n,2}|}{2}\right|\beta_n\right]\\&=\sum_{k=1}^{\infty}\left[\frac{(1-j)|a_{n,1}|\alpha_n+(1-j)|b_{n,1}|\beta_n}{2}+\frac{(1+j)|a_{n,2}|\alpha_n+(1+j)|b_{n,2}|\beta_n}{2}\right]\\&=\frac{(1-j)}{2}\sum_{k=1}^{\infty}\left(|a_{n,1}|\alpha_n+|b_{n,1}|\beta_n\right)+\frac{(1+j)}{2}\sum_{k=1}^{\infty}\left(|a_{n,2}|\alpha_n+|b_{n,2}|\beta_n\right)\\&\le\frac{(1-j)}{2}+\frac{(1+j)}{2}=1.\end{aligned}\tag{36}$$

Hence, $F_i\in\mathscr{M}^{**}_{\mathscr{H}}[A,B]$.

Data Availability

Data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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Research Article

On Certain Class of Bazilevič Functions Associated with the Lemniscate of Bernoulli

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Making use of the principle of subordination, we introduce a certain class of multivalently Bazilevic functions involving the Lemniscate of Bernoulli. Also, we obtain subordination properties, inclusion relationship, convolution result, coefficients estimate, and Fekete–Szegö problem for this class.

1. Introduction

Let $\mathscr{H}(\mathbb{U})$ be the class of analytic functions in the open unit disk

$$\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}, \tag{1}$$

and let $\mathscr{A}_p$ denote the subclass of $\mathscr{H}(\mathbb{U})$ consisting of functions of the form:

$$f(\zeta) = \zeta^p + \sum_{k=p+1}^{\infty} a_k \zeta^k (p \in \mathbb{N} = \{1, 2, 3, \cdots\}). \tag{2}$$

We write $\mathscr{A}_1 = \mathscr{A}$. For $f_1, f_2 \in \mathscr{H}(\mathbb{U})$, we say that $f_1(\zeta)$ is subordinate to $f_2(\zeta)$, written symbolically, $f_1 \prec f_2$ in $\mathbb{U}$ or $f_1(\zeta) \prec f_2(\zeta) (\zeta \in \mathbb{U})$, if there exists a Schwarz function $\omega(\zeta)$, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(\zeta)| < 1 (\zeta \in \mathbb{U})$ such that $f_1(\zeta) = f_2(\omega(\zeta))$ $(\zeta \in \mathbb{U})$. Further more, if the function $f_2(\zeta)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [1, 2]):

$$f_1(\zeta) \prec f_2(\zeta)\ (\zeta \in \mathbb{U}) \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathbb{U}) \subset f_2(\mathbb{U}). \tag{3}$$

Let $\phi : \mathbb{C}^2 \times \mathbb{U} \longrightarrow \mathbb{C}$ and $h(\zeta)$ be univalent in $\mathbb{U}$. If $g(\zeta)$ is analytic in $\mathbb{U}$ and satisfies the first order differential subordination:

$$\phi\left(g(\zeta), \zeta g'(\zeta); \zeta\right) \prec h(\zeta), \tag{4}$$

then $g(\zeta)$ is a solution of the differential subordination (4). The univalent function $q(\zeta)$ is called a dominant of the solutions of the differential subordination (4) if $g(\zeta) \prec q(\zeta)$ for all $g(\zeta)$ satisfying (4). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (4) is called the best dominant.

Sokól and Stankiewicz [3] introduced the class $\mathscr{SL}^*$ consisting of analytic functions $f \in \mathscr{A}$ satisfying the following condition

$$\left| \left[\frac{\zeta f'(\zeta)}{f(\zeta)} \right]^2 - 1 \right| < 1, \tag{5}$$

which is equivalent to

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \prec q(\zeta) = \sqrt{1+\zeta}, \tag{6}$$

where the function

$$q(\zeta) = \sqrt{1+\zeta}\,(\zeta \in \mathbb{U}), \tag{7}$$

maps $\mathbb{U}$ onto the domain $\mathcal{O} = \{w \in \mathbb{C} : \Re w > 0, |w^2 - 1| < 1\}$, and its boundary $\partial\mathcal{O}$ is the right-half of the lemniscate of Bernoulli $(x^2+y^2)^2 - 2(x^2-y^2) = 0$. Several geometric properties of $\mathcal{SL}^*$ were investigated done by many authors in ([4–7]).

