Abstract and Applied Analysis

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Editorial

Fuzzy Linear and Nonlinear Integral Equations: Numerical Methods

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Integral equations are one of the most useful mathematical tools in both pure and applied mathematics. They have enormous applications in many real problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations.

Indeed, modeling such problems using integral equations with the exact parameters is not only easy but also impossible in the real problems. For this purpose, one way is using some uncertainty measures for handling such lack of information. One of the most and recent approaches is using Zadeh's fuzzy concept. So, instead of using deterministic models, we provide fuzzy integral equations of both linear and nonlinear forms.

In fact, obtaining the exact solutions of such fuzzy integral equations is not possible in all cases because of the inherited restrictions form application of fuzzy concepts in these problems. So, in this special issue, we intend to consider the numerical methods to solve fuzzy integral equations and the related topics with real applications. These topics include fuzzy linear and nonlinear integral equations with numerical methods, investigating the convergence, stability, and consistency of numerical approaches, numerically modeling the real problems associated with numerical methods, considering the differences between deterministic and fuzzy numerical methods to solve fuzzy integral equations, numerically solving fuzzy differential equations of arbitrary order using the equivalence fuzzy integral equations, obtaining some approximations of the solutions via ranking approaches, and applications in real-world problems with numerical techniques.

Our special issue contains few papers in which different numerical techniques are employed. The paper "*A simplified Milstein scheme for SPDEs with multiplicative noise*" replaces the exponential term with a Padé approximation of order 1 and denotes the resulting scheme by simplified Milstein scheme. The paper "*On properties of pseudointegrals based on pseudoaddition decomposable measures*" discussed pseudointegrals based on a pseudoaddition decomposable measure. Particularly, the definition of the pseudointegral for a measurable function based on a strict pseudoaddition decomposable measure by generalizing the definition of the pseudointegral of a bounded measurable function was stated. The paper "*Quadrature rules and iterative method for numerical solution of two-dimensional fuzzy integral equations*" introduced some generalized quadrature rules to approximate two-dimensional, Henstock integral of fuzzy-number-valued functions. Also, it gave error bounds for mappings of bounded variation in terms of uniform modulus of continuity. Moreover, it proposed an iterative procedure based on quadrature formula to solve two-dimensional linear fuzzy Fredholm integral equations of the second kind (2DFFLIE2) and presented the error estimation of the proposed method. The paper "*On solution of integrodifferential equation with delay parameter by Sinc basis functions*" is considered. For this purpose, a numerical solution is obtained for an integrodifferential equation with an integral boundary

condition and delay parameter. This type of problems arises in mathematical physics, mechanics, population growth, and other fields of physics and mathematical chemistry. Then, convergence of this approach is discussed by presenting a theorem which gives exponential type convergence rate and guarantees the accuracy of that. The paper "*A new reconstruction of variational iteration method and its application to nonlinear Volterra integrodifferential equations*" is proposed. Indeed, it reconstructed the variational iteration method, that is, the so-called parametric iteration method (PIM). The proposed method was applied for solving nonlinear Volterra integrodifferential equations (NVIDEs). The paper "*Approximating the solution of the linear and nonlinear fuzzy Volterra integrodifferential equations using expansion method*" is considered. To this end, it introduced an innovative method applying power series to solve numerically the linear and nonlinear fuzzy integrodifferential equation systems.

We hope the papers published in this special issue will provide a useful guide to a large community of researchers and will give way to development of new innovative theories and numerical approaches in the fields of modeling and approximating fuzzy integral equations and the related topics.

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Reza Ezzati
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Research Article

A Simplified Milstein Scheme for SPDEs with Multiplicative Noise

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This paper deals with a research question raised by Jentzen and Röckner (A Milstein scheme for SPDEs, arXiv:1001.2751v4 (2012)), whether the exponential term in their introduced scheme can be replaced by a simpler mollifier. This replacement can lead to more simplification and computational reduction in simulation. So, in this paper, we essentially replace the exponential term with a Padé approximation of order 1 and denote the resulting scheme by simplified Milstein scheme. The convergence analysis for this scheme is carried out and it is shown that even with this replacement the order of convergence is maintained, while the resulting scheme is easier to implement and slightly more efficient computationally. Some numerical tests are given that confirm the order of accuracy and also computational cost reduction.

1. Introduction

Many models in engineering, physics, complex phenomena, and so forth are described by stochastic partial differential equations (SPDEs); for example, see [1–6]. Since the exact solutions of these equations are rarely known, the numerical analysis of SPDEs has been recently the subject of many papers; for example, see [7–14], for more detailed discussion on this topic and many examples in applied sciences. In this paper, we consider strong approximation (see [4, Section 9.3]) of SPDEs of evolutionary type. To demonstrate the results of this paper clearly, we focus on the following example of SPDE:

$$dX_t(x) = \left[\lambda \frac{\partial^2}{\partial x^2} X_t(x) + f(x, X_t(x))\right] dt + b(x, X_t(x))\, dW_t(x), \tag{1}$$

with initial condition $X_0(x) = \xi(x)$ and Dirichlet boundary conditions $X_t(0) = X_t(1) = 0$ for all $x \in (0,1)$ and $t \in [0,T]$, where $\lambda \in (0,\infty)$. Let (Ω, F, P) be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $H = L^2((0,1),\mathbb{R})$ be the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0,1)$ to $\mathbb{R}$, and let $f, b : (0,1)\times\mathbb{R} \to \mathbb{R}$ be two appropriate smooth and regular functions with globally bounded derivatives. Let $W : [0,T]\times\Omega \to H$ be a standard Q-Wiener process with regard to $(\mathcal{F}_t)_{t\in[0,T]}$, with a trace class operator $Q : H \to H$ and $\xi : [0,1] \to \mathbb{R}$ with $\xi(0) = \xi(1) = 0$ being a smooth function. The covariance operator $Q : H \to H$ has orthonormal basis $g_j \in H$, $j \in \mathbb{N}$, of eigenfunctions with summable eigenvalues $\mu_j \in [0,\infty)$ $j \in \mathbb{N}$. Under the previous assumption, the SPDE (1) has a unique mild solution. Specifically, there exists an up to unpredictable unique adapted stochastic process $X : [0,T]\times\Omega \to H$ with continuous sample path which satisfies

$$X_t = e^{At}\xi + \int_0^t e^{A(t-s)} F(X_s)\, ds + \int_0^t e^{A(t-s)} B(X_s)\, dW_s, \tag{2}$$

for all $t \in [0,T]$, where $A : D(A) \subset H \to H$ is the Laplacian with Dirichlet boundary conditions times the constant $\lambda \in (0,\infty)$ and where $F : H \to H$ and $B : H \to HS(U,H)$ are given by $(F(v))(x) = f(x, v(x))$ and $(B(v)u)(x) = b(x, v(x)) \cdot u(x)$ for all $x \in (0,1)$, $v \in H$ and all $u \in U_0$ where $U_0 = Q^{1/2}$ with $\langle v, w\rangle_{U_0} = \langle Q^{-1/2}v, Q^{-1/2}w\rangle$ for all $v, w \in U_0$ is the image

$\mathbb{R}$-Hilbert space of $Q^{1/2}$ (see [15, Appendix C]); note that A and Q commutate in our example SPDE (2). Now we are concerned about the strong approximation of the SPDE (1). More formally we want to compute numerical approximation $Y : \Omega \to H$ such that

$$\left(E\left\|X_T - Y\right\|_H^2\right)^{1/2} := \left(E\left[\int_{(0,1)} \left|X_T(x) - Y(x)\right|^2 dx\right]\right)^{1/2} < \varepsilon \tag{3}$$

holds for a given precision $\varepsilon > 0$ with the least possible computational effort. To simulate the numerical approximation on a computer, one has to discretize both the time interval $[0,T]$ and the infinite dimensional space $H = L^2((0,1),\mathbb{R})$. In this paper we consider spectral Galerkin for spatial discretization and difference method for temporal discretization. A simple full discretization for (1) is the linear implicit Euler method combined with spectral Galerkin method which is given by

$$\begin{aligned}\overline{Y}_{k+1}^N = {} & P_N\left(I - \frac{T}{N^3}A\right)^{-1} \\ & \times\left(\overline{Y}_k^N + \frac{T}{N^3} f\left(\cdot, \overline{Y}_k^N\right)\right. \\ & \left. + b\left(\cdot, \overline{Y}_k^N\right)\left(W_{(k+1)T/N^3}^N - W_{kT/N^3}^N\right)\right),\end{aligned} \tag{4}$$

$k = 0, 1, \ldots, N^3 - 1$, and all $N \in \mathbb{N}$, with $\overline{Y}_0^N = \xi^N$ and $\xi^N = P_N(\xi)$, where P_N is a bounded linear operator such that $P_N : H \to H$ with

$$\left(P_N(v)\right)(x) = \sum_{n=1}^{N} 2\sin(n\pi x)\int_0^1 \sin(n\pi y)\, v(y)\, dy \tag{5}$$

for all $x \in (0,1)$, $v \in H$, and $N \in \mathbb{N}$, and the finite dimensional Wiener processes $W^N : [0,T] \times \Omega \to H$, $N \in \mathbb{N}$, are given by $W_t^N(\omega) = P_N(W_t(\omega))$ for all $t \in [0,T]$, $\omega \in \Omega$, and $N \in \mathbb{N}$. According to the analysis of [16], for method (4) with $k = N^3$, there exist real numbers $C_r > 0$ that for small $r \in (0, 3/2)$

$$\left(E\left\|X_T - Y_{N^3}^N\right\|_H^2\right)^{1/2} \le C_r N^{r-3/2} \tag{6}$$

holds for all $N \in \mathbb{N}$. This means that the method has overall convergence 3/8− (for a real number $\beta \in (0,\infty)$, we write $\beta-$ for the convergence order if the convergence order is higher in order than $\beta - \epsilon$ for all arbitrary small $\epsilon \in (0,\beta)$). In [16] Jentzen and Rockner proposed an infinite dimensional analog of Milstein type scheme for (1) given by $\widetilde{Y}_0^N = \xi_0^N = P_N(\xi)$ and

$$\begin{aligned}\widetilde{Y}_{k+1}^N = {} & P_N e^{A(T/N^2)} \\ & \times\left(\widetilde{Y}_k^N + \frac{T}{N^2} f\left(\cdot, \widetilde{Y}_k^N\right)\right. \\ & + b\left(\cdot, \widetilde{Y}_k^N\right)\left(W_{(k+1)T/N^2}^N - W_{kT/N^2}^N\right) \\ & + \frac{1}{2}\left(\frac{\partial}{\partial y} b\right)\left(\cdot, \widetilde{Y}_k^N\right) b\left(\cdot, \widetilde{Y}_k^N\right) \\ & \left.\times\left(\left(W_{(k+1)T/N^2}^N - W_{kT/N^2}^N\right)^2 - \frac{T}{N^2}\sum_{i=1}^{N}\eta_i g_i^2\right)\right)\end{aligned} \tag{7}$$

for all $k \in \{0, 1, \ldots, N^2 - 1\}$ and $N \in \mathbb{N}$. Here we use the notations $v \cdot w : (0,1) \to \mathbb{R}$, $v^2 : (0,1) \to \mathbb{R}$, and $(\phi(\cdot, v))(x) = \phi(x, v(x))$ for all $x \in (0,1)$ and all functions $v, w : (0,1) \to \mathbb{R}$, $\phi : (0,1) \times \mathbb{R} \to \mathbb{R}$. Method (7) gives a break of complexity of the numerical approximation of nonlinear SPDE with multiplicative trace class noise. More precisely, it is shown in [16] that N^2 time steps in contrast to N^3 time steps for the linear implicit Euler scheme (4) are required to achieve (6). That is the Milstein type scheme (7) with N^2 time steps guarantees that for real numbers $C_r > 0$, $r \in (0, 3/2)$, such that

$$\left(E\left\|X_T - Y_{N^2}^N\right\|_H^2\right)^{1/2} \le C_r N^{r-3/2} \tag{8}$$

holds for all $N \in \mathbb{N}$. Thus the scheme has the overall convergence order of 1/2−. Consequently scheme (7) increases the overall convergence order from 3/8− to 1/2−. As mentioned before, in this paper essentially the exponential term in the Milstein type scheme [16] is replaced by a first order approximation which makes the scheme easier to implement and slightly more efficient computationally while preserving the order of convergence. The analysis and implementation will be carried out as follows. In Section 2 the required setting and assumptions are formulated. In Section 3 the simplified Milstein scheme is formulated. In Section 4 we state and prove the main result of this section concerning the convergence of the simplified Milstein scheme. Finally in Section 5 numerical example for a stochastic reaction diffusion equation is presented to show numerically the order of convergence and computational costs. The numerical simulations will be carried out in MATLAB environment on a PC with CPU 2.66 GHz.

2. Setting and Assumptions

Throughout this paper suppose that the setting and following assumptions are fulfilled. Fix $T \in (0,\infty)$. Let (Ω, F, P) be a probability space with a normal filtration $\{F_t\}_{t\in[0,T]}$ and let $(H, \langle,\rangle, \|\cdot\|_H)$ and $(U, \langle,\rangle, \|\cdot\|_U)$ be two separable $\mathbb{R}$-Hilbert spaces. Moreover, let $W : [0,T] \times \Omega \to U$ be a standard Q-Wiener process with respect to $\{F_t\}_{t\in[0,T]}$, with a trace class operator $Q : U \to U$.

Assumption 1 (linear operator A). Let $A : D(A) \subset H \to H$ be a linear operator such that

$$Av = -\sum_{i\in\mathbb{N}} \lambda_i \langle e_i, v\rangle_H e_i \tag{9}$$

for every $v \in D(A)$ with $D(A) = \{w \in H \mid \sum_{i\in\mathbb{N}} |\lambda_i|^2 |\langle e_i, w\rangle_H|^2 < \infty\}$.

Here $(\lambda_i)_{i\in\mathbb{N}}$ is a family of real numbers with $\inf_{i\in\mathbb{N}}\lambda_i \in (0,\infty)$ and $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of H. By $V_r := D((-A)^r)$ equipped with the norm $\|v\|_{V_r} = \|(-A)^r v\|_H$ for all $v \in V_r$, $0 \le r$, we denote the $\mathbb{R}$-Hilbert space of domains of fractional powers of the linear operator $-A : D(A) \to H$.

Assumption 2 (drift term F). Let $\beta \in [0,1)$ be a real number and let $F : V_\beta \to H$ be a globally Lipschitz continuous; that is, $\sup_{v,w\in H}(\|F(v) - F(w)\|_H/\|v - w\|_H) < \infty$ and $\|F(v)\|_H < c(1 + \|v\|_{V_\beta})$, $c \in (0,\infty)$; in addition

$$\begin{gathered}\sup_{v\in V_\beta}\left\|F'(v)\right\|_{L(H)} < \infty,\\ \sup_{v\in V_\beta}\left\|F''(v)\right\|_{L^{(2)}(V_\beta,H)} < \infty.\end{gathered} \tag{10}$$

Assumption 3 (diffusion term B). Let $B : V_\beta \to HS(U_0, H)$ be a globally Lipschitz continuous mapping and twice continuously Frechet differentiable mapping with $\sup_{v\in V_\beta}\|B'(v)\|_{L(H,HS(U_0,H))} < \infty$ and $\sup_{v\in V_\beta}\|B''(v)\|_{L^{(2)}(V_\beta,HS(U_0,H))} < \infty$. In addition $\alpha \in (0,\infty)$, $\delta, \theta \in (0, 1/2)$ with $\beta \le \delta + 1/2$, $\gamma \in [\max(\delta,\beta), \delta + 1/2)$, and $c \in (0,\infty)$ is a real number such that $B(V_\delta) \subset HS(U_0, V_\delta)$ and

$$\|B(u)\|_{HS(U_0,V_\delta)} \le c\left(1 + \|u\|_{V_\delta}\right), \tag{11}$$

$$\left\|B'(v)B(v) - B'(w)B(w)\right\|_{HS^{(2)}(U_0,H)} \le c\|v - w\|_H, \tag{12}$$

$$\left\|(-A)^{-\vartheta}B(v)Q^{-\alpha}\right\|_{HS(U_0,H)} \le c\left(1 + \|v\|_{V_\gamma}\right), \tag{13}$$

hold for all $u \in V_\delta$ and $v, w \in V_\gamma$. Additionally, let the bilinear Hilbert-Schmidt operator $B'(v)B(v) \in HS^{(2)}(U_0, H)$ be symmetric for all $v \in V_\beta$. Note that the operator $B'(v)B(v) : U_0 \times U_0 \to H$, given by

$$\left(B'(v)B(v)\right)(u,\tilde{u}) = \left(B'(v)(B(v)u)\right)\tilde{u} \tag{14}$$

for all $u, \tilde{u} \in U_0$, is a bilinear Hilbert-Schmidt operator in $HS^{(2)}(U_0, H)$ for all $v \in V_\beta$.

The assumed symmetry of $B'(v)B(v) \in HS^{(2)}(U_0, H)$ thus reads as [16, Remark 1].

Assumption 4 (initial value ξ). Let $\xi : \Omega \to V_\gamma$ be an $F_0/B(V_\gamma)$-measurable mapping with $E\|\xi\|^2_{V_\gamma} < \infty$.

Proposition 5 (existence of the mild solution). *Let $T > 0$. Then under Assumptions 1–4, there exists an up to modifications unique predictable stochastic process $X : [0,T]\times\Omega \to V_\gamma$ which fulfills $\sup_{t\in[0,T]}E\|X_t\|^2_{V_\gamma} < \infty$,*

$$\begin{gathered}\sup_{t\in[0,T]} E\left\|B(X_t)\right\|^2_{HS(U_0,V_\delta)} < \infty,\\ X_t = e^{At}\xi + \int_0^t e^{A(t-s)}F(X_s)\,ds\\ \quad + \int_0^t e^{A(t-s)}B(X_s)\,dW_s,\end{gathered} \tag{15}$$

for all $t \in [0,T]$; moreover, we have

$$\sup_{t_1,t_2\in[0,t]} \frac{\left(E\left\|X_{t_2} - X_{t_1}\right\|^2_H\right)^{1/2}}{\left|t_2 - t_1\right|^{\min(\gamma,1/2)}} < \infty. \tag{16}$$

Proposition 5 immediately follows from Theorem 1 in [17].

3. The Proposed Simplified Milstein Scheme

We construct the simplified Milstein scheme for nonlinear stochastic partial differential equations. For this work first we use Taylor formula in Banach space for coefficients B and F for the problem (2). More formally using $F(X_s) \approx F(X_0)$ and $B(X_s) \approx B(X_0) + B'(X_0)(X_s - X_0)$ for small $s \in [0,T]$ shows

$$\begin{aligned}X_t &= e^{At}\xi + \int_0^t e^{A(t-s)}F(X_0)\,ds\\ &\quad + \int_0^t e^{A(t-s)}B'(X_0)(X_s - X_0)\,dW_s\end{aligned} \tag{17}$$

for small $t \in [0,T]$. Using the approximation $X_s \approx X_0 + \int_0^s B(X_0)dW_u$ for small $s \in [0,T]$ gives

$$\begin{aligned}X_t &\approx e^{At}X_0 + te^{At}F(X_0)\\ &\quad + \int_0^t e^{At}B(X_0)\,dW_s\\ &\quad + \int_0^t e^{At}B'(X_0)\left(\int_0^s B(X_0)\,dW_u\right)dW_s.\end{aligned} \tag{18}$$

We then substitute $e^{At} \approx (I - tA)^{-1}$ for small $t \in [0,T]$ to obtain

$$\begin{aligned}X_t \approx S_t\Big(&X_0 + tF(X_0)\\ &+ \int_0^t B(X_0)\,dW_s\\ &+ \int_0^t B'(X_0)\left(\int_0^s B(X_0)\,dW_u\right)dW_s\Big),\end{aligned} \tag{19}$$

where $S_t = (I - tA)^{-1}$. Combining the temporal approximation (19) and spatial discretization in (4) suggests the numerical scheme given by $Y_0^N = P_N(\xi) = \xi^N$ and

$$\begin{aligned} Y_{k+1}^N &= P_N S_{T/N^2} \\ &\quad \times \Bigg(Y_k^N + \frac{T}{N^2} F\left(Y_k^N\right) \\ &\qquad + B\left(Y_k^N\right)\left(W_{(k+1)T/N^2}^N - W_{kT/N^2}^N\right) \\ &\qquad + \int_{kT/N^2}^{(k+1)T/N^2} B'\left(Y_k^N\right) \\ &\qquad\qquad \times \left(\int_{kT/N^2}^{s} B\left(Y_k^N\right) dW_u^N \right) dW_s^N \Bigg), \end{aligned} \tag{20}$$

for all $k \in \{0, 1, \ldots, N^2 - 1\}$ and all $N \in \mathbb{N}$, where $S_{T/N^2} = (I-(T/N^2)A)^{-1}$. The difficulty in this formula is working with the term corresponding to the double integral. As suggested by Jentzen and Rockner (see [16, Subsection 6.7]), this double integral can be replaced by

$$\begin{aligned} &\int_{kT/N^2}^{(k+1)T/N^2} B'\left(Y_k^N\right) \left(\int_{kT/N^2}^{s} B\left(Y_k^N\right) dW_u^N \right) dW_s^N \\ &\quad = \frac{1}{2} \left(\frac{\partial}{\partial y} b \right)\left(\cdot, Y_k^N\right) b\left(\cdot, Y_k^N\right) \\ &\qquad \times \left(\left(W_{(k+1)T/N^2}^N - W_{kT/N^2}^N\right)^2 - \frac{T}{N^2} \sum_{i=1}^{N} \eta_i g_i^2 \right). \end{aligned} \tag{21}$$

By using (21), the numerical scheme (20) thus reduces to

$$\begin{aligned} Y_{k+1}^N &= P_N S_{T/N^2}^N \\ &\quad \times \Bigg(Y_k^N + \frac{T}{N^2} f\left(\cdot, Y_k^N\right) + b\left(\cdot, Y_k^N\right) \\ &\qquad \times \left(W_{(k+1)T/N^2}^N - W_{kT/N^2}^N\right) \\ &\qquad + \frac{1}{2} \left(\frac{\partial}{\partial y} b \right)\left(\cdot, Y_k^N\right) b\left(\cdot, Y_k^N\right) \\ &\qquad \times \left(\left(W_{(k+1)T/N^2}^N - W_{kT/N^2}^N\right)^2 - \frac{T}{N^2} \sum_{i=1}^{N} \eta_i g_i^2 \right) \Bigg), \end{aligned} \tag{22}$$

where $S_{T/N^2}^N = (I-(T/N^2)A)^{-N}$ and for all $k \in \{0, 1, \ldots, N^2 - 1\}$, $N \in \mathbb{N}$.

For the simplified Milstein scheme (22) applied to (1), the main result of this paper, that is, Theorem 7, will show that with $K = N^2$

$$\left(E\left\| X_T - Y_{N^2}^N \right\|_H^2 \right)^{1/2} \le C_r N^{r-3/2}, \quad 0 < r < \frac{3}{2}. \tag{23}$$

Similar to scheme (7), the numerical method (22) can be simulated quite easily. The term $(T/N^2)\sum_{i=1}^N \eta_i g_i^2$ in (22) can be computed once in advance for which $O(N^2)$ computational operations are needed. With the term $(T/N^2)\sum_{i=1}^N \eta_i g_i^2$ at hand, $O(N \log N)$ further computational operations and independent standard normal random variables are needed to compute Y_{k+1}^N from Y_k^N by using fast Fourier transform. Therefore, since N^2 time steps are used, $O(N^3 \log N)$ computational operations and random variables are required to obtain $Y_{N^2}^N$. The logarithmic term in $O(N^3 \log N)$ arises from fast Fourier transform computations, due to the nonlinearities of f and b. Taking into account the convergence order $3/2-$ in (23), one can show that scheme (22) shares the same overall convergence order of $1/2-$, which is greater than the overall convergence order $3/8-$ of the linear implicit scheme (4). We then take a more closer look at schemes (7) and (22) at each step. It is obvious that the Milstein scheme (7) requires evaluation of exponential term, while the simplified Milstein scheme needs one simple mollifier $(I - (T/N^2)A)^{-1}$ instead of exponential term. The CPU time for one path simulation by the simplified Milstein scheme (22) applied to (1) is less than that for (7). For example, see Table 1; for $N = 1024$, one path simulation of the simplified Milstein scheme (22) requires 834.842365 CPU seconds, while this simulation by Milstein scheme (7) needs 842.699400 CPU seconds. This difference is due to the fact that evaluation of the exponential term takes more time than that of the simple mollifier term. A natural question thus arises on whether such substitution can maintain the high convergence order of (7). In this paper we investigate this issue and prove that the simplified Milstein scheme maintains the expected order of convergence.

Table 1: Runtime (seconds) for one path simulation using three methods $\overline{Y}_{N^3}^N$, $\widetilde{Y}_{N^2}^N$, and $Y_{N^2}^N$ for $N = 64, 128, 256, 512, 1024$.

N	Implicit Euler scheme	Milstein scheme	Simplified Milstein scheme
64	38.316292	0.703536	0.690610
128	381.021714	3.628195	3.581998
256	3874.629760	19.811083	19.405515
512	46885.088426	126.663048	124.849040
1024	$5.1574e+005$	842.699400	834.842365

4. Convergence Analysis

Let $(g_j)_{j\in\mathbb{N}} \subset U$ be an orthonormal basis consisting of the eigenfunctions of $Q : U \to U$, and let $(\mu_j)_{j\in\mathbb{N}} \subset [0, \infty)$ be their corresponding eigenvalues with $Q : U \to U$ as a trace class operator; that is,

$$Qu = \sum_{j\in\mathbb{N}} \mu_j \left\langle g_j, u \right\rangle_U g_j \tag{24}$$

for all $u \in U$. We define the linear projection operator

$$P_N : H \longrightarrow H \quad \text{by } P_N v = \sum_{j=1}^{N} \left\langle e_j, v \right\rangle_H e_j. \tag{25}$$

Furthermore, we define Wiener processes $W^N : [0, T] \times \Omega \to U_0$ by $W_t^N(\omega) = \sum_{j=1}^N \langle g_j, W_t(\omega)\rangle_U g_j$ for all $t \in [0, T]$,

$\omega \in \Omega$, and $N \in \mathbb{N}$. Let $T = M\Delta t$, $M = N^2$, and Δt be the time discretization step, and let $H = U = L^2((0,1),\mathbb{R})$ be the $\mathbb{R}$-Hilbert space of equivalence classes of $\beta(0,1)/\beta(\mathbb{R})$-measurable and Lebesgue square integrable functions from $(0,1)$ to $\mathbb{R}$ with the scalar product $\langle u,v\rangle_H = \int_0^1 u(x)v(x)dx$ and the norm $\|v\|_H = (\int_0^1 |v(x)|^2 dx)^{1/2}$ for $u,v \in H = U$. Now we start our investigation to analyze the proposed method for SPDE fulfilling Assumptions 1–4. Based on (1) we then consider $A = \lambda(\partial^2/\partial x^2)$ and $e_k(x) = \sqrt{2}\sin(k\pi x)$, $k \in \mathbb{N}$, as the orthonormal basis of $H = L^2((0,1),\mathbb{R})$, which satisfy

$$Ae_k = -\lambda_k e_k, \quad \lambda_k = \lambda\pi^2 k^2. \tag{26}$$

For the drift term to fulfill Assumption 2, let $f : (0,1)\times\mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $\int_0^1 |f(x,0)|^2 dx < \infty$ and

$$\sup_{y\in\mathbb{R}} \left|\frac{\partial f}{\partial y}(x,y)\right| < \infty. \tag{27}$$

Then, the operator $F : H \to H$ given by $(F(v))(x) = f(x,v(x))$, for $x \in (0,1)$ and $v \in H$, satisfies Assumption 2. For the diffusion term to satisfy Assumption 3, we consider $b : (0,1)\times\mathbb{R} \to \mathbb{R}$ to be a twice continuously differentiable function with

$$\begin{gathered} |b(x,0)| \le R, \qquad \left|\frac{\partial b}{\partial x}(x,y)\right| \le R, \\ \left\|\frac{\partial b}{\partial x}(x,y)\right\|_{L(\mathbb{R},\mathbb{R})} \le R, \end{gathered} \tag{28}$$

and also

$$\begin{gathered} E\|B(X_t)\|^2_{HS(U_0,V_\delta)} \le R, \qquad \|F'(v)\|_{L(H)} \le R, \\ \|F''(v)\|_{L^{(2)}(V_\beta,H)} \le R, \qquad E\|F(X_t)\|^2_H \le R, \\ \|B'(v)\|_{L(H,HS(U_0,V_\delta))} \le R, \qquad \|B''(v)\|_{L^{(2)}(V_\beta,HS(U_0,V_\delta))} \le R, \\ E\|(-A)^\gamma X_t\|^2_H = E\|X_t\|^2_{V_\gamma} \le R, \\ E\|X_{t_2} - X_{t_1}\|^4_{V_\beta} \le R|t_2 - t_1|^{\min(4(\gamma-\beta),2)}, \end{gathered} \tag{29}$$

for all $x \in (0,1)$ and some given $R \in (0,\infty)$. Let $b : H \to HS(U_0,H)$ be the operator $(B(v)u)(x) = b(x,v(x))\cdot u(x)$ for all $x \in (0,1)$.

It has been shown in [16] that $B : H \to HS(U_0,H)$ fulfills Assumption 3. For the initial value to satisfy Assumption 4, we assume that $x_0 : (0,1) \to \mathbb{R}$ is a twice continuously differentiable function with $x_0|_{\partial(0,1)} = 0$. Then the mapping $\xi : \Omega \to V_\gamma$ given by $\xi(\omega) = x_0$ for all $\omega \in \Omega$ fulfills Assumption 4 for all $\gamma \in (0,1)$. With the above setting, the SPDE (1) reduces to

$$\begin{aligned} dX_t(x) = {} & [\lambda\Delta X_t(x) + f(x,X_t(x))]\,dt \\ & + b(x,X_t(x))\,dW_t(x), \end{aligned} \tag{30}$$

with $X_t(0) = X_t(1) = 0$ and $X_0 = x_0(x)$ for $t \in [0,T]$, $x \in (0,1)$, and $\Delta = \partial^2/\partial x^2$.

Moreover, we define a family $\beta^j(\omega) = (1/\sqrt{\mu_j})\langle g_j, W_t(\omega)\rangle_U$ for all $\omega \in \Omega$, $t \in [0,T]$ and all $j \in \mathbb{N}$, and we consider the mappings $\Delta W_k^N : \Omega \to U_0$, $k \in \{0,1,\ldots,N^2-1\}$ by $\Delta W_k^N(\omega) = W^N_{(k+1)\Delta t}(\omega) - W^N_{k\Delta}(\omega)$.

Using these notations, the SPDE (30) can be rewritten as

$$\begin{aligned} dX_t(x) = {} & \left[\lambda\frac{\partial}{\partial x^2}X_t(x) + f(x,X_t(x))\right]dt \\ & + \sum_{j=1}^N \left[b(x,X_t(x))\sqrt{\mu_j}g_j(x)\right]d\beta_t^j, \end{aligned} \tag{31}$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = x_0(x)$ for $t \in [0,T]$ and $x \in (0,1)$.

Scheme (21)-(22) applied to the SPDE (30) reduces to

$$\begin{aligned} Y^N_{k+1} = {} & P_N\Bigg(S_{\Delta t}Y^N_k + \Delta t S_{\Delta t}F(Y^N_k) \\ & + \int_{k\Delta t}^{(k+1)\Delta t} S_{\Delta t}B(Y^N_k)\,dW^N_s \\ & + \int_{k\Delta t}^{(k+1)\Delta t} S_{\Delta t}B'(Y^N_k) \\ & \times \left(\int_{k\Delta t}^s B(Y^N_k)\,dW^N_u\right)dW^N_s\Bigg), \end{aligned} \tag{32}$$

where $S_{\Delta t} = (I - \Delta t A)^{-1}$ and $\Delta t = T/N^2$, $k \in \{0,1,\ldots,N^2-1\}$, $N \in \mathbb{N}$. Therefore the numerical method (32) satisfies

$$\begin{aligned} Y^N_k = {} & S^k_{\Delta t}Y^N_0 + P_N\left(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} S^{k-l}_{\Delta t}F(Y^N_l)\,ds\right) \\ & + P_N\left(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} S^{k-l}_{\Delta t}B(Y^N_l)\,dW^N_s\right) \\ & + P_N\Bigg(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} S^{k-l}_{\Delta t}B'(Y^N_l) \\ & \times \left(\int_{l\Delta t}^s B(Y^N_l)\,dW^N_u\right)dW^N_s\Bigg), \end{aligned} \tag{33}$$

where $S^k_{\Delta t} = (I - \Delta t A)^{-k}$ and for all $k \in \{0,1,\ldots,N^2\}$, $N \in \mathbb{N}$. The following inequalities are classical and one can easily prove them by using the spectral decomposition of A [1]:

$$\|(-A)^s S^l_{\Delta t}\|_{L(H)} \le M t_l^{-s}, \quad l \ge 1,\ s \in [0,1], \tag{34}$$

$$\|(-A)^s e^{tA}\|_{L(H)} \le M t^{-s}, \quad t > 0, \tag{35}$$

$$\|S^l_{\Delta t} - e^{l\Delta t A}\|_{L(H)} \le \frac{M}{l}, \tag{36}$$

$$\|(-A)^{-s}(e^{tA} - I)\|_{L(H)} \le M t^s, \tag{37}$$

$$\|(-A)^{-s}(S_{\Delta t} - I)\|_{L(H)} \le M(\Delta t)^s, \quad s \in [0,1]. \tag{38}$$

To give the order of the L^2 convergence for the simplified Milstein approximation of the evolution equation, we need the following version of the Gronwall lemma.

Lemma 6. *Let $\{\alpha_n\}_{n\geq 0}, \{\beta_n\}_{n\geq 0}$ be two sequences of nonnegative numbers such that $\alpha_0 = \beta_0 = 0$ and such that there exists a positive constant L such that*

$$\alpha_n \leq L\sum_{k=0}^{n-1}\alpha_k + \beta_n, \quad \forall n \geq 1; \tag{39}$$

then

$$\alpha_n \leq \sum_{k=0}^{n-1} e^{(n-k-1)L}\left(\beta_{(k+1)} - \beta_k\right), \quad \forall n \geq 1. \tag{40}$$

Proof. By Mathematical induction with respect to n using $L \leq e^L - 1$. □

From the above lemma we can deduce that

$$\forall n \geq 1, \quad \alpha_n \leq e^{(n-1)L}\beta_n. \tag{41}$$

The main result of this section is stated below.

Theorem 7. *Let $T > 0$, $\Delta t = T/N^2$, and $X_0 \in L^2(\Omega, H)$. Suppose that X is the solution of (2) on $[0,T]$. Let Assumptions 1–4 hold, and let $\{Y_l^N\}_{l\geq 0}$ be the numerical approximations obtained by scheme (33). Then there exists a positive constant C such that*

$$\begin{aligned}\left(E\left\|e_k^N\right\|_H^2\right)^{1/2} &\leq C\left(\left(\inf_{j>N+1}\lambda_j\right)^{-\gamma} + \frac{\left(E\left\|X_0\right\|_H^2\right)^{1/2}}{k}\right.\\ &\quad\left.+\left(\sup_{j>N+1}\mu_j\right)^{\alpha} + (\Delta t)^{\min(2(\gamma-\beta),\gamma)}\right).\end{aligned} \tag{42}$$

Proof. To start the proof, we first note that the exact solution of SPDE (2) satisfies

$$\begin{aligned}X_{k\Delta t} &= e^{At}X_0 + \int_0^{k\Delta t} e^{A(t-s)}F\left(X_s\right)ds\\ &\quad+\int_0^{k\Delta t} e^{A(t-s)}B\left(X_s\right)dW_s\\ &= e^{At}X_0 + \sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} e^{A(t-s)}F\left(X_s\right)ds\\ &\quad+\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} e^{A(t-s)}B\left(X_s\right)dW_s,\end{aligned} \tag{43}$$

where $t = k\Delta t$. In particular, (43) shows

$$\begin{aligned}P_N\left(X_{k\Delta t}\right) &= e^{At}X_0^N + P_N\left(\int_0^{k\Delta t} e^{A(t-s)}F\left(X_s\right)ds\right)\\ &\quad+P_N\left(\int_0^{k\Delta t} e^{A(t-s)}B\left(X_s\right)dW_s\right)\\ &= e^{At}X_0^N + P_N\left(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} e^{A(t-s)}F\left(X_s\right)ds\right)\\ &\quad+P_N\left(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} e^{A(t-s)}B\left(X_s\right)dW_s\right).\end{aligned} \tag{44}$$

Let

$$\begin{aligned}e_k^N &= X_{k\Delta t} - Y_k^N\\ &= X_{k\Delta t} - P_N\left(X_{k\Delta t}\right) + P_N\left(X_{k\Delta t}\right) - Y_k^N\\ &= \bar{e}_k^N + \hat{e}_k^N,\end{aligned} \tag{45}$$

where

$$\bar{e}_k^N = X_{k\Delta t} - P_N\left(X_{k\Delta t}\right), \qquad \hat{e}_k^N = P_N\left(X_{k\Delta t}\right) - Y_k^N. \tag{46}$$

For the spatial discretization error $\bar{e}_k^N$, we have

$$\begin{aligned}E\left\|\bar{e}_k^N\right\|_H^2 &= E\left\|X_{k\Delta t} - P_N\left(X_{k\Delta t}\right)\right\|_H^2\\ &= E\left\|\left(I - P_N\right)X_{k\Delta t}\right\|_H^2\\ &= E\left\|(-A)^{-\gamma}\left(I - P_N\right)(-A)^{\gamma}X_{k\Delta t}\right\|_H^2\\ &\leq \left\|(-A)^{-\gamma}\left(I - P_N\right)\right\|_{L(H)}^2 E\left\|X_{k\Delta t}\right\|_{V_\gamma}^2\\ &\leq R\left(s_N\right)^2;\end{aligned} \tag{47}$$

the real numbers $(s_N)_{N\in\mathbb{N}}$ are given by (see [16])

$$s_N := \left\|(-A)^{-\gamma}\left(I - P_N\right)\right\|_{L(H)} = \left(\inf_{j>N+1}\lambda_j\right)^{-\gamma}. \tag{48}$$

For the $\hat{e}_k^N$ with respect to (33) and (44), we have

$$\begin{aligned}\hat{e}_k^N &= e^{At}X_0^N - S_{\Delta t}^k Y_0^N\\ &\quad+P_N\left(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left(e^{A(t-s)}F\left(X_s\right)\right.\right.\\ &\qquad\left.\left.-S_{\Delta t}^{k-l}F\left(Y_l^N\right)\right)ds\right)\\ &\quad+P_N\left(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left(e^{A(t-s)}B\left(X_s\right)dW_s\right.\right.\end{aligned}$$

$$
\begin{aligned}
&\quad - S_{\Delta t}^{k-l} B\left(Y_l^N\right) dW_s^N\Big) \\
&- \sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} S_{\Delta t}^{k-l} B'\left(Y_l^N\right) \\
&\qquad \times \left(\int_{l\Delta t}^{s} B\left(Y_l^N\right) dW_u^N\right) dW_s^N\Bigg) \\
&= I + II + III,
\end{aligned} \tag{49}
$$

where $I = e^{At} X_0^N - S_{\Delta t}^k Y_0^N$ and II and III are, respectively, the other terms under P_N operator. From (36), the first term of (49) can be easily estimated by

$$
\begin{aligned}
&E\|I\|_H^2 \\
&\quad = \left(E\left\|\left(e^{At} - S_{\Delta t}^k\right) X_0^N + S_{\Delta t}^k \left(X_0^N - Y_0^N\right)\right\|_H^2\right) \\
&\quad \le C\left(\frac{E\left\|X_0^N\right\|_H^2}{k^2} + E\left\|\widehat{e}_0^N\right\|_H^2\right).
\end{aligned} \tag{50}
$$

Let C denote a constant which may depend on A, f, b, R, N, M, or T. We now treat the second term of (49)

$$
\begin{aligned}
II &= P_N\left(\sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} \left(e^{A(t-s)} F\left(X_s\right)\right.\right. \\
&\qquad \left.\left. - S_{\Delta t}^{k-l} F\left(Y_l^N\right)\right) ds\right) \\
&= P_N\left(\sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} \left(e^{A(t-s)} - S_{\Delta t}^{k-l}\right) F\left(X_s\right) ds\right) \\
&\quad + P_N\left(\sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} S_{\Delta t}^{k-l} \left(F\left(X_s\right) - F\left(X_{l\Delta t}\right)\right) ds\right) \\
&\quad + P_N\left(\sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} S_{\Delta t}^{k-l} \left(F\left(X_{l\Delta t}\right) - F\left(Y_l^N\right)\right) ds\right) \\
&= II_1 + II_2 + II_3.
\end{aligned} \tag{51}
$$

According to Assumption 2 and the fact that $\|P_N(v)\|_H \leqslant \|v\|_H$ for all $v \in H$, we get

$$
\begin{aligned}
\left\|II_1\right\|_H &\le C \sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} \left\|e^{A(t-s)} - S_{\Delta t}^{k-l}\right\|_{L(H)} \\
&\qquad \times \left(1 + \left\|X_s\right\|_{V_\gamma}\right) ds \\
&\le C \int_{(k-1)\Delta t}^{k\Delta t} \left\|e^{A(t-s)} - S_{\Delta t}\right\|_{L(H)} \left(1 + \left\|X_s\right\|_{V_\gamma}\right) ds \\
&\quad + C \sum_{l=0}^{k-2} \int_{l\Delta t}^{(l+1)\Delta t} \left\|e^{A(t-s)} - S_{\Delta t}^{k-l}\right\|_{L(H)} \\
&\qquad \times \left(1 + \left\|X_s\right\|_{V_\gamma}\right) ds.
\end{aligned} \tag{52}
$$

For the first term on the right-hand side of inequality (52), we have

$$
\begin{aligned}
&E\left(\int_{(k-1)\Delta t}^{k\Delta t} \left\|e^{A(t-s)} - S_{\Delta t}\right\|_{L(H)} \left(1 + \left\|X_s\right\|_{V_\gamma}\right) ds\right)^2 \\
&\quad \le M(\Delta t)^2 \left(1 + \sup_{t\in[0,T]} E\left\|X_t\right\|_{V_\gamma}^2\right).
\end{aligned} \tag{53}
$$

To estimate the second term of (52), one should note that

$$
\begin{aligned}
\left\|e^{A(t-s)} - S_{\Delta t}^{k-l}\right\|_{L(H)} &\le \left\|e^{A(t-s)} - e^{A(k-l-1)\Delta t}\right\|_{L(H)} \\
&\quad + \left\|e^{A(k-l-1)\Delta t} - S_{\Delta t}^{k-l}\right\|_{L(H)}
\end{aligned} \tag{54}
$$

and from (35) and (37), we have

$$
\begin{aligned}
\left\|e^{A(t-s)} - e^{A(k-l-1)\Delta t}\right\|_{L(H)} &\le \left\|(-A)^\gamma e^{A(k-l-1)\Delta t}\right\|_{L(H)} \\
&\quad \times \left\|(-A)^{-\gamma}\left(e^{A(l+1)\Delta t - s} - I\right)\right\|_{L(H)} \\
&\le C \frac{((l+1)\Delta t - s)^\gamma}{((k-l-1)\Delta t)^\gamma}.
\end{aligned} \tag{55}
$$

Therefore, from (35), (38), and (36), we have

$$
\begin{aligned}
&\left\|e^{A(k-l-1)\Delta t} - S_{\Delta t}^{k-l}\right\|_{L(H)} \\
&\quad \le \left\|(-A)^\gamma \left(e^{A(k-l-1)\Delta t}\right) (-A)^{-\gamma} \left(I - S_{\Delta t}\right)\right\|_{L(H)} \\
&\qquad + \left\|S_{\Delta t}\left(e^{A(k-l-1)\Delta t} - S_{\Delta t}^{k-l-1}\right)\right\|_{L(H)} \\
&\quad \le M\left(\frac{\Delta t^\gamma}{((k-l-1)\,\Delta t)^\gamma} + \frac{1}{k-l-1}\right) \\
&\quad \le C \frac{(\Delta t)^\gamma}{((k-l-1)\Delta t)^\gamma};
\end{aligned} \tag{56}
$$

thus we have

$$
\left\|e^{A(t-s)} - S_{\Delta t}^{k-l}\right\|_{L(H)} \le M\left(\frac{(\Delta t)^\gamma}{((k-l-1)\,\Delta t)^\gamma}\right). \tag{57}
$$

Therefore, by taking the expectation of (52) to the power of 2 and using (53) and (57), we get

$$\begin{aligned} &E\|II_1\|_H^2 \\ &\quad\le C\left(\sum_{l=0}^{k-2}\int_{l\Delta t}^{(l+1)\Delta t}\frac{(\Delta t)^{\gamma}}{((k-l-1)\Delta t)^{\gamma}}\left(1+E\|X_s\|_{V_\gamma}^2\right)ds\right)^2 \\ &\qquad + C(\Delta t)^2, \end{aligned} \tag{58}$$

in which the summation can be estimated as

$$\begin{aligned} \sum_{l=0}^{k-2}\frac{\Delta t}{((k-l-1)\Delta t)^{d}} &\le \sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\frac{dt}{t^d} \\ &\le \frac{1}{1-d}T^{1-d}, \quad d<1. \end{aligned} \tag{59}$$

For the second term of (51), Proposition 5 and Assumption 2 lead to

$$\begin{aligned} E\|II_2\|_H^2 &\le \sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left\|S_{\Delta t}^{k-l}\right\|_{L(H)}^2(\Delta t)^{2\min\{\gamma,1/2\}}ds \\ &\le C(\Delta t)^{2\min\{\gamma,1/2\}}. \end{aligned} \tag{60}$$

For the third term of (51), by using Assumption 2, we have

$$\begin{aligned} \|II_3\|_H &\le \sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left\|S_{\Delta t}^{k-l}\left(f\left(X_{l\Delta t}\right)-f\left(Y_l^N\right)\right)\right\|_H ds \\ &\le \sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left\|S_{\Delta t}^{k-l}\right\|_{L(H)}\left\|e_l^N\right\|_H ds \\ &\le C\sum_{l=0}^{k-1}\Delta t\left\|e_l^N\right\|_H \end{aligned} \tag{61}$$

which leads to

$$E\|II_3\|_H^2 \le C\left(\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|e_l^N\right\|_H^2\right)^{1/2}\right)^2, \tag{62}$$

where we have used the Minkowski inequality in (62).

