# Analysis and Models in Interdisciplinary Mathematics 

Guest Editors: Lucas Jódar Sánchez, Benito Chen-Charpentier, Juan Carlos Coriés López, Francisco Javier Solís Lozano, and Rafael Jacinto Villanueva Micó


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## Abstract and Applied Analysis

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Peixuan Weng, China
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## Contents

Analysis and Models in Interdisciplinary Mathematics, L. Jódar, Benito M. Chen-Charpentier, J.-C. Cortés, Francisco J. Solís, and R.-J. Villanueva

Volume 2014, Article ID 408980, 2 pages
Effect of Diffusion and Cross-Diffusion in a Predator-Prey Model with a Transmissible Disease in the
Predator Species, Guohong Zhang and Xiaoli Wang
Volume 2014, Article ID 167856, 12 pages
Identification of V-Formations and Circular and Doughnut Formations in a Set of Moving Entities with Outliers, Francisco Javier Moreno Arboleda, Jaime Alberto Guzmán Luna, and Sebastian Alonso Gomez Arias
Volume 2014, Article ID 241684, 11 pages
Global Positive Periodic Solutions for Periodic Two-Species Competitive Systems with Multiple Delays and Impulses, Zhenguo Luo, Liping Luo, Liu Yang, and Yunhui Zeng Volume 2014, Article ID 785653, 23 pages

The Influence Analysis of Number of Functional Logistics Service Providers on Quality Supervision Game in LSSC with Compensation Strategy, Weihua Liu, Yijia Wang, Zhicheng Liang, and Xiaoyan Liu Volume 2014, Article ID 956569, 15 pages

A Multicriteria Framework to Evaluate Supplier's Greenness, Elham Falatoonitoosi, Shamsuddin Ahmed, and Shahryar Sorooshian
Volume 2014, Article ID 396923, 12 pages
Choice Overload, Satisficing Behavior, and Price Distribution in a Time Allocation Model, Francisco Álvarez, José-Manuel Rey, and Raúl G. Sanchis Volume 2014, Article ID 569054, 9 pages

Modelling Oil-Spill Detection with Swarm Drones, F. Aznar, M. Sempere, M. Pujol, R. Rizo, and M. J. Pujol Volume 2014, Article ID 949407, 14 pages

Characterization of Consistent Completion of Reciprocal Comparison Matrices, Julio Benítez, Laura Carrión, Joaquín Izquierdo, and Rafael Pérez-García Volume 2014, Article ID 349729, 12 pages

A Study about the Integration of the Elliptical Orbital Motion Based on a Special One-Parametric Family of Anomalies, José Antonio López Ortí, Francisco José Marco Castillo, and María José Martínez Usó Volume 2014, Article ID 162060, 11 pages

Resolution of the Generalized Eigenvalue Problem in the Neutron Diffusion Equation Discretized by the Finite Volume Method, Álvaro Bernal, Rafael Miró, Damián Ginestar, and Gumersindo Verdú Volume 2014, Article ID 913043, 15 pages

Global Stability of Multigroup SIRS Epidemic Model with Varying Population Sizes and Stochastic Perturbation around Equilibrium, Xiaoming Fan Volume 2014, Article ID 154725, 14 pages

Error Analysis of Some Demand Simplifications in Hydraulic Models of Water Supply Networks, Joaquín Izquierdo, Enrique Campbell, Idel Montalvo, Rafael Pérez-García, and David Ayala-Cabrera Volume 2013, Article ID 169670, 13 pages

Exact and Analytic-Numerical Solutions of Lagging Models of Heat Transfer in a Semi-Infinite
Medium, M. A. Castro, F. Rodríguez, J. Escolano, and J. A. Martín
Volume 2013, Article ID 397053, 6 pages
Forecasting Latin America's Country Risk Scores by Means of a Dynamic Diffusion Model,
R. Cervelló-Royo, J.-C. Cortés, A. Sánchez-Sánchez, F.-J. Santonja, and R.-J. Villanueva

Volume 2013, Article ID 264657, 11 pages
Hot Topic Propagation Model and Opinion Leader Identifying Model in Microblog Network, Yan Lin, Huaxian Li, Xueqiao Liu, and Suohai Fan
Volume 2013, Article ID 893961, 13 pages
Forecasting $\mathrm{SO}_{2}$ Pollution Incidents by means of Elman Artificial Neural Networks and ARIMA Models, Antonio Bernardo Sánchez, Celestino Ordóñez, Fernando Sánchez Lasheras, Francisco Javier de Cos Juez, and Javier Roca-Pardiñas
Volume 2013, Article ID 238259, 6 pages
Preliminary Orbit Determination of Artificial Satellites: A Vectorial Sixth-Order Approach, Carlos Andreu, Noelia Cambil, Alicia Cordero, and Juan R. Torregrosa
Volume 2013, Article ID 960582, 10 pages
The Perturbed Dual Risk Model with Constant Interest and a Threshold Dividend Strategy, Fanzi Zeng and Jisheng Xu
Volume 2013, Article ID 981076, 9 pages
Mathematical Modeling of the Propagation of Democratic Support of Extreme Ideologies in Spain: Causes, Effects, and Recommendations for Its Stop, E. De la Poza, L. Jódar, and A. Pricop
Volume 2013, Article ID 729814, 8 pages
Turing Patterns in a Predator-Prey System with Self-Diffusion, Hongwei Yin, Xiaoyong Xiao, and Xiaoqing Wen
Volume 2013, Article ID 891738, 10 pages
Mathematical Modeling of the Consumption of Low Invasive Plastic Surgery Practices: The Case of Spain, E. De la Poza, M. S. S. Alkasadi, and L. Jódar
Volume 2013, Article ID 169253, 8 pages
Dealing with Dependent Uncertainty in Modelling: A Comparative Study Case through the Airy
Equation, J.-C. Cortés, J.-V. Romero, M.-D. Roselló, and R.-J. Villanueva
Volume 2013, Article ID 279642, 12 pages
Epidemic Random Network Simulations in a Distributed Computing Environment, J. Villanueva-Oller, L. Acedo, J. A. Moraño, and A. Sánchez-Sánchez

Volume 2013, Article ID 462801, 10 pages

Analytic-Numerical Solution of Random Boundary Value Heat Problems in a Semi-Infinite Bar, M.-C. Casabán, J.-C. Cortés, B. García-Mora, and L. Jódar Volume 2013, Article ID 676372, 9 pages

A Structural Equation Model for Analysis of Factors Associated with the Choice of Engineering Degrees in a Technical University, Antonio Hervás, Joan Guàrdia Olmos, Maribel Peró Cebollero, Roberto Capilla Lladró, and Pedro Pablo Soriano Jiménez
Volume 2013, Article ID 368529, 7 pages
Two-Dimension Hydrodynamic Dispersion Equation with Seepage Velocity and Dispersion Coefficient as Function of Space and Time, Abdon Atangana and S. C. Oukouomi Noutchie Volume 2013, Article ID 206942, 7 pages

## Editorial

# Analysis and Models in Interdisciplinary Mathematics 

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People understand analysis concluding with numbers better than discourse analysis, because writing well is not easy and, to be exact, writing is only a benefit of genius. Analysis without numbers is always under the risk of ambiguity and lack of accuracy. Comparison with numbers is easier than with words. Mathematics is a powerful quantification tool, but we need a quantifiable model, a mathematical model.

The transit from the real problem to the mathematical model is the task of mathematical modelling. Social sciences usually waste the mathematical power of analysis, because they do not use mathematical models, just some statistics that mainly provide fixed pictures of the problem under study.

It is true that human problems are not easy because of their emotional and not always rational nature. But as humans, we are creatures of habits, we practice human herding, and we are mimetic and imitate other fellows. These characteristics make the modelling of human behaviour possible.

Mathematical innovation does not need to find new theorems or algorithms, but they may be found in the mathematical model construction that should be an acceptable representation of the real problem. Not every model is acceptable; it must be reliable in the sense that although restricted by some hypotheses, these must be clearly exposed. Important factors cannot be disregarded, and real data must be obtained.

Probably the main difference between abstract and applied analysis lies in the construction or not of a new model and the interaction with another discipline where
the problem is originated. Without interaction with other disciplines or without real data, the innovation becomes empty and unreliable.

The boundary between innovation and mathematical irrelevance is subtle. Correct results could be zero innovative; just true theorems with inaccessible hypotheses are irrelevant. Thus, the innovation or novelty can arrive from the mathematical model construction or from the method to compute and simulate the model. Clear hypotheses, conclusions, and recommendations are suitable.

This special issue contains both types of contributions, those coming from the model construction and others that, starting from a given model, provide insight in the computing and simulation of the mathematical model.

This special issue contains interdisciplinary models related to the following:
(i) the propagation of opinions in microblog networks,
(ii) finding influence factors on the Spanish students selecting an engineering degree in the university,
(iii) the behaviour of consumer satisfaction,
(iv) quality supervising efficiency in logistic service providers,
(v) the risk related to threshold dividend strategies,
(vi) the identification of several formations in a set of moving entities with outliers,
(vii) forecasting Latin America's country risk scores,
(viii) the propagation of democratic support of extreme ideologies in Spain,
(ix) oil-spill detection with swarm drones,
(x) forecasting $\mathrm{SO}_{2}$ pollution incidents,
(xi) propagation of the consumption of low invasive plastic surgery in Spain,
(xii) the effect of diffusion in a predator-prey system when predator species suffers a transmissible disease.

Methods for solving interdisciplinary models include the following:
(i) random and deterministic analytic-numerical methods for heat problems,
(ii) comparison of random methods for random ordinary differential equations,
(iii) vector algebraic algorithms to determine satellite orbits,
(iv) solving of hydrodynamic dispersion equation,
(v) error analysis for hydraulic models in water supply networks,
(vi) epidemic random network simulation in a distributed computing environment,
(vii) generalized eigenvalue problem in the neutron diffusion equation,
(viii) integration of elliptic orbital motion,
(ix) two species competitive systems with multiple delays and impulses,
(x) consistent completion of reciprocal comparison matrices,
(xi) global stability of multigroup SIRS epidemic models with varying population sizes and stochastic perturbation around equilibrium,
(xii) evaluation of supplier greenness.

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L. Jódar

Benito M. Chen-Charpentier
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R.-J. Villanueva

## Research Article

# Effect of Diffusion and Cross-Diffusion in a Predator-Prey Model with a Transmissible Disease in the Predator Species 

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#### Abstract

We study a Lotka-Volterra type predator-prey model with a transmissible disease in the predator population. We concentrate on the effect of diffusion and cross-diffusion on the emergence of stationary patterns. We first show that both self-diffusion and crossdiffusion can not cause Turing instability from the disease-free equilibria. Then we find that the endemic equilibrium remains linearly stable for the reaction diffusion system without cross-diffusion, while it becomes linearly unstable when cross-diffusion also plays a role in the reaction-diffusion system; hence, the instability is driven solely from the effect of cross-diffusion. Furthermore, we derive some results for the existence and nonexistence of nonconstant stationary solutions when the diffusion rate of a certain species is small or large.


## 1. Introduction

The study of the dynamic relationship between predator and prey has long been one of the most important themes in population dynamics because of its universal existence in nature and many different phenomena have been observed (see [1-13] and references therein). At the same time, since species need to interact with the environment, they are always subject to diseases in the natural world. So it is necessary and interesting to combine demographic as well as epidemic aspects in the standard classical population models. This type of systems is now known as ecoepidemic model.

In fact, the importance of disease influence on the dynamics of plant as well as animal populations has been recognized and several such studies are reviewed in a number of recent publications. However, most of the previous researches on ecoepidemic models assume that the distribution of the predators and prey is homogeneous, which leads to the ODE system (see [14-23] and references therein). As we know, both predators and prey have the natural tendency to diffuse to areas of smaller population concentration. At the same time, some prey species always congregate and form a huge group to protect themselves from the attack of infected predator. So it is important to take into account the inhomogeneous
distribution of the predators and prey within a fixed bounded domain $\Omega$ and consider the effect of diffusion and crossdiffusion.

In order to construct the corresponding reactiondiffusion type model, we first propose the following assumptions, which are proper in biological background.
(H1) The disease spreads among the predator species only by contact and the disease incidence follows the simple law of mass action.
(H2) In the absence of predators, the prey population $u_{1}$ grows logistically with the intrinsic growth rate $r>$ 0 and carrying capacity $r / B$, in which $B$ measures intraspecific competition of the prey.
(H3) The sound predator population $u_{2}$ has no other food sources, and $\mu>0$ represents natural mortality. The infected predator population $u_{3}$ cannot recover and their total death rate $d>0$ encompasses natural and disease-related mortality. The conversion factor of a consumed prey into a sound or infected predator is $0<e<1$.
(H4) The sound and infected predators hunt the prey with different searching efficiencies, denoted, respectively,
by $m$ and $p m$, with $0<p<1$. This is due to the fact that sound predators are more efficient to catch the prey than the infected ones, weakened by the infection.
(H5) Both predators and prey have the natural tendency to diffuse to areas of smaller population concentration and the natural dispersive forces of movements of the prey, sound predators, and infected predators are $d_{1}$, $d_{2}$, and $d_{3}$, respectively.
(H6) The prey species congregate and form a huge group to protect themselves from the attack of infected predator.

With the above assumptions, our model takes the following form, in which all parameters are assumed to be positive:

$$
\begin{align*}
& \frac{\partial u_{1}}{\partial t}-d_{1} \Delta u_{1}=r u_{1}-B u_{1}^{2}-m u_{1} u_{2}-p m u_{1} u_{3} \\
& \text { in } \Omega \times(0, \infty), \\
& \frac{\partial u_{2}}{\partial t}-d_{2} \Delta u_{2}=e m u_{1} u_{2}-a u_{2} u_{3}-\mu u_{2} \\
& \text { in } \Omega \times(0, \infty), \\
& \begin{aligned}
\frac{\partial u_{3}}{\partial t}-d_{3} \Delta\left(u_{3}+d_{4} u_{1} u_{3}\right)=a u_{2} u_{3}+e m p u_{1} u_{3}-\delta u_{3} \\
\partial_{\nu} u_{1}=\partial_{\gamma} u_{2}=\partial_{\gamma} u_{3}=0 \\
\text { on } \partial \Omega \times(0, \infty), \\
\text { in } \Omega \times(0, \infty), \\
\left(u_{1}(0, x), u_{2}(0, x), u_{3}(0, x)\right) \geq(0,0,0)
\end{aligned} \\
& \text { on } \Omega,
\end{align*}
$$

where $\Omega$ is a bounded domain in $R^{N}(N \geq 1$ is an integer) with a smooth boundary $\partial \Omega$ and $\nu$ is the outward unit rector on $\partial \Omega$. The homogeneous Neumann boundary condition indicates that there is zero population flux across the boundary. In the diffusion terms, the constant $d_{i}(i=$ $1,2,3$ ), which is usually termed self-diffusion coefficient, represents the natural dispersive force of movement of an individual. The constant $d_{3} d_{4}$ could be referred to as crossdiffusion pressure, which describes a mutual interference between individuals.

In fact, it is easy to see that the infected predator $u_{3}$ diffuses with flux:

$$
\begin{align*}
\mathbf{J} & =-\nabla\left(d_{3} u_{3}+d_{3} d_{4} u_{1} u_{3}\right) \\
& =-d_{3} d_{4} u_{3} \nabla u_{1}-\left(d_{3}+d_{3} d_{4} u_{1}\right) \nabla u_{3} \tag{2}
\end{align*}
$$

As $d_{3} d_{4} u_{3}<0$, the part $-d_{3} d_{4} u_{3} \nabla u_{1}$ of the flux is directed toward the decreasing population density of the prey $u_{1}$, which means that the prey species congregate and form a huge group to protect themselves from the attack of infected predator. We remark that this kind of nonlinear diffusion was
first introduced by Shigesada et al. [24] and has been used in different type of population models [25-28]. We also point out that the corresponding ODE system of (1) with delay has been studied by [29], and they mainly investigate the stability and bifurcations related to the two most important equilibria of the ecoepidemic system, namely, the endemic equilibrium and the disease-free one.

Since the first example of stationary patterns in a predator-prey system arising solely from the effect of crossdiffusion is introduced by Pang and Wang [30], recently, more attention has been given to investigate the effect of crossdiffusion in reaction-diffusion systems; see, for example, [3136] and references therein. Here we point out that, to our knowledge, there is little work about ecoepidemic models with diffusion and cross-diffusion was discussed.

In our work here, one of the main purposes is to study the existence of positive stationary solutions of (1) by using degree theory, which are the positive solutions of

$$
\begin{gather*}
-d_{1} \Delta u_{1}=r u_{1}-B u_{1}^{2}-m u_{1} u_{2}-p m u_{1} u_{3} \quad \text { in } \Omega, \\
-d_{2} \Delta u_{2}=e m u_{1} u_{2}-a u_{2} u_{3}-\mu u_{2} \quad \text { in } \Omega, \\
-d_{3} \Delta\left(u_{3}+d_{4} u_{1} u_{3}\right)=a u_{2} u_{3}+e m p u_{1} u_{3}-\delta u_{3} \tag{3}
\end{gather*}
$$

$$
\partial_{\nu} u_{1}=\partial_{\nu} u_{2}=\partial_{\nu} u_{3}=0 \quad \text { on } \partial \Omega
$$

in $\Omega$,

Hence we are interested in nonconstant positive solutions of (3), which correspond to coexistence states of prey and predators. For convenience, we denote $\Lambda=(r, B, m, p, e, a, \mu, \delta)$. By a direct computation, we can show that (3) has a semitrivial constant steady state $\mathbf{u}_{0}^{*}=\left(u_{01}^{*}, u_{02}^{*}, u_{03}^{*}\right)=(\mu / e m)$, erm $\left.\mu B) / e m^{2}, 0\right)$ if $\mathrm{erm}>\mu B$ and has a positive constant steady state $\mathbf{u}^{*}=\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}\right)$, where

$$
\begin{gather*}
u_{1}^{*}=\frac{r a-m(\delta-\mu p)}{a B}, \quad u_{2}^{*}=\frac{\delta-e m p u^{*}}{a}  \tag{4}\\
u_{3}^{*}=\frac{e m u^{*}-\mu}{a}
\end{gather*}
$$

provided that

$$
\begin{gather*}
a r>m(\delta-\mu p) \\
\frac{\delta}{e m p}>\frac{r a-m(\delta-\mu p)}{a B}>\frac{\mu}{e m} \tag{5}
\end{gather*}
$$

Here we remark that the semitrivial constant steady state $\mathbf{u}_{0}^{*}$ and the positive constant steady state $\mathbf{u}^{*}$ are also called disease-free equilibrium and endemic equilibrium, respectively, in endemic models.

The rest of this paper is organized as follows. In Section 2, we will investigate the stability of disease-free equilibrium $\mathbf{u}_{0}^{*}$ and the endemic equilibrium $\mathbf{u}^{*}$ and show that the cross-diffusion destabilizes a uniform equilibrium which is stable for the kinetic and self-diffusion reaction systems. In Section 3, a priori upper bounds and lower bounds for the nonconstant positive solutions of (3) are given. In Section 4,
we study nonexistence of nonconstant positive solutions of model (3) when considering only the self-diffusion. Finally, in Section 5, we investigate the existence of the nonconstant positive solutions of (3) by using the Leray-Schauder degree theory, which explains why shrub ecosystem generates patterns.

## 2. Stability Analysis of the Constant Solutions $\mathbf{u}_{0}^{*}$ and $\mathbf{u}^{*}$

In order to study the stability of the constant steady states $\mathbf{u}_{0}^{*}$ and $\mathbf{u}^{*}$ of (1), we first set up the following notation.

Notation 1. Consider the following.
(i) $0=\mu_{0}<\mu_{1}<\mu_{2}<\cdots$ are the eigenvalues of $-\Delta$ in $\Omega$ under homogeneous Neumann boundary condition.
(ii) $S\left(\mu_{i}\right)$ is the set of eigenfunctions corresponding to $\mu_{i}$.
(iii) $\mathbf{X}_{i j}:=\mathbf{c} \varphi_{i j}: \mathbf{c} \in \mathbb{R}^{3}$, where $\varphi_{i j}$ are orthonormal basis of $S\left(\mu_{i}\right)$ for $j=1, \ldots, \operatorname{dim}\left[S\left(\mu_{i}\right)\right]$.
(iv) $\mathbf{X}:=\left\{\left(u_{1}, u_{2}, u_{3}\right) \in\left[C^{1}(\bar{\Omega})\right]^{3}: \partial u_{1} / \partial v=\partial u_{2} / \partial v=\right.$ $\partial u_{3} / \partial v=0$ on $\left.\partial \Omega\right\}$, and so $\mathbf{X}=\oplus_{i=1}^{\infty} \oplus_{j=1}^{\operatorname{dim}\left[S\left(\mu_{i}\right)\right]} \mathbf{X}_{i j}$.

Now, we first consider system (1) without cross-diffusion and introduce the following system:

$$
\begin{aligned}
& \frac{\partial u_{1}}{\partial t}-d_{1} \Delta u_{1}=r u_{1}-B u_{1}^{2}-m u_{1} u_{2}-p m u_{1} u_{3} \\
& \text { in } \Omega \times(0, \infty), \\
& \frac{\partial u_{2}}{\partial t}-d_{2} \Delta u_{2}=e m u_{1} u_{2}-a u_{2} u_{3}-\mu u_{2} \\
& \text { in } \Omega \times(0, \infty), \\
& \begin{array}{r}
\frac{\partial u_{3}}{\partial t}-d_{3} \Delta u_{3}=a u_{2} u_{3}+e m p u_{1} u_{3}-\delta u_{3} \\
\text { in } \Omega \times(0, \infty), \\
\partial_{\gamma} u_{1}=\partial_{\gamma} u_{2}=\partial_{\gamma} u_{3}=0 \\
\text { on } \partial \Omega \times(0, \infty), \\
\left(u_{1}(0, x), u_{2}(0, x), u_{3}(0, x)\right) \geq(0,0,0) \\
\text { on } \Omega .
\end{array}
\end{aligned}
$$

Obviously, system (6) has the same disease-free equilibrium $\mathbf{u}_{0}^{*}$ and endemic equilibrium $\mathbf{u}^{*}$ with system (1). From (6), we can get the following theorem.

Theorem 1. (i) If erm $>\mu B$ and $(\mathrm{erm}+\mu B) / \mathrm{em}^{2}<$ $\delta / a$, the disease-free equilibrium $\mathbf{u}_{0}^{*}$ of system (6) is locally asymptotically stable.
(ii) Assume that (5) holds. The endemic equilibrium $\mathbf{u}^{*}$ of system (6) is locally asymptotically stable.

Proof. (i) For simplicity, throughout this paper, we denote

$$
\begin{align*}
\mathbf{G}(\mathbf{u}) & =\left(\begin{array}{l}
G_{1}(\mathbf{u}) \\
G_{2}(\mathbf{u}) \\
G_{3}(\mathbf{u})
\end{array}\right) \\
& =\left(\begin{array}{c}
r u_{1}-B u_{1}^{2}-m u_{1} u_{2}-p m u_{1} u_{3} \\
e m u_{1} u_{2}-a u_{2} u_{3}+\mu u_{2} \\
a u_{2} u_{3}+e m p u_{1} u_{3}-\delta u_{3}
\end{array}\right) . \tag{7}
\end{align*}
$$

By a direct calculation, we obtain

$$
\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right)=\left(\begin{array}{ccc}
-B u_{1}^{*} & -m u_{01}^{*} & -p m u_{01}^{*}  \tag{8}\\
e m u_{02}^{*} & 0 & -a u_{02}^{*} \\
0 & 0 & a u_{02}^{*}-\delta
\end{array}\right)
$$

The linearization of (6) at $\mathbf{u}_{0}^{*}$ can be expressed by

$$
\begin{equation*}
\mathbf{u}_{t}=\left(\mathscr{D} \Delta+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right)\right) \mathbf{u} \tag{9}
\end{equation*}
$$

where

$$
\mathscr{D}=\left(\begin{array}{ccc}
d_{1} & 0 & 0  \tag{10}\\
0 & d_{2} & 0 \\
0 & 0 & d_{3}
\end{array}\right)
$$

According to Notation 1, $\mathbf{X}_{i}$ is invariant under the operator $\mathscr{D} \Delta+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right)$, and $\lambda$ is an eigenvalue of this operator on $\mathbf{X}_{i}$ if and only if it is an eigenvalue of the matrix $-\mu_{i} \mathscr{D}+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right)$.

A direct calculation shows that the characteristic polynomial of $-\mu_{i} \mathscr{D}+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right)$ can be given by

$$
\begin{align*}
\phi_{0 i}(\lambda)= & \left(\lambda+\delta-a u_{02}^{*}+d_{3} \mu_{i}\right) \\
& \times\left[\lambda^{2}+\left(B u_{01}^{*}+d_{1} \mu_{i}+d_{2} \mu_{i}\right) \lambda+e m^{2} u_{01}^{*} u_{02}^{*}\right] . \tag{11}
\end{align*}
$$

It follows from (11) that, if $(e r m+\mu B) / e m^{2}<\delta / a$, the corresponding eigenvalues have negative real parts for all $i \geq$ 1 , so we know that $u_{0}^{*}$ is locally asymptotically stable.
(ii) Since $\mathbf{G}\left(\mathbf{u}^{*}\right)=0$, it follows from (7) that

$$
\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)=\left(\begin{array}{ccc}
-B u_{1}^{*} & -m u_{1}^{*} & -p m u_{1}^{*}  \tag{12}\\
e m u_{2}^{*} & 0 & -a u_{2}^{*} \\
e m p u_{3}^{*} & a u_{3}^{*} & 0
\end{array}\right)
$$

The linearization of (6) at $\mathbf{u}^{*}$ can be expressed by

$$
\begin{equation*}
\mathbf{u}_{t}=\left(\mathscr{D} \Delta+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right) \mathbf{u} \tag{13}
\end{equation*}
$$

where the matrix $\mathscr{D}$ is defined in (10). Direct calculation shows that the characteristic polynomial of $-\mu_{i} \mathscr{D}+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$ is given by

$$
\begin{equation*}
\phi_{i}(\lambda)=\lambda^{3}+c_{1} \lambda^{2}+c_{2} \lambda+c_{3}, \tag{14}
\end{equation*}
$$

where

$$
\begin{align*}
& c_{1}=c_{1}\left(\mu_{i}\right)=\left(d_{1}+d_{2}+d_{3}\right) \mu_{i}+B u_{1}^{*}, \\
& c_{2}=c_{2}\left(\mu_{i}\right)=\left(d_{1} d_{2}+d_{1} d_{3}+d_{2} d_{3}\right) \mu_{i}^{2}+\left(d_{2}+d_{3}\right) B u_{1}^{*} \mu_{i} \\
& +\left(e m^{2} u_{1}^{*} u_{2}^{*}+e m^{2} p^{2} u_{1}^{*} u_{3}^{*}+a^{2} u_{2}^{*} u_{3}^{*}\right), \\
& c_{3}= \\
& c_{3}\left(\mu_{i}\right)=d_{1} d_{2} d_{3} \mu_{i}^{3}+d_{2} d_{3} B u_{1}^{*} \mu_{i}^{2} \\
& +\left(d_{1} a^{2} u_{2}^{*} u_{3}^{*}+d_{2} e m^{2} p^{2} u_{1}^{*} u_{3}^{*}+d_{3} e m^{2} u_{1}^{*} u_{2}^{*}\right) \mu_{i}  \tag{15}\\
& +B a^{2} u_{1}^{*} u_{2}^{*} u_{3}^{*} .
\end{align*}
$$

It is easy to see that $c_{1}, c_{2}$, and $c_{3}$ are positive.
Notice that

$$
\begin{align*}
c_{1} c_{2}-c_{3}= & \left(d_{2}+d_{3}\right)\left(d_{1}^{2}+d_{1} d_{2}+d_{1} d_{3}+d_{2} d_{3}\right) \mu_{i}^{3} \\
& +B u_{1}^{*}\left(d_{2}+d_{3}\right)\left(2 d_{1}+d_{2}+d_{3}\right) \mu_{i}^{2} \\
& +\left[\left(B u_{1}^{*}\right)^{2}\left(d_{2}+d_{3}\right)\right. \\
& +d_{1}\left(e m^{2} u_{1}^{*} u_{2}^{*}+e m^{2} p^{2} u_{1}^{*} u_{3}^{*}\right)  \tag{16}\\
& +d_{2}\left(e m^{2} u_{1}^{*} u_{2}^{*}+a^{2} u_{2}^{*} u_{3}^{*}\right) \\
& \left.+d_{3}\left(e m^{2} p^{2} u_{1}^{*} u_{3}^{*}+a^{2} u_{2}^{*} u_{3}^{*}\right)\right] \mu_{i} \\
& +B u_{1}^{*}\left(e m^{2} u_{1}^{*} u_{2}^{*}+e m^{2} p^{2} u_{1}^{*} u_{3}^{*}\right)>0
\end{align*}
$$

Then by the Routh-Hurwitz criterion, we know that, for each $i \geq 1$, all the three roots $\lambda_{i, 1}, \lambda_{i, 2}$, and $\lambda_{i, 3}$ of characteristic equation $\phi_{i}(\lambda)=0$ have negative real parts. Now we can prove that there exists a positive constant $\delta$ such that

$$
\begin{equation*}
\operatorname{Re}\left\{\lambda_{i 1}\right\}, \quad \operatorname{Re}\left\{\lambda_{i 1}\right\}, \quad \operatorname{Re}\left\{\lambda_{i 1}\right\} \leq-\delta, \quad i \geq 1 \tag{17}
\end{equation*}
$$

In fact, let $\lambda=\mu_{i} \xi$; then we have

$$
\begin{equation*}
\phi_{i}(\lambda)=\mu_{i}^{3} \xi^{3}+c_{1} \mu_{i}^{2} \xi^{2}+c_{2} \mu_{i} \xi+c_{3} \triangleq \widetilde{\phi}_{i}(\xi) \tag{18}
\end{equation*}
$$

Note that $\mu_{i} \mapsto \infty$ as $i \mapsto \infty$. It follows that

$$
\begin{align*}
\lim _{i \mapsto \infty} \frac{\tilde{\phi}_{i}(\xi)}{\mu_{i}^{3}}= & \xi^{3}+\left(d_{1}+d_{2}+d_{3}\right) \xi^{2} \\
& +\left(d_{1} d_{2}+d_{1} d_{3}+d_{2} d_{3}\right) \xi+d_{1} d_{2} d_{3} \triangleq \widetilde{\phi}(\xi) \tag{19}
\end{align*}
$$

Using the Routh-Hurwitz criterion again, we can see that all the three roots $\xi_{1}, \xi_{2}$, and $\xi_{3}$ of equation $\widetilde{\phi}(\xi)=0$ have negative real parts. Thus, there exists a positive constant such that

$$
\begin{equation*}
\operatorname{Re}\left\{\xi_{1}\right\}, \quad \operatorname{Re}\left\{\xi_{2}\right\}, \quad \operatorname{Re}\left\{\xi_{3}\right\} \leq-\bar{\delta} \tag{20}
\end{equation*}
$$

By continuity, we know that there exists $i_{0} \in N$ such that the three roots $\xi_{i 1}, \xi_{i 2}$, and $\xi_{i 3}$ of $\widetilde{\phi}_{i}(\xi)=0$ satisfy

$$
\begin{equation*}
\operatorname{Re}\left\{\xi_{i 1}\right\}, \quad \operatorname{Re}\left\{\xi_{i 2}\right\}, \quad \operatorname{Re}\left\{\xi_{i 3}\right\} \leq \frac{-\bar{\delta}}{2}, \quad i \geq i_{0} \tag{21}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\operatorname{Re}\left\{\lambda_{i 1}\right\}, \quad \operatorname{Re}\left\{\lambda_{i 2}\right\}, \quad \operatorname{Re}\left\{\lambda_{i 3}\right\} \leq \frac{-\mu_{i} \bar{\delta}}{2} \leq \frac{-\bar{\delta}}{2}, \quad i \geq i_{0} \tag{22}
\end{equation*}
$$

Let

$$
\begin{equation*}
-\widetilde{\delta}=\max _{1 \leq i \leq i_{0}}\left\{\operatorname{Re}\left\{\lambda_{i 1}\right\}, \operatorname{Re}\left\{\lambda_{i 2}\right\}, \operatorname{Re}\left\{\lambda_{i 3}\right\}\right\} \tag{23}
\end{equation*}
$$

then $\widetilde{\delta}>0$, and (17) holds for $\delta=\min \{\widetilde{\delta}, \bar{\delta} / 2\}$. Thus the proof is completed by Theorem 5.1.1 of Henry [37].

Remark 2. From Theorem 1, we can see that if only the free diffusion is introduced to the corresponding ODE system of (1), the uniform positive stationary solution is also locally stable, which means that only self-diffusion cannot induce Turing instability.

We now consider the effect of the cross-diffusion and introduce the following theorem, which give the necessary conditions for the existence of nonconstant positive solution of system (3).

Theorem 3. Consider the following.
(i) If erm $>\mu B$ and $(e r m+\mu B) / \mathrm{em}^{2}<\delta / a$, the disease-free equilibrium $\mathbf{u}_{0}^{*}$ of system (1) is locally asymptotically stable.
(ii) Assume that (5) holds and $d_{4}>0$ in (1). Suppose that $b_{2}<0$ and $b_{2}^{2}-4 b_{1} b_{3}>0$, where $b_{i}$ is given in (31). If $\mu_{2}^{*} \in\left(\mu_{i}, \mu_{i+1}\right)$ and $\mu_{3}^{*} \in\left(\mu_{j}, \mu_{j+1}\right)$ for some $j>i>$ 0 , where $\mu_{2}^{*}$ and $\mu_{3}^{*}$ are defined in (34), there exists a positive constant $d_{3}^{*}$ such that the uniform stationary solution $\mathbf{u}^{*}$ of (1) is unstable when $d_{3} \geq d_{3}^{*}$.

Proof. (i) For simplicity, we denote that $\Phi(\mathbf{u})=\left(d_{1} u_{1}, d_{2} u_{2}\right.$, $\left.d_{3}\left(u_{3}+d_{4} u_{4}\right)\right)^{\mathrm{T}}$. Then the linearized system of system (1) at $\mathbf{u}_{0}^{*}$ is

$$
\begin{equation*}
\mathbf{u}_{t}=\left(\Phi_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right) \Delta+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right)\right) \mathbf{u} \tag{24}
\end{equation*}
$$

where

$$
\Phi_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right)=\left(\begin{array}{ccc}
d_{1} & 0 & 0  \tag{25}\\
0 & d_{2} & 0 \\
0 & 0 & d_{3}+d_{3} d_{4} u_{01}^{*}
\end{array}\right)
$$

By some calculations, the characteristic polynomial of $-\mu_{i} \Phi_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right)+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}_{0}^{*}\right)$ can be given by

$$
\begin{align*}
\psi_{0 i}(\lambda)= & \left(\lambda+\delta-a u_{02}^{*}+d_{3} \mu_{i}+d_{3} d_{4} u_{01}^{*} \mu_{i}\right) \\
& \times\left(\lambda^{2}+\left(B u_{01}^{*}+d_{1} \mu_{i}+d_{2} \mu_{i}\right) \lambda+e m^{2} u_{01}^{*} u_{02}^{*}\right) \tag{26}
\end{align*}
$$

It is easy to see that, if $(e r m+\mu B) / e m^{2}<\delta / a$, all the corresponding eigenvalues of $\phi_{0 i}(\lambda)=0$ have negative real parts for all $i \geq 1$, which implies that $u_{0}^{*}$ is locally asymptotically stable.
(ii) The linearized system of system (1) at $\mathbf{u}^{*}$ is

$$
\begin{equation*}
\mathbf{u}_{t}=\left(\Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right) \Delta+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right) \mathbf{u} \tag{27}
\end{equation*}
$$

where

$$
\Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right)=\left(\begin{array}{ccc}
d_{1} & 0 & 0  \tag{28}\\
0 & d_{2} & 0 \\
d_{3} d_{4} u_{3}^{*} & 0 & d_{3}+d_{3} d_{4} u_{1}^{*}
\end{array}\right)
$$

By some calculations, the characteristic polynomial of $-\mu_{i} \Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right)+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$ can be given by

$$
\begin{equation*}
\Psi_{i}(\lambda)=\lambda^{3}+\bar{c}_{1} \lambda^{2}+\bar{c}_{2} \lambda+\bar{c}_{3} \tag{29}
\end{equation*}
$$

where

$$
\begin{align*}
& \bar{c}_{1}=\left(d_{1}+d_{2}+d_{3}+d_{3} d_{4} u_{1}^{*}\right) \mu_{i}+B u_{1}^{*} \\
\bar{c}_{2}= & {\left[d_{1} d_{2}+d_{1} d_{3}+d_{2} d_{3}+\left(d_{1} d_{3} d_{4}+d_{2} d_{3} d_{4}\right) u_{1}^{*}\right] \mu_{i}^{2} } \\
& +\left[B\left(d_{2}+d_{3}+d_{3} d_{4}\right)\left(u_{1}^{*}\right)^{2}-d_{3} d_{4} p m u_{1}^{*} u_{2}^{*}\right] \mu_{i} \\
& +e m^{2} u_{1}^{*} u_{2}^{*}+e m^{2} p^{2} u_{1}^{*} u_{3}^{*}+a^{2} u_{2}^{*} u_{3}^{*}, \\
\bar{c}_{3}= & d_{1} d_{2} d_{3}\left(1+d_{4} u_{1}^{*}\right) \mu_{i}^{3} \\
+ & {\left[B u_{1}^{*}\left(d_{2} d_{3}+d_{2} d_{3} d_{4} u_{1}^{*}\right)-d_{2} d_{3} d_{4} p m u_{1}^{*} u_{3}^{*}\right] \mu_{i}^{2} } \\
+ & {\left[\left(d_{1} a^{2} u_{2}^{*} u_{3}^{*}\right)+d_{2} e m^{2} p^{2} u_{1}^{*} u_{3}^{*}\right.} \\
& \left.+\left(d_{3}+d_{3} d_{4} u_{1}^{*}\right) e m^{2} u_{1}^{*} u_{2}^{*}+d_{3} d_{4} a m u_{1}^{*} u_{2}^{*} u_{3}^{*}\right] \mu_{i} \\
& +B a^{2} u_{1}^{*} u_{2}^{*} u_{3}^{*} \\
\triangleq & Q_{3} \mu_{i}^{3}+Q_{2} \mu_{i}^{2}+Q_{1} \mu_{i}+Q_{0} . \tag{30}
\end{align*}
$$

Let $\lambda_{i, 1}, \lambda_{i, 2}$, and $\lambda_{i, 3}$ be the three roots of $\Psi_{i}(\lambda)=0$; then $\lambda_{i, 1} \lambda_{i, 2} \lambda_{i, 3}=-\bar{c}_{3}$. In order to have at least one $\operatorname{Re} \lambda_{i, j}>0(j=$ $1,2,3)$, it is sufficient to prove that $\bar{c}_{3}<0$.

In the following we will find out the conditions such that $\bar{c}_{3}<0$. Let $\widetilde{Q}(\mu)=Q_{3} \mu^{3}+Q_{2} \mu^{2}+Q_{1} \mu+Q_{0}$ and let $\widetilde{\mu}_{1}, \widetilde{\mu}_{2}$, and $\widetilde{\mu}_{3}$ be the three roots of $\widetilde{Q}(\mu)=0$ with $\operatorname{Re}\left(\widetilde{\mu}_{1}\right) \leq \operatorname{Re}\left(\widetilde{\mu}_{2}\right) \leq$ $\operatorname{Re}\left(\widetilde{\mu}_{3}\right)$. Notice that $Q_{0}>0$ and $Q_{3}>0$. Then $\widetilde{\mu}_{1} \widetilde{\mu}_{2} \widetilde{\mu}_{3}=$ $-Q_{0} / Q_{3}<0$. Thus, one of the three roots $\tilde{\mu}_{1}, \widetilde{\mu}_{2}$, and $\widetilde{\mu}_{3}$ is real and negative, and the product of the other two is positive.

Consider the following limits:

$$
\begin{gather*}
\lim _{d_{3} \mapsto \infty} \frac{Q_{0}}{d_{3}}=0 \\
\lim _{d_{3} \mapsto \infty} \frac{Q_{1}}{d_{3}}=\left(1+d_{4} u_{1}^{*}\right) e m^{2} u_{1}^{*} u_{2}^{*}+d_{4} a m u_{1}^{*} u_{2}^{*} u_{3}^{*} \triangleq b_{1}  \tag{31}\\
\lim _{d_{3} \mapsto \infty} \frac{Q_{2}}{d_{3}}=B u_{1}^{*}\left(d_{2}+d_{2} d_{4} u_{1}^{*}\right)-d_{2} d_{4} p m u_{1}^{*} u_{3}^{*} \triangleq b_{2} \\
\lim _{d_{3} \mapsto \infty} \frac{Q_{3}}{d_{3}}=d_{1} d_{2}\left(1+d_{4} u_{1}^{*}\right) \triangleq b_{3}
\end{gather*}
$$

It is easy to see that $b_{1}>0$ and $b_{3}>0$.

Note that

$$
\begin{equation*}
\lim _{d_{3} \mapsto \infty} \frac{\widetilde{\mathbb{Q}}(\mu)}{d_{3}}=b_{3} \mu^{3}+b_{2} \mu^{2}+b_{1} \mu=\mu\left(b_{3} \mu^{2}+b_{2} \mu+b_{1}\right) . \tag{32}
\end{equation*}
$$

It follows that equation $b_{3} \mu^{2}+b_{2} \mu+b_{1}=0$ has two strictly positive solutions when the following conditions hold:

$$
\begin{equation*}
b_{2}<0, \quad b_{2}^{2}-4 b_{1} b_{3}>0 \tag{33}
\end{equation*}
$$

By a continuity argument, we know that, when $d_{3}$ is large enough, $\widetilde{\mu}_{1}$ is real and negative, and $\widetilde{\mu}_{2}$ and $\widetilde{\mu}_{3}$ are real and positive as $\tilde{\mu}_{2} \tilde{\mu}_{3}>0$. Furthermore, we have

$$
\begin{gather*}
\lim _{d_{3} \mapsto \infty} \tilde{\mu}_{1}=0, \\
\lim _{d_{3} \mapsto \infty} \tilde{\mu}_{2}=\frac{-b_{2}-\sqrt{b_{2}^{2}-4 b_{1} b_{3}}}{2 b_{3}} \triangleq \mu_{2}^{*}>0,  \tag{34}\\
\lim _{d_{3} \mapsto \infty} \tilde{\mu}_{3}=\frac{-b_{2}+\sqrt{b_{2}^{2}-4 b_{1} b_{3}}}{2 b_{3}} \triangleq \mu_{3}^{*}>0
\end{gather*}
$$

So there exists a positive number $d_{3}^{*}$ such that, when $d_{3}>d_{3}^{*}$, the following hold:

$$
\begin{align*}
& -\infty<\tilde{\mu}_{1}<0<\widetilde{\mu}_{2}<\tilde{\mu}_{3} ; \\
& \widetilde{Q}(\mu)<0 \quad \text { if } \mu \in\left(-\infty, \widetilde{\mu}_{1}\right) \cup\left(\widetilde{\mu}_{2}, \widetilde{\mu}_{3}\right) \text {; }  \tag{35}\\
& \widetilde{Q}(\mu)>0 \quad \text { if } \mu \in\left(\widetilde{\mu}_{1}, \widetilde{\mu}_{2}\right) \cup\left(\widetilde{\mu}_{3}, \infty\right) .
\end{align*}
$$

Since $\mu_{2}^{*} \in\left(\mu_{i}, \mu_{i+1}\right)$ and $\mu_{3}^{*} \in\left(\mu_{j}, \mu_{j+1}\right)$ for some $j>i>0$, we have $\widetilde{\mathrm{Q}}\left(\mu_{k}\right)<0$ when $i+1<k<j+1$. Thus we know that $\bar{c}_{3}<0$, and the proof is completed.

## 3. A Priori Estimates to the Positive Solution of (3)

In this section, we will give a priori estimates to the positive solution of (3). Let us first introduce two lemmas and we remark that the first lemma is due to Lou and Ni [38].

Lemma 4 (maximum principle). Suppose that $g \in C(\bar{\Omega} \times \mathbb{R})$.
(i) Assume that $w \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ and satisfies

$$
\begin{gather*}
\Delta w(x)+g(x, w(x)) \geq 0 \quad \text { in } \Omega, \\
\partial_{\gamma} w \leq 0 \quad \text { on } \partial \Omega . \tag{36}
\end{gather*}
$$

$$
\text { If } w\left(x_{0}\right)=\max _{\bar{\Omega}} w(x), \text { then } g\left(x_{0}, w\left(x_{0}\right)\right) \geq 0
$$

(ii) Assume that $w \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ and satisfies

$$
\begin{gather*}
\Delta w(x)+g(x, w(x)) \leq 0 \quad \text { in } \Omega  \tag{37}\\
\partial_{\nu} w \geq 0 \quad \text { on } \partial \Omega
\end{gather*}
$$

Next, we state the second lemma which is due to Lin et al. [39].

Lemma 5 (Harnack inequality). Assume that $c(x) \in C(\bar{\Omega})$. Let $w \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ and satisfy

$$
\begin{gather*}
\Delta w(x)+g(x, w(x))=0 \quad \text { in } \Omega, \\
\partial_{\gamma} w=0 \quad \text { on } \partial \Omega . \tag{38}
\end{gather*}
$$

Then there exists a positive constant $C$, depending only on $\|c(x)\|_{C(\bar{\Omega})}$ such that

$$
\begin{equation*}
\max _{\bar{\Omega}} w(x) \leq C \min _{\bar{\Omega}} w(x) \tag{39}
\end{equation*}
$$

Our results are the following theorems.
Theorem 6 (upper bounds). Any positive solution $\mathbf{u}(x)=$ $\left(u_{1}(x), u_{2}(x), u_{3}(x)\right)^{T}$ of (3) satisfies

$$
\begin{gather*}
\max _{\bar{\Omega}} u_{1}(x) \leq \bar{M}_{1}, \quad \max _{\bar{\Omega}} u_{2}(x) \leq \bar{M}_{2},  \tag{40}\\
\max _{\bar{\Omega}} u_{3}(x) \leq \bar{M}_{3},
\end{gather*}
$$

where

$$
\begin{gather*}
\bar{M}_{1}=\frac{r}{B}, \quad \bar{M}_{2}=\frac{e r d_{1}}{B d_{2}}+\frac{e r^{2}}{\mu B} \\
\bar{M}_{3}=\frac{e r}{d_{3} B}\left[d_{1}+e+\frac{d_{2} r}{4 \mu}+\frac{d_{3} r}{4 \delta}\left(1+\frac{d_{4} r}{B}\right)\right] . \tag{41}
\end{gather*}
$$

Proof. Let $x_{0} \in \bar{\Omega}$ such that $u_{1}\left(x_{0}\right)=\max _{\bar{\Omega}} u_{1}(x)$. Then by Lemma 4, it is clear that

$$
\begin{align*}
& r u_{1}\left(x_{0}\right)-B u_{1}^{2}\left(x_{0}\right)-m u_{1}\left(x_{0}\right) u_{2}\left(x_{0}\right) \\
& -p m u_{1}\left(x_{0}\right) u_{3}\left(x_{0}\right) \geq 0, \tag{42}
\end{align*}
$$

and so

$$
\begin{equation*}
u_{1}\left(x_{0}\right)=\max _{\bar{\Omega}} u_{1}(x) \leq \frac{r}{B} \triangleq \bar{M}_{1} . \tag{43}
\end{equation*}
$$

Define $y(x)=e d_{1} u_{1}(x)+d_{2} u_{2}(x)$; then $y(x)$ satisfies

$$
\begin{equation*}
-\Delta y=e r u_{1}-e B u_{1}^{2}-e p m u_{1} u_{3}-a u_{2} u_{3}-\mu u_{2} \tag{44}
\end{equation*}
$$

in $\Omega$,

$$
\partial_{\nu} y=0 \quad \text { on } \partial \Omega
$$

Let $x_{1} \in \bar{\Omega}$ such that $y\left(x_{1}\right)=\max _{\bar{\Omega}} y(x)$. Then, by Lemma 4 , we can get

$$
\begin{gather*}
e r u_{1}\left(x_{1}\right)-e B u_{1}^{2}\left(x_{1}\right)-e p m u_{1}\left(x_{1}\right) u_{3}\left(x_{1}\right)  \tag{45}\\
-a u_{2}\left(x_{1}\right) u_{3}\left(x_{1}\right)-\mu u_{2}\left(x_{1}\right) \geq 0
\end{gather*}
$$

which implies

$$
\begin{equation*}
u_{2}\left(x_{1}\right) \leq \frac{1}{\mu}\left(e r u_{1}\left(x_{1}\right)-e B u_{1}^{2}\left(x_{1}\right)\right) \leq \frac{e r^{2}}{4 \mu B} \tag{46}
\end{equation*}
$$

So, by the definition of $y(x)$, we have

$$
\begin{align*}
\max _{\bar{\Omega}} u_{2}(x) & \leq \frac{1}{d_{2}} \max y(x)=\frac{1}{d_{2}} y\left(x_{1}\right) \\
& =\frac{e d_{1}}{d_{2}} u_{1}\left(x_{1}\right)+u_{2}\left(x_{1}\right) \leq \frac{e r d_{1}}{B d_{2}}+\frac{e r^{2}}{4 \mu B} \triangleq \bar{M}_{2} . \tag{47}
\end{align*}
$$

Let $\omega(x)=u_{3}(x)+d_{4} u_{1}(x) u_{3}(x)$; then $u_{3}(x)=\omega(x) /(1+$ $\left.d_{4} u_{1}(x)\right)$. Define $z(x)=e d_{1} u_{1}(x)+d_{2} u_{2}(x)+d_{3} \omega(x)$; then $z(x)$ satisfies
$-\Delta z=e r u_{1}(x)-e B u_{1}^{2}(x)-\mu u_{2}(x)-\frac{\delta \omega(x)}{1+d_{4} u_{1}(x)}$
in $\Omega$,

$$
\partial_{\gamma} z=0 \quad \text { on } \partial \Omega
$$

Let $x_{2} \in \bar{\Omega}$ such that $z\left(x_{2}\right)=\max _{\bar{\Omega}} z(x)$. Then, by using Lemma 4 again, we can obtain

$$
\begin{equation*}
e r u_{1}\left(x_{2}\right)-e B u_{1}^{2}\left(x_{2}\right)-\mu u_{2}\left(x_{2}\right)-\frac{\delta \omega\left(x_{2}\right)}{1+d_{4} u_{1}\left(x_{2}\right)} \geq 0 \tag{49}
\end{equation*}
$$

which implies

$$
\begin{align*}
\omega\left(x_{2}\right) & \leq \frac{1}{\delta}\left(1+d_{4} u_{1}\left(x_{2}\right)\right)\left(e r u_{1}\left(x_{2}\right)-e B u_{1}^{2}\left(x_{2}\right)\right) \\
& \leq \frac{e r^{2}}{4 \delta B}\left(1+\frac{d_{4} r}{B}\right) \tag{50}
\end{align*}
$$

It follows that

$$
\begin{align*}
\max _{\bar{\Omega}} u_{3}(x) & \leq \max _{\bar{\Omega}} \omega(x) \leq \frac{1}{d_{3}} \max _{\bar{\Omega}} z(x)=\frac{1}{d_{3}} z\left(x_{2}\right) \\
& =\frac{1}{d_{3}}\left(e d_{1} u_{1}\left(x_{2}\right)+d_{2} u_{2}\left(x_{2}\right)+d_{3} \omega\left(x_{2}\right)\right) \\
& \leq \frac{e r}{d_{3} B}\left[d_{1}+e+\frac{d_{2} r}{4 \mu}+\frac{d_{3} r}{4 \delta}\left(1+\frac{d_{4} r}{B}\right)\right] \triangleq \bar{M}_{3} . \tag{51}
\end{align*}
$$

Then we obtain the three upper bounds in (40).
Theorem 7. There exist three positive constants $C_{1}$ (depending on $r / d_{1}, \Omega$ ), $C_{2}$ (depending on emr $/ d_{2} B, \Omega$ ), and $C_{3}$ (depending on $\left.\Lambda, d_{i}, \Omega\right)$ such that any positive solution $\mathbf{u}(x)=$ $\left(u_{1}(x), u_{2}(x), u_{3}(x)\right)$ of (3) satisfies

$$
\begin{align*}
& \frac{\max _{\bar{\Omega}} u_{1}(x)}{\min _{\bar{\Omega}} u_{1}(x)} \leq C_{1}, \\
& \frac{\max _{\bar{\Omega}} u_{2}(x)}{\min _{\bar{\Omega}} u_{2}(x)} \leq C_{2}, \tag{52}
\end{align*}
$$

$$
\frac{\max _{\bar{\Omega}} u_{3}(x)}{\min _{\bar{\Omega}} u_{3}(x)} \leq C_{3}, \quad(i=1,2,3) .
$$

Proof. It is easy to see that $u_{i}(x)(i=1,2)$ satisfies

$$
\begin{gather*}
\Delta u_{i}(x)+\frac{c_{i}(x)}{d_{i}} u_{i}(x)=0 \quad \text { in } \Omega  \tag{53}\\
\partial_{\nu} u_{i}(x)=0 \quad \text { on } \partial \Omega
\end{gather*}
$$

where $c_{1}(x)=r-B u_{1}-m u_{1} u_{2}-p m u_{1} u_{3}$ and $c_{2}(x)=e m u_{1}-$ $a u_{3}-\mu$. By (40), we know that

$$
\begin{equation*}
\left\|\frac{c_{1}(x)}{d_{1}}\right\|_{C(\bar{\Omega})} \leq \frac{r}{d_{1}}, \quad\left\|\frac{c_{2}(x)}{d_{2}}\right\|_{C(\bar{\Omega})} \leq \frac{e m r}{d_{2} B} \tag{54}
\end{equation*}
$$

So by Lemma 5, we know that the first two inequalities of (52) hold. Define $\varphi(x)=d_{3} u_{3}(x)+d_{3} d_{4} u_{1}(x) u_{3}(x)$; we have

$$
\begin{gather*}
\Delta \varphi(x)+c_{3}(x) \varphi(x)=0 \quad \text { in } \Omega \\
\partial_{\nu} \varphi(x)=0 \quad \text { on } \partial \Omega \tag{55}
\end{gather*}
$$

where $c_{3}(x)=\left(e m p u_{1}+a u_{2}-\delta\right) / d_{3}\left(1+d_{4} u_{1}\right)$. By (40), we know that

$$
\begin{equation*}
\left\|c_{3}(x)\right\|_{C(\bar{\Omega})} \leq \frac{e m p r}{d_{3} B}+\frac{e r d_{1}}{B d_{2} d_{3}}+\frac{e r^{2}}{4 \mu B d_{3}} \tag{56}
\end{equation*}
$$

Then Lemma 5 yields $\max _{\bar{\Omega}} \varphi(x) / \min _{\bar{\Omega}} \varphi(x) \leq C_{3}^{*}$ for some positive constant $C_{3}^{*}\left(\Lambda, d_{1}, d_{2}, d_{3}, \Omega\right)$, and

$$
\begin{align*}
\frac{\max _{\bar{\Omega}} u_{3}}{\min _{\bar{\Omega}} u_{3}} & \leq \frac{\max _{\bar{\Omega}} \varphi \max _{\bar{\Omega}}\left(1+d_{4} u_{1}\right)}{\min _{\bar{\Omega}} \varphi \min _{\bar{\Omega}}\left(1+d_{4} u_{1}\right)}  \tag{57}\\
& \leq C_{3}^{*} \frac{\max _{\bar{\Omega}} u_{1}}{\min _{\bar{\Omega}} u_{1}} \leq C_{3}^{*} C_{1} \triangleq C_{3} .
\end{align*}
$$

The proof is completed.
Theorem 8 (lower bounds). Let $\Lambda, D_{1}, D_{2}, D_{3}$, and $D_{4}$ be fixed positive constants. Assume that

$$
\begin{equation*}
\min \left\{\frac{e m p}{B}\left(r-m \bar{M}_{2}\right), \frac{e m p}{B}\left(r-p m \bar{M}_{3}\right)\right\}>\delta \tag{58}
\end{equation*}
$$

where $\bar{M}_{2}$ and $\bar{M}_{2}$ are given in (47) and (51). Then there exists a positive constant $\underline{C}=\underline{C}\left(\Lambda, D_{1}, D_{2}, D_{3}, D_{4}\right)$ such that, when $\left(d_{1}, d_{2}, d_{3}\right) \in\left[D_{1}, \infty\right) \times\left[D_{2}, \infty\right) \times\left[D_{3}, \infty\right)$ and $d_{4} \in$ [ $0, D_{4}$ ], any positive solution $\mathbf{u}(x)=\left(u_{1}(x), u_{2}(x), u_{3}(x)\right)$ of (3) satisfies

$$
\begin{equation*}
\min _{\bar{\Omega}} u_{i}(x) \geq \underline{C}, \quad i=1,2,3 \tag{59}
\end{equation*}
$$

Proof. Suppose that (59) fails. Then there exist sequences $\left\{d_{1, i}, d_{2, i}, d_{3, i}, d_{4, i}\right\}_{i=1}^{\infty}$ with $\left(d_{1, i}, d_{2, i}, d_{3, i}\right) \in\left[D_{1}, \infty\right) \times$ $\left[D_{2}, \infty\right) \times\left[D_{3}, \infty\right)$ and $d_{4, i} \in\left[0, D_{4}\right]$ such that the corresponding positive solutions ( $u_{1, i}, u_{2, i}, u_{3, i}$ ) of (3) satisfy

$$
\begin{equation*}
\max _{\bar{\Omega}} u_{1 i} \longmapsto 0, \quad \text { or } \quad \max _{\bar{\Omega}} u_{2 i} \longmapsto 0, \quad \text { or } \quad \max _{\bar{\Omega}} u_{3 i} \longmapsto 0 \tag{60}
\end{equation*}
$$

By a direct application of the maximum principle to the first equation of (3), we can obtain $u_{1 i} \leq r / B$. Integrating by parts, we obtain that

$$
\begin{gather*}
\int_{\Omega}\left(r u_{1 i}-B u_{1 i}^{2}-m u_{1 i} u_{2 i}-p m u_{1 i} u_{3 i}\right) d x=0, \\
\int_{\Omega}\left(e m u_{1} u_{2 i}-a u_{2 i} u_{3 i}-\mu u_{2 i}\right) d x=0,  \tag{61}\\
\int_{\Omega}\left(a u_{2 i} u_{3 i}+e m p u_{1 i} u_{3 i}-\delta u_{3 i}\right) d x=0,
\end{gather*}
$$

for $i=1,2, \ldots$. By the standard regularity theorem for the elliptic equations, we know that there exists a subsequence of $\left\{u_{1, i}, u_{2, i}, u_{3, i}\right\}_{i=1}^{\infty}$, which we will still denote by $\left\{u_{1, i}, u_{2, i}, u_{3, i}\right\}_{i=1}^{\infty}$, and three nonnegative functions $u_{1}, u_{2}, u_{3} \in$ $C^{2}(\bar{\Omega})$ such that

$$
\begin{align*}
& \left(u_{1, i}, u_{2, i}, u_{3, i}\right) \longmapsto\left(u_{1}, u_{2}, u_{3}\right) \\
& \quad \text { in }\left[C^{2}(\bar{\Omega})\right]^{3} \text { as } i \longmapsto \infty \tag{62}
\end{align*}
$$

By (60), we know that

$$
\begin{equation*}
u_{1} \equiv 0, \quad \text { or } \quad u_{2} \equiv 0, \quad \text { or } \quad u_{3} \equiv 0 \tag{63}
\end{equation*}
$$

Furthermore, we assume that $\left(d_{1, i}, d_{2, i}, d_{3, i}, d_{4, i}\right) \mapsto\left(\underline{d}_{1}\right.$, $\left.\underline{d}_{2}, \underline{d}_{3}, \underline{d}_{4}\right) \in\left[D_{1}, \infty\right) \times\left[D_{2}, \infty\right) \times\left[D_{3}, \infty\right) \times\left[0, D_{4}\right]$. Let $i \mapsto \infty$ in (61); we obtain

$$
\begin{gather*}
\int_{\Omega}\left(r u_{1}-B u_{1}^{2}-m u_{1} u_{2}-p m u_{1} u_{3}\right) d x=0 \\
\int_{\Omega}\left(e m u_{1} u_{2}-a u_{2} u_{3}-\mu u_{2}\right) d x=0  \tag{64}\\
\int_{\Omega}\left(a u_{2} u_{3}+e m p u_{1} u_{3}-\delta u_{3}\right) d x=0
\end{gather*}
$$

Now, we consider the following three cases, respectively.
Case $1\left(u_{1} \equiv 0\right)$. Note that $u_{1 i} \mapsto u_{1}$, as $i \mapsto \infty$. Then we know that

$$
\begin{equation*}
e m u_{1 i}-a u_{3 i}-\mu<0 \quad \text { on } \bar{\Omega}, \forall i \gg 1 \tag{65}
\end{equation*}
$$

Integrating the differential equation for $u_{2 i}$ over $\Omega$ by parts, we have

$$
\begin{align*}
0 & =-d_{2 i} \int_{\partial \Omega} \partial_{\nu} u_{2 i} d x=-d_{2 i} \int_{\Omega} \Delta u_{2 i} d x \\
& =\int_{\Omega} u_{2 i}\left(e m u_{1 i}-a u_{3 i}-\mu\right) d x<0, \quad \forall i \gg 1 \tag{66}
\end{align*}
$$

which is a contradiction.
Case $2\left(u_{2} \equiv 0, u_{1} \neq 0\right.$ on $\left.\bar{\Omega}\right)$. By using Hopf boundary lemma, we know $u_{1}>0$ on $\bar{\Omega}$. Then $u_{1}$ and $u_{3}$ satisfy the following equation:

$$
\begin{gather*}
-\underline{d}_{1} \Delta u_{1}=u_{1}\left(r-B u_{1}-p m u_{3}\right) \quad \text { in } \Omega, \\
\partial_{\nu} u_{1}=0 \quad \text { on } \partial \Omega \tag{67}
\end{gather*}
$$

Let $u_{1}\left(x_{0}\right)=\min _{\bar{\Omega}} u_{1}(x)$. It follows from Lemma 4 and (67) that

$$
\begin{equation*}
r-B u_{1}\left(x_{0}\right)-p m u_{3}\left(x_{0}\right) \leq 0 \tag{68}
\end{equation*}
$$

that is,

$$
\begin{equation*}
u_{1}\left(x_{0}\right) \geq \frac{r}{B}-\frac{p m}{B} u_{3}\left(x_{0}\right) \geq \frac{r}{B}-\frac{p m \bar{M}_{3}}{B} . \tag{69}
\end{equation*}
$$

By using the assumption $(e m p / B)\left(r-p m \bar{M}_{3}\right)>\delta$, we know that

$$
\begin{equation*}
e m p u_{1 i}-\delta>0 \quad \text { on } \bar{\Omega}, \forall i \gg 1 \tag{70}
\end{equation*}
$$

Integrating the differential equation for $u_{3 i}$ over $\Omega$ by parts, we have

$$
\begin{align*}
0 & =-d_{3 i} \int_{\partial \Omega}\left[\left(1+d_{4} u_{1 i}\right) \partial_{\gamma} u_{3 i}+d_{4} u_{3 i} \partial_{\gamma} u_{1 i}\right] d x \\
& =-d_{3 i} \int_{\Omega} \Delta\left(u_{3 i}+d_{4} u_{1 i} u_{3 i}\right) d x  \tag{71}\\
& =\int_{\Omega} u_{3 i}\left(e m p u_{1 i}+a u_{2 i}-\delta\right) d x>0, \quad \forall i \gg 1
\end{align*}
$$

which is a contradiction.
Case 3 ( $u_{3} \equiv 0, u_{1} \neq 0$, and $u_{2} \neq 0$ on $\left.\bar{\Omega}\right)$. By using Hopf boundary lemma, we know $u_{1}>0$ and $u_{2}>0$ on $\bar{\Omega}$.

Then $u_{1}$ and $u_{2}$ satisfy the following equation:

$$
\begin{gather*}
-\underline{d}_{1} \Delta u_{1}=u_{1}\left(r-B u_{1}-m u_{2}\right) \quad \text { in } \Omega, \\
\partial_{\gamma} u_{1}=0 \quad \text { on } \partial \Omega . \tag{72}
\end{gather*}
$$

Let $u_{1}\left(x_{1}\right)=\min _{\bar{\Omega}} u_{1}(x)$. It follows from Lemma 4 and (72) that

$$
\begin{equation*}
r-B u_{1}\left(x_{1}\right)-m u_{2}\left(x_{1}\right) \leq 0 ; \tag{73}
\end{equation*}
$$

that is,

$$
\begin{equation*}
u_{1}\left(x_{1}\right) \geq \frac{r}{B}-\frac{m}{B} u_{2}\left(x_{1}\right) \geq \frac{r}{B}-\frac{m \bar{M}_{2}}{B} . \tag{74}
\end{equation*}
$$

By using the assumption $(e m p / B)\left(r-m \bar{M}_{2}\right)>\delta$, we know that

$$
\begin{equation*}
e m p u_{1 i}-\delta>0 \quad \text { on } \bar{\Omega}, \forall i \gg 1 \tag{75}
\end{equation*}
$$

Integrating the differential equation for $u_{3 i}$ over $\Omega$ by parts, we have

$$
\begin{align*}
0 & =-d_{3 i} \int_{\partial \Omega}\left[\left(1+d_{4} u_{1 i}\right) \partial_{\gamma} u_{3 i}+d_{4} u_{3 i} \partial_{\gamma} u_{1 i}\right] d x \\
& =-d_{3 i} \int_{\Omega} \Delta\left(u_{3 i}+d_{4} u_{1 i} u_{3 i}\right) d x  \tag{76}\\
& =\int_{\Omega} u_{3 i}\left(e m p u_{1 i}+a u_{2 i}-\delta\right) d x>0, \quad \forall i \gg 1
\end{align*}
$$

which is a contradiction. The proof is completed.

## 4. Nonexistence of Nonconstant Positive Solution of System (3) without Cross-Diffusion

In order to discuss the effect of cross-diffusion on the existence of nonconstant positive solution of system (3), we first give a nonexistence result when the cross-diffusion term is absent, which shows that the cross-diffusion coefficients do play important roles. The mathematical technique to be employed here is the energy method.

Theorem 9. Suppose that $d_{4}=0$ and $d_{1} \geq r / \mu_{2} \triangleq D_{1}^{*}$, where $\mu_{2}$ is given in Notation 1. There exist positive constants $D_{2}^{*}$ and $D_{3}^{*}$, depending on $\Lambda, \varepsilon, \Omega$ such that (3) has no nonconstant positive solution provided that $d_{2} \geq D_{2}^{*}$ and $d_{3} \geq D_{3}^{*}$. Furthermore, one has

$$
\begin{equation*}
u_{i} \equiv \bar{u}_{i}, \quad \text { where } \bar{u}_{i}=\frac{1}{\left|u_{i}\right|} \int_{\Omega} u_{i} d x, i=1,2,3 . \tag{77}
\end{equation*}
$$

Proof. Assume that $\mathbf{u}=\left(u_{1}, u_{2}, u_{3}\right)$ is a positive solution of (3) with $d_{4}=0$. Multiplying the $i$ th equation of (3) by $u_{i}-\bar{u}_{i}$ and integrating the results over $\Omega$ by parts, we have

$$
\begin{aligned}
& d_{1} \int_{\Omega}\left|\nabla u_{1}\right|^{2} d x=\int_{\Omega}\left(u_{1}-\bar{u}_{1}\right) \\
& \times\left(r u_{1}-B u_{1}^{2}-m u_{1} u_{2}-p m u_{1} u_{3}\right. \\
& -r \bar{u}_{1}+B \bar{u}_{1}^{2}+m \bar{u}_{1} \bar{u}_{2} \\
& \left.+p m \bar{u}_{1} \bar{u}_{3}\right) d x \\
& =\int_{\Omega}\left(u_{1}-\bar{u}_{1}\right)^{2} \\
& \times\left(r-B u_{1}-B \bar{u}_{1}-m u_{2}-p m u_{3}\right) d x \\
& -\int_{\Omega} m \bar{u}_{1}\left(u_{1}-\bar{u}_{1}\right)\left(u_{2}-\bar{u}_{2}\right) d x \\
& -\int_{\Omega} p m \bar{u}_{1}\left(u_{1}-\bar{u}_{1}\right)\left(u_{3}-\bar{u}_{3}\right) d x ; \\
& d_{2} \int_{\Omega}\left|\nabla u_{2}\right|^{2} d x=\int_{\Omega}\left(u_{2}-\bar{u}_{2}\right) \\
& \times\left(e m u_{1} u_{2}-a u_{2} u_{3}-\mu u_{2}\right. \\
& \left.-e m \bar{u}_{1} \bar{u}_{2}+a \bar{u}_{2} \bar{u}_{3}+\mu \bar{u}_{2}\right) d x \\
& =\int_{\Omega}\left(u_{2}-\bar{u}_{2}\right)^{2}\left(e m u_{1}-a u_{3}-\mu\right) d x \\
& +\int_{\Omega} e m \bar{u}_{2}\left(u_{1}-\bar{u}_{1}\right)\left(u_{2}-\bar{u}_{2}\right) d x \\
& -\int_{\Omega} a \bar{u}_{2}\left(u_{2}-\bar{u}_{2}\right)\left(u_{3}-\bar{u}_{3}\right) d x ;
\end{aligned}
$$

$$
\begin{align*}
& d_{3} \int_{\Omega}\left|\nabla u_{3}\right|^{2} d x= \int_{\Omega}\left(u_{3}-\bar{u}_{3}\right) \\
& \times\left(e m p u_{1} u_{3}+a u_{2} u_{3}-\delta u_{3}\right. \\
&\left.-e m p \bar{u}_{1} \bar{u}_{3}-a \bar{u}_{2} \bar{u}_{3}+\delta \bar{u}_{2}\right) d x \\
&= \int_{\Omega}\left(u_{3}-\bar{u}_{3}\right)^{2}\left(e m p u_{1}-a u_{2}-\delta\right) d x \\
&+\int_{\Omega} e m p \bar{u}_{3}\left(u_{1}-\bar{u}_{1}\right)\left(u_{3}-\bar{u}_{3}\right) d x \\
&+\int_{\Omega} a \bar{u}_{2}\left(u_{2}-\bar{u}_{2}\right)\left(u_{3}-\bar{u}_{3}\right) d x . \tag{78}
\end{align*}
$$

Then it follows from (78) that

$$
\begin{aligned}
& d_{1} \int_{\Omega}\left|\nabla u_{1}\right|^{2} d x+d_{2} \int_{\Omega}\left|\nabla u_{2}\right|^{2} d x+d_{3} \int_{\Omega}\left|\nabla u_{3}\right|^{2} d x \\
&= \int_{\Omega}\left(u_{1}-\bar{u}_{1}\right)^{2} \\
& \times\left(r-B u_{1}-B \bar{u}_{1}-m u_{2}-p m u_{3}\right) d x \\
&+\int_{\Omega}\left(u_{2}-\bar{u}_{2}\right)^{2}\left(e m u_{1}-a u_{3}-\mu\right) d x \\
&+\int_{\Omega}\left(u_{3}-\bar{u}_{3}\right)^{2}\left(e m p u_{1}-a u_{2}-\delta\right) d x \\
&+\int_{\Omega}\left(e m \bar{u}_{2}-m \bar{u}_{1}\right)\left(u_{1}-\bar{u}_{1}\right)\left(u_{2}-\bar{u}_{2}\right) d x \\
&+\int_{\Omega}\left(e m p \bar{u}_{3}-p m \bar{u}_{1}\right)\left(u_{1}-\bar{u}_{1}\right)\left(u_{3}-\bar{u}_{3}\right) d x \\
&+\int_{\Omega}\left(a \bar{u}_{3}-a \bar{u}_{2}\right)\left(u_{2}-\bar{u}_{2}\right)\left(u_{3}-\bar{u}_{3}\right) d x .
\end{aligned}
$$

By Cauchy inequality with $\varepsilon$, we can get from (79) that

$$
\begin{aligned}
& d_{1} \int_{\Omega}\left|\nabla u_{1}\right|^{2} d x+d_{2} \int_{\Omega}\left|\nabla u_{2}\right|^{2} d x+d_{3} \int_{\Omega}\left|\nabla u_{3}\right|^{2} d x \\
& \leq \int_{\Omega}\left(u_{1}-\bar{u}_{1}\right)^{2} \\
& \quad \times\left(r-B u_{1}-B \bar{u}_{1}-m u_{2}-p m u_{3}+2 \varepsilon\right) d x \\
& \quad+\int_{\Omega}\left(u_{2}-\bar{u}_{2}\right)^{2} \\
& \quad \times\left(e m u_{1}-a u_{3}-\mu+\frac{\left(e m \bar{u}_{2}-m \bar{u}_{1}\right)^{2}}{4 \varepsilon}\right. \\
& \left.\quad+\frac{\left(a \bar{u}_{3}-a \bar{u}_{2}\right)^{2}}{4 \varepsilon}\right) d x
\end{aligned}
$$

$$
\begin{align*}
& +\int_{\Omega}\left(u_{3}-\bar{u}_{3}\right)^{2} \\
& \quad \times\left(e m p u_{1}-a u_{2}-\delta+\frac{\left(e m p \bar{u}_{3}-m p \bar{u}_{1}\right)^{2}}{4 \varepsilon}\right. \\
& \left.\quad+\frac{\left(a \bar{u}_{3}-a \bar{u}_{2}\right)^{2}}{4 \varepsilon}\right) d x . \tag{80}
\end{align*}
$$

On the other hand, applying Poincaré inequality, we know that

$$
\begin{align*}
& d_{1} \int_{\Omega}\left|\nabla u_{1}\right|^{2} d x+d_{2} \int_{\Omega}\left|\nabla u_{2}\right|^{2} d x+d_{3} \int_{\Omega}\left|\nabla u_{3}\right|^{2} d x \\
& \quad \geq \int_{\Omega} d_{1} \mu_{2}\left(u_{1}-\bar{u}_{1}\right)^{2}+\int_{\Omega} d_{2} \mu_{2}\left(u_{2}-\bar{u}_{2}\right)^{2}  \tag{81}\\
& \quad+\int_{\Omega} d_{3} \mu_{2}\left(u_{3}-\bar{u}_{3}\right)^{2}
\end{align*}
$$

Then, by assumption, we can choose a sufficiently small positive constant $\varepsilon_{0}$ such that

$$
\begin{equation*}
d_{1} \mu_{2}>r-B u_{1}-B \bar{u}_{1}-m u_{2}-p m u_{3}+2 \varepsilon_{0} . \tag{82}
\end{equation*}
$$

So by taking

$$
\begin{align*}
& D_{2}^{*}> \frac{1}{\mu_{2}}\left(e m u_{1}-a u_{3}-\mu\right. \\
&\left.+\frac{\left(e m \bar{u}_{2}-m \bar{u}_{1}\right)^{2}}{4 \varepsilon_{0}}+\frac{\left(a \bar{u}_{3}-a \bar{u}_{2}\right)^{2}}{4 \varepsilon_{0}}\right) \\
& D_{3}^{*}>\frac{1}{\mu_{2}}\left(e m p u_{1}-a u_{2}-\delta\right.  \tag{83}\\
&+\frac{\left(e m p \bar{u}_{3}-m p \bar{u}_{1}\right)^{2}}{4 \varepsilon_{0}} \\
&\left.+\frac{\left(a \bar{u}_{3}-a \bar{u}_{2}\right)^{2}}{4 \varepsilon_{0}}\right)
\end{align*}
$$

we can conclude that, when $d_{4}=0$, (3) has only the positive constant solution $u_{i} \equiv \bar{u}_{i}$ for $i=1,2,3$. The proof is completed.

## 5. Existence of Nonconstant Positive Solution of System (3)

From Theorem 9 we know that, when the cross-diffusion $d_{3} d_{4} u_{1} u_{3}$ is absent, (3) has no nonconstant positive solution under some conditions. In the following, we will discuss the effect of cross-diffusion on the existence of nonconstant positive solution of system (3) for certain values of diffusion coefficient $d_{3}$, while the other parameters are fixed.

Our main findings are the following theorem, which shows that the presence of cross-diffusion creates nonhomogeneous solution.

Theorem 10. Let $d_{1}, d_{2}$, and $d_{4}$ be fixed and satisfy (33) and (58), and let $\mu_{2}^{*}$ and $\mu_{3}^{*}$ be defined in (34). If $\mu_{2}^{*} \in\left(\mu_{i}, \mu_{i+1}\right)$ and $\mu_{3}^{*} \in\left(\mu_{j}, \mu_{j+1}\right)$ for some $j>i \geq 1$ and the sum $\sum_{n=i+1}^{j} m\left(\mu_{n}\right)$ is odd, then there exists a positive constant $d_{3}^{*}$ such that, ifd $\geq d_{3}^{*}$,
(3) admits at least one nonconstant positive solution.

In order to prove the above theorem by using LeraySchauder theory, we start with some preliminary results. Throughout this section, Notation 1 and $\mathbf{G}(\mathbf{u})$ defined in Section 2 will be used again. Define the set

$$
\begin{gather*}
\mathbf{X}^{+}=\left\{\mathbf{u}=\left(u_{1}, u_{1}, u_{1}\right)^{T} \in \mathbf{X}: u_{i}>0 \text { on } \bar{\Omega}, i=1,2,3\right\}, \\
\mathbf{B}(\mathbf{C})=\left\{\mathbf{u}=\left(u_{1}, u_{1}, u_{1}\right)^{T}, \underline{C} \leq u_{i} \leq \bar{C} \text { on } \bar{\Omega}, i=1,2,3\right\}, \tag{84}
\end{gather*}
$$

where $\bar{C}=\max \left\{\bar{M}_{1}, \bar{M}_{2}, \bar{M}_{3}\right\}$ and $\underline{C}$ is given in Theorem 8 . Then we will look for nonconstant positive solutions of (3) in the set $\mathbf{B}(\mathbf{C})$. Let $\Phi(\mathbf{u})=\left[d_{1} u_{1}, d_{2} u_{2}, d_{3}\left(u_{3}+d_{4} u_{1} u_{3}\right)\right]^{T}$. Then (3) can be written as

$$
\begin{gather*}
-\Delta \Phi(\mathbf{u})=\mathbf{G}(\mathbf{u}) \quad \text { in } \Omega  \tag{85}\\
\partial_{\gamma} \mathbf{u}=0 \\
\text { on } \partial \Omega .
\end{gather*}
$$

Noting that the determinant of $\Phi_{\mathbf{u}}(\mathbf{u})$ is positive for all $\mathbf{u} \in$ $\mathbf{X}^{+}$, we know that $\Phi_{\mathbf{u}}^{-1}(\mathbf{u})$ exists and $\operatorname{det} \Phi_{\mathbf{u}}^{-1}(\mathbf{u})$ is positive. Then, $\mathbf{u}$ is a positive solution to (85) if and only if

$$
\begin{align*}
\mathbf{F}(\mathbf{u})= & \mathbf{u} \\
& -(\mathbf{I}-\Delta)^{-1}  \tag{86}\\
& \times\left\{\Phi_{\mathbf{u}}^{-1}(\mathbf{u})\left[\mathbf{G}(\mathbf{u})+\nabla \mathbf{u} \Phi_{u u}(\mathbf{u}) \nabla \mathbf{u}+\mathbf{u}\right]\right\}=0
\end{align*}
$$

$$
\text { in } \mathbf{X}^{+},
$$

where $(\mathbf{I}-\Delta)^{-1}$ is the inverse of $\mathbf{I}-\Delta$ in $\mathbf{X}$ with the noflux boundary condition. As $\mathbf{F}(\cdot)$ is a compact perturbation of the identity operator, for any $\mathbf{B}=\mathbf{B}(\mathbf{C})$, the Leray-Schauder degree $\operatorname{deg}(\mathbf{F}(\cdot), 0, \mathbf{B})$ is well defined if $\mathbf{F}(\mathbf{u}) \neq 0$ on $\partial \mathbf{B}$. Note that

$$
\begin{equation*}
\mathbf{D}_{\mathbf{u}} \mathbf{F}\left(\mathbf{u}^{*}\right)=\mathbf{I}-(\mathbf{I}-\Delta)^{-1}\left\{\Phi_{\mathbf{u}}^{-1}\left(\mathbf{u}^{*}\right)\left[\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)+\mathbf{I}\right]\right\} . \tag{87}
\end{equation*}
$$

If $\mathbf{D}_{\mathbf{u}} \mathbf{F}\left(\mathbf{u}^{*}\right)$ is invertible, the index of $\mathbf{F}$ at $\mathbf{u}^{*}$ is defined as index $\left(\mathbf{F}(\cdot), \mathbf{u}^{*}\right)=(-1)^{\gamma}$, where $\gamma$ is the multiplicity of negative eigenvalues of $\mathbf{D}_{\mathbf{u}} \mathbf{F}\left(\mathbf{u}^{*}\right)$ [40, Theorem 2.8.1]. For the sake of convenience, we denote

$$
\begin{align*}
\mathbf{H}(\mu) & =\mathbf{H}\left(\mathbf{u}^{*}, \mu\right) \triangleq \operatorname{det}\left[\mu \mathbf{I}-\Phi_{\mathbf{u}}^{-1}\left(\mathbf{u}^{*}\right) \mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right]  \tag{88}\\
& =\operatorname{det}\left[\Phi_{\mathbf{u}}^{-1}\left(\mathbf{u}^{*}\right)\right] \operatorname{det}\left[\mu \Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right]
\end{align*}
$$

By arguments similar to those in [41], we can conclude that the following proposition holds.

Proposition 11. Suppose that, for all $n \geq 1$, the matrix $\mu_{n} \mathbf{I}-$ $\Phi_{\mathbf{u}}^{-1}\left(\mathbf{u}^{*}\right) \mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$ is nonsingular. Then index $\left(\mathbf{F}(\cdot), \mathbf{u}^{*}\right)=(-1)^{\sigma}$, where $\sigma=\sum_{n \geq 1, \mathbf{H}\left(\mu_{n}\right)<0} \operatorname{dim} S\left(\mu_{n}\right)$.

From Proposition 11, we can see that, in order to compute index $\left(\mathbf{F}(\cdot), \mathbf{u}^{*}\right)$, it is necessary to consider carefully the sign of $\mathbf{H}\left(\mu_{i}\right)$. Noting that $\operatorname{det} \Phi_{\mathbf{u}}^{-1}\left(\mathbf{u}^{*}\right)$ is positive, then we only need to consider the sign of $\operatorname{det}\left[\mu \Phi_{\mathbf{u}^{*}}\left(\mathbf{u}^{*}\right)-\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right]$. In fact, the direct calculation gives that the value of $\bar{c}_{3}$, which is given in (30), is equal to $\operatorname{det}\left[\mu \Phi_{\mathbf{u}^{*}}\left(\mathbf{u}^{*}\right)-\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right]$. To study the existence of the positive solution of (3) with respect to the cross-diffusion constant $d_{3} d_{4}$, we will concentrate on the dependence of $\mathbf{H}\left(\mu_{i}\right)$ on $d_{3}$, and let $d_{1}, d_{2}$, and $d_{4}$ be fixed. Hence, from Theorem 3, we first introduce the following proposition.

Proposition 12. Assume that $d_{4}>0$ and that (33) holds. Then there exists a positive number $d_{3}^{*}$ such that, for all $d_{3} \geq d_{3}^{*}$, all the three roots $\tilde{\mu}_{1}, \tilde{\mu}_{2}$, and $\tilde{\mu}_{3}$ of $\operatorname{det}\left[\mu \Phi\left(\mathbf{u}^{*}\right)-\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right]=0$ are real and satisfy

$$
\begin{gather*}
-\infty<\widetilde{\mu}_{1}<0<\widetilde{\mu}_{2}<\widetilde{\mu}_{3} ; \\
\operatorname{det}\left[\mu \Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right]<0 \\
\text { if } \mu \in\left(-\infty, \widetilde{\mu}_{1}\right) \cup\left(\widetilde{\mu}_{2}, \widetilde{\mu}_{3}\right)  \tag{89}\\
\operatorname{det}\left[\mu \Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right]>0 \\
\quad \text { if } \mu \in\left(\widetilde{\mu}_{1}, \widetilde{\mu}_{2}\right) \cup\left(\widetilde{\mu}_{3}, \infty\right)
\end{gather*}
$$

Proof of Theorem 10. By Proposition 12 and our assumptions, there exists a positive constant $d_{3}^{*}$ such that, when $d_{3} \geq d_{3}^{*}$, (89) holds and

$$
\begin{equation*}
\mu_{i}<\widetilde{\mu}_{2}<\mu_{i+1}, \quad \mu_{j}<\widetilde{\mu}_{3}<\mu_{j+1} . \tag{90}
\end{equation*}
$$

Now, we show that, for any $d_{3} \geq d_{3}^{*}$, (3) has at least one nonconstant positive solution. The proof, which will be accomplished by a contradict argument, is based on the homotopy invariance of the topological degree.

Suppose, on the contrary, that the assertion is not true. Let $\widehat{d}_{i}=D_{i}^{*}(i=1,2,3,4)$, where $D_{i}^{*}$ is defined in Theorem 9. For $t \in[0,1]$, define

$$
\begin{align*}
\Phi(t ; \mathbf{u})= & {\left[\widehat{d}_{1} u_{1}+t\left(d_{1}-\widehat{d}_{1}\right) u_{1}, \widehat{d}_{2} u_{2}+t\left(d_{2}-\widehat{d}_{2}\right) u_{2}\right.} \\
& \left.\widehat{d}_{3}+t\left(d_{3}-\widehat{d}_{3}\right)\left(u_{3}+t d_{4} u_{1} u_{3}\right)\right]^{T} \tag{91}
\end{align*}
$$

Now we consider the following problem:

$$
\begin{gather*}
-\Delta \Phi(t ; \mathbf{u})=\mathbf{G}(\mathbf{u}) \quad \text { in } \Omega, 0 \leq t \leq 1, \\
\partial_{\gamma} \mathbf{u}=0  \tag{92}\\
\text { on } \partial \Omega .
\end{gather*}
$$

Then $\mathbf{u}$ is a positive solution of (3) if and only if it is a positive solution of (92) for $t=1$. For $0 \leq t \leq 1$, it is obvious that $\mathbf{u}^{*}$ is the unique positive constant solution of (92) and $\mathbf{u}$ is a positive solution of (92) if and only if

$$
\begin{align*}
\mathbf{F}(t ; \mathbf{u})= & \mathbf{u}-(\mathbf{I}-\Delta)^{-1} \\
& \times\left\{\Phi_{\mathbf{u}}^{-1}(t ; \mathbf{u})\left[\mathbf{G}(\mathbf{u})+\nabla \mathbf{u} \Phi_{u u}(t ; \mathbf{u}) \nabla \mathbf{u}\right]+\mathbf{u}\right\}=0 \\
& \text { in } \mathbf{X}^{+} . \tag{93}
\end{align*}
$$

Clearly, $\mathbf{F}(1 ; \mathbf{u})=\mathbf{F}(\mathbf{u})$. Theorem 9 shows that the only positive solution of $\mathbf{F}(0 ; \mathbf{u})$ is $\mathbf{u}^{*}$ in $\mathbf{B}(\mathbf{C})$. By a direct computation, we have

$$
\begin{equation*}
\mathbf{D}_{\mathbf{u}} \mathbf{F}\left(\mathbf{t} ; \mathbf{u}^{*}\right)=\mathbf{I}-(\mathbf{I}-\Delta)^{-1}\left\{\Phi_{\mathbf{u}}^{-1}\left(t ; \mathbf{u}^{*}\right) \mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)+\mathbf{I}\right\} . \tag{94}
\end{equation*}
$$

In particular,

$$
\begin{gather*}
\mathbf{D}_{\mathbf{u}} \mathbf{F}\left(\mathbf{0} ; \mathbf{u}^{*}\right)=\mathbf{I}-(\mathbf{I}-\Delta)^{-1}\left\{\mathfrak{D}^{-1} \mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)+\mathbf{I}\right\} \\
\mathbf{D}_{\mathbf{u}} \mathbf{F}\left(\mathbf{1} ; \mathbf{u}^{*}\right)=\mathbf{I}-(\mathbf{I}-\Delta)^{-1}\left\{\Phi_{\mathbf{u}}^{-1}\left(\mathbf{u}^{*}\right) \mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)+\mathbf{I}\right\} \tag{95}
\end{gather*}
$$

where $\mathfrak{D}=\operatorname{diag}\left(\widehat{d}_{1}, \widehat{d}_{2}, \widehat{d}_{3}\right)$. In view of Proposition 12 and (90), it follows that

$$
\begin{gather*}
\mathbf{H}\left(\mu_{1}\right)=\mathbf{H}(0)>0, \\
\mathbf{H}\left(\mu_{n}\right)<0, \quad i+1 \leq n \leq j,  \tag{96}\\
\mathbf{H}\left(\mu_{n}\right)>0, \quad 1<n \leq i \text { or } n \geq j+1 .
\end{gather*}
$$

Therefore, zero is not an eigenvalue of the matrix $\mu_{n} I-$ $\Phi_{\mathbf{u}}^{-1}\left(\mathbf{u}^{*}\right) \mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$ for all $n \geq 1$, and

$$
\begin{equation*}
\sum_{n \geq 1, H\left(\mu_{n}\right)<0} m\left(\mu_{n}\right)=\sum_{n=i+1}^{j} m\left(\mu_{n}\right)=\text { an odd number. } \tag{97}
\end{equation*}
$$

Then Proposition 11 yields

$$
\begin{equation*}
\operatorname{index}\left(\mathbf{I}-\mathbf{F}(\mathbf{1} ; \cdot), \mathbf{u}^{*}\right)=(-1)^{\gamma}=-1 . \tag{98}
\end{equation*}
$$

Similarly we can get that

$$
\begin{equation*}
\operatorname{index}\left(\mathbf{I}-\mathbf{F}(\mathbf{0} ; \cdot), \mathbf{u}^{*}\right)=(-1)^{0}=1 \tag{99}
\end{equation*}
$$

According to Theorems 9 and (40) and (59), there exist positive constants $\bar{C}$ and $\underline{C}$ such that, for all $0 \leq t \leq 1$, the positive solutions of (3) satisfy $\underline{C} \leq u_{1}, u_{2}, u_{3} \leq \bar{C}$. Therefore, $\mathbf{F}(t ; \mathbf{u}) \neq 0$ on $\partial \mathbf{B}(\mathbf{C})$ for all $0 \leq t \leq 1$. By the homotopy invariance of the topological degree, we have

$$
\begin{equation*}
\operatorname{deg}(\mathbf{I}-\mathbf{F}(\mathbf{1} ; \cdot), 0, \mathbf{B}(\mathbf{C}))=\operatorname{deg}(\mathbf{I}-\mathbf{F}(\mathbf{0} ; \cdot), 0, \mathbf{B}(\mathbf{C})) . \tag{100}
\end{equation*}
$$

On the other hand, under our assumptions, the only positive solution of both $\mathbf{F}(\mathbf{1} ; \mathbf{u})=0$ and $\mathbf{F}(\mathbf{0} ; \mathbf{u})=0$ in $\mathbf{B}(\mathbf{C})$ is $\mathbf{u}^{*}$, and hence, by (98) and (99),

$$
\begin{gather*}
\operatorname{deg}(\mathbf{I}-\mathbf{F}(\mathbf{0} ; \cdot), 0, \mathbf{B}(\mathbf{C}))=\operatorname{index}\left(\mathbf{I}-\mathbf{F}(\mathbf{0} ; \cdot), \mathbf{u}^{*}\right)=1 \\
\operatorname{deg}(\mathbf{I}-\mathbf{F}(\mathbf{1} ; \cdot), 0, \mathbf{B}(\mathbf{C}))=\operatorname{index}\left(\mathbf{I}-\mathbf{F}(\mathbf{1} ; \cdot), \mathbf{u}^{*}\right)=-1 \tag{101}
\end{gather*}
$$

which contradicts (100). The proof is completed.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Identification of V-Formations and Circular and Doughnut Formations in a Set of Moving Entities with Outliers 

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#### Abstract

Diverse movement patterns may be identified when we study a set of moving entities. One of these patterns is known as a Vformation for it is shaped like the letter V. Informally, a set of entities shows a V-formation if the entities are located on one of their two characteristic lines. These lines meet in a position where there is just one entity considered the leader of the formation. Another movement pattern is known as a circular formation for it is shaped like a circle. Informally, circular formations present a set of entities grouped around a center in which the distance from these entities to the center is less than a given threshold. In this paper we present a model to identify V-formations and circular formations with outliers. An outlier is an entity which is part of a formation but is away from it. We also present a model to identify doughnut formations, which are an extension of circular formations. We present formal rules for our models and an algorithm for detecting outliers. The model was validated with NetLogo, a programming and modeling environment for the simulation of natural and social phenomena.


## 1. Introduction

Diverse movement patterns may be identified when we study a set of moving entities, for example, a flock of birds [1] and a school of fish [2]. One of these patterns is known as a V-formation for it is shaped like the letter V; see Figure 1. Another movement pattern is known as a circular formation for it is shaped like a circle; see Figure 2.

Informally, a set of entities shows a V-formation if the entities are located on one of their two characteristic lines. The lines meet in a position where there is just one entity considered the leader of the formation [3]. Several authors have analyzed V-formations. In $[4,5]$, there is an attempt to explain from a physical point of view the reasons why certain species of birds, such as Canadian geese (Branta canadensis), red knots (Calidris canutus), and plovers (Calidris alpina), tend to fly this way.

Other authors try to simulate V-formations at a computational level. For instance, Nathan and Barbosa [6] propose a model based on rules that allows us to generate V-formations depending on specific parameters. The authors validated their
model using NetLogo [7], a programming and modeling environment to simulate natural and social phenomena.

On the other hand, a circular formation is a set of entities grouped around a common center and where the entities' distance to the center is less than a given threshold. Regarding related works with circular formations, we identified the following.

In [8] the authors experimented with a set of data referring to the movement of different animal species. It was found that despite being in different ecosystems, species follow similar behavioral patterns. The authors also tried to model general grouping behaviors of fish, birds, insects, and even people. One of these behaviors is circular formation in which they identified physical forces: attraction, repulsion, alignment, and frontal interaction.

On the other hand, researchers in the field of robotics and in control theory, inspired by social grouping phenomena and by the patterns of birds and fish, have developed applications to coordinate the movement of multivehicle systems. Among these patterns are circular [9] and V-formations; see Figure 3.


Figure 1: V-formation in birds. Source: [9].


Figure 2: Circular formation in a fish bank. Source: [21].

On the other hand, regardless of the type of formation, Reynolds [2, 10] proposes a computer model of coordinated animal motion for bird flocks and fish schools. His model considers three simple rules that act upon individuals: (i) separation: steer to avoid crowding local flockmates, (ii) alignment: steer towards the average heading of local flockmates, and (iii) cohesion: steer to move toward the average position of local flockmates.

Although the previous works allow the simulation of a set of moving entities, they are not aimed at the explicit identification of V-formations and circular formations. The identification of these types of formations may be useful in fields as zoology, to analyze the movement of birds [3] and fish [11], and in the military and videogames where squadrons of combat planes, ships, and robots usually assume these types of formations [3, 12]. V-formations also usually appear in stock markets (stock prices) [13]. In this direction the model of Andersson et al. [14], although it is not aimed at the explicit identification of formations, identifies an entity leader in a set of moving entities. We use their model as the basis for identifying V-formations and circular formations, as we will show in our paper.

This paper is organized as follows. In Section 2, we present our models for V-formations and circular and doughnut formations, which are an extension of circular formations. In Section 3, we present support for outliers in our models. In Section 4, we present experiments. In Section 5, we conclude the paper and propose future works.

## 2. V-Formations and Circular and Doughnut Formations

2.1. Andersson's Model. Next, we present the essential elements of Andersson's model [14].

Consider a set $E$ of $n$ entities $\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ that move in a space, usually in a region during a time interval $\left[t_{1}, t_{f}\right]$. This space is represented by the Euclidian plane. Time is represented continuously. Tp denotes an infinite set of timepoints: $\left\{t \mid t \in\left[t_{1}, t_{f}\right]\right\}$. On the other hand, Ts denotes a finite set of discrete time-points $\left\{t_{1}, t_{2}, \ldots, t_{f}\right\}$. Every $t_{i} \in T s$ represents a time-step and corresponds to a time-point in which the position of a moving entity (and perhaps other data) was recorded. A unit-time-interval is an open interval between two consecutive time-steps, that is, $\left(t_{i-1}, t_{i}\right)$, for all $t_{i} \in T s, i \neq 1$.

The coordinates of an entity at a time-point are given by a pair of functions xpos and ypos, both having the signature $E \times T p \rightarrow$ Real. Andersson assumes that between two consecutive time-steps an entity moves along a straight line with constant direction and constant velocity. This assumption has been adopted in other works (see [15-17] for a discussion). Therefore, in such a model in order to be practical enough, the recording frequency of the positions of moving objects should be adapted to the nature of each domain. For example, when detecting bird formations the elapsed time between two consecutive time-steps should not be longer than 30 seconds.

Consider an entity $e_{i}$ at a time-point $t$, where $t_{x-1}<t<$ $t_{x}, t_{x} \in T s, x \neq 1$; then the direction of $e_{i}$ at $t$, denoted by $d\left(e_{i}, t\right)$, is the angle specified by the straight line segment that goes from $\left(x p o s\left(e_{i}, t_{x-1}\right), y p o s\left(e_{i}, t_{x-1}\right)\right)$ to $\left(x p o s\left(e_{i}, t_{x}\right)\right.$, $\left.y p o s\left(e_{i}, t_{x}\right)\right)$; see Figure 4 . The angle $d\left(e_{i}, t\right)$ is between $[0,2 \pi)$. Andersson declares the direction of an entity at a time-step $t_{x}$ to be undefined, because at a time-step an entity might change its direction (for more details see [14]).

The front region of an entity $e_{i}$ is a region associated with $e_{i}$ at a time-point $t$, where $t_{x-1}<t<t_{x}, t_{x} \in T s, x \neq 1$, which represents the perception region of an entity, for example, a visual or an auditive range. The front region of $e_{i}$ is defined as follows: consider three straight line segments $s_{0}, s_{1}$, and, $s_{2}$, each of length $r$. Each segment has an endpoint at $\operatorname{xpos}\left(e_{i}\right.$, $\left.t), y \operatorname{pos}\left(e_{i}, t\right)\right)$. The direction of the segment $s_{0}$ is $d\left(e_{i}, t\right)$. Segments $s_{1}$ and $s_{2}$ form angles of $\alpha / 2$ and $-\alpha / 2(0 \leq \alpha<2 \pi)$ with regard to segment $s_{0}$, respectively. The circular sector (a wedge-shaped region) with a radius $r(r \geq 0)$, delimited by $s_{1}$ and $s_{2}$, makes up the front region of $e_{i}$ at $t$ and it is denoted by front $\left(e_{i}, t\right)$; see Figure 5.

An entity $e_{j}$ is in front of an entity $e_{i}$, at a time-point $t$ $(t \in T p, t \notin T s)$, where $t_{x-1}<t<t_{x}, t_{x} \in T s, x \neq 1$, if $e_{j}$ is in the front region of $e_{i}$, that is, if $\left(x p o s\left(e_{j}, t\right), y p o s\left(e_{j}, t\right)\right)$ is inside front $\left(e_{i}, t\right)$; this is denoted $e_{j} \in \operatorname{front}\left(e_{i}, t\right)$ and we say that $e_{i}$ follows $e_{j}\left(e_{i}\right.$ is a follower of $\left.e_{j}\right)$ at $t$. We consider an additional restriction to establish that one entity follows another: let $\beta \in[0, \pi]$; then entity $e_{i}$ follows entity $e_{j}$ at $t$ if (i) $e_{j} \in \operatorname{front}\left(e_{i}, t\right)$ and (ii) $\left\|d\left(e_{i}, t\right)-d\left(e_{j}, t\right)\right\| \leq \beta$; see Figure 6. An entity $e_{i}$ is said to follow an entity $e_{j}$ during a time interval [ $t_{a}, t_{b}$ ], where $t_{a}$ and $t_{b}$ are time-points in $T p$, if and only if $e_{i}$ follows $e_{j}$ at $t$, for all $t \in\left[t_{a}, t_{b}\right], t \notin T s$.


Figure 3: Robots in circular formation and planes in V-formation. Sources: [11, 12].


Figure 4: $d\left(e_{i}, t\right)$ : angle of entity $e_{i}$ at a time-point $t$.


Figure 5: Front region of an entity $e_{i}$ at a time-point $t$.


Figure 6: Entity $e_{i}$ follows entity $e_{j}$ at a time-point $t$.


Figure 7: An example of a V-formation at a time-point $t$.

An entity $e_{i}$ is said to be a leader of a formation during a time interval $I=\left[t_{a}, t_{b}\right]$, where $t_{a}$ and $t_{b}$ are time-points in $T p$, if (i) $e_{i}$ does not follow any entity during $I$ and (ii) $e_{i}$ is at least followed by $m$ entities at each time-point $t \in I, t \notin T s$. It is said that there is a leadership pattern if $e_{i}$ is a leader of at least $m$ entities for at least $k$ unit-time-intervals.
2.2. V-Formations. Consider the formation in Figure 7 where entities displayed a V-formation at a time-point $t$, and $e_{4}$ is the leader entity. Note that if we applied Andersson's model [14] to this formation, we would observe that no entity contains $e_{4}$ in its front region. Therefore, this example shows that Andersson's model [14] is inadequate to identify leadership patterns in these types of formations, unless we consider a wide front region for each entity, as shown in Figure 8. Thus, the entities on the extremes, $e_{1}$ and $e_{7}$, will require a wide front region to be able to contain the leader entity $e_{4}$; this is not a realistic assumption in V-formations since an entity's visual field is not usually that wide [4].

In the following section, we extend Andersson's model to identify leadership patterns in V-formations.


Figure 8: A V-formation: entities $e_{1}$ and $e_{7}$ would require a wide front region in order to contain the entity leader $e_{4}$.
2.3. $V$-Formation Model. Let $F=\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ be a set of moving entities at a time-point $t, \operatorname{card}(F) \geq 3 . F$ is a V formation at $t$ if the following hold.
(i) There exist nonempty subsets $A$ and $B$ of $F$ such that $\operatorname{card}(A) \geq 2, \operatorname{card}(B) \geq 2, A \cup B=F$, and $A \cap B=$ $\left\{e_{k}\right\}$, where $e_{k}$ is said to be the leader entity of the formation.
(ii) Entities in $A$ tend to form a straight line $l_{1}$ at $t$.
(iii) Entities in $B$ tend to form a straight line $l_{2}$ at $t$.
(iv) Straight lines $l_{1}$ and $l_{2}$ meet at position $\left(x p o s\left(e_{k}, t\right)\right.$, $\left.y p o s\left(e_{k}, t\right)\right)$.
(v) $L a p_{t}>0$ (the smallest angle defined by straight lines $l_{1}$ and $l_{2}$ at $t$ ).
Regarding conditions (ii) and (iii), to establish if a set of entities tend to form a straight line, we use Pearson's correlation coefficient $r$ [18]. Thus, given a set of points $\left\{\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right), \ldots,\left(x_{n}, y_{n}\right)\right\}, r$ indicates how much they adjust to a straight line (linearity degree). $r \in(-1,1)$; if $|r| \approx 1$, then points tend to form a straight line. $r$ is calculated as shown in (1). A threshold $\mu_{p}$ may be specified by the user to indicate the linearity degree required for the lines of the formation; that is, $|r| \geq \mu_{p}$ :

$$
\begin{equation*}
r=\frac{n *\left(\sum_{i=1}^{n} x_{i} * y_{i}\right)-\sum_{i=1}^{n} x_{i} * \sum_{i=1}^{n} y_{i}}{\sqrt{\left[n * \sum_{i=1}^{n} x_{i}^{2}-\left(\sum_{i=1}^{n} x_{i}\right)^{2}\right] *\left[n * \sum_{i=1}^{n} y_{i}^{2}-\left(\sum_{i=1}^{n} y_{i}\right)^{2}\right]}} . \tag{1}
\end{equation*}
$$

To obtain the equation for each straight line $(y=m x+b)$ characteristic of the formation $\left(l_{1}\right.$ and $\left.l_{2}\right)$ we may apply the equations that correspond to the straight line which most suits a set of points $\left\{\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right), \ldots,\left(x_{n}, y_{n}\right)\right\}$; see (2). Table 1 presents the results for the formation in Figure 9:

$$
\begin{gather*}
m=\frac{n *\left(\sum_{i=1}^{n} x_{i} * y_{i}\right)-\sum_{i=1}^{n} x_{i} * \sum_{i=1}^{n} y_{i}}{n * \sum_{i=1}^{n} x_{i}^{2}-\left(\sum_{i=1}^{n} x_{i}\right)^{2}},  \tag{2}\\
b=\frac{\sum_{i=1}^{n} y_{i}-m * \sum_{i=1}^{n} x_{i}}{n}
\end{gather*}
$$



Figure 9: A formation of entities at a time-point $t$.

Regarding condition (v), $L a p_{t}$ is calculated as follows: we get straight lines $l_{1}$ and $l_{2}$ from the formation and find the smallest angle between them as follows.
$a$ is the positive angle in $[0,2 \pi)$ that is specified by the line segment that goes from $p_{k}$ to $p_{i}$, where $p_{k}$ is the position of $e_{k}$ (leader entity) according to equation for $l_{1}$ and $p_{i}$ is the position of $e_{i} \in A, e_{i} \neq e_{k}$ according to equation for $l_{1}$.
$b$ is the positive angle in $[0,2 \pi)$ that is specified by the line segment that goes from $p_{k}$ to $p_{j}$, where $p_{k}$ is the position of $e_{k}$ (leader entity) according to equation for $l_{2}$ and $p_{j}$ is the position of $e_{j} \in B, e_{j} \neq e_{k}$, according to equation for $l_{2}$.

Let $w=|a-b|$; then $\angle a p_{t}=w$ if $w \leq \pi$, and $\angle a p_{t}=2 \pi-$ $w$, otherwise. For example, in the formation in Figure 9, $a=$ $0.7 \mathrm{rad}\left(40.36^{\circ}\right), b=5.14 \mathrm{rad}\left(294.44^{\circ}\right)$, and $w=4.43 \mathrm{rad}$ (254.08 ${ }^{\circ}$ ); thus, $\angle a p_{t}=1.84 \mathrm{rad}\left(105.91^{\circ}\right)$.

In an analogous way to leadership patterns, we say that there is a V-formation pattern if the set $F$ of moving entities shows a $V$-formation for at least $k$ unit-time-intervals.
2.4. Circular and Doughnut Formations. In this section, we present a formal model to identify circular formations. We also present doughnut formations, which are an extension of circular formations.
2.4.1. Circular Formation Model. Let $F=\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ be a set of moving entities at a time-point $t$. Let $C$ be the minimum circle that encloses the entities in $F$ and let $C_{r}$ be its radius and let $\left(C_{x}, C_{y}\right)$ be its center (Welzl [19] and Megiddo [20] have showed that this circle can be found in linear time). $F$ is a circular formation at $t$ if
(i) $C_{r} \leq R$ (a user-given radius with center at $\left(C_{x}, C_{y}\right)$ ),
(ii) the minimum number of members in the formation is $N_{\text {min }}$.

In Figure 10, we show a circular formation with 9 members.
2.4.2. Doughnut Formation Model. Let $F=\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ be a set of moving entities at a time-point $t$. Let $C$ be the minimum circle that encloses the entities in $F$ and let $C_{r}$ be its radius and $\left(C_{x}, C_{y}\right)$ its center. $F$ is a doughnut formation at $t$ if

Table 1: Applying our model to the formation in Figure 9.

| Characteristic line | Entities that make up the <br> characteristic line | Characteristic line <br> equation | Pearson's coefficient |
| :--- | :---: | :---: | :---: | | Entity coordinates: original and <br> calculated using the <br> characteristic line equation |
| :---: |
| $l_{1}$ |
|  |



Figure 10: A circular formation.


Figure 11: A doughnut formation.
(i) $C_{r} \leq R_{\text {ext }}$ (external radius, a user-given radius with center at $\left.\left(C_{x}, C_{y}\right)\right)$,
(ii) for each entity of the formation its distance $d$ regarding $\left(C_{x}, C_{y}\right)$ is greater than $R_{\text {int }}$ (internal radius, a user-given radius with center at $\left.\left(C_{x}, C_{y}\right), R_{\text {int }} \leq C_{r}\right)$,
(iii) the minimum number of members in the formation is $N_{\text {min }}$.

Note that a circular formation is a particular case of a doughnut formation, where $R_{\mathrm{int}}=0$. In Figure 11, we show a doughnut formation with 8 members.

Circular and doughnut patterns are also defined in an analogous way to a V-formation pattern.

## 3. Outliers

Informally, for V-formations an outlier is an entity which is away from its characteristic lines, and for circular formations it is an entity found beyond the radius of the formation $(R)$.
3.1. Outliers in $V$-Formations. There are sets of entities which tend to display a V-formation; they may have at a time-point $t$ entities which are away from their characteristic lines and which, therefore, affect Pearson's coefficient. These entities are called outliers [22, 23].

There are numerous methods to detect outliers in different domains [24]. Listing 1 presents an algorithm that receives an array of $m$ entities (lineMembers) which form
a characteristic line of a formation at a time-point $t$. The algorithm determines if, after removing a maximum number of entities on the given array, the Pearson's coefficient surpasses a given threshold $\mu_{p}$. For example, if it is permitted to remove a maximum of two entities from a characteristic line of the formation, it is considered that the entities in Figure 12 display a V-formation having two outliers on each characteristic line. Then, the algorithm receives the minimum value of Pearson's coefficient $\mu_{p}$ which should be met and a maximum percentage of entities (percentageOutliers) which are permitted to be removed from the array of entities. This percentage is calculated with regard to the total number of entities $m$.

Example 1. Consider the set of entities in Figure 13 which would form a straight line if entities $e_{2}$ and $e_{4}$ were not considered; that is, $e_{2}$ and $e_{3}$ are outliers.

Input algorithm parameters:
lineMembers $=\left[e_{1}, e_{2}, \ldots, e_{8}\right], \mu_{p}=0.99$ and percentageOutliers $=20 \%$.

Step 1. Equation of $l: y=-0.754 x+2.01$.
Step 2. nbrOutliers $=\lceil 8 * 0.2\rceil=2$.
Step 3. $e_{2}$ and $e_{4}$.
Step 4. lineMembers $=\left[e_{1}, e_{3}, e_{5}, e_{6}, e_{7}, e_{8}\right]$.

## ALGORITHM: Outlier detection on a characteristic line

INPUT: lineMembers $=\left[e_{1}, e_{2}, \ldots, e_{m}\right] / /$ Array of $m$ entities
$\mu_{p} / /$ Threshold for Pearson's coefficient
percentageOutliers //Maximum percentage of outliers permitted on lineMembers
OUTPUT: outliers = [] // Array of outliers
BEGIN
(1) Obtain the equation of the line $l$ which most suits the positions $(x, y)$ of the entities in lineMembers, see (2).
(2) Get the maximum number of outliers permitted on lineMembers: nbrOutliers $=\lceil m *$ percentageOutliers $/ 100\rceil$.
(3) Find the nbrOutliers entities in lineMembers which have the maximum distance to $l$.
(4) Remove from lineMembers the entities found in Step 3.
(5) Calculate Pearson's coefficient using positions ( $x, y$ ) of each entity in lineMembers: Pearson $=$ PearsonCoefficient (lineMembers).
(6) If Pearson $\geq \mu_{p}$ then return in outliers the entities found in Step 3.

END

Listing 1: Outlier detection algorithm.


Figure 12: A V-formation with two outliers.


Figure 13: Characteristic line of a V-formation with two outliers.