Now, we define a class $\mathcal{B}_p(\lambda, \alpha)$ of Bazilevic functions associated with lemniscate of Bernoullia by using the principle of differential subordination as follows.

Definition 1. A function $f \in \mathcal{A}_p$ is said to be the class $\mathcal{B}_p(\lambda, \alpha)$ if it satisfies the following subordination condition:

$$(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} + \lambda \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} \prec \sqrt{1+\zeta}, \tag{8}$$

all the powers are principal values and throughout the paper unless otherwise mentioned the parameters λ, α, and p are constrained as $\lambda \in \mathbb{C}$, $\alpha > 0$, $p \in \mathbb{N}$, and $\zeta \in \mathbb{U}$.

We note that

(1) $\mathcal{B}_1(\lambda, \alpha) = \mathcal{B}(\lambda, \alpha) = \{f \in \mathcal{A} : (1-\lambda)(f(\zeta)/\zeta)^{\alpha} + \lambda(\zeta f'(\zeta)/f(\zeta))(f(\zeta)/\zeta)^{\alpha} \prec \sqrt{1+\zeta}\}$

(2) $\mathcal{B}_p(\lambda, 1) = \mathcal{B}_p(\lambda) = \{f \in \mathcal{A}_p : (1-\lambda)(f(\zeta)/\zeta^p) + \lambda(f'(\zeta)/p\zeta^{p-1}) \prec \sqrt{1+\zeta}\}$ and $\mathcal{B}_1(\lambda) = \mathcal{B}(\lambda)$

(3) $\mathcal{B}_p(1, \alpha) = \mathcal{B}_p(\alpha) = \{f \in \mathcal{A}_p : (\zeta f'(\zeta)/pf(\zeta))(f(\zeta)/\zeta^p)^{\alpha} \prec \sqrt{1+\zeta}\}$ and $\mathcal{B}_1(\alpha) = \mathcal{B}(\alpha)$

(4) $\mathcal{B}_p(1, 0) = \mathcal{SL}_p^* = \{f \in \mathcal{A}_p : (\zeta f'(\zeta)/pf(\zeta)) \prec \sqrt{1+\zeta}\}$ and $\mathcal{SL}_1^* = \mathcal{SL}^*$

In order to establish our main results, we need the following lemmas.

Lemma 2 [8]. *Let the function h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function $g(\zeta)$ given by*

$$g(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + \cdots, \tag{9}$$

is analytic in $\mathbb{U}$. If

$$g(\zeta) + \frac{\zeta g'(\zeta)}{\gamma} \prec h(\zeta)\,(\Re(\gamma) \geq 0; \gamma \neq 0; \zeta \in \mathbb{U}), \tag{10}$$

then

$$g(\zeta) \prec q(\zeta) = \gamma\zeta^{-\gamma}\int_0^{\zeta} h(t)t^{\gamma-1}\quad dt \prec h(\zeta), \tag{11}$$

and $q(\zeta)$ is the best dominant.

Lemma 3. *[9]. For real or complex numbers $a, b, c (c \neq 0, -1, -2, \cdots)$ and $\zeta \in \mathbb{U}$,*

$$\begin{aligned}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t\zeta)^{-a}dt &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a, b; c; \zeta) \quad (\Re(c) > \Re(b) > 0), \\ {}_2F_1(a, b; c; \zeta) &= (1-\zeta)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{\zeta}{\zeta-1}\right).\end{aligned} \tag{12}$$