Therefore, from (58), (60), and (62), we get

$$\begin{aligned} E\|II\|_H^2 \le C\Bigg(&(\Delta t)^{2\gamma}+(\Delta t)^{2\min\{\gamma,1/2\}} \\ &+\left(\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|e_l^N\right\|_H^2\right)^{1/2}\right)^2\Bigg). \end{aligned} \tag{63}$$

Now for the last term of (49), we obtain

$$\begin{aligned} III &= P_N\Bigg(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\Big(e^{A(t-s)}B\left(X_s\right)dW_s \\ &\qquad - S_{\Delta t}^{k-l}B\left(Y_l^N\right)dW_s^N\Big) \\ &\qquad - \sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}S_{\Delta t}^{k-l}B'\left(Y_l^N\right) \\ &\qquad \times\left(\int_{l\Delta t}^{s}B\left(Y_l^N\right)dW_u^N\right)dW_s^N\Bigg) \\ &= III_1+III_2+III_3+III_4, \end{aligned} \tag{64}$$

where

$$\begin{aligned} III_1 &= P_N\Bigg(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}e^{A(t-s)} \\ &\qquad \times B\left(X_s\right)\left(dW_s-dW_s^N\right)\Bigg), \\ III_2 &= P_N\Bigg(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right) \\ &\qquad \times B\left(X_s\right)dW_s^N\Bigg), \\ III_3 &= P_N\Bigg(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}S_{\Delta t}^{k-l} \\ &\qquad \times\left(B\left(X_{l\Delta t}\right)-B\left(Y_l^N\right)\right)dW_s^N\Bigg), \\ III_4 &= P_N\Bigg(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}S_{\Delta t}^{k-l} \\ &\qquad \times\left(B\left(X_s\right)-B\left(X_{l\Delta t}\right)\right)dW_s^N \\ &\qquad - \sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}S_{\Delta t}^{k-l}B'\left(Y_l^N\right) \\ &\qquad \times\left(\int_{l\Delta t}^{s}B\left(Y_l^N\right)dW_u^N\right)dW_s^N\Bigg). \end{aligned} \tag{65}$$

Using the fact that $\|P_N(v)\|_H \leqslant \|v\|_H$ for all $v\in H$, for III_1, we have

$$\begin{aligned} &E\left\|\int_0^t e^{A(t-s)}B\left(X_s\right)d\left(W_s-W_s^N\right)\right\|_H^2 \\ &\quad = E\left\|\sum_{j=N+1}^{\infty}\left(\int_0^t e^{A(t-s)}B\left(X_s\right)g_j d\left\langle g_j,W_s\right\rangle\right)\right\|_H \\ &\quad = \sum_{j=N+1}^{\infty}\mu_j\left(\int_0^t E\left\|e^{A(t-s)}B\left(X_s\right)g_j\right\|_H^2 ds\right), \end{aligned} \tag{66}$$

and therefore

$$\begin{aligned} &E\left\|\int_0^t e^{A(t-s)}B(X_s)\,d\left(W_s-W_s^N\right)\right\|_H^2 \\ &\quad=\sum_{j=N+1}^{\infty}\mu_j\left(\int_0^t E\left\|e^{A(t-s)}B(X_s)Q^{-\alpha}Q^{\alpha}g_j\right\|_H^2 ds\right) \\ &\quad=\sum_{j=N+1}^{\infty}(\mu_j)^{2\alpha+1}\left(\int_0^t E\left\|e^{A(t-s)}B(X_s)Q^{-\alpha}g_j\right\|_H^2 ds\right). \end{aligned} \tag{67}$$

Thus

$$\begin{aligned} &E\left\|\int_0^t e^{A(t-s)}B(X_s)\,d\left(W_s-W_s^N\right)\right\|_H^2 \\ &\quad\le\left(\sup_{j>N+1}\mu_j\right)^{2\alpha} \\ &\qquad\times\sum_{j=N+1}^{\infty}\mu_j\left(\int_0^t E\left\|e^{A(t-s)}B(X_s)Q^{-\alpha}g_j\right\|_H^2 ds\right) \\ &\quad\le\left(\sup_{j>N+1}\mu_j\right)^{2\alpha} \\ &\qquad\times\sum_{j=1}^{\infty}\mu_j\left(\int_0^t E\left\|e^{A(t-s)}B(X_s)Q^{-\alpha}g_j\right\|_H^2 ds\right) \end{aligned} \tag{68}$$

which means

$$\begin{aligned} &E\left\|\int_0^t e^{A(t-s)}B(X_s)\,d\left(W_s-W_s^N\right)\right\|_H^2 \\ &\quad\le\left(\sup_{j>N+1}\mu_j\right)^{2\alpha} \\ &\qquad\times\left(\int_0^t E\left\|e^{A(t-s)}B(X_s)Q^{-\alpha}g_j\right\|_{HS(U_0,H)}^2 ds\right) \\ &\quad\le\left(\sup_{j>N+1}\mu_j\right)^{2\alpha} \\ &\qquad\times\left(\int_0^t (t-s)^{2\theta}E\left\|(-A)^{-\theta}B(X_s)Q^{-\alpha}\right\|_{HS(U_0,H)}^2 ds\right). \end{aligned} \tag{69}$$

Therefore, from (13), we get

$$\begin{aligned} &E\left\|\int_0^t e^{A(t-s)}B(X_s)\,d\left(W_s-W_s^N\right)\right\|_H^2 \\ &\quad\le C\left(\sup_{j>N+1}\mu_j\right)^{2\alpha} \\ &\qquad\times\left(\int_0^t (t-s)^{-2\theta}E\left[\left(1+\|X_s\|_{V_\gamma}\right)^2\right]ds\right) \\ &\quad\le C\left(\sup_{j>N+1}\mu_j\right)^{2\alpha} \\ &\qquad\times\left(\int_0^t (t-s)^{-2\theta}\left[\left(1+E\|X_s\|_{V_\gamma}\right)^2\right]ds\right) \\ &\quad\le C\left(\sup_{j>N+1}\mu_j\right)^{2\alpha}\left(\int_0^t (t-s)^{-2\theta}ds\right), \end{aligned} \tag{70}$$

which leads to

$$\begin{aligned} E\|III_1\|_H^2 &\le E\left\|\int_0^t e^{A(t-s)}B(X_s)\,d\left(W_s-W_s^N\right)\right\|_H^2 \\ &\le C\left(\sup_{j>N+1}\mu_j\right)^{2\alpha}. \end{aligned} \tag{71}$$

For III_2, by Proposition 5, it is seen that

$$\begin{aligned} &E\left\|\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right)B(X_s)\,dW_s^N\right\|_H^2 \\ &\quad\le\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}E\left\|\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right)B(X_s)\right\|_{HS(U_0,H)}^2 ds \\ &\quad\le\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left\|(-A)^{-\delta}\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right)\right\|_{L(H)}^2 \\ &\qquad\times\|B(X_s)\|_{HS(U_0,V_\delta)}^2 ds \\ &\quad\le R\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left\|(-A)^{-\delta}\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right)\right\|_{L(H)}^2 ds, \end{aligned} \tag{72}$$

in which

$$\begin{aligned} &\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left\|(-A)^{-\delta}\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right)\right\|_{L(H)}^2 ds \\ &\quad=\sum_{l=0}^{k-2}\int_{l\Delta t}^{(l+1)\Delta t}\left\|(-A)^{-\delta}\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right)\right\|_{L(H)}^2 ds \\ &\qquad+\int_{(k-1)\Delta t}^{k\Delta t}\left\|(-A)^{-\delta}\left(e^{A(t-s)}-S_{\Delta t}\right)\right\|_{L(H)}^2 ds. \end{aligned} \tag{73}$$

Now for the first term on the right-hand side of (73), we need to estimate

$$\begin{aligned} &\left\|(-A)^{-\delta}\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right)\right\|_{L(H)}^2 \\ &\quad\le\left\|(-A)^{-\delta}\left(e^{A(t-s)}-e^{A(k-l-1)\Delta t}\right)\right\|_{L(H)} \\ &\qquad+\left\|(-A)^{-\delta}\left(e^{A(k-l-1)\Delta t}-S_{\Delta t}^{k-l}\right)\right\|_{L(H)} \end{aligned} \tag{74}$$

and then by (34) and (36), we obtain

$$\begin{aligned}&\left\|(-A)^{-\delta}\left(e^{A(t-s)}-e^{A(k-l-1)\Delta t}\right)\right\|_{L(H)}\\&\quad\le\left\|(-A)^{1-\delta}e^{A(k-l-1)\Delta t}\right\|_{L(H)}\\&\qquad\times\left\|(-A)^{-1}\left(e^{A(l+1)\Delta t-s}-I\right)\right\|_{L(H)}\\&\quad\le C\frac{((k+1)\Delta t-s)}{((k-l-1)\Delta t)^{1-\delta}}.\end{aligned}\tag{75}$$

Similarly from (34), (35), we obtain

$$\begin{aligned}&\left\|(-A)^{-\delta}\left(e^{A(k-l-1)\Delta t}-S_{\Delta t}^{k-l}\right)\right\|_{L(H)}\\&\quad\le\left\|(-A)^{1/2-\delta}e^{A(k-l-1)\Delta t}(-A)^{-1/2}\left(I-S_{\Delta t}\right)\right\|_{L(H)}\\&\qquad+\left\|(-A)^{-\delta}S_{\Delta t}\left(e^{A(k-l-1)\Delta t}-S_{\Delta t}^{k-l-1}\right)\right\|_{L(H)}\\&\quad\le C\left(\frac{(\Delta t)^{1/2}}{((k-l-1)\Delta t)^{1/2-\delta}}+\frac{1}{k-l-1}\right)\\&\quad\le C\left(\frac{(\Delta t)^{1/2}}{((k-l-1)\Delta t)^{1/2-\delta}}+\frac{(\Delta t)^{\gamma_1}}{((k-l-1)\Delta t)^{\gamma_1}}\right),\end{aligned}\tag{76}$$

where $\gamma_1>0$ is such that

$$\tilde{\gamma}<\gamma_1<\frac{1}{2},\tag{77}$$

which is possible since

$$\tilde{\gamma}<\frac{1}{2}.\tag{78}$$

Therefore,

$$\begin{aligned}&\left\|(-A)^{-\delta}\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right)\right\|_{L(H)}^2\\&\quad\le C\left(\frac{(\Delta t)}{((k-l-1)\Delta t)^{1-2\delta}}+\frac{(\Delta t)^{2\gamma_1}}{((k-l-1)\Delta t)^{2\gamma_1}}\right).\end{aligned}\tag{79}$$

For the second term on the right-hand side of (73), we can write

$$\begin{aligned}&\left\|(-A)^{-\delta}\left(e^{A(t-s)}-S_{\Delta t}\right)\right\|_{L(H)}^2\\&\quad\le\left\|(-A)^{-\delta}\left(e^{A(t-s)}-I\right)+\left(I-S_{\Delta t}\right)\right\|_{L(H)}\\&\quad\le C\left((t-s)^{\delta}+(\Delta t)^{\delta}\right),\end{aligned}\tag{80}$$

from which we get

$$\int_{(k-1)\Delta t}^{k\Delta t}\left\|(-A)^{-\delta}\left(e^{A(t-s)}-S_{\Delta t}\right)\right\|_{L(H)}^2ds\le C(\Delta t)^{1+2\delta}.\tag{81}$$

Thus from (73), after replacing (79) and (81) in (72), we obtain

$$\begin{aligned}&E\left\|\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\left(e^{A(t-s)}-S_{\Delta t}^{k-l}\right)B\left(X_s\right)dW_s^N\right\|_H^2\\&\quad\le C\left((\Delta t)+(\Delta t)^{2\gamma_1}+(\Delta t)^{1+2\delta}\right),\end{aligned}\tag{82}$$

which gives

$$E\left\|III_2\right\|_H^2\le C(\Delta t)^{2\gamma_1}\le C(\Delta t)^{2\tilde{\gamma}},\quad\text{since }\tilde{\gamma}<\gamma_1.\tag{83}$$

For the third term III_3, we have

$$\begin{aligned}&E\left\|\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}S_{\Delta t}^{k-l}\left(B\left(X_{l\Delta t}\right)-B\left(Y_l^N\right)\right)dW_s^N\right\|_H^2\\&\quad\le\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}E\left\|S_{\Delta t}^{k-l}\left(B\left(X_{l\Delta t}\right)-B\left(Y_l^N\right)\right)\right\|_{HS(U_0,H)}ds\\&\quad\le\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}E\left\|S_{\Delta t}^{k-l}\right\|_{L(H)}\\&\qquad\times\left\|\left(B\left(X_{l\Delta t}\right)-B\left(Y_l^N\right)\right)\right\|_{HS(U_0,H)}ds\\&\quad\le C\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}E\left\|e_l^N\right\|_H^2ds.\end{aligned}\tag{84}$$

This implies that

$$\begin{aligned}&E\left\|III_3\right\|_H^2\\&\quad\le E\left\|\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}S_{\Delta t}^{k-l}\left(B\left(X_{l\Delta t}\right)-B\left(Y_l^N\right)\right)dW_s^N\right\|_H^2\\&\quad\le C\sum_{l=0}^{k-1}(\Delta t)E\left\|e_l^N\right\|_H^2.\end{aligned}\tag{85}$$

Finally for III_4, the last term of (65), we should recall

$$\begin{aligned}III_4=P_N\Bigg(&\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}S_{\Delta t}^{k-l}\\&\times\left(B\left(X_s\right)-B\left(X_{l\Delta t}\right)\right)dW_s^N\\&-\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}S_{\Delta t}^{k-l}B'\left(Y_l^N\right)\\&\times\left(\int_{l\Delta t}^{s}B\left(Y_l^N\right)dW_u^N\right)dW_s^N\Bigg).\end{aligned}\tag{86}$$

Using the fact that

$$\begin{aligned}&B\left(X_s\right)-B\left(X_{l\Delta t}\right)\\&\quad=B'\left(X_{l\Delta t}\right)\left(X_s-X_{l\Delta t}\right)\\&\qquad+\int_0^1B\left(X_{l\Delta t}+r\left(X_s-X_{l\Delta t}\right)\right)\left(X_s-X_{l\Delta t},X_s-X_{l\Delta t}\right)\\&\qquad\times(1-r)\,dr\end{aligned}\tag{87}$$

for all $s \in [l\Delta t, (l+1)\Delta t]$, $l \in \{0, 1, 2, \ldots, N^2 - 1\}$, $N \in \mathbb{N}$, and $r \in (0, 1)$ and using the inequality

$$(a_1 + a_2 + \cdots + a_n)^2 \leq n\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \tag{88}$$

for all $a_i \in \mathbb{R}$, $i = 1, \ldots, N$, for III_4, we obtain

$$\begin{aligned} &E\left\|III_4\right\|_H^2 \\ &\quad \leq 2E\left\|\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} S_{\Delta t}^{k-l} B'\left(X_{l\Delta t}\right) \right. \\ &\qquad\qquad \left. \times\left(X_s - X_{\Delta t} - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right) dW_s^N\right\|_H^2 \\ &\quad + 2E\left\|\int_{l\Delta t}^{(l+1)\Delta t} S_{\Delta t}^{k-l}\int_0^1 B''\left(X_{l\Delta t} + r\left(X_s - X_{l\Delta t}\right)\right)\right. \\ &\qquad\qquad \times\left(X_s - X_{l\Delta t}, X_s - X_{l\Delta t}\right) \\ &\qquad\qquad \left. \times (1-r)\, dr\, dW_s^N\right\|_H^2. \end{aligned} \tag{89}$$

To estimate the first term of (89), we first approximate

$$E\left\|X_s - X_{l\Delta t} - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right\|_H^2 \tag{90}$$

for all $l\Delta t, s \in [0, T]$, with $l\Delta t \leq s$ and all $N \in \mathbb{N}$. More precisely, with respect to (88), we have

$$\begin{aligned} &E\left\|X_s - X_{l\Delta t} - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right\|_H^2 \\ &\quad \leq 5E\left\|\left(e^{A(s-l\Delta t)} - I\right) X_{l\Delta t}\right\|_H^2 \\ &\qquad + 5E\left\|\int_{l\Delta t}^{s} e^{A(s-u)} F\left(X_u\right) du\right\|_H^2 \\ &\qquad + 5E\left\|\int_{l\Delta t}^{s} e^{A(s-u)} B\left(X_u\right) d\left(W_u - W_u^N\right)\right\|_H^2 \\ &\qquad + 5E\left\|\int_{l\Delta t}^{s}\left(e^{A(s-u)} - I\right) B\left(X_u\right) dW_u^N\right\|_H \\ &\qquad + 5E\left\|\int_{l\Delta t}^{s}\left(B\left(X_u\right) - B\left(X_s\right)\right) dW_u^N\right\|_H^2. \end{aligned} \tag{91}$$

By using (71), we have

$$\begin{aligned} &E\left\|X_s - X_{l\Delta t} - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right\|_H^2 \\ &\quad \leq 5\left\|(-A)^{-\gamma}\left(e^{A(s-l\Delta t)} - I\right)\right\|_{L(H)}^2 E\left\|(-A)^{\gamma} X_{l\Delta t}\right\|_H^2 \\ &\qquad + 5\left(s - l\Delta t\right)\left(\int_{l\Delta t}^{s} E\left\|e^{A(s-u)} F\left(X_u\right)\right\|_H^2 du\right) \\ &\qquad + C\left(\sup_{j>N+1}\eta_j\right)^{2\alpha} \\ &\qquad + 5\left(\int_{l\Delta t}^{s} E\left\|\left(e^{A(s-u)} - I\right) B\left(X_u\right)\right\|_{HS(U_0,H)}^2 du\right) \\ &\qquad + 5\left(\int_{l\Delta t}^{s} E\left\|B\left(X_u\right) - B\left(X_{l\Delta t}\right)\right\|_{HS(U_0,H)}^2 du\right). \end{aligned} \tag{92}$$

This implies that

$$\begin{aligned} &E\left\|X_s - X_{l\Delta t} - \int_{l\Delta t}^{s} B(X_{l\Delta t}) dW_u^N\right\|_H^2 \\ &\quad \leq 5R(s - l\Delta t)^{2\gamma} + 5\left(s - l\Delta t\right) \\ &\qquad \times\left(\int_{l\Delta t}^{s} E\left\|F\left(X_u\right)\right\|_H^2 du\right) + C\left(\sup_{j>N+1}\eta_j\right)^{2\alpha} \\ &\qquad + 5\left(\int_{l\Delta t}^{s}\left\|(-A)^{-\delta}\left(e^{A(s-u)} - I\right)\right\|_{L(H)}^2 \right. \\ &\qquad\qquad \left. \times E\left\|(-A)^{\delta} B\left(X_u\right)\right\|_{HS(U_0,H)}^2 du\right) \\ &\qquad + 5R^2\left(\int_{l\Delta t}^{s} E\left\|X_u - X_{l\Delta t}\right\|_H^2 du\right) \\ &\quad \leq 5R(s - l\Delta t)^{2\gamma} + 5R(s - l\Delta t)^2 + C\left(\sup_{j>N+1}\eta_j\right)^{2\alpha} \\ &\qquad + 5\left(\int_{l\Delta t}^{s}\left\|(-A)^{-\delta}\left(e^{A(s-u)} - I\right)\right\|_{L(H)}^2 \right. \\ &\qquad\qquad \left. \times E\left\|(-A)^{\delta} B\left(X_u\right)\right\|_{HS(U_0,H)}^2 du\right) \\ &\qquad + 5R^2\left(\int_{l\Delta t}^{s} E\left\|X_u - X_{l\Delta t}\right\|_H^2 du\right) \\ &\quad \leq 5R(s - l\Delta t)^{2\gamma} \\ &\qquad + 5R(s - l\Delta t)^2 + C\left(\sup_{j>N+1}\eta_j\right)^{2\alpha} \\ &\qquad + 5\left(\int_{l\Delta t}^{s}(s-u)^{2\delta} E\left\|(-A)^{\delta} B\left(X_u\right)\right\|_{HS(U_0,H)}^2 du\right) \\ &\qquad + 5R^2\left(\int_{l\Delta t}^{s}\left\|(-A)^{-\beta}\right\|_{L(H)} E\left\|X_u - X_s\right\|_{V_\beta}^2 du\right) \end{aligned} \tag{93}$$

for all $l\Delta t, s \in [0,T]$, with $l\Delta t \le s$, and all $k \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
&E\left\|X_s - X_{l\Delta t} - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right\|_H^2 \\
&\quad \le 5R(s-l\Delta t)^{2\gamma} + 5R(s-l\Delta t)^2 \\
&\qquad + C\left(\sup_{j>N+1} \eta_j\right)^{2\alpha} + 5R\left(\int_{l\Delta t}^{s} (s-u)^{2\delta} du\right) \\
&\qquad + 5R^4\left(\int_{l\Delta t}^{s} E\left\|X_u - X_{l\Delta t}\right\|_{V_\beta}^2 du\right) \\
&\quad \le 10R^3(s-l\Delta t)^{2\gamma} + C\left(\sup_{j>N+1} \eta_j\right)^{2\alpha} \\
&\qquad + 5R(s-l\Delta t)^{1+2\delta} \\
&\qquad + 5R^4\left(\int_{l\Delta t}^{s} E\left\|X_u - X_{l\Delta t}\right\|_{V_\beta}^2 du\right) \\
&\quad \le 15R^3(s-l\Delta t)^{2\gamma} + C\left(\sup_{j>N+1} \eta_j\right)^{2\alpha} \\
&\qquad + 5R^4\left(\int_{l\Delta t}^{s} \left(E\left\|X_u - X_{l\Delta t}\right\|_{V_\beta}^4\right)^{1/2} du\right) \\
&\quad \le 15R^3(s-l\Delta t)^{2\gamma} + C\left(\sup_{j>N+1} \eta_j\right)^{2\alpha} \\
&\qquad + 5R^4\left(\int_{l\Delta t}^{s} R(u-l\Delta t)^{\min(4(\gamma-\beta),2)}\right)^{1/2} du
\end{aligned} \tag{94}$$

and hence

$$\begin{aligned}
&E\left\|X_s - X_{l\Delta t} - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right\|_H^2 \\
&\quad \le 15R^3\left((s-l\Delta t)^{2\gamma} + C\left(\sup_{j>N+1} \eta_j\right)^{2\alpha}\right. \\
&\qquad \left. + 5R^5\left(\int_{l\Delta t}^{s} (u-l\Delta t)^{\min(4(\gamma-\beta),2)}\right) du\right) \\
&\quad \le 15R^3\left((s-l\Delta t)^{2\gamma} + C\left(\sup_{j>N+1} \eta_j\right)^{2\alpha}\right. \\
&\qquad \left. + 5R^5(s-l\Delta t)^{1+\min(4(\gamma-\beta),2)}\right) \\
&\quad \le C\left((s-l\Delta t)^{\min(4(\gamma-\beta),2\gamma)} + \left(\sup_{j>N+1} \eta_j\right)^{2\alpha}\right)
\end{aligned} \tag{95}$$

for all $l\Delta t, s \in [0,T]$, with $l\Delta t \le s$.

Therefore,

$$\begin{aligned}
&E\left\|\sum_{l=0}^{k-1} \int_{l\Delta t}^{l+1\Delta t} S_{\Delta t}^{k-l} B'\left(X_{l\Delta t}\right)\right. \\
&\qquad \left. \times\left(X_s - X_{l\Delta t} - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right) dW_s^N\right\|_H^2 \\
&\quad \le \sum_{l=0}^{k-1} \int_{l\Delta t}^{l+1\Delta t} E \\
&\qquad \times\left\|B'\left(X_{l\Delta t}\right)\right. \\
&\qquad\qquad \times\left(X_s - X_{l\Delta t}\right. \\
&\qquad\qquad\qquad \left.\left. - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right) dW_s^N\right\|_{HS(U_0,H)}^2 ds \\
&\quad \le R^2\left(\sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} E\right. \\
&\qquad \left. \times\left\|X_s - X_{l\Delta t} - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right\|_H^2 ds\right)
\end{aligned} \tag{96}$$

and hence from (95), we get

$$\begin{aligned}
&E\left\|\sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} S_{\Delta t}^{k-l} B'\left(X_{l\Delta t}\right)\right. \\
&\qquad \left. \times\left(X_s - X_{l\Delta t} - \int_{l\Delta t}^{s} B\left(X_{l\Delta t}\right) dW_u^N\right) dW_s^N\right\|_H^2 \\
&\quad \le C\left(\sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} \left((s-l\Delta t)^{\min(4(\gamma-\beta),2\gamma)}\right.\right. \\
&\qquad\qquad \left.\left. + \left(\sup_{j>N+1} \eta_j\right)^{2\alpha}\right) ds\right) \\
&\quad \le C\left(\sum_{l=0}^{k-1} \int_{l\Delta t}^{(l+1)\Delta t} (s-l\Delta t)^{\min(4(\gamma-\beta),2\gamma)} ds\right. \\
&\qquad \left. + \left(\sup_{j>N+1} \eta_j\right)^{2\alpha}\right) \\
&\quad \le C\left(N^2(\Delta t)^{(1+\min(4(\gamma-\beta),2\gamma))} + \left(\sup_{j>N+1} \eta_j\right)^{2\alpha}\right) \\
&\quad \le C\left((\Delta t)^{\min(4(\gamma-\beta),2\gamma)} + \left(\sup_{j>N+1} \eta_j\right)^{2\alpha}\right).
\end{aligned} \tag{97}$$

And for the second term of (89), we have

$$\begin{aligned}
&E\left\|\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} S_{\Delta t}^{k-l}\int_0^1 B''\left(X_{l\Delta t}+r\left(X_s-X_{l\Delta t}\right)\right)\right.\\
&\qquad\times\left(X_s-X_{l\Delta t},X_s-X_{l\Delta t}\right)\\
&\qquad\left.\times(1-r)\,dr\,dW_s^N\right\|_H^2 \qquad (98)\\
&\le\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t}\int_0^1 E\left\|B''\left(X_{l\Delta t}+r\left(X_s-X_{l\Delta t}\right)\right)\right.\\
&\qquad\times\left(X_s-X_{l\Delta t},\right.\\
&\qquad\left.\left.X_s-X_{l\Delta t}\right)\right\|_{HS(U_0,H)}^2 dr\,ds\\
&\le\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} E\left[\left(R\left\|X_s-X_{l\Delta t}\right\|_{V_\beta}^2\right)^2\right]ds\\
&\le R^2\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} E\left\|X_s-X_{l\Delta t}\right\|_{V_\beta}^4 ds \qquad (99)\\
&\le R^2\left(\sum_{l=0}^{k-1}\int_{l\Delta t}^{(l+1)\Delta t} R(S-l\Delta t)^{\min(4(\gamma-\beta),2)}ds\right.\\
&\qquad\left.\le R^3\sum_{l=0}^{k-1}(\Delta t)^{1+\min(4(\gamma-\beta),2)}\right)\\
&\le C(\Delta t)^{\min(4(\gamma-\beta),2)}.
\end{aligned}$$

Therefore, from (71), (83), (85), (89), (97), and (98), we obtain

$$\begin{aligned}
&E\|III\|_H^2\\
&\le C\left(\left(\sup_{j>N+1}\mu_j\right)^{2\alpha}+(\Delta t)^{2\widehat{\gamma}}+\sum_{l=0}^{k-1}(\Delta t)\,E\left\|\widehat{e}_l^N\right\|_H^2\right)\\
&\quad+(\Delta t)^{\min(4(\gamma-\beta),2\gamma)}+\left(\sup_{j>N+1}\mu_j\right)^{2\alpha}\\
&\quad+(\Delta t)^{\min(4(\gamma-\beta),2\gamma)} \qquad (100)\\
&\le C\left(\left(\sup_{j>N+1}\mu_j\right)^{2\alpha}+(\Delta t)^{\min(4(\gamma-\beta),2\gamma)}\right.\\
&\qquad\left.+\sum_{l=0}^{k-1}(\Delta t)\,E\left\|\widehat{e}_l^N\right\|_H^2\right).
\end{aligned}$$

Hence from (50), (63), and (100), we obtain

$$\begin{aligned}
&\left(E\left\|\widehat{e}_k^N\right\|_H^2\right)^{1/2}\\
&\le C\left(\frac{\left(E\left\|X_0\right\|_H^2\right)^{1/2}}{k}+\left(E\left\|\widehat{e}_0^N\right\|_H^2\right)^{1/2}+(\Delta t)^{\gamma}\right.\\
&\quad+(\Delta t)^{\min\{\gamma,1/2\}}+\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|e_k^N\right\|_H^2\right)^{1/2}\\
&\quad+\left(\sup_{j>N+1}\mu_j\right)^{\alpha}\\
&\quad\left.+(\Delta t)^{\min(2(\gamma-\beta),\gamma)}+\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|\widehat{e}_k^N\right\|_H^2\right)^{1/2}\right). \qquad (101)
\end{aligned}$$

Now we take an integer $q\ge 1$ and use the Holder inequality for the two summations in the last estimation to get

$$\begin{aligned}
&\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|\widehat{e}_l^N\right\|_H^2\right)^{1/2}\\
&\le\left(\sum_{l=0}^{k-1}\Delta t\right)^{(2q-1)/2q}\\
&\quad\times\left(\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|\widehat{e}_l^N\right\|_H^2\right)^{2q/2}\right)^{1/2q} \qquad (102)\\
&\le C\left(\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|\widehat{e}_l^N\right\|_H^2\right)^{q}\right)^{1/2q}.
\end{aligned}$$

Therefore, with using (97) and (98), we obtain

$$\begin{aligned}
&\left(E\left\|\widehat{e}_k^N\right\|_H^2\right)^{q}\\
&\le R\left(\left(\frac{\left(E\left\|X_0\right\|_H^2\right)^{1/2}}{k}+\left(E\left\|\widehat{e}_0^N\right\|_H^2\right)^{1/2}\right.\right.\\
&\qquad\left.\left.+(\Delta t)^{\min(2(\gamma-\beta),\gamma)}+\left(\sup_{j>N+1}\mu_j\right)^{\alpha}\right)^{2q}\right. \qquad (103)\\
&\quad+\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|\widehat{e}_k^N\right\|_H^2\right)^{q}\\
&\quad\left.+\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|e_k^N\right\|_H^2\right)^{q}\right).
\end{aligned}$$

We then have

$$\begin{aligned}
&\left(E\left\|\widehat{e}_k^N\right\|_H^2\right)^{q}\\
&\le C\left(\left(\frac{\left(E\left\|X_0\right\|_H^2\right)^{1/2}}{k}+\left(E\left\|\widehat{e}_0^N\right\|_H^2\right)^{1/2}\right.\right.
\end{aligned}$$

$$+ (\Delta t)^{\min(2(\gamma-\beta),\gamma)} + \left(\sup_{j>N+1} \mu_j\right)^{\alpha}\Bigg)^{2q}$$
$$+ (1+\Delta t)\sum_{l=0}^{k-1}(\Delta t)\left(E\left\|\widehat{e}_k^N\right\|_H^2\right)^q\Bigg). \tag{104}$$

Hence, we conclude from (41) that

$$\left(E\left\|\widehat{e}_k^N\right\|_H^2\right)^{1/2} \le C\left(\frac{\left(E\left\|X_0\right\|_H^2\right)^{1/2}}{k} + \left(\sup_{j>N+1}\mu_j\right)^{\alpha} + (\Delta t)^{\min(2(\gamma-\beta),\gamma)}\right). \tag{105}$$

Finally, with respect to (47) and (105), we obtain

$$\left(E\left\|e_k^N\right\|_H^2\right)^{1/2} \le C\left(\left(\inf_{j>N+1}\lambda_j\right)^{-\gamma} + \frac{\left(E\left\|X_0\right\|_H^2\right)^{1/2}}{k} + \left(\sup_{j>N+1}\mu_j\right)^{\alpha} + (\Delta t)^{\min(2(\gamma-\beta),\gamma)}\right) \tag{106}$$

which completes the proof of the theorem. □

5. Simulation Results

In this section we consider SPDE (1) and solve it by numerical scheme (22). More formally, let $k = 1/100$ and $\xi : [0,1] \to \mathbb{R}$ be given by $\xi(x) = 0$ for all $x \in [0,1]$ and suppose that $f, b : (0,1)\times\mathbb{R} \to \mathbb{R}$ are given by $f(x,y) = 1-y$ and $b(x,y) = (1-y)/(1+y^2)$ for all $x \in (0,1)$, $y \in \mathbb{R}$. The SPDE (1) reduces to

$$dX_t(x) = \left[\frac{1}{100}\frac{\partial^2}{\partial x^2}X_t(x) + 1 - X_t(x)\right]dt + \frac{1-X_t(x)}{1+X_t^2(x)}dW_t(x) \tag{107}$$

with $X_t(0) = X_t(1) = 0$ and $X_0 = 0$ for $x \in (0,1)$ and $t \in [0,1]$. We also assume that the SPDE (107) should be solved with a precision of, say, two decimals, that is, with the precision $\varepsilon = 0.01$ in (3). To confirm numerically our theoretical founding in Theorem 7, we recall that for SPDE (107) there should exist some real number $C_r \in (0,\infty)$ such that

$$\left(E\left\|X_T - Y_{N^2}^N\right\|_H^2\right)^{1/2} \leqslant C_r N^{r-3/2} \tag{108}$$

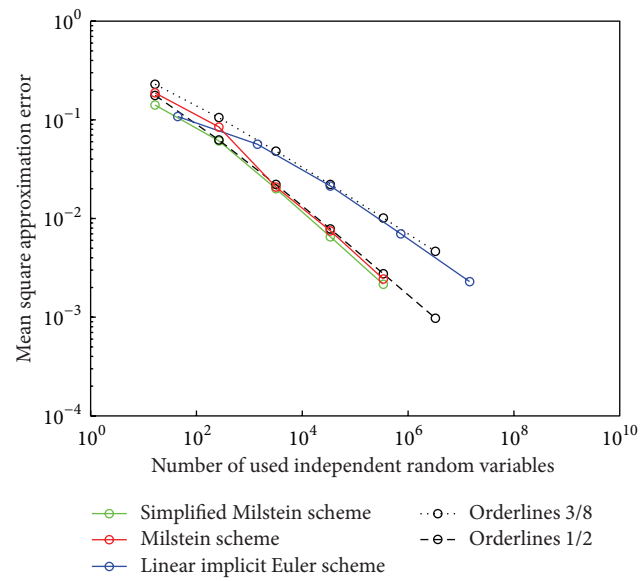

Figure 1: Approximation error in the sense of (22) of the linear implicit Euler and simplified Milstein and Milstein type schemes against the precise number of independent standard normal random variables needed to compute the corresponding approximation for $N \in \{2,4,8,16,32,64\}$.

holds for each small $r \in (0,3/4)$. The overall convergence order of the linear implicit Euler method (4) is $3/8-$ (see [16]), while the overall convergence of the simplified Milstein scheme (22) and Milstein scheme (7) is $1/2$. In Figure 1 the approximation error in the sense of (6) of the linear implicit Euler approximation $\overline{Y}_{N^3}^N$, obtained by (4), of the approximation $\widetilde{Y}_{N^2}^N$, obtained by Milstein scheme (7), and of the approximation $Y_{N^2}^N$, obtained by simplified Milstein scheme (22), is plotted against the precise number of independent standard normal random variables that is needed to compute the corresponding approximation for $N \in \{2,4,8,\ldots,128\}$ on a log-log scale. Figure 1 confirms the order of convergence of our scheme and compares with the other two schemes. Besides, the simplified Milstein scheme (22) and the Milstein scheme (7) produce nearly the same approximation errors. Numerical results also show that the linear simplified Milstein scheme (21) and the Milstein type scheme (7) are much computationally effective than the linear implicit Euler scheme (4). To simulate one path $\overline{Y}_{32^3}^{32}$, one needs to generate $32^4 = 1048576$ independent normal random variables, but this amount for simulation of $\widetilde{Y}_{32^2}^{32}$ and $Y_{32^2}^{32}$ reduces to $32^3 = 32768$. From the numerical results reported in Table 1 and Figure 1, we conclude that the simplified Milstein scheme is more effective than implicit Euler method and slightly better than Milstein scheme.

6. Conclusions

A simplified Milstein scheme for solving stochastic partial differential equations of the form (1) with multiplicative trace class noise was theoretically and numerically investigated.

This scheme has advantages to some other methods such as linear implicit Euler and Milstein schemes. We have shown the L^2 convergence of this method under the stated conditions and then we have illustrated the effectiveness of the simplified Milstein scheme numerically.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On Properties of Pseudointegrals Based on Pseudoaddition Decomposable Measures

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We mainly discussed pseudointegrals based on a pseudoaddition decomposable measure. Particularly, we give the definition of the pseudointegral for a measurable function based on a strict pseudoaddition decomposable measure by generalizing the definition of the pseudointegral of a bounded measurable function. Furthermore, we got several important properties of the pseudointegral of a measurable function based on a strict pseudoaddition decomposable measure.