Step 5. Pearson $=$ PearsonCoefficient $($ lineMembers $)=1$.
Step 6. $1 \geq 0.99$; then return outliers $=\left[e_{2}, e_{4}\right]$.
The algorithm determines that entities $e_{2}$ and $e_{4}$ are outliers. Then, we conclude that entities $e_{1}, e_{3}, e_{5}, e_{6}, e_{7}$, and $e_{8}$ tend to form a straight line with Pearson's coefficient greater than 0.99 .
3.2. Outliers in Circular Formations. Let $F=\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ be a set of moving entities at a time-point $t$. Consider an entity $e_{i} \in F$ such that $F-\left\{e_{i}\right\}$ is a circular formation and


Figure 14: A circular formation with two outliers.
the distance from $e_{i}$ to $\left(C_{x}, C_{y}\right)$ (center of the minimum circle that encloses the entities in $\left.F-\left\{e_{i}\right\}\right)$ is greater than $R$ at $t$. This entity may be considered as a member of the formation which is temporally away from it; that is, $e_{i}$ is an outlier. To identify this type of entities, we introduce a parameter $R_{\text {max Outlier }}$, where $R_{\max O u t l i e r}>R$. An entity is considered an outlier at a time-point $t$ if its distance $d$ to $\left(C_{x}, C_{y}\right)$ is greater than $R$ and less than $R_{\max O u t l i e r}$; see Figure 14.

Since the distancing of the outlier entity from the formation is temporal, an analyst may introduce a second parameter $T_{\text {maxTimeSeparation }}$ to control the maximum continuous time of permitted distancing. That is, if an entity separates from a circular formation at a time-point $t$, then, to be considered an outlier, it will have to reincorporate $(d \leq R)$ to the formation before $t+T_{\text {maxTimeSeparation }}$. This same aspect can also be considered for outliers in $V$-formations.

An analyst can also specify a maximum permitted number of outliers maxNumberOutliers in the formation. This value can be calculated from a percentage (percentageOutliers) with regard to the total number of entities in the formation.
3.3. Outliers in Doughnut Formations. Let $F=\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ be a set of moving entities at a time-point $t$. Consider an entity


Figure 15: Outliers in doughnut formations.

Table 2: Parameters to generate V-formations in NetLogo using Nathan's model.

| Number of entities | 15 |
| :--- | :---: |
| Vision parameters |  |
| $\quad$ Vision distance | 9 patches |
| Vision cone | $1.79 \mathrm{rad}\left(103^{\circ}\right)$ |
| Obstruction cone | $0.75 \mathrm{rad}\left(43^{\circ}\right)$ |
| Movement parameters |  |
| Base speed | 0.2 |
| Speed change factor | 0.15 |
| Updraft distance | 9 patches |
| Too close | 3.1 |
| Max. turn | $0.14 \mathrm{rad}\left(8^{\circ}\right)$ |

$e_{i} \in F$ such that $F-\left\{e_{i}\right\}$ is a doughnut formation. For doughnut formations we consider two types of outliers: external and internal. An entity $e_{i}$ is considered an external outlier at $t$ if its distance $d$ to $\left(C_{x}, C_{y}\right)$ (center of the minimum circle that encloses the entities in $\left.F-\left\{e_{i}\right\}\right)$ is greater than $R_{\text {ext }}$ and less than $R_{\max \text { Outlier }}$, where $R_{\max \text { Outlier }}>R_{\text {ext }}$. On the other hand, $e_{i}$ is considered an internal outlier at $t$ if $d$ is less than $R_{\text {int }}$; see Figure 15.

Analogously to circular formations, $T_{\text {maxTimeSeparation }}$ and maxNumberOutliers values may be specified, not only for external outliers but also for internal outliers.

## 4. Experiments and Discussion

4.1. Experiment for V-Formations. For our experiment, we used Nathan's model [6] implemented in NetLogo in order to generate V-formations. NetLogo is a programmable modeling environment for simulating natural and social phenomena that have been used in previous works cite9, 26.

We worked with a population of 15 entities. The dimensions of the Euclidian plane where the entities move were $x \in[-35,35]$ and $y \in[-35,35]$, where the unit of measure for distance in NetLogo is a patch. To generate V-formations, we used the parameters in Table 2. These parameters allow us to define aspects related to vision and movement of the entities in Nathan's model.

For outlier detection, we considered for our model the following parameters: $\mu_{p}=0.90$ and percentageOutliers $=$ $20 \%$. Our experiment began from tick 40199 and ended at tick 40400, because in Nathan's model the generation of V-formations requires a stabilization time (tick 40199 in our experiment, a tick is the unit of measure for time in NetLogo). Time-steps were taken in intervals of 20 ticks, for if we consider a smaller step, a change in the position and direction of an entity would be practically imperceptible. Thus, the total number of steps was 10 . Figure 16 shows the position of entities at ticks 40200, 40220, 40380, and 40400 and Table 3 summarizes our results.
4.1.1. Analysis of V-Formation Results. According to the results shown in Table 3, our model classified several sets of entities as V-formations. These results agree with what was expected since Nathan's model had already stabilized and Vformations were being generated effectively for those times. Moreover, it was necessary to apply the outlier detection algorithm just in two ticks (at ticks 40380 and 40400, see entities in brackets in Table 3, second column). On the other hand, we found that formations showed a high linearity degree since Pearson's coefficients in all cases were greater than 0.92 .

We also applied Andersson's model. We considered $m=2$ and for the front region of each entity the parameters in Table 2; that is, $s_{0}=9$ patches (vision distance) and $\alpha=1.79$ rad ( $103^{\circ}$ ) (vision cone). Andersson's model only identified formations of two and three entities. This is reasonable as explained in Section 2.2 (see Figure 8). In addition, his model does not give information about the shape of the formation that it identifies; even if we considered a wider front region, his model would not indicate that a formation is, for example, $V$-shaped.
4.2. Experiments for Circular and Doughnut Formations. For experiments with circular and doughnut formations, we also worked in NetLogo and used Wilensky's model [26] (a model inspired in Reynolds' work [10]), which generates randomized formations of entities in NetLogo.

We worked with a population of 102 entities. The dimensions of the Euclidian plane where the entities move were $x \in[-35,35]$ and $y \in[-35,35]$. To generate formations, we used the parameters in Table 4.

The model was executed on 1200 consecutive ticks (one run) and we conducted an analysis of circular formations and doughnut formations every 400 ticks. We considered that a set of entities showed a circular/doughnut pattern if the set showed a circular/doughnut formation during all the run (1200 ticks). A total of 100 runs were conducted. The parameters used to detect circular and doughnut formations are shown in Tables 5 and 6 . Table 7 summarizes our results for the 100 runs. Figure 17 shows the results for one of the runs.

### 4.2.1. Analysis of Experiments for Circular and Doughnut Formations. According to the results shown in Table 7,


(a)

(c)

(b)

(d)

FIgure 16: V-formations. Position of entities at ticks (a) 40200, (b) 40220, (c) 40380, and (d) 40400.
the model classified several sets of entities as circular formations and around three remained throughout the 1200 ticks of a run. Regarding doughnut formations, the model detected around two formations that remained during the 1200 ticks of a run. These results agree with what was expected since Wilensky's model had already stabilized in NetLogo.

With respect to Andersson's model, we considered $m=$ 3 and the parameters in Table 2. His model identified 752 formations during the 100 runs, a value that is greater than the total number of circular and doughnut formations identified during the 100 runs. This is reasonable because with $m=3$ (note that the average number of entities in his formations was four) it is expected that any set of at least three entities that are close to each other (considering $s_{0}=9$ patches) will be identified as a formation. However, as our previous experiments with V-formations, his model is unable to indicate the shape of a formation that it identifies.

## 5. Conclusions and Future Work

In this paper, we propose two models:
(I) a model to identify V-formations with outliers. The model considers the location of entities to determine if they form this type of formation during a time interval;
(II) a model to identify circular formations with outliers. The model considers the location of entities to determine if they form this type of formation during a time interval. In addition, we proposed an extension to identify doughnut formations with outliers.
The rules for model (I) are flexible since they allow Vformations which are not necessarily perfectly aligned, in accordance with the real world. Furthermore, we consider outliers in V -, circular, and doughnut formations, that is, members of the formation that could be temporarily away from it. Our experimental results in NetLogo showed that our models identified these types of formations in such simulation environment where they were generated. We also showed that when we applied Andersson's model, his model did not give information about the shape of the formation that it identified; that is, his model suffers from lack of semantic information about shape formations.

Regarding future work, we plan to conduct a series of experiments in the stock market where V-formations usually

Table 3: V-formations: results of the experiment in NetLogo.

| Tick | Formations identified by our model (leader bolded, outliers in brackets) | Pearson's coefficient straight line 1 | Pearson's coefficient straight line 2 | Opening angle | V-formation | Formations identified by Andersson's model (leader bolded) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 40200 | $\{1,2,3,5,13,14\}$ | $\{1,2,3\} \rightarrow 0.99$ | $\{3,5,13,14\} \rightarrow 0.99$ | $1.5 \mathrm{rad}\left(86.49^{\circ}\right)$ | Yes | $\begin{aligned} & \{2,3,5\} \\ & \{13,14\} \end{aligned}$ |
|  | $\{7,9,12\}$ | $\{7,9\} \rightarrow 1$ | $\{9,12\} \rightarrow 1$ | $1.33 \mathrm{rad}\left(75.92^{\circ}\right.$ ) | Yes | $\{7,9,12\}$ |
|  | $\{0,4,6,8\}$ | $\{0,4,6\} \rightarrow 0.98$ | $\{6,8\} \rightarrow 1$ | $0.52 \mathrm{rad}\left(29.97^{\circ}\right)$ | Yes | $\begin{aligned} & \{0, \mathbf{4}\} \\ & \{\mathbf{6}, 8\} \end{aligned}$ |
|  | $\{10,11\}$ | NA | $\{10,11\} \rightarrow 1$ | NA | No | \{10,11\} |
| 40220 | $\{1,2,3,5,13,14\}$ | $\{1,2,3\} \rightarrow 0.99$ | $\{3,5,13,14\} \rightarrow 0.99$ | $1.51 \mathrm{rad}\left(86.49^{\circ}\right)$ | Yes | $\begin{aligned} & \{2,3,5\} \\ & \{13, \mathbf{1 4}\} \end{aligned}$ |
|  | $\{7,9,12\}$ | $\{7,9\} \rightarrow 1$ | $\{9,12\} \rightarrow 1$ | $1.33 \mathrm{rad}\left(75.92^{\circ}\right.$ ) | Yes | $\{7,9,12\}$ |
|  | $\{0,4,6,8\}$ | $\{0,4,6\} \rightarrow 0.98$ | $\{6,8\} \rightarrow 1$ | $0.52 \mathrm{rad}\left(29.97^{\circ}\right)$ | Yes | $\begin{aligned} & \{0, \mathbf{4}\} \\ & \{\mathbf{6}, 8\} \end{aligned}$ |
|  | $\{10,11\}$ | NA | $\{10,11\} \rightarrow 1$ | NA | No | \{10,11\} |
|  |  |  | . |  |  |  |
| 40380 | $\{2,[3], 13,1,5,14\}$ | $\{2,13,1\} \rightarrow 1$ | $\{1,5,14\} \rightarrow 0.95$ | $0.79 \mathrm{rad}\left(45.13^{\circ}\right)$ | Yes | \{1, 5\} |
|  | $\{7,9,12\}$ | $\{7,9\} \rightarrow 1$ | $\{9,12\} \rightarrow 1$ | $1.33 \mathrm{rad}\left(75.92^{\circ}\right)$ | Yes | \{7, 9, 12\} |
|  | $\{0,4,8\}$ | $\{0,4\} \rightarrow 1$ | $\{4,8\} \rightarrow 1$ | $2.68 \mathrm{rad}\left(153.74^{\circ}\right)$ | Yes | No |
|  | $\{10,11\}$ | NA | $\{10,11\} \rightarrow 1$ | NA | No | \{10,11\} |
|  | \{6\} | NA | NA | NA | NA | NA |
| 40400 | $\{[1], 5,3,2,13,[15]\}$ | $\{5,3\} \rightarrow 0.93$ | $\{3,2,13\} \rightarrow 0.98$ | $0.86 \mathrm{rad}\left(49.5^{\circ}\right)$ | Yes | $\begin{aligned} & \{2, \mathbf{3}\} \\ & \{\mathbf{1}, 5\} \end{aligned}$ |
|  | $\{7,9,12\}$ | $\{7,9\} \rightarrow 1$ | $\{9,12\} \rightarrow 1$ | $1.33 \mathrm{rad}\left(75.92^{\circ}\right.$ ) | Yes | \{7, 9, 12\} |
|  | $\{0,4\}$ | NA | $\{0,4\} \rightarrow 1$ | NA | No | No |
|  | $\{10,11\}$ | NA | $\{10,11\} \rightarrow 1$ | NA | No | \{10,11\} |
|  | $\{6,8\}$ | NA | $\{6,8\} \rightarrow 1$ | NA | No | No |

Table 4: Parameters to generate formations in NetLogo using Wilensky's model.

| Number of entities | 102 |
| :--- | :---: |
| $\quad$ Vision parameters | 3 patches |
| Vision distance | 1 patch |
| $\quad$ Minimum Separation |  |
| Movement parameters | $0.08 \mathrm{rad}\left(4.75^{\circ}\right)$ |
| Maximum angle of rotation | $0.04 \mathrm{rad}\left(2.50^{\circ}\right)$ |
| Maximum following angle | $0.06 \mathrm{rad}\left(3.5^{\circ}\right)$ |
| Maximum angle of separation |  |

Table 5: Parameters for circular formation detection in NetLogo.

| Minimum number of entities $\left(N_{\min }\right)$ | 5 |
| :--- | :---: |
| Maximum distance $(R)$ from the center of the | 15 patches |
| minimum enclosing circle to the entities |  |
| Maximum percentage of outliers permitted | $30 \%$ |
| (percentageOutliers) | 30 patches |
| $R_{\text {maxOutlier }}$ | 400 ticks |
| $T_{\text {maxTimeSeparation }}$ |  |

Table 6: Parameters for doughnut formation detection in NetLogo.

| Minimum number of entities $\left(N_{\text {min }}\right)$ | 5 |
| :--- | :---: |
| Maximum distance $\left(R_{\text {ext }}\right)$ from the center of the <br> minimum enclosing circle to the entities | 30 patches |
| Maximum distance $\left(R_{\text {int }}\right)$ from the center of the | 10 patches |
| minimum enclosing circle to the entities | $30 \%$ |
| Maximum percentage of outliers permitted | 50 patches |
| $R_{\text {maxOutlier }}$ | 400 ticks |

appear [13]. Moreover, we plan to extend our models to identify isolated entities, that is, entities that even if they are considered members of a formation, they follow their own path and do not influence the path of other entities [27]. We also plan to propose models to identify patterns such as convergence, that is, a set of entities that converge or approach a place; divergence, that is, a set of entities that disperse or move away from a place [28], and self-organization, that is, a set of entities that move as a formation without there being a leader or an entity guiding the rest [29] or that this leader is unknown to the members [30-32].


Figure 17: Circular and doughnut formations. Position of entities at ticks (a) 0, (b) 400, (c) 800, and (d) 1200.

Table 7: Circular and doughnut formations: results of the experiments in NetLogo

|  | Circular | Doughnut | Andersson's <br> model |
| :--- | :---: | :---: | :---: |
| Total number of <br> patterns identified <br> during the 100 runs | 332 | 198 | 752 |
| Average number of <br> patterns identified in <br> each run (1200 ticks) | 3 | 2 | 8 |
| Average number of <br> entities in each pattern | 13 | 22 | 4 |
| Average number of <br> outliers | 2 | Internal: 3 <br> External: 2 | NA |

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Global Positive Periodic Solutions for Periodic Two-Species Competitive Systems with Multiple Delays and Impulses 

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A set of easily verifiable sufficient conditions are derived to guarantee the existence and the global stability of positive periodic solutions for two-species competitive systems with multiple delays and impulses, by applying some new analysis techniques. This improves and extends a series of the well-known sufficiency theorems in the literature about the problems mentioned previously.

## 1. Introduction

Throughout this paper, we make the following notation and assumptions:
let $\omega>0$ be a constant and
$C_{\omega}=\{x \mid x \in C(R, R), x(t+\omega)=x(t)\}$, with the norm being defined by $|x|_{0}=\max _{t \in[0, \omega]}|x(t)|$;
$C_{\omega}^{1}=\left\{x \mid x \in C^{1}(R, R), x(t+\omega)=x(t)\right\}$, with the norm being defined by $\|x\|=\max _{t \in[0, \omega]}\left\{|x|_{0},\left|x^{\prime}\right|_{0}\right\}$;
$P C=\left\{x \mid x: R \rightarrow R^{+}, \lim _{s \rightarrow t} x(s)=x(t)\right.$, if $t \neq t_{k}, \lim _{t \rightarrow t_{k}^{-}} x(t)=x\left(t_{k}\right), \lim _{t \rightarrow t_{k}^{+}} x(t)$ exists, $k \in$ $\left.Z^{+}\right\}$;
$P C^{1}=\left\{x \mid x: R \rightarrow R^{+}, x^{\prime} \in P C\right\} ;$
$P C_{\omega}=\{x \mid x \in P C, x(t+\omega)=x(t)\}$, with the norm being defined by $|x|_{0}=\max _{t \in[0, \omega]}|x(t)|$;
$P C_{\omega}^{1}=\left\{x \mid x \in P C^{1}, x(t+\omega)=x(t)\right\}$, with the norm being defined by $\|x\|=\max _{t \in[0, \omega]}\left\{|x|_{0},\left|x^{\prime}\right|_{0}\right\}$;
then those spaces are all Banach spaces. We also denote that

$$
\begin{gather*}
\bar{f}=\frac{1}{\omega} \int_{0}^{\omega} f(t) d t, \quad f^{L}=\min _{t \in[0, \omega]} f(t), \\
f^{M}=\max _{t \in[0, \omega]} f(t) \tag{1}
\end{gather*}
$$

for any $f \in P C_{\omega}$.

In this paper, we investigate the existence, uniqueness, and global stability of the positive periodic solution for two corresponding periodic Lotka-Volterra competitive systems involving multiple delays and impulses:

$$
\begin{aligned}
& x_{1}^{\prime}(t)=x_{1}(t)\left[r_{1}(t)-a_{1}(t) x_{1}(t)\right. \\
&+\sum_{i=1}^{n} b_{1 i}(t) x_{1}\left(t-\tau_{i}(t)\right) \\
&\left.-\sum_{j=1}^{m} c_{1 j}(t) x_{2}\left(t-\delta_{j}(t)\right)\right]
\end{aligned}
$$

$$
\begin{align*}
& x_{2}^{\prime}(t)=x_{2}(t)\left[r_{2}(t)-a_{2}(t) x_{2}(t)\right. \\
& +\sum_{j=1}^{m} b_{2 j}(t) x_{2}\left(t-\eta_{j}(t)\right) \\
& \left.-\sum_{i=1}^{n} c_{2 i}(t) x_{1}\left(t-\sigma_{i}(t)\right)\right], \quad t \neq t_{k}, \\
& \Delta x_{l}(t)=x_{l}\left(t^{+}\right)-x_{l}(t)=\theta_{l k} x_{l}(t), \\
& l=1,2, \quad k=1,2, \ldots, \quad t=t_{k}, \\
& x_{1}^{\prime}(t)=x_{1}(t)\left[r_{1}(t)-a_{1}(t) x_{1}(t)\right.  \tag{2}\\
& -\sum_{i=1}^{n} b_{1 i}(t) x_{1}\left(t-\tau_{i}(t)\right) \\
& \left.-\sum_{j=1}^{m} c_{1 j}(t) x_{2}\left(t-\delta_{j}(t)\right)\right] \text {, } \\
& x_{2}^{\prime}(t)=x_{2}(t)\left[r_{2}(t)-a_{2}(t) x_{2}(t)\right.  \tag{3}\\
& -\sum_{j=1}^{m} b_{2 j}(t) x_{2}\left(t-\eta_{j}(t)\right) \\
& \left.-\sum_{i=1}^{n} c_{2 i}(t) x_{1}\left(t-\sigma_{i}(t)\right)\right], \quad t \neq t_{k}, \\
& \Delta x_{l}(t)=x_{l}\left(t^{+}\right)-x_{l}(t)=\theta_{l k} x_{l}(t), \\
& l=1,2, \quad k=1,2, \ldots, \quad t=t_{k},
\end{align*}
$$

with initial conditions

$$
\begin{gather*}
x_{l}(\xi)=\phi_{l}(\xi), \quad x_{l}^{\prime}(\xi)=\phi_{l}^{\prime}(\xi), \\
\xi \in[-\tau, 0], \quad \phi_{l}(0)>0,  \tag{4}\\
\phi_{l} \in C\left([-\tau, 0), R^{+}\right) \bigcap C^{1}\left([-\tau, 0), R^{+}\right), \quad l=1,2,
\end{gather*}
$$

where $a_{1}(t), a_{2}(t), b_{1 i}(t), b_{2 j}(t), c_{1 j}(t)$, and $c_{2 i}(t)$ are all in $P C_{\omega}$. Also $\tau_{i}(t), \delta_{j}(t), \eta_{j}(t)$, and $\sigma_{i}(t)$ are all in $P C_{\omega}^{1}$ with $\tau_{i}(t)>0, \delta_{j}(t)>0, \eta_{j}(t)>0, \sigma_{i}(t)>0, t \in[0, \omega], \tau=$ $\max \left\{\tau_{i}(t), \delta_{j}(t), \eta_{j}(t), \sigma_{i}(t)\right\}, \tau_{i}^{\prime}(t)<1, \delta_{j}^{\prime}(t)<1, \eta_{j}^{\prime}(t)<$ $1, \sigma_{i}^{\prime}(t)<1(i=1,2, \ldots, n, j=1,2, \ldots, m)$. Furthermore, the intrinsic growth rates $r_{1}(t), r_{2}(t) \in P C_{\omega}$ are with $\int_{0}^{\omega} r_{l}(t) d t>$ $0,(l=1,2)$. For the ecological justification of (2) and (3) and similar types refer to [1-10].

In [1], Freedman and Wu proposed the following periodic single-species population growth models with periodic delay:

$$
\begin{equation*}
y^{\prime}(t)=y(t)[r(t)-a(t) y(t)+b(t) y(t-\tau(t))] . \tag{5}
\end{equation*}
$$

They had assumed that the net birth $r(t)$, the selfinhibition rate $a(t)$, and the delay $\tau(t)$ are continuously differentiable $\omega$-periodic functions, and $r(t)>0, a(t)>0$, $b(t) \geq 0$, and $\tau(t) \geq 0$ for $t \in R$. The positive feedback term $b(t) y(t-\tau(t))$ in the average growth rate of species has a positive time delay (the sign of the time delay term is positive), which is a delay due to gestation (see [1, 2]). They had established sufficient conditions which guarantee that system (5) has a positive periodic solution which is globally asymptotically stable.

In [3], Fan and Wang investigated the following periodic single-species population growth models with periodic delay:

$$
\begin{equation*}
y^{\prime}(t)=y(t)[r(t)-a(t) y(t)-b(t) y(t-\tau(t))] . \tag{6}
\end{equation*}
$$

They had assumed that the net birth $r(t)$, the selfinhibition rate $a(t)$, and the delay $\tau(t)$ are continuously differentiable $\omega$-periodic functions, and $r(t)>0, a(t)>0$, $b(t) \geq 0$, and $\tau(t) \geq 0$ for $t \in R$. The negative feedback term $-b(t) y(t-\tau(t))$ in the average growth rate of species has a negative time delay (the sign of the time delay term is negative), which can be regarded as the deleterious effect of time delay on a species growth rate (see [4-6]). They had derived sufficient conditions for the existence and global attractivity of positive periodic solutions of system (6). But the discussion of global attractivity is only confined to the special case when the periodic delay is constant.

Alvarez and Lazer [7] and Ahmad [8] have studied the following two-species competitive system without delay:

$$
\begin{align*}
& y_{1}^{\prime}(t)=y_{1}(t)\left[r_{1}(t)-a_{1}(t) y_{1}(t)-c_{1}(t) y_{2}(t)\right],  \tag{7}\\
& y_{2}^{\prime}(t)=y_{2}(t)\left[r_{2}(t)-a_{2}(t) y_{2}(t)-c_{2}(t) y_{1}(t)\right] .
\end{align*}
$$

They had derived sufficient conditions for the existence and global attractivity of positive periodic solutions of system (7) by using differential inequalities and topological degree, respectively. In fact, in many practical situations the time delay occurs so often. A more realistic model should include some of the past states of the system. Therefore, in [10], Liu et al. considered two corresponding periodic Lotka-Volterra competitive systems involving multiple delays:

$$
\begin{gather*}
y_{1}^{\prime}(t)=y_{1}(t)\left[r_{1}(t)-a_{1}(t) y_{1}(t)+\sum_{i=1}^{n} b_{1 i}(t) y_{1}\left(t-\tau_{i}(t)\right)\right. \\
\left.-\sum_{j=1}^{m} c_{1 j}(t) y_{2}\left(t-\rho_{j}(t)\right)\right], \\
y_{2}^{\prime}(t)=y_{2}(t)\left[r_{2}(t)-a_{2}(t) y_{2}(t)+\sum_{j=1}^{m} b_{2 j}(t) y_{2}\left(t-\eta_{j}(t)\right)\right. \\
\left.-\sum_{i=1}^{n} c_{2 i}(t) y_{1}\left(t-\sigma_{i}(t)\right)\right] \tag{8}
\end{gather*}
$$

$$
\begin{gather*}
y_{1}^{\prime}(t)=y_{1}(t)\left[r_{1}(t)-a_{1}(t) y_{1}(t)-\sum_{i=1}^{n} b_{1 i}(t) y_{1}\left(t-\tau_{i}(t)\right)\right. \\
\left.-\sum_{j=1}^{m} c_{1 j}(t) y_{2}\left(t-\rho_{j}(t)\right)\right] \\
y_{2}^{\prime}(t)=y_{2}(t)\left[r_{2}(t)-a_{2}(t) y_{2}(t)-\sum_{j=1}^{m} b_{2 j}(t) y_{2}\left(t-\eta_{j}(t)\right)\right. \\
\left.-\sum_{i=1}^{n} c_{2 i}(t) y_{1}\left(t-\sigma_{i}(t)\right)\right] \tag{9}
\end{gather*}
$$

where $b_{1 i}(t), b_{2 j}(t) \in C(R,[0,+\infty)), a_{1}(t), a_{2}(t), c_{1 j}(t)$, $c_{2 i}(t) \in C(R,[0,+\infty)), \tau_{i}(t), \rho_{j}(t), \eta_{j}(t)$, and $\sigma_{i}(t) \in$ $C^{1}(R,[0,+\infty))(i=1,2, \ldots, n ; j=1,2, \ldots, m)$ are $\omega$ periodic functions. Here, the intrinsic growth rates $r_{k}(t) \in$ $C(R, R)$ are $\omega$-periodic functions with $\int_{0}^{\omega} r_{k}(t) d t>0(k=$ $1,2)$. They had derived the same criteria for the existence and globally asymptotic stability of positive periodic solutions of the above two competitive systems by using Gaines and Mawhin's continuation theorem of coincidence degree theory and by means of a suitable Lyapunov functional.

However, the ecological system is often deeply perturbed by human exploitation activities such as planting, harvesting, and so on, which makes it unsuitable to be considered continually. For having a more accurate description of such a system, we need to consider the impulsive differential equations. The theory of impulsive differential equations not only is richer than the corresponding theory of differential equations without impulses, but also represents a more natural framework for mathematical modeling of many real world phenomena [11-13]. Recently, some impulsive equations have been recently introduced in population dynamics in relation to population ecology [14-26] and chemotherapeutic treatment $[27,28]$. However, to the best of the authors' knowledge, to this day, few scholars have done works on the existence, uniqueness, and global stability of positive periodic solution of (2) and (4). One could easily see that systems (5)-(9) are all special cases of systems (2) and (3). Therefore, we propose and study the systems (2) and (3) in this paper.

For the sake of generality and convenience, we always make the following fundamental assumptions.
$\left(H_{1}\right) a_{1}(t), a_{2}(t), b_{1 i}(t), b_{2 j}(t), c_{1 j}(t), c_{2 i}(t), r_{1}(t)$, and $r_{2}(t)$ are all in $P C_{\omega} ; \tau_{i}(t), \delta_{j}(t), \eta_{j}(t)$, and $\sigma_{i}(t)$ are all in $P C_{\omega}^{1}$ with $\tau_{i}(t)>0, \delta_{j}(t)>0, \eta_{j}(t)>0, \sigma_{i}(t)>0, t \in$ $[0, \omega], \tau=\max \left\{\tau_{i}(t), \delta_{j}(t), \eta_{j}(t), \sigma_{i}(t)\right\}, \tau_{i}^{\prime}(t)<1$, $\delta_{j}^{\prime}(t)<1, \eta_{j}^{\prime}(t)<1$, and $\sigma_{i}^{\prime}(t)<1(i=1,2, \ldots, n ; j=$ $1,2, \ldots, m)$.
$\left(H_{2}\right)\left[t_{k}\right]_{k \in N}$ satisfies $0<t_{1}<t_{2}<\cdots<t_{k}<\cdots$, $\lim _{k \rightarrow \infty} t_{k}=+\infty, \theta_{l k}(i=1,2)$ are constants, and there exists a positive integer $q>0$ such that $t_{k+q}=$ $t_{k}+\omega, \theta_{l(k+q)}=\theta_{l k}$. Without loss of generality, we can assume that $t_{k} \neq 0$ and $[0, \omega] \cap\left\{t_{k}\right\}=t_{1}, t_{2}, \ldots, t_{m}$, and then $q=m$.
$\left(H_{3}\right)\left\{\theta_{l k}\right\}$ is a real sequence such that $\theta_{l k}+1>0$, $\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right), l=1,2$ is an $\omega$-periodic function.

Definition 1. A function $x_{l}: R \rightarrow(0,+\infty), l=1,2$ is said to be a positive solution of (2) and (3), if the following conditions are satisfied:
(a) $x_{l}(t)$ is absolutely continuous on each $\left(t_{k}, t_{k+1}\right)$;
(b) for each $k \in Z_{+}, x_{l}\left(t_{k}^{+}\right)$and $x_{l}\left(t_{k}^{-}\right)$exist, and $x_{l}\left(t_{k}^{-}\right)=$ $x_{l}\left(t_{k}\right)$;
(c) $x_{l}(t)$ satisfies the first equation of (2) and (3) for almost everywhere (for short a.e.) in $[0, \infty] \backslash\left\{t_{k}\right\}$ and satisfies $x_{l}\left(t_{k}^{+}\right)=\left(1+\theta_{l k}\right) x_{l}\left(t_{k}\right)$ for $t=t_{k}, k \in Z_{+}=$ $\{1,2, \ldots\}$.

Under the above hypotheses $\left(H_{1}\right)-\left(H_{3}\right)$, we consider the following nonimpulsive delay differential equation:

$$
\begin{align*}
& y_{1}^{\prime}(t)=y_{1}(t)\left[r_{1}(t)-A_{1}(t) y_{1}(t)\right. \\
& +\sum_{i=1}^{n} B_{1 i}(t) y_{1}\left(t-\tau_{i}(t)\right) \\
& \left.-\sum_{j=1}^{m} C_{1 j}(t) y_{2}\left(t-\delta_{j}(t)\right)\right],  \tag{10}\\
& y_{2}^{\prime}(t)=x_{2}(t)\left[r_{2}(t)-A_{2}(t) y_{2}(t)\right. \\
& +\sum_{j=1}^{m} B_{2 j}(t) y_{2}\left(t-\eta_{j}(t)\right) \\
& \left.-\sum_{i=1}^{n} C_{2 i}(t) y_{1}\left(t-\sigma_{i}(t)\right)\right] \text {, } \\
& y_{1}^{\prime}(t)=y_{1}(t)\left[r_{1}(t)-A_{1}(t) y_{1}(t)\right. \\
& -\sum_{i=1}^{n} B_{1 i}(t) y_{1}\left(t-\tau_{i}(t)\right) \\
& \left.-\sum_{j=1}^{m} C_{1 j}(t) y_{2}\left(t-\delta_{j}(t)\right)\right],  \tag{11}\\
& y_{2}^{\prime}(t)=x_{2}(t)\left[r_{2}(t)-A_{2}(t) y_{2}(t)\right. \\
& -\sum_{j=1}^{m} B_{2 j}(t) y_{2}\left(t-\eta_{j}(t)\right) \\
& \left.-\sum_{i=1}^{n} C_{2 i}(t) y_{1}\left(t-\sigma_{i}(t)\right)\right] \text {, }
\end{align*}
$$

with the initial conditions

$$
\begin{gather*}
y_{l}(t)=\varphi_{l}(\xi), \quad \xi \in[-\tau, 0], \\
\tau=\max \left\{\tau_{i}(t), \delta_{j}(t), \eta_{j}(t), \sigma_{i}(t)\right\},  \tag{12}\\
\varphi_{l}(0)>0, \quad \varphi_{l} \in C\left([-\tau, 0], R_{+}\right),
\end{gather*}
$$

where

$$
\begin{gather*}
A_{l}(t)=a_{l}(t) \prod_{0<t_{k}<t}\left(1+\theta_{l k}\right) \\
B_{1 i}(t)=b_{1 i}(t) \prod_{0<t_{k}<t-\tau_{i}(t)}\left(1+\theta_{1 k}\right), \\
B_{2 j}(t)=b_{2 j}(t) \prod_{0<t_{k}<t-\eta_{j}(t)}\left(1+\theta_{2 k}\right),  \tag{13}\\
C_{1 j}(t)=c_{1 j}(t) \prod_{0<t_{k}<t-\rho_{j}(t)}\left(1+\theta_{2 k}\right), \\
C_{2 i}(t)=c_{2 i}(t) \prod_{0<t_{k}<t-\sigma_{i}(t)}\left(1+\theta_{1 k}\right) \\
l=1,2 ; \quad i=1,2, \ldots, n ; \quad j=1,2, \ldots, m .
\end{gather*}
$$

The following lemmas will be used in the proofs of our results. The proof of Lemma 2 is similar to that of Theorem 1 in [25].

Lemma 2. Suppose that $\left(H_{1}\right)-\left(H_{3}\right)$ hold; then
(1) if $y(t)=\left(y_{1}(t), y_{2}(t)\right)^{T}$ is a solution of (10)-(12) on $[-\tau,+\infty)$, then $x_{l}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right) y_{l}(t)(l=1,2)$ is a solution of (2)-(4) on $[-\tau,+\infty)$;
(2) if $x(t)=\left(x_{1}(t), x_{2}(t)\right)^{T}$ is a solution of (2)-(4) on $[-\tau,+\infty)$, then $y_{l}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right)^{-1} x_{l}(t)(l=$ $1,2)$ is a solution of $(10)-(12)$ on $[-\tau,+\infty)$.

Proof. (1) It is easy to see that $x_{l}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right) y_{l}(t)(l=$ $1,2)$ is absolutely continuous on every interval ( $\left.t_{k}, t_{k+1}\right]$, $t \neq t_{k}, k=1,2, \ldots$, and

$$
\begin{aligned}
& x_{1}^{\prime}(t)-x_{1}(t)\left[r_{1}(t)-a_{1}(t) x_{1}(t)\right. \\
& \quad+\sum_{i=1}^{n} b_{1 i}(t) x_{1}\left(t-\tau_{i}(t)\right) \\
& \left.\quad-\sum_{j=1}^{m} c_{1 j}(t) x_{2}\left(t-\delta_{j}(t)\right)\right]
\end{aligned}
$$

$$
\begin{align*}
&= \prod_{0<t_{k}<t}\left(1+\theta_{1 k}\right) y_{1}^{\prime}(t)-\prod_{0<t_{k}<t}\left(1+\theta_{1 k}\right) y_{1}(t) \\
& \times {\left[r_{1}(t)-a_{1}(t) \prod_{0<t_{k}<t}\left(1+\theta_{1 k}\right) y_{1}(t)\right.} \\
& \quad+\sum_{i=1}^{n} b_{1 i}(t) \prod_{0<t_{k}<t-\tau_{i}(t)}\left(1+\theta_{1 k}\right) y_{1}\left(t-\tau_{i}(t)\right) \\
& \quad-\sum_{j=1}^{m} c_{1 j}(t) \prod_{0<t_{k}<t-\rho_{j}(t)}\left(1+\theta_{2 k}\right) \\
&\left.\quad \times y_{2}\left(t-\delta_{j}(t)\right)\right] \\
&=\prod_{0<t_{k}<t}\left(1+\theta_{1 k}\right)\left(y_{1}^{\prime}(t)-y_{1}(t)\right) \\
& \times {\left[r_{1}(t)-A_{1}(t) y_{1}(t)-\sum_{i=1}^{n} B_{1 i}(t) y_{1}\left(t-\tau_{i}(t)\right)\right.} \\
&\left.\quad-\sum_{j=1}^{m} C_{1 j}(t) y_{2}\left(t-\rho_{j}(t)\right)\right]=0 . \tag{14}
\end{align*}
$$

On the other hand, for any $t=t_{k}, k=1,2, \ldots$,

$$
\begin{align*}
x_{1}\left(t_{k}^{+}\right) & =\lim _{t \rightarrow t_{k}^{+}} \prod_{0<t_{j}<t}\left(1+\theta_{1 k}\right) y_{1}(t) \\
& =\prod_{0<t_{j} \leq t_{k}}\left(1+\theta_{1 k}\right) y_{1}\left(t_{k}\right),  \tag{15}\\
x_{1}\left(t_{k}\right) & =\prod_{0<t_{j}<t_{k}}\left(1+\theta_{1 k}\right) y_{1}\left(t_{k}\right) ;
\end{align*}
$$

thus

$$
\begin{equation*}
\Delta x_{1}\left(t_{k}^{+}\right)=\left(1+\theta_{1 k}\right) y_{1}\left(t_{k}\right) \tag{16}
\end{equation*}
$$

which implies that $x_{1}(t)$ is a solution of (2); similarly, we can prove that $x_{2}(t)$ is also a solution of (3). Therefore, $x_{l}(t)$, $l=1,2$ are solutions of $(2)-(4)$ on $[-\tau,+\infty)$. Similarly, if $y(t)=\left(y_{1}(t), y_{2}(t)\right)^{T}$ is a solution of (10)-(12) on $[-\tau,+\infty)$, we can prove that $x_{l}(t)(l=1,2)$ are solutions of (2)-(4) on $[-\tau,+\infty)$.
(2) Since $x_{l}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right) y_{l}(t)(l=1,2)$ is absolutely continuous on every interval $\left(t_{k}, t_{k+1}\right], t \neq t_{k}, k=$ $1,2, \ldots$, and in view of (15), it follows that for any $k=1,2, \ldots$,

$$
\begin{aligned}
y_{1}\left(t_{k}^{+}\right) & =\prod_{0<t_{j} \leq t_{k}}\left(1+\theta_{1 k}\right)^{-1} x_{1}\left(t_{k}^{+}\right) \\
& =\prod_{0<t_{j}<t_{k}}\left(1+\theta_{1 k}\right)^{-1} x_{1}\left(t_{k}\right)=y_{1}\left(t_{k}\right),
\end{aligned}
$$

$$
\begin{align*}
y_{1}\left(t_{k}^{-}\right) & =\prod_{0<t_{j}<t_{k}}\left(1+\theta_{1 k}\right)^{-1} x_{1}\left(t_{k}^{-}\right) \\
& =\prod_{0<t_{j} \leq t_{k}^{-}}\left(1+\theta_{1 k}\right)^{-1} x_{1}\left(t_{k}^{-}\right)=y_{1}\left(t_{k}\right), \tag{17}
\end{align*}
$$

which implies that $y_{1}(t)$ is continuous on $[-\tau,+\infty)$. It is easy to prove that $y_{1}(t)$ is absolutely continuous on $[-\tau,+\infty)$. Similarly, we can prove that $y_{2}(t)$ is absolutely continuous on $[-\tau,+\infty)$. Similar to the proof of (1), we can check that $y_{l}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right)^{-1} x_{l}(t)(l=1,2)$ are solutions of $(10)-$ (12) on $[-\tau,+\infty)$. If $x(t)=\left(x_{1}(t), x_{2}(t)\right)^{T}$ is a solution of (2) $-(4)$ on $[-\tau,+\infty)$ by the same method, we can prove that $y_{l}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right)^{-1} x_{l}(t)(l=1,2)$ are solutions of $(10)-$ (12) on $[-\tau,+\infty)$. The proof of Lemma 2 is completed.

From Lemma 2, if we want to discuss the existence and global asymptotic stability of positive periodic solutions of systems (2)-(4), we only discuss the existence of the existence and global asymptotic stability of positive periodic solutions of systems (10)-(12).

The organization of this paper is as follows. In Section 2, we introduce several useful definitions and lemmas. In Section 3, first, we study the existence of at least one periodic solution of systems (2)-(4) by using continuation theorem proposed by Gaines and Mawhin (see [9]). Second, we investigate the global asymptotic stability of positive periodic solutions of the above systems by using the method of Lyapunov functional. As applications in Section 4, we study some particular cases of systems (2)-(4) which have been investigated extensively in the references mentioned previously.

## 2. Preliminaries

In this section, we will introduce some concepts and some important lemmas which are useful for the next section.

Let $X, Z$ be two real Banach spaces, let $L: \operatorname{Dom} L \subset X \rightarrow$ $Z$ be a linear mapping, and let $N: X \rightarrow Z$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\operatorname{dim} \operatorname{Ker} L=$ condimIm $L<+\infty$ and $\operatorname{Im} L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero and there exist continuous projectors $P: X \rightarrow X$ and $Q: Z \rightarrow$ $Z$ such that $\operatorname{Im} P=\operatorname{Ker} L, \operatorname{Ker} Q=\operatorname{Im} L=\operatorname{Im}(I-Q)$, it follows that $\left.L\right|_{\text {Dom } L \cap \operatorname{Ker} P}:(I-P) X \rightarrow \operatorname{Im} L$ is invertible; we denote the inverse of that map by $K_{p}$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\bar{\Omega}$ if $Q N(\bar{\Omega})$ is bounded and $K_{p}(I-Q) N: \bar{\Omega} \rightarrow X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exist isomorphisms $J: \operatorname{Im} Q \rightarrow \operatorname{Ker} L$. Let $P C_{\omega}$ denote the space of $\omega$-periodic functions $\Psi: J \rightarrow R$ which are continuous for $t \neq t_{k}$, are continuous from the left for $t \in R$, and have discontinuities of the first kind at point $t=t_{k}$. We also denote that $P C_{\omega}^{1}=$ $\left\{\Psi \in P C_{\omega}: \Psi^{\prime} \in P C_{\omega}\right\}$.

Definition 3 (see [11]). The set $F \in P C_{\omega}$ is said to be quasiequicontinuous in $[0, \omega$ ] if for any $\epsilon>0$ there exists
$\delta>0$ such that if $x \in F, k \in N^{+}, t_{1}, t_{2} \in\left(t_{k-1}, t_{k}\right) \cap[0, \omega]$, and $\left|t_{1}-t_{2}\right|<\delta$, then $\left|x\left(t_{1}\right)-x\left(t_{2}\right)\right|<\epsilon$.

Definition 4. Let $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ be a strictly positive periodic solution of (2)-(4). One says that $x^{*}(t)$ is globally attractive if any other solution $x(t)=\left(x_{1}(t), x_{2}(t)\right)^{T}$ of (2)(4) has the property $\lim _{t \rightarrow+\infty}\left|x_{i}^{*}(t)-x_{i}(t)\right|=0, i=1,2$.

Lemma 5. The region $R_{+}^{2}=\left\{\left(x_{1}, x_{2}\right): x_{1}(0)>0, x_{2}(0)>0\right\}$ is the positive invariable region of the systems (2)-(4).

Proof. By the definition of $x_{l}(t)(l=1,2)$ we have $x_{l}(0)>0$. In view of having

$$
\begin{aligned}
& x_{1}(t)=x_{1}(0) \exp \left\{\int _ { 0 } ^ { t } \left[r_{1}(\xi)-a_{1}(\xi) x_{1}(\xi)\right.\right. \\
& +\sum_{i=1}^{n} b_{1 i}(\xi) x_{1}\left(\xi-\tau_{i}(\xi)\right) \\
& \left.\left.-\sum_{j=1}^{m} c_{1 j}(\xi) x_{2}\left(\xi-\delta_{j}(\xi)\right)\right] d \xi\right\} \text {, } \\
& t \in\left[0, t_{1}\right], \\
& x_{1}(t)=x_{1}\left(t_{k}\right) \exp \left\{\int _ { 0 } ^ { t } \left[r_{1}(\xi)-a_{1}(\xi) x_{1}(\xi)\right.\right. \\
& +\sum_{i=1}^{n} b_{1 i}(\xi) x_{1}\left(\xi-\tau_{i}(\xi)\right) \\
& \left.\left.-\sum_{j=1}^{m} c_{1 j}(\xi) x_{2}\left(\xi-\delta_{j}(\xi)\right)\right] d \xi\right\} \text {, } \\
& t \in\left(t_{k}, t_{k+1}\right], \\
& x_{1}\left(t_{k}^{+}\right)=\left(1+\theta_{1 k}\right) x_{1}\left(t_{k}\right)>0, \quad k \in N ; \\
& x_{2}(t)=x_{2}(0) \exp \left\{\int _ { 0 } ^ { t } \left[r_{2}(\xi)-a_{2}(\xi) x_{2}(\xi)\right.\right. \\
& +\sum_{j=1}^{m} b_{2 j}(\xi) x_{2}\left(\xi-\eta_{j}(\xi)\right) \\
& \left.\left.-\sum_{i=1}^{n} c_{2 i}(\xi) x_{1}\left(\xi-\sigma_{i}(\xi)\right)\right] d \xi\right\}, \\
& t \in\left[0, t_{1}\right] \text {, } \\
& x_{2}(t)=x_{2}\left(t_{k}\right) \exp \left\{\int _ { 0 } ^ { t } \left[r_{2}(\xi)-a_{2}(\xi) x_{2}(\xi)\right.\right. \\
& +\sum_{j=1}^{m} b_{2 j}(\xi) x_{2}\left(\xi-\eta_{j}(\xi)\right) \\
& \left.\left.-\sum_{i=1}^{n} c_{2 i}(\xi) x_{1}\left(\xi-\sigma_{i}(\xi)\right)\right] d \xi\right\}, \\
& t \in\left(t_{k}, t_{k+1}\right],
\end{aligned}
$$

$$
\begin{aligned}
& x_{2}\left(t_{k}^{+}\right)=\left(1+\theta_{2 k}\right) x_{2}\left(t_{k}\right)>0, \quad k \in N, \\
& x_{1}(t)=x_{1}(0) \exp \left\{\int _ { 0 } ^ { t } \left[r_{1}(\xi)-a_{1}(\xi) x_{1}(\xi)\right.\right. \\
& \quad-\sum_{i=1}^{n} b_{1 i}(\xi) x_{1}\left(\xi-\tau_{i}(\xi)\right) \\
& \left.\left.\quad-\sum_{j=1}^{m} c_{1 j}(\xi) x_{2}\left(\xi-\delta_{j}(\xi)\right)\right] d \xi\right\}, \\
& t \in\left[0, t_{1}\right],
\end{aligned}
$$

$$
x_{1}(t)=x_{1}\left(t_{k}\right) \exp \left\{\int _ { 0 } ^ { t } \left[r_{1}(\xi)-a_{1}(\xi) x_{1}(\xi)\right.\right.
$$

$$
-\sum_{i=1}^{n} b_{1 i}(\xi) x_{1}\left(\xi-\tau_{i}(\xi)\right)
$$

$$
\left.\left.-\sum_{j=1}^{m} c_{1 j}(\xi) x_{2}\left(\xi-\delta_{j}(\xi)\right)\right] d \xi\right\}
$$

$$
t \in\left(t_{k}, t_{k+1}\right]
$$

$$
x_{1}\left(t_{k}^{+}\right)=\left(1+\theta_{1 k}\right) x_{1}\left(t_{k}\right)>0, \quad k \in N ;
$$

$$
x_{2}(t)=x_{2}(0) \exp \left\{\int _ { 0 } ^ { t } \left[r_{2}(\xi)-a_{2}(\xi) x_{2}(\xi)\right.\right.
$$

$$
-\sum_{j=1}^{m} b_{2 j}(\xi) x_{2}\left(\xi-\eta_{j}(\xi)\right)
$$

$$
\left.\left.-\sum_{i=1}^{n} c_{2 i}(\xi) x_{1}\left(\xi-\sigma_{i}(\xi)\right)\right] d \xi\right\}
$$

$$
t \in\left[0, t_{1}\right],
$$

$$
x_{2}(t)=x_{2}\left(t_{k}\right) \exp \left\{\int _ { 0 } ^ { t } \left[r_{2}(\xi)-a_{2}(\xi) x_{2}(\xi)\right.\right.
$$

$$
-\sum_{j=1}^{m} b_{2 j}(\xi) x_{2}\left(\xi-\eta_{j}(\xi)\right)
$$

$$
\left.\left.-\sum_{i=1}^{n} c_{2 i}(\xi) x_{1}\left(\xi-\sigma_{i}(\xi)\right)\right] d \xi\right\},
$$

$$
t \in\left(t_{k}, t_{k+1}\right]
$$

$$
\begin{equation*}
x_{2}\left(t_{k}^{+}\right)=\left(1+\theta_{2 k}\right) x_{2}\left(t_{k}\right)>0, \quad k \in N . \tag{19}
\end{equation*}
$$

Then the solution of (2)-(4) is positive. The proof of Lemma 5 is completed.

Lemma 6 (see [19, 29]). Suppose that $\sigma \in C_{\omega}^{1}$ and $\sigma^{\prime}(t)<1$, $t \in[0, \omega]$. Then the function $t-\sigma(t)$ has a unique inverse $\mu(t)$ satisfying $\mu \in C(R, R)$ with $\mu(a+\omega)=\mu(a)+\omega \forall a \in R$, and if $g \in P C_{\omega}, \tau^{\prime}(t)<1, t \in[0, \omega]$, then $g(\mu(t)) \in P C_{\omega}$.

Proof. Since $\sigma^{\prime}(t)<1, t \in[0, \omega]$, and $t-\sigma(t)$ is continuous on $R$, it follows that $t-\sigma(t)$ has a unique inverse function $\mu(t) \in C(R, R)$ on $R$. Hence, it suffices to show that $\mu(a+$ $\omega)=\mu(a)+\omega, \forall a \in R$. For any $a \in R$, by the condition $\sigma^{\prime}(t)<1$, one can find that $t-\sigma(t)=a$ exists as a unique solution $t_{0}$ and $t-\sigma(t)=a+\omega$ exists as a unique solution $t_{1}$; that is, $t_{0}-\sigma\left(t_{0}\right)=a$ and $t_{1}-\sigma\left(t_{1}\right)=a+\omega$; that is, $\mu(a)=t_{0}=\sigma\left(t_{0}\right)+a$ and $\mu(a+\omega)=t_{1}$.

As

$$
\begin{align*}
a+\omega & +\sigma\left(t_{0}\right)-\sigma\left(a+\omega+\sigma\left(t_{0}\right)\right) \\
& =a+\omega+\sigma\left(t_{0}\right)-\sigma\left(a+\sigma\left(t_{0}\right)\right)  \tag{20}\\
& =a+\omega+\sigma\left(t_{0}\right)-\sigma\left(t_{0}\right)=a+\omega
\end{align*}
$$

it follows that $t_{1}=a+\omega+\sigma\left(t_{0}\right)$. Since $\mu(a+\omega)=t_{1}$, we have $\mu(a+\omega)=t_{1}=a+\omega+\sigma\left(t_{0}\right)$ and $\mu(a+\omega)=t_{1}=\mu(a)+\omega$. We can easily obtain that if $g \in P C_{\omega}, \tau^{\prime}(t)<1, t \in[0, \omega]$, then $g(\mu(t+\omega))=g(\mu(t)+\omega)=g(\mu(t)), t \in R$, where $\mu(t)$ is the unique inverse function of $t-\tau(t)$, which together with $\mu \in$ $C(R, R)$ implies that $g(\mu(t)) \in P C_{\omega}$. The proof of Lemma 6 is completed.

Lemma 7 (see [9]). Let $X$ and $Z$ be two Banach spaces, and let $L: \operatorname{Dom} L \subset X \rightarrow Z$ be a Fredholm operator with index zero. $\Omega \subset X$ is an open bounded set, and let $N: \bar{\Omega} \rightarrow Z$ be L-compact on $\bar{\Omega}$. Suppose that
(a) $L x \neq \lambda N x$ for each $\lambda \in(0,1)$ and $x \in \partial \Omega \cap \operatorname{Dom} L$;
(b) $Q N x \neq 0$ for each $x \in \partial \Omega \cap \operatorname{ker} L$;
(c) $\operatorname{deg}\{J Q N, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then, equation $L x=N x$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

Lemma 8 (see [11]). The set $F \subset P C_{\omega}$ is relatively compact if only if
(1) $F$ is bounded, that is, $\|x\| \leq M$, for each $x$, and some $M>0$;
(2) $F$ is quasiequicontinuous in $[0, \omega]$.

Lemma 9 (see [30]). Assume that $f(t), g(t)$ are continuous nonnegative functions defined on the interval $[\alpha, \beta]$; then there exists $\xi \in[\alpha, \beta]$ such that $\int_{\alpha}^{\beta} f(t) g(t) d t=f(\xi) \int_{\alpha}^{\beta} g(t) d t$.

Lemma 10 (see $[20,31]$ ). Suppose that $\phi(t)$ is a differently continuous $\omega$-periodic function on $R$ with ( $\omega>0$ ); then, for any $t^{*} \in R$, the following inequality holds:

$$
\begin{equation*}
\max _{t \in\left[t^{*}, t^{*}+\omega\right]} \Phi(t) \leq\left|\Phi\left(t^{*}\right)\right|+\frac{1}{2}\left[\int_{0}^{\omega}\left|\Phi^{\prime}(t)\right| d t\right] . \tag{21}
\end{equation*}
$$

Lemma 11 (see Barbalat's Lemma [32]). Let $f(t)$ be a nonnegative function defined on $[0,+\infty)$ such that $f(t)$ is integrable and uniformly continuous on $[0,+\infty)$; then $\lim _{t \rightarrow+\infty} f(t)=0$.

In the following section, we only discuss the existence and global asymptotic stability of positive periodic solutions of systems (10)-(12).

## 3. Existence and Global Asymptotic Stability

Since $\tau_{i}^{\prime}(t)<1, \delta_{j}^{\prime}(t)<1, \eta_{j}^{\prime}(t)<1, \sigma_{i}^{\prime}(t)<1, t \in[0, \omega]$, by Lemma 6, we see that all $t-\tau_{i}(t)$ have their inverse functions. Throughout the following part, we set $\alpha_{i}(t), \beta_{i}(t), \mu_{j}(t)$, and $v_{j}(t)$ to represent the inverse function of $t-\tau_{i}(t), t-\sigma_{i}(t), t-$ $\delta_{j}(t)$, and $t-\eta_{j}(t)$, respectively. Obviously, $\alpha_{i}(t), \beta_{i}(t), \mu_{j}(t)$, $v_{j}(t) \in P C_{\omega}^{1}$. We also denote that

$$
\begin{align*}
& F_{1}(t)=A_{1}(t)-\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(t)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)}, \\
& F_{2}(t)=A_{2}(t)-\sum_{j=1}^{m} \frac{B_{2 j}\left(v_{j}(t)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(t)\right)}, \\
& F_{1}^{*}(t)=A_{1}(t)+\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(t)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)}, \\
& F_{2}^{*}(t)=A_{2}(t)+\sum_{j=1}^{m} \frac{B_{2 j}\left(v_{j}(t)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(t)\right)},  \tag{22}\\
& G_{1}(t)=\sum_{j=1}^{m} \frac{C_{1 j}\left(\mu_{j}(t)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(t)\right)}, \\
& G_{2}(t)=\sum_{i=1}^{n} \frac{C_{2 i}\left(\beta_{i}(t)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(t)\right)} .
\end{align*}
$$

Theorem 12. In addition to $\left(H_{1}\right)-\left(H_{3}\right)$, assume that one of the following conditions hold:

$$
\left(H_{4}\right) \overline{r_{1}} F_{1}^{* M}<\overline{r_{2}} G_{2}^{L}, \overline{r_{2}} F_{2}^{* M}<\overline{r_{1}} G_{1}^{L} ;
$$

$$
\left(H_{5}\right) \overline{r_{1}} F_{1}^{* L}>\overline{r_{2}} G_{2}^{M}, \overline{r_{2}} F_{2}^{* L}>\overline{r_{1}} G_{1}^{M}
$$

Then systems (3) and (4) have at least one positive $\omega$ periodic solution, where $F_{1}^{*}(t), F_{2}^{*}(t), G_{1}(t)$, and $G_{2}(t)$ are defined in (22).
Proof. Since the solutions of systems (11) and (12) remain positive for $t \geq 0$, we carry out the change of variable $u_{i}(t)=$ $\ln y_{i}(t)(i=1,2)$; then (11) can be transformed to

$$
\begin{align*}
u_{1}^{\prime}(t)= & r_{1}(t)-A_{1}(t) e^{u_{1}(t)}-\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)} \\
& -\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)},  \tag{23}\\
u_{2}^{\prime}(t)= & r_{2}(t)-A_{2}(t) e^{u_{2}(t)}-\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)} \\
& \quad-\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}
\end{align*}
$$

It is easy to see that if system (23) has one $\omega$-periodic solution $\left(u_{1}^{*}(t), u_{2}^{*}(t)\right)^{T}$, then $\left(y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}=\left(e^{u_{1}^{*}(t)}, e^{u_{2}^{*}(t)}\right)^{T}$ is a positive $\omega$-periodic solution of system (10); that is to say, $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}=\left(\prod_{0<t_{k}<t}\left(1+\theta_{1 k}\right) e^{u_{1}^{*}(t)}, \prod_{0<t_{k}<t}(1+\right.$ $\left.\left.\theta_{2 k}\right) e^{u_{2}^{*}(t)}\right)^{T}$ is a positive $\omega$-periodic solution of system (2). Therefore, it suffices to prove that system (23) has a $\omega$ periodic solution. In order to use Lemma 7 to (23), we take

$$
\begin{align*}
& X=Z=\left\{u(t)=\left(u_{1}(t), u_{2}(t)\right)^{T} \mid u_{i}(t)\right. \\
&\left.\in C\left(R, R^{2}\right): u_{i}(t+\omega)=u_{i}(t), l=1,2\right\} \tag{24}
\end{align*}
$$

and define

$$
\begin{align*}
\|u\| & =\sum_{i=1}^{2} \sup _{t \in[0, \omega]}\left|u_{l}(t)\right|  \tag{25}\\
u(t) & =\left(u_{1}(t), u_{2}(t)\right)^{T} \in X(\text { or } Z) .
\end{align*}
$$

Then $X$ and $Z$ are Banach spaces when they are endowed with the norm $\|\cdot\|$. Let $N: X \rightarrow Z$ with

$$
N u=\left[\begin{array}{c}
r_{1}(t)-A_{1}(t) e^{u_{1}(t)}-\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}  \tag{26}\\
r_{2}(t)-A_{2}(t) e^{u_{2}(t)}-\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}-\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}
\end{array}\right]=\left[\begin{array}{l}
f_{1}(t) \\
f_{2}(t)
\end{array}\right] \quad \text { for any } u \in X
$$

and define

$$
\begin{gather*}
L u=u^{\prime} ; \quad P u=\frac{1}{\omega} \int_{0}^{\omega} u(t) d t, \quad u \in X  \tag{27}\\
\mathrm{Q} z=\frac{1}{\omega} \int_{0}^{\omega} z(t) d t, \quad z \in Z
\end{gather*}
$$

It is not difficult to show that

$$
\begin{gather*}
\operatorname{Ker} L=\left\{u \in X \mid u=h \in R^{2}\right\} \\
\operatorname{Im} L=\left\{z \in Z \mid \int_{0}^{\omega} z(s) d s=0\right\} \tag{28}
\end{gather*}
$$

and $\operatorname{dimKer} L=2=$ codimIm $L$. So, $\operatorname{Im} L$ is closed in $Z$, and $L$ is a Fredholm mapping of index zero. It is trivial to show that $P, Q$ are continuous projectors such that

$$
\begin{equation*}
\operatorname{Im} P=\operatorname{Ker} L, \quad \operatorname{Ker} Q=\operatorname{Im} L=\operatorname{Im}(I-Q) . \tag{29}
\end{equation*}
$$

Furthermore, the generalized inverse (to $L$ ) $K_{P}: \operatorname{Im} L \rightarrow$ $\operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$
\begin{equation*}
K_{P} z=\int_{0}^{t} z(s) d s-\frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} z(s) d s d t . \tag{30}
\end{equation*}
$$

Thus, for $u \in X$

$$
\begin{gather*}
Q N u=\binom{\frac{1}{\omega} \int_{0}^{\omega}\left[r_{1}(t)-A_{1}(t) e^{u_{1}(t)}-\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right] d t}{\frac{1}{\omega} \int_{0}^{\omega}\left[r_{2}(t)-A_{2}(t) e^{u_{2}(t)}-\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}-\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}\right] d t},  \tag{31}\\
K_{P}(I-Q) N u=\binom{\int_{0}^{t} f_{1}(\xi) d \xi}{\int_{0}^{t} f_{2}(\xi) d \xi}-\binom{\frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} f_{1}(\xi) d \xi d t}{\frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} f_{2}(\xi) d \xi d t}-\binom{\left(\frac{t}{\omega}-\frac{1}{2}\right) \int_{0}^{\omega} f_{1}(\xi) d \xi}{\left(\frac{t}{\omega}-\frac{1}{2}\right) \int_{0}^{\omega} f_{2}(\xi) d \xi} . \tag{32}
\end{gather*}
$$

Clearly, $Q N$ and $K_{P}(I-Q) N$ are continuous. By applying Ascoli-Arzela theorem, one can easily show that $\mathrm{QN}(\bar{\Omega})$, $K_{P}(I-Q) N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Moreover, $\mathrm{QN}(\bar{\Omega})$ is obviously bounded. Thus, $N$ is $L$-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$. Now, we reach the position to search for an appropriate open bounded set $\Omega \subset X$ for the application of Lemma 7. Corresponding to the operating equation $L u=\lambda N u, \lambda \in(0,1)$, we have

$$
\begin{gather*}
u_{1}^{\prime}(t)=\lambda\left[r_{1}(t)-A_{1}(t) e^{u_{1}(t)}-\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}\right. \\
\left.-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right],  \tag{33}\\
u_{2}^{\prime}(t)=\lambda\left[r_{2}(t)-A_{2}(t) e^{u_{2}(t)}-\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}\right. \\
\left.-\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}\right] .
\end{gather*}
$$

Since $u(t)=\left(u_{1}(t), u_{2}(t)\right)^{T}$ is a $\omega$-periodic function, we need only to prove the result in the interval [ $0, \omega$ ]. Integrating (33) over the interval $[0, \omega]$ leads to the following:

$$
\begin{gathered}
\int_{0}^{\omega}\left[r_{1}(t)-A_{1}(t) e^{u_{1}(t)}-\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}\right. \\
\left.-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right] d t=0
\end{gathered}
$$

$$
\begin{gather*}
\int_{0}^{\omega}\left[r_{2}(t)-A_{2}(t) e^{u_{2}(t)}-\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}\right. \\
\left.-\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}\right] d t=0 . \tag{34}
\end{gather*}
$$

Hence, we have

$$
\begin{align*}
& \int_{0}^{\omega}\left[\begin{array}{l}
\text { A }
\end{array} \quad \begin{array}{l}
\text { t }) e^{u_{1}(t)}+\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)} \\
\left.\quad+\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right] d t=\overline{r_{1}} \omega, \\
\int_{0}^{\omega}\left[A_{2}(t) e^{u_{2}(t)}+\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}\right. \\
\left.\quad+\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}\right] d t=\overline{r_{2}} \omega .
\end{array} .\right.
\end{align*}
$$

Note that $u(t)=\left(u_{1}(t), u_{2}(t)\right) \in X$, and then there exists $\zeta_{l}, \xi_{l} \in[0, \omega](l=1,2)$ such that

$$
\begin{equation*}
u_{l}\left(\zeta_{l}\right)=\inf _{t \in[0, \omega]} u_{l}(t), \quad u_{l}\left(\xi_{l}\right)=\sup _{t \in[0, \omega]} u_{l}(t), \quad l=1,2 . \tag{36}
\end{equation*}
$$

Since $\tau_{i}^{\prime}(t)<1$, we can let $s=t-\tau_{i}(t)$, that is, $t=\alpha_{i}(s)(i=$ $1,2, \ldots, n)$; then

$$
\begin{equation*}
\int_{0}^{\omega} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)} d t=\int_{-\tau_{i}(0)}^{\omega-\tau_{i}(\omega)} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)} d s \tag{37}
\end{equation*}
$$

According to Lemma 7, we know that $\left(\left(B_{1 i}\left(\alpha_{i}(s)\right)\right) /(1-\right.$ $\left.\left.\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)\right)\right) e^{u_{1}(s)} \in C_{\omega}$. Thus,

$$
\begin{equation*}
\int_{-\tau_{i}(0)}^{\omega-\tau_{i}(\omega)} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)} d s=\int_{0}^{\omega} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)} d s \tag{38}
\end{equation*}
$$

By (37) and (38), we have

$$
\begin{equation*}
\int_{0}^{\omega} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)} d t=\int_{0}^{\omega} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)} d s \tag{39}
\end{equation*}
$$

Similarly, we obtain

$$
\begin{align*}
& \int_{0}^{\omega} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)} d t=\int_{0}^{\omega} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} e^{u_{2}(s)} d s \\
& \int_{0}^{\omega} B_{2 j}(t) e^{u_{1} 2\left(t-\eta_{j}(t)\right)} d t=\int_{0}^{\omega} \frac{B_{2 j}\left(v_{j}(s)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(s)\right)} e^{u_{2}(s)} d s  \tag{40}\\
& \int_{0}^{\omega} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)} d t=\int_{0}^{\omega} \frac{C_{2 i}\left(\beta_{i}(s)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(s)\right)} e^{u_{1}(s)} d s
\end{align*}
$$

It follows from (35), (39), and (40) that we get

$$
\begin{gather*}
\int_{0}^{\omega}\left[\left(A_{1}(s)+\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)}\right) e^{u_{1}(s)}\right. \\
\left.+\sum_{j=1}^{m} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} e^{u_{2}(s)}\right] d s=\overline{r_{1}} \omega,  \tag{41}\\
\int_{0}^{\omega}\left[\left(A_{2}(s)+\sum_{j=1}^{m} \frac{B_{2 j}\left(v_{j}(s)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(s)\right)}\right) e^{u_{2}(s)}\right. \\
\left.\quad+\sum_{i=1}^{n} \frac{C_{2 i}\left(\beta_{i}(s)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(s)\right)} e^{u_{1}(s)}\right] d s=\overline{r_{2}} \omega
\end{gather*}
$$

Thus, from (41) we get

$$
\begin{align*}
& \int_{0}^{\omega} F_{1}^{*}(s) e^{u_{1}(s)} d s+\int_{0}^{\omega} G_{1}(s) e^{u_{2}(s)} d s=\overline{r_{1}} \omega  \tag{42}\\
& \int_{0}^{\omega} F_{2}^{*}(s) e^{u_{2}(s)} d s+\int_{0}^{\omega} G_{2}(s) e^{u_{1}(s)} d s=\overline{r_{2}} \omega
\end{align*}
$$

where $F_{1}^{*}(s), F_{2}^{*}(s), G_{1}(s)$, and $G_{2}(s)$ are defined by (22). On the other hand, by Lemma 7 , we can see that $\alpha_{i}(\omega)=\alpha_{i}(0)+\omega$, so we can derive

$$
\begin{align*}
\int_{0}^{\omega} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} d s & =\int_{\alpha_{i}(0)}^{\alpha_{i}(\omega)} \frac{B_{1 i}(t)\left(1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)} d t  \tag{43}\\
& =\int_{0}^{\omega} B_{1 i}(t) d t=\overline{B_{1 i}} \omega
\end{align*}
$$

Thus, from (43) we get

$$
\begin{align*}
& \overline{F_{1}^{*}} \omega=\int_{0}^{\omega} F_{1}^{*}(s) d s=\int_{0}^{\omega}\left[A_{1}(s)+\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)}\right] d s \\
&=\left(\overline{A_{1}}-\sum_{i=1}^{n} \overline{B_{1 i}}\right) \omega, \\
& \overline{F_{2}^{*}} \omega=\int_{0}^{\omega} F_{2}^{*}(s) d s=\int_{0}^{\omega}\left[A_{2}(s)+\sum_{i=1}^{n} \frac{B_{2 j}\left(v_{j}(s)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(s)\right)}\right] d s \\
&=\left(\overline{A_{2}}+\sum_{i=1}^{n} \overline{B_{2 j}}\right) \omega, \\
& \overline{G_{1}} \omega=\int_{0}^{\omega} G_{1}(s) d s \\
&=\int_{0}^{\omega} \sum_{j=1}^{m} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} d s=\sum_{j=1}^{m} \overline{C_{1 j}} \omega, \\
& \overline{G_{2}} \omega=\int_{0}^{\omega} G_{2}(s) d s \\
&=\int_{0}^{\omega} \sum_{i=1}^{n} \frac{C_{2 i}\left(\beta_{i}(s)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(s)\right)} d s=\sum_{i=1}^{n} \overline{C_{2 i}} \omega .  \tag{44}\\
& \text { On one hand, by }(42), \text { we have } \\
& G_{1}^{L} \int_{0}^{\omega} e^{u_{2}(s)} d s \leq \int_{0}^{\omega} e^{u_{1}(s)} d s \leq \int_{0}^{\omega} G_{2}(s) e^{u_{1}(s)} d s \leq \overline{r_{2}} \omega,  \tag{45}\\
& G_{1}(s) e^{u_{2}(s)} d s \leq \overline{r_{1}} \omega, \\
& \hline
\end{align*}
$$

which implies that

$$
\begin{align*}
& \int_{0}^{\omega} e^{u_{2}(s)} d s \leq \frac{\overline{r_{1}} \omega}{G_{1}^{L}}, \\
& \int_{0}^{\omega} e^{u_{1}(s)} d s \leq \frac{\overline{r_{2}} \omega}{G_{2}^{L}} . \tag{46}
\end{align*}
$$

On the other hand, by (42), the integral mean value theorem that there are $\lambda_{1}, \lambda_{2}, \rho_{1}$, and $\rho_{2} \in[0, \omega]$ such that

$$
\begin{align*}
& F_{1}^{*}\left(\lambda_{1}\right) \int_{0}^{\omega} e^{u_{1}(s)} d s+G_{1}\left(\rho_{1}\right) \int_{0}^{\omega} e^{u_{2}(s)} d s=\overline{r_{1}} \omega  \tag{47}\\
& F_{2}^{*}\left(\lambda_{2}\right) \int_{0}^{\omega} e^{u_{2}(s)} d s+G_{2}\left(\rho_{2}\right) \int_{0}^{\omega} e^{u_{1}(s)} d s=\overline{r_{2}} \omega
\end{align*}
$$

By $\left(H_{4}\right)$, we have $G_{1}^{L} G_{2}^{L}>F_{1}^{* M} F_{2}^{* M}$, which, together with (47), lead to

$$
\begin{align*}
\int_{0}^{\omega} e^{u_{1}(s)} d s & =\frac{\overline{r_{2}} \omega G_{1}\left(\rho_{1}\right)-\overline{r_{1}} \omega F_{2}^{*}\left(\lambda_{2}\right)}{G_{1}\left(\rho_{1}\right) G_{2}\left(\rho_{2}\right)-F_{1}^{*}\left(\lambda_{1}\right) F_{2}^{*}\left(\lambda_{2}\right)} \\
& \geq \frac{\overline{r_{2}} \omega G_{1}^{L}-\overline{r_{1}} \omega F_{2}^{* M}}{G_{1}^{M} G_{2}^{M}-F_{1}^{* L} F_{2}^{* L}}:=\Gamma_{1} \omega, \\
\int_{0}^{\omega} e^{u_{2}(s)} d s & =\frac{\overline{r_{1}} \omega G_{2}\left(\rho_{2}\right)-\overline{r_{2}} \omega F_{1}^{*}\left(\lambda_{1}\right)}{G_{1}\left(\rho_{1}\right) G_{2}\left(\rho_{2}\right)-F_{1}^{*}\left(\lambda_{1}\right) F_{2}^{*}\left(\lambda_{2}\right)}  \tag{48}\\
& \geq \frac{\overline{r_{1}} \omega G_{2}^{L}-\overline{r_{2}} \omega F_{1}^{* M}}{G_{1}^{M} G_{2}^{M}-F_{1}^{* L} F_{2}^{* L}}:=\Gamma_{2} \omega .
\end{align*}
$$

Again, by $\left(H_{4}\right)$, one can deduce that the following inequalities:

$$
\begin{aligned}
& \frac{\overline{r_{2}} \omega}{G_{2}^{L}} \geq \frac{\overline{r_{2}} \omega G_{1}^{L}-\overline{r_{1}} \omega F_{2}^{* M}}{G_{1}^{M} G_{2}^{M}-F_{1}^{* L} F_{2}^{* L}}:=\Gamma_{1} \omega>0, \\
& \frac{\overline{r_{1}} \omega}{G_{1}^{L}} \geq \frac{\overline{r_{1}} \omega G_{2}^{L}-\overline{r_{2}} \omega F_{1}^{* M}}{G_{1}^{M} G_{2}^{M}-F_{1}^{* L} F_{2}^{* L}}:=\Gamma_{2} \omega>0 .
\end{aligned}
$$

It follows from (46), (48), and (49) that

$$
\begin{align*}
& \Gamma_{2} \omega \leq \int_{0}^{\omega} e^{u_{2}(s)} d s \leq \frac{\overline{r_{1}} \omega}{G_{1}^{L}}, \\
& \Gamma_{1} \omega \leq \int_{0}^{\omega} e^{u_{1}(s)} d s \leq \frac{\overline{r_{2}} \omega}{G_{2}^{L}}, \tag{50}
\end{align*}
$$

which together with (36) yield

$$
\begin{array}{ll}
\Gamma_{2} \leq e^{u_{2}\left(\zeta_{2}\right)}, & e^{u_{2}\left(\xi_{2}\right)} \leq \frac{\overline{r_{1}}}{G_{1}^{L}}, \\
\Gamma_{1} \leq e^{u_{1}\left(\zeta_{1}\right)}, & e^{u_{1}\left(\xi_{1}\right)} \leq \frac{\overline{r_{2}}}{G_{2}^{L}},
\end{array}
$$

which implies that

$$
\begin{array}{ll}
\ln \Gamma_{2} \leq u_{2}\left(\xi_{2}\right), & u_{2}\left(\zeta_{2}\right) \leq \ln \frac{\overline{r_{1}}}{G_{1}^{L}} \\
\ln \Gamma_{1} \leq u_{1}\left(\xi_{1}\right), & u_{1}\left(\zeta_{1}\right) \leq \ln \frac{\overline{r_{2}}}{G_{2}^{L}}
\end{array}
$$

From the first equation of (32), we get

$$
\begin{aligned}
& \int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| d t \\
& =\lambda \int_{0}^{\omega} \mid r_{1}(t)-A_{1}(t) e^{u_{1}(t)}-\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)} \\
& \quad-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)} \mid d t \\
& \begin{aligned}
\leq & \int_{0}^{\omega}\left|r_{1}(t)\right| d t
\end{aligned} \\
& \quad+\int_{0}^{\omega}\left[A_{1}(t) e^{u_{1}(t)}+\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}\right. \\
& \left.\quad+\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right] d t \\
& = \\
& \quad \int_{0}^{\omega}\left|r_{1}(t)\right| d t \\
& \quad+\int_{0}^{\omega}\left[A_{1}(s) e^{u_{1}(s)}+\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)}\right. \\
& \left.\quad+\sum_{j=1}^{m} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} e^{u_{2}(s)}\right] d s
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{0}^{\omega}\left|r_{1}(t)\right| d t \\
& +\int_{0}^{\omega}\left[\left(A_{1}(s)+\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)}\right) e^{u_{1}(s)}\right. \\
& \left.\quad+\sum_{j=1}^{m} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} e^{u_{2}(s)}\right] d s
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{0}^{\omega}\left|r_{1}(t)\right| d t \\
& +\int_{0}^{\omega}\left[F_{1}^{*}(s) e^{u_{1}(s)}+G_{1}(s) e^{u_{2}(s)}\right] d s \\
\leq & \overline{R_{1}} \omega+F_{1}^{* M} \int_{0}^{\omega} e^{u_{1}(s)} d s+G_{1}^{M} \int_{0}^{\omega} e^{u_{2}(s)} d s
\end{aligned}
$$

where $\overline{R_{1}}=(1 / \omega) \int_{0}^{\omega}\left|r_{1}(t)\right| d t, F_{1}^{*}(s), G_{1}(s)$ are defined by (22). By (46) and (53), we obtain

$$
\begin{equation*}
\int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| d t \leq \overline{R_{1}} \omega+F_{1}^{* M} \frac{\overline{r_{2}} \omega}{G_{2}^{L}}+G_{1}^{M} \frac{\overline{r_{1}} \omega}{G_{1}^{L}}:=\Delta_{1} \tag{54}
\end{equation*}
$$

Similarly, by the second equation of (32), we get

$$
\begin{equation*}
\int_{0}^{\omega}\left|u_{2}^{\prime}(t)\right| d t \leq \overline{R_{2}} \omega+F_{2}^{* M} \frac{\overline{r_{1}} \omega}{G_{1}^{L}}+G_{2}^{M} \frac{\overline{r_{2}} \omega}{G_{2}^{L}}:=\Delta_{2} \tag{55}
\end{equation*}
$$

where $\overline{R_{2}}=(1 / \omega) \int_{0}^{\omega}\left|r_{2}(t)\right| d t, F_{2}^{*}(s), G_{2}(s)$ are defined by (22). From (52), (54), and (55) and Lemma 10, it follows that for $t \in[0, \omega]$ that

$$
\begin{align*}
& u_{1}(t) \leq u_{1}\left(\zeta_{1}\right)+\frac{1}{2} \int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| d t \leq \ln \frac{\overline{r_{2}}}{G_{2}^{L}}+\frac{1}{2} \Delta_{1} \\
& u_{2}(t) \leq u_{2}\left(\zeta_{1}\right)+\frac{1}{2} \int_{0}^{\omega}\left|u_{2}^{\prime}(t)\right| d t \leq \ln \frac{\overline{r_{1}}}{G_{1}^{L}}+\frac{1}{2} \Delta_{2}  \tag{56}\\
& u_{1}(t) \geq u_{1}\left(\xi_{1}\right)-\frac{1}{2} \int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| d t \geq \ln \Gamma_{1}-\frac{1}{2} \Delta_{1} \\
& u_{2}(t) \geq u_{2}\left(\xi_{1}\right)-\frac{1}{2} \int_{0}^{\omega}\left|u_{2}^{\prime}(t)\right| d t \geq \ln \Gamma_{2}-\frac{1}{2} \Delta_{2}
\end{align*}
$$

Let

$$
\begin{align*}
& R_{1}=\max \left\{\left|\ln \frac{\overline{r_{2}}}{G_{2}^{L}}+\frac{1}{2} \Delta_{1}\right|,\left|\ln \Gamma_{1}-\frac{1}{2} \Delta_{1}\right|\right\} \\
& R_{2}=\max \left\{\left|\ln \frac{\overline{r_{1}}}{G_{1}^{L}}+\frac{1}{2} \Delta_{2}\right|,\left|\ln \Gamma_{2}-\frac{1}{2} \Delta_{2}\right|\right\} \tag{57}
\end{align*}
$$

It follows from (56)-(57) that

$$
\begin{align*}
& \sup _{t \in[0, \omega]}\left|u_{1}(t)\right| \leq R_{1}  \tag{58}\\
& \sup _{t \in[0, \omega]}\left|u_{2}(t)\right| \leq R_{2}
\end{align*}
$$

Clearly, $\Gamma_{l}, \Delta_{l}, R_{l}(l=1,2)$ are independent of $\lambda$, respectively. Note that $\int_{0}^{\omega} F_{l}(t) d t \leq F_{l}^{M} \omega, \int_{0}^{\omega} G_{l}(t) d t \leq G_{l}^{L} \omega(l=1,2)$. From (44), we have

$$
\begin{array}{ll}
\overline{A_{1}}+\sum_{i=1}^{n} \overline{B_{1 i}}=\overline{F_{1}^{*}} \leq F_{1}^{* M}, & G_{1}^{L} \leq \overline{G_{1}}=\sum_{j=1}^{m} \overline{C_{1 j}} ; \\
\overline{A_{2}}+\sum_{j=1}^{m} \overline{B_{2 j}}=\overline{F_{2}^{*}} \leq F_{2}^{* M}, \quad G_{2}^{L} \leq \overline{G_{2}}=\sum_{i=1}^{n} \overline{C_{2 i}}, \tag{59}
\end{array}
$$

which deduces that

$$
\begin{aligned}
\overline{r_{1}}\left(\overline{A_{1}}+\sum_{i=1}^{n} \overline{B_{1 i}}\right) & =\overline{r_{1}} F_{1}^{*} \leq \overline{r_{1}} F_{1}^{* M}<\overline{r_{2}} G_{2}^{L} \\
& \leq \overline{r_{2}} \overline{G_{2}}=\overline{r_{2}} \sum_{i=1}^{n} \overline{C_{2 i}} ; \\
\overline{r_{2}}\left(\overline{A_{2}}+\sum_{j=1}^{m} \overline{B_{2 j}}\right) & =\overline{r_{2}} \overline{F_{2}^{*}} \leq \overline{r_{2}} F_{2}^{* M}<\overline{r_{1}} G_{1}^{L} \\
& \leq \overline{r_{1}} \overline{G_{1}}=\overline{r_{1}} \sum_{j=1}^{m} \overline{C_{1 j}},
\end{aligned}
$$

which implies that

$$
\begin{align*}
& \overline{r_{1}}\left(\overline{A_{1}}+\sum_{i=1}^{n} \overline{B_{1 i}}\right) \leq \overline{r_{2}} \sum_{i=1}^{n} \overline{C_{2 i}} \\
& \overline{r_{2}}\left(\overline{A_{2}}+\sum_{j=1}^{m} \overline{B_{2 j}}\right) \leq \overline{r_{1}} \sum_{j=1}^{m} \overline{C_{1 j}} \tag{61}
\end{align*}
$$

Hence

$$
\begin{equation*}
\left(\overline{A_{1}}+\sum_{i=1}^{n} \overline{B_{1 i}}\right)\left(\overline{A_{2}}+\sum_{j=1}^{m} \overline{B_{2 j}}\right) \leq \sum_{j=1}^{m} \overline{C_{1 j}} \sum_{i=1}^{n} \overline{C_{2 i}} \tag{62}
\end{equation*}
$$

From (61) and (62), it is easy to show that the system of algebraic equations

$$
\begin{align*}
& \overline{r_{1}}-\left(\overline{A_{1}}+\sum_{i=1}^{n} \overline{B_{1 i}}\right) e^{u_{1}}-\sum_{j=1}^{m} \overline{C_{1 j}} e^{u_{2}}=0 \\
& \overline{r_{2}}-\left(\overline{A_{2}}+\sum_{j=1}^{m} \overline{B_{2 j}}\right) e^{u_{2}}-\sum_{i=1}^{n} \overline{C_{2 i}} e^{u_{1}}=0 \tag{63}
\end{align*}
$$

has a unique solution $\left(u_{1}^{*}, u_{2}^{*}\right) \in R^{2}$. In view of (58), we can take sufficiently large $R$ such that $R>R_{1}+R_{2}, R>\left|u_{1}^{*}\right|+\left|u_{2}^{*}\right|$ and define $\Omega=\left\{u(t)=\left(u_{1}(t), u_{2}(t)\right)^{T} \in X:\|u\|<R\right\}$, and it is clear that $\Omega$ satisfies condition (a) of Lemma 7. Letting $u \in \partial \Omega \cap \operatorname{Ker} L=\partial \Omega \cap R^{2}$, then $u$ is a constant vector in $R^{2}$ with $\|u\|=R$. Then

$$
\begin{equation*}
Q N u=\binom{\overline{r_{1}}-\left(\overline{A_{1}}+\sum_{i=1}^{n} \overline{B_{1 i}}\right) e^{u_{1}}-\sum_{j=1}^{m} \overline{C_{1 j}} e^{u_{2}}}{\overline{r_{2}}-\left(\overline{A_{2}}+\sum_{j=1}^{m} \overline{B_{2 j}}\right) e^{u_{2}}-\sum_{i=1}^{n} \overline{C_{2 i}} e^{u_{1}}} \neq 0 \tag{64}
\end{equation*}
$$

that is, condition (b) of Lemma 7 holds. In order to verify condition (c) in the Lemma 7, by (62) and the formula for Brouwer degree, a straightforward calculation shows that

$$
\operatorname{deg}\{J Q N u, \operatorname{Ker} L \cap \partial \Omega, 0\}
$$

$$
\begin{align*}
&=\operatorname{sign}\left\{\left(\sum_{j=1}^{m} \overline{C_{1 j}} \sum_{i=1}^{n} \overline{C_{2 i}}-\left(\overline{A_{1}}+\sum_{i=1}^{n} \overline{B_{1 i}}\right)\right.\right.  \tag{65}\\
&\left.\left.\times\left(\overline{A_{2}}+\sum_{j=1}^{m} \overline{B_{2 j}}\right)\right) e^{\left(u_{1}^{*}+u_{2}^{*}\right)}\right\} \neq 0
\end{align*}
$$

By now we have proved that all the requirements in Lemma 7 hold. Hence system (32) has at least one $\omega$-periodic solution, say $\left(u_{1}^{*}, u_{2}^{*}\right)^{T}$. Setting $y_{1}^{*}(t)=e^{u_{1}^{*}(t)}, y_{2}^{*}(t)=e^{u_{2}^{*}(t)}$, then $\left(y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}$ has at least one positive $\omega$-periodic solution of systems (11) and (12). Furthermore, setting $x_{1}^{*}(t)=$ $\prod_{0<t_{k}<t}\left(1+\theta_{1 k}\right) y_{1}^{*}(t), x_{2}^{*}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{2 k}\right) y_{2}^{*}(t)$, then $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ has at least one positive $\omega$-periodic solution
of systems (3) and (4). If $\left(H_{5}\right)$ holds, similarly, we can prove that systems (2) and (4) have at least one positive $\omega$-periodic solution. The proof of Theorem 12 is complete.

We now proceed to the discussion on the uniqueness and global stability of the $\omega$-periodic solution $x^{*}(t)$ in Theorem 12. It is immediate that if $x^{*}(t)$ is globally asymptotically stable, then $x^{*}(t)$ is unique in fact.

Theorem 13. In addition to $\left(H_{1}\right)-\left(H_{3}\right)$, assume further that

$$
\left(H_{6}\right) F_{1}^{* L} F_{2}^{* L}>G_{1}^{M} G_{2}^{M} .
$$

Then systems (3) and (4) have a unique positive $\omega$ periodic solution $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ which is globally asymptotically stable.

Proof. Letting $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ be a positive $\omega$ periodic solution of (3) and (4), then $y^{*}(t)=\left(y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}$, $\left(y_{l}^{*}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right)^{-1} x_{l}^{*}(t)\right)(l=1,2)$ is the positive $\omega$-periodic solution of system (11) and (12), and let $y_{l}(t)=$ $\left(y_{1}(t), y_{2}(t)\right)^{T}$ be any positive solution of system (11) with the initial conditions (12). It follows from Theorem 12 that there exist positive constants $T, r_{l}, R_{l}$, such that, for all $t \geq T$,

$$
\begin{equation*}
r_{l} \leq y_{l}^{*}(t) \leq R_{l}, \quad l=1,2 . \tag{66}
\end{equation*}
$$

By the assumptions of Theorem 12, we can obtain $F_{1}^{* L} F_{2}^{* L}>$ $G_{1}^{M} G_{2}^{M}$, and then there exist constants $\alpha_{1}>0, \alpha_{2}>0$; we can choose a positive constant $\varepsilon$ such that

$$
\begin{equation*}
F_{1}^{* L} \alpha_{1}-G_{2}^{M} \alpha_{2}=\varepsilon, \quad F_{2}^{* L} \alpha_{2}-G_{1}^{M} \alpha_{1}=\varepsilon . \tag{67}
\end{equation*}
$$

In the following, we always assume that $\alpha_{1}$ and $\alpha_{2}$ satisfy (67). We define

$$
\begin{align*}
V_{1}(t)= & \alpha_{1}\left|\ln y_{1}(t)-\ln y_{1}^{*}(t)\right|  \tag{68}\\
& +\alpha_{2}\left|\ln y_{2}(t)-\ln y_{2}^{*}(t)\right| .
\end{align*}
$$

Calculating the upper right derivative of $V_{1}(t)$ along solutions of (11), it follows that

$$
\begin{aligned}
& D^{+} V_{1}(t) \\
& =\sum_{l=1}^{2} \alpha_{l}\left(\frac{\dot{y}_{l}(t)}{y_{l}(t)}-\frac{\dot{y}_{l}^{*}(t)}{y_{l}^{*}(t)}\right) \operatorname{sgn}\left(y_{l}(t)-y_{l}^{*}(t)\right) \\
& \leq \operatorname{sgn}\left(y_{1}(t)-y_{1}^{*}(t)\right) \alpha_{l}
\end{aligned}
$$

$$
\begin{align*}
& \times\left\{-A_{1}(t)\left(y_{1}(t)-y_{1}^{*}(t)\right)\right. \\
& -\sum_{i=1}^{n} B_{1 i}(t)\left(y_{1}\left(t-\tau_{i}(t)\right)-y_{1}^{*}\left(t-\tau_{i}(t)\right)\right) \\
& \left.-\sum_{j=1}^{m} C_{1 j}(t)\left(y_{2}\left(t-\delta_{j}(t)\right)-y_{2}^{*}\left(t-\delta_{j}(t)\right)\right)\right\} \\
& +\operatorname{sgn}\left(y_{2}(t)-y_{2}^{*}(t)\right) \alpha_{2} \\
& \times\left\{-A_{2}(t)\left(y_{2}(t)-y_{2}^{*}(t)\right)\right. \\
& -\sum_{j=1}^{m} B_{2 j}(t)\left(y_{2}\left(t-\eta_{j}(t)\right)-y_{2}^{*}\left(t-\eta_{j}(t)\right)\right) \\
& \left.-\sum_{i=1}^{n} C_{2 i}(t)\left(y_{1}\left(t-\sigma_{i}(t)\right)-y_{1}^{*}\left(t-\sigma_{i}(t)\right)\right)\right\} \\
& \leq-\alpha_{l} A_{1}(t)\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& +\sum_{i=1}^{n} \alpha_{l} B_{1 i}(t)\left|y_{1}\left(t-\tau_{i}(t)\right)-y_{1}^{*}\left(t-\tau_{i}(t)\right)\right| \\
& +\sum_{j=1}^{m} \alpha_{l} C_{1 j}(t)\left|y_{2}\left(t-\delta_{j}(t)\right)-y_{2}^{*}\left(t-\delta_{j}(t)\right)\right| \\
& -\alpha_{2} A_{2}(t)\left|y_{2}(t)-y_{2}^{*}(t)\right|+\sum_{j=1}^{m} \alpha_{2} B_{2 j}(t) \\
& \times\left|y_{2}\left(t-\eta_{j}(t)\right)-y_{2}^{*}\left(t-\eta_{j}(t)\right)\right| \\
& +\sum_{i=1}^{n} \alpha_{2} C_{2 i}(t)\left|y_{1}\left(t-\sigma_{i}(t)\right)-y_{1}^{*}\left(t-\sigma_{i}(t)\right)\right| . \tag{69}
\end{align*}
$$

We also define

$$
\begin{align*}
V_{2}(t)= & \sum_{i=1}^{n} \alpha_{1} \int_{t-\tau_{i}(t)}^{t} \frac{B_{1 i}\left(\alpha_{i}(\xi)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(\xi)\right)}\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right| d \xi \\
& +\sum_{j=1}^{m} \alpha_{1} \int_{t-\delta_{j}(t)}^{t} \frac{C_{1 j}\left(\mu_{j}(\xi)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(\xi)\right)}\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right| d \xi \\
& +\sum_{j=1}^{m} \alpha_{2} \int_{t-\rho_{j}(t)}^{t} \frac{B_{2 j}\left(v_{j}(\xi)\right)}{1-\rho_{j}^{\prime}\left(v_{j}(\xi)\right)}\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right| d \xi \\
& +\sum_{i=1}^{n} \alpha_{2} \int_{t-\sigma_{i}(t)}^{t} \frac{C_{2 i}\left(\beta_{i}(\xi)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(\xi)\right)}\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right| d \xi \tag{70}
\end{align*}
$$

Calculating the upper right derivative of $V_{2}(t)$ along solutions of (11), it follows that

$$
\begin{align*}
D^{+} V_{2}(t)= & \sum_{i=1}^{n} \alpha_{1} \frac{B_{1 i}\left(\alpha_{i}(t)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)}\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& -\sum_{i=1}^{n} \alpha_{1} \frac{B_{1 i}(t)}{1-\tau_{i}^{\prime}(t)}\left(1-\tau_{i}^{\prime}(t)\right) \\
& \times\left|y_{1}\left(t-\tau_{i}(t)\right)-y_{1}^{*}\left(t-\tau_{i}(t)\right)\right| \\
& +\sum_{j=1}^{m} \alpha_{1} \frac{C_{1 j}\left(\mu_{j}(t)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(t)\right)}\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& -\sum_{j=1}^{m} \alpha_{1} \frac{C_{1 j}(t)}{1-\delta_{j}^{\prime}(t)}\left(1-\delta_{j}^{\prime}(t)\right) \\
& \times\left|y_{2}\left(t-\delta_{j}(t)\right)-y_{2}^{*}\left(t-\delta_{j}(t)\right)\right| \\
& +\sum_{j=1}^{m} \alpha_{2} \frac{B_{2 j}\left(v_{j}(t)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(t)\right)}\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& -\sum_{j=1}^{m} \alpha_{1} \frac{B_{2 j}(t)}{1-\eta_{j}^{\prime}(t)}\left(1-\eta_{j}^{\prime}(t)\right) \\
& \times\left|y_{2}\left(t-\eta_{j}(t)\right)-y_{2}^{*}\left(t-\eta_{j}(t)\right)\right| \\
& +\sum_{i=1}^{n} \alpha_{2} \frac{C_{2 i}\left(\beta_{i}(t)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(t)\right)} \\
& \times\left|y_{1}(t)-y_{1}^{*}(t)\right|-\sum_{i=1}^{n} \alpha_{2} \frac{C_{2 i}(t)}{1-\sigma_{i}^{\prime}(t)} \\
& \times\left(1-\sigma_{i}^{\prime}(t)\right)\left|y_{1}\left(t-\sigma_{i}(t)\right)-y_{1}^{*}\left(t-\sigma_{i}(t)\right)\right| \tag{71}
\end{align*}
$$

We define a Lyapunov functional $V(t)$ as follows:

$$
\begin{equation*}
V(t)=V_{1}(t)+V_{2}(t) \tag{72}
\end{equation*}
$$

Calculating the upper right derivative of $V(t)$ along solutions of (11), it follows that

$$
\begin{aligned}
D^{+} V(t)= & -\alpha_{l} A_{1}(t)\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& +\sum_{i=1}^{n} \alpha_{1} \frac{B_{1 i}\left(\alpha_{i}(t)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)}\left|y_{1}(t)-y_{1}^{*}(t)\right|
\end{aligned}
$$

$$
\begin{align*}
& \quad+\sum_{j=1}^{m} \alpha_{1} \frac{C_{1 j}\left(\mu_{j}(t)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(t)\right)}\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& \\
& -\alpha_{2} A_{2}(t)\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& \\
& \quad+\sum_{j=1}^{m} \alpha_{2} \frac{B_{2 j}\left(v_{j}(t)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(t)\right)}\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& \\
& \quad+\sum_{i=1}^{n} \alpha_{2} \frac{C_{2 i}\left(\beta_{i}(t)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(t)\right)}\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& \\
& \quad-\left[\alpha_{l} A_{1}(t)-\sum_{i=1}^{n} \alpha_{1} \frac{B_{1 i}\left(\alpha_{i}(t)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)}\right. \\
& \quad-\left[\sum_{i=1}^{n} \alpha_{2} \frac{C_{2 i}\left(\beta_{i}(t)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(t)\right)}\right]\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
&  \tag{73}\\
& \leq\left(\alpha_{l}(t)-\sum_{j=1}^{m} \alpha_{1} \frac{C_{1 j}\left(\mu_{j}(t)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(t)\right)}\right. \\
& \left.\quad-\alpha_{2} G_{2}^{M}\right)\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& \quad-\left(\alpha_{2} F_{2}^{L}-\alpha_{1} G_{1}^{M}\right)\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& =-\varepsilon\left(\left|y_{1}(t)-y_{1}^{*}(t)\right|+\left|y_{2}(t)-y_{2}^{*}(t)\right|\right) .
\end{align*}
$$

So by (73), we have

$$
\begin{gather*}
\varepsilon \int_{0}^{t}\left(\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right|+\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right|\right) d \xi  \tag{74}\\
+V(t) \leq V(0)<+\infty, \quad t \geq 0
\end{gather*}
$$

where

$$
V(0)=\alpha_{1}\left|\ln y_{1}(0)-\ln y_{1}^{*}(0)\right|+\alpha_{2}\left|\ln y_{2}(0)-\ln y_{2}^{*}(0)\right|
$$

$$
\begin{align*}
& +\sum_{i=1}^{n} \alpha_{1} \int_{-\tau_{i}(0)}^{0} \frac{B_{1 i}\left(\alpha_{i}(\xi)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(\xi)\right)}\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right| d \xi \\
& +\sum_{j=1}^{m} \alpha_{1} \int_{-\delta_{j}(0)}^{0} \frac{C_{1 j}\left(\mu_{j}(\xi)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(\xi)\right)}\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right| d \xi \\
& +\sum_{j=1}^{m} \alpha_{2} \int_{-\rho_{j}(0)}^{0} \frac{B_{2 j}\left(v_{j}(\xi)\right)}{1-\rho_{j}^{\prime}\left(v_{j}(\xi)\right)} \\
& \times\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right| d \xi \\
& +\sum_{i=1}^{n} \alpha_{2} \int_{-\sigma_{i}(0)}^{0} \frac{C_{2 i}\left(\beta_{i}(\xi)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(\xi)\right)}\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right| d \xi \tag{75}
\end{align*}
$$

which implies that

$$
\begin{equation*}
\int_{0}^{t}\left(\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right|+\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right|\right) d \xi \leq \frac{V(0)}{\varepsilon} \tag{76}
\end{equation*}
$$

By (76), it is obvious that $\left|y_{1}(t)-y_{1}^{*}(t)\right|+\left|y_{2}(t)-y_{2}^{*}(t)\right|$ is bounded.

On the other hand, we know that

$$
\begin{gather*}
\alpha_{1}\left|\ln y_{1}(t)-\ln y_{1}^{*}(t)\right|+\alpha_{2}\left|\ln y_{2}(t)-\ln y_{2}^{*}(t)\right| \\
\leq V(t) \leq V(0)<+\infty, \quad t \geq 0 \tag{77}
\end{gather*}
$$

which implies that

$$
\begin{align*}
& \left|\ln y_{1}(t)-\ln y_{1}^{*}(t)\right| \leq \frac{V(0)}{\alpha_{1}}  \tag{78}\\
& \left|\ln y_{2}(t)-\ln y_{2}^{*}(t)\right| \leq \frac{V(0)}{\alpha_{2}},
\end{align*}
$$

which, together with (66), yield

$$
\begin{align*}
& r_{1} e^{-V(0) / \alpha_{1}} \leq y_{1}(t) \leq R_{1} e^{V(0) / \alpha_{1}}<+\infty \\
& r_{2} e^{-V(0) / \alpha_{2}} \leq y_{2}(t) \leq R_{2} e^{V(0) / \alpha_{2}}<+\infty \tag{79}
\end{align*}
$$

From (66) and (79), it follows that $y_{l}(t)(l=1,2)$ is bounded for $t \geq 0$. Hence, $y_{1}(t)-y_{1}^{*}(t), y_{2}(t)-y_{2}^{*}(t)$, and their derivatives remain bounded on $[0,+\infty)$. So $\left|y_{1}(t)-y_{1}^{*}(t)\right|$, $\left|y_{2}(t)-y_{2}^{*}(t)\right|$ are uniformly continuous on $[0,+\infty)$. By Lemma 11, we have

$$
\begin{align*}
\lim _{t \rightarrow+\infty} & \left|y_{l}(s)-y_{l}^{*}(s)\right| \\
& =\lim _{t \rightarrow+\infty}\left[\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right)^{-1}\left|x_{l}^{*}(s)-x_{l}(s)\right|\right]=0, \tag{80}
\end{align*}
$$

$$
l=1,2 .
$$

Therefore

$$
\begin{equation*}
\lim _{t \rightarrow+\infty}\left|x_{l}(s)-x_{l}^{*}(s)\right|=0, \quad l=1,2 . \tag{81}
\end{equation*}
$$

By Theorems 7.4 and 8.2 in [30], we know that the periodic positive solution $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ is uniformly asymptotically stable. The proof of Theorem 13 is completed.