Lemma 4. *[10]. Let F be analytic and convex in $\mathbb{U}$. If $f, g \prec F$, then*

$$\lambda f + (1-\lambda)g \prec F\,(0 \leq \lambda \leq 1). \tag{13}$$

Lemma 5 [11]. *Let $f(\zeta) = \sum_{k=1}^{\infty} a_k\zeta^k$ be analytic in $\mathbb{U}$ and $g(\zeta) = \sum_{k=1}^{\infty} b_k\zeta^k$ be analytic and convex in $\mathbb{U}$. If $f \prec g$, then*

$$|a_k| < |b_1|(k \in \mathbb{N}). \tag{14}$$

Lemma 6 [12]. *Let $g(\zeta) = 1 + \sum_{k=1}^{\infty} c_k\zeta^k \in \mathcal{P}$, i.e., let g be analytic in $\mathbb{U}$ and satisfy $\Re\{g(\zeta)\} > 0$ for $\zeta \in \mathbb{U}$, then the following sharp estimate holds*

$$\left|c_2 - \nu c_1^2\right| \leq 2\max\{1, |2\nu - 1|\}\,\text{for all}\,\nu \in \mathbb{C}. \tag{15}$$

The result is sharp for the functions given by

$$g(\zeta) = \frac{1+\zeta^2}{1-\zeta^2}\text{ or }g(\zeta) = \frac{1+\zeta}{1-\zeta}. \tag{16}$$

Lemma 7. *[12]. If $g(\zeta) = 1 + \sum_{k=1}^{\infty} c_k\zeta^k \in \mathcal{P}$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \left\{\begin{array}{lll} -4\nu + 2 & \text{if} & \nu \leq 0, \\ 2 & \text{if} & 0 \leq \nu \leq 1, \\ 4\nu - 2 & \text{if} & \nu \geq 1, \end{array}\right\}, \tag{17}$$

when $\upsilon < 0$ or $\nu > 1$, the equality holds if and only if $g(\zeta) = (1+\zeta)/(1-\zeta)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $g(\zeta) = (1+\zeta^2)/(1-\zeta^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$g(\zeta) = \left(\frac{1+\lambda}{2}\right)\frac{1+\zeta}{1-\zeta} + \left(\frac{1-\lambda}{2}\right)\frac{1-\zeta}{1+\zeta}(0 \leq \lambda \leq 1), \tag{18}$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if g is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$.

Also, the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$\begin{aligned} \left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2 &\leq 2\left(0 \leq \nu \leq \frac{1}{2}\right), \\ \left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 &\leq 2\left(\frac{1}{2} \leq \nu \leq 1\right). \end{aligned} \tag{19}$$

In the present paper, we obtain subordination properties, inclusion relationship, convolution result, coefficients estimate, and Fekete–Szegö inequalities for the class $\mathscr{B}_p(\lambda, \alpha)$.

2. Main Results

We begin by presenting our first subordination property given by Theorem 8.

Theorem 8. *If $f \in \mathscr{B}_p(\lambda, \alpha)$ with $\Re(\lambda) > 0$, then*

$$\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} \prec Q(\zeta) \prec \sqrt{1+\zeta}, \tag{20}$$

where the function $Q(\zeta)$ given by

$$Q(\zeta) = (1+\zeta)^{1/2}{}_2F_1\left(-\frac{1}{2}, 1; \frac{p\alpha}{\lambda} + 1; \frac{\zeta}{1+\zeta}\right), \tag{21}$$

is the best dominant.

Proof. Let $f \in \mathscr{B}_p(\lambda, \alpha)$ and suppose that

$$g(\zeta) = \left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} \quad (\zeta \in \mathbb{U}). \tag{22}$$

Then, the function $g(\zeta)$ is of the form (9), analytic in $\mathbb{U}$, and $g(0) = 1$. By taking the derivatives in the both sides of (22), we get

$$(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} + \lambda\frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} = g(\zeta) + \frac{\lambda}{p\alpha}\zeta g'(\zeta). \tag{23}$$

Since $f \in \mathscr{B}_p(\lambda, \alpha)$, we have

$$g(\zeta) + \frac{\lambda}{p\alpha}\zeta g'(\zeta) \prec \sqrt{1+\zeta}. \tag{24}$$