1. Introduction

The classical measure theory is one of the most important theories in mathematics [1, 2]. Although the additive measures are widely used, they do not allow modelling many phenomena involving interaction between criteria. For this reason, the fuzzy measure proposed by Sugeno is an extension of classical measure in which the additivity is replaced by a weaker condition, that is, monotonicity [3, 4]. Therefore, fuzzy measure and the corresponding integrals, for example, Choquet and Sugeno, are introduced [5–10].

So far, there have been many different fuzzy measures, such as the decomposable measure, the λ-additive measure, the belief measure, the possibility measure, and the plausibility measure. Among the fuzzy measures mentioned before, the decomposable measure was independently introduced by Dubois and Prade [11] and Weber [12]. Since the close relations with the classical measure theory, further developments of decomposable measures and related integrals have been extensive [13–18]. Decomposable measures include several well-known fuzzy measures such as the λ-additive measure and probability and possibility measures, and they provide a natural setting for relaxing probabilistic assumptions regarding the modeling of uncertainty [19, 20]. Decomposable measures and the corresponding integrals are very useful in decision theory and the theory of nonlinear differential and integral equations [21–24].

In many problems with uncertainty as in the theory of probabilistic metric spaces [20, 25, 26], multivalued logics [27, 28], and general measures [1, 4] often we work with many operations different from the usual addition and multiplication of reals. Some of them are triangular norms, triangular conorms, pseudoadditions, pseudomultiplications, and so forth [21, 29]. Based on the above-mentioned measures, pseudoanalysis as a generalization of the classical analysis is developed, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, +\infty]$ endowed with pseudoaddition $\oplus$ and with pseudomultiplication $\odot$ (see [13, 19, 30–33]). The families of the pseudooperations generated by a function g turn out to be solutions of well-known nonlinear functional equations [22–24].

In this paper, we will discuss pseudointegrals based on pseudoaddition decomposable measures. In Section 2, we recall the concepts of the pseudoaddition $\oplus$ and the pseudomultiplication $\odot$, which form a real semiring on the interval $[a, b] \subset [-\infty, +\infty]$ and the notion of the σ-$\oplus$-decomposable measure. Then we will give the definition of the pseudointegral of a measurable function based on a strict pseudoaddition decomposable measure by generalizing the definition of the pseudointegral of a bounded measurable function. In Section 3, we will discuss several important properties of the pseudointegral of a measurable function based on the strict pseudoaddition decomposable measure.

2. Preliminaries

Let $[a,b]$ be a closed subinterval of $\mathbb{R}$ (in some cases we will also take semiclosed subintervals). The total order on $[a,b]$ will be denoted by $\preceq$. This can be the usual order of the real line, but it can also be another order. We will denote by Δ maximum element on $[a,b]$ (usually Δ is either a or b) with respect to this total order.

Definition 1 (see [34]). Let $\{x_n\}$ be a sequence from $[a,b]$.

(1) If $x_m \preceq x_n$ whenever $n > m$, then we say that the sequence $\{x_n\}$ is an increasing sequence.

(2) If $x_m \prec x_n$ whenever $n > m$, then we say that the sequence $\{x_n\}$ is a strict increasing sequence.

(3) If $x_n \preceq x_m$ whenever $n > m$, then we say that the sequence $\{x_n\}$ is a decreasing sequence.

(4) If $x_n \prec x_m$ whenever $n > m$, then we say that the sequence $\{x_n\}$ is a strict decreasing sequence.

Let X be a nonempty set; we will denote by $\mathcal{S}$, $\mathcal{A}$, and $\mathcal{B}_X$ algebra, σ-algebra, and Borel σ-algebra of subsets of a set X, respectively.

Denote by $\mathcal{F}(X)$ the set of all functionals from X to $[a,b]$. For each $\lambda \in [a,b]$ the constant functional in $\mathcal{F}(X)$ with value λ will also be denoted by λ. It will be clear from the context which usage is intended. A functional $f \in \mathcal{F}(X)$ is said to be finite if $f(x) \prec \Delta$ for all $x \in X$. The functional $f \in \mathcal{F}(X)$ is said to be bounded if there exists $\Omega \prec \Delta$, such that $f(x) \preceq \Omega$ for all $x \in X$. Denote by $\mathcal{B}(X)$ the set of all bounded functionals.

Let f and h be two functions defined on X and with values in $[a,b]$ and let $\star$ be arbitrary binary operation on $[a,b]$. Then, we define for any $x \in X$

$$(f \star h)(x) = f(x) \star h(x), \tag{1}$$

and for any $\lambda \in [a,b]$, $(\lambda \star f)(x) = \lambda \star f(x)$. Let $\mathcal{A}$ be a subset of $\mathcal{F}(X)$. If $f \star h \in \mathcal{A}$ for all $f, h \in \mathcal{A}$, then $\mathcal{A}$ is $\star$-closed. The total order $\preceq$ on $[a,b]$ induces a partial order $\preceq$ on $\mathcal{F}(X)$ defined pointwise by stipulating that $f \preceq h$ if and only if $f(x) \preceq h(x)$ for all $x \in X$. Thus $(\mathcal{F}(X), \preceq)$ is a poset, and whenever we consider $\mathcal{F}(X)$ as a poset then it will always be with respect to this partial order. Let $\mathcal{S}[\lambda \prec f] = \{x \mid x \in X, \lambda \prec f(x), f \in \mathcal{F}(X)\}$.

Definition 2 (see [35]). A binary operation $\oplus : [a,b] \times [a,b] \to [a,b]$ is called a pseudoaddition, if it satisfies the following conditions, for all $x, y, z, w \in [a,b]$:

(1) $\mathbf{0} \oplus x = x$, where $\mathbf{0}$ is a zero element (usually $\mathbf{0}$ is either a or b) (boundary condition);

(2) $x \oplus z \preceq y \oplus w$ whenever $x \preceq y$ and $z \preceq w$ (monotonicity);

(3) $x \oplus y = y \oplus x$ (commutativity);

(4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ (associativity).

A pseudoaddition $\oplus$ is said to be continuous if it is a continuous function in $[a,b]^2$; a pseudoaddition $\oplus$ is called strict if $\oplus$ is continuous and strictly monotone. The following are examples of pseudoadditions: $x \vee_{\oplus} y = y$ if and only if $x \preceq y$; $x \oplus y = g^{-1}(g(x) + g(y))$, where $g : [a,b] \to [0,1]$ is a strictly monotone and continuous generator surjective function and $x \preceq y$ if and only if $g(x) \leq g(y)$. It is obvious that $\Delta \oplus x = \Delta$ for all $x \in [a,b]$.

Let $[a,b]_+ = \{x \mid x \in [a,b], \mathbf{0} \preceq x\}$. In this paper, we assume $[a,b] = [a,b]_+$.

Definition 3 (see [35]). A binary operation $\odot : [a,b] \times [a,b] \to [a,b]$ is called a pseudomultiplication, if it satisfies the following conditions, for all $x, y, z, w \in [a,b]$:

(1) $\mathbf{1} \odot x = x$, where $\mathbf{1} \in [a,b]$ is a unit element (boundary condition);

(2) $x \odot z \preceq y \odot w$ whenever $x \preceq y$ and $z \preceq w$ (monotonicity);

(3) $x \odot y = y \odot x$ (commutativity);

(4) $(x \odot y) \odot z = x \odot (y \odot z)$ (associativity).

A pseudomultiplication $\odot$ is said to be continuous if it is a continuous function in $[a,b]^2$. The following are examples of pseudomultiplications: $x \wedge_{\odot} y = x$ if and only if $x \preceq y$; $x \odot_g y = g^{-1}(g(x) \cdot g(y))$, where $g : [a,b] \to [0,1]$ is a strictly monotone and continuous generator surjective function and $x \preceq y$ if and only if $g(x) \leq g(y)$. It is obvious that $g(\mathbf{0}) = 0$.

We assume also that $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudomultiplication with respect to $\oplus$; that is,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z). \tag{2}$$

The structure $([a,b], \oplus, \odot)$ is called a real semiring.

Because of the associative property of the pseudoaddition $\oplus$, it can be extended by induction to n-ary operation by setting

$$\bigoplus_{i=1}^{n} x_i = \left(\bigoplus_{i=1}^{n-1} x_i\right) \oplus x_n. \tag{3}$$

Due to monotonicity, for each sequence $\{x_i\}_{i \in \mathbb{N}}$ of elements of $[a,b]$, the following limit can be considered:

$$\bigoplus_{i=1}^{\infty} x_i = \lim_{n \to \infty} \bigoplus_{i=1}^{n} x_i. \tag{4}$$

Definition 4 (see [36]). Let A be a nonempty set and $\oplus$ a pseudoaddition. A binary operation $d_{\oplus} : A \times A \to [a,b]$ is called a pseudometric on A, if it satisfies the following conditions, for all $x, y, z \in A$:

(1) $d_{\oplus}(x,y) = \mathbf{0}$ if and only if $x = y$;

(2) $d_{\oplus}(x,y) = d_{\oplus}(y,x)$;

(3) there exists $\lambda \in [a,b]$ such that

$$d_{\oplus}(x,y) \preceq \lambda \odot (d_{\oplus}(x,z) \oplus d_{\oplus}(z,y)), \tag{5}$$

where $\odot$ is a distributive pseudomultiplication with respect to $\oplus$.

Let $\{x_n\}_{n\geq 1}$ be a sequence from $[a,b]$. The sequence $\{x_n\}_{n\geq 1}$ is said to be convergent, if for any $\mathbf{0} \prec \varepsilon$, there exists positive integer $N(\varepsilon)$, such that $d_\oplus(x_n, x) \prec \varepsilon$ for all $n \geq N(\varepsilon)$, denoted by $x = \lim_{n\to\infty} x_n$, and x is said to be the limit of the sequence $\{x_n\}_{n\geq 1}$;

$$\underline{\lim}_{n\to\infty} x_n = \bigvee_{n=1}^{\infty}{}_{\oplus} \left(\bigwedge_{k\geq n}{}_{\odot} x_k \right) \tag{6}$$

is said to be the lower limit of the sequence $\{x_n\}_{n\geq 1}$;

$$\overline{\lim}_{n\to\infty} x_n = \bigwedge_{n=1}^{\infty}{}_{\odot} \left(\bigvee_{k\geq n}{}_{\oplus} x_k \right) \tag{7}$$

is said to be the upper limit of the sequence $\{x_n\}_{n\geq 1}$. It is obvious that $\underline{\lim}_{n\to\infty} x_n \preceq \overline{\lim}_{n\to\infty} x_n$. Let $\{f_n\}_{n\geq 1}$ be a sequence from $\mathscr{F}(X)$. The sequence $\{f_n\}_{n\geq 1}$ is said to be convergent, if for any $\mathbf{0} \prec \varepsilon$, and for each point $x_0 \in X$, there exists positive integer $N(\varepsilon, x_0)$, such that $d_\oplus(f_n(x_0), f(x_0)) \prec \varepsilon$ for all $n \geq N(\varepsilon, x_0)$, denoted by $f = \lim_{n\to\infty} f_n$, and f is said to be the limit functional of the functionals sequence $\{f_n\}_{n\geq 1}$.

Let $\mathscr{A}$ be a subset of $\mathscr{F}(X)$. The poset $\mathscr{A}$ is said to be upper complete if $\lim_{n\to\infty} f_n \in \mathscr{A}$ for each increasing sequence $\{f_n\}_{n\geq 1}$ from $\mathscr{A}$; the poset $\mathscr{A}$ is said to be lower complete if $\lim_{n\to\infty} f_n \in \mathscr{A}$ for each decreasing sequence $\{f_n\}_{n\geq 1}$ from $\mathscr{A}$; the poset $\mathscr{A}$ is said to be complete if $\lim_{n\to\infty} f_n \in \mathscr{A}$ for each sequence $\{f_n\}_{n\geq 1}$ from $\mathscr{A}$, where the limit of the sequence of functionals $\{f_n\}_{n\geq 1}$ is given by $(\lim_{n\to\infty} f_n)(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$.

For any continuous pseudoaddition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, there exists at least one point $z \in [a,b]$ such that $y = x \oplus z$. If pseudoaddition $\oplus$ is strict, then there exists only one point $z \in [a,b]$ such that $y = x \oplus z$ for all $x, y \in [a,b]$ with $x \prec \Delta$. Thus we have the following concepts.

Definition 5 (see [34]). For any continuous pseudoaddition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, the paracomplement set $y -_\oplus x$ is a nonempty set of all points z such that $y = x \oplus z$.

Example 6. Let the total order $\preceq$ on $[0,1]$ be the usual order of the real line and let the pseudoaddition $\oplus$ be the usual multiplication of the real numbers. It is obvious that zero element is 1. If $x = 0$, then $y = 0$ and $y -_\oplus x = [0,1]$. If $x \neq 0$, then for any $0 \leq y < x$, we have $y -_\oplus x = \{y/x\} \subseteq [0,1]$.

Definition 7 (see [34]). For any continuous pseudoaddition $\oplus$, if $f, h \in \mathscr{F}(X)$, then define the paracomplement set $|f -_\oplus h|$ as the set of all those functionals φ such that

$$\varphi(x) = \begin{cases} f(x) -_\oplus h(x), & \text{if } h(x) \preceq f(x), \\ h(x) -_\oplus f(x), & \text{if } f(x) \prec h(x), \end{cases} \tag{8}$$

for all $x \in X$.

Definition 8 (see [34]). For any strict pseudoaddition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, the complement $y -'_\oplus x$ is defined as

$$y -'_\oplus x = \begin{cases} z \in [a,b], & \text{such that } y = x \oplus z, \text{ if } x \prec \Delta, \\ \mathbf{0}, & \text{otherwise.} \end{cases} \tag{9}$$

Definition 9 (see [34]). For any strict pseudoaddition $\oplus$, if $f, h \in \mathscr{F}(X)$, then define the complement functional $|f -'_\oplus h|$ pointwise as

$$\left| f -'_\oplus h \right| (x) = \begin{cases} f(x) -'_\oplus h(x), & \text{if } h(x) \preceq f(x), \\ h(x) -'_\oplus f(x), & \text{if } f(x) \prec h(x), \end{cases} \tag{10}$$

for all $x \in X$.

Definition 10 (see [34]). For any pseudoaddition $\oplus$, a nonempty subset $\mathscr{K}$ of $\mathscr{F}(X)$ is said to be a functional space with respect to $\oplus$, denoted by $(\mathscr{K}, \oplus)$, if $(\lambda \odot f) \oplus (\mu \odot h) \in \mathscr{K}$ for all $f, h \in \mathscr{K}$ and $\lambda, \mu \in [a,b]$, where $\odot$ is a distributive pseudomultiplication with respect to $\oplus$.

It is clear that $(\mathscr{F}(X), \oplus)$ is the greatest functional space with respect to any pseudoaddition $\oplus$. Thus the functional space $(\mathscr{K}, \oplus)$ with $\mathscr{K} \subseteq \mathscr{F}(X)$ is also called a subspace of $(\mathscr{F}(X), \oplus)$. If $(\mathscr{K}, \oplus)$ is a functional space with respect to $\oplus$, then we just write $\mathscr{K}$ instead of $(\mathscr{K}, \oplus)$ whenever $\oplus$ can be determined from the context.

Definition 11 (see [34]). For each subset $\mathscr{A}$ of $\mathscr{F}(X)$ the upper closure of $\mathscr{A}$, denoted by $\widehat{\mathscr{A}}$, is the set of all elements of $\mathscr{F}(X)$ having the form $\lim_{n\to\infty} f_n$ for some increasing sequence $\{f_n\}_{n\geq 1}$ from $\mathscr{A}$.

It follows from Definition 11 that $\mathscr{A} \subseteq \widehat{\mathscr{A}}$ and $\mathscr{A} = \widehat{\mathscr{A}}$ if and only if $\mathscr{A}$ is upper complete.

Definition 12 (see [34]). For any continuous pseudoaddition $\oplus$, a subspace $(\mathscr{K}, \oplus)$ will be called paracomplemented if $|f -_\oplus h| \subseteq \mathscr{K}$ for all $f, h \in \mathscr{K}$; for any strict pseudoaddition $\oplus$, a subspace $(\mathscr{K}, \oplus)$ will be called complemented if $|f -'_\oplus h| \in \mathscr{K}$ for all $f, h \in \mathscr{K}$.

Definition 13 (see [34]). For any continuous pseudoaddition $\oplus$, a paracomplemented subspace $(\mathscr{K}, \oplus)$ is regular if it contains $\mathbf{1}$ and is closed under $\vee_\oplus$; for any strict pseudoaddition $\oplus$, a complemented subspace $(\mathscr{K}, \oplus)$ is normal if it contains $\mathbf{1}$ and is closed under $\vee_\oplus$.

Note that $(f \vee_\oplus h) \oplus (f \wedge_\odot h) = f \oplus h$ for all $f, h \in \mathscr{F}(X)$ and thus a paracomplemented subspace of $\mathscr{F}(X)$ is $\wedge_\odot$-closed if and only if it is $\vee_\oplus$-closed. It is obvious that regular and normal are closed under $\wedge_\odot$.

Definition 14 (see [37]). The pseudocharacteristic function of a set $E \subseteq X$ is defined with

$$I_E(x) = \begin{cases} \mathbf{0}, & x \notin E, \\ \mathbf{1}, & x \in E, \end{cases} \tag{11}$$

where $\mathbf{0}$ is zero element for $\oplus$ and $\mathbf{1}$ is unit element for $\odot$.

Definition 15 (see [21]). A functional $\varphi \in \mathscr{F}(X)$ is said to be elementary if it has the following representation:

$$\varphi = \bigoplus_{i=1}^{n} \lambda_i \odot I_{E_i}, \tag{12}$$

for each $\lambda_i \in [a,b]$ and $E_i \in \mathscr{A}$ pairwise disjoint and with $X = \bigcup_{i=1}^{n} E_i$, and the set of such elementary functionals will be denoted by $\mathscr{E}(X)$. It is obvious that $I_E \in \mathscr{E}(X)$, for all $E \subseteq X$.

Definition 16 (see [21]). A set function $m : \mathscr{A} \to [a,b]$ (or semiclosed interval) is called a σ-$\oplus$-decomposable measure if it satisfies the following conditions:

(1) $m(\emptyset) = \mathbf{0}$;

(2) $m(E) \preceq m(F)$ for all $E, F \in \mathscr{A}$ with $E \subset F$;

(3) $m(E \cup F) = m(E) \oplus m(F)$ for all $E, F \in \mathscr{A}$ and $E \cap F = \emptyset$;

(4) $m(\bigcup_{i=1}^{\infty} E_i) = \oplus_{i=1}^{\infty} m(E_i)$ for any sequence $\{E_i\}_{i\geq 1}$ of pairwise disjoint sets from $\mathscr{A}$.

A pair $(X, \mathscr{A})$ consisting of a nonempty set X and a σ-algebra of subsets of X is called a measurable space. A functional $f : X \to [a,b]$ is said to be a measurable functional if $f^{-1}(\mathscr{B}_{[a,b]}) \subseteq \mathscr{A}$. Let $\mathscr{M}(\mathscr{A})$ be the set of all measurable mappings from $(X, \mathscr{A})$ to $([a,b], \mathscr{B}_{[a,b]})$; that is,

$$\mathscr{M}(\mathscr{A}) = \left\{ f \in \mathscr{F}(X) \mid f^{-1}(\mathscr{B}_{[a,b]}) \subseteq \mathscr{A} \right\}. \tag{13}$$

Then $\mathscr{E}(\mathscr{S})$ will denote the set of those elements $f \in \mathscr{E}(X)$ for which $f^{-1}(\lambda) = \{x \in X \mid f(x) = \lambda\} \in \mathscr{S}$ for each $\lambda \in f(X)$. In particular, this means that $\mathscr{E}(\mathscr{A}) = \mathscr{M}(\mathscr{A}) \cap \mathscr{E}(X)$. Denote by $\mathscr{B}(\mathscr{A})$ the set of all bounded measurable functionals.

Definition 17 (see [38]). Let $\oplus$ be a continuous pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functionals of a.e. pseudofinite on X. If there exists a measurable functional f of a.e. pseudofinite on X, such that

$$\lim_{n\to\infty} m\mathscr{S}\left[\sigma \preceq d_{\oplus}(f_n, f)\right] = \mathbf{0}, \tag{14}$$

for arbitrary $\mathbf{0} \prec \sigma \prec \Delta$, then the functionals sequence $\{f_n\}_{n\geq 1}$ is said to be convergent to f with respect to $\oplus$-measure, denoted by $f_n \Rightarrow f$. If the functionals sequence $\{f_n\}_{n\geq 1}$ does not converge to f with respect to $\oplus$-measure, denote by $f_n \nRightarrow f$.

Definition 18 (see [35]). Let $\oplus$ be a continuous pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure.

(i) If $m(X) \prec \Delta$, then the pseudointegral of an elementary measurable function $\varphi : X \to [a,b]$ is defined by

$$\int_X^{\oplus} \varphi \odot dm = \bigoplus_{i=1}^{n} \lambda_i \odot m(E_i), \tag{15}$$

for $\lambda_i \in [a,b]$ and $E_i \in \mathscr{A}$ pairwise disjoint and with $X = \bigcup_{i=1}^{n} E_i$.

(ii) If $m(X) \prec \Delta$ and $\{\varphi_n\}$ is the sequence of elementary measurable functions such that, for each $x \in X$,

$$d_{\oplus}(\varphi_n(x), f(x)) \longrightarrow \mathbf{0} \quad \text{uniformly as } n \longrightarrow \infty, \tag{16}$$

where a sequence of elementary functions $\{\varphi_n\}$ from the previous definition is constructed in [34], then the pseudointegral of a bounded measurable function $f : X \to [a,b]$ is defined by

$$\int_X^{\oplus} f \odot dm = \lim_{n\to\infty} \int_X^{\oplus} \varphi_n \odot dm. \tag{17}$$

If there exists an increasing sequence of sets $\{E_n\} \subset \mathscr{A}$ with $m(E_n) \prec \Delta$, $n = 1, 2, \ldots$, such that $X = \bigcup_{n=1}^{\infty} E_n$, then we say that X is σ-finite set of $\oplus$-measure and $\{E_n\}$ is a $\oplus$-measure finite and monotone cover of X. The sequence of bounded measurable functionals $[f]_n$ is given by

$$[f]_n(x) = \begin{cases} f(x), & \text{if } f(x) \preceq \mu_n, \\ \mu_n, & \text{if } \mu_n \prec f(x), \end{cases} \tag{18}$$

$\mathbf{0} \prec \mu_1 \prec \mu_2 \prec \cdots \prec \mu_n \prec \cdots$, $\mu_n \oplus \mu_n = \mu_{2n}$ and $\lim_{n\to\infty} \mu_n = \Delta$. It is obvious that $\{[f]_n\}$ is an increasing functionals sequence.

Definition 19. Let $\oplus$ be a strict pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If X is σ-finite of $\oplus$-measure and $\{E_n\}$ is a $\oplus$-measure finite and monotone cover of X, then the pseudointegral of a measurable function $f : X \to [a,b]$ is defined by

$$\int_X^{\oplus} f \odot dm = \lim_{n\to\infty} \int_{E_n}^{\oplus} [f]_n \odot dm. \tag{19}$$

3. Main Results

Lemma 20 (see [21]). *Let $\oplus$ be a continuous pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $m(X) \prec \Delta$, then for all $f, h \in \mathscr{B}(\mathscr{A})$, we have*

(1) $\int_X^{\oplus} (f \vee_{\oplus} h) \odot dm = \int_X^{\oplus} f \odot dm \vee_{\oplus} \int_X^{\oplus} h \odot dm$;

(2) $\int_X^{\oplus} (f \wedge_{\odot} h) \odot dm = \int_X^{\oplus} f \odot dm \wedge_{\odot} \int_X^{\oplus} h \odot dm$;

(3) *If $f \oplus h \in \mathscr{B}(\mathscr{A})$, then*

$$\int_X^{\oplus} (f \oplus h) \odot dm = \int_X^{\oplus} f \odot dm \oplus \int_X^{\oplus} h \odot dm; \tag{20}$$

(4) $f \preceq h \Rightarrow \int_X^{\oplus} f \odot dm \preceq \int_X^{\oplus} h \odot dm$;

(5) $\int_{X_1 \cup X_2}^{\oplus} f \odot dm = \int_{X_1}^{\oplus} f \odot dm \oplus \int_{X_2}^{\oplus} f \odot dm$, *where $X_1, X_2 \in \mathscr{A}$ with $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$;*

(6) $\int_E^{\oplus} f \odot dm = \mathbf{0}$ *whenever $E \in \mathscr{A}$ with $m(E) = \mathbf{0}$.*

Theorem 21. *Let $\oplus$ be a strict pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If X is σ-finite of $\oplus$-measure and $f \in \mathscr{M}(\mathscr{A})$. Let $\{E_n^{(i)}\}$ $(i = 1, 2)$ be two different $\oplus$-measure finite and monotone covers of X and let $\{k_n^{(j)}\}$ $(j = 1, 2)$ be two different positive integer sequences with $\lim_{n\to\infty} k_n^{(j)} = +\infty$. Then*

$$\lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm = \lim_{n\to\infty} \int_{E_n^{(2)}}^{\oplus} [f]_{k_n^{(2)}} \odot dm. \tag{21}$$

Proof. Let $s = \lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm$. Since $\{\int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm\}$ is an increasing sequence, we have

$$\int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm \preceq s, \tag{22}$$

for every positive integer n. Let $F \in \mathscr{A}$ with $m(F) \prec \Delta$ and k is an arbitrary positive integer. If $k_n^{(1)} > k$, then we have

$$\begin{aligned}
&\int_F^{\oplus} [f]_k \odot dm \\
&\quad = \int_{F\cap E_n^{(1)}}^{\oplus} [f]_k \odot dm \oplus \int_{F-E_n^{(1)}}^{\oplus} [f]_k \odot dm \\
&\quad \preceq \int_{F\cap E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm \oplus \left(\mu_k \odot m\left(F - E_n^{(1)}\right)\right) \\
&\quad \preceq \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm \oplus \left(\mu_k \odot m\left(F - E_n^{(1)}\right)\right) \\
&\quad \preceq s \oplus \left(\mu_k \odot m\left(F - E_n^{(1)}\right)\right).
\end{aligned} \tag{23}$$

Since $\{F - E_n^{(1)}\}$ is a decreasing sequence and

$$\bigcap_{n=1}^{\infty}\left(F - E_n^{(1)}\right) = F - \bigcup_{n=1}^{\infty} E_n^{(1)} = F - X = \emptyset, \tag{24}$$

by Theorem 3.3 in [38], we have

$$\lim_{n\to\infty} m\left(F - E_n^{(1)}\right) = m\left(\lim_{n\to\infty}\left(F - E_n^{(1)}\right)\right) = \mathbf{0}, \tag{25}$$

which implies that

$$\begin{aligned}
\int_F^{\oplus} [f]_k \odot dm &\preceq s \oplus \left(\mu_k \odot \lim_{n\to\infty} m\left(F - E_n^{(1)}\right)\right) \\
&= s = \lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm.
\end{aligned} \tag{26}$$

In particular, let $F = E_l^{(2)}$ and $k = k_l^{(2)}$. Then we have

$$\int_{E_l^{(2)}}^{\oplus} [f]_{k_l^{(2)}} \odot dm \preceq \lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm, \tag{27}$$

for every positive integer l. Hence, we get that

$$\lim_{l\to\infty} \int_{E_l^{(2)}}^{\oplus} [f]_{k_l^{(2)}} \odot dm \preceq \lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm. \tag{28}$$

On the contrary, using a similar argument, we can obtain

$$\lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm \preceq \lim_{l\to\infty} \int_{E_l^{(2)}}^{\oplus} [f]_{k_l^{(2)}} \odot dm. \tag{29}$$

□

In Theorem 21, put $k_n^{(1)} = n$ and $k_l^{(2)} = l$. Then we can easily see that the pseudointegral in Definition 19 has a unique value. In particular, we can get some elementary properties of the pseudointegral in the following theorem.

Theorem 22. *Let $\oplus$ be a strict pseudoaddition and $m : \mathscr{A} \to [a, b]$ a σ-$\oplus$-decomposable measure. If there exists an increasing sequence of sets $\{E_n\} \subset \mathscr{A}$ with $m(E_n) \prec \Delta$, $n = 1, 2, \ldots$, such that $X = \bigcup_{n=1}^{\infty} E_n$, then for all $f, h \in \mathscr{M}(\mathscr{A})$, we have*

(1) $\int_X^{\oplus}(f\vee_{\oplus}h) \odot dm = \int_X^{\oplus} f \odot dm \vee_{\oplus} \int_X^{\oplus} h \odot dm$;

(2) $\int_X^{\oplus}(f\wedge_{\odot}h) \odot dm = \int_X^{\oplus} f \odot dm \wedge_{\odot} \int_X^{\oplus} h \odot dm$;

(3) $\int_X^{\oplus}(f \oplus h) \odot dm = \int_X^{\oplus} f \odot dm \oplus \int_X^{\oplus} h \odot dm$;

(4) $f \preceq h \Rightarrow \int_X^{\oplus} f \odot dm \preceq \int_X^{\oplus} h \odot dm$;

(5) $\int_{X_1\cup X_2}^{\oplus} f \odot dm = \int_{X_1}^{\oplus} f \odot dm \oplus \int_{X_2}^{\oplus} f \odot dm$, *where* $X_1, X_2 \in \mathscr{A}$ *with* $X_1 \cup X_2 = X$ *and* $X_1 \cap X_2 = \emptyset$;

(6) $\int_E^{\oplus} f \odot dm = \mathbf{0}$ *whenever* $E \in \mathscr{A}$ *with* $m(E) = \mathbf{0}$.

Proof. For (1) and (2), we only prove (1) holds. By a similar proof, we can prove (2) holds. Since

$$\begin{aligned}
[f]_n(x) &= \begin{cases} f(x), & \text{if } f(x) \preceq \mu_n, \\ \mu_n, & \text{if } \mu_n \prec f(x), \end{cases} \\
[h]_n(x) &= \begin{cases} h(x), & \text{if } h(x) \preceq \mu_n, \\ \mu_n, & \text{if } \mu_n \prec h(x), \end{cases}
\end{aligned} \tag{30}$$

$n = 1, 2, \ldots$, we get that

$$([f]_n \vee_{\oplus} [h]_n)(x) = \begin{cases} (f\vee_{\oplus}h)(x), & \text{if } (f\vee_{\oplus}h)(x) \preceq \mu_n, \\ \mu_n, & \text{if } \mu_n \prec (f\vee_{\oplus}h)(x), \end{cases} \tag{31}$$

which implies that

$$[f\vee_{\oplus}h]_n = ([f]_n \vee_{\oplus} [h]_n). \tag{32}$$

Thus, by (1) of Lemma 20, we have

$$\begin{aligned}
&\int_X^{\oplus} (f\vee_{\oplus}h) \odot dm \\
&\quad = \lim_{n\to\infty} \int_{E_n}^{\oplus} [f\vee_{\oplus}h]_n \odot dm \\
&\quad = \lim_{n\to\infty} \int_{E_n}^{\oplus} ([f]_n \vee_{\oplus} [h]_n) \odot dm \\
&\quad = \lim_{n\to\infty} \int_{E_n}^{\oplus} [f]_n \odot dm \vee_{\oplus} \lim_{n\to\infty} \int_{E_n}^{\oplus} [h]_n \odot dm \\
&\quad = \int_X^{\oplus} f \odot dm \vee_{\oplus} \int_X^{\oplus} h \odot dm.
\end{aligned} \tag{33}$$

(3) Since

$$[f \oplus h]_n(x) = \begin{cases} (f \oplus h)(x), & \text{if } (f \oplus h)(x) \preceq \mu_n, \\ \mu_n, & \text{if } \mu_n \prec (f \oplus h)(x), \end{cases}$$

$$([f]_n \oplus [h]_n)(x) = \begin{cases} (f \oplus h)(x), & \text{if } (f \vee_{\oplus} h)(x) \preceq \mu_n, \\ \mu_n \oplus (f \wedge_{\odot} h)(x), & \text{if } (f \wedge_{\odot} h)(x) \preceq \mu_n \prec (f \vee_{\oplus} h)(x), \\ \mu_n \oplus \mu_n = \mu_{2n}, & \text{if } \mu_n \prec (f \wedge_{\odot} h)(x), \end{cases} \tag{34}$$

$n = 1, 2, \ldots$, we get that

$$[f \oplus h]_n \preceq [f]_n \oplus [h]_n \preceq [f \oplus h]_{2n}. \tag{35}$$

Thus, we have

$$\begin{aligned} \int_{E_n}^{\oplus} [f \oplus h]_n \odot dm &\preceq \int_{E_n}^{\oplus} ([f]_n \oplus [h]_n) \odot dm \\ &\preceq \int_{E_n}^{\oplus} [f \oplus h]_{2n} \odot dm \\ &\preceq \int_{E_{2n}}^{\oplus} [f \oplus h]_{2n} \odot dm. \end{aligned} \tag{36}$$

By (3) of Lemma 20, we have

$$\int_{E_n}^{\oplus} ([f]_n \oplus [h]_n) \odot dm = \int_{E_n}^{\oplus} [f]_n \odot dm \oplus \int_{E_n}^{\oplus} [h]_n \odot dm, \tag{37}$$

which implies that

$$\begin{aligned} \int_{E_n}^{\oplus} [f \oplus h]_n \odot dm &\preceq \int_{E_n}^{\oplus} [f]_n \odot dm \oplus \int_{E_n}^{\oplus} [h]_n \odot dm \\ &\preceq \int_{E_{2n}}^{\oplus} [f \oplus h]_{2n} \odot dm. \end{aligned} \tag{38}$$

Hence, we get that

$$\begin{aligned} &\lim_{n\to\infty} \int_{E_n}^{\oplus} [f \oplus h]_n \odot dm \\ &\quad \preceq \lim_{n\to\infty} \left(\int_{E_n}^{\oplus} [f]_n \odot dm \oplus \int_{E_n}^{\oplus} [h]_n \odot dm \right) \\ &\quad = \lim_{n\to\infty} \int_{E_n}^{\oplus} [f]_n \odot dm \oplus \lim_{n\to\infty} \int_{E_n}^{\oplus} [h]_n \odot dm \\ &\quad \preceq \lim_{n\to\infty} \int_{E_{2n}}^{\oplus} [f \oplus h]_{2n} \odot dm, \end{aligned} \tag{39}$$

which implies that

$$\begin{aligned} \int_X^{\oplus} (f \oplus h) \odot dm &\preceq \int_X^{\oplus} f \odot dm \oplus \int_X^{\oplus} h \odot dm \\ &\preceq \int_X^{\oplus} (f \oplus h) \odot dm; \end{aligned} \tag{40}$$

that is,

$$\int_X^{\oplus} (f \oplus h) \odot dm = \int_X^{\oplus} f \odot dm \oplus \int_X^{\oplus} h \odot dm. \tag{41}$$

(4) If $f \preceq h$, then $[f]_n \preceq [h]_n$, $n = 1, 2, \ldots$. Thus, by (4) of Lemma 20, we have

$$\int_{E_n}^{\oplus} [f]_n \odot dm \preceq \int_{E_n}^{\oplus} [h]_n \odot dm. \tag{42}$$

Hence, we get that

$$\lim_{n\to\infty} \int_{E_n}^{\oplus} [f]_n \odot dm \preceq \lim_{n\to\infty} \int_{E_n}^{\oplus} [h]_n \odot dm, \tag{43}$$

that is,

$$\int_X^{\oplus} f \odot dm \preceq \int_X^{\oplus} h \odot dm; \tag{44}$$

(5) Since $X = \bigcup_{n=1}^{\infty} E_n$ with $m(E_n) \prec \Delta$, we have $X_1 = \bigcup_{n=1}^{\infty} (E_n \cap X_1)$ with $m(E_n \cap X_1) \prec \Delta$ and $X_2 = \bigcup_{n=1}^{\infty} (E_n \cap X_2)$ with $m(E_n \cap X_2) \prec \Delta$. By (5) of Lemma 20, we have

$$\int_{E_n}^{\oplus} [f]_n \odot dm = \int_{E_n \cap X_1}^{\oplus} [f]_n \odot dm \oplus \int_{E_n \cap X_2}^{\oplus} [f]_n \odot dm, \tag{45}$$

which implies that

$$\begin{aligned} &\int_X^{\oplus} f \odot dm \\ &\quad = \lim_{n\to\infty} \int_{E_n}^{\oplus} [f]_n \odot dm \\ &\quad = \lim_{n\to\infty} \int_{E_n \cap X_1}^{\oplus} [f]_n \odot dm \oplus \int_{E_n \cap X_2}^{\oplus} [f]_n \odot dm \\ &\quad = \lim_{n\to\infty} \int_{E_n \cap X_1}^{\oplus} [f]_n \odot dm \oplus \lim_{n\to\infty} \int_{E_n \cap X_2}^{\oplus} [f]_n \odot dm \\ &\quad = \int_{X_1}^{\oplus} f \odot dm \oplus \int_{X_2}^{\oplus} f \odot dm. \end{aligned} \tag{46}$$

(6) Since $X = \bigcup_{n=1}^{\infty} E_n$, we have $E = \bigcup_{n=1}^{\infty} (E_n \cap E)$. By the monotonicity of σ-$\oplus$-decomposable measure m, we get that if $m(E) = \mathbf{0}$, then $m(E_n \cap E) = \mathbf{0}$. By (6) of Theorem 22, we have

$$\int_{E \cap E_n}^{\oplus} [f]_n \odot dm = \mathbf{0}, \tag{47}$$

which implies that

$$\int_E^{\oplus} f \odot dm = \lim_{n\to\infty} \int_{E \cap E_n}^{\oplus} [f]_n \odot dm = \mathbf{0}. \tag{48}$$

□

Theorem 23. *Let $\oplus$ be a strict pseudoaddition and $m : \mathscr{A} \to [a, b]$ a σ-$\oplus$-decomposable measure.*

(1) *If $f \in \mathscr{M}(\mathscr{A})$ and $E \in \mathscr{A}$ is a σ-finite set of $\oplus$-measure, then $\int_E^{\oplus} f \odot dm = \mathbf{0}$ if and only if $f = \mathbf{0}$ a.e. on E.*

(2) *If $f \in \mathscr{M}(\mathscr{A})$, then for any $E \in \mathscr{A}$, $\lim_{mE \to \mathbf{0}} \int_E^{\oplus} f \odot dm = \mathbf{0}$.*

Proof. (1) Suppose $\int_E^{\oplus} f \odot dm = \mathbf{0}$. For arbitrary $\mathbf{0} \prec \delta$, let $E_\delta = \{x \in E \mid \delta \preceq f(x)\} \in \mathscr{A}$. Then we get that

$$\begin{aligned}\delta \odot m(E_\delta) &\preceq \int_{E_\delta}^{\oplus} f \odot dm \preceq \int_{E_\delta}^{\oplus} f \odot dm \oplus \int_{E-E_\delta}^{\oplus} f \odot dm \\ &= \int_E^{\oplus} f \odot dm = \mathbf{0}.\end{aligned} \tag{49}$$

Thus, we have $m(E_\delta) = \mathbf{0}$. Since $\mathbf{0} \prec \delta$ is arbitrary, we have $m(\mathscr{S}[\mathbf{0} \prec f] \cap E) = \mathbf{0}$.

Suppose $f = \mathbf{0}$ a.e. on E, that is, $m(E \cap \mathscr{S}[\mathbf{0} \prec f]) = \mathbf{0}$. By (6) of Theorem 22, we have

$$\int_E^{\oplus} f \odot dm = \int_{E \cap \mathscr{S}[\mathbf{0} \prec f]}^{\oplus} f \odot dm \oplus \int_{E \cap \mathscr{S}[f = \mathbf{0}]}^{\oplus} f \odot dm = \mathbf{0}. \tag{50}$$

(2) If there exists $\Omega \prec \Delta$, such that $f(x) \preceq \Omega$ for all $x \in E$, then

$$\int_E^{\oplus} f \odot dm \preceq \Omega \odot m(E), \quad \text{i.e.,} \lim_{mE \to \mathbf{0}} \int_E^{\oplus} f \odot dm = \mathbf{0}. \tag{51}$$

For any $f \in \mathscr{M}(\mathscr{A})$, we have

$$\int_E^{\oplus} f \odot dm = \lim_{n \to \infty} \int_E^{\oplus} [f]_n \odot dm, \tag{52}$$

which implies that

$$\begin{aligned}\lim_{mE \to 0} \int_E^{\oplus} f \odot dm &= \lim_{mE \to 0} \lim_{n \to \infty} \int_E^{\oplus} [f]_n \odot dm \\ &= \lim_{n \to \infty} \lim_{mE \to 0} \int_E^{\oplus} [f]_n \odot dm = \mathbf{0}.\end{aligned} \tag{53}$$

□

Lemma 24 (see [38]). *Let $\oplus$ be a strict pseudoaddition. The function $d_\oplus : [a,b]^2 \to [a,b]$ given by*

$$d_\oplus(x,y) = \left| x -'_\oplus y \right| = \begin{cases} y -'_\oplus x, & \text{if } x \preceq y, \\ x -'_\oplus y, & \text{if } y \prec x, \end{cases} \tag{54}$$

is a pseudometric on $[a,b]$ with $\lambda = \mathbf{1}$.