Theorem 14. In addition to $\left(H_{1}\right)-\left(H_{3}\right)$, assume that one of the following conditions holds:

$$
\begin{aligned}
& \left(H_{7}\right) \overline{r_{1}} F_{1}^{M}<\overline{r_{2}} G_{2}^{L}, \quad \overline{r_{2}} F_{2}^{M}<\overline{r_{1}} G_{1}^{L} \\
& \left(H_{8}\right) \overline{r_{1}} F_{1}^{L}>\overline{r_{2}} G_{2}^{M}, \overline{r_{2}} F_{2}^{L}>\overline{r_{1}} G_{1}^{M}
\end{aligned}
$$

Then systems (2) and (4) have at least one positive $\omega$ periodic solution, where $F_{1}(t), F_{2}(t), G_{1}(t)$, and $G_{2}(t)$ are defined in (22).
Proof. Since the solutions of systems (10) and (12) remain positive for $t \geq 0$, we carry out the change of variable $u_{i}(t)=$ $\ln y_{i}(t)(i=1,2)$, and then (10) can be transformed to

$$
\begin{align*}
u_{1}^{\prime}(t)= & r_{1}(t)-A_{1}(t) e^{u_{1}(t)}+\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)} \\
& -\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}, \\
u_{2}^{\prime}(t)= & r_{2}(t)-A_{2}(t) e^{u_{2}(t)}+\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}  \tag{82}\\
& -\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)} .
\end{align*}
$$

It is easy to see that if system (82) has one $\omega$-periodic solution $\left(u_{1}^{*}(t), u_{2}^{*}(t)\right)^{T}$, then $\left(y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}=\left(e^{u_{1}^{*}(t)}, e^{u_{2}^{*}(t)}\right)^{T}$ is a positive $\omega$-periodic solution of systems (10) and (12); that is to say, $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}=\left(\prod_{0<t_{k}<t}\left(1+\theta_{1 k}\right) e^{u_{1}^{*}(t)}, \prod_{0<t_{k}<t}(1+\right.$ $\left.\left.\theta_{2 k}\right) e^{u_{2}^{*}(t)}\right)^{T}$ is a positive $\omega$-periodic solution of systems (2) and (4). Therefore, it suffices to prove that system (82) has a $\omega$ periodic solution. In order to use Lemma 6 for (81), we take

$$
\begin{gather*}
X=Z=\left\{u(t)=\left(u_{1}(t), u_{2}(t)\right)^{T} \mid u_{i}(t) \in C\left(R, R^{2}\right)\right.  \tag{83}\\
\left.: u_{i}(t+\omega)=u_{i}(t), l=1,2\right\}
\end{gather*}
$$

and define

$$
\begin{align*}
\|u\| & =\sum_{i=1}^{2} \sup _{t \in[0, \omega]}\left|u_{l}(t)\right|  \tag{84}\\
u(t) & =\left(u_{1}(t), u_{2}(t)\right)^{T} \in X(\text { or } Z) .
\end{align*}
$$

Then $X$ and $Z$ are Banach spaces when they are endowed with the norm $\|\cdot\|$. Let $N: X \rightarrow Z$ with

$$
N u=\left[\begin{array}{l}
r_{1}(t)-A_{1}(t) e^{u_{1}(t)}+\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}  \tag{85}\\
r_{2}(t)-A_{2}(t) e^{u_{2}(t)}+\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}-\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}
\end{array}\right]=\left[\begin{array}{l}
g_{1}(t) \\
g_{2}(t)
\end{array}\right] \quad \text { for any } u \in X,
$$

and define

$$
\begin{gather*}
L u=u^{\prime} ; \quad P u=\frac{1}{\omega} \int_{0}^{\omega} u(t) d t, \quad u \in X ;  \tag{86}\\
Q z=\frac{1}{\omega} \int_{0}^{\omega} z(t) d t, \quad z \in Z .
\end{gather*}
$$

It is not difficult to show that

$$
\begin{gathered}
\operatorname{Ker} L=\left\{u \in X \mid u=h \in R^{2}\right\} \\
\operatorname{Im} L=\left\{z \in Z \mid \int_{0}^{\omega} z(s) d s=0\right\}
\end{gathered}
$$

and $\operatorname{dimKer} L=2=$ codimIm$L$. So, $\operatorname{Im} L$ is closed in $Z$, and $L$ is a Fredholm mapping of index zero. It is trivial to show that $P, Q$ are continuous projectors such that $\operatorname{Im} P=$ $\operatorname{Ker} L, \operatorname{Ker} Q=\operatorname{Im} L=\operatorname{Im}(I-Q)$. Furthermore, the generalized inverse (to $L$ ) $K_{P}: \operatorname{Im} L \rightarrow \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$
\begin{gather*}
\mathrm{QNu}=\binom{\frac{1}{\omega} \int_{0}^{\omega}\left[r_{1}(t)-A_{1}(t) e^{u_{1}(t)}+\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right] d t}{\frac{1}{\omega} \int_{0}^{\omega}\left[r_{2}(t)-A_{2}(t) e^{u_{2}(t)}+\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}-\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}\right] d t},  \tag{89}\\
K_{P}(I-Q) N u=\binom{\int_{0}^{t} g_{1}(\xi) d \xi}{\int_{0}^{t} g_{2}(\xi) d \xi}-\binom{\frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} g_{1}(\xi) d \xi d t}{\frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} g_{2}(\xi) d \xi d t}-\binom{\left(\frac{t}{\omega}-\frac{1}{2}\right) \int_{0}^{\omega} g_{1}(\xi) d \xi}{\left(\frac{t}{\omega}-\frac{1}{2}\right) \int_{0}^{\omega} g_{2}(\xi) d \xi} .
\end{gather*}
$$

Clearly, $Q N$ and $K_{P}(I-Q) N$ are continuous. By applying Ascoli-Arzela theorem, one can easily show that $\mathrm{QN}(\bar{\Omega})$, $K_{P}(I-Q) N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Moreover, $\mathrm{QN}(\bar{\Omega})$ is obviously bounded. Thus, $N$ is $L$-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$. Now, we reach the position to search for an appropriate open bounded set $\Omega \subset X$ for the application of Lemma 6 . Corresponding to the operating equation $L u=\lambda N u, \lambda \in(0,1)$, we have

$$
\begin{gather*}
u_{1}^{\prime}(t)=\lambda\left[r_{1}(t)-A_{1}(t) e^{u_{1}(t)}+\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}\right. \\
\left.-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right] \\
u_{2}^{\prime}(t)=\lambda\left[r_{2}(t)-A_{2}(t) e^{u_{2}(t)}+\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}\right. \\
\left.-\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}\right] \tag{90}
\end{gather*}
$$

Since $u(t)=\left(u_{1}(t), u_{2}(t)\right)^{T}$ is a $\omega$-periodic function, we need only to prove the result in the interval $[0, \omega]$. Integrating (90) over the interval $[0, \omega]$ leads to the following:

$$
\begin{gather*}
\int_{0}^{\omega}\left[r_{1}(t)-A_{1}(t) e^{u_{1}(t)}+\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}\right. \\
\left.-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right] d t=0 \\
\int_{0}^{\omega}\left[\begin{array}{c}
r_{2}(t)-A_{2}(t) e^{u_{2}(t)}+\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)} \\
\left.-\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}\right] d t=0
\end{array} .\right.
\end{gather*}
$$

Hence, we have

$$
\begin{align*}
& \int_{0}^{\omega}\left[A_{1}(t) e^{u_{1}(t)}-\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}\right. \\
& \left.+\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right] d t=\overline{r_{1}} \omega, \\
& \int_{0}^{\omega}\left[A_{2}(t) e^{u_{2}(t)}-\sum_{j=1}^{m} B_{2 j}(t) e^{u_{2}\left(t-\eta_{j}(t)\right)}\right. \\
& \left.+\sum_{i=1}^{n} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)}\right] d t=\overline{r_{2}} \omega . \tag{92}
\end{align*}
$$

Noting that $u(t)=\left(u_{1}(t), u_{2}(t)\right) \in X$, then there exists $\zeta_{l}, \xi_{l} \in$ $[0, \omega](l=1,2)$ such that

$$
\begin{equation*}
u_{l}\left(\zeta_{l}\right)=\inf _{t \in[0, \omega]} u_{l}(t), \quad u_{l}\left(\xi_{l}\right)=\sup _{t \in[0, \omega]} u_{l}(t), \quad l=1,2 \tag{93}
\end{equation*}
$$

Since $\tau_{i}^{\prime}(t)<1$, we can let $s=t-\tau_{i}(t)$, that is, $t=\alpha_{i}(s)$, $i=1,2, \ldots, n$, and then

$$
\begin{equation*}
\int_{0}^{\omega} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)} d t=\int_{-\tau_{i}(0)}^{\omega-\tau_{i}(\omega)} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)} d s \tag{94}
\end{equation*}
$$

According to Lemma 6, we know that $\left(\left(B_{1 i}\left(\alpha_{i}(s)\right)\right) /(1-\right.$ $\left.\left.\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)\right)\right) e^{u_{1}(s)} \in C_{\omega}$. Thus,

$$
\begin{equation*}
\int_{-\tau_{i}(0)}^{\omega-\tau_{i}(\omega)} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)} d s=\int_{0}^{\omega} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)} d s \tag{95}
\end{equation*}
$$

By (37) and (38), we have

$$
\begin{equation*}
\int_{0}^{\omega} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)} d t=\int_{0}^{\omega} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)} d s . \tag{96}
\end{equation*}
$$

Similarly, we obtain

$$
\begin{align*}
& \int_{0}^{\omega} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)} d t=\int_{0}^{\omega} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} e^{u_{2}(s)} d s \\
& \int_{0}^{\omega} B_{2 j}(t) e^{u_{1} 2\left(t-\eta_{j}(t)\right)} d t=\int_{0}^{\omega} \frac{B_{2 j}\left(v_{j}(s)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(s)\right)} e^{u_{2}(s)} d s  \tag{97}\\
& \int_{0}^{\omega} C_{2 i}(t) e^{u_{1}\left(t-\sigma_{i}(t)\right)} d t=\int_{0}^{\omega} \frac{C_{2 i}\left(\beta_{i}(s)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(s)\right)} e^{u_{1}(s)} d s
\end{align*}
$$

It follows from (92), (96), and (97) that we get

$$
\begin{align*}
& \int_{0}^{\omega}\left[\left(A_{1}(s)-\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)}\right) e^{u_{1}(s)}\right. \\
& \left.\quad+\sum_{j=1}^{m} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} e^{u_{2}(s)}\right] d s=\overline{r_{1}} \omega, \\
& \int_{0}^{\omega}\left[\left(A_{2}(s)-\sum_{j=1}^{m} \frac{B_{2 j}\left(v_{j}(s)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(s)\right)}\right) e^{u_{2}(s)}\right.  \tag{98}\\
& \left.\quad+\sum_{i=1}^{n} \frac{C_{2 i}\left(\beta_{i}(s)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(s)\right)} e^{u_{1}(s)}\right] d s=\overline{r_{2}} \omega .
\end{align*}
$$

Thus from (98) we get

$$
\begin{align*}
& \int_{0}^{\omega} F_{1}(s) e^{u_{1}(s)} d s+\int_{0}^{\omega} G_{1}(s) e^{u_{2}(s)} d s=\overline{r_{1}} \omega,  \tag{99}\\
& \int_{0}^{\omega} F_{2}(s) e^{u_{2}(s)} d s+\int_{0}^{\omega} G_{2}(s) e^{u_{1}(s)} d s=\overline{r_{2}} \omega .
\end{align*}
$$

where $F_{1}(s), F_{2}(s), G_{1}(s)$, and $G_{2}(s)$ are defined by (22). On the other hand, by Lemma 6 , we can see that $\alpha_{i}(\omega)=\alpha_{i}(0)+\omega$, so we can derive

$$
\begin{align*}
\int_{0}^{\omega} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} d s & =\int_{\alpha_{i}(0)}^{\alpha_{i}(\omega)} \frac{B_{1 i}(t)\left(1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)\right.}{1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)} d t  \tag{100}\\
& =\int_{0}^{\omega} B_{1 i}(t) d t=\overline{B_{1 i}} \omega
\end{align*}
$$

Thus, from (99) and (100), we get

$$
\begin{align*}
\overline{F_{1}} \omega & =\int_{0}^{\omega} F_{1}(s) d s=\int_{0}^{\omega}\left[A_{1}(s)-\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)}\right] d s \\
& =\left(\overline{A_{1}}-\sum_{i=1}^{n} \overline{B_{1 i}}\right) \omega, \\
\overline{F_{2}} \omega & =\int_{0}^{\omega} F_{2}(s) d s=\int_{0}^{\omega}\left[A_{2}(s)-\sum_{i=1}^{n} \frac{B_{2 j}\left(v_{j}(s)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(s)\right)}\right] d s \\
& =\left(\overline{A_{2}}-\sum_{i=1}^{n} \overline{B_{2 j}}\right) \omega, \\
\overline{G_{1}} \omega & =\int_{0}^{\omega} G_{1}(s) d s=\int_{0}^{\omega} \sum_{j=1}^{m} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} d s \\
& =\sum_{j=1}^{m} \overline{C_{1 j}} \omega, \\
\overline{G_{2}} \omega & =\int_{0}^{\omega} G_{2}(s) d s=\int_{0}^{\omega} \sum_{i=1}^{n} \frac{C_{2 i}\left(\beta_{i}(s)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(s)\right)} d s \\
& =\sum_{i=1}^{n} \overline{C_{2 i}} \omega . \tag{101}
\end{align*}
$$

By (99), on one hand, we have

$$
\begin{align*}
& G_{1}^{L} \int_{0}^{\omega} e^{u_{2}(s)} d s \leq \int_{0}^{\omega} G_{1}(s) e^{u_{2}(s)} d s \leq \overline{r_{1}} \omega \\
& G_{2}^{L} \int_{0}^{\omega} e^{u_{1}(s)} d s \leq \int_{0}^{\omega} G_{2}(s) e^{u_{1}(s)} d s \leq \overline{r_{2}} \omega \tag{102}
\end{align*}
$$

which implies that

$$
\begin{align*}
& \int_{0}^{\omega} e^{u_{2}(s)} d s \leq \frac{\overline{r_{1}} \omega}{G_{1}^{L}}, \\
& \int_{0}^{\omega} e^{u_{1}(s)} d s \leq \frac{\overline{r_{2}} \omega}{G_{2}^{L}} . \tag{103}
\end{align*}
$$

On the other hand, by (99) the integral mean value theorem that there is $\lambda_{1}, \lambda_{2}, \rho_{1}$, and $\rho_{2} \in[0, \omega]$ such that

$$
\begin{align*}
& F_{1}\left(\lambda_{1}\right) \int_{0}^{\omega} e^{u_{1}(s)} d s+G_{1}\left(\rho_{1}\right) \int_{0}^{\omega} e^{u_{2}(s)} d s=\overline{r_{1}} \omega \\
& F_{2}\left(\lambda_{2}\right) \int_{0}^{\omega} e^{u_{2}(s)} d s+G_{2}\left(\rho_{2}\right) \int_{0}^{\omega} e^{u_{1}(s)} d s=\overline{r_{2}} \omega \tag{104}
\end{align*}
$$

By $\left(H_{7}\right)$, we have $G_{1}^{L} G_{2}^{L}>F_{1}^{M} F_{2}^{M}$, which together with (104), lead to the following:

$$
\begin{align*}
\int_{0}^{\omega} e^{u_{2}(s)} d s & =\frac{\overline{r_{1}} \omega G_{2}\left(\rho_{2}\right)-\overline{r_{2}} \omega F_{1}\left(\lambda_{1}\right)}{G_{1}\left(\rho_{1}\right) G_{2}\left(\rho_{2}\right)-F_{1}\left(\lambda_{1}\right) F_{2}\left(\lambda_{2}\right)} \\
& \geq \frac{\overline{r_{1}} \omega G_{2}^{L}-\overline{r_{2}} \omega F_{1}^{M}}{G_{1}^{M} G_{2}^{M}-F_{1}^{L} F_{2}^{L}}:=\Gamma_{4} \omega,  \tag{105}\\
\int_{0}^{\omega} e^{u_{1}(s)} d s & =\frac{\overline{r_{2}} \omega G_{1}\left(\rho_{1}\right)-\overline{r_{1}} \omega F_{2}\left(\lambda_{2}\right)}{G_{1}\left(\rho_{1}\right) G_{2}\left(\rho_{2}\right)-F_{1}\left(\lambda_{1}\right) F_{2}\left(\lambda_{2}\right)} \\
& \geq \frac{\overline{r_{2}} \omega G_{1}^{L}-\overline{r_{1}} \omega F_{2}^{M}}{G_{1}^{M} G_{2}^{M}-F_{1}^{L} F_{2}^{L}}:=\Gamma_{3} \omega .
\end{align*}
$$

Again, by $\left(H_{7}\right)$, one can deduce that the following inequalities:

$$
\begin{align*}
& \frac{\overline{r_{1}} \omega}{G_{1}^{L}} \geq \frac{\overline{r_{1}} \omega G_{2}^{L}-\overline{r_{2}} \omega F_{1}^{M}}{G_{1}^{M} G_{2}^{M}-F_{1}^{L} F_{2}^{L}}:=\Gamma_{4} \omega>0 \\
& \frac{\overline{r_{2}} \omega}{\overline{G_{2}^{L}}} \geq \frac{\overline{r_{2}} \omega G_{1}^{L}-\overline{r_{1}} \omega F_{2}^{M}}{G_{1}^{M} G_{2}^{M}-F_{1}^{L} F_{2}^{L}}:=\Gamma_{3} \omega>0 \tag{106}
\end{align*}
$$

It follows from (103), (105), and (106) that

$$
\begin{align*}
& \Gamma_{4} \omega \leq \int_{0}^{\omega} e^{u_{2}(s)} d s \leq \frac{\overline{r_{1}} \omega}{G_{1}^{L}}, \\
& \Gamma_{3} \omega \leq \int_{0}^{\omega} e^{u_{1}(s)} d s \leq \frac{\overline{r_{2}} \omega}{G_{2}^{L}}, \tag{107}
\end{align*}
$$

which, together with (92) yield

$$
\begin{array}{ll}
\Gamma_{4} \leq e^{u_{2}\left(\zeta_{2}\right)}, & e^{u_{2}\left(\xi_{2}\right)} \leq \frac{\overline{r_{1}}}{G_{1}^{L}}, \\
\Gamma_{3} \leq e^{u_{1}\left(\zeta_{1}\right)}, & e^{u_{1}\left(\xi_{1}\right)} \leq \frac{\overline{r_{2}}}{G_{2}^{L}}
\end{array}
$$

which implies that

$$
\begin{array}{ll}
\ln \Gamma_{4} \leq u_{2}\left(\xi_{2}\right), & u_{2}\left(\zeta_{2}\right) \leq \ln \frac{\overline{r_{1}}}{G_{1}^{L}} \\
\ln \Gamma_{3} \leq u_{1}\left(\xi_{1}\right), & u_{1}\left(\zeta_{1}\right) \leq \ln \frac{\overline{r_{2}}}{G_{2}^{L}}
\end{array}
$$

From the first equation of (90), we get

$$
\begin{aligned}
& \int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| d t \\
& =\lambda \int_{0}^{\omega} \mid r_{1}(t)-A_{1}(t) e^{u_{1}(t)}+\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)} \\
& \quad-\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)} \mid d t \\
& \begin{aligned}
\leq & \int_{0}^{\omega}\left|r_{1}(t)\right| d t
\end{aligned} \\
& \quad+\int_{0}^{\omega}\left[A_{1}(t) e^{u_{1}(t)}+\sum_{i=1}^{n} B_{1 i}(t) e^{u_{1}\left(t-\tau_{i}(t)\right)}\right. \\
& \left.\quad+\sum_{j=1}^{m} C_{1 j}(t) e^{u_{2}\left(t-\delta_{j}(t)\right)}\right] d t \\
& =\int_{0}^{\omega}\left|r_{1}(t)\right| d t \\
& \quad+\int_{0}^{\omega}\left[A_{1}(s) e^{u_{1}(s)}+\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)} e^{u_{1}(s)}\right. \\
& \left.\quad+\sum_{j=1}^{m} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} e^{u_{2}(s)}\right] d s
\end{aligned}
$$

$$
\begin{aligned}
& =\int_{0}^{\omega}\left|r_{1}(t)\right| d t \\
& \quad+\int_{0}^{\omega}\left[\left(A_{1}(s)+\sum_{i=1}^{n} \frac{B_{1 i}\left(\alpha_{i}(s)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(s)\right)}\right) e^{u_{1}(s)}\right. \\
& \quad \\
& \left.\quad+\sum_{j=1}^{m} \frac{C_{1 j}\left(\mu_{j}(s)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(s)\right)} e^{u_{2}(s)}\right] d s
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{0}^{\omega}\left|r_{1}(t)\right| d t \\
& +\int_{0}^{\omega}\left[F_{1}^{*}(s) e^{u_{1}(s)}+G_{1}(s) e^{u_{2}(s)}\right] d s \\
\leq & \overline{R_{1}} \omega+F_{1}^{* M} \int_{0}^{\omega} e^{u_{1}(s)} d s+G_{1}^{M} \int_{0}^{\omega} e^{u_{2}(s)} d s
\end{aligned}
$$

where $\overline{R_{1}}=(1 / \omega) \int_{0}^{\omega}\left|r_{1}(t)\right| d t, F_{1}^{*}(s), G_{1}(s)$ are defined by (22). By (103) and (110), we obtain

$$
\begin{equation*}
\int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| d t \leq \overline{R_{1}} \omega+F_{1}^{* M} \frac{\overline{r_{2}} \omega}{G_{2}^{L}}+G_{1}^{M} \frac{\overline{r_{1}} \omega}{G_{1}^{L}}:=\Delta_{3} . \tag{111}
\end{equation*}
$$

Similarly, by the second equation of (90), we get

$$
\begin{equation*}
\int_{0}^{\omega}\left|u_{2}^{\prime}(t)\right| d t \leq \overline{R_{2}} \omega+F_{2}^{* M} \frac{\overline{r_{1}} \omega}{G_{1}^{L}}+G_{2}^{M} \frac{\overline{r_{2}} \omega}{G_{2}^{L}}:=\Delta_{4} \tag{112}
\end{equation*}
$$

where $\overline{R_{2}}=(1 / \omega) \int_{0}^{\omega}\left|r_{2}(t)\right| d t, F_{2}^{*}(s), G_{2}(s)$ are defined by (22). From (109), (111), and (112) and Lemma 10, it follows that for $t \in[0, \omega]$

$$
\begin{align*}
& u_{1}(t) \leq u_{1}\left(\zeta_{1}\right)+\frac{1}{2} \int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| d t \leq \ln \frac{\overline{r_{2}}}{G_{2}^{L}}+\frac{1}{2} \Delta_{3},  \tag{113}\\
& u_{2}(t) \leq u_{2}\left(\zeta_{1}\right)+\frac{1}{2} \int_{0}^{\omega}\left|u_{2}^{\prime}(t)\right| d t \leq \ln \frac{\overline{r_{1}}}{G_{1}^{L}}+\frac{1}{2} \Delta_{4} \\
& u_{1}(t) \geq u_{1}\left(\xi_{1}\right)-\frac{1}{2} \int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| d t \geq \ln \Gamma_{1}-\frac{1}{2} \Delta_{3} \\
& u_{2}(t) \geq u_{2}\left(\xi_{1}\right)-\frac{1}{2} \int_{0}^{\omega}\left|u_{2}^{\prime}(t)\right| d t \geq \ln \Gamma_{2}-\frac{1}{2} \Delta_{4} \tag{114}
\end{align*}
$$

Let

$$
\begin{align*}
& R_{3}=\max \left\{\left|\ln \frac{\overline{r_{2}}}{G_{2}^{L}}+\frac{1}{2} \Delta_{3}\right|,\left|\ln \Gamma_{3}-\frac{1}{2} \Delta_{3}\right|\right\},  \tag{115}\\
& R_{4}=\max \left\{\left|\ln \frac{\overline{r_{1}}}{G_{1}^{L}}+\frac{1}{2} \Delta_{4}\right|,\left|\ln \Gamma_{4}-\frac{1}{2} \Delta_{4}\right|\right\} .
\end{align*}
$$

It follows from (113)-(115) that

$$
\begin{align*}
& \sup _{t \in[0, \omega]}\left|u_{1}(t)\right| \leq R_{3},  \tag{116}\\
& \sup _{t \in[0, \omega]}\left|u_{2}(t)\right| \leq R_{4} .
\end{align*}
$$

Clearly, $\Gamma_{l}, \Delta_{l}, R_{l}(l=3,4)$ are independent of $\lambda$, respectively. Note that $\int_{0}^{\omega} F_{l}(t) d t \leq F_{l}^{M} \omega, \int_{0}^{\omega} G_{l}(t) d t \leq G_{l}^{L} \omega, l=1,2$. From (44), we have

$$
\begin{array}{ll}
\overline{A_{1}}-\sum_{i=1}^{n} \overline{B_{1 i}}=\overline{F_{1}} \leq F_{1}^{M}, & G_{1}^{L} \leq \overline{G_{1}}=\sum_{j=1}^{m} \overline{C_{1 j}} ; \\
\overline{A_{2}}-\sum_{j=1}^{m} \overline{B_{2 j}}=\overline{F_{2}} \leq F_{2}^{M}, & G_{2}^{L} \leq \overline{G_{2}}=\sum_{i=1}^{n} \overline{C_{2 i}} \tag{117}
\end{array}
$$

which deduces that

$$
\begin{aligned}
\overline{r_{1}}\left(\overline{A_{1}}-\sum_{i=1}^{n} \overline{B_{1 i}}\right) & =\overline{r_{1}} F_{1} \leq \overline{r_{1}} F_{1}^{M} \\
& <\overline{r_{2}} G_{2}^{L} \leq \overline{r_{2}} \overline{G_{2}}=\overline{r_{2}} \sum_{i=1}^{n} \overline{C_{2 i}} \\
\overline{r_{2}}\left(\overline{A_{2}}-\sum_{j=1}^{m} \overline{B_{2 j}}\right) & =\overline{r_{2}} \overline{F_{2}} \leq \overline{r_{2}} F_{2}^{M} \\
& <\overline{r_{1}} G_{1}^{L} \leq \overline{r_{1}} \overline{G_{1}}=\overline{r_{1}} \sum_{j=1}^{m} \overline{C_{1 j}}
\end{aligned}
$$

which implies that

$$
\begin{align*}
& \overline{r_{1}}\left(\overline{A_{1}}-\sum_{i=1}^{n} \overline{B_{1 i}}\right) \leq \overline{r_{2}} \sum_{i=1}^{n} \overline{C_{2 i}} ; \\
& \overline{r_{2}}\left(\overline{A_{2}}-\sum_{j=1}^{m} \overline{B_{2 j}}\right) \leq \overline{r_{1}} \sum_{j=1}^{m} \overline{C_{1 j}} . \tag{119}
\end{align*}
$$

Hence

$$
\begin{equation*}
\left(\overline{A_{1}}-\sum_{i=1}^{n} \overline{B_{1 i}}\right)\left(\overline{A_{2}}-\sum_{j=1}^{m} \overline{B_{2 j}}\right) \leq \sum_{j=1}^{m} \overline{C_{1 j}} \sum_{i=1}^{n} \overline{C_{2 i}} . \tag{120}
\end{equation*}
$$

From (119) and (120), it is easy to show that the system of algebraic equations

$$
\begin{align*}
& \overline{r_{1}}-\left(\overline{A_{1}}-\sum_{i=1}^{n} \overline{B_{1 i}}\right) e^{u_{1}}-\sum_{j=1}^{m} \overline{C_{1 j}} e^{u_{2}}=0, \\
& \overline{r_{2}}-\left(\overline{A_{2}}-\sum_{j=1}^{m} \overline{B_{2 j}}\right) e^{u_{2}}-\sum_{i=1}^{n} \overline{C_{2 i}} e^{u_{1}}=0 \tag{121}
\end{align*}
$$

has a unique solution $\left(u_{1}^{*}, u_{2}^{*}\right) \in R^{2}$. In view of (116), we can take sufficiently large $R$ such that $R>R_{3}+R_{4}, R>\left|u_{1}^{*}\right|+\left|u_{2}^{*}\right|$ and define $\Omega=\left\{u(t)=\left(u_{1}(t), u_{2}(t)\right)^{T} \in X:\|u\|<R\right\}$, and it is clear that $\Omega$ satisfies condition (a) of Lemma 7. Letting $u \in \partial \Omega \cap \operatorname{Ker} L=\partial \Omega \cap R^{2}$, then $u$ is a constant vector in $R^{2}$ with $\|u\|=R$. Then

$$
\begin{equation*}
Q N u=\binom{\overline{r_{1}}-\left(\overline{A_{1}}-\sum_{i=1}^{n} \overline{B_{1 i}}\right) e^{u_{1}}-\sum_{j=1}^{m} \overline{C_{1 j}} e^{u_{2}}}{\overline{r_{2}}-\left(\overline{A_{2}}-\sum_{j=1}^{m} \overline{B_{2 j}}\right) e^{u_{2}}-\sum_{i=1}^{n} \overline{C_{2 i}} e^{u_{1}}}, \tag{122}
\end{equation*}
$$

$$
\neq 0
$$

That is, condition (b) of Lemma 7 holds. In order to verify condition (c) in the Lemma 7, by (120) and the formula for Brouwer degree, a straightforward calculation shows that $\operatorname{deg}\{J Q N u, \operatorname{Ker} L \cap \partial \Omega, 0\}$

$$
\begin{align*}
&=\operatorname{sign}\left\{\left(\sum_{j=1}^{m} \overline{C_{1 j}} \sum_{i=1}^{n} \overline{C_{2 i}}-\left(\overline{A_{1}}-\sum_{i=1}^{n} \overline{B_{1 i}}\right)\right.\right.  \tag{123}\\
&\left.\left.\times\left(\overline{A_{2}}-\sum_{j=1}^{m} \overline{B_{2 j}}\right)\right) e^{\left(u_{1}^{*}+u_{2}^{*}\right)}\right\} \neq 0 .
\end{align*}
$$

By now we have proved that all requirements in Lemma 7 hold. Hence system (82) has at least one $\omega$-periodic solution, say $\left(u_{1}^{*}, u_{2}^{*}\right)^{T}$. Setting $y_{1}^{*}(t)=e^{u_{1}^{*}(t)}, y_{2}^{*}(t)=e^{u_{2}^{*}(t)}$, then $\left(y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}$ has at least one positive $\omega$-periodic solution of systems (10) and (12). Furthermore, setting $x_{1}^{*}(t)=$ $\prod_{0<t_{k}<t}\left(1+\theta_{1 k}\right) y_{1}^{*}(t), x_{2}^{*}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{2 k}\right) y_{2}^{*}(t)$, then
$\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ has at least one positive $\omega$-periodic solution of systems (2) and (4). If ( $H_{8}$ ) holds, similarly we can prove that systems (2) and (4) have at least one positive $\omega$-periodic solution. The proof of Theorem 14 is complete.

We now proceed to the discussion on the uniqueness and global stability of the $\omega$-periodic solution $x^{*}(t)$ in Theorem 14. It is immediate that if $x^{*}(t)$ is globally asymptotically stable, then $x^{*}(t)$ is unique in fact.

Theorem 15. In addition to $\left(H_{1}\right)-\left(H_{3}\right)$, assume further that

$$
\left(H_{9}\right) F_{1}^{L} F_{2}^{L}>G_{1}^{M} G_{2}^{M} .
$$

Then systems (2) and (4) have a unique positive $\omega$ periodic solution $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ which is globally asymptotically stable.

Proof. Letting $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ be a positive $\omega$-periodic solution of (2) and (4), then $y^{*}(t)=$ $\left(y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}\left(\right.$ where $y_{l}^{*}(t)=\prod_{0<t_{k}<t}\left(1+\theta_{l k}\right)^{-1} x_{l}^{*}(t)$, $l=1,2$ ) is the positive $\omega$-periodic solution of systems (10) and (12), and let $y_{l}(t)=\left(y_{1}(t), y_{2}(t)\right)^{T}$ be any positive solution of system (10) with the initial conditions (12). It follows from Theorem 14 that there exist positive constants $T, r_{l}, R_{l}$, such that for all $t \geq T$

$$
\begin{equation*}
r_{l} \leq y_{l}^{*}(t) \leq R_{l}, \quad l=1,2 . \tag{124}
\end{equation*}
$$

By the assumptions of Theorem 14, we can obtain $F_{1}^{L} F_{2}^{L}>$ $G_{1}^{M} G_{2}^{M}$; then there exist constants $\alpha_{3}>0, \alpha_{4}>0$; we can choose a positive constant $\varepsilon$ such that

$$
\begin{equation*}
F_{1}^{L} \alpha_{3}-G_{2}^{M} \alpha_{4}=\varepsilon, \quad F_{2}^{L} \alpha_{4}-G_{1}^{M} \alpha_{3}=\varepsilon \tag{125}
\end{equation*}
$$

In the following, we always assume that $\alpha_{3}$ and $\alpha_{4}$ satisfy (67). We define

$$
\begin{equation*}
V_{1}(t)=\alpha_{3}\left|\ln y_{1}(t)-\ln y_{1}^{*}(t)\right|+\alpha_{4}\left|\ln y_{2}(t)-\ln y_{2}^{*}(t)\right| . \tag{126}
\end{equation*}
$$

Calculating the upper right derivative of $V_{1}(t)$ along solutions of (10), it follows that

$$
\begin{aligned}
& D^{+} V_{1}(t) \\
& =\alpha_{3}\left(\frac{\dot{y}_{1}(t)}{y_{1}(t)}-\frac{\dot{y}_{1}^{*}(t)}{y_{1}^{*}(t)}\right) \operatorname{sgn}\left(y_{1}(t)-y_{1}^{*}(t)\right) \\
& \quad+\alpha_{4}\left(\frac{\dot{y}_{2}(t)}{y_{2}(t)}-\frac{\dot{y}_{2}^{*}(t)}{y_{2}^{*}(t)}\right) \operatorname{sgn}\left(y_{2}(t)-y_{2}^{*}(t)\right) \\
& \leq \\
& \operatorname{sgn}\left(y_{1}(t)-y_{1}^{*}(t)\right) \alpha_{3} \\
& \quad \times\left\{-A_{1}(t)\left(y_{1}(t)-y_{1}^{*}(t)\right)\right. \\
& \quad+\sum_{i=1}^{n} B_{1 i}(t)\left(y_{1}\left(t-\tau_{i}(t)\right)-y_{1}^{*}\left(t-\tau_{i}(t)\right)\right)
\end{aligned}
$$

$$
\begin{align*}
& -\sum_{j=1}^{m} C_{1 j}(t)\left(y_{2}\left(t-\delta_{j}(t)\right)\right. \\
& \left.\left.-y_{2}^{*}\left(t-\delta_{j}(t)\right)\right)\right\} \\
& +\operatorname{sgn}\left(y_{2}(t)-y_{2}^{*}(t)\right) \alpha_{4} \\
& \times\left\{-A_{2}(t)\left(y_{2}(t)-y_{2}^{*}(t)\right)\right. \\
& +\sum_{j=1}^{m} B_{2 j}(t)\left(y_{2}\left(t-\eta_{j}(t)\right)-y_{2}^{*}\left(t-\eta_{j}(t)\right)\right) \\
& -\sum_{i=1}^{n} C_{2 i}(t)\left(y_{1}\left(t-\sigma_{i}(t)\right)\right. \\
& \left.\left.-y_{1}^{*}\left(t-\sigma_{i}(t)\right)\right)\right\} \\
& \leq-\alpha_{3} A_{1}(t)\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& +\sum_{i=1}^{n} \alpha_{3} B_{1 i}(t)\left|y_{1}\left(t-\tau_{i}(t)\right)-y_{1}^{*}\left(t-\tau_{i}(t)\right)\right| \\
& +\sum_{j=1}^{m} \alpha_{3} C_{1 j}(t)\left|y_{2}\left(t-\delta_{j}(t)\right)-y_{2}^{*}\left(t-\delta_{j}(t)\right)\right| \\
& -\alpha_{4} A_{2}(t)\left|y_{2}(t)-y_{2}^{*}(t)\right|+\sum_{j=1}^{m} \alpha_{4} B_{2 j}(t) \\
& \times\left|y_{2}\left(t-\eta_{j}(t)\right)-y_{2}^{*}\left(t-\eta_{j}(t)\right)\right| \\
& +\sum_{i=1}^{n} \alpha_{4} C_{2 i}(t)\left|y_{1}\left(t-\sigma_{i}(t)\right)-y_{1}^{*}\left(t-\sigma_{i}(t)\right)\right| . \tag{127}
\end{align*}
$$

We also define

$$
\begin{align*}
V_{2}(t)= & \sum_{i=1}^{n} \alpha_{3} \int_{t-\tau_{i}(t)}^{t} \frac{B_{1 i}\left(\alpha_{i}(\xi)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(\xi)\right)}\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right| d \xi \\
& +\sum_{j=1}^{m} \alpha_{3} \int_{t-\delta_{j}(t)}^{t} \frac{C_{1 j}\left(\mu_{j}(\xi)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(\xi)\right)}\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right| d \xi \\
& +\sum_{j=1}^{m} \alpha_{4} \int_{t-\rho_{j}(t)}^{t} \frac{B_{2 j}\left(v_{j}(\xi)\right)}{1-\rho_{j}^{\prime}\left(v_{j}(\xi)\right)}\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right| d \xi \\
& +\sum_{i=1}^{n} \alpha_{4} \int_{t-\sigma_{i}(t)}^{t} \frac{C_{2 i}\left(\beta_{i}(\xi)\right)}{1-\sigma_{j}^{\prime}\left(\beta_{i}(\xi)\right)}\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right| d \xi \tag{128}
\end{align*}
$$

Calculating the upper right derivative of $V_{2}(t)$ along solutions of (10), it follows that

$$
\begin{aligned}
D^{+} V_{2}(t)= & \sum_{i=1}^{n} \alpha_{3} \frac{B_{1 i}\left(\alpha_{i}(t)\right)}{1-\tau_{j}^{\prime}\left(\alpha_{i}(t)\right)}\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& -\sum_{i=1}^{n} \alpha_{1} \frac{B_{1 i}(t)}{1-\tau_{j}^{\prime}(t)}\left(1-\tau_{j}^{\prime}(t)\right) \\
& \times\left|y_{1}\left(t-\tau_{i}(t)\right)-y_{1}^{*}\left(t-\tau_{i}(t)\right)\right| \\
& +\sum_{j=1}^{m} \alpha_{3} \frac{C_{1 j}\left(\mu_{j}(t)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(t)\right)}\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& -\sum_{j=1}^{m} \alpha_{1} \frac{C_{1 j}(t)}{1-\delta_{j}^{\prime}(t)}\left(1-\delta_{j}^{\prime}(t)\right) \\
& \times\left|y_{2}\left(t-\delta_{j}(t)\right)-y_{2}^{*}\left(t-\delta_{j}(t)\right)\right| \\
& +\sum_{j=1}^{m} \alpha_{4} \frac{B_{2 j}\left(v_{j}(t)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(t)\right)}\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& -\sum_{j=1}^{m} \alpha_{1} \frac{B_{2 j}(t)}{1-\eta_{j}^{\prime}(t)}\left(1-\eta_{j}^{\prime}(t)\right) \\
& \times\left|y_{2}\left(t-\eta_{j}(t)\right)-y_{2}^{*}\left(t-\eta_{j}(t)\right)\right| \\
& +\sum_{i=1}^{n} \alpha_{4} \frac{C_{2 i}\left(\beta_{i}(t)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(t)\right)}\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& -\sum_{i=1}^{n} \alpha_{2} \frac{C_{2 i}(t)}{1-\sigma_{i}^{\prime}(t)}\left(1-\sigma_{i}^{\prime}(t)\right) \\
& \times\left|y_{1}\left(t-\sigma_{i}(t)\right)-y_{1}^{*}\left(t-\sigma_{i}(t)\right)\right| .
\end{aligned}
$$

We define a Lyapunov functional $V(t)$ as follows:

$$
\begin{equation*}
V(t)=V_{1}(t)+V_{2}(t) . \tag{130}
\end{equation*}
$$

Calculating the upper right derivative of $V(t)$ along solutions of (10), it follows that

$$
\begin{aligned}
D^{+} V(t)= & -\alpha_{3} A_{1}(t)\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& +\sum_{i=1}^{n} \alpha_{3} \frac{B_{1 i}\left(\alpha_{i}(t)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)}\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& +\sum_{j=1}^{m} \alpha_{1} \frac{C_{1 j}\left(\mu_{j}(t)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(t)\right)}\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& -\alpha_{4} A_{2}(t)\left|y_{2}(t)-y_{2}^{*}(t)\right|
\end{aligned}
$$

$$
\begin{align*}
& \quad+\sum_{j=1}^{m} \alpha_{4} \frac{B_{2 j}\left(v_{j}(t)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(t)\right)}\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& \\
& +\sum_{i=1}^{n} \alpha_{2} \frac{C_{2 i}\left(\beta_{i}(t)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(t)\right)}\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& =- \\
& \quad\left[\alpha_{3} A_{1}(t)-\sum_{i=1}^{n} \alpha_{3} \frac{B_{1 i}\left(\alpha_{i}(t)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(t)\right)}\right. \\
& \\
& \left.\quad-\quad-\sum_{i=1}^{n} \alpha_{4} \frac{C_{2 i}\left(\beta_{i}(t)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(t)\right)}\right]\left|y_{1}(t)-y_{1}^{*} A_{2}(t)\right| \\
& \quad-\sum_{j=1}^{m} \alpha_{1} \frac{C_{1 j}\left(\mu_{j}(t)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(t)\right)} \\
& \left.\quad-\sum_{j=1}^{m} \alpha_{4} \frac{B_{2 j}\left(v_{j}(t)\right)}{1-\eta_{j}^{\prime}\left(v_{j}(t)\right)}\right]\left|y_{2}(t)-y_{2}^{*}(t)\right|  \tag{131}\\
& \leq\left(\alpha_{3} F_{1}^{L}-\alpha_{4} G_{2}^{M}\right)\left|y_{1}(t)-y_{1}^{*}(t)\right| \\
& \\
& \quad-\left(\alpha_{4} F_{2}^{L}-\alpha_{3} G_{1}^{M}\right)\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& =-\varepsilon\left(\left|y_{1}(t)-y_{1}^{*}(t)\right|+\left|y_{2}(t)-y_{2}^{*}(t)\right|\right) .
\end{align*}
$$

So by (131), we have

$$
\begin{gather*}
\varepsilon \int_{0}^{t}\left(\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right|+\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right|\right) d \xi  \tag{132}\\
+V(t) \leq V(0)<+\infty, \quad t \geq 0
\end{gather*}
$$

where

$$
\begin{align*}
V(0)= & \alpha_{3}\left|\ln y_{1}(0)-\ln y_{1}^{*}(0)\right|+\alpha_{4}\left|\ln y_{2}(0)-\ln y_{2}^{*}(0)\right| \\
& +\sum_{i=1}^{n} \alpha_{3} \int_{-\tau_{i}(0)}^{0} \frac{B_{1 i}\left(\alpha_{i}(\xi)\right)}{1-\tau_{i}^{\prime}\left(\alpha_{i}(\xi)\right)} \\
& \times\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right| d \xi \\
& +\sum_{j=1}^{m} \alpha_{3} \int_{-\delta_{j}(0)}^{0} \frac{C_{1 j}\left(\mu_{j}(\xi)\right)}{1-\delta_{j}^{\prime}\left(\mu_{j}(\xi)\right)} \\
& \times\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right| d \xi+\sum_{j=1}^{m} \alpha_{4} \\
& \times \int_{-\rho_{j}(0)}^{0} \frac{B_{2 j}\left(v_{j}(\xi)\right)}{1-\rho_{j}^{\prime}\left(v_{j}(\xi)\right)}\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right| d \xi \\
& +\sum_{i=1}^{n} \alpha_{4} \int_{-\sigma_{i}(0)}^{0} \frac{C_{2 i}\left(\beta_{i}(\xi)\right)}{1-\sigma_{i}^{\prime}\left(\beta_{i}(\xi)\right)} \\
& \times\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right| d \xi \tag{133}
\end{align*}
$$

which implies that

$$
\begin{equation*}
\int_{0}^{t}\left(\left|y_{1}(\xi)-y_{1}^{*}(\xi)\right|+\left|y_{2}(\xi)-y_{2}^{*}(\xi)\right|\right) d \xi \leq \frac{V(0)}{\varepsilon} \tag{134}
\end{equation*}
$$

By (134), it is obvious that $\left|y_{1}(t)-y_{1}^{*}(t)\right|+\left|y_{2}(t)-y_{2}^{*}(t)\right|$ is bounded.

On the other hand, we know that

$$
\begin{gather*}
\alpha_{3}\left|\ln y_{1}(t)-\ln y_{1}^{*}(t)\right|+\alpha_{4}\left|\ln y_{2}(t)-\ln y_{2}^{*}(t)\right| \\
\leq V(t) \leq V(0)<+\infty, \quad t \geq 0, \tag{135}
\end{gather*}
$$

which implies that

$$
\begin{align*}
& \left|\ln y_{1}(t)-\ln y_{1}^{*}(t)\right| \leq \frac{V(0)}{\alpha_{3}} \\
& \left|\ln y_{2}(t)-\ln y_{2}^{*}(t)\right| \leq \frac{V(0)}{\alpha_{4}} \tag{136}
\end{align*}
$$

which, together with (123), yield

$$
\begin{align*}
& r_{1} e^{-V(0) / \alpha_{3}} \leq y_{1}(t) \leq R_{1} e^{V(0) / \alpha_{3}}<+\infty, \\
& r_{2} e^{-V(0) / \alpha_{4}} \leq y_{2}(t) \leq R_{2} e^{V(0) / \alpha_{4}}<+\infty . \tag{137}
\end{align*}
$$

From (124) and (137), it follows that $y_{l}(t)(l=1,2)$ are bounded for $t \geq 0$. Hence, $y_{1}(t)-y_{1}^{*}(t), y_{2}(t)-y_{2}^{*}(t)$, and their derivatives remain bounded on $[0,+\infty)$. So $\mid y_{1}(t)-$ $y_{1}^{*}(t)\left|,\left|y_{2}(t)-y_{2}^{*}(t)\right|\right.$ are uniformly continuous on $[0,+\infty)$. By Lemma 11, we have

$$
\begin{align*}
\lim _{t \rightarrow+\infty} & \left|y_{l}(s)-y_{l}^{*}(s)\right| \\
& =\lim _{t \rightarrow+\infty}\left[\prod_{0<t_{k}<t}\left(1+\theta_{i k}\right)^{-1}\left|x_{l}^{*}(s)-x_{l}(s)\right|\right]=0, \tag{138}
\end{align*}
$$

$$
l=1,2 .
$$

Therefore

$$
\begin{equation*}
\lim _{t \rightarrow+\infty}\left|x_{l}(s)-x_{l}^{*}(s)\right|=0, \quad l=1,2 . \tag{139}
\end{equation*}
$$

By Theorems 7.4 and 8.2 in [30], we know that the periodic positive solution $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ is uniformly asymptotically stable. The proof of Theorem 15 is completed.

## 4. Applications

In this section, for some applications of our main results, we will consider some special cases of systems (2) and (3), which have been investigated extensively in [10].

Application 1. consider the following equations:

$$
\begin{aligned}
& y_{1}^{\prime}(t)=y_{1}(t)\left[r_{1}(t)-a_{1}(t) y_{1}(t)+\sum_{i=1}^{n} b_{1 i}(t) y_{1}\left(t-\tau_{i}(t)\right)\right. \\
& \left.-\sum_{j=1}^{m} c_{1 j}(t) y_{2}\left(t-\rho_{j}(t)\right)\right], \\
& y_{2}^{\prime}(t)=y_{2}(t)\left[r_{2}(t)-a_{2}(t) y_{2}(t)+\sum_{j=1}^{m} b_{2 j}(t) y_{2}\left(t-\eta_{j}(t)\right)\right. \\
& y_{i}(0)>0, \quad i=1,2, \\
& \left.-\sum_{i=1}^{n} c_{2 i}(t) y_{1}\left(t-\sigma_{i}(t)\right)\right], \\
& y_{1}^{\prime}(t)=y_{1}(t)\left[r_{1}(t)-a_{1}(t) y_{1}(t)-\sum_{i=1}^{n} b_{1 i}(t) y_{1}\left(t-\tau_{i}(t)\right)\right. \\
& \quad[140) \\
& \left.-\sum_{j=1}^{m} c_{1 j}(t) y_{2}\left(t-\rho_{j}(t)\right)\right], \\
& y_{2}^{\prime}(t)=y_{2}(t)\left[\begin{array}{r}
r_{2}(t)-a_{2}(t) y_{2}(t)-\sum_{j=1}^{m} b_{2 j}(t) y_{2}\left(t-\eta_{j}(t)\right)
\end{array}\right.
\end{aligned}
$$

$$
\begin{equation*}
y_{i}(0)>0, \quad i=1,2, \tag{141}
\end{equation*}
$$

which are special cases of systems (2) and (3) without impulse, respectively. By applying Theorems $12-15$ to systems (140) and (141), respectively, we obtain the following theorems.

Theorem 16. In addition to $\left(H_{1}\right)$, assume that the following conditions hold:

$$
\left(H_{10}\right) \overline{r_{1}} F_{1}^{L}>\overline{r_{2}} G_{2}^{M}, \quad \overline{r_{2}} F_{2}^{L}>\overline{r_{1}} G_{1}^{M}
$$

Then system (140) has a unique positive $\omega$-periodic solution $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ which is globally asymptotically stable, where $F_{1}(t), F_{2}(t), G_{1}(t)$, and $G_{2}(t)$ are defined in (22).

Proof. It is similar to the proof of Theorems 12 and 13, so we omit the details here.

Theorem 17. In addition to $\left(H_{1}\right)$, assume further that

$$
\left(H_{11}\right) \overline{r_{1}} F_{1}^{* L}>\overline{r_{2}} G_{2}^{M}, \overline{r_{2}} F_{2}^{* L}>\overline{r_{1}} G_{1}^{M} .
$$

Then system (140) has a unique positive $\omega$-periodic solution $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ which is globally asymptotically stable, where $F_{1}^{*}(t), F_{2}^{*}(t), G_{1}(t)$, and $G_{2}(t)$ are defined in (22).

Proof. It is similar to the proof of Theorems 14 and 15 , so we omit the details here.

We consider the following systems:

$$
\begin{align*}
& x^{\prime}(t)=x(t)\left[r(t)-a(t) x(t)+\sum_{i=1}^{n} b_{i}(t) x\left(t-\tau_{i}(t)\right)\right],  \tag{142}\\
& x^{\prime}(t)=x(t)\left[r(t)-a(t) x(t)-\sum_{i=1}^{n} b_{i}(t) x\left(t-\tau_{i}(t)\right)\right], \tag{143}
\end{align*}
$$

which are special cases of systems (140) and (141), respectively. From Theorems 17 and 18, we have the following corollary.

Corollary 18. In addition to $\left(H_{1}\right)$, assume that the following condition holds:

$$
\left(H_{12}\right) a(t)-\sum_{i=1}^{n}\left(b_{i}\left(\mu_{i}(t)\right)\right) /\left(1-\tau_{i}^{\prime}\left(\mu_{i}(t)\right)\right)>0 .
$$

Then systems (142) and (143) have a unique positive $\omega$ periodic solution $x^{*}(t)$ which is globally asymptotically stable, where $\mu_{i}(t)$ are the inverses of functions $t-\tau_{i}(t)$.

Proof. It is similar to the proof of Theorems 12 and 13, so we omit the details here.
Application 2. Let us consider two delayed two-species competitive systems:

$$
\begin{align*}
& y_{1}^{\prime}(t)=y_{1}(t)\left[r_{1}(t)-a_{1}(t) y_{1}(t)\right. \\
& \left.+b_{1}(t) y_{1}(t-\tau(t))-c_{1}(t) y_{2}(t-\delta(t))\right], \\
& y_{2}^{\prime}(t)=y_{2}(t)\left[r_{2}(t)-a_{2}(t) y_{2}(t)\right. \\
& \left.+b_{2}(t) y_{2}(t-\eta(t))-c_{2}(t) y_{1}(t-\sigma(t))\right], \\
& y_{i}(0)>0, \quad i=1,2, \\
& y_{1}^{\prime}(t)=y_{1}(t)\left[r_{1}(t)-a_{1}(t) y_{1}(t)\right. \\
& \left.-b_{1}(t) y_{1}(t-\tau(t))-c_{1}(t) y_{2}(t-\delta(t))\right], \\
& y_{2}^{\prime}(t)=y_{2}(t)\left[r_{2}(t)-a_{2}(t) y_{2}(t)\right. \\
& \left.-b_{2}(t) y_{2}(t-\eta(t))-c_{2}(t) y_{1}(t-\sigma(t))\right], \\
& y_{i}(0)>0, \quad i=1,2, \tag{145}
\end{align*}
$$

which are special cases of systems (2) and (3) without impulse and $i=j=1$, respectively. By applying Theorems 12-15 to systems (144) and (145), respectively, we obtain the following theorems.

Theorem 19. In addition to $\left(H_{1}\right)$, assume that the following conditions hold:

$$
\left(H_{12}\right) \overline{r_{1}} C_{1}^{L}>\overline{r_{2}} D_{2}^{M}, \overline{r_{2}} C_{2}^{L}>\overline{r_{1}} D_{1}^{M} .
$$

Then system (144) has a unique positive $\omega$-periodic solution $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ which is globally asymptotically stable, where $C_{1}(t), C_{2}(t), D_{1}(t)$, are $D_{2}(t)$ are defined as follow:

$$
\begin{array}{ll}
C_{1}(t)=a_{1}(t)-\frac{b_{1}(\alpha(t))}{1-\tau^{\prime}(\alpha(t))}, & D_{1}(t)=\frac{C_{1}(\mu(t))}{1-\delta^{\prime}(\mu(t))} \\
C_{2}(t)=a_{2}(t)-\frac{b_{2}(\nu(t))}{1-\eta^{\prime}(\nu(t))}, & D_{2}(t)=\frac{C_{2}(\beta(t))}{1-\sigma^{\prime}(\beta(t))} \tag{146}
\end{array}
$$

And $\alpha(t), \beta(t), \mu(t)$, and $\nu(t)$ represent the inverse function of $t-\tau(t), t-\sigma(t), t-\delta(t)$, and $t-\eta(t)$, respectively.

Proof. It is similar to the proof of Theorems 12 and 13, so we omit the details here.

Theorem 20. In addition to $\left(H_{1}\right)$, assume further that

$$
\left(H_{13}\right) \overline{r_{1}} C_{1}^{* L}>\overline{r_{2}} D_{2}^{M}, \overline{r_{2}} C_{2}^{* L}>\overline{r_{1}} G_{1}^{M} .
$$

Then system (145) has a unique positive $\omega$-periodic solution $x^{*}(t)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)^{T}$ which is globally asymptotically stable, where $C_{1}^{*}(t), C_{2}^{*}(t), D_{1}(t)$, and $D_{2}(t)$ are defined as follow:

$$
\begin{array}{ll}
C_{1}^{*}(t)=a_{1}(t)+\frac{b_{1}(\alpha(t))}{1-\tau^{\prime}(\alpha(t))}, & D_{1}(t)=\frac{C_{1}(\mu(t))}{1-\delta^{\prime}(\mu(t))}, \\
C_{2}^{*}(t)=a_{2}(t)+\frac{b_{2}(\nu(t))}{1-\eta^{\prime}(\nu(t))}, & D_{2}(t)=\frac{C_{2}(\beta(t))}{1-\sigma^{\prime}(\beta(t))} . \tag{147}
\end{array}
$$

And $\alpha(t), \beta(t), \mu(t)$, and $\nu(t)$ represent the inverse function of $t-\tau(t), t-\sigma(t), t-\delta(t)$, and $t-\eta(t)$, respectively.

Proof. It is similar to the proof of Theorems 14 and 15, so we omit the details here.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# The Influence Analysis of Number of Functional Logistics Service Providers on Quality Supervision Game in LSSC with Compensation Strategy 

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#### Abstract

There is a close link between the number of suppliers and the quality of products including service. However, there is a research gap in this field. Particularly, the problem whether this link would be affected by different supply chain profit distribution policies is lack of in-depth research. In this paper, a basic game model for quality supervision game without FLSPs' competition (Model I) in a logistics service supply chain (LSSC) is established first. Model I adopts a mixed payment contract and is composed of a logistics service integrator (LSI) and functional logistics service provider (FLSP). The mixed-strategy Nash equilibrium of Model I is presented. Model II considering competition among FLSPs is then built based on Model I, and the new mixedstrategy Nash equilibrium is provided. Results show that under competition the ordinary mixed payment contract cannot make quality supervision game parameters all optimized. Therefore, Model III under a compensation mechanism is established based on Model II, and the range of the compensation value is calculated. Furthermore, we propose three kinds of concrete compensation mechanisms, which are fixed, linear, and nonlinear compensation mechanism. And the optimal compensation mechanism for a LSI with different numbers of FLSPs is provided.


## 1. Introduction

In recent years, service outsourcing such as IT service outsourcing, financial service outsourcing, and logistics service outsourcing is becoming increasingly popular. Many service integrators have established long-term partnership with service providers to provide customers with integrated services. In the process, service supply chains have been formed. Take logistics services outsourcing as an example; a logistics service integrator (LSI) integrates multiple functional logistics service providers' (FLSP) service capabilities, providing customers with integrated logistics services, and the LSI and FLSP form the logistics service supply chain [1]. For example, P\&G Logistics Group Co., Ltd., in Guangzhou, as the largest LSI in China, successfully undertakes the logistics business of its customers (Procter \& Gamble, Philips, etc.) by integrating more than 500 warehousing suppliers, 1,200 road transport suppliers, and 500 manpower loading and unloading operation teams.

A supply chain is a typical system that requires coordination, and LSSC is not an exception. LSSC is a service supply chain whose core is capacity cooperation [1]. The LSI strengthening the quality supervision is the key factor to achieving successful cooperation within a service supply chain [2]. As the LSI usually has to integrate the service capabilities of multiple FLSPs when providing logistics services for customers, not only would the number of FLSPs affect whether the demand for services can be satisfied successfully but also the competition among them would directly affect the LSI's quality supervision results. The existing supply chain quality management research results show that there is a close link between the number of suppliers and the quality of products (services) [3]. On the other hand, there is a research gap in the relationship between the number of suppliers and the quality supervising strength of manufacturers (or service integrators). In particular, an in-depth research has not yet been conducted on the question whether different supply
chain profit distribution policies would have an impact on these two factors.

The necessity of this study in the industrial practice has also been confirmed. From the industrial practice, many LSIs in China, such as Baogong logistics company, Tianjin Baoyun Logistics company, and Beijing Leader logistics company, obtain a high level of service quality via reinforcing the competition among multiple FLSPs in the process of completing logistics service orders in the goal of achieving optimal service performance. The competition among multiple FLSPs indeed helps to reduce the LSI's supervision difficulty level of logistics service quality, but it does not significantly improve the FLSPs' logistics service quality and the cooperation initiatives among them. P\&G Logistics Group Co., Ltd., for example, had to replace at least $10 \%$ of the FLSPs by the end of each year in order to improve the service quality the following year [4]. This leads us to an interesting question: why the competition among multiple FLSPs does not effectively improve the logistics service quality? Can we introduce a kind of profit compensation mechanism to reduce the LSI's supervision difficulty level as well as improve FLSPs' service quality? Assuming this kind of profit compensation mechanism exists, what is the relationship between the kind of mechanism and the number of FLSPs? These issues, which have not been studied and discussed in the industrial practice, are to be discussed in depth in this paper.

Considering the difficulty of measuring service, a model (Model I) for quality supervision game in LSSC without FLSPs' competition was initially established in this study based on the model developed by Liu and Xie [5]. Model I assumes that the supply chain adopts a mixed payment contract and the optimal supervision probability of the LSI and the optimal compliance probability of FLSP $i$ are obtained. Based on Model I and taking the impact of competition among FLSPs on the effect of quality supervision game into consideration [3], we introduced the competition factor to establish Model II (the quality supervision game model for LSSC with FLSPs' competition). By solving and analyzing Model II, we found that, under competition, the ordinary mixed payment contract cannot make quality supervision game parameters all optimized. Thus, we introduced compensation mechanism to optimize the effect of supply chain quality supervision and established Model III: the quality supervision game model with FLSPs' competition under a compensation mechanism. We have obtained some important findings from the study of Model III. For example, when the number of FLSPs is small, the LSI should choose nonlinear compensation mechanism; when the number of FLSPs is large, the LSI should choose linear compensation mechanism; when the number of FLSPs is in other intervals, no optimal compensation mechanism exists. In the third scenario, the LSI has to make a trade-off between the LSI's optimal supervision probability and FLSPs' optimal compliance probability when choosing a compensation mechanism. Figure 1 is the research process in this paper.

This paper is organized as follows. Section 2 provides the literature review of this paper. The quality supervision game model for LSSC without FLSPs' competition (Model I) is presented in Section 3 and the Nash equilibrium is obtained.

In this section, the quality supervision game model for LSSC with FLSPs' competition (Model II) is presented by introducing competition among FLSPs. Via introducing compensation mechanism, the quality supervision game model with FLSPs' competition under a compensation mechanism (Model III) is presented in Section 4. MATLAB 8.0 software is utilized for numerical analysis in Section 5 to verify the main conclusions. The conclusions and management implications of this paper are presented in Section 6.

## 2. Literature Review

The existing research on the impact of the number of competitors on the quality supervision game for a supply chain mainly focuses on the manufacturing supply chain. However, research on the service supply chain, especially LSSC, is relatively inadequate. This section (Literature Review) mainly includes three aspects: (1) review of the literature on supply chain quality supervision game, (2) summary of research on competition among providers for manufacturing supply chain, and (3) review of the literature on compensation contract.
2.1. Supply Chain Quality Supervision Game. Quality issues in the supply chain, mostly those with regard to the significance and importance of quality in the manufacturing supply chain $[6,7]$ and the contract design of quality control, have been discussed extensively in relevant literature. The International Journal of Production Economics published a special issue for supply chain quality management including eight papers that focused on theoretical models and empirical studies [4]. The risk involved in quality control game has also been discussed in recent years. For example, Baiman et al. discussed quality control game based on the moral hazard in supply chain cooperation [8]. Lim established a quality control model with asymmetric information [9]. Kaya and Özer discussed the quality risk of outsourcing [10]. Chao et al. discussed two contractual agreements by which product recall costs can be shared between the manufacturer and suppliers to induce quality improvement for cases where information on the quality of suppliers' products is not accessible to the manufacturer [11].

The increase in the number of studies on service supply chain has caused quality control game in LSSC to be continuously studied. Hertz and Macquet established a Nash equilibrium model for the third-party logistics service providers and subcontractors and analyzed the impact on quality control as contract parameters change [12]. Jayaram and Tan believed that supervising the service quality of the third-party logistics (TPL) is important [13]. Considering that the quality of logistics services is difficult to measure, Liu et al. developed a multiperiod quality control game model in twoechelon LSSC and a model in three-echelon LSSC [2]. Liu and Xie studied the quality decision problems of LSIs and FLSPs when FLSPs provide service quality guarantee [5]. However, these studies did not consider the influence of the number of FLSPs on quality decisions.


Figure 1: The research process of this paper.

### 2.2. The Impact of FLSPs' Competition on the Supply Chain

 Performance. The study on the impact of providers' competition on the supply chain mainly focuses on the decision of the providers' number. Some literature shows that the integrator gains advantage by owning multiple providers whether requirements are determined or stochastic [14-16]. Many scholars discussed how to determine the optimal number of suppliers, and the common modeling methods include decision trees $[17,18]$ and goal programming method $[19,20]$. In recent years, many literatures have discussed the supplier quantity decision problem in a competitive environment [21-23], considering that the number of businesses of each supplier is related to that of others.As for service integrators, at present, researches mainly focus on selection and evaluation of the service outsourcing suppliers, and the selection of multiservice outsourcing suppliers is considered as well as the multisourcing suppliers selection [24, 25]. On the basis of the basic quality control game model in a two-echelon logistics service supply chain, Liu et al. considered the impact of the competition among FLSPs on the quality control and found that the quality control can be strengthened by reinforcing monitoring and control in the process of logistics cooperation and establishing a competition mechanism among FLSPs and so forth [2]. But there is little literature regarding how to design a contract to improve the efficiency of the supply chain with a certain number of providers.
2.3. Compensation Contract. Compensation contract is widely used in supply chain coordination. It can be used to overcome information asymmetry between buyers and sellers [26], to motivate retailers to increase the order quantity [27], to increase sales effort [28, 29], and to improve the performance of decentralized supply chains [30-32]. In addition, Starbird studied the award, punishment, and supervision strategies that the buyer applies to suppliers' quality, and he found that reward and punishment can be
substituted for each other to motivate providers [33]. Zhang et al. compared buyback contract, target compensation contracts, and incremental buyback contract when the retailer is risk-averse [34]. In the model presented by Taylor, there were two forms of compensation, linear (pay a certain amount of compensation per unit) and nonlinear (pay a certain amount of compensation per unit when a certain sales target is reached) [29]. The study showed that compensation would improve the retailer's sales effort. However, these studies about compensation have rarely been applied to service supply chain, and few scholars have applied compensation contract to improve the supply chain performance when considering competition among providers.

Based on the literature review above, we have found that the existing research has two deficiencies as follows.
(1) Research on supply chain quality supervision game mainly focuses on product supply chain, and research on quality supervision-oriented service supply chain is inadequate, especially the research on the impact of multiple FLSPs' competition on the effect of quality supervision in service supply chain.
(2) The existing research on the quality supervision game in LSSC explored the impact of quality behaviors (such as quality commitment behavior) of a LSI and a FLSP on the performance of the supply chain but did not consider the impact of multiple FLSPs' competition on the performance of the supply chain [2, 5].

This study aims to address the two problems mentioned above. We have established a quality supervision game model for LSSC with FLSPs' competition. Then by introducing compensation mechanism, the quality supervision game model with FLSPs' compensation under a compensation mechanism is presented, and the relationship between the number of FLSPs and the type of compensation mechanisms
is discussed. The results of this research can serve as a scientific reference for quality decision making of LSIs.

## 3. The Quality Supervision Game Models for LSSC without Compensation

Some of the important assumptions and variables of Models I and II are provided in Section 3.1. The quality supervision game model for LSSC without FLSPs' competition (model I) is established in Section 3.2. The section also presents the optimal supervision probability of the LSI and the optimal compliance probability of the FLSP. Based on Model I, the quality supervision game model for the LSSC with FLSPs' competition (Model II) is studied in Section 3.3. And the analytical results of Model II are compared with that of Model II in this section.
3.1. Assumptions and Parameters. We assume that a LSSC is composed of a LSI and a FLSP, and the LSI is in dominant position. The logistics capacity required by the LSI is completely provided by the FLSP, and the FLSP has the ability to complete the LSI's outsourced tasks. The FLSP's effort level determines its service quality, and its service quality directly determines the total revenue of the LSSC. Some of the assumptions of the model are described below.

Assumption 1. The FLSP provides logistics services for the LSI. The total revenue $\pi(p)$ of the LSSC is a function of the FLSP's service quality, and the FLSP's service quality is a function of the FLSP's effort level $p$. We assume that $\pi(p)=$ $g(p)+\varepsilon$ and $\varepsilon \sim N\left(0, \sigma_{\varepsilon}^{2}\right)$ in which $g(p)$ is a function of $p$ and $\varepsilon$ is the impact of the environment on total revenue.

Assumption 2. The FLSP in cooperation with the LSI has two choices: comply with or break the contract. The service cost $c\left(p_{i}\right)$ of the FLSP is a function of its effort level. When the FLSP complies with the contract, the effort level is $p_{1}$, the total revenue of the LSSC is $\pi\left(p_{1}\right)=g\left(p_{1}\right)+\varepsilon$, and the service cost of the FLSP is $c\left(p_{1}\right)$. When the FLSP opts to break the contract, the effort level is $p_{2}$, the total revenue of the LSSC is $\pi\left(p_{2}\right)=g\left(p_{2}\right)+\varepsilon$, and the service cost of the FLSP is $c\left(p_{2}\right)$. The FLSP can acquire a certain amount of self-interest $\varphi$ from breaking the contract.

Assumption 3. The LSI also has two choices: to supervise the service quality of the FLSP or not. When the LSI supervises quality, the supervision cost $f(q)$ is not only associated with its supervision effort but is also affected by the external environment. We let $f(q)=h(q)+\eta$ and $\eta \sim N\left(0, \sigma_{\eta}^{2}\right)$, where $h(q)$ is a function of $q, q$ is the supervision effort level of the LSI, and $\eta$ is the impact of the environment on the LSI's supervision cost. Given that service is difficult to measure [35], breach of the contract by the FLSP may not be discovered by the LSI. The probability of the FLSP breaking the contract and being discovered is assumed to be $\rho$, and the penalty of this situation is $\theta$.

Assumption 4. We assume that the LSI provides a mixed payment contract to the FLSP. The revenue of the FLSP in the contract can be divided into two parts. One is a fixed payment, and the other is a specific percentage of the LSSC's total net income, which is equal to $\omega+r[\pi(p)-\omega]$. Here, $p$ represents the FLSP's effort level, $\omega$ is the fixed payment, and $r$ is the coefficient of the revenue that the FLSP obtains from cooperation. Accordingly, the income of the LSI is (1-$r)[\pi(p)-\omega]$.

Assumption 5. $\varepsilon$ and $\eta$ are independent of each other; that is, $\operatorname{Cov}(\varepsilon, \eta)=0$.

The notations for the model are summarized in the Appendix.
3.2. Model I: The Quality Supervision Game Model for LSSC without FLSPs' Competition. When the FLSP opts to comply with the contract, the effort level of the FLSP is $p_{1}$, the service cost is $c\left(p_{1}\right)$, the total revenue of the LSSC is $\pi\left(p_{1}\right)$, and the expected revenue of the FLSP is $(1-r) \omega+r \pi\left(p_{1}\right)-c\left(p_{1}\right)$. When the LSI supervises service quality, the supervision cost is $f(q)$ and expected revenue is $(1-r)\left[\pi\left(p_{1}\right)-\omega\right]-f(q)$; otherwise, the expected revenue is $(1-r)\left[\pi\left(p_{1}\right)-\omega\right]$.

When the FLSP opts to break the contract, the effort exerted by the FLSP is $p_{2}$, the service cost is $c\left(p_{2}\right)$, the total revenue of the LSSC is $\pi\left(p_{2}\right)$, and the FLSP obtains personal interests $\varphi$. When the LSI supervises service quality, the supervision cost is $f(q)$, the probability of the FLSP's breach of the contract being discovered is $\rho$, and the FLSP's penalty from the LSI is $\theta$. The expected revenue of the FLSP is $(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi-\rho \theta$, and the expected revenue of the LSI is $(1-r)\left[\pi\left(p_{2}\right)-\omega\right]-f(q)+\rho \theta$. When the LSI does not supervise service quality, the expected revenue of the FLSP is $(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi$, and the expected revenue of the LSI is $(1-r)\left[\pi\left(p_{2}\right)-\omega\right]$. The expected payoff matrix is shown in Table 1, in which $(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi>$ $(1-r) \omega+r \pi\left(p_{1}\right)-c\left(p_{1}\right)$.

From Table 1, we can see that a pure strategy Nash equilibrium does not exist. Therefore, we proceed to find a mixed strategy Nash equilibrium. We assume that the supervision probability of the LSI is $x$, nonsupervision probability is $(1-x)$, compliance probability of the FLSP is $y$, and noncompliance probability is $(1-y)$. The expected revenue of the LSI is

$$
\begin{align*}
& \Pi_{I}^{0}(x, y) \\
&= x\left\{y\left[(1-r)\left(\pi\left(p_{1}\right)-\omega\right)-f(q)\right]\right. \\
&\left.\quad+(1-y)\left[(1-r)\left(\pi\left(p_{2}\right)-\omega\right)-f(q)+\rho \theta\right]\right\} \\
&+(1-x) \\
& \times\left\{y\left[(1-r)\left(\pi\left(p_{1}\right)-\omega\right)\right]+(1-y)\left[(1-r)\left(\pi\left(p_{2}\right)-\omega\right)\right]\right\} \\
&= {[\rho \theta-f(q)] x+(1-r)\left[\pi\left(p_{1}\right)-\pi\left(p_{2}\right)\right] y } \\
&-\rho \theta x y+(1-r) \pi\left(p_{2}\right)-(1-r) \omega . \tag{1}
\end{align*}
$$

The maximum expected revenue of the LSI meets the condition that $\partial \Pi_{I}^{0}(x, y) / \partial x=0$; that is,

$$
\begin{equation*}
y_{0}^{*}=\frac{\rho \theta-f(q)}{\rho \theta}=\frac{A}{\rho \theta} \tag{2}
\end{equation*}
$$

Equation (2) shows that $y_{0}^{*}$ is positively correlated to $\rho$ and $\theta$ and negatively correlated to $f(q)$.

The expected revenue of the FLSP is

$$
\begin{align*}
\Pi_{F}^{0} & (x, y) \\
= & x\left\{y\left[(1-r) \omega+r \pi\left(p_{1}\right)-c\left(p_{1}\right)\right]\right. \\
& \left.+(1-y)\left[(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi-\rho \theta\right]\right\} \\
+ & (1-x) \\
\times & \left\{y\left[(1-r) \omega+r \pi\left(p_{1}\right)-c\left(p_{1}\right)\right]+(1-y)\right. \\
& \left.\times\left[(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi\right]\right\} \\
= & -\rho \theta x+\left\{r\left[\pi\left(p_{1}\right)-\pi\left(p_{2}\right)\right]-\left[c\left(p_{1}\right)-c\left(p_{2}\right)\right]-\varphi\right\} y \\
& +\rho \theta x y \\
& +(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi . \tag{3}
\end{align*}
$$

The maximum expected revenue of the FLSP meets the condition that $\partial \Pi_{F}^{0}(x, y) / \partial y=0$; that is,

$$
\begin{equation*}
x_{0}^{*}=\frac{\varphi-r\left[\pi\left(p_{1}\right)-\pi\left(p_{2}\right)\right]+\left[c\left(p_{1}\right)-c\left(p_{2}\right)\right]}{\rho \theta}=\frac{-T}{\rho \theta} . \tag{4}
\end{equation*}
$$

Equation (4) shows that $x_{0}^{*}$ is negatively correlated to $\rho$, $\theta, r$, and $\left[\pi\left(p_{1}\right)-\pi\left(p_{2}\right)\right]$ and positively correlated to $\varphi$ and $\left[c\left(p_{1}\right)-c\left(p_{2}\right)\right]$.

### 3.3. Model II: The Quality Supervision Game Model for LSSC

 with FLSPs' Competition. Now the model will be extended to the case of multiple FLSPs. It is assumed that the LSI has multiple upstream FLSPs, and there is mutual influence among FLSPs. We assume that the LSI takes the following punishment strategy: if FLSP $i$ cannot achieve the required service quality according to agreement, then the LSI will choose other FLSPs, reduce the cooperation level with FLSP $i$, and even ultimately abandon the cooperation with FLSP $i$. Therefore, the presence of competitors makes any FLSP's default cost increase and expected revenue decrease accordingly. This loss of FLSP $i$ is $F_{i}(N)=\gamma N b R$, where $b$ denotes the competitive coefficient, that is, the degree of mutual influence among FLSPs, and $0<b<1$. A large $b$ means that the business competition among FLSPs is fierce; in particular, there is not any mutual effect among FLSPs when $b=0$. $R$ indicates the service quality level of FLSP $i$, which can be informed from the FLSP's past experience of cooperation or credibility. It is assumed in this paper that all the FLSPs are homogeneous; that is, their service qualities are at the same level. The expected revenue matrix of the LSI and FLSP $i$ considering FLSP's competition is shown in Table 2.The expected revenue function of the LSI in this case is

$$
\begin{align*}
\Pi_{I}^{1}(x, y)= & \Pi_{I}^{0}(x, y) \\
= & {[\rho \theta-f(q)] x+(1-r)\left[\pi\left(p_{1}\right)-\pi\left(p_{2}\right)\right] y }  \tag{5}\\
& -\rho \theta x y+(1-r) \pi\left(p_{2}\right)-(1-r) \omega .
\end{align*}
$$

We let $\partial \Pi_{I}^{1}(x, y) / \partial x=0$ to obtain

$$
\begin{equation*}
y_{1}^{*}=\frac{A}{\rho \theta} \tag{6}
\end{equation*}
$$

The expected revenue function of FLSP $i$ is

$$
\begin{align*}
& \Pi_{F_{i}}^{1}(x, y) \\
& =\begin{aligned}
&= \Pi_{I}^{0}(x, y)-(1-y) F_{i}(N) \\
&=-\rho \theta x+ \\
& \quad\left\{r\left[\pi\left(p_{1}\right)-\pi\left(p_{2}\right)\right]-\left[c\left(p_{1}\right)-c\left(p_{2}\right)\right]\right. \\
&\quad-\varphi+\gamma N b R\} y \\
&+\rho \theta x y+ \\
& \quad(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi-\gamma N b R .
\end{aligned}
\end{align*}
$$

We let $\partial \Pi_{F_{i}}^{1}(x, y) / \partial y=0$ to obtain

$$
\begin{equation*}
x_{1}^{*}=\frac{-T-\gamma N b R}{\rho \theta} \tag{8}
\end{equation*}
$$

Comparing (2), (4), (6), and (8), we find that when taking into consideration the competition among FLSPs, the mixed payment contract reduces the LSI's optimal supervision probability, but it does not increase FLSP i's optimal compliance probability.

## 4. Model III: The Quality Supervision Game Model with FLSPs' Competition under a Compensation Mechanism

The analysis in Section 3.3 shows that, under the abovementioned mixed payment contract, the introduction of FLSPs' competition makes the LSI's optimal supervision probability decrease, but FLSP i's optimal compliance probability remains unchanged. In Section 4.1, we derive a compensation mechanism which optimizes the LSI's optimal supervision probability, FLSP $i$ 's optimal compliance probability, and the expected revenue of the LSI and FLSP $i$. In Section 4.2, we design three specific compensation mechanisms, that is, fixed compensation mechanism, linear compensation mechanism, and nonlinear compensation mechanism, and calculate their respective range of application. Then a comparison is made among the three compensation mechanisms in Section 4.3, and the results can serve as a scientific reference for compensation decision making in the LSI.
4.1. Modeling and Solving. It is assumed that $\alpha(\alpha>0)$ is the compensation that the LSI pays to FLSP $i$ when FLSP $i$ complies with the contract (including the situation that FLSP $i$ 's breach of the contract is not discovered and that FLSP $i$

Table 1: Expected revenue matrix of the LSI and the FLSP.

| LSI | FLSP |  |
| :--- | :--- | :--- |
|  | Complies with the contract | Breaks the contract |
| Supervises | $(1-r) \omega+r \pi\left(p_{1}\right)-c\left(p_{1}\right)$, | $(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi-\rho \theta$, |
|  | $(1-r)\left[\pi\left(p_{1}\right)-\omega\right]-f(q)$ | $(1-r)\left[\pi\left(p_{2}\right)-\omega\right]-f(q)+\rho \theta$ |
| Does not supervise | $(1-r) \omega+r \pi\left(p_{1}\right)-c\left(p_{1}\right)$, | $(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi$, |
|  | $(1-r)\left[\pi\left(p_{1}\right)-\omega\right]$ | $(1-r)\left[\pi\left(p_{2}\right)-\omega\right]$ |

Table 2: Expected revenue matrix of the LSI and FLSP $i$ considering FLSP's competition.

| LSI | FLSP |  |
| :--- | :--- | :--- |
|  | Complies with the contract | Breaks the contract |
| Supervises | $(1-r) \omega+r \pi\left(p_{1}\right)-c\left(p_{1}\right)$, | $(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi-\rho \theta-\gamma N b R$, <br> $(1-r)\left[\pi\left(p_{1}\right)-\omega\right]-f(q)$ <br> Does not supervise |

breaks the contract under no supervision of the LSI). The expected payoff matrix of the LSI and FLSP $i$ is shown in Table 3.

The expected revenue function of the LSI is

$$
\begin{align*}
& \Pi_{I}^{2}(x, y) \\
&= \Pi_{I}^{1}(x, y) \\
&-\{x[y+(1-y)(1-\rho)]+(1-x)[y+(1-y)]\} \alpha  \tag{9}\\
&=(A+\alpha \rho) x+(1-r) \Delta \pi y-(\theta+\alpha) \rho x y \\
&+(1-r) \pi\left(p_{2}\right)-(1-r) \omega-\alpha .
\end{align*}
$$

We let $\partial \Pi_{I}^{2}(x, y) / \partial x=0$ to obtain

$$
\begin{equation*}
y_{2}^{*}=\frac{A+\alpha \rho}{(\theta+\alpha) \rho} . \tag{10}
\end{equation*}
$$

The expected revenue function of FLSP $i$ is

$$
\begin{align*}
& \Pi_{F_{i}}^{2}(x, y) \\
&= \Pi_{I}^{1}(x, y) \\
&+\{x[y+(1-y)(1-\rho)]+(1-x)[y+(1-y)]\} \alpha  \tag{11}\\
&=-(\theta+\alpha) \rho x+(T+\gamma N b R) y+(\theta+\alpha) \rho x y \\
&+(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi+\alpha-\gamma N b R .
\end{align*}
$$

We let $\partial \Pi_{F_{i}}^{2}(x, y) / \partial y=0$ to obtain

$$
\begin{equation*}
x_{2}^{*}=\frac{-T-\gamma N b R}{(\theta+\alpha) \rho} . \tag{12}
\end{equation*}
$$

Comparing (6), (8), (10), and (12), we find that the introduction of compensation mechanism makes the LSI's optimal supervision probability decrease and FLSP i's optimal compliance probability increase.

Next, we will find out the range of the compensation value $\alpha$. Obviously, in order to obtain the LSI's optimal supervision probability, FLSP $i$ 's optimal compliance probability, and the expected revenue of the LSI and FLSP $i$ at the same time, (13) must be satisfied,

$$
\begin{align*}
y_{2}^{*} & \geq y_{1}^{*} \\
x_{2}^{*} & \leq x_{1}^{*} \\
\Pi_{I}^{2}\left(x_{2}^{*}, y_{2}^{*}\right) & \geq \Pi_{I}^{1}\left(x_{1}^{*}, y_{1}^{*}\right),  \tag{13}\\
\Pi_{F_{i}}^{2}\left(x_{2}^{*}, y_{2}^{*}\right) & \geq \Pi_{F_{i}}^{1}\left(x_{1}^{*}, y_{1}^{*}\right) .
\end{align*}
$$

According to (6), (8), (10), and (12), the first two inequalities in (13) are always satisfied.

From the third inequality in (13), after calculation, we know that $\alpha \leq((1-r) \Delta \pi f(q) / \rho \theta)-\theta$. To make $\alpha \geq 0$, $(1-r) \Delta \pi f(q) / \rho \theta \geq \theta$ must be satisfied; that is, $(1-r) \Delta \pi f(q) \geq$ $\rho \theta^{2}$.

From the forth inequality in (13), after calculation we know that $\alpha \geq 0$.

In conclusion, the range of $\alpha$ is

$$
\begin{equation*}
0 \leq \alpha \leq \frac{(1-r) \Delta \pi f(q)}{\rho \theta}-\theta \tag{14}
\end{equation*}
$$

4.2. Design of Compensation Mechanism and the Optimal $N$. The compensation value $\alpha$ is in the interval of [ $0,((1-$ $r) \Delta \pi f(q) / \rho \theta)-\theta]$ according to Section 4.1. In practical applications, when implementing compensation mechanism, the LSI has to consider the specific methods of compensating FLSPs as well as the compensation value. In this section, we design three specific compensation mechanisms, that is, fixed compensation mechanism, linear compensation mechanism, and nonlinear compensation mechanism, and calculate their range of application, respectively.
4.2.1. Compensation Mechanism 1: Fixed Compensation Mechanism. The LSI compensates $\alpha=\alpha_{0}$ to FLSP $i$ when FLSP $i$

TAbLe 3: Expected revenue matrix of LSI and FLSP $i$ with FLSPs' competition under a compensation mechanism.

| LSI |  |  |
| :--- | :--- | :--- |
|  | Complies with the contract |  |
| Supervises | $(1-r) \omega+r \pi\left(p_{1}\right)-c\left(p_{1}\right)+\alpha$, | Breaks the contract |
|  | $(1-r)\left[\pi\left(p_{1}\right)-\omega\right]-f(q)-\alpha$ | $(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi-\rho \theta-\gamma N b R+(1-\rho) \alpha$, |
| Does not supervise | $(1-r) \omega+r \pi\left(p_{1}\right)-c\left(p_{1}\right)+\alpha$, | $(1-r)\left[\pi\left(p_{2}\right)-\omega\right]-f(q)+\rho \theta-(1-\rho) \alpha$ |
|  | $(1-r)\left[\pi\left(p_{1}\right)-\omega\right]-\alpha$ | $(1-r) \omega+r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi-\gamma N b R+\alpha$, |
|  |  | $(1-r)\left[\pi\left(p_{2}\right)-\omega\right]-\alpha$ |

complies with the contract (including the situation that FLSP $i$ breaks the contract but is not discovered and that FLSP $i$ breaks the contract without the LSI's supervision).
(1) To adopt compensation mechanism 1, the number of FLSPs should meet

$$
\begin{equation*}
0 \leq \alpha_{0} \leq \frac{(1-r) \Delta \pi f(q)}{\rho \theta}-\theta \tag{15}
\end{equation*}
$$

Thus

$$
\begin{equation*}
N \in Z \tag{16}
\end{equation*}
$$

Proposition 6. As long as $0 \leq \alpha_{0} \leq((1-r) \Delta \pi f(q) / \rho \theta)-\theta$ is satisfied, the LSI can adopt fixed compensation mechanism, no matter how many FLSPs it has.

Proposition 6 shows that the LSI does not need to care about the number of FLSPs when it adopts the fixed compensation mechanism, as long as the compensation it pays to each FLSP is no more than $((1-r) \Delta \pi f(q) / \rho \theta)-\theta$.
(2) The impact of $N$ on the efficiency of the supply chain collaboration is as follows.

The optimal supervision probability of the LSI is

$$
\begin{align*}
x_{2}^{* 1} & =\frac{-T-\gamma N b R}{(\theta+\alpha) \rho}=\frac{-T-\gamma N b R}{\left(\theta+\alpha_{0}\right) \rho} \\
\frac{\partial x_{2}^{* 1}}{\partial N} & =-\frac{\gamma b R}{\left(\theta+\alpha_{0}\right) \rho}<0 . \tag{17}
\end{align*}
$$

The optimal compliance probability of FLSP $i$ is

$$
\begin{align*}
y_{2}^{* 1} & =\frac{A+\alpha \rho}{(\theta+\alpha) \rho}=\frac{A+\alpha_{0} \rho}{\left(\theta+\alpha_{0}\right) \rho} \\
\frac{\partial y_{2}^{* 1}}{\partial N} & =0 \tag{18}
\end{align*}
$$

According to the analysis above, when using the fixed compensation mechanism, the LSI's optimal supervision probability is negatively correlated to $N$, while FLSP $i$ 's optimal compliance probability is not related to $N$. Thus, Proposition 7 is obtained.

Proposition 7. A large $N$ will lead to high efficiency of the supply chain cooperation (a lower optimal supervision probability of the LSI and a higher optimal compliance probability of FLSP i). Thus, when adopting the fixed compensation mechanism, the efficiency of the supply chain cooperation reaches the maximum when $N=\infty$.

Propositions 6 and 7 show that the LSI can always adopt the fixed compensation mechanism no matter how many FLSPs it has, and the efficiency of the fixed compensation mechanism improves as $N$ increases.
4.2.2. Compensation Mechanism 2: Linear Compensation Mechanism. The LSI compensates $\alpha=\alpha_{1}+b_{1} N$ to FLSP $i$ when FLSP $i$ complies with the contract (including the situation that FLSP $i$ breaks the contract but is not discovered and that FLSP $i$ breaks the contract without the LSI's supervision).
(1) To adopt compensation mechanism 2, the number of FLSPs should meet

$$
\begin{equation*}
0 \leq \alpha_{1}+b_{1} N \leq \frac{(1-r) \Delta \pi f(q)}{\rho \theta}-\theta \tag{19}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
0 \leq N \leq \overline{N_{1}} \tag{20}
\end{equation*}
$$

in which $\overline{N_{1}}=\left((1-r) \Delta \pi f(q)-\rho \theta^{2}-\rho \theta \alpha_{1}\right) / \rho \theta b_{1}$.
Proposition 8. If $\overline{N_{1}}>0$, the LSI can adopt the linear compensation mechanism when the number of FLSPs satisfies $N \in\left(0, \overline{N_{1}}\right]$; otherwise, the LSI's optimal supervision probability, FLSP i's optimal compliance probability, and the expected revenue of the LSI and FLSP $i$ cannot be simultaneously optimized by using the linear compensation mechanism.

Proposition 8 shows that the linear compensation mechanism only applies to cases that the number of FLSPs is small, and it does not apply to those LSIs who have a large number of FLSPs.

In the following analysis, we assume that $\overline{N_{1}}>0$, which means that the LSI can adopt the linear compensation mechanism when $N \in\left(0, \overline{N_{1}}\right]$.
(2) The impact of $N$ on the efficiency of the supply chain collaboration is as follows.

The optimal supervision probability of the LSI is

$$
\begin{align*}
x_{2}^{* 2} & =\frac{-T-\gamma N b R}{(\theta+\alpha) \rho}=\frac{-T-\gamma N b R}{\left[\theta+\left(\alpha_{1}+b_{1} N\right)\right] \rho} \\
\frac{\partial x_{2}^{* 2}}{\partial N} & =\frac{b_{1} T-\left(\theta+\alpha_{1}\right) \gamma b R}{\left(\theta+\alpha_{1}+b_{1} N\right)^{2} \rho}<0 \tag{21}
\end{align*}
$$

The optimal compliance probability of FLSP $i$ is

$$
\begin{align*}
y_{2}^{* 2} & =\frac{A+\alpha \rho}{(\theta+\alpha) \rho}=\frac{A+\left(\alpha_{1}+b_{1} N\right) \rho}{\left[\theta+\left(\alpha_{1}+b_{1} N\right)\right] \rho} \\
\frac{\partial y_{2}^{* 2}}{\partial N} & =\frac{f(q) b_{1}}{\left(\theta+\alpha_{1}+b_{1} N\right)^{2} \rho}>0 . \tag{22}
\end{align*}
$$

According to the analysis above, when using the linear compensation mechanism, the LSI's optimal supervision probability is negatively correlated to $N$, while FLSP $i$ 's optimal compliance probability is positively correlated to $N$. Thus, Proposition 9 is obtained.

Proposition 9. A large $N$ will lead to high efficiency of the supply chain cooperation (a lower optimal supervision probability of the LSI and a higher optimal compliance probability of FLSP i). Thus, when adopting the linear compensation mechanism, the efficiency of the supply chain cooperation reaches the maximum when $N=\left\lfloor\overline{N_{1}}\right\rfloor$.

Propositions 8 and 9 show that the LSI can adopt the linear compensation mechanism when the number of FLSPs is small ( $\left.N \in\left(0, \overline{N_{1}}\right]\right)$, and the efficiency of the linear compensation mechanism achieves the maximum when $N=$ $\left\lfloor\overline{N_{1}}\right\rfloor$.
4.2.3. Compensation Mechanism 3: Nonlinear Compensation Mechanism. The LSI compensates $\alpha=\alpha_{2}+\left(b_{2} / N\right)$ to FLSP $i$ when FLSP $i$ complies with the contract (including the situation that FLSP $i$ breaks the contract but is not discovered and that FLSP $i$ breaks the contract without the LSI's supervision).
(1) To adopt compensation mechanism 3, the number of FLSPs should meet

$$
\begin{equation*}
0 \leq \alpha_{2}+\frac{b_{2}}{N} \leq \frac{(1-r) \Delta \pi f(q)}{\rho \theta}-\theta \tag{23}
\end{equation*}
$$

Thus

$$
\begin{equation*}
N \geq \underline{N_{2}}, \tag{24}
\end{equation*}
$$

in which $N_{2}=\rho \theta b_{2} /\left((1-r) \Delta \pi f(q)-\rho \theta^{2}-\rho \theta \alpha_{2}\right)$.
Proposition 10. If $N_{2}>0$, the LSI can adopt the nonlinear compensation mechanism when the number of FLSPs satisfies $N \in\left[N_{2}, \infty\right)$; otherwise, the LSI can adopt the nonlinear compensation mechanism when the number of FLSPs is a positive integer.

Proposition 10 shows that the nonlinear compensation mechanism applies to cases that the number of FLSPs is large, and it may not apply to those LSIs who have a small number of FLSPs.

In the following analysis, we assume that $N_{2}>0$, which means that the LSI can adopt the nonlinear compensation mechanism when $N \in\left[N_{2}, \infty\right)$.
(2) The impact of $N \overline{\text { on }}$ the efficiency of the supply chain collaboration is as follows.

The optimal supervision probability of the LSI is

$$
\begin{equation*}
x_{2}^{* 3}=\frac{-T-\gamma N b R}{(\theta+\alpha) \rho}=-\frac{N T+\gamma N^{2} b R}{\left(N \theta+N \alpha_{2}+b_{2}\right) \rho} \tag{25}
\end{equation*}
$$

Since $\partial^{2} x_{2}^{* 3} / \partial N^{2}=-\left(2 \gamma b R b_{2}^{2}-2\left(\theta+\alpha_{2}\right) b_{2} T\right) /$ $\left(N \theta+N \alpha_{2}+b_{2}\right)^{3} \rho<0, x_{2}^{* 3}$ reaches the maximum when $\partial x_{2}^{* 3} / \partial N=0$; that is, $b_{2} T+\gamma N b R\left(N \theta+N \alpha_{2}+2 b_{2}\right)=0$.

The optimal compliance probability of FLSP $i$ is

$$
\begin{align*}
y_{2}^{* 3} & =\frac{A+\alpha \rho}{(\theta+\alpha) \rho}=\frac{N A+\left(N \alpha_{2}+b_{2}\right) \rho}{\left(N \theta+N \alpha_{2}+b_{2}\right) \rho} \\
\frac{\partial y_{2}^{* 3}}{\partial N} & =-\frac{f(q) b_{2}}{\left(N \theta+N \alpha_{2}+b_{2}\right)^{2} \rho}<0 . \tag{26}
\end{align*}
$$

According to the analysis above, when using the nonlinear compensation mechanism, the LSI's optimal supervision probability is firstly positively and then negatively correlated to $N$, while FLSP $i$ 's optimal compliance probability is negatively correlated to $N$. Thus, Proposition 11 is obtained.

Proposition 11. A small $N$ will lead to high efficiency of the supply chain cooperation (a lower optimal supervision probability of the LSI and a higher optimal compliance probability of FLSP $i$ ). Thus, when adopting the nonlinear compensation mechanism, the efficiency of the supply chain cooperation reaches the maximum when $N=\left\lceil\underline{N_{2}}\right\rceil$.

Propositions 10 and 11 show that the LSI can adopt the nonlinear compensation mechanism when the number of FLSPs is large $\left(N \in\left[N_{2}, \infty\right)\right.$ ), and the efficiency of the nonlinear compensation mechanism achieves the maximum when $N=\left\lceil N_{2}\right\rceil$.
4.3. Comparative Analysis of Compensation Mechanisms. According to the analysis in Section 4.2, the three kinds of compensation mechanisms apply to different scopes. For LSIs who have different number of FLSPs, which mechanism is the most suitable one? This issue is discussed in this section.
4.3.1. The Conditions When Compensation Mechanism 1 (Fixed Compensation Mechanism) Is the Optimal Choice. When compensation mechanism 1 has the highest efficiency, (27) is satisfied:

$$
\begin{align*}
& x_{2}^{* 1} \leq x_{2}^{* 2} \\
& x_{2}^{* 1} \leq x_{2}^{* 3} \\
& y_{2}^{* 1} \geq y_{2}^{* 2}  \tag{27}\\
& y_{2}^{* 1} \geq y_{2}^{* 3}
\end{align*}
$$

Therefore, when compensation mechanism 1 (fixed compensation mechanism) is the optimal choice, (28) is satisfied:

$$
\begin{cases}\frac{b_{2}}{\alpha_{0}-\alpha_{2}} \leq N \leq \frac{\alpha_{0}-\alpha_{1}}{b_{1}}, & \text { if }\left(\alpha_{0}>\alpha_{2}\right) \cap\left(\alpha_{0}>\alpha_{1}\right) \cap  \tag{28}\\ & \left(\frac{b_{2}}{\alpha_{0}-\alpha_{2}} \leq \frac{\alpha_{0}-\alpha_{1}}{b_{1}}\right) \\ 0, & \text { else. }\end{cases}
$$

4.3.2. The Conditions When Compensation Mechanism 2 (Linear Compensation Mechanism) Is the Optimal Choice. When compensation mechanism 2 has the highest efficiency, (29) is satisfied:

$$
\begin{align*}
& x_{2}^{* 2} \leq x_{2}^{* 1} \\
& x_{2}^{* 2} \leq x_{2}^{* 3} \\
& y_{2}^{* 2} \geq y_{2}^{* 1}  \tag{29}\\
& y_{2}^{* 2} \geq y_{2}^{* 3}
\end{align*}
$$

Equation (29) could be calculated and changed into

$$
N \geq\left\{\begin{array}{c}
\frac{\alpha_{0}-\alpha_{1}}{b_{1}}  \tag{30}\\
N_{4}
\end{array}\right.
$$

in which it is assumed that $N_{3}=\left(-\left(\alpha_{1}-\alpha_{2}\right)-\right.$ $\sqrt{\left.\left(\alpha_{1}-\alpha_{2}\right)^{2}+4 b_{1} b_{2}\right)} / 2 b_{1}<0, N_{4}=\left(-\left(\alpha_{1}-\alpha_{2}\right)+\right.$ $\sqrt{\left.\left(\alpha_{1}-\alpha_{2}\right)^{2}+4 b_{1} b_{2}\right)} / 2 b_{1}>0$.

Therefore, when compensation mechanism 2 (linear compensation mechanism) is the optimal choice, (31) is satisfied:

$$
N \geq \begin{cases}\frac{\alpha_{0}-\alpha_{1}}{b_{1}}, & \text { if } \frac{\alpha_{0}-\alpha_{1}}{b_{1}}>N_{4}  \tag{31}\\ N_{4}, & \text { if } \frac{\alpha_{0}-\alpha_{1}}{b_{1}} \leq N_{4}\end{cases}
$$

When adopting linear compensation mechanism, there is an upper limit $\overline{N_{1}}$ for the number of FLSPs. In this paper, we assume $\overline{N_{1}} \geq N_{4}$ and $\overline{N_{1}} \geq\left(\alpha_{0}-\alpha_{1}\right) / b_{1}$ to ensure that the linear compensation mechanism can take every value in the optimal interval.
4.3.3. The Conditions When Compensation Mechanism 3 (Nonlinear Compensation Mechanism) Is the Optimal Choice. When compensation mechanism 3 has the highest efficiency, (32) is satisfied:

$$
\begin{align*}
& x_{2}^{* 3} \leq x_{2}^{* 1} \\
& x_{2}^{* 3} \leq x_{2}^{* 2} \\
& y_{2}^{* 3} \geq y_{2}^{* 1}  \tag{32}\\
& y_{2}^{* 3} \geq y_{2}^{* 2}
\end{align*}
$$

Therefore, when compensation mechanism 3 (nonlinear compensation mechanism) is the optimal choice, the nether inequalities are satisfied:

$$
0<N \leq \begin{cases}\frac{b_{2}}{\alpha_{0}-\alpha_{2}}, & \left(\alpha_{0}>\alpha_{2}\right) \cap\left(N_{4} \geq \frac{b_{2}}{\alpha_{0}-\alpha_{2}}\right)  \tag{33}\\ N_{4}, & \left(\alpha_{0} \leq \alpha_{2}\right) \\ & \cup\left(\alpha_{0}>\alpha_{2} \cap N_{4}<\frac{b_{2}}{\alpha_{0}-\alpha_{2}}\right) .\end{cases}
$$

When adopting nonlinear compensation mechanism, there is a lower limit $N_{2}$ for the number of FLSPs. In this paper, we assume $\underline{N_{2}} \leq N_{4}$ and $\underline{N_{2}} \leq b_{2} /\left(\alpha_{0}-\alpha_{2}\right)$ to ensure that the nonlinear compensation mechanism can take every value in the optimal interval.

Therefore, when compensation mechanism 3 (nonlinear compensation mechanism) is the optimal choice, (34) is satisfied:

$$
\underline{N_{2}}<N \leq \begin{cases}\frac{b_{2}}{\alpha_{0}-\alpha_{2}}, & \left(\alpha_{0}>\alpha_{2}\right) \cap\left(N_{4} \geq \frac{b_{2}}{\alpha_{0}-\alpha_{2}}\right)  \tag{34}\\ N_{4}, & \left(\alpha_{0} \leq \alpha_{2}\right) \\ & \cup\left(\alpha_{0}>\alpha_{2} \cap N_{4}<\frac{b_{2}}{\alpha_{0}-\alpha_{2}}\right)\end{cases}
$$

4.3.4. The Optimal Compensation Mechanism with Different Number of FLSPs. According to the analysis in Sections 4.3.1 to 4.3.3, the optimal compensation mechanism with different number of FLSPs is shown in Table 4.

Table 4 shows that LSIs should follow the principles below.
(1) If $\alpha_{0} \leq \alpha_{2}$, the LSI should adopt the nonlinear compensation mechanism when $N \in\left(0, N_{4}\right]$.
(2) If $\alpha_{0}>\alpha_{2}$, the LSI should adopt the fixed compensation mechanism when $N \in\left[b_{2} /\left(\alpha_{0}-\alpha_{2}\right),\left(\alpha_{0}-\alpha_{1}\right) / b_{1}\right]$ (if $b_{2} /\left(\alpha_{0}-\alpha_{2}\right) \leq\left(\alpha_{0}-\alpha_{1}\right) / b_{1}$ and $\left.\alpha_{0}>\alpha_{1}\right)$ and adopt the nonlinear compensation mechanism when $N \in\left[N_{2}, b_{2} /\left(\alpha_{0}-\alpha_{2}\right)\right]$ (if $N_{4} \geq b_{2} /\left(\alpha_{0}-\alpha_{2}\right)$ ) or $N \in\left[\overline{N_{2}}, N_{4}\right]\left(\right.$ if $\left.N_{4}<b_{2} /\left(\alpha_{0}-\alpha_{2}\right)\right)$.
(3) The LSI should adopt the linear compensation mechanism when $N \in\left[\left(\alpha_{0}-\alpha_{1}\right) / b_{1}, \overline{N_{1}}\right]$ (if $N_{4}<\left(\alpha_{0}-\right.$ $\left.\alpha_{1}\right) / b_{1}$ ) or $N \in\left[N_{4}, \overline{N_{1}}\right]$ (if $\left.N_{4} \geq\left(\alpha_{0}-\alpha_{1}\right) / b_{1}\right)$.

The fixed compensation mechanism is never the optimal choice when $\alpha_{0} \leq \alpha_{1}$ or $\alpha_{0} \leq \alpha_{2}$. Therefore, we assume $\alpha_{0}>$ $\alpha_{1}$ and $\alpha_{0}>\alpha_{2}$ in the following analysis. Moreover, we assume $\overline{N_{1}} \geq b_{2} /\left(\alpha_{0}-\alpha_{2}\right)$ and $\underline{N_{2}} \leq\left(\alpha_{0}-\alpha_{1}\right) / b_{1}$ to simplify the analysis.

Since the size relation of $N_{4},\left(\alpha_{0}-\alpha_{1}\right) / b_{1}$ and $b_{2} /\left(\alpha_{0}-\alpha_{2}\right)$ remains uncertain, there are six possible relations according to the Permutations and Combinations Theory. Their size relation influences the optimal intervals of the three compensation mechanisms. Thus, the optimal compensation mechanism under different intervals of $N$ is shown in Figures $2,3,4,5,6$, and 7 .

Table 4: The optimal compensation mechanism with different number of FLSPs.

| Condition | The interval of $N$ | The optimal compensation <br> mechanism |
| :--- | :--- | :--- |
| $\alpha_{0}>\alpha_{2}$ | $\alpha_{0} \leq \alpha_{2}$ | $\left(0, N_{4}\right]$ |
| Nonlinear compensation <br> mechanism |  |  |
| $N_{0}-\alpha_{2}$ | $\frac{\alpha_{0}-\alpha_{1}}{b_{1}}$ and $\alpha_{0}>\alpha_{1}$ | $\left[\frac{b_{2}}{\alpha_{0}-\alpha_{2}}\right.$ |
| $\left.N_{4}<\frac{b_{2}}{\alpha_{0}-\alpha_{2}}, \frac{\alpha_{0}-\alpha_{1}}{b_{1}}\right]$ | Fixed compensation <br> mechanism |  |
| $N_{4}<\frac{\alpha_{0}-\alpha_{1}}{b_{1}}$ | $\left[\frac{\left.N_{2}, \frac{b_{2}}{\alpha_{0}-\alpha_{2}}\right]}{\begin{array}{l}\text { Nonlinear compensation } \\ \text { mechanism }\end{array}}\right.$ |  |
| $N_{4} \geq \frac{\alpha_{0}-\alpha_{1}}{b_{1}}$ | $\left[\underline{N_{2}}, N_{4}\right]$ | Nonlinear compensation <br> mechanism |

The following three conclusions can be obtained from Figures 2-7.
(1) The LSI should adopt nonlinear compensation mechanism when the number of FLSPs is small ( $N \leq N_{4}$ and $\left.N \leq b_{2} /\left(\alpha_{0}-\alpha_{2}\right)\right)$. This suggests that those LSIs who ask for customized or upscale services from FLSPs should give priority to nonlinear compensation mechanism. However, if the number of FLSPs is smaller than the lower limit $N_{2}$, the LSI can only select from the fixed compensation mechanism and the linear compensation mechanism.
(2) The LSI should adopt linear compensation mechanism when the number of FLSPs is large ( $N \geq N_{4}$ and $\left.N \geq\left(\alpha_{0}-\alpha_{1}\right) / b_{1}\right)$. This suggests that those LSIs who ask for mass or humble services from FLSPs should give priority to linear compensation mechanism. However, if the number of FLSPs is larger than the higher limit $\overline{N_{1}}$, the LSI can only select from the fixed compensation mechanism and the nonlinear compensation mechanism.
(3) It can be seen from Figures $2-7$ that there is no optimal compensation mechanism when the number of FLSPs is in other intervals. This is because in these cases, the optimal supervision probability of the LSI and the optimal compliance probability of FLSPs cannot be optimized at the same time. Thus, the LSI has to balance the importance of these two variables in the selection of these compensation mechanisms.

## 5. Numerical Analysis

Section 5 conducts numerical analysis with MATLAB 8.0 software, which verifies the propositions provided above.


Figure 2: The optimal compensation mechanism in different intervals when $N_{4}<b_{2} /\left(\alpha_{0}-\alpha_{2}\right)<\left(\alpha_{0}-\alpha_{1}\right) / b_{1}$.


Figure 3: The optimal compensation mechanism in different intervals when $N_{4}<\left(\alpha_{0}-\alpha_{1}\right) / b_{1}<b_{2} /\left(\alpha_{0}-\alpha_{2}\right)$.

Some of the parameters involved in the numerical analysis are as follows:

$$
\begin{gathered}
\omega=25, \quad \rho=0.6, \\
\theta=40, \quad f(q)=22, \quad r=0.3 \\
\pi\left(p_{1}\right)=175, \quad \pi\left(p_{2}\right)=80, \\
c\left(p_{1}\right)=25, \quad c\left(p_{2}\right)=10, \\
\varphi=38, \quad \gamma=0.2, \\
b=0.2, \quad R=40 \\
\alpha_{0}=25, \quad \alpha_{1}=10
\end{gathered}
$$



Figure 4: The optimal compensation mechanism in different intervals when $b_{2} /\left(\alpha_{0}-\alpha_{2}\right)<N_{4}<\left(\alpha_{0}-\alpha_{1}\right) / b_{1}$.


Figure 5: The optimal compensation mechanism in different intervals when $b_{2} /\left(\alpha_{0}-\alpha_{2}\right)<\left(\alpha_{0}-\alpha_{1}\right) / b_{1}<N_{4}$.

$$
\begin{gather*}
b_{1}=4, \quad \alpha_{2}=20, \quad b_{2}=40 ; \\
\Delta \pi=\pi\left(p_{1}\right)-\pi\left(p_{2}\right)=95, \\
A=\rho \theta-f(q)=2, \\
T=r \Delta \pi-\left[c\left(p_{1}\right)-c\left(p_{2}\right)\right]-\varphi=-24.5 . \tag{35}
\end{gather*}
$$

Substituting the values of some of the parameters provided above into (17), (18), (21), (22), (25), and (26), the changes of $x_{2}^{*}$ and $y_{2}^{*}$ as $N$ changes can be obtained, as shown in Figures 8, 9, and 10.
$x_{2}^{* 1}<x_{2}^{* 2}$ at the left side of $A(3.75,0.47)$, which means that compensation mechanism 1 is better than compensation mechanism 2; $x_{2}^{* 1}>x_{2}^{* 2}$ at the right side of $A$, which means compensation mechanism 2 is better than compensation mechanism 1. $y_{2}^{* 1}>y_{2}^{* 2}$ at the left side of $B(3.75,0.44)$, which means compensation mechanism 1 is better than compensation mechanism 2; $y_{2}^{* 1}<y_{2}^{* 2}$ at the right side of $B$, which means compensation mechanism 2 is better than compensation mechanism 1. Consequently, compensation mechanism 1 is better than compensation mechanism 2 when $N \in(0,3.75]$, and compensation mechanism 2 is better than compensation mechanism 1 when $N \in[3.75,12]$.
$x_{2}^{* 1}>x_{2}^{* 3}$ at the left side of $C(8,0.3)$, which means compensation mechanism 3 is better than compensation mechanism 1; $x_{2}^{* 1}<x_{2}^{* 3}$ at the right side of $C$, which means compensation mechanism 1 is better than compensation mechanism 3. $y_{2}^{* 1}<y_{2}^{* 3}$ at the left side of $D(8,0.44)$, which means compensation mechanism 3 is better than compensation mechanism 1; $y_{2}^{* 1}>y_{2}^{* 3}$ at the right side of $D$, which means compensation mechanism 1 is better than compensation mechanism 3. Consequently, compensation mechanism 3 is better than compensation mechanism 1 when $N \in(0,8]$, and compensation mechanism 1 is better than compensation mechanism 3 when $N \in[8,12]$.
$x_{2}^{* 2}>x_{2}^{* 3}$ at the left side of $E(4.65,0.41)$, which means compensation mechanism 3 is better than compensation mechanism 2; $x_{2}^{* 2}<x_{2}^{* 3}$ at the right side of $E$, which means compensation mechanism 2 is better than compensation


Figure 6: The optimal compensation mechanism in different intervals when $\left(\alpha_{0}-\alpha_{1}\right) / b_{1}<N_{4}<b_{2} /\left(\alpha_{0}-\alpha_{2}\right)$.


Figure 7: The optimal compensation mechanism in different intervals when $\left(\alpha_{0}-\alpha_{1}\right) / b_{1}<b_{2} /\left(\alpha_{0}-\alpha_{2}\right)<N_{4}$.
mechanism 3. $y_{2}^{* 2}<y_{2}^{* 3}$ at the left side of $F(4.65,0.47)$, which means compensation mechanism 3 is better than compensation mechanism 2; $y_{2}^{* 2}>y_{2}^{* 3}$ at the right side of $F$, which means compensation mechanism 2 is better than compensation mechanism 3. Consequently, compensation mechanism 3 is better than compensation mechanism 2 when $N \in(0,4.65]$, and compensation mechanism 2 is better than compensation mechanism 3 when $N \in[4.65,12]$.

In conclusion, compensation mechanism 1 is better than compensation mechanism 2 when $N \in(0,3.75]$, and compensation mechanism 2 is better than compensation mechanism 1 when $N \in[3.75,12]$; compensation mechanism 3 is better than compensation mechanism 1 when $N \in(0,8]$, and compensation mechanism 1 is better than compensation mechanism 3 when $N \in[8,12]$; compensation mechanism 3 is better than compensation mechanism 2 when $N \in$ $(0,4.65]$, and compensation mechanism 2 is better than compensation mechanism 3 when $N \in[4.65,12]$.

In consequence, compensation mechanism 2 is better than compensation mechanisms 1 and 3 when $N \in[4.65,12]$, compensation mechanism 3 is better than compensation mechanisms 1 and 2 when $N \in(0,4.65]$, and compensation mechanism 1 is not the best choice in any interval. (According to the analysis in Section 4.3.4, if $\alpha_{0}>\alpha_{2}$, the LSI should adopt the fixed compensation mechanism when $N \in$ $\left[b_{2} /\left(\alpha_{0}-\alpha_{2}\right),\left(\alpha_{0}-\alpha_{1}\right) / b_{1}\right]\left(\right.$ if $b_{2} /\left(\alpha_{0}-\alpha_{2}\right) \leq\left(\alpha_{0}-\alpha_{1}\right) / b_{1}$ and $\left.\alpha_{0}>\alpha_{1}\right)$, while in this section, we assume $b_{2} /\left(\alpha_{0}-\alpha_{2}\right)=8$ and $\left(\alpha_{0}-\alpha_{1}\right) / b_{1}=3.75$, which leads to $b_{2} /\left(\alpha_{0}-\alpha_{2}\right)>\left(\alpha_{0}-\alpha_{1}\right) / b_{1}$, so compensation mechanism 1 is not the best choice in any interval.) In other words, the LSI should adopt the nonlinear compensation mechanism when the number of FLSPs ranges from 1 to 4 and adopt the linear compensation mechanism when the number of FLSPs ranges from 5 to 12 .

## 6. Main Conclusions and Management Implications

6.1. Main Conclusions. This paper studies the impact of the number of competitors on the effect of quality supervision game in a supply chain and finds that the original mixed payment contract cannot optimize the LSI's optimal supervision probability and FLSP i's optimal compliance


Figure 8: The changes of $x_{2}^{*}$ and $y_{2}^{*}$ as $N$ changes in compensation mechanisms 1 and 2.


Figure 9: The changes of $x_{2}^{*}$ and $y_{2}^{*}$ as $N$ changes in compensation mechanisms 1 and 3.
probability at the same time. Therefore, the compensation mechanism is introduced into our model and it is proved that the compensation mechanism can lead to Pareto optimality. Based on this, the application conditions of three different compensation mechanisms are discussed, and it is proved that the compensation mechanism choice of the LSI is influenced by the number of FLSPs.

Firstly, the LSI should adopt nonlinear compensation mechanism when the number of FLSPs is small ( $N \leq N_{4}$ and $\left.N \leq b_{2} /\left(\alpha_{0}-\alpha_{2}\right)\right)$, but if the number of FLSPs is smaller than the lower limit $N_{2}$, LSI can only select from the fixed compensation mechanism and the linear compensation mechanism.

Secondly, LSI should adopt linear compensation mechanism when the number of FLSPs is large ( $N \geq N_{4}$ and $\left.N \geq\left(\alpha_{0}-\alpha_{1}\right) / b_{1}\right)$, but if the number of FLSPs is larger than the higher limit $\overline{N_{1}}$, LSI can only select from the fixed compensation mechanism and the nonlinear compensation mechanism.


Figure 10: The changes of $x_{2}^{*}$ and $y_{2}^{*}$ as $N$ changes in compensation mechanisms 2 and 3 .

Thirdly, there is no optimal compensation mechanism when the number of FLSPs is in other intervals, so LSI has to balance the importance of two variables, which are the LSI's optimal supervision probability and FLSP is optimal compliance probability, in the selection of compensation mechanism.
6.2. Management Implications. In this paper, we use game theory to obtain the mixed strategy Nash equilibrium of the quality supervision game, study the impact of the number of FLSPs on the quality supervision game in LSSC, and obtain applicable intervals for different compensation mechanisms. Our research provides a new idea for further studies on LSSC quality supervision game with competition among FLSPs. For instance, researchers can further explore, in addition to compensation mechanism, that other kinds of mechanism exist to improve the effectiveness of the quality supervision game in the supply chain. Also, researchers can further study on how to express the competitive relations and improve the effectiveness of the quality supervision game if the competitors are heterogeneous. In addition, this study provides a necessary theoretical basis for future empirical studies on the impact of the number of FLSPs on the quality supervision game of LSSC.

From the manager's perspective, our conclusions have significance in improving the effectiveness of quality supervision game in LSSC. LSIs, as the leader in the supply chain, can consider using compensation mechanisms to improve the effectiveness of quality supervision game when they have multiple competing FLSPs. LSIs that require different types of service need different numbers of FLSPs; correspondingly, the compensation mechanisms they should adopt are different. For example, the LSI who asks for customized or upscale services from a small number of FLSPs ( $N \leq N_{4}$ and $\left.N \leq b_{2} /\left(\alpha_{0}-\alpha_{2}\right)\right)$ should give priority to the nonlinear compensation mechanism; however, if the number of FLSPs is smaller than the lower limit $N_{2}$, the LSI can only select from the fixed compensation $\overline{m e c h a n i s m}$ and the linear
compensation mechanism. The LSI who asks for mass or humble services from a large number of FLSPs ( $N \geq N_{4}$ and $\left.N \geq\left(\alpha_{0}-\alpha_{1}\right) / b_{1}\right)$ should give priority to linear compensation mechanism; however, if the number of FLSPs is larger than the higher limit $\overline{N_{1}}$, the LSI can only select from the fixed compensation mechanism and the nonlinear compensation mechanism.

Furthermore, there is no optimal compensation mechanism in some cases. This is because in these cases, the optimal supervision probability of the LSI and the optimal compliance probability of FLSP $i$ cannot be optimized at the same time. Thus, the LSI has to balance the importance of these two variables in the selection of compensation mechanism.

This paper attempts to analyze the impact of the number of FLSPs on the quality supervision game of LSSC and provide a theoretical reference for the study on the quality supervision game. There are still some limitations worth to be improved in future studies. For instance, we assume that $\gamma, b$, and $R$ are the same for all FLSPs, but they may not be exactly the same in practice. In addition, the follow-up study might concentrate on empirical research to explore the impact of the number of FLSPs on the quality supervision game of LSSC in the reality.

## Appendix

## Notations for the Model

$p_{i}$ : Effort level of the FLSP: $i=1$ is the FLSP's effort level when it complies with the contract; $i=2$ is the FLSP's effort level when it does not comply with the LSI.
$\omega$ : Fixed payment that the FSLP gains from the LSI.
$r$ : Coefficient of revenue that the FLSP obtains from cooperation.
$c\left(p_{i}\right)$ : Cost of the FLSP's service.
$\pi\left(p_{i}\right)$ : Total revenue of the LSSC, which is a function of the FLSP's effort level, $\pi(p)=g(p)+\varepsilon$, in which $g(p)$ is a function of $p$.
$\varepsilon$ : External environment's impact on the total revenue of the LSSC, $\varepsilon \sim N\left(0, \sigma_{\varepsilon}^{2}\right)$.
$f(q)$ : Cost of the LSI when it supervises the FLSP's effort level, $f(q)=h(q)+\eta$, where $h(q)$ is a function of $q$ and $q$ is the supervision effort level of the LSI.
$\eta$ : External environment's impact on the supervision cost of the LSI, $\eta \sim N\left(0, \sigma_{\eta}^{2}\right)$.
$\varphi$ : Personal interests that the FLSP gains when it opts not to comply with the contract.
$\rho$ : Probability that the FLSP's noncompliance with the contract is discovered when the LSI supervises the FLSP's service quality: $\rho$ represents the measurability of the service quality; $\rho$ increases as the difficulty level of measuring service quality decreases.
$\theta$ : Penalty that the FLSP takes when it breaks the contract and is discovered by the LSI.
$x$ : Supervision probability of the LSI: $(1-x)$ is the nonsupervision probability of the LSI.
$y$ : Compliance probability of the FLSP: $(1-y)$ is the noncompliance probability of the FLSP.
$\Pi_{I}^{0}(x, y)$ : The expected revenue of the LSI in the quality supervision game model for the LSSC without FLSPs' competition: it is a function of $x$ and $y$.
$\Pi_{F}^{0}(x, y)$ : The expected revenue of the FLSP in the quality supervision game model for the LSSC without FLSPs' competition: it is a function of $x$ and $y$.
$\Pi_{I}^{1}(x, y)$ : The expected revenue of the LSI in the quality supervision game model for the LSSC with FLSPs' competition: it is a function of $x$ and $y$.
$\Pi_{F_{i}}^{1}(x, y)$ : The expected revenue of FLSP $i$ in the quality supervision game model for the LSSC with FLSPs' competition: it is a function of $x$ and $y$.
$F_{i}(N)$ : The expected loss of FLSP $i$ due to the competition among FLSPs. $F_{i}(N)=\gamma N b R$.
$\gamma$ : Competitive impact coefficient.
$N$ : The number of FLSPs, $N \geq 2$.
$b$ : Competitive coefficient, that is, the degree of mutual influence among FLSPs, and $0<b<1$ : a large $b$ means that the business competition among FLSPs is fierce; in particular, there is not any mutual effect among FLSPs when $b=0$.
$R$ : The service quality level of the FLSP, which can be informed from the FLSP's past experience of cooperation or credibility: it is assumed in this paper that all the FLSPs are homogeneous; that is, their service qualities are at the same level.
$\Pi_{I}^{2}(x, y)$ : The expected revenue of the LSI in the quality supervision game model with FLSPs' competition under a compensation mechanism: it is a function of $x$ and $y$.
$\Pi_{F_{i}}^{2}(x, y)$ : The expected revenue of FLSP $i$ in the quality supervision game model with FLSPs' competition under a compensation mechanism: it is a function of $x$ and $y$.
$\alpha$ : The compensation that the LSI pays to FLSP $i$ when FLSP $i$ complies with the contract (including that FLSP $i$ breaks the contract but is not discovered and that FLSP $i$ breaks the contract under no supervision of the LSI): $\alpha>0$.
$\Delta \pi: \Delta \pi=\pi\left(p_{1}\right)-\pi\left(p_{2}\right)$ presents the difference between the total revenues of the LSSC when the FLSP adopts the strategy of compliance and noncompliance.
$A$ : $A=\rho \theta-f(q)$ is equal to the net income of the LSI when it has supervision cost and penalty income; we can obtain that $0 \leq A \leq \rho \theta$ from (2) if $y^{*} \geq 0$.
$T: T=r \Delta \pi-\left[c\left(p_{1}\right)-c\left(p_{2}\right)\right]-\varphi:$ if $T>0$, then $r \pi\left(p_{1}\right)-$ $c\left(p_{1}\right)>r \pi\left(p_{2}\right)-c\left(p_{2}\right)+\varphi$, which means that, without considering the LSI's fixed payment and penalty to the

FLSP, the total revenue when the FLSP complies with the contract is larger than that when the FLSP breaks the contract. On the contrary, $T<0$ means that, without considering the LSI's fixed payment and penalty to the FLSP, the total revenue when the FLSP complies with the contract is smaller than that when the FLSP breaks the contract. We can obtain that $0 \leq-T \leq$ $\rho \theta$ from (4).
$M: M=T+\gamma N b R$ : we can obtain that $-T-\gamma N b R \geq 0$ from (8).

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# A Multicriteria Framework to Evaluate Supplier's Greenness 

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#### Abstract

Environmental protection is becoming more and more important for enterprises because of stronger public awareness, competitors and communities, and government regulations. For this purpose, some programs have become more popular for raising environmental awareness including total quality environmental management and green supply chain management. Reducing the environmental pollution from upstream to downstream during procuring raw materials, producing, distribution, selling products, and products depreciation is the most important goal of Green Supply Chain Management (GSCM). The main contribution of this study is introducing the main factors in green supply chain management that are very important in environmental attributes by providing an evaluation framework to select the most eligible green suppliers by examining the influential and important criteria and subcriteria among ten elements of two main GSCM practices, namely, green logistics and environmental protection. First, these factors are divided into two groups, that is, green logistics and environmental protection, and then by applying DEMATEL technique, the complex causal relationship between all factors dependencies and feedbacks among them is examined. Finally, by drawing the impact relationship map the most important and influential factors are determined for improving green supply chain environmental aspects.