Now, by using Lemma 2 for $\gamma = p\alpha/\lambda$, we deduce that

$$\begin{aligned} \left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} \prec Q(\zeta) &= \frac{p\alpha}{\lambda}\zeta^{-(p\alpha/\lambda)}\int_0^{\zeta} t^{(p\alpha/\lambda)-1}(1+t)^{1/2}\,dt \\ &= \frac{p\alpha}{\lambda}\int_0^1 u^{(p\alpha/\lambda)-1}(1+\zeta u)^{1/2}du \\ &= (1+\zeta)^{1/2}{}_2F_1\left(-\frac{1}{2}, 1; \frac{p\alpha}{\lambda} + 1; \frac{\zeta}{1+\zeta}\right), \end{aligned} \tag{25}$$

where we have made a change of variables followed by the use of identities in Lemma 3 with $a = -1/2$, $b = p\alpha/\lambda n$, and $c = b + 1$. This completes the proof of Theorem 8.

For a function $f \in \mathscr{A}(p)$ given by (2), the generalized Bernardi-Libera-Livingston integral operator $F_{p,\mu} : \mathscr{A}(p) \longrightarrow \mathscr{A}(p)$, with $\mu > -p$, is defined by (see [13–16])

$$F_{p,\mu}f(\zeta) = \frac{\mu+p}{\zeta^{\mu}}\int_0^{\zeta} t^{\mu-1}f(t)\,dt(\mu>-p). \tag{26}$$

It is easy to verify that for all $f \in \mathscr{A}(p)$, we have

$$\zeta\left(F_{p,\mu}f(\zeta)\right)' = (\mu+p)f(\zeta) - \mu F_{p,\mu}f(\zeta). \tag{27}$$

Theorem 9. *If the function $f \in \mathscr{A}(p)$ satisfies the subordination condition*

$$(1-\lambda)\left(\frac{F_{p,\mu}f(\zeta)}{\zeta^p}\right)^{\alpha} + \lambda\frac{f(\zeta)}{F_{p,\mu}f(\zeta)}\left(\frac{F_{p,\mu}f(\zeta)}{\zeta^p}\right)^{\alpha} \prec \sqrt{1+\zeta}, \tag{28}$$

and $F_{p,\mu}$ is the integral operator defined by (26), then

$$\left(\frac{F_{p,\mu}f(\zeta)}{\zeta^p}\right)^{\alpha} \prec K(\zeta) \prec \sqrt{1+\zeta}, \tag{29}$$

where the function K given by

$$K(\zeta) = (1+\zeta)^{1/2}{}_2F_1\left(-\frac{1}{2}, 1; \frac{\alpha(p+\mu)}{\lambda} + 1; \frac{\zeta}{1+\zeta}\right), \tag{30}$$

is the best dominant of (28).

Proof. Let

$$g(\zeta) = \left(\frac{F_{p,\mu}f(\zeta)}{\zeta^p}\right)^{\alpha}(\zeta \in \mathbb{U}), \tag{31}$$

then g is analytic in $\mathbb{U}$. Differentiating (31) with respect to ζ and using the identity (28) in the resulting relation, we get

$$\begin{aligned} &(1-\lambda)\left(\frac{F_{p,\mu}f(\zeta)}{\zeta^p}\right)^{\alpha} + \lambda\frac{f(\zeta)}{F_{p,\mu}f(\zeta)}\left(\frac{F_{p,\mu}f(\zeta)}{\zeta^p}\right)^{\alpha} \\ &\quad = g(\zeta) + \frac{\lambda\zeta g'(\zeta)}{\alpha(p+\mu)} \prec \sqrt{1+\zeta}. \end{aligned} \tag{32}$$

Employing the same technique that we used in the proof of Theorem 8, the remaining part of the theorem can be proved similarly.