Theorem 25. *Let $\oplus$ be a strict pseudoaddition and X a σ-finite set of $\oplus$-measure. If $m : \mathscr{A} \to [a,b]$ is a σ-$\oplus$-decomposable measure, then for any $f, h \in \mathscr{M}(\mathscr{A})$,*

$$\left| \int_X^{\oplus} f \odot dm -'_\oplus \int_X^{\oplus} h \odot dm \right| \preceq \int_X^{\oplus} \left| f -'_\oplus h \right| \odot dm. \tag{55}$$

Proof. Let $E = \{x \mid h(x) \preceq f(x), x \in X\}$ and $F = \{x \mid f(x) \prec h(x), x \in X\}$. Then E and F are two $\oplus$-measure σ-finite sets of X. By (4) of Theorem 22, we have

$$\int_E^{\oplus} h \odot dm \preceq \int_E^{\oplus} f \odot dm, \qquad \int_F^{\oplus} f \odot dm \preceq \int_F^{\oplus} h \odot dm. \tag{56}$$

Thus, by (3) of Theorem 22, we have

$$\begin{aligned}\int_E^{\oplus} f \odot dm &= \int_E^{\oplus} \left(\left| f -'_\oplus h \right| \oplus h \right) \odot dm \\ &= \int_E^{\oplus} \left| f -'_\oplus h \right| \odot dm \oplus \int_E^{\oplus} h \odot dm, \\ \int_F^{\oplus} h \odot dm &= \int_F^{\oplus} \left(\left| f -'_\oplus h \right| \oplus f \right) \odot dm \\ &= \int_F^{\oplus} \left| f -'_\oplus h \right| \odot dm \oplus \int_F^{\oplus} f \odot dm,\end{aligned} \tag{57}$$

which implies that

$$\begin{aligned}&\int_F^{\oplus} \left| f -'_\oplus h \right| \odot dm \oplus \int_X^{\oplus} f \odot dm \\ &\quad = \int_E^{\oplus} \left| f -'_\oplus h \right| \odot dm \oplus \int_X^{\oplus} h \odot dm.\end{aligned} \tag{58}$$

If $\int_X^{\oplus} h \odot dm \preceq \int_X^{\oplus} f \odot dm$, then we have

$$\begin{aligned}\int_X^{\oplus} f \odot dm &= \left| \int_X^{\oplus} f \odot dm -'_\oplus \int_X^{\oplus} h \odot dm \right| \oplus \int_X^{\oplus} h \odot dm \\ &\preceq \int_E^{\oplus} \left| f -'_\oplus h \right| \odot dm \oplus \int_X^{\oplus} h \odot dm \\ &\preceq \int_X^{\oplus} \left| f -'_\oplus h \right| \odot dm \oplus \int_X^{\oplus} h \odot dm,\end{aligned} \tag{59}$$

which implies that

$$\left| \int_X^{\oplus} f \odot dm -'_\oplus \int_X^{\oplus} h \odot dm \right| \preceq \int_X^{\oplus} \left| f -'_\oplus h \right| \odot dm. \tag{60}$$

Similarly, if $\int_X^{\oplus} f \odot dm \prec \int_X^{\oplus} h \odot dm$, we can also get this conclusion. □

Theorem 26. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If*

(1) *$\{f_n\} \subset \mathscr{M}(\mathscr{A})$;*

(2) *$f_n \preceq F$ a.e. on X, $n = 1, 2, \ldots$, and $F \in \mathscr{M}(\mathscr{A})$;*

(3) *$f_n \Rightarrow f$,*

then $f \in \mathscr{M}(\mathscr{A})$ and

$$\lim_{n \to \infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{61}$$

Proof. Since $f_n \Rightarrow f$ on X, by Theorem 3.8 in [38], there exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ that a.e. converges to f on X. By Theorem 3.5 in [38], we have $f \in \mathscr{M}(\mathscr{A})$.

(I) Suppose $m(X) \prec \Delta$. By (2) of Theorem 23, for arbitrary $\mathbf{0} \prec \varepsilon = \varepsilon_1 \oplus \varepsilon_1$, there exists $\mathbf{0} \prec \delta$ such that if $E \subset X$ with $m(E) \prec \delta$, we have

$$\int_E^{\oplus} F \odot dm \prec \varepsilon_1. \tag{62}$$

Since $f_n \Rightarrow f$, there exists a natural number $N > 0$, such that $m(\mathscr{S}[\sigma \preceq |f -'_{\oplus} f_n|]) \prec \delta$ for all $n \geq N$, where $\varepsilon_1 = \sigma \odot m(X)$. Thus, we get that

$$\int_{\mathscr{S}[\sigma \preceq |f -'_{\oplus} f_n|]}^{\oplus} F \odot dm \prec \varepsilon_1. \tag{63}$$

Hence, by Theorem 25, we have

$$\begin{aligned}
&\left|\int_X^{\oplus} f \odot dm -'_{\oplus} \int_X^{\oplus} f_n \odot dm\right| \\
&\quad \preceq \int_X^{\oplus} \left|f -'_{\oplus} f_n\right| \odot dm \\
&\quad = \int_{\mathscr{S}[\sigma \preceq |f -'_{\oplus} f_n|]}^{\oplus} \left|f -'_{\oplus} f_n\right| \odot dm \\
&\qquad \oplus \int_{\mathscr{S}[|f -'_{\oplus} f_n| \prec \sigma]}^{\oplus} \left|f -'_{\oplus} f_n\right| \odot dm \\
&\quad \preceq \int_{\mathscr{S}[\sigma \preceq |f -'_{\oplus} f_n|]}^{\oplus} F \odot dm \\
&\qquad \oplus \left(\sigma \odot m\left(\mathscr{S}\left[\left|f -'_{\oplus} f_n\right| \prec \sigma\right]\right)\right) \\
&\quad \prec \varepsilon_1 \oplus (\sigma \odot m(X)) = \varepsilon_1 \oplus \varepsilon_1 = \varepsilon.
\end{aligned} \tag{64}$$

By Lemma 24, we obtain that

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{65}$$

(II) Suppose $m(X) = \Delta$. For arbitrary $\mathbf{0} \prec \varepsilon = \varepsilon_1 \oplus \varepsilon_1$, there exists $E_k \subseteq X$ with $m(E_k) \prec \Delta$, such that

$$\int_X^{\oplus} F \odot dm \prec \int_{E_k}^{\oplus} [F]_k \odot dm \oplus \varepsilon_1. \tag{66}$$

Thus, we have

$$\begin{aligned}
&\int_{E_k}^{\oplus} [F]_k \odot dm \oplus \int_{X-E_k}^{\oplus} F \odot dm \\
&\quad \preceq \int_{E_k}^{\oplus} F \odot dm \oplus \int_{X-E_k}^{\oplus} F \odot dm \\
&\quad = \int_X^{\oplus} F \odot dm \prec \int_{E_k}^{\oplus} [F]_k \odot dm \oplus \varepsilon_1;
\end{aligned} \tag{67}$$

that is, $\int_{X-E_k}^{\oplus} F \odot dm \prec \varepsilon_1$. Since the measurable functionals sequence $\{|f -'_{\oplus} f_n|\}$ satisfies

(i) $|f -'_{\oplus} f_n| \preceq F$ a.e. on E_k;

(ii) $|f -'_{\oplus} f_n| \Rightarrow \mathbf{0}$ on E_k,

by (I), we get that there exists a natural number $N > 0$, such that

$$\int_{E_k}^{\oplus} \left|f -'_{\oplus} f_n\right| \odot dm \prec \varepsilon_1, \tag{68}$$

for all $n > N$. Hence, by Theorem 25, we have

$$\begin{aligned}
&\left|\int_X^{\oplus} f \odot dm -'_{\oplus} \int_X^{\oplus} f_n \odot dm\right| \\
&\quad \preceq \int_X^{\oplus} \left|f -'_{\oplus} f_n\right| \odot dm \\
&\quad = \int_{X-E_k}^{\oplus} \left|f -'_{\oplus} f_n\right| \odot dm \oplus \int_{E_k}^{\oplus} \left|f -'_{\oplus} f_n\right| \odot dm \\
&\quad \preceq \int_{X-E_k}^{\oplus} F \odot dm \oplus \varepsilon_1 \prec \varepsilon_1 \oplus \varepsilon_1 = \varepsilon.
\end{aligned} \tag{69}$$

Consequently, we obtain that

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{70}$$

□

Corollary 27. *If the condition (3) of Theorem 26 is replaced by $f_n \to f$ a.e. on X, then the conclusion of Theorem 26 holds.*

Proof. Since $f_n \to f$ a.e. on X, by Theorem 3.5 in [38], we have $f \in \mathscr{M}(\mathscr{A})$.

(I) Suppose $m(X) \prec \Delta$. By Theorem 3.9 in [38], if $f_n \to f$ a.e. on X, then $f_n \Rightarrow f$. By Theorem 26 (I), we have

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{71}$$

(II) Suppose $m(X) = \Delta$. Since X is σ-finite set of $\oplus$-measure, there exists an increasing sequence of sets $\{E_n\} \subset \mathscr{A}$ with $m(E_n) \prec \Delta$, $n = 1, 2, \ldots$, such that $X = \bigcup_{n=1}^{\infty} E_n$. For any E_k, $k = 1, 2, \ldots$, the sequence of measurable functionals $\{|f -'_{\oplus} f_n|\}$ satisfies

(i) $|f -'_{\oplus} f_n| \preceq F$ a.e. on E_k, $k = 1, 2, \ldots$;

(ii) $|f -'_{\oplus} f_n| \to \mathbf{0}$ a.e. on E_k, $k = 1, 2, \ldots$.

By Theorem 3.9 in [38], we have

(ii)$'$ $|f -'_{\oplus} f_n| \Rightarrow \mathbf{0}$ on E_k, $k = 1, 2, \ldots$.

By (I) and proof of Theorem 26 (II), we have

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{72}$$

□

Lemma 28. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{x_n\}$ is a monotone sequence, then the sequence $\{x_n\}$ is convergence.*

Proof. If $\{x_n\}$ is an increasing sequence, then

$$\begin{aligned} \underline{\lim}_{n\to\infty} x_n &= \bigvee_{\oplus,n=1}^{\infty}\left(\bigwedge_{\odot,k\geq n} x_k\right) = \bigvee_{\oplus,n=1}^{\infty} x_n, \\ \overline{\lim}_{n\to\infty} x_n &= \bigwedge_{\odot,n=1}^{\infty}\left(\bigvee_{\oplus,k\geq n} x_k\right) = \bigvee_{\oplus,n=1}^{\infty} x_n. \end{aligned} \tag{73}$$

If $\{x_n\}$ is a decreasing sequence, then

$$\begin{aligned} \underline{\lim}_{n\to\infty} x_n &= \bigvee_{\oplus,n=1}^{\infty}\left(\bigwedge_{\odot,k\geq n} x_k\right) = \bigwedge_{\odot,n=1}^{\infty} x_n, \\ \overline{\lim}_{n\to\infty} x_n &= \bigwedge_{\odot,n=1}^{\infty}\left(\bigvee_{\oplus,k\geq n} x_k\right) = \bigwedge_{\odot,n=1}^{\infty} x_n. \end{aligned} \tag{74}$$

Thus, we have

$$\underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n. \tag{75}$$

By Theorem 3.2 in [38], we get that the sequence $\{x_n\}$ is convergent. □

Theorem 29. *Let $\oplus$ be a strict pseudoaddition and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{f_n\}$ is an increasing sequence of measurable functionals on X, then*

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} \lim_{n\to\infty} f_n \odot dm. \tag{76}$$

Proof. Let $\{f_n\}$ be an increasing sequence of measurable functionals on X. By Lemma 28, we get that the sequence of measurable functionals $\{f_n\}$ is convergent. Let $f = \lim_{n\to\infty} f_n$. By Theorem 3.5 in [38], we have $f \in \mathscr{M}(\mathscr{A})$ with $f_n \preceq f$ on X. By (4) of Theorem 22, we get that

$$\int_X^{\oplus} f_n \odot dm \preceq \int_X^{\oplus} f \odot dm, \tag{77}$$

which implies that

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm \preceq \int_X^{\oplus} f \odot dm. \tag{78}$$

On the contrary, since X is σ-finite set of $\oplus$-measure, there exists an increasing sequence of sets $\{E_n\} \subset \mathscr{A}$ with $m(E_n) \prec \Delta$, $n = 1, 2, \ldots$, such that $X = \bigcup_{n=1}^{\infty} E_n$. For any given integer $k > 0$, $\{[f_n]_k\}_{n\geq k}$ is an increasing sequence of measurable functionals and $[f_n]_k \preceq f$ on X, for all $n \geq k$. Now we show that

$$\lim_{n\to\infty} [f_n]_k = [f]_k. \tag{79}$$

For arbitrary $x_0 \in X$,

(i) if $f_n(x_0) \preceq \mu_k$, that is, $[f_n]_k(x_0) = f_n(x_0)$ for all $n \geq k$, then $f(x_0) \preceq \mu_k$, that is, $[f]_k(x_0) = f(x_0)$. Thus, we have

$$\lim_{n\to\infty} [f_n]_k(x_0) = [f]_k(x_0); \tag{80}$$

(ii) if there exists $n_0 \geq k$, such that $\mu_k \prec f_{n_0}(x_0)$, then $\mu_k \prec f_n(x_0)$; that is, $[f_n]_k(x_0) = \mu_k$ for all $n \geq n_0$; it follows that $\mu_k \preceq f(x_0)$; that is, $[f]_k(x_0) = \mu_k$. Thus, we have

$$\lim_{n\to\infty} [f_n]_k(x_0) = [f]_k(x_0) = \mu_k. \tag{81}$$

Hence, by Corollary 27, we get that

$$\int_{E_k}^{\oplus} [f]_k \odot dm = \lim_{n\to\infty} \int_{E_k}^{\oplus} [f_n]_k \odot dm \preceq \lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm, \tag{82}$$

which implies that

$$\int_X^{\oplus} f \odot dm = \lim_{k\to\infty} \int_{E_k}^{\oplus} [f]_k \odot dm \preceq \lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm. \tag{83}$$

Consequently, we obtain that

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm = \int_X^{\oplus} \lim_{n\to\infty} f_n \odot dm. \tag{84}$$

□

Theorem 30. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{f_n\}$ is a decreasing sequence of finite measurable functionals and pseudointegral of f_1 is finite on X, then*

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} \lim_{n\to\infty} f_n \odot dm. \tag{85}$$

Proof. Let $\{f_n\}$ be a decreasing sequence of measurable functionals on X. By Lemma 28, we get that the sequence of measurable functionals $\{f_n\}$ is convergent. Let $f = \lim_{n\to\infty} f_n$. By Theorem 3.5 in [38], we have $f \in \mathscr{M}(\mathscr{A})$. Since $\{f_1 -'_{\oplus} f_n\}$ is an increasing sequence of measurable functionals, by Theorem 29, we have

$$\lim_{n\to\infty} \int_X^{\oplus} \left(f_1 -'_{\oplus} f_n\right) \odot dm = \int_X^{\oplus} \lim_{n\to\infty} \left(f_1 -'_{\oplus} f_n\right) \odot dm. \tag{86}$$

Since $f_1 = (f_1 -'_{\oplus} f_n) \oplus f_n$ and $\oplus$ is continuous, we have

$$f_1 = \lim_{n\to\infty} \left(f_1 -'_{\oplus} f_n\right) \oplus \lim_{n\to\infty} f_n = \lim_{n\to\infty} \left(f_1 -'_{\oplus} f_n\right) \oplus f. \tag{87}$$

Since $f_1 = (f_1 -'_{\oplus} f) \oplus f \prec \Delta$ and $\oplus$ is strict, we get that

$$\lim_{n\to\infty} \left(f_1 -'_{\oplus} f_n\right) = f_1 -'_{\oplus} f, \tag{88}$$

which implies that

$$\lim_{n\to\infty}\int_X^{\oplus}\left(f_1 -'_{\oplus} f_n\right)\odot dm = \int_X^{\oplus}\left(f_1 -'_{\oplus} f\right)\odot dm. \tag{89}$$

By (3) of Theorem 22, we have

$$\begin{aligned}\int_X^{\oplus} f_1\odot dm &= \int_X^{\oplus}\left(f_1 -'_{\oplus} f_n\right)\odot dm \oplus \int_X^{\oplus} f_n\odot dm,\\ \int_X^{\oplus} f_1\odot dm &= \int_X^{\oplus}\left(f_1 -'_{\oplus} f\right)\odot dm \oplus \int_X^{\oplus} f\odot dm.\end{aligned} \tag{90}$$

Thus, we get that

$$\begin{aligned}&\int_X^{\oplus} f_1\odot dm\\ &\quad= \lim_{n\to\infty}\int_X^{\oplus}\left(f_1 -'_{\oplus} f_n\right)\odot dm \oplus \lim_{n\to\infty}\int_X^{\oplus} f_n\odot dm\\ &\quad= \int_X^{\oplus}\left(f_1 -'_{\oplus} f\right)\odot dm \oplus \lim_{n\to\infty}\int_X^{\oplus} f_n\odot dm.\end{aligned} \tag{91}$$

Since $\int_X^{\oplus} f_1\odot dm \prec \Delta$ and $\oplus$ is strict, we obtain that

$$\lim_{n\to\infty}\int_X^{\oplus} f_n\odot dm = \int_X^{\oplus} f\odot dm = \int_X^{\oplus}\lim_{n\to\infty} f_n\odot dm. \tag{92}$$

□

Theorem 31. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{f_n\}$ is a sequence of measurable functionals on X, then*

$$\int_X^{\oplus}\left(\bigoplus_{n=1}^{\infty} f_n\right)\odot dm = \bigoplus_{n=1}^{\infty}\int_X^{\oplus} f_n\odot dm. \tag{93}$$

Proof. Let $h_n = \oplus_{i=1}^{n} f_i$, $n = 1,2,\ldots$. Then $\{h_n\}$ is an increasing sequence of measurable functionals on X. By Theorem 29, we have

$$\lim_{n\to\infty}\int_X^{\oplus} h_n\odot dm = \int_X^{\oplus}\lim_{n\to\infty} h_n\odot dm. \tag{94}$$

By (3) of Theorem 22, we have

$$\int_X^{\oplus} h_n\odot dm = \int_X^{\oplus}\bigoplus_{i=1}^{n} f_i\odot dm = \bigoplus_{i=1}^{n}\int_X^{\oplus} f_i\odot dm; \tag{95}$$

that is,

$$\lim_{n\to\infty}\int_X^{\oplus} h_n\odot dm = \bigoplus_{n=1}^{\infty}\int_X^{\oplus} f_n\odot dm. \tag{96}$$

Since $\lim_{n\to\infty} h_n = \oplus_{n=1}^{\infty} f_n$, we have

$$\int_X^{\oplus}\left(\bigoplus_{n=1}^{\infty} f_n\right)\odot dm = \bigoplus_{n=1}^{\infty}\int_X^{\oplus} f_n\odot dm. \tag{97}$$

□

Theorem 32. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If f is a measurable functional on X,*

$$\int_X^{\oplus} f\odot dm = \bigoplus_{n=1}^{\infty}\int_{E_n}^{\oplus} f\odot dm, \tag{98}$$

for any sequence $\{E_n\}$ of pairwise disjoint sets from $\mathscr{A}$ with $X = \bigcup_{n=1}^{\infty} E_n$.

Proof. A functionals sequence $[f]_n$ is given by

$$f_n(x) = \begin{cases} f(x), & \text{if } x\in E_n,\\ \mathbf{0}, & \text{if } x\in X - E_n,\end{cases}\qquad n = 1,2,\ldots, \tag{99}$$

then $f = \oplus_{n=1}^{\infty} f_n$ and

$$\int_X^{\oplus} f_n\odot dm = \int_{E_n}^{\oplus} f_n\odot dm \oplus \int_{X-E_n}^{\oplus} f_n\odot dm = \int_{E_n}^{\oplus} f\odot dm. \tag{100}$$

By Theorem 31, we have

$$\int_X^{\oplus}\bigoplus_{n=1}^{\infty} f_n\odot dm = \bigoplus_{n=1}^{\infty}\int_X^{\oplus} f_n\odot dm. \tag{101}$$

Hence, we obtain that

$$\int_X^{\oplus} f\odot dm = \bigoplus_{n=1}^{\infty}\int_{E_n}^{\oplus} f\odot dm. \tag{102}$$

□

Theorem 33. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{f_n\}$ is a sequence of measurable functionals on X, then*

$$\int_X^{\oplus}\underline{\lim}_{n\to\infty} f_n\odot dm \preceq \underline{\lim}_{n\to\infty}\int_X^{\oplus} f_n\odot dm. \tag{103}$$

Proof. Let $h_n = \wedge_{\odot k=n}^{\infty} f_k$, $n = 1,2,\ldots$. Then $\{h_n\}$ is an increasing sequence of measurable functionals on X. By proof of Theorem 29, we have

$$\lim_{n\to\infty} h_n = \bigvee_{\oplus n=1}^{\infty} h_n = \bigvee_{\oplus n=1}^{\infty}\bigwedge_{\odot k=n}^{\infty} f_k = \underline{\lim}_{n\to\infty} f_n. \tag{104}$$

By Theorem 29, we have

$$\int_X^{\oplus}\lim_{n\to\infty} h_n\odot dm = \lim_{n\to\infty}\int_X^{\oplus} h_n\odot dm, \tag{105}$$

which implies that

$$\int_X^{\oplus}\underline{\lim}_{n\to\infty} f_n\odot dm = \lim_{n\to\infty}\int_X^{\oplus} h_n\odot dm. \tag{106}$$

By (4) of Theorem 22 and $h_n \preceq f_k$ for all $k \geq n$, we have

$$\int_X^{\oplus} h_n\odot dm \preceq \int_X^{\oplus} f_k\odot dm, \tag{107}$$

for all $k \geq n$, which implies that

$$\int_X^{\oplus} h_n \odot dm \preceq \bigwedge_{k=n}^{\infty}{}_{\odot} \int_X^{\oplus} f_k \odot dm. \tag{108}$$

By (4) of Theorem 22 and the monotonicity of $\{h_n\}$, we have $\{\int_X^{\oplus} h_n \odot dm\}$ is an increasing sequence. Thus, by proof of Theorem 29, we have

$$\lim_{n\to\infty} \int_X^{\oplus} h_n \odot dm = \bigvee_{n=1}^{\infty}{}_{\oplus} \int_X^{\oplus} h_n \odot dm. \tag{109}$$

Hence, we obtain that

$$\int_X^{\oplus} \underline{\lim}_{n\to\infty} f_n \odot dm \preceq \bigvee_{n=1}^{\infty}{}_{\oplus} \bigwedge_{k=n}^{\infty}{}_{\odot} \int_X^{\oplus} f_k \odot dm = \underline{\lim}_{n\to\infty} \int_X^{\oplus} f_n \odot dm. \tag{110}$$

□

Example 34. Let the total order $\preceq$ on $[0, +\infty)$ be the usual order of the real line and the pseudoaddition $\oplus$ is defined by

$$x \oplus y = \begin{cases} \dfrac{x+y}{2}, & \text{if } x, y \in (0, \infty), \\ \max\{x, y\}, & \text{if } x = 0 \text{ or } y = 0, \end{cases} \tag{111}$$

and the pseudomultiplication $\odot$ is the usual multiplication of the real numbers. It is obvious that zero element is 0 and unit element is 1. Let the decomposable measure m be Lebesgue measure on $[0, 1]$. We know that the pseudointegral is

$$\int_{[0,1]}^{\oplus} f \odot dm = \frac{1}{2} \int_0^1 f(x)\, dx, \tag{112}$$

for each $f \in \mathscr{M}(\mathscr{A}([0, 1]))$, where the right hand side is the Lebesgue integral. Let

$$f_n(x) = \begin{cases} n, & \dfrac{1}{2n} \leq x \leq \dfrac{1}{n}, \\ 0, & \dfrac{1}{n} < x \leq 1 \text{ or } 0 \leq x < \dfrac{1}{2n}. \end{cases} \tag{113}$$

Then, we get that

$$\begin{aligned} \int_{[0,1]}^{\oplus} f_n \odot dm = \int_{[0,1/2n]}^{\oplus} 0 \odot dm \oplus \int_{[1/2n,1/n]}^{\oplus} n \odot dm \\ \oplus \int_{[1/n,1]}^{\oplus} 0 \odot dm = \frac{1}{4}; \end{aligned} \tag{114}$$

that is, $\underline{\lim}_{n\to\infty} \int_{[0,1]}^{\oplus} f_n \odot dm = 1/4$ and $\underline{\lim}_{n\to\infty} f_n = 0$, which implies that $\int_{[0,1]}^{\oplus} \underline{\lim}_{n\to\infty} f_n \odot dm = 0$. Hence, we obtain that

$$\int_{[0,1]}^{\oplus} \underline{\lim}_{n\to\infty} f_n \odot dm \leq \underline{\lim}_{n\to\infty} \int_{[0,1]}^{\oplus} f_n \odot dm. \tag{115}$$

4. Conclusions

In this paper, we mainly discussed pseudointegral based on pseudoaddition decomposable measure. Particularly, we have given the definition of the pseudointegral of a measurable function based on a strict pseudoaddition decomposable measure by generalizing the definition of the pseudointegral of a bounded measurable function. Furthermore, we have derived several important properties of the pseudointegral of a measurable function based on strict pseudoaddition decomposable measure. Finally, we have obtained that some theorems on the integral and the limit can be exchanged.

Recently, pseudoanalysis has obtained rapid development in the mechanical, chemical, biological, medical, and computer fields and has solved some uncertainty problems of knowledge. Pseudoanalysis theory has important applications in the field of computer image processing [39, 40]; for example, it can analyze and grasp the variation range of the image gray value, solve the relationship between the grey value and image color change, and take appropriate grey value to achieve better image processing effect. With the development of computer technology, pseudoanalysis will also get more and more widely used in computer science. We also hope that our results in this paper may lead to significant, new, and innovative results in other related fields.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Quadrature Rules and Iterative Method for Numerical Solution of Two-Dimensional Fuzzy Integral Equations

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We introduce some generalized quadrature rules to approximate two-dimensional, Henstock integral of fuzzy-number-valued functions. We also give error bounds for mappings of bounded variation in terms of uniform modulus of continuity. Moreover, we propose an iterative procedure based on quadrature formula to solve two-dimensional linear fuzzy Fredholm integral equations of the second kind (2*DFFLIE*2), and we present the error estimation of the proposed method. Finally, some numerical experiments confirm the theoretical results and illustrate the accuracy of the method.

1. Introduction

The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh and others. The topic of fuzzy integrations is discussed in [1]. The Henstock and Riemann integral for fuzzy-number-valued functions was introduced and studied in [2, 3]. Their numerical computation was also proposed; see, for example, [3–6]. In [6], the authors obtained the upper estimates of error of some fuzzy quadrature rules for mappings of bounded variation and of Lipschitz type and gave some applications. In [7], the authors studied the Gaussian quadrature rules for fuzzy integrals. Also, in [8], Wu presented some optimal fuzzy quadrature formula for classes of fuzzy-number-valued functions of Lipschitz type. To study other works, see [9–12].

Since many real-valued problems in engineering and mechanics can be brought in the form of two-dimensional fuzzy integral equations, it is important that we develop quadrature rules and numerical methods for such integral equations. In this paper, we introduce two-dimensional fuzzy integrals and propose some generalized quadrature rules and their dependent theorems for mappings of bounded variation. Also, we present the conditions for existence of unique solution for *2DFFLIE2*. Finally, we introduce an iterative method for solving *2DFFLIE2*. The rest of the paper is organized as follows. In Section 2, we give basic information about the fuzzy set theory and develop them to two-dimensional space. Also, we define two-dimensional fuzzy integral equation and some other properties of it in this section. In Section 3, we derive the proposed method to obtain numerical solutions of *2DFFLIE2* based on an iterative procedure. The error estimation of the introduced method is presented in Section 4 in terms of uniform modulus of continuity to prove the convergence of the method. Some numerical experiments are presented in Section 5.

2. Preliminaries

In this section, we review some necessary basic definitions on fuzzy numbers, fuzzy-number-valued functions, and fuzzy integrals.

Definition 1 (see [13, 14]). A fuzzy number is a function $u : R \to [0, 1]$ having the following properties:

(i) u is normal; that is, $\exists x_0 \in R$, such that $u(x_0) = 1$;

(ii) u is fuzzy convex set (i.e., $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$, for all $x, y \in R, \lambda \in [0, 1]$);

(iii) u is upper semicontinuous on R;

(iv) the support $\overline{\{x \in R : u(x) > 0\}}$ is a compact set, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy numbers is denoted by R_F. According to [2], any real number $\alpha \in R$ can be interpreted as a fuzzy number $\alpha = \chi_{\{\alpha\}}$, and therefore $R \subset R_F$. Also, the neutral element with respect to $\oplus$ in R_F is denoted by $\widetilde{0} = \chi_{\{0\}}$.

Definition 2 (see [2, 15]). For any $0 < r \leq 1$, an arbitrary fuzzy number is represented in parametric form, by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, which satisfies the following properties:

(i) $\underline{u}(r)$ is bounded left continuous nondecreasing function over $[0, 1]$;

(ii) $\overline{u}(r)$ is bounded left continuous nonincreasing function over $[0, 1]$;

(iii) $\underline{u}(r) \leq \overline{u}(r)$.

Moreover, the addition and scalar multiplication of fuzzy numbers in R_F are defined as follows:

(i)

$$(u \oplus v)(r) = \left(\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)\right), \tag{1}$$

(ii)

$$(\lambda \odot v)(r) = \begin{cases} \left(\lambda\underline{u}(r), \lambda\overline{u}(r)\right) & \lambda \geq 0, \\ \left(\lambda\overline{u}(r), \lambda\underline{u}(r)\right) & \lambda < 0. \end{cases} \tag{2}$$

Also, according to [2, 16], the following algebraic properties for any $u, v, w \in R_F$ hold:

(i) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$;

(ii) $u \oplus \widetilde{0} = \widetilde{0} \oplus u = u$;

(iii) with respect to $\widetilde{0}$, none of $u \in (R_F - R)$, $u \neq \widetilde{0}$ has opposite in $(R_F, +)$;

(iv) $(a \oplus b) \odot u = a \odot u \oplus b \odot u$, for all $a, b \in R$ with $ab \geq 0$ or $ab \leq 0$;

(v) $a \odot (u \oplus v) = a \odot u \oplus a \odot v$, for all $a \in R$;

(vi) $a \odot (b \odot u) = (ab) \odot u$, for all $a \in R$ and $1 \odot u = u$.

Definition 3 (see [2, 17]). For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$, the quantity $D(u, v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$ is the distance between u and v. Also, the following properties hold [6]:

(i) (R_F, D) is a complete metric space;

(ii) $D(u \oplus w, v \oplus w) = D(u, v)$ for all $u, v, w \in R_F$;

(iii) $D(k \odot u, k \odot v) = |k| D(u, v)$ for all $u, v \in R_F$ for all $k \in R$;

(iv) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)$ for all $u, v, w, e \in R_F$;

(v) $D(k_1 \odot u, k_2 \odot u) = |k_1 - k_2| D(u, \widetilde{0})$ for all $k_1, k_2 \in R$ with $k_1 k_2 \geq 0$ and for all $u \in R_F$.

Throughout this paper, we denote that $\|\cdot\|_F = D(\cdot, 0)$.

Theorem 4 (see [14]). *(i) (R_F, D) is a complete metric space.*

(ii) The pair (R_F, D) is a commutative semigroup with $\widetilde{0} = \chi_0$ zero elements but cannot be a group for pure fuzzy numbers.

(iii) $\|\cdot\|_F$ has the properties of a usual norm on R_F; that is, $\|\cdot\|_F = 0$ if and only if $u = 0$, $\|\lambda \odot u\|_F = |\lambda| \|u\|_F$, and $\|u \oplus v\|_F \leq \|u\|_F + \|v\|_F$.

(iv) $|\|u\|_F - \|v\|_F| \leq D(u, v)$ and $D(u, v) \leq \|u\|_F + \|v\|_F$ for any $u, v \in R_F$.

In [2], the authors introduced the concept of the Henstock integral for a fuzzy-number-valued function. We present a generalized definition of this concept for two-dimensional Henstock integrability for bivariate fuzzy-number-valued functions.

Definition 5. Suppose that $f : [a, b] \times [c, d] \to R_F$ is a bounded mapping, and then the function $\omega_{[a,b]\times[c,d]}(f, \cdot) : R_+ \cup 0 \to R_+$ defined by

$$\begin{aligned} \omega_{[a,b]\times[c,d]}(f, \delta) = \sup \Big\{ & D\left(f(x, y), f(s, t)\right); \\ & x, s \in [a, b]; y, t \in [c, d]; \\ & \sqrt{(x - s)^2 + (y - t)^2} \leq \delta \Big\} \end{aligned} \tag{3}$$

is called the modulus of oscillation of f on $[a, b] \times [c, d]$.

Also, if $f \in C_F([a, b] \times [c, d])$ (i.e., $f : [a, b] \times [c, d] \to R_F$ is continuous on $[a, b] \times [c, d]$), then $\omega_{[a,b]\times[c,d]}(f, \delta)$ is called uniform modulus of continuity of f. The following properties will be very useful in what follows. The proofs of these properties in one-dimensional case are presented in [14] and those in two-dimensional case will be obtained in a similar way.

Theorem 6. *The following properties hold:*

(i) $D(f(x, y), f(s, t)) \leq \omega_{[a,b]\times[c,d]}(f, \sqrt{(x - s)^2 + (y - t)^2})$ *for any $x, s \in [a, b]$ and $y, t \in [c, d]$;*

(ii) $\omega_{[a,b]\times[c,d]}(f, \delta)$ *is a nondecreasing mapping in δ;*

(iii) $\omega_{[a,b]\times[c,d]}(f, 0) = 0$;

(iv) $\omega_{[a,b]\times[c,d]}(f, \delta_1 + \delta_2) \leq \omega_{[a,b]\times[c,d]}(f, \delta_1) + \omega_{[a,b]\times[c,d]}(f, \delta_2)$ *for any $\delta_1, \delta_2 \geq 0$;*

(v) $\omega_{[a,b]\times[c,d]}(f, n\delta) \leq n\omega_{[a,b]\times[c,d]}(f, \delta)$ *for any $\delta \geq 0$ and $n \in N$;*

(vi) $\omega_{[a,b]\times[c,d]}(f, \lambda\delta) \leq (\lambda + 1)\omega_{[a,b]\times[c,d]}(f, \delta)$ *for any $\delta, \lambda \geq 0$.*

Definition 7. Let $f : [a, b] \times [c, d] \to R_F$, for $\Delta_x^n : a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta_y^n : c = y_0 < y_1 < \cdots < y_n = d$, be two partitions of the intervals $[a, b]$ and $[c, d]$, respectively. Let one consider the intermediate points $\xi_i \in [x_{i-1}, x_i]$ and $\eta_i \in [y_{j-1}, y_j]$, $i = 1, \ldots, n$; $j = 1, \ldots, n$, and $\delta : [a, b] \to R_+$ and $\sigma : [c, d] \to R_+$. The divisions $P_x = ([x_{i-1}, x_i]; \xi_i)$, $i = 1, \ldots, n$, and $P_y = ([y_{j-1}, y_j]; \eta_j)$, $j = 1, \ldots, n$, denoted shortly by $P_x = (\Delta^n, \xi)$ and $P_y = (\Delta^n, \eta)$ are said to be δ-fine

and σ-fine, respectively, if $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $[y_{j-1}, y_j] \subseteq (\eta_j - \sigma(\eta_j), \eta_j + \sigma(\eta_j))$.

The function f is said to be two-dimensional Henstock integrable to $I \in R_F$ if for every $\varepsilon > 0$ there are functions $\delta : [a,b] \to R_+$ and $\sigma : [c,d] \to R_+$ such that for any δ-fine and σ-fine divisions we have $D(\sum_{i=0}^{n}\sum_{j=0}^{n}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j), I) < \epsilon$, where $\sum$ denotes the fuzzy summation. Then, I is called the two-dimensional Henstock integral of f and is denoted by $I(f) = (FH)\int_c^d \int_a^b f(s,t)ds\,dt$.

If the above δ and σ are constant functions, then one recaptures the concept of Riemann integral. In this case, $I \in R_F$ will be called two-dimensional integral of f on $[a,b] \times [c,d]$ and will be denoted by $(FR)\int_c^d \int_a^b f(s,t)ds\,dt$.

Corollary 8. *In [13], the authors proved that if $f \in C_F[a,b]$, its definite integral exists, also $\underline{(FR)\int_a^b f(t;r)dt} = \int_a^b \underline{f}(t,r)dt$, and $\overline{(FR)\int_a^b f(t;r)dt} = \int_a^b \overline{f}(t,r)dt$. In a similar way, we can prove that if $f \in C_F([a,b] \times [c,d])$, its definite integral exists, and one has*

$$\begin{aligned} \underline{(FR)\int_c^d \int_a^b f(s,t;r)\,ds\,dt} &= \int_c^d \int_a^b \underline{f}(s,t,r)\,ds\,dt, \\ \overline{(FR)\int_c^d \int_a^b f(s,t;r)\,ds\,dt} &= \int_c^d \int_a^b \overline{f}(s,t,r)\,ds\,dt. \end{aligned} \tag{4}$$

Theorem 9. *If f and g are Henstock integrable mappings on $[a,b] \times [c,d]$ and if $D(f(s,t), g(s,t))$ is Lebesgue integrable, then*

$$\begin{aligned} &D\left((FH)\int_c^d \int_a^b f(s,t)\,ds\,dt, (FH)\int_c^d \int_a^b g(s,t)\,ds\,dt\right) \\ &\quad \le (L)\int_c^d \int_a^b D\left(f(s,t), g(s,t)\right)ds\,dt. \end{aligned} \tag{5}$$

Proof. In [2, 17], the authors demonstrated that for any integrable functions $h, r : [\alpha, \beta] \to R_F$ we have $D((FH)\int_\alpha^\beta h(x)dx, (FH)\int_\alpha^\beta r(x)dx) \le (L)\int_\alpha^\beta D(h(x), r(x))dx$, and, clearly, we obtain

$$\begin{aligned} &D\left((FH)\int_c^d \int_a^b f(s,t)\,ds\,dt, (FH)\int_c^d \int_a^b g(s,t)\,ds\,dt\right) \\ &\quad \le (L)\int_c^d D\left((FH)\int_a^b f(s,t)\,ds, (FH)\int_a^b g(s,t)\,ds\right)dt \\ &\quad \le (L)\int_c^d \int_a^b D\left(f(s,t), g(s,t)\right)ds\,dt, \end{aligned} \tag{6}$$

which completes the proof. □

Theorem 10. *If $f : [a,b] \times [c,d] \to R_F$ is an integrable bounded mapping, then for any fixed $u \in [a,b]$ and $v \in [c,d]$ the function $\varphi_{uv} : [a,b] \times [c,d] \to R_+$, defined by $\varphi_{uv}(s,t) = D(f(u,v), f(s,t))$, is Lebesgue integrable on $[a,b] \times [c,d]$.*

Proof. Regarding [6], Lemma 1, part (ii), it is easy to see that if f is two-dimensional Henstock integrable and bounded on $[a,b] \times [c,d]$, then $f_-^r(s,t)$ and $f_+^r(s,t)$ as real functions of $(s,t) \in [a,b] \times [c,d]$ are two-dimensional integrable and uniformly bounded with respect to $r \in [0,1]$; that is, $f_-^r(s,t)$ and $f_+^r(s,t)$ are Lebesgue measurable (as functions of (s,t)) and uniformly bounded with respect to $r \in [0,1]$ by

$$\begin{aligned} \varphi_{uv}(s,t) &= D\left(f(u,v), f(s,t)\right) \\ &= \sup_{r\in[0,1]} \max\left\{\left|f_-^r(u,v) - f_-^r(s,t)\right|, \right. \\ &\qquad \left.\left|f_+^r(u,v) - f_+^r(s,t)\right|\right\} \\ &= \sup_{r_n\in[0,1]} \max\left\{\left|f_-^{r_n}(u,v) - f_-^{r_n}(s,t)\right|, \right. \\ &\qquad \left.\left|f_+^{r_n}(u,v) - f_+^{r_n}(s,t)\right|\right\}, \end{aligned} \tag{7}$$

where $r_n, n \in N$, represent all the rational numbers in $[0,1]$. By Lebesgue's theorem of dominated convergence, it follows that $\varphi_{uv}(s,t)$ is Lebesgue integrable on $[a,b] \times [c,d]$, and this ends the proof. □

Definition 11. A function $f : [a,b] \times [c,d] \to R_F$ is said to be bounded if there exists M such that $\|f(x,y)\|_F \le M$ for any $(x,y) \in [a,b] \times [c,d]$.

Definition 12. A function $f : [a,b] \times [c,d] \to R_F$ is said to be of bounded variation if

$$\sup_{(x,y)\in[a,b]\times[c,d]} V_{\Delta_{xy}^n} < \infty, \tag{8}$$

where

$$V_{\Delta_{xy}^n} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} D\left(f\left(x_{i+1}, y_{j+1}\right), f\left(x_i, y_j\right)\right) \tag{9}$$

is the variation of f related to partitions Δ_x^n, Δ_y^n. The total variation of f is defined to be, in this case, the number

$$\bigvee(f) = \sup_{(x,y)\in[a,b]\times[c,d]} V_{\Delta_{xy}^n} \in R. \tag{10}$$

It is known also that a function of bounded variation is Riemann integrable (see [18]), so it is Henstock integrable too.

Theorem 13. *(i) If $[a,b] \times [c,d] \subseteq [e,f] \times [g,h]$, then $\omega_{[a,b]\times[c,d]}(f,\delta) \le \omega_{[e,f]\times[g,h]}(f,\delta)$ for all $\delta \ge 0$.*

(ii) If f is of bounded variation, then $\omega_{[a,b]\times[c,d]}(f,\delta) \le \bigvee(f)$ for all $\delta \ge 0$.