## 1. Introduction

The emergence of Green Supply Chain Management (GSCM) is one of the most significant environmental developments in the past decade, offering the opportunity for companies to align their supply chains in accordance with environmental and sustainability goals. More attention, recently, has been paid to these environmental problems while the supply chain operation combined with sustainable consideration has increasingly been a major issue [1-4]. The most important challenge for enterprises has been integration of social, environment, and economic performances to obtain sustainable improvement [5]; in other words, if firms want to survive in the global market, they cannot disregard environmental issues. While traditional supplier selection focused on price, quality, and delivery on time [6] or focused only on the requirements of single organizations and ignored considering the whole supply chain [7], green supplier selection processes have to be focused on improving environmental factors
in whole supply chain through organizational performance and activities, consumption, logistics, customer service, and financial performance concurrently $[4,8]$. Today, buyers are willing to purchase products and services from suppliers that manage to provide them with high quality, low cost, and short lead time with environmental responsibility at the same time due to increased environmental attentiveness. By the way, one of the most obvious gaps by considering previous studies is lack of direct consideration to environmental protection and its aspects in proposed evaluation models in this area. In the green foundation, the first target that is extremely egregious is environment conversation. On the other hand, another powerful category named green logistics that has a direct relation to improving green supply chain management [9], so combination of the two main clusters, green logistics and environmental protection, can play an important role in improving GSCM.

In addition, previous studies have worked just on quantitative models that have been applied for selecting green
suppliers such as analytic hierarchy process technique [10, 11], fuzzy comprehensive evaluation [12], comprehensive grade model [13], and grey widespread evaluation [14] with no consideration to the complex causal relationship between criteria of the system along with dependencies and feedbacks among criteria and alternatives simultaneously [15]. Therefore, interactions between main features of green supplier selection could not be considered and any evaluation would not be accurate and reliable. Therefore, this study proposes the causal evaluation model which consists of two main green categories and applies the DEMATEL technique to evaluate the model and examine the relationship and interaction between the GSCM factors including their direct and indirect effects on each other. Finally, dimensions are divided into cause and effect groups. By improving aspects in cause group, improvements in the whole GSCM will come to pass.

In next part, we present some related background regarding green supply chain and its significant related features and methods.

## 2. Literature Review

2.1. Green Supply Chain Management. Green supply Chain Management (GSCM) is a multiple business network and collaborative relationship for better environment; it is not a business chain with business-to-business or one-to- one relationship [16]. Successful integration coordination and management across members is required in the supply chain including manufacturers, raw material supplier's recyclers, distributors, and users [17]. Three green supply chain management general characteristics are known as strategy, environment, and logistics. The environmental and green purchasing which was assumed to be a subset of green supply chain management accounted for the environmental and green issues whilst disregarded the social conditions [18]. Reference [19], however, gave its complete definition; according to the report, sustainability of supply chain management is regarded as the material management, capital flow information, and company's cooperation along the supply chain, while goals are taken from all the dimensions of sustainable development, including social, economic, and environmental, into account which met the stakeholder and customer requirements. In the supply chain sustainability, social and environmental criteria are required to be fulfilled by the members to retain the supply chain, while anticipating the maintenance of competitiveness by meeting the customer requirements and criteria for related economy. The definition, however, includes the green supply chain management and social responsibility of supply chain management [20,21].

For the companies transferring to "greenness" in their supply chain, motivations should be different. Although some of the motivations are unclear, according to [22], some of the organizations do this due to the fact that it is the right thing for the environment. Although some are more radical for the change of the environment, others may not [22]. Researchers have reported that reduction in the cost and profitability are some of the major business motivations to become "green" in the supply chain [23, 24].
2.2. Significance of Green Supplier Selection. Reference [22] has shown that in the corporate value chain, every element engages in minimizing the firms' environmental impact from the onset till the end of supply chain management; however, this differed compared to the traditional supply chain management. To be precise, green supply chain needs not only the companies to be alert throughout the entire process while considering the traditional supply chain management including timeliness of delivery, material cost flexibility, and other factors, but also it should be mindful of procurement of materials and reduced production, marketing, and other environmental negative impacts (e.g., harmful substances, packaging recycling). As a result of this, the green supply selection is important [25]. Organizational assessment is the most common GSCM practice involved in the performance of the suppliers' environment, which requires suppliers to undertake measures that ensure product environmental quality, evaluating the wastage in the cost of their system of operation [26]. Environmental high level of performance achieved by a firm may be broken down due to poor level of environmental management via its support. Next section discusses the method and proposes an evaluation framework by regarding green supplier selection.

## 3. Material and Method

This study proposes a causal model based on the DEMATEL methodologies to support green supply chain management strategic decisions and green supplier selection. Firstly, two main criteria, namely, green logistics and environmental protection, are identified and subcriteria that are related to them are defined. In the next step, vectors indicate direct relationships and loops indicate interdependencies inside the two clusters. According to previous studies, five elements for each cluster due to improving green supply chain management have been identified [4, 27-29].
3.1. Proposed Causal Model for Green Supplier Selection. The general view of the proposed green supplier causal model is exposed in Figure 1. Green supplier selection problem is a sort of complex multiple criteria decision making problem including both quantitative and qualitative factors, which may be inconsistent and uncertain. Due to the nature of supplier selection, multicriteria decision making (MCDM) methods are required to handle and solve the problem effectively [30]. The techniques of MCDM are comprehensibly derived to manage this kind of problems [31, 32].
3.2. Criteria for Green Supplier Evaluation. According to the review of previous researches, two main clusters for improving green supply chain management have been identified: green logistics (GL) and environmental protection (EP).
3.2.1. Green Logistics (GL). GL is the management activities to pursue customer satisfaction and social development goals, connecting the main body of green supply and demand and overcoming space and time obstacles to achieve efficient and rapid movement of goods and services. It inhibited the


FIgUre 1: Causal evaluation model methodology.
damage to environmental to achieve the purification of the logistics environment and the best use of logistics resources. GL is a multilevel concept which includes both the green logistics business activities and social activities for green logistics management, standard and control [33].

The main elements of green logistics dimension include procurement, production, distribution, packaging, and reverse logistics [6, 27, 34, 35].
3.2.2. Environmental Protection. Protection of the environment is a major area of green supply chain which compares the system of environmental protection like controlling all kinds of pollutions, design products according to green image, ISO 14001, and so forth. Major indicators are engaged in the recovery of resources conservation and resource utilization and resource type, environmental governance, environmental pollution, and reinvented ability. Moreover, it is made up of packaging materials, transport, emissions, garbage, and emissions [36-38]. In the US, Air pollution Control Act and Water Pollution Control Act 1955 and 1948 were introduced by the Environmental Protection Agency (EPA). Environmental protection dimensions consist of green design of products, ISO 14001, Eco-labeling, life cycle, and pollution control [28, 29].

### 3.2.3. Green Logistics Dimensions

(i) Production. The anticipation is that $80 \%$ of environmental impact related products are determined in the design phase; therefore, combining early environmental considerations in the developmental cycle for the product design is the best active means of decreasing their impact [39]. However, the major elements of the stage design include selection of materials and design products [39]. Production has great effects on
the green supply chain via the production process and design. Suppliers should aim at enhancing the raw material and packaging type for use in green environmental protection and green supply chain advancement. Moreover, supplier's production capabilities can be divided into many areas which include manufacturing quality equipment, highly developed equipment, and the number of workers with qualified experience, and the technical levels is comprised of three components: the ability to design green products and services and services, the level of technical security, and higher technological degrees [37]. The use of hazardous or restricted raw material through the entire manufacturing protocol, one important issue that should be addressed during the process is waste reduction [40-42]. In this function, total quality environmental management, closed-loop manufacturing, source reduction, and demanufacturing have some value added contribution, although some also influenced other functional areas [35].
(ii) Procurement. The initial stage in the chain of environmental friendliness is supplier procurement and selection. Reduction in purchase or procurement and using materials that are reusable, or have been recycled, have a significant role to play in green supply chain [9].
(iii) Distribution. Distribution is one of the protocols that show effects on the green supply chain. Different items including delivery services, customer distance and transporter capacity, fuel type used, and shipping frequency or transport [9] influence the green distribution performance. Moreover, the forward and reverse logistic network is influenced by many decisions of transportation and also transportation type and shipping, location of distribution channel,
just-in-time (JIT) strategy, and system control. The distribution and transportation network process are not only essential attributes of operation with major effects on the green supply chain, but also they are most closely attached to the features and requirements of the custom. Therefore, involvement of customers in the distribution design system and growth is likely to give an active network distribution. For instance, joining decision location to those vendors and customs will enhance JIT system [4].
(iv) Reverse Logistics. A major GSCM trend is the strategic recognition of reverse logistics. From the environmental perspective, reverse logistics means focusing primarily on the return of reusable and recyclable products and material into the forward supply chain.Effective design of RL network is the key in economic benefit provision required to start and sustain the initiatives of GSCM on a larger scale [ 9 , 43]. According to [44], reverse logistics was primarily motivated by economic condition, but not concerns on ecosystem protection $[45,46]$ have stated that reverse logistics is capable of bringing profitability only, reducing wastage and advert. Reverse logistics may be seen as the opposite of traditional or forward logistics [47]. According to [48, 49], reverse Logistics is a process where manufacturers initially accept shipping products from a point of consumption for possible recycling and remanufacturing. The basis of reverse logistics is illustrated by [47] as works around RL include the flow material management towards recycling and remanufacturing, which minimizes the cost of developing new products [49]. Reverse logistics is widely applied in automobile companies including General Motors and BMW [50]. Studies on logistics are thought of as part of the process of reverse logistics [51], which identified some stages in the channel of reverse logistics which include separation, collection, transitional process, densification, integration, and delivery.
(v) Packaging. Characteristics of packaging, for example, material size and shape, have effects on distribution as a result of their effect on the product transport characteristics. Reorganized loading patterns and good packages can minimize material usage and raise utilization of space in warehouses and in trucks as well as reduction in the level of handling. For systems where returnable packaging is encouraged, stronger customer supplier relationship with active channel for reverse logistics is required [9]. An active program for green packaging is decisive to the general environmental program success [52].

### 3.2.4. Environmental Protection Dimensions

(i) Green Design. Green design means product design or services with consideration to some environmental awareness which is a subset of green supply chain management. If companies desire good image, they
should design their products to be in line with the requirements of environmentally friendliness, and also as a result of limitations in resources and modern society environmental degradation [36, 37]. Green design is highly associated with environmental conditions as product safety, risk management, waste reduction and pollution avoidance, and resource protection to prevent being damaged. Moreover, analysis of lifecycle is needed in green design conception. Resource related and environmental related products are assessed by LCA through the protocol of production. Alternatively, green design has been adopted with the objective of lowering product impact on the environment throughout the entire life cycle of product such as raw material choice, production, marketing, development disposal, and use of products meaning "the systematic incorporation of environmental factors into the design of products and development" [28]. Many researchers have argued the green design; in 1998, Barros et al., which are related to the product reuse with the Dutch government's support, suggested a two-level model location. The activity of purchasing in reverse logistic system and design in 1998 was assessed [44]. Applied logarithms were created by [53] for product recovery design system. The study demonstrated that "core algorithm" and allocation "algorithms" are two system schedules that can assist in waste reduction. Measurement designs by Green include tracking materials and product reverse flow from raw material retrieval from the environment to product disposal to the environment [54].
(ii) ISO 14001. International organization for Standardization (ISO) is the world's international standard publisher. ISO is a national standard institute network of 161 countries, with its secretariat in Geneva, Switzerland, which runs the system. It is a nongovernmental organization that acts as a mediator between private and public sectors. Vendors who possess ISO 14000 certification are preferred in the purchase and procurement decisions; hence risk of the environment is lowered [4]. ISO 14000 is a recognized standard for the management of environment. Environmental management can interact between environment and modern human society. A good of researches have stressed the need for ISO 14000 in green supply chain management improvement [55].
(iii) Pollution Control. Major indicators are engaged in the recovery of resources conservation and resource utilization and resource type, environmental governance, environmental pollution, and reinvented ability. Moreover, it is made up of packaging materials, transport, emissions, and garbage [36, 37]. In the US, Air pollution Control Act and Water Pollution Control Act 1955 and 1948 were introduced by the Environmental Protection Agency (EPA). According to [56], the rules and standards for greenhouse gas prevention effects have been violated routinely. In the US discharge of greenhouse gas has risen by $14 \%$ from

1990 to 2008 (EPA 2008). Transport and generation of electricity are the major gas emitters. The total greenhouse discharge of gas worldwide from human activities increased by $26 \%$ from 1990 to 2005 [57].
(iv) Eco-Labeling. Eco-labeling is an approach that voluntarily to the certification of environmental performance. It is practices all over the world. A product that fulfils certain criteria effectiveness or standard is identified by "eco-label." It is awarded by a third-party organization for services or product that is ready to fulfill certain environmental criteria. Different organizations such as nonprofit and profit organizations, government have built eco-label program. There are differences in the issues addressed by eco-labelled programs; for example, energy star focuses on the use of energy during the operation of equipment, whereas others address environmental life cycle problems and ergonomic and worker and issues of safety and health. The decision to use eco-labels by purchasers to help in purchase decisions should review carefully the criteria to ensure that program reflects on their specific problems. The meaning of environmentally preferable purchase (EPP) is that items are identified and purchased with eco-labelled services and products; for example, "Green" from USA, "Blue Angel" from Germany, Ecomark from Japan, and Environmental Choice from Canada are known ecolabeled services or product certifications [28]. Ecolabeling is included in environmental attributes considered by consumer's recyclability, upgradability, low acoustic levels, energy efficiency, ease of serviceability, and limitation of hazardous material content [58].
(v) Life Cycle. Life cycle analysis is a major subconcept in green design analysis of life cycle introduced for environment and resource related product measurement of the production process [43]. The measurements include steps in the raw material production, extraction, distribution and remanufacturing, and recycling and disposal. According to [59] report that the analysis of lifecycle "examines and qualifies the energy and material used, wasted and assesses the impact of product on the environment." Regulation by government is also a measure that organizations use for the analysis of life cycle. Life cycle analysis framework has been discussed by [54]. The stage of life cycle product will essentially influence the supply chain greening. For instance, from the initial stages, the design seriously affected the product and environmental issues design, playing pivotal roles at this stage for the decline and maturity product life cycle stages, and process improvement, with an active system of reverse logistics, while the organizational environmental practices will be affected. In the analysis of multiproduct, the decision of environmental management becomes highly complicated. However


Figure 2: The causal evaluation model.
in the company's product portfolio variable environmental strategies and life cycle product development foci will depend on product maturity life cycle [4].

## 4. Causal Model for Improving Green Supply Chain

Based on criteria identification phase, direct and indirect effects and interactions between all elements are detected and causal model for green supplier selection is provided in Figure 2. To ensure the validity of the model, in this phase, five researchers from University Putra Malaysia (UPM), who work on SCM, are consulted.

In Figure 2, arrows (A and B) indicate the interactions between two elements and loops ( C and D ) show interdependencies between elements inside the clusters. As an instance, when the factors of a cluster "green logistics" depend on factors of "environmental protection" cluster, this relation is represented by an arrow from component of "green logistics" to element of "environmental protection."

In addition, Figure 3 demonstrates detailed causal evaluation model consisting of all elements and main criteria.

## 5. Application and Testing of the Model

For testing the model, as same as our previous research attempt [60], "case study with expert interview technique" is applied for this research. The objects were 10 professional experts who are working in supply chain departments of Iran Khodro Co. Each interview has been done individually by each expert and took time between minimum 30 minutes to maximum 45 minutes for each of them. First, the evaluation model along with all components and interactions between criteria was described for each of them. Next step in interview is determining relations between concepts according to loops and arrows. In this step, consolation committee with experts determines the relations among influential factors in causal evaluation model. Each expert performs pairwise comparisons between factors and gives the score from 0 to 4 according to their experiences and believes that factor $i$ affects factor $j$. For this purpose, a group of engineers are selected from Iran Khodro supply chain department SAPCO (Supplying Automotive Parts, Co.), the most important supplier and the main subset for Khorasan Iran Khodro Company. In fact, to apply DEMATEL technique, using expert's opinion among and within the elements to paired comparison analysis is required.


Figure 3: Detailed causal evaluation model.


Figure 4: Graphical initial direct impact relation between EP and GL.

## 6. Case Analysis Method

Green supplier selection problem is a sort of complex multiple criteria decision making problem including both quantitative and qualitative factors, which may be inconsistent and uncertain. Due to the nature of supplier selection, MCDM methods are required to handle and solve the problem effectively [60]. Multiple criteria decision making (MCDM) using the decision making trial and evaluation laboratory (DEMATEL) was proposed in this case analysis. The method of DEMATEL was chosen to assess the interdependence level existence of green supply chain management, for selected practice indicators [61, 62]. In DEMATEL structure, each factor or part may exert an effect and obtain from other higher or lower level factors. The entire factors establish worth and importance of factors instead of considering only specific factors [62, 63].
6.1. The Procedures of the DEMATEL Technique. There are 5 main steps for applying DEMATEL, [64]. The procedures of the DEMATEL method can be expressed as follows.

Step 1 (finding the direct-relation (average) matrix). At first we have four scales that determine the values of relationships between different factors according to the experts' opinion:
$0=$ no influence;
1 = low influence;

2 = high influence;
3 = very high influence.
There are $H$ experts and $n$ factor (criteria) to be considered. Each expert answers the certain questions to illustrate the degree of a criterion $i$ effect criterion $j$ due to her or his beliefs. For now $a_{i j}$ denotes pairwise comparisons between any two criteria and it is assigned integer score ranging from $0,1,2,3$, and 4 . The scores are given by each expert and $X^{1} X^{2} \cdots X^{H}$ are the answers of each of them that make the $n \times n$ nonnegative matrix $X^{k}=\left[x_{i j}^{k}\right]_{n \times n}$, with $1<k<H$. A high score indicates a belief that greater improvement in $i$ is required to improve $j$. Then it is possible to calculate the $n \times n$ average matrix $A$ on account of all experts' opinions by averaging the $H$ their scores as follows:

$$
\begin{equation*}
\left[a_{i j}\right]_{n \times n}=\frac{1}{H} \sum_{k=1}^{H}\left[x_{i j}^{k}\right]_{n \times n} . \tag{1}
\end{equation*}
$$

The average matrix $\left[a_{i j}\right]_{n \times n}$ is also called the initial directinfluenced matrix which indicates the initial direct effects each criterion exerts on and receives from other criteria. Moreover in this step, obtaining the causal effect between each pair of criteria in a system by drawing an influence map will be possible. Also consider the following.

If $a_{i j} \leq 1$ (independence is identified among all criteria; otherwise, nonindependence will be identified).

Table 1: Total-relation matrix of environmental protection and green logistics.

|  | Environmental protection and green logistics |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Procurement | Production | Distribution | Reverse logistics | Packaging | $R$ |
| Green design | 0.679 | 0.787 | 0.59 | 0.408 | 0.521 | 2.985 |
| ISO 14001 | 0.578 | 0.751 | 0.539 | 0.387 | 0.463 | 2.718 |
| Pollution control | 0.429 | 0.639 | 0.557 | 0.371 | 0.317 | 2.313 |
| Eco-labeling | 0.691 | 0.818 | 0.653 | 0.41 | 0.52 | 3.092 |
| Life cycle | 0.564 | 0.692 | 0.51 | 0.328 | 0.288 | 2.382 |
| C | 2.941 | 3.687 | 2.849 | 1.904 | 2.109 |  |

Table 2: Total-relation matrix of green logistics and environmental protection.

|  | Green logistics and environmental protection |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Green design | ISO 14001 | Pollution control | Eco-labeling | Life cycle | $R$ |
| Production | 0.42 | 0.065 | 0.365 | 0.282 | 0.338 | 1.47 |
| Procurement | 0.454 | 0.144 | 0.571 | 0.345 | 0.36 | 1.874 |
| Distribution | 0.062 | 0.024 | 0.341 | 0.095 | 0.046 | 0.568 |
| Reverse logistics | 0.237 | 0.014 | 0.281 | 0.134 | 0.062 | 0.728 |
| Packaging | 0.024 | 0.018 | 0.098 | 0.079 | 0.011 | 0.23 |
| $C$ | 1.197 | 0.265 | 1.656 | 0.935 | 0.817 |  |

Table 3: Ranking dimensions: environmental protection on green logistics.

|  | Ranking of affected factors | Ranking of influential factors |
| :--- | :---: | :---: |
| 1 | Production | Eco-labeling |
| 2 | Procurement | Green design |
| 3 | Distribution | ISO 14001 |
| 4 | Packaging | Life cycle |
| 5 | Reverse logistics | Pollution control |

TAble 4: Ranking dimensions based on I relation.

|  | Ranking of affected factors | Ranking of influential factors |
| :--- | :---: | :---: |
| 1 | Pollution control | Procurement |
| 2 | Green design | Production |
| 3 | Eco-labeling | Reverse logistics |
| 4 | Life cycle | Distribution |
| 5 | ISO 14001 | Packaging |

The structural relations between the criteria of a system are converted to intelligible and logical map of the system.

Step 2 (normalizing the initial direct-relation matrix). By normalizing the average matrix $A$, normalized initial direct relation matrix $D$ is obtained in the following formulation:

$$
\begin{gather*}
S=\max \left\{\max \sum_{j=1}^{n} a_{i j}, \max \sum_{i=1}^{n} a_{i j}\right\},  \tag{2}\\
D=\frac{A}{S}
\end{gather*}
$$

Consequently total direct effects that criterion $i$ exerts on the other criteria are obtained by sum of each row $i$ of matrix
$A$; the sum of each column $j$ represents most direct effects on others by total direct effects of the criterion. Likewise, since the sum of each column $j$ of matrix $A$ represents the total direct effects received from other criteria by criterion $i$; max $\sum_{i=1}^{n} a_{i j}$ represents the total direct effects that the criterion $j$ receives, the most direct effects, from other criteria and the positive numerical $s$ takes the smaller of the two as the upper bound, and the matrix $D$ is obtained by dividing each element of $A$ by the scalar $s$. Each element $d_{i j}$ of matrix $D$ is between zero and less than 1: $0<d_{i j}<1$.

Step 3 (calculating the total-relation matrix). A continuous reduction of the indirect effects of problems beside the powers of matrix $D$, like to an engrossing Markov chain matrix, guarantees convergent solutions to the matrix inversion.

Note that

$$
\begin{align*}
& D^{2}, D^{3}, \ldots, D^{\infty} \\
& \lim _{m \rightarrow \infty} D^{m}=[0]_{n \times n} \tag{3}
\end{align*}
$$

$$
[0]_{n \times n} \text { is a } n \times n \text { null matrix. }
$$

The total relation matrix $T_{n \times n}$ is achieved as follows:

$$
\begin{align*}
\sum_{m=1}^{\infty} D_{i} & =D+D^{2}+D^{3} \cdots D^{m} \\
& =D\left(I+D+D^{2}+\cdots+D^{m-1}\right)  \tag{4}\\
& =D(I-D)^{-1}(I-D)\left(I+D+D^{2}+\cdots+D^{m-1}\right) \\
& =D(1-D)^{-1}\left(I-D^{m}\right)=D(I-D)^{-1}
\end{align*}
$$

where $I$ is identity matrix and $T$ is total relation matrix ( $[T]_{n \times n}$ ).

Table 5: Total-relation matrix for elements of environmental protection.

|  | Environmental protection |  |  |  |  |  |  | EP4 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | EP1 | EP2 | EP3 | EP4 | EP5 | $R$ | $R+C$ | $R-C$ |
| Green design (EP1) | 0.677 | 0.75 | 0.923 | 0 | 0 | 2.35 | 5.46 | -0.76 |
| ISO 14001 (EP2) | 0.815 | 0.5 | 0.85 | 0.218 | 0.154 | 2.537 | 5.063 | 0.011 |
| Pollution control (EP3) | 0.514 | 0.443 | 0.358 | 0.189 | 0.133 | 1.637 | 4.946 | -1.672 |
| Eco-labeling (EP4) | 0.54 | 0.401 | 0.568 | 0.172 | 0.121 | 1.802 | 2.553 | 1.051 |
| Life cycle (EP5) | 0.564 | 0.432 | 0.61 | 0.172 | 0.121 | 1.899 | 2.428 | 1.37 |
| $C$ | 3.11 | 2.526 | 3.309 | 0.751 | 0.529 |  |  |  |

Table 6: Total-relation matrix for elements of green logistics.

|  | Green logistics |  |  |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | LG1 | LG2 | LG3 | LG4 | LG5 | $R$ | $R+C$ | $R-C$ |
| Procurement (GL1) | 0.523 | 0.491 | 0.924 | 0.924 | 0.734 | 3.596 | 6.977 | 0.215 |
| Production (G2) | 0.671 | 0.471 | 0.916 | 0.914 | 0.74 | 3.712 | 6.606 | 0.818 |
| Distribution (GL3) | 0.816 | 0.747 | 0.892 | 0.888 | 0.949 | 4.292 | 8.914 | -0.33 |
| Reverse logistics (GL4) | 0.645 | 0.557 | 0.847 | 0.844 | 0.688 | 3.581 | 8.191 | -1.029 |
| Packaging (GL5) | 0.726 | 0.628 | 1.043 | 1.04 | 0.667 | 4.104 | 7.882 | 0.326 |
| $C$ | 3.381 | 2.894 | 4.622 | 4.61 | 3.778 |  |  |  |

## 7. Case Analysis Result and Discussion

In this phase, at first in part one, both direct and indirect influences will be achieved by applying DEMATEL technique according to the arcs (A and B). After that in part two, direct and indirect influences and also interdependencies will be found out inside clusters based on loops (C and D).

Part One. Direct and Indirect Influences between Elements of Two Clusters.

Step 1. At first, a committee was formed including the ten experts from supply chain department in an automotive company which is named Iran Khodro.

Step 2 (finding the initial direct-relation (average) matrix). Figure 4 illustrates the values of relationships between elements that are determined by pairwise comparisons between any two criteria according to the experts' opinions and they are assigned integer score ranging from $0,1,2,3$, and 4 . Figure 4, shows network relation among the elements of two clusters. The values in Figure 4 are calculated according to (1).

Step 3. Normalized initial direct relation: direct-relation (average) matrix $D$ is obtained by normalizing the initial matrix $A$ according to (2).

Step 4. Calculate the total-relationships matrix $T$ for causal relation by achieving the $D(I-D)^{-1}$ according to (4). Tables 1 and 2 indicate the total-relation matrix $T$.

According to Table 1, eco-labeling has the highest degree of dispatching impacts ( $R$ ) on aspects of green logistics as 3.092. After eco-labeling, green design has major impact on green logistics cluster with 2.985 . Indeed, production,
procurement, distribution packaging, and reverse logistics get the most impact from eco-labeling and green design, respectively.

According to Table 2, in this interaction first, production and then procurement have the highest degree of dispatching impacts ( $R$ ) on aspects environmental protection factors as 1.874 and 1.47 , respectively. In contrast, pollution control receives the maximum effects from factors of green logistics as 1.656 , both direct and indirect.

## 8. Ranking Dimensions Based on Impact of Environmental Protection on Green Logistics

In this step all factors are prioritized according to their power in sending impact or receiving effect to other factors. According to Table 3, the maximum total effects both direct and indirect which are received from factors of environmental protection are related to production. As a result, to improve the green logistics cluster based on this relation, eco-labeling and green design have to be considered more than other factors in environmental protection group.

## 9. Ranking Dimensions Based on Impact of Green Logistics on Environmental Protection

Table 4 indicates the priority of all factors according to their power in sending impact or receiving effect from other factors based on Impact of GL on EP. According to prioritizing the factors, to develop the environmental protection cluster based on this relation, production and then procurement

Table 7: Ranking dimensions of environmental protection.

|  | Ranking of important factors | Ranking of affected factors | Ranking of influential factors |
| :--- | :---: | :---: | :---: |
| 1 | Green design | Pollution control | ISO 14001 |
| 2 | ISO 14001 | Green design | Green design |
| 3 | Pollution control | ISO 14001 | Life cycle |
| 4 | Eco-labeling | Eco-labeling | Eco-labeling |
| 5 | Life cycle | Life cycle | Pollution control |

Table 8: Ranking dimensions of green logistics.

|  | Ranking of important factors | Ranking of affected factors | Ranking of influential factors |
| :--- | :---: | :---: | :---: |
| 1 | Distribution | Distribution | Production |
| 2 | Reverse logistics | Reverse logistics | Distribution |
| 3 | Packaging | Packaging | Packaging |
| 4 | Procurement | Production | Reverse logistics |
| 5 | Production | Procurement | Procurement |



Figure 5: Graphical initial direct impact relation inside environmental protection.
have to be considered more than other factors in green logistics group.

Part Two. Direct and Indirect Influences between Elements Inside of Two Clusters. In this part all steps are same as part one, but calculations are based on loops. Figures 5 and 6 show network relations inside the elements of two clusters. The values in Figures 4 and 5, are calculated according to (1).

Normalized initial direct relation matrix $D$ is obtained by normalizing the initial matrix $A$ according to (2).

Total-Relation Matrix for Environmental Protection. The totalrelationships matrix $T$ for causal relation by achieving the $D(I-D)^{-1}$ according to (4). Tables 5 and 6 indicate the totalrelation matrix $T$.

Table 5 demonstrates the total-relationships matrix for aspects of environmental protection cluster. To the degree that Table 5, both direct and indirect impacts among dimensions of environmental protection are emerged.


Figure 6: Graphical initial direct impact relation inside green logistics.

According to the result of Table 5, the maximum degree of influential impact $R(2.537)$ and maximum degree of receiving impact $C$ (3.309) are related to ISO 14001 and pollution control, respectively. It means that ISO 14001 has the highest impacts on other aspects and pollution control is the most influenced factor in this category. In addition, by considering $(R+C)$ scores, the highest value belongs to green design with 5.46 . Hence, green design plays a central role between dimensions and it catches the most important position in the group. Nevertheless, the $(R-C)$ value of green design is negative $(-0.76)$, fairly below zero and it is an effect factor.

Based on the result of Table 6, the maximum values of $R$ (4.292) and $C$ (4.622) are related to distribution. It means that distribution has the greatest impacts on other factors and also it is influenced by other factors mostly at the same time. Similarly, by considering the amount of $(R+C)$, the highest value belongs to distribution. In spite of the considerable degree of importance of the role that distribution has in


Figure 7: Impact relationship diagram.
the system, it receives more impacts from the whole green logistics group.

Ranking Dimensions of Environmental Protection and Green Logistics. The priority of factor importance, factor influenced, and factor influencing are illustrated in Tables 7 and 8, respectively. According to Table 7, green design plays the most important role in EP cluster, but ISO 14001 has the highest impact on other 4 dimension in EP cluster. On the other hand, pollution control receives the most impact from other factors inside the EP cluster.

By considering Table 8, distribution plays the most important role in GL cluster, but it is the most affected factor in GL cluster. Production is the most powerful factor inside the GL cluster as well.

Impact-Relation Map of Environmental Protection and Green Logistics. The impact relationship diagram is provided by mapping a data series of $(R+C, R-C)$ in Figure 7. The causal diagram represents the importance of each of environmental protection and green logistic factors based results that are indicated in Tables 5 and 6 . Elements that are located in the positive part of the graph belong to the cause group and the others that are posited in the negative part of the chart belong to the effect group. It would be necessary to focus on the cause group dimensions which will be discussed in conclusion. According to the causal diagram in Figure 7, life cycle, eco-labeling, and ISO 14001, production, packaging, and procurement should be paid more attention to develop the system.

## 10. Conclusion

This study proposes the multilevel causal framework to improve green supply chain by applying the DEMATEL method to analyze and prioritize two essential green image groups and their elements in automotive industries. The results of this research enable enterprises to find out which suppliers are suitable by considering the environmental practices in proposed multilevel causal model. Finally, all concepts inside the clusters are divided into cause and effect groups by applying DEMATEL method. By considering Tables 5 and 6, in EP cluster cause group consists of "life cycle," "eco-labeling," and "ISO 14001" and effect group includes
"pollution control" and "green design" while in the GL cluster cause group contains "production," "packaging," and "procurement" and effect group includes "distribution" and "reverse logistics." To develop the whole system, enterprises and suppliers have to concentrate on the cause group because elements in cause group have considerable impacts on the system, especially on elements in effect group. By improving the cause group, the whole system will be developed automatically. In spite of the considerable degree of importance of the role that green design plays in the system, it gets more impacts through the environmental protection system; it suggests that although green design is just slightly influenced by other system aspects, it has significant influence on the other factors and plays the central role in environmental protection cluster at the same time. As a result it could not be the most powerful factor in this category. Therefore, the proposed model can be applied for two purposes: first, it would help enterprisers to find out which factors are more effective and important to select the best possible green supplier with regard to both direct and indirect relations between elements. On the other hand, if a company is a supplier for other companies and wants to implement environmental practices in its own manufacturing system, it can apply the model for the whole the system.

In fact, the case study of this research finds criteria that influence green supplier selection. The current study finds that both eco-labeling and procurement have a considerable impact on the other criteria, so by improving these 2 criteria, other criteria will be improved automatically. This study suggests further researches to extend the scope of this study or add green image group to proposed causal model for improving green supply chain practices more acquire.

## Conflict of Interests

The authors declare that this research work has no possible conflict of interests.

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## Research Article

# Choice Overload, Satisficing Behavior, and Price Distribution in a Time Allocation Model 

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#### Abstract

Recent psychological research indicates that consumers that search exhaustively for the best option of a market product-known as maximizers-eventually feel worse than consumers who just look for something good enough-called satisficers. We formulate a time allocation model to explore the relationship between different distributions of prices of the product and the satisficing behavior and the related welfare of the consumer. We show numerically that, as the number of options becomes large, the maximizing behavior produces less and less welfare and eventually leads to choice paralysis-these are effects of choice overload-whereas satisficing conducts entail higher levels of satisfaction and do not end up in paralysis. For different price distributions, we provide consistent evidence that maximizers are better off for a low number of options, whereas satisficers are better off for a sufficiently large number of options. We also show how the optimal satisficing behavior is affected when the underlying price distribution varies. We provide evidence that the mean and the dispersion of a symmetric distribution of prices-but not the shape of the distributioncondition the satisficing behavior of consumers. We also show that this need not be the case for asymmetric distributions.


## 1. Introduction

Facing a consumer decision in the market in the Western world today entails considering a very large number of options. From shopping at a modern supermarket to buying online, daily experience seems to highlight that a consumer choice is quite a demanding task in terms of processing all the available information in the relevant market.

Psychological research [1, 2] has recently revealed several adverse effects associated with such overload of choices, mainly the so-called paradox of choice and the paralysis effect. The former suggests the fact that an explosion of choice affects consumers' welfare in a way that "more is less." This is manifested in the shape of an inverted $U$ for the welfare with respect to number of options, what implies that a consumer is less satisfied when facing a choice problem with a large number of options. The paralysis effect refers to situations in which the overload is perceived so important that consumers choose not to choose. Rational time allocation was proposed
in [3] as an underlying framework to account for choice overload effects in a market with an increasing number of options. While choice overload is not considered a universal phenomenon [4], the analysis of the model in [3] reproduced the effects described above under certain natural conditions, for example, about consumer's preferences over time uses.

Further research from both economics and psychology $[5,6]$ suggests the following provoking idea: consumers who exhaustively search for the best product option in the market-maximizers-eventually feel worse than consumers who just look for something "good enough"-called satisficers, despite the fact that maximizers get a better deal. To analyze the "satisficing claim" above a more precise definition of "satisficer" is required, which seems to be lacking in the literature. We propose in this paper a definition for satisficer that fits well within the time allocation model in [3].

We consider a typical profile of consumer, that is, with a balanced preference among different uses of time and with a linear search cost (see Section 2 and [3] for more
details). Then, given an underlying probability distribution for the prices of the product, the welfare issue is analyzed by scanning different degrees of satisficer behavior for an increasing sequence of the number of product options/prices.

The basic inputs of the model thus are the price distribution and a (growing) sequence of product options, and the main output is a classification of satisficing behaviors in terms of welfare for each number of options. In particular, the analysis yields a whole of satisficer conducts-varying with the number of given product options-that are better off than the corresponding maximizer. Different price distributions are considered and the satisficing responses are analyzed. Similar qualitative results are obtained for different distributions that confirm the "satisficing claim." According to our analysis, satisficers tend to be better off than maximizers beyond certain number of options, whereas maximizers are better off for a relatively small number of options. The sensitivity of the results with respect to statistical parameters of the distribution (mean and dispersion of prices) as well as with respect to the shape of distribution itself is also addressed. A significant finding here is that, for symmetric distributions, changes in the shape of the distribution that preserve mean and dispersion do not influence the satisficing behavior. It is also shown that for asymmetric distributions that need not be the case.

The main features of the model are introduced in Section 2, the results obtained from the model analysis and their discussion are presented in Section 3, and some conclusions are gathered in Section 4.

## 2. Time Allocation, Price Distribution, and Satisficing Behavior

Our approach to the problem of choice overload and the "satisficing claim" relies on a version of the model of rational time allocation among competing uses of time that was introduced in [3]. A consumer decides how to spend her total available time $(T)$ in three different rival uses of time; so that

$$
\begin{equation*}
T_{s}+T_{f}+T_{w}=T, \quad T_{s}, T_{f}, T_{w} \geq 0 \tag{1}
\end{equation*}
$$

where $T_{s}$ is shopping time, $T_{w}$ is working time, and $T_{f}$ is nonworking time, or free time.

The consumer typically faces a large number of given market options for every product. We focus on her decision about acquiring a single product among the many versions of the product offered in the market. Let $N$ be the total number of product options that are available and that she could inspect to make her buying decision. Let $0 \leq a \leq 1$ denote the fraction of the total number $N$ that a consumer may decide to check, so that $n=a N$ is the number of options that she actually looks over. We may think of $a$ as the strategy adopted ex-ante by the consumer to solve her shopping problem, so that $a=1$ corresponds to a maximizer-she explores all possible market options-whereas $0<a<1$ refers to an $a$-satisficer-the one that looks only at a fraction $a$ of the options before making her buying decision.

Consumer's total expenditure is bounded from below by some quantity $G$ which is a function of the number of checked
options $n, G:=G(n)$. The consumer's decision is subject to the budget constraint

$$
\begin{equation*}
G(n) \leq w T_{w}+V \tag{2}
\end{equation*}
$$

where $w$ is the wage rate per unit of working time $\left(T_{w}\right)$ and $V$ is nonworking income or savings. Since $G(n)$ represents the best deal when $n$ options have actually been considered, it typically depends on $n$ in a nonincreasing fashion.

We assume that a set of $N$ prices of the different market options has first been obtained as a random sample of length $N$ of an underlying distribution $\mathscr{F}$, so that the consumer has to make her decision about checking a subsample of length $0 \leq n \leq N$ of the $N$ market options. The cases of $\mathscr{F}$ normal and uniform will be first considered below.

We also adopt the point of view that a representative consumer searching for the best deal in fact faces $G(n)=$ $\mathbf{E}\left[\min \left\{p_{1}, p_{2}, \ldots, p_{n}\right\}\right]$, that is, she solves her decision problem with $G(n)$ defined as the expected minimum price from a subsample of length $n$ of the available prices.

Since the expected best price decreases as the number of seen options increases, there is incentive to look for more options and consequently to spend more time searching in the market. Nevertheless, searching for more options is time consuming. Let $\tau=\tau(n)$ denote the minimum time that is required to find and evaluate $n$ options of the desired product. The consumer thus faces the time constraint

$$
\begin{equation*}
T_{s} \geq \tau(n) \tag{3}
\end{equation*}
$$

Notice that the shopping time floor defined by $\tau(n)$ may depend on the search efficiency of the consumer and also on the organization of the market of the product. In general, it may be assumed that $\tau(n)$ is a nondecreasing function of $n$. In this paper, we assume a linear form for the cost of inspection time, that is, $\tau(n)=c n$, so that $c>0$ is the inspection time per unit.

Under the standard assumption of rational behavior, the consumer seeks to maximize her welfare. Since this depends on the way she decides to use her time, her welfare can be written as a function

$$
\begin{equation*}
U\left(T_{s}, T_{f}, T_{w}\right) \tag{4}
\end{equation*}
$$

Therefore, the consumer choice problem is an optimization problem in which she determines the time distribution ( $T_{s}, T_{f}, T_{w}$ ) that maximizes her welfare (4) subject to the constraints (1), (2), and (3). Given the number $N$ of market prices, the optimal search strategy $n^{*}$ for the satisficer's problem-along with suboptimal strategies that beat the maximizer's welfare status-is obtained as a by-product of the time allocation solution.

## 3. Results and Discussion

We consider a standard profile of consumer; that is, she has balanced preferences among different uses of time in addition to linear inspection cost (see case \#1 in [3], where other profiles are considered). To illustrate the solution output, the model is first solved numerically for two natural


Figure 1: Price distribution and expected minimum prices, case of normal distribution $N(500,50)$.

(a) Satisficer correspondence (a-range better than maximizer for each $N$ ) and best satisficer curve (in red)

(b) Utility as $n$ increases $(N=150)$

Figure 2: Satisficing behavior and welfare, case of normal distribution $N(500,50)$.
price distributions, namely, normal and uniform. Table 1 summarizes the essential functional forms and parameter values for the reference case that will remain fixed for the sequel.

We first consider the case that prices are normally distributed. Synthetic price data are displayed in Figure 1(a) corresponding to $N=150$ along with the theoretical density, whereas the estimated expected minimum price as a function of $N$ is shown in Figure 1(b).

Figures 2(a) and 2(b) show the output of the numerical analysis of the model. Optimal welfare as a function of $N$
can be seen in Figure 2(b): it is apparent that welfare exhibits a shape of inverted- $U$, which indicates that the consumer suffers from choice overload: although her welfare increases when the number of product options is low, it starts to decrease when the market size $N$ reaches certain value $N^{*}$ (in this case, $N^{*}=22$ ). The welfare analysis for different satisficer strategies is represented in Figure 2(a). The picture represents the graph of the correspondence that, for each $N$, gives the range of $a$-satisficers (looking over a sample of $a N$ options out of the total $N$ ) that are better off than a maximizer (looking over the $N$ options). Specifically, let $U^{*}(n)$ be


Figure 3: Price distribution and expected minimum prices, case of uniform distribution with mean $\mu=500$ and standard distribution $\sigma=50$.

Table 1: Essential parameter values.

| Concept | Function | Parameter values |
| :--- | :---: | :---: |
| Total time | See (1) | $T=200$ |
| Budget | See (2) | $(V, w)=(0,6.5)$ |
| Utility | $a_{2} \ln T_{f}+a_{1} \ln T_{s}+$ | $\left(a_{1}, a_{2}, a_{3}\right)=$ |
| Search | $\tau(n)=c n$ | $(0.25,0.5,0.25)$ |
| cost |  | $c=2$ |
| Market <br> options | $10 \leq N \leq 150$ |  |

the optimal welfare of the consumer problem as a function of the number $n$ of checked options. Then Figure 2(a) shows the graph of the satisficer correspondence, defined by

$$
\begin{align*}
A(N)=\left\{a \in[0,1]: U^{*}(a N)\right. & \left.\geq U^{*}(N)\right\},  \tag{5}\\
10 & \leq N \leq 150 .
\end{align*}
$$

The red curve $a^{*}=a^{*}(N)$ inside the shaded region corresponds to the best satisficer, which enjoys maximal welfare for each value of $N$. The fading from red to green in the graph of $A(N)$ corresponds with the decrease in welfare as $a$ moves away from the red curve $a^{*}$. When $N$ is so large that the feasible set of time uses is empty, say, for $N>N_{p}$, there is not a solution for the consumer problem, what may be interpreted as choice paralysis, as it was explained in [3]. This effect may be noticed in the blank region on the upper right corner of Figure 2(a): from $N=N_{p}=67$ on, it is no longer feasible to explore more options and thus the upper contour of the graph of the correspondence decays as $N_{p} / N$.

Figures 3(a)-3(b) and 4(a)-4(b) show the numerical input and output of the model in the case of a uniform price
distribution with the same mean $(\mu=50)$ and standard deviation ( $\sigma=50$ ) as the normal distribution considered above.

From the numerical analysis above it is apparent that the model output does not differ qualitatively whether prices are distributed normally or uniformly. The graphs in Figures 2(a) and 4(a) show that a maximizing strategy gives maximal welfare for low values of $N$, whereas a whole variety of satisficing strategies produces more welfare than the maximizing strategy for larger values of $N$, due to choice overload.

Under choice overload effects, the typical graph of the satisficer correspondence $A(N)$ is as shown in Figure 5. Specifically, for values of $N$ below $N^{*}$, the graph of the satisficer correspondence is in fact the curve corresponding to $a=1$-the maximizer's curve-because checking all options is the best strategy. This defines the maximizer zone in Figure 5. Then choice overload enters the scene and maximizing is no longer the best strategy in terms of welfare. Rather, for each $N$ beyond $N^{*}$, there is an interval of $a$-satisficers that are better off than maximizers. Within the satisficer zone I in Figure 5 the size of the satisficer intervals and the height of the graph increase for $N$ between $N^{*}$ and $N_{p}$. Beyond the number $N_{p}$ of options, the maximizer experiences paralysis-no time allocation is feasibleand the fraction of options that can be checked decreases as $N$ increases. In this region-the satisficer zone II in Figure 5-the width of the correspondence graph decreases as the number of options increases.

Notice that Figures 2(a) and 4(a) look strikingly similar. In fact, the threshold numbers between the three regions of the graph are $N^{*}=22$ and $N_{p}=67$ in both cases, which is remarkable. The resemblance between the two satisficer correspondences suggests that satisficing behaviors may not respond to the way in which prices are distributed, as long


Figure 4: Satisficing behavior and welfare, case of uniform distribution with $\mu=500$ and $\sigma=50$.


Figure 5: Model output: typical satisficer correspondence $A(N)$ and best satisficer curve $a^{*}(N)$. The two curves $\bar{a}$ and $\underline{a}$ limit the contour of the graph of $A(N)$. They define three different satisficer regions. In the maximizer zone where $\underline{a}(N)=\bar{a}(N)$, welfare is optimal when all options are checked and the best satisficer is the maximizer. In the satisficer zone I , the consumer is able to choose as a maximizer but optimal welfare is obtained looking over a fraction $a^{*}$ of $N$ which is decreasing as $N$ increases. In the satisficer zone II, the optimal fraction is also decreasing but the consumer cannot act as a maximizer, who is experiencing choice paralysis.

Table 2: Shocks $a_{h}$ below are independent and uniformly distributed in $[-\theta, \theta]$ where $\theta$ is adjusted as $H$ changes so as to keep $\sigma$ constant.

|  | $\mu$ | $\sigma$ | $\mathscr{F}$ | $H$ |
| :--- | :---: | :---: | :---: | :---: |
| Green line |  |  |  | $H=2$ |
|  | 700 | 150 | $\mu+\frac{1}{H} \sum_{h} a_{h}$ | $H=20$ <br> Red line |
|  |  |  | $H=30$ |  |

as they have a common mean and standard deviation, as in the two cases studied above. To further explore this claim,
we depart from a normal distribution $N(\mu=700, \sigma=150)$ as the benchmark case and consider a family of distributions $\mathscr{F}_{H}$ obtained as an average of $H$ identical shocks around $\mu$, which are independent and uniformly distributed. This allows us to rank $\mathscr{F}_{H}$ from the uniform distribution (when $H=1$ ) up to an asymptotically normal distribution (when $H$ is very large). Specifically, we consider that each price observation $p$ is obtained as

$$
\begin{equation*}
p=\mu+\frac{1}{H} \sum_{h} a_{h} \tag{6}
\end{equation*}
$$

where $\left\{a_{1}, \ldots, a_{H}\right\}$ are the iid shocks uniformly distributed on some interval $[-\theta, \theta]$. Notice that $\mu$ is the mean of $\mathscr{F}_{H}$ regardless of the number of shocks. Additionally, $\theta$ is changed as $H$ varies so as to keep the variance of $\mathscr{F}_{H}$ at its benchmark value. It is clear that $\mathscr{F}_{H}$ remains symmetric for any $H$. Roughly, the distance to normality is parameterized by the number of shocks. We consider three distributions generated by intermediate values of $H$ between the uniform and the normal cases (see Table 2). The output of the model for this set of distributions is displayed in Figures 6(a) and $6(\mathrm{~b})$. The analysis produces best satisficer curves $a^{*}(N)$ and satisficer correspondences $A(N)$ that are all practically indistinguishable. Also, no significant difference is perceived in the related best satisficer's utilities, as it can be seen in Figure 6(b). This numerical exercise supports our claim that different symmetrical distributions with common mean and dispersion have the same influence either in the satisficer behavior or her optimal welfare.

Observe that, once the essential parameters of the model have been fixed, the satisficer correspondence $A(N)$ and the best satisficer curve $a^{*}(N)$ depend only on the price distribution $\mathscr{F}$. The numerical experiment above strongly suggests that if $\mathscr{F}_{1}$ and $\mathscr{F}_{2}$ are different symmetric distributions with the same means $\mu_{1}=\mu_{2}$ and standard deviations $\sigma_{1}=\sigma_{2}$


Figure 6: Satisficing behavior and induced optimal welfare for a set of distributions with different shapes preserving mean and dispersion (see Table 2).

Table 3

|  | $\mu$ | $\sigma$ | $\mathscr{F}$ |
| :--- | :---: | :---: | :---: |
| Green line | 500 |  |  |
|  | 600 | 50 | Normal |
| Red line | 700 |  |  |

TAble 4

|  | $\mu$ | $\sigma$ | $\mathscr{F}$ |
| :--- | :---: | :---: | :---: |
| Green line |  | 50 |  |
|  | 700 | 100 | Normal |
| Red line |  | 150 |  |

then the curves $a^{*}(N)$ and correspondence $A(N)$ generated by each distribution coincide. In order to test the reversed implication and, at the same time, to study the response of the consumer behavior to a change in the parameters of the distribution, we compare $a^{*}(N)$ and $A(N)$ for dispersionpreserving distributions with different means and for meanpreserving distributions with different dispersions.

We first take the set of normal distributions $N(\mu, \sigma=$ 50) with means varying in the range $\mu \in\{500,600,700\}$, while all other parameters are kept fixed. Table 3 summarizes the parameter values for this exercise. Figures 7(a) and 7(b) show the output of the model for this set of distributions. It is apparent that, as the mean increases, both the best satisficer curves $a^{*}(N)$ and the satisficer correspondences $A(N)$ shift downwards. Thus, as the mean price increases, the best satisficer looks over a fewer proportion of price quotes no matter the size of available prices. Also, as it could be expected, the welfare of the best satisficer diminishes uniformly as the mean price increases.

Next we study the changes in the satisficer graphs when a mean preserving normal price distribution becomes more dispersed. The set of normal distributions $N(\mu=700, \sigma)$ with $\sigma \in\{50,100,150\}$ is considered (see Table 4). The results are displayed in Figures 8(a) and 8(b). In contrast with the response with a dispersion-preserving mean shift, now it is observed that both the satisficer correspondences and the best satisficer curve shift upwards when the dispersion increases. Also, when there is a sufficiently large number of price data so that increasing dispersion is noticeable, the utilities of the best satisficer are ordered according to the corresponding dispersions (see Figure 8(b)). Thus, when prices become more dispersed without increasing in mean, the happiest consumer looks over a bigger number of batch prices and in turn her welfare increases.

The normal distribution has been selected above as a benchmark to illustrate the analysis of the model. It can be checked that the qualitative results above also hold for other symmetric distributions. In general, an increase in the mean shifts the best satisficer curve $a^{*}(N)$ downwards whereas an increase in the dispersion has the opposite effect: it shifts the curve $a^{*}(N)$ upwards. Also, the maximizer region and the satisficer zone I (see Figure 5) are affected by changes in the mean and standard deviation: if the price mean increases the maximizer region shrinks and satisficer conducts appear for a lower number of options; whereas if the dispersion increases the satisficer zone I expands and the maximizer behavior remains for a larger number of options.

The numerical study above indicates the following invariance property of the model: the satisficer behaviordescribed by $A(N)$ and $a^{*}(N)$-induced by two different symmetric price distributions changes if and only if the distributions differ either in their means or in their standard deviations-but not in their shapes.


Figure 7: Analysis of satisficing behavior for a dispersion-preserving mean shift under a normal distributions Color of graphs shifts from green to red as $\mu$ increases (see Table 3).


Figure 8: Analysis of satisficing behavior and welfare for mean-preserving dispersion shift under a normal distribution Color of graphs shifts from green to red as $\sigma$ increases (see Table 4).

Given that finding a lower price expands the set of affordable time allocations and in turn it might improve welfare, the consumer has an incentive to search for low prices. Since they are to be found at the left tail of the distribution, the symmetry of the distribution may play a crucial role in the invariance property described above. It can thus be suspected that distributions with the same mean and standard deviation but that are asymmetrically dispersed around the mean do not satisfy the invariance property. In order to confirm this claim, we analyze the satisficer
behavior for a set of asymmetric distributions with common mean and dispersion. Specifically, we consider split-normal distributions $\mathrm{SN}\left(m, \sigma_{1}, \sigma_{2}\right)$. A split-normal distribution $\mathscr{F}$ is characterized by three moments- $m, \sigma_{1}$, and $\sigma_{2}$-such that an arbitrary realization of $\mathscr{F}$, say $p$, is distributed $N\left(m, \sigma_{1}\right)$ if $p \leq m$, whereas it is distributed $N\left(m, \sigma_{2}\right)$ if $p>m$. Clearly, the normal (symmetric) distribution is the particular case $\sigma_{1}=\sigma_{2}$, for which $m$ is the mean and the mode. If $\sigma_{1} \neq \sigma_{2}$, then $m$ is mode, while the mean lies to the left (right) of $m$ whenever $\sigma_{1}$ is larger (smaller) than $\sigma_{2}$. Below we


Figure 9: Histogram for a split-normal distribution $\mathrm{SN}\left(m=559.82, \sigma_{1}=50, \sigma_{2}=225.68\right)$ with $\mu=700$ and $\sigma=150$ (see Table 5).


Figure 10: Analysis of satisficing behavior for dispersion and mean preserving split-normal distributions as left dispersion $\sigma_{1}$ increases. Color of graphs shifts from green to red as $\sigma_{1}$ increases (see Table 5).

Table 5: The parameters $m$ and $\sigma_{2}$ below are adjusted as $\sigma_{1}$ changes so as to keep $\mu$ and $\sigma$ constant.

|  | $\mu$ | $\sigma$ | $\mathscr{F}$ | $\sigma_{1}$ |
| :--- | :---: | :---: | :---: | :---: |
| Green line |  |  |  | 50 |
|  | 700 | 150 | $\mathrm{SN}\left(m, \sigma_{1}, \sigma_{2}\right)$ | 100 |
| Red line |  |  |  | 150 (Normal) |

denote the mode by $m$, while we keep $\mu$ and $\sigma$ for mean and standard deviation of $\mathscr{F}$, respectively. A typical asymmetric split-normal histogram can be seen in Figure 9.

The model is solved for a set of three split-normal distributions $\mathrm{SN}\left(m, \sigma_{1}, \sigma_{2}\right)$ with $\sigma_{1} \in\{50,100,150\}$ and with $m$ and $\sigma_{2}$ adjusted in each case to keep the same mean $\mu=700$ and standard deviation $\sigma=150$. The results are displayed in Figures 10 (a) and $10(\mathrm{~b})$. Observe that the qualitative changes of the satisficer graphs $A(N)$ and $a^{*}(N)$ as $\sigma_{1}$ increases mimic those of the corresponding graphs in Figures 8(a) and 8(b). That is, as $\sigma_{1}$ increases-in turn the left tail of the distribution becomes thicker-the satisficer graphs shift upwards and the best satisficer improves her welfare. These results show that it is the dispersion on the left tail of the distribution what ultimately modifies the search behavior of satisficers and in
turn causes the positive effect on her welfare. Furthermore, it confirms that the symmetry of the distribution is required for the invariance property of the model to hold.

## 4. Conclusions

We provide a consumer behavioral model based on rational allocation of time which supports several empirical facts from psychological research. First, the consumer may typically not look over all the product options in the market when searching for a best deal. Second, for a small number of options, consumers who exhaustively look at all product options (maximizers) are better off than consumers who do not consider all options and look over a smaller sample (satisficers). For a large number of options a wide range of satisficers are better off than maximizers. Given a consumer profile and a price distribution for the product options, the model supplies the class of satisficer's strategies that are better off compared with the maximizer's strategy.

Our model analysis confirms a choice overload phenomenon: the higher the number of available options, the lower the fraction of them; a consumer should actually check when seeking her optimal welfare. Remarkably, this is so regardless how prices are distributed in the market. The numerical analysis of the model strongly supports an invariance property with respect to symmetric price distributions, namely, that satisficing conducts depend on the mean and dispersion of prices but not on the shape of the distribution. It is shown that either an increase in the mean or a decrease in the dispersion of prices implies that a relatively less exhaustive search behavior is a better strategy, in terms of welfare, for the consumer. It is also shown that changes in the distribution that preserve mean and standard deviation may alter the satisficer response if the dispersion of the distribution increases only asymmetrically on the left side of the distribution.

The results in this paper support the idea that choice overload phenomena are a behavioral trait of consumers which depends heavily on the number of product options and also on the mean of prices and on the dispersion of low prices. If the distribution of prices in the market is symmetric, its shape plays a limited role in the consumer's satisficing behavior.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Modelling Oil-Spill Detection with Swarm Drones 

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Nowadays, swarm robotics research is having a great increase due to the benefits derived from its use, such as robustness, parallelism, and flexibility. Unlike distributed robotic systems, swarm robotics emphasizes a large number of robots, and promotes scalability. Among the multiple applications of such systems we could find are exploring unstructured environments, resource monitoring, or distributed sensing. Two of these applications, monitoring, and perimeter/area detection of a given resource, have several ecological uses. One of them is the detection and monitoring of pollutants to delimit their perimeter and area accurately. Maritime activity has been increasing gradually in recent years. Many ships carry products such as oil that can adversely affect the environment. Such products can produce high levels of pollution in case of being spilled into sea. In this paper we will present a distributed system which monitors, covers, and surrounds a resource by using a swarm of homogeneous low cost drones. These drones only use their local sensory information and do not require any direct communication between them. Taking into account the properties of this kind of oil spills we will present a microscopic model for a swarm of drones, capable of monitoring these spills properly. Furthermore, we will analyse the proper macroscopic operation of the swarm. The analytical and experimental results presented here show the proper evolution of our system.

## 1. Introduction

Maritime activity has been increasing gradually in recent years. For example, around 42000 vessels (excluding the fishing ones) pass throughout the North Sea, carrying products that can adversely affect the environment, such as oil, which can produce high levels of pollution in case of being spilled into sea. Moreover, many pollutants are accidentally spilled from ships during "normal" operations. These spills are probably small but become significant due to the large number of ships.

Dramatic incidences of marine pollution, such as the Prestige oil spill off the Spanish north coast [1-3], have highlighted the potential for human-caused environmental damage. In attempting to mitigate or avoid future damage to valuable natural resources caused by marine pollution, research has been undertaken by the scientific community to study the processes affecting the fate and distribution of marine pollution and especially to model and simulate these processes. Furthermore, active systems, able to detect and
track such kind of spills, are an invaluable tool to help to locate and clean the affected resources [4-6].

Moreover, swarm robotics is an approach to solve problems inspired by the collective behaviour of social animals and it is focused on the interaction of multiple robots. It is a different approach to classical artificial intelligence, where the main goal is usually to develop behaviours that mimic human brain function. Swarm robotics is based on the metaphor of social insects and emphasizes aspects like decentralized control, limited communication between agents, local information, emergence of a global behaviour, and robustness [7-10]. As it has been shown [10] it is different from other multirobotic systems, because, for example, the swarm robots are relatively simple.

However, the distributed nature of these systems makes it difficult to develop an architecture that correctly models a swarm and coordinates it to perform complex tasks. In order to design a swarm behaviour, a mathematical model must be provided for both the individual agents and the whole swarm [11, 12]. These models will be tested to evaluate
the performance of the swarm before its deployment in real UAVs. This is mainly because this kind of systems has emergent properties that make it difficult to predict the overall operation of the swarm when only the local behaviour of the agents is being analysed.

In this paper, we will present a swarm behaviour to be used with robotic drones in order to cover the perimeter or the area of an oil spill. We will use GNOME [13], a very realistic environmental simulation system for marine pollutants. With the data provided by this simulation, using real maritime information and air currents predictions, we will create a virtual oil spill in the Spanish coast. We will show that our behaviour is able to both detect and track the spill for a given period of time. Moreover, we will also show, through the use of the macroscopic model derived by the local rules of the robots, that the swarm will track the oil pollutants. A Fokker-Plank equation will be derived from the microscopic behaviour in order to predict the probability that a drone stays in a given position at time $t$.

## 2. Pollutant Dispersion Model

Whether the trajectory model is accurate, adequate, or correct is often questioned. The accuracy and adequacy of a model are associated with the data used for the calculations and modelled physical processes. The modelling of an oil spill is not a simple task, mainly due to the number of factors influencing the trajectories of pollutants: sea currents, winds, and even the gravitational force or surface tension of water.

There are several applications that model and simulate pollutant discharges into the sea. These applications are based on the modelling of the most important forces that interact on an oil slick. In this paper, one of these applications has been used to obtain realistic data of an oil spill off Spanish coast. As it will be seen in subsequent sections, we use both ocean currents and wind data. These data will be used for testing a realistic oil spill using a swarm of flying robots (drones).

More specifically, we have used GNOME, an interactive environmental simulation system designed for the rapid modelling of pollutant trajectories in the marine environment. GNOME simulates oil movement due to winds, currents, tides, and spreading. It was developed by the Hazardous Materials Response Division (HAZMAT) of the National Oceanic and Atmospheric Administration Office of Response and Restoration (NOAA OR\&R). HAZMAT uses GNOME during spill events to calculate the spill trajectory and the associated uncertainty in that trajectory. As it has been mentioned in [14], GNOME can be used for predicting how winds, currents, and other processes might move and spread oil on the water, for studying how predicted oil trajectories are affected by inexactness (uncertainty) in current and wind observations and forecasts, and for predicting the chemical and physical changes of the oil spill during the time that it remains on the water surface.

One of the benefits of this tool is the possibility of using external files containing information on tides, currents, and so forth. If this information is added, then GNOME uses it to make the trajectory prediction of the spill (even using
uncertainty during simulation) in the specified region. The output from the model consists of graphics, movies, and data files for postprocessing in a geographic information system (GIS) or NOAA Emergency Response Division's (ERD1) inhouse model GNOME Analyst.

The model used by this tool is general and applies to trajectory problems. It is an Eulerian/Lagrangian model that is two-dimensional (2-D) in space, but vertically isolated layered systems can be modelled. Shoreline maps are inputs to the model, so any area can be modelled. The model automatically handles either hemisphere or east or west longitude.

More specifically, random spreading (diffusion) is calculated by a simple random walk with a square unit probability. The random walk is based on the diffusion value, $D$, that represents the horizontal turbulent diffusion of the spill in the water. During a spill, the value is calibrated as based on overflight data. In this way diffusion and spreading are treated as stochastic processes. Gravitational and surface tension effects are ignored, as these are only important during the first moments of a spill. Complex representation of subgrid diffusion and spreading effects are ignored.

The main diffusion equation used in GNOME is presented in [13], where $D_{x}$ and $D_{y}$ are the scalar diffusion coefficients in $x$ and $y$ directions and $C$ is the pollutant concentration:

$$
\begin{equation*}
\frac{\partial C}{\partial t}=D_{x} \cdot \frac{\partial^{2} C}{\partial x^{2}}+D_{y} \cdot \frac{\partial^{2} C}{\partial y^{2}} \tag{1}
\end{equation*}
$$

GNOME simulates this diffusion with a random walk with any distribution, with the resulting diffusion coefficient being one-half the variance of the distribution of each step divided by the time step:

$$
\begin{equation*}
D_{x}=\frac{1}{2} \cdot \frac{\sigma^{2}}{\Delta t}, \tag{2}
\end{equation*}
$$

where $\sigma^{2}$ is the variance of the distribution of diffused points and $\Delta t$ is the time elapsed between time steps.

Evaporation in GNOME is modelled by a simplistic 3phase evaporation algorithm where the pollutant is treated as a three-component substance with independent half-lives [15]:

$$
\begin{align*}
X_{\text {prob }}= & \left(P_{1} \cdot\left(2^{-t_{i} / H_{1}}-2^{\left(t_{i-1}-2 t_{i}\right) / H_{1}}\right)\right. \\
& +P_{2} \cdot\left(2^{-t_{i} / H_{2}}-2^{\left(t_{i-1}-2 t_{i}\right) / H_{2}}\right) \\
& \left.+P_{3} \cdot\left(2^{-t_{i} / H_{3}}-2^{\left(t_{i-1}-2 t_{i}\right) / H_{3}}\right)\right)  \tag{3}\\
& \times\left(P_{1} \cdot 2^{-t_{i} / H_{1}}+P_{2} \cdot 2^{-t_{i} / H_{2}}+P_{3} \cdot 2^{-t_{i} / H_{3}}\right)^{-1}
\end{align*}
$$

where $t_{i}$ and $t_{i-1}$ are the time elapsed (age; in hours) at time step $i$ and the previous time step $t-1$, respectively, since the LEs release. $H_{1}, H_{2}$, and $H_{3}$ are the half-lives of each constituent (in hours) for the pollutant, and $P_{1}, P_{2}$, and $P_{3}$ are the percentages of each constituent (as decimals) for the pollutant.

Spilled substances are modelled as point masses (up to $10,000^{4}$ ) called LEs (Lagrangian elements) or "splots" (from "spill dots"). Spills can be initialized as one-time or continuous releases and as point or line sources or evenly spaced in a grid on the map for diagnostic purposes.

Once GNOME executes a simulation, the solution is produced in the form of a trajectory. GNOME provides two solutions to an oil spill scenario, the best estimate trajectory and the uncertainty trajectory. The best estimate solution shows the model result with all of the input data assumed to be correct. The uncertainty solution allows the model to predict other possible trajectories that are less likely to occur, but which may have higher associated risks. In this paper we will use the uncertainty solution of pollutant particles (represented by its LEs) for generating a continuous pollutant map. More details of this mathematical model could be obtained in [13].

## 3. Swarm Design

In this section we will analyse the features that must have a robotic swarm to detect and follow an oil spill in effective way. We will also propose a specific microscopic behaviour for this task. As mentioned above, the modelling of this kind of pollutants is a complex task because of the interaction of many factors. However, it is easy to extract a set of characteristics that a system needs to locate this kind of spills.

On the one hand, the swarm must be able to detect and follow pollutants that can change and move in time mainly by the action of advection and diffusion. Depending on the application, two situations can be found: the origin of the spill is known or the swarm initially makes the detection process (in this work it is assumed that detection is one of the tasks to be performed). On the other hand, the appearance of several polluting slicks is very probable due, among other factors, to the transport and deposit of sediments on the coast while the oil slick disperses and evaporates.

The behaviour of the swarm must be highly robust and tolerant to failures and should be totally distributed. Therefore, all agents must have the same importance when performing the behaviour, without the existence of any agent more prominent than another. Finally, the behaviour should be highly scalable for two reasons: robust issues and performance of the behaviour, since as a first step it may be beneficial to use a reduced number of agents until they find any evidence of a particular discharge.

Although in this paper behaviours will be analysed in a simulated way, the features of agents are directly based on flying robots (our intention is to use these algorithms in drones). These drones will have a video camera that will use a vision algorithm to determine the existence of any oil slick in its visual field (as presented in [4, 16]). For security reasons, we assume that drones will fly at different heights, so that collisions in a position $(x, y)$ are avoided. We also assume that due to flight altitude (about 500 m above sea level) the differences in the visual field caused by different flying altitudes are negligible. All tests, unless otherwise specified,
will be carried out by a swarm of 200 drones. It is a mediumsized swarm, appropriate for simulation.

Our main goal, once the behaviour is designed, is to determine the ability of the swarm to locate, converge, and follow an oil spill. Therefore, a macroscopic model to predict the global behaviour of the swarm and to verify its performance will be specified. More specifically, we propose a homogeneous behaviour, executed by all agents, consisting of three states. Initially, drones look for any trace of the spill in the environment. Once the spill is detected, the drone will head to it. Finally, the drone will try to stay inside it (to cover the slick) or in its perimeter (to mark it).

Figure 1 shows the finite state machine (FSM) that governs the behaviour of each robot. The initial state is Wander, since we initially assume that the position of the spill is unknown. The transition from Wander state is performed when the agent's visual sensor detects an oil slick. In this case, the new state will be Resource. This state (Resource) has two possible transitions: the spill is not detected (transition $d$ ) so the system returns to Wander state or the amount of oil detected is $>80 \%$ of the image (transition $b$ ), so, the system switches to InResource state. The agent remains in this state if the detected oil percentage is $>80 \%$; otherwise, the system returns to Resource state.

Next, each behaviour specified in the FSM will be described in more detail.

At the beginning of the execution agents start at Wander state. At that time they have no idea where to find an oil spill. Therefore, all directions are equally valid and they will move randomly. The velocity of the drone at time $t$ is defined as

$$
\begin{equation*}
\mathbf{v}_{w}(t)=\mathbf{v}_{w}(t-1)+\text { rand } \cdot \mu_{1} \tag{4}
\end{equation*}
$$

where rand is a random uniform vector defined within the interval of the maximum and minimum velocity of the drone, and $\mu_{1}$ is the coefficient of variability on the current velocity. With values close to 1 , the robot will move in a totally random way.

If a drone detects the resource, it heads to the resource to position itself at the perimeter or over it, depending on the desired behaviour. The velocity of the agent is defined by three factors:

$$
\begin{equation*}
\mathbf{v}=\alpha_{1} \times \mathbf{v}_{c}+\alpha_{2} \times \mathbf{v}_{o}+\alpha_{3} \times \mathbf{v}_{r}, \tag{5}
\end{equation*}
$$

where $\alpha_{1}+\alpha_{2}+\alpha_{3}=1$ and these values define the intensity of each factor.

More specifically, $\mathbf{v}_{c}$ specifies the direction of the robot. This direction is determined by the area with the larger intensity resource average:

$$
\begin{equation*}
\mathbf{v}_{c}=\gamma(S) \times\left\|\sum_{s \in S}((\boldsymbol{\operatorname { p o s }}(s)-\operatorname{pos}(\mathrm{rob})) \cdot s)\right\|, \tag{6}
\end{equation*}
$$

where $S$ is sensor readings that detect the resource at a given time, $\operatorname{pos}(s)$ is the position vector of a reading, and pos(rob) is the position of the robot. We assume that the intensity of the resource is in the range $[0,1]$, where 0 is the complete lack of resource and 1 is the unambiguous detection of it.


Figure 1: Finite state machine (FSM) that governs the operation of each agent. The initial state is Wander. The transition $a$ is triggered when the agent's visual sensor detects an oil slick. Transition $b$ occurs when the amount of oil detected is $>80 \%$ of the image. Transition $c$ is triggered when the amount of oil is $\leq 80 \%$ of the image. Finally, transition $d$ is triggered if no oil is detected.
$\gamma$ determines the direction of the velocity vector, depending on whether the robot is outside or inside de resource. The aim of this variable is, therefore, to keep the robot on the perimeter of the resource:

$$
\gamma(S)= \begin{cases}1 & \frac{\Sigma_{s \in S} s}{|S|} \leq \eta  \tag{7}\\ -1 & \frac{\Sigma_{s \in S} s}{|S|}>\eta\end{cases}
$$

where $\eta$ is a threshold that determines, from the quantity of resource detected ( 0 means no resource and 1 maximum quantity of resource), if the agent is located on the perimeter. If the main objective of the system is that drones cover the pollute slick, then $\gamma(S)$ will be defined as 1 for any set of readings $S$ at a given time.
$\mathbf{v}_{\mathbf{o}}$ specifies an avoidance vector with respect to all robots detected at a given moment:

$$
\begin{equation*}
\mathbf{v}_{\mathbf{o}}=\left\|\sum_{i=1}^{|R|}\left(\boldsymbol{\operatorname { p o s }}\left(r_{i}\right)-\boldsymbol{\operatorname { p o s }}(\mathrm{rob})\right)\right\|, \tag{8}
\end{equation*}
$$

where $R$ is the set of detected robots, $\operatorname{pos}\left(r_{i}\right)$ is the position of the detected robot $i$, and pos(rob) is the position of the current robot.

Moreover, we will take into account the accuracy of the transmitted locations: there are several factors that could make these locations not to be optimal. We will include, therefore, a random component to model this uncertainty in the movement of the robot: $\mathbf{v}_{\mathbf{r}}(t)=\mathbf{v}_{r}(t-1)+\mathbf{r a n d} \cdot \mu_{2}$, where $\mu_{2}$ is the coefficient of variability on the velocity.

Finally, when drones are located inside the spill and, therefore, the borders of the resource are not detected, we assume that drones will develop a random wander behaviour until they find water again because there is no other information about which direction is better to follow:

$$
\begin{equation*}
\mathbf{v}_{s}(t)=\mathbf{v}_{s}(t-1)+\text { rand } \cdot \mu_{3}, \tag{9}
\end{equation*}
$$

where $\mu_{3}$ is the coefficient of variability on the velocity.

In this section we have defined a microscopic behaviour for the detection and tracking of an oil spill. Each agent of the swarm executes this behaviour locally. It is a simple behaviour that can be run in low processing capacity systems. This behaviour does not require any communication between agents or a global positioning system.

## 4. Experimentation

We will now present a set of experiments in order to test the operation of the proposed microscopic behaviour. As we will see, these experiments use simulations of oil spills based on real data.

We have simulated an oil spill in the coastal area of "Sant Antoni de Portmany," on Ibiza island, using an area of approximately $350 \mathrm{~km}^{2}$. This area has been chosen because of the various currents that affect it and its proximity to the peninsula. Real weather data have been used for both sea and air currents, considering the start date of the spill on April 10,2013 , at $0: 00 \mathrm{~h}$. We use these real data to run a simulation for seven days from the start of the spill, simulating a ship loaded with 100000 barrels of oil. Figure 7 shows the map of the main area affected and the initial place of impact. In order to work with data generated by GNOME we use the average of the points of discharge generated by the application (More specifically, $M(t)^{\prime}=G\left(M(t) \oplus s_{\text {shape }}\right.$ ) where $M(t)$ is the map obtained with GNOME at time $t, G$ is a Gaussian filter with $\sigma=10|S|$, and $\oplus$ is a morphological binary operator) (splots in GNOME), making uniform the continuity between nearby contaminated areas and simplifying the simulation. This allows us to reduce the number of splots needed in the simulation:

$$
\begin{gathered}
\text { Robots }=200 \\
\mu_{1}=0.3 \\
\mu_{2}=0.3 \\
\mu_{3}=0.3
\end{gathered}
$$


(c)

FIGURE 2: Distribution of agents with respect to time (in seconds) for the task of detecting the perimeter of an oil spill: (a) geographical location of the spill and initial position of agents, (b) distribution of agents at time $t=15000 \mathrm{~s}$, and (c) distribution of agents at time $t=30000 \mathrm{~s}$.

$$
\begin{gather*}
\eta=0.8 \\
\alpha_{1}=0.6 \\
\alpha_{2}=0.2 \\
\alpha_{3}=0.2 . \tag{10}
\end{gather*}
$$

The previous equation shows default parameters used in simulations for the microscopic model. These parameters have been adjusted taking into account the microscopic behaviour experiments.

The simulation of the microscopic model has been developed using the software MASON [17]. We use a swarm of 200 agents randomly distributed by the environment that moves uniformly at $60 \mathrm{~km} / \mathrm{h}$. The simulation uses small size drones $\left(<3 \mathrm{~m}^{2}\right)$ that are able to visually detect an area of $1 \mathrm{~km}^{2}$.

Using these parameters, several tests will be performed to verify the correct operation of our model at local level. Initially, first tests check the convergence of the swarm for a single static slick. Using the same example, $\gamma$ will be modified to cover the slick instead of marking its perimeter. Next tests will verify the convergence of the swarm choosing an instant with several active slicks. Finally, we will check the tracking of the movement of the spill.
4.1. Spill Detection. First tests have been developed to detect and monitor a static slick. On the one hand, the swarm will mark the perimeter of the slick. Figure 2 shows the initial distribution of agents with their geographical location (a) and the position of agents at time $t=15000 \mathrm{~s}$ and $t=30000 \mathrm{~s}$
((b) and (c)). At time $t=30000 \mathrm{~s}$ the swarm completely surrounds the oil slick. Figure 3 presents the percentage of agents over an oil slick for ten independent simulations. As it can be appreciated, this percentage increases progressively.

Moreover, as it has been discussed in the definition of the microscopic model of the swarm, when $\gamma(S)=1$ the swarm will cover the slick instead of marking its perimeter. Figure 4 shows the results of the swarm for this case, where 10 independent simulations have been carried out.

The experiments in Figure 2 show that the operation of the swarm is correct: $50 \%$ of agents are able to detect the spill in less than 4 hours since the beginning of the execution, marking correctly the perimeter of the spill. However, in a real environment the appearance of several slicks is common. This case will be analysed in the next section.
4.2. Several Spills. When an oil spill occurs, the oil may spread across kilometres generating several oil slicks. For this reason, it is very important to verify the correct operation of the swarm with different slicks that can spread out over a wide area. In the simulation, at early hours, several areas containing large proportions of pollutant are produced. For example, underwater currents can move oil slick close to the coast, while the ship continues spilling oil into the sea.

In order to check the operation of the microscopic model in these cases, we have chosen an instant of the simulation where the slick has a complex structure. Figure 5 shows the evolution of the distribution of robots. As it can be seen in this figure, even in this complex case and without previous data that indicates the spill trend, our model is able to locate and mark the perimeter of the slicks. More specifically,


Figure 3: Percentage of agents on an oil slick with respect to time (in seconds). Ten different experiments have been carried out, showing the average and variance.

(c)

Figure 4: Distribution of agents for the task of covering the spill $(\gamma(S)=1)$ : (a) geographical location of the slick and initial position of agents, (b) position of agents at time $t=15000 \mathrm{~s}$, (c) position of agents at time $t=30000 \mathrm{~s}$.

Figure 6 presents the percentage of agents that are located on a polluted area. In this figure it can be seen that the number of agents increases with time, therefore distributing uniformly on the slick.
4.3. Spill Movement. Figure 5 tests have shown that the swarm is able to detect and mark the perimeter/area with complex slicks. However, in a real spill, finding the location of the spill is as important as effective monitoring is. In this section we will analyse the behaviour of the swarm taking into account that the spill will spread as time progresses.

A simulation of 168 hours of the evolution of the spill has been performed, starting April 10, 2013, at 0:00 h. In
order to simplify the simulation, we capture an image of the state of the spill every 4.5 hours. Although the spill will experience changes in this period, the swarm can deal with these changes. Figure 7 shows the origin point of the spill and some captures of its evolution.

As in previous experiments, the swarm is distributed randomly by the environment. In this case, in the simulation process, the location of the resource obtained by GNOME is updated every 4.5 hours. Figure 8 shows the evolution of the swarm with respect to the state of the oil spill.

Figure 9 shows the percentage of drones that are on top of a resource at a given time. In order to produce this graph, 10 different simulations have been carried out. The graph shows


Figure 5: Distribution of agents for the task of detecting the perimeter of an oil spill on a map with several slicks: (a) geographical location of the spill and initial position of agents, (b) distribution of agents at time $t=15000 \mathrm{~s}$, and (c) distribution of agents at time $t=30000 \mathrm{~s}$.


Figure 6: Percentage of agents on an oil slick with respect to time (in seconds) for a complex slick. Ten different experiments have been carried out, showing the average and variance.
the average and variance of these simulations. In this graph a series of steps produced artificially by the simulator (because of the sudden change from one state to the next) can be seen.

Nonetheless, the evolution of the swarm is correct. The swarm is able to track the spill and mark each of the slicks.

## 5. Macroscopic Model

Once the microscopic behaviour has been described, it is interesting to see the global behaviour of the swarm. There are several techniques used to analyse this behaviour [18] such as the use of recurrence equations, generated from a microscopic behaviour defined by a PFSM or the definition of differential equations. However, most of these methods allow only the analysis of the evolution of the state transitions globally.

In this work, we consider the framework proposed in [19] in order to obtain the probability distribution of the swarm
position for any time $t$. This will enable us to predict, in great detail, the behaviour of the overall system. As described by [19], once the microscopic behaviour has been defined, the global behaviour of the system can be calculated using the Fokker-Planck equation:

$$
\begin{equation*}
\frac{\partial \rho(\mathbf{r}, t)}{\partial t}=-\nabla(\mathbf{A}(\mathbf{r}, t) \rho(\mathbf{r}, t))+\frac{1}{2} Q \nabla^{2}\left(B^{2}(\mathbf{r}, t) \rho(\mathbf{r}, t)\right) \tag{11}
\end{equation*}
$$

where $Q$ is the displacement by a collision. $\rho(\mathbf{r}, t) d r_{x} d r_{y}$ is the probability of encountering a robot at position $r$ within the rectangle defined by $d r_{x}$ and $d r_{y}$ at time $t$.

This equation provides a method to statistically model a swarm of robots based on modelling techniques of multiparticle systems from the field of quantum physics. From a Langevin equation that represents the behaviour of a single particle, the Fokker-Planck equation is derived for all the system.


Figure 7: Origin and temporal evolution of the spill. Several snapshots that show the position of the spill at different instants, measured in hours from the beginning of the spill, are shown.

As we have already seen in [20], the Fokker-Planck equation implements the necessary abstraction of microscopic details as described above and treats rapidly changing parameters such as noise. The equation is still exact if this noise is generated by a Gaussian process, that is, if it is fully determined by the first two moments. It gives the temporal evolution of the probability density describing the positions of the agents.

Initially, the swarm designer must specify the functions $A$ and $B$ of (11), in accordance with the desired microscopic behaviour. Function $\mathbf{A}$ is a direction and describes the deterministic motion based on information provided by the environment and the information indirectly provided by other robots via the environment. Function $B$ describes the random component of the motion. $A$ and $B$ are characterized by the underlying control algorithm. $B$ typically incorporates influences by other robots that disturb the robot, for example, by the need of collision avoidance. A might incorporate an external influence such as a light gradient.

The most difficult task is the definition of these functions. This design is not unique and it requires finding two functions that correctly describe the behaviour of the swarm. These functions will be analytically presented and their performance will be tested in following sections.

A function determines the displacement of the swarm. A depends primarily on a vector representing the directional information. A potential field $P$ is commonly used to define it. In our case, we need to establish a function that takes
into account the following things based on the proposed microscopic model: the random motion states of the robot, the probability that a movement of an agent fails in its execution (e.g., due to a collision), and the potential field where the robots move. Although it is possible to model a probability distribution for each state, as our microscopic model has no interaction between agents (except purely physical, as collisions) and the behaviour of states is relatively simple, the macroscopic behaviour of the swarm can be comprised in a single distribution:

$$
\begin{align*}
\mathbf{A}(\mathbf{r}, t)= & \left(1-b_{r}\right) \cdot(1-\rho(\mathbf{r}, t))^{\mu_{4}}  \tag{12}\\
& \cdot(\Gamma(P(\mathbf{r}, t)) \cdot K(\nabla P(\mathbf{r}, t))),
\end{align*}
$$

where $P$ is directly related to the sensor readings in point $\mathbf{r}$ at time $t$ and $\mu_{5}$ is a normalization term:

$$
\begin{equation*}
P(\mathbf{r}, t)=\mu_{5} \cdot s(\mathbf{r}, t) . \tag{13}
\end{equation*}
$$

In (12), a function $\mathbf{A}$ that takes into account previous aspects is proposed. When the density of agents increases, the probability of collision also increases and, therefore, this situation reduces the rate at which robots are directed by vector specified by A. More specifically, consider the following.
$b_{r}$ is the probability that comprises the influences of all states that develop a random or pseudorandom state.


FIgURE 8: Evolution of the swarm with respect to the oil slick. Multiple snapshots at different instants in time (measured in hours) are shown.
$P(\mathbf{r}, t)$ as commented before, is defined as the probability of encountering a robot at position $\mathbf{r}$ at time $t$.
$\Gamma$ is a function applied to the potential field that produces, for each position of $P$, the direction to be taken by drones, as a single agent would do using the function $\gamma$ In our case, we use a "slidingneighbourhood" filter as commented, for example, in [21] to perform the same calculation on $P$ as on the microscopic model, changing the sign of the displacement when a ratio greater than $80 \%$ of pollutant is detected.
$K$ is a convolution operator. Being $G$ a square gaussian kernel of size $d$, generated with $\sigma=4 \cdot|S|$, and assuming two-dimensional coordinates so that $\mathbf{r}=$ $(i, j)$, the sum of the components of this kernel is $\sum_{i=1}^{d} \sum_{j=1}^{d} G(i, j) . F$ is defined as this sum if it is different from 0 ; otherwise, it will be defined as 1 . In this way,

$$
\begin{equation*}
K_{i, j}=\frac{1}{F} \cdot \sum_{i=1}^{d} \sum_{j=1}^{d} G(i, j) \cdot \nabla P((i, j), t) \tag{14}
\end{equation*}
$$



Figure 9: Percentage of agents over an oil slick with respect to time (in seconds). Ten different experiments have been carried out, showing the average and variance.

Function $B$ describes the nondeterministic motion and, therefore, it takes into account the random motion of agents. Two forces, which must be considered, take part in the microscopic behaviour. On the one hand, some influences derived from agents that are on Wander and InResource states. These states have a random motion, depending on the intensity of parameters $\mu_{1}, \mu_{3}$. On the other hand, the behaviour itself causes that the environment has areas with a higher density of agents. In these areas the probability of collision can be increased depending on the density of agents at a given time:

$$
\begin{equation*}
B(\mathbf{r}, t)=b_{r} \cdot \rho(\mathbf{r}, t)^{\mu_{4}} . \tag{15}
\end{equation*}
$$

Thus, in (15) two terms can be observed: $b_{r}$ comprises the influences of all states that develop a random or pseudorandom state, as previously commented, and $\rho(\mathbf{r}, t)^{\mu_{4}}$ is a term that defines the connection between the density of robots and their probability of collision.
5.1. Spill Detection. The operation of the microscopic behaviour has been analysed in the previous section. It has been shown that the swarm is able to locate and mark the perimeter of an oil spill with relatively simple rules. Several tests have been developed to verify the validity of the presented model in various cases.

In addition to the tests above, it is possible to establish, for a given discharge, the areas of the map with the highest probability of containing a robot, independently of the number of agents (on condition that you use enough of them), by using the macroscopic model. This provides a more accurate visualization of the behaviour for large swarms without being limited by the number of agents to be simulated.

This section, using the previously presented macroscopic definition, presents how the swarm is able to locate and mark, in a given time, an oil spill:

$$
\begin{gather*}
\text { Robots }=200 \\
\mu_{4}=5 \\
\mu_{5}=0.95  \tag{16}\\
b_{r}=0.5 .
\end{gather*}
$$

The previous equation shows default parameters used in simulations for the macroscopic model. These parameters have been adjusted taking into account the microscopic behaviour experiments. We will present the analysis of the macroscopic behaviour of the swarm for two time instants $t=(140 \mathrm{~h}, 168 \mathrm{~h})$.

The simulation process is simple, once the functions $\mathbf{A}$ and $B$ of Fokker-Planck equation have been defined. Initially, an origin point for the swarm must be established. Although in microscopic simulations it is possible to establish different origin points (as setting randomly the position of agents), Fokker-Planck equation requires a single point. Nonetheless, several simulations for different origin points have been performed, observing that the results for large values of $t$ are very similar: only slight variations can be seen if the origin point is located just on an oil slick. In this case, the probability of this slick in relation to the rest of the spill can be higher.

The same origin point has been used for all tests, discretizing the simulation area in $100 \times 100$ units. The origin point is established in Fokker-Planck equation as $(x, y)=(40,25)$.

Once the Fokker-Planck equation has been defined, the probability distribution that a certain agent is in a position of the environment at a given time can be obtained in an iterative way. This distribution can be calculated iterating for each instant of time $t$ all the positions of the map, updating for each position the new probability described by the equation.


Figure 10: (a) Map (M) generated from GNOME data for $t=140 \mathrm{~h}$. (b) Probability that a robot is in a certain position of the space at $t=140 \mathrm{~h}$. Sampled using the macroscopic model of the swarm with the Fokker-Planck equation.

The macroscopic state of the swarm is presented in Figure 10 for $t=140 \mathrm{~h}$. A clear convergence in the perimeter of the spill can be observed. A three-dimensional representation of the probability distribution that an agent is in a certain position of the environment at $t=168 \mathrm{~h}$ is presented in Figure 11. As it can be observed, the macroscopic model correctly predicts the behaviour presented at the microscopic level of the swarm.
5.2. Model Comparison. In the previous section the behaviour of the macroscopic model and how this model predicts the overall behaviour of the swarm have been presented. Now, we will compare predictions of microscopic model and macroscopic model for a specific case.

Figure 12(a) shows a probability distribution obtained from the microscopic model. In order to do this, we have discretized the simulation environment and, with the same parameters used in previous sections, we have developed a simulation with 200 agents at $t=140 \mathrm{~h}$. Along the simulation, we save the number of agents that pass on a discrete cell (in order to compare it with the macroscopic model that we have also discretized in $100 \times 100$ units) and then we calculate the probability that an agent is in this cell at a given time.

In the same way, Figure 12(b) shows the probability distribution that predicts the macroscopic model. In this figure it can be seen that the area of interest is covered in both models. There are minor differences within the models, due to the deficiencies of microscopic simulation that, among other things, depends on the number of agents used in the simulation.

Nevertheless, we can compare both approaches by multiplying both distributions. In this way only high probabilities remain and, therefore, it is easier to observe if the areas of the spill are correctly identified in both models. Bearing this in mind, we have slightly rectified the microscopic distribution. When we use a limited number of agents in the simulation, we discover, in some cases, that high probabilities hide important information in the distribution. In order to avoid this loss of
information, we have used the square root of the distribution in order to compare both models.

The product of macroscopic distribution and the square root of microscopic distribution are presented in Figure 12(c). This figure shows how the most important parts of the spill are detected with both distributions, predicting the macroscopic model the same perimeter areas detected by the microscopic model.

## 6. Discussion

This paper describes a microscopic model that is able to locate and mark the perimeter of an oil spill. The microscopic behaviour presented does not require direct communication between agents. This limitation can cause that the convergence of the swarm on the spill takes more time, depending on the number of drones and the size of the spill. However, this behaviour is versatile and easy to implement and develop, even in areas without GPS coverage. It is important to highlight that a swarm system, which requires direct communication between agents, is a limited system because of the maximum range of each agent and the saturation of the radiofrequency space if the system needs a large number of agents.

Moreover, we have demonstrated that the process of locating and marking the perimeter of the spill without communication is robust and efficient. We have shown that the swarm system is able to completely delimit the spill if the number of agents is sufficient. In order to achieve this task, an agent must be able to detect drones that are nearby. There are several ways, as, for example, using a camera or transmitting the GPS position.

We propose the use of signal intensity (at a given frequency) for obstacle avoidance tasks. This strategy may show some problems (we have implemented it by using a reactive behaviour); however, it has several advantages. Many countries require that drones broadcast a continuous signal indicating their position. Europe uses 433 MHz frequency


FIgure 11: Three-dimensional representation of the probability that a robot is in a particular position at $t=168 \mathrm{~h}$. Sampled using the macroscopic model of the swarm with the Fokker-Planck equation. A map for the probability 0.002 where the environment and the state of the spill at $t=168 \mathrm{~h}$ are represented is added.


Figure 12: Comparison of the microscopic prediction and macroscopic model for the map $M$ at $t=140 \mathrm{~h}$. (a) $\left(\log \left(P_{\text {micro }}\right)\right)$ Logarithm of the probability distribution obtained simulating 200 agents for 30,000 seconds. Logarithmic distribution is used to highlight the states with low probability. (b) ( $P_{\text {macro }}$ ) Probability distribution obtained with Fokker-Planck equation. (c) Product of both distributions, decreasing the importance of high values in the microscopic simulation (specifically $P_{\text {macro }} \times P_{\text {micro }}^{1 / 2}$ ).
for this purpose. The intensity of the signal in a particular area can be detected by using the same infrastructure. If the intensity of the signal grows with the movement of the agent, this agent must change its direction. We emphasize that, as a swarm approach, this is not a communication between agents but simply a beacon that we can use, if necessary, to know the position of drones.

The proposed macroscopic model demonstrates that the tendency of the swarm, for a sufficient number of drones, is the same that can be perceived in the microscopic model. The connection of both models has been tested for a complex spill, generated with GNOME. These experiments have shown that the fundamental characteristics of the behaviour (detection and monitoring) are reflected in both models. It is advisable not to forget the differences between the two models.

The microscopic model defines the individual behaviour. Because of this it is easy to understand at local level. However, this model does not define the behaviour of all the swarm. In order to analyse the global behaviour, a set of tests can be defined for a large number of agents. However, these tests can be expensive and difficult and are not exempt from problems.

The macroscopic model defines the global behaviour of the swarm. It allows us to verify the emergent behaviour from the interaction between all agents that run the microscopic model. The macroscopic model demonstrates the tendency of the swarm for a large number of agents. The analysis of this model is complex, because of the use of differential equations that, for example, force us to choose a single point to start the simulation. Even so, this model has remarkable advantages [20], for example, continuous analysis for any point of the environment, time of the probability that an agent is located in a given location, and simulation time negligible compared to microscopic model.

We are currently working on the implementation of this system in a real swarm of drones. Our immediate future research focuses on this real swarm, since it allows us to adjust the algorithms for a real system.

We are already in the testing phase for small swarms ( 5 drones), obtaining satisfactory results in our preliminary tests. We are using low-cost, custom-developed hexacopters to test this behaviour. The low computational needs required by this behaviour make it possible to use cheap Arduino control boards to control the UAVs. We have developed several tests for our swarm behaviors using artificial color landmarks. While flying our agents can localize (by wandering) a landmark and follow it (if moving) in real time.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Characterization of Consistent Completion of Reciprocal Comparison Matrices 

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#### Abstract

Analytic hierarchy process (AHP) is a leading multi-attribute decision-aiding model that is designed to help make better choices when faced with complex decisions involving several dimensions. AHP, which enables qualitative analysis using a combination of subjective and objective information, is a multiple criteria decision analysis approach that uses hierarchical structured pairwise comparisons. One of the drawbacks of AHP is that a pairwise comparison cannot be completed by an actor or stakeholder not fully familiar with all the aspects of the problem. The authors have developed a completion based on a process of linearization that minimizes the matrix distance defined in terms of the Frobenius norm (a strictly convex minimization problem). In this paper, we characterize when an incomplete, positive, and reciprocal matrix can be completed to become a consistent matrix. We show that this characterization reduces the problem to the solution of a linear system of equations-a straightforward procedure. Various properties of such a completion are also developed using graph theory, including explicit calculation formulas. In real decisionmaking processes, facilitators conducting the study could use these characterizations to accept an incomplete comparison body given by an actor or to encourage the actor to further develop the comparison for the sake of consistency.


## 1. Introduction

The so-called analytic hierarchy process (AHP) [1, 2] has been accepted as a leading multiattribute decision-aiding model both by practitioners and academicians, since it is designed to make better choices when faced with complex decisions involving several dimensions. As a multiple criteria decision analysis (MCDA) technique [3], AHP solves optimization discrete decision problems that involve choosing one of several alternatives. In many fields, decision making (DM) has become very complex due to a large number of alternatives and multiple goals that sometimes conflict with each other. The AHP approach, which enables qualitative analysis using a combination of subjective and objective information/data, is an MCDA approach that uses hierarchically structured pairwise comparisons.

One of the weaknesses of AHP, which AHP shares with other decision models, comes from the fact that typically the input is static. In other words, users must provide all the preference data at the same time, and the criteria must be completely defined from the start. Nevertheless, changing scenarios are currently more than frequent, due to various sources of uncertainty, and pairwise comparison cannot be successfully completed when there are many alternatives to be considered and/or the comparison is required from an actor or stakeholder not fully familiar with all the aspects of the problem. The current trend toward greater interactive involvement of citizens in policy making is unavoidable and highly desirable. It is generally agreed that better decisions are implemented with less conflict and more success when they are driven by stakeholders [4]. Public participation is also likely to improve the quality of decisions; using a wider
pool of knowledge and understanding can prevent obstacles that would obstruct effective implementation of a particular decision $[5,6]$.

For example, environmental (field of expertise of the authors) projects and programs are likely to be more relevant, successful, and sustainable if their actors are involved in planning, implementation, and evaluation [7, 8]. Moreover, integrating water resource management is a reference framework for water management in many countries [9]. For example, the Water Framework Directive (WFD) [10] was enforced in Europe in 2000. Taking the example down a level in concretion, in [11] the authors describe a management model to address the main needs of an aquifer, with this being a problem of environmental, social and public policy; the relevant decisions are enriched, when incorporating the interests of the parties concerned, including academics, users, and administrators of the aquifer.

However, some actors may not be completely familiar with one or more of the elements about which they have to issue their judgement or opinion. As a result, it is difficult to gather complete information about the preferences of such a stakeholder at a given moment. It seems reasonable to enable such actors to express their preferences several times at their own convenience. Meanwhile, partial results based on partial preference data may be generated from data collected at various times-and this data may eventually be consolidated when the information is complete. Based on a process of linearization [12] that minimizes a matrix distance defined in terms of the Frobenius norm (a strictly convex minimization problem based on the best one rank approximation) in [13] the authors have initiated a line towards a dynamic model of AHP by addressing the problem of adding new criteria or deleting obsolete criteria. In [14] this research line was extended to propose a framework that enables users to provide data on their preferences in a partial and/or incomplete way and at different times. The consistent completion of a reciprocal matrix as a mechanism to obtain a consistent body of opinion issued in an incomplete manner by a specific actor was addressed. This feature can help stakeholders not fully problem acquainted participate in processes.

Public participation is, however, not a panacea. Collaboration and participation cannot solve every problem and should not be used as a surrogate for other systematic attempts to plan and manage decision issues [15, 16]. Public participation efforts must be responsive to the needs of the stakeholders. It is also critical to recognize that participation processes require a flexible approach that is appropriate to user conditions. As a result, the need to integrate multiple criteria and uncertainty demands a systematic framework to suitably represent and handle the information [17].

In particular, uncertainty produced by a lack of comprehensive knowledge on the part of the stakeholders must be handled suitably. In a real participatory DM process, the facilitator in charge of conducting the study may have to face an incomplete body of opinion given by an actor. In this event, he or she needs robust criteria to either accept the opinion or encourage the actor to further develop the comparison so
that the judgment is eventually completed consistently, thus helping ensure an optimal decision.

In this paper, we provide a solution to this issue by solving the following problem: to characterize when an incomplete, positive, reciprocal matrix can be completed to become a consistent matrix. If the incomplete judgment given by an actor passes this test, then the facilitator will feel confident about accepting it. Otherwise, the facilitator should keep on working with the actor until the characterization criterion is met. We show that this characterization reduces the consistent completion of an incomplete, positive, reciprocal matrix to the solution of a linear system of equations-a straightforward procedure. In addition, the uniqueness of the completion is studied using graph theory and we offer several ways to find such completion when it exists.

This paper is organized as follows. In Section 2 we give a formal statement of the problem and provide some prerequisites. Sections 3 and 4, respectively, address the main objectives of the paper, namely, the characterization and the uniqueness, together with the computation of the consistent completion (when it exists). In Section 5, these results are applied to two comparison matrices for deciding suitable leakage policies within the framework of water supply management. Section 6 contains the conclusions and closes the paper.

## 2. Prerequisites and Formal Statement of the Problem

2.1. A Brief Review of $A H P$. AHP $[1,2,18]$ is a multiple attribute decision method that uses structured pairwise comparisons with numerical judgements from an absolute scale of numbers. The fundamentals of AHP, including its hierarchical, multilevel structure with goals, criteria, and alternatives, the way in which judgement is compiled into positive reciprocal matrices, the estimation of the relative weights of the decision elements, the use of prioritization techniques, and the way in which aggregation is performed to obtain a final composite vector of priorities, can be found in any handbook and many papers about the subject (see, e.g., [1, 17, 18]).

In this paper, we consider a nine-point scale developed by Saaty [1,2], with the possibility of including intermediate numerical values on this scale. Thus, as a result of the comparison performed, an $n \times n$ matrix $A=\left(a_{i j}\right)$, is formed, with $n$ being the number of the decision elements and $a_{i j}$ measuring the relative importance of element $i$ over element $j$. To extract priority vectors from the comparison matrices, the eigenvector method, which was first proposed by Saaty in his seminal paper [1], is used in this paper.

A comparison matrix, $A$, exhibits a basic property, namely reciprocity:

$$
\begin{equation*}
a_{i j}=\frac{1}{a_{j i}}, \quad \forall i, j \in\{1, \ldots, n\} \tag{1}
\end{equation*}
$$

This property implies the homogeneity: if elements $i$ and $j$ are considered equally important, then $a_{i j}=1$. In particular, one has $a_{i i}=1$.

Definition 1. An $n \times n$ matrix $A$ is reciprocal when $a_{i j}>0$ for any $i, j$ and the condition (1) is satisfied.

In addition to the reciprocity property, another property, consistency, should theoretically be desirable for a comparison matrix.

Definition 2. An $n \times n$ matrix $A$ is consistent when $a_{i j}>0$ for any $i, j$ and

$$
\begin{equation*}
a_{i j} a_{j k}=a_{i k}, \quad \forall i, j, k \in\{1, \ldots, n\} \tag{2}
\end{equation*}
$$

is satisfied.
Consistency expresses the coherence that may exist between judgements about the elements of a set. Since preferences are expressed in a subjective manner it is reasonable for some kind of incoherence to exist. When dealing with intangibles, judgements are rarely consistent unless they are forced in some artificial manner.

In addition, a comparison matrix is not generally consistent because it contains comparison values obtained through numerical judgement using a fixed scale. For most problems, estimates of these values by an expert are assumed to be small perturbations of the "right" values. This implies small perturbations for the eigenvalues and eigenvectors (see, e.g., [19]). For a consistent matrix $A$, the leading eigenvalue and the principal (Perron) eigenvector of $A$ provide information to deal with complex decisions, with the normalized Perron eigenvector giving the sought priority vector [18]. In the general case, however, as said, $A$ is not consistent. For nonconsistent matrices, the problem to solve is the eigenvalue problem $A \mathbf{w}=\lambda_{\max } \mathbf{w}$, where $\lambda_{\max }$ is the unique largest eigenvalue of $A$ that gives the Perron eigenvector as an estimate of the priority vector.
2.2. Notations and Basic Facts. The set of $n \times m$ real matrices is denoted by $\mathbb{R}_{n, m}$. We write $\mathbb{R}_{n, m}^{+}=\left\{A=\left(a_{i j}\right) \in \mathbb{R}_{n, m}: a_{i j}>\right.$ 0 for all $i, j\}$. If $A$ is a matrix, then $\operatorname{tr}(A)$ and $A^{T}$ will denote the trace and the transpose of $A$, respectively. The standard basis of $\mathbb{R}^{n}$ is denoted by $\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}\right\}$. Any vector of $\mathbb{R}^{n}$ will be considered a column. The vector $(1, \ldots, 1)^{T} \in \mathbb{R}^{n}$ will be denoted by $\mathbf{1}_{n}$.

As can be seen from (1) and (2), any consistent matrix is reciprocal. Also, it can be easily proven that the rank of any consistent matrix is 1 .

We will use the mappings $L: \mathbb{R}_{n, m}^{+} \rightarrow \mathbb{R}_{n, m}$ and $E:$ $\mathbb{R}_{n, m} \rightarrow \mathbb{R}_{n, m}^{+}$given by $L(A)=\left(\log \left(a_{i j}\right)\right)$ and $E(A)=$ $\left(\exp \left(a_{i j}\right)\right)$, respectively, where $A=\left(a_{i j}\right)$. Evidently, for $A \in$ $\mathbb{R}_{n, n}^{+}$,
$A$ is reciprocal $\Longleftrightarrow L(A)$ is skew symmetric.
The image by $L$ of the set of consistent matrices will play an important role in the sequel. Precisely, we define

$$
\begin{equation*}
\mathscr{L}_{n}=\left\{L(A): A \in \mathbb{R}_{n, n}^{+} \text {is consistent }\right\} \tag{4}
\end{equation*}
$$

A basic property of $\mathscr{L}_{n}$ is established in the next result.

Theorem 3 (Theorem 2.2 of [12] ). If we define

$$
\begin{equation*}
\phi_{n}: \mathbb{R}^{n} \longrightarrow \mathbb{R}_{n, n}, \quad \phi_{n}(\mathbf{v})=\mathbf{v} \mathbf{1}_{n}^{T}-\mathbf{1}_{n} \mathbf{v}^{T} \tag{5}
\end{equation*}
$$

then $\phi_{n}$ is linear, $\operatorname{ker} \phi_{n}=\operatorname{span}\left\{\mathbf{1}_{n}\right\}$, $\operatorname{Im} \phi_{n}=\mathscr{L}_{n}$, and $\operatorname{dim} \mathscr{L}_{n}=n-1$.
2.3. Problem Definition. The purpose of this paper is to characterize when a reciprocal and incomplete matrix can be completed to become a consistent matrix. We state the precise formulation of the completion problem.

Problem 4. Let $n \in \mathbb{N}$ and $1 \leq i_{1}, j_{1}, \ldots, i_{q}, j_{q} \leq n$ be such that $i_{r}<j_{r}$ for $r=1, \ldots, q$. Let $\alpha_{1}, \ldots, \alpha_{q}$ be given positive numbers. Can we find a consistent matrix $A=\left(a_{i j}\right) \in \mathbb{R}_{n, n}^{+}$ such that $a_{i_{r} j_{r}}=\alpha_{r}$ and $a_{j_{r} i_{r}}=-\alpha_{r}$ for any $r=1, \ldots, q$ ? In case that the answer to this question is affirmative, find all matrices $A$ verifying the condition of the aforementioned question.

Although the following result can be dealt by means of the general characterization given in Theorem 7, we will prove it using Theorem 3.

Theorem 5. Let $A \in \mathbb{R}_{n, n}^{+}$. The following statements are equivalent:
(i) There exist $A_{1} \in \mathbb{R}_{n, m}^{+}, A_{2} \in \mathbb{R}_{m, n}^{+}$, and $A_{3} \in \mathbb{R}_{m, m}^{+}$ such that $B=\left[\begin{array}{cc}A & A_{1} \\ A_{2} & A_{3}\end{array}\right]$ is consistent.
(ii) $A$ is consistent.

Proof. (i) $\Rightarrow$ (ii): since $B$ is consistent, the relations given in (2) hold for all indices and in particular for a subset of them.
(ii) $\Rightarrow$ (i): if $A$ is consistent, then, by Theorem 3, there exists $\mathbf{u} \in \mathbb{R}^{n}$ such that $L(A)=\mathbf{u} \mathbf{1}_{n}^{T}-\mathbf{1}_{n} \mathbf{u}^{T}$. Let $\mathbf{w}$ be an arbitrary vector of $\mathbb{R}^{m}$, and define $\mathbf{v} \in \mathbb{R}^{n+m}$ in such a way that $\mathbf{v}^{T}=\left[\mathbf{u}^{T} \mathbf{w}^{T}\right]$. Let us define

$$
\begin{array}{ll}
A_{1}=E\left(\mathbf{u} \mathbf{1}_{m}^{T}-\mathbf{1}_{n} \mathbf{w}^{T}\right), & A_{2}=E\left(\mathbf{w} \mathbf{1}_{n}^{T}-\mathbf{1}_{m} \mathbf{u}^{T}\right) \\
A_{3}=E\left(\mathbf{w} \mathbf{1}_{m}^{T}-\mathbf{1}_{m} \mathbf{w}^{T}\right), & B=\left[\begin{array}{cc}
A & A_{1} \\
A_{2} & A_{3}
\end{array}\right] \tag{6}
\end{array}
$$

We get

$$
\begin{align*}
& L(B)=\left[\begin{array}{cc}
\mathbf{u} \mathbf{1}_{n}^{T}-\mathbf{1}_{n} \mathbf{u}^{T} & \mathbf{u} \mathbf{1}_{m}^{T}-\mathbf{1}_{n} \mathbf{w}^{T} \\
\mathbf{w} \mathbf{1}_{n}^{T}-\mathbf{1}_{m} \mathbf{u}^{T} & \mathbf{w} \mathbf{1}_{m}^{T}-\mathbf{1}_{m} \mathbf{w}^{T}
\end{array}\right],  \tag{7}\\
& =\left[\begin{array}{c}
\mathbf{u} \\
\mathbf{w}
\end{array}\right]\left[\begin{array}{ll}
\mathbf{1}_{n}^{T} & \mathbf{1}_{m}^{T}
\end{array}\right]-\left[\begin{array}{l}
\mathbf{1}_{n} \\
\mathbf{1}_{m}
\end{array}\right]\left[\begin{array}{ll}
\mathbf{u}^{T} & \mathbf{w}^{T}
\end{array}\right] .
\end{align*}
$$

Hence $L(B)=\mathbf{v} \mathbf{1}_{n+m}^{T}-\mathbf{1}_{n+m} \mathbf{v}^{T}$, showing that $B$ is consistent.

Observe that the proof of the former theorem enables us to give the general solution of the following problem: find all consistent completions of the matrix

$$
B=\left[\begin{array}{cccc}
A & \star & \cdots & \star  \tag{8}\\
\star & 1 & \cdots & \star \\
\vdots & \vdots & \ddots & \vdots \\
\star & \star & \cdots & 1
\end{array}\right] \in \mathbb{R}_{n+m, n+m}
$$

where $A \in \mathbb{R}_{n, n}$ is consistent. Since $A$ is consistent, there exists $\mathbf{u} \in \mathbb{R}^{n}$ such that $L(A)=\phi_{n}(\mathbf{u})$. It is now enough to pick any $\mathbf{w} \in \mathbb{R}^{m}$ and define $B$ as in (6).

To motivate the notation and the precise definition of the problem considered in this paper, let us consider the following example. Let

$$
A=\left[\begin{array}{cccc}
1 & 2 & 3 & \star  \tag{9}\\
\frac{1}{2} & 1 & 3 & 4 \\
\frac{1}{3} & \frac{1}{3} & 1 & \star \\
\star & \frac{1}{4} & \star & 1
\end{array}\right] .
$$

By taking logarithms of the entries of the matrix, the aforementioned completion problem can be managed. Since the image by $L$ of any reciprocal matrix is skew symmetric, in order to find a consistent completion of an incomplete reciprocal matrix, it is enough to restrict ourselves to the subset of reciprocal matrices of order n. From (9) we obtain

$$
L(A)=\left[\begin{array}{cccc}
0 & \log 2 & \log 3 & \star  \tag{10}\\
-\log 2 & 0 & \log 3 & \log 4 \\
-\log 3 & -\log 3 & 0 & \star \\
\star & -\log 4 & \star & 0
\end{array}\right],
$$

then any skew symmetric completion of $L(A)$ is of the form

$$
\begin{align*}
C(\lambda, \mu)= & {\left[\begin{array}{cccc}
0 & \log 2 & \log 3 & \lambda \\
-\log 2 & 0 & \log 3 & \log 4 \\
-\log 3 & -\log 3 & 0 & \mu \\
-\lambda & -\log 4 & -\mu & 0
\end{array}\right] } \\
= & {\left[\begin{array}{cccc}
0 & \log 2 & \log 3 & 0 \\
-\log 2 & 0 & \log 3 & \log 4 \\
-\log 3 & -\log 3 & 0 & 0 \\
0 & -\log 4 & 0 & 0
\end{array}\right] }  \tag{11}\\
& +\lambda\left[\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array}\right]+\mu\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{array}\right],
\end{align*}
$$

where $\lambda, \mu \in \mathbb{R}$.
From now on, we define for $1 \leq i<j \leq n$ the following skew symmetric matrices

$$
\begin{equation*}
B_{i j}=\mathbf{e}_{i} \mathbf{e}_{j}^{T}-\mathbf{e}_{j} \mathbf{e}_{i}^{T} . \tag{12}
\end{equation*}
$$

Thus, with this notation, the skew symmetric completion considered in equalities (9), (10), and (11) takes the simpler form

$$
\begin{equation*}
C(\lambda, \mu)=C_{0}+\lambda B_{14}+\mu B_{34}, \tag{13}
\end{equation*}
$$

where

$$
C_{0}=\left[\begin{array}{cccc}
0 & \log 2 & \log 3 & 0  \tag{14}\\
-\log 2 & 0 & \log 3 & \log 4 \\
-\log 3 & -\log 3 & 0 & 0 \\
0 & -\log 4 & 0 & 0
\end{array}\right] .
$$

Furthermore, observe that matrix $C_{0}$ appearing in (13) can be written as

$$
\begin{equation*}
C_{0}=\sum_{(i, j) \in N_{4} \backslash\{(1,4),(3,4)\}} \rho_{i j} B_{i j}, \tag{15}
\end{equation*}
$$

where $N_{n}=\{(i, j): 1 \leq i<j \leq n\}$ and $\rho_{i j}$ are real numbers that can be easily determined from the incomplete matrix $A$ given in (9). In an informal way, we can think of $C_{0}$ as the incomplete skew symmetric matrix to be completed, and $(1,4)$, and ( 3,4 ) -and their symmetric positions with respect to the principal diagonal-as the void positions that must be filled.