Theorem 10. *If $\lambda_2 \geq \lambda_1 \geq 0$, then*

$$\mathscr{B}_p(\lambda_2, \alpha) \subset \mathscr{B}_p(\lambda_1, \alpha). \tag{33}$$

Proof. Suppose that $f \in \mathscr{B}_p(\lambda_2, \alpha)$. We know that

$$(1-\lambda_2)\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} + \lambda_2 \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} \prec \sqrt{1+\zeta}, \quad (34)$$

Thus, the assertion of Theorem 10 holds for $\lambda_2 = \lambda_1 \geq 0$. If $\lambda_2 > \lambda_1 \geq 0$, by Theorem 8 and (34), we have

$$\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} \prec \sqrt{1+\zeta}. \quad (35)$$

At the same time, we have

$$\begin{aligned}(1-\lambda_1)&\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} + \lambda_1 \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} \\ &= \left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} + \frac{\lambda_1}{\lambda_2}\left[(1-\lambda_2)\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha}\right. \\ &\quad \left. + \lambda_2 \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha}\right].\end{aligned} \quad (36)$$

Moreover, since $0 \leq \lambda_1/\lambda_2 < 1$, and the function $\sqrt{1+\zeta}$ $(\zeta \in \mathbb{U})$ is analytic and convex in $\mathbb{U}$.

Combining (34)–(36) and Lemma 4, we find that

$$(1-\lambda_1)\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} + \lambda_1 \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} \prec \sqrt{1+\zeta}, \quad (37)$$

that is $f \in \mathscr{B}_p(\lambda_1, \alpha)$, which implies that the assertion (33) of Theorem 10 holds.

Theorem 11. *If $f \in \mathscr{A}_p$, then $f \in \mathscr{B}_p(\lambda, \alpha)$ if and only if*

$$\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} * \left[\frac{1-L\zeta+M\zeta^2}{(1-\zeta)^2}\right] \neq 0 \quad (\zeta \in \mathbb{U}), \quad (38)$$

where

$$\begin{aligned} L &= \left(1+\frac{\lambda}{\alpha p}\right)e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+2 \\ M &= e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+1. \end{aligned} \quad (39)$$

Proof. For any function $f \in \mathscr{A}_p$, we can verify that

$$\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} = \left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} * \frac{1}{1-\zeta}, \quad (40)$$

$$\frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} = \left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} * \frac{1-(1-1/p\alpha)\zeta}{(1-\zeta)^2}. \quad (41)$$

First, in order to prove that (38) holds, we will write (8) by using the principle of subordination, that is,

$$(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} + \lambda \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} = \sqrt{1+w(\zeta)}, \quad (42)$$

where $w(\zeta)$ is a Schwarz function, hence

$$(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} + \lambda \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} \neq \sqrt{1+e^{i\theta}}, \quad (43)$$

for all $\zeta \in \mathbb{U}$ and $\theta \in 0, 2\pi)$. From (40) and (41), the relation (43) may be written as

$$\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} * \left[\frac{1-\sqrt{1+e^{i\theta}}-\left(1-(\lambda/\alpha p)-2\sqrt{1+e^{i\theta}}\right)\zeta-\sqrt{1+e^{i\theta}}\zeta^2}{(1-\zeta)^2}\right] \neq 0, \quad (44)$$

which is equivalent to

$$\left(\frac{f(\zeta)}{\zeta^p}\right)^{\alpha} * \left[\frac{1-\left[(1+(\lambda/p\alpha))e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+2\right]\zeta+\left[e^{-i\theta}\left(1+\sqrt{1+e^{i\theta}}\right)+1\right]\zeta^2}{(1-\zeta)^2}\right] \neq 0, \quad (45)$$

that is (38).

Reversely, suppose that $f \in \mathscr{A}_p$ satisfy the condition (38). Like it was previously shown, the assumption (38) is equivalent to (41), that is,

$$(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha + \lambda \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha \neq \sqrt{1+e^{i\theta}} \quad (\zeta \in \mathbb{U}). \tag{46}$$

Denoting

$$\varphi(\zeta) = (1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha + \lambda \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha \text{ and } \psi(\zeta) = \sqrt{1+\zeta}, \tag{47}$$

the relation (46) could be written as $\varphi(\mathbb{U}) \cap \psi(\partial\mathbb{U}) = \varnothing$. Therefore, the simply connected domain $\varphi(\mathbb{U})$ is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U})$. From this fact, using that $\varphi(0) = \psi(0) = 1$ together with the univalence of the function ψ, it follows that $\varphi(\zeta) \prec \psi(\zeta)$, that is $f \in \mathscr{B}_p(\lambda, \alpha)$.