Proof. (i) It is easy to see that

$$\begin{aligned}
&\sup\left\{D\left(f\left(x,y\right),f\left(s,t\right)\right) \mid x,s\in\left[a,b\right],y,t\in\left[c,d\right],\right.\\
&\qquad\left.\sqrt{\left(x-s\right)^2+\left(y-t\right)^2}\le\delta\right\}\\
&\quad\le\sup\left\{D\left(f\left(x,y\right),f\left(s,t\right)\right) \mid x,s\in\left[e,f\right],y,t\in\left[g,h\right],\right.\\
&\qquad\left.\sqrt{\left(x-s\right)^2+\left(y-t\right)^2}\le\delta\right\},
\end{aligned}\tag{11}$$

and, therefore, we obtain the required inequality.

(ii) Let $x,s\in[a,b]$ and $y,t\in[c,d]$; assume that $a<x<s<b$, $c<y<t<d$, $V_{\Delta x}=a=x_0<x_1=x<x_2=s<b$, and $V_{\Delta y}=c=y_0<y_1=y<y_2=t<d$. Taking supremum for any $x,s\in[a,b]$ and $y,t\in[c,d]$ with $\sqrt{(x-s)^2+(y-t)^2}\le\delta$, we obtain the required inequality. It is obvious now that under this condition f is bounded; therefore, we obtain

$$\begin{aligned}
\left\|f(x,y)\right\|_F &= D\left(f\left(x,y\right),\widetilde{0}\right)\\
&\le D\left(f\left(x,y\right),f\left(a,c\right)\right)+d\left(f\left(a,c\right),\widetilde{0}\right)\\
&\le \bigvee\left(f\right)+\left\|f\left(a,c\right)\right\|_F,
\end{aligned}\tag{12}$$

which completes the proof. □

Definition 14. A function $f:[a,b]\times[c,d]\to R_F$ is said to be *L-Lipschitz*, if

$$D\left(f\left(x,y\right),f\left(s,t\right)\right)\le L\sqrt{\left(x-s\right)^2+\left(y-t\right)^2},\tag{13}$$

for any $x,s\in[a,b]$ and $y,t\in[c,d]$.

Definition 15. A function $f:[a,b]\times[c,d]\to R_F$ is said to be *M-Condition*, if

$$D\left(f\left(x,y\right),f\left(s,t\right)\right)\le M\left(b-a\right)\left(d-c\right),\tag{14}$$

for any $x,s\in[a,b]$ and $y,t\in[c,d]$.

Remark 16. We see that if f is *M-Condition* function, then f is of bounded variation and

$$\bigvee\left(f\right)\le M\left(b-a\right)\left(d-c\right).\tag{15}$$

Indeed, we have

$$\begin{aligned}
V_{\Delta^n_{xy}} &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}D\left(f\left(x_{i+1},y_{j+1}\right),f\left(x_i,y_j\right)\right)\\
&\le \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left(M\left(x_{i+1}-x_i\right)\left(y_{j+1}-y_j\right)\right)\\
&= M\left(b-a\right)\left(d-c\right),
\end{aligned}\tag{16}$$

and since

$$\bigvee\left(f\right)=\sup_{(x,y)\in[a,b]\times[c,d]}V_{\Delta^n_{xy}},\tag{17}$$

we obtain the required result.

3. Quadrature Rules for Two-Dimensional (2D) Henstock Integrals

In this section, we present some quadrature rules for 2D Henstock integral. The following theorem gives a unified approach to quadrature rules in 2D Henstock integrals.

Theorem 17. *Let $f:[c,d]\times[c,d]\to R_F$ be Henstock integrable, bounded mappings. Then, for any divisions $a=x_0<x_1<\cdots<x_n=b$ and $c=y_0<y_1<\cdots<y_n=d$ and any points $\xi_i\in[x_{i-1},x_i]$ and $\eta_j\in[y_{j-1},y_j]$, one has*

$$\begin{aligned}
&D\left((FH)\int_c^d(FH)\int_a^b f\left(s,t\right)ds\,dt,\right.\\
&\qquad\left.\sum_{j=1}^{n}\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\odot f\left(\xi_i,\eta_j\right)\right)\\
&\quad\le\sum_{j=1}^{n}\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}\\
&\qquad\cdot\left(f,\sqrt{\left(x_i-x_{i-1}\right)^2+\left(y_j-y_{j-1}\right)^2}\right).
\end{aligned}\tag{18}$$

Proof. It is known that the Henstock integrals are additive related to interval. This leads us to

$$\begin{aligned}
&D\left((FH)\int_c^d(FH)\int_a^b f\left(s,t\right)ds\,dt,\right.\\
&\qquad\left.\sum_{j=1}^{n}\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\odot f\left(\xi_i,\eta_j\right)\right)\\
&\quad= D\left((FH)\int_c^d\left(\sum_{i=1}^{n}(FH)\int_{x_{i-1}}^{x_i}f\left(s,t\right)ds\right)dt,\right.\\
&\qquad\left.\sum_{j=1}^{n}\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\odot f\left(\xi_i,\eta_j\right)\right)\\
&\quad= D\left((FH)\sum_{j=1}^{n}\int_{y_{j-1}}^{y_j}\left(\sum_{i=1}^{n}(FH)\int_{x_{i-1}}^{x_i}f\left(s,t\right)ds\right)dt,\right.\\
&\qquad\left.\sum_{j=1}^{n}\sum_{i=1}^{n}(FH)\int_{y_{j-1}}^{y_j}(FH)\int_{x_{i-1}}^{x_i}f\left(\xi_i,\eta_j\right)ds\,dt\right),
\end{aligned}\tag{19}$$

and, by Definition 3 part (iv) and Theorem 9, we have

$$\begin{aligned}
&D\left(\sum_{j=1}^{n}(FH)\int_{y_{j-1}}^{y_j}\left(\sum_{i=1}^{n}(FH)\int_{x_{i-1}}^{x_i} f(s,t)\,ds\right)dt,\right.\\
&\qquad\left.\sum_{j=1}^{n}\sum_{i=1}^{n}(FH)\int_{y_{j-1}}^{y_j}(FH)\int_{x_{i-1}}^{x_i} f(\xi_i,\eta_j)\,ds\,dt\right)\\
&\le \sum_{j=1}^{n} D\left((FH)\int_{y_{j-1}}^{y_j}\sum_{i=1}^{n}(FH)\int_{x_{i-1}}^{x_i} f(s,t)\,ds\,dt,\right.\\
&\qquad\left.\sum_{i=1}^{n}(FH)\int_{y_{j-1}}^{y_j}(FH)\int_{x_{i-1}}^{x_i} f(\xi_i,\eta_j)\,ds\,dt\right)\\
&\le \sum_{j=1}^{n}\sum_{i=1}^{n} D\left((FH)\int_{y_{j-1}}^{y_j}(FH)\int_{x_{i-1}}^{x_i} f(s,t)\,ds\,dt,\right.\\
&\qquad\left.(FH)\int_{y_{j-1}}^{y_j}(FH)\int_{x_{i-1}}^{x_i} f(\xi_i,\eta_j)\,ds\,dt\right)\\
&\le \sum_{j=1}^{n}\sum_{i=1}^{n}(L)\int_{y_{j-1}}^{y_j}(L)\int_{x_{i-1}}^{x_i} D\left(f(s,t),f(\xi_i,\eta_j)\right)ds\,dt.
\end{aligned}\tag{20}$$

From part (i) of Theorem 6, we conclude that

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{i=1}^{n}(L)\int_{y_{j-1}}^{y_j}(L)\int_{x_{i-1}}^{x_i} D\left(f(s,t),f(\xi_i,\eta_j)\right)ds\,dt\\
&\quad\le \sum_{j=1}^{n}\sum_{i=1}^{n}(x_i-x_{i-1})(y_j-y_{j-1})\,\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}\\
&\qquad\cdot\left(f,\sqrt{(x_i-x_{i-1})^2+(y_j-y_{j-1})^2}\right),
\end{aligned}\tag{21}$$

which completes the proof. □

From the above inequality, we infer some generalization of well-known trapezoidal-type, midpoint-type, and three-point-type inequalities with error estimations.

Corollary 18. *Assume that $f:[a,b]\times[c,d]\to R_F$ is a Henstock integrable, bounded mapping. Then, with the notation*

$$\omega_{\overline{xy}\times\overline{zt}}=\omega_{[x,y]\times[z,t]}\left(f,\sqrt{(y-x)^2+(t-z)^2}\right),\tag{22}$$

one has

(i)

$$\begin{aligned}
&D\left((FH)\int_c^d(FH)\int_a^b f(s,t)\,ds\,dt,\right.\\
&\qquad\left.(b-a)(d-c)\odot f(x,y)\right)\\
&\le (x-a)(y-c)\,w_{\overline{ax}\times\overline{cy}}+(b-x)(y-c)\,w_{\overline{ab}\times\overline{cy}}\\
&\quad+(x-a)(d-y)\,w_{\overline{ax}\times\overline{yd}}\\
&\quad+(b-x)(d-y)\,w_{\overline{xb}\times\overline{yd}},
\end{aligned}\tag{23}$$

for any $(x,y)\in[a,b]\times[c,d]$;

(ii)

$$\begin{aligned}
&D\left((FH)\int_c^d(FH)\int_a^b f(s,t)\,ds\,dt,\right.\\
&\qquad[(x-a)(y-c)\\
&\qquad\odot f(u,\alpha)\oplus(x-a)(d-y)\\
&\qquad\odot f(u,\beta)\oplus(b-x)(d-y)\\
&\qquad\odot f(c,\beta)\oplus(b-x)(d-y)\\
&\qquad\left.\odot f(v,\beta)]\right)\\
&\le (x-a)(y-c)\,w_{\overline{ax}\times\overline{cy}}+(b-x)(y-c)\,w_{\overline{xb}\times\overline{cy}}\\
&\quad+(x-a)(d-y)\,w_{\overline{ax}\times\overline{yd}}\\
&\quad+(b-x)(d-y)\,w_{\overline{xb}\times\overline{yd}},
\end{aligned}\tag{24}$$

for any $x\in[a,b]$, $y\in[c,d]$, $u\in[a,x]$, $v\in[x,b]$, $\alpha\in[c,y]$, and $\beta\in[y,d]$;

(iii)

$$\begin{aligned}
&D\left((FH)\int_c^d(FH)\int_a^b f(s,t)\,ds\,dt,\right.\\
&\qquad[(\alpha-a)(\theta-c)\odot f(u,r)\\
&\qquad\oplus(\alpha-a)(\gamma-\theta)\odot f(u,p)\\
&\qquad\oplus(\alpha-a)(d-\gamma)\odot f(u,z)\\
&\qquad\oplus(\beta-\alpha)(\theta-c)\odot f(v,r)\\
&\qquad\oplus(\beta-\alpha)(\gamma-\theta)\odot f(v,p)
\end{aligned}$$

$$\oplus (\beta-\alpha)(d-\gamma)\odot f(u,z)$$
$$\oplus (b-\beta)(\theta-c)\odot f(w,r)$$
$$\oplus (b-\beta)(\gamma-\theta)\odot f(w,p)$$
$$\oplus (b-\beta)(d-\gamma)\odot f(w,z)]\Bigg)$$
$$\leq (\alpha-a)(\theta-c)\,\omega_{\overline{a\alpha}\times\overline{c\theta}}+(\alpha-a)(\gamma-\theta)\,\omega_{\overline{a\alpha}\times\overline{\theta\gamma}}$$
$$+(\alpha-a)(d-\gamma)\,\omega_{\overline{a\alpha}\times\overline{\gamma d}}$$
$$+(\beta-\alpha)(\theta-c)\,\omega_{\overline{\alpha\beta}\times\overline{c\theta}}+(\beta-\alpha)(\gamma-\theta)\,\omega_{\overline{\alpha\beta}\times\overline{\theta\gamma}}$$
$$+(\beta-\alpha)(d-\gamma)\,\omega_{\overline{\alpha\beta}\times\overline{\gamma d}}+(b-\beta)(\theta-c)\,\omega_{\overline{\beta b}\times\overline{c\theta}}$$
$$+(b-\beta)(\gamma-\theta)\,\omega_{\overline{\beta b}\times\overline{\theta\gamma}}+(b-\beta)(d-\gamma)\,\omega_{\overline{\beta b}\times\overline{\gamma d}}, \tag{25}$$

for any u, v, w, α, β, r, p, θ, γ, and z with $a<u<\alpha<v<\beta<w<b$ and $c<r<\theta<p<\gamma<z<d$.

Proof. (i) Taking in the previous theorem that $n=2$, $x_1=\xi_1=\xi_2=x$, and $y_1=\eta_1=\eta_2=y$, we obtain the required inequality. Indeed,

$$D\Bigg((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,$$
$$\sum_{j=1}^{2}\sum_{i=1}^{2}(x_i-x_{i-1})(y_j-y_{j-1})\odot f(\xi_i,\eta_j)\Bigg)$$
$$=D\Bigg((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,$$
$$\sum_{j=1}^{2}[(x-a)(y_j-y_{j-1})\odot f(x,y)$$
$$\oplus (b-x)(y_j-y_{j-1})\odot f(x,\eta_j)]\Bigg)$$
$$=D\Bigg((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,$$
$$\sum_{j=1}^{2}[(x-a)(y-c)\odot f(x,y)$$
$$\oplus (b-x)(y-c)\odot f(x,y)$$
$$\oplus (x-a)(d-y)\odot f(x,y)$$
$$\oplus (b-x)(d-y)\odot f(x,y)]\Bigg)$$
$$=D\Bigg((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,$$
$$\sum_{j=1}^{2}[(b-a)(d-c)\odot f(x,y)]\Bigg)$$
$$\leq \sum_{j=1}^{n}\sum_{i=1}^{n}(x_i-x_{i-1})(y_j-y_{j-1})\,\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}$$
$$\cdot\left(f,\sqrt{(x_i-x_{i-1})^2+(y_j-y_{j-1})^2}\right)$$
$$=(x-a)(y-c)\,\omega_{\overline{ax}\times\overline{cy}}+(b-x)(y-c)\,\omega_{\overline{xb}\times\overline{cy}}$$
$$+(x-a)(d-y)\,\omega_{\overline{ax}\times\overline{yd}}+(b-x)(d-y)\,\omega_{\overline{xb}\times\overline{yd}}. \tag{26}$$

(ii) Taking that $n=2$, $x_1=x$, $\xi_1=u$, $\xi_2=v$, $y_1=y$, $\eta_1=\alpha$, and $\eta_2=\beta$ in Theorem 17, we obtain the required inequality. Indeed,

$$D\Bigg((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,$$
$$\sum_{j=1}^{2}\sum_{i=1}^{2}(x_i-x_{i-1})(y_j-y_{j-1})\odot f(\xi_i,\eta_j)\Bigg)$$
$$\leq D\Bigg((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,$$
$$\sum_{j=1}^{2}[(x-a)(y_j-y_{j-1})\odot f(u,\eta_j)$$
$$\oplus (b-x)(y_j-y_{j-1})\odot f(v,\eta_j)]\Bigg)$$
$$=D\Bigg((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,$$
$$[(x-a)(y-c)\odot f(u,\alpha)$$
$$\oplus (x-a)(d-y)\odot f(u,\beta)$$
$$\oplus (b-x)(y-c)\odot f(v,\alpha)$$
$$\oplus (b-x)(d-y)\odot f(v,\beta)]\Bigg)$$
$$\leq \sum_{j=1}^{2}\sum_{i=1}^{2}(x_i-x_{i-1})(y_j-y_{j-1})\,\omega_{\overline{x_{i-1},x_i}\times\overline{y_{j-1},y_j}}$$
$$=(x-a)(y-c)\,\omega_{\overline{ax}\times\overline{cy}}+(b-x)(y-c)\,\omega_{\overline{xb}\times\overline{cy}}$$
$$+(x-a)(d-y)\,\omega_{\overline{ax}\times\overline{yd}}+(b-x)(d-y)\,\omega_{\overline{xb}\times\overline{yd}}. \tag{27}$$

(iii) Considering $n = 4$ and performing the similar way in part (ii), it is obvious that the inequality in previous theorem becomes the inequality stated above. □

Corollary 19. *Let $f : [a,b] \times [c,d] \to R_F$ be a two-dimensional Henstock integrable, bounded mapping. Then, the following inequalities hold:*

$$(i)\ D\left((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,\ (b-a)(d-c)\odot f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right) \le (b-a)(d-c)\,w_{[a,b]\times[c,d]}\left(f,\frac{(b-a)(d-c)}{4}\right),$$

$$(ii)\ D\left((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,\ \frac{(b-a)(d-c)}{4}\odot\left[f(a,c)\oplus f(a,d)\oplus f(b,c)\oplus f(b,d)\right]\right) \le (b-a)(d-c)\,w_{[a,b]\times[c,d]}\left(f,\frac{(b-a)(d-c)}{4}\right),$$

$$(iii)\ D\left((FH)\int_a^b f(s,t)\,ds\,dt,\ \frac{(b-a)(d-c)}{36}\odot\left[f(a,c)\oplus f(a,d)\oplus 4\odot f\left(a,\frac{c+d}{2}\right)\oplus 4\odot f\left(\frac{a+b}{2},c\right)\oplus 16\odot f\left(\frac{a+b}{2},\frac{a+c}{2}\right)\oplus 4\odot f\left(\frac{a+b}{2},d\right)\oplus 4\odot f\left(b,\frac{c+d}{2}\right)\oplus f(b,c)\oplus f(b,d)\right]\right) \le (b-a)(d-c)\,w_{[a,b]\times[c,d]}\left(f,\frac{(b-a)(d-c)}{36}\right). \tag{28}$$

Proof. (i) If we take $x = (a+b)/2$ and $y = (c+d)/2$ in the assertion (i) of Corollary 18, we obtain the required inequality. In other words, we have

$$\begin{aligned} &D\left((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,\ (b-a)(d-c)\odot f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right)\\ &\le \left(\frac{a+b}{2}-a\right)\left(\frac{c+d}{2}-c\right)\omega_{\overline{a((a+b)/2)}\times\overline{c((c+d)/2)}}\\ &\quad+\left(b-\frac{a+b}{2}\right)\left(\frac{c+d}{2}-c\right)\omega_{\overline{((a+b)/2)b}\times\overline{c((c+d)/2)}}\\ &\quad+\left(\frac{a+b}{2}-a\right)\left(d-\frac{c+d}{2}\right)\omega_{\overline{a((a+b)/2)}\times\overline{((c+d)/2)d}}\\ &\quad+\left(b-\frac{a+b}{2}\right)\left(d-\frac{c+d}{2}\right)\omega_{\overline{((a+b)/2)b}\times\overline{((c+d)/2)d}}\\ &=\frac{(b-a)(d-c)}{4}\times\left[\omega_{\overline{a((a+b)/2)}\times\overline{c((c+d)/2)}}+\omega_{\overline{((a+b)/2)b}\times\overline{c((c+d)/2)}}+\omega_{\overline{a((a+b)/2)}\times\overline{((c+d)/2)d}}+\omega_{\overline{((a+b)/2)b}\times\overline{((c+d)/2)d}}\right]\\ &\le (b-a)(d-c)\,\omega_{[a,b]\times[c,d]}\left(f,\frac{(b-a)(d-c)}{4}\right). \end{aligned} \tag{29}$$

(ii) Taking $x = (a+b)/2$, $y = (c+d)/2$, $u = a$, $v = b$, $\alpha = c$, and $\beta = d$ in the assertion (ii) of the previous corollary, we obtain

$$\begin{aligned} &D\left((FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt,\ \left(\frac{a+b}{2}-a\right)\left(\frac{c+d}{2}-c\right)\odot f(a,c)\oplus\left(\frac{a+b}{2}-a\right)\left(d-\frac{c+d}{2}\right)\odot f(a,d)\oplus\left(b-\frac{a+b}{2}\right)\left(\frac{c+d}{2}-c\right)\odot f(b,c)\oplus\left(b-\frac{a+b}{2}\right)\left(d-\frac{c+d}{2}\right)\odot f(a,d)\right)\\ &=\frac{(b-a)(d-c)}{4}\times\left[\omega_{\overline{a((a+b)/2)}\times\overline{c((c+d)/2)}}+\omega_{\overline{((a+b)/2)b}\times\overline{c((c+d)/2)}}+\omega_{\overline{a((a+b)/2)}\times\overline{((c+d)/2)d}}+\omega_{\overline{((a+b)/2)b}\times\overline{((c+d)/2)d}}\right]\\ &\le (b-a)(d-c)\,\omega_{[a,b]\times[c,d]}\left(f,\frac{(b-a)(d-c)}{4}\right). \end{aligned} \tag{30}$$

(iii) It is easy to see that the inequality follows from the corresponding assertion (iii) of the previous corollary by

taking $n = 4$, $\alpha = (5a + b)/6$, $\beta = (a + 5b)/6$, $u = a$, $v = (a + b)/2$, $w = b$, $\theta = (5c + d)/6$, $\gamma = (c + 5d)/6$, $r = c$, $p = (c + d)/2$, and $z = d$. Indeed, we have

$$\begin{aligned} D\Bigg(& (FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt, \\ & \frac{(b-a)(d-c)}{36} \\ & \odot\Bigg[f(a,c)\oplus f(a,d)\oplus 4\odot f\left(a,\frac{c+d}{2}\right) \\ & \oplus 4\odot f\left(\frac{a+b}{2},c\right) \\ & \oplus 16\odot f\left(\frac{a+b}{2},\frac{a+c}{2}\right)\oplus 4\odot f\left(\frac{a+b}{2},d\right) \\ & \oplus 4\odot f\left(b,\frac{c+d}{2}\right)\oplus f(b,c)\oplus f(b,d)\Bigg]\Bigg) \\ \le{} & (b-a)(d-c)\,\omega_{[a,b]\times[c,d]}\left(f,\frac{(b-a)(d-c)}{36}\right). \end{aligned} \tag{31}$$

□

The next corollaries present simpler error estimation for the inequality stated in Theorem 17.

Corollary 20. *Let $f : [a,b]\times[c,d] \to R_F$ be a two-dimensional Henstock integrable, bounded mapping. Then, for any divisions $\Delta_x^n : a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta_y^n : c = y_0 < y_1 < \cdots < y_n = d$, $\xi_i \in [x_{i-1}, x_i]$ and $\eta_i \in [y_{j-1}, y_j]$, $i = 1,\ldots,n$; $j = 1,\ldots,n$, one has*

$$\begin{aligned} D\Bigg(& (FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt, \\ & \sum_{j=1}^n\sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})\odot f(\xi_i,\eta_j)\Bigg) \\ & \le (b-a)(d-c)\,\omega_{[a,b]\times[c,d]}\left(f, v(\Delta_{xy})\right), \end{aligned} \tag{32}$$

where $v(\Delta_{xy}) = \max_{i,j=1,\ldots,n}\{(x_i - x_{i-1})(y_j - y_{j-1})\}$ is the norm of the divisions Δ_x^n and Δ_y^n.

Proof. Considering Theorem 17 and parts (i), (ii) of Theorem 13 and by regarding the definition of $\bigvee(f)$, we infer that

$$\begin{aligned} D\Bigg(& (FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt, \\ & \sum_{j=1}^n\sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})\odot f(\xi_i,\eta_j)\Bigg) \\ & \le \sum_{j=1}^n\sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})\,\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]} \\ & \cdot\left(f, \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}\right) \\ & \le \sum_{j=1}^n\sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})\,\omega_{[a,b]\times[c,d]}\left(f, v(\Delta_{xy})\right) \\ & = (b-a)(c-d)\,\omega_{[a,b]\times[c,d]}\left(f, v(\Delta_{xy})\right). \end{aligned} \tag{33}$$

□

Corollary 21. *Let $f : [a,b]\times[c,d] \to R_F$ be a two-dimensional Henstock integrable, bounded mapping. Then, for any divisions $\Delta_x^n : a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta_y^n : c = y_0 < y_1 < \cdots < y_n = d$, $\xi_i \in [x_{i-1}, x_i]$ and $\eta_i \in [y_{j-1}, y_j]$, $i = 1,\ldots,n$; $j = 1,\ldots,n$, one has*

$$\begin{aligned} D\Bigg(& (FH)\int_c^d (FH)\int_a^b f(s,t)\,ds\,dt, \\ & \sum_{j=1}^n\sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})\odot f(\xi_i,\eta_j)\Bigg) \\ & \le v(\Delta_{xy})\sum_{j=1}^n\sum_{i=1}^n \omega_{[a,b]\times[c,d]} \\ & \cdot\left(f, \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}\right). \end{aligned} \tag{34}$$

Proof. Since $v(\Delta_{xy})$ is the least upper bound of partitions Δ_x^n and Δ_y^n, we conclude that $(x_i - x_{i-1})(y_j - y_{j-1}) \le v(\Delta_{xy})$ for any $i, j = \overline{1,n}$. Hence, the required inequality holds. □

Remark 22. If $f : [a,b]\times[c,d] \to R_F$ is a two-dimensional Riemann integrable function, it is also Henstock integrable function. Therefore, the above quadrature rules hold for Riemann integrable function too.

Theorem 23. *Let $f : [a,b]\times[c,d] \to R_F$ be a mapping of bounded variation. Then, for any divisions $\Delta_x^n : a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta_y^n : c = y_0 < y_1 < \cdots < y_n = d$, $\xi_i \in [x_{i-1}, x_i]$ and $\eta_i \in [y_{j-1}, y_j]$, $i = 1,\ldots,n$; $j = 1,\ldots,n$, one has*

$$\begin{aligned} D\Bigg(& (FR)\int_c^d (FR)\int_a^b f(s,t)\,ds\,dt, \\ & \sum_{j=1}^n\sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})\odot f(\xi_i,\eta_j)\Bigg) \\ & \le v(\Delta_{xy})\cdot\bigvee(f). \end{aligned} \tag{35}$$

Proof. If we define $\varphi_{xy} : [a,b]\times[c,d] \to R_+$ such that $\varphi_{xy}(s,t) = D(f(s,t), f(x,y))$ for any $(x,y) \in [a,b]\times[c,d]$, we see that φ is of bounded variation and we have

$$\bigvee \varphi_{xy}(s,t) \le \bigvee(f), \quad (x,y) \in [a,b]\otimes[c,d]; \tag{36}$$

in other words,

$$\begin{aligned} &V_{\Delta^n_{xy}}\left(D\left(f\left(s,t\right),f\left(x,y\right)\right)\right)\\ &=\sum_{k=0}^{n-1}\sum_{m=0}^{n-1}\left|D\left(f\left(t_{k+1},s_{m+1}\right),f\left(x,y\right)\right)\right.\\ &\qquad\left.-D\left(f\left(t_k,s_m\right),f\left(x,y\right)\right)\right|\\ &\le\sum_{k=0}^{n-1}\sum_{m=0}^{n-1}\left(D\left(f\left(t_{k+1},s_{m+1}\right),f\left(x,y\right)\right)\right)=V_{\Delta^n_{xy}}\left(f\right). \end{aligned} \tag{37}$$

Considering Theorem 13, Theorem 17, Corollary 21, and [18] and since any real valued function of bounded variation is Lebesgue integrable, we observe that

$$\begin{aligned} &\sum_{j=1}^{n}\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}\\ &\quad\cdot\left(f,\sqrt{\left(x_i-x_{i-1}\right)^2+\left(y_j-y_{j-1}\right)^2}\right)\\ &\le v\left(\Delta_{xy}\right)\sum_{j=1}^{n}\sum_{i=1}^{n}\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}\\ &\quad\cdot\left(f,\sqrt{\left(x_i-x_{i-1}\right)^2+\left(y_j-y_{j-1}\right)^2}\right)\\ &=v\left(\Delta_{xy}\right)\sum_{j=1}^{n}\sum_{i=1}^{n}\bigvee_{[x_{i-1},x_i]}^{[y_{j-1},y_j]}\left(f\right)\\ &=v\left(\Delta_{xy}\right)\cdot\bigvee\left(f\right). \end{aligned} \tag{38}$$

□

Theorem 24. *If $f:[a,b]\times[c,d]\to R_F$ is L-Lipschitz mapping, then, for any divisions $\Delta^n_x: a=x_0<x_1<\cdots<x_n=b$ and $\Delta^n_y: c=y_0<y_1<\cdots<y_n=d$, $\xi_i\in[x_{i-1},x_i]$ and $\eta_i\in[y_{j-1},y_j]$, $i=1,\ldots,n$; $j=1,\ldots,n$, one has*

$$\begin{aligned} &D\left(\left(FR\right)\int_c^d\left(FR\right)\int_a^b f\left(s,t\right)ds\,dt,\right.\\ &\qquad\left.\sum_{j=1}^{n}\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\odot f\left(\xi_i,\eta_j\right)\right)\\ &\le L\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\left(x_i-x_{i-1}\right)^2\left(y_j-y_{j-1}\right)^2\right). \end{aligned} \tag{39}$$

Proof. Analogous to the proof of Theorem 17 and by definition of *L-Lipschitz* mapping, we infer that

$$\begin{aligned} &D\left(\left(FR\right)\int_c^d\left(FR\right)\int_a^b f\left(s,t\right)ds\,dt,\right.\\ &\qquad\left.\sum_{j=1}^{n}\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\odot f\left(\xi_i,\eta_j\right)\right)\\ &\le\sum_{j=1}^{n}\sum_{i=1}^{n}\left(L\right)\int_{y_{j-1}}^{y_j}\left(L\right)\int_{x_{i-1}}^{x_i}D\left(f\left(s,t\right),f\left(\xi_i,\eta_j\right)\right)ds\,dt\\ &\le L\sum_{j=1}^{n}\sum_{i=1}^{n}\left(L\right)\int_{y_{j-1}}^{y_j}\left(L\right)\int_{x_{i-1}}^{x_i}\sqrt{\left(s-\xi_i\right)^2+\left(t-\eta_j\right)^2}\,ds\,dt\\ &\le L\sum_{j=1}^{n}\sum_{i=1}^{n}\left(L\right)\int_{y_{j-1}}^{y_j}\left(L\right)\\ &\qquad\cdot\int_{x_{i-1}}^{x_i}\left(\left(x_i-x_{i-1}\right)^2\right.\\ &\qquad\left.+\left(y_j-y_{j-1}\right)^2\right)^{1/2}ds\,dt\\ &=L\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\left(x_i-x_{i-1}\right)^2+\left(y_j-y_{j-1}\right)^2\right). \end{aligned} \tag{40}$$

□

4. 2D Fuzzy Fredholm Integral Equations

Here, we consider the two-dimensional fuzzy Fredholm integral equations as follows:

$$F\left(s,t\right)=f\left(s,t\right)\oplus\lambda\odot\int_c^d\int_a^b K\left(s,t,x,y\right)\odot F\left(x,y\right)dx\,dy, \tag{41}$$

where $\lambda>0$, $K(s,t,x,y)$ is an arbitrary positive kernel on $[a,b]\times[c,d]\times[a,b]\times[c,d]$ and $f:[a,b]\times[c,d]\to R_F$. We assume that K is continuous, and therefore it is uniformly continuous with respect to (s,t). This property implies that there exists $M>0$ such that

$$M=\max_{\substack{a\le s,x\le b\\ c\le t,y\le d}}\left|K\left(s,t,x,y\right)\right|. \tag{42}$$

Now, we will prove the existence and uniqueness of the solution of (41) by the method of successive approximations. Let $X=\{f:[a,b]\times[c,d]\to R_F; f \text{ is continuous}\}$ be the space of two-dimensional fuzzy continuous functions with the metric

$$D^*\left(f,g\right)=\sup_{\substack{a\le s\le b\\ c\le t\le d}}D\left(f\left(s,t\right),g\left(s,t\right)\right) \tag{43}$$

that is called the uniform distance between two-dimensional fuzzy-number-valued functions. We define the operator $A : X \to X$ by

$$A(F)(s,t) = f(s,t) \oplus \lambda \odot \int_c^d \int_a^b K(s,t,x,y) \odot F(x,y)\,dx\,dy, \quad \forall (s,t) \in [a,b] \times [c,d], \quad \forall f \in X. \tag{44}$$

Sufficient conditions for the existence of a unique solution of (41) are given in the following result.

Theorem 25. *Let $K(s,t,x,y)$ be continuous and positive for $a \le s, x \le b$, and $c \le t, y \le d$, and let $f : [a,b] \times [c,d] \to R_F$ be continuous on $[a,b] \times [c,d]$. If $B = \lambda M(b-a)(d-c) < 1$, then the iterative procedure*

$$F_0(s,t) = f(s,t),$$
$$F_m(s,t) = f(s,t) \oplus \lambda \odot (FR) \cdot \int_c^d (FR) \int_a^b K(s,t,x,y) \odot F_{m-1}(x,y)\,dx\,dy, \quad m \ge 1, \tag{45}$$

converges to the unique solution F^* of (41).

Moreover, the following error bound holds:

$$D^*(F^*, F_m) \le \frac{B^{m+1}}{1-B} M_1, \tag{46}$$

where

$$M_1 = \sup_{\substack{a \le s \le b \\ c \le t \le d}} \|F(s,t)\|_F. \tag{47}$$

Proof. To prove this theorem, we investigate the conditions of the Banach fixed point principle. We first show that A maps X into X (i.e., $A(X) \subset X$). To the end, we show that the operator A is uniformly continuous. Since f is continuous on compact set of $[a,b] \times [c,d]$, we deduce that it is uniformly continuous, and hence for $\varepsilon_1 > 0$ exists $\delta_1 > 0$ such that

$$D(f(s_1,t_1), f(s_2,t_2)) < \varepsilon_1 \quad \text{whenever } \sqrt{(t_2-t_1)^2 + (s_2-s_1)^2} < \delta_1, \quad \forall s_1, s_2 \in [a,b], \quad \forall t_1, t_2 \in [c,d]. \tag{48}$$

As mentioned above, K also is uniformly continuous; thus, for $\varepsilon_2 > 0$ exists $\delta_2 > 0$ such that

$$|K(s_1,t_1,x,y) - K(s_2,t_2,x,y)| < \varepsilon_2 \quad \text{whenever } \sqrt{(t_2-t_1)^2 + (s_2-s_1)^2} < \delta_2, \quad \forall s_1, s_2 \in [a,b], \quad \forall t_1, t_2 \in [c,d]. \tag{49}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, $s_1, s_2 \in [a,b]$, and $t_1, t_2 \in [c,d]$, with $\sqrt{(t_2-t_1)^2 + (s_2-s_1)^2} < \delta$. According to Definition 3 and Theorem 9, we obtain

$$\begin{aligned}
&D(A(F)(s_1,t_1), A(F)(s_2,t_2)) \\
&\quad \le D(f(s_1,t_1), f(s_2,t_2)) \\
&\qquad + D\Bigg(\lambda \odot (FR) \int_c^d (FR) \int_a^b K(s_1,t_1,x,y) \odot F(x,y)\,dx\,dy, \\
&\qquad\qquad \lambda \odot (FR) \int_c^d (FR) \int_a^b K(s_2,t_2,x,y) \odot F(x,y)\,dx\,dy\Bigg) \\
&\quad \le \varepsilon_1 + \lambda \odot (FR) \int_c^d D\Bigg((FR) \int_a^b K(s_1,t_1,x,y) \odot F(x,y)\,dx, \\
&\qquad\qquad (FR) \int_a^b K(s_2,t_2,x,y) \odot F(x,y)\,dx\Bigg)\,dy \\
&\quad \le \varepsilon_1 + \lambda \odot (FR) \cdot \int_c^d (FR) \int_a^b D(K(s_1,t_1,x,y) \odot F(x,y), \\
&\qquad\qquad K(s_2,t_2,x,y) \odot F(x,y))\,dx\,dy \\
&\quad \le \varepsilon_1 + \lambda |K(s_1,t_1,x,y) - K(s_2,t_2,x,y)| \\
&\qquad \odot (FR) \int_c^d (FR) \int_a^b (F(x,y), \widetilde{0})\,dx\,dy \\
&\quad \le \varepsilon_1 + \lambda\varepsilon_2 \odot (FR) \int_c^d (FR) \int_a^b \|F(x,y)\|_F\,dx\,dy \\
&\quad \le \varepsilon_1 + \lambda\varepsilon_2 (b-a)(d-c) \|F(x,y)\|_F \\
&\quad \le \varepsilon_1 + \lambda (b-a)(d-c) M_1 \varepsilon_2 \\
&\qquad \text{where } M_1 = \sup_{\substack{a \le s \le b \\ c \le t \le d}} \|F(s,t)\|_F,
\end{aligned} \tag{50}$$

and by choosing $\varepsilon_1 = \varepsilon/2$ and $\varepsilon_2 = (1/2M_1\lambda(b-a)(d-c))\varepsilon$, we derive

$$D(A(F)(s_1,t_1), A(F)(s_2,t_2)) \le \varepsilon. \tag{51}$$

This shows that $A(F)$ is uniformly continuous for any $F \in X$ and so continuous on $[a,b] \times [c,d]$, and hence $A(X) \subset X$.

Now, we prove that the operator A is a contraction map. So, for $F_1, F_2 \in X$, $s \in [a,b]$, and $t \in [c,d]$, we have

$$\begin{aligned}
&D\left(A\left(F_1\right)(s,t), A\left(F_2\right)(s,t)\right) \\
&\quad \le D\left(f(s,t), f(s,t)\right) \\
&\qquad + D\Bigg(\lambda \odot (FR) \int_c^d (FR) \int_a^b K(s,t,x,y) \\
&\qquad\qquad \odot F_1(x,y)\, dx\, dy, \\
&\qquad\qquad \lambda \odot (FR) \int_c^d (FR) \int_a^b K(s,t,x,y) \\
&\qquad\qquad \odot F_2(x,y)\, dx\, dy\Bigg) \\
&\quad \le \lambda \odot (FR) \int_c^d (FR) \int_a^b D\left(K(s,t,x,y) \odot F_1(x,y),\right. \\
&\qquad\qquad \left. K(s,t,x,y) \odot F_2(x,y)\right) dx\, dy \\
&\quad = \lambda \left|K(s,t,x,y)\right| \odot (FR) \\
&\qquad \cdot \int_c^d (FR) \int_a^b D\left(F_1(x,y), F_2(x,y)\right) dx\, dy \\
&\quad \le \lambda M \odot (FR) \int_c^d (FR) \int_a^b D\left(F_1(x,y),\right. \\
&\qquad\qquad \left. F_2(x,y)\right) dx\, dy \\
&\quad \le \lambda M \odot (FR) \int_c^d (FR) \int_a^b D^*\left(F_1, F_2\right) dx\, dy \\
&\quad = \lambda M (b-a)(d-c) D^*\left(F_1, F_2\right) = B D^*\left(F_1, F_2\right).
\end{aligned} \tag{52}$$

Therefore, we obtained

$$D^*\left(A\left(F_1\right)(s,t), A\left(F_2\right)(s,t)\right) \le B D^*\left(F_1, F_2\right). \tag{53}$$

Since $B < 1$, the operator A is a contraction on Banach space (X, D^*). Consequently, Banach's fixed point principle implies that (41) has a unique solution F^* in X and we also have

$$\begin{aligned}
&D\left(F^*(s,t), F_m(s,t)\right) \\
&\quad \le D^*\left(F^*, F_m\right) \\
&\quad \le \lambda M (b-a)(d-c) D^*\left(F^*, F_{m-1}\right) \\
&\quad = B D^*\left(F^*, F_{m-1}\right) \le B D^*\left(F^*, F_m\right) + B D^*\left(F_{m-1}, F_m\right) \\
&\quad \le B D^*\left(F^*, F_m\right) + B^m D^*\left(F_0, F_1\right);
\end{aligned} \tag{54}$$

therefore,

$$D^*\left(F^*, F_m\right) \le \frac{B^m}{1-B} D^*\left(F_0, F_1\right); \tag{55}$$

on the other hand,

$$\begin{aligned}
&D^*\left(F_0, F_1\right) \\
&\quad = \sup_{\substack{a \le s \le b \\ c \le t \le d}} D\Bigg(f(s,t) \oplus \widetilde{0}, f(s,t) \oplus \lambda \otimes (FR) \\
&\qquad\qquad \cdot \int_c^d (FR) \int_a^b K(s,t,x,y) \\
&\qquad\qquad \odot F_0(x,y)\, dx\, dy\Bigg) \\
&\quad \le \sup_{\substack{a \le s \le b \\ c \le t \le d}} \lambda \odot (FR) \\
&\qquad \cdot \int_c^d (FR) \int_a^b D\left(\widetilde{0}, K(s,t,x,y) \odot F_0(x,y)\, dx\, dy\right) \\
&\quad \le M\lambda \odot (FR) \int_c^d (FR) \int_a^b \sup_{\substack{a \le s \le b \\ c \le t \le d}} D\left(\widetilde{0}, F_0(x,y)\right) dx\, dy \\
&\quad = \lambda M (b-a)(d-c) M_1 = M_1 B,
\end{aligned} \tag{56}$$

so by (55) and (56), we obtained inequality (46), which completes the proof. □

Now, we introduce a numerical method to solve (41). We consider (41) with continuous kernel $K(s,t,x,y)$ having positive sign on $[a,b] \times [c,d] \times [a,b] \times [c,d]$ and uniform partitions

$$\begin{aligned}
D_x &: a = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = b, \\
D_y &: b = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = d,
\end{aligned} \tag{57}$$

with $s_i = a + ih$, $t_j = c + jh'$, where $h = (b-a)/n$, $h' = (d-c)/n$. Then, the following iterative procedure gives the approximate solution of (41) in point (s,t):

$$\begin{aligned}
u_0(s,t) &= f(s,t), \\
u_m(s,t) &= f(s,t) \oplus \frac{\lambda h h'}{4} \\
&\quad \odot \Bigg[\Bigg(K(s,t,s_0,t_0) \odot u_{m-1}(s_0,t_0) \\
&\qquad \oplus K(s,t,s_0,t_n) \odot u_{m-1}(s_0,t_n) \\
&\qquad \oplus K(s,t,s_n,t_0) \odot u_{m-1}(s_n,t_0) \\
&\qquad \oplus K(s,t,s_n,t_n) \\
&\qquad \odot u_{m-1}(s_n,t_n)\Bigg)
\end{aligned}$$

$$\oplus 2\left(\sum_{i=0}^{n-1} K(s,t,s_i,t_0) \odot u_{m-1}(s_i,t_0) \oplus \sum_{j=0}^{n-1} K(s,t,s_0,t_j) \odot u_{m-1}(s_0,t_j)\right)$$
$$\oplus 4\left(\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} K(s,t,s_i,t_j) \odot u_{m-1}(s_i,t_j)\right)\Bigg]. \tag{58}$$

The above recursive relation can be written as follows:

$$\begin{aligned} u_0(s,t) &= f(s,t), \\ u_m(s,t) &= f(s,t) \oplus \frac{\lambda hh'}{4} \\ &\quad \odot \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big(K(s,t,s_i,t_j) \odot u_{m-1}(s_i,t_j) \\ &\quad \oplus K(s,t,s_i,t_{j+1}) \\ &\quad \odot u_{m-1}(s_i,t_{j+1}) \\ &\quad \oplus K(s,t,s_{i+1},t_j) \\ &\quad \odot u_{m-1}(s_{i+1},t_j) \\ &\quad \oplus K(s,t,s_{i+1},t_{j+1}) \\ &\quad \odot u_{m-1}(s_{i+1},t_{j+1})\Big). \end{aligned} \tag{59}$$

4.1. Error Estimation. Here, we obtain an error estimate between the exact solution and the approximate solution for the given fuzzy Fredholm integral equation (41).