We need the following lemma [14].
Lemma 6. Let the matrices $B_{i j}$ be defined as in (12) and let the mapping $\phi_{n}$ be defined as in (5). Then

$$
\begin{gather*}
\operatorname{tr}\left(B_{p q}^{T} B_{i j}\right)= \begin{cases}2 & \text { if }(p, q)=(i, j), \\
0 & \text { if }(p, q) \neq(i, j),\end{cases}  \tag{16}\\
\operatorname{tr}\left(B_{i j}^{T} \phi_{n}(\mathbf{v})\right)=2\left(\mathbf{e}_{i}-\mathbf{e}_{j}\right)^{T} \mathbf{v}
\end{gather*}
$$

for any $\mathbf{v} \in \mathbb{R}^{n}$.

## 3. Characterization of the Completion of a Reciprocal Matrix

Now we are ready to establish the first main result of this paper. In the statement of the next theorem we use two sets of indices of matrix $A, I$, and $J$. The unspecified entries of $A$ above the main diagonal will be located at the indices belonging to $I$. For example, for the matrix $A$ given in (9) we have $I=\{(1,4),(3,4)\}$ and $J=N_{4} \backslash I$.

Theorem 7. Let $1 \leq i_{1}, j_{1}, \ldots, i_{k}, j_{k} \leq n$ be indices such that $i_{r}<j_{r}$ for $r=1, \ldots, k$. Denote $I=\left\{\left(i_{1}, j_{1}\right), \ldots,\left(i_{k}, j_{k}\right)\right\}$ and $J=N_{n} \backslash I$. Let $C_{0}=\sum_{(i, j) \in J} \rho_{i j} B_{i j}$. The following statements are equivalent.
(i) There exist $\lambda_{1}, \ldots, \lambda_{k} \in \mathbb{R}$ such that $C_{0}+\sum_{r=1}^{k} \lambda_{r} B_{i_{r} j_{r}} \in$ $\mathscr{L}_{n}$.
(ii) There exists $\mathbf{w}=\left(w_{1}, \ldots, w_{n}\right)^{T} \in \mathbb{R}^{n}$ such that $w_{1}+$ $\cdots+w_{n}=0$ and $\rho_{p q}=w_{p}-w_{q}$ for any $(p, q) \in J$.
(iii) There exists $\mathbf{w}=\left(w_{1}, \ldots, w_{n}\right)^{T} \in \mathbb{R}^{n}$ such that $\rho_{p q}=$ $w_{p}-w_{q}$ for any $(p, q) \in J$.
Furthermore, in the case that the statements hold, then

$$
\begin{equation*}
\lambda_{r}=w_{i_{r}}-w_{j_{r}}, \quad \forall r \in\{1, \ldots, k\} . \tag{17}
\end{equation*}
$$

Proof. (i) $\Rightarrow$ (ii): by Theorem 3, there exists $\mathbf{v} \in \mathbb{R}^{n}$ such that

$$
\begin{equation*}
\sum_{(i, j) \in J} \rho_{i j} B_{i j}+\sum_{r=1}^{k} \lambda_{r} B_{i_{r} j_{r}}=\phi_{n}(\mathbf{v}) . \tag{18}
\end{equation*}
$$

Note that any vector and in particular $\mathbf{v}$ can be splitted as $\mathbf{v}=\alpha \mathbf{1}_{n}+\mathbf{w}$, where $\alpha \in \mathbb{R}$ and $\mathbf{w}$ is orthogonal to $\mathbf{1}_{n}$. Here
we assume that the inner product in $\mathbb{R}^{n}$ is the standard inner product; that is, $\langle\mathbf{x}, \mathbf{y}\rangle=\mathbf{x}^{T} \mathbf{y}$. Hence, if $\mathbf{w}=\left(w_{1}, \ldots, w_{n}\right)^{T}$, then $w_{1}+\cdots+w_{n}=0$. By Theorem 3 we obtain $\phi_{n}(\mathbf{v})=\phi_{n}(\mathbf{w})$. Let us recall that the trace is a linear operator. If $(p, q) \in J$, then (18) and Lemma 6 yield

$$
\begin{equation*}
\rho_{p q}=\left(\mathbf{e}_{p}-\mathbf{e}_{q}\right)^{T} \mathbf{w}=w_{p}-w_{q} . \tag{19}
\end{equation*}
$$

Let $r \in\{1, \ldots, k\}$. We obtain from (18) and Lemma 6

$$
\begin{equation*}
\lambda_{r}=\left(\mathbf{e}_{i_{r}}-\mathbf{e}_{j_{r}}\right)^{T} \mathbf{w}=w_{i_{r}}-w_{j_{r}} \tag{20}
\end{equation*}
$$

(ii) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (i): let us define $\lambda_{r}=w_{i_{r}}-w_{j_{r}}$ for any $r \in\{1, \ldots, k\}$ and

$$
\begin{equation*}
R=\sum_{(i, j) \in J} \rho_{i j} B_{i j}+\sum_{r=1}^{k} \lambda_{r} B_{i_{r} j_{r}}-\phi_{n}(\mathbf{w}) . \tag{21}
\end{equation*}
$$

Observe that from Lemma 6 one has

$$
\text { if }(p, q) \in J, \text { then } \operatorname{tr}\left(B_{p q}^{T} R\right)=2 \rho_{p q}-2\left(w_{p}-w_{q}\right)=0
$$

if $r \in\{1, \ldots, k\}$, then $\operatorname{tr}\left(B_{i_{r} j_{r}}^{T} R\right)=2 \lambda_{r}-2\left(w_{i_{r}}-w_{j_{r}}\right)=0$.

Let us now recall the following.
(a) $\phi_{n}(\mathbf{w})$ is skew symmetric. This fact can be easily deduced from (5) by proving $\phi_{n}(\mathbf{v})^{T}=-\phi_{n}(\mathbf{v})$. Hence (21) and the fact that any $B_{i j}$ is also skew symmetric prove that $R$ is skew symmetric.
(b) The set $\mathbb{R}_{n, n}$ can be endowed with the inner product $\langle M, N\rangle=\operatorname{tr}\left(M^{T} N\right)$.
(c) The matrices $B_{i j}$ for $1 \leq i<j \leq n$ form a basis for the subset of $\mathbb{R}_{n, n}$ composed of skew symmetric matrices.

Previous item (c) and implications (22) lead to the fact that $R$ is orthogonal to any skew symmetric matrix. By previous item (a) we obtain $\langle R, R\rangle=0$. Thus, $R=0$. Hence, by Theorem 3, we obtain $C_{0}+\sum_{r=1}^{k} \lambda_{r} B_{i_{r}, j_{r}}=\phi_{n}(\mathbf{w}) \in \operatorname{Im} \phi_{n}=$ $\mathscr{L}_{n}$.

Note 1. Observe that we have reduced the completion problem to study the linear system occurring in item (ii)—or (iii) -of the previous theorem. We shall see how this theorem works in two specific situations.

Example 8. We will apply Theorem 7 in order to show that matrix $A$ in (9) cannot be completed to be consistent. If this completion was feasible, then by Theorem 7 and (14), there would exist $\mathbf{w}=\left(w_{1}, w_{2}, w_{3}, w_{4}\right)^{T} \in \mathbb{R}^{4}$ such that

$$
\begin{array}{ll}
\log 2=w_{1}-w_{2}, & \log 3=w_{1}-w_{3} \\
\log 3=w_{2}-w_{3}, & \log 4=w_{2}-w_{4} . \tag{23}
\end{array}
$$

It can be quickly shown that this linear system has no solution.

Example 9. We will see if

$$
A=\left[\begin{array}{lll}
1 & \star & \frac{1}{3}  \tag{24}\\
\star & 1 & \frac{2}{3} \\
3 & \frac{3}{2} & 1
\end{array}\right]
$$

has a consistent completion. By taking logarithms, we construct

$$
C_{0}=\left[\begin{array}{ccc}
0 & 0 & -\log 3  \tag{25}\\
0 & 0 & \log 2-\log 3 \\
\log 3 & \log 3-\log 2 & 0
\end{array}\right]
$$

If there is a consistent completion, by item (iii) of Theorem 7, then there will exist $\mathbf{w}=\left(w_{1}, w_{2}, w_{3}\right)^{T} \in \mathbb{R}^{3}$ such that

$$
\begin{equation*}
-\log 3=w_{1}-w_{3}, \quad \log 2-\log 3=w_{2}-w_{3} \tag{26}
\end{equation*}
$$

This system, clearly, is solvable. Hence, the completion is possible. We will see how Theorem 7 enables us to find such completion(s). The general solution of (26) is

$$
\begin{gather*}
w_{1}=-\log 3+\alpha, \quad w_{2}=\log 2-\log 3+\alpha \\
w_{3}=\alpha, \quad \alpha \in \mathbb{R} \tag{27}
\end{gather*}
$$

If $X$ is any consistent completion of $A$, then item (i) of Theorem 7 guarantees that there exists $\lambda \in \mathbb{R}$ such that $L(X)=C_{0}+\lambda B_{12}$, and such $\lambda$ can be obtained from (17) obtaining $\lambda=w_{1}-w_{2}=-\log 2$. Thus, $L(X)=C_{0}-\log 2 \cdot B_{12}$. We conclude that there is a unique consistent completion of $A$ and it is given by

$$
X=\left[\begin{array}{lll}
1 & \frac{1}{2} & \frac{1}{3}  \tag{28}\\
2 & 1 & \frac{2}{3} \\
3 & \frac{3}{2} & 1
\end{array}\right]
$$

## 4. Completion of Reciprocal Matrices and Graph Theory: Uniqueness of the Completion

In this section we develop several useful results that enable us to study the uniqueness of the consistent completion and to compute in a straightforward manner all possible completions. Let us start by having a deeper look at the linear systems (23) and (26).

We have, respectively,

$$
\begin{align*}
& {\left[\begin{array}{cccc}
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 1 & 0 & -1
\end{array}\right]\left[\begin{array}{l}
w_{1} \\
w_{2} \\
w_{3} \\
w_{4}
\end{array}\right]=\left[\begin{array}{l}
\log 2 \\
\log 3 \\
\log 3 \\
\log 4
\end{array}\right],}  \tag{29}\\
& {\left[\begin{array}{ccc}
1 & 0 & -1 \\
0 & 1 & -1
\end{array}\right]\left[\begin{array}{l}
w_{1} \\
w_{2} \\
w_{3}
\end{array}\right]=\left[\begin{array}{c}
-\log 3 \\
\log 2-\log 3
\end{array}\right] .}
\end{align*}
$$



Figure 1: Directed graphs corresponding to Examples 8 and 9.


Figure 2: A disconnected graph and its incidence matrix.

The transposes of the matrices of the above systems are the incidence matrices of the directed graphs of Figure 1.

For an arbitrary $n \times n$ incomplete reciprocal matrix $A=$ $\left(a_{i j}\right)$, we use the following procedure to construct a directed graph, denoted by $G_{A}$ : If $i \geq j$, then there is no arrow from $i$ to $j$. If $i<j$ and we do not know the entry $a_{i j}$, then there is no arrow from $i$ to $j$. If $i<j$ and we know the entry $a_{i j}$, then there is an arrow from $i$ to $j$. Now, we easily construct the incidence matrix of $G_{A}$, denoted in the sequel by $M_{A}$. Observe that any incidence matrix has exactly two nonzero entries, namely, 1 and -1 , in each column.

To describe the linear system that appears in item (iii) of Theorem 7, we also define, for an incomplete reciprocal matrix $A \in \mathbb{R}_{n, n}$, the vector $\mathbf{b}_{A}=\left(b_{1}, \ldots, b_{m}\right)^{T} \in \mathbb{R}^{m}$ by the following procedure.
(a) $m$ is the number of columns of $M_{A}$; that is, $m$ is the number of arrows of the directed graph $G_{A}$. Explicitly, $m=n(n-1) / 2-k$, where the meaning of $k \in \mathbb{N}$ is given in Theorem 7: $2 k$ is the number of entries of $A$ to be filled.
(b) For $r=1, \ldots, m$, let us pay attention to the $r$ th column of $M_{A}$ and let $i, j$ be the unique indices such that the entry $(i, r)$ of $M_{A}$ is 1 and the entry $(j, r)$ of $M_{A}$ is -1 . We set $b_{r}=\log \left(a_{i j}\right)$.
Observe that $M_{A}$ and $\mathbf{b}_{A}$ are unique up to a permutation. Theorem 7 can be rephrased as follows.

Theorem 10. If $A$ is an incomplete reciprocal matrix, then $A$ can be completed to be a consistent matrix if and only if the system $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is consistent.

We present here an m-file that can be executed in Matlab or Octave. This file checks if an incomplete reciprocal matrix can be completed to be consistent. We use Theorem 10 and the following criterion: the linear system $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is solvable if and only if $M_{A}^{T}\left(M_{A}^{T}\right)^{\dagger} \mathbf{b}_{A}=\mathbf{b}_{A}$, where the superscript $\dagger$ means the Moore-Penrose inverse (see Algorithm 1).

There is a trivial bound for $k$ (let us recall that $2 k$ is the number of entries of $A$ to be filled). Since the diagonal entries
of $A \in \mathbb{R}_{n, n}$ must be equal to 1 , then $2 k \leq n^{2}-n$. The following result gives a sufficient condition for a reciprocal incomplete matrix to be completed to be consistent.

Theorem 11. Let $A \in \mathbb{R}_{n, n}$ be a reciprocal incomplete matrix and let $2 k$ be the number of void entries (located up and down the main diagonal of $A$ ). If $G_{A}$ has $p$ connected components and $2 k \geq n^{2}-3 n+2 p$, then A can be completed to be consistent.

Proof. We denote $m=n(n-1) / 2-k$ and we construct $\mathbf{b}_{A} \in$ $\mathbb{R}^{m}$ as the previous procedure states. Evidently, $M_{A} \in \mathbb{R}_{n, m}$. Since $G_{A}$ has $p$ connected components, then the rank of $M_{A}$ is $n-p$ (see, e.g., [20, Th. 2.3]). Let $\mathscr{R}\left(M_{A}^{T}\right)$ be the range space of $M_{A}^{T}$, which obviously satisfies $\mathscr{R}\left(M_{A}^{T}\right) \subset \mathbb{R}^{m}$. From $\operatorname{dim} \mathscr{R}\left(M_{A}^{T}\right)=\operatorname{rank}\left(M_{A}^{T}\right)=\operatorname{rank}\left(M_{A}\right)=n-p \geq n(n-1) / 2-$ $k=m$ we obtain $\mathscr{R}\left(M_{A}^{T}\right)=\mathbb{R}^{m}$; hence $\mathbf{b}_{A} \in \mathscr{R}\left(M_{A}^{T}\right)$, which shows that the linear system $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is consistent.

Example 12. In this example we will see that the graph $G_{A}$ can be disconnected. Let $a \in \mathbb{R}$ be positive, and let us consider

$$
A=\left[\begin{array}{ccc}
1 & \star & \star  \tag{30}\\
\star & 1 & a \\
\star & \frac{1}{a} & 1
\end{array}\right] .
$$

The graph $G_{A}$ and its incidence matrix $M_{A}$ are shown in Figure 2. Obviously, $G_{A}$ has two connected components.

To find all possible consistent completions of $A$, we must consider the system $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ :

$$
\left[\begin{array}{lll}
0 & 1 & -1
\end{array}\right]\left[\begin{array}{l}
w_{1}  \tag{31}\\
w_{2} \\
w_{3}
\end{array}\right]=\log a
$$

Its general solution is $w_{1}, w_{2} \in \mathbb{R}, w_{3}=w_{2}-\log a$.
If $X$ is any consistent completion of $A$, then Theorem 7 ensures that $\lambda_{1}=w_{1}-w_{2}$ and $\lambda_{2}=w_{1}-w_{3}=w_{1}-w_{2}+\log a$ are such that

$$
\begin{align*}
L(X) & =\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & \log a \\
0 & -\log a & 0
\end{array}\right]+\lambda_{1} B_{12}+\lambda_{2} B_{13} \\
& =\left[\begin{array}{ccc}
0 & w_{1}-w_{2} & w_{1}-w_{2}+\log a \\
-w_{1}+w_{2} & 0 & \log a \\
-w_{1}+w_{2}-\log a & -\log a & 0
\end{array}\right] . \tag{32}
\end{align*}
$$

By denoting $b=\exp \left(w_{1}-w_{2}\right)$ we obtain

$$
X=\left[\begin{array}{ccc}
1 & b & a b  \tag{33}\\
\frac{1}{b} & 1 & a \\
\frac{1}{(a b)} & \frac{1}{a} & 1
\end{array}\right]
$$

Observe that the consistent completion of $A$ is not unique since $b \in \mathbb{R}^{+}$is arbitrary.

```
function [M, b, N] = test(A, P)
% A is the matrix to be completed (if we do not know a_{ij}, then A(i, j)=1
% P is a (0, 1)-matrix such that if we do not know a_{ij}, then P(i,j)=1.
% if we know a_{ij}, then P(i, j)=0.
% Use: [M, b, N] = test(A, P)
% M = M_A; b = b_A ; N = pseudo-inverse of the transpose of M_A
[n, m]=size(A);
n1 = (n^2-n)/2;
n2 = sum(sum(P))/2;
n3 = n1-n2;
M= zeros(n3, n);
k=1;
b = zeros(n3, 1);
for i=1:n
        for j=i+1:n
            If P(i, j)==0
                M(k, i)=1;M(k, j)=-1;
                b(k)= log(A(i, j));
                k=k+1;
            end
        end
end
M=M ';
N=pinv(M');
% Since there are numerical errors, a tolerance is used
if norm(M'*N*b-b) > 10^(-10)
    disp('There is no consistent completion')
else
    disp('There is consistent completion')
end
```

Algorithm 1

In the case that the completion is possible, we will show that it is unique in a particular situation: the corresponding graph is connected. We will denote by $\mathcal{N}(C)$ the null space of matrix $C$, that is, all vectors $\mathbf{x}$ such that $C \mathbf{x}=\mathbf{0}$. For future use, we will need the following lemma.

Lemma 13. Let $G$ be a graph and $M$ its incidence matrix. Then

$$
\begin{align*}
\mathcal{N}\left(M^{T}\right)=\{\mathbf{x} & =\left(x_{1}, \ldots, x_{n}\right)^{T}: x_{i}=x_{j} \\
& \Longleftrightarrow i \text { and } j \text { are in the same } \tag{34}
\end{align*}
$$

connected component of $G\}$.
Hence, the dimension of $\mathcal{N}\left(M^{T}\right)$ is the number of connected components of graph G. In particular, if the graph is connected, then $\mathcal{N}\left(M^{T}\right)=\operatorname{span}\left\{\mathbf{1}_{n}\right\}$.

Proof. Let $\mathbf{x} \in \mathbb{R}^{n}$ be such that $M^{T} \mathbf{x}=\mathbf{0}$. This implies that "if there is an edge connecting nodes $i$ and $j$, then $x_{i}=x_{j}$." Hence $x_{i}=x_{j}$ if and only if $i$ and $j$ are in the same connected component of $G$.

In the next result, the hypothesis $2 k<n(n-1)$ serves to guarantee $m>0$.

Theorem 14. Let $A \in \mathbb{R}_{n, n}$ be a reciprocal incomplete matrix and $2 k$ be the number of void entries (located $u p$ and down the main diagonal of $A$ ). If $2 k<n(n-1), G_{A}$ is connected and there exists a consistent completion of $A$, then this completion is unique.

Proof. Let us define $M_{A}, \mathbf{b}_{A}$, and $m$ as in Theorem 11. Since there is a consistent completion of $A$, then the linear system $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is consistent. Let $\mathbf{w}_{0}=\left(c_{1}, \ldots, c_{n}\right)^{T}$ be a solution. It is simple to prove that $\left\{\mathbf{w} \in \mathbb{R}^{n}: M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}\right\}=\mathbf{w}_{0}+\mathscr{N}\left(M_{A}^{T}\right)$. Using this fact and Lemma 13 we have that if $\mathbf{w}$ is any solution of $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$, then there exists $\alpha \in \mathbb{R}$ such that

$$
\begin{equation*}
\mathbf{w}=\mathbf{w}_{0}+\alpha \mathbf{1}_{n}=\left(c_{1}+\alpha, \ldots, c_{n}+\alpha\right)^{T} \tag{35}
\end{equation*}
$$

Any possible completion of $A$ must be computed using the values $\lambda_{1}, \ldots, \lambda_{k}$ from (17); then $\lambda_{r}=w_{i_{r}}-w_{j_{r}}=c_{i_{r}}-c_{j_{r}}$ for $r=1, \ldots, k$. This shows that the completion is unique.

Observe that if there is a consistent completion of $A$, then the general solution of $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is given by $\mathbf{w}_{0}+\mathscr{N}\left(M_{A}^{T}\right)$, where $\mathbf{w}_{0}$ is a particular solution of $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$. It is simple to prove that if $N$ is a matrix satisfying $M_{A}^{T} N M_{A}^{T}=M_{A}^{T}$,

Table 1: Notation.

|  | Incomplete <br> matrix $A$ | Directed <br> graph $G_{A}$ | Incidence <br> matrix $M_{A}$ |
| :---: | :---: | :---: | :---: |
| $n$ | Size of $A$ | Points of $G_{A}$ | Rows of $M_{A}$ |
| $m$ | Arrows of $G_{A}$ | Columns of $M_{A}$ |  |
| $2 k$ | Entries of <br> $A$ to be filled |  |  |
| $p$ |  | Connected components <br> of $G_{A}$ |  |

then $N \mathbf{b}_{A}$ verifies the system $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$. Hence the general solution of this latter system is

$$
\begin{equation*}
\left\{N \mathbf{b}_{A}+\mathbf{x}: \mathbf{x} \in \mathcal{N}\left(M_{A}^{T}\right)\right\} \tag{36}
\end{equation*}
$$

where $N$ is an arbitrary but fixed matrix such that $M_{A}^{T} N M_{A}^{T}=$ $M_{A}^{T}$. We can choose $N=\left(M_{A}^{T}\right)^{\dagger}$, where the superindex $\dagger$ means the Moore-Penrose inverse of a matrix. Now, observe that $\left(M_{A}^{T}\right)^{\dagger}=\left(M_{A}^{\dagger}\right)^{T}$ and in [20, Ch. 2] a method for finding the Moore-Penrose inverse of an incidence matrix is given.

Another result that will be useful is the following (see [21, Ch. 2]). "Let $A$ be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^{m}$ such that the system $A \mathbf{x}=\mathbf{b}$ is consistent. If $N$ is any matrix satisfying $A N A=A$, then the general solution of the $A \mathbf{x}=\mathbf{b}$ is given by $N \mathbf{b}+(I-N A) \mathbf{y}$ for arbitrary $\mathbf{y} \in \mathbb{R}^{n \prime \prime}$. This result enables us to find the general solution of $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ without computing the null space of $M_{A}^{T}$ as in (36).

Finally, notice that, to find the consistent completion of $A$ (as distilled in the proof of Theorem 14) when the corresponding graph $G_{A}$ is connected, we can discard the vector $\mathbf{x}$ in the null space of $\mathcal{N}\left(M_{A}^{T}\right)$ appearing in (36).

Example 15 (this is the revisited previous Example 9). Let $A$ be the incomplete matrix given in (24). Following the notation of Table 1 we have $k=1, n=3, m=2$, and $p=1$. By Theorem 11 we obtain that there is a consistent completion. A solution of $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is (by employing $N$ as the MoorePenrose inverse of $M_{A}^{T}$ )

$$
\begin{align*}
\mathbf{w} & =\left(M_{A}^{T}\right)^{\dagger} \mathbf{b}_{A}=\frac{1}{3}\left[\begin{array}{cc}
2 & -1 \\
-1 & 2 \\
-1 & -1
\end{array}\right]\left[\begin{array}{c}
-\log 3 \\
\log 2-\log 3
\end{array}\right] \\
& =\frac{1}{3}\left[\begin{array}{c}
-\log 2-\log 3 \\
2 \log 2-\log 3 \\
-\log 2+2 \log 3
\end{array}\right] \tag{37}
\end{align*}
$$

and $\lambda=w_{1}-w_{2}=-\log 2$. This example finishes as before.
Example 16 (this is the revisited Example 12). Let $A$ be the incomplete matrix given in (30). Following the notation of

Table 1 we have $n=3, m=1, k=2$, and $p=2$. Any solution of $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is given by

$$
\mathbf{w}=\left(M_{A}^{T}\right)^{\dagger} \mathbf{b}_{A}+\mathbf{x}=\left[\begin{array}{c}
0  \tag{38}\\
\frac{1}{2} \\
-\frac{1}{2}
\end{array}\right] \log a+\mathbf{x}
$$

with $\mathbf{x} \in \mathcal{N}\left(M_{A}^{T}\right)$. But any vector of $\mathcal{N}\left(M_{A}^{T}\right)$ is of the form $(x, y, y)^{T}$, where $x, y \in \mathbb{R}$. Hence

$$
\mathbf{w}=\left[\begin{array}{c}
x  \tag{39}\\
y+\frac{\log a}{2} \\
y-\frac{\log a}{2}
\end{array}\right]
$$

Theorem 5 ensures that $\lambda_{1}=w_{1}-w_{2}=x-y-\log a / 2$ and $\lambda_{2}=w_{1}-w_{3}=x-y+\log a / 2$ satisfy that $Y=\log a \cdot B_{23}+$ $\lambda_{1} \cdot B_{12}+\lambda_{2} \cdot B_{13}$ is a matrix such that $E(Y)$ is any consistent completion of $A$. By denoting $b=\exp (x-y) / \sqrt{a}$ we obtain the same solution of the Example 12 obtained previously.

Another way of obtaining the same solution is by means of

$$
\begin{align*}
\mathbf{w} & =\left(M_{A}^{T}\right)^{\dagger} \mathbf{b}_{A}+\left(I-\left(M_{A}^{T}\right)^{\dagger} M_{A}^{T}\right) \mathbf{y} \\
& =\left[\begin{array}{c}
0 \\
\frac{1}{2} \\
-\frac{1}{2}
\end{array}\right] \log a+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{array}\right]\left[\begin{array}{l}
y_{1} \\
y_{2} \\
y_{3}
\end{array}\right] . \tag{40}
\end{align*}
$$

Obviously, one obtains the same solution as in (39) by assigning $y_{1} \rightarrow x$ and $y_{1}+y_{2} \rightarrow y$.

Let us observe that the linear system $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is consistent if and only if $\mathbf{b}_{A} \in \mathscr{R}\left(M_{A}^{T}\right)$. But standard linear algebra produces $\mathscr{R}\left(M_{A}^{T}\right)=\mathscr{N}\left(M_{A}\right)^{\perp}$. Hence we have that the linear system $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is consistent if and only $\mathbf{b}_{A}^{T} x=0$ for any $x \in \mathcal{N}\left(M_{A}\right)$.

In next result we find the null space of $\mathcal{N}\left(M_{A}\right)$ for some types of graphs. To this end, we recall the concept of cycle in a graph. A cycle is a chain starting at a point and finishing at the same point. For instance, the left graph of Figure 1 has a cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ (observe that the edge $3 \rightarrow 1$ is oriented in the opposite direction).

Let $G$ be an oriented graph (with the notation established in Table 1) whose edges are denoted by $e_{1}, \ldots, e_{m}$, with the enumeration of the edges being dictated by the incidence matrix. For example, for the left graph of Figure 1, the edge $2 \rightarrow 3$ is the third one. For a of set $\mathcal{S}$ of edges (maybe the orientation of these edges can be distinct to the one given by the graph) we define the following vector $\mathbf{v}=\left(v_{1}, \ldots, v_{m}\right)^{T} \in$ $\mathbb{R}^{m}$ : set $v_{i}=1$ if $e_{i} \in \mathcal{S}$ and the orientation of $e_{i}$ agrees with
the corresponding one of $\mathcal{S}$. Set $v_{i}=-1$ if $e_{i} \in \mathcal{S}$ and the orientation of $e_{i}$ is the opposite of the corresponding one of $\mathcal{S}$. Set $v_{i}=0$ if $e_{i} \notin \mathcal{S}$. For example, for the left graph of Figure 1 we associate the cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ to $(1,-1,1,0)^{T}$ because the first and third edges $(1 \rightarrow 2$ and $2 \rightarrow 3)$ are in the loop; the second edge $(1 \rightarrow 3)$ appears in the cycle but in reverse and the fourth edge $(2 \rightarrow 4)$ does not appear in the cycle.

Some further properties of the consistent completion of $A$ can be deduced if the associated graph $G_{A}$ is planar (see, e.g., [22] or any textbook on graph theory for a deeper insight of planar graphs). Let us remark that $G_{A}$ is not necessarily planar as the following example shows. Let

$$
\begin{align*}
& B=\left[\begin{array}{lll}
1 & \star & \star \\
\star & 1 & \star \\
\star & \star & 1
\end{array}\right], \quad C=\left[\begin{array}{lll}
a & b & c \\
d & e & f \\
g & h & i
\end{array}\right], \\
& D=\left[\begin{array}{lll}
a^{-1} & d^{-1} & g^{-1} \\
b^{-1} & e^{-1} & h^{-1} \\
c^{-1} & f^{-1} & i^{-1}
\end{array}\right], \tag{41}
\end{align*}
$$

with $a, b, \ldots, i$ being positive numbers. The matrix $A=$ $\left[\begin{array}{ll}B & C \\ D & B\end{array}\right]$ leads to a nonplanar graph, namely, the complete $(3,3)$ bipartite graph. Also, observe that a planar graph divides the plane into several faces; there is only one unbounded face; and any bounded face corresponds to a cycle. If $f$ denotes the number of bounded faces, the very well-known Euler's formula holds:

$$
\begin{equation*}
n-m+f=p \tag{42}
\end{equation*}
$$

It is a known fact (see, e.g., [22, Ch. 2]) that any vector $\mathbf{x}$ corresponding to a cycle belongs to the null space of the incidence matrix of the graph.

Theorem 17. Let $G$ be a planar oriented graph and $M$ its incidence matrix. If $\mathbf{x}_{1}, \ldots, \mathbf{x}_{f}$ correspond to the bounded faces of the graph, then $\left\{\mathbf{x}_{1}, \ldots, \mathbf{x}_{f}\right\}$ is a basis of $\mathcal{N}(M)$.

Proof. By the previous paragraph we have $\mathbf{x}_{1}, \ldots, \mathbf{x}_{f} \in$ $\mathcal{N}(M)$. We shall prove the next affirmation by induction on $f$. "If $\mathbf{x}_{1}, \ldots, \mathbf{x}_{f}$ correspond to the bounded faces of a graph, then $\mathbf{x}_{1}, \ldots, \mathbf{x}_{f}$ are linearly independent". For $f=1$ the theorem obviously is true since $\mathbf{x}_{1} \neq \mathbf{0}$. Assume that the claim is true for $f-1$ and let $\mathbf{x}_{1}, \ldots, \mathbf{x}_{f}$ correspond to the $f$ bounded faces of a graph such that $\sum_{j=1}^{f} \alpha_{j} \mathbf{x}_{j}=\mathbf{0}$. There must be at least one edge from the unbounded face. This edge, let us say the $i$ th edge, belongs to only one face (any edge is a boundary of two faces; but if one face is unbounded, the other face must be bounded), let us say $\mathbf{x}_{k}$ this face. Looking at the $i$ th coordinate of $\sum_{j=1}^{f} \alpha_{j} \mathbf{x}_{j}=\mathbf{0}$ we conclude that $\alpha_{k}=0$. Now, the faces of the graph obtained by deleting the $i$ th edge in the original graph are $\mathbf{x}_{1}, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_{f}$. Since $\sum_{j=1, j \neq k}^{f} \alpha_{j} \mathbf{x}_{j}=\mathbf{0}$, by the induction hypothesis we have that $\alpha_{1}=\cdots=\alpha_{f}=0$.

To finish the proof of the theorem, let us recall that the rank of the incidence matrix is $n-p$, where $n$ is the number of points and $p$ the number of connected components. It will be also useful to remark that the incidence matrix has $n$ rows
and $m$ columns, with $m$ being the number of edges. By Euler's formula (42) we have

$$
\begin{equation*}
\operatorname{dim} \mathcal{N}(M)=m-\operatorname{rk}(M)=m-(n-p)=f \tag{43}
\end{equation*}
$$

Since $\mathbf{x}_{1}, \ldots, \mathbf{x}_{f}$ are independent vectors in a $f$-dimensional subspace, these vectors form a basis of this subspace.

Corollary 18. Let $A$ be an incomplete reciprocal matrix. If $G_{A}$ is planar and has no bounded faces, then there exists a consistent completion of $A$. Observe that by construction, $G_{A}$ is always oriented.

Example 19 (Example 8 revisited). Looking at the left graph of Figure 1, we see that the only bounded face corresponds to the cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$. Hence a basis of $\mathcal{N}\left(M_{A}\right)$ is $\left\{(1,-1,1,0)^{T}\right\}$. Since the matrix $A$ of Example 8 has a consistent completion if and only if $\mathbf{b}_{A} \in \mathcal{N}\left(M_{A}\right)^{\perp}, \mathbf{b}_{A}=$ $(\log 2, \log 3, \log 3, \log 4)^{T}$, and $(1,-1,1,0)^{T} \mathbf{b}_{A}=\log 2 \neq 0$, matrix $A$ cannot be completed to a consistent matrix.

Let us now consider an incomplete reciprocal matrix that cannot be completed to be consistent. How can the known entries be modified to complete the matrix to be consistent? The answer will be clear if we recall the following summary.

For an incomplete reciprocal matrix $A$, the following affirmations are equivalent.
(i) There exists a consistent completion of $A$.
(ii) The linear system $M_{A}^{T} \mathbf{w}=\mathbf{b}_{A}$ is consistent.
(iii) $\mathbf{b}_{A}^{T} \mathbf{x}=0$ for any $\mathbf{x} \in \mathscr{N}\left(M_{A}\right)$.

Example 20. If we want to modify some entries of matrix $A$ given in (9) in order to have a consistent completion, let us start by writing

$$
A=\left[\begin{array}{cccc}
1 & a_{1} & a_{2} & \star  \tag{44}\\
a_{1}^{-1} & 1 & a_{3} & a_{4} \\
a_{2}^{-1} & a_{3}^{-1} & 1 & \star \\
\star & a_{4}^{-1} & \star & 1
\end{array}\right], \quad \mathbf{b}_{A}=\left[\begin{array}{l}
\log a_{1} \\
\log a_{2} \\
\log a_{3} \\
\log a_{4}
\end{array}\right] .
$$

Now we can choose the entries $a_{1}, \ldots, a_{4}$ by using one of the above items. But as we know the null space of $M_{A}$-in fact, from Example 19 we know that a basis of $\mathcal{N}\left(M_{A}\right)$ is spanned by $\mathbf{x}=(1,-1,1,0)^{T}$-we choose item (iii).

There is a consistent completion of $A$

$$
\begin{align*}
& \Longleftrightarrow \mathbf{b}_{A}^{T} \mathbf{x}=0  \tag{45}\\
& \Longleftrightarrow \log a_{1}-\log a_{2}+\log a_{3}=0 \Longleftrightarrow a_{1} a_{3}=a_{2}
\end{align*}
$$

It is noteworthy that the value of $a_{4}$ is arbitrary.

## 5. Application to Leakage Control in a Water Supply System

By way of illustration, in this section we use some examples corresponding to a problem that challenges water supply
managers, and in which the main objective is to achieve a suitable leakage policy to minimize water loss (see, e.g., [23]). The criteria considered are given in the examples. Finally, various management alternatives for leakage control are considered: ranging from active leakage control (ALC) to passive leakage control (PLC). ALC involves taking actions in supply systems or individual hydrometric district areas, to identify and repair detectable leaks that have not been reported. PLC refers to repairing only reported or evident leaks.

We use the characterization presented in this paper as a decision support tool to assess the consistency of incomplete judgments given by specific stakeholders.

Example 21. We consider here the following multiple criteria to decide on the alternatives:
$\mathrm{C}_{1}$ : planning development cost and its implementation,
$\mathrm{C}_{2}$ : budgets and payment appropriations,
$\mathrm{C}_{3}$ : investment recovery,
$\mathrm{C}_{4}$ : social cost,
$\mathrm{C}_{5}$ : environmental cost.
In this example, we use the point of view of an employee who develops water supply projects in a company in Spain. The actor has given an incomplete body of opinion and is unable to provide a robust viewpoint regarding some criteria, particularly comparisons regarding social $\left(\mathrm{C}_{4}\right)$ and environmental $\left(\mathrm{C}_{5}\right)$ costs, as well as planning development $\operatorname{cost}\left(C_{1}\right)$.

Let $A$ be the following incomplete reciprocal matrix that represents the incomplete judgment given by the stakeholder:

$$
A=\left[\begin{array}{ccccc}
1 & \frac{1}{2} & 1 & \star & \star  \tag{46}\\
2 & 1 & 2 & 7 & \star \\
1 & \frac{1}{2} & 1 & 4 & \frac{1}{2} \\
\star & \frac{1}{7} & \frac{1}{4} & 1 & \frac{1}{8} \\
\star & \star & 2 & 8 & 1
\end{array}\right]
$$

By running the file given in Section 4, we find that there is no consistent completion of $A$. However, the associated directed graph of this incomplete matrix is relatively easy to obtain (see Figure 3). We can see that this graph is planar. Thus, Theorem 17 enables us to very quickly obtain a basis of $\mathcal{N}\left(M_{A}\right)$ : the basis corresponds to cycles $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$, $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$, and $3 \rightarrow 4 \rightarrow 5 \rightarrow 3$. Explicitly we have that the following vectors

$$
\begin{align*}
& \mathbf{x}_{1}=(1,-1,1,0,0,0,0)^{T}, \quad \mathbf{x}_{2}=(0,0,1,-1,1,0,0)^{T}, \\
& \mathbf{x}_{3}=(0,0,0,0,1,-1,1)^{T} \tag{47}
\end{align*}
$$

In this case, a slight modification of some entries of $A=$ $\left(a_{i j}\right)$ makes a consistent completion feasible. In effect, by forcing $\mathbf{b}_{A}^{T} \mathbf{x}_{1}=\mathbf{b}_{A}^{T} \mathbf{x}_{2}=\mathbf{b}_{A}^{T} \mathbf{x}_{3}=0$, we find that the three following conditions must be simultaneously satisfied:

$$
\begin{equation*}
a_{12} a_{23}=a_{13}, \quad a_{23} a_{34}=a_{24}, \quad a_{34} a_{45}=a_{35} \tag{48}
\end{equation*}
$$

Observe that the first and the third equalities from (48) are satisfied by the original matrix $A$, whereas the second equality does not hold.

The stakeholder considers that some entries can be changed. Since $a_{23}$ and $a_{34}$ appear in the first and third equalities, the stakeholder is asked to modify only $a_{24}$, and his answer is affirmative. By modifying entry $a_{24}$ to 8 , we obtain a matrix that is consistently completed. This alteration does not change the opinion of the stakeholder, because the old value, 7 , and the new value, 8 , correspond to similar verbal judgements of preference in Saaty's scale. Let us rename $A$, the new reciprocal incomplete matrix,

$$
A=\left[\begin{array}{ccccc}
1 & \frac{1}{2} & 1 & \star & \star  \tag{49}\\
2 & 1 & 2 & 8 & \star \\
1 & \frac{1}{2} & 1 & 4 & \frac{1}{2} \\
\star & \frac{1}{8} & \frac{1}{4} & 1 & \frac{1}{8} \\
\star & \star & 2 & 8 & 1
\end{array}\right] .
$$

We execute the m -file in order to find $M_{A}, \mathbf{b}_{A}$, and $N=$ $\left(M_{A}^{T}\right)^{\dagger}$. This m-file also ensures that there is a consistent completion of $B$ (this fact was previously known because (48) are satisfied). This completion is unique by Theorem 14. As we mentioned, this unique completion of $A$ can be obtained from (36) by discarding the vectors of $\mathcal{N}\left(M_{A}^{T}\right)$. By executing $>_{\mathrm{W}}=\mathrm{N} * \mathrm{~b}$
where $N=\left(M_{A}^{T}\right)^{\dagger}$ and $\mathrm{b}=\mathbf{b}_{A}$, we obtain
w =
0.0000
0.6931
0.0000
$-1.3863$

```
>> [i j] = find(P);
>> r=length(i);
>> for index=1:r
    A(i(index),j(index)) = exp(w(i(index))-w (j(index)));
end
```

Algorithm 2


Figure 3: Directed graph corresponding to Example 21.

Theorem 7 enables us to obtain the unique completion. It can be quickly found by using the calls in Algorithm 2.

Thus obtaining a consistent completion of the matrix which is satisfactory for the stakeholder

| $>\mathrm{A}$ |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- |
| $\mathrm{A}=$ |  |  |  |  |
|  |  |  |  |  |
| 1.0000 | 0.5000 | 1.0000 | 4.0000 | 0.5000 |
| 2.0000 | 1.0000 | 2.0000 | 8.0000 | 1.0000 |
| 1.0000 | 0.5000 | 1.0000 | 4.0000 | 0.5000 |
|  |  |  |  |  |
| 0.2500 | 0.1250 | 0.2500 | 1.0000 | 0.1250 |
| 2.0000 | 1.0000 | 2.0000 | 8.0000 | 1.0000 |

Example 22. A final example shows an incomplete judgment that cannot be consistently completed. The criteria used to decide between the alternatives may be varied and take into account economic, environmental, and social impacts. In addition to the five criteria shown in Example 21, two more criteria are used:
$\mathrm{C}_{6}$ : construction of tanks and reservoirs,
$\mathrm{C}_{7}: \mathrm{CO}_{2}$ emissions.

Let $A$ be the matrix that corresponds to an incomplete judgment.

$$
A=\left[\begin{array}{ccccccc}
1 & 2 & 5 & 1 & 7 & 5 & \star  \tag{50}\\
\frac{1}{2} & 1 & 2 & \frac{1}{2} & \star & 2 & 1 \\
\frac{1}{5} & \frac{1}{2} & 1 & \frac{1}{5} & 2 & 1 & \frac{1}{2} \\
1 & 2 & 5 & 1 & \star & 3 & \star \\
\frac{1}{7} & \star & 0.5 & \star & 1 & \frac{1}{2} & \star \\
\frac{1}{5} & \frac{1}{2} & 1 & \frac{1}{3} & 2 & 1 & \frac{1}{2} \\
\star & 1 & 2 & \star & \star & 2 & 1
\end{array}\right] .
$$

By running the previous file, we find that there is no consistent completion of $A$. We can obtain the directed graph corresponding to this example. This graph is more complex than the other example. However, since we know the entries $a_{i, j}$ for $i \in\{2,3,6\}$ and $j \in\{1,4,7\}$, then the graph associated with the matrix of this example contains the complete $(3,3)$ bipartite graph. Hence, the graph associated with $A$ is not planar. In this case, the facilitator should invite the stakeholder to work further in order to reach a point where a consistent completion is possible.

## 6. Conclusions

MCDA, in particular AHP, has emerged as a decision support tool to integrate technical information and stakeholder values. Over the last decade, there has been a significant growth in MCDA applications in many fields, including decision support tools [24]. These techniques provide a systematic approach to combine information inputs with benefit/cost information and decision maker or stakeholder views to rank alternatives [25].

Since vast investments are frequently at stake, DM must be performed with extreme care. In this regard, facilitators conducting participatory processes need robust tools that enable them to make discerning decisions when collecting opinions from various stakeholders.

In this paper, by characterizing incomplete comparison matrices that can be consistently completed we have provided a robust tool to decide if an incomplete comparison body given by an actor is acceptable or, on the contrary, the actor must further develop the comparison for the sake of
consistency. This tool is very simple since it reduces to the solution of a linear set of equations. Using graph theory we have developed several useful results that enable us to study the uniqueness of the consistent completion and to compute all the possible completions in a straightforward manner. Finally, in Section 5 we have provided an illustrative example corresponding to the (incomplete) opinion given by a stakeholder in a DM process regarding the most suitable water leak management policy in a water distribution system. As can be observed, the calculations are simple and clear and can be easily implemented in any decision support tool based on AHP.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# A Study about the Integration of the Elliptical Orbital Motion Based on a Special One-Parametric Family of Anomalies 

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#### Abstract

This paper aimed to address the study of a new family of anomalies, called natural anomalies, defined as a one-parameter convex linear combination of the true and secondary anomalies, measured from the primary and the secondary focus of the ellipse, and its use in the study of analytical and numerical solutions of perturbed two-body problem. We take two approaches: first, the study of the analytical development of the basic quantities of the two-body problem to be used in the analytical theories of the planetary motion and second, the study of the minimization of the errors in the numerical integration by an appropriate choice of parameters in our family for each value of the eccentricity. The use of an appropriate value of the parameter can improve the length of the developments in the analytical theories and reduce the errors in the case of the numerical integration.


## 1. Introduction

One of the main problems in celestial mechanics is the study of the motion in the solar system. In this regard, it is interesting to study the planetary theories and the motion of an artificial satellite around the Earth.

To construct a planetary theory, we can follow two main ways: one, the use of a numerical integrator, and two, the use of analytical methods to integrate the problem [1-4]. This paper is focused on planetary theories in the second sense. The analytical and semianalytical planetary theories involve the management of very long Poisson series developments [5] using, as temporal variables, the mean longitudes of the planets. In this sense, the study of a change of the temporal variable in order to obtain developments that are more compact than the ones obtained if the mean longitudes (or mean anomalies) are used [6-9] can be interesting. To construct an analytical or semianalytical planetary theory, it is necessary to obtain a set of developments for the twobody problem, and from that, we can evaluate the inverse of the distance between the two planets and so to develop
the second member of the differential equations of motion. Obtaining precise developments of the inverse of the distance is one of most important problems to construct the analytical theories.

To study the motion of an artificial satellite, it is more convenient to use a numerical integration method. An excellent overview on numerical integrators can be seen in [10] containing a theory of symplectic and symmetric integrator, including Runge-Kutta (composition, splitting, ...) and multistep and specially designed integrators, also their construction and practical merits are discussed. However, the history of modern numerical integrators begins long ago. Adams introduced a multistep method to study the perturbed motion of the couple Jupiter-Saturn, Kutta introduced at the beginning of the 20th century the well known family of Runge-Kutta integrators, Bulirsch and Stoer developed extrapolation methods that were used by the Institute of Applied Astronomy of St. Petersburg to obtain the minor planet ephemerides, and so forth.

Numerical integrators are appropriate to study the planetary theories and the satellite motion. In the first case,
it is very important for the errors of the methods to be bounded for a very large time span and for this purpose, symplectic integrators can be adequate. One of the most important problems in the study of a satellite motion is the case of very high eccentricities. The main problem with using the mean anomaly in this case is the non uniformity on the point distribution on the ellipse due to the second Kepler law; this, for a constant step size, implies a minor concentration of points in the perigee and so larger errors in this region. In this case, it can be convenient to use a technique known as analytical regularization of step size. This technique involves a change of the temporal variable in order to have,for the new variable, a point distribution containing a major density of points in the orbital regions, where the velocity is higher. In this context, Sundman [11] introduces a new temporal variable $\tau$ related to the time $t$ through $d t=C r d \tau$. Nacozy [12], Janin and Bond [13, 14], and Velez and Hilinski [15] extend this technique defining a new one-parameter family of transformations $\alpha$ called generalized Sundman transformations:

$$
\begin{equation*}
d t=Q(r, \alpha) d \tau_{\alpha} \tag{1}
\end{equation*}
$$

where $Q(r, \alpha)=C_{\alpha} r^{\alpha}$ is the called partition function. A more complicated transformation was introduced by Ferrándiz et al. [16] $Q(r)=r^{2 / 3}\left(a_{0}+a_{1} r\right)^{-1 / 2}$ and Brumberg [17] who proposed the use of the regularized length of arc:

$$
\begin{equation*}
Q(r)=\frac{1}{\sqrt{2 G M}} \frac{\sqrt{r}}{\sqrt{1-r / 2 a}} \tag{2}
\end{equation*}
$$

where GM is the spaceflight constant of the Earth and $a$ the major semiaxe of the osculating elliptic orbit of the satellite.

In this paper, we follow this technique. In particular, we propose a new one-parametric family of anomalies, called natural anomalies, defined by a convex linear combination $\Psi_{\alpha}=\alpha f+(1-\alpha) f^{\prime}$, where $f$ is the true anomaly and $f^{\prime}$ the polar angle referred to the secondary focus called secondary true anomaly.

The rest of this article is organized as follows.
In Section 2, we introduce the general background. This section contains the equations of the perturbed motion in two ways: first, to study the analytical planetary theories and second, to use appropriate numerical methods.

In Section 3, the properties of natural family of anomalies are also described. These properties include the connection between the quantities $r$ and $r^{\prime}$ with $f^{\prime}$, the connection between the natural anomaly $\Psi_{\alpha}$ and the eccentric anomaly $E$, and the study of the partition function denoted by $Q_{\alpha}(r)$ for this case.

In Section 4, the analytical developments according to the natural anomaly are studied. This study contains the developments of $E, \sin E, \cos E$, and $1 / r$ with respect to the natural anomaly. So, an appropriate use of them is enough to obtain the development of the mean anomaly according to the natural anomaly, that is the Kepler equation. In this section, we show the length of the analytical development of the inverse of the distance for the couple Jupiter-Saturn using several values of the $\alpha$.

In Section 5, we obtain the differential equations of motion using an arbitrary anomaly from the natural family. A set of numerical examples about the two-body problem are analyzed. We carry out a perturbed problem in order to analyze the robustness of the method.

In Section 6, the main conclusions and remarks of this paper are exposed.

## 2. Basic Equations

The analytical methods are based on the solution of the twobody problem (Sun planet) through a set of orbital elements, for example, the third set of Brouwer and Clemence [18] $(a, e, i, \Omega, \omega, M)$, where $M=M_{0}+n\left(t-t_{0}\right), n$ is the mean motion, and $t_{0}$ is the initial epoch being ( $a, e, i, \Omega, \omega, n$ ) constants in the unperturbed two-body problem. This solution can be used as a first approximation of the perturbed problem and we can use the Lagrange method of variation of constants to replace the first elements by the osculating ones given by the Lagrange planetary equations [19]:

$$
\begin{gather*}
\frac{d a}{d t}=\frac{2}{n a} \frac{\partial R}{\partial \sigma} \\
\frac{d e}{d t}=-\frac{\sqrt{1-e^{2}}}{n a^{2} e} \frac{\partial R}{\partial \omega}+\frac{1-e^{2}}{n a^{2} e} \frac{\partial R}{\partial \sigma} \\
\frac{d i}{d t}=-\frac{1}{n a^{2} \sqrt{1-e^{2}} \sin i} \frac{\partial R}{\partial \Omega}+\frac{\operatorname{ctg} i}{n a^{2} \sqrt{1-e^{2}}} \frac{\partial R}{\partial \omega}, \\
\frac{d \Omega}{d t}=\frac{1}{n a^{2} \sqrt{1-e^{2}} \sin i} \frac{\partial R}{\partial i},  \tag{3}\\
\frac{d \omega}{d t}=\frac{\sqrt{1-e^{2}}}{n a^{2} e} \frac{\partial R}{\partial e}-\frac{\cos i}{n a^{2} \sqrt{1-e^{2}} \sin i} \frac{\partial R}{\partial i} \\
\frac{d \sigma}{d t}=-\frac{2}{n a} \frac{\partial R}{\partial a}-\frac{1-e^{2}}{n a^{2} e} \frac{\partial R}{\partial e},
\end{gather*}
$$

where $\sigma$ is a new variable defined by the equation:

$$
\begin{equation*}
M=\sigma+\int_{t_{0}}^{t} n d t \tag{4}
\end{equation*}
$$

and it coincides with $M_{0}$ in the case of the unperturbed motion. $R$ is the disturbing potential $R=\sum_{k=1}^{N} R_{i}$ due to the disturbing bodies $i=1, \ldots, N$. It is defined as [19]

$$
\begin{equation*}
R=\sum_{k=1}^{N} G m_{k}\left[\left(\frac{1}{\Delta_{k}}\right)-\frac{x \cdot x_{k}+y \cdot y_{k}+z \cdot z_{k}}{r_{k}^{3}}\right] \tag{5}
\end{equation*}
$$

where $\vec{r}=(x, y, z)$ and $\vec{r}_{k}=\left(x_{k}, y_{k}, z_{k}\right)$ are the heliocentric vector position of the secondary body and the $k$ th disturbing body, respectively, $\Delta_{k}$ is the distance between the secondary body and the disturbing body, and $m_{k}$ is the mass of the disturbing body.

In order to integrate the Lagrange planetary equations through analytical methods, it is necessary to develop the second member of the Lagrange planetary equations as truncated Fourier series, which is a classical problem in celestial mechanics $[4,5,18,20,21]$. The analytical methods are appropriated to study planetary motion because the eccentricities of the planetary orbits are small. Despite this, analytical methods provide very long series solution and it is convenient to obtain more compact developments using as temporal variable an appropriate anomaly [6, 7].

To obtain the developments with respect to an anomaly $\Psi_{i}$, it is necessary to obtain for each planet $i$ the developments of the coordinates and the inverse of the radius in Fourier series of $\Psi_{i}$. Then, the integration of the Lagrange planetary equations with respect to the $\Psi_{i}$ anomalies requires to obtain solution of the corresponding Kepler equation $M_{i}=M_{i}\left(\Psi_{i}\right)$, for each planet.

To use numerical integration methods it is more appropriate to consider the equation of motion in the form of the second Newton law. To study the efficiency of the numerical integration methods using an anomaly $\Psi$ as temporal variable, we select the problem of the motion of an artificial satellite around the Earth. The relative motion of the secondary with respect to the Earth is defined by the second order differential equations

$$
\begin{equation*}
\frac{d^{2} \vec{r}}{d t^{2}}=-G M \frac{\vec{r}}{r^{3}}-\vec{\nabla} U-\vec{F}, \tag{6}
\end{equation*}
$$

where $\vec{r}$ is the radius vector of the satellite, $U$ the potential from which the perturbative conservative forces are derived, and $\vec{F}$ includes the nonconservative forces. To integrate system (6), it is necessary to know the initial values of the radius vector $\vec{r}_{0}$ and velocity $\vec{v}_{0}$.

In order to uniformize the truncation errors when a numerical integrator is used, there are three main ways: the use of a very small stepsize, the use of a adaptive stepsize method, and the use of a change in the temporal variable to get an appropriate distribution of the points in the orbit so that the points are mostly concentrated in the regions where the speed and curvature are maxima. In this paper, we follow the third way, as previously stated.

## 3. Natural Anomalies

In this section, a new family of anomalies depending on a parameter is defined. Let us represent in Figure 1 the elliptic orbit corresponding to the motion of the two-body problem. This ellipse is defined by its major semiaxis $a=\overline{O Q}$ and its eccentricity $e=c / a, 0 \leq e<1$, where $c$ is the focal semidistance $c=\overline{F F^{\prime}} / 2$, and the minor semiaxis $b$ is defined as $b=a \sqrt{1-e^{2}}$. Let $O$ be the center of the ellipse, let $F$ be the primary focus, let $F^{\prime}$ be the secondary focus (also called equality point), let $Q$ be the pericenter, and let $P$ be the position of the secondary in the orbit. Let us define the coordinates $(\xi, \eta)$ referred to the primary focus and let $r$ and $r^{\prime}$ be the distance between the secondary $P$ and the primary focus and the secondary focus, $F$ and $F^{\prime}$, respectively.


Figure 1: Elliptic motion.

The angle $f$ is called the true anomaly and for the angle $f^{\prime}$, we propose the name "secondary true anomaly".

The quantities $r$ and $r^{\prime}$ satisfy

$$
\begin{equation*}
r+r^{\prime}=2 a, \quad r=\frac{a^{2}\left(1-e^{2}\right)}{1+e \cos f}, r^{\prime}=\frac{a^{2}\left(1-e^{2}\right)}{1-e \cos f^{\prime}} . \tag{7}
\end{equation*}
$$

On the other hand, the coordinates of the secondary with respect to the primary, in the orbital plane, are

$$
\begin{gather*}
\xi=r \sin f=r^{\prime} \sin f^{\prime}=a(e-\cos E), \\
\eta=r \cos f=r^{\prime} \cos f^{\prime}-2 a e=a \sqrt{1-e^{2}} \sin E . \tag{8}
\end{gather*}
$$

The spatial orbital coordinates $\vec{r}=(x, y, x)^{t}$ are related to the orbital coordinates $(\xi, \eta, 0)^{t}$ by means of

$$
\begin{equation*}
\vec{r}=A \vec{r}_{\text {orb }}, \tag{9}
\end{equation*}
$$

where $A=R_{1}(-\Omega) R_{3}(-i) R_{1}(-\omega) . R_{i}$ defines a rotation around the $i$-axis.

From (7) we have

$$
\begin{gather*}
d r+d r^{\prime}=0, \quad d r=-\frac{a^{2}\left(1-e^{2}\right) \sin f d f}{(1+e \cos f)^{2}} \\
d f^{\prime}=\frac{a^{2}\left(1-e^{2}\right) \sin f^{\prime} d f^{\prime}}{\left(1-e \cos f^{\prime}\right)^{2}} \tag{10}
\end{gather*}
$$

and from (10), it is easy to obtain $r^{2} \sin f d f=r^{\prime 2} \sin f^{\prime} d f^{\prime}$. Taking into account (8), we obtain

$$
\begin{equation*}
r d f=r^{\prime} d f^{\prime} \tag{11}
\end{equation*}
$$

The radii $r$ and $r^{\prime}$ are related to the eccentric anomaly $E$ through

$$
\begin{equation*}
r=a(1-e \cos E), \quad r^{\prime}=a(1+e \cos E) . \tag{12}
\end{equation*}
$$

The anomalies $f$ and $f^{\prime}$ are related to the eccentric anomaly $E$ by

$$
\begin{equation*}
\tan \frac{f}{2}=\sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}, \quad \tan \frac{f^{\prime}}{2}=\sqrt{\frac{1-e}{1+e}} \tan \frac{E}{2} . \tag{13}
\end{equation*}
$$

To relate the anomaly $f^{\prime}$ to the mean anomaly $M$ we use the integral of areas $d f=\left(a^{2} \sqrt{1-e^{2}} / r^{2}\right) d M$, and after replacing in (11), we get

$$
\begin{equation*}
d f^{\prime}=\frac{a^{2} \sqrt{1-e^{2}}}{r r^{\prime}} d M=\frac{\sqrt{1-e^{2}}}{1-e^{2} \cos ^{2} E} d M \tag{14}
\end{equation*}
$$

To compare $f^{\prime}$ and $M$ up to second order in $e$ we have

$$
\begin{equation*}
f^{\prime}=M+\frac{e^{2}}{4} \sin 2 M+O\left(e^{3}\right) \tag{15}
\end{equation*}
$$

and so, for small values of eccentricity, the anomaly $f^{\prime}$ is near $M$.

Let us define the natural family of anomalies $\Psi_{\alpha}$ as

$$
\begin{equation*}
\Psi_{\alpha}=\alpha f+(1-\alpha) f^{\prime}, \quad 0 \leq \alpha \leq 1 \tag{16}
\end{equation*}
$$

this family includes, for $\alpha=0$, the secondary true anomaly $f^{\prime}$ and, for $\alpha=1$, the true anomaly $f$. The anomaly $\Psi_{\alpha}$ is related to the mean anomaly by

$$
\begin{equation*}
d M=Q_{\alpha}(r) d \Psi_{\alpha} \tag{17}
\end{equation*}
$$

where the function $Q_{\alpha}(r)$ is defined as

$$
\begin{equation*}
Q_{\alpha}(r)=\frac{1}{a^{2} \sqrt{1-e^{2}}} r\left[\frac{\alpha}{r}+\frac{1-\alpha}{2 a-r}\right]^{-1} \tag{18}
\end{equation*}
$$

## 4. Analytical Developments

To integrate the planetary Lagrange equations, it is necessary to develop the second member of the Lagrange planetary equations as Fourier series with respect to the selected anomalies for each couple of planets. For this task, it is necessary to obtain for each planet the developments according to the selected anomaly of the orbital coordinates $\xi$, and $\eta$, the inverse of the radius $1 / r$ and the mean anomaly $M$.

For this purpose, first of all, we obtain the development of an arbitrary anomaly $\Psi(\alpha)$ in the natural anomalies family. In the future, we will denote $\Psi(\alpha)$ as $\Psi$, if it does not lead to confusion. The developments of the orbital coordinates $\xi, \eta$ (8) are completely determined by the ones of $\sin E$ and $\cos E$. To obtain these developments, it is necessary first to expand $\Psi$ with respect to $E$ [22].

To relate the natural anomaly to the eccentric anomaly, it is convenient to obtain the development of the natural anomaly as Fourier series of eccentric anomaly. To this aim, it is convenient to use the classic formula [4]

$$
\begin{equation*}
\tan \frac{f}{2}=\sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}=\frac{1+q}{1-q} \tan \frac{E}{2} \tag{19}
\end{equation*}
$$

where $q=\tan (\phi / 2), e=\sin \phi$ and so

$$
\begin{equation*}
f=E+\sum_{k=1}^{\infty} \frac{2}{k} q^{k} \sin k E . \tag{20}
\end{equation*}
$$

The secondary true anomaly verifies

$$
\begin{equation*}
\tan \frac{f^{\prime}}{2}=\sqrt{\frac{1-e}{1+e}} \tan \frac{E}{2}=\frac{1-q}{1+q} \tan \frac{E}{2} \tag{21}
\end{equation*}
$$

and consequently

$$
\begin{equation*}
f^{\prime}=E+\sum_{k=1}^{\infty}(-1)^{k} \frac{2}{k} q^{k} \sin k E . \tag{22}
\end{equation*}
$$

Replacing (20) and (22) in (16) we obtain

$$
\begin{equation*}
\Psi=E+\sum_{k=1}^{\infty}\left[1+\left(-1+(-1)^{k}\right) q^{k} \alpha\right] \frac{2}{k} \sin k E \tag{23}
\end{equation*}
$$

From this development, we can obtain up to an arbitrary order $k$ in the developments, with respect to $\Psi$, of $E$ and an arbitrary analytical function $f(E)$ using the Deprit inversion algorithm [23]. As an example, up to fourth order in $q$, we have

$$
\begin{align*}
E= & \Psi+\left\{2(1-2 \alpha) q+\left(8 \alpha^{3}-12 \alpha^{2}+4 \alpha\right) q^{3}\right\} \sin \Psi \\
& +\left\{\left(8 \alpha^{2}-8 \alpha+1\right) q^{2}\right. \\
& \left.+\left(-\frac{128}{3} \alpha^{4}+\frac{256 \alpha^{3}}{3}-\frac{160 \alpha^{2}}{3}+\frac{32 \alpha}{3}\right) q^{4}\right\} \\
& \times \sin 2 \Psi \\
& +\left\{-24 \alpha^{3}+36 \alpha^{2}-\frac{40 \alpha}{3}+\frac{2}{3}\right\} q^{3} \sin 3 \Psi \\
& +\left\{\frac{256 \alpha^{4}}{3}-\frac{512 \alpha^{3}}{3}+\frac{320 \alpha^{2}}{3}-\frac{64 \alpha}{3}+\frac{1}{2}\right\} q^{4} \\
& \times \sin 4 \Psi \tag{24}
\end{align*}
$$

$\times \sin 3 \Psi$

$$
\begin{aligned}
\sin E=\{1+ & \left(-2 \alpha^{2}+2 \alpha-1\right) q^{2} \\
& \left.+\left(\frac{4 \alpha^{4}}{3}-\frac{8 \alpha^{3}}{3}+\frac{8 \alpha^{2}}{3}-\frac{4 \alpha}{3}\right) q^{4}\right\} \sin \Psi \\
+ & \{(1-2 \alpha) q \\
& \left.+\left(\frac{32 \alpha^{3}}{3}-16 \alpha^{2}+\frac{22 \alpha}{3}-1\right) q^{3}\right\} \sin 2 \Psi \\
+ & \left\{\left(6 \alpha^{2}-6 \alpha+1\right) q^{2}\right. \\
& \left.+\left(-54 \alpha^{4}+108 \alpha^{3}-72 \alpha^{2}+18 \alpha-1\right) q^{4}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& +\left\{-\frac{64}{3} \alpha^{3}+32 \alpha^{2}-\frac{38 \alpha}{3}+1\right\} q^{3} \sin 4 \Psi \\
& +\left\{\frac{250 \alpha^{4}}{3}-\frac{500 \alpha^{3}}{3}+\frac{320 \alpha^{2}}{3}-\frac{70 \alpha}{3}+1\right\} q^{4} \\
& \times \sin 5 \Psi
\end{aligned}
$$

$$
\begin{equation*}
\cos E=(1-2 \alpha) q \tag{25}
\end{equation*}
$$

$$
\begin{aligned}
& +\left\{1+\left(-1+6 \alpha-6 \alpha^{2}\right) q^{2}\right. \\
& \left.\quad+\left(\frac{20 \alpha^{4}}{3}-\frac{40 \alpha^{3}}{3}+\frac{28 \alpha^{2}}{3}-\frac{8}{3}\right) q^{4}\right\} \cos \Psi \\
& +\left\{q+\left(\frac{64 \alpha^{3}}{3}-32 \alpha^{2}+\frac{38 \alpha}{3}-2 \alpha q-1\right) q^{3}\right\} \\
& \times \cos 2 \Psi \\
& +\left\{\left(1-6 \alpha+6 \alpha^{2}\right) q\right. \\
& \left.\quad+\left(-90 \alpha^{4}+180 \alpha^{3}-116 \alpha^{2}+26 \alpha-1\right) q^{4}\right\}
\end{aligned}
$$

$\times \cos 3 \Psi$
$+\left\{-\frac{64}{3} \alpha^{3}+32 \alpha^{2}-\frac{38 \alpha}{3}+1\right\} q^{3} \cos (4 \Psi)$
$+\left\{\frac{250 \alpha^{4}}{3}-\frac{500 \alpha^{3}}{3}+\frac{320 \alpha^{2}}{3}-\frac{70 \alpha}{3}+1\right\} q^{4}$
$\times \cos (5 \Psi)$.

The mean anomaly $M$ is related to the eccentric anomaly through the Kepler equation $M=E-e \sin E$. Replacing (24) and (25) in the Kepler equation we obtain

$$
\begin{align*}
M= & \Psi+\left\{-4 \alpha q+\left(8 \alpha^{3}-8 \alpha^{2}+4\right) q^{3}\right\} \sin \Psi \\
& +\left\{\left(8 \alpha^{2}-4 \alpha-1\right) q\right. \\
& \left.+\left(-\frac{128}{3} \alpha^{4}+64 \alpha^{3}-\frac{64 \alpha^{2}}{3} 8 \alpha+4\right) q^{4}\right\} \sin 2 \Psi \\
& +\left\{-24 \alpha^{3}+24 \alpha^{2}-\frac{4 \alpha}{3}-\frac{4}{3}\right\} q^{3} \sin 3 \Psi \\
& +\left\{\frac{256 \alpha^{4}}{3}-128 \alpha^{3}+\frac{128 \alpha^{2}}{3}+4 \alpha-\frac{3}{2}\right\} q^{4} \sin 4 \Psi . \tag{27}
\end{align*}
$$

This is the Kepler equation for the natural anomaly $\Psi$ and it is necessary to integrate the second member of planetary equation of Lagrange [8, 24].

The parameter $q$ is related to $e$ by $q=\left(1-\sqrt{1-e^{2}}\right) / e$ an so, $q$ can be developed as

$$
\begin{equation*}
q=\sum_{k=1}^{\infty} \frac{(2 k-1)!!}{2^{k} k!} e^{2 k-1} \tag{28}
\end{equation*}
$$

To develop $a / r$ with respect to $\Psi$, it is more convenient to use (7) and then we have

$$
\begin{equation*}
\frac{a}{r}=\frac{1+e \cos f}{1-e^{2}} \tag{29}
\end{equation*}
$$

It is easy to see from (19) and (21) that

$$
\begin{equation*}
\tan \frac{f^{\prime}}{2}=\frac{1-e}{1+e} \tan \frac{f}{2} \tag{30}
\end{equation*}
$$

and so

$$
\begin{equation*}
f^{\prime}=f+\sum_{k=1}^{\infty}(-1)^{k} \frac{2}{k} e^{k} \sin k f . \tag{31}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
\Psi=f+\sum_{k=1}^{\infty}(-1)^{k}(1-\alpha) \frac{2}{k} q^{k} \sin k f . \tag{32}
\end{equation*}
$$

Applying Deprit algorithm [23], we obtain the development of $\cos f$ up to an arbitrary order in $e$. As an example, up to fourth order in $e$, we have

$$
\begin{align*}
\cos f= & (\alpha-1) e \\
& +\left\{1+\left(\frac{5 \alpha}{2}-\frac{3 \alpha^{2}}{2}+1\right) e^{2}\right. \\
& \left.+\left(\frac{5 \alpha^{4}}{12}-\frac{7 \alpha^{3}}{6}+\frac{13 \alpha^{2}}{12}-\frac{\alpha}{3}\right) e^{4}\right\} \cos \Psi \\
& +\left\{(1-\alpha) e+\left(\frac{8 \alpha^{3}}{3}-6 \alpha^{2}+\frac{13 \alpha}{3}-1\right) e^{3}\right\} \cos 2 \Psi \\
& +\left\{\left(\frac{3 \alpha^{2}}{2}-\frac{5 \alpha}{2}+\right) e^{2}\right. \\
& \left.+\left(-\frac{45}{8} \alpha^{4}+\frac{63 \alpha^{3}}{4}-\frac{127 \alpha^{2}}{8}+\frac{27 \alpha}{4}-1\right) e^{4}\right\} \\
& +\cos 3 \Psi \\
& \left\{-\frac{8}{3} \alpha^{3}+6 \alpha^{2}-\frac{13 \alpha}{3}+1\right\} e^{3} \cos 4 \Psi \\
& +\left\{\frac{125 \alpha^{4}}{24}-\frac{175 \alpha^{3}}{12}+\frac{355 \alpha^{2}}{24}-\frac{77 \alpha}{12}+1\right\} e^{4} \cos 5 \Psi . \tag{33}
\end{align*}
$$

Replacing (33) in (29) finally we obtain

$$
\begin{align*}
\frac{a}{r}= & 1+\alpha e^{4}+\alpha e^{2} \\
& +\left\{e+\left(-\frac{3}{2} \alpha^{2}+\frac{5 \alpha}{2}\right) e^{3}\right\} \cos \Psi \\
& +\left\{(1-\alpha) e^{2}+\left(\frac{8 \alpha^{3}}{3}-6 \alpha^{2}+\frac{10 \alpha}{3}\right) e^{4}\right\} \cos 2 \Psi \\
& +\left\{\frac{3 \alpha^{2}}{2}-\frac{5 \alpha}{2}+1\right\} e^{3} \cos 3 \Psi \\
& +\left\{-\frac{8}{3} \alpha^{3}+6 \alpha^{2}-\frac{13 \alpha}{3}+1\right\} e^{4} \cos 4 \Psi \tag{34}
\end{align*}
$$

Table 1: Length of the $1 / \Delta$ using $M, \Psi_{0}, \Psi_{0.25}, \Psi_{0.5}, \Psi_{0.75}$, and $\Psi_{1.0}$.

| Number of iteration | $M$ | $\alpha=0.0$ | $\alpha=0.25$ | $\alpha=0.50$ | $\alpha=0.75$ | $\alpha=1.0$ | err |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 2197 | 1597 | 1438 | 1298 | 1095 | 419 |  |
| 1 | 2518 | 1882 | 1717 | 1570 | 1379 | 1172 | $1.01 \cdot 10^{-2}$ |
| 2 | 2728 | 2144 | 1988 | 1855 | 1728 | 1707 | $1.36 \cdot 10^{-3}$ |
| 3 | 2841 | 2294 | 2182 | 2102 | 2028 | 2056 | $3.67 \cdot 10^{-5}$ |
| 4 | 2613 | 2052 | 1988 | 1957 | 1924 | 1912 | $3.50 \cdot 10^{-8}$ |
| 5 | 2627 | 2112 | 2036 | 2007 | 1940 | 1952 | $6.27 \cdot 10^{-14}$ |


(a) $\alpha=0.0$

(c) $\alpha=0.75$

(b) $\alpha=0.5$

(d) $\alpha=1.0$

Figure 2: Points distribution for $e=0.7, \alpha=0.0,0.5,0.75,1.0$.


Figure 3: Optimal value of $\alpha$ for each value of $e \in[0,0.95]$.

The convergence of these developments is poor when the eccentricity is near to one. Fortunately, in the case of the planetary motion, the eccentricity values are small and the convergence rate is appropriate. On the other hand, for
intermediate values of eccentricity, the length of the series using an appropriate anomaly $\Psi_{\alpha}$ is lower than if the mean anomaly is used as temporal variable.

The main problem to develop the second member of the Lagrange planetary equations is to develop the inverse of the distance. This development can be obtained using the Kovalevsky algorithm [25]. Table 1 shows, for the couple Jupiter-Saturn, the performance of the algorithm. This table contains the length of the expansion of the inverse of the distance using the mean anomaly $M$ and the natural anomaly for $\alpha=0,0.25,0.50,0.75,1.0$, and the estimated error in astronomical units for each iteration. The planetary elements are taken from [26].

## 5. Numerical Examples

In general, the perturbative forces are small, for this reason, it is convenient to test the methods by applying them to the well known two-body problem, referred to the orbital


Figure 4: Local integration errors distribution $e=0.7, \alpha=0.0$.


Figure 5: Local integration errors distribution $e=0.7, \alpha=0.5$.


Figure 6: Local integration errors distribution $e=0.7, \alpha=0.75$.

(a) $e r x$

(c) ervx

(b) ery

(d) ervy

Figure 7: Distribution of the local integration errors $e=0.7, \alpha=1.0$.
coordinate system $(x, y, 0)$, to select an appropriate new temporal variable in order to minimize the distribution of the truncation errors in the orbit. Let us define a generic family $\Psi_{\alpha}$ of anomalies depending on a parameter $\alpha$ as $d t=$ $Q_{\alpha}(r) d \Psi_{\alpha}$. For each $\alpha$ we have

$$
\begin{equation*}
\frac{d}{d t}=n \frac{d}{d M}=n \frac{d}{d \Psi_{\alpha}^{\prime}} \frac{d \Psi_{\alpha}}{d M}=\frac{n}{Q_{\alpha}(r)} \frac{d}{d \Psi_{\alpha}} \tag{35}
\end{equation*}
$$

So,

$$
\begin{array}{ll}
\frac{d x}{d \Psi_{\alpha}}=\frac{Q_{\alpha}(r)}{n} v_{x}, & \frac{d v_{x}}{d \Psi_{\alpha}}=-\frac{Q_{\alpha}(r)}{n}\left[G M \frac{x}{r^{3}}+\frac{\partial V}{\partial x}-F_{x}\right], \\
\frac{d y}{d \Psi_{\alpha}}=\frac{Q_{\alpha}(r)}{n} v_{y}, & \frac{d v_{y}}{d \Psi_{\alpha}}=-\frac{Q_{\alpha}(r)}{n}\left[G M \frac{y}{r^{3}}+\frac{\partial V}{\partial y}-F_{y}\right] \tag{36}
\end{array}
$$

In order to test the performance of this method, we use a fictitious artificial satellite with the same elements than HEOS II used by Brumberg [17] ( $a=118363.47 \mathrm{Km}$, $e=0.942572319, i=28^{\circ} .16096, \Omega=185^{\circ} .07554, \omega=$ $270^{\circ} .07151, M_{0}=0^{\circ}$ ), except for its eccentricity, that it is changed to study the optimum value of $\alpha$ depending on the value of the eccentricity $e$. In Figure 2, we show a sample of twenty points for $\Psi_{\alpha}$ with homogeneous distribution over the orbit.

Table 2 shows the values of the different $\alpha$, where the minimum of the errors for this fictitious satellite with the same $(a, i, \Omega, \omega, M)$ elements than HEOSII and different values of eccentricity ( $e=0.0,0.05, \ldots, 0.95$ ) is reached. This table has been carried out using a classic Runge Kutta integrator of fourth order with 1000 uniform steps. In this table, we can see that the value of $\alpha$, where the errors in position reach their minimum, depends on the eccentricity.

A least square analytical approach to the optimal value of $\alpha$ can be written as

$$
\begin{align*}
\alpha= & 2.10826 e^{5}-4.77809 e^{4}+3.92513 e^{3} \\
& -1.71554 e^{2}+0.71606 e+0.72724 \tag{37}
\end{align*}
$$

Figure 3 shows the value of $\alpha$, where the error $|\Delta \vec{r}|$ reaches its minimum for each value of $e \in[0,0.95]$ and its analytical approximation.

Figures 4, 5, 6 and 7 show the local integration errors, in position and velocity, for a satellite with $a=118363.47 \mathrm{Km}$ and $e=0.7$ for the values of $\alpha=0,0.5,0.75,1.0$. These errors have been obtained solving for each value of $\Psi_{\alpha}$ the equation $\Psi_{\alpha}=\Psi_{\alpha}(E)$. From the value of $E$, we compute, using the two-body equations, the exact solution for the position and velocity vectors. These values are the initial conditions for the numerical integrator. The local truncation errors are exactly evaluated by comparing the values obtained through integration in the next step with the corresponding ones evaluated solving the equation $\Psi_{\alpha}+h=\Psi_{\alpha}(E)$.

The solution of the equation $\Psi_{\alpha}=\Psi_{\alpha}(E)$ is obtained solving, for each value of $\Psi_{\alpha}=i * h,(16)$ and (13).

To test the robustness of the method, we will study a perturbed case included in the planar restricted three-body
problem, this problem includes the Earth, the Moon, and an artificial satellite with the same semiaxe that Heos II and eccentricity 0.95 . The problem uses the following approach to the motion of the satellite perturbed by the Moon.
(1) The satellite is in the orbital plane of the Moon.
(2) The Moon motion is approached through a circular motion around the Earth with a sidereal period of 27 days 4 h 43 m 11.5 s , orbital radius of the Moon $R_{L}=$ 384.400 Km , and mass 0, 01231 in units of Earth mass.
(3) The couple Earth-Moon is not perturbed by the satellite motion.

Table 3 shows the number of steps necessary to provide an accuracy of $10^{-4} \mathrm{Km}$ in position in an integration over 100 days using as integrator a classic Runge-Kutta of fourth order and a Runge-Kutta of order eight [27]. To evaluate the global error, we have compared the position results using $N$ and 1.1 N steps. In this table, we can see that the integration can be improved by the use of an appropriate choice of $\alpha$ in the natural family of anomalies given by $\alpha(e)$ (37). It is important to emphasize that all the anomalies of this family improve the values obtained by the use of the mean anomaly.

## 6. Concluding Remarks

In this paper, a new one-parametric family of anomalies, called natural anomalies, has been defined.

This family of anomalies is adequate to be used in the construction of the analytical theories of planetary motion. In this sense, we have described a method to obtain the analytical development as Fourier series of the natural anomaly up an arbitrary order in eccentricity of the most common quantities (25), (26), and (29) of the two-body problem, and from these developments, we can obtain the development of the second member of the Lagrange planetary equation.

To test the efficiency of the use of the variables $\Psi_{\alpha}$ as independent variables in the analytical developments, we show the performance of the Kovalevsky algorithm to obtain the inverse of the distance for the couple Jupiter-Saturn, obtaining more compact developments for values of $\alpha$ near to values given by (37). To integrate the Lagrange planetary equations by analytical methods using a generalized anomaly as temporal variable, it is necessary to use the generalized Kepler equation (27).

The natural anomalies conform to a parametric family of anomalies that includes the true anomaly. For $\alpha=0$, we have the secondary true anomaly, its value being near to the mean anomaly for small values of the eccentricity and, in the case of an eccentricity that is not small, the spatial points distribution over the orbit is the most appropriate, if the mean anomaly is used.

In the symmetrical case, $\Psi_{1 / 2}$ defines a point distribution over the orbit with a major concentration in the perigee than if the eccentric anomaly $E$ is used.

The two-body problem has been used to test the performance of the numerical integrators. In this study, the local and global errors depend on the value of $\alpha$. The global

Table 2: Optimal $\alpha$ for $e|\Delta \vec{r}|$ in $10^{-5} \mathrm{Km}|\Delta \vec{v}|$ in $10^{-8} \mathrm{Km} / \mathrm{s}$.

| $e$ | $\alpha$ | $\|\Delta \vec{r}\|$ | $\|\Delta \vec{v}\|$ | $e$ | $\alpha$ | $\|\Delta \vec{r}\|$ | $\|\Delta \vec{v}\|$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.00 | 0.7284 | $3.730 \cdot 10^{-2}$ | $2.893 \cdot 10^{-4}$ | 0.50 | 0.9127 | $2.357 \cdot 10^{-2}$ | $8.684 \cdot 10^{-2}$ |
| 0.05 | 0.7581 | $3.555 \cdot 10^{-2}$ | $3.426 \cdot 10^{-3}$ | 0.55 | 0.9225 | $2.111 \cdot 10^{-2}$ | $9.858 \cdot 10^{-2}$ |
| 0.10 | 0.7839 | $3.451 \cdot 10^{-2}$ | $8.409 \cdot 10^{-3}$ | 0.60 | 0.9310 | $1.852 \cdot 10^{-2}$ | $1.091 \cdot 10^{-1}$ |
| 0.15 | 0.8066 | $3.374 \cdot 10^{-2}$ | $1.483 \cdot 10^{-2}$ | 0.65 | 0.9381 | $1.643 \cdot 10^{-2}$ | $1.170 \cdot 10^{-1}$ |
| 0.20 | 0.8268 | $3.303 \cdot 10^{-2}$ | $2.253 \cdot 10^{-2}$ | 0.70 | 0.9435 | $1.479 \cdot 10^{-2}$ | $1.199 \cdot 10^{-1}$ |
| 0.25 | 0.8450 | $3.229 \cdot 10^{-2}$ | $3.143 \cdot 10^{-2}$ | 0.75 | 0.9463 | $1.353 \cdot 10^{-2}$ | $1.094 \cdot 10^{-1}$ |
| 0.30 | 0.8613 | $3.116 \cdot 10^{-2}$ | $4.118 \cdot 10^{-2}$ | 0.80 | 0.9461 | $1.264 \cdot 10^{-2}$ | $6.894 \cdot 10^{-2}$ |
| 0.35 | 0.8761 | $2.980 \cdot 10^{-2}$ | $5.179 \cdot 10^{-2}$ | 0.85 | 0.9454 | $6.821 \cdot 10^{-3}$ | $4.313 \cdot 10^{-2}$ |
| 0.40 | 0.8896 | $2.810 \cdot 10^{-2}$ | $6.319 \cdot 10^{-2}$ | 0.90 | 0.9512 | $6.064 \cdot 10^{-2}$ | $3.600 \cdot 10^{-1}$ |
| 0.45 | 0.9017 | $2.599 \cdot 10^{-2}$ | $7.483 \cdot 10^{-2}$ | 0.95 | 0.9661 | $1.121 \cdot 10^{0}$ | $2.185 \cdot 10^{0}$ |

Table 3: Number of steps to get a precision of $|\Delta \vec{r}|<10^{-4} \mathrm{Km}$.

|  | $N$ | $\alpha=0$ | $\alpha=0.25$ | $\alpha=0.5$ | $\alpha=0.75$ | $\alpha=1.0$ | $\alpha(e)$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| RK4 | 7084079 | 3292627 | 401504 | 251229 | 190193 | 158367 | 146393 |
| RK8 | 305033 | 94362 | 14303 | 10833 | 9025 | 6840 | 6630 |

integration errors in a revolution depend on $\alpha$. This value increases with the eccentricity, and it is near 0.72 for lower eccentricities and 0.9661 for higher ones ( $e=0.95$ ).

To test the robustness of the method, a simplified problem, included in the restricted three-body problem, has been analyzed. In this problem, we study the number of steps that is necessary to get a precision of $1 \cdot 10^{-4} \mathrm{Km}$ in $|\Delta \vec{r}|$ using a classical method of Runge-Kutta of fourth order and a RungeKutta of eighth order. The number of steps is minimum when we take for each step the value of $\alpha$ that minimizes the error in the osculating two-body problem (37). The result is similar using a classic RK4 method and using a RK8 method, obviously with a minor number of steps in the last case.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Resolution of the Generalized Eigenvalue Problem in the Neutron Diffusion Equation Discretized by the Finite Volume Method 

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#### Abstract

Numerical methods are usually required to solve the neutron diffusion equation applied to nuclear reactors due to its heterogeneous nature. The most popular numerical techniques are the Finite Difference Method (FDM), the Coarse Mesh Finite Difference Method (CFMD), the Nodal Expansion Method (NEM), and the Nodal Collocation Method (NCM), used virtually in all neutronic diffusion codes, which give accurate results in structured meshes. However, the application of these methods in unstructured meshes to deal with complex geometries is not straightforward and it may cause problems of stability and convergence of the solution. By contrast, the Finite Element Method (FEM) and the Finite Volume Method (FVM) are easily applied to unstructured meshes. On the one hand, the FEM can be accurate for smoothly varying functions. On the other hand, the FVM is typically used in the transport equations due to the conservation of the transported quantity within the volume. In this paper, the FVM algorithm implemented in the ARB Partial Differential Equations solver has been used to discretize the neutron diffusion equation to obtain the matrices of the generalized eigenvalue problem, which has been solved by means of the SLEPc library.


## 1. Introduction

The neutron diffusion equation is used to calculate the neutron flux distribution, which is one of the most important variables in a Nuclear Power Reactor (NPR). This equation is a simplification of the neutron transport equation using Fick's Law, as discussed by Stacey [1]. Nevertheless, the use of the neutron diffusion equation is justified by the lower computational time and relatively low heterogeneity of commercial NPR.

In order to calculate the spatial distribution of the neutron flux, the steady state of the neutron diffusion equation is considered by transforming the neutron diffusion equation into a generalized eigenvalue problem, explained in Section 2.1.

The greatest eigenvalue is the most important one and it has a special interest for nuclear reactors safety. As a result, most methods used to calculate only this eigenvalue and utilizing iterative methods to avoid solving the generalized
eigenvalue problem. Nevertheless, the calculation of several eigenvalues and eigenvectors is important for different applications as the modal analysis of nuclear reactors and BWR instabilities analysis, as discussed elsewhere [2,3].

However, the resolution of this generalized eigenvalue problem could be a difficult task due to the large and sparse nature of the matrices. In this paper, the SLEPc library solves this problem. Actually, the emphasis of this library is on methods appropriate for problems in which the associated matrices are sparse, such as those arising after the discretization of partial differential equations, as discussed by Hernandez et al. [4].

On the other hand, numerical methods are usually required to solve the neutron diffusion equation applied to nuclear reactors due to its heterogeneous nature, discretizing the partial differential terms. The most popular numerical techniques are the Finite Difference Method (FDM), the Coarse Mesh Finite Difference Method (CFMD), the Nodal

TABLE 1: 2D homogeneous reactor cross sections.

| $D_{1}(\mathrm{~cm})$ | $D_{2}(\mathrm{~cm})$ | $\Sigma_{a 1}\left(\mathrm{~cm}^{-1}\right)$ | $\Sigma_{a 2}\left(\mathrm{~cm}^{-1}\right)$ | $v \Sigma_{f 1}\left(\mathrm{~cm}^{-1}\right)$ | $v \Sigma_{f 2}\left(\mathrm{~cm}^{-1}\right)$ | $\Sigma_{12}\left(\mathrm{~cm}^{-1}\right)$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 1.28205128205 | 0.666667 | 0.01 | 0.1 | 0.01 | 0.109017634020268 | 0.075 |

Table 2: 2D homogeneous reactor results.

|  | Structured mesh |  |  | Unstructured mesh |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Mesh length $(\mathrm{cm})$ | 10 | 5 | 2 | 10 | 5 | 2 |
| Number of cells | 72 | 288 | 1620 | 234 | 666 | 4054 |
| Computational time (s) | 0.38 | 0.85 | 4.11 | 0.74 | 1.52 | 10.41 |
| Eigenvalue-1 error (pcm) | 139.903827 | 35.083823 | 5.733821 | 29.653822 | 12.923822 | 2.053821 |
| Eigenvalue-2 error (pcm) | 309.789649 | 77.398527 | 14.010938 | 117.773860 | 38.609075 | 6.124990 |
| Eigenvalue-3 error (pcm) | 1207.774879 | 246.016129 | 47.219525 | 437.256280 | 179.905306 | 18.743235 |
| Eigenvalue-4 error $(\mathrm{pcm})$ | 1587.045813 | 462.009487 | 74.560259 | 221.291968 | 128.800577 | 20.948371 |
| Eigenvalue-5 error $(\mathrm{pcm})$ | 1948.227429 | 492.735800 | 80.710670 | 436.132079 | 185.870476 | 29.279108 |



Figure 1: Virtual fluxes example.

Expansion Method (NEM), and the Nodal Collocation Method (NCM), used virtually in all the neutronic diffusion codes, which give accurate results in structured meshes. However, the application of these methods in unstructured meshes to deal with complex geometries is not straightforward and it may cause problems of stability and convergence of the solution, as discussed by Hoffmann and Chiang [5]. In fact, the use of unstructured meshes is justified by the thermal hydraulic-neutronic coupled calculation, which sometimes uses the unstructured meshes. By contrast, the Finite Element Method (FEM) and the Finite Volume Method (FVM) are easily applied to unstructured meshes. On the one hand, the FEM can be accurate for smoothly varying functions. On the other hand, the FVM is typically used in the transport equations due to the conservation of the transported quantity within the volume.

In this paper, the FVM algorithm implemented in the ARB Partial Differential Equations solver has been used to discretize the neutron diffusion equation to obtain the matrices of the generalized eigenvalue problem. The strength of ARB is that a fully implicit numerical formulation is generated and formulated easily, for an arbitrary set of equations, which are input by the user using pseudomathematical expressions, as discussed by Harvie [6].


Figure 2: Adjacent cells.


Figure 3: Partial and global gradients.

TABLE 3: Numbering of the 2D homogeneous reactor nodes.

| 7 | 8 | 9 |
| :--- | :--- | :--- |
| 4 | 5 | 6 |
| 1 | 2 | 3 |

The outline of the paper is as follows. Section 2 presents the discretization of the equations and the methodology used. Section 3 describes the reactors used to validate the method and their results. Section 4 contains few comments and conclusions about the results.

## 2. Materials and Methods

2.1. Discretization of the Diffusion Equation Using the Finite Volume Method. Although several approaches in terms of energy could be applied to the neutron diffusion equation,

Table 4: 2D homogeneous reactor power errors (\%) corresponding to the first eigenvalue.

| Node | Structured mesh |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm | 10 cm | Unstructured mesh |  |
| 1 | 0.009206 | 0.000081 | 0.000069 | 0.094629 | 0.131237 | 0.018789 |
| 2 | 0.005681 | 0.001217 | 0.000559 | 0.005927 | 0.002009 | 0.005058 |
| 3 | 0.002318 | 0.002675 | 0.001150 | 0.098431 | 0.136212 | 0.015750 |
| 4 | 0.010802 | 0.001601 | 0.000126 | 0.091223 | 0.125323 | 0.007585 |
| 5 | 0.014169 | 0.003062 | 0.000684 | 0.000774 | 0.005144 | 0.001788 |
| 6 | 0.017709 | 0.004348 | 0.001148 | 0.073893 | 0.111893 | 0.015444 |
| 7 | 0.009206 | 0.000081 | 0.000058 | 0.092636 | 0.096630 | 0.020416 |
| 8 | 0.005672 | 0.001208 | 0.000577 | 0.006823 | 0.000913 | 0.001457 |
| 9 | 0.002318 | 0.002686 | 0.001182 | 0.082769 | 0.141282 | 0.007689 |

Table 5: 2D homogeneous reactor power errors (\%) corresponding to the second eigenvalue.

| Node | Structured mesh |  |  | Unstructured mesh |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm | 10 cm | 5 cm | 2 cm |
| 1 | 0.010699 | 0.002026 | 0.000177 | 0.064440 | 0.131450 | 0.023185 |
| 2 | - | - | - | - | - | - |
| 3 | 0.011767 | 0.000900 | 0.000735 | 0.218325 | 0.056848 | 0.013567 |
| 4 | 0.013305 | 0.002670 | 0.000291 | 0.370735 | 0.142994 | 0.009263 |
| 5 | - | - | - | - | - | - |
| 6 | 0.012219 | 0.003781 | 0.001195 | 0.270132 | 0.000779 | 0.021504 |
| 7 | 0.010706 | 0.002026 | 0.000135 | 0.116602 | 0.091269 | 0.018453 |
| 8 | - | - | - | - | - | - |
| 9 | 0.011767 | 0.000900 | 0.000693 | 0.228939 | 0.057844 | 0.029325 |



Figure 4: 2D structured mesh.


Figure 5: 2D unstructured mesh.
the most widely used for Light Water Reactors (LWR) is the 2energy group neutron diffusion approximation, as discussed by Stacey [1]:

$$
\begin{aligned}
& \frac{1}{v_{1}} \frac{d \phi_{1}(\vec{r}, t)}{d t} \\
&=-\vec{\nabla}\left(-D_{1} \nabla \phi_{1}(\vec{r}, t)\right)-\Sigma_{a, 1}(\vec{r}) \phi_{1}(\vec{r}, t) \\
& \quad-\Sigma_{s, 1 \rightarrow 2}(\vec{r}) \phi_{1}(\vec{r}, t) \\
& \quad+\left(v \Sigma_{f, 1}(\vec{r}) \phi_{1}(\vec{r}, t)+v \Sigma_{f, 2}(\vec{r}) \phi_{2}(\vec{r}, t)\right) \\
& \frac{1}{v_{2}} \frac{d \phi_{2}(\vec{r}, t)}{d t}=-\vec{\nabla}\left(-D_{2} \nabla \phi_{2}(\vec{r}, t)\right)-\Sigma_{a, 2}(\vec{r}) \phi_{2}(\vec{r}, t) \\
&+\Sigma_{s, 1 \rightarrow 2}(\vec{r}) \phi_{1}(\vec{r}, t)
\end{aligned}
$$

In order to calculate the spatial distribution of the neutron flux, the steady state of the previous differential equations system is considered by putting the temporal derivation of (1) equal to 0 . However, if the geometry of the problem is fixed, the steady state will be accomplished only for certain set of diffusion coefficients and cross sections, and vice versa. These nuclear parameters depend on the materials and are the following coefficients of (1): $D_{1}, D_{2}, \Sigma_{a, 1}, \Sigma_{a, 2}, v \Sigma_{f, 1}, v \Sigma_{f, 2}$, and $\Sigma_{s, 1 \rightarrow 2}$.

Therefore, the problem is transformed into an eigenvalue problem to assure the steady state accomplishment, where the eigenvectors are the spatial distribution of the neutron flux and the inverse of the first eigenvalue $\left(\mathbf{k}_{\mathbf{1}}\right)$ represents


Figure 6: 2D homogeneous reactor power corresponding to the first eigenvalue and the 2 cm structured mesh.


Figure 7: 2D homogeneous reactor power corresponding to the first eigenvalue and the 2 cm unstructured mesh.
a measure of the steady state condition, numbering $\mathbf{k}$ in descending order:

$$
\begin{align*}
& \vec{\nabla}\left(-D_{1} \nabla \phi_{1}(\vec{r}, t)\right)+\Sigma_{a, 1}(\vec{r}) \phi_{1}(\vec{r}, t)+\Sigma_{s, 1 \rightarrow 2}(\vec{r}) \phi_{1}(\vec{r}, t) \\
& \quad-\frac{1}{\mathbf{k}}\left(v \Sigma_{f, 1}(\vec{r}) \phi_{1}(\vec{r}, t)+v \Sigma_{f, 2}(\vec{r}) \phi_{2}(\vec{r}, t)\right)=0, \\
& \vec{\nabla}\left(-D_{2} \nabla \phi_{2}(\vec{r}, t)\right)+\Sigma_{a, 2}(\vec{r}) \phi_{2}(\vec{r}, t) \\
& \quad-\Sigma_{s, 1 \rightarrow 2}(\vec{r}) \phi_{1}(\vec{r}, t)=0 . \tag{2}
\end{align*}
$$

Moreover, the geometry should be discretized in elements where the neutron diffusion equation will be applied, due to the reactor heterogeneity and the cross-section spatial dependence. Consequently, a set of equations will be obtained for each element. Then, these equations are integrated in each element volume $\left(V_{i}\right)$ and the Divergence Theorem is applied, so that the divergence term could be avoided:

$$
\begin{aligned}
& \int\left(-D_{1} \nabla \phi_{1}(\vec{r}, t)\right) d S_{V_{i}}+\int \Sigma_{a, 1}(\vec{r}) \phi_{1}(\vec{r}, t) d V_{i} \\
&+\int \Sigma_{s, 1 \rightarrow 2}(\vec{r}) \phi_{1}(\vec{r}, t) d V_{i} \\
&= \frac{1}{\mathbf{k}} \int\left(v \Sigma_{f, 1}(\vec{r}) \phi_{1}(\vec{r}, t)+v \Sigma_{f, 2}(\vec{r}) \phi_{2}(\vec{r}, t)\right) d V_{i}
\end{aligned}
$$



Figure 8: 2D homogeneous reactor power corresponding to the second eigenvalue and the 2 cm structured mesh.

$$
\begin{gather*}
\int\left(-D_{2} \nabla \phi_{2}(\vec{r}, t)\right) d S_{V_{i}}+\int \Sigma_{a, 2}(\vec{r}) \phi_{2}(\vec{r}, t) d V_{i} \\
-\int \Sigma_{s, 1 \rightarrow 2}(\vec{r}) \phi_{1}(\vec{r}, t) d V_{i}=0 \tag{3}
\end{gather*}
$$

Finally, considering surface and volume averaged values and dividing by the volume of the element $\left(V_{i}\right)$, the following equations will be obtained:

$$
\begin{align*}
& \sum_{j}\left(\frac{S_{j}}{V_{i}} u_{j}\left(-D_{1}^{V_{i}} \overline{\nabla \phi_{1 j}}\right)\right)+\Sigma_{a, 1}^{V_{i}} \phi_{1, i}+\Sigma_{s, 1 \rightarrow 2}^{V_{i}} \phi_{1, i} \\
& =\frac{1}{\mathbf{k}}\left(v \Sigma_{f, 1}^{V_{i}} \phi_{1, i}+v \Sigma_{f, 2}^{V_{i}} \phi_{2, i}\right),  \tag{4}\\
& \sum_{j}\left(\frac{S_{j}}{V_{i}} u_{j}\left(-D_{2}^{V_{i}} \overline{\nabla \phi_{2 j}}\right)\right)+\Sigma_{a, 2}^{V_{i}} \phi_{2, i}-\Sigma_{s, 1 \rightarrow 2}^{V_{i}} \phi_{1, i}=0
\end{align*}
$$

where $j$ represents each face surrounding the element $i, V_{i}$ is the volume of the element $i, S_{j}$ is the area of the face $j$, $u_{j}$ could be -1 or 1 depending on the direction of the face $j$ with respect to the direction of the neutron flux gradient at this face, $D_{1}^{V_{i}}$ and $D_{2}^{V_{i}}$ are the first and second energy group diffusion coefficients volume averaged values for the element $i, \Sigma_{a, 1}^{V_{i}}$ and $\Sigma_{a, 2}^{V_{i}}$ are the first and second energy group absorption macroscopic cross-section volume averaged values for the element $i, v \Sigma_{f, 1}^{V_{i}}$ and $v \Sigma_{f, 2}^{V_{i}}$ are the first and second energy group nufission macroscopic cross-section volume averaged values for the element $i, \Sigma_{s, 1 \rightarrow 2}^{V_{i}}$ is the scattering macroscopic cross-section volume averaged values for the element $i, \phi_{1, i}$ and $\phi_{2, i}$ are the first and second energy group volume averaged values of the neutron flux for the element $i$, and $\overline{\nabla \phi_{1 j}}$ and $\overline{\nabla \phi_{2}}$ are the first and second energy group surface averaged values of the neutron flux gradient for the face $j$.

With regard to $\overline{\nabla \phi_{1}}$ and $\overline{\nabla \phi_{2}}$, although they are not known, moving Least-Squares reproducing kernel methods can express them as a weighted sum of $\phi_{1, n}$ and $\phi_{2, n}$, respectively, by considering the elements $n$ surrounding face $j$, as discussed elsewhere [6, 7]:

$$
\begin{equation*}
\overline{\nabla \phi_{g_{j}}}=\sum_{n} k_{n, j}^{\mathrm{grad}} \phi_{g, n}, \quad g=1,2 \tag{5}
\end{equation*}
$$

Table 6: Biblis reactor cross sections.

| Material | $D_{1}(\mathrm{~cm})$ | $D_{2}(\mathrm{~cm})$ | $\Sigma_{a 1}\left(\mathrm{~cm}^{-1}\right)$ | $\Sigma_{a 2}\left(\mathrm{~cm}^{-1}\right)$ | $\Sigma_{12}\left(\mathrm{~cm}^{-1}\right)$ | $v \Sigma_{f 1}\left(\mathrm{~cm}^{-1}\right)$ | $v \Sigma_{f 2}\left(\mathrm{~cm}^{-1}\right)$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1.4360 | 0.3635 | 0.0095042 | 0.0750580 | 0.017754 | 0.0058708 | 0.0960670 |
| 2 | 1.4366 | 0.3636 | 0.0096785 | 0.0784360 | 0.017621 | 0.0061908 | 0.1035800 |
| 3 | 1.3200 | 0.2772 | 0.0026562 | 0.0715960 | 0.023106 | 0.0 | 0.0 |
| 4 | 1.4389 | 0.3638 | 0.0103630 | 0.0914080 | 0.017101 | 0.0074527 | 0.1323600 |
| 5 | 1.4381 | 0.3665 | 0.0100030 | 0.0848280 | 0.017290 | 0.0061908 | 0.1035800 |
| 6 | 1.4385 | 0.3665 | 0.0101320 | 0.0873140 | 0.017192 | 0.0064285 | 0.1091100 |
| 7 | 1.4389 | 0.3679 | 0.0101650 | 0.0880240 | 0.017125 | 0.0061908 | 0.1035800 |
| 8 | 1.4393 | 0.3680 | 0.0102940 | 0.0905100 | 0.017027 | 0.0064285 | 0.1091100 |

Table 7: Biblis reactor meshes.

|  | Structured |  |  | mesh | Unstructured mesh |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Mesh length (cm) | 10 | 5 | 2 | 10 | 5 | 2 |  |
| Number of cells | 1028 | 2416 | 9252 | 1808 | 4542 | 23972 |  |



Figure 9: 2D homogeneous reactor power corresponding to the second eigenvalue and the 2 cm unstructured mesh.
where $n$ is each element surrounding the face $j$ and $k_{n, j}^{\mathrm{grad}}$ is the weighting factor of the element $n$ with respect to the face $j$, which is called kernel by Harvie [6].

Therefore, the final equations for each element will be

$$
\begin{align*}
& \sum_{j}\left(\frac{S_{j}}{V_{i}} u_{j}\left(-D_{1}^{V_{i}} \sum_{n} k_{n, j}^{\mathrm{grad}} \phi_{1, n}\right)\right)+\Sigma_{a, 1}^{V_{i}} \phi_{1, i}+\Sigma_{s, 1 \rightarrow 2}^{V_{i}} \phi_{1, i} \\
& \quad=\frac{1}{\mathbf{k}}\left(v \Sigma_{f, 1}^{V_{i}} \phi_{1, i}+v \Sigma_{f, 2}^{V_{i}} \phi_{2, i}\right)  \tag{6}\\
& \sum_{j}\left(\frac{S_{j}}{V_{i}} u_{j}\left(-D_{2}^{V_{i}} \sum_{n} k_{n, j}^{\mathrm{grad}} \phi_{2, n}\right)\right)+\Sigma_{a, 2}^{V_{i}} \phi_{2, i} \\
& \quad-\Sigma_{s, 1 \rightarrow 2}^{V_{i}} \phi_{1, i}=0
\end{align*}
$$

The boundary conditions typically used in nuclear reactors are face boundary conditions, and consequently these boundary conditions should be expressed in terms of the volume averaged values of the neutron flux, which are the unknown values. The boundary conditions most widely used


Figure 10: Biblis reactor: geometry and material composition.
are zero flux or reflective flux expressed mathematically by (7) and (8), respectively,

$$
\begin{gather*}
\phi_{g}^{\mathrm{BC}}=0, \quad g=1,2,  \tag{7}\\
\nabla \phi_{g}^{\mathrm{BC}}=0, \quad g=1,2 . \tag{8}
\end{gather*}
$$

Therefore, the same procedure as the neutron flux gradient could be used to obtain these face averaged values in terms of the volume averaged values:

$$
\begin{gather*}
\phi_{g}^{\mathrm{BC}}=\sum_{n} k_{n, j} \phi_{g, n}=0, \quad g=1,2,  \tag{9}\\
\nabla \phi_{g}^{\mathrm{BC}}=\sum_{n} k_{n, j}^{\mathrm{grad}} \phi_{g, n}=0, \quad g=1,2, \tag{10}
\end{gather*}
$$

where $n$ represents each element surrounding the face $j$ that is part of the boundary condition BC.

As a result, the previous equations will compose a generalized eigenvalue problem considering (6) for each element and (9) and/or (10) for each boundary:

$$
\left(\begin{array}{cc}
L_{11} & 0  \tag{11}\\
L_{21} & L_{22} \\
L_{31} & L_{32}
\end{array}\right)\binom{\phi_{1}}{\phi_{2}}=\frac{1}{\mathbf{k}}\left(\begin{array}{cc}
M_{11} & M_{12} \\
0 & 0 \\
0 & 0
\end{array}\right)\binom{\phi_{1}}{\phi_{2}} .
$$

Table 8: Biblis reactor computational time (minutes:seconds).

|  | Structured mesh |  |  | Unstructured mesh |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm | 10 cm | 5 cm | 2 cm |
| CELL+ | $0: 3.17$ | $0: 6.78$ | $0: 40.31$ | $0: 4.02$ | $0: 12.79$ | $3: 33.1$ |
| CELL- | $0: 2.73$ | $0: 6.75$ | $0: 40.48$ | $0: 4.03$ | $0: 12.6$ | $3: 42.44$ |
| HOM | $0: 2.71$ | $0: 6.79$ | $0: 40.74$ | $0: 4.43$ | $0: 12.64$ | $3: 34.98$ |
| LIN | $0: 2.73$ | $0: 6.94$ | $0: 40.15$ | $0: 3.99$ | $0: 12.9$ | $3: 44.03$ |

Table 9: Biblis reactor eigenvalue-1 error (pcm).

|  | Structured mesh |  |  | Unstructured mesh |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm | 10 cm | 5 cm | 2 cm |
| CELL+ | 97.882178 | 42.210104 | 11.208553 | 49.516637 | 19.919813 | 4.536099 |
| CELL- | 40.941948 | 7.784530 | 3.521573 | 15.305675 | 0.097551 | 1.951010 |
| HOM | 67.641521 | 23.968160 | 3.463043 | 31.411263 | 9.452644 | 1.160851 |
| LIN | 69.182819 | 24.797339 | 3.726429 | 32.250198 | 9.862356 | 1.258402 |



Figure 11: Biblis reactor power corresponding to the first eigenvalue and the 2 cm structured mesh and HOM.


FIGURE 12: Biblis reactor power corresponding to the first eigenvalue and the 2 cm unstructured mesh and HOM.

Table 10: Numbering of the Biblis reactor nodes.

| 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | - |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | - |
| 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | - |
| 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | - |
| 18 | 19 | 20 | 21 | 22 | 23 | 24 | - | - |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | - |  |
| 5 | 6 | 7 | 8 | 9 | 10 | - | - |  |
| 1 | 2 | 3 | 4 | - | - | - |  |  |
| - | - | - | - | - |  |  |  |  |



Figure 13: 3D homogeneous reactor power corresponding to the first eigenvalue and the 2 cm structured mesh.
$L_{11}$ entries correspond to the first equation left side of (6), $M_{11}$ and $M_{12}$ entries correspond to the first equation right side of (6), while $L_{21}$ and $L_{22}$ entries correspond to the second equation of (6), and $L_{31}$ and $L_{32}$ entries correspond to (9) and/or (10).

On the other hand, $\phi_{1}$ and $\phi_{2}$ are vectors containing the flux volume averaged values in each element for the first and second energy group, respectively. Nonetheless, the addition of boundary conditions equations implies an equations excess, and therefore more unknown values have to be taken into account. In order to solve this problem, some virtual fluxes are considered, in particular, the same number as the boundary conditions equations. These virtual fluxes have no physical meaning and (6) is not applied to them, but they are needed to evaluate the boundary conditions equations and obtain the same number of equations and unknowns. In addition, these virtual fluxes have the same centroid as the faces where they are defined, and their volume values are the area values of these faces.

In Figure 1, an example of the virtual fluxes is shown. In this example the eigenvectors are

$$
\begin{gathered}
\phi_{g}=\left(\phi_{g}^{\text {elements }}, \phi_{g}^{\text {virtual }}\right)^{T}, \quad g=1,2 \\
\phi_{g}^{\text {elements }}=\left(\phi_{g}^{1}, \phi_{g}^{2}, \phi_{g}^{3}, \phi_{g}^{4}, \phi_{g}^{5}, \phi_{g}^{6}, \phi_{g}^{7}, \phi_{g}^{8}, \phi_{g}^{9}\right)^{T}, \quad g=1,2
\end{gathered}
$$



Figure 14: 3D homogeneous reactor power corresponding to the second eigenvalue and the 2 cm structured mesh.