Theorem 12. *If $f(\zeta)$ given by (2) belongs to $\mathscr{B}_p(\lambda, \alpha)$, then*

$$|a_{p+1}| \leq \frac{p}{2|p\alpha + \lambda|}. \tag{48}$$

Proof. Combining (2) and (8), we obtain

$$\begin{aligned}(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha &+ \lambda \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha \\ &= 1 + \left(\frac{p\alpha+\lambda}{p}\right)a_{p+1}\zeta + \cdots . \prec \sqrt{1+\zeta} \\ &= 1 + \frac{1}{2}\zeta - \frac{1}{8}\zeta^2 + \cdots .\end{aligned} \tag{49}$$

An application of Lemma 5 to (49) yields

$$\left|\left(\frac{p\alpha+\lambda}{p}\right)a_{p+1}\right| < \frac{1}{2}. \tag{50}$$

Thus, from (50), we easily obtain (48) asserted by Theorem 12.

3. Fekete–Szegö Problem

Many authors have considered the Fekete–Szegö problem for many subclasses of analytic functions (see, for instance, [17–21]). In this section, we evaluate the Fekete–Szegö inequalities for the class $\mathscr{B}_p(\lambda, \alpha)$.

Theorem 13. *If f given by (2) belongs to the class $\mathscr{B}_p(\lambda, \alpha)$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p}{2(\alpha p + 2\lambda)} \max\left\{1; \frac{1}{4}\left|1 + \frac{p(\alpha p + 2\lambda)(\alpha - 1 + 2\mu)}{(\alpha p + \lambda)^2}\right|\right\}. \tag{51}$$

The result is sharp.

Proof. If $f \in \mathscr{B}_p(\lambda, \alpha)$, then there is a Schwarz function ω in $\mathbb{U}$ such that

$$(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha + \lambda \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha = \phi(\omega(\zeta)), \tag{52}$$

where $\phi(\zeta) = \sqrt{1+\zeta}$. Define the function $g(\zeta)$ by

$$g(\zeta) = \frac{1+\omega(\zeta)}{1-\omega(\zeta)} = 1 + c_1\zeta + c_2\zeta^2 + \cdots . \tag{53}$$

Since $\omega(\zeta)$ is a Schwarz function, we see that $g \in \mathscr{P}$ with $g(0) = 1$. Therefore,

$$\begin{aligned}\phi(\omega(\zeta)) &= \phi\left(\frac{g(\zeta)-1}{g(\zeta)+1}\right) = \sqrt{\frac{2g(\zeta)}{g(\zeta)+1}} \\ &= 1 + \frac{1}{4}c_1\zeta + \left(\frac{1}{4}c_2 - \frac{5}{32}c_1^2\right)\zeta^2 + . \cdots\end{aligned} \tag{54}$$

Now by substituting (54) in (52), we have

$$\begin{aligned}(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha &+ \lambda \frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha \\ &= 1 + \frac{1}{4}c_1\zeta + \left(\frac{1}{4}c_2 - \frac{5}{32}c_1^2\right)\zeta^2 + . \cdots\end{aligned} \tag{55}$$

Equating the coefficients of ζ and ζ^2, we obtain

$$\begin{aligned}a_{p+1} &= \frac{p}{4(\alpha p + \lambda)}c_1, \\ a_{p+2} &= \frac{p}{4(\alpha p + 2\lambda)}c_2 - \frac{p}{32}\left(\frac{5}{(\alpha p + 2\lambda)} + \frac{p(\alpha-1)}{(\alpha p + \lambda)^2}\right)c_1^2.\end{aligned} \tag{56}$$

Therefore,

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p}{4(\alpha p + 2\lambda)}\left\{c_2 - \nu c_1^2\right\}, \tag{57}$$

where

$$\nu = \frac{1}{8}\left[5 + \frac{p(\alpha p + 2\lambda)(\alpha - 1 + 2\mu)}{(\alpha p + \lambda)^2}\right]. \quad (58)$$

Our result now follows by an application of Lemma 6. The result is sharp for the functions

$$\begin{aligned} &(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha + \lambda\frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha = \phi\left(\zeta^2\right), \\ &(1-\lambda)\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha + \lambda\frac{\zeta f'(\zeta)}{pf(\zeta)}\left(\frac{f(\zeta)}{\zeta^p}\right)^\alpha = \phi(\zeta). \end{aligned} \quad (59)$$

This completes the proof of Theorem 13.