Theorem 26. *Consider the 2DFFLIE2* (41) *with continuous kernel* $K(s,t,x,y)$ *having positive sign on* $[a,b]\times[c,d]\times[a,b]\times[c,d]$ *and suppose that* f *is continuous on* $[a,b]\times[c,d]$. *If* $B = \lambda M(b-a)(d-c) < 1$, *where* $M = \max_{\substack{a\le s,x\le b\\ c\le t,y\le d}} |K(s,t,x,y)|$, *then the iterative procedure* (59) *converges to the unique solution of* (41), F, *and the following error estimate holds true:*

$$D^*(F,u_m) \le \left(\frac{B^{m+1}}{1-m}\right)\Gamma_0 + \left(\frac{B}{4(1-B)}\right)\omega_{[a,b]\times[c,d]}(f,hh') + \left(\frac{\mu B^2 + 4\tau B}{4M(1-B)}\right)\omega_{st}(K,h+h'), \tag{60}$$

where

$$\omega_{st}(K,\delta) = \sup\left\{\left|K(s_1,t_1,x,y) - K(s_2,t_2,x,y)\right|; \sqrt{(s_2-s_1)^2+(t_2-t_1)^2} \le \delta\right\} \tag{61}$$

$$\begin{aligned} &\forall \delta \ge 0, \quad a \le s_1, s_2 \le b, \quad c \le t_1, t_2 \le d, \\ &M_k = \sup_{(s,t)\in[a,b]\times[c,d]} \|u_k(s,t)\|_F, \\ &\Gamma_k = \sup_{(s,t)\in[a,b]\times[c,d]} \|F_k(s,t)\|_F, \end{aligned} \tag{62}$$

$$\begin{aligned} &\tau = \max_{i=0,1,\dots,m-1}\{M_i\}, \\ &\mu = \max_{i=0,1,\dots,m-2}\{\Gamma_i\}. \end{aligned} \tag{63}$$

Proof. Considering iterative procedure (59), for all $(s,t) \in [a,b]\times[c,d]$, we have

$$\begin{aligned} &D(F_1(s,t), u_1(s,t)) \\ &= D(f(s,t), f(s,t)) \\ &\quad + D\Bigg(\lambda \odot (FR) \cdot \int_c^d (FR) \int_a^b K(s,t,x,y) \odot F_0(x,y)\,dx\,dy, \\ &\quad \frac{\lambda hh'}{4} \odot \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big[K(s,t,s_i,t_j) \odot F_0(s_i,t_j) \\ &\quad \oplus K(s,t,s_i,t_{j+1}) \odot F_0(s_i,t_{j+1}) \\ &\quad \oplus K(s,t,s_{i+1},t_j) \odot F_0(s_{i+1},t_j) \\ &\quad \oplus K(s,t,s_{i+1},t_{j+1}) \\ &\quad \odot F_0(s_{i+1},t_{j+1})\Big]\Bigg) \\ &= D\Bigg(\lambda \odot \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (FR) \cdot \int_{s_i}^{s_{i+1}} (FR) \int_{t_j}^{t_{j+1}} K(s,t,x,y) \odot f(x,y)\,dx\,dy, \end{aligned}$$

$$\begin{aligned}
&\frac{\lambda hh'}{4} \odot \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big[K\left(s,t,s_i,t_j\right) \odot f\left(s_i,t_j\right) \\
&\qquad \oplus K\left(s,t,s_i,t_{j+1}\right) \odot f\left(s_i,t_{j+1}\right) \\
&\qquad \oplus K\left(s,t,s_{i+1},t_j\right) \odot f\left(s_{i+1},t_j\right) \\
&\qquad \oplus K\left(s,t,s_{i+1},t_{j+1}\right) \\
&\qquad \odot f\left(s_{i+1},t_{j+1}\right)\Big]\Bigg) \\
&\le \lambda \odot \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} D\Bigg((FR)\int_{s_i}^{s_{i+1}} (FR)\int_{t_j}^{t_{j+1}} K(s,t,x,y) \\
&\qquad \odot f(x,y)\,dx\,dy, \\
&\qquad \frac{hh'}{4} \odot \Big[K\left(s,t,s_i,t_j\right) \odot f\left(s_i,t_j\right) \\
&\qquad \oplus K\left(s,t,s_i,t_{j+1}\right) \odot f\left(s_i,t_{j+1}\right) \\
&\qquad \oplus K\left(s,t,s_{i+1},t_j\right) \odot f\left(s_{i+1},t_j\right) \\
&\qquad \oplus K\left(s,t,s_{i+1},t_{j+1}\right) \\
&\qquad \odot f\left(s_{i+1},t_{j+1}\right)\Big]\Bigg) \\
&\le \lambda \odot \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} D\Bigg((FR)\int_{s_i}^{s_{i+1}} (FR)\int_{t_j}^{t_{j+1}} K(s,t,x,y) \\
&\qquad \odot f(x,y)\,dx\,dy, \\
&\qquad \frac{hh'}{4} \odot \Big[K(s,t,x,y) \odot f\left(s_i,t_j\right) \\
&\qquad \oplus K(s,t,x,y) \odot f\left(s_i,t_{j+1}\right) \\
&\qquad \oplus K(s,t,x,y) \odot f\left(s_{i+1},t_j\right) \\
&\qquad \oplus K(s,t,x,y) \\
&\qquad \odot f\left(s_{i+1},t_{j+1}\right)\Big]\Bigg) \\
&+ \lambda \odot \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} D\Bigg(\frac{hh'}{4}\Big[K(s,t,x,y) \odot f\left(s_i,t_j\right) \\
&\qquad \oplus K(s,t,x,y) \odot f\left(s_i,t_{j+1}\right) \\
&\qquad \oplus K(s,t,x,y) \odot f\left(s_{i+1},t_j\right) \\
&\qquad \oplus K(s,t,x,y) \odot f\left(s_{i+1},t_{j+1}\right)\Big], \\
&\qquad \frac{hh'}{4} \odot \Big[K\left(s,t,s_i,t_j\right) \odot f\left(s_i,t_j\right) \\
&\qquad \oplus K\left(s,t,s_i,t_{j+1}\right) \odot f\left(s_i,t_{j+1}\right) \\
&\qquad \oplus K\left(s,t,s_{i+1},t_j\right) \odot f\left(s_{i+1},t_j\right) \\
&\qquad \oplus K\left(s,t,s_{i+1},t_{j+1}\right) \\
&\qquad \odot f\left(s_{i+1},t_{j+1}\right)\Big]\Bigg) \\
&\le \lambda \odot \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \left|K(s,t,x,y)\right| \\
&\times D\Bigg((FR)\int_{s_i}^{s_{i+1}} (FR)\int_{t_j}^{t_{j+1}} f(x,y)\,dx\,dy, \\
&\qquad \frac{hh'}{4} \odot \Big[f\left(s_i,t_j\right) \oplus f\left(s_i,t_{j+1}\right) \\
&\qquad \oplus f\left(s_{i+1},t_j\right) \oplus f\left(s_{i+1},t_{j+1}\right)\Big]\Bigg) \\
&+ \frac{\lambda hh'}{4} \odot \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big[D\left(K(s,t,x,y) \odot f\left(s_i,t_j\right),\right. \\
&\qquad \left. K\left(s,t,s_i,t_j\right) \odot f\left(s_i,t_j\right)\right) \\
&+ D\left(K(s,t,x,y) \odot f\left(s_i,t_{j+1}\right),\right. \\
&\qquad \left. K\left(s,t,s_i,t_{j+1}\right) \odot f\left(s_i,t_{j+1}\right)\right) \\
&+ D\left(K(s,t,x,y) \odot f\left(s_{i+1},t_j\right),\right. \\
&\qquad \left. K\left(s,t,s_{i+1},t_j\right) \odot f\left(s_{i+1},t_j\right)\right) \\
&+ D\left(K(s,t,x,y) \odot f\left(s_{i+1},t_{j+1}\right),\right. \\
&\qquad \left. K\left(s,t,s_{i+1},t_{j+1}\right) \odot f\left(s_{i+1},t_{j+1}\right)\right)\Big].
\end{aligned} \tag{64}$$

Using part (ii) of Corollary 19, part (v) of Definition 3, and part (i) of Theorem 13, we obtain

$$\begin{aligned}
&D\left(F_1(s,t), u_1(s,t)\right) \\
&\quad \le \frac{\lambda M hh'}{4} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \left(4\omega_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}\left(f, \frac{hh'}{4}\right)\right) \\
&\qquad + \frac{\lambda hh'}{4} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big[\left|K(s,t,x,y) - K\left(s,t,s_i,t_j\right)\right| \\
&\qquad \times D\left(f\left(s_i,t_j\right), \tilde{0}\right) \\
&\qquad + \left|K(s,t,x,y) - K\left(s,t,s_i,t_{j+1}\right)\right| \\
&\qquad \times D\left(f\left(s_i,t_{j+1}\right), \tilde{0}\right) \\
&\qquad + \left|K(s,t,x,y) - K\left(s,t,s_{i+1},t_j\right)\right| \\
&\qquad \times D\left(f\left(s_{i+1},t_j\right), \tilde{0}\right) \\
&\qquad + \left|K(s,t,x,y) - K\left(s,t,s_i,t_j\right)\right| \\
&\qquad \times D\left(f\left(s_{i+1},t_{j+1}\right), \tilde{0}\right)\Big].
\end{aligned} \tag{65}$$

By part (ii) of Theorem 6 and direct computation, it follows that

$$\begin{aligned}
&D\left(F_1(s,t), u_1(s,t)\right) \\
&\quad \le \frac{\lambda M(b-a)(d-c)}{4}\omega\left(f, hh'\right) \\
&\qquad + \lambda hh' \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left(\left|K(s,t,x,y) - K\left(s,t,s_i,t_j\right)\right|\right) \\
&\qquad \cdot \sup_{(s,t)\in[a,b]\times[c,d]} D\left(f(s,t), \tilde{0}\right) \\
&\quad \le \frac{\lambda M(b-a)(d-c)}{4}\omega\left(f, hh'\right) + \lambda(b-a)(d-c) \\
&\quad M_0\omega_{st}\left(K, h+h'\right) = \frac{B}{4}\omega\left(f, hh'\right) + \frac{B}{M}M_0\omega_{st}\left(K, h+h'\right);
\end{aligned} \tag{66}$$

therefore, we obtain

$$D\left(F_1(s,t), u_1(s,t)\right) \le \frac{B}{4}\omega\left(f, hh'\right) + \frac{B}{M}M_0\omega_{st}\left(K, h+h'\right). \tag{67}$$

Now, since $F_2(s,t) = f(s,t) + \lambda \odot (FR)\int_c^d (FR)\int_a^b K(s,t,x,y) \odot F_1(x,y)dx\,dy$, we infer that

$$\begin{aligned}
&D\left(F_2(s,t), u_2(s,t)\right) \\
&\quad = D\left(f(s,t), f(s,t)\right) \\
&\qquad + \lambda D\Bigg((FR)\int_c^d (FR)\int_a^b K(s,t,x,y)\odot F_1(x,y)\,dx\,dy, \\
&\qquad\qquad \frac{hh'}{4}\odot\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Big[K\left(s,t,s_i,t_j\right)\odot F_1\left(s_i,t_j\right) \\
&\qquad\qquad\qquad \oplus K\left(s,t,s_i,t_{j+1}\right)\odot F_1\left(s_i,t_{j+1}\right) \\
&\qquad\qquad\qquad \oplus K\left(s,t,s_{i+1},t_j\right)\odot F_1\left(s_{i+1},t_j\right) \\
&\qquad\qquad\qquad \oplus K\left(s,t,s_{i+1},t_{j+1}\right) \\
&\qquad\qquad\qquad \odot F_1\left(s_{i+1},t_{j+1}\right)\Big]\Bigg) \\
&\quad \le \lambda\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Bigg[D\Bigg((FR)\int_{s_i}^{s_{i+1}}(FR)\int_{t_j}^{t_{j+1}} K(s,t,x,y) \\
&\qquad\qquad\qquad \odot F_1(x,y)\,dx\,dy, \\
&\qquad\qquad \frac{hh'}{4}\odot K(s,t,x,y) \\
&\qquad\qquad \odot\Big[F_1\left(s_i,t_j\right)\oplus F_1\left(s_i,t_{j+1}\right)\oplus F_1\left(s_{i+1},t_j\right) \\
&\qquad\qquad\qquad \oplus F_1\left(s_{i+1},t_{j+1}\right)\Big]\Bigg) \\
&\qquad + \frac{hh'}{4}D\Bigg(K(s,t,x,y) \\
&\qquad\qquad \Big[F_1\left(s_i,t_j\right)\oplus F_1\left(s_i,t_{j+1}\right) \\
&\qquad\qquad\quad \oplus F_1\left(s_{i+1},t_j\right)\oplus F_1\left(s_{i+1},t_{j+1}\right)\Big], \\
&\qquad\qquad K(s,t,x,y) \\
&\qquad\qquad \times\Big[u_1\left(s_i,t_j\right) \\
&\qquad\qquad\quad \oplus u_1\left(s_i,t_{j+1}\right)\oplus u_1\left(s_{i+1},t_j\right) \\
&\qquad\qquad\quad \oplus u_1\left(s_{i+1},t_{j+1}\right)\Big]\Bigg) \\
&\qquad + \frac{hh'}{4}D\Bigg(K(s,t,x,y) \\
&\qquad\qquad \times\Big[u_1\left(s_i,t_j\right) \\
&\qquad\qquad\quad \oplus u_1\left(s_i,t_{j+1}\right)\oplus u_1\left(s_{i+1},t_j\right) \\
&\qquad\qquad\quad \oplus u_1\left(s_{i+1},t_{j+1}\right)\Big], \\
&\qquad\qquad \Big[K\left(s,t,s_i,t_j\right)\odot u_1\left(s_i,t_j\right) \\
&\qquad\qquad\quad \oplus K\left(s,t,s_i,t_{j+1}\right)\odot u_1\left(s_i,t_{j+1}\right) \\
&\qquad\qquad\quad \oplus K\left(s,t,s_{i+1},t_j\right)\odot u_1\left(s_{i+1},t_j\right) \\
&\qquad\qquad\quad \oplus K\left(s,t,s_{i+1},t_{j+1}\right) \\
&\qquad\qquad\quad \odot u_1\left(s_{i+1},t_{j+1}\right)\Big]\Bigg)\Bigg] \\
&\quad \le \frac{\lambda M(b-a)(d-c)}{4}\omega_{[a,b]\times[c,d]}\left(F_1, hh'\right) \\
&\qquad + \frac{\lambda M(b-a)(d-c)}{4}\Big[D\left(F_1\left(s_i,t_j\right), u_1\left(s_i,t_j\right)\right) \\
&\qquad\qquad + D\left(F_1\left(s_i,t_{j+1}\right), u_1\left(s_i,t_{j+1}\right)\right) \\
&\qquad\qquad + D\left(F_1\left(s_{i+1},t_j\right), u_1\left(s_{i+1},t_j\right)\right) \\
&\qquad\qquad + D\left(F_1\left(s_{i+1},t_{j+1}\right), \right. \\
&\qquad\qquad\qquad \left. u_1\left(s_{i+1},t_{j+1}\right)\right)\Big] \\
&\qquad + \lambda(b-a)(d-c)M_1\omega_{st}\left(K, h+h'\right);
\end{aligned} \tag{68}$$

therefore, we have

$$\begin{aligned} &D\left(F_2(s,t), u_2(s,t)\right) \\ &\quad \le \frac{B}{4}\omega_{[a,b]\times[c,d]}\left(F_1, hh'\right) \\ &\qquad + \frac{B}{4}\left[D\left(F_1\left(s_i,t_j\right), u_1\left(s_i,t_j\right)\right)\right. \\ &\qquad + D\left(F_1\left(s_i,t_{j+1}\right), u_1\left(s_i,t_{j+1}\right)\right) \\ &\qquad + D\left(F_1\left(s_{i+1},t_j\right), u_1\left(s_{i+1},t_j\right)\right) \\ &\qquad \left.+D\left(F_1\left(s_{i+1},t_{j+1}\right), u_1\left(s_{i+1},t_{j+1}\right)\right)\right] \\ &\qquad + \lambda(b-a)(d-c)M_1\omega_{st}\left(K, h+h'\right). \end{aligned} \tag{69}$$

By induction for $m \ge 3$, using (45), (46), (59), and (62), we see that

$$\begin{aligned} &D\left(F_m(s,t), u_m(s,t)\right) \\ &\quad \le \frac{B}{4}\omega_{[a,b]\times[c,d]}\left(F_1, hh'\right) \\ &\qquad + \frac{B}{4}\left[D\left(F_{m-1}\left(s_i,t_j\right), u_{m-1}\left(s_i,t_j\right)\right)\right. \\ &\qquad\quad + D\left(F_1\left(s_i,t_{j+1}\right), u_1\left(s_i,t_{j+1}\right)\right) \\ &\qquad\quad + D\left(F_{m-1}\left(s_{i+1},t_j\right), u_{m-1}\left(s_{i+1},t_j\right)\right) \\ &\qquad\quad \left.+ D\left(F_{m-1}\left(s_{i+1},t_{j+1}\right), u_{m-1}\left(s_{i+1},t_{j+1}\right)\right)\right] \\ &\qquad + \frac{B}{M}M_{m-1}\omega_{st}\left(K, h+h'\right); \end{aligned} \tag{70}$$

taking supremum for $(t,s) \in [a,b]\times[c,d]$ from (70), we conclude that

$$\begin{aligned} &D^*\left(F_m, u_m\right) \\ &\quad \le \frac{B}{4}\omega_{[a,b]\times[c,d]}\left(F_{m-1}, hh'\right) \\ &\qquad + BD^*\left(F_{m-1}, u_{m-1}\right) + \frac{B}{M}M_{m-1}\omega_{st}\left(K, h+h'\right), \\ &D^*\left(F_{m-1}, u_{m-1}\right) \\ &\quad \le \frac{B}{4}\omega_{[a,b]\times[c,d]}\left(F_{m-2}, hh'\right) \\ &\qquad + BD^*\left(F_{m-2}, u_{m-2}\right) + \frac{B}{M}M_{m-2}\omega_{st}\left(K, h+h'\right), \\ &D^*\left(F_{m-2}, u_{m-2}\right) \\ &\quad \le \frac{B}{4}\omega_{[a,b]\times[c,d]}\left(F_{m-3}, hh'\right) \\ &\qquad + BD^*\left(F_{m-3}, u_{m-3}\right) + \frac{B}{M}M_{m-3}\omega_{st}\left(K, h+h'\right), \\ &\vdots \\ &D^*\left(F_1, u_1\right) \\ &\quad \le \frac{B}{4}\omega_{[a,b]\times[c,d]}\left(F_0, hh'\right) \\ &\qquad + BD^*\left(F_0, u_0\right) + \frac{B}{M}M_0\omega_{st}\left(K, h+h'\right), \end{aligned} \tag{71}$$

and multiplying the above inequalities by $1, B, B^2, \ldots, B^{m-1}$, respectively, and summing them, we obtain

$$\begin{aligned} &D^*\left(F_m, u_m\right) \\ &\quad \le \frac{B}{4}\left(\omega_{[a,b]\times[c,d]}\left(F_{m-1}, hh'\right) + B\omega_{[a,b]\times[c,d]}\right. \\ &\qquad \times\left(F_{m-2}, hh'\right) \\ &\qquad \left.+ \cdots + B^{m-1}\omega_{[a,b]\times[c,d]}\left(f, hh'\right)\right) \\ &\qquad + \frac{B}{M}\omega_{st}\left(K, h+h'\right)\left(M_{m-1} + BM_{m-2} + B^2M_{m-3}\right. \\ &\qquad\qquad \left.+ \cdots + B^{m-1}M_0\right). \end{aligned} \tag{72}$$

Since, for $(s_1,t_1), (s_2,t_2) \in [a,b]\times[c,d]$ with $|s_1 - s_2| \le h$, $|t_1 - t_2| \le h'$, we have

$$\begin{aligned} &D\left(F_m(s_1,t_1), F_m(s_2,t_2)\right) \\ &\quad = D\left(f(s_1,t_1) \oplus \lambda\right. \\ &\qquad \otimes \int_c^d\int_a^b K(s_1,t_1,x,y) \\ &\qquad \otimes F_{m-1}(x,y)\,dx\,dy, \\ &\qquad f(s_2,t_2) \oplus \lambda \otimes \int_c^d\int_a^b K(s_2,t_2,x,y) \\ &\qquad\qquad \left.\otimes F_{m-1}(x,y)\,dx\,dy\right) \\ &\quad \le D\left(f(s_1,t_1), f(s_2,t_2)\right) \oplus \lambda \\ &\qquad \otimes \int_c^d\int_a^b \left|K(s_1,t_1,x,y) - K(s_2,t_2,x,y)\right| \\ &\qquad\quad \times D\left(F_{m-1}(x,y), \widetilde{0}\right)dx\,dy \\ &\quad \le D\left(f(s_1,t_1), f(s_2,t_2)\right) \\ &\qquad + \frac{B}{M}w_{st}\left(K, h+h'\right)\Gamma_{m-1}, \end{aligned} \tag{73}$$

we infer that

$$\begin{aligned} &\omega_{[a,b]\times[c,d]}\left(F_m, hh'\right) \\ &\quad \le \omega_{[a,b]\times[c,d]}\left(f, hh'\right) \\ &\qquad + \frac{B}{M}\omega_{st}\left(K, h+h'\right)\Gamma_{m-1}. \end{aligned} \tag{74}$$

By this inequality and (72), we see that

$$\begin{aligned}
&D^*(F_m, u_m)\\
&\quad\le \frac{B}{4}\left(1+B+B^2+\cdots+B^{m-1}\right)\omega_{[a,b]\times[c,d]}\left(f, hh'\right)\\
&\qquad+\frac{B}{4M}\omega_{st}\left(K, h+h'\right)\left(B\Gamma_{m-2}+B^2\Gamma_{m-3}+\cdots+B^{m-1}\Gamma_0\right)\\
&\qquad+\frac{B}{M}\omega_{st}\left(K, h+h'\right)\left(M_{m-1}+BM_{m-2}+B^2M_{m-3}\right.\\
&\qquad\qquad\left.+\cdots+B^{m-1}M_0\right)\\
&\quad=\frac{B}{4}\left(\frac{1-B^m}{1-B}\right)\omega_{[a,b]\times[c,d]}\left(f, hh'\right)+\frac{B}{4M}\omega_{st}\left(K, h+h'\right)\\
&\qquad\times\left[\left(B\Gamma_{m-2}+B^2\Gamma_{m-3}+\cdots+B^{m-1}\Gamma_0\right)\right.\\
&\qquad\quad\left.+4\left(M_{m-1}+BM_{m-2}+B^2M_{m-3}+\cdots+B^{m-1}M_0\right)\right].
\end{aligned}\tag{75}$$

By (62) and (63) and since $B<1$, we obtain

$$\begin{aligned}
&D^*(F_m, u_m)\\
&\quad\le \frac{B}{4}\left(\frac{1-B^m}{1-B}\right)\omega_{[a,b]\times[c,d]}\left(f, hh'\right)\\
&\qquad+\frac{B}{4M}\omega_{st}\left(K, h+h'\right)\left(\frac{B\left(1-B^m\right)}{1-B}\mu+\frac{4\left(1-B^m\right)}{1-B}\tau\right)\\
&\quad\le \frac{B}{4(1-B)}\omega_{[a,b]\times[c,d]}\left(f, hh'\right)\\
&\qquad+\frac{B}{4M}\omega_{st}\left(K, h+h'\right)\left(\frac{\mu B+4\tau}{1-B}\right);
\end{aligned}\tag{76}$$

therefore, we obtain

$$\begin{aligned}
&D^*(F_m, u_m)\\
&\quad\le\left(\frac{B}{4(1-B)}\right)\omega_{[a,b]\times[c,d]}\left(f, hh'\right)\\
&\qquad+\left(\frac{\mu B^2+4\tau B}{4M(1-B)}\right)\omega_{st}\left(K, h+h'\right).
\end{aligned}\tag{77}$$

By inequalities (77) and (46), we deduce that

$$\begin{aligned}
&D^*(F, u_m)\\
&\quad\le D^*(F, F_m)+D^*(F_m, u_m)\\
&\quad\le\left(\frac{B^{m+1}}{1-B}\right)\Gamma_0+\left(\frac{B}{4(1-B)}\right)\omega_{[a,b]\times[c,d]}\left(f, hh'\right)\\
&\qquad+\left(\frac{\mu B^2+4\tau B}{4M(1-B)}\right)\omega_{st}\left(K, h+h'\right).
\end{aligned}\tag{78}$$

□

Remark 27. Since $B<1$, it is easy to see that

$$\lim_{\substack{m\to\infty\\ h,h'\to 0}} D^*(F, u_m)=0,\tag{79}$$

which shows the convergence of the method.

5. Numerical Experiments

The proposed iterative method of successive approximations was tested on three numerical examples to provide the accuracy and the convergence of the method and illustrate the correctness of the theoretical results. In these examples, we assumed that $[a,b]\times[c,d]=[0,1]\times[0,1]$, $\lambda=1$, and we performed the algorithm in point $[s_0,t_0]=[0.5,0.5]$.

Example 1. Assume that

$$F(s,t)=f(s,t)\oplus\iint_0^1 K(s,t,x,y)\odot F(x,y)\,dx\,dy,\tag{80}$$

where

$$\begin{aligned}
f(s,t,r)&=\left(\underline{f}(s,t,r),\overline{f}(s,t,r)\right),\\
\underline{f}(s,t,r)&=\left(r^2+r\right)s\sin\frac{t}{2},\\
\overline{f}(s,t,r)&=\left(4-r^3-r\right)s\sin\frac{t}{2},\\
K(s,t,x,y)&=s^2tx;
\end{aligned}\tag{81}$$

the exact solution is given by

$$\begin{aligned}
F(s,t,r)&=\left(\underline{F}(s,t,r),\overline{F}(s,t,r)\right),\\
\underline{F}(s,t,r)&=\left(r^2+r\right)\left(s\sin\frac{t}{2}-\frac{16}{21}\left(\cos\frac{1}{2}-1\right)s^2t\right),\\
\overline{F}(s,t,r)&=\left(4-r^3-r\right)\left(s\sin\frac{t}{2}-\frac{16}{21}\left(\cos\frac{1}{2}-1\right)s^2t\right).
\end{aligned}\tag{82}$$

To obtain numerical solution, we apply the proposed method. To compare numerical and exact solutions, see Table 1.

Example 2. Consider (80) with

$$\begin{aligned}
f(s,t,r)&=\left(\underline{f}(s,t,r),\overline{f}(s,t,r)\right),\\
\underline{f}(s,t,r)&=r\left(\frac{1}{3}r+\frac{8}{3}\right)\left(1+s+t-\frac{7}{12}st\right),\\
\overline{f}(s,t,r)&=\left(2r^2-4r+5\right)\left(1+s+t-\frac{7}{12}st\right),\\
K(s,t,x,y)&=stxy
\end{aligned}\tag{83}$$

Table 1: Numerical results on the level sets for Example 1 in $(s_0, t_0) = (0.5, 0.5)$.

r-level	$m = 5, n = 10$		$m = 5, n = 20$		$m = 7, n = 10$		$m = 7, n = 20$	
	$\lvert\underline{F} - \underline{u}_m\rvert$	$\lvert\overline{F} - \overline{u}_m\rvert$	$\lvert\underline{F} - \underline{u}_m\rvert$	$\lvert\overline{F} - \overline{u}_m\rvert$	$\lvert\underline{F} - \underline{u}_m\rvert$	$\lvert\overline{F} - \overline{u}_m\rvert$	$\lvert\underline{F} - \underline{u}_m\rvert$	$\lvert\overline{F} - \overline{u}_m\rvert$
0.00	0.000000	0.000657	0.000000	0.000661	0.000000	0.000008	0.000000	0.000003
0.25	0.000067	0.000427	0.000051	0.000617	0.000011	0.000053	0.000001	0.000005
0.50	0.000086	0.000586	0.000024	0.000258	0.000066	0.000815	0.000043	0.000001
0.75	0.000150	0.000423	0.000022	0.000367	0.000069	0.000499	0.000008	0.000006
1.00	0.000229	0.000329	0.000131	0.000221	0.000154	0.000154	0.000005	0.000005

Table 2: Numerical results on the level sets for Example 2 in $(s_0, t_0) = (0.5, 0.5)$.

r-level	$m = 4, n = 10$		$m = 8, n = 20$	
	$\lvert\underline{F} - \underline{u}_m\rvert$	$\lvert\overline{F} - \overline{u}_m\rvert$	$\lvert\underline{F} - \underline{u}_m\rvert$	$\lvert\overline{F} - \overline{u}_m\rvert$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.000003	0.000008	0.000000	0.000002
0.4	0.000007	0.000011	0.000000	0.000008
0.6	0.000004	0.000006	0.000001	0.000000
0.8	0.000000	0.000005	0.000000	0.000000
1.0	0.000000	0.000000	0.000000	0.000000

Table 3: Numerical results on the level sets for Example 3 in $(s_0, t_0) = (0.5, 0.5)$.

r-level	$m = 5, n = 20$		$m = 5, n = 40$	
	$\lvert\underline{F} - \underline{u}_m\rvert$	$\lvert\overline{F} - \overline{u}_m\rvert$	$\lvert\underline{F} - \underline{u}_m\rvert$	$\lvert\overline{F} - \overline{u}_m\rvert$
0.0	0.000047	0.000028	0.000008	0.000003
0.2	0.000039	0.000086	0.000006	0.000009
0.4	0.000009	0.000008	0.000003	0.000000
0.6	0.000004	0.000005	0.000001	0.000000
0.8	0.000012	0.000033	0.000007	0.000002
1.0	0.000006	0.000004	0.000000	0.000000

and exact solution

$$\begin{aligned} F(s,t,r) &= \left(\underline{F}(s,t,r), \overline{F}(s,t,r)\right), \\ &= \left(r\left(\frac{1}{3}r + \frac{8}{3}\right)(s+t+1), \right. \\ &\qquad \left.\left(2r^2 - 4r + 5\right)(s+t+1)\right). \end{aligned} \tag{84}$$

We perform the proposed method and obtain numerical solution. Comparison of these two results is presented in Table 2.

Example 3. The integral equation (80) with

$$\begin{aligned} f(s,t,r) &= \left(\underline{f}(s,t,r), \overline{f}(s,t,r)\right), \\ \underline{f}(s,t,r) &= (2r\cos(1-r) - 1)\left(1 + s^2 + t - \frac{13}{24}(s+t)\right), \\ \overline{f}(s,t,r) &= \left(2 - \sin\frac{r\pi}{2}\right)\left(1 + s^2 + t - \frac{13}{24}(s+t)\right), \\ K(s,t,x,y) &= (s+t)xy \end{aligned} \tag{85}$$

has the exact solution

$$\begin{aligned} F(s,t,r) &= \left(\underline{F}(s,t,r), \overline{F}(s,t,r)\right), \\ &= \left((2r\cos(1-r) - 1)\left(s^2 + t + 1\right), \right. \\ &\qquad \left.\left(2 - \sin\frac{r\pi}{2}\right)\left(s^2 + t + 1\right)\right). \end{aligned} \tag{86}$$

For this linear example, we apply our proposed iterative method and obtain numerical results that can be viewed in Table 3.

6. Conclusions

In this paper, we introduced 2D fuzzy mappings and defined 2D fuzzy integrals. Quadrature rules to approximate the solution of 2D fuzzy integrals are given. We established the theorem of existence of unique solution of *2DFFLIE2*, and we have proved it by using Banach's fixed point principle. Moreover, to approximate the solution of *2DFFLIE2*, we have proposed an iterative algorithm based on method of successive approximations. The convergence to the unique solution in our iterative method is investigated. The presented numerical experiments show that the method applies well for *2DFFLIE2*.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On Solution of Integrodifferential Equation with Delay Parameter by Sinc Basis Functions

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We want to find a numerical solution for an integrodifferential equation with an integral boundary condition and delay parameter. This type of problems arises in mathematical physics, mechanics, population growth, and other fields of physics and mathematical chemistry. So, convergence of this approach is discussed by presenting a theorem which gives exponential type convergence rate and guarantees the accuracy of that. Finally, by some numerical examples, we show the efficiency and accuracy of this numerical method.

1. Introduction

Discussing integrodifferential equations with integral boundary condition consisting of delay parameter is a worthy and significant branch of nonlinear applied mathematics. It is important that integrodifferential equation with delay parameter is generated often in investigations connected with chemical engineering, mathematical physics, underground water flow, engineering, and so on (see [1, 2]). Note that the problems with integral boundary conditions have various applications in applied fields such as population growth problems and blood flow problems. For a detailed description of the integrodifferential equations with delay parameter and problems with integral boundary conditions, the reader can refer to references of [3–7].

In this paper we discuss the following problem which shows a first-order integrodifferential equation with integral boundary condition consisting of delay parameter. The existence and uniqueness of solution for this problem are proved in [8]. But analytic solving and reaching an exact solution are impossible. Then in this paper we approximate the exact solution by numerical method

$$\begin{gathered}\frac{dy}{dx} = g\left(x, y(x), y\left(\gamma_1(x)\right), Uy(x), Vy(x)\right) \equiv Gy(x), \\ y(0) = \eta_1 y(\kappa) + \eta_2 \int_0^b \omega\left(r, y(r)\right) dr + a,\end{gathered} \tag{1}$$

where $x \in I = [0, b]$ $(b > 0)$, $g \in C(I \times \mathbb{R}^4, \mathbb{R})$, where C is family of all continuous functions, and $\gamma_1 \in C(I, I)$, $\kappa \in (0, b]$, $\omega \in C(I \times \mathbb{R}, \mathbb{R})$, $\eta_1, \eta_2, a \in \mathbb{R}$.

And

$$\begin{gathered}(Uy)(x) = \int_0^{\gamma_2(x)} k(x, r)\, y(B(r))\, dr, \\ (Vy)(x) = \int_0^b h(x, r)\, y(D(r))\, dr.\end{gathered} \tag{2}$$

Here $\gamma_2, B, D \in C(I, I)$, $k(x, r) \in C[A, \mathbb{R}^+]$, and $h(x, r) \in C[A_0, \mathbb{R}^+]$ that

$$\begin{gathered}A = \left\{(x, r) \in \mathbb{R}^2 \mid 0 \le r \le \gamma_2(x), x \in I\right\}, \\ A_0 = \left\{(x, r) \in \mathbb{R}^2 \mid 0 \le r \le x, x \in I\right\}.\end{gathered} \tag{3}$$

Here if $\eta_1 = 1$, $\eta_2 = a = 0$, and $\kappa = b$, then we have a problem with boundary condition of kind periodic, and if $\eta_1 = 0$, then we have a problem with integral boundary condition, and if $\eta_1 = \eta_2 = 0$, we have a problem with an initial condition. So, problem (1) is general type of these cases.

Now, the following functional integral equation can be easily concluded from (1):

$$y(x) = \eta_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\, dr + a + \int_0^x g\left(t, y(t), y(\gamma_1(t)), \int_0^{\gamma_2(t)} k(t,r)\, y(B(r))\, dr, \int_0^b h(t,r)\, y(D(r))\, dr\right) dt \tag{4}$$

and in this paper we want to discuss solution of (1) in point of equivalent integral equation (4). But we know the fact that we cannot solve this integral equation to give an exact solution, so numerical approaches are used to reach an approximated solution.

Numerical approaches for estimated solution of integrodifferential and integral equation and search for existence and uniqueness of solution for some problems have been researched by many authors and reader can see these methods in [9–15]. In these references authors use methods on an estimate by basis function such as wavelets, polynomials, and so forth or use some quadrature formulas. But these technics usually have convergence rate of polynomial order with respect to M where M represents the cardinal of terms of sum in the expansion or the cardinal of points of the quadrature formula. In [16] author showed that if we employ the Sinc approach, the convergence rate is exponential order such as $O(\exp(-cM^{1/2}))$ with some $c > 0$. We know the exponential rate is much faster than that of polynomial rate. So, in this paper, we employ the Sinc function instead of base function and an iterative technique to estimate exact solution of (4) in marked points. Our approach dose not contain of changing the solution of (4) to a system of algebraic equations by expanding $y(x)$ as basis function with unknown coefficients, so this technique has computations less than other methods and exponential rate in accuracy. Also in this present paper, we prove a theorem to guarantee the convergence of numerical technique.

2. Main Results

In this section, we introduce basic requirements and theorem to prove existence and uniqueness of solution for first-order integrodifferential equation with integral boundary condition consisting of delay parameter (1). For detailed descriptions, we refer the reader to [17, 18].

Definition 1. A function as $y \in C^1(I, I)$ is called a lower solution of (1) if

$$\frac{dy}{dx} \le Gy(x), \quad x \in I, \qquad y(0) \le \eta_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\, dr + a \tag{5}$$

and it is an upper solution of (1) if

$$\frac{dy}{dx} \ge Gy(x), \quad x \in I, \qquad y(0) \ge \eta_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\, dr + a. \tag{6}$$

Theorem 2. *Let $u_0, v_0 \in C^1(I, \mathbb{R})$ be lower and upper solutions of (1), respectively, and $u_0(x) \le v_0(x)$, $x \in I$.*

In addition consider the following.

(L_1): *$g \in C(I \times \mathbb{R}^4, \mathbb{R})$, $\gamma_1, \gamma_2, B, D \in C(I, I)$, $\gamma_1(x), \gamma_2(x), B(x), D(x) \le x$, $\forall x \in I$, $\kappa \in (0, b)$, $\omega \in C(I \times \mathbb{R}, \mathbb{R})$, and $\eta_1, \eta_2 \ge 0$.*

(L_2): *there are nonnegative bounded integrable functions $M_1(x)$, $M_2(x)$, $M_3(x)$, $M_4(x)$ on I that*

$$\int_0^b \left[M_1(t) + M_2(x) + M_3(x) \int_0^{\gamma_2(x)} k(x,r)\, dr + M_4(x) \int_0^b h(x,r)\, dr \right] dx \le 1 \tag{7}$$

such that

$$\begin{aligned} g(x, \phi_1, \phi_2, U\phi_1, V\phi_1) &- g(x, \psi_1, \psi_2, U\psi_1, V\psi_1) \\ &\ge -M_1(x)(\phi_1 - \psi_1) - M_2(x)(\phi_2 - \psi_2) \\ &\quad - M_3(x) U(\phi_1 - \psi_1) - M_4(x) V(\phi_1 - \psi_1) \end{aligned} \tag{8}$$

if $u_0 \le \psi_1 \le \phi_1 \le v_0$, $u_0(\gamma_1(x)) \le \psi_2 \le \phi_2 \le v_0(\gamma_1(x))$.