Absorbent
Figure 15: Axial plane of Langenbuch reactor.

$$
\begin{align*}
& \phi_{g}^{\text {virtual }} \\
& \qquad \begin{array}{r}
=\left(\phi_{g}^{10}, \phi_{g}^{11}, \phi_{g}^{12}, \phi_{g}^{13}, \phi_{g}^{14}, \phi_{g}^{15}, \phi_{g}^{16}, \phi_{g}^{17}, \phi_{g}^{18}, \phi_{g}^{19}, \phi_{g}^{20}, \phi_{g}^{21}\right)^{T} \\
g=1,2
\end{array}
\end{align*}
$$

In conclusion, the following formula represents the generalized eigenvalue problem:

$$
\left.\left.\begin{array}{l}
\left(\begin{array}{ll}
L_{11} & 0 \\
L_{21} & L_{22} \\
L_{31} & L_{32}
\end{array}\right)\binom{\left[\begin{array}{c}
\phi_{1}^{\text {elements }} \\
\phi_{1}^{\text {virtual }}
\end{array}\right]}{\left[\begin{array}{c}
\text { elements } \\
\phi_{2}^{\text {virtual }}
\end{array}\right]}  \tag{13}\\
=\frac{1}{\mathbf{k}}\left(\begin{array}{cc}
M_{11} & M_{12} \\
0 & 0 \\
0 & 0
\end{array}\right)\left(\begin{array}{c}
\phi_{1}^{\text {elements }} \\
\phi_{1}^{\text {virtual }}
\end{array}\right] \\
\phi_{2}^{\text {elements }} \\
\phi_{2}^{\text {virtual }}
\end{array}\right]\right) . . .
$$

2.2. Neutron Current Condition. In the neutron diffusion theory, the partial neutron current calculated at the face

Table 11: Biblis reactor power errors (\%) corresponding to HOM.

| Node | Structured mesh |  |  | Unstructured mesh |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm | 10 cm | 5 cm | 2 cm |
| 1 | 2.284672 | 0.001124 | 0.367491 | 0.879946 | 0.030794 | 0.180845 |
| 2 | 2.304411 | 0.135778 | 0.367554 | 0.824867 | 0.142893 | 0.191021 |
| 3 | 2.376446 | 0.134061 | 0.376089 | 0.864341 | 0.162517 | 0.180412 |
| 4 | 2.376446 | 0.134061 | 0.376089 | 0.864341 | 0.162517 | 0.180412 |
| 5 | 2.488581 | 0.003299 | 0.489467 | 0.429311 | 0.390778 | 0.228766 |
| 6 | 1.074203 | 0.220179 | 0.038275 | 0.588963 | 0.054508 | 0.060416 |
| 7 | 0.934102 | 0.176358 | 0.083588 | 0.479943 | 0.043084 | 0.086734 |
| 8 | 1.469022 | 0.483910 | 0.021256 | 0.658828 | 0.161010 | 0.032085 |
| 9 | 0.027694 | 0.558208 | 0.369578 | 0.276270 | 0.457187 | 0.183278 |
| 10 | 1.969739 | 0.070576 | 0.335412 | 0.446243 | 0.289564 | 0.156511 |
| 11 | 2.755899 | 0.126755 | 0.433999 | 0.793181 | 0.334266 | 0.202575 |
| 12 | 1.849153 | 0.858648 | 0.268538 | 1.049713 | 0.424649 | 0.110011 |
| 13 | 0.442460 | 0.372698 | 0.185346 | 0.282549 | 0.266245 | 0.089598 |
| 14 | 1.537896 | 0.814179 | 0.251776 | 0.831365 | 0.408647 | 0.095628 |
| 15 | 1.077245 | 0.592678 | 0.167885 | 0.500974 | 0.281762 | 0.078132 |
| 16 | 0.221870 | 0.218571 | 0.199870 | 0.032188 | 0.187027 | 0.063343 |
| 17 | 2.075160 | 0.300133 | 0.189122 | 0.755988 | 0.026896 | 0.045875 |
| 18 | 2.755899 | 0.126755 | 0.433972 | 0.840364 | 0.366588 | 0.198116 |
| 19 | 0.014891 | 0.540713 | 0.404170 | 0.298276 | 0.443251 | 0.188347 |
| 20 | 2.219547 | 0.768456 | 0.099073 | 1.019476 | 0.302488 | 0.036096 |
| 21 | 0.295151 | 0.629288 | 0.353496 | 0.274481 | 0.436990 | 0.164800 |
| 22 | 1.672244 | 0.729190 | 0.169991 | 0.865452 | 0.342074 | 0.059898 |
| 23 | 0.882870 | 0.491431 | 0.140994 | 0.374551 | 0.237709 | 0.074700 |
| 24 | 0.221876 | 0.218560 | 0.199864 | 0.043428 | 0.188407 | 0.065455 |
| 25 | 1.969739 | 0.070576 | 0.335412 | 0.473057 | 0.317933 | 0.151979 |
| 26 | 2.189996 | 0.490979 | 0.095543 | 0.948384 | 0.057490 | 0.051558 |
| 27 | 0.351107 | 0.582043 | 0.464038 | 0.056930 | 0.475770 | 0.206069 |
| 28 | 2.368009 | 0.709176 | 0.016244 | 1.057085 | 0.220311 | 0.005560 |
| 29 | 0.223202 | 0.671285 | 0.391755 | 0.271355 | 0.448080 | 0.172726 |
| 30 | 1.672244 | 0.729190 | 0.169991 | 0.855210 | 0.346483 | 0.059600 |
| 31 | 1.077245 | 0.592678 | 0.167868 | 0.535406 | 0.269561 | 0.079436 |
| 32 | 0.027702 | 0.558224 | 0.369601 | 0.231229 | 0.486828 | 0.179438 |
| 33 | 2.488581 | 0.003299 | 0.489478 | 0.525898 | 0.439935 | 0.231726 |
| 34 | 0.717648 | 0.496628 | 0.533502 | 0.053316 | 0.522542 | 0.251418 |
| 35 | 2.951741 | 0.772375 | 0.049890 | 1.260491 | 0.198974 | 0.034441 |
| 36 | 0.382809 | 0.717441 | 0.545718 | 0.150071 | 0.595033 | 0.251171 |
| 37 | 2.368009 | 0.709176 | 0.016217 | 1.041303 | 0.221315 | 0.004546 |
| 38 | 0.295151 | 0.629279 | 0.353451 | 0.301348 | 0.431097 | 0.163978 |
| 39 | 1.537890 | 0.814179 | 0.251782 | 0.790510 | 0.415010 | 0.094983 |
| 40 | 1.469022 | 0.483910 | 0.021268 | 0.718255 | 0.135214 | 0.031668 |
| 41 | 2.376439 | 0.134061 | 0.376089 | 1.033266 | 0.223136 | 0.182892 |
| 42 | 3.220990 | 0.751593 | 0.089823 | 1.413971 | 0.171813 | 0.054183 |
| 43 | 0.764614 | 0.595606 | 0.606721 | 0.003153 | 0.592573 | 0.276228 |
| 44 | 2.951741 | 0.772375 | 0.049916 | 1.244259 | 0.210189 | 0.034626 |
| 45 | 0.351107 | 0.582043 | 0.464013 | 0.098390 | 0.470016 | 0.204001 |
| 46 | 2.219547 | 0.768446 | 0.099092 | 0.984197 | 0.309352 | 0.036133 |
| 47 | 0.442460 | 0.372698 | 0.185365 | 0.357257 | 0.261884 | 0.086875 |
| 48 | 0.943520 | 0.185705 | 0.074237 | 0.575157 | 0.042333 | 0.074883 |

Table 11: Continued.

| Node | Structured mesh |  |  |  | Unstructured mesh |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm | 10 cm | 5 cm | 2 cm |
| 49 | 2.304417 | 0.135772 | 0.367517 | 1.003710 | 0.178850 | 0.186330 |
| 50 | 0.861206 | 0.553378 | 0.631204 | 0.085637 | 0.536703 | 0.290387 |
| 51 | 3.220990 | 0.751593 | 0.089823 | 1.405356 | 0.184740 | 0.055653 |
| 52 | 0.717648 | 0.496628 | 0.533502 | 0.092312 | 0.518655 | 0.248030 |
| 53 | 2.189996 | 0.490979 | 0.095543 | 0.910271 | 0.061734 | 0.046650 |
| 54 | 0.014881 | 0.540713 | 0.404161 | 0.357925 | 0.437896 | 0.182018 |
| 55 | 1.849153 | 0.858642 | 0.268519 | 0.983297 | 0.433686 | 0.112597 |
| 56 | 2.284672 | 0.001134 | 0.367482 | 1.015056 | 0.047411 | 0.185290 |

Table 12: 3D homogeneous reactor results.

|  | Structured mesh |  |  |
| :--- | :---: | :---: | :---: |
| Mesh length (cm) | 10 | 5 | 2 |
| Number of cells | 1080 | 8640 | 135000 |
| Computational time (h:min:s) | $0: 0: 8.56$ | $0: 2: 51.75$ | $12: 41: 9$ |
| Eigenvalue-1 error (pcm) | 148.492003 | 37.141902 | 5.927090 |
| Eigenvalue-2 error (pcm) | 174.642897 | 43.473709 | 6.912315 |
| Eigenvalue-3 error (pcm) | 276.887015 | 68.855621 | 10.939931 |
| Eigenvalue-4 error (pcm) | 390.546564 | 97.622804 | 15.555099 |
| Eigenvalue-5 error (pcm) | 416.479277 | 103.762637 | 16.498151 |



Figure 16: Frontal plane of Langenbuch reactor.
separating 2 cells must be the same in both cells. Therefore, if Figure 2 is considered, the neutron current condition at the face separating these cells will be

$$
\begin{equation*}
J_{g}^{+}=J_{g}^{-}, \quad g=1,2 \tag{14}
\end{equation*}
$$

According to Fick's Law,

$$
\begin{equation*}
D_{g}^{+} \nabla \phi_{g}^{+}=D_{g}^{-} \nabla \phi_{g}^{-}, \quad g=1,2, \tag{15}
\end{equation*}
$$

where $D_{g}^{+}$is the $g$ energy group diffusion coefficient for the left cell, $D_{g}^{-}$is the $g$ energy group diffusion coefficient for the right cell, $\nabla \phi_{g}^{+}$is the $g$ energy group partial neutron flux gradient for the left cell, and $\nabla \phi_{g}^{-}$is the $g$ energy group partial neutron flux gradient for the right cell.


Figure 17: Langenbuch reactor power corresponding to the first eigenvalue and the 5 cm structured mesh and HOM.

If both cells are composed of same material, $D_{g}^{+}$and $D_{g}^{-}$ will be the same and the partial neutron flux gradient will be the same; therefore, the neutron current condition will be accomplished. Nevertheless, if the materials are different, the accomplishment of this condition is not assured.

On the other hand, the method proposed in this paper does not include the calculation of the partial gradients, only the global gradient calculation, and it depends on the face but not on the cell. These gradients are exposed in Figure 3. As a result, a global diffusion coefficient $\left(D_{g}^{j}\right)$ is needed in order to accomplish the neutron current condition by using the global gradient:

$$
\begin{equation*}
D_{g}^{j} \nabla \phi_{g}=D_{g}^{+} \nabla \phi_{g}^{+}=D_{g}^{-} \nabla \phi_{g}^{-} . \tag{16}
\end{equation*}
$$

In this paper, 4 approaches of this global diffusion coefficient have been considered.
(i) Case 1 (CELL+): take the diffusion coefficient of the left cell as the global diffusion coefficient:

$$
\begin{equation*}
D_{g}^{j}=D_{g}^{+} . \tag{17}
\end{equation*}
$$

TABLE 13: Numbering of 3D homogeneous reactor nodes.

| 1st axial plane |  |  |  |  |  |  |  |  |  | 2nd axial plane |  |  |  | 3rd axial plane |  |  | 4th axial plane |  | 5th axial plane |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 7 | 8 | 9 | 16 | 17 | 18 | 25 | 26 | 27 | 34 | 35 | 36 | 43 | 44 | 45 |  |  |  |  |  |  |
| 4 | 5 | 6 | 13 | 14 | 15 | 22 | 23 | 24 | 31 | 32 | 33 | 40 | 41 | 42 |  |  |  |  |  |  |
| 1 | 2 | 3 | 10 | 11 | 12 | 19 | 20 | 21 | 28 | 29 | 30 | 37 | 38 | 39 |  |  |  |  |  |  |



Figure 18: Langenbuch reactor power corresponding to the second eigenvalue and the 5 cm structured mesh and HOM.
(ii) Case 2 (CELL-): take the diffusion coefficient of the right cell as the global diffusion coefficient:

$$
\begin{equation*}
D_{g}^{j}=D_{g}^{-} . \tag{18}
\end{equation*}
$$

(iii) Case 3 (HOM): take a weighted sum of the diffusion coefficients of both cells as the global diffusion coefficient. In this case, the weighting factors are based on the kernels mentioned in Section 2.1:
$D_{i}^{j}=D_{i}^{+} \cdot \frac{\left|k_{+}^{\text {grad }}\right|}{\left|k_{+}^{\text {grad }}\right|+\left|k_{-}^{\text {grad }}\right|}+D_{i}^{-} \cdot \frac{\left|k_{-}^{\text {grad }}\right|}{\left|k_{+}^{\text {grad }}\right|+\left|k_{-}^{\text {grad }}\right|}$.
(iv) Case 4 (LIN): by considering the neutron flux is a line function of the distance between the centroids of the cell and face. In addition, the distances have been replaced by the kernels mentioned in Section 2.1 due to the fact that kernels depend on distances:

$$
\begin{align*}
D_{i}^{j}= & \left(D_{i}^{+} \cdot D_{i}^{-}\right) \\
& \times\left(D_{i}^{-} \cdot \frac{\left|k_{+}^{\text {grad }}\right|}{\left|k_{+}^{\text {grad }}\right|+\left|k_{-}^{\text {grad }}\right|}\right.  \tag{20}\\
& \left.+D_{i}^{+} \cdot \frac{\left|k_{-}^{\text {grad }}\right|}{\left|k_{+}^{\text {grad }}\right|+\left|k_{-}^{\text {grad }}\right|}\right)^{-1}
\end{align*}
$$

$k_{+}^{\text {grad }}$ is the kernel of the left cell to calculate the contribution to the neutron flux gradient at the interface. It may be negative, so the absolute value is considered to avoid a null dividing.
$k_{-}^{\text {grad }}$ is the kernel of the right cell to calculate the contribution to the neutron flux gradient at the interface. It may be negative, so the absolute value is considered to avoid a null dividing.

Therefore, (6) is transformed into (21), and this last equation should be used instead of (6) to obtain the matrices entries of (13):

$$
\begin{align*}
& \sum_{j}\left(\frac{S_{j}}{V_{i}} u_{j}\left(-D_{1}^{j} \sum_{n} k_{n, j}^{\mathrm{grad}} \phi_{1, n}\right)\right)+\Sigma_{a, 1}^{V_{i}} \phi_{1, i}+\Sigma_{s, 1 \rightarrow 2}^{V_{i}} \phi_{1, i} \\
& \quad=\frac{1}{\mathbf{k}}\left(v \Sigma_{f, 1}^{V_{i}} \phi_{1, i}+v \Sigma_{f, 2}^{V_{i}} \phi_{2, i}\right), \\
& \sum_{j}\left(\frac{S_{j}}{V_{i}} u_{j}\left(-D_{2}^{j} \sum_{n} k_{n, j}^{\mathrm{grad}} \phi_{2, n}\right)\right)+\Sigma_{a, 2}^{V_{i}} \phi_{2, i}-\Sigma_{s, 1 \rightarrow 2}^{V_{i}} \phi_{1, i}=0, \tag{21}
\end{align*}
$$

where $D_{1}^{j}$ and $D_{2}^{j}$ are the first and second energy group global diffusion coefficients for the face $j$.
2.3. Calculation Methodology. The following steps of the calculation have been done by the next codes:
(i) geometry meshing by means of Gmsh developed by Geuzaine and Remacle [8];
(ii) discretization of the neutron diffusion equations with the Finite Volume Method by means of Arb developed by Harvie [6];
(iii) solution of the generalized eigenvalue problem by means of SLEPc developed by Hernández et al. [4, 9].

A code has been developed to do these steps automatically.

## 3. Results and Discussions

In this section, 4 different cases are exposed: homogeneous and heterogeneous 2D and 3D reactors, with the aim of showing the capabilities of the method for 2 D and 3 D reactors. In fact, the homogeneous reactors check the discretization of the equations without taking into account the use of the global diffusion coefficient and the heterogeneous reactors check the different approaches of the global diffusion coefficient developed in this study. Regarding the calculation, 5 eigenvalues

Table 14:3D homogeneous reactor power errors (\%) corresponding to the first eigenvalue.

| Node | Structured mesh |  |  |
| :---: | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm |
| 1 | 0.982818 | 0.283377 | 0.044883 |
| 2 | 1.021300 | 0.279512 | 0.044440 |
| 3 | 0.982822 | 0.283386 | 0.044883 |
| 4 | 1.067796 | 0.290603 | 0.045257 |
| 5 | 0.991369 | 0.277265 | 0.044169 |
| 6 | 1.067802 | 0.290610 | 0.045211 |
| 7 | 0.982818 | 0.283364 | 0.044932 |
| 8 | 1.021300 | 0.279545 | 0.044217 |
| 9 | 0.982822 | 0.283373 | 0.044914 |
| 10 | 1.020863 | 0.283123 | 0.044656 |
| 11 | 1.011269 | 0.282063 | 0.044683 |
| 12 | 1.020863 | 0.283113 | 0.044617 |
| 13 | 1.042381 | 0.284688 | 0.044685 |
| 14 | 1.043681 | 0.285467 | 0.044857 |
| 15 | 1.042373 | 0.284664 | 0.044693 |
| 16 | 1.020863 | 0.283142 | 0.044694 |
| 17 | 1.011261 | 0.282119 | 0.044569 |
| 18 | 1.020863 | 0.283103 | 0.044712 |
| 19 | 1.015342 | 0.281914 | 0.044589 |
| 20 | 1.017327 | 0.283325 | 0.044599 |
| 21 | 1.015328 | 0.281928 | 0.044560 |
| 22 | 1.036297 | 0.283471 | 0.044532 |
| 23 | 1.050299 | 0.286768 | 0.044747 |
| 24 | 1.036291 | 0.283471 | 0.044526 |
| 25 | 1.015350 | 0.281900 | 0.044489 |
| 26 | 1.017315 | 0.283308 | 0.044682 |
| 27 | 1.015342 | 0.281893 | 0.044503 |
| 28 | 1.015342 | 0.281921 | 0.044582 |
| 29 | 1.017327 | 0.283337 | 0.044599 |
| 30 | 1.015328 | 0.281921 | 0.044567 |
| 31 | 1.036297 | 0.283453 | 0.044502 |
| 32 | 1.050299 | 0.286774 | 0.044753 |
| 33 | 1.036291 | 0.283471 | 0.044526 |
| 34 | 1.015350 | 0.281893 | 0.044483 |
| 35 | 1.017315 | 0.283290 | 0.044688 |
| 36 | 1.015350 | 0.281900 | 0.044510 |
| 37 | 1.020863 | 0.283142 | 0.044656 |
| 38 | 1.011269 | 0.282063 | 0.044674 |
| 39 | 1.020863 | 0.283113 | 0.044626 |
| 40 | 1.042381 | 0.284688 | 0.044677 |
| 41 | 1.043681 | 0.285467 | 0.044857 |
| 42 | 1.042381 | 0.284672 | 0.044685 |
| 43 | 1.020863 | 0.283132 | 0.044712 |
| 44 | 1.011261 | 0.282103 | 0.044569 |
| 45 | 1.020863 | 0.283093 | 0.044712 |
| 46 | 0.982822 | 0.283373 | 0.044888 |
| 47 | 1.021306 | 0.279519 | 0.044453 |
| 48 | 0.982809 | 0.283377 | 0.044883 |
| 49 | 1.067796 | 0.290610 | 0.045244 |

Table 14: Continued.

| Node | Structured mesh |  |  |
| :--- | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm |
| 50 | 0.991362 | 0.277277 | 0.044169 |
| 51 | 1.067802 | 0.290616 | 0.045197 |
| 52 | 0.982822 | 0.283364 | 0.044914 |
| 53 | 1.021306 | 0.279545 | 0.044210 |
| 54 | 0.982822 | 0.283373 | 0.044905 |

have been calculated in each case. With respect to the results, the following magnitudes will be used so as to evaluate them:

$$
\begin{gather*}
\text { Power error }(\%)=\frac{\mid \text { Power }- \text { Power }_{\text {reference }} \mid}{\text { Power }_{\text {reference }}} * 100 \\
\text { Eigenvalue error }(\mathrm{pcm})=\frac{\left|\mathbf{k}-\mathbf{k}_{\text {reference }}\right|}{\mathbf{k}_{\text {reference }}} * 10^{5}  \tag{22}\\
\text { Power }=\sum_{i}\left(v \Sigma_{f, 1}^{V_{i}} \phi_{1, i}+v \Sigma_{f, 2}^{V_{i}} \phi_{2, i}\right)
\end{gather*}
$$

In addition, power is normalized to attain that mean power equals the unity, calculated with the following formula:

$$
\begin{equation*}
\text { Mean Power }=\frac{\sum_{i=1}^{N} \operatorname{Power}_{i} V_{i}}{\sum_{i=1}^{N} V_{i}}, \tag{23}
\end{equation*}
$$

where $V_{i}$ is volume or area of the element $i$, depending on 3D or 2D geometry, respectively, and considering only the elements with not null power.
3.1. 2D Homogeneous Reactor. The reactor considered is composed only of one material, whose cross sections are shown in Table 1. In addition, it has a rectangular shape and its dimensions are $100 \mathrm{~cm} \times 60 \mathrm{~cm}$. Furthermore, zero flux boundary conditions have been considered.

On the other hand, this study includes the results for structured and unstructured meshes, each one with several sizes. In particular, 3 different mesh lengths have been used: $10 \mathrm{~cm}, 5 \mathrm{~cm}$, and 2 cm . Table 2 contains the number of cells of structured and unstructured meshes. Furthermore, Figures 4 and 5 show the structured and unstructured mesh, respectively, for the 5 cm mesh length.

Moreover, this problem has analytical solution and it will be the reference solution. The analytical eigenvalues $\mathbf{k}$ are $0.99999996179,0.94332347153,0.85966257142$, 0.85451196366 , and 0.81030009136 . In this case, the reactor is in steady state since the greatest eigenvalue $\mathbf{k}$ is virtually 1.

The results corresponding to computational time and eigenvalue errors are exposed in Table 2. On the other hand, power errors are evaluated in 9 rectangular nodes of the same dimensions $33.33 \mathrm{~cm} \times 20 \mathrm{~cm}$, which are shown in Table 3, and they are presented in Tables 4 and 5, but only those corresponding to the first and second eigenvalues. In addition, the power corresponding to the first and second eigenvalues and the 2 cm structured and unstructured meshes is shown in Figures 6, 7, 8, and 9.

Table 15: 3D homogeneous reactor power errors (\%) corresponding to the second eigenvalue.

| Node | Structured mesh |  |  |
| :---: | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm |
| 1 | 0.996377 | 0.284899 | 0.045155 |
| 2 | 1.019201 | 0.279652 | 0.044237 |
| 3 | 0.996370 | 0.284906 | 0.045097 |
| 4 | 1.062091 | 0.289487 | 0.045069 |
| 5 | 1.008268 | 0.280020 | 0.044300 |
| 6 | 1.062091 | 0.289473 | 0.045105 |
| 7 | 0.996377 | 0.284871 | 0.045155 |
| 8 | 1.019187 | 0.279666 | 0.044165 |
| 9 | 0.996377 | 0.284856 | 0.045155 |
| 10 | 1.037403 | 0.284821 | 0.044523 |
| 11 | 1.008552 | 0.282182 | 0.044436 |
| 12 | 1.037403 | 0.284835 | 0.044473 |
| 13 | 1.034915 | 0.283264 | 0.044419 |
| 14 | 1.064620 | 0.288742 | 0.044959 |
| 15 | 1.034915 | 0.283235 | 0.044389 |
| 16 | 1.037410 | 0.284806 | 0.044658 |
| 17 | 1.008552 | 0.282277 | 0.044626 |
| 18 | 1.037403 | 0.284813 | 0.044729 |
| 19 | 1.031311 | 0.283454 | 0.044715 |
| 20 | 1.015537 | 0.283716 | 0.044796 |
| 21 | 1.031304 | 0.283440 | 0.044693 |
| 22 | 1.027935 | 0.281797 | 0.044396 |
| 23 | 1.072478 | 0.290229 | 0.045138 |
| 24 | 1.027935 | 0.281790 | 0.044459 |
| 25 | 1.031311 | 0.283412 | 0.044765 |
| 26 | 1.015558 | 0.283695 | 0.044655 |
| 27 | 1.031318 | 0.283412 | 0.044686 |
| 28 | 1.031311 | 0.283462 | 0.044715 |
| 29 | 1.015537 | 0.283716 | 0.044782 |
| 30 | 1.031304 | 0.283440 | 0.044686 |
| 31 | 1.027928 | 0.281804 | 0.044403 |
| 32 | 1.072478 | 0.290224 | 0.045144 |
| 33 | 1.027928 | 0.281797 | 0.044452 |
| 34 | 1.031311 | 0.283419 | 0.044772 |
| 35 | 1.015558 | 0.283695 | 0.044655 |
| 36 | 1.031311 | 0.283419 | 0.044679 |
| 37 | 1.037403 | 0.284813 | 0.044516 |
| 38 | 1.008552 | 0.282182 | 0.044442 |
| 39 | 1.037395 | 0.284828 | 0.044466 |
| 40 | 1.034915 | 0.283258 | 0.044419 |
| 41 | 1.064620 | 0.288751 | 0.044959 |
| 42 | 1.034915 | 0.283235 | 0.044389 |
| 43 | 1.037403 | 0.284806 | 0.044650 |
| 44 | 1.008552 | 0.282289 | 0.044626 |
| 45 | 1.037403 | 0.284799 | 0.044729 |
| 46 | 0.996377 | 0.284892 | 0.045155 |
| 47 | 1.019208 | 0.279659 | 0.044244 |
| 48 | 0.996384 | 0.284913 | 0.045097 |
| 49 | 1.062091 | 0.289473 | 0.045076 |

Table 15: Continued.

| Node | Structured mesh |  |  |
| :--- | :---: | :---: | :---: |
|  | 10 cm | 5 cm | 2 cm |
| 50 | 1.008250 | 0.280026 | 0.044300 |
| 51 | 1.062091 | 0.289458 | 0.045119 |
| 52 | 0.996377 | 0.284871 | 0.045162 |
| 53 | 1.019194 | 0.279659 | 0.044151 |
| 54 | 0.996377 | 0.284871 | 0.045155 |

For the second eigenvalue, there is not error in nodes 2, 5 , and 8 , since the power in these nodes is 0 , so the error in these nodes has been represented with "-" in Table 5. Actually, it can be seen in Figures 8 and 9 that power in nodes 2,5 , and 8 are virtually 0 . Moreover, there are high errors in structured mesh of 10 cm in length for the third and fourth eigenvalue, and consequently this mesh is not acceptable. It can be noted that unstructured mesh errors are higher than structured mesh ones. Furthermore, the results show an error decrease as the mesh is finer. In any case, the maximum power error is $0.370735 \%$, corresponding to the second eigenvalue, 10 cm unstructured mesh, and node 4.
3.2. Biblis Reactor. Biblis is a 2 D heterogeneous reactor composed of 8 materials. Its geometry and cross sections are described in Figure 10 and Table 6, respectively. A quarter of the reactor has been simulated, and therefore reflective flux has been assumed at west and north boundaries, and zero flux at east and south boundaries.

Moreover, the same meshes as in Section 3.1 have been used.

On the other hand, the reference solution considered was the solution obtained by Müller and Weiss [10]. This solution was obtained by means of PANIC analytic nodal code using a $4 * 4$ mesh, but only the results for the first eigenvalue are available. Therefore, only the results for the first eigenvalue will be exposed in this section, although the computational time corresponds to the calculation of five eigenvalues. Number of elements, computational times and eigenvalue errors are shown in Tables 7, 8, and 9, respectively. The reference solution of the first eigenvalue $\left(\mathbf{k}_{\mathbf{1}}\right)$ is 1.025110 .

In this case, the 4 approaches of the global diffusion coefficient have been used since this reactor is heterogeneous.

With respect to power errors, the results are evaluated at the nodes of Figure 10, but without taking into account the nodes of material 3 , since the power in these nodes is 0 . In particular, the nomenclature exposed in Table 10 will be used.

Only power errors corresponding to HOM are exposed in Table 11, because they are the most accurate results. The power corresponding to the first eigenvalue and the 2 cm structured and unstructured meshes is shown in Figures 11 and 12.

The finer the mesh, the lower the errors. In addition, unstructured mesh errors are lower than structured mesh ones. Regarding the different approaches of the global diffusion coefficient, the lowest power errors correspond to HOM and LIN. However, the lowest eigenvalue error corresponds to

Table 16: Langenbuch reactor cross sections.

| Material | $D_{1}(\mathrm{~cm})$ | $D_{2}(\mathrm{~cm})$ | $\Sigma_{a 1}\left(\mathrm{~cm}^{-1}\right)$ | $\Sigma_{a 2}\left(\mathrm{~cm}^{-1}\right)$ | $\Sigma_{12}\left(\mathrm{~cm}^{-1}\right)$ | $v \Sigma_{f 1}\left(\mathrm{~cm}^{-1}\right)$ | $v \Sigma_{f 2}\left(\mathrm{~cm}^{-1}\right)$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Comb. | 1.423913 | 0.356306 | 0.01040206 | 0.08766217 | 0.0175555 | 0.006477691 | 0.1127328 |
| Comb.2 | 1.425611 | 0.350574 | 0.01099263 | 0.09925634 | 0.01717768 | 0.007503284 | 0.1378004 |
| Absorbent | 1.423913 | 0.356306 | 0.01095206 | 0.09146217 | 0.0175555 | 0.006477691 | 0.11273228 |
| Reflector | 1.634227 | 0.264002 | 0.002660573 | 0.04936351 | 0.02759693 | 0.0 | 0.0 |

Table 17: Langenbuch reactor meshes.

|  | Structured mesh |  |
| :--- | :---: | :---: |
| Mesh length $(\mathrm{cm})$ | 10 | 5 |
| Number of cells | 2800 | 18720 |

Table 18: Langenbuch reactor computational time (h:min:s).

|  | Structured mesh |  |
| :--- | :---: | :---: |
|  | 10 cm | 5 cm |
| CELL+ | $0: 0: 25.69$ | $0: 10: 54.5$ |
| CELL- | $0: 0: 25.68$ | $0: 11: 23.11$ |
| HOM | $0: 0: 25.77$ | $0: 10: 55.77$ |
| LIN | $0: 0: 25.73$ | $0: 10: 43.59$ |

Table 19: Langenbuch reactor eigenvalue-1 error (pcm).

|  | Structured mesh |  |
| :--- | :---: | :---: |
|  | 10 cm | 5 cm |
| CELL+ | 83.3771 | 12.5047 |
| CELL- | 212.8518 | 63.0611 |
| HOM | 143.1450 | 22.6874 |
| LIN | 150.0886 | 25.2636 |

Table 20: Langenbuch reactor eigenvalue-2 error (pcm).

|  | Structured mesh |  |
| :--- | :---: | :---: |
|  | 10 cm | 5 cm |
| CELL+ | 226.1219 | 7.7211 |
| CELL- | 350.5162 | 80.2113 |
| HOM | 280.9989 | 39.9871 |
| LIN | 292.1948 | 44.2362 |

Table 21: Numbering of the Langenbuch reactor nodes.

| 2nd axial plane |  |  |  |  |  | 5th axial plane |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| - | - | - | - | - |  | - | - | - | - | - |  |
| 21 | 22 | 23 | 24 | - | - | 45 | 46 | 47 | 48 | - | - |
| 16 | 17 | 18 | 19 | 20 | - | 40 | 41 | 42 | 43 | 44 | - |
| 11 | 12 | 13 | 14 | 15 | - | 35 | 36 | 37 | 38 | 39 | - |
| 6 | 7 | 8 | 9 | 10 | - | 30 | 31 | 32 | 33 | 34 | - |
| 1 | 2 | 3 | 4 | 5 | - | 25 | 26 | 27 | 28 | 29 | - |

CELL- in coarse meshes, yet this error is virtually the same for all the approaches in the finest mesh.
3.3. $3 D$ Homogeneous Reactor. The reactor considered is composed of the same material as the 2D homogeneous reactor, it is a parallelepiped of the next dimensions: $100 \mathrm{~cm} \times$ $60 \mathrm{~cm} \times 180 \mathrm{~cm}$. In addition, the boundary conditions applied are zero flux. Regarding the meshes, the same mesh lengths as in the previous sections have been used again, but only the structured mesh.

Moreover, the analytical solution exists, and therefore it will be the reference solution. On the one hand, eigenvalue errors and computational time are shown in Table 12. On the other hand, power errors are evaluated in 54 parallelepiped nodes of the same dimensions $33.33 \mathrm{~cm} \times 20 \mathrm{~cm} \times 30 \mathrm{~cm}$, which are exposed in Table 13, and they are shown in Tables 14 and 15 , but only those corresponding to the first and second eigenvalues. In addition, the power corresponding to the first and second eigenvalues and the 2 cm structured mesh is shown in Figures 13 and 14.

The analytical eigenvalues $\mathbf{k}$ are 0.99391916952 , $0.97602952377,0.94734259138,0.93778667636$, and 0.92148598185 . In this case, the reactor is in steady state since the greatest eigenvalue $\mathbf{k}$ is virtually 1 .

In conclusion, the finer the mesh, the more accurate the results. However, the maximum error of the coarse mesh is about $1 \%$, which is acceptable. On the other hand, the computational time of the finest mesh is not practical.
3.4. Langenbuch Reactor. Langenbuch is a 3D heterogeneous reactor composed of 4 materials. Its cross sections are exposed in Table 16. Its geometry is described in Figures 15 and 16. A quarter of the reactor has been simulated, and therefore reflective flux has been assumed at west and south boundaries, and zero flux at east, north, top, and bottom boundaries.

In this case, only structured meshes have been used, in particular the 10 cm and 5 cm mesh length. Table 17 shows the number of cells for each mesh.

Moreover, the reference solution has been obtained with VALKIN code developed by Miró et al. [3], which was called MODKIN in the cited reference. VALKIN is a nodal modal code that is able to calculate several eigenvalues and their respective eigenvectors, which are the neutronic fluxes. In this case, the reference solution was obtained for 5 eigenvalues at nodes of Figures 15 and 16. Furthermore, the 4 approaches of the global diffusion coefficient have been used owing to the heterogeneous nature of this reactor. The reference solution of the first and second eigenvalues ( $\mathbf{k}_{\mathbf{1}}$ and $\mathbf{k}_{\mathbf{2}}$ ) are 0.994881227 and 0.948210698 .

Regarding the results, Table 18 exposes the computational time for each mesh. Moreover, eigenvalue errors are shown in

Table 22: Langenbuch reactor power errors (\%) corresponding to the first eigenvalue and HOM.

| Node | Structured mesh |  | Node | Structured mesh |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | 10 cm | 5 cm |  | 10 cm | 5 cm |
| 1 | 0,88732759 | 0,083512937 | 1 | 2,748470704 | 0,472742419 |
| 2 | 0,413636006 | 0,169843064 | 2 | 1,064854909 | 0,505072706 |
| 3 | 0,285898499 | 0,26806574 | 3 | 0,379827902 | 0,520035451 |
| 4 | 1,650354128 | 0,463970268 | 4 | 0,846862035 | 0,527032189 |
| 5 | 7,308155289 | 2,443154619 | 5 | 6,400564495 | 2,241810405 |
| 6 | 0,413578042 | 0,169901321 | 6 | 1,064870054 | 0,505064685 |
| 7 | 0,218154684 | 0,211939733 | 7 | 0,500339771 | 0,504531289 |
| 8 | 0,89582742 | 0,279213792 | 8 | 1,13600946 | 0,496430539 |
| 9 | 2,440629291 | 0,498489677 | 9 | 2,54996541 | 0,525233749 |
| 10 | 8,245123598 | 2,368554084 | 10 | 8,338474277 | 2,124761105 |
| 11 | 0,285938524 | 0,268114276 | 11 | 0,379868741 | 0,519985959 |
| 12 | 0,895860783 | 0,279229236 | 12 | 1,135973249 | 0,496377465 |
| 13 | 1,485778349 | 0,264874283 | 13 | 1,719135726 | 0,413130915 |
| 14 | 3,250361067 | 0,503872312 | 14 | 3,303282255 | 0,426080951 |
| 15 | 8,254395078 | 1,996055827 | 15 | 8,233916013 | 1,633618147 |
| 16 | 1,650484996 | 0,46408882 | 16 | 0,846698163 | 0,526850063 |
| 17 | 2,440737883 | 0,498585195 | 17 | 2,549801498 | 0,525079406 |
| 18 | 3,250407371 | 0,5039109 | 18 | 3,303171711 | 0,426002352 |
| 19 | 6,540873894 | 1,269229237 | 19 | 6,461736033 | 0,930426868 |
| 20 | 10,49036132 | 3,000440824 | 20 | 10,39493074 | 2,481116214 |
| 21 | 7,308460522 | 2,44345405 | 21 | 6,400329869 | 2,24158495 |
| 22 | 8,245387401 | 2,368777266 | 22 | 8,338216136 | 2,124537285 |
| 23 | 8,25461267 | 1,99626084 | 23 | 8,233749447 | 1,633473241 |
| 24 | 10,49053455 | 3,000606939 | 24 | 10,39486316 | 2,481044325 |
| 25 | 5,466731896 | 0,931224354 | 25 | 8,712874235 | 0,839916928 |
| 26 | 4,140848872 | 0,716286587 | 26 | 6,327127 | 0,784501724 |
| 27 | 3,657496267 | 0,698504374 | 27 | 5,950753659 | 0,862124448 |
| 28 | 2,237579532 | 0,474563288 | 28 | 4,44225414 | 0,601601562 |
| 29 | 3,189694202 | 1,268772356 | 29 | 0,062727543 | 0,461900883 |
| 30 | 4,140836529 | 0,716252343 | 30 | 6,327225132 | 0,78459806 |
| 31 | 2,96775873 | 0,633299552 | 31 | 4,313496832 | 0,803439342 |
| 32 | 2,476161123 | 0,618678432 | 32 | 3,885076646 | 0,873376122 |
| 33 | 0,910178355 | 0,364113213 | 33 | 2,432393528 | 0,789928103 |
| 34 | 4,548866137 | 1,200011218 | 34 | 2,461696614 | 0,258876027 |
| 35 | 3,657467753 | 0,698474984 | 35 | 5,951087173 | 0,86245633 |
| 36 | 2,476131091 | 0,618647924 | 36 | 3,885366412 | 0,873674968 |
| 37 | 2,967442105 | 0,56874817 | 37 | 4,2237795 | 0,779967598 |
| 38 | 0,292240082 | 0,412798339 | 38 | 1,90610207 | 0,962348979 |
| 39 | 4,267581802 | 0,724868524 | 39 | 2,22742644 | 0,28534924 |
| 40 | 2,237496931 | 0,474473118 | 40 | 4,44301111 | 0,602394933 |
| 41 | 0,910102518 | 0,364043283 | 41 | 2,433102829 | 0,790642934 |
| 42 | 0,292200002 | 0,412751132 | 42 | 1,906524369 | 0,962753237 |
| 43 | 2,930647399 | 0,281476818 | 43 | 0,944598463 | 0,649099892 |
| 44 | 6,41914469 | 1,657371938 | 44 | 4,201989073 | 0,422463819 |
| 45 | 3,189876024 | 1,268969435 | 45 | 0,061633966 | 0,460821104 |
| 46 | 4,549015577 | 1,2001458 | 46 | 2,460613715 | 0,257845259 |
| 47 | 4,267727516 | 0,724995924 | 47 | 2,226705013 | 0,286064257 |
| 48 | 6,419233002 | 1,657446439 | 48 | 4,20167533 | 0,422178482 |

Tables 19 and 20, but only those corresponding to the first and second eigenvalues. On the other hand, power errors have been calculated in the nodes of Figures 15 and 16, but only the nodes numbered in Table 21 are exposed in this paper due to the results extent. These power errors are shown in Tables 22 and 23 , but only those corresponding to the first and second eigenvalues and HOM global diffusion coefficient approach. In addition, the power corresponding to the first and second eigenvalues and the 5 cm structured mesh is shown in Figures 17 and 18.

With respect to the meshes used, the 10 cm mesh length implies the highest errors and the 5 cm mesh length the lowest ones. Furthermore, the highest errors are located near the reflector. Regarding the global diffusion coefficient approaches, HOM and LIN give better results than CELL+ and CELL-. It is important to remark the computational time dependence on the mesh that implies 2 cm mesh is not practical owing to the fact that it has an order of magnitude of hours.

## 4. Conclusions

A method has been developed to solve the steady state of the 2-energy group neutron diffusion equation for LWR in any reactor configuration, using the Finite Volume Method and calculating several eigenvalues.

This method supplies accurate results for 2D reactors and low computational times, about seconds. However, 3D reactors computational times are higher that could be about hours for fine meshes. Moreover, 3D reactor results are less accurate than 2D reactor ones.

If the global diffusion coefficient had not been used, the power errors corresponding to the first eigenvalue would have been about $15-20 \%$, although this study does not show these results. Consequently, the global diffusion coefficient has to be used to obtain acceptable results.

With reference to future work, the method will include the parallelization of both geometry processing and eigenvalue calculation to reduce the computational time. Furthermore, more global diffusion coefficients approaches will be developed, and another alternatives to evaluate the face averaged gradient flux are being considered as the implementation of high-order schemes. Regarding the nuclear applications, the transitory state calculation will be developed in order to evaluate any reactor condition. Finally, the advanced thermal-hydraulic coupling will be the final step to take into account the thermal hydraulic influence in the neutronic calculation.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Research Article 

# Global Stability of Multigroup SIRS Epidemic Model with Varying Population Sizes and Stochastic Perturbation around Equilibrium 

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#### Abstract

We discuss multigroup SIRS (susceptible, infectious, and recovered) epidemic models with random perturbations. We carry out a detailed analysis on the asymptotic behavior of the stochastic model; when reproduction number $\mathscr{R}_{0}>1$, we deduce the globally asymptotic stability of the endemic equilibrium by measuring the difference between the solution and the endemic equilibrium of the deterministic model in time average. Numerical methods are employed to illustrate the dynamic behavior of the model and simulate the system of equations developed. The effect of the rate of immunity loss on susceptible and recovered individuals is also analyzed in the deterministic model.


## 1. Introduction

To curb the spread and impact of viruses, it is important to study their feature, propagating methods, means, and limitation. Many studies about the outbreak and spread of disease have been done by means of establishing epidemic models. These researches provided some useful and valid reference for the characteristic of disease transmission. Based on these results of theoretical analysis, one can predict the future course of an outbreak and evaluate strategies to control an epidemic. In 1927, Kermack and McKendrick created an SIR model in which they considered a fixed population with only three compartments of three classes: susceptible $S(t)$, infected $I(t)$, and removed $R(t) . S(t)$ represents the number of individuals not yet infected with the disease at time $t$, or those susceptible to the disease. $I(t)$ denotes the number of individuals who have been infected with the disease and are capable of spreading the disease to those in the susceptible class. $R(t)$ is the compartment used for those individuals who have been infected and then removed from the disease, either due to immunization or due to death. Those in this class are not able to be infected again or to transmit the infection
to others. A single group SIRS model is an extension of the SIR model. The SIRS model for infections that do not confer permanent immunity, that is, an infection that does not leave a long lasting immunity: thus individuals that have recovered return to being susceptible again, moving back into the $S(t)$ compartment. The only difference between SIR and SIRS is that SIRS model allows members of the recovered class to be free of infection and rejoin the susceptible class.

Considering different contact patterns, a distinct number of sexual partners, or different geography, and so forth, it is more appropriate to divide individual hosts into groups in modeling epidemic disease [1]. Therefore, it is reasonable to propose multigroup models to describe the transmission dynamics of viruses in heterogeneous host populations on epidemic models. In fact, there are already many scholars focusing their study on various forms of multigroup epidemic models (see [2-7]). They have also proved the global stability of the unique endemic equilibrium through Lyapunov function, which is one of the main mathematical challenges in analyzing multigroup models. Recently, Guo et al. have paid more attention to multigroup models [8-10]. They first proposed a graph-theoretic approach to the method
of global Lyapunov functions and used it to establish the global stability of the interior equilibrium for more general models [8]. Muroya et al. considered a class of $n$-group ( $n \geqslant 2$ ) SIRS epidemic models described by the following system of equations [11]:

$$
\begin{gather*}
\dot{S}_{k}=\Lambda_{k}-d_{k}^{S} S_{k}-\sum_{j=1}^{n} \beta_{k j} S_{k}(t) I_{j}(t)+\delta_{k} R_{k}, \\
\dot{I}_{k}=\sum_{j=1}^{n} \beta_{k j} S_{k}(t) I_{j}(t)-\left(d_{k}^{I}+\gamma_{k}\right) I_{k},  \tag{1}\\
\dot{R}_{k}=\gamma_{k} I_{k}-\left(d_{k}^{R}+\delta_{k}\right) R_{k}, \quad k=1, \ldots, n .
\end{gather*}
$$

Nakata et al. [12] and Enatsu et al. [13] proposed an idea to extend Lyapunov functional techniques in McCluskey [14] for SIR epidemic models to SIRS epidemic models. By extending well-known Lyapunov function techniques, Muroya et al. [11] succeeded to prove the global stability of system (1) without use of the grouping technique by graph theory 4 in Guo et al. [10]. Note that because of environmental noises, the deterministic approach has some limitations in the mathematical modeling transmission of an infectious disease, several authors began to consider the effect of white noise in epidemic models [15-18]. Beretta et al. proved the stability of epidemic model with stochastic time delays influenced by probability under certain conditions [19]. Such type of stochastic perturbations firstly was proposed in [19, 20] and later was successfully used in many other papers for many other different systems (see, e.g., [21-27]). Yuan et al. in [28] and Yu et al. in [29] all investigated epidemic models with fluctuations around the positive equilibrium and they proved locally stochastically asymptotic stability of the positive equilibrium. Ji et al. also discuss a multigroup SIR model with stochastic perturbation and deduce the globally asymptotic stability of the disease-free equilibrium when $R_{0} \leq 1$, which means the disease will die out. When $R_{0}>1$, they derive the disease will prevail, which is measured through the difference between the solution and the endemic equilibrium of the deterministic model in time average [1]. Imhof and Walcher [30] considered a stochastic chemostat model and they proved that the stochastic model led to extinction even though the deterministic counterpart predicts persistence. In our previous work, we considered an SEIR epidemic model with constant immigration and random fluctuation around the endemic equilibrium, we carried out a detailed analysis on the asymptotic behavior of the stochastic model [31], and we also investigated a twogroup epidemic model with distributed delays and random perturbation [32]. In addition, we investigated multigroup SEIQR models with and without random perturbation in computer network [33]. In the present paper, based on system (1), to examine the influence of white noise on system (1) we also consider a stochastic version of the SIRS model by perturbing the deterministic system (1) by a white noise and assume that the perturbations are around the positive endemic equilibrium of epidemic models.

This paper is organized as follows. We begin in Section 2 with necessary background with respect to the deterministic
multigroup SIRS model. We establish the global dynamics determined by the basic reproduction number $\mathscr{R}_{0}$ and introduce some results of graph theory used by Guo et al. We also review the important results in Theorem 2 on deterministic model (1) by Muroya et al. In Section 3 we derive the stochastic version from the deterministic model (1). In Section 4 we analyse the asymptotic behavior of the stochastic model by means of the method of Lyapunov functions and the theories of stochastic differential equation in Theorem 5. Numerical methods are employed to simulate the dynamic behavior of the model, and the effect of the rate of immunity loss on the recovered is also analyzed in the deterministic models and the corresponding stochastic models in Section 5. Finally, we give the conclusion of our paper in Section 6.

## 2. Global Stability of Deterministic Multigroup SIRS Models

First, let us review some theories and results on deterministic multigroup SIRS models; we summarize the parameters in the model (1) by the following list:
$\Lambda_{k}$ : the recruitment rate of the population,
$\beta_{k j}$ : transmission coefficient between compartments $S_{k}$ and $I_{j}$,
$d_{k}^{S}, d_{k}^{I}, d_{k}^{R}$ : the natural death rates of susceptible, infected, and recovered individuals in city $k$, respectively,
$\delta_{k}$ : the rate of immunity loss in the $k$ th group,
$\gamma_{k}$ : the natural recovery rate of the infected individuals in city $k$.

We assume $d_{k}^{S}, d_{k}^{I}, d_{k}^{R}, \Lambda_{k}>0$ and the rest of the parameters are nonnegative for all $k$. In particular, $\beta_{k j}=0$ if there is no transmission of the disease between compartments $S_{k}$ and $I_{j}$.

By the biological meanings, we may assume that

$$
\begin{equation*}
d_{k}^{S} \leqslant \min \left\{d_{k}^{I}, d_{k}^{R}\right\}, \quad k=1,2, \ldots, n . \tag{2}
\end{equation*}
$$

For each $k$, adding the three equations in (1) gives $\left(S_{k}+I_{k}+R_{k}\right)^{\prime} \leqslant \Lambda_{k}-d_{k}^{S}\left(S_{k}+I_{k}+R_{k}\right)$. Hence, $\lim \sup _{t \rightarrow \infty}\left(S_{k}+I_{k}+R_{k}\right) \leqslant \Lambda_{k} / d_{k}^{S}$. Therefore, omega limit sets of system (1) are contained in the following bounded region in the non-negative cone of $R^{3 n}$ :

$$
\begin{gather*}
\Gamma=\left\{\left(S_{1}, I_{1}, R_{1}, \ldots, S_{n}, I_{n}, R_{n}\right) \in R_{+}^{3 n} \mid S_{k} \leqslant S_{k}^{0},\right. \\
\left.S_{k}+I_{k}+R_{k} \leqslant \frac{\Lambda_{k}}{d_{k}^{S}}, 1 \leqslant k \leqslant n\right\} . \tag{3}
\end{gather*}
$$

It can be verified that region $\Gamma$ is positively invariant and system (1) always has the disease-free equilibrium $P_{0}=$ $\left(S_{1}^{0}, 0,0,0,0, \ldots, S_{n}^{0}, 0,0,0,0\right)$, where $S_{k}^{0}=\Lambda_{k} / d_{k}^{S}$ is the equilibrium of the $S_{k}$ population in the absence of disease ( $I_{1}=I_{2}=\cdots=I_{n}=0$ ). Let $\Gamma$ denote the interior of $\Gamma$. An endemic equilibrium $P^{*}=\left(S_{1}^{*}, I_{1}^{*}, R_{1}^{*}, \ldots, S_{n}^{*}, I_{n}^{*}, R_{n}^{*}\right)$ belongs to $\stackrel{\circ}{\Gamma}$ satisfying the equilibrium equations

$$
\begin{gather*}
\Lambda_{k}=\sum_{j=1}^{n} \beta_{k j} S_{k}^{*} I_{j}^{*}-d_{k}^{S} S_{k}^{*}+\delta_{k} R_{k}^{*}  \tag{4}\\
\left(d_{k}^{I}+\gamma_{k}\right) I_{k}^{*}=\sum_{j=1}^{n} \beta_{k j} S_{k}^{*} I_{j}^{*}  \tag{5}\\
\left(d_{k}^{I}+\delta_{k}\right) R_{k}^{*}=\gamma_{k} I_{k}^{*} . \tag{6}
\end{gather*}
$$

Let

$$
\begin{equation*}
\mathscr{R}_{0}=\rho\left(M_{0}\right) \tag{7}
\end{equation*}
$$

denote the spectral radius of the matrix

$$
\begin{equation*}
M_{0}=\left(\frac{\beta_{k j} S_{k}^{0}}{d_{k}^{I}+\gamma_{k}}\right)_{1 \leqslant k, j \leqslant n} \tag{8}
\end{equation*}
$$

The parameter $\mathscr{R}_{0}$ is referred to as the basic reproduction number. Its biological significance is that if $\mathscr{R}_{0}<1$, the disease dies out while if $\mathscr{R}_{0}>1$, the disease becomes endemic [9]. In the following theorem, we show that the multi-group model (1) has at least one endemic equilibrium $P^{*}$ when $\mathscr{R}_{0}>$ 1 , and $P^{*}$ is globally stable [9].

The matrix $B=\left(\beta_{k j}\right)$ denotes the contact matrix. Associated with $B$, one can construct a directed graph $\mathfrak{L}=$ $G(B)$ whose vertex $k$ represents the $k$ th group, $k=1, \ldots, n$. A directed edge exists from vertex $k$ to vertex $j$ if and only if $\beta_{k j}>0$. Throughout the paper, we assume that $B$ is irreducible. This is equivalent to $G(B)$ being strongly connected. Biologically, this is the same as assuming that any two groups $k$ and $j$ have a direct or indirect route of transmission. More specifically, individuals in $I_{j}$ can infect ones in $S_{k}$ directly or indirectly. Now consider the linear system

$$
\begin{equation*}
\overline{\mathfrak{B}} \ell=0, \tag{9}
\end{equation*}
$$

where

$$
\overline{\mathfrak{B}}=\left[\begin{array}{cccc}
\sum_{i \neq 1}^{\bar{\beta}_{1 i}} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n 1}  \tag{10}\\
-\bar{\beta}_{12} & \sum_{i \neq 2} \bar{\beta}_{2 i} & \cdots & -\bar{\beta}_{n 2} \\
\vdots & \vdots & \ddots & \vdots \\
-\bar{\beta}_{1 n} & \bar{\beta}_{2 n} & \cdots & \sum_{i \neq n} \bar{\beta}_{n i}
\end{array}\right]
$$

and $\bar{\beta}_{k j}=\beta_{k j} S_{k}^{*} I_{j}^{*}, \bar{\beta}_{k j}>0,1 \leqslant k, j \leqslant n$. Let $\mathbb{Z}=G(B)$ denote the directed graph associated with matrix $B$ (and $\left(\bar{\beta}_{k j}\right)$ ), and let $C_{j k}$ denote the cofactor of the $(j, k)$ entry of $\overline{\boldsymbol{B}}$.

We have the following fundamental lemma[8].

Lemma 1 (Kirchhoff's Matrix-Tree Theorem). Assume that $\left(\bar{\beta}_{k j}\right)_{n \times n}$ is irreducible and $n \geqslant 2$. Then the following results hold.
(1) The solution space of system (9) has dimension 1, with $\operatorname{abasis}\left(\ell_{1}, \ell_{2}, \ldots, \ell_{n}\right)=\left(C_{11}, C_{22}, \ldots, C_{n n}\right)$.
(2) For $1 \leqslant k \leqslant n$,

$$
\begin{equation*}
C_{k k}=\sum_{T \in \mathbb{T}_{k}} W(T)=\sum_{T \in \mathbb{T}_{k}} \prod_{(r, m) \in E(T)} \bar{\beta}_{r m}>0, \tag{11}
\end{equation*}
$$

where $\mathbb{T}_{k}$ is the set of all directed spanning subtrees of $\mathfrak{Z}$ that are rooted at vertex $k, W(T)$ is the weight of a directed tree $T$, and $E(T)$ denotes the set of directed arcs in a directed tree $T$.
Y. Muroya et al. proved the following theorem.

Theorem 2. Assume that $B=\left(\beta_{k j}\right)$ is irreducible and inequality (2) holds. If $\mathscr{R}_{0}>1$ and $d_{k}^{S} S^{*}-\delta_{k} R_{k}^{*} \geqslant 0$, then system (1) has at least one endemic equilibrium $P^{*}$ in $\Gamma$, and $P^{*}$ is globally asymptotically stable.

## 3. Multigroup Stochastic SIRS Model

Under assumptions that $\mathscr{R}_{0}>1, d_{k}^{S} S^{*}-\delta_{k} R_{k}^{*} \geqslant 0, d_{k}^{S} \leqslant$ $\min \left\{d_{k}^{I}, d_{k}^{R}\right\}$, and $\mathfrak{B}=\left(\beta_{k j}\right)$ is irreducible in Theorem 2, we know from Section 2 that there exists a positive endemic equilibrium $P^{*}$ in $\stackrel{\circ}{\Gamma}$. Furthermore, we assume stochastic perturbations are of white noise type, which are directly proportional to distances $S_{k}(t), I_{k}(t)$, and $R_{k}(t)$ from values of $S_{k}^{*}, I_{k}^{*}, R_{k}^{*}$, influence on the $S_{k}(t), I_{k}(t)$, and $R_{k}(t)$, respectively. So system (1) results in

$$
\begin{gather*}
\dot{S}_{k}=\Lambda_{k}-d_{k}^{S} S_{k}-\sum_{j=1}^{n} \beta_{k j} S_{k}(t) I_{j}(t)+\delta_{k} R_{k} \\
+\sigma_{1 k}\left(S_{k}-S_{k}^{*}\right) \dot{B}_{1 k} \\
\dot{I}_{k}=\sum_{j=1}^{n} \beta_{k j} S_{k}(t) I_{j}(t)-\left(d_{k}^{I}+\gamma_{k}\right) I_{k}+\sigma_{2 k}\left(I_{k}-I_{k}^{*}\right) \dot{B}_{2 k} \\
\dot{R}_{k}=\gamma_{k} I_{k}-\left(d_{k}^{R}+\delta_{k}\right) R_{k}+\sigma_{3 k}\left(R_{k}-R_{k}^{*}\right) \dot{B}_{3 k} \\
k=1, \ldots, n \tag{12}
\end{gather*}
$$

where $B_{1 k}(t), B_{2 k}(t)$, and $B_{3 k}(t)$ are independent standard Brownian motions and $\sigma_{i k}^{2}>0$ represent the intensities of $B_{i k}(t)(i=1,2,3)$, respectively. Obviously, stochastic system (12) has the same equilibrium points as system (1). Hence, when the stochastic model (12) satisfies the conditions of Theorem 2, we can ensure the existence of the positive equilibrium point of the stochastic model (12). Note that because of the influence of the white noise, the components of solutions to (12) can also be negative at some time. But because it is impossible for populations to be less than zero physically, once the negative sections appear on components of solutions of (12), we only need to consider the nonnegative
section of the components of solutions. In the next section, we will investigate asymptotic stability of the equilibrium $P^{*}$ of system (12). To this end, we will construct a class of different Lyapunov functions to achieve our proof under certain conditions.

## 4. Stochastic Stability of the Endemic Equilibrium

Let us now proceed to discuss asymptotic stability of the system (12). In this paper, unless otherwise specified, let $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}_{t \geq t_{0}}, P\right)$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geqslant t_{0}}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_{0}$ contains all $P$ null sets). Let $B_{i k}(t)$ be the Brownian motions defined on this probability space. If $\mathscr{R}_{0}>1$, then the stochastic system (12) can be centered at its endemic equilibrium $P^{*}=$ $\left(S_{1}^{*}, I_{1}^{*}, R_{1}^{*}, \ldots, S_{n}^{*}, I_{n}^{*}, R_{n}^{*}\right)$; by the change of variables

$$
\begin{equation*}
\mathbf{u}_{k}=S_{k}-S_{k}^{*}, \quad \mathbf{v}_{k}=I_{k}-I_{k}^{*}, \quad \mathbf{w}_{k}=R_{k}-R_{k}^{*}, \tag{13}
\end{equation*}
$$

we obtain the following system:

$$
\begin{align*}
& \dot{\mathbf{u}}_{k}=-d_{k}^{S} \mathbf{u}_{k}-\sum_{j=1}^{n} \beta_{k j} S_{k}^{*} \mathbf{v}_{j}-\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{j} \\
&-\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} I_{j}^{*}+\delta_{k} \mathbf{w}_{k}+\sigma_{1 k} \mathbf{u}_{k} \dot{B}_{1 k}, \\
& \dot{\mathbf{v}}_{k}=\sum_{j=1}^{n} \beta_{k j} S_{k}^{*} \mathbf{v}_{j}+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{j}  \tag{14}\\
&+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} I_{j}^{*}-\left(d_{k}^{I}+\gamma_{k}\right) \mathbf{v}_{k}+\sigma_{2 k} \mathbf{v}_{k} \dot{B}_{2 k}, \\
& \dot{\mathbf{w}}_{k}= \gamma_{k} \mathbf{v}_{k}-\left(d_{k}^{R}+\delta_{k}\right) \mathbf{w}_{k}+\sigma_{3 k} \mathbf{w}_{k} \dot{B}_{3 k}, \\
& \quad k=1, \ldots, n .
\end{align*}
$$

It is clear that the stability of equilibrium of system (12) is equivalent to the stability of zero solution of system (14). Considering the $d$-dimensional stochastic differential equation

$$
\begin{equation*}
d \mathbf{x}(t)=f(\mathbf{x}(t), t) d t+g(\mathbf{x}(t), t) d B(t), \quad t \geqslant t_{0} . \tag{15}
\end{equation*}
$$

If the assumptions of the existence and uniqueness theorem are satisfied, then for any given initial value $\mathbf{x}\left(t_{0}\right)=x_{0} \in \mathbb{R}^{d}$, (15) has a unique global solution denoted by $\mathbf{x}\left(t ; t_{0}, x_{0}\right)$. For the purpose of stability, we assume in this section $f(0, t)=0$ and $g(\mathbf{0}, t)=0$ for all $t \geqslant t_{0}$. So (15) admits a solution $\mathbf{x}(t) \equiv \mathbf{0}$ which is called the trivial solution or equilibrium position. We let $C^{2,1}\left(\mathbb{R}^{d} \times\left[t_{0}, \infty\right) ; \mathbb{R}_{+}\right)$denote the family of all nonnegative functions $V(\mathbf{x}, t)$ on $\mathbb{R}^{d} \times\left[t_{0}, \infty\right)$ which are
continuously twice differentiable in $\mathbf{x}$ and once in $t$. Define the differential operator $L$ associated with (15) by

$$
\begin{equation*}
L=\frac{\partial}{\partial t}+\sum_{i=1}^{d} f_{i}(\mathbf{x}, t) \frac{\partial}{\partial x_{i}}+\frac{1}{2} \sum_{i, j=1}^{d}\left[g^{T}(\mathbf{x}, t) g(\mathbf{x}, t)\right]_{i j} \frac{\partial^{2}}{\partial x_{i} x_{j}} . \tag{16}
\end{equation*}
$$

If $L$ acts on a function $V \in C^{2,1}\left(\mathbb{R}^{d} \times\left[t_{0}, \infty\right)\right.$; $\left.\mathbb{R}_{+}\right)$, then

$$
\begin{align*}
L V(\mathbf{x}, t)= & V_{t}(\mathbf{x}, t)+V_{t}(\mathbf{x}, t) f(\mathbf{x}, t) \\
& +\frac{1}{2} \operatorname{trace}\left[g^{T}(\mathbf{x}, t) V_{x x}(\mathbf{x}, t) g(\mathbf{x}, t)\right] . \tag{17}
\end{align*}
$$

Definition 3. (1) The trivial solution of (15) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in(0,1)$ and $r>0$, there exists a $\delta=\delta\left(\varepsilon, r, t_{0}\right)>0$ such that

$$
\begin{equation*}
P\left\{\left|\mathbf{x}\left(t ; t_{0}, x_{0}\right)\right|<r \forall t \geqslant t_{0}\right\} \geqslant 1-\varepsilon \tag{18}
\end{equation*}
$$

whenever $\left|x_{0}\right|<\delta$. Otherwise, it is said to be stochastically unstable.
(2) The trivial solution is said to be stochastically asymptotically stable if it is stochastically stable and for every $\varepsilon \in$ $(0,1)$, there exists a $\delta_{0}=\delta_{0}\left(\varepsilon, t_{0}\right)>0$ such that

$$
\begin{equation*}
P\left\{\lim _{t \rightarrow \infty} \mathbf{x}\left(t ; t_{0}, x_{0}\right)=0\right\} \geqslant 1-\varepsilon \tag{19}
\end{equation*}
$$

whenever $\left|x_{0}\right|<\delta_{0}$.
(3) The trivial solution is said to be stochastically asymptotically stable in the large if it is stochastically stable and for all $x_{0} \in \mathbb{R}^{d}$

$$
\begin{equation*}
P\left\{\lim _{t \rightarrow \infty} \mathbf{x}\left(t ; t_{0}, x_{0}\right)=0\right\}=1 \tag{20}
\end{equation*}
$$

Before proving the main theorem we put forward a lemma in [34].

Lemma 4 (see [34]). If there exists a positive-definite decrescent radially unbounded function $V(\mathbf{x}, t) \in C^{2,1}\left(\mathbb{R}^{d} \times\right.$ $\left.\left[t_{0}, \infty\right) ; \mathbb{R}_{+}\right)$such that $L V(\mathbf{x}, t)$ is negative-definite, then the trivial solution of (15) is stochastically asymptotically stable in the large.

From the above lemma, we can obtain the stochastically asymptotical stability of equilibrium as following.

Theorem 5. Assume that $\mathfrak{B}=\left(\beta_{k j}\right)$ is irreducible and inequality (2) holds and $\mathscr{R}_{0}>1, d_{k}^{S} S_{k}^{*}-\delta_{k} R_{k}^{*} \geqslant 0$; then if the following condition is satisfied:




$$
\begin{array}{cc}
\cdots-- & S_{2}(t) \\
\cdots & I_{2}(t) \\
\cdots \cdots & R_{2}(t)
\end{array}
$$

FIGURE 1: Deterministic trajectories of SIRS model (1) for initial condition $S_{1}(0)=2, I_{1}(0)=4, R_{1}(0)=6, S_{2}(0)=9, I_{2}(0)=4$, and $R_{2}(0)=0.5$.



$$
\begin{array}{cc}
---S_{1}(t) & ---S_{2}(t) \\
\cdots-I_{1}(t) & -\cdots-I_{2}(t) \\
\cdots \cdots R_{1}(t) & \cdots \cdots R_{2}(t)
\end{array}
$$

FIGURE 2: Stochastic trajectories of SIRS model (12) for $\sigma_{11}=0.4, \sigma_{21}=0.73, \sigma_{31}=0.45, \sigma_{12}=0.5, \sigma_{22}=0.67, \sigma_{32}=0.5$, and $\Delta t=10^{-3}$.


FIgure 3: Stochastic trajectories of SIRS model (12) for $\sigma_{11}=0.7, \sigma_{21}=1.2, \sigma_{31}=1.3, \sigma_{12}=0.8, \sigma_{22}=0.8, \sigma_{32}=1.4$, and $\Delta t=10^{-3}$.

$\begin{array}{cc}\cdots & S_{1}(t) \\ \cdots & I_{1}(t) \\ \ldots \ldots & R_{1}(t)\end{array}$
$I_{2}(t)$

FIGURE 4: Stochastic trajectories of SIRS model (12) for $\sigma_{11}=40.5, \sigma_{21}=32.73, \sigma_{31}=36.92, \sigma_{12}=33.4, \sigma_{22}=39.67$, and $\sigma_{32}=30.5$.


Figure 5: The relation between $S_{k}$ and $R_{k}$ for the deterministic SIRS model.


Figure 6: The relation between $S_{k}$ and $R_{k}$ for the stochastic SIRS model.

$$
\begin{array}{rlr}
\sigma_{1 k}^{2}<2 d_{k}^{S}, & \sigma_{2 k}^{2}<\frac{2\left(d_{k}^{I}+\gamma_{k}\right) \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{\sum_{j=1}^{n} \beta_{k j} I_{j}^{*}+d_{k}^{S}+d_{k}^{I}+\gamma_{k}}, & \\
\sigma_{3 k}^{2}<2\left(d_{k}^{R}+\delta_{k}\right), & & \frac{2 \delta_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{M_{k}\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)} \\
\frac{\delta_{k}^{2}}{\left(d_{k}^{S}-(1 / 2) \sigma_{1 k}^{2}\right)\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)} & & \frac{\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right) M_{k}}{2 \gamma_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}},
\end{array}
$$



Figure 7: Comparison of relationships of $S_{k}$ and $R_{k}$ for the different values $\delta_{k}$ in the deterministic SIRS model.
the endemic equilibrium $P^{*}$ of system (12) is stochastically asymptotically stable in the large, where

$$
\begin{equation*}
M_{k}=2\left(d_{k}^{I}+\gamma_{k}\right) \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}-\left(\sum_{j=1}^{n} \beta_{k j} I_{j}^{*}+d_{k}^{S}+d_{k}^{I}+\gamma_{k}\right) \sigma_{2 k}^{2} \tag{22}
\end{equation*}
$$

Proof. Set

$$
\overline{\mathfrak{B}}=\left[\begin{array}{cccc}
\sum_{i \neq 1}^{\bar{\beta}_{1 i}} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n 1}  \tag{23}\\
-\bar{\beta}_{12} & \sum_{i \neq 2} \bar{\beta}_{2 i} & \cdots & -\bar{\beta}_{n 2} \\
\vdots & \vdots & \ddots & \vdots \\
-\bar{\beta}_{1 n} & \bar{\beta}_{2 n} & \cdots & \sum_{i \neq n} \bar{\beta}_{n i}
\end{array}\right]
$$

and $\bar{\beta}_{k j}=\beta_{k j} S_{k}^{*} I_{j}^{*}, \bar{\beta}_{k j}>0,1 \leqslant k, j \leqslant n$.
Note that $\overline{\mathfrak{B}}$ is the Laplacian matrix of the matrix $\left(\bar{\beta}_{k j}\right)_{n \times n}$ (see Lemma 1). Since $\beta_{k j}$ is irreducible, the matrices $\left(\bar{\beta}_{k j}\right)_{n \times n}$ and $\overline{\mathfrak{B}}$ are also irreducible. Let $C_{k j}$ denote the cofactor of the $(k, j)$ entry of $\overline{\mathfrak{B}}$. We know that the system $\overline{\mathfrak{B}} \ell=0$ has a positive solution $\ell=\left(\ell_{1}, \ell_{2}, \ldots, \ell_{n}\right)$, where $\ell_{k}=C_{k k}>0$ for $k=1,2, \ldots, n$, by Lemma 1 .

It is easy to see that we only need to prove the zero solution of (14) is stochastically asymptotically stable in the large. Let $\mathbf{x}_{k}(t)=\left(\mathbf{u}_{k}(t), \mathbf{v}_{k}(t), \mathbf{w}_{k}(t)\right)^{T} \in R_{+}^{3}, k=1, \ldots, n$ and $\mathbf{x}(t)=$
$\left(\mathbf{x}_{1}(t), \ldots, \mathbf{x}_{n}(t)\right)^{T} \in R_{+}^{3 n}$. We define the Lyapunov function $V(\mathbf{x}(t))$ as follows:

$$
\begin{equation*}
V(\mathbf{x})=\frac{1}{2} \sum_{k=1}^{n}\left(a_{k} \mathbf{v}_{k}^{2}+b_{k}\left(\mathbf{u}_{k}+\mathbf{v}_{k}\right)^{2}+c_{k} \mathbf{w}_{k}^{2}\right) \tag{24}
\end{equation*}
$$

where $a_{k}>0, b_{k}>0, c_{k}>0$ are real positive constants to be chosen later. Then it can be described as the quadratic form

$$
\begin{equation*}
V(\mathbf{x})=\frac{1}{2} \sum_{k=1}^{n} \mathbf{x}_{k}^{T} Q \mathbf{x}_{k}, \tag{25}
\end{equation*}
$$

where

$$
Q=\left(\begin{array}{ccc}
b_{k} & b_{k} & 0  \tag{26}\\
b_{k} & a_{k}+b_{k} & 0 \\
0 & 0 & c_{k}
\end{array}\right)
$$

is a symmetric positive-definite matrix. So it is obvious that $V(\mathbf{x})$ is positive-definite decrescent radially unbounded. For the sake of simplicity, (24) may be divided into three functions: $V(\mathbf{x})=V_{1}(\mathbf{x})+V_{2}(\mathbf{x})+V_{3}(\mathbf{x})$, where

$$
\begin{gather*}
V_{1}(\mathbf{x})=\frac{1}{2} a_{k} \mathbf{v}_{k}^{2}, \quad V_{2}(\mathbf{x})=\frac{1}{2} b_{k}\left(\mathbf{u}_{k}+\mathbf{v}_{k}\right)^{2}, \\
V_{3}(\mathbf{x})=\frac{1}{2} c_{k} \mathbf{w}_{k}^{2} . \tag{27}
\end{gather*}
$$

Using Itô's formula and (5), we compute

$$
\begin{aligned}
L V_{1}=\sum_{k=1}^{n} a_{k} \mathbf{v}_{k}( & \sum_{j=1}^{n} \beta_{k j} s_{k}^{*} \mathbf{v}_{j}+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{j} \\
& \left.+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} I_{j}^{*}-\left(d_{k}^{I}+\gamma_{k}\right) \mathbf{v}_{k}\right)+\frac{1}{2} \sum_{k=1}^{n} a_{k} \sigma_{2 k}^{2} \mathbf{v}_{k}^{2}
\end{aligned}
$$

$$
=\sum_{k=1}^{n} a_{k} \mathbf{v}_{k}\left(\sum_{j=1}^{n} \beta_{k j} S_{k}^{*} \mathbf{v}_{j}+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{j}\right.
$$

$$
\left.+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} I_{j}^{*}-\frac{\sum_{j=1}^{n} \beta_{k j} s_{k}^{*} I_{j}^{*}}{I_{k}^{*}} \mathbf{v}_{k}\right)
$$

$$
+\frac{1}{2} \sum_{k=1}^{n} a_{k} \sigma_{2 k}^{2} \mathbf{v}_{k}^{2}
$$

$$
=\sum_{k=1}^{n} a_{k}\left(\sum_{j=1}^{n} \beta_{k j} S_{k}^{*} I_{k}^{*} I_{j}^{*} \frac{\mathbf{v}_{k}}{I_{k}^{*}} \frac{\mathbf{v}_{j}}{I_{j}^{*}}-\sum_{j=1}^{n} \beta_{k j} s_{k}^{*} I_{k}^{*} I_{j}^{*}\left(\frac{\mathbf{v}_{k}}{I_{k}^{*}}\right)^{2}\right)
$$

$$
+\sum_{k=1}^{n} a_{k}\left(\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j}+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} I_{j}^{*}\right)
$$

$$
+\frac{1}{2} \sum_{k=1}^{n} a_{k} \sigma_{2 k}^{2} \mathbf{v}_{k}^{2}
$$

$$
=\sum_{k=1}^{n} a_{k}\left(\sum_{j=1}^{n} \bar{\beta}_{k j} I_{k}^{*} \frac{\mathbf{v}_{k}}{I_{k}^{*}} \frac{\mathbf{v}_{j}}{I_{j}^{*}}-\sum_{j=1}^{n} \bar{\beta}_{k j} I_{k}^{*}\left(\frac{\mathbf{v}_{k}}{I_{k}^{*}}\right)^{2}\right)
$$

$$
+\sum_{k=1}^{n} a_{k}\left(\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j}+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} I_{j}^{*}\right)
$$

$$
+\frac{1}{2} \sum_{k=1}^{n} a_{k} \sigma_{2 k}^{2} \mathbf{v}_{k}^{2}
$$

$$
\leqslant \sum_{k=1}^{n} a_{k}\left(\frac{1}{2} \sum_{j=1}^{n} \bar{\beta}_{k j} I_{k}^{*}\left(\left(\frac{\mathbf{v}_{k}}{I_{k}^{*}}\right)^{2}+\left(\frac{\mathbf{v}_{j}}{I_{j}^{*}}\right)^{2}\right)\right.
$$

$$
\left.-\sum_{j=1}^{n} \bar{\beta}_{k j} J_{k}^{*}\left(\frac{\mathbf{v}_{k}}{I_{k}^{*}}\right)^{2}\right)
$$

$$
+\sum_{k=1}^{n} a_{k}\left(\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j}+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} I_{j}^{*}\right)+\frac{1}{2} \sum_{k=1}^{n} a_{k} \sigma_{2 k}^{2} \mathbf{v}_{k}^{2}
$$

$$
=\frac{1}{2}\left(\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \bar{\beta}_{k j} I_{k}^{*}\left(\frac{\mathbf{v}_{j}}{I_{j}^{*}}\right)^{2}-\sum_{k=1}^{n} \sum_{j=1}^{n} \bar{\beta}_{k j} I_{k}^{*}\left(\frac{\mathbf{v}_{k}}{I_{k}^{*}}\right)^{2}\right)
$$

$$
\begin{equation*}
+\sum_{k=1}^{n} a_{k}\left(\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j}+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} I_{j}^{*}\right)+\frac{1}{2} \sum_{k=1}^{n} a_{k} \sigma_{2 k}^{2} \mathbf{v}_{k}^{2} . \tag{28}
\end{equation*}
$$

Let $a_{k}=C_{k k} / I_{k}^{*}$, so $\ell_{k}=a_{k} I_{k}^{*}$. It follows from $\mathfrak{B} \ell=0$ and $\bar{\beta}_{j k}=\beta_{j k} S_{j}^{*} I_{k}^{*}$ that

$$
\begin{equation*}
\sum_{j=1}^{n} \bar{\beta}_{j k} \ell_{j}=\sum_{k=1}^{n} \bar{\beta}_{k j} \ell_{k} \tag{29}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \bar{\beta}_{k j} I_{k}^{*}\left(\frac{\mathbf{w}_{j}}{I_{j}^{*}}\right)^{2}=\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \bar{\beta}_{k j} I_{k}^{*}\left(\frac{\mathbf{w}_{k}}{I_{k}^{*}}\right)^{2} \tag{30}
\end{equation*}
$$

Hence, inequality (28) becomes

$$
\begin{align*}
L V_{1} \leqslant & \sum_{k=1}^{n} a_{k}\left(\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j}+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} I_{j}^{*}\right)  \tag{31}\\
& +\frac{1}{2} \sum_{k=1}^{n} a_{k} \sigma_{2 k}^{2} \mathbf{v}_{k}^{2}
\end{align*}
$$

Similarly, from Itô's formula, we obtain

$$
\begin{align*}
L V_{2}= & \sum_{k=1}^{n} b_{k}\left(\mathbf{u}_{k}+\mathbf{v}_{k}\right)\left(-d_{k}^{S} \mathbf{u}_{k}-\left(d_{k}^{I}+\gamma_{k}\right) \mathbf{v}_{k}+\delta_{k} \mathbf{w}_{k}\right) \\
& +\frac{1}{2} \sum_{k=1}^{n} b_{k}\left(\sigma_{1 k}^{2} \mathbf{u}_{k}^{2}+\sigma_{2 k}^{2} \mathbf{v}_{k}^{2}\right) \\
= & -\sum_{k=1}^{n} b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2} \\
& -\sum_{k=1}^{n} b_{k}\left(d_{k}^{I}+\gamma_{k}-\frac{1}{2} \sigma_{2 k}^{2}\right) \mathbf{v}_{k}^{2} \\
& -\sum_{k=1}^{n} b_{k}\left(d_{k}^{S}+d_{k}^{I}+\gamma_{k}\right) \mathbf{u}_{k} \mathbf{v}_{k}  \tag{32}\\
& +\sum_{k=1}^{n} b_{k} \delta_{k}\left(\mathbf{u}_{k}+\mathbf{v}_{k}\right) \mathbf{w}_{k}, \\
L V_{3}= & \sum_{k=1}^{n} c_{k} \mathbf{w}_{k}\left(\gamma_{k} \mathbf{v}_{k}-\left(d_{k}^{R}+\delta_{k}\right) \mathbf{w}_{k}\right) \\
& +\frac{1}{2} \sum_{k=1}^{n} c_{k} \sigma_{3 k}^{2} \mathbf{w}_{k}^{2} \\
= & -\sum_{k=1}^{n} c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right) \mathbf{w}_{k}^{2}+\sum_{k=1}^{n} c_{k} \gamma_{k} \mathbf{v}_{k} \mathbf{w}_{k}
\end{align*}
$$

Moreover, using Cauchy inequality, we can obtain

$$
\begin{gathered}
c_{k} \gamma_{k} \mathbf{v}_{k} \mathbf{w}_{k} \leqslant \\
\frac{1}{2} c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right) \mathbf{w}_{k}^{2} \\
+\frac{c_{k} \gamma_{k}^{2}}{2\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)} \mathbf{v}_{k}^{2}, \\
b_{k} \delta_{k} \mathbf{u}_{k} \mathbf{w}_{k} \leqslant \\
\frac{1}{2} b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2} \\
\\
+\frac{b_{k} \delta_{k}^{2}}{2\left(d_{k}^{S}-(1 / 2) \sigma_{1 k}^{2}\right)} \mathbf{w}_{k}^{2}, \\
b_{k} \delta_{k} \mathbf{v}_{k} \mathbf{w}_{k} \leqslant+\frac{b_{k}}{4 \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}} M_{k} \mathbf{v}_{k}^{2}+\frac{b_{k} \delta_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{M_{k}} \mathbf{w}_{k}^{2},
\end{gathered}
$$

where

$$
\begin{align*}
M_{k}= & 2\left(d_{k}^{I}+\gamma_{k}\right) \sum_{j=1}^{n} \beta_{k j} I_{j}^{*} \\
& -\left(\sum_{j=1}^{n} \beta_{k j} I_{j}^{*}+d_{k}^{S}+d_{k}^{I}+\gamma_{k}\right) \sigma_{2 k}^{2} . \tag{34}
\end{align*}
$$

Hence, we can calculate

$$
\begin{aligned}
& L V=L V_{1}+L V_{2}+L V_{3} \\
&=\sum_{k=1}^{n}\left\{a_{k}\left(\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j}+\sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} I_{j}^{*}\right)\right. \\
&+\frac{1}{2} a_{k} \sigma_{2 k}^{2} \mathbf{v}_{k}^{2}-b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2} \\
&-b_{k}\left(d_{k}^{I}+\gamma_{k}-\frac{1}{2} \sigma_{2 k}^{2}\right) \mathbf{v}_{k}^{2}-b_{k}\left(d_{k}^{S}+d_{k}^{I}+\gamma_{k}\right) \mathbf{u}_{k} \mathbf{v}_{k} \\
&+b_{k} \delta_{k}\left(\mathbf{u}_{k}+\mathbf{v}_{k}\right) \mathbf{w}_{k}-c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right) \mathbf{w}_{k}^{2} \\
&\left.+c_{k} \gamma_{k} \mathbf{v}_{k} \mathbf{w}_{k}\right\} \\
& \leqslant \sum_{k=1}^{n}\left\{\begin{array}{l}
a_{k} \sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} I_{j}^{*}+\frac{1}{2} a_{k} \sigma_{2 k}^{2} \mathbf{v}_{k}^{2} \\
\end{array}\right.-b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2}-b_{k}\left(d_{k}^{I}+\gamma_{k}-\frac{1}{2} \sigma_{2 k}^{2}\right) \mathbf{v}_{k}^{2}
\end{aligned}
$$

$$
\begin{aligned}
& -b_{k}\left(d_{k}^{S}+d_{k}^{I}+\gamma_{k}\right) \mathbf{u}_{k} \mathbf{v}_{k}-c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right) \mathbf{w}_{k}^{2} \\
& +\frac{1}{2} c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right) \mathbf{w}_{k}^{2} \\
& +\frac{c_{k} \gamma_{k}^{2}}{2\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)} \mathbf{v}_{k}^{2} \\
& +\frac{1}{2} b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2}+\frac{b_{k} \delta_{k}^{2}}{2\left(d_{k}^{S}-(1 / 2) \sigma_{1 k}^{2}\right)} \mathbf{w}_{k}^{2} \\
& \left.+\frac{b_{k}}{4 \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}} M_{k} \mathbf{v}_{k}^{2}+\frac{b_{k} \delta_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{M_{k}} \mathbf{w}_{k}^{2}\right\}
\end{aligned}
$$

$$
+\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j}
$$

$$
=\sum_{k=1}^{n}\left\{-\frac{1}{2} b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2}\right.
$$

$$
+\left(a_{k} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}-b_{k}\left(d_{k}^{S}+d_{k}^{I}+\gamma_{k}\right)\right) \mathbf{u}_{k} \mathbf{v}_{k}
$$

$$
+\frac{1}{2} a_{k} \sigma_{2 k}^{2} v_{k}^{2}
$$

$$
-b_{k}\left(d_{k}^{I}+\gamma_{k}-\frac{1}{2} \sigma_{2 k}^{2}\right) \mathbf{v}_{k}^{2}+\frac{b_{k}}{4 \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}} M_{k} \mathbf{v}_{k}^{2}
$$

$$
+\frac{c_{k} \gamma_{k}^{2}}{2\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)} v_{k}^{2}
$$

$$
-\frac{1}{2} c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right) \mathbf{w}_{k}^{2}
$$

$$
\left.+\frac{b_{k} \delta_{k}^{2}}{2\left(d_{k}^{S}-(1 / 2) \sigma_{1 k}^{2}\right)} \mathbf{w}_{k}^{2}+\frac{b_{k} \delta_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{M_{k}} \mathbf{w}_{k}^{2}\right\}
$$

$$
\begin{equation*}
+\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j} . \tag{35}
\end{equation*}
$$

Set

$$
\begin{equation*}
a_{k}=\frac{\left(d_{k}^{S}+d_{k}^{I}+\gamma_{k}\right)}{\sum_{j=1}^{n} \beta_{k j} j_{j}^{I}} b_{k} ; \tag{36}
\end{equation*}
$$

then we have

$$
\begin{gathered}
L V \leqslant \sum_{k=1}^{n}\left\{\begin{array}{c}
-\frac{1}{2} b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2} \\
-\frac{b_{k}}{2 \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}
\end{array}\right.
\end{gathered}
$$

$$
\begin{align*}
& \times\left(2\left(d_{k}^{I}+\gamma_{k}\right) \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}\right. \\
& \left.-\left(\sum_{j=1}^{n} \beta_{k j} I_{j}^{*}+d_{k}^{S}+d_{k}^{I}+\gamma_{k}\right) \sigma_{2 k}^{2}\right) \mathbf{v}_{k}^{2} \\
& +\frac{b_{k}}{4 \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}} M_{k} \mathbf{v}_{k}^{2}+\frac{c_{k} \gamma_{k}^{2}}{2\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)} \mathbf{v}_{k}^{2} \\
& -\frac{1}{2} c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right) \mathbf{w}_{k}^{2} \\
& +\frac{b_{k} \delta_{k}^{2}}{2\left(d_{k}^{S}-(1 / 2) \sigma_{1 k}^{2}\right)} \mathbf{w}_{k}^{2} \\
& \left.+\frac{b_{k} \delta_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{M_{k}} \mathbf{w}_{k}^{2}\right\}+\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j} \\
& =\sum_{k=1}^{n}\left\{-\frac{1}{2} b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2}-\frac{b_{k}}{4 \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}} M_{k} \mathbf{v}_{k}^{2}\right. \\
& +\frac{c_{k} \gamma_{k}^{2}}{2\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)} \mathbf{v}_{k}^{2} \\
& -\frac{1}{2} c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right) \mathbf{w}_{k}^{2} \\
& \left.+\frac{b_{k} \delta_{k}^{2}}{2\left(d_{k}^{S}-(1 / 2) \sigma_{1 k}^{2}\right)} \mathbf{w}_{k}^{2}+\frac{b_{k} \delta_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{M_{k}} \mathbf{w}_{k}^{2}\right\} \\
& +\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j} \\
& =\sum_{k=1}^{n}\left\{-\frac{1}{2} b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2}\right. \\
& -\left(\frac{b_{k}}{4 \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}} M_{k}\right. \\
& \left.-\frac{c_{k} \gamma_{k}^{2}}{2\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)}\right) \mathbf{v}_{k}^{2} \\
& -\left(\frac{1}{2} c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right)-\frac{b_{k} \delta_{k}^{2}}{2\left(d_{k}^{S}-(1 / 2) \sigma_{1 k}^{2}\right)}\right. \\
& \left.\left.-\frac{b_{k} \delta_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{M_{k}}\right) \mathbf{w}_{k}^{2}\right\}+\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j} \\
& =L V_{0}+\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j}, \tag{37}
\end{align*}
$$

where

$$
\begin{align*}
L V_{0}=\sum_{k=1}^{n}\{ & -\frac{1}{2} b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right) \mathbf{u}_{k}^{2} \\
& -\left(\frac{b_{k}}{4 \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}} M_{k}\right. \\
& \left.-\frac{c_{k} \gamma_{k}^{2}}{2\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)}\right) \mathbf{v}_{k}^{2} \\
& -\left(\frac{1}{2} c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right)-\frac{b_{k} \delta_{k}^{2}}{2\left(d_{k}^{S}-(1 / 2) \sigma_{1 k}^{2}\right)}\right. \\
& \left.\left.\quad-\frac{b_{k} \delta_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{M_{k}}\right) \mathbf{w}_{k}^{2}\right\} . \tag{38}
\end{align*}
$$

We let

$$
\begin{gather*}
\mathscr{A}_{k}=\frac{1}{2} b_{k}\left(d_{k}^{S}-\frac{1}{2} \sigma_{1 k}^{2}\right), \\
\mathscr{B}_{k}= \\
\frac{b_{k}}{4 \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}} M_{k}-\frac{c_{k} \gamma_{k}^{2}}{2\left(d_{k}^{R}+\delta_{k}-(1 / 2) \sigma_{3 k}^{2}\right)},  \tag{39}\\
\mathscr{C}_{k}= \\
\frac{1}{2} c_{k}\left(d_{k}^{R}+\delta_{k}-\frac{1}{2} \sigma_{3 k}^{2}\right)-\frac{b_{k} \delta_{k}^{2}}{2\left(d_{k}^{S}-(1 / 2) \sigma_{1 k}^{2}\right)} \\
\\
-\frac{b_{k} \delta_{k}^{2} \sum_{j=1}^{n} \beta_{k j} I_{j}^{*}}{M_{k}} .
\end{gather*}
$$

The proofs above show that if the condition (21) is satisfied, $\mathscr{A}_{k}, \mathscr{B}_{k}$, and $\mathscr{C}_{k}$ are positive constants. Let $\lambda=$ $\min _{k \in\{1, \ldots, n\}}\left\{\mathscr{A}_{k}, \mathscr{B}_{k}, \mathscr{C}_{k}\right\}$; then $\lambda>0$. From (37) and (38), one sees that

$$
\begin{align*}
L V \leqslant & L V_{0}+\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j} \\
= & -\sum_{k=1}^{n}\left(\mathscr{A}_{k} \mathbf{u}_{k}^{2}+\mathscr{B}_{k} \mathbf{v}_{k}^{2}+\mathscr{C}_{k} \mathbf{w}_{k}^{2}\right) \\
& +\sum_{k=1}^{n} a_{k} \sum_{j=1}^{n} \beta_{k j} \mathbf{u}_{k} \mathbf{v}_{k} \mathbf{v}_{j}  \tag{40}\\
= & -\lambda \sum_{k=1}^{n}\left(\left|\mathbf{x}_{k}(t)\right|^{2}+o\left(\left|\mathbf{x}_{k}(t)\right|^{2}\right)\right) \\
= & -\lambda|\mathbf{x}(t)|^{2}+o\left(|\mathbf{x}(t)|^{2}\right),
\end{align*}
$$

where $\left|\mathbf{x}_{k}(t)\right|=\sqrt{\mathbf{u}_{k}^{2}(t)+\mathbf{v}_{k}^{2}(t)+\mathbf{w}_{k}^{2}(t)}, \quad|\mathbf{x}(t)|=$ $\left(\sum_{k=1}^{n}\left|\mathbf{x}_{k}(t)\right|\right)^{1 / 2}$ and $o\left(|\mathbf{x}(t)|^{2}\right)$ is an infinitesimal of higher order of $|\mathbf{x}(t)|^{2}$ for $t \rightarrow \infty$. Hence, $L V(\mathbf{x}, t)$ is negativedefinite for $t \geqslant 0$. According to Lemma 4, we therefore conclude that the zero solution of (12) is stochastically asymptotically stable in the large. The proof is complete.

## 5. Numerical Simulation

Numerical methods are employed to solve the system (1) and (12) and to depict the behavior of the susceptible, infectious, and recovered nodes with respect to time. We numerically simulate the solution of system (1) and (12) with $n=2$. In this case, we have

$$
\begin{align*}
& M_{0}=\left[\begin{array}{ll}
\frac{\beta_{11}\left(\Lambda_{1} / d_{1}^{S}\right)}{d_{1}^{I}+\gamma_{1}} & \frac{\beta_{12}\left(\Lambda_{1} / d_{1}^{S}\right)}{d_{1}^{I}+\gamma_{1}} \\
\frac{\beta_{21}\left(\Lambda_{2} / d_{2}^{S}\right)}{d_{2}^{I}+\gamma_{2}} & \frac{\beta_{22}\left(\Lambda_{2} / d_{2}^{S}\right)}{d_{2}^{I}+\gamma_{2}}
\end{array}\right]:=\left[\begin{array}{ll}
\beta_{11} K_{1} & \beta_{12} K_{1} \\
\beta_{21} K_{2} & \beta_{22} K_{2}
\end{array}\right], \\
& \mathscr{R}_{0} \\
& =\rho\left(M_{0}\right) \\
& =\frac{\beta_{11} K_{1}+\beta_{22} K_{2}+\sqrt{\left(\beta_{11} K_{1}-\beta_{22} K_{2}\right)^{2}+4 \beta_{12} \beta_{21} K_{1} K_{2}}}{2} . \tag{41}
\end{align*}
$$

Let

$$
\begin{gather*}
\Lambda_{1}=2.5, \quad \beta_{11}=0.55, \quad \beta_{12}=0.35, \quad d_{1}^{S}=0.15 \\
d_{1}^{I}=0.2, \quad d_{1}^{R}=0.25, \quad \delta_{1}=0.01, \quad \gamma_{1}=0.4 \\
\Lambda_{2}=1.0, \quad \beta_{21}=0.5, \quad \beta_{22}=0.2, \quad d_{2}^{S}=0.1, \quad d_{2}^{I}=0.15 \\
d_{2}^{R}=0.2, \quad \delta_{2}=0.012, \quad \gamma_{2}=0.15 . \tag{42}
\end{gather*}
$$

Hence we obtain $S_{1}^{*}=0.7205, I_{1}^{*}=4.0914, R_{1}^{*}=6.2945$, $S_{2}^{*}=0.3663, I_{2}^{*}=3.3048, R_{2}^{*}=2.3383$. We can calculate

$$
\begin{gather*}
\mathscr{R}_{0}=\frac{250}{9}>1, \quad d_{1}^{S} S_{1}^{*}-\delta_{1} R_{1}^{*}=0.04513>0  \tag{43}\\
d_{2}^{S} S_{2}^{*}-\delta_{2} R_{2}^{*}=0.0085704>0
\end{gather*}
$$

In the absence of noise, we simulate the global stability of the endemic equilibrium of deterministic system (1) in Figure 1, and we always choose initial value $S_{1}(0)=2.0, I_{1}(0)=4.0$, $R_{1}(0)=6.0, S_{2}(0)=9.0, I_{2}(0)=4.0$, and $R_{2}(0)=0.5$.

Next, we consider the effect of stochastic fluctuations of environment to the endemic equilibrium of the corresponding deterministic system. Given the discretization of system (12) for $t=0, \Delta t, 2 \Delta t, \ldots, n \Delta t$, and $k=1,2$,

$$
\begin{align*}
S_{k, i+1}= & S_{k, i}+\left(\Lambda_{k}-d_{k}^{S} S_{k, i}-\beta_{k 1} S_{k, i} I_{1, i}\right. \\
& \left.-\beta_{k 2} S_{k, i} I_{2, i}+\delta_{k} R_{k, i}\right) \Delta t \\
& +\sigma_{1 k}\left(S_{k, i}-S_{k}^{*}\right) \sqrt{\Delta t} \varepsilon_{1 k, i}, \\
I_{k, i+1}= & I_{k, i}+\left(\beta_{k 1} S_{k, i} I_{1, i}+\beta_{k 2} S_{k, i} I_{2, i}-\left(d_{k}^{I}+\gamma_{k}\right) I_{k, i}\right) \Delta t \\
& +\sigma_{2 k}\left(I_{k, i}-I_{k}^{*}\right) \sqrt{\Delta t} \varepsilon_{2 k, i} \\
R_{k, i+1}= & R_{k, i}+\left(\gamma_{k} I_{k, i}-\left(d_{k}^{I}+\delta_{k}\right) R_{k, i}\right) \Delta t \\
& +\sigma_{3 k}\left(R_{k, i}-R_{k}^{*}\right) \sqrt{\Delta t} \varepsilon_{3 k, i}, \tag{44}
\end{align*}
$$

where time increment $\Delta t>0$, and $\varepsilon_{1 k, i}, \varepsilon_{2 k, i}, \varepsilon_{3 k, i}$ are $N(0,1)$-distributed independent random variables which can be generated numerically by pseudorandom number generators.

As mentioned above, there is an endemic equilibrium $P^{*}=\left(S_{1}^{*}, I_{1}^{*}, R_{1}^{*}, S_{2}^{*}, I_{2}^{*}, R_{2}^{*}\right)$ of system (1) when $\mathscr{R}_{0}>1$, where

$$
\begin{array}{lll}
S_{1}^{*}=0.7205, & I_{1}^{*}=4.0914, & R_{1}^{*}=6.2945 \\
S_{2}^{*}=0.3663, & I_{2}^{*}=3.3048, & R_{2}^{*}=2.3383 \tag{45}
\end{array}
$$

Figure 2 corresponds to $\sigma_{11}=0.4, \sigma_{21}=0.73, \sigma_{31}=0.45$, $\sigma_{12}=0.5, \sigma_{22}=0.67$, and $\sigma_{32}=0.5$; the numerical simulation shows that the endemic equilibrium of stochastic system (12) is global asymptotically stable under the condition (21), while Figure 3 corresponds to $\sigma_{11}=0.7, \sigma_{21}=1.2, \sigma_{31}=1.3$, $\sigma_{12}=0.8, \sigma_{22}=0.8$, and $\sigma_{32}=1.4$. Moreover, comparison of Figures 2 and 3 suggests that fluctuations enhance as the noise level increases. Note that the condition (21) is just a sufficient condition. When this condition is not satisfied, the stochastic system (12) may be or may not be stable. For instance, the intensities of Brownian motions $\sigma_{11}=0.7$, $\sigma_{21}=1.2, \sigma_{31}=1.3, \sigma_{12}=0.8, \sigma_{22}=0.8$, and $\sigma_{32}=$ 1.4 do not meet the condition (21), but we can see from Figure 3 that the stochastic system (12) is still asymptotically stable. If we choose $\sigma_{11}=40.5, \sigma_{21}=32.73, \sigma_{31}=36.92$, $\sigma_{12}=33.4, \sigma_{22}=39.67$, and $\sigma_{32}=30.5$, then the solution of the stochastic system (12) is not asymptotically stable but explodes to infinity at the finite time (see Figure 4).

The relationships between $S_{k}$ and $R_{k}$ are also observed and are depicted in Figures 5 and 6. Note that these two curves of Figure 5 show that the number of susceptible individuals sharply declines to near $S_{k}^{*}$ as the number of the recovery individuals increases slightly and then increases with the number of the increasing recovered individuals gently and goes on increasing with the number of the decreasing recovered individuals, and finally both reach the steady state values. For the stochastic version, Figure 6 corresponds to $\sigma_{11}=0.4$, $\sigma_{21}=0.73, \sigma_{31}=0.45, \sigma_{12}=0.5, \sigma_{22}=0.67$, and $\sigma_{32}=0.5$; oscillation appears under environmental driving forces which

Table 1: Values of $\left(S_{1}^{*}, R_{1}^{*}\right)$, when fixing $\delta_{2}=0.012$ and changing $\delta_{1}$.

| $\delta_{1}=0.1$ | $\delta_{1}=0.025$ | $\delta_{1}=0.02$ | $\delta_{1}=0.01$ |
| :--- | :---: | :---: | :---: |
| $S_{1}^{*}=0.7627$ | $S_{1}^{*}=0.7290$ | $S_{1}^{*}=0.7263$ | $S_{1}^{*}=0.7205$ |
| $R_{1}^{*}=5.6130$ | $R_{1}^{*}=6.1694$ | $R_{1}^{*}=6.2105$ | $R_{1}^{*}=6.2945$ |

Table 2: Values of $\left(S_{2}^{*}, R_{2}^{*}\right)$, when fixing $\delta_{1}=0.01$ and changing $\delta_{2}$.

| $\delta_{2}=0.35$ | $\delta_{2}=0.25$ | $\delta_{2}=0.15$ | $\delta_{2}=0.012$ |
| :--- | :---: | :---: | :---: |
| $S_{2}^{*}=0.4678$ | $S_{2}^{*}=0.4502$ | $S_{2}^{*}=0.4254$ | $S_{2}^{*}=0.3663$ |
| $R_{2}^{*}=1.2710$ | $R_{2}^{*}=1.4692$ | $R_{2}^{*}=1.7408$ | $R_{2}^{*}=2.3383$ |

actually affect the deterministic curves shown in Figure 5. But Figure 6 still maintains the same total trend as Figure 5 and reaches the same equilibrium point as deterministic version. Simulation result agrees with the real life situation.

Figure 7 and Tables 1 and 2 show that when the rate of immunity loss $\delta_{k}$ gradually reduces, $S_{k}^{*}$ decreases, but $R_{k}^{*}$ increases; the lower the rate of immunity loss $\delta_{k}$ is, the lower the steady state value of the susceptible is, the higher the steady state value of the recovered is. Thus, it will be of great importance for health management to control the rate of immunity loss to keep it in a lower level; for example, when antibody concentrations of recovered individuals are at a low level or are zero, they can be required to undergo vaccination to achieve the protective antibody levels.

## 6. Conclusion

This paper presented a mathematical study describing the dynamical behavior of an SIRS epidemic model with stochastic perturbations. Our purpose was based on analyzing this behavior using a stochastic model. When the reproduction number $\mathscr{R}_{0}$ is greater than one, we obtained sufficient conditions for stochastic stability of the endemic equilibrium $P^{*}$ by using a suitable Lyapunov function and other techniques of stochastic analysis. The investigation of this stochastic model revealed that the stochastic stability of $P^{*}$ depends on the magnitude of the intensity of noise as well as the parameters involved within the model system. Finally, numerical simulations are given to validate the theoretical results. The proposed model, a more accurate epidemic model, can help us to understand the dynamic behavior of virus. Moreover, the theoretical results may provide some useful guidance for making effective countermeasures on virus propagation.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Error Analysis of Some Demand Simplifications in Hydraulic Models of Water Supply Networks 

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#### Abstract

Mathematical modeling of water distribution networks makes use of simplifications aimed to optimize the development and use of the mathematical models involved. Simplified models are used systematically by water utilities, frequently with no awareness of the implications of the assumptions used. Some simplifications are derived from the various levels of granularity at which a network can be considered. This is the case of some demand simplifications, specifically, when consumptions associated with a line are equally allocated to the ends of the line. In this paper, we present examples of situations where this kind of simplification produces models that are very unrealistic. We also identify the main variables responsible for the errors. By performing some error analysis, we assess to what extent such a simplification is valid. Using this information, guidelines are provided that enable the user to establish if a given simplification is acceptable or, on the contrary, supplies information that differs substantially from reality. We also develop easy to implement formulae that enable the allocation of inner line demand to the line ends with minimal error; finally, we assess the errors associated with the simplification and locate the points of a line where maximum discrepancies occur.


## 1. Introduction

In the task of mathematical modeling of such complex structures as water distribution networks (WDNs) the use of simplifications aimed to optimize the development and use of the models is unavoidable. Such simplifications stem from the complexity of the modeled infrastructure and, at the same time, are related to the large spatial distribution typical of WDNs. These models are applied in all the areas of hydraulics-including urban hydraulics [1, 2]. Currently, with the generalized use of geographic information systems, models containing even hundreds of thousands of pipes and nodes are being built [3].

Extremely detailed modeling of real WDNs, even under the unrealistic hypothesis in which uncertainty can be ignored, produces a substantial amount of data and requires sophisticated computational tools and mechanisms to reliably interpret the obtained results in terms of what occurs in the system. Current computational power can be used
to build hydraulic simulation models capable of providing a very detailed and accurate model but not an improved understanding of the main structure of the underlying system. Also, as such models would require very dedicated calculations, certain aspects must be considered to ensure efficient implementations. Specifically, optimization of WDNs, whether used for planning or operational purposes, often requires many iterations [4], each involving computationally expensive simulations and huge computer memory.

Let us mention just a few of the most typical simplifications: the use of one equivalent pipe to represent two or more parallel or series pipes; the removal of short pipe segments including dead ends, service connections, and hydrants; the distribution of emitter exponents without considering leakage profiles along the WDN; the assumption of friction factor values without a detailed consideration of the pipes state; model calibration assuming values of one or more of the calibration elements as fixed; the use of a single friction factor for the entire WDN or for an entire sector; or the use of
a single emitter leakage exponent for the entire network. The last two simplifications, in particular, are inescapable when working with EPANET [5], a tool of water network analysis in general use worldwide. Also, some simplifications can totally undermine studies of hydraulic transient phenomena in WDNs. See, for example, [6, 7].

Among the various simplifications used in the analysis of water distribution systems some are assumed in a generalized way without additional introspection. They are simplifications derived from the different levels of granularity at which a network can be considered. As mentioned before, real water distribution networks, especially those of large cities, cannot be efficiently modeled in their entirety. In approximate models some granulation reduction actions are performed such as skeletonization, grouping, pruning, and clustering. See, among others, [8-18].

In particular, one of these simplifications is the grouping of the consumptions associated with a line in one or both ends of the line. These points concentrate all the existing consumption points (users) within the line. The need for implementing this specific simplification is evident because a faithful consideration of demand would imply the inclusion of a large number of nodes equal to the number of consumption points. In a branched network with few consumers, this would not represent a major problem. However, problems arise in WDNs with up to 30 connections per line (e.g., a street pipeline). In a large WDN (e.g., a 500 km WDN) it would amount to considering about 150,000 nodes, which is impractical when it comes to the construction of the network model, the performance of the calculations, and the display and understanding of the results. This simplification copes with the continuity principle or conservation of mass. However, the energy aspects are completely ignored.

The methods for calculating demand load in the models are one of the least studied aspects despite the obvious importance for defining model reliability. Recently, Giustolisi et al. [19] have addressed this problem from a global perspective and have developed a matrix transformation approach that changes the classical solution of the nonlinear system of equations describing a WDN based on the conjugate gradient [20], into what they call enhanced global gradient algorithm (EGGA). Note that the solution in [20] is implemented within the widely available freeware EPANET 2, used in a generalized way by many engineers and practitioners around the world. EGGA reduces the size of the mathematical problem through a transformed topological representation of the original network model that preserves both mass and energy balance equations and improves the numerical stability of the solution procedure. According to the authors, EGGA significantly improves the model's computational efficiency without sacrificing its hydraulic accuracy.

However, this solution, although technically impeccable from a theoretical point of view, exhibits a number of practical drawbacks. Firstly, those matrix transformations are not implemented within EPANET and, as a result, are not available to the huge community of its users. Secondly, the transformations are too complex for most users of this program and, especially, too complex to be incorporated into the EPANET toolkit, given the in-depth knowledge of
programming techniques that their implementation would require. Thirdly, the transformation matrices must previously be explicitly written; even though obtaining these matrices is relatively straightforward, the computational efficiency, at least in terms of memory needs, is not evident. One has to use several additional (very sparse) matrices of sizes or the order of the number of resulting pipes times the number of original pipes, and the number of resulting pipes times the number of unknown head nodes, are explicitly used. When solving real-world problems with hundreds of thousands of nodes and pipes, this may become a serious problem. Fourthly, for networks already modeled by excluding those intermediate demand nodes, the solution of matrix transformation will be useless. Fifthly, when planning and designing a new network starting from the household demand distribution, it would be perhaps desirable to start building the model by performing the simplification from the outset, in order to avoid later complications.

Compelled by those drawbacks we have addressed the problem from another, we claim, more practical point of view. This paper analyzes the possible errors from the effect of using certain types of simplifications when loading demands in models, specifically, the widespread $50 \%$ rule, which allocates half of the in-line demand to each line end. We analyze to what extent this simplification is acceptable or, on the contrary, supply information that differs substantially from reality. Also, we obtain formulae that enable to allocate inner line demand to the line ends with minimal error. Finally, a calculation of the maximum head point discrepancy associated is provided.

Our proposal involves simple, direct methods that can be easily applied by any user of any WDN analysis package, since emphasis is not placed on programming ability, but on how to make a decision about the technical aspect of simplifying the model and, thus, load the demand properly. Users, having already developed models of their networks, may revise the allocation rule used and replace it, if necessary, with the values provided by the new formulae, what will enable them to obtain more reliable results. Also, users starting the model of a new network may make an a priori decision about how to simplify the network and suitably implement the associated simplifications.

We first present a simple case that enables us to shed light onto the problem: a single line with variable distribution and granulation consumption is considered. Then an example of a real network is analyzed using the lessons learned. The paper closes with a conclusions section.

## 2. Line with Associated Demand

Let us first consider the case of a single line associated with some internal consumption under steady state condition. Such a line is representative of the simplest installation (a line between two nodes) with a given inflow rate. The characteristics of the line are
(i) length: $L$,
(ii) diameter: $D$,
(iii) upstream head (boundary condition at the upstream node): $H_{0}$,
(iv) friction factor: $f$, with its associated line resistance: $K=8 f /\left(g \pi^{2} D^{5}\right)$,
(v) inflow: $Q_{\text {in }}$.

Various demand scenarios of consumption in the line may be tested. Such scenarios are associated with two characteristics:
(i) total demand in the line with regard to inflow,
(ii) specific distribution of the demand along the line.

Let us assume that the flow consumed within the line (total in-line demand) represents a percentage of the line inflow. If this fraction is represented by $F_{\mathrm{Q}}, 0<F_{\mathrm{Q}} \leq 1$, the actual demand in the line is given by the expression

$$
\begin{equation*}
Q_{d}=F_{\mathrm{Q}} Q_{\mathrm{in}} \tag{1}
\end{equation*}
$$

2.1. Uniformly Distributed Demand along the Line. To start with the study, we will assume that the actual demand of the line is uniformly distributed into a number $n$ of equally spaced interior points (nodes); $n$ can take a value ranging from 1 (in the case of a line with a single demand node in the middle) to a large integer number (in the case of an equally distributed demand throughout the line). Observe that we do not consider any demand at the end nodes, since we are only interested in the line demand associated with the interior nodes.

Figure 1 shows various distributions of piezometric head corresponding to values of $F_{Q}$ equal to $1,0.8,0.6$, and 0.4 , for a set of values of $L, D, H_{0}, f$, and $Q_{\text {in }}$. To build Figure 1 we have used the specific values $L=500 \mathrm{~m}, D=300 \mathrm{~mm}, H_{0}=50 \mathrm{~m}$, $f=0.018$, and $Q_{\mathrm{in}}=0.25 \mathrm{~m}^{3} / \mathrm{s}$. As mentioned before, the demand has been equally distributed among $n$ equally spaced interior nodes. Specifically, in Figure 1, $n$ takes the values $1,3,7,11,19$, and "infinity." The "infinity" case represents a uniform continuous demand. The various curves in Figure 1 have straightforward interpretation.

For the polygonal hydraulic grade lines (HGLs) made out of segments between consumption points, the calculations correspond to the usual hydraulic calculation of losses. The polygonals start at the boundary condition $\left(0, H_{0}\right)$, the other vertices being the $n+1$ points as follows:

$$
\begin{array}{r}
\left(j \frac{L}{n+1}, H_{0}-K \frac{L}{n+1} Q_{\text {in }}^{2} \sum_{k=1}^{j}\left(1-\frac{k-1}{n} F_{\mathrm{Q}}\right)^{2}\right)  \tag{2}\\
j=1, \ldots, n+1
\end{array}
$$

Let us call $H_{R}\left(F_{Q}, n ; x\right)$ these HGLs, the subindex " $R$ " standing for "real" distribution of piezometric head along the line, as real demands are used to calculate (2).

The calculation for the ideal HGL corresponding to a uniform continuous demand, which is used here as the limit for $n \rightarrow \infty$ of the discrete uniform distribution of demands, is performed by integrating the differential loss

$$
\begin{equation*}
\Delta H=-K\left(\mathrm{Q}_{\mathrm{in}}-\frac{\mathrm{Q}_{d}}{L} x\right)^{2} \Delta x \tag{3}
\end{equation*}
$$

along the line. The value $Q_{d} / L$ is the (constant) demand per unit length and $x$ is the distance to the upstream node. By integrating, and using (1), the piezometric head corresponding to this continuous loss is given by

$$
\begin{equation*}
H_{R}\left(F_{\mathrm{Q}} ; x\right)=H_{0}-K \frac{L}{3} \frac{Q_{\mathrm{in}}^{2}}{F_{\mathrm{Q}}}\left[1-\left(1-F_{\mathrm{Q}} \frac{x}{L}\right)^{3}\right] \tag{4}
\end{equation*}
$$

which corresponds to the upper curve, a cubic, in each of the graphs in Figure 1.

It becomes clear that the greatest discrepancies occur for values of $F_{\mathrm{Q}}$ close to 1 (e.g., when a high percentage of the inflow is consumed along the line).

As mentioned before, these "real" HGLs in Figure 1 have been calculated according to the demand distribution at the various inner points in the line. However, models of large WDNs do not usually take intermediate demands into account; in contrast, the demand of each line is allocated to the end nodes of the line, the $50 \%$ rule being generally used.
2.2. Allocation of In-Line Demand to the Line Ends: Is the 50\% Rule Adequate? Let $F_{\mathrm{Q}_{d}}$ be the factor that allocates a part of the line distributed demand, $Q_{d}$, to its upstream end. Thus, the demand assigned to this upstream node is $Q_{0}=F_{Q_{d}} Q_{d}$. As a result, $Q_{l}=Q_{\mathrm{in}}-Q_{0}$ is the flowrate through the line. Note that, using (1),

$$
\begin{equation*}
\mathrm{Q}_{l}=\mathrm{Q}_{\mathrm{in}}\left(1-F_{\mathrm{Q}} F_{\mathrm{Q}_{d}}\right) \tag{5}
\end{equation*}
$$

Then, the calculated head value $H_{C}$ for a given value of $F_{\mathrm{Q}_{d}}$ is

$$
\begin{equation*}
H_{C}\left(F_{Q_{d}} ; x\right)=H_{0}-K Q_{l}^{2} x \tag{6}
\end{equation*}
$$

The HGL obtained is, thus, a straight line that connects the point $\left(0, H_{0}\right)$ with the point $\left(L, H_{C}\left(F_{Q_{d}} ; L\right)\right)$. This last value corresponds to the calculated head at $L$, the downstream node.