Putting $\lambda = 1$ and $\alpha = 0$ in Theorem 13, we obtain the following corollary.

Corollary 14. *If f given by (2) belongs to the class $\mathscr{B}_p(\alpha)$, then*

$$\begin{aligned} \left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p}{2(\alpha p + 2)} \max\left\{1; \frac{1}{4}\right| 1 \\ + \frac{p(\alpha p + 2)(\alpha - 1 + 2\mu)}{(\alpha p + 1)^2}\left|\right\}. \end{aligned} \quad (60)$$

The result is sharp.

Putting $\lambda = 1$ and $\alpha = 0$ in Theorem 13, we obtain the following corollary.

Corollary 15. *If f given by (2) belongs to the class $\mathcal{SL}_p^*$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p}{4} \max\left\{1; \frac{|1 + 2p(2\mu - 1)|}{4}\right\}. \quad (61)$$

The result is sharp.

Putting $p = \lambda = 1$ and $\alpha = 0$ in Theorem 13, we obtain the following corollary.

Corollary 16. *If f given by (2) (with $p = 1$) belongs to the class $\mathcal{SL}^*$, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{4} \max\left\{1; \frac{|4\mu - 1|}{4}\right\}. \quad (62)$$

The result is sharp.

Applying Lemma 7 to (57) and (58), we obtain the following theorem.

Theorem 17. *Let*

$$\begin{aligned} \sigma_1 &= \frac{p(\alpha p + 2\lambda)(1 - \alpha) - 5(\alpha p + \lambda)^2}{2p(\alpha p + 2\lambda)}, \\ \sigma_2 &= \frac{p(\alpha p + 2\lambda)(1 - \alpha) + 3(\alpha p + \lambda)^2}{2p(\alpha p + 2\lambda)}, \\ \sigma_3 &= \frac{p(\alpha p + 2\lambda)(1 - \alpha) - (\alpha p + \lambda)^2}{2p(\alpha p + 2\lambda)}. \end{aligned} \quad (63)$$

If f given by (2) belongs to the class $\mathscr{B}_p(\lambda, \alpha)$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} -\dfrac{p}{8(\alpha p + 2\lambda)}\left[1 + \dfrac{p(\alpha p + 2\lambda)(\alpha - 1 + 2\mu)}{(\alpha p + \lambda)^2}\right] & (\mu \le \sigma_1), \\ \dfrac{p}{2(\alpha p + 2\lambda)} & (\sigma_1 \le \mu \le \sigma_2), \\ \dfrac{p}{8(\alpha p + 2\lambda)}\left[1 + \dfrac{p(\alpha p + 2\lambda)(\alpha - 1 + 2\mu)}{(\alpha p + \lambda)^2}\right] & (\mu \ge \sigma_2). \end{cases} \quad (64)$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, then

$$\begin{aligned} &\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{2}\left[\frac{5(\alpha p + \lambda)^2}{p(\alpha p + 2\lambda)} + \alpha - 1 + 2\mu\right]\left|a_{p+1}\right|^2 \\ &\le \frac{p}{2(\alpha p + 2\lambda)}. \end{aligned} \quad (65)$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\begin{aligned} &\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{2}\left[\frac{3(\alpha p + \lambda)^2}{p(\alpha p + 2\lambda)} - \alpha + 1 - 2\mu\right]\left|a_{p+1}\right|^2 \\ &\le \frac{p}{2(\alpha p + 2\lambda)}. \end{aligned} \quad (66)$$

Putting $\lambda = 1$ in Theorem 17, we obtain the following result.