(L_3): *there is $\theta(x) \in C(I, \mathbb{R}^+)$ such that $\omega(x, \psi) - \omega(x, \psi^-) \ge \theta(x)(\psi - \psi^-)$, and if $u_0(t) \le \psi^- \le \psi \le v_0(x)$. then problem (1) has extremal solutions $u, v \in [u_0, v_0]$. In addition, there are monotone sequences $u_n(x), v_n(x) \subset [u_0, v_0]$ such that $u_n \to u$, $v_n \to v$ for $n \to +\infty$ and these are convergent uniformly on $x \in I$, where $u_n(x)$, $v_n(x)$ are defined as*

$$\begin{aligned} u_n(x) = &\int_0^x e^{-\int_r^x M_1(s)ds} \\ &\times [g(r, u_{n-1}(r), u_{n-1}(\gamma_1(r)), \\ &\quad Uu_{n-1}(r), Vu_{n-1}(r)) + M_1(r) u_{n-1}(r) \\ &- M_2(r)(u_n - u_{n-1})(\gamma_1(r)) \\ &- M_3(r) U(u_n - u_{n-1})(r) \\ &- M_4(r) V(u_n - u_{n-1})(r)]\, dr \\ &+ e^{-\int_0^x M_1(s)ds} \left[\eta_1 u_{n-1}(\kappa) + \eta_2 \int_0^b \omega(r, u_{n-1}(r))\, dr + a \right]; \\ &\forall x \in I, \quad n = 1, 2, 3, \ldots, \end{aligned}$$

$$\begin{aligned} v_n(x) = & \int_0^x e^{-\int_r^x M_1(s)ds} \\ & \times [g(r, v_{n-1}(r), v_{n-1}(\gamma_1(r)), \\ & \quad Uv_{n-1}(r), Vv_{n-1}(r)) + M_1(r) z_{n-1}(r) \\ & - M_2(r)(v_n - v_{n-1})(\gamma_2(r)) \\ & - M_3(r) U(v_n - v_{n-1})(r) \\ & - M_4(r) V(v_n - v_{n-1})(r)] dr \\ & + e^{-\int_0^x M_1(s)ds} \left[\eta_1 v_{n-1}(\kappa) \right. \\ & \left. + \eta_2 \int_0^b \omega(r, v_{n-1}(r)) dr + a \right]; \\ & \forall x \in I, \quad n = 1, 2, 3, \ldots, \\ u_0 \le u_1 \le \cdots \le u_n \le \cdots \le u \le v \le \cdots \le v_n \\ \le \cdots \le v_1 \le v_0. \end{aligned} \tag{9}$$

Proof. See [18]. □

Theorem 3. *Consider that assumptions of Theorem 2 hold. Moreover, consider the following.*

(L_4): *there are nonnegative bounded functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\alpha_4(x)$ on I, such that*

$$\begin{aligned} & g(x, \phi_1, \phi_2, U\phi_1, V\phi_1) - g(x, \psi_1, \psi_2, U\psi_1, V\psi_1) \\ & \le \alpha_1(x)(\phi_1 - \psi_1) + \alpha_2(x)(\phi_2 - \psi_2) \\ & + \alpha_3(x) U(\phi_1 - \psi_1) + \alpha_4(x) V(\phi_1 - \psi_1) \end{aligned} \tag{10}$$

if $u_0(x) \le \psi_1 \le \phi_1 \le v_0(x)$, $u_0(\gamma_1(x)) \le \psi_2 \le \phi_2 \le v_0(\gamma_1(x))$.

(L_5): *there is $\beta(x) \in \mathscr{C}(I, \mathbb{R}^+)$ such that $\omega(x, \psi) - \omega(x, \psi^-) \le \beta(x)(\psi - \psi^-)$ if $u_0(t) \le \psi^- \le \psi \le v_0(x)$.*

Then problem (1) has a unique solution $u^- \in [u_0, v_0]$. In addition, there are sequences $u_n(x), v_n(x) \subset [u_0, v_0]$ that these are monotone and $u_n \to u^-$, $v_n \to u^-$ for $n \to +\infty$. This convergence is uniformly on $x \in I$, where $u_n(x)$, $v_n(x)$ are defined as (9) such that $\|u_n - u^-\|_c \le L^n \|v_0 - u_0\|_c$, $n \in \mathbb{N}$, where

$$\begin{aligned} L = \eta_1 + \int_0^b \Big[& \eta_2 \beta(x) + \alpha_1(x) + M_1(x) + \alpha_2(x) + M_2(x) \\ & + (\alpha_3(x) + M_3(x)) \int_0^{\gamma_2(x)} K(x, r) dr \\ & + (\alpha_4(x) + M_4(x)) \int_0^b h(x, r) dr \Big] dx < 1. \end{aligned} \tag{11}$$

Proof. See [18]. □

3. Sinc Function

In this section, we recall the basis function and some of its applicabilities. In here, definition of sinc(x) function is followed by

$$\operatorname{Sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}; & x \ne 0, \\ 1; & x = 0. \end{cases} \tag{12}$$

Now, for $h > 0$ and integer j, we define jth Sinc function with step size h by

$$S(j, h)(x) = \frac{\sin(\pi(x - jh)/h)}{\pi(x - jh)/h}. \tag{13}$$

3.1. Sinc Estimation on $[a, b]$. Let $x = \varphi(w)$ be a transformation that denotes a conformal transformation which transfers the simply connected domain A onto a strip region A_d such that

$$\begin{gathered} \varphi((a, b)) = (-\infty, \infty), \qquad \lim_{x \to a} \varphi(x) = -\infty, \\ \lim_{x \to b} \varphi(x) = \infty. \end{gathered} \tag{14}$$

In here ∂A is boundary of A and in order to have the Sinc estimation on (a, b) conformal transformation is applied as follows:

$$\varphi(t) = \ln\left(\frac{t - a}{b - t}\right). \tag{15}$$

This function transfers the eye-shaped complex region

$$\left\{ w = x + iy : \left| \arg\left(\frac{w - a}{b - w}\right) \right| < d \le \frac{\pi}{2} \right\} \tag{16}$$

onto A_d that it is a strip region:

$$A_d = \left\{ \sigma = \alpha + \beta i : |\beta| < d < \frac{\pi}{2} \right\}. \tag{17}$$

The basis functions on finite interval (a, b) are given by

$$S(j, h) \circ \varphi(x) = \frac{\sin(\pi(\varphi(x) - jh)/h)}{\pi(\varphi(x) - jh)/h}, \tag{18}$$

and also, Sinc function for interpolation points $x_j = jh$ is given by

$$S(j, h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1; & j = k, \\ 0; & j \ne k. \end{cases} \tag{19}$$

Then, $S(j, h) \circ \varphi(x)$ shows behavior of Kronecker delta function on the network points

$$x_j = \varphi^{-1}(jh) = \frac{a + be^{jh}}{1 + e^{jh}}. \tag{20}$$

The approximation of $g(x)$ by interpolation and quadrature formulas for $\int_a^b g(x)dx$ is

$$g(x) \approx \sum_{j=-M}^{M} g(x_j) S(j,h) \circ \varphi(x),$$
$$\int_a^b g(x)\,dx \approx h \sum_{j=-M}^{M} \frac{g(x_j)}{\varphi'(x_j)}. \tag{21}$$

Theorem 4. *Consider that, for a map $w = \varphi^{-1}(\xi)$, the map $g(\varphi^{-1}(\xi))$ satisfies*

(1) *$g \in H^1(D_d)$, for $d > 0$,*

(2) *g decays exponentially on the real line such that*

$$|g(x)| \le \alpha \exp(-\beta |x|), \quad \forall x \in \mathbb{R},\ \alpha, \beta > 0 \tag{22}$$

with some α, β, and d. Then one has

$$\sup_{a<x<b} \left| g(x) - \sum_{j=-M}^{M} g\left(\varphi^{-1}(jh)\right) S(j,h) \circ \varphi(x) \right| \le C\sqrt{M} \exp\left(-\sqrt{\pi d \beta M}\right). \tag{23}$$

That there is some C and h is $h = \sqrt{\pi d/\beta M}$.

Proof. See [16]. □

Definition 5. Let $L_\alpha(A)$ be the set of all analytic functions g, for which there exists a constant C, such that

$$|g(z)| \le C \frac{|e^{\varphi(z)}|^\alpha}{(1 + |e^{\varphi(z)}|)^{2\alpha}}; \quad z \in A,\ 0 < \alpha \le 1. \tag{24}$$

Theorem 6. *Let $g/\varphi' \in L_\alpha(A)$, with $0 < \alpha \le 1$ and $0 < d \le \pi$; also let $h = \sqrt{\pi d/\alpha M}$. Then there exists a constant C_1, which is independent of M, such that*

$$\left| \int_a^{x_j} g(t)\,dt - h \sum_{k=-M}^{M} \delta_{jk}^{(-1)} \frac{g(x_j)}{\varphi'(x_k)} \right| \le C_1 e^{-\sqrt{\pi d \alpha M}}, \tag{25}$$

where

$$\delta_{jk}^{(-1)} = \frac{1}{2} + \int_0^{j-k} \frac{\sin(\pi t)}{\pi t} dt, \tag{26}$$

and $\varphi(x)$, x_k are defined as above.

Proof. See [16]. □

3.2. Sinc-Qudrature Method. In this section, for solving equation

$$\begin{aligned} y(x) = {} & \lambda_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\,dr + a \\ & + \int_0^x g\Big(s, y(s), y(\gamma_1(s)), \\ & \qquad \int_0^{\gamma_2(s)} k(s,r)\, y(B(r))\,dr, \\ & \qquad \int_0^b h(s,r)\, y(D(s))\,dr\Big)\,ds, \end{aligned} \tag{27}$$

we try to discrete integral equation by quadrature formula as

$$\begin{aligned} \int_a^b g(s)\,ds &= h \sum_{j=-M}^{M} \frac{g(s_j)}{\varphi'(s_j)} + O\left(\exp\left(-\frac{2\pi d M}{\log(2\pi d M/\beta)}\right)\right) \\ \int_a^s f(x)\,dx &= h \sum_{j=-M}^{M} \frac{g(x_j)}{\varphi'(x_j)} \eta_{h,j}(s) \\ &\quad + O\left(\frac{\log M}{M} \exp\left(-\frac{\pi d M}{\log(\pi d M/\beta)}\right)\right), \end{aligned} \tag{28}$$

where

$$\eta_{h,j}(s) = \frac{1}{2} + \frac{1}{\pi} r_i\left(\pi \frac{r - jh}{h}\right); \quad r_i = \int_0^x \frac{\sin(t)}{t} dt \tag{29}$$

with $x_j = s_j = (a + be^{jh})/(1 + jh)$, $j = -M, \ldots, M$ and $h = (1/M)\log(\pi d M/\beta)$ (see [16]).

Now, by substituting quadrature formulas in the integral equation (4), we have

$$\begin{aligned} y_M(x) = {} & \eta_1 y(\kappa) + \eta_2 h \sum_{j=-M}^{M} \frac{\omega(s_j, y(s_j))}{\varphi'(s_j)} + a \\ & + h \sum_{j=-M}^{M} g\Big(s_j, y(s_j), y(\alpha(s_j)), \\ & \qquad h \sum_{i=-M}^{M} \frac{K(s_j, r_i)\, y(B(r_i))}{\varphi'(r_i)} \eta_{h,i}(t), \\ & \qquad h \sum_{i=-M}^{M} \frac{h(s_j, r_i)\, y(D(r_i))}{\varphi'(r_i)}\Big) \\ & \times (\varphi'(s_j))^{-1} \eta_{h,j}(x). \end{aligned} \tag{30}$$

Now, for $n = 1, 2, 3, \ldots$, let

$$y_{1,M}(x) = \eta_1 y(\kappa),$$

$$\begin{aligned} y_{n+1,M}(x) &= \eta_1 y_{n,M}(\kappa) + \eta_2 h \sum_{j=-M}^{M} \frac{\omega(s_j, y_{n,M}(s_j))}{\varphi'(s_j)} + a \\ &\quad + h \sum_{j=-M}^{M} g\Bigg(s_j, y_{n,M}(s_j), y_{n,M}(\alpha(s_j)), \\ &\qquad h \sum_{i=-M}^{M} \frac{a(s_j, r_i)\, y_{n,M}(B(r_i))}{\varphi'(r_i)} \eta_{h,i}(x), \\ &\qquad h \sum_{i=-M}^{M} \frac{h(s_j, r_i)\, y_{n,M}(\delta(r_i))}{\varphi'(r_i)} \Bigg) \\ &\quad \times (\varphi'(s_j))^{-1} \eta_{h,j}(x). \end{aligned} \tag{31}$$

4. Convergence of Method

In this section, we present a theorem that shows a bound for $y(x) - y_n(x)$ with the real norm where $y(x)$ is the exact solution of problem (4) and $y_n(x)$ is an estimation for $y(x)$ by using Sinc function in interpolaion and quadrature formula. The result is shown as follows.

Theorem 7. *Under the assumptions* (L_1)–(L_5), *iterative approximation approach* (31) *is convergent to exact solution if* $y_1(x)$ *is closed enough to the it and*

$$N_2 \le \frac{1 - (\eta_1 + \eta_2 b N_1)}{b(2 + N_3 + N_4)}, \tag{32}$$

where $N_1 = \sup\{\beta(x); x \in I\}$, $N_2 = \max\{\alpha_i(x); i = 1, 2, 3, 4, x \in I\}$, $N_3 = \sup\{K(x, r); x, r \in I\}$, *and* $N_4 = \sup\{h(x, r); x, r \in I\}$.

Proof. For a fixed M let

$$\begin{aligned} y_{n+1,M}(x) &= \eta_1 x_{n,M}(\tau) + \eta_2 \int_0^b \omega(r, y_{n,M}(r))\, dr + a \\ &\quad + \int_0^x g(s, y_{n,M}(s), y_{n,M}(\gamma_1(s)), U y_{n,M}(s), \\ &\qquad V y_{n,M}(s))\, ds, \\ y(x) &= \eta_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\, dr + a \\ &\quad + \int_0^x g(s, x(s), y(\gamma_1(s)), U y(s), V y(s))\, ds. \end{aligned} \tag{33}$$

Now

$$\begin{aligned} &|y_{n+1,M}(x) - y(x)| \\ &\le \eta_1 |y_n(\kappa) - y(\kappa)| \\ &\quad + \eta_2 \left| \int_0^b (\omega(r, y_n(r)) - \omega(r, y(r)))\, dr \right| \\ &\quad + \Bigg| \int_0^x \Bigg[g\Bigg(s, y_n(s), y_n(\gamma_1(s)), \\ &\qquad \int_0^{\gamma_2(s)} K(s, r)\, y_n(B(r))\, dr, \\ &\qquad \int_0^b h(s, r)\, y_n(D(r))\, dr \Bigg) \\ &\qquad - f\Bigg(s, y(s), y(\gamma_1(s)), \\ &\qquad \int_0^{\gamma_2(s)} K(s, r)\, y(B(r))\, dr, \\ &\qquad \int_0^b h(s, r)\, y(D(r))\, dr \Bigg) \Bigg] ds \Bigg| \\ &\le \eta_1 |y_n(\kappa) - y(\kappa)| + \eta_2 b N_1 |y_n(x) - y(x)| \\ &\quad + b N_2 |y_n(x) - y(x)| + b N_2 |y_n(\gamma_1(x)) - y(\gamma_1(x))| \\ &\quad + b N_2 N_3 |x_n(B(r)) - y(B(r))| \\ &\quad + b N_2 M_4 |y_n(D(r)) - y(D(r))| \\ &\le (\eta_1 + \eta_2 b N_1 + b N_2 (2 + N_3 + N_4)) |y_{n,M}(x) - y(x)|. \end{aligned} \tag{34}$$

Then

$$\begin{aligned} |y_{n+1,M}(x) - y(x)| &\le (\eta_1 + \eta_2 b N_1 + b N_2 (2 + N_3 + N_4))^n \\ &\quad \times |y_{1,M}(x) - y(x)|. \end{aligned} \tag{35}$$

Now, by assumption in theorem, we have $\eta_1 + \eta_2 b N_1 + b N_2 (2 + N_3 + N_4) < 1$.

Because $\eta_1 + \eta_2 b N_1 + b N_2 (2 + N_3 + N_4) < \eta_1 + \eta_2 b N_1 + b((1 - (\eta_1 + \eta_2 b N_1))/b(2 + N_3 + N_4))(2 + N_3 + N_4) = 1$ and then $\lim_{n \to +\infty} y_{n+1,M}(x) = y(x)$, $0 \le x \le b$. □

5. Illustrative Examples

In this section, for showing efficiency of the iterative scheme and in order to show the facts of the exact solution, we give some examples below. All routines have been written in Mathematica 7 and a Dual-Core CPU 2.00 GHz is used to run the programs. Also, about efficiency and accuracy of the proposed numerical method, we present absolute errors for different examples. Therefore numerical results are shown in figures to illustrate the efficiency of this scheme.

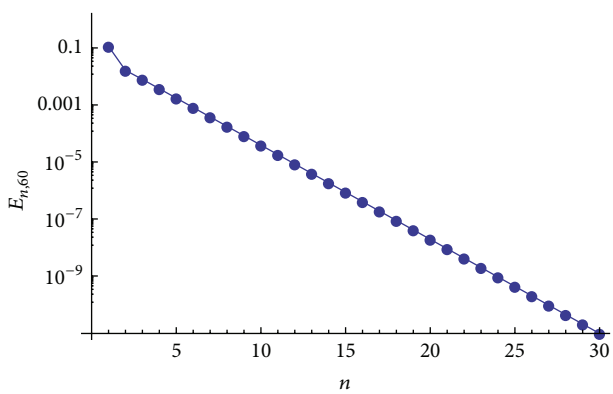

Figure 1: Absolute errors of Example 1. For different values of iteration n.

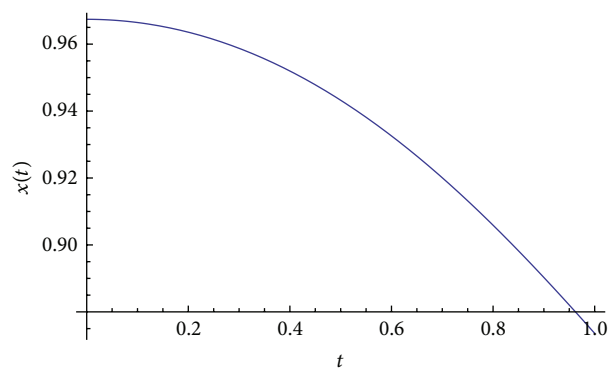

Figure 2: Approximate solution for Example 1.

Example 1. Consider the following problem:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{15}x^3\left[x - y(x)\right] - \frac{1}{5}xy\left(\gamma_1(x)\right) + \frac{1}{100}x^2y^2\left(\gamma_1(x)\right)\\ &+ \frac{1}{500}x\left[x^3 - \int_0^{(1/2)x} 8xry\left(r^2\right)dr\right]^5\\ &- \frac{1}{600}x^2\left[x^3 - \int_0^{(1/2)x} 8xry(r^2)dr\right]^6\\ &+ \frac{1}{700}x^2\left[x^2 - \int_0^1 2x^2ry\left(r^2\right)dr\right]^7\\ &- \frac{1}{800}x^3\left[x^2 - \int_0^1 2x^2ry(r^2)dr\right]^8 \equiv Gy(x),\\ &\qquad x \in I = [0,1],\\ y(0) &= \frac{1}{4}y(\kappa) + \frac{1}{4}\int_0^1 \left(2r + y(r)\right)dr + \frac{1}{4},\end{aligned} \tag{36}$$

where $\kappa \in (0,1]$, $\gamma_1 \in C(I,I)$, and $\gamma_1(x) \le x$ on I.

Now, we want to show that assumptions (L_1)–(L_5) and convergence criterion which was prepared in Theorem 7 are satisfied for integral equation (4).

Noting that $\gamma_2(x) = (1/2)x$, $B(x) = D(x) = x^2$, $\forall x \in I$, then (L_1)–(L_5) is true. For details see [6].

In addition, we have $\beta(x) = 1$, $\alpha_1(x) = 0$, $\alpha_2(x) = (1/50)x^2$, $\alpha_3(x) = \alpha_4(x) = (1/100)x^{17}$, $K(x,r) = 8xr$, $h(x,r) = 2x^2r$, and $\eta_1 = \eta_2 = 1/4$; then $N_1 = 1$, $N_2 = 1/50$, $N_3 = 8$, and $N_4 = 2$, and then

$$N_2 = \frac{1}{50} \le \frac{1 - \left(\eta_1 + \eta_2 b N_1\right)}{b\left(2 + N_3 + N_4\right)} = \frac{1}{24}. \tag{37}$$

Therefore, the iterative method (31) converges to the exact solution of this equation. Now, based on Sinc quadrature scheme we can have some successive approximations for solution of this example for $\kappa = 1/2$. Absolute error for nth approximation and N points quadrature method is defined by

$$e_{n,M} = \max_{x\in[0,1]}\left\{\left|y_{n+1,M}(x) - y_{n,M}(x)\right|\right\}; \quad n = 1,2,3,\ldots. \tag{38}$$

For different values of n absolute errors are depicted in Figure 1 and approximate solution for $n = 30$ is depicted in Figure 2.

Example 2. In Example 1 of [19], authors considered the integrodifferential equation

$$\begin{aligned}\frac{dy}{dx} &= 1 - \frac{1}{3}x^3 + \int_0^1 x^3y^2(z)\,dz,\\ y(0) &= 0.\end{aligned} \tag{39}$$

The exact solution is $y(x) = x$. Maximum absolute error for each iteration and different values of quadrature points are depicted in Figures 3 and 4. By comparing these results with the numerical results given in [19] in Table 2, efficiency and accuracy of current approach are guaranteed.

6. Conclusion

In this paper, we apply a numerical approach by Sinc function for reaching the estimated solution of integrodifferential equation with enteral boundary condition and with delay parameter. To reach this aim we change this problem to a functional enteral equation. The Sinc estimation has exponential convergence rate such as $O(-ce^{M^{1/2}})$ that this property is an advantage. so we applied it to solve our problem by using collocation method. Finally, some examples are solved by this numerical method to show the efficiency and accuracy of Sinc estimation. It is worthy to note that this method can

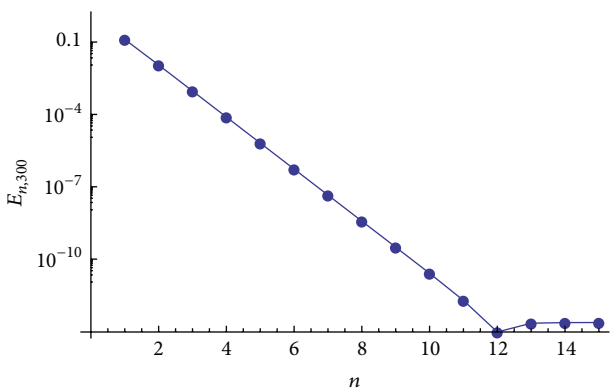

Figure 3: Maximum absolute error related to each iteration in Example 2.

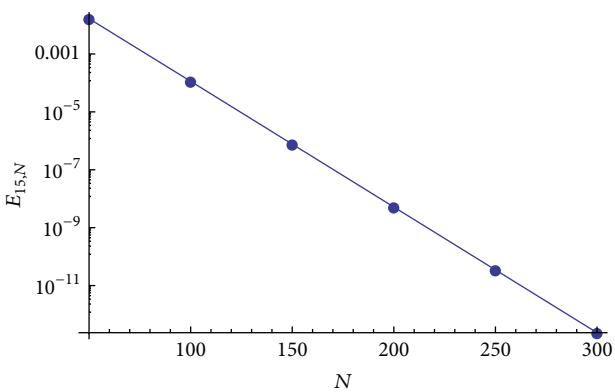

Figure 4: Maximum error related to different quadrature points in Example 2.

be used for solving integrodifferential equations with integral boundary conditions with deviating arguments arising in all sciences such as chemistry, physics, and other fields of applied mathematics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A New Reconstruction of Variational Iteration Method and Its Application to Nonlinear Volterra Integrodifferential Equations

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We reconstruct the variational iteration method that we call, parametric iteration method (PIM). The purposed method was applied for solving nonlinear Volterra integrodifferential equations (NVIDEs). The solution process is illustrated by some examples. Comparisons are made between PIM and Adomian decomposition method (ADM). Also exact solution of the 3rd example is obtained. The results show the simplicity and efficiency of PIM. Also, the convergence of this method is studied in this work.

1. Introduction

It is well known that many events in scientific fields deal with integrodifferential equations. The nonlinear Volterra integrodifferential equations play a major role in many physical processes such as nanohydrodynamic [1], dropwise condensation [2], and biologic [3]. The various numerical methods exist for solving NVIDEs, for example, variational iteration method (VIM), Adomian decomposition method [4, 5], Chebyshev polynomials [6], and Bernstein's approximation [7]. First, Liao purposed homotopy analysis method [8] and it is applied in many scientific problems [9, 10]; then, VIM was purposed by He [11]. In this paper, we reconstruct the VIM that we call PIM. PIM was applied successfully for solving boundary value problems [12]. We consider nonlinear integrodifferential equations as follows:

$$u^{(m)}(x) = f(x) + \lambda \int_0^x k(x,t) F(u(t))\, dt, \quad a \le x \le b, \tag{1}$$

$$u^{(m)}(x) = f(x) + \lambda \int_0^x k(x,t) F\left(u, u'\right) dt, \quad a \le x \le b, \tag{2}$$

and the initial value for both of the two equations is as follows:

$$u^{(i)}\left(x_0\right) = y_i, \quad (i = 0, 1, \ldots, m-1). \tag{3}$$

In this work, the numerical solution of (1) and (2) is possible by PIM when f, k, u are continuous and F is continuous operator. Parametric iteration method provides solution for NVIDEs as a sequence of iterations. In this study, some examples are given and we solve them using parametric iteration method and compare the obtained results with ADM results. In all these cases, the present technique worked excellently, as it will be shown in this study.

2. The Basic Idea of the Parametric Iteration Method

In this section, we describe PIM for solving nonlinear Volterra integrodifferential equations. Then, the local convergence is discussed.

2.1. Parametric Iteration Method. The PIM provides the solution for (1) and (2) as a sequence of approximations. This method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise, approximations can be used for numerical purposes. We assume L and N are the linear and nonlinear operators on $C^m[a,b]$. To explain the basic idea of PIM, we consider (1) and (2) as follows:

$$L(u) + N(u) = f(x), \tag{4}$$

where L with the property $L(g) \equiv 0$ when $g \equiv 0$ denotes the so-called auxiliary linear operator with respect to u, N is a nonlinear continuous operator with respect to u, $f(x)$ is the known continuous function, and $u \in C^m[a,b]$. The basic essence of this method is to construct a family of iterative processes for (1) and (2) as [13]

$$L\left[u_{k+1}(x) - u_k(x)\right] = hH(x)\,A\left[u_k(x)\right], \tag{5}$$

with the initial conditions

$$u^{(i)}(x_0) = y_i, \qquad (i = 0, 1, \ldots, m-1), \tag{6}$$

where

$$\begin{aligned} A\left[u_k(x)\right] &= L\left[\left(u_k(x)\right] + N\left[u_k(x)\right] - f(x) \\ &= u_k^{(m)} - \lambda \int_0^x k(x,t)\,u_k^n\,dt - f(x), \end{aligned} \tag{7}$$

or

$$\begin{aligned} A\left[u_k(x)\right] &= L\left[\left(u_k(x)\right] + N\left[u_k(x)\right] - f(x) \\ &= u_k^{(m)} - \lambda \int_0^x F\left(x, t, u_k, u_k'\right) dt - f(x), \end{aligned} \tag{8}$$

and $u_0(x)$ is the initial guess which can be chosen arbitrarily, but the suitable selection is positively affect for the rate of convergence [13], or it can also be solved from its corresponding linear homogeneous equation $L[u_0(x)] = 0$ or linear nonhomogeneous equation $L[u_0(x)] = f(x)$. The parameter $h \neq 0$ and function $H(x) \neq 0$ denote the so-called auxiliary parameter and auxiliary function. The selection of h, $H(x)$ was described in [13]. Also, we are free to choose the auxiliary linear operator L, the auxiliary parameter h, the auxiliary function $H(x)$, and the initial approximation $u_0(x)$. Therefore, if the successive approximations $u_k(x)$, $k \geq 0$ are obtained by PIM in terms of the auxiliary parameter h, then exact solution may be given by $u(x) = \lim_{k\to\infty} u_k(x)$. According to [13], Let $V = \{u : u \in C^m[a,b]\}$ be the solution space and let $\{e_j(x) : e_j(x) \in V, j = 0, 1, \ldots\}$ denote the set of base functions. Hence, we can represent the solution in the series $u(x) = \sum_{j=0}^{\infty} \alpha_j e_j(x)$, where α_j is a coefficient belonging to real numbers. As long as the set of base functions is determined, the auxiliary linear operator L, the initial approximation $u_0(x)$, and the auxiliary function $H(x)$ must be chosen in such a way that all solutions of the corresponding PIM equations (4) exist and it can be expressed by this set of base functions. Now, in order to avoid expensive computational works for solving (1) and (2) via PIM, it is straightforward to use the following set of base functions:

$$\left\{(x-a)^j \mid j = 0, 1, \ldots\right\}; \tag{9}$$

that is,

$$u(x) = \sum_{j=0}^{\infty} \alpha_j (x-a)^j, \tag{10}$$

where $\alpha_j \in R$ are unknown coefficients to be determined and a is a constant belonging to real numbers. Now, we set the auxiliary operator L as follows:

$$L\left[u(x)\right] = u^{(m)}(x). \tag{11}$$

The initial guess is to form combination of m-terms of (9); that is,

$$u_0(x) = \alpha_0 + \alpha_1(x-a) + \alpha_2(x-a)^2 + \cdots + \alpha_j(x-a)^j. \tag{12}$$

According to (11) and the initial conditions (3) and with due attention to $L[u_0(x)] = f(x)$, the coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_m$ will be determined. Also, we set $H(x) = 1$. The selecting of $H(x)$ is arbitrary, but the suitable selection depends on the base functions for solution [13], and we use the PIM processes to compute the approximation solutions of (1) and (2).

2.2. The Valid Region of h. Assume that we gain a family of solution series in terms of the auxiliary parameter h by means of PIM. We consider this solution as a function in terms of h, x; then, we derive (once or more) this function with respect to x in $x = \beta$ that $\beta \in [a,b]$; that is, let $U = G(x,h)$ be the solution of (1) or (2); then, we set

$$\Omega = \left.\frac{\partial^i G}{\partial x^i}\right|_{x=\beta}, \quad \beta \in [a,b], \quad (i = 1, 2, \ldots), \tag{13}$$

therefore, Ω will be in terms of h; now we plot Ω curve, and according to these h curves, it is easy to discover the valid region of h, which corresponds to the line segments nearly parallel to the horizontal axis. This region is called valid region of h which we denote by R_h. We ensure that the solution series converge for any $h \in R_h$.

2.3. Analysis of Convergence of the Parametric Iteration Formula. In this section, we study the local convergence of approximate solution provided by PIM for solving (1). The convergence of approximate solution for (2) is similar to (1).

Initially, let $u^{(i)}(0) = 0$, $(i = 0, 1, \ldots, m-1)$ and set $L[u(x)] = u^{(m)}(x)$; therefore, we have from (5) the following parametric iteration formula:

$$\begin{aligned} &u_{k+1}(x) \\ &= (1+h)\,u_k(x) - \frac{h}{(m-1)!} \\ &\quad \times \int_0^x (x-t)^{m-1} \left[g(t) + \lambda \int_0^t k(t,s)\,F\left(u_k(s)\right) ds\right] dt. \end{aligned} \tag{14}$$

The iterative formula (14) expressed by sequence makes a recurrence sequence $\{u_k(x)\}$. Obviously, the limit of the sequence will be the solution of (1) if the sequence is convergent.

In order to prove that the sequence $\{u_k(x)\}$ is convergent, we construct a series:

$$u_0(x) + [u_1(x) - u_0(x)] + \cdots + [u_k(x) - u_{k-1}(x)] + \cdots. \tag{15}$$

Noticing that

$$\begin{aligned} s_{k+1} &= u_0(x) + [u_1(x) - u_0(x)] + \cdots \\ &\quad + [u_k(x) - u_{k-1}(x)] = u_k(x), \end{aligned} \tag{16}$$

the sequence $\{u_k(x)\}$ will be convergent if the series is convergent.

Theorem 1. *If $F(u(t))$ is Lipschitz-continuous in $[a,b]$ and $g(x) \in C[a,b]$ and $|\lambda| \le 1/MN$ where M, N are positive real numbers, then the series of* (15) *is convergent; that is, the sequence $\{u_k(x)\}$ is convergent for $x \in [a,b]$.*

Proof. According to (14), note that

$$\begin{aligned} &|u_1(x) - u_0(x)| \\ &= \left| h \left[u_0(x) - \frac{1}{(m-1)!} \right.\right. \\ &\qquad \left.\left. \times \int_0^x (x-t)^{m-1} \left[g(t) + \lambda \int_0^t k(t,s) F(u_0(s))\, ds \right] dt \right] \right| \\ &\le |h| \left[L_1 + \frac{1}{(m-1)!} \left(L_2 L_3 b + \frac{L_2 L_4}{2N} b^2 \right) \right] = |h|\, r, \end{aligned} \tag{17}$$

where

$$\begin{gathered} r := \left(L_1 + \frac{1}{(m-1)!} \right) \left(L_2 L_3 b + \frac{L_2 L_4}{2N} b^2 \right), \\ M = \max_{a \le t \le x \le b} |k(x,t)|, \qquad L_1 = \max_{a \le t \le x \le b} |u_0(t)|, \\ L_2 = \max_{a \le t \le x \le b} \left| (x-t)^{m-1} \right|, \qquad L_3 = \max_{a \le t \le x \le b} |g(t)|, \\ L_4 = \max_{a \le t \le x \le b} |F(u_0(t))|. \end{gathered} \tag{18}$$

From (14) and (17) and the assumption that $|F(u_{k+1}) - F(u_k)| \le N |u_{k+1} - u_k|$ where N denotes the Lipschitz constant of $F(u(t))$, it follows that

$$\begin{aligned} &|u_2 - u_1| \\ &= \left| \left[(1+h)(u_1 - u_0) - \frac{h}{(m-1)!} \right.\right. \\ &\qquad \left.\left. \times \int_0^x (x-t)^{m-1} \left[\lambda \int_0^t k(t,s) (F(u_1) - F(u_0))\, ds \right] dt \right] \right| \\ &\le \left| |1+h| |u_1 - u_0| + \frac{|h|}{(m-1)!} \right. \\ &\qquad \left. \times \int_0^x (x-t)^{(m-1)} \left[\lambda N \int_0^t k(t,s) |u_1 - u_0|\, ds \right] dt \right| \\ &\le |h|\, r \left[|1+h| + \frac{L_2 |h|}{2(m-1)!} b^2 \right] \\ &= |h|\, r \sum_{n=0}^{1} \binom{1}{n} |1+h|^{1-n} \left(\frac{L_2 |h|}{2(m-1)!} b^2 \right)^n, \end{aligned} \tag{19}$$

$$\begin{aligned} &|u_3 - u_2| \\ &= \left| \left[(1+h)(u_2 - u_1) - \frac{h}{(m-1)!} \right.\right. \\ &\qquad \left.\left. \times \int_0^x (x-t)^{m-1} \left[\lambda \int_0^t k(t,s) (F(u_2) - F(u_1))\, ds \right] dt \right] \right| \\ &\le \left| |1+h| |u_2 - u_1| + \frac{|h|}{(m-1)!} \right. \\ &\qquad \left. \times \int_0^x (x-t)^{(m-1)} \left[\lambda N \int_0^t k(t,s) |u_2 - u_1|\, ds \right] dt \right| \\ &\le |h|\, r \left[|1+h|^2 + \frac{2L_2 |h| |1+h|}{2(m-1)!} b^2 + \frac{{L_2}^2 |h|^2}{(2(m-1)!)^2} b^4 \right] \\ &= |h|\, r \sum_{n=0}^{2} \binom{2}{n} |1+h|^{1-n} \left(\frac{L_2 |h|}{2(m-1)!} b^2 \right)^n \\ &\vdots \end{aligned} \tag{20}$$

$$|u_{k+1} - u_k| \le (|h|\, r) \sum_{n=0}^{k} \binom{k}{n} |1+h|^{k-n} \left(\frac{L_2 |h| b^2}{2(m-1)!} \right)^n. \tag{21}$$

In view of (21), the convergence of the series (15) can be concluded for the solution domain $x > 0$ and $|1+h| < 1$ with the help of some mathematical software. Therefore, the series of (15) is absolute convergence; that is, the sequence $\{u_k(x)\}$ is convergent for $x \in [a,b]$. □

3. Illustrative Examples

Now, we use PIM to solve two examples of the kind of (1) and (2) and compare the obtained results with ADM [5] to show the efficiency of PIM.

Example 1. The first example is a nonlinear Volterra integrodifferential equation of the second kind as follows:

$$\begin{gathered} u'(x) = -1 + \int_0^x u^2(t)\, dt, \quad 0 \le x \le 1 \\ u(0) = 0. \end{gathered} \tag{22}$$

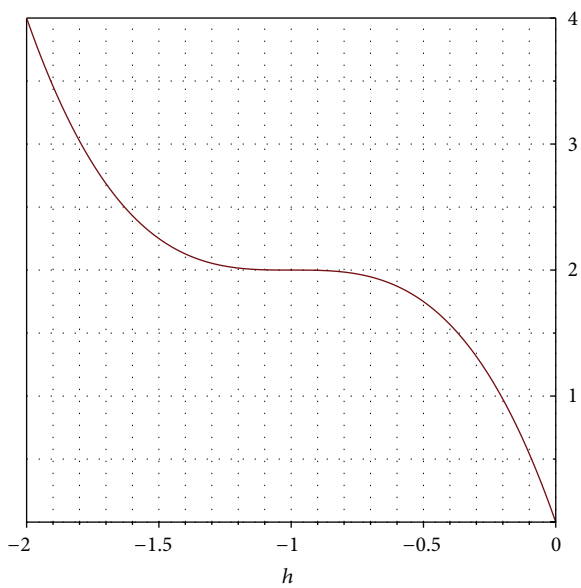

Figure 1: The valid region of h. Example 1 shows $-1.2 \le h \le -0.8$.

According to PIM proceeding, we define

$$L[u(x)] = u'(x),$$
$$N[u(x)] = \int_0^x u^2(t)\,dt, \qquad f(x) = -1. \tag{23}$$

Then, we have from (10) that $u_0(x) = \alpha_0 + \alpha_1 x$. Using initial condition and $L[u_0(x)] = f(x)$ gives us, that $\alpha_0 = 0$, $\alpha_1 = -1$ that is, $u_0(x) = -x$. Now, If we set $H(x) = 1$, then we obtain from (5) and (8) that

$$u'_{k+1}(x) - u'_k(x) = h\left(u'_k(x) - \int_0^x u_k^2(t)\,dt + 1\right). \tag{24}$$

Thus we integrate from both sides of (25); then the PIM equations are as follows:

$$u_{k+1}(x) = u_k(x) + h\int_0^x \left[\left(u'_k(t) - \int_0^t u_k^2(s)\,ds + 1\right)\right] dx, \quad (k = 0, 1, \ldots). \tag{25}$$

Some of the iterations obtained from (25) are as follows and the other iterations have also been calculated by Maple 13:

$$\begin{aligned} u_1(x) &= -x + h\left(-\frac{1}{12}x^4\right), \\ u_2(x) &= -x - \frac{1}{12}hx^4 \\ &\quad + h\left(-\frac{1}{12960}h^2x^{10} - \frac{1}{252}hx^7 + \frac{1}{4}\left(-\frac{1}{3}h - \frac{1}{3}\right)x^4\right). \\ &\vdots \end{aligned} \tag{26}$$

The obtained result for the 3rd iteration was shown in Table 1; also the valid region of h that is R_h was presented in Figure 1.

Table 1: The results of Example 1 for $u_3(x)$ with $h = -1$.

x	ADM	PIM
0.0000	0.000000000	0.0000000
0.0938	−0.0937935	−0.0937935
0.2188	−0.2186090	−0.2186091
0.3125	−0.3117060	−0.3117064
0.4062	−0.4039390	−0.4039385
0.5000	−0.4948230	−0.4948225
0.6250	−0.6124310	−0.6124306
0.7188	−0.6969410	−0.6969414
0.8125	−0.7770900	−0.7770900
0.9062	−0.8519340	−08519338
1.0000	−0.9204760	−0.9204746

Example 2. We consider the nonlinear Volterra integrodifferential equation of the second kind as follows:

$$u'(x) = 1 + \int_0^x u(t)\,u'(t)\,dt, \quad 0 \le x \le 1, \tag{27}$$

$$u(0) = 0. \tag{28}$$

According to PIM procedure we set

$$L[u(x)] = u'(x),$$
$$N[u(x)] = \int_0^x u(t)\,u'(t)\,dt, \qquad f(x) = 1. \tag{29}$$

Then, we have from (10) $u_0(x) = \alpha_0 + \alpha_1 x$, and (29) gives us that $\alpha_0 = 0$, $\alpha_1 = 1$; that is, $u_0(x) = x$. Now similar to Example 1, the iteration scheme is as follows:

$$u_{k+1}(x) = u_k(x) + h\int_0^x \left[\left(u'_k(t) - \int_0^t u_k(s)\,u'_k(s)\,ds - 1\right)\right] dx, \quad (k = 0, 1, \ldots). \tag{30}$$

Some of the iterations obtained from (30) are as follows:

$$\begin{aligned} u_1(x) &= x - \frac{1}{6}hx^3, \\ u_2(x) &= x - \frac{1}{6}hx^3 \\ &\quad + h\left(-\frac{1}{504}h^2x^7 + \frac{1}{30}h\,x^5 + \frac{1}{3}\left(-\frac{1}{2}h - \frac{1}{2}\right)x^3\right). \\ &\vdots \end{aligned} \tag{31}$$

The results of Example 2 are available in Table 2. R_h for (27) is presented in Figure 2.