In Figure 2, dashed lines have been added to the first two charts of Figure 1. These new HGLs have been calculated to give the same piezometric head at the downstream node as the line corresponding to a demand concentrated in the middle point, $n=1$, for the case $F_{\mathrm{Q}}=1$ (left chart), and as the line corresponding to a continuous demand for $F_{\mathrm{Q}}=0.8$ (right chart). These lines have been obtained by allocating a fraction of the interior line demand to the upstream node and the rest to the downstream node. Analogous dashed lines can be obtained for other combinations of $F_{Q}$ and $n$. If the allocated fractions to the line ends are different, the corresponding (straight) lines (the lines given by the numerical model with lumped demands at the ends of the line) will also be different.

Two problems arise at this point.
(a) Firstly, it would be desirable to know the best allocation of the total in-line demand to the line ends, that is to say, to know the value $F_{\mathrm{Q}_{d}}$ that solves the following problem:

$$
\begin{equation*}
\underset{F_{\mathrm{Q}_{d}}}{\operatorname{Minimize}}\left\|H_{C}\left(F_{\mathrm{Q}_{d}} ; x\right)-H_{R}\left(F_{\mathrm{Q}}, * x\right)\right\| \tag{7}
\end{equation*}
$$



Figure 1: Hydraulic grade lines in one single line for various uniform distributions of demand.



Figure 2: Examples of discrepancy between distributed and lumped demand models.
for certain functional norm $\|\cdot\|$, where $H_{C}\left(F_{Q_{d}} ; x\right)$ is calculated by (6), according to the lumped demand allocation to both line ends, and $H_{R}\left(F_{\mathrm{Q}},{ }^{*} ; x\right)$ accounts for the real demand distribution, either calculated by (2) or (4) (or any other more general expression corresponding to not uniformly distributed demand,
which we address later). The asterisk denotes other parameters, such as $n$ in (2), which may appear in the expression of $H_{R}$.
(b) Secondly, after having made a suitable line-end allocation decision, it is of interest to know the actual distribution of piezometric head errors on the line

Table 1: Values of $F_{\mathrm{Q}_{d}}$ as a function of $F_{\mathrm{Q}}$ and $n$ when solving (9).

| (order) |  |  | $F_{\mathrm{Q}}$ |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| $1(1)$ | 0.472 | 0.438 | 0.397 | 0.349 | 0.293 |
| $3(2)$ | 0.485 | 0.466 | 0.442 | 0.413 | 0.376 |
| $7(3)$ | 0.488 | 0.473 | 0.455 | 0.432 | 0.402 |
| $11(4)$ | 0.489 | 0.475 | 0.457 | 0.435 | 0.407 |
| $19(5)$ | 0.490 | 0.477 | 0.461 | 0.441 | 0.415 |
| $\inf (6)$ | 0.491 | 0.479 | 0.465 | 0.446 | 0.423 |

and, specifically, to identify at what point or points the maximum head discrepancy occurs:

$$
\begin{equation*}
\underset{x}{\operatorname{Maximize}}\left\|H_{C}\left(F_{\mathrm{Q}_{d}} ; x\right)-H_{R}\left(F_{\mathrm{Q}},{ }^{*} ; x\right)\right\| \tag{8}
\end{equation*}
$$

We start by discussing (7), and then address (8) in Section 2.4.

The solution of (7) is here constrained by the nature of the problem: we have to adhere to the fact that one or more lines (pipes) may be connected to the downstream end of the considered line. The connected lines need the correct piezometric head at $L$-upstream end for themto suitably perform their respective calculations. It means that the piezometric head at $L$, given by $H_{C}$ and $H_{R}$, must coincide. That is to say, (7) reduces, in our case, to

$$
\begin{equation*}
\text { Solve } H_{C}\left(F_{\mathrm{Q}_{d}} ; L\right)=H_{R}\left(F_{\mathrm{Q}},{ }^{*} ; L\right) \quad \text { for } F_{\mathrm{Q}_{d}} . \tag{9}
\end{equation*}
$$

By solving (9) the following expressions for $F_{\mathrm{Q}_{d}}$ are obtained.
(i) Case of continuous demand:

$$
\begin{equation*}
F_{\mathrm{Q}_{d}}^{\infty}\left(F_{\mathrm{Q}}\right)=\frac{1}{F_{\mathrm{Q}}}\left(1-\sqrt{\frac{1-\left(1-F_{\mathrm{Q}}\right)^{3}}{3 F_{\mathrm{Q}}}}\right) . \tag{10}
\end{equation*}
$$

(ii) Case of demand equally distributed among $n$ equally distributed nodes:

$$
\begin{equation*}
F_{\mathrm{Q}_{d}}^{(n)}\left(F_{\mathrm{Q}}\right)=\frac{1}{F_{\mathrm{Q}}}\left(1-\sqrt{\frac{1}{n+1} \sum_{k=1}^{n+1}\left(1-\frac{k-1}{n} F_{\mathrm{Q}}\right)^{2}}\right) \tag{11}
\end{equation*}
$$

In Table 1 and in two-dimensional Figure 3, values for (10) and (11) for $F_{Q}$ values between 0.2 and 1 and for $n$ varying along the previously used values, namely, $1,3,7,11,19$, and infinity, are presented. Note that values 1 to 6 on the frontal axis of Figure 3 symbolize, as shown in Table 1, the values $n=$ $1,3,7,11,19$, and infinity, respectively.

The following facts are remarkable.
(i) These values are independent of the problem data, namely, $H_{0}, Q_{\mathrm{in}}, L, D$, and $f$, and depend only on $F_{\mathrm{Q}}$ and $n$, in the case of (11); that is to say, they depend on the magnitude and the pattern of the distributed demand. This is a very remarkable result, since the present study thus becomes nondimensional and, as a result, completely general.


Figure 3: Solution of (9) for $F_{\mathrm{Q}_{d}}$.
(ii) The values of $F_{\mathrm{Q}_{d}}$ range from approximately 0.3 to 0.5 . This means that about $30 \%-50 \%$ of the total demand must be allocated to the upstream node and the remainder to the downstream node.
(iii) The lowest $F_{\mathrm{Q}_{d}}$ values correspond to the most awkward cases: the rate of demand is close to or equals the total inflow in the line, and the demand is highly concentrated at a few points (upper right corner of the table, right front of the figure).
(iv) The highest $F_{\mathrm{Q}_{d}}$ values, closer to $50 \%$, correspond to the less problematic cases, meaning little total distributed demand in relation to the total inflow in the line, and widely distributed demand (lower left corner of the table, bottom left of the figure). This value approaches $50 \%$ as the rate of inflow consumed in the line approaches zero. (Observe that for both (10) and (11) $\lim _{F_{\mathrm{Q}} \rightarrow 0} F_{\mathrm{Q}_{d}}^{(\cdot)}\left(F_{\mathrm{Q}}\right)=0.5$.)
(v) It is also worth noting that $\lim _{n \rightarrow \infty} F_{\mathrm{Q}_{d}}^{n}=F_{\mathrm{Q}_{d}}^{\infty}$ for all $F_{\mathrm{Q}}$; that is to say, the monotonic sequence of continuous functions $\left\{F_{\mathrm{Q}_{d}}^{n}\right\}_{n=1}^{\infty}$ converges to the continuous function $F_{\mathrm{Q}_{d}}^{\infty}$ uniformly in [0,1] (Dini theorem; see, e.g., [21, 22]). This has a direct interpretation as the continuous demand is the limit of a uniformly distributed demand among an increasing number of equally distributed points on the pipe.

As mentioned before, it is common practice in mathematical modeling of WSN engineering to distribute the line flow into two parts: $50 \%$ for the upstream node and $50 \%$ for the downstream node, which approximately coincides with what is observed in Table 1 and Figure 3, except for cases where the demand line is highly concentrated and represents a large percentage of the total flow through the line.

As a result of what has been presented so far, it can be said that, for uniformly distributed in-line demand, the usual $50 \%$ rule seems, in principle, an appropriate solution provided
that the inner demand of the line is small compared with the pipe inflow and that such a demand is widely distributed. However, equal demand allocation to the end nodes of the line may produce important discrepancies, since the study highlights the need for other assignments in certain cases.
2.3. Arbitrary Demand along the Line. To state the problem in its more general form, let us now consider a demand distribution on the line whose accumulated demand is given by a function $Q(x)=Q_{d} q(x)$, where $q(x)$ is the accumulated demand ratio, a function increasing monotonically from 0 to 1 . While $H_{C}$ is calculated as in (6), $H_{R}$ is calculated by integrating the loss $\Delta H=-K\left(Q_{\text {in }}-Q(x)\right)^{2} \Delta x$ through the line $[0, L]$ as follows:

$$
\begin{equation*}
H_{R}\left(F_{\mathrm{Q}} ; q(x) ; x\right)=H_{0}-K \int_{0}^{x}\left(Q_{\mathrm{in}}-Q(u)\right)^{2} \mathrm{~d} u \tag{12}
\end{equation*}
$$

Observe that this function is monotonically decreasing and concave upwards.

Example 1. $q(x)=x / L$, for the case of continuous uniform demand, which, by using (1), gives (4).

Using (12) and the expression (6) for $H_{C}$ in $L$, written, using (5), as

$$
\begin{equation*}
H_{C}\left(F_{\mathrm{Q}_{d}} ; L\right)=H_{0}-K L Q_{\mathrm{in}}^{2}\left(1-F_{\mathrm{Q}_{d}} F_{\mathrm{Q}}\right)^{2} \tag{13}
\end{equation*}
$$

the equation $H_{C}=H_{R}$ in $L$ may be rewritten as

$$
\begin{equation*}
-K L Q_{\mathrm{in}}^{2}\left(1-F_{\mathrm{Q}_{d}} F_{\mathrm{Q}}\right)^{2}=-K \int_{0}^{L}\left(\mathrm{Q}_{\mathrm{in}}-\mathrm{Q}(x)\right)^{2} \mathrm{~d} x \tag{14}
\end{equation*}
$$

By substituting

$$
\begin{equation*}
Q(x)=Q_{\mathrm{in}} F_{\mathrm{Q}} q(x), \tag{15}
\end{equation*}
$$

one gets

$$
\begin{equation*}
L\left(1-F_{\mathrm{Q}_{d}} F_{\mathrm{Q}}\right)^{2}=\int_{0}^{L}\left(1-F_{\mathrm{Q}} q(x)\right)^{2} \mathrm{~d} x \tag{16}
\end{equation*}
$$

from where the following expression is readily obtained:

$$
\begin{equation*}
F_{\mathrm{Q}_{d}}\left(F_{\mathrm{Q}}\right)=\frac{1}{F_{\mathrm{Q}}}\left(1-\sqrt{\frac{1}{L} \int_{0}^{L}\left(1-F_{\mathrm{Q}} q(x)\right)^{2} \mathrm{~d} x}\right) . \tag{17}
\end{equation*}
$$

The general solution for the problem at hand, when an arbitrary demand through the line is considered, may only be solved after having a specific expression for $q(x)$. As a consequence, for arbitrary demands we will restrict ourselves to the case of discrete demands on a finite number of points of the pipe, as happens in real life.

Example 2. Let us start by considering a single demand withdrawn at a specific point of the line. That is to say, let us consider a demand distribution given by

$$
\begin{equation*}
q(x)=\delta\left(x-x_{1}\right) \tag{18}
\end{equation*}
$$



Figure 4: Demand allocation fraction to the upstream node depending on the location of a single withdrawal in the line.
which concentrates the whole demand $Q_{d}$ at $x_{1}$, where $\delta(\cdot)$ is the well-known Dirac delta.

In this case, (12) is written as

$$
\begin{align*}
H_{R}( & \left.F_{\mathrm{Q}} ; \delta\left(x-x_{1}\right) ; L\right) \\
& =H_{0}-K \int_{0}^{L}\left[Q_{\mathrm{in}}-Q_{d} \delta\left(u-u_{1}\right)\right]^{2} \mathrm{~d} u \\
& =H_{0}-K \int_{0}^{x_{1}} \mathrm{Q}_{\mathrm{in}}^{2} \mathrm{~d} u-K \int_{x_{1}}^{L}\left(\mathrm{Q}_{\mathrm{in}}-\mathrm{Q}_{d}\right)^{2} \mathrm{~d} u  \tag{19}\\
& =H_{0}-K x_{1} Q_{\mathrm{in}}^{2}-K\left(L-x_{1}\right)\left(\mathrm{Q}_{\mathrm{in}}-\mathrm{Q}_{d}\right)^{2} \\
& =H_{0}-K L Q_{\mathrm{in}}^{2}\left(\lambda+(1-\lambda)\left(1-F_{\mathrm{Q}}\right)^{2}\right)
\end{align*}
$$

where $\lambda=x_{1} / L$ is the fraction of the pipe where the concentrated demand is located from the origin.

Equating again $H_{C}=H_{R}$ at $L$ gives

$$
\begin{equation*}
F_{\mathrm{Q}_{d}}\left(F_{\mathrm{Q}}, \lambda\right)=\frac{1}{F_{\mathrm{Q}}}\left(1-\sqrt{\lambda+(1-\lambda)\left(1-F_{\mathrm{Q}}\right)^{2}}\right) \tag{20}
\end{equation*}
$$

As could be expected, these values not only depend on $F_{\mathrm{Q}}$, as in the case of uniform demand, but also strongly depend on $\lambda$. In Figure 4 we have plotted these values as a function of $\lambda$ for various instances of $F_{\mathrm{Q}}$, namely, 1.0, $0.8,0.6,0.4$, and 0.2.

As expected, the worst cases occur once more for withdrawals representing a large percentage of the inflow to the line. For small in-line demands (see, e.g., the curve for $F_{Q}=$ 0.2 ; also calculate the limit of (20) for $F_{\mathrm{Q}}$ approaching to 0 ), the demand should be allocated to the end nodes almost linearly proportional to the relative distance of the demand point to the downstream end, as is completely natural. In contrast, this rule does not apply to large in-line demands, as their corresponding curves show, by becoming less and less linear. As an extreme case, let us consider the case of $F_{\mathrm{Q}}=1$. Various HGLs have been plotted in Figure 5 by


Figure 5: HGLs depending on the location of a single withdrawal in the line and HGL (dashed line) corresponding to the $50 \%$ allocation rule.
varying the location of the withdrawal point, specifically for values $\lambda=0.1,0.25,0.5$, and 0.9 . The (straight dashed) line, corresponding to the $50 \%$ allocation rule, has also been represented. We can observe that only for $\lambda=0.25$ does the dashed line match the correct piezometric head at the downstream end (also, observe that the lower curve in Figure 4, corresponding to $F_{Q}=1$, contains the point $\left.\left(\lambda, F_{\mathrm{Q}_{d}}\right)=(0.25,0.5)\right)$. In the other cases, disagreements are not only important at the downstream end but all along the line.

In the general case, we consider $q(x)=\sum_{k=1}^{n} d_{k} \delta\left(x-x_{k}\right)$, where $d_{k}$ are the demands at points $x_{k}$, with $0<x_{1}<x_{2}<$ $\cdots<x_{n-1}<x_{n}<L$, such that $\sum_{k=1}^{n} d_{k}=Q_{d}$.

Example 3. $q(x)=(1 / n) \sum_{k=1}^{n} \delta(x-(k /(n+1)) L)$ in the case of uniform demand at $n$ equally distributed points in the pipe. This demand produces the expression in (2).

In the general case, $H_{R}$ is calculated by

$$
\begin{aligned}
& H_{R}\left(F_{\mathrm{Q}}, L\right) \\
& =H_{0}-K x_{1} \mathrm{Q}_{\mathrm{in}}^{2}-K\left(x_{2}-x_{1}\right)\left(\mathrm{Q}_{\mathrm{in}}-d_{1}\right)^{2} \\
& \quad-K\left(x_{3}-x_{2}\right)\left(\mathrm{Q}_{\mathrm{in}}-\left(d_{1}+d_{2}\right)\right)^{2} \\
& \quad-\cdots-K\left(x_{n}-x_{n-1}\right)\left(\mathrm{Q}_{\mathrm{in}}-\sum_{k=1}^{n-1} d_{k}\right)^{2} \\
& \\
& \quad-K\left(L-x_{n}\right)\left(\mathrm{Q}_{\mathrm{in}}-\sum_{k=1}^{n} d_{k}\right)^{2} .
\end{aligned}
$$

By denoting

$$
\begin{array}{r}
\mu_{i}=\frac{d_{i}}{Q_{d}}, \quad \mu_{0}=0, \quad \lambda_{i}=\frac{x_{i}}{L}, \quad \lambda_{0}=0,  \tag{22}\\
\lambda_{n+1}=1, \quad \text { for } i=1, \ldots, n
\end{array}
$$

this expression can be written as

$$
\begin{equation*}
H_{R}\left(F_{\mathrm{Q}}, L\right)=H_{0}-K L Q_{\mathrm{in}}^{2} \sum_{k=0}^{n}\left(\lambda_{k+1}-\lambda_{k}\right)\left(1-F_{\mathrm{Q}} \sum_{j=0}^{k} \mu_{j}\right)^{2} . \tag{23}
\end{equation*}
$$

Then, equating again $H_{C}=H_{R}$ at $L$ gives

$$
\begin{align*}
& F_{\mathrm{Q}_{d}}\left(F_{\mathrm{Q}}, \lambda, \mu\right) \\
& \quad=\frac{1}{F_{\mathrm{Q}}}\left(1-\sqrt{\sum_{k=0}^{n}\left(\lambda_{k+1}-\lambda_{k}\right)\left(1-F_{\mathrm{Q}} \sum_{j=0}^{k} \mu_{j}\right)^{2}}\right) . \tag{24}
\end{align*}
$$

This expression can be easily calculated using, for example, a worksheet, as in Figure 6.

In this figure, we consider a demand distribution at the (inner) points given by the values of $\lambda(0.246$, $0.338, \ldots, 0.954$ ), the demand values given by the values ( $0.171,0.084, \ldots, 0.329$ ) of $\mu$, representing demand fractions at those points, according to (22). In the worksheet we can also read, besides the specific variable values used for the calculations, the value of $F_{\mathrm{Q}}=0.8$ used, meaning that a demand of $Q_{d}=0.8 \cdot Q_{\mathrm{in}}=0.8 \cdot 0.25=0.2 \mathrm{~m}^{3} / \mathrm{s}$ is extracted in the line. By using formula (24), implemented in the cell below $F_{\mathrm{Q}_{d}}$, we obtain the rate of demand that must be allocated to the upstream end to get the correct piezometric head at the downstream end. The graph in Figure 6 represents this situation. It can be clearly observed that the use of (24) provides a calculated HGL for the considered example (mid line) that perfectly matches the right end of the line representing the real HGL (lower polygonal). On the other hand, the application of the $50 \%$ rule (upper HGL) produces unacceptable errors.
2.4. Maximum Head Discrepancy When Using the Proposed Formula. As mentioned before, allocation of in-line demands to the end nodes of a line may be of great interest in order to reduce the size of the mathematical model of a WDN. In the previous section, we have given formulae to obtain allocation values that zero the piezometric head error at the downstream end of the line, a mandatory condition for correct calculation on the line(s) connected to this end. However, this reduction of the model size is at the price of making some piezometric head errors at the inner points of the line. The engineer analyzing a WDN should be aware of the magnitude of those discrepancies, in order to have a better representation and understanding of the problem. In this section we answer this question by solving problem (8).


FIGURE 6: Calculation of $F_{\mathrm{Q}_{d}}$ for a nonuniform distribution of demand within a line and comparison among hydraulic grade lines.

In the (ideal) case of a continuous demand along the line, it is easy to show that (8) reduces to find the point $x$ where the derivatives with respect to $x$ of

$$
\begin{gather*}
H_{R}\left(F_{\mathrm{Q}} ; x\right)=H_{0}-K \int_{0}^{x}\left(Q_{\mathrm{in}}-\mathrm{Q}(u)\right)^{2} \mathrm{~d} u  \tag{25}\\
H_{C}\left(F_{\mathrm{Q}_{d}} ; x\right)=H_{0}-K x \mathrm{Q}_{\mathrm{in}}^{2}\left(1-F_{\mathrm{Q}_{d}} F_{\mathrm{Q}}\right)^{2} \tag{26}
\end{gather*}
$$

coincide.
Let us observe that $H_{R}\left(F_{Q} ; x\right)$ is a differentiable function since $Q(x)$ is continuous and that $H_{C}\left(F_{Q_{d}} ; x\right)$ is a linear function of $x$. Thus, it is easily seen that the maximum discrepancy is located at the point $x$ given by

$$
\begin{equation*}
q(x)=F_{Q_{d}} \tag{27}
\end{equation*}
$$

Example 4. In the case of uniform continuous demand the solution of (8) is given by

$$
\begin{equation*}
x_{\infty}=L F_{\mathrm{Q}_{d}}^{\infty} \tag{28}
\end{equation*}
$$

obtained using $q(x)=x / L$, as in Example 1, and (10).
In the (real) case of a discrete demand along the line, the maximum discrepancy occurs at one of the points $x_{k}$, since the real HGL is a decreasing concave upwards polygonal (see Figure 6). Then, the problem reduces to identify the first $x_{k 0}$ for which the next section of the polygonal has a slope equal to or lower than the slope of $H_{C}\left(F_{\mathrm{Q}_{d}} ; x\right)$ in absolute values (if equal, all the points between $x_{k 0}$ and $x_{k 0+1}$ will provide the maximum since the mentioned section of the polygonal and
$H_{C}\left(F_{Q_{d}} ; x\right)$ will run parallel between both points). So we have to solve the following problem.

Find the first point $x_{k 0}$ such that

$$
\begin{equation*}
K\left(Q_{\mathrm{in}}-\left(\sum_{i=1}^{k 0} d_{i}\right)\right)^{2} \leq K Q_{\mathrm{in}}^{2}\left(1-F_{\mathrm{Q}_{d}} F_{\mathrm{Q}}\right)^{2} \tag{29}
\end{equation*}
$$

This inequality may be simplified and written, in terms of $\mu_{i}$, as

$$
\begin{equation*}
\left(1-F_{\mathrm{Q}}\left(\sum_{i=1}^{k 0} \mu_{i}\right)\right)^{2} \leq\left(1-F_{\mathrm{Q}_{d}} F_{\mathrm{Q}}\right)^{2} \tag{30}
\end{equation*}
$$

which gives

$$
\begin{equation*}
\sum_{i=1}^{k 0} \mu_{i} \geq F_{\mathrm{Q}_{d}} \tag{31}
\end{equation*}
$$

in terms of $\mu_{i}=d_{i} / Q_{d}$.
That is to say, the problem may be rewritten as follows:

$$
\begin{equation*}
\text { Find the first point } x_{k 0} \text { such that } \sum_{i=1}^{k 0} \mu_{i} \geq F_{\mathrm{Q}_{d}} . \tag{32}
\end{equation*}
$$

Example 5. In the case of uniform demand on $n$ equally spaced points, this gives the following:

$$
\begin{equation*}
\text { Find the first point } x_{k 0} \text { such that } \sum_{i=1}^{k 0} \frac{1}{n}=\frac{k 0}{n}=\frac{x_{k 0}^{n}}{L} \geq F_{Q_{d}}^{n} \text {. } \tag{33}
\end{equation*}
$$

Observe that $\lim _{n \rightarrow \infty}\left(x_{k 0}^{n} / L\right) \geq \lim _{n \rightarrow \infty} F_{\mathrm{Q}_{d}}^{n}=F_{\mathrm{Q}_{d}}^{\infty}$, since $F_{\mathrm{Q}_{d}}^{n}$ converges to $F_{\mathrm{Q}_{d}}^{\infty}$. We also have that $\lim _{n \rightarrow \infty}\left(x_{k 0-1}^{n} / L\right) \leq$ $\lim _{n \rightarrow \infty} F_{\mathrm{Q}_{d}}^{n}=F_{\mathrm{Q}_{d}}^{\infty}$. But obviously these two limits are the same. As a result, one has from (28) that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{x_{k 0}^{n}}{L}=F_{\mathrm{Q}_{d}}^{\infty}=\frac{x_{\infty}}{L} \tag{34}
\end{equation*}
$$

Observe that $x_{k 0}^{n}$ is the point where the polygonals (2) exhibit the biggest head discrepancy with the calculated head given by (9) corresponding to the allocation factor $F_{\mathrm{Q}_{d}}^{n}$. Also, $x_{\infty}$ is the point of maximum head discrepancy between the HGL corresponding to the uniform continuous demand and the calculated head corresponding to the allocation factor $F_{\mathrm{Q}_{d}}^{\infty}$. As we could expect,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} x_{k 0}^{n}=x_{\infty} . \tag{35}
\end{equation*}
$$

Expression (32) does not need any additional extra work in the general case if the calculations have been organized as shown in Figure 6. In effect, in this figure the first value of the accumulated rated demand that exceeds $F_{\mathrm{Q}_{d}}=0.261$ is $\sum_{i=1}^{k 0} \mu_{i}=0.272$, which corresponds to the third inner demand point $x_{k 0}=\lambda L$, with $\lambda=0.604$. This maximum is

$$
\begin{equation*}
H_{C}\left(0.261 ; x_{k 0}\right)-H_{R}\left(0.8 ; x_{k 0}\right)=92.767-90.763=2.003 \tag{36}
\end{equation*}
$$

## 3. Illustrative Example on a Real-World Network

This section is aimed to present some significant results obtained when applying the lessons learned in the previous section to a practical situation. The assessment is carried out in a real water supply network from Tegucigalpa City (the capital of Honduras). The network, which is intended to be a district metered area, is supplied by one main pipe connected to one of the water tanks administered by the local water authority (SANAA).

The network has 203 nodes and 211 pipes. Figure 7 shows the network model in EPANET. To illustrate the aspects previously studied, three scenarios have been put forward. For each scenario, intermediate demand nodes (mid nodes) are placed in the main pipe. In each case, the demand values loaded in the intermediates nodes, the distribution along the line, or the pipe diameter vary as it is indicated as follows.

Case A. As it may be seen in Figure 8, an internal demand representing $60 \%$ of the total inflow in the main line is fixed in two nodes placed along the line. The first one, loaded with $25 \%$ of the total internal demand, is located at $50 \%$ of the total length. The second intermediate node is located at $80 \%$ of the total length and the remaining $75 \%$ of the internal demand is allocated to it.

Case B. As shown in Figure 9, one single node is placed along the main line, at $50 \%$ of its total length. As it can be expected, $100 \%$ of the total internal demand is allocated to it. In this case the internal demand represents $80 \%$ of the total inflow.

Case C. This case is fairly similar to Case B. However, in this case, the diameter of the main line is reduced from 600 mm to 450 mm .
3.1. Result (Case A). Figure 8 clearly reflects the characteristics and the results obtained for Case A. First, a hydraulic simulation was conducted with the internal demand loaded in the mid nodes. For that (real) case, the pressure in node B (downstream node) is 42.35 water column meters (wcm). When the internal demand is reallocated to the upstream and downstream nodes following the $50 \%$ rule, the result varies up to 12.2 wcm in comparison with the initial case. Through the application of formula (24), a more suited demand allocation rule is found ( $22.45 \%$ for the upstream node and $77.55 \%$ for the downstream node). When this rule is applied, the resulting pressure value is, as expected, exactly the same as in the (real) case with the internal demand located in the mid nodes.
3.2. Result (Case B and Case C). Given the fact that between Cases B (Figure 9) and C (Figure 10) there is only one difference, namely, the pipe diameter, the allocation rule in both cases is the same; $34.86 \%$ of the internal demand should be allocated to the upstream node and the rest to the downstream node. Nevertheless, the differences between the initial pressure values and the pressure obtained when applying the $50 \%$ allocation rule are larger in Case C than in Case B, since in Case C the head loss is higher than that in Case B. If Case A is included in the same comparison (higher head loss than Cases B and C), the importance of a demand allocation based on the proposed methodology over a $50 \%$ 50\% becomes evident.

Finally, to better visualize the negative impact of using the common $50 \%$ demand distribution rule instead of the suited rule (24) developed in this paper, resulting pressure values obtained in four nodes of the example network (see Figure 11) are compared for Case A: first, when the demand is allocated using the $50 \%$ rule, and then, when the suited rule, given by formula (24), is implemented. The results of such comparison may be seen in Table 2.

## 4. Conclusions

The complexity of the interaction of all the input data in a mathematical model makes it impossible to include all the data accurately. The reality is very complex [2], and the use of simplifications in order to make the model feasible is therefore inevitable. The use of such tools should be, in any case, accompanied by a clear understanding of the consequences of such assumptions. It is obvious that the lower level of simplification corresponds with more complex tools. In this sense, software packages available in the market devoted to the analysis, design, and, in general, the simulation of the various states in a WDN should be used with the necessary caution.

This research focuses on the study of the influence that the concentration of a distributed demand in a line on the line ends represents in modeling steady state conditions in

Table 2: Comparison among scenarios.

| Control point | Scenario A.1. Real <br> case (demand loaded <br> in mid nodes) | Scenario A.2. <br> Demand allocated <br> using the suited rule <br> $(24)$ | Scenario A.3. <br> Demand allocated <br> using the $50 \%-50 \%$ <br> rule | Differences <br> between A.1 and <br> A.2 | Differences <br> between A.1 and |
| :--- | :---: | :---: | :---: | :---: | :---: |
| A | 41.35 | 41.35 | 53.56 | 0 | A.3 |
| B | 20.72 | 20.72 | 32.93 | 0 | 12.21 |
| C | 53.39 | 53.39 | 65.60 | 0 | 12.21 |
| D | 32.70 | 44.91 | 0 | 12.21 |  |



Figure 7: Water supply system used for the practical example.


Figure 8: Details and results for Case A.


Figure 9: Details and results for Case B.


Figure 10: Details and results for Case C.


Figure 11: Pressure control points.

WDNs. The research work, using a practical approach, shows the importance of the relation between the total inflow and the flow that is extracted due to the load of demands in the line. The distribution of the consumptions along the line also greatly influences the validity of the model. After assessing the error that may derive from using the generalized $50 \%$ rule, we obtain the general formula (24) that enables to allocate an arbitrary inner line demand to the line ends with zero error at the downstream end of the line. In addition, we provide a means to calculate the associated maximum head point discrepancy.

In Figure 6 we have shown that our proposal is straightforward and involves direct methods that can be easily applied by any user of any WDN analysis package, for example, using a very simple worksheet. Existing network models may be revised to replace, if necessary, the used allocation values with he values provided by (24), which produce more reliable results. For a new network model, users may make an a priori decision on implementing the right simplifications so that a suitable tradeoff between model simplification and accuracy is obtained.

This study applies only to branched networks, in which the flow direction is predefined. The results do not strictly apply to looped networks in which the definition of upstream and downstream nodes cannot be known a priori and even may vary depending on the hourly demand in the network and on the demand pattern used. Nevertheless, a number of facts make the obtained results in this paper applicable to looped networks as well, with a little extra work from the technician in charge of the analysis. In effect, firstly the long lines susceptible of the considered simplification must be selected; this task will not represent any problem for the expert. Secondly, the expert will also be able to have a clear idea about the flow direction in most of those lines. In case of doubt, he or she can use an initial analysis approach over specific lines following the $50 \%$ rule, and, after the analysis, go back to the design to check the actual flow direction and make a decision about the final direction of the flow and the values for $F_{\mathrm{Q}_{d}}$. Perhaps a little more of iteration will enable to get the final design. Thirdly, if the expert is not happy with the obtained results for one of the lines, one has the opportunity to include in the model an additional point of the line, where the maximum head discrepancy occurs at the point given by condition (32). This will divide the line into two new lines to which the same criteria may be applied. Fourthly, this solution of including an interior point of one line into the model will be also useful in case if a line is fed by both ends.

In a future work, we will consider the inclusion of variable values for the factor $F_{\mathrm{Q}_{d}}$ in the demand curves assigned at the ends of the challenging lines, so that variations of the flow direction in certain lines during the day may be considered when running extended period simulations (EPSs).

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Exact and Analytic-Numerical Solutions of Lagging Models of Heat Transfer in a Semi-Infinite Medium 

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Different non-Fourier models of heat conduction have been considered in recent years, in a growing area of applications, to model microscale and ultrafast, transient, nonequilibrium responses in heat and mass transfer. In this work, using Fourier transforms, we obtain exact solutions for different lagging models of heat conduction in a semi-infinite domain, which allow the construction of analytic-numerical solutions with prescribed accuracy. Examples of numerical computations, comparing the properties of the models considered, are presented.

## 1. Introduction

Non-Fourier models of heat conduction have increasingly been considered in recent years to model microscale and ultrafast, transient, nonequilibrium responses in heat and mass transfer, where thermal lags and nonclassical phenomena are present (see, e.g., [1] and references therein). The growing area of applications of these models include, among other examples, the processing of thin-film engineering structures with ultrafast lasers [2,3], the transfer of heat in nanofluids [4,5], or the exchange of heat in biological tissues [6-8].

In the Dual-Phase-Lag (DPL) model [9-11], the equation relating the heat flux vector $\mathbf{q}$ and the temperature $T$, for time $t$ and spatial point $\mathbf{r}$,

$$
\begin{equation*}
\mathbf{q}\left(\mathbf{r}, t+\tau_{q}\right)=-k \nabla T\left(\mathbf{r}, t+\tau_{T}\right) \tag{1}
\end{equation*}
$$

where $k>0$ is the thermal conductivity, incorporates two lags, $\tau_{q}$ for the heat flux and $\tau_{T}$ for the temperature gradient. When both lags are zero, the Fourier law is recovered, while for $\tau_{q}>0$ and $\tau_{T}=0$, it reduces to the Single-Phase-Lag (SPL) model [12].

Combining (1) with the principle of energy conservation,

$$
\begin{equation*}
-\nabla \cdot \mathbf{q}(\mathbf{r}, t)+Q(\mathbf{r}, t)=C_{p} T_{t}(\mathbf{r}, t), \tag{2}
\end{equation*}
$$

where $C_{p}$ is the volumetric heat capacity and $Q$, the volumetric heat source, in the absence of heat sources a partial differential equation with delay is obtained $[13,14]$ as

$$
\begin{equation*}
T_{t}\left(\mathbf{r}, t+\tau_{q}\right)=\alpha \Delta T\left(\mathbf{r}, t+\tau_{T}\right) \tag{3}
\end{equation*}
$$

where $\alpha=k / C_{p}$ is the thermal diffusivity. When both lags are zero, the diffusion equation, a parabolic partial differential equation which represents the classical model for heat conduction and other transport phenomena, is obtained.

Using first-order approximations in (1),

$$
\begin{equation*}
\mathbf{q}(\mathbf{r}, t)+\tau_{q} \frac{\partial \mathbf{q}}{\partial t}(\mathbf{r}, t) \cong-k\left\{\nabla T(\mathbf{r}, t)+\tau_{T} \frac{\partial}{\partial t} \nabla T(\mathbf{r}, t)\right\} \tag{4}
\end{equation*}
$$

a hyperbolic equation is derived, commonly referred to as the DPL model [9], here denoted as $\operatorname{DPL}(1,1)$,

$$
\begin{equation*}
T_{t}(\mathbf{r}, t)+\tau_{q} T_{t t}(\mathbf{r}, t)=\alpha\left\{\Delta T(\mathbf{r}, t)+\tau_{T} \Delta T_{t}(\mathbf{r}, t)\right\} \tag{5}
\end{equation*}
$$

which for $\tau_{T}=0$ reduces to the Cattaneo-Vernotte (CV) model [15-17].

Approximations in (1) up to order two in $\tau_{q}$ and/or $\tau_{T}$ have also been considered [18, 19], leading to models that will be denoted as $\operatorname{DPL}(2,1)$,

$$
\begin{align*}
T_{t}(\mathbf{r}, t) & +\tau_{q} T_{t t}(\mathbf{r}, t)+\frac{\tau_{q}^{2}}{2} T_{t t t}(\mathbf{r}, t)  \tag{6}\\
= & \alpha\left\{\Delta T(\mathbf{r}, t)+\tau_{T} \Delta T_{t}(\mathbf{r}, t)\right\}
\end{align*}
$$



Figure 1: Temperature evolution, at $x=10$, for DPL, DH, and classical diffusion (Diff) models with Dirichlet boundary conditions and parameters $\tau_{T}=0, \tau_{q}=1$, and initial function $\varphi(x, t)=2(1-\cos (x)) /(\pi x)$, for $\alpha=0.1$ (a) and $\alpha=0.8$ (b).
and $\operatorname{DPL}(2,2)$,

$$
\begin{align*}
& T_{t}(\mathbf{r}, t)+\tau_{q} T_{t t}(\mathbf{r}, t)+\frac{\tau_{q}^{2}}{2} T_{t t t}(\mathbf{r}, t) \\
& \quad=\alpha\left\{\Delta T(\mathbf{r}, t)+\tau_{T} \Delta T_{t}(\mathbf{r}, t)+\frac{\tau_{T}^{2}}{2} \Delta T_{t t}(\mathbf{r}, t)\right\} \tag{7}
\end{align*}
$$

From the original formulation of the DPL model, as given in (1), for $\tau=\tau_{q}-\tau_{T}>0$, a retarded partial differential equation is obtained $[13,14]$, referred to as the delayed heat conduction model (DH),

$$
\begin{equation*}
T_{t^{\prime}}\left(\mathbf{r}, t^{\prime}\right)=\alpha \Delta T\left(\mathbf{r}, t^{\prime}-\tau\right) \tag{8}
\end{equation*}
$$

where $t^{\prime}=t+\tau_{q}$.
Exact solutions for some particular DPL models in different settings have been discussed (e.g., [11, 13, 14, 20-22]), and many specific methods to construct numerical solutions, usually in finite domains using finite difference schemes, have been developed (see, e.g., [23-27]).

In semi-infinite domains, some particular problems have also been considered. Solutions for heat propagation according to $\operatorname{DPL}(1,1)$ model in a semi-infinite solid, produced by suddenly raising the temperature at the boundary, were obtained in [11, 20], using Laplace and Fourier transforms. Relations between the local values of heat flux and temperature, in the form of integral equations, in a semi-infinite solid were considered in [13, 28].

In this work, using Fourier transforms, explicit solutions for lagging models of heat conduction in a semi-infinite domain, with different types of boundary conditions, are obtained, allowing the construction of numerical solutions with bounded errors.

It should be noted that Fourier transforms can also be used in time-dependent problems (e.g., $[29,30]$ ), and
the approach of this work could also be useful for timedependent DPL models, which have already been proposed [31].

## 2. Solutions of DPL Models in a Semi-Infinite Domain

Consider a plate of infinite thickness, $x \in[0, \infty]$, that can be heated either at its surface, $x=0$, or up to a certain depth, $x \in$ $[0, l]$. We will consider, for $t \geq 0$, either Dirichlet, $T(0, t)=0$, or Neumann, $T_{x}(0, t)=0$, boundary conditions and also that

$$
\begin{equation*}
\lim _{x \rightarrow \infty} T(x, t)=0, \quad t \geq 0 \tag{9}
\end{equation*}
$$

Appropriate initial conditions must be provided for the different models. Thus, for $\operatorname{DPL}(1,1)$ initial values for temperature and its time derivative have to be specified,

$$
\begin{equation*}
T(x, 0)=\varphi(x, 0), \quad T_{t}(x, 0)=\phi(x, 0), \quad 0 \leq x<\infty \tag{10}
\end{equation*}
$$

while for $\operatorname{DPL}(2,1)$ and $\operatorname{DPL}(2,2)$, its second derivative also has to be given,

$$
\begin{equation*}
T_{t t}(x, 0)=\psi(x, 0), \quad 0 \leq x<\infty \tag{11}
\end{equation*}
$$

and for the DH model, the initial condition for the temperature has to be specified for a time interval of $\tau$ amplitude,

$$
\begin{equation*}
T(x, t)=\varphi(x, t), \quad 0 \leq t \leq \tau, \quad 0 \leq x<\infty . \tag{12}
\end{equation*}
$$

For a wide class of initial functions [32,33], the method of Fourier transform can be used to eliminate derivatives in the spatial domain and to obtain expressions for the exact solutions in the form of an infinite integral, either using Fourier sine transforms for Dirichlet conditions,

$$
\begin{equation*}
T(x, t)=\frac{2}{\pi} \int_{0}^{\infty} \mathscr{T}(w, t) \sin (w x) d w \tag{13}
\end{equation*}
$$



Figure 2: Differences from classical diffusion for models DPL and DH, for the data shown in Figure 1.


Figure 3: Temperature evolution, for $(x, t) \in[10,20] \times[0,10]$, for the DH model with Dirichlet boundary conditions and parameters $\tau_{T}=0$, $\tau_{q}=1, \alpha=0.8$, and initial function $\varphi(x, t)=2(1-\cos (x)) /(\pi x)(\mathrm{a})$, and differences from DH of $\mathrm{DPL}(1,1)$ and $\mathrm{DPL}(2,1)$ at $x=10(\mathrm{~b})$.
or cosine transforms for Neumann conditions,

$$
\begin{equation*}
T(x, t)=\frac{2}{\pi} \int_{0}^{\infty} \mathscr{T}(w, t) \cos (w x) d w \tag{14}
\end{equation*}
$$

where the functions $\mathscr{T}(w, t)$, which are the corresponding Fourier sine or cosine transforms of $T(x, t)$, are obtained as solutions of the transformed temporal problems, depending on the continuous set of eigenvalues $w^{2}$.

For the family of DPL approximations, the transformed problems are initial-value problems for linear differential equations with constant coefficients. Thus, for DPL $(1,1)$, one gets

$$
\begin{gather*}
\tau_{q} \mathscr{T}^{\prime \prime}(w, t)+\left(1+w^{2} \alpha \tau_{T}\right) \mathscr{T}^{\prime}(w, t)+w^{2} \alpha \mathscr{T}(w, t)=0 \\
\mathscr{T}(w, 0)=F(w), \quad \mathscr{T}^{\prime}(w, 0)=G(w) \tag{15}
\end{gather*}
$$



FIGURE 4: Temperature evolution for DPL, DH, and classical diffusion (Diff) models (left), and differences from DH of DPL approximations (right), at $x=10$, with Dirichlet boundary conditions, initial function $\varphi(x, t)=2(1-\cos (x)) /(\pi x)$, and parameters $\alpha=0.8, \tau_{T}=1$, and $\tau_{q}=1$ (top), or $\tau_{T}=19$ and $\tau_{q}=20$ (down).
for $\operatorname{DPL}(2,1)$, the problem reads

$$
\begin{align*}
& \frac{\tau_{q}^{2}}{2} \mathscr{T}^{\prime \prime \prime}(w, t)+\tau_{q} \mathscr{T}^{\prime \prime}(w, t)+\left(1+w^{2} \alpha \tau_{T}\right) \mathscr{T}^{\prime}(w, t)  \tag{16}\\
& \quad+w^{2} \alpha \mathscr{T}(w, t)=0,
\end{align*}
$$

with initial conditions

$$
\begin{gather*}
\mathscr{T}(w, 0)=F(w), \quad \mathscr{T}^{\prime}(0)=G(w),  \tag{17}\\
\mathscr{T}^{\prime \prime}(0)=H(w),
\end{gather*}
$$

and, with the same initial conditions as in $\operatorname{DPL}(2,1)$, for $\operatorname{DPL}(2,2)$, one gets

$$
\begin{align*}
& \left(\frac{\tau_{q}^{2}}{2}\right) \mathscr{T}^{\prime \prime \prime}(w, t)+\left(\frac{\tau_{q}+w^{2} \alpha \tau_{T}^{2}}{2}\right) \mathscr{T}^{\prime \prime}(w, t)  \tag{18}\\
& \quad+\left(1+w^{2} \alpha \tau_{T}\right) \mathscr{T}^{\prime}(w, t)+w^{2} \alpha \mathscr{T}(w, t)=0
\end{align*}
$$

where $F(w), G(w)$, and $H(w)$ are the Fourier sine or cosine transforms, according to the type of boundary conditions, of the initial functions $\varphi(x, t), \phi(x, t)$, and $\psi(x, t)$, respectively.

Hence, these problems can be solved, obtaining expressions for $\mathscr{T}(w, t)$ in terms of the roots of the corresponding characteristic equation, and thus explicit expressions for the exact solutions for these models, in the form of (13) or (14), can be obtained.

For the DH model, the transformed temporal problems are initial-value problems for delay differential equations with general initial functions,

$$
\begin{gather*}
\mathscr{T}^{\prime}(w, t)+w^{2} \alpha \mathscr{T}(w, t-\tau)=0, \quad t>\tau \\
\mathscr{T}(w, t)=F(w, t), \quad 0 \leq t \leq \tau \tag{19}
\end{gather*}
$$

where $F(w, t)$ is the appropriate Fourier transform, according to the boundary conditions, of the initial function $\varphi(x, t)$. To obtain constructive solutions for this problem, a combination of the steps method and a convolution integral can be applied [34, 35], producing the following expression, for $t \in[p \tau$, ( $p+$ 1) $\tau$ ],

$$
\begin{align*}
\mathscr{T}(w, t)= & F(w, \tau)+F(w, 0) \sum_{k=1}^{p} \frac{\left(-w^{2}\right)^{k} \alpha^{k}(t-k \tau)^{k}}{k!} \\
& +\sum_{k=1}^{p-1} \frac{\left(-w^{2}\right)^{k} \alpha^{k}}{k!} \int_{0}^{\tau}(t-k \tau-s)^{k} F_{s}(w, s) d s \\
& +\frac{\left(-w^{2}\right)^{p} \alpha^{p}}{p!} \int_{0}^{t-p \tau}(t-p \tau-s)^{p} F_{s}(w, s) d s . \tag{20}
\end{align*}
$$

The solutions obtained with the Fourier transforms, as given in (13) or (14), can be shown to converge and provide exact solutions under adequate integrability and regularity conditions on the initial functions. Numerical integration is required in general to compute numerical approximations of these solutions, with errors that can be bounded in finite spatial and temporal domains by controlling errors in the numerical integrators or by appropriately truncating the infinite integrals. However, for some particular initial functions, the solutions given by (13) or (14) may reduce to finite integrals.

## 3. Numerical Examples

Numerical examples are presented in the following figures, where, in order to properly compare DPL and DH models, the initial interval for DH , where the initial function $\varphi(x, t)$ is given, is displaced to [ $-\tau, 0$ ], and the initial functions for DPL models are set so that the values of temperature and its first derivative at $t=0$, and also its second derivative for $\operatorname{DPL}(2,1)$ and $\operatorname{DPL}(2,2)$, are matched to those of the DH model. The classical diffusion model, whose solution is available and readily obtained [36], is also included as reference.

First, we consider models with $\tau_{T}=0$, so that $\operatorname{DPL}(2,1)$ and $\operatorname{DPL}(2,2)$ are equal, and an initial function with damped temperature oscillations, thought to be the result of a modulated heat source that is switched off at $t=0$, showing the transient behavior for the different DPL models for different
values of $\alpha$ (Figure 1), as well as their differences from classical diffusion (Figure 2).

In Figure 3, a more detailed view of the spatiotemporal behavior of the DH model (Figure 3(a)) and differences from DH of $\operatorname{DPL}(1,1)$ and $\operatorname{DPL}(2,1)$ (Figure $3(b))$ are presented.

In Figure 4, different values of $\tau_{T}$ and $\tau_{q}$, such that $\tau=$ $\tau_{q}-\tau_{T}$ is kept constant, are used, so that variations in the temperature evolution are observed in the DPL approximate models, but not in the DH model, which only depends on the value of $\tau$.

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## Research Article

# Forecasting Latin America's Country Risk Scores by Means of a Dynamic Diffusion Model 

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#### Abstract

Over the last years, worldwide financial market instability has shaken confidence in global economies. Global financial crisis and changes in sovereign debts ratings have affected the Latin American financial markets and their economies. However, Latin American's relative resilience to the more acute rise in risk seen in other regions like Europe during last years is offering investors new options for improving risk-return trade-offs. Therefore, forecasting the future of economic situation involves high levels of uncertainty. The Country Risk Score (CRS) represents a broadly used indicator to measure the current situation of a country regarding measures of economic, political, and financial risk in order to determine country risk ratings. In this contribution, we present a diffusion model to study the dynamics of the CRS in 18 Latin American countries which considers both the endogenous effect of each country policies and the contagion effect among them. The model predicts quite well the evolution of the CRS in the short term despite the economic and political instability. Furthermore, the model reproduces and forecasts a slight increasing trend, on average, in the CRS dynamics for almost all Latin American countries over the next months.


## 1. Introduction and Motivation

Worldwide financial market instability has shaken confidence in global economies. This loss of confidence has a strong influence on the capital flows [1,2] and the investors' attitude towards some countries [3-9].

In Latin America and until the first half of 2011, the South and Central American economies expanded at a high pace. However, since late 2011, this strong growth started to slow down. This growth was particularly strong in South America (Chile, Brazil, Peru, Colombia, Uruguay, Paraguay, Argentina, Bolivia, Venezuela, Ecuador, Guyana, Suriname, Trinidad, and Tobago) due to strong economic demand, better external financing conditions, and higher commodity exportation prices, whereas in Central America (Mexico, Panama, Costa Rica, El Salvador, Guatemala, Honduras, Nicaragua, and Belize), growth has been subdued but also accelerated due to the recovery of domestic demand and a stronger agricultural expansion in Mexico. On the other
hand, most of the Caribbean economies (Dominican Republic, Haiti, or Cuba) growth remained weak.

Global financial crisis and changes in sovereign debts ratings and confidence have also affected the Latin American financial markets and their economies. In fact, experiences from former currency crashes (Argentina, Mexico, etc.) are fueling avid interest in the Eurozone crisis, with the corresponding cross-country contagion. Traditional concepts of risk, solvency, liquidity, or foreign investment grade allow us to understand this issue. This is the main reason why country risk ratings have become a topic of major concern for the international financial community over the last two decades. Kaminsky and Schmmkler state: "the effects of rating and outlook changes are stronger during crisis, in nontransparent economies, and in neighboring countries" [10]. Therefore, following the series of currency crisis, policies like bailouts by both international institutions and governments have also been blamed for financial volatility and financial excesses [11, 12]. The work presented in [13] suggests that
procyclical behavior (upgrading countries in good times and downgrading them in bad times) may have magnified the boom-bust pattern in stock markets. This effect seems to be stronger in emerging markets, where problems of asymmetric information and transparency are more severe. Furthermore, upgrades or downgrades in one country are likely to affect other similar economies.

The importance of country ratings is also underscored by the existence of several major country risk rating agencies [14]. The country risk literature argues that country risk ratings have a direct impact on the cost of borrowings as they reflect the probability of debt default by a country. An improvement in country risk ratings, or country creditworthiness, will lower a country's cost of borrowing and debt servicing obligations and vice versa $[9,15]$. For this purpose, Country Risk Scores (CRS) are built in order to measure several factors, both quantitative and qualitative. In our study we use the CRS underscored by the Euromoney Agency [16], which combines the following categories: political risk, economic performance, debt indicators, structural assessments, access to bank finance/capital markets, and credit ratings. The categories involve the following information.
(C1) Political risk
Corruption: a measure of how corruption affects country risk;
Government nonpayments/nonrepatriation: a measure of the risk Government policies and actions pose to financial transfers;
Government stability: a measure of how stable a government is;
Information access/transparency: a measure of the accessibility and reliability of information and statistics;
Institutional risk: a measure of the independence and efficiency of state institutions;
Regulatory and policy environment: a measure of the quality of the regulatory environment and how well policy is formulated/implemented.
(C2) Economic performance
Bank stability/risk: a measure of banking sector strength;
Economic Gross National Product (GNP) outlook: a measure of optimism/pessimism for the economic growth outlook;
Employment/unemployment: a measure of the risk posed to the economy by unemployment;
Monetary policy/currency stability: a measure of monetary policy effectiveness/exchange rate risk;
Government finances: a measure of a country's fiscal strength.
(C3) Debt indicators: Calculated using the following ratios from the World Bank's Global Development Finance
figures: total debt stocks to GNP (A), debt service to exports (B), and current account balance to GNP (C). Developing countries which do not report complete debt data get a score of zero.
(C4) Structural assessments
Demographics: a measure of the impact of the demographic profile on economic growth and political stability;
Hard infrastructure: a measure of the adequacy of a country's physical infrastructure;
Labour market/industrial relations: a measure of the suitability of the labour environment for economic growth and political stability;
Soft infrastructure: a measure of the health of the economic, medical, and cultural/social institutions of a country.
(C5) Access to bank finance/capital markets: Participants rate each country's accessibility to international markets.
(C6) Credit ratings: Nominal values are assigned to sovereign ratings from Moody's, Standard and Poor's, and Fitch IBCA.

Thus, CRS can represent a complete indicator of the current situation of a country regarding measures of economic, political, and financial risk in order to determine country risk ratings. In the case of Euromoney Agency, the overall (ECR) Euromoney Risk Score is obtained by assigning to the six categories introduced above the following weights:
(i) Three qualitative expert opinions: political risk ( $30 \%$ weighting), economic performance ( $30 \%$ weighting) and structural assessment ( $10 \%$ weighting),
(ii) Three quantitative values: debt indicators (10\% weighting), credit ratings ( $10 \%$ weighting), and access to bank finance/capital markets ( $10 \%$ weighting).

When talking about financial crises there is a lot of literature that takes into account several reasons for crises to appear in clusters [ $6,8,17,18$ ]. Therefore, we consider that in Latin America a crisis in one Latin American country may focus investors' attention on other Latin American countries with similar trends and general structural similarities and vulnerabilities. This effect is widely known as common weakness contagion [19-22]. According to this, a cluster analysis has been performed in our study. The contagion is usually modelled using epidemiological and/or diffusion techniques. Both are in close connection and let us study the dynamics of CRS using these modelling techniques. Our objective is to predict the CRS trends over the next months, providing prediction tools for policy makers and investors. Thus, with these tools, these policy makers and investors are able to design strategies, simulate different situations, and analyse the effect of changes in order to improve the economic situation.

This paper is organized as follows. Section 2 is addressed to introduce the available data, to perform a cluster analysis,


Figure 1: Scheme summarizing the techniques applied through this paper.
and to construct and justify the mathematical model used to describe the dynamics of the CRS of 18 Latin American countries. First part of Section 3 is devoted to model parameter estimation. Since uncertainty and variability are the rules when dealing with modelling real problems, in the second subsection, we provided predictions by means of confidence intervals using a cross-validation technique. In the third part the model is validated and the results are discussed in Section 3.3. Figure 1 summarizes the aforementioned processes developed in this approach. Conclusions are drawn in Section 4.

## 2. Modelling

This section is addressed to construct and justify the mathematical model used to describe the dynamics of the CRS of the 18 most important Latin American countries (listed in Table 1). To perform the study we have considered a total of 25 available values of CRSs for every country corresponding to different dates starting from February 6, 2012, to August 6, 2012 [16]. Table 2 collects the CRS numerical values corresponding to the starting date and all CRS values are graphically represented as black points in Figures 4 and 5.
2.1. Clustering. As we said previously, in finance and especially in country risk assessment, it is useful to group the different countries sharing similar economic characteristics. The clustering technique allows us to gather the different countries into homogeneous groups. Thus, before constructing our dynamic diffusion model, we have performed a clustering. As we will see later, an additional advantage of the clustering is the reduction of the number of model parameters to be estimated. In order to deal with this task, we have

Table 1: Clustering obtained by non-hierarchical clustering technique.

| $i$ | Country name | Cluster | $i$ | Country name | Cluster |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | 8 | Honduras | 3 |
|  |  |  | 9 | Ecuador | 3 |
| 1 | Chile | 1 | 10 | Nicaragua | 3 |
|  |  |  | 11 | Dominican Republic | 3 |
|  |  |  | 12 | Trinidad and Tobago | 3 |
| 2 | Brazil | 2 | 13 | Costa Rica | 4 |
| 3 | Mexico | 2 | 14 | Paraguay | 4 |
| 4 | Peru | 2 | 15 | El Salvador | 4 |
| 5 | Colombia | 2 | 16 | Argentina | 4 |
| 6 | Uruguay | 2 | 17 | Bolivia | 4 |
| 7 | Panama | 2 | 18 | Venezuela | 4 |

used the non-hierarchical clustering (also termed $k$-means clustering) [23, 24]. This method separates $i$ observations into $k$ clusters in which each observation belongs to the cluster with the nearest mean. Applying this technique, Latin American countries $(i=1,2, \ldots, 18)$ have been grouped into four clusters considering the available data corresponding to the six categories C1-C6 introduced in the previous section. The results are reported in Table 1.

The first cluster gathers Chile, the safest and most prosperous South American economy. It should be also remarked that it leads Latin American nations to human development, competitiveness, income per capita, globalization, economic freedom, and low perception of corruption [25]. The second cluster gathers both South American (Brazil, Peru, Colombia, and Uruguay) and Central American countries (Mexico and Panama) economies which have done better in the last 10

Table 2: Initial CRS data corresponding to February 6, 2012 (Euromoney, 2012 [16]).

| Country name | Initial CRS | Country name | Initial CRS |
| :--- | :---: | :---: | :---: |
|  |  | Honduras | 37.95 |
|  |  | Ecuador | 34.25 |
| Chile | 75.07 | Nicaragua | 31.43 |
|  |  | Dominican Rep. | 35.40 |
|  |  | Trinidad Tobago | 51.59 |
| Brazil | 62.76 | Costa Rica | 52.27 |
| Mexico | 58.93 | Paraguay | 41.40 |
| Peru | 55.76 | El Salvador | 42.73 |
| Colombia | 59.61 | Argentina | 59.61 |
| Uruguay | 50.61 | Bolivia | 35.38 |
| Panama | 57.78 | Venezuela | 35.55 |

years, with Brazil and Mexico being the leaders of the group. The third cluster includes Central American (Honduras and Nicaragua), South American (Ecuador), and Caribbean (Dominican Republic and Trinidad and Tobago) countries which have suffered a weaker growth in the last 10 years. The fourth cluster includes the considered less safe attractive investing and also most unstable South American (Argentina, Bolivia, Venezuela, and Paraguay) and Central American (Costa Rica and El Salvador) countries due to its politicalsecurity and/or economic situation.

In the following, for our dynamic diffusion model, we are going to assume that the obtained clustering does not change over the time. This hypothesis is reasonable because, as we said previously, we are going to predict CRS evolution in a short time and there will be very few countries that may move out from one cluster to another in the studied period. Furthermore, the obtained clusters represent quite accurately the current economic Latin American groups.
2.2. Mathematical Model. Once the clusters have been established, we propose a diffusion dynamic model to study the evolution of the CRS of each Latin American country. Diffusion dynamic models have been demonstrated to be powerful tools to study a wide range of applied problems in different areas including economics and its related fields [2628]. Although complex models have been proposed based on this approach, they all follow the next pattern:

$$
\begin{equation*}
x^{\prime}(t)=\beta(t) x(t), \tag{1}
\end{equation*}
$$

where $x(t)$ represents the magnitude under modelling (in our case, the CRS to each Latin American country at time $t), x^{\prime}(t)$ denotes the derivative with respect to the time $t$ (it may also be denoted by $d x(t) / d t$ ), and $\beta(t)$ is the so-called time-dependent diffusion coefficient, which may involve the unknown $x(t)$. This coefficient is basically what differentiates one specific model from another. Our model considers $\beta(t)$ as a linear function of the unknowns which can be decomposed into two factors: the first one represents, through CRS, the autonomous economic behavior of each country and the second one the contagion effect for loss or gain of confidence
both between and within clusters for each country. As we will see, in our case the resulting model will be nonlinear.

For the sake of clarity in the model setting, we will identify each one of the 18 Latin American countries with the index $i=1,2, \ldots, 18$ according to the same order obtained after clustering (see Table 1). Now, it is convenient to denote the indexes $j_{k}$ and the values $J_{k}, 1 \leq k \leq 4$, as follows:

$$
\begin{gather*}
1 \leq j_{1} \leq J_{1}=1, \quad 1 \leq j_{2} \leq J_{2}=6, \\
1 \leq j_{3} \leq J_{3}=5, \quad 1 \leq j_{4} \leq J_{4}=6,  \tag{2}\\
\sum_{k=1}^{4} J_{k}=18,
\end{gather*}
$$

where $J_{k}$ is the number of countries in cluster $k$. We assume that CRS variation rate of a country, modelled by its derivative $C_{i}^{\prime}(t), 1 \leq i \leq 18$, is a mixture of an autonomous term, related to its endogenous economic politics, and several diffusion terms, related to the exogenous economic influences of other Latin American countries belonging to the same or different cluster. In the following, we describe in detail these terms.

Autonomous Behavior. Each Latin American country develops endogenous politics that may result in an increase or decrease in its own CRS. We model this autonomous behavior by $\alpha_{i} C_{i}(t), 1 \leq i \leq 18$. The coefficient $\alpha_{i}$ can take positive or negative values which are identified, respectively, with suitable and unsuitable domestic politics, with respect to its CRS.

Transmission Behavior. In an economically globalized world, the economic situation of a specific country could eventually influence other countries. This statement is even more realistic for a set of countries sharing similar economic characteristics, like Latin American countries. Initially, this might motivate to establish a dynamic diffusion model by considering a full contagion between each pair of countries. However this approach entails the introduction of a large number of parameters, what would make the model parameter estimation computationally unfeasible. In order to reduce the number of model parameters, we take advantage of previous clustering classification to consider as a balanced CRS' indicator of each cluster the following average value:

$$
\begin{equation*}
\bar{C}_{k}(t)=\frac{1}{J_{k}} \sum_{j_{k}=1}^{J_{k}} C_{j_{k}}(t), \quad 1 \leq k \leq 4 \tag{3}
\end{equation*}
$$

Based on the CRS, we propose to model the influence of the economic policies of the countries belonging to the cluster $k$, $1 \leq k \leq 4$, on the $i$ th Latin American country, $1 \leq i \leq 18$, according to the following term ( $t$ measured in years):

$$
\begin{equation*}
\beta_{k(i), k} C_{i}(t)\left(\bar{C}_{k}(t)-C_{i}(t)\right), \quad \beta_{k(i), k}>0,1 \leq k \leq 4 \tag{4}
\end{equation*}
$$

where the first subindex $k(i)$ of coefficient $\beta_{k(i), k}$ denotes the cluster to which the $i$ th country belongs:

$$
k(i)= \begin{cases}1 & \text { if } 1 \leq i \leq 1=J_{1},  \tag{5}\\ 2 & \text { if } 2 \leq i \leq 7=J_{1}+J_{2}, \\ 3 & \text { if } 8 \leq i \leq 12=J_{1}+J_{2}+J_{3}, \\ 4 & \text { if } 13 \leq i \leq 18=J_{1}+J_{2}+J_{3}+J_{4} .\end{cases}
$$

In order to include all the possible influences of countries belonging to every cluster on the $i$ th country $(1 \leq i \leq 18)$, we consider the sum of all the transmission terms (4) as follows.

$$
\begin{array}{r}
C_{i}(t) \sum_{1 \leq k, k(i) \leq 4} \beta_{k(i), k}\left(\bar{C}_{k}(t)-C_{i}(t)\right),  \tag{6}\\
\beta_{k(i), k}>0,1 \leq k \leq 4
\end{array}
$$

where $k(i)$ is given by (5). The two factors that determine each summand in (6) can be interpreted as follows.
(i) The term $\bar{C}_{k}(t)-C_{i}(t)$ measures the CRS difference (positive or negative) between the $i$ th country at time $t$, denoted by $C_{i}(t)$, and the CRS average of countries belonging to cluster $k$ at the same time $t$, denoted by $\bar{C}_{k}(t)$. So, we assume that this factor contributes positively (negatively) to the transmission term (4) when the $i$ th country has a CRS lower (higher) than the CRS average of countries in cluster $k$. Thus, we suppose that countries with lower (higher) CRS than those belonging to a cluster with higher (lower) average CRS tend to increase (decrease) their CRS influenced by the countries belonging to cluster $k$.
(ii) The factors responsible for the contagion effect are embedded in the $\beta_{k(i), k}$ coefficients, and they modulate the weight of the differences $\bar{C}_{k}(t)-C_{i}(t)$ in the transmission terms. Notice that, once the subindex $k$ is fixed, we are assuming that this coefficient $\beta_{k(i), k}$ is the same for every country belonging to the same cluster $k(i)$. This reduces the total number of $\beta_{k(i), k}$ up to 16.

Taking into account the previous exposition that includes both autonomous and transmission behavior, we propose the following diffusion dynamic model, based on a coupled system of 18 nonlinear differential equations, one per country, to study the dynamic evolution of the CRS's Latin American countries:

$$
\begin{align*}
C_{i}^{\prime}(t)= & \alpha_{i} C_{i}(t)+\sum_{1 \leq k, k(i) \leq 4} \beta_{k(i), k} C_{i}(t)  \tag{7}\\
& \times\left(\bar{C}_{k}(t)-C_{i}(t)\right), \quad 1 \leq i \leq 18
\end{align*}
$$

where $k(i)$ is defined by (5) and $\alpha_{i} \in \mathbb{R}$ and $\beta_{k(i), k}>0$ are the $18+16=34$ model parameters.

## 3. Probabilistic Predictions over the Next Few Months

This section is divided into three subsections. The first one, is devoted to model parameters estimation. Since uncertainty

Table 3: Estimation of the autonomous model parameters, $\alpha_{i}$, separated by clusters.

| Country name | $\alpha_{i}$ | Country name | $\alpha_{i}$ |
| :--- | :---: | :---: | :---: |
|  |  | Honduras | -0.0440 |
| Chile | Ecuador | -0.5731 |  |
|  | 1.6844 | Nicaragua | -0.7551 |
|  |  | Dominican Rep. | -0.4340 |
|  |  | Trinidad Tobago | 1.3411 |
| Brazil | 1.8554 | Costa Rica | 1.0763 |
| Mexico | 1.2784 | Paraguay | -0.0203 |
| Peru | 1.1126 | El Salvador | 0.4658 |
| Colombia | 1.5382 | Argentina | -0.4093 |
| Uruguay | 0.0696 | Bolivia | -0.5837 |
| Panama | 1.2563 | Venezuela | -0.4590 |

Table 4: Estimated values of the contagion model parameters. The value of model parameter $\beta_{k(i), k}$ measures the contagion effect transmitted by the countries belonging to cluster $k(1 \leq k \leq 4)$ on country $i(1 \leq i \leq 18)$ belonging to cluster $k(i)$ according to assignment (5). Figures indicate that countries in clusters $k=3,4$ have a remarkable influence on the others, being lower on Chile (columns 3 and 4). In addition, it can be observed that Chile (cluster 1) has a strong influence on countries which belong to cluster 2 (element ( 2,1 ), whose value is 0.01122 ).

| $\beta_{k(i), k}$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
| :--- | :---: | :---: | :---: | :---: |
| $k(i)=1,1 \leq i \leq 1$ | 0 | 0.01122 | 0.02525 | 0.01736 |
| $k(i)=2,2 \leq i \leq 7$ | 0.05222 | 0.00007 | 0.04198 | 0.07828 |
| $k(i)=3,8 \leq i \leq 12$ | 0.00047 | 0.00180 | 0.03809 | 0.04129 |
| $k(i)=4,13 \leq i \leq 18$ | 0.00004 | 0.00283 | 0.04903 | 0.04771 |

and variability are the rules when dealing with modelling real problems, in the second subsection, we provide predictions by means of confidence intervals obtained using a crossvalidation technique. In the third one, we validate and discuss the obtained results.
3.1. Parameter Estimation. As we have previously pointed out, this subsection is firstly addressed to estimate the parameters of model (7). This task has been performed by fitting the model in the mean square sense to the available data using optimization techniques. Computations have been carried out with Mathematica 8.0 [29].

The system of differential equations (7) is numerically solved by taking as initial conditions the CRS data of February 6,2012 (corresponding to $t=0$ ), that is, according to Table 2. Tables 3 and 4 report the estimation of the autonomous $\left(\alpha_{i}\right)$ and contagion $\left(\beta_{k(i), k}\right)$ model parameters.

For the sake of clarity, in Figure 2 we have depicted the estimation of the autonomous model parameters in four plots according to previous clustering. Notice that countries in cluster 2 have similar values for autonomous decision except for Uruguay (identified by number 5 in cluster 2) whose gross public sector debt contracted in the second quarter of 2010, after five consecutive periods of sustained increase and reached the equivalent to $59.5 \%$ of the GDP [30]. Analogously, in cluster 3 except for Trinidad and Tobago (identified by number 5), which is considered one of the wealthiest

Cluster 1

(a)

Cluster 3

(c)

Cluster 2

(b)

Cluster 4

(d)

Figure 2: Representation of the estimation of the autonomous model parameters, $\alpha_{i}$, per cluster. Notice that countries in cluster 2 have similar values for autonomous decision except for Uruguay (identified by number 5 in cluster 2) whose gross public sector debt contracted in the second quarter of 2010, after five consecutive periods of sustained increase, and reached the equivalent to $59.5 \%$ of the GDP [30]. Analogously, in cluster 3 except for Trinidad and Tobago (identified by number 5), which is considered one of the wealthiest and most developed nations in the Caribbean [31]. The countries are in the same order as they appear in Table 3.
and most developed nations in the Caribbean [31]. Also, looking at Table 4, observe that the estimated transmission parameters suggest that except for Chile (cluster 1) countries in clusters $k=3,4$ have a remarkable influence on the rest of the countries. In addition, it can be observed that Chile (cluster 1) has a strong influence on countries which belong to cluster 2 (see Figure 3).

In Figure 4, we can see graphically the model fitting for every Latin American country. The solid lines plotted represent the model approximation for all the countries of each cluster, namely, cluster 1: Chile; cluster 2: Brazil, Mexico, Peru, Colombia, Uruguay, and Panama; cluster 3: Honduras, Ecuador, Nicaragua, Dominican Rep., and Trinidad Tobago; cluster 4: Costa Rica, Paraguay, El Salvador, Argentina, Bolivia, and Venezuela. The black points represent the available data of CRS which have been used to fit the model. Taking into account the scale in vertical axis, the model fits accurately the available data.
3.2. Introducing Uncertainty into the Model: Predictions over the Next Few Months Using Confidence Intervals. Randomness can be attributed not only to sampling errors in the data
but also to the inherent complexity of the phenomenon under study. This statement particularly holds in dealing with economic problems such as forecasting CRS, since there are large jumps in CRS points in several countries with a difference of a week (until 4.2 CRS points in Trinidad and Tobago). Therefore, it is more realistic to construct predictions by confidence intervals. To calculate these intervals, let us use an adaptation of the statistical technique usually referred to as Cross-Validation or rotation estimation [32, 33]. CrossValidation is a versatile statistical method of evaluating and comparing learning algorithms by dividing data into two segments: one used to learn or train the model and the other used to validate the model. Apart from its basic formulation, several variations of Cross-Validation have been proposed to get other statistical goals including the estimation of model parameters [34, 35].

The version of the Cross-Validation process we propose is the following:
(i) we have 18 CRS data sets, one for each country, for 25 different time instants between February 6, 2012, and August 6, 2012 (1st data set, the black points in Figures 4 and 5) and for 13 different time instants between


Figure 3: In this figure we use a grayscale to represent the level of contagion from cluster in column $j$ to cluster in row $i(1 \leq i, j \leq 4)$ (the darker the more contagion).

August 13, 2012 and November 5, 2012, for further model validation (2nd data set, the green points in Figure 5);
(ii) we take the CRS data corresponding to $h$ time instants among the 25 available in the 1st data set. There are a lot of combinations for taking $h$ time instants among 25 . However, in case that the number of combinations is greater than 5,000, we take randomly 5,000. Let $n_{h}$ be the number of selected combinations with $h$ data;
(iii) we fit the model with each one of the $n_{h}$ selected combinations of $h$ data taking as the model initial condition the earliest ordered data, obtaining a set of $n_{h}$ estimations of the model parameters;
(iv) we substitute each one of the obtained $n_{h}$ estimated parameters into the model and solve it numerically;
(v) we compute the model outputs for the $n_{h}$ model solutions in the known 25 time instants of the 1st data set and the 13 time instants of the 2nd data set;
(vi) for each time instant, we have $n_{h}$ model output values and we can obtain the $95 \%$ confidence interval by computing percentiles 2.5 and 97.5 of the $n_{h}$ data.

Among all the values of $h$ used, we determine as the best one which makes the maximum number of CRS data in 1st and 2 nd set lie inside the $95 \%$ confidence interval. Then, we will be able to provide reliable probabilistic predictions extending the $95 \%$ confidence intervals 6 months after November 5, 2012 (end of the 2nd data test). After testing all the combinations, the best obtained value is $h=3$, where $98.44 \%$ of the 1st data set ( 443 out of 450 ) and $73.08 \%$ of the 2nd data set ( 171 out of 234) lie inside their corresponding $95 \%$ confidence intervals.

Considering the relevance of each cluster for our study, we have selected at least one representative country taking into account the different performance of CRS in the countries belonging to each cluster. Thus, we selected Chile as representative of cluster 1; Mexico, Brazil, and Uruguay of cluster 2; Ecuador in case of cluster 3; and Paraguay, Argentina, Bolivia, and Venezuela as representatives of cluster 4.

In Figure 5 we can see graphically the probabilistic prediction of the CRS of the aforementioned countries extending the $95 \%$ confidence intervals 6 months after November 5, 2012. Additionally, the 1st data set (black points) and the 2nd data set for validation (green points) are presented. To obtain Figure 5 we calculate $95 \%$ confidence intervals (red lines) and their mean (dashed line) from April 2, 2012, to May 5, 2013, with time jumps of 0.01 .
3.3. Validation and Discussion. According to the obtained results, we should remark the difficult task of forecasting the complex economic dynamics of 18 Latin American countries considering the current international scenario, added to the uncertainty in the global economy. The CRS data behavior reflects these circumstances, since for some CRS data, we can observe sudden large jumps in a short time.

Now, we proceed to predict and validate the model. With this aim, we gather new actual and current CRS data, in particular, from August 13, 2012, to November 5, 2012 (the 2nd data set), data that we did not use initially to fit the model because they were not available at that time. These new data allow us to compare and validate the obtained predictions from our model with the new real data.

Looking at Figure 5, in general, there is a certain stability or a slight increasing trend in the CRS for all the countries over the next few months with uncertainties given by $95 \%$ confidence intervals. More precisely, considering the average prediction (dashed lines) there is less uncertainty in the lower range of the confidence intervals than in the upper ones.

The $95 \%$ confidence intervals provide a quite accurate forecasting for almost all considered countries, being not wider than 5 CRS points for all the countries for early May 2013. In fact, the confidence intervals contain almost all the black points $(98.44 \%)$ and a high percentage of the green points ( $73.08 \%$ ). Also, most of the points outside the $95 \%$ confidence intervals are relatively close to them. On average, the total percentage of points that lie inside the confidence intervals is up to $89.77 \%$, a high rate taking into account the uncertainty of economic and political situation of some countries.

Argentina and Venezuela are the countries where the forecasting has been least valid. In particular, in the case of Argentina, it can be noted that from late January 2012 (42.29 CRS points) to early November 2012 (34.46 CRS points), there is a drop of 8 CRS points due to the tensions derived from the government interference and expropriation. However, the $95 \%$ confidence intervals forecast the period August 13th until September 24 when a drop of a CRS points is experienced because of its rising expropriation risk and its nonpayment debt risk. A right prediction of 6 weeks in Argentina is not a minor issue due to its instability. However, we only predict correctly 3 weeks for Venezuela.


FIgURE 4: CRS fitting for the 18 selected Latin American countries. The solid line is the model solution for the parameters of Tables 3 and 4 that best fit the model. The points are the known CRS data from February 6, 2012, to August 6, 2012. Notice the different vertical scales in the plots for each country.


Figure 5: Probabilistic Country Risk Score forecasting for selected Latin American countries belonging to each cluster (cluster 1: Chile; cluster 2: Brazil, Mexico, and Uruguay; cluster 3: Ecuador; cluster 4: Paraguay, Argentina, Bolivia, and Venezuela). The dashed line is the mean of the $95 \%$ confidence interval and the red lines correspond to $95 \%$ confidence intervals between April 2, 2012, and May 5, 2013. The drawn black points are the known CRS of the 1st data set. The green points are the actual and current available data (from August 13, 2012, to November 5,2012 ) used in order to validate the model, the 2nd data set. Notice that every plot has its own numerical range in the vertical axis.

In countries like Uruguay, Ecuador, and Bolivia, we also have a right prediction of 6 weeks, when an increasing jump of 1.2-1.9 CRS points arose the last week of September. However, we should take into account that, if the jumps had happened 1-2 months later, the corresponding $95 \%$ confidence intervals would have captured these jumps. In some way, the jumps are predicted by the model with a delay. This does not happen with Argentina or Venezuela. On the other hand, quite accurate predictions are obtained for countries with an economic and political stability such as Chile, Brazil, Mexico, and Paraguay since the $95 \%$ confidence intervals provided by the Cross-Validation contain all the points considered for validation. In fact, it is possible that, for these stable countries, our forecasting keeps valid longer than the 3 months of available data (2nd data set).

Thus, our model allows for predicting a longer period of time for most of the countries. However, the economic and
political instability of some countries involves jumps of some CRS points which might not permit the predictions to be as accurate as in the more stable countries.

## 4. Conclusions

Worldwide financial crisis and changes in sovereign debts ratings have also affected the Latin American financial markets and their economies. However, Latin America's relative resilience to more acute rise in risk seen in other regions like Europe during last years is offering investors new options to improve risk-return trade-offs. Country Risk Score (CRS) represents a measure of the level of confidence on each country and a measure of its economic health. Latin America, a regional grouping of several countries, has also invariably succumbed to increased risk this year, according to Euromoney's Country Risk Survey, in line with the global trend.

In this work, we present a dynamic diffusion model to study the evolution of the Country Risk Score (CRS), for a total of 18 Latin American countries, which considers both the endogenous effect of each country politics and the contagion effect among them. Using data of CRS, we fit the model with the data estimating unknown autonomous behavior and transmission parameters. Then, we use an adapted Cross-Validation technique in order to provide probabilistic predictions over the next months (August 2012 until May 2013) taking into account that most of CRS data should be inside the confidence intervals corresponding to their time instants.

The obtained results depict quite well the evolution of the CRS for most of the countries, despite the jumps and uncertainty in the CRS data within some periods. Chile is still holding its own as the darling of the region, and Brazil remains the second safest. However, whereas seems to be more confidence in Uruguay and Ecuador, faith in Venezuela and Argentina has diminished alarmingly. The increased perception of risk stems from a range of domestic and external factors, from political and economic policy failings in Argentina to worries about the impact of dissipating global growth prospects for the region's exports.

As we have remarked, it should be pointed out that mathematical modelling with probabilistic predictions may be a powerful tool where policy makers and investors are able to design strategies, modify the model parameters in order to simulate them, and analyse the effect of changes. Looking back at the last year, the Latin American score decline is relatively mild in comparison with the falls seen in countries which belong to the Eurozone and Central and Eastern Europe. As we can check in our results even over next months horizon, Latin America might held up fairly well, despite an average score loss driven by drops for Argentina and Venezuela.

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## Research Article

# Hot Topic Propagation Model and Opinion Leader Identifying Model in Microblog Network 

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#### Abstract

As the network technique is fast developing, the microblog has been a significant carrier representing the social public opinions. Therefore, it is important to investigate the propagation characteristics of the topics and to unearth the opinion leaders in Micro-blog network. The propagation status of the hot topics in the Micro-blog is influenced by the authority of the participating individuals. We build a time-varying model with the variational external field strength to simulate the topic propagation process. This model also fits for the multimodal events. The opinion leaders are important individuals who remarkably influence the topic discussions in its propagation process. They can help to guide the healthy development of public opinion. We build an AHP model based on the influence, the support, and the activity of a node, as well as a microblog-rank algorithm based on the weighted undirected network, to unearth and analyze the opinion leaders' characteristics. The experiments in the data, collected from the Sina Micro-blog from October 2012 to November 2012 and from January 2013 to February 2013, show that our models predict the trend of hot topic efficiently and the opinion leaders we found are reasonable.


## 1. Introduction

Microblog is another important network information interactive and propagative platform after blog. It is based on the network and communication technology. There are considerable advantages on the speed and space of information propagation as well as on the breadth and the depth of reports. Microblog opinion leaders rely on their microblog amount and quality to raise a drastic group debate through setting discussion topics on this free and open platform. They even cause the attitude shaping, turning, and action following. According to the statistics, among the Chinese Internet users, microblog users older than 19 years old occupy $88.81 \%$ until September 20, 2012. The number of the microblog users is about 327 millions [1]. Microblog has been a crucial network tool for information propagation. Therefore, it is important to predict topic law and propagation trend in microblog network and study the opinion leaders in topics. It will contribute to design corresponding mechanisms to guide and control the propagation process.

Nowadays, researches about topic diffusion law have obtained high attention, which are mainly related to the time
varying model [2, 3]. Zhao et al. [2] put forward a propagation model in discrete time based on the node popularity and liveness. Zhang et al. [3] used epidemic model for reference to deduce both the BBS and the blog multimodal topic propagation models as well as the multimodal ones. Yan et al. [4] proposed an extended susceptible-infected (SI) propagation model to incorporate bursty and limited attention. Chen and Gao [5] defined some authority nodes that release anti-rumor information as the prevention strategy to control the rumor in a directed microblog user network. And some works predicted diffusion probabilities by independent cascade (IC) model [6, 7]. Afrasiabi and Benyoucef [8] observed that the effect on propagation of people who are not either in a friendship network or a subscription network is higher than that of friends or subscribers. Yoganarasimhan [9] studied how the size and structure of the local network around a node affect the aggregate diffusion of products seeded by it.

Identification of opinion leaders has been widely concerned. Zhai et al. [10] gave many kinds of recognition methods in their work, while there are three research methods in opinion leader recognition: firstly, an analytical method based on the characteristic attribute, for instance, AHP
method [11] and TOPSIS method [12], and an improved mix framework for opinion leader identification [13]; secondly, a method based on the cluster analysis, such as opinion leader recognition with K-means clustering method [14]; thirdly, a method based on social network analysis, including the PageRank algorithm [15-17] and HITS algorithm [17]. However, the propagation model [2] simulated certain topic propagation process accurately. Its effects are not satisfactory when topics contain subevents. PageRank algorithm [17] only considered the interactive relationship between users but did not consider a user's own authority.

This paper censuses and analyzes data of four hot topics of Sina Microblog, which has the most users in China. We first build a time-varying model with the variational external field strength to simulate the topic propagation process in Section 2. This model also fits for the multimodal events. Then, we build an AHP model based on the influence, the support, and the activity of a node, as well as a microblogrank algorithm based on the weighted undirected network to unearth and analyze the opinion leaders' characteristics in Section 3. The experiments in the data, collected from the Sina Microblog, show that our models predict the trend of hot topic efficiently and the opinion leaders we found are reasonable.

## 2. The Hot Topic Propagation Model

Hot topic refers to the hot issue that the public most care about within a certain time and range. In recent years, most issues come to public attention through the Internet. This paper takes Sina Microblog as the background and takes the hot topic as research object. This research observes the characteristics of the dynamic propagation process and may digs the opinion leaders.
2.1. The Hot Topic Propagation. The propagation velocity of hot topic is wide and quick. In order to collect the real-time and more complete microblog data, we use Rweibo to grab the Sina Microblog data automatically. Rweibo is a software development kit of R language, which implements the interface provided by Sina microblog. The data refers to the numbers of talking about these hot events on Sina Microblog. We analyze the quantity change of 4 events, 40 days after happening. The 1st event is about Yuan Lihai's adopting those abandoned babies and orphans. The 2nd event is about the PM2.5 haze in China. The 3rd event is concerning the Diaoyu Islands. And the 4th event is concerning the 2012 Nobel Prize for literature which Mo Yan was awarded.

As shown in Figure 1, after these incidents, the rate of the amount of daily posting is easily seen. The figure's horizontal axis shows the days of these events and the vertical axis shows the percentage of the amount of daily posting and the total number of microblogs that the users participate in discussing one topic in the network.

In event 1, the number of its microblog posts peaks in a day, which shows the timeliness of microblog. The number of microblog postings on event 2 shows the first peak from the 5th day to the 7th day. The National Meteorological Center


Figure 1: The rate of the amount of daily posting.
of CMA issued a haze alert so that the second peak occurred after the 22 day. The number of microblog posting peaked in the 16th, since Japan deployed fighter plane to prevent Chinese plane from flying in the Diaoyu Islands on the 14th day, and the USA has long interfered with this event. After Mo Yan was awarded the 2012 Nobel Prize for literature, the number of postings on the event doubled. We can see the development trend of the event through the number of microblog every day. The data we collected is completely matched with the actual situation. As shown in Figure 1, event 1, event 3 and event 4 belong to the single-peak events. They meet at the peak, and the propagation rate spread slowly so they died in about 30 days. Otherwise, event 2 belongs to the multipeak event. Its propagation rate has two peaks, and the first one is higher than the second one. Therefore, the data collection of identifying the opinion leaders' needs to last for at least 30 days after the first peak appeared.
2.2. The Hot Topic Propagation Model. Let the undirected graph $G=\{V, E, W\}$ represent the actual propagation network, where $V$ is the set of microblog nodes, $E$ is the set of the edges of connecting the users, and $W$ is the set of authority value. We suppose that any two nodes can communicate with each other and the microblog network is a fully connected undirected graph. Zhao et al. [2] proposed a discrete time dynamic model for bursty propagation of incidental events. We build a time-varying model based on Zhao's model with the variational external field strength to simulate the topic propagation process.

Assume that $N_{\text {MAX }}$ represents the total number of microblogs that participate in discussing one topic in the network. Let $t_{0}$ be the initial time and let $t_{n}$ be the $n$ unit time. Let $I\left(t_{n}\right)$ be the posted microblog numbers at $t_{n}$ and let $r\left(t_{n}\right)$ be the new posting microblog number in $\left(t_{n-1}, t_{n}\right]$. Namely,

$$
\begin{equation*}
I\left(t_{n}\right)=I\left(t_{n-1}\right)+r\left(t_{n}\right) \tag{1}
\end{equation*}
$$

We mainly discuss the statistical properties of $r\left(t_{n}\right)$ and the change trend of $r\left(t_{n}\right)$ by the simulation.

The authority value of the user in the actual network is average value through the normalization of friends count, fans count, and microblog count. After checking the actual
data of four events, we know the authority value follows the power-law distribution. Let the authority value of the user $i$ be $w_{i}$. Its distribution is $p(w)$ which follows the power-law distribution, and the power law is at $[-1.3,-1.9]$. Therefore, the authority probability density function is defined as

$$
\begin{equation*}
p(w)=(1+\beta w)^{-\alpha} \tag{2}
\end{equation*}
$$

where $\alpha$ is 1.5 at $[1.3,1.9]$ and $\beta$ is a parameter.
The node state is divided into the published microblog and the unpublished microblog. The function $\delta_{i}\left(t_{n}\right)$ represents the state of microblog $i$ at $t_{n}$. Consider the following:

$$
\delta_{i}\left(t_{n}\right)= \begin{cases}1, & \text { the published microblog; }  \tag{3}\\ 0, & \text { the unpublished microblog. }\end{cases}
$$

The topic field strength formed by internal nodes in the network is defined as

$$
\begin{equation*}
B_{1}\left(t_{n}\right)=\sum_{j=1}^{N_{\max }} \delta_{j}\left(t_{n-1}\right) w_{j} \tag{4}
\end{equation*}
$$

where $w_{i}$ is the authority value of node $i$.
In fact, we can obtain the topic from the external network information. With the time passing, the external field strength will improve over time above a fundamental level and then tend to be stable. Because the external field strength is limited to the environmental capacity, we assume that the external field strength follows the logistic model partly. Suppose $B_{0}$ is a parameter related to the rate of the initial external field strength changing and $B_{m}$ is the fundamental level. The external field strength formula is as follows:

$$
\begin{equation*}
B_{2}\left(t_{n}\right)=B_{m}+\frac{1}{1+\left(\left(1 / B_{0}\right)-1\right) e^{-t_{n-1}}} \tag{5}
\end{equation*}
$$

In practice, some events contain two or more subevents. For example, event 2 contains two sub-events: "The National Meteorological Center of CMA issued a yellow haze alert on the 5th day" and "haze is enshrouded in eastern and midland China on the 21st day." The subevent can lead to a high propagation rate.

Therefore, on the event day, the simulation system is reset by the certain proportion. Namely, we turn some of nodes' state from published to unpublished when the first day of the second sub-event of each event comes. According to the actual situation, the occurred event time is known, saying that to set the occurred sub-event time is reasonable.

If the microblog $i$ gets the topic information from the network at $t_{n}$, the probability of the unpublished state transformed into the published state is

$$
\begin{equation*}
P_{i}\left(t_{n}\right)=1-(1-\lambda)^{w_{i}\left(B_{1}\left(t_{n}\right)+B_{2}\left(t_{n}\right)\right)} \tag{6}
\end{equation*}
$$

The different topics have some differences on the microblog number. In order to see the trend, we perform normalization to the propagation data; namely,

$$
\begin{equation*}
r_{0}\left(t_{n}\right)=\frac{r_{0}\left(t_{n}\right)}{N_{\mathrm{MAX}}} \tag{7}
\end{equation*}
$$

In order to judge the simulation effect, we define the mean square error as the error function:

$$
\begin{equation*}
\sigma=\sqrt{\frac{1}{N} \sum_{i=1}^{N}\left(x\left(t_{i}\right)-\mu\right)^{2}} \tag{8}
\end{equation*}
$$

where $\mu$ represents the actual normalized data,

$$
\begin{equation*}
\mu=\frac{x\left(t_{n}\right)}{\sum_{i=1}^{n} x\left(t_{i}\right)} . \tag{9}
\end{equation*}
$$

2.3. Simulation. We set the following steps in Algorithm 1 to simulate the process of the topic dynamic propagation.

After collecting the real data of event 1 to event 4, we use the computer program to estimate optimal parameters within a reasonable range of parameters. The result is shown in Table 1.

Zhao's algorithm [2] aims at the sudden accidents that do not contain sub-events. Accordingly, we give out the parameters in this algorithm, as listed in Table 2.

We work out the average error and minimum error of our algorithm and Zhao's algorithm in 1000 tests. Figure 2 and Table 3 are the algorithm comparison of events 1,3, and 4.

The two algorithms have the better results in unimodal topic propagation. Event 2 contains sub-event, so the result has the obvious difference. As shown in Figure 3 and Table 4, our algorithm has better results on the precision.

## 3. Opinion Leader Identifying Model of Topics Network

Now, microblog, which is known as the most deadly public opinion carrier in network, creates a new era of the Internet media. With the emergence and prosperity, microblog not only provides a new platform to the traditional opinion leaders but also provides the fertile soil for the growth of the emerging opinion leaders.
3.1. Microblog Dataset. From the section above, we discuss the topics of how to propagate in the microblog network. We know that a topic will last for about 30 days. So the opinion leaders may appear in 30 days after the incident occurred. Therefore, we only dig out information in that period on the web. The data we use in this paper is about 3 hot topics in January 2013 and the event that Mo Yan awarded the 2012 Nobel Prize for literature, as shown in Table 5.

The details information of each microblog is as follows:
(1) microblog: ID of microblog, the number of comments, the number of forward, the text of microblogs, the length of microblog, the posting time;
(2) author: ID of user, the number of fans, the number of friends, the number of microblogs;
in addition, we also collect information of comments about the event 4;
(3) comment: ID of comment, the text of comment, and the length of comment, the posting time.

```
Function Topic Propagation\{
Initialize
    \(N \_M A X, w_{i}, B_{m}, B_{0}, N\).
    \(\delta_{i}\left(t_{0}\right)=0\).
    \(n=1\).
While \((n \leq N)\) \{
    \(r\left(t_{n}\right)=0\).
    if (the new sub-event occurs) \(\{\)
    if \((\operatorname{rand}()<0.5)\{\)
    \(\delta_{i}\left(t_{k}\right)=0, k=n, n+1, \ldots, N \_\)MAX \(\left., i=1,2, \ldots, N \_M A X.\right\}\)
    \}
    \(i=1\).
While ( \(i \leq N\) _MAX \()\{\)
        \(P_{i}\left(t_{n}\right)=1-(1-\lambda)^{w_{i}\left(B_{1}\left(t_{n}\right)+B_{2}\left(t_{n}\right)\right)}\).
        if \(\left(\operatorname{rand}()<P_{i}\left(t_{n}\right) \& \delta_{i}\left(t_{n}\right)==0\right)\{\)
        \(\delta_{i}\left(t_{k}\right)=1, k=n, n+1, \ldots, N_{-}\)MAX.
        \(\left.r\left(t_{n}\right)=r\left(t_{n}\right)+1.\right\}\)
        \(i=i+1\).
    \}
    \(r_{0}\left(t_{n}\right)=r\left(t_{n}\right) / N \_M A X\).
    \(n=n+1\).
\}
    \(\sigma=\sqrt{\frac{1}{N} \sum_{i=1}^{N}\left(x\left(t_{i}\right)-\mu\right)^{2}}\).
Return
    \(r_{0}\left(t_{n}\right), \sigma\).
\}
```

Algorithm 1: Topic Propagation.

TABLE 1: The algorithm parameters settings.

|  | $\alpha$ | $\beta$ | $B_{m}$ | $B_{0}$ | $\lambda$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Event 1 | 1.5 | 0.1 | 900 | 0.1 | $1.8 * 10^{-6}$ |
| Event 2 | 1.5 | 0.8 | 20 | 0.1 | $8 * 10^{-5}$ |
| Event 3 | 1.5 | 0.7 | 10 | 0.5 | $3 * 10^{-5}$ |
| Event 4 | 1.5 | 0.82 | 400 | 0.5 | $1.6 * 10^{-4}$ |

Table 2: The parameter settings in Zhao's algorithm.

|  | $\beta$ | $B_{2}$ | $\lambda$ |
| :--- | :---: | :---: | :---: |
| Event 1 | 0.82 | 450 | $1.6 * 10^{-5}$ |
| Event 2 | 0.1 | 2 | $6 * 10^{-4}$ |
| Event 3 | 0.3 | 10 | $1 * 10^{-4}$ |
| Event 4 | 0.5 | 800 | $1.3 * 10^{-4}$ |

Through Figure 4, we can see that the number of forward and the number of comments satisfy the power-law distribution and the exponent is in $[-1.55,-1.30]$. It proves that the communication networks of these events are scale-free networks, and only a few users have much focus, so opinion leaders possibly exist.
3.2. The Method of Identifying Opinion Leader. Although the theory of opinion leader has been widely used in different fields, the judgment standards of opinion leaders are
divergent. There are three traditional methods of finding opinion leaders: questionnaire, self-report, and observation, but the cost of these methods is too high. Sina Microblog is a platform for information exchanging, so users can show their opinions to others by commenting and forwarding microblog. Users communicate with each other through commenting and forwarding microblog. Interaction provides a lot of data to support our research on opinion leaders. According to the definition of opinion leaders proposed by Paul Lazarsfeld, opinion leaders should be very active and have much influence in some topics. Therefore, we should analyze microblog opinion leaders from three aspects: influence, support, and activity. The more influence the users have, the more response they obtain by posting information and influence for the other users accordingly. In addition, opinion leaders should take an active part in discussing any topics and interact with other users such that it is more likely to show their own ideals to others.

In this section, considering these three aspects and combining the characteristics of microblog spreading, we extract features of opinion leaders. Then, we identify and analyze opinion leaders using methods based on the PageRank algorithm and the analytic hierarchy process (AHP).
3.2.1. AHP. In this assessment system, we set 3 one-class indexes and 7 two-class targets, as shown in Table 6. The value of two-class targets is normalization of the actual data.


Figure 2: Comparison of the mean square error of simulation data and real data of events 1,3 , and 4 .

Table 3: Comparison of events 1, 3, and 4.

|  |  | Our algorithm |  | Zhao's algorithm |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  | Average error | Minimum error | Average error | Minimum error |  |
| Event 1 | 0.0219 | 0.0176 | 0.0260 | 0.0220 |  |
| Event 3 | 0.0181 | 0.0109 | 0.0144 | 0.0095 |  |
| Event 4 | 0.0146 | 0.0105 | 0.01251 | 0.0094 |  |



FIGURE 3: Algorithm comparison: (a) demonstration of the simulation results of our algorithm and (b) demonstration of the simulation results of Zhao's algorithm.

Table 4: Comparison of event 2.

|  | Our algorithm |  | Zhao's algorithm |  |
| :---: | :---: | :---: | :---: | :---: |
|  | Average error | Minimum error | Average error | Minimum error |
| Event 2 | 0.0251 | 0.0191 | 0.0346 | 0.0246 |
| Table 5: Dataset. |  |  |  |  |
| Event | Time | The quantity of real data | The quantity of collected data | The rate of collected |
| 1 | 2013/1/4-2013/2/2 | 818 | 794 | 97.07\% |
| 2 | 2013/1/9-2013/2/7 | 1196 | 1138 | 95.15\% |
| 3 | 2013/1/4-2013/2/2 | 1036 | 992 | 95.75\% |
| 4 | 2012/10/11-2012/11/9 | 703 | 703 | 100\% |



Figure 4: The logarithmic graphs of comment amount and forward amount.

Table 6: The assessment system.

| First level indicator | Second level indicator |
| :--- | :---: |
|  | Fans amount $I_{1}$ |
| Influence $I$ | Friends amount $I_{2}$ |
|  | Weibo amount $I_{3}$ |
| Support $S$ | Forward amount $F$ |
|  | Comment amount $C$ |
| Activity $A$ | Release time $T$ |
|  | Micro-blog length $L$ |

Since each two-class target of the same one class target is equally important, equations of 1-class targets are as follows:

$$
\begin{gathered}
I=\frac{1}{3}\left(I_{1}+I_{2}+I_{3}\right), \quad S=\frac{1}{2}(F+C), \\
A=\frac{1}{2}(T+L) .
\end{gathered}
$$

Every two-class target is a normalization of actual data. The formula of normalizing is

$$
\begin{equation*}
x_{0}=\frac{x-x_{\min }}{x_{\max }-x_{\min }} \quad \text { or } \quad x_{0}=\frac{\log _{10} x}{\log _{10} x_{\max }} \tag{11}
\end{equation*}
$$

where $x$ is original data of two-class target. Before normalizing the $\operatorname{target} T$, we should use an equation to measure it. The equation of posting time is $T_{i}=e^{-b|t(i)-t(0)|}$. And we set $t(0)$ be be Jan. 4th, 2013. Supposing $b$ is a parameter, we make it 0.01 . Therefore, the value of assessment about user $i$ is

$$
\begin{equation*}
\operatorname{AHP}(i)=w_{I} I(i)+w_{S} S(i)+w_{A} A(i) \tag{12}
\end{equation*}
$$

where $w$ is the vector of weight and $w=\left(w_{I}, w_{S}, w_{A}\right)$ (see Algorithm 2).
3.2.2. Microblog-Rank Algorithm. The recognition method of PageRank algorithm is a method based on graph theory. It identifies whether the users are opinion leaders through studying the comments and reviewed numbers among the

```
Function AHP{
Initialize
    M:The number of nodes
    I(i): the value of influence about user i.
    S(i): the power of support about user i.
    A(i): the value of activity about user i.
    w=(\mp@subsup{w}{I}{},\mp@subsup{w}{S}{},\mp@subsup{w}{A}{}).
    i=1.
While (i\leqM)
    AHP(i)= w
    i=i+1.
Return
    The Top N opinion leaders N=1%\cdotM.
}
```

Algorithm 2: AHP.
microblog users and considering the influence of the microblog users. Thus, the microblog opinion leaders are those users who have higher influence, get more comments to their microblogs, actively comment on others' microblogs and form a frequent interaction with surrounding people.

According to the above description, a microblog network on a certain topic can be defined as an undirected network $G=(V, E, W, P)$ with edge weight $W$ and node strength $P$. A node in set $V$ means a microblog user. Set $E$ is an edge set, where edge $\left\langle v_{i}, v_{j}\right\rangle \in E$, which means a relationship of the comments between the user $v_{i}$ and the user $v_{j}$. The $w_{i j}$ means the edge weight between the node $v_{i}$ and the node $v_{j}$, which is the number of comments between the user $v_{i}$ and the user $v_{j}$. Meanwhile, in the actual network, the users have different influential power, such as friends count, fans count, and microblog count, so we should add a node strength $p(i)$ to measure it. As shown in Figure 5.

PageRank algorithm is one of the top ten classical algorithms in data mining. It assigns a numerical weighting to each element of a hyperlinked set of documents, such as the World Wide Web, with the purpose of "measuring" its relative importance within the set. Assume that user $i$ in the microblog network has interactive behavior with others; we define the user's opinion leader value (Microblog-Rank, MR) as follows.

Microblog interactive network is a weighted and undirected network. Firstly, we like to give out the weight of links and nodes. The formula is

$$
\begin{equation*}
w_{i j}=p(i) \cdot N_{i j} . \tag{13}
\end{equation*}
$$

In (13), $p(i)$ is their own influence value measures by normalization of initial data, such as the number of fans, the number of friends, and the number of microblogs. Then, we get the sum of them. $N_{i j}$ is the number of communications between the user $i$ and the user $j$. the Microblog-Rank value for any node $i$ can be expressed as follows:

$$
\begin{equation*}
\operatorname{MR}(i)=(1-\alpha)+\alpha \sum_{\text {edge }\left\langle v_{j}, v_{i}\right\rangle} \frac{\operatorname{MR}(j) w_{j i}}{\sum_{\text {edge }\left\langle v_{j}, v_{k}\right\rangle} w_{j k}} . \tag{14}
\end{equation*}
$$



Figure 5: The relationship of comments among users.

This section is based on the weighted network, so we calculate the MR value by the weight. In addition, because dangling links exist in the actual network, which has no reply link, it will lead the algorithms to be not convergent. Therefore, we add the damping factor $\alpha$, and this factor should be set between 0 and 1 . And $\alpha$ is always 0.85 (see [18]). By the iteration, we can get all the users' MR values (see Algorithm 3).

### 3.3. Actual Examples

3.3.1. The Event of "Mo Yan Being Awarded the Nobel Prize". On October 11, 2012, Beijing time 19 o'clock, the 2012 Nobel Prize for literature was announced and Chinese writer Mo Yan was awarded. This event has received wide attention in China. We try to explore the influence of the emergencies among college students. We collected 703 microblogs in total. The data set covers 698 pairs of comment relationships and involves 1171 users. Then, we establish a microblog interactive network based on reply relationship. Figure 6 is the degree distribution of the network, the abscissa is the number of degrees, and the ordinate is the percentage of each degree in the network.

In Figure 6, we see that the microblogs network of reply relationship is a scale-free network, and it satisfies the powerlaw distribution. Isolated users that did not participate in any replies account for nearly $45 \%$, and only one person received 40 replies.

Using MATLAB R2009a, we calculate the MR value for each user and pick up the opinion leaders who are the users whose MR value is in the top $1 \%$; the others are general users. Furthermore, the opinion leaders are visualized in the interactive network by UCINET6.0. Table 7 gives out the opinion leaders of the event "Mo Yan."

In Figure 7, blue nodes represent general users, while red nodes represent opinion leaders.

In order to analyze the relationship between scale and influence of opinion leaders, we draw a picture to show that. In Figure 9, the influence increases quickly, when there are less than 15 opinion leaders. If the number is more than 30 , the influence is not changing obviously.

From Figure 8, we know that, when $N$ is more than 25, the value of each parameter of opinion leaders tends to be

```
    Function Micro-blog-rank{
    Initialize
    e: The accuracy of convergencee = 10-20}\mathrm{ .
    M: The number of nodes.
    p(i): Influence of user i.
    N
    wij}\mathrm{ : Weight of edge }i\mathrm{ to }j\cdot\mp@subsup{w}{ij}{}=p(i)\cdot\mp@subsup{N}{ij}{}\mathrm{ .
    MR
While}(\mp@subsup{\sum}{i=1}{M}|\mp@subsup{MR}{n}{\prime}(i)-\mp@subsup{MR}{n-1}{(i)|}\mp@subsup{|}{}{2}>e
    MR
Return
    The Top N opinion leaders, N=1%\cdotM.
}
```

Algorithm 3: Microblog-rank.

Table 7: Top 12 ranked users by Micro-blog-Rank from Sina.

| Ranking | MR value | Serial number | ID | Identification |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 17.955 | 329 | Youth Literary Digest | Yes |
| 2 | 12.869 | 515 | Jinan University | Yes |
| 3 | 11.108 | 982 | Entrepreneur magazine | Yes |
| 4 | 10.189 | 1097 | 1011 Zhang Chi | Yes |
| 5 | 8.8108 | 104 | Solitary guest Zhu Qiqi | No |
| 6 | 7.257 | 594 | Chongqing reed | Yes |
| 7 | 5.594 | 958 | Oriental Morning Post | Yes |
| 8 | 5.157 | 481 | China News Week | Yes |
| 9 | 5.135 | 245 | Zhou Jiangong | Yes |
| 10 | 4.944 | 52 | The Sichuan Channel broadcasts at the special ten oclock | Yes |
| 11 | 4.675 | 14 | Micro-blog broadcast | Yes |
| 12 | 4.216 | 945 | New culture network | Yes |



Figure 6: The relationship of comments among users.
stable. When $N$ is less than 10 , each parameter value changes greatly. Therefore, parameter $N$ should be from 10 to 20 . It


Figure 7: The interactive network of "Mo Yan."
is reasonable to let $N$ be 12 and our results in Table 7 are reasonable.

Through the above analysis, we found that the opinion leaders of microblog in an accident should have high value in the number of fans, and the number of forward, the number of comments. Because the more fans an author has, the more users can see the microblog. And high numbers of forward and comments mean that the microblog will get much attention on the Internet. So the result is reasonable.



Figure 8: Diversity comparison for opinion leaders.


Figure 9: The interactive network of "Mo Yan".
3.3.2. Three Hot Topics in January, 2013. Opinion leaders must be those who can give guides in topic discussions and attract more attention. Therefore, we set the weight of the Support to the maximum. In addition, opinion leaders should be those who are active in the topic discussions. Therefore, we set the active to the second most important parameter. The detailed weights are set as the above Table 8.

In order to measure the effectiveness of the algorithm, we use AHP and TOPSIS method [12] to obtain the Top 10 opinion leaders in these three events. The results are shown in Tables 9 and 10.

According to our analysis, the opinion leaders are all those who possess prominent values on one or more attributes (Figures 10, 11, 12, and 13). Their integrated ranks are prior to others.

In event 1 , users of the top 10 opinion leaders are in this list all the time. But their ranks have a little difference. All opinion leaders perform outstandingly on more than one attribute.

TABLE 8: Comparison table of the weights of the indicators.

| First level indicator |  | Second level indicator |  |
| :--- | :---: | :---: | :---: |
| Influence $I$ | $5 \%$ | Fans amount | $1.67 \%$ |
|  |  | Friends amount | $1.67 \%$ |
|  | Microblog amount | $1.67 \%$ |  |
| Support $S$ |  | Forward amount | $42.5 \%$ |
|  |  | Comment amount | $42.5 \%$ |
| Activity $A$ | $10 \%$ | Release time | $5 \%$ |
|  |  | Microblog length | $5 \%$ |

In event 2, the first leader and the second one performs outstandingly on many attributes; however, others merely possess high values on the last two attributes. Moreover, values of parameters of the last six leaders are close to each other.

In event 3, opinion leaders that we obtained all perform outstandingly on "release time" attribute and "microblog length" attribute. From Figure 13, we can come to the conclusion that current affairs such as "Diaoyu Islands" are related more closely to the time and opinion leaders that often appear in the several days after the topic just occurred.

Above all, the results obtained by these two methods are similar. So it proves that the AHP method is cogent and effective. In the TOPSIS, we need firstly to find out the positive ideal solution and the negative ideal solution [5], but this is not needed in the AHP. Therefore, the AHP is simpler and more convenient.
3.4. Opinion Leaders. From the results we recognized, we know that opinion leaders consist of the following kinds of users.
(1) Official microblog users of mass media, including magazines, newspapers, and TV stations such as "Youth Digest," "Entrepreneurial state magazine," "Oriental Morning Post," and "China News Weekly," all belong to the news media or the literature media. Mass media's understanding to the events is more authoritative and deeper than others and could attract more attention from web surfers.
(2) Public figures, such as the radio program host "Guo Chendong," the chairman of the HIERSUN diamond agency "Li Houlin," the radio program host "1011 Zhang Chi," magazine editor "Zhou Jiangong", and the litigant of the "a post-90s girl who showed off her books" "Chongqing Weizi," possess certain social influence and their expressions in microblog attract more attention from others. Thus, their possibilities to be opinion leaders are much bigger than common users.
(3) Microblog users in fields related to the emergencies. "Yuan Lihai's adoption" are about public welfare assistance; therefore, public welfare microblog user "powerful mouse v" exists in opinion leaders; "PM2.5 haze in China" is an event about environment problem; thus, microblog users on environmental protection such as "Moruier Air Purifier" and "Sina Environmental Protection" exist in opinion leaders; "Chinese Diaoyu Island" is politics military hot topics; therefore, "Nothing God 2430" in the field of current affairs and "Nucleon Submarine Chaser" on military field also come to

Table 9: The top 10 opinion leaders based on the AHP.

| Rank | Event 1 |  | Event 2 |  | Event 3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | Southern Urban Daily | Media | Global Hot Topic | Entertainment | Nothing God 2430 | Current politics |
| 2 | powerful mouse v | Public welfare | Li Houlin | Management | Sina <br> Finance | Finance |
| 3 | In and out of the law | Law | Li Danyang | Entertainment | Zheng <br> Hongsheng | Writer |
| 4 | Chen Li | Current politics | Car Words-Car <br> Consumption <br> Choice Expert | Automobile | SingNet | Media |
| 5 | Guo Chendong | Media | Sina Environment Protection | Environment protection | Xi'an Qiangzhe | Media |
| 6 | Xinhua News Agency China Internet Event | Media | Moruier Air <br> Purifier | Environment protection | Wealth Key | Economics |
| 7 | Yu-Yan | Law | Chengyang Police | Government | Hanyi <br> Romeo | Common people |
| 8 | Civil law Li Jianwei | Law | HexunNet | Economics | Gao Weiwei | Media |
| 9 | China <br> Newsweek | Media | Hebei Release | Media | Nucleon Submarine Chaser | Media |
| 10 | CCTV News | Media | Sina Real Estate | Real estate | Cloud <br> Pillow Mist <br> Clothes | Writer |

Table 10: The top 10 opinion leaders based on the TOPSIS.

| Rank | Event 1 |  | Event 2 |  | Event 3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | Southern Urban Daily | Media | Global Hot Topic | Entertainment | Nothing God 2430 | Current politics |
| 2 | Powerful mouse $v$ | Public welfare | Li Houlin | Management | Sina <br> Finance | Finance |
| 3 | In and out of the law | Law | Shelley Xiao Mo | Common people | Zheng <br> Hongsheng | Writer |
| 4 | Chen Li | Current politics | Any officer | IT | SingNet | Media |
| 5 | Guo Chendong | Media | Sina Environment Protection | Environment protection | Hanyi Romeo | people |
| 6 | Yu-Yan | Law | Moruier Air Purifier | Environment protection | Xi'an Qiangzhe | Media |
| 7 | Xinhua News Agency China | Media | The girl on the seaside | Common people | Hanyi Romeo | Common people |
| 8 | China Jianwei | Media | ChinaVenture | Economics | Yugen | Writer |
| 9 | Civil law Li Newsweek | Law | Serenity 347 | Common people | Whitelock Chen Guodong | Common people |
| 10 | CCTV News | Media | Xinmin evening news Estate | Media | Enjoyment HAPPY-XI | Environment |

be opinion leaders. Because the event "Mo Yan won the Nobel Prize" belongs to the topics on the cultural fields, students are more concerned about it. Thus universities' official microblog user such as "Jinan University" may be an opinion leader in this field. They are more authoritative, their understandings are more deeper, and they possess more prestige so they attract attention more easily than common users.


Figure 10: The attribute legend of AHP.


Figure 11: Top 10 opinion leaders of event 1.


Figure 12: The attribute legend of event 2.


Figure 13: The attribute legend of event 3.