Corollary 18. *Let*

$$\delta_1 = \frac{p(\alpha p+2)(1-\alpha) - 5(\alpha p+1)^2}{2p(\alpha p+2)},$$

$$\delta_2 = \frac{p(\alpha p+2)(1-\alpha) + 3(\alpha p+1)^2}{2p(\alpha p+2)},$$

$$\delta_3 = \frac{p(\alpha p+2)(1-\alpha) - (\alpha p+1)^2}{2p(\alpha p+2)}. \tag{67}$$

If f given by (2) belongs to the class $\mathscr{B}_p(\alpha)$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} -\dfrac{p}{8(\alpha p+2)}\left[1 + \dfrac{p(\alpha p+2)(\alpha-1+2\mu)}{(\alpha p+1)^2}\right] & (\mu \le \delta_1), \\ \dfrac{p}{2(\alpha p+2)} & (\delta_1 \le \mu \le \delta_2), \\ \dfrac{p}{8(\alpha p+2)}\left[1 + \dfrac{p(\alpha p+2)(\alpha-1+2\mu)}{(\alpha p+1)^2}\right] & (\mu \ge \delta_2). \end{cases} \tag{68}$$

Further, if $\delta_1 \le \mu \le \delta_3$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{2}\left[\frac{5(\alpha p+1)^2}{p(\alpha p+2)} + \alpha - 1 + 2\mu\right]\left|a_{p+1}\right|^2 \le \frac{p}{2(\alpha p+2)}. \tag{69}$$

If $\delta_3 \le \mu \le \delta_2$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{2}\left[\frac{3(\alpha p+1)^2}{p(\alpha p+2)} - \alpha + 1 - 2\mu\right]\left|a_{p+1}\right|^2 \le \frac{p}{2(\alpha p+2)}. \tag{70}$$

Putting $\lambda = 1$ and $\alpha = 0$ in Theorem 17, we obtain the following result for the subclass $\mathcal{SL}_p^*$.

Corollary 19. *If f given by (2) belongs to the class $\mathcal{SL}_p^*$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} -\dfrac{p[1+2p(2\mu-1)]}{16} & \left(\mu \le \dfrac{2p-5}{4p}\right), \\ \dfrac{p}{4} & \left(\dfrac{2p-5}{4p} \le \mu \le \dfrac{2p+3}{4p}\right), \\ \dfrac{p[1+2p(2\mu-1)]}{16} & \left(\mu \ge \dfrac{2p+3}{4p}\right). \end{cases} \tag{71}$$

Further, if $((2p-5)/4p) \le \mu \le ((2p-1)/4p)$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{4}\left(\frac{5}{p} - 2 + 4\mu\right)\left|a_{p+1}\right|^2 \le \frac{p}{4}. \tag{72}$$

If $((2p-1)/4p) \le \mu \le ((2p+3)/4\text{p})$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{4}\left(\frac{3}{p} + 2 - 4\mu\right)\left|a_{p+1}\right|^2 \le \frac{p}{4}. \tag{73}$$

Putting $\lambda = p = 1$ and $\alpha = 0$ in Theorem 17, we obtain the following result obtained by ([18], Theorem 2.1).

Corollary 20. *([18], Theorem 2.1). If f given by (2) (with $p = 1$) belongs to the class $\mathcal{SL}^*$, then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} -\dfrac{1}{16}(4\mu-1) & \left(\mu \le -\dfrac{3}{4}\right), \\ \dfrac{1}{4} & \left(-\dfrac{3}{4} \le \mu \le \dfrac{5}{4}\right), \\ \dfrac{1}{16}(4\mu-1) & \left(\mu \ge \dfrac{5}{4}\right). \end{cases} \tag{74}$$

Further, if $-(3/4) \le \mu \le 1/4$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{4}(3+4\mu)\left|a_2\right|^2 \le \frac{1}{4}. \tag{75}$$

If $1/4 \leq \mu \leq 5/4$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{4}(5 - 4\mu)\left|a_2\right|^2 \leq \frac{1}{4}. \tag{76}$$

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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