Table 2: The results of Example 2 for $u_3(x)$ with $h = -1$.

x	ADM	PIM
0.0000	0.0000000	0.0000000
0.0938	0.0938065	0.0939377
0.2188	0.2189910	0.2205626
0.3125	0.3132980	0.3176876
0.4062	0.4084910	0.4177517
0.5000	0.5053030	0.5219304
0.6250	0.6381770	0.6691415
0.7188	0.7422990	0.7878390
0.8125	0.8518530	0.9155129
0.9062	0.9691440	1.0546355
1.0000	1.0973700	1.2083812

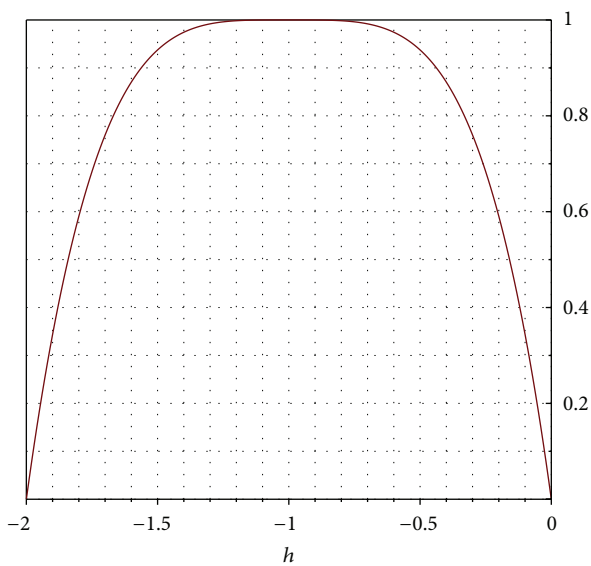

Figure 2: The valid region of h. Example 2 shows $-1.2 \le h \le -0.8$.

Example 3. We consider the nonlinear Volterra integrodifferential equation of the second kind as follows:

$$u'(x) = 1 - \frac{x}{2} + \frac{xe^{-x^2}}{2} + \int_0^x xte^{-u^2(t)}dt, \quad 0 \le x \le 1, \tag{32}$$

$$u(0) = 0. \tag{33}$$

Similar to Examples 1 and 2

$$L[u(x)] = u'(x), \tag{34}$$

$$N[u(x)] = \int_0^x xte^{-u^2(t)}dt, \qquad f(x) = 1 - \frac{x}{2} + \frac{xe^{-x^2}}{2}. \tag{35}$$

Now, we consider the initial solution as $u_0(x) = \alpha_0 + \alpha_1 x$, and (33) and (34) give us that $\alpha_0 = 0$, $\alpha_1 = 1$; that is, $u_0(x) = x$. According to PIM iterative formula for this example we have

$$\begin{aligned} u_{k+1}(x) = (1+h)\,u_k(x) &+ h\left[\frac{1}{4} + x - \frac{1}{4}x^2 - \frac{1}{4}e^{x^2}\right. \\ &\left. + \left(\int_0^x \int_0^t tse^{-u_k^2(s)}ds\right)\right]dt \\ &(k = 0, 1, \ldots). \end{aligned} \tag{36}$$

Starting with $k = 0$ and initial solution that is $u_0(x) = x$, the exact solution of this example, which is $u(x) = x$, is achieved. The valid value of parameter for (36) is $h = -1$.

4. Results

In this section, we present the results of Examples 1 and 2 in two tables and plot the h-curve to determine R_h. All the computations have been done with Maple 13.

5. Conclusion

In this paper, we reconstruct the VIM that we call parametric iteration method, and PIM was applied to solve the nonlinear Volterra integrodifferential equations. In order to illustrate the method, we solve three examples. PIM results compared to ADM show that the former is easier in practice and more accurate for NVIDEs. For the 3rd example exact solution was achieved. Further, the convergence of PIM for solving NVIDEs in the valid region of h (R_h) was presented. Additionally, if we increase the number of iterations by PIM scheme, it seems that the results will have more accuracy in solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Approximating the Solution of the Linear and Nonlinear Fuzzy Volterra Integrodifferential Equations Using Expansion Method

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The present research study introduces an innovative method applying power series to solve numerically the linear and nonlinear fuzzy integrodifferential equation systems. Finally, it ends with some examples supporting the idea.

1. Introduction

Fuzzy integrodifferential equations have attracted great interests in recent years since they play a major role in different areas of theory such as control theory. For the first time, Chang and Zadeh have introduced fuzzy numbers as well as the related arithmetic operations [1, 2]. Furthermore, applying the operators on fuzzy numbers has been developed by Mizumoto and Tanaka [3]. It should be mentioned that the concept of LR fuzzy numbers was expressed by Dubois and Prade [4]. In this regard, they made a significant contribution by providing a computational formula for operations on fuzzy numbers. After that, the notation of fuzzy derivative was presented by Seikkala [5]. However, Goetschel, Jr., and Voxman have proposed the Riemann integral-type approach [6]. Some mathematicians have separately worked on the existence and having a unique solution of fuzzy Volterra integrodifferential equation [7–9]. Recently, numerical methods have been applied to solve the linear as well as nonlinear differential equation fuzzy integral equation and fuzzy integrodifferential equation [7, 8, 10–13].

In this paper, we use the power series method of the exact solution of linear or nonlinear fuzzy integrodifferential equations, which is obtained by recursive procedure as follows.

We consider the following system of fuzzy integrodifferential equations:

$$\widetilde{X}'(s) = G\left(s, \widetilde{X}(s)\right) \oplus \int_0^s k\left(s, t, \widetilde{X}(t), \widetilde{X}'(t)\right) dt, \quad (1)$$

with initial condition $\widetilde{F}(0) = \widetilde{a}$, and

$$\begin{aligned} \widetilde{X} &= \left[\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right]^T, \\ \widetilde{G} &= \left[\widetilde{g}_1, \widetilde{g}_2, \ldots, \widetilde{g}_n\right]^T, \\ \widetilde{K} &= \left[\widetilde{k}_{ij}\right], \qquad i, j = 1, 2, \ldots, n, \\ \widetilde{a} &= \left[\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n\right]^T. \end{aligned} \quad (2)$$

In (1), $\widetilde{G}$ and $\widetilde{K}$ are given fuzzy functions and, also, $\widetilde{a}$ is fuzzy vector and vector fuzzy function $\widetilde{X}$ is solution of (1), which will be determined.

2. Basic Concepts

Here basic definitions of a fuzzy number are given as follows [14–19].

Let E be a set of all triangular fuzzy numbers.

Definition 1. An arbitrary fuzzy number $\widetilde{u} \in E$ in the parametric form is represented by an ordered pair of functions $(\underline{u}, \overline{u})$ which satisfy the following requirements.

(i) $\overline{u} : r \to \underline{u}(r) \in \mathbb{R}$ is a bounded left-continuous nondecreasing function over $[0, 1]$.

(ii) $\underline{u} : r \to \overline{u}(r) \in \mathbb{R}$ is a bounded left-continuous nonincreasing function over $[0, 1]$.

(iii) $\underline{u} \le \overline{u}, \quad 0 \le r \le 1$.

Definition 2. For arbitrary fuzzy numbers $\widetilde{u}, \widetilde{v} \in E$, one uses the distance (Hausdorff metric) [6]

$$D(u(r), v(r)) = \max\left\{\sup_{r\in[0,1]} \left|\underline{u}(r) - \underline{v}(r)\right|, \sup_{r\in[0,1]} \left|\overline{u}(r) - \overline{v}(r)\right|\right\}, \tag{3}$$

and it is shown in [6] that (E, D) is a complete metric space and the following properties are well known:

$$\begin{gathered} D(\widetilde{u} + \widetilde{w}, \widetilde{v} + \widetilde{w}) = D(\widetilde{u}, \widetilde{v}), \quad \forall \widetilde{u}, \widetilde{v} \in E, \\ D(k\widetilde{u}, k\widetilde{v}) = |k| D(\widetilde{u}, \widetilde{v}), \quad \forall k \in \mathbb{R}, \ \widetilde{u}, \widetilde{v} \in E, \\ D(\widetilde{u} + \widetilde{v}, \widetilde{w} + \widetilde{e}) \le D(\widetilde{u}, \widetilde{w}) + D(\widetilde{v}, \widetilde{e}), \quad \forall \widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{e} \in E. \end{gathered} \tag{4}$$

Definition 3. A triangular fuzzy number is defined as a fuzzy set in E, which is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \le b \le c$ such that $[u]^r = [\underline{u}(r), \overline{u}(r)]$ are the endpoints of r-level sets for all $r \in [0, 1]$, where $\underline{u}(r) = a + (b - a)r$ and $\overline{u}(r) = c - (c - b)r$. Here, $\underline{u}(0) = a$, $\overline{u}(0) = c$, $\underline{u}(1) = \overline{u}(1) = b$, which is denoted by u^1.

Definition 4. A fuzzy number $\widetilde{A}$ is of LR type if there exist shape functions L (for left), R (for right), and scalar $\alpha \ge 0$, $\beta \ge 0$ with

$$\widetilde{\mu}_A(x) = \begin{cases} L\left(\dfrac{a - x}{\alpha}\right) & x \le a, \\ R\left(\dfrac{x - a}{\beta}\right) & x \ge a; \end{cases} \tag{5}$$

the mean value of $\widetilde{A}$, a is a real number, and α, β are called the left and right spreads, respectively. $\widetilde{A}$ is denoted by (a, α, β).

Definition 5. Let $\widetilde{M} = (m, \alpha, \beta)_{LR}$, $\widetilde{N} = (n, \gamma, \delta)_{LR}$, and $\lambda \in \mathbb{R}^+$. Then,

(1) $\lambda\widetilde{M} = (\lambda m, \lambda\alpha, \lambda\beta)_{LR}$

(2) $\lambda\widetilde{M} = (-\lambda m, \lambda\beta, \lambda\alpha)_{LR}$

(3) $\widetilde{M} \oplus \widetilde{N} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}$

(4) $\widetilde{M} \odot \widetilde{N}$

$$\simeq \begin{cases} (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR} & \widetilde{M}, \widetilde{N} > 0 \\ (mn, n\alpha - m\delta, n\beta - m\gamma)_{LR} & \widetilde{M} > 0, \widetilde{N} < 0 \\ (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{LR} & \widetilde{M}, \widetilde{N} < 0. \end{cases} \tag{6}$$

Definition 6. The integral of a fuzzy function was defined in [6] by using the Riemann integral concept.

Let $f : [a, b] \to E^1$, for each partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_i - 1, t_i]$, $1 \le i \le n$, and suppose

$$\begin{gathered} R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}), \\ \Delta := \max\{|t_i - t_{i-1}|, 1 \le i \le n\}. \end{gathered} \tag{7}$$

The definite integral of $f(t)$ over $[a, b]$ is

$$\int_a^b f(t)\,dt = \lim_{\Delta \to 0} R_p, \tag{8}$$

provided that this limit exists in the metric D.

If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists [17], and also

$$\begin{gathered} \underline{\left(\int_a^b f(t,r)\,dt\right)} = \int_a^b \underline{f}(t,r)\,dt, \\ \overline{\left(\int_a^b f(t,r)\,dt\right)} = \int_a^b \overline{f}(t,r)\,dt. \end{gathered} \tag{9}$$

Definition 7 (see [14]). Consider $\widetilde{x}, \widetilde{y} \in E$. If there exists $\widetilde{z} \in E$ such that $\widetilde{x} = \widetilde{y} + \widetilde{z}$, then $\widetilde{z}$ is called the H-difference of $\widetilde{x}$ and $\widetilde{y}$ and is denoted by $\widetilde{x} \ominus \widetilde{y}$.

Proposition 8 (see [14]). *If $\widetilde{f} : (a, b) \to E$ is a continuous fuzzy-valued function, then $\widetilde{g}(x) = \int_a^x \widetilde{f}(t)$ is differentiable, with derivative $\widetilde{g}'(x) = \widetilde{f}(x)$.*

Definition 9 (see [20]). Let $f : R \to E$ be a fuzzy valued function. If, for arbitrary fixed $t_0 \in R$ and $\epsilon > 0$, a $\delta > 0$ such that

$$|t - t_0| < \delta \Longrightarrow d(f(t), f(t_0)) < \epsilon, \tag{10}$$

f is said to be continuous.

Definition 10 (see [21]). Let be $\widetilde{f} : (a, b) \to E$ and $x_0 \in (a, b)$. One says that $\widetilde{f}$ is differentiable at x_0 if

(1) there exists an element $\widetilde{f}'(x_0) \in E$ such that, for all $h > 0$ sufficiently near to 0, there are $\widetilde{f}(x_0 + h) \ominus \widetilde{f}(x_0)$, $\exists \widetilde{f}(x_0) \ominus \widetilde{f}(x_0 - h)$, and the limits

$$\begin{aligned} &\lim_{h \to 0^+} \frac{\widetilde{f}(x_0 + h) \ominus \widetilde{f}(x_0)}{h} \\ &\quad = \lim_{h \to 0^+} \frac{\widetilde{f}(x_0) \ominus \widetilde{f}(x_0 - h)}{h} = \widetilde{f}'(x_0) \end{aligned} \tag{11}$$

or

(2) there exists an element $\tilde{f}'(x_0) \in E$ such that, for all $h < 0$, sufficiently near to 0, there are $\tilde{f}(x_0+h) \ominus \tilde{f}(x_0)$, $\exists \tilde{f}(x_0) \ominus \tilde{f}(x_0-h)$, and the limits

$$\lim_{h\to 0^-} \frac{\tilde{f}(x_0+h) \ominus \tilde{f}(x_0)}{h} = \lim_{h\to 0^-} \frac{\tilde{f}(x_0) \ominus \tilde{f}(x_0-h)}{h} = \tilde{f}'(x_0). \tag{12}$$

Lemma 11 (see [14]). *For $\tilde{x}_0 \in R$ the fuzzy differential equation*

$$\tilde{y}' = \tilde{f}(x,y), \qquad \tilde{y}(x_0) = \tilde{y}_0 \in E, \tag{13}$$

where $\tilde{f} : R \times E \to E$ is supposed to be continuous, if equivalent to one of the integral equations:

$$\tilde{y}(x) = \tilde{y}_0 + \int_{x_0}^{x} f(t, \tilde{y}(t)), \quad \forall x \in [x_0, x_1] \tag{14}$$

or

$$\tilde{y}(x) = \tilde{y}_0 + (-1)\int_{x_0}^{x} f(t, \tilde{y}(t)), \quad \forall x \in [x_0, x_1]. \tag{15}$$

On some interval (x_0, x_1) under the differentiability condition, (i) or (ii), respectively.

Definition 12 (see [22]). For fuzzy number $\tilde{u}(r) = (\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$, one writes (1) $\tilde{u} > 0$, if $\underline{u}(r) > 0$, (2) $\tilde{u} \ge 0$, if $\underline{u}(r) \ge 0$, (3) $\tilde{u} < 0$, if $\overline{u} < 0$, and (4) $\tilde{u} \le 0$, if $\overline{u} \le 0$.

Theorem 13 (see [14]). *Let $c \in E$ and $g : (a,b) \to R$. If g is differentiable on x_0, then the function $f : (a,b) \to R$, defined by $f(x) = c \odot g(x)$, $\forall x \in (a,b)$, is differentiable on x_0 and one has $f'(x_0) = c \odot g'(x_0)$.*

Corollary 14 (see [14]). *Let $c \in E$ and $g : (a,b) \to R$. And define $f : (a,b) \to R$ by $f(x) = c \odot g(x)$, $\forall x \in (a,b)$. If g is differentiable on (a,b) and g' is differentiable on $x_0 \in (a,b)$, then f is differentiable on (a,b) and twice differentiable on x_0, with $f''(x_0) = c \odot g''(x_0)$.*

Remark 15 (see [14]). In general, if the above g is $n-1$ times differentiable on (a,b) and g^{n-1} is differentiable on x_0, then $f(x) = c \odot g(x)$ is differentiable of order n on x_0 and $f^n(x_0) = c \odot g^n(x_0)$.

Theorem 16 (see [21]). *Let $\tilde{f} : (a,b) \to E$ be a function and denote $[\tilde{f}(x)]^r = [\underline{f}(r), \overline{f}(r)]$, for each $r \in [0,1]$. Then one has the following.*

(i) *If $\tilde{f}$ is differentiable in the first form (Definition 10), then $\underline{f}$, $\overline{f}$ are differentiable functions and*

$$\left[\tilde{f}'(x)\right]^r = \left[\underline{f}'(x), \overline{f}'(x)\right]. \tag{16}$$

(ii) *If $\tilde{f}$ is differentiable in the second form (Definition 10), then $\underline{f}$, $\overline{f}$ are differentiable functions and*

$$\left[\tilde{f}'(x)\right]^r = \left[\overline{f}'(x), \underline{f}'(x)\right]. \tag{17}$$

3. Approximation Based on the Expansion Method

Since s is positive so all derivatives of $\tilde{X}(s)$ in (18) are in case (i) in Definition 10.

Suppose the solution of the system of fuzzy integrodifferential equations (1) is as follows:

$$\tilde{X}_i(s) = \sum_{j=0}^{m} \tilde{e}_{ij} s^j, \quad i = 1, 2, \ldots, n, \tag{18}$$

where $\tilde{e}_{ij} \in E$, for all $i = 1, 2, \ldots, n$. By using initial conditions, we have

$$\tilde{e}_{i0} = \tilde{x}_i(0), \quad i = 1, 2, \ldots, n. \tag{19}$$

The coefficients of (18) are computed step by step. Firstly, the solution of problem (1) is considered as

$$\tilde{X}(s) = \tilde{e}_0 \oplus \tilde{e}_1 \odot s, \tag{20}$$

where $\tilde{e}_{ij}$, $i = 1, 2, \ldots, n$, and $\tilde{e}_1$ are unknown. With derivative of (20) we have $\tilde{X}'(s) = \tilde{e}_1$ and by substituting $\tilde{X}'(s)$, (20) into (1), we have

$$\tilde{e}_1 = G(s, \tilde{e}_0 \oplus \tilde{e}_1 \odot s) \oplus \int_0^s k(s, t, \tilde{e}_0 \oplus \tilde{e}_1 \odot s, \tilde{e}_1)\, dt, \tag{21}$$

where by integration and sort of terms of above equation we obtain the following system:

$$\left(A_1 \odot \tilde{e}_1 \ominus \tilde{b}_1\right) \oplus \tilde{Q}_1(s) = \tilde{0}, \tag{22}$$

where A_1 is $n \times n$ constant matrix, $\tilde{b}_1$ is $n \times 1$ fuzzy vector, $\tilde{Q}_1(s) = [\tilde{q}_{i1}(s)]$, $i = 1, 2, \ldots, n$, and $\tilde{q}_{i1}(s)$ are polynomials of order equal or greater than 1. If $s = 0$ by neglecting $\tilde{Q}_1(s)$, we have fuzzy linear equations system of $\tilde{e}_1$. By solving this system, the coefficient of $\tilde{x}$ in (20) can be determined.

In the second step, we assume that

$$\tilde{X}(s) = \tilde{e}_0 \oplus \tilde{e}_1 \odot s \oplus \tilde{e}_2 \odot s^2, \tag{23}$$

where $\tilde{e}_0$ and $\tilde{e}_1$ are known and $\tilde{e}_2$ is unknown. With derivative of (23) we have

$$\text{case (1), if } \tilde{e}_1 > 0 : \tilde{X}'(s) = \tilde{e}_1 \oplus 2\tilde{e}_2 \odot s;$$
$$\text{case (2), if } \tilde{e}_1 < 0 : \tilde{X}'(s) = \ominus\tilde{e}_1 \oplus 2\tilde{e}_2 \odot s,$$

and by substituting $\tilde{X}'(s)$, (23) into (1), we have

$$\tilde{e}_1 \oplus 2\tilde{e}_2 \odot s = G\left(s, \tilde{e}_0 \oplus \tilde{e}_1 \odot s \oplus \tilde{e}_2 \odot s^2\right) \oplus \int_0^s k\left(s, t, \tilde{e}_0 \oplus \tilde{e}_1 \odot s \oplus \tilde{e}_2 \odot s^2, \tilde{e}_1 \oplus 2\tilde{e}_2 \odot s\right) dt \tag{24}$$

or

$$\ominus\tilde{e}_1 \oplus 2\tilde{e}_2 \odot s = G\left(s, \tilde{e}_0 \oplus \tilde{e}_1 \odot s \oplus \tilde{e}_2 \odot s^2\right) \oplus \int_0^s k\left(s, t, \tilde{e}_0 \oplus \tilde{e}_1 \odot s \oplus \tilde{e}_2 \odot s^2, \ominus\tilde{e}_1 \oplus 2\tilde{e}_2 \odot s\right) dt, \tag{25}$$

where by integration and sort of terms of above equation we obtain the following system:

$$\left(A_2 \odot \tilde{e}_2 \ominus \tilde{b}_2\right) \oplus \widetilde{Q}_2(s) = \tilde{0}, \tag{26}$$

where A_2 is $n \times n$ constant matrix, $\tilde{b}_2$ is $n \times 1$ fuzzy vector A_1, and $\widetilde{Q}_2(s) = [\tilde{q}_{i2}(s)]$, $i = 1, 2, \ldots, n$, and $\tilde{q}_{i2}(s)$ are polynomials of order greater than unity, where by neglecting $\widetilde{Q}_2(s)$, we have again fuzzy system of linear equations of $\tilde{e}_2$ and by solving this system, coefficients of s^2 in (23) can be determined. This procedure can be repeated till the arbitrary order coefficients of power series of the solution for the problem are obtained.

The following theorem shows convergence of the method. Without loss of generality, we prove it for $n = 1$.

Theorem 17. *Let $\widetilde{X} = \widetilde{F}(s)$ be the exact solution of the following fuzzy integrodifferential equation:*

$$\widetilde{X}'(s) = G\left(s, \widetilde{X}(s)\right) \oplus \int_0^s k\left(s, t, \widetilde{X}(t), \widetilde{X}'(t)\right) dt, \qquad \widetilde{X}(0) = \tilde{a}. \tag{27}$$

Assume that $\tilde{f}(s)$ has a power series representation. Then,

$$\lim_{m \to \infty} \tilde{f}_m(s) = \tilde{f}(s). \tag{28}$$

Proof. According to the proposed method, we assume that the approximate solution to (27) is as follows:

$$\tilde{f}_m(s) = \tilde{e}_0 \oplus \tilde{e}_1 \odot s \oplus \tilde{e}_2 \odot s^2 \oplus \cdots \oplus \tilde{e}_m \odot s^m. \tag{29}$$

Hence, it is sufficient that we only prove

$$\text{if: } \tilde{e}_m > 0, \quad \tilde{e}_m = \frac{\tilde{f}^{(m)}(0)}{m!}, \qquad \text{if: } \tilde{e}_m < 0, \quad \ominus\tilde{e}_m = \frac{\tilde{f}^{(m)}(0)}{m!} \tag{30}$$

for $m = 1, 2, 3, \ldots$.

Note that, for $m = 0$, the initial condition gives

$$\tilde{e}_0 = \tilde{f}(0) = \tilde{a}. \tag{31}$$

Moreover, for $m = 1$, if we set $\widetilde{X} = \tilde{f}(s)$ and $s = 0$ in (27), we obtain

$$\tilde{f}'(0) = g\left(0, \tilde{f}(0)\right) \oplus \tilde{0}. \tag{32}$$

On the other hand, from (29) and (31), we have

$$\tilde{f}_1(s) = \tilde{e}_0 \oplus \tilde{e}_1 \odot s. \tag{33}$$

By substituting (33) into (27) and setting $s = 0$, we get

$$\text{if: } \tilde{e}_1 > 0, \quad \tilde{e}_1 = g\left(0, \tilde{f}(0)\right) \oplus \tilde{0} = \tilde{f}'(0), \qquad \text{if: } \tilde{e}_1 < 0, \quad \ominus\tilde{e}_1 = g\left(0, \tilde{f}(0)\right) \oplus \tilde{0} = \tilde{f}'(0). \tag{34}$$

For $m = 2$, differentiating (27) with respect to s, we have

$$\tilde{f}''(s) = \frac{\sigma}{\sigma s} g\left(s, \tilde{f}(s)\right) \oplus \frac{\sigma}{\sigma s} g\left(s, \tilde{f}(s)\right) \tilde{f}'(s) \oplus K\left(s, \tilde{f}(s), \tilde{f}'(s)\right) \oplus \int_0^s \frac{\sigma}{\sigma s} K\left(s, t, \tilde{f}(t), \tilde{f}'(t)\right) dt. \tag{35}$$

Setting $s = 0$ in (35), we get

$$\tilde{f}''(0) = \frac{\sigma}{\sigma s} g\left(0, \tilde{f}(0)\right) \oplus \frac{\sigma}{\sigma s} g\left(0, \tilde{f}(0)\right) \tilde{f}'(0) \oplus K\left(0, \tilde{f}(0), \tilde{f}'(0)\right). \tag{36}$$

According to (29), (31), and (34), let

$$\tilde{f}_2(s) = \tilde{f}(0) \oplus \tilde{f}'(0) \odot s \oplus \tilde{e}_2 \odot s^2. \tag{37}$$

By substituting (37) into (35) and setting $s = 0$, we obtain

$$\begin{aligned}
&\text{if: } \tilde{e}_2 > 0,\ \tilde{e}_1 > 0,\\
&2\tilde{e}_2 = \frac{\sigma}{\sigma s} g\left(0, \tilde{e}_0\right) \oplus \frac{\sigma}{\sigma s} g\left(0, \tilde{e}_0\right) \tilde{e}_1 \oplus K\left(0, \tilde{e}_0, \tilde{e}_1\right)\\
&\text{if: } \tilde{e}_2 > 0,\ \tilde{e}_1 < 0,\\
&2\tilde{e}_2 = \frac{\sigma}{\sigma s} g\left(0, \tilde{e}_0\right) \oplus \frac{\sigma}{\sigma s} g\left(0, \tilde{e}_0\right) \left(\ominus\tilde{e}_1\right) \oplus K\left(0, \tilde{e}_0, \ominus\tilde{e}_1\right)\\
&\text{if: } \tilde{e}_2 < 0,\ \tilde{e}_1 > 0,\\
&\ominus 2\tilde{e}_2 = \frac{\sigma}{\sigma s} g\left(0, \tilde{e}_0\right) \oplus \frac{\sigma}{\sigma s} g\left(0, \tilde{e}_0\right) \tilde{e}_1 \oplus K\left(0, \tilde{e}_0, \tilde{e}_1\right)\\
&\text{if: } \tilde{e}_2 < 0,\ \tilde{e}_1 < 0,\\
&\ominus 2\tilde{e}_2 = \frac{\sigma}{\sigma s} g\left(0, \tilde{e}_0\right) \oplus \frac{\sigma}{\sigma s} g\left(0, \tilde{e}_0\right) \left(\ominus\tilde{e}_1\right) \oplus K\left(0, \tilde{e}_0, \ominus\tilde{e}_1\right).
\end{aligned} \tag{38}$$

So, with comparison (36) and (38), we conclude that

$$2\tilde{e}_2 = \tilde{f}''(0) \Longrightarrow \tilde{e}_2 = \frac{\tilde{f}''(0)}{2!} \tag{39}$$

or

$$\ominus 2\tilde{e}_2 = \tilde{f}''(0) \Longrightarrow \ominus\tilde{e}_2 = \frac{\tilde{f}''(0)}{2!}. \tag{40}$$

By constituting the above procedure, we can easily prove (30) for $m = 3, 4, \ldots$. □

4. Numerical Illustrations

Example 18. Consider the following system of fuzzy linear Volerra integrodifferential equations:

$$\begin{aligned}\tilde{x}_1'(s) &= g_1(s,\tilde{x}_1(s),\tilde{x}_2(s)) \\ &\quad \ominus \int_0^s k_1(s,t,\tilde{x}_1(t),\tilde{x}_2(t))\,dt, \\ \tilde{x}_2'(s) &= g_2(s,\tilde{x}_1(s),\tilde{x}_2(s)) \\ &\quad \ominus \int_0^s k_2(s,t,\tilde{x}_1(t),\tilde{x}_2(t))\,dt,\end{aligned} \tag{41}$$

with initial conditions

$$\begin{aligned}&\tilde{x}_1(0) = (1+0.5(r-1),1-0.5(r-1)), \\ &\tilde{x}_2(0) = (-1+0.5(r-1),-1-0.5(r-1)), \\ &g_1(s,\tilde{x}_1(s),\tilde{x}_2(s)) = 1+s+s^2 \ominus \tilde{x}_2(s), \\ &g_2(s,\tilde{x}_1(s),\tilde{x}_2(s)) = -1-s \oplus \tilde{x}_1(s), \\ &k_1(s,t,\tilde{x}_1(t),\tilde{x}_2(t)) = \tilde{x}_1(t) \oplus \tilde{x}_2(t), \\ &k_2(s,t,\tilde{x}_1(t),\tilde{x}_2(t)) = \tilde{x}_1(t) \ominus \tilde{x}_2(t).\end{aligned} \tag{42}$$

From the initial conditions

$$\tilde{e}_0 = \begin{bmatrix} 1+0.5(r-1), 1-0.5(r-1) \\ -1+0.5(r-1), -1-0.5(r-1) \end{bmatrix}^T. \tag{43}$$

Let the solution of (41) be

$$\begin{aligned}\tilde{x}_1(s) &= \tilde{e}_{10} \oplus \tilde{e}_{11} \odot s \\ &= [1+0.5(r-1),1-0.5(r-1)] \oplus \tilde{e}_{11} \odot s, \\ \tilde{x}_2(s) &= \tilde{e}_{20} \oplus \tilde{e}_{21} \odot s \\ &= [-1+0.5(r-1),-1-0.5(r-1)] \oplus \tilde{e}_{21} \odot s.\end{aligned} \tag{44}$$

For obtaining $\tilde{e}_{11}$, $\tilde{e}_{21}$, we substitute (44) into (41); then we will have

$$\begin{aligned}&(\underline{e}_{11} - 2 - 0.5(r-1)) \\ &\quad + \left(-s - s^2 + \overline{e}_{21}s + (1-r)s - \frac{\overline{e}_{11}s^2}{2} - \frac{\underline{e}_{21}s^2}{2}\right) = 0, \\ &(\overline{e}_{11} - 2 + 0.5(r-1)) \\ &\quad + \left(-s - s^2 + \underline{e}_{21}s + (1-r)s - \frac{\underline{e}_{11}s^2}{2} - \frac{\overline{e}_{21}s^2}{2}\right) = 0, \\ &(\underline{e}_{21} - 0.5(r-1)) + \left(3s - \underline{e}_{11}s + \frac{\overline{e}_{11}s^2}{2} + \frac{\underline{e}_{21}s^2}{2}\right) = 0, \\ &(\overline{e}_{21} + 0.5(r-1)) + \left(3s - \overline{e}_{11}s + \frac{\underline{e}_{11}s^2}{2} + \frac{\overline{e}_{21}s^2}{2}\right) = 0, \\ &(\underline{e}_{11} - 2 - 0.5(r-1)) + \underline{q}_{11}(s) = 0, \\ &(\overline{e}_{11} - 2 + 0.5(r-1)) + \overline{q}_{11}(s) = 0, \\ &(\underline{e}_{21} - 0.5(r-1)) + \underline{q}_{21}(s) = 0, \\ &(\overline{e}_{21} + 0.5(r-1)) + \overline{q}_{21}(s) = 0,\end{aligned} \tag{45}$$

where $\underline{q}_{11}(s)$, $\overline{q}_{11}(s)$, $\underline{q}_{21}(s)$, $\overline{q}_{21}(s)$ are $O(s)$ and by neglecting them, we have

$$A_1 \odot \tilde{e}_1 = \tilde{b}_1, \tag{46}$$

where

$$\begin{aligned}A_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \tilde{b}_1 &= \begin{bmatrix} (2+0.5(r-1),2-0.5(r-1)) \\ (0.5(r-1),-0.5(r-1)) \end{bmatrix}, \\ \tilde{e}_1 &= \begin{bmatrix} (\underline{e}_{11},\overline{e}_{11}) \\ (\underline{e}_{21},\overline{e}_{21}) \end{bmatrix}.\end{aligned} \tag{47}$$

So,

$$\tilde{e}_1 = \begin{bmatrix} (2+0.5(r-1),2-0.5(r-1)) \\ (0.5(r-1),-0.5(r-1)) \end{bmatrix}. \tag{48}$$

And then

$$\begin{aligned}\tilde{x}_1(s) &= [1+0.5(r-1),1-0.5(r-1)] \\ &\quad \oplus [2+0.5(r-1),2-0.5(r-1)] \odot s, \\ \tilde{x}_2(s) &= [-1+0.5(r-1),-1-0.5(r-1)] \\ &\quad \oplus [0.5(r-1),-0.5(r-1)] \odot s.\end{aligned} \tag{49}$$

We go to next step. Let

$$\begin{aligned}\tilde{x}_1(s) &= [1+0.5(r-1),1-0.5(r-1)] \\ &\quad \oplus [2+0.5(r-1),2-0.5(r-1)] \odot s \oplus \tilde{e}_{12}s^2, \\ \tilde{x}_2(s) &= [-1+0.5(r-1),-1-0.5(r-1)] \\ &\quad \oplus [0.5(r-1),-0.5(r-1)] \odot s \oplus \tilde{e}_{22}s^2.\end{aligned} \tag{50}$$

Similar to previous step, by substituting (50) into (41), we have

$$\begin{aligned}&(2\underline{e}_{12} - 1.5(r-1) - 1)s \\ &\quad + \left(\underline{e}_{22}s^2 - (r-1)\frac{s^2}{2} + \frac{\overline{e}_{12}s^3}{3} + \frac{\overline{e}_{22}s^3}{3}\right) = 0,\end{aligned}$$

$$\begin{aligned}
&\left(2\overline{e}_{12}+1.5\left(r-1\right)-1\right)s\\
&\quad+\left(\underline{e}_{22}s^{2}+\left(r-1\right)\frac{s^{2}}{2}+\frac{\underline{e}_{12}s^{3}}{3}+\frac{\underline{e}_{22}s^{3}}{3}\right)=0,\\
&\left(2\underline{e}_{22}-1.5\left(r-1\right)+1\right)s\\
&\quad+\left(s^{2}-\underline{e}_{22}s^{2}+\left(r-1\right)\frac{s^{2}}{2}+\frac{\overline{e}_{12}s^{3}}{3}+\frac{\overline{e}_{22}s^{3}}{3}\right)=0,\\
&\left(2\overline{e}_{22}+1.5\left(r-1\right)+1\right)s\\
&\quad+\left(s^{2}+\underline{e}_{22}s^{2}+\left(r-1\right)\frac{s^{2}}{2}+\frac{\underline{e}_{12}s^{3}}{3}+\frac{\underline{e}_{22}s^{3}}{3}\right)=0.
\end{aligned}\tag{51}$$

So,

$$\begin{aligned}
&\left(2\underline{e}_{12}-1.5\left(r-1\right)-1\right)s+\underline{q}_{12}\left(s\right)=0,\\
&\left(2\overline{e}_{12}+1.5\left(r-1\right)-1\right)s+\overline{q}_{12}\left(s\right)=0,\\
&\left(2\underline{e}_{22}-1.5\left(r-1\right)+1\right)s+\underline{q}_{22}\left(s\right)=0,\\
&\left(2\overline{e}_{22}+1.5\left(r-1\right)+1\right)s+\overline{q}_{22}\left(s\right)=0.
\end{aligned}\tag{52}$$

By neglecting $\underline{q}_{12}(s)$, $\overline{q}_{12}(s)$, $\underline{q}_{22}(s)$, $\overline{q}_{22}(s)$ which are $O(s^2)$ and solve system $A_2 \odot \widetilde{e}_2 = \widetilde{b}_2$, we obtain

$$\widetilde{e}_2=\begin{bmatrix}\left(\frac{3}{4}\left(r-1\right)+\frac{1}{2},-\frac{3}{4}\left(r-1\right)+\frac{1}{2}\right)\\ \left(\frac{3}{4}\left(r-1\right)-\frac{1}{2},-\frac{3}{4}\left(r-1\right)-\frac{1}{2}\right)\end{bmatrix}\tag{53}$$

and then in a similar way go to next step and we have

$$\begin{aligned}
&\left(3\underline{e}_{13}-\frac{5}{4}\left(r-1\right)-0.5\right)s^{2}+\underline{q}_{13}\left(s\right)=0,\\
&\left(3\overline{e}_{13}-\frac{5}{4}\left(r-1\right)-0.5\right)s^{2}+\overline{q}_{13}\left(s\right)=0,\\
&\left(3\underline{e}_{23}-\frac{5}{4}\left(r-1\right)+0.5\right)s^{2}+\underline{q}_{23}\left(s\right)=0,\\
&\left(3\overline{e}_{23}+\frac{5}{4}\left(r-1\right)+0.5\right)s^{2}+\overline{q}_{23}\left(s\right)=0.
\end{aligned}\tag{54}$$

And

$$\widetilde{e}_3=\begin{bmatrix}\left(\frac{5}{12}\left(r-1\right)+\frac{1}{6},-\frac{5}{12}\left(r-1\right)+\frac{1}{6}\right)\\ \left(\frac{5}{12}\left(r-1\right)+\frac{1}{6},-\frac{5}{12}\left(r-1\right)+\frac{1}{6}\right)\end{bmatrix}.\tag{55}$$

Example 19. As second example we consider the following nonlinear fuzzy integrodifferential equation:

$$\begin{aligned}
\widetilde{x}'\left(s\right)&=\left(1,0.1,0.4\right)\oplus\int_{0}^{s}\widetilde{x}\left(t\right)\odot\widetilde{x}'\left(t\right)dt,\\
\widetilde{x}\left(0\right)&=\left(0,0.2,0.6\right).
\end{aligned}\tag{56}$$

Typically, we use the power series method for obtaining the solution of problem. From the initial condition, $\widetilde{e}(0) = (0, 0.2, 0.6)$, let the solution of (56) be the form

$$\widetilde{x}\left(s\right)=\widetilde{e}_{0}\oplus\widetilde{e}_{1}\odot s=\left(0,0.2,0.6\right)\oplus\widetilde{e}_{1}\odot s.\tag{57}$$

For obtaining $\widetilde{e}_1 = (m, \alpha, \beta)$, we substitute (57) into (56); we will have

$$\left(m,\alpha,\beta\right)=\left(1+\frac{ms^{2}}{2},0.1+\frac{\alpha s^{2}}{2}+0.2s,0.4+\frac{\beta s^{2}}{2}+0.3s\right),\tag{58}$$

or

$$\left(m,\alpha,\beta\right)=\left(1+q_{1}\left(s\right),0.1+q_{2}\left(s\right),0.4+q_{3}\left(s\right)\right).\tag{59}$$

By neglecting q_1, q_2, q_3 which are $O(s)$, we obtain $\widetilde{e}_1 = (1, 0.1, 0.4)$ and then

$$\widetilde{x}\left(s\right)=\left(0,0.2,0.6\right)\oplus\left(1,0.1,0.4\right)\odot s.\tag{60}$$

For the next step, we assume that

$$\widetilde{x}\left(s\right)=\widetilde{e}_{0}\oplus\widetilde{e}_{1}\odot s\oplus\widetilde{e}_{2}\odot s^{2}.\tag{61}$$

By substituting (61) into (56), we have

$$\begin{aligned}
&\left(1+2ms,0.1+\alpha s,0.4+\beta s\right)\\
&\quad=\left(1+\frac{s^{2}}{2},0.1+0.2s+q_{1}\left(s\right),0.4+0.6s+q_{2}\left(s\right)\right).
\end{aligned}\tag{62}$$

From above relation and by neglecting $s^2/2$, $q_1(s)$, $q_2(s)$, we have $\widetilde{e}_2 = (0, 0.1, 0.3)$.

By repeating this method, we can compute more coefficients of the solution.

Example 20. Consider the following nonlinear fuzzy integrodifferential equation:

$$\begin{aligned}
\widetilde{x}'\left(s\right)&=e^{s}-\frac{1}{3}e^{3s}+\left(\frac{1}{3},0.1,0.3\right)\oplus\int_{0}^{s}\widetilde{x}^{3}\left(t\right)dt,\\
\widetilde{x}\left(0\right)&=\left(1,0.4,0.4\right).
\end{aligned}\tag{63}$$

Again, we use the power series method for obtaining the solution of the problem. From the initial condition, $\widetilde{e}_0 = (1, 0.4, 0.4)$. Assume that the solution of (63) is the form

$$\widetilde{x}\left(s\right)=\widetilde{e}_{0}\oplus\widetilde{e}_{1}\odot s=\left(1,0.4,0.4\right)\oplus\widetilde{e}_{1}\odot s.\tag{64}$$

By substituting (64) into (63), we obtain

$$\left(m,\alpha,\beta\right)=\left(1+q_{1}\left(s\right),0.4+q_{2}\left(s\right),0.4+q_{3}\left(s\right)\right).\tag{65}$$

By neglecting $q_1(s)$, $q_2(s)$, $q_3(s)$, we obtain $\widetilde{e}_1 = (1, 0.4, 0.4)$ and then

$$\widetilde{x}\left(s\right)=\left(1,0.4,0.4\right)\oplus\left(1,0.4,0.4\right)\odot s.\tag{66}$$

For the next step, we assume that

$$\widetilde{x}\left(s\right)=\left(1,0.4,0.4\right)\oplus\left(1,0.4,0.4\right)s\oplus\widetilde{e}_{2}\odot s^{2}.\tag{67}$$

And by substituting it into (63), we have

$$\begin{aligned}&\left(1+2ms, 0.4+2\alpha s, 0.4+2\beta s\right)\\&\quad=\left(1+s+q_1(s), 0.4+0.6s+q_2(s), 0.4+0.6s+q_3(s)\right).\end{aligned} \tag{68}$$

From the above relation and by neglecting $q_1(s)$, $q_2(s)$, $q_3(s)$, we have

$$\widetilde{e}_2=\left(\frac{1}{2}, 0.3, 0.3\right). \tag{69}$$

By continuing this procedure, more coefficients of the solution can be computed.

5. Conclusion

In summary, this study has exploited power series to find a numerical solution for linear as well as nonlinear fuzzy Volterra integrodifferential equations. In effect, using power series can provide an approximate solution for the mentioned integral equations. Since there are challenging issues to solve the nonlinear integrodifferential equations, the presented method can be simply applied to find an appropriate solution for this kind of equations that is regarded as a considerable benefit of this method undoubtedly.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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