## 4. Conclusion

A research of topic propagation characteristics and identification of opinion leaders is important to the guidance of public opinion and rumor control. In the business world, this influence can be put to commercial use. This paper constitutes time-varying hot topic propagation model and models of identifying opinion leaders based on AHP and PageRank algorithm. We use Sina microblog's data of four events to validate and get rational results. However, there are several points that should be improved. We can extend in the following aspects.
(1) On the spread of topics, the number of parameters is a bit big so it is hard to find out accurate value of parameter to fit. We can consider more about the connection between parameters and actual data and simplify the parameters.
(2) This paper considers the opinion leader identification from the aspect of microblog users rather than microblog contents. Therefore, text recognition can be added to truly reflect users' attitude to topics in the future.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Forecasting $\mathrm{SO}_{2}$ Pollution Incidents by means of Elman Artificial Neural Networks and ARIMA Models 

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#### Abstract

An $\mathrm{SO}_{2}$ emission episode at coal-fired power station occurs when the series of bihourly average of $\mathrm{SO}_{2}$ concentration, taken at 5minute intervals, is greater than a specific value. Advance prediction of these episodes of pollution is very important for companies generating electricity by burning coal since it allows them to take appropriate preventive measures. In order to forecast $\mathrm{SO}_{2}$ pollution episodes, three different methods were tested: Elman neural networks, autoregressive integrated moving average (ARIMA) models, and a hybrid method combining both. The three methods were applied to a time series of $\mathrm{SO}_{2}$ concentrations registered in a control station in the vicinity of a coal-fired power station. The results obtained showed a better performance of the hybrid method over the Elman networks and the ARIMA models. The best prediction was obtained 115 minutes in advance by the hybrid model.


## 1. Introduction

Coal-fired power stations are a major source of atmospheric pollutants, $\mathrm{SO}_{2}$ being one of the most significant. Mixed with rain, with $\mathrm{SO}_{2}$ is transformed into sulfuric acid producing acid rain. The wind helps to transport this element thousands of kilometers until it settles on the ground causing various negative effects. Sulfuric acid causes respiratory irritation, sometimes leading to damage to lung tissue.

European legislation on air pollution from coal-fired power stations establishes limits for the emissions of $\mathrm{SO}_{2}$. Specifically, it imposes a limit on the average of 24 consecutive concentrations of $\mathrm{SO}_{2}$ taken at 5-minute intervals. An emission episode is said to occur when the series of biannual averages is greater than a threshold set by current regulations. The interest is in predicting emission episodes with at least one hour in advance.

Forecasting $\mathrm{SO}_{2}$ levels can be addressed through mathematical models such as autoregressive-moving-average (ARMA) or artificial neural networks (ANNs) models. An ARMA model was used by Hassanzadeh et al. in 2009 [1]
to forecast $\mathrm{SO}_{2}$ levels for five stations. According to their results, an ARMA $(2,2)$ model provides reliable predictions. Kandya and Mohan [2] study the forecasting of $\mathrm{SO}_{2}$ (and other pollutants) using five statistical techniques. Although each technique has its own advantages and limitations, they found that the one-day prediction autoregressive integrated moving average (ARIMA) [3] technique scores well over the other techniques. Goyal et al. [4] pointed out that linear models such as multilinear regression MLR and ARIMA fail to predict extreme concentrations of pollutants.

Neural networks have also been used to predict concentrations of $\mathrm{SO}_{2}$ and other pollutants emitted by a power plant. Mok and Tam [5] used three-layered feed-forward artificial neural networks to predict the daily $\mathrm{SO}_{2}$ concentration 5 days in advance. Nunnari et al. [6] compared multilayer perceptron (MLP) models with a neurofuzzy approach and an autoregressive-moving-average model with exogenous inputs (ARMAX) model. The results confirmed the superiority of the MLP model over the other. Pérez et al. [7] compared the forecasting produced by three different methods (MLP, multiple linear regression (MLR), and persistence methods).

They concluded that the MLP models achieved more accurate regression results and better predictions. Fernández de Castro et al. [8] found good results for predicting $\mathrm{SO}_{2}$ levels half an hour in advance in the neighborhood of a power plant using neural networks. Cortina et al. (2008) compared an adaptive linear neural network (ADALINE) and a generalized regression neural network (GRNN) for the prediction of pollution levels due to the chemical industry and electricity generation in Salamanca (Spain). Prediction experiments were carried out for 1,12 , and 24 hours in advance. They concluded that a linear regression network needs less adjustment of parameters than a nonlinear regression network, thus facilitating its implementation; however, to obtain better results with a linear regression network, they need to search for a pattern scheme. Abdul-Wahab and Al-Alawi [9] used a neural network model to forecast $\mathrm{SO}_{2}$ concentration levels at a refinery in Oman; they also analyzed the effect of five meteorological parameters that were expected to affect the $\mathrm{SO}_{2}$ concentrations. Zhang et al. [10] compared the performance of several statistical methods in $\mathrm{SO}_{2}$ forecasting. The results showed that all the methods mentioned can be used in time series analysis of air pollutants although the denoising BP neural network has some advantages, mainly relating to its strong memory and learning ability.

Hybrid ARIMA-ANN models have also been applied in atmospheric pollutant forecasting. Tseng et al. [11] showed that the hybrid performed better than ARIMA or ANN alone. Díaz-Robles et al. [12] developed a hybrid model combining ARIMA and ANN to predict extreme events of particle emission in a city in Chile. They concluded that the hybrid model performs better than either of the models used separately.

In this study we analyzed the usefulness of Elman recurrent neural networks in forecasting $\mathrm{SO}_{2}$ emission episodes in a coal-fired power station, given the capacity of this ANN to work with temporary data and compared to an ensemble method that improves the prediction of an ARIMA model using Elman recurrent neural networks. The method proposed is completely different from those combining ARIMA and ANN models previously exposed. The differences are twofold: (1) the kind of neural network used is different (Elman neural network instead of back-propagation recurrent neural network) and (2) the way the network is used, not for an adjustment of the ARIMA model residuals but for the prediction of $\mathrm{SO}_{2}$ concentration in half an hour before the time of interest.

## 2. Materials and Methods

2.1. The Database. The data for the present research was obtained from a coal-fired power station in northern Spain. It contains the average 5 -minute concentrations of $\mathrm{SO}_{2}$ measured during the year 2012 in a control station located in the neighborhood of the power station. Figure 1 shows the time series for the whole year. As be appreciated, there are some $\mathrm{SO}_{2}$ peaks in the line representing the $\mathrm{SO}_{2}$ concentrations registered in the control station, some of them corresponding to emission episodes.


Figure 1: Time series of $\mathrm{SO}_{2}$ concentrations registered on the control station. Data from July and December were used to validate the models.

The data corresponding from January 1 to June 30 was used to construct one model, while that corresponding to July was used for the referred model validation. Also, another test was carried out. For this test, models were trained with data from January 1 to November 30 and $\mathrm{SO}_{2}$ records for December were used for validation. The purpose of using these two sets of data for the models training was to compare the results obtained with different amounts of training information and also to perform the $\mathrm{SO}_{2}$ concentration prediction in two different seasons (summer and winter, resp.).

### 2.2. Methods

2.2.1. The ARIMA Models. ARIMA models are the most generally used class of models for forecasting time series that can be stationarized by transformations such as differencing and logging [13]. The acronym ARIMA stands for auto-regressive integrated moving average. lags of the differenced series appearing in the forecasting equation are called autoregressive terms, while lags of the forecast errors are called moving average terms. A time series which needs to be differenced to be made stationary is said to be an integrated version of a stationary series.

A nonseasonal ARIMA model [13] is classified as an $\operatorname{ARIMA}(p, d, q)$ model, where
(i) $p$ is the number of autoregressive terms,
(ii) $d$ is the number of non-seasonal differences,
(iii) $q$ is the number of lagged forecast errors in the prediction equation.

The generalized form of ARIMA can be described as follows [14]:

$$
\begin{equation*}
\emptyset(B) \cdot \Phi\left(B^{S}\right) \cdot(1-B)^{d} \cdot(1-B)^{D} \cdot Y_{t}=\theta(B) \cdot \Theta\left(B^{S}\right) \cdot Z_{t} \tag{1}
\end{equation*}
$$

where $B$ is the backward shift operator, $d$ is non-seasonal order of differences, $D$ is seasonal order of differences, and $\emptyset, \Phi, \theta$, and $\Theta$ are polynomials in $B$ and $B^{S}$.


Figure 2: Scheme of a recursive Elman neural network with one hidden layer.

Forecasting based on ARIMA (autoregressive integrated moving averages) models, commonly known as the BoxJenkins approach, comprises the following stages:
(i) model identification,
(ii) parameter estimation,
(iii) diagnostic checking.

These stages are repeated until a suitable model for the given data has been identified. In this research we have used a variation of the Hyndman and Khandakar algorithm [15] which combines unit root tests and minimization of the AIC and MLE to obtain the ARIMA models. The use of these algorithms speeds up the model identification process.
2.2.2. Recurrent Neural Networks Models. For the present research a kind of partially recurrent neural network (RNN) called an Elman network [16] was employed. An Elman RNN is a network with an initial configuration based on a regular feedforward neural network.

An Elman network has a layer called a context layer. The neurons in the context layer, which are called context neurons, hold a copy of the output given by the neurons of the hidden layer to the output one (Figure 2). In the following computing step, information that was given as an output by the hidden layer is used as a new input information for this layer.

The strength of the relationships between neurons in an Elman RNN is indicated by their weights. For this kind of neural network, the weight values of the neurons are chosen randomly when the process is initiated, and their values are changed during the model training in order to optimize them, with the exception of the weights from the hidden layer which do not change during the training process because the values of the context neurons need to receive the output information of the hidden layer as it is calculated.

The dynamics of the Elman RNN are described by the following equations [17]:

$$
\begin{gathered}
X_{h} \cdot(k+1)=S \cdot\left[W_{c} \cdot X_{c}(k+1) \cdot W_{1} \cdot U(k)\right], \\
X_{c} \cdot(k+1)=W_{t} \cdot X_{h} \\
Y \cdot(k+1)=W_{o} \cdot X_{h}(k+1),
\end{gathered}
$$

where $S$ is hyperbolic tangent function, $U(k)$ is an input of the network at a discrete time $k, Y(k)$ is output of the network at a discrete time $k, X_{c}$ is nodes of the context layer, $X_{h}$ nodes of the hidden layer, $W_{c}$ is weight matrix of the context-hidden layer, $W_{I}$ is weight matrix of the input-hidden layer, $W_{r}$ is weight matrix of the hidden-context layer, and $W_{o}$ is weight matrix of the hidden-output layer.

The training of the Elman recurrent neural networks models was made using the Levenberg-Marquardt algorithm. This procedure is a modification of Gauss-Newton's method, which was designed in order to minimize the sum of squares of nonlinear functions combining this technique with the steepest-descent algorithm. The Levenberg-Marquardt algorithm, whose application is currently very common for RNN, was chosen as it does not suffer the slow convergence problems that were reported in the methods [18] from which it is derived.
2.2.3. The Proposed Hybrid Model. Since their introduction in the 1970s [3], ARIMA models have been used for the forecasting of linear time series. As has already been reported in the previous studies, these kinds of models have a poor performance as regards prediction in nonlinear problems [19]. In order to overcome their limitations, a hybrid model is proposed in the present research. This hybrid model consists of two main steps:
(i) training of an Elman recurrent neural network in order to mimic the temporal linear behavior of the $\mathrm{SO}_{2}$ time series, predicting some output values that will be the input of the ARIMA model,
(ii) selection of an ARIMA model that will model the time variation of the $\mathrm{SO}_{2}$ concentration, using as input values the prediction of the recurrent neural network.

Therefore, as may be observed in Figure 3, the proposed hybrid model uses the $\mathrm{SO}_{2}$ concentrations predicted by the Elman recurrent neural network model two to three hours in advance as input for an ARIMA model in order to achieve a more accurate prediction of the $\mathrm{SO}_{2}$ concentration. The main goal of this hybrid model is not only to achieve a general more accurate prediction of the $\mathrm{SO}_{2}$ concentration at all times, but also to improve the detection of pollution incidents. In other


Figure 3: Scheme of the hybrid model.
words, this hybrid model was chosen in order to improve the capability of the previous model to detect incidences of pollution as early as possible. The $\mathrm{SO}_{2}$ concentration in each moment is calculated as the sum of the concentration 5 minutes before ( $t 5$ ) and in the mentioned moment $(t)$ given by the recurrent neural network, plus the increase in the $\mathrm{SO}_{2}$ concentration from the previous moment, and divided by two.

## 3. Results and Discussion

3.1. Results of the ARIMA Model. The best model found using the Hyndman and Khandakar algorithm was the ARIMA ( $5,0,3$ ). Figure 4 shows the autocorrelation functions (a) and the residual partial autocorrelation (b). The root-meansquare error (RMSE) obtained for the set of training data (data from January to November) applying the mentioned ARIMA model was 8.7446. The Ljung-Box statistic [20] was used to check the adequacy of model. The $P$ value for the Ljung-Box statistic was 0.9821 , and therefore it can be stated that the data in the residuals were independently distributed or, in other words, that the residuals from the ARIMA model have no correlation. The error obtained when the ARIMA $(5,0,3)$ model was applied to the set of validation data set corresponding to the month of December was of 8.9629. Similarly, when the equivalent model with data from January to June was trained, the RMSE obtained was 8.9101. In this case, the $P$ value of the Ljung-Box statistic was 0.8347 while the RMSE value of the model applied to the data of July gave a value of 10.60121. As it can be observed in Figure 5, the main


Figure 4: Autocorrelation function variable $\mathrm{SO}_{2}$ (a) and residual partial correlation (b).


Figure 5: ARIMA $(5,0,3)$ : December 2012. Predicted $\mathrm{SO}_{2}$ concentration.
weakness of the ARIMA model is that in spite of its good average, it seems to be unable to predict pollution incidents because it is unable to reproduce the real peaks of $\mathrm{SO}_{2}$ concentration (see Figure 1).
3.2. Results of the Recurrent Neural Network Model. Some Elman recurrent neural networks architectures were tested in order to find the best generalization characteristics of the data. The input variables employed were the $\mathrm{SO}_{2}$ concentrations 2 and 3 hours before the current time and recorded each five minutes.

The best configuration resulted in an Elman neural network with 11 neurons in the hidden layer. The activation


Figure 6: Hybrid model: December 2012. Real $\mathrm{SO}_{2}$ concentration (a) versus predicted concentration (b).
function employed was the hyperbolic tangent [21], while the learning rate was 0.1 and the momentum 0.9 . The results obtained using the information from January to November as training data gave us an RMSE of 5.5225 while convergence was achieved after 934 epochs in the case of the model trained with data from January to June; convergence was achieved after 632 epochs with an RMSE of the training data set of 7.8013. The validation results corresponding to the dataset of December, trained with the data from January to November were 7.6121 while the results of the data corresponding to the month of July using the model trained with the data from January to June gave us an RMSE result of 8.0347.
3.3. Results of the Hybrid Model. The proposed hybrid model was applied to the same database. First of all, a hybrid model using the values of the $\mathrm{SO}_{2}$ concentration from January to November as input data was trained. The RMSE value of the training data set was 5.0238 while the RMSE obtained using the validation subset corresponding to the month of December was 6.6850 . Finally, another model was trained using data from January to June, obtaining an RMSE of 6.6014 while the RMSE obtained when the model was validated with data from the month of July, the value obtained was 6.5356. In this case, the number of epochs that were necessary for the Elman recurrent neural networks convergence was from 1023 and 712, respectively. Figure 6 shows the results obtained of the application of the hybrid model to the
month of December. This figure represents a total of 8,928 measurements, showing their real value and the predicted one. It can be observed how the hybrid model is able to predict the $\mathrm{SO}_{2}$ pollution incidences although in some cases it is not able to predict their maximum values.

## 4. Conclusions

In the present research, the utility of three different mathematical models (ARIMA, Elman recurrent neural network, and a hybrid model) to predict $\mathrm{SO}_{2}$ emission episodes of a coal-fired station was analyzed. Emission episodes correspond to peaks in the time series of $\mathrm{SO}_{2}$ concentration.

The ARIMA model was not able to reproduce emission episodes, just the general trend of the time series. The Elman recurrent neural network performed better given its capacity to detect emission episodes. However, the best results were obtained with a hybrid model that applies the ARIMA model to the Elman neural network output.

The results obtained with the hybrid model made it possible to predict emission episodes 115 minutes in advance, which is a sufficient response time to take preventive measures.

We would like to remark that the main advantages of the method proposed in the present research are, on the one hand, linked with the capability of Elman recurrent neural networks to perform sequence prediction that is beyond the power of a standard backpropagation recurrent neural network, and, on the other hand, with how its capabilities are used in our hybrid model. Therefore, in the proposed hybrid model, the predictions obtained from the Elman recurrent neural network are used as input values for an ARIMA model that, having information corresponding from 30 to 5 minutes in advance, is able to predict the $\mathrm{SO}_{2}$ concentration 55 minutes in advance.

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## Research Article

# Preliminary Orbit Determination of Artificial Satellites: A Vectorial Sixth-Order Approach 

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#### Abstract

A modified classical method for preliminary orbit determination is presented. In our proposal, the spread of the observations is considerably wider than in the original method, as well as the order of convergence of the iterative scheme involved. The numerical approach is made by using matricial weight functions, which will lead us to a class of iterative methods with a sixth local order of convergence. This is a process widely used in the design of iterative methods for solving nonlinear scalar equations, but rarely employed in vectorial cases. The numerical tests confirm the theoretical results, and the analysis of the dynamics of the problem shows the stability of the proposed schemes.


## 1. Introduction

The analysis of linear systems has a well-developed mathematical and computational theory. However, many of the applied problems in science and engineering are nonlinear. This situation is more complicated than the linear one, and the estimation of its solution needs a numerical treatment.

While computational engineering has achieved significant maturity, computational costs can be extremely large when high accuracy simulations are required. The development of a practical high-order solution method could diminish this problem by significantly decreasing the computational time required to achieve an acceptable error level (see, for instance, [1]).

The existence of an extensive literature on higher order methods reveals that they are only limited by the nature of the problem to be solved: in particular, the numerical solution of nonlinear equations and systems is needed in the study of dynamical models of chemical reactors [2] or in radioactive transfer [3]. Moreover, many of numerical applications use high precision in their computations; in [4], high-precision calculations are used to solve interpolation problems in astronomy; in [5], the authors describe the use of arbitrary precision computations to improve the results obtained in climate simulations; the results of these numerical experiments show that the high order methods associated
with a multiprecision arithmetic floating point are very useful, because it yields a clear reduction in iterations. A motivation for an arbitrary precision in interval methods can be found in [6], in particular for the calculation of zeros of nonlinear functions.

In last decades, many researchers have proposed different iterative methods to improve Newton's one, which is still the most used scheme in practice. These variants of Newton's method have been designed by means of different techniques, providing in the most of cases multistep schemes. Some of them are extensions of one-dimensional schemes (see, e.g., [ 7,8$]$ ), and others come from Adomian decomposition (see e.g., $[9,10]$ ), specifically the methods proposed by Darvishi and Barati in $[11,12]$ with super cubic convergence and the schemes proposed by Cordero et al. in [13] with order of convergence 4 and 5 . Another procedure to develop iterative methods for nonlinear systems is the replacement of the second derivative in Chebyshev-type methods by some approximation. In [14], Traub presented a family of multipoint methods based on approximating the second derivative that appears in the iterative formula of Chebyshev's scheme, and Babajee et al. in [15] designed two Chebyshevlike methods free from second derivatives.

A common way to generate new schemes is the direct composition of known methods with a later treatment to


Figure 1: Size, shape, and anomalies in orbital plane 2-dimensional coordinate system.
reduce the number of functional evaluations (see e.g., [1619]). A variant of this technique is the so called pseudocomposition, introduced in [20,21]. Let us note that if the initial approximation or any of the successive estimations make the jacobian matrix almost singular, the convergence is not guaranteed. In some of these cases, the problem can be avoided by using some kind of pseudoinverse to solve the linear system involved in each step of the iterative process (see, for instance, [22, 23]).

Recently, for $n=1$, the weight-function procedure has been used to increase the order of convergence of known methods [7]. This technique can be also used, with some restrictions, in the development of high order iterative methods for systems: see, for example the papers of Sharma et al. [24, 25] and Abad et al. [26], where the authors apply the designed method to the software improvement of the Global Positioning System.
1.1. Preliminary Orbit Determination. A classical reference in preliminary orbit determination is F. Gauss (1777-1855), who deduced the orbit of the minor planet Ceres, discovered in 1801 and afterwards lost. The calculation of its trajectory by means of the procedure designed by Gauss marked the international recognition of Gauss and his work.

The first step in orbit determination methods is to obtain preliminary orbits, as the motion analyzed is under the premises of the two bodies problem. It is possible to set a two-dimensional coordinate system (see Figure 1), where the $X$ axis points to the perigee of the orbit, the closest point of the elliptical orbit to the focus and center of the system, the Earth. In this picture, the true anomaly $\nu$ and the eccentric anomaly $E$ can be observed. In order to place this orbit in the celestial sphere and determine completely the position of a body in the orbit, some elements (called orbital or keplerian elements) must be determined. These orbital elements are as follows.
(i) $\Omega$ (right ascension of the ascending node): defined as the equatorial angle between the Vernal point $\gamma$
and the ascending node $N$; it orients the orbit in the equatorial plane.
(ii) $\omega$ (argument of the perigee): defined as the angle of the orbital plane, centered at the focus, between the ascending node $N$ and the perigee of the orbit; it orients the orbit in its plane.
(iii) $i$ (inclination): dihedral angle between the equatorial and the orbital planes.
(iv) $a$ (semimajor axis): which sets the size of the orbit.
(v) $e$ (eccentricity): which gives the shape of the ellipse.
(vi) $T_{0}$ (perigee epoch): time for the passing of the object over the perigee, to determine a reference origin in time. It can be denoted by a exact date, in Julian days, or by the amount of time ago the object was over the perigee.

The so-called Gauss' method is based on the rate $y$ between the triangle and the ellipse sector defined by two position vectors, $\vec{r}_{1}$ and $\overrightarrow{r_{2}}$, from astronomical observations. This proportion is related with the geometry of the orbit and the observed position by

$$
\begin{equation*}
y=1+X(l+x) \tag{1}
\end{equation*}
$$

where

$$
\begin{gather*}
l=\frac{r_{1}+r_{2}}{4 \sqrt{r_{1} r_{2}} \cos \left(\left(v_{2}-v_{1}\right) / 2\right)}-\frac{1}{2} \\
x=\sin ^{2}\left(\frac{E_{2}-E_{1}}{4}\right)  \tag{2}\\
X=\frac{E_{2}-E_{1}-\sin \left(E_{2}-E_{1}\right)}{\sin ^{3}\left(\left(E_{2}-E_{1}\right) / 2\right)} .
\end{gather*}
$$

The angles $E_{i}, v_{i}, i=1,2$, are the eccentric and true anomalies, respectively, associated to the observed positions $\overrightarrow{r_{1}}$ and $\overrightarrow{r_{2}}$ (let us denote by $r_{i}$ the modulus of vector $\overrightarrow{r_{1}}, i=1,2$ ).

Equation (1) is, actually, the composition of the first Gauss equation

$$
\begin{equation*}
y^{2}=\frac{m}{l+x} \tag{3}
\end{equation*}
$$

and the second Gauss equation

$$
\begin{equation*}
y^{2}(y-1)=m X \tag{4}
\end{equation*}
$$

where $m=\mu \tau^{2} /\left[2 \sqrt{r_{1} r_{2}} \cos \left(\left(v_{2}-v_{1}\right) / 2\right)\right]^{3}, \mu$ is the gravitational parameter of the motion, and $\tau$ is a modified time variable.

The original scheme used by Gauss (see [27]) was based on applying fixed point method on unified (1). By using the initial estimation $y_{0}=1$, the classical Gauss procedure gets to calculate the orbital elements only if the range of the true anomalies corresponding to the observed positions is lower than $\pi / 4$. Our aim is to widen the admissible range of true anomalies up to $\pi$ by solving the same problem but using the nonlinear system formed by (3) and (4), being the unknowns $y$ and $E_{2}-E_{1}$.

In the second section of this paper, we present a class of sixth-order Newton-type methods by using the composition technique and matricial weight functions. The convergence results of this family have been obtained by means of the $n$ dimensional Taylor expansion of the involved functions, by using the notation introduced in [28]. Section 3 is devoted to analyze the efficiency of the proposed methods applied on preliminary orbit determination and other nonlinear problems, compared with the classical Newton', Traub' [14], and Jarratt's [29] methods. In Section 4, the preliminary orbit determination problem is revisited in order to analyze the stability of the mentioned schemes by means of dynamical planes, comparing the set of starting points that leads each of the methods to converge to the solution. We finish the paper with some conclusions and the references used in it.

## 2. The New Family and Its Convergence

In this section, we present our three-step iterative methods designed from Newton's one composing with itself twice, once with a "frozen" function and other one with a "frozen" Jacobian matrix. By using matricial weight functions in the second and third step, we proof that the methods of the proposed family have order of convergence six, under certain conditions of function $F$ and of weight functions.

Let us consider the following three-step iterative method which makes use of the weight functions:

$$
\begin{gather*}
y^{(k)}=x^{(k)}-\left[F^{\prime}\left(x^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right) \\
z^{(k)}=y^{(k)}-H\left(\mu^{(k)}\right)\left[F^{\prime}\left(y^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right)  \tag{5}\\
x^{(k+1)}=z^{(k)}-G\left(\mu^{(k)}\right)\left[F^{\prime}\left(y^{(k)}\right)\right]^{-1} F\left(z^{(k)}\right)
\end{gather*}
$$

where $\mu^{(k)}=\left[F^{\prime}\left(y^{(k)}\right)\right]^{-1} F^{\prime}\left(x^{(k)}\right)$ and $H, G: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ are matricial functions.

Theorem 1. Let $\alpha \in D$ be a zero of a sufficiently differentiable function $F: D \subseteq \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ in a convex set $D$ with nonsingular Jacobian in $\alpha$. Let $H$ and $G$ be any function with the following conditions: $H(I)=0, H^{\prime}(I)=(1 / 2) I, H^{\prime \prime}(I)=0$ and $G(I)=I, G^{\prime}(I)=0, G^{\prime \prime}(I)=(1 / 2) I$, being $I$ the identity matrix. Then, the scheme defined in (5) provides sixth order of convergence, whose error equation is given by

$$
\begin{align*}
& e^{(k+1)} \\
& =\left[-\frac{3}{2} C_{3} C_{2} C_{3}+\frac{1}{4} C_{2} C_{3}^{2}+6 C_{3} C_{2}^{3}\right.  \tag{6}\\
& \left.\quad-C_{2} C_{3} C_{2}^{2}+C_{2}^{3} C_{3}-4 C_{2}^{5}\right] e^{(k)^{6}}+O\left(e^{(k)^{7}}\right),
\end{align*}
$$

where $C_{k}=(1 / k!)\left[F^{\prime}(\alpha)\right]^{-1} F^{(k)}(\alpha), k=2,3,4, \ldots$, and $e^{(k)}=$ $x^{(k)}-\alpha$.

Proof. If we use Taylor expansion of $F\left(x^{(k)}\right)$ and $F^{\prime}\left(x^{(k)}\right)$ around $\alpha$, we obtain

$$
\begin{align*}
F\left(x^{(k)}\right)= & F^{\prime}(\alpha)\left\{e^{(k)}+C_{2} e^{(k)^{2}}+C_{3} e^{(k)^{3}}\right. \\
& \left.+C_{4} e^{(k)^{4}}+C_{5} e^{(k)^{5}}+C_{6} e^{(k)^{6}}\right\} \\
& +O\left(e^{(k)^{7}}\right)  \tag{7}\\
F^{\prime}\left(x^{(k)}\right)= & F^{\prime}(\alpha)\left\{I+2 C_{2} e^{(k)}+3 C_{3} e^{(k)^{2}}+4 C_{4} e^{(k)^{3}}\right. \\
& \left.+5 C_{5} e^{(k)^{4}}+6 C_{6} e^{(k)^{5}}\right\}+O\left(e^{(k)^{6}}\right)
\end{align*}
$$

From (7), we calculate the expression of the inverse

$$
\begin{align*}
& {\left[F^{\prime}\left(x^{(k)}\right)\right]^{-1}} \\
& \quad=\left\{I+X_{2} e^{(k)}+X_{3} e^{(k)^{2}}+X_{4} e^{(k)^{3}}+X_{5} e^{(k)^{4}}+X_{6} e^{(k)^{5}}\right\} \\
& \quad \times\left[F^{\prime}(\alpha)\right]^{-1}+O\left(e^{(k)^{6}}\right), \tag{8}
\end{align*}
$$

where $X_{2}=-2 C_{2}, X_{3}=4 C_{2}^{2}-3 C_{3}, X_{4}=6 C_{3} C_{2}-6 C_{2} C_{3}-$ $8 C_{2}^{3}-4 C_{4}, X_{5}=-5 C_{5}+8 C_{2} C_{4}+8 C_{4} C_{2}-12 C_{2}^{2} C_{3}-12 C_{3} C_{2}^{2}-$ $12 C_{2} C_{3} C_{2}+9 C_{3}^{2}+16 C_{2}^{4}$, and $X_{6}=-6 C_{6}+10 C_{2} C_{5}+10 C_{5} C_{2}-$ $16 C_{2}^{2} C_{4}-16 C_{4} C_{2}^{2}-16 C_{2} C_{4} C_{2}+12 C_{3} C_{4}+12 C_{4} C_{3}-18 C_{3} C_{2} C_{3}-$ $18 C_{2} C_{3}^{2}-18 C_{3}^{2} C_{2}+24 C_{2}^{3} C_{3}+24 C_{2}^{2} C_{3} C_{2}+24 C_{2} C_{3} C_{2}^{2}+24 C_{3} C_{2}^{3}-$ $32 C_{2}^{5}$.

These values have been obtained by imposing the conditions

$$
\begin{equation*}
\left[F^{\prime}\left(x^{(k)}\right)\right]^{-1} F^{\prime}\left(x^{(k)}\right)=F^{\prime}\left(x^{(k)}\right)\left[F^{\prime}\left(x^{(k)}\right)\right]^{-1}=I \tag{9}
\end{equation*}
$$

Then, the error expression in the first step of the method is

$$
\begin{align*}
y^{(k)}-\alpha= & C_{2} e^{(k)^{2}}+\left(-2 C_{2}^{2}+2 C_{3}\right) e^{(k)^{3}} \\
& +\left(4 C_{2}^{3}-4 C_{2} C_{3}-3 C_{3} C_{2}+4 C_{2}^{3}\right) e^{(k)^{4}} \\
& +\left(4 C_{5}-6 C_{2} C_{4}-4 C_{4} C_{2}+8 C_{2}^{2} C_{3}\right.  \tag{10}\\
& \left.+6 C_{3} C_{2}^{2}+6 C_{2} C_{3} C_{2}-6 C_{3}^{2}-8 C_{2}^{4}\right) e^{(k)^{5}} \\
& +O\left(e^{(k)^{6}}\right)
\end{align*}
$$

Furthermore, we know that

$$
\begin{align*}
F\left(y^{(k)}\right)= & F^{\prime}(\alpha)\left\{\left(y^{(k)}-\alpha\right)+C_{2}\left(y^{(k)}-\alpha\right)^{2}\right. \\
& \left.+C_{3}\left(y^{(k)}-\alpha\right)^{3}+C_{4}\left(y^{(k)}-\alpha\right)^{4}\right\}  \tag{11}\\
& +O\left(\left(y^{(k)}-\alpha\right)^{5}\right)
\end{align*}
$$

and if we replace in (11) the powers of $\left(y^{(k)}-\alpha\right)$, we obtain after some operations

$$
\begin{align*}
F\left(y^{(k)}\right)= & F^{\prime}(\alpha)\left\{C_{2} e^{(k)^{2}}+2\left(-C_{2}^{2}+C_{3}\right) e^{(k)^{3}}\right. \\
& +\left(3 C_{4}-4 C_{2} C_{3}-3 C_{3} C_{2}+5 C_{2}\right) e^{(k)^{4}} \\
& +\left(4 C_{5}-6 C_{2} C_{4}-4 C_{4} C_{2}+8 C_{2}^{3} C_{3}\right. \\
& +6 C_{3} C_{2}^{2}+6 C_{2} C_{3} C_{2}+2 C_{2}^{2} C_{3} \\
& \left.\left.+2 C_{2} C_{3} C_{2}-6 C_{3}^{2}-12 C_{2}^{4}\right) e^{(k)^{5}}\right\} \\
& +O\left(e^{(k)^{6}}\right) \tag{12}
\end{align*}
$$

and also

$$
\begin{align*}
& F^{\prime}\left(y^{(k)}\right) \\
& \begin{aligned}
&=F^{\prime}(\alpha)\{I+2 C_{2}^{2} e^{(k)^{2}}+4\left(C_{2} C_{3}-C_{2}^{3}\right) e^{(k)^{3}} \\
&+\left(6 C_{2} C_{4}-8 C_{2}^{2} C_{3}-6 C_{2} C_{3} C_{2}\right. \\
&\left.+8 C_{2}^{4}+3 C_{3} C_{2}^{2}\right) e^{(k)^{4}} \\
&+\left(8 C_{2} C_{5}-12 C_{2}^{2} C_{4}-8 C_{2} C_{4} C_{2}+16 C_{2}^{3} C_{3}\right. \\
&+12 C_{2} C_{3} C_{2}^{2}+12 C_{2}^{2} C_{3} C_{2}-12 C_{3} C_{2}^{3} \\
&-12 C_{2} C_{3}^{2}+6 C_{3} C_{2} C_{3} \\
&\left.\left.+6 C_{3}^{2} C_{2}-16 C_{2}^{5}\right) e^{(k)^{5}}\right\} \\
&+O\left(e^{(k)^{6}}\right) .
\end{aligned}
\end{align*}
$$

In a similar way as in (8),

$$
\begin{align*}
& {\left[F^{\prime}\left(y^{(k)}\right)\right]^{-1}} \\
& \quad=\left\{I+Y_{2} e^{(k)}+Y_{3} e^{(k)^{2}}+Y_{4} e^{(k)^{3}}+Y_{5} e^{(k)^{4}}+Y_{6} e^{(k)^{5}}\right\}  \tag{14}\\
& \quad \times\left[F^{\prime}(\alpha)\right]^{-1}+O\left(e^{(k)^{6}}\right)
\end{align*}
$$

where $Y_{2}=0, Y_{3}=-2 C_{2}^{2}, Y_{4}=4\left(C_{2}^{3}-C_{2} C_{3}\right), Y_{5}=$ $-6 C_{2} C_{4}+8 C_{2}^{2} C_{3}+6 C_{2} C_{3} C_{2}-4 C_{2}^{4}-3 C_{3} C_{2}^{2}$, and $Y_{6}=-8 C_{2} C_{5}+$ $12 C_{2}^{2} C_{4}+8 C_{2} C_{4} C_{2}-16 C_{2}^{3} C_{3}-12 C_{2} C_{3} C_{2}^{2}-12 C_{2}^{2} C_{3} C_{2}+$ $12 C_{3} C_{2}^{3}+8 C_{2}^{3} C_{3}+8 C_{2} C_{3} C_{2}^{2}+12 C_{2} C_{3}^{2}-6 C_{3} C_{2} C_{3}-6 C_{3}^{2} C_{2}$.

So,

$$
\begin{align*}
& {\left[F^{\prime}\left(y^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right)} \\
& \quad=e^{(k)}+C_{2} e^{(k)^{2}}+\left(-2 C_{2}^{2}+C_{3}\right) e^{(k)^{3}} \\
& \quad+\left(C_{4}+2 C_{2}^{3}-4 C_{2} C_{3}\right) e^{(k)^{4}} \\
& \quad+\left(C_{5}+6 C_{2}^{2} C_{3}+2 C_{2} C_{3} C_{2}-3 C_{3} C_{2}^{2}-6 C_{2} C_{4}\right) e^{(k)^{5}} \\
& \quad+O\left(e^{(k)^{6}}\right) . \tag{15}
\end{align*}
$$

On the other hand, we get the Taylor expansion of $\mu^{(k)}$ by using (7) and (15):

$$
\begin{align*}
& \mu^{(k)} \\
&= {\left[F^{\prime}\left(y^{(k)}\right)\right]^{-1} F^{\prime}\left(x^{(k)}\right) } \\
&= I+2 C_{2} e^{(k)}+\left(-2 C_{2}^{2}+3 C_{3}\right) e^{(k)^{2}}+4\left(C_{4}-C_{2} C_{3}\right) e^{(k)^{3}} \\
&+\left(5 C_{5}+2 C_{2}^{2} C_{3}-2 C_{2} C_{3} C_{2}\right. \\
&\left.\quad-3 C_{3} C_{2}^{2}+4 C_{2}^{4}-6 C_{2} C_{4}\right) e^{(k)^{4}} \\
&+\left(6 C_{6}+4 C_{2}^{2} C_{4}-4 C_{2} C_{4} C_{2}+4 C_{2}^{3} C_{3}+4 C_{2}^{2} C_{3} C_{2}\right. \\
&+8 C_{2} C_{3} C_{2}^{2}+6 C_{3} C_{2}^{3}-6 C_{3} C_{2} C_{3} \\
&\left.-6 C_{3}^{2} C_{2}-8 C_{2}^{5}-8 C_{2} C_{5}\right) e^{(k)^{5}}+O\left(e^{(k)^{6}}\right) \tag{16}
\end{align*}
$$

Let us note that $\mu^{(k)}$ tends to the identity matrix when $x^{(k)}$ and $y^{(k)}$ tend to $\alpha$, thereby the second order polynomial approximation of the weight function, $H\left(\mu^{(k)}\right)$, is

$$
\begin{equation*}
H\left(\mu^{(k)}\right) \approx H(I)+H^{\prime}(I)\left(\mu^{(k)}-I\right)+\frac{H^{\prime \prime}(I)}{2}\left(\mu^{(k)}-I\right)^{2} . \tag{17}
\end{equation*}
$$

Let us consider now the following notation:

$$
\begin{equation*}
H(I)=H_{0}, \quad H^{\prime}(I)=H_{1}, \quad \frac{H^{\prime \prime}(I)}{2}=H_{2} \tag{18}
\end{equation*}
$$

So,

$$
\begin{aligned}
z^{(k)} & -\alpha \\
= & y^{(k)}-\alpha-H\left(\mu^{(k)}\right)\left[F^{\prime}\left(y^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right) \\
= & -H_{0} e^{(k)}+\left(I-H_{0}-2 H_{1}\right) C_{2} e^{(k)^{2}} \\
& +\left[\left(-2 I+2 H_{0}-4 H_{2}\right) C_{2}^{2}+\left(2 I-H_{0}-3 H_{1}\right) C_{3}\right] e^{(k)^{3}}
\end{aligned}
$$

$$
\begin{align*}
+ & {\left[\left(3 I-H_{0}-4 H_{1}\right) C_{4}+\left(4 I-2 H_{0}+6 H_{1}+4 H_{2}\right) C_{2}^{3}\right.} \\
& +\left(-4 I+4 H_{0}+2 H_{1}-6 H_{2}\right) C_{2} C_{3} \\
& \left.+\left(-3 I-3 H_{1}-6 H_{2}\right) C_{3} C_{2}\right] e^{(k)^{4}} \\
+ & {\left[\left(4 I-H_{0}-5 H_{1}\right) C_{5}+\left(-6 I-8 H_{2}+6 H_{0}+4 H_{1}\right)\right.} \\
& \times C_{2} C_{4}+\left(-4 I-8 H_{2}-4 H_{1}\right) C_{4} C_{2} \\
& +\left(8 I+10 H_{2}-6 H_{0}+8 H_{1}\right) C_{2}^{2} C_{3} \\
& +\left(6 I+3 H_{0}+9 H_{1}\right) C_{3} C_{2}^{2} \\
& +\left(6 I-2 H_{0}+2 H_{2}+6 H_{1}\right) C_{2} C_{3} C_{2} \\
& +\left(-6 I-9 H_{2}-3 H_{1}\right) C_{3}^{2} \\
& \left.+\left(-8 I+12 H_{2}-12 H_{1}\right) C_{2}^{4}\right] e^{(k)^{5}} \\
+ & O\left(e^{(k)^{6}}\right) . \tag{19}
\end{align*}
$$

Then,

$$
\begin{aligned}
& F\left(z^{(k)}\right) \\
& =F^{\prime}(\alpha)\left\{-H_{0} e^{(k)}+\left(I-H_{0}-2 H_{1}+H_{0}^{2}\right) C_{2} e^{(k)^{2}}\right. \\
& +\left[\left(-2 I+2 H_{0}^{2}-4 H_{2}+4 H_{0} H_{1}\right) C_{2}^{2}\right. \\
& \left.+\left(2 I-H_{0}-3 H_{1}-H_{0}^{3}\right) C_{3}\right] e^{(k)^{3}} \\
& +\left[\left(3 I-H_{0}-4 H_{1}+H_{0}^{4}\right) C_{4}\right. \\
& +\left(5 \mathrm{I}+2 \mathrm{H}_{1}+4 \mathrm{H}_{2}-3 \mathrm{H}_{0}^{2}+8 \mathrm{H}_{0} \mathrm{H}_{2}\right. \\
& \left.+4 H_{1}^{2}+4 H_{0} H_{1}\right) C_{2}^{3} \\
& +\left(-4 I+2 H_{1}-6 H_{2}+2 H_{0}^{2}+6 H_{0} H_{1}\right) C_{2} C_{3} \\
& +\left(-3 I-3 H_{1}-6 H_{2}+3 H_{0}^{2}\right. \\
& \left.\left.-6 H_{0}^{2} H_{1}-3 H_{0}^{3}\right) C_{3} C_{2}\right] e^{(k)^{4}} \\
& +\left[\left(4 I-H_{0}-H_{0}^{5}-5 H_{1}\right) C_{5}\right. \\
& +\left(-6 I+4 H_{1}-8 H_{2}+2 H_{0}^{2}+8 H_{0} H_{1}\right) C_{2} C_{4} \\
& +\left(-4 I-4 H_{1}-8 H_{2}-4 H_{0}^{3}+4 H_{0}^{4}+8 H_{0}^{3} H_{1}\right) \\
& \times \mathrm{C}_{4} \mathrm{C}_{2}+\left(10 I-\mathrm{H}_{0}+\mathrm{H}_{1}+10 \mathrm{H}_{2}+\mathrm{H}_{0} H_{1}\right. \\
& \left.+12 H_{0} H_{2}-7 H_{0}^{2}+6 H_{1}^{2}\right) C_{2}^{2} C_{3} \\
& +\left(6 I+9 H_{1}+12 H_{0} H_{1}-12 H_{0}^{2} H_{1}\right. \\
& \left.-12 H_{0} H_{1}^{2}-12 H_{0}^{2} H_{2}+3 H_{0}^{3}\right) C_{3} C_{2}^{2}
\end{aligned}
$$

$$
\begin{align*}
&+\left(8 I+H_{0}-H_{1}+2 H_{2}+11 H_{0} H_{1}\right. \\
&\left.+12 H_{0} H_{2}+H_{0}^{2}+6 H_{1}^{2}\right) C_{2} C_{3} C_{2} \\
&+\left(-6 I+6 H_{0}^{2}-3 H_{0}^{3}-3 H_{1}\right. \\
&\left.-9 H_{0}^{2} H_{1}-9 H_{2}\right) C_{3}^{2} \\
&+\left(-12 I-4 H_{1}-20 H_{0} H_{1}\right. \\
&\left.\left.\left.+4 H_{2}+16 H_{1} H_{2}\right) C_{2}^{4}\right] e^{(k)^{5}}\right\} \\
&+O\left(e^{(k)^{6}}\right) \tag{20}
\end{align*}
$$

Let us consider the truncated Taylor expansion of order two of the weight function $G\left(\mu^{(k)}\right)$,

$$
\begin{equation*}
G\left(\mu^{(k)}\right) \approx G(I)+G^{\prime}(I)\left(\mu^{(k)}-I\right)+\frac{G^{\prime \prime}(I)}{2}\left(\mu^{(k)}-I\right)^{2} \tag{21}
\end{equation*}
$$

and let us denote by

$$
\begin{equation*}
G(I)=G_{0}, \quad G^{\prime}(I)=G_{1}, \quad \frac{G^{\prime \prime}(I)}{2}=G_{2} \tag{22}
\end{equation*}
$$

Finally, the error equation is expressed as

$$
\begin{align*}
& e^{(k+1)} \\
& \qquad \begin{aligned}
=H_{0}\left(I-G_{0}\right) e^{(k)}+ & {\left[\left(I-H_{0}-G_{0}+G_{0} H_{0}+2 G_{1} H_{0}\right.\right.} \\
& \left.\left.-G_{0} H_{0}^{2}-2 H_{1}+2 G_{0} H_{1}\right) C_{2}\right] e^{(k)^{2}} \\
+ & {\left[\left(-2 I+2 G_{0}-2 G_{1}+2 H_{0}-2 G_{0} H_{0}\right.\right.} \\
& +4 G_{2} H_{0}-2 G_{0} H_{0}^{2}-2 G_{1} H_{0}^{2} \\
& +4 G_{1} H_{1}-4 G_{0} H_{0} H_{1} \\
& \left.-4 H_{2}+4 G_{0} H_{2}\right) C_{2}^{2} \\
+ & \left(2 I-H_{0}-3 H_{1}-2 G_{0}+G_{0} H_{0}\right. \\
& \left.\left.+3 G_{0} H_{1}+G_{0} H_{0}^{3}+3 G_{1} H_{0}\right) C_{3}\right]
\end{aligned} \\
& \times e^{(k)^{3}}+O\left(e^{\left.(k)^{4}\right) .}\right.
\end{align*}
$$

If we choose $H_{0}=0, H_{1}=(1 / 2) I$, and $H_{2}=0$, we obtain

$$
\begin{align*}
e^{(k+1)}= & {\left[\left(-2 I+2 G_{0}\right) C_{2}^{2}+\left(\frac{1}{2} I-\frac{G_{0}}{2}\right) C_{3}\right] e^{(k)^{3}} }  \tag{24}\\
& +O\left(e^{(k)^{4}}\right)
\end{align*}
$$

So, to increase the order of convergence up to four, the value of $G_{0}$ must be $G_{0}=I$, and then,

$$
\begin{equation*}
e^{(k+1)}=G_{1}\left[-C_{3}+4 C_{2}^{2}\right] e^{(k)^{4}}+O\left(e^{(k)^{5}}\right) \tag{25}
\end{equation*}
$$

Table 1: Orbital elements and temporal distance between $\vec{r}_{1}$ and $\overrightarrow{r_{2}}$. Reference Orbit I.

| $a=4.0$ e.r. | $i=15^{\circ}$ | $\Delta t=0.01044412$ D.J. |
| :--- | :---: | :---: |
| $e=0.2$ e.r. | $\Omega=30^{\circ}$ | $\Delta \nu=12.232^{\circ}$ |
| $T=$ January $1,19640^{\mathrm{hr}}, 0^{\text {min }}, 0^{\text {seg }}$ | $\omega=10^{\circ}$ |  |

Table 2: Orbital elements and temporal distance between $\vec{r}_{1}$ and $\vec{r}_{2}$. Tundra Orbit.

| $a=6.62$ e.r. | $i=63.43^{\circ}$ | $\Delta t=0.399753$ D.J. |
| :--- | :---: | :---: |
| $e=0.27$ e.r. | $\Omega=290.2^{\circ}$ | $\Delta v=171^{\circ}$ |
| $M=144^{\circ}$ | $\omega=270^{\circ}$ |  |

Moreover, in order to reach fifth order of convergence, $G_{1}$ must be null. Therefore, the error equation is

$$
\begin{equation*}
e^{(k+1)}=\left[\left(I-2 G_{2}\right) C_{2}^{2} C_{3}+\left(-4 I+8 G_{2}\right) C_{2}^{4}\right] e^{(k)^{5}}+O\left(e^{(k)^{6}}\right) . \tag{26}
\end{equation*}
$$

Finally, if $G_{2}=(1 / 2) I$, we have

$$
\begin{gather*}
e^{(k+1)}=\left[-\frac{3}{2} C_{3} C_{2} C_{3}+\frac{1}{4} C_{2} C_{3}^{2}+6 C_{3} C_{2}^{3}-C_{2} C_{3} C_{2}^{2}\right.  \tag{27}\\
\left.+C_{2}^{3} C_{3}-4 C_{2}^{5}\right] e^{(k)^{6}}+O\left(e^{(k)^{7}}\right)
\end{gather*}
$$

and the theorem is proved.

From the previous theorem, (5) defines a family of sixth-order methods. We can find different weight functions satisfying the conditions of the theorem. Specifically, we propose the following examples for the next sections.

Example 2. An element of our family of sixth-order is given by the weight functions

$$
\begin{gather*}
H(t)=\frac{1}{2}(t-I),  \tag{28}\\
G(t)=[I+t]^{-1}\left(2 I-t+t^{2}\right),
\end{gather*}
$$

which is called NAJC1.
Example 3. Another combination of weight functions that can be used is

$$
\begin{gather*}
H(t)=\frac{1}{2}(t-I),  \tag{29}\\
G(t)=I+\frac{1}{2}(t-I)^{2} .
\end{gather*}
$$

We will refer to this element of the class as NAJC2.

## 3. Numerical Results

In this section, we analyze the computational efficiency of our methods and compare them with other classical ones in the problem of preliminary orbit determination as well as in academic examples. The classical methods used are Newton, Traub', and Jarratt's ones of convergence order 2, 3, and 4, respectively, whose iterative expressions are as follows.
(i) Newton (N)

$$
\begin{equation*}
x^{(k+1)}=x^{(k)}-\left[F^{\prime}\left(x^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right) . \tag{30}
\end{equation*}
$$

(ii) Traub (T)

$$
\begin{gather*}
y^{(k)}=x^{(k)}-\left[F^{\prime}\left(x^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right)  \tag{31}\\
x^{(k+1)}=y^{(k)}-\left[F^{\prime}\left(x^{(k)}\right)\right]^{-1} F\left(y^{(k)}\right)
\end{gather*}
$$

(iii) Jarratt (J)

$$
\begin{align*}
& z^{(k)}=x^{(k)}-\frac{2}{3}\left[F^{\prime}\left(x^{(k)}\right)\right]^{-1} F\left[x^{(k)}\right] \\
& x^{(k+1)}= x^{(k)}-\frac{1}{2}\left[3 F^{\prime}\left(z^{(k)}\right)-F^{\prime}\left(x^{(k)}\right)\right]^{-1} \\
& \times\left(3 F^{\prime}\left(z^{(k)}\right)\right)-F^{\prime}\left(x^{(k)}\right)\left[F^{\prime}\left(x^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right) . \tag{32}
\end{align*}
$$

In our tests, we have used the following numerical settings: variable precision arithmetics of two hundred and fifty digits in Mathematica 8.0; moreover, in each iterative method, we have used the stopping criterium $\left\|F\left(x^{(k+1)}\right)\right\|+$ $\left\|x^{(k+1)}-x^{(k)}\right\|<10^{-100}$, and the approximated computational order of convergence $\rho$ (see [30]) has been calculated by using the formula:

$$
\begin{equation*}
p \approx \rho=\frac{\ln \left(\left\|x^{(k+1)}-x^{(k)}\right\| /\left\|x^{(k)}-x^{(k-1)}\right\|\right)}{\ln \left(\left\|x^{(k)}-x^{(k-1)}\right\| /\left\|x^{(k-1)}-x^{(k-2)}\right\|\right)} \tag{33}
\end{equation*}
$$

From this value, we have designed two practical indices to measure the computational efficiency: the approximated efficiency index

$$
\begin{equation*}
\tilde{I}=\rho^{1 / d} \tag{34}
\end{equation*}
$$

and the approximated computational index

$$
\begin{equation*}
\tilde{I}_{c}=\rho^{1 / \mathrm{op}} \tag{35}
\end{equation*}
$$

$d$ and op being the number of functional evaluations and the number of operations (products and quotients) per iteration, respectively.

In Tables 3, 4, 5, 6, and 7, we show the number of iterations, the previously defined indices, and the absolute errors committed between theoretical and practical values that we denote by $\epsilon$.

Two reference orbits have been used in the test for the preliminary orbit determination. The first can be found in

Table 3: Results of reference Orbit I.

| Method | Iter | $\rho$ | $\widetilde{I}$ | $\tilde{I}_{c}$ | $\varepsilon_{a}$ | $\varepsilon_{e}$ | $\varepsilon_{i}$ | $\varepsilon_{w}$ | $\varepsilon_{\Omega}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Time $(\mathrm{s})$ |  |  |  |  |  |  |  |  |  |
| N | 7 | 1.9999 | 1.1112 | 1.1112 | $3.2757 e-109$ | $4.8982 e-110$ | $7.3653 e-109$ | $2.6237 e-108$ | 0 |
| T | 5 | 2.9995 | 1.1396 | 1.1102 | $2.0466 e-120$ | $3.0603 e-121$ | $4.6017 e-120$ | $1.6393 e-119$ | 0 |
| J | 4 | 4.0000 | 1.1264 | 1.0771 | $4.8431 e-200$ | $8.8034 e-201$ | $3.9324 e-200$ | $1.4008 e-199$ | 0 |
| $\mathrm{NAJC1}$ | 3 | 5.7569 | 1.1347 | 1.0485 | $3.6731 e-148$ | $4.3049 e-148$ | $1.5905 e-146$ | $5.6659 e-146$ | 0 |
| NAJC 2 | 3 | 5.7821 | 1.1352 | 1.0487 | $3.9057 e-149$ | $4.6174 e-149$ | $1.7089 e-147$ | $6.0879 e-147$ | 0 |

Table 4: Results of tundra Orbit.

| Method | Iter | $\rho$ | $\widetilde{I}$ | $\widetilde{I}_{c}$ | $\varepsilon_{a}$ | $\varepsilon_{e}$ | $\varepsilon_{i}$ | $\varepsilon_{w}$ | $\varepsilon_{\Omega}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| T | 6 | 2.0121 | 1.1041 | 1.1041 | $3.1284 e-16$ | $1.6038 e-17$ | $2.4321 e-16$ | $6.5148 e-15$ | 0 |
| T | 5 | 2.9852 | 1.1331 | 1.1051 | $3.1284 e-16$ | $1.6038 e-17$ | $2.4321 e-16$ | $6.5148 e-15$ | 0 |
| J | 3 | 3.9931 | 1.1409 | 1.0859 | $3.1284 e-16$ | $1.6038 e-17$ | $2.4321 e-16$ | $6.5148 e-15$ | 0 |
| NAJC1 | 3 | 4.9478 | 1.1336 | 1.0482 | $3.1284 e-16$ | $1.6038 e-17$ | $2.4321 e-16$ | $6.5148 e-15$ | 0 |
| NAJC2 | 3 | 5.2465 | 1.1337 | 1.0482 | $3.1284 e-16$ | $1.6038 e-17$ | $2.4321 e-16$ | $6.5148 e-15$ | 0.005493 |

TABLE 5: Results of system (a), $x^{(0)}=(4,-3)^{T}$.

| Method | Iter | $\rho$ | $\widetilde{I}$ | $\widetilde{I}_{c}$ | $\varepsilon_{\alpha_{1}}$ | $\varepsilon_{\alpha_{2}}$ | 0 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| N | 8 | 1.9999 | 1.1734 | 1.1078 | $7.5993 e-174$ | 0.004658 |  |
| T | 6 | 3.0000 | 1.3841 | 1.1251 | $5.8838 e-206$ | 0 |  |
| J | 4 | 3.9887 | 1.1174 | 1.0715 | $2.0217 e-113$ | 0 |  |
| NAJC1 | 4 | 6.0051 | 1.1259 | 1.0451 | 0 | 0.006098 |  |
| NAJC2 | 4 | 6.0028 | 1.1289 | 1.0462 | 0 | 0.004620 |  |

TABLE 6: Results of system (b), $x^{(0)}=(12,-2,-1)^{T}$.

| Method | Iter | $\rho$ | $\widetilde{I}$ | $\widetilde{I}_{c}$ | $\varepsilon_{\alpha_{1}}$ | $\varepsilon_{\alpha_{2}}$ | $\varepsilon_{\alpha_{3}}$ | Time $(\mathrm{s})$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| N | 13 | 1.9948 | 1.0281 | 1.0201 | $3.4121 e-125$ | $2.7759 e-125$ | $1.0794 e-125$ |  |
| T | - | - | - | - | - | - | - | 0.007766 |
| J | 8 | 3.9940 | 1.0301 | 1.0158 | 0 | 0 | 0 |  |
| $\mathrm{NAJC1}$ | 5 | 4.9496 | 1.0646 | 1.0171 | 0 | 0 | 0 |  |
| NAJC2 | 6 | 4.9329 | 1.0716 | 1.0190 | 0 | 0 | 0.009805 |  |

TABLE 7: Results of system (c), $x^{(0)}=(5,5,5,-1)^{T}$.

| Method | Iter | $\rho$ | $\widetilde{I}$ | $\widetilde{I}_{c}$ | $\varepsilon_{\alpha_{1}}$ | $\varepsilon_{\alpha_{2}}$ | $\varepsilon_{\alpha_{3}}$ | $\varepsilon_{\alpha_{4}}$ | Time $(\mathrm{s})$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| N | 10 | 2.0244 | 1.0249 | 1.0249 | $6.5830 e-102$ | $6.5830 e-102$ | $6.5830 e-102$ | $2.7466 e-103$ | 0.007272 |
| T | 7 | 3.0909 | 1.0356 | 1.0162 | $2.9862 e-145$ | $2.9862 e-145$ | $2.9862 e-145$ | $7.8319 e-147$ | 0.009802 |
| J | 5 | 4.1871 | 1.0349 | 1.0141 | $6.5830 e-102$ | $6.5830 e-102$ | $6.5830 e-102$ | $2.7466 e-103$ | 0.007952 |
| NAJC1 | 5 | 6.4193 | 1.0493 | 1.0104 | 0 | 0 | 0 | 0 | 0.017668 |
| NAJC2 | 5 | 6.1729 | 1.0520 | 1.0110 | 0 | 0 | 0 | 0 | 0.014849 |

[27], and the second one is a commercial real orbit called Tundra. As the orbital elements of each one are known, the vector positions (measured in Earth radius) at the instants $t_{1}$ and $t_{2}$ have been recalculated with 500 exact digits. These vector positions are

$$
\begin{align*}
\vec{r}_{2} \approx[ & 1.98804155574820,2.50333354505224 \\
& 0.31455350605251] \tag{36}
\end{align*}
$$

for Reference Orbit I and

$$
\vec{r}_{1} \approx[-2.02862564034533,-0.74638890547506
$$

$$
-4.322222156844465]
$$

$$
\begin{align*}
& \vec{r}_{2} \approx[4.24372000256074,-1.689387746496 \\
&6.79724893784587] \tag{37}
\end{align*}
$$

for Tundra Orbit. Then, our aim is to gain from these positions the orbital elements showed in Tables 1 and 2 with a precision as high as possible by means of proposed iterative schemes.

If we look at the numerical results of Table 3, our methods NAJC1 and NAJC2 have the least number of iterations, the highest order of convergence with the highest efficiency index. For this case, the absolute error committed by Jarratt's method is lower than in our procedures, which is due to use of an initial estimation very close to the solution and a very small temporal distance.

If we observe the numerical results of Tundra Orbit (Table 4), we note that the absolute error is stabilized, and we maintain the good results of the Reference Orbit I. We can conclude that working with the two equations provided by Gauss as a system, we improve the original procedure which has the restriction that the difference of true anomalies cannot be greater than $\pi / 4$. We are able to increase the range of the difference of true anomalies associated to the observations until values close to $\pi$.
3.1. Other Nonlinear Problems. In order to continue checking the computational efficiency of the proposed schemes, NAJC1 and NAJC2 we use, in the following, some academic examples.
(a) $F_{1}(x)=\left(f_{1}(x), f_{2}(x)\right)^{T}: x=\left(x_{1}, x_{2}\right)^{T}$ and $f_{i}: \mathbb{R}^{2} \rightarrow$ $\mathbb{R}, i=1,2, \alpha \approx(3.47063096,-2.47063096)$,

$$
\begin{gather*}
f_{1}(x)=e^{x_{1}} e^{x_{2}}+x_{1} \cos x_{2}  \tag{38}\\
f_{2}(x)=x_{1}+x_{2}-1
\end{gather*}
$$

(b) $F_{2}(x)=\left(f_{1}(x), f_{2}(x), f_{3}(x)\right)^{T}: x=\left(x_{1}, x_{2}, x_{3}\right)^{T}$ and $f_{i}: \mathbb{R}^{3} \rightarrow \mathbb{R}, i=1,2,3, \alpha \approx(2.14025,-2.09029$, $-0.223525)$,

$$
\begin{gather*}
f_{1}(x)=x_{1}^{2}+x_{2}^{2}+x_{3}^{2}-9 \\
f_{2}(x)=x_{1} x_{2} x_{3}-1  \tag{39}\\
f_{3}(x)=x_{1}+x_{2}-x_{3}^{2} .
\end{gather*}
$$

(c) $F_{3}(x)=\left(f_{1}(x), f_{2}(x), f_{3}(x), f_{4}(x)\right)^{T}: x=\left(x_{1}, x_{2}\right.$, $\left.x_{3}, x_{4}\right)^{T}$ and $f_{i}: \mathbb{R}^{4} \rightarrow \mathbb{R}, i=1, \ldots, 4, \alpha=$ $( \pm 1 / \sqrt{3}, \pm 1 / \sqrt{3}, \pm 1 / \sqrt{3}, \pm 2 / \sqrt{3})$,

$$
\begin{gathered}
f_{1}(x)=x_{2} x_{3}+x_{4}\left(x_{2}+x_{3}\right) \\
f_{2}(x)=x_{1} x_{3}+x_{4}\left(x_{1}+x_{3}\right) \\
f_{3}(x)=x_{1} x_{2}+x_{4}\left(x_{1}+x_{2}\right) \\
f_{4}(x)=x_{1} x_{2}+x_{1} x_{3}+x_{2} x_{3}-1
\end{gathered}
$$

The results are showed in Tables 5, 6, and 7, where $\varepsilon_{\alpha_{i}}$ denotes $\left|f_{i}(\alpha)\right|$. From these tables, the best schemes in terms of precision are NAJC1 and NAJC2, even when the initial estimation is far from the solution. In system (b), Traub's method does not converge after five hundred iterations from this initial estimation.

## 4. Dynamical Planes

From the numerical results presented in Section 3, our proposed methods show to be competitive with respect to existing ones. Nevertheless, under the dynamical point of view, we have checked that they can have better behavior in terms of stability and wideness of the region of convergence.

For the representation of the convergence basins of our procedures and classical methods, we have used the software described in [31]. We draw a mesh with two thousand points per axis; each point of the mesh is a different initial estimation which we introduce in each procedure. If the method reaches the final solution in less than five hundred iterations, this point is drawn in orange. The color will be more intense when the number of iterations is lower. Otherwise, if the method arrives at the maximum of iterations, the point will be drawn in black. In each axis, we will represent each of the variables with which we work. The ratio sector triangle is represented in the abscissas and the difference of eccentric anomalies in the ordinates. In addition, we will use the reference orbit I, which is defined in Table 1, and the solution of the nonlinear system is in this case around ( $1,0.1$ ). For this reason, we choose $[0,3] \times[-1,1]$ as the region of representation.

In Figure 2, we show the dynamical planes of the classical methods. It can be observed that, in general, higher order means lower stability. If we focus our attention on the attraction basins of each plane, the method with the greatest stability is Newton, and the procedure with the lowest number of iterations is Jarratt.

As we can see in Figure 3, both NAJC1 and NAJC2 have large areas of stability, similar to Newton's one, but with order of convergence six. For the intensity of the orange in the attraction basins, the two schemes will have the least number of iterations. Moreover, if we compare both procedures, the attraction basins of NAJCl are more disperse than the convergence basins of NAJC2 which makes the first one more unstable than the second one.

## 5. Conclusions

The classical Gauss' method for preliminary orbit determination has been improved, introducing a new performance by means of a nonlinear system. This fact increases the admissible spread of the observations (in order to ensure the convergence) from $\pi / 4$ to $\pi$ and reduces the number of iterations of the process.

The new sixth-order methods NAJC1 and NAJC2 belonging to the class of methods designed by using matricial weight functions have good global properties of convergence and stability, even for initial estimations far from the solution.


Figure 2: Dynamical planes from classical methods and Reference Orbit I.


Figure 3: Dynamical planes from new methods and Reference Orbit I.

It is a well-known fact that the size of the area of convergence is inversely proportional to the order of convergence. However, our methods hold a basin of attraction comparable with Newton's one in spite of their sixth-order of convergence.

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# Research Article 

# The Perturbed Dual Risk Model with Constant Interest and a Threshold Dividend Strategy 

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#### Abstract

We consider the perturbed dual risk model with constant interest and a threshold dividend strategy. Firstly, we investigate the moment-generation function of the present value of total dividends until ruin. Integrodifferential equations with certain boundary conditions are derived for the present value of total dividends. Furthermore, using techniques of sinc numerical methods, we obtain the approximation results to the expected present value of total dividends. Finally, numerical examples are presented to show the impact of interest on the expected present value of total dividends and the absolute ruin probability.


## 1. Introduction

In insurance mathematics, the classical risk model has been the center of focus for decades. The surplus in the classical model at time $t$ can be expressed as

$$
\begin{equation*}
U(t)=u+c t-S(t), \quad t \geq 0 \tag{1}
\end{equation*}
$$

where $u \geq 0$ is the initial surplus, $c>0$ is the premium rate, and $S(t)$ is the aggregate claims by time $t$, usually modeled by a compound Poisson process. In recent years, quite a few interesting papers have been written on a model which is dual to the classical insurance risk model. See, for example, Avanzi et al. [1], Avanzi and Gerber [2]. The surplus of a dual classical risk model at time $t$ is

$$
\begin{equation*}
U(t)=u-c t+S(t), \quad t \geq 0 \tag{2}
\end{equation*}
$$

In this model, the premium rate should be viewed as an expense rate and claims should be viewed as profits or gains. While not very popular in insurance mathematics, this model has appeared in various literature (see Cramér [3, Section 5.13] and Seal [4, pages 116-119] and Takács [5, pages $152-154]$ ). There were many possible interpretations for this model. For example, we can treat the surplus as the amount of the capital of a business engaged in research and development. The company paid expenses for research, and occasional profit of random amounts (such as the award of
a patent or a sudden increase in sales) arises according to a Poisson process. A similar model was used by Bayraktar and Egami [6] to model the capital of a venture capital investment. Another model was an annuity business. The company issues payments continuously to annuitants, while the gross reserve of an annuitant was released as emerging profit when he died. Yang and Zhu [7] generalized the dual model into a regime-switching setting and calculated bounds for ruin probabilities. One of the current topics of interest in insurance mathematics is the problem of maximizing the expected total discounted dividends until ruin, which goes back to de Finetti [8], and it had also been studied by Bühlmann [9, Section 6.4], Gerber [10, Sections 7 and 8], and Gerber [11, Section 10.1]. In Avanzi et al. [1], the authors studied the expected total discounted dividends until ruin for the dual model under the barrier strategy by means of integrodifferential equations. They derived explicit formulas when profits or gains followed an exponential or a mixture of exponential distributions and showed that the optimal value of the dividend barrier under the dual model was independent of the initial surplus. Avanzi and Gerber [2] studied a dual model perturbed by diffusion and discussed how the optimal value of the dividend barrier can be determined. Albrecher et al. [12] studied a dual model that also paid taxes when the surplus was at a running maximum and calculated the expected total discounted dividends before ruin for exponentially distributed profits.

Considering the perturbed dual risk model, the surplus of an insurer has the following form

$$
\begin{equation*}
U(t)=u-c t+\sum_{i=1}^{N(t)} X_{i}+\sigma B(t), \quad t \geq 0 \tag{3}
\end{equation*}
$$

where $u \geq 0$ is the initial surplus and $c>0$ is the constant rate. $\left\{X_{i}, i=1,2, \ldots\right\}$ is a sequence of independent income size random variables with a common distribution function $F(x)$ with $F(0)=0$ and density function $f(x) .\{N(t), t \geq 0\}$ is the Poisson income-number process with an intensity $\lambda>0$ and is defined as $N(t)=\sup \left\{k: T_{1}+\cdots+T_{k} \leq t\right\}$, where the i.i.d. interincome times $\left\{T_{i}\right\}_{i=1}^{\infty}$ have a common exponential distribution with a parameter $\lambda .\{B(t), t \geq 0\}$ is a standard Brownian motion with $B(0)=0$ and $\sigma>0$ is a constant, representing the diffusion volatility parameter. In addition, $\left\{X_{i}, i=1,2, \ldots\right\},\{N(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$, are mutually independent.

Dividend strategy for insurance risk models was first proposed by de Finetti [8] to reflect more realistically the surplus cash flowed in an insurance portfolio. Various threshold strategies have been studied by many authors, including Gerber and Landry [13], Ng [14], and Lin and Pavlova [15]. Among them, Ng [14] showed that the threshold strategy was optimal when the dividend rate was bounded and the individual claim distribution was exponential. Recently, the threshold dividend strategy has been considered in the class of compound Poisson process perturbed by diffusion and its generalizations; readers may refer to Avanzi and Gerber [2], Gao and Liu [16], Wan [17], and the references therein.

Then, we consider the modification of the surplus process by a threshold strategy with a threshold level $b$. When $U(t)$ is below $b$, no dividends are paid and the surplus decreases at the original rate $c_{1}$. But when $U(t)$ is above $b$, the surplus would decrease at a different rate $c_{2}>c_{1}$ and dividends are paid at rate $\mathcal{c}_{2}-c_{1}$. Incorporating the threshold strategy into (3) yields the surplus process $\left\{U_{b}(t), t \geq 0\right\}$ which can be expressed by

$$
d U_{b}(t)=\left\{\begin{array}{l}
-c_{1} d t+d S(t)+\sigma d B(t), \quad U_{b}(t)<b  \tag{4}\\
-c_{2} d t+d S(t)+\sigma d B(t), \quad U_{b}(t) \geq b
\end{array}\right.
$$

where $U_{b}(0)=u, 0<b<\infty$, and $S(t)=\sum_{i=1}^{N(t)} X_{i}$.
In this paper, we consider that the insurance company earns credit interest with a constant force $r>0$. In the mean time, the insurer will receive interest. Incorporating interest into (4) yields the surplus process $\left\{U_{b}(t), t \geq 0\right\}$ which can be expressed by

$$
d U_{b}(t)=\left\{\begin{array}{l}
-c_{1} d t+r U_{b}(t) d t+d S(t)+\sigma d B(t), U_{b}(t)<b  \tag{5}\\
-c_{2} d t+r U_{b}(t) d t+d S(t)+\sigma d B(t), \quad U_{b}(t) \geq b
\end{array}\right.
$$

Let $D(t)$ denote the cumulative amount of dividends paid out up to time $t$ and $\delta>0$ the force of interest; then

$$
\begin{equation*}
D_{u, b}=\int_{0}^{T_{b}} e^{-\delta t} d D(t) \tag{6}
\end{equation*}
$$

is the present value of all dividends until $T_{b}$, where $T_{b}$ denoted by $T_{b}=\inf \left\{t \geq 0: U_{b}(t) \leq 0\right\}$ is the time of ruin. An alternative expression for $D_{u, b}$ is

$$
\begin{equation*}
D_{u, b}=\left(c_{2}-c_{1}\right) \int_{0}^{T_{b}} e^{-\delta t} I\left(U_{b}(t)>b\right) d t \tag{7}
\end{equation*}
$$

with $I(\cdot)$ denoting the indicator function. It is obvious that $0<D_{u, b} \leq\left(c_{2}-c_{1}\right) / \delta$.

In the sequel, we will be interested in the following moment generating function:

$$
\begin{equation*}
M(u, y ; b)=E\left[e^{y D_{u, b}}\right] \tag{8}
\end{equation*}
$$

(for those values of $y$ where it exist) and the expected total discounted dividends until ruin is

$$
\begin{equation*}
V(u ; b)=E\left[D_{u, b} \mid U_{b}(0)=u\right] . \tag{9}
\end{equation*}
$$

Throughout this paper, we assume that $M(u, y ; b)$ and $V(u ; b)$ are sufficiently smooth functions in $u$ and $y$, respectively.

## 2. Integrodifferential Equations

2.1. Integrodifferential Equations for $M(u, y ; b)$. In this section, we will give the integrodifferential equations satisfied by the moment generating function $M(u, y ; b)$, respectively. It is easy to see that $M(u, y ; b)$ behaves differently with different initial surplus. Hence, for notation convenience, we set

$$
M(u, y ; b)= \begin{cases}M_{1}(u, y ; b), & \text { if } 0<u<b  \tag{10}\\ M_{2}(u, y ; b), & \text { if } b \leq u<\infty\end{cases}
$$

In the case of $\sigma=0$, we write $M(u, y ; b)$ as $M^{0}(u, y ; b)$, and the other one is similar.

In the following, we firstly derive the integrodifferential equations satisfied by $M^{0}(u, y ; b)$.

Theorem 1. When $0<u<b$,

$$
\begin{align*}
& \left(r u-c_{1}\right) \frac{\partial M_{1}^{0}(u, y ; b)}{\partial u} \\
& =y \delta \frac{\partial M_{1}^{0}(u, y ; b)}{\partial y}+\lambda M_{1}^{0}(u, y ; b)  \tag{11}\\
& \quad-\lambda \int_{0}^{b-u} M_{1}^{0}(u+x, y ; b) f(x) d x \\
& \quad-\lambda \int_{b-u}^{\infty} M_{2}^{0}(u+x, y ; b) f(x) d x
\end{align*}
$$

and for $b \leq u<\infty$,

$$
\begin{align*}
& \left(r u-c_{2}\right) \frac{\partial M_{2}^{0}(u, y ; b)}{\partial u} \\
& \quad=y \delta \frac{\partial M_{2}^{0}(u, y ; b)}{\partial y}+\lambda M_{2}^{0}(u, y ; b)  \tag{12}\\
& \quad-\lambda \int_{0}^{\infty} M_{1}^{0}(u+x, y ; b) f(x) d x
\end{align*}
$$

with the following boundary conditions:

$$
\begin{align*}
M_{1}^{0}(b-, y ; b) & =M_{2}^{0}(b, y ; b),  \tag{13}\\
\lim _{u \rightarrow \infty} M_{2}^{0}(u, y ; b) & =e^{\left(\left(c_{2}-c_{1}\right) / y\right) \delta},  \tag{14}\\
\left.\left(r b-c_{1}\right) \frac{\partial M_{1}^{0}(u, y ; b)}{\partial u}\right|_{u=b-} & =\left.\left(r b-c_{2}\right) \frac{\partial M_{2}^{0}(u, y ; b)}{\partial u}\right|_{u=b} \tag{15}
\end{align*}
$$

Proof. When $0<u<b$, consider $t>0$ such that the surplus cannot reach level $b$ by time; that is, $h_{\delta}(u, \tau)=e^{r \tau}\left(u-\left(c_{1} / r\right)\right)+$ $\left(c_{1} / r\right)>0$. By conditioning on the time and amount of first claim, if it occurs by $t$, and whether the claim causes ruin, one gets

$$
\begin{align*}
M_{1}^{0}(u, y ; b)= & (1-\lambda \tau) M_{1}^{0}\left(h_{\delta}(u, \tau), y e^{-\delta \tau} ; b\right) \\
& +\lambda \tau E\left[M_{1}^{0}\left(h_{\delta}(u, \tau)+x, y e^{-\delta \tau} ; b\right)\right] . \tag{16}
\end{align*}
$$

By Taylor's expansion,

$$
\begin{align*}
M_{1}^{0} & \left(h_{\delta}(u, \tau), y e^{-\delta \tau} ; b\right) \\
= & M_{1}^{0}(u, y ; b)+\left(r u-c_{1}\right) \tau \frac{\partial M_{1}^{0}(u, y ; b)}{\partial u}  \tag{17}\\
& -y \delta \tau \frac{\partial M_{1}^{0}(u, y ; b)}{\partial y}+o(\tau) .
\end{align*}
$$

Also, due to

$$
\begin{align*}
E & {\left[M_{1}^{0}\left(h_{\delta}(u, \tau), y e^{-\delta \tau} ; b\right)\right] } \\
& =\int_{0}^{\infty} M_{1}^{0}\left(h_{\delta}(u, \tau)+x, y e^{-\delta \tau} ; b\right) f(x) d x \tag{18}
\end{align*}
$$

Substituting the above expansion into (16), dividing both sides of (16) by $t$, and letting $t \rightarrow 0$, we can get (11). Using the same way as (11), we can easily prove (12) when $u \rightarrow \infty$, $T_{b}=\infty$ so the condition (12) is correct, by the method used in Gao and Liu [16], and it is easy to check that the boundary conditions hold. This ends the proof of Theorem 1.

Now we consider the case of $\sigma \neq 0$. For $t \geq 0$, define $W(t)=\sigma \int_{0}^{t} e^{-r s} d B(s)$. Thus, $W(t)$ is an Itô stochastic integral. Denote by $\{\langle W\rangle(t), t \geq 0\}$ the variance process of $\{W(t), t \geq 0\}$. We have $\langle W\rangle(t)=\sigma^{2} \int_{0}^{t} e^{-2 r s} d s=(2 r)^{-1} \sigma^{2}(1-$ $\left.e^{-2 r s}\right)$ for $t \geq 0$. Let $v(s)=\inf \{t:\langle W\rangle(t)>s\}$; then

$$
\begin{equation*}
v(s)=\frac{1}{2} \ln \frac{\sigma^{2}}{\sigma^{2}-2 r s}, \quad 0<s<\frac{\sigma^{2}}{2 r} . \tag{19}
\end{equation*}
$$

Set $W(t)=B(v(t))$ for $t \geq 0, i=1,2$. By the time change of Brownian motion, we have that $W_{i}$ is local standard Brownian motion with $W_{i}(0)=0$ running for an amount of time $\sigma^{2} / 2 r$.

Let $\phi(u, s)=e^{r s}\left(u+c_{1} \int_{0}^{s} e^{-r t} d t\right)$ and $\phi_{1}(u, s)=e^{r s}(u+$ $\left.c_{2} \int_{0}^{\infty} e^{-r t} d t\right)$. Then, we have the following results.

Theorem 2. When $0<u<b$,

$$
\begin{align*}
& \frac{1}{2} \sigma^{2} \frac{\partial^{2} M_{1}(u, y ; b)}{\partial u^{2}}+\left(r u-c_{1}\right) \frac{\partial M_{1}(u, y ; b)}{\partial u} \\
& \quad=y \delta \frac{\partial M_{1}(u, y ; b)}{\partial y}+\lambda M_{1}(u, y ; b)  \tag{20}\\
& \quad-\lambda \int_{0} b-u M_{1}(u+x, y ; b) f(x) d x \\
& \quad+\lambda \int_{b-u}^{\infty} M_{2}(u+x, y ; b) f(x) d x
\end{align*}
$$

and when $b \leq u<\infty$,

$$
\begin{align*}
& \frac{1}{2} \sigma^{2} \frac{\partial^{2} M_{2}(u, y ; b)}{\partial u^{2}}+\left(r u-c_{2}\right) \frac{\partial M_{2}(u, y ; b)}{\partial u} \\
& \quad=y \delta \frac{\partial M_{2}(u, y ; b)}{\partial y}+\left(\lambda-\left(c_{2}-c_{1}\right)\right) M_{2}(u, y ; b)  \tag{21}\\
& \quad+\lambda \int_{0}^{\infty} M_{2}(u+x, y ; b) f(x) d x
\end{align*}
$$

with the following boundary conditions:

$$
\begin{gather*}
M_{1}(0, y ; b)=1  \tag{22}\\
M_{1}(b-, y ; b)=M_{2}(b, y ; b)  \tag{23}\\
\lim _{u \rightarrow \infty} M_{2}(u, y ; b)=e^{\left(\left(c_{2}-c_{1}\right) / y\right) \delta}  \tag{24}\\
\left.\left(r b-c_{1}\right) \frac{\partial M_{1}(u, y ; b)}{\partial u}\right|_{u=b-} \\
=\left.\left(r b-c_{2}\right) \frac{\partial M_{2}(u, y ; b)}{\partial u}\right|_{u=b} . \tag{25}
\end{gather*}
$$

Proof. For $0<u<b$, assume that $\epsilon, t>0$ such that $\epsilon<u<b$. Define $T_{0}=T_{t}^{\epsilon} \wedge T_{1}$ and $T_{t}^{\epsilon}=\inf \{s>0: \phi(u+W(s), s) \notin$ $(\epsilon, b)\} \wedge t$. We have $P\left(T_{0}<\infty\right)=1$ for all $s \in\left(0, T_{0}\right)$. Then, by the strong Markov property, we get

$$
\begin{align*}
M_{1}(u, y ; b)= & E^{u}\left[e^{y D_{\mathrm{U}, b}}\right] \\
= & E^{u}\left[M_{1}\left(U_{b}\left(T_{0}\right), y e^{-\delta T_{0}} ; b\right)\right] \\
= & E\left[I\left(T_{1}>t\right) M_{1}\left(U_{b}\left(T_{t}^{\mathscr{E}}\right), y e^{-\delta T_{t}^{\mathscr{E}}} ; b\right)\right] \\
& +E\left[I\left(T_{1} \leq t\right) M_{1}\left(U_{b}\left(T_{T_{1}}^{\mathscr{E}}\right), y e^{-\delta T_{T_{1}}^{\mathscr{}}} ; b\right)\right] \\
= & \mathbf{I}(t)+\mathbf{I I}(t) . \tag{26}
\end{align*}
$$

By the assumption of independence, one gets

$$
\begin{align*}
& \mathbf{I}(t)=e^{-\lambda t} E\left[M_{1}\left(U_{b}\left(T_{t}^{\mathscr{E}}\right), y e^{-\delta T_{t}^{\mathscr{E}}} ; b\right)\right], \\
& \mathbf{I I}(t)= \int_{0}^{t} \lambda e^{-\lambda s} E\left[I\left(T_{t}^{\mathscr{E}}=s\right) M_{1}\left(U_{b}(s), y e^{-\delta T_{t}^{\mathscr{E}}} ; b\right)\right] d s \\
&+\int_{0}^{t} \lambda e^{-\lambda s} E\left[I\left(T_{t}^{\mathscr{E}}<s\right) M_{1}\left(U_{b}(s), y e^{-\delta T_{t}^{\mathscr{E}}} ; b\right)\right] d s \\
&= \int_{0}^{t} \lambda e^{-\lambda s} E\left[I\left(T_{t}^{\mathscr{E}}=s\right) g_{s}(\phi(u+W(s), s))\right] d s \\
&+\int_{0}^{t} \lambda e^{-\lambda s} E\left[I\left(T_{t}^{\mathscr{E}}<s\right) M_{1}\left(U_{b}(s), y e^{-\delta T_{t}^{\mathscr{E}}} ; b\right)\right] d s \tag{27}
\end{align*}
$$

where

$$
\begin{align*}
& g_{s}(\phi(u+W(s), s)) \\
& \quad=\int_{0}^{b-\phi(u+W(s), s)} M_{1}\left(\phi(u+W(s), s)+x, y e^{-\delta s} ; b\right) d s \\
& \quad+\int_{b-\phi(u+W(s), s)}^{\infty} M_{2}\left(\phi(u+W(s), s)+x, y e^{-\delta s} ; b\right) d s . \tag{28}
\end{align*}
$$

Noting that $\lim _{t \rightarrow 0}\left(T_{t}^{\epsilon}=t\right)=1, \lim _{t \rightarrow 0}\left(T_{t}^{\epsilon}<t\right)=0$ a.s. and using Itô's formula, we have

$$
\begin{align*}
& \lim _{t \rightarrow 0} \frac{\mathbf{I}(t)-M_{1}(u, y ; b)}{t} \\
&= \frac{1}{2} \sigma^{2} \frac{\partial^{2} M_{1}^{0}(u, y ; b)}{\partial u^{2}}+\left(r u-c_{1}\right) \frac{\partial M_{1}^{0}(u, y ; b)}{\partial u} \\
&-y \delta \frac{\partial M_{1}^{0}(u, y ; b)}{\partial y}-\lambda M_{1}^{0}(u, y ; b), \\
& \lim _{t \rightarrow 0} \frac{\mathbf{I I}(t)}{t}= \lambda \int_{0}^{b-u} M_{1}^{0}(u+x, y ; b) f(x) d x \\
&+ \lambda \int_{b-u}^{\infty} M_{2}^{0}(u+x, y ; b) f(x) d x . \tag{29}
\end{align*}
$$

From (26) and (29), we can get (20) for all $u \in(\epsilon$,$) . Hence,$ (20) holds in $(0, b)$.

Similarly, using the above method, we get (21). Conditions (22) and (23) are obvious. By (14) and (15) and by the weak convergence method used in [17], it is easy to check that the boundary conditions (24) and (25) hold. This ends the proof of Theorem 2.

### 2.2. Integrodifferential Equations for $V(u, b)$

Theorem 3. When $0<u<b$,

$$
\begin{align*}
& \frac{1}{2} \sigma^{2} V_{1}^{\prime \prime}(u)+\left(r u-c_{1}\right) V_{1}^{\prime}(u)-(\lambda+\delta) V_{1}(u) \\
& \quad+\lambda \int_{0}^{b-u} V_{1}(u+x ; b) f(x) d x  \tag{30}\\
& \quad+\lambda \int_{b-u}^{\infty} V_{2}(u+x ; b) f(x) d x=0
\end{align*}
$$

and when $b \leq u<\infty$,

$$
\begin{align*}
& \frac{1}{2} \sigma^{2} V_{2}^{\prime \prime}(u)+\left(r u-c_{2}\right) V_{2}^{\prime}(u) \\
& \quad-(\lambda+\delta) V_{2}(u)+\lambda \int_{0}^{\infty} V_{2}(u+x ; b) f(x) d x  \tag{31}\\
& \quad+c_{2}-c_{1}=0
\end{align*}
$$

with the following boundary conditions:

$$
\begin{gather*}
V_{1}(0 ; b)=0 \\
V_{1}(b-; b)=V_{2}(b ; b), \\
\lim _{u \rightarrow \infty} V_{2}(u ; b)=\frac{\left(c_{2}-c_{1}\right)}{\delta},  \tag{32}\\
\left.\left(r b-c_{1}\right) \frac{\partial V_{1}(u ; b)}{\partial u}\right|_{u=b-}=\left.\left(r b-c_{2}\right) \frac{\partial V_{2}(u ; b)}{\partial u}\right|_{u=b}
\end{gather*}
$$

## 3. Numerical Analysis

The second order system of integrodifferential equations such as in Theorem 3 does not have an explicit solution except in some special cases. In this section, we propose the approximate solution of this equation via the use of a collocation based on sinc methods. The sinc method is highly efficient numerical method developed by Frank Stenger. It is widely used in various fields of numerical analysis such as interpolation, quadrature, approximation of transforms, and solution of integral, ordinary differential, and partial differential equations. An introduction to the sinc approximation theory can be found in the Appendix.
3.1. Numerical Solution of the Expected Present Value of Total Dividends $V(u ; b)$. To construct an approximation on the interval $(0, \infty)$, we consider the following conformal map:

$$
\begin{equation*}
\phi(z)=\log z . \tag{33}
\end{equation*}
$$

The function $\phi$ also provides a one-to-one transformation of $(0, \infty)$ onto the real line $\mathbb{R}$. The sinc grid points $z_{k}$ are defined for $h>0$ and $k=0, \pm 1, \pm 2, \ldots$ by

$$
\begin{equation*}
z_{k}=\phi^{-1}(k h)=e^{k h} . \tag{34}
\end{equation*}
$$

In order to adopt the sinc method procedure, we arrange the systems in Theorem 3 into the following integrodifferential equation:

$$
\begin{align*}
& \frac{1}{2} \sigma^{2} V^{\prime \prime}(u)+\left[\left(r u-c_{2}\right) I(u \geq b)+\left(r u-c_{1}\right) I(0<u<b)\right] \\
& \quad \times V^{\prime}(u)-(\lambda+\delta) V(u)+\lambda \int_{0}^{\infty} V(u+x ; b) \\
& \quad \times f(x) d x+\left(c_{2}-c_{1}\right) I(u>b)=0 . \tag{35}
\end{align*}
$$

Using the properties of convolution, the above equation can be further written as

$$
\begin{align*}
& \frac{1}{2} \sigma^{2} V^{\prime \prime}(u)+\left[\left(r u-c_{2}\right) I(u \geq b)\right. \\
& \left.\quad+\left(r u-c_{1}\right) I(0<u<b)\right] V^{\prime}(u) \\
& \quad-(\lambda+\delta) V(u)+\lambda \int_{u}^{\infty} V(x ; b) f(x-u) d x  \tag{36}\\
& \quad+\left(c_{2}-c_{1}\right) I(u>b)=0
\end{align*}
$$

with the following boundary conditions:

$$
\begin{equation*}
V(0 ; b)=0, \quad \lim _{u \rightarrow \infty} V(u ; b)=\frac{c_{2}-c_{1}}{\delta} . \tag{37}
\end{equation*}
$$

Set $W(u)=V(u ; b)-L V(u ; b)=V(u ; b)-(u /(1+u))\left(\left(c_{2}-\right.\right.$ $\left.\left.c_{1}\right) / \delta\right)$; then $W(u) \in L_{\tilde{\alpha}, \widetilde{\beta}, d}(\phi)$ and satisfies

$$
\begin{align*}
& W^{\prime \prime}(u)+\mu(u) W^{\prime}(u)-\frac{2}{\sigma^{2}}(\lambda+\delta) W(u) \\
& \quad+\frac{2}{\sigma^{2}} \lambda \int_{u}^{\infty} W(x) f(u-x) d x  \tag{38}\\
& \quad+R(u)=0,
\end{align*}
$$

with the following boundary conditions:

$$
\begin{equation*}
W(0)=0, \quad \lim _{u \rightarrow \infty} W(u)=0 \tag{39}
\end{equation*}
$$

where

$$
\begin{aligned}
\mu(u)= & \frac{2}{\sigma^{2}}\left[\left(r u-c_{2}\right) I(u \geq b)\right. \\
& \left.+\left(r u-c_{1}\right) I(0<u<b)\right] \\
R(u)= & \frac{2}{\sigma^{2}}\left(c_{2}-c_{1}\right) I(u \geq b) \\
& -\frac{2}{(1+u)^{3}} \frac{c_{2}-c_{1}}{\delta} \\
& +\mu(u) \frac{1}{(1+u)^{2}} \frac{c_{2}-c_{1}}{\delta} \\
& -\frac{2}{\sigma^{2}}(\lambda+\delta) \frac{u}{1+u} \frac{c_{2}-c_{1}}{\delta} \\
& +\lambda \frac{2}{\sigma^{2}} \frac{c_{2}-c_{1}}{\delta} \int_{u}^{\infty} \frac{x}{1+x} f(x-u) d x
\end{aligned}
$$

Let $\phi$ be defined by (33). The sinc grid points are defined for $h>0$ by

$$
\begin{equation*}
u_{k}=e^{k h}, \quad k=0, \pm 1, \pm 2, \ldots \tag{41}
\end{equation*}
$$

Then, by using Theorems A. 3 and A. 6 in the Appendix, we have

$$
\begin{gather*}
\int_{u}^{\infty} f(x-u) W(x) d x \approx \sum_{l=-M}^{N} \sum_{i=-M}^{N} w_{i}^{*} A_{i l} W_{l},  \tag{42}\\
W(x) \approx \widetilde{W}(x)=\sum_{l=-M}^{N} w_{l} W_{l}, \tag{43}
\end{gather*}
$$

where $A=Y F(S) Y^{-1}, A_{i l}$ is the $(i, l)$ th element of matrix $A$. $W_{l}$ denotes an approximate value of $W\left(u_{l}\right)$, and $\phi(x)=\log x$, $\phi(0)=-\infty, \phi(+\infty)=+\infty$, and $\phi^{\prime}(x)=1 / x \cdot w_{j}$ and $w_{j}^{*}$ for $j=-M, \ldots, N$ are defined as Definition A. 2 in the Appendix.

Since $W(x) \in L_{\tilde{\alpha}, \tilde{\beta}, d}(\phi)$, by Theorem A. 3 in the Appendix, then it is convenient to take $w_{j}=\gamma_{j}=S(j, h) \circ \phi, j=$ $-M, \ldots, N$. Moreover, by some simple calculations, we have $w_{j}^{*}\left(u_{k}\right)=S(j, h) \circ \phi\left(u_{k}\right), j, k=-M, \ldots, N$ with $\phi$ and $u_{k}$ being defined in (33) and (41), respectively.

Having replaced the integral term on the left-hand side of system (38) with the right-hand side of (42) and having substituted $u=u_{k}$ for $k=-M, \ldots, N$, where $u_{k}$ are sinc grid points and also by replacing $W(u)$ by $\widetilde{W}(u)$ as in (43), we obtain the following collocation result:

$$
\begin{align*}
& \widetilde{W}^{\prime \prime}\left(u_{k}\right)+\mu\left(u_{k}\right) \widetilde{W}^{\prime}\left(u_{k}\right)-\frac{2}{\sigma^{2}}(\lambda+\delta) \widetilde{W}\left(u_{k}\right) \\
& \quad+\frac{2}{\sigma^{2}} \lambda \sum_{l=-M i=-M}^{N} \sum^{N} S(j, h) \circ \phi\left(u_{k}\right) A_{i l} W_{l}+R\left(u_{k}\right)=0 \tag{44}
\end{align*}
$$

where $\widetilde{W}\left(u_{k}\right)=\sum_{l=-M}^{N} W_{l} S(l, h) \circ \phi\left(u_{k}\right)$.
Then, by Theorem A.7, we have

$$
\begin{align*}
& \widetilde{W}\left(u_{k}\right)=\sum_{l=-M}^{N} W_{l} \delta_{l k}^{(0)} \\
& \begin{aligned}
\widetilde{W}^{\prime}\left(u_{k}\right) & =\sum_{l=-M}^{N} W_{l} \phi^{\prime}\left(u_{k}\right) h^{-1} \delta_{l k}^{(1)} \\
\widetilde{W}^{\prime \prime}\left(u_{k}\right) & =\sum_{l=-M}^{N} W_{l}\left[\phi^{\prime \prime}\left(u_{k}\right) h^{-1} \delta_{l k}^{(1)}\right. \\
& \left.+\left(\phi^{\prime}\left(u_{k}\right)\right)^{2} h^{-2} \delta_{l k}^{(2)}\right]
\end{aligned} \tag{45}
\end{align*}
$$

Replacing (45) in (44), we rewrite (44) as

$$
\begin{align*}
\sum_{l=-M}^{N} W_{l}\left\{\phi^{\prime \prime}\right. & \left(u_{k}\right) h^{-1} \delta_{l k}^{(1)} \\
& +\left(\phi^{\prime}\left(u_{k}\right)\right)^{2} h^{-2} \delta_{l k}^{(2)}+\mu\left(u_{k}\right) \phi^{\prime}\left(u_{k}\right) h^{-1} \delta_{l k}^{(1)} \\
& -\frac{2}{\sigma^{2}}(\lambda+\delta) \delta_{l k}^{(0)}+\frac{2}{\sigma^{2}} \lambda \\
& \left.\times \sum_{i=-M}^{N} S(j, h) \circ \phi\left(u_{k}\right) A_{i l}\right\}=-R\left(u_{k}\right) \tag{46}
\end{align*}
$$

Having multiplied the resulting equations by $h^{2} /$ $\left(\phi^{\prime}\left(u_{k}\right)\right)^{2}$, we have

$$
\begin{align*}
& \sum_{l=-M}^{N} W_{l}\left\{\delta_{l k}^{(2)}+h\left[\frac{\phi^{\prime \prime}\left(u_{k}\right)}{\left(\phi^{\prime}\left(u_{k}\right)\right)^{2}}+\frac{\mu\left(u_{k}\right)}{\phi^{\prime}\left(u_{k}\right)}\right] \delta_{l k}^{(1)}\right. \\
& \quad-\frac{2 h^{2}(\lambda+\delta)}{\sigma^{2}\left(\phi^{\prime}\left(u_{k}\right)\right)^{2}} \delta_{l k}^{(0)} \\
& \left.\quad+\frac{2 \lambda h^{2}}{\sigma^{2}\left(\phi^{\prime}\left(u_{k}\right)\right)^{2}} \sum_{i=-M}^{N} S(j, h) \circ \phi\left(u_{k}\right) A_{i l}\right\}  \tag{47}\\
& =-\frac{h^{2} R\left(u_{k}\right)}{\left(\phi^{\prime}\left(u_{k}\right)\right)^{2}} .
\end{align*}
$$

Now, since $\delta_{k l}^{(0)}=\delta_{l k}^{(0)}, \delta_{k l}^{(1)}=-\delta_{l k}^{(1)}, \delta_{k l}^{(2)}=\delta_{l k}^{(2)}$, and $\phi^{\prime \prime}\left(u_{k}\right) /\left(\phi^{\prime}\left(u_{k}\right)\right)^{2}=-\left(1 / \phi^{\prime}\left(u_{k}\right)\right)^{\prime}$, we obtain the collocation result as

$$
\begin{gather*}
\sum_{l=-M}^{N} W_{l}\left\{\delta_{k l}^{(2)}+h\left[\left(\frac{1}{\phi^{\prime}\left(u_{k}\right)}\right)^{\prime}-\frac{\mu\left(u_{k}\right)}{\phi^{\prime}\left(u_{k}\right)}\right] \delta_{k l}^{(1)}\right. \\
\quad-\frac{2 h^{2}(\lambda+\delta)}{\sigma^{2}}\left(\phi^{\prime}\left(u_{k}\right)\right)^{2} \delta_{k l}^{(0)}+\frac{2 \lambda h^{2}}{\sigma^{2}\left(\phi^{\prime}\left(u_{k}\right)\right)^{2}} \\
\left.\quad \times \sum_{i=-M}^{N} S(j, h) \circ \phi\left(u_{k}\right) A_{i l}\right\} \\
=-\frac{h^{2} R\left(u_{k}\right)}{\left(\phi^{\prime}\left(u_{k}\right)\right)^{2}}, \quad k=-M, \ldots, N . \tag{48}
\end{gather*}
$$

We set $I^{(m)}=\left[\delta_{k l}^{(m)}\right]$ and $m=-1,0,1,2$, where $\delta_{k l}^{(m)}$ denotes the $(k, l)$ th element of the matrix $I^{(m)}$. Also, we denote $\bar{\Delta}=\left[S(i, h) \circ \phi\left(u_{k}\right)\right]$ and $I^{(m)}, m=-1,0,1,2$ are square matrices of order $(N+M+1) \times(N+M+1)$. So, the system (48) can be given in matrix form as

$$
\begin{equation*}
B W=F, \tag{49}
\end{equation*}
$$



Figure 1: The expected present value of total dividends $V(u ; 2)$.
where $W=\left[W_{-M}, \ldots, W_{N}\right]^{T}, F=\left[-\left(h^{2} H\left(u_{-M}\right) /\right.\right.$ $\left.\left.\left(\phi^{\prime}\left(u_{-M}\right)\right)^{2}\right), \ldots,-\left(h^{2} H\left(u_{N}\right) /\left(\phi^{\prime}\left(u_{N}\right)\right)^{2}\right)\right]^{T}$, and

$$
\begin{align*}
B= & I^{(2)}+h D_{m}\left[\left(\frac{1}{\phi^{\prime}}\right)^{\prime}-\frac{\mu}{\phi^{\prime}}\right] I^{(1)} \\
& -\frac{2 h^{2}}{\sigma^{2}}(\lambda+\delta) D_{m} \frac{1}{\left(\phi^{\prime}\right)^{2}} I^{(0)}+\frac{2 h^{2} \lambda}{\sigma^{2}} D_{m} \frac{1}{\left(\phi^{\prime}\right)^{2}} \bar{\Delta} A . \tag{50}
\end{align*}
$$

The above linear system contains $(N+M+1)$ equations with $(N+M+1)$ unknown coefficient $W_{k}, k=-M, \ldots, N$. Solving this linear system, we obtain the approximate solution of the system as follows:

$$
\begin{gather*}
W(u) \approx \widetilde{W}(u)=\sum_{l=-M}^{N} W_{l} S(l, h) \circ \phi(u),  \tag{51}\\
V(u ; b)=W(u)+\frac{u}{1+u} \frac{c_{2}-c_{1}}{\delta} .
\end{gather*}
$$

## 4. Numerical Example

In this section, we consider some numerical samples to illustrate the performance of sinc method and investigate how much the values of $V(u ; b)$ are affected by interest $r$. An example is solved under the assumption that the claim size density is given by $f(y)=\eta e^{-\eta y} I(y>0)$.

Example 1. Let $c_{1}=0.1, c_{2}=0.2, \mu=5, \alpha=\beta=\pi / 4, \lambda=3$, $b=2, \sigma=0.05, \delta=0.006, \widetilde{\alpha}=1 / 2, \widetilde{\beta}=1 / 4, d=1 / 4000$, and $r=0.05$, the result is shown in Figure 1.

From Figure 1, we see that $V(u ; 2)$ is an increasing function with respect to $u$.

## Appendix

The sinc method is a highly efficient numerical method developed by Stenger, the pioneer of this field, people in his school, and others [18-22]. It is widely used in various fields of numerical analysis such as interpolation, quadrature, approximation of transforms, and solution of integral, ordinary differential, and partial differential equations.
sinc methods are based on the use of the cardinal function, $C(f, h)$, which is sinc expansion of function $f$, defined by

$$
\begin{equation*}
C(f, h)(x)=\sum_{k \in \mathbb{N}} f(k h) \operatorname{sinc}\left\{\frac{x}{h}-k\right\}, \quad-\infty<x<\infty \tag{A.1}
\end{equation*}
$$

where the step size $h>0$, and the function sinc is defined on the whole real line, $-\infty<x<\infty$, by

$$
\operatorname{sinc}(z)= \begin{cases}\frac{\sin (\pi z)}{\pi z}, & \text { if } z \neq 0  \tag{A.2}\\ 1, & \text { if } z=0\end{cases}
$$

For any $h>0$, the translated sinc functions with evenly spaced nodes are given as

$$
\begin{equation*}
S(j, h)(z)=\operatorname{sinc}\left\{\frac{z}{h}-j\right\}, \quad j=0, \pm 1, \pm 2, \ldots \tag{A.3}
\end{equation*}
$$

The sinc functions are cardinal for the interpolating points $k h$ in the sense that

$$
S(j, h)(k h)=\delta_{j k}^{(0)}= \begin{cases}0, & \text { if } k \neq j  \tag{A.4}\\ 1, & \text { if } k=j\end{cases}
$$

Definition A. 1 (see [19, P73, De. 1.5.2]). Let $\phi$ denote a smooth one-to-one transformation of an $\operatorname{arc} \Gamma$ with end-point $t_{1}$ and $t_{2}$ onto $\mathbb{R}$, such that $\phi\left(t_{1}\right)=-\infty$ and $\phi\left(t_{2}\right)=\infty$. Let $\psi=\phi^{-1}$ denote the inverse map, so that

$$
\begin{equation*}
\Gamma=\{z \in \mathbb{C}: z=\psi(u), u \in \mathbb{R}\} . \tag{A.5}
\end{equation*}
$$

Given $\phi, \psi$, and a positive number $h$, define the sinc points $z_{k}$ by

$$
\begin{equation*}
z_{k}=z_{k}(h)=\psi(k h), \quad k=0, \pm 1, \pm 2, \ldots, \tag{A.6}
\end{equation*}
$$

and a function $\rho$ by

$$
\begin{equation*}
\rho(z)=e^{\phi(z)} \tag{A.7}
\end{equation*}
$$

Observe that $\rho(z)$ increases from 0 to $\infty$ as $z$ traverses $\Gamma$ from $t_{1}$ to $t_{2}$.

Corresponding to positive numbers $\widetilde{\alpha}, \widetilde{\beta}$, and $d$, let $L_{\tilde{\alpha}, \widetilde{\beta}, d}(\phi)$ denote the family of all functions $f$ defined on $\Gamma$ for which

$$
f(z)= \begin{cases}\mathcal{O}\left(\rho(z)^{\tilde{\alpha}}\right), & \text { if } z \longrightarrow t_{1}  \tag{A.8}\\ \mathcal{O}\left(\rho(z)^{-\mathcal{\beta}}\right), & \text { if } z \longrightarrow t_{2}\end{cases}
$$

such that the Fourier transform $\left\{f \circ \phi^{-1}\right\}^{\sim}$ satisfies the following relation:

$$
\begin{equation*}
\left\{f \circ \phi^{-1}\right\}^{\sim}(\zeta)=\mathcal{O}\left(e^{-d|\zeta|}\right) \tag{A.9}
\end{equation*}
$$

for all $\zeta \in \mathbb{R}$.
Another important family of functions is $M_{\tilde{\alpha}, \widetilde{\beta}, d}(\phi)$, with $0<\widetilde{\alpha}, \widetilde{\beta} \leq 1$, and $0<d<\pi$. It consists of all those functions $f$ defined on $\Gamma$ such that $g=f-L f \in L_{\tilde{\alpha}, \widetilde{\beta}, d}(\phi)$ and where $L f$ is defined by $L f=\left(f\left(t_{1}\right)+\rho(x) f\left(t_{2}\right)\right) /(1+\rho(x))$.

Let $N$ and $M$ denote two positive integers, such that $M=[\widetilde{\beta} N / \widetilde{\alpha}]$, or $N=[\widetilde{\alpha} M / \widetilde{\beta}]$ where $[\cdot]$ denotes the greatest integer function. Moreover, we denote $m=M+N+1$.

Definition A.2. Given three positive integers $N, M$, and $m$ as above, let $D_{m}$ and $V_{m}$ denote linear operators acting on functions $f$ defined on $\Gamma$ given by

$$
\begin{gather*}
D_{m} f=\operatorname{diag}\left[f\left(z_{-M}\right), \ldots, f\left(z_{N}\right)\right], \\
V_{m} f=\left(f\left(z_{-M}\right), \ldots, f\left(z_{N}\right)\right)^{T} \tag{A.10}
\end{gather*}
$$

where $z_{j}=\psi(j h)$ denote the sinc points. Set

$$
\begin{gather*}
h=\left(\frac{\pi d}{\widetilde{\beta} N}\right)^{1 / 2}, \\
\gamma_{j}=S(j, h) \circ \phi, \quad j=-M, \ldots, N, \\
w_{j}=w_{j}^{*}=\gamma_{j}, \quad j=-M+1, \ldots, N-1, \\
w_{-M}=\frac{1}{1+\rho}-\sum_{j=-M+1}^{N} \frac{\gamma_{j}}{1+e^{j h}}, \\
w_{N}=\frac{\rho}{1+\rho}-\sum_{j=-M}^{N-1} \frac{e^{j h} \gamma_{j}}{1+e^{j h}},  \tag{A.11}\\
w_{-M}^{*}=\left(1+e^{-M h}\right) w_{-M}, \\
w_{N}^{*}=\left(1+e^{-N h}\right) w_{N}, \\
\mathscr{E}_{N}=N^{1 / 2} e^{-(\pi d \tilde{\beta} N)^{1 / 2}}, \\
\Delta_{m}=\left(w_{-M}, \ldots, w_{N}\right), \\
\Delta_{m}^{*}=\left(w_{-M}^{*}, \ldots, w_{N}^{*}\right), \\
\delta_{i j}^{(-1)}=\frac{1}{2}+\int_{0}^{i-j} \operatorname{sinc}(t) d t,
\end{gather*}
$$

and we then define an $m \times m$ matrix $I^{(-1)}$ by $I^{(-1)}=\left[\delta_{i j}^{(-1)}\right]$, with $\delta_{i j}^{(-1)}$ denoting the $(i, j)$ th element of $I^{(-1)}$. Finally, let $\|\cdot\|$ denote the uniform norm on $\Gamma$ that is,

$$
\begin{equation*}
\|f\|=\sup _{x \in \Gamma}|f(x)| . \tag{A.12}
\end{equation*}
$$

Theorem A. 3 (see [19, P85, Th. 1.5.13]). Let $f \in M_{\tilde{\alpha}, \widetilde{\beta}, d}(\phi)$; then, as $N \rightarrow \infty$,

$$
\begin{equation*}
\left\|f-\Delta_{m} V_{m} f\right\|=\mathcal{O}\left(\mathscr{E}_{N}\right) \tag{A.13}
\end{equation*}
$$

If $f \in L_{\tilde{\alpha}, \widetilde{\beta}, d}(\phi)$, then it is convenient to take $w_{j}=\gamma_{j}=S(j, h)$ 。 $\phi, j=-M, \ldots, N$.

Theorem A. 4 (see [19, P87, Th. 1.5.14]). Let $f \in M_{\widetilde{\alpha}, \widetilde{\beta}, d}(\phi)$; then,

$$
\begin{equation*}
\left\|\left(\frac{h}{\phi^{\prime}}\right)^{k}\left[f^{(k)}-\Delta_{m}^{(k)} V_{m} f\right]\right\|=\mathcal{O}\left(\mathscr{E}_{N}\right), \quad N>1 \tag{A.14}
\end{equation*}
$$

Theorem A. 5 (see [19, P93, Th. 1.5.19]). Let $f / \phi^{\prime} \in L_{\tilde{\alpha}, \widetilde{\beta}, d}(\phi)$; then, for all $N>1$,

$$
\begin{align*}
\left\|\mathscr{J}^{+} f-\mathscr{J}_{m}^{+} f\right\| & =\mathcal{O}\left(\mathscr{E}_{N}\right),  \tag{A.15}\\
\left\|\mathscr{J}^{-} f-\mathscr{J}_{m}^{-} f\right\| & =\mathcal{O}\left(\mathscr{E}_{N}\right),
\end{align*}
$$

where

$$
\begin{array}{cc}
\left(\mathscr{J}^{+} f\right)(x)=\int_{t_{1}}^{x} f(t) d t, & \left(\mathscr{J}^{-} f\right)(x)=\int_{x}^{t_{2}} f(t) d t, \\
\left(\mathscr{J}_{m}^{+} f\right)(x)=\Delta_{m}(x) A^{+} V_{m} f, & A^{+}=h I^{(-1)} D_{m}\left(\frac{1}{\phi^{\prime}}\right), \\
\left(\mathscr{J}_{m}^{-} f\right)(x)=\Delta_{m}(x) A^{-} V_{m} f, & A^{-}=h\left(I^{(-1)}\right)^{T} D_{m}\left(\frac{1}{\phi^{\prime}}\right), \tag{A.16}
\end{array}
$$

with $\left(I^{(-1)}\right)^{T}$ denoting the transpose of $I^{(-1)}$.
Theorem A. 6 (see [19, P95-96, Th. 1.5.20]). Let g/ $\phi^{\prime} \in$ $L_{\tilde{\alpha}, \widetilde{\beta}, d}(\phi)$; suppose that $A^{ \pm}$can be diagonalized with $A^{+}=$ $X S X^{-1}$ and with $A^{-}=Y S Y^{-1}$ and that $S$ is a diagonal matrix. If for some positive $c^{\prime}$ independent of $N$, one has $F^{\prime}(s) \leq c^{\prime}$ for all $\Re s \geq 0$, then there exists a constant $C$ independent of $N$ such that

$$
\begin{align*}
& \left\|p-F\left(\mathscr{f}_{m}^{+}\right) g\right\| \leq C \mathscr{E}_{N},  \tag{A.17}\\
& \left\|q-F\left(\mathscr{F}_{m}^{-}\right) g\right\| \leq C \mathscr{E}_{N},
\end{align*}
$$

where

$$
\begin{gather*}
F(s)=\int_{0}^{c^{\prime \prime}} e^{-t / s} f(t) d t \\
p(x)=\int_{t_{1}}^{x} f(x-t) g(t) d t  \tag{A.18}\\
q(x)=\int_{x}^{t_{2}} f(t-x) g(t) d t, \\
\left(F\left(\mathscr{J}_{m}^{+}\right) g\right)(x)=\Delta_{m}^{*}(x) X F(S) X^{-1} V_{m} g \\
\left(F\left(\mathscr{J}_{m}^{-}\right) g\right)(x)=\Delta_{m}^{*}(x) Y F(S) Y^{-1} V_{m} g, \tag{A.19}
\end{gather*}
$$

with $c^{\prime \prime} \geq t_{2}-t_{1}$.

Theorem A. 7 (see [18, P106, Th. 4.1]). Let $\phi$ be a conformal one-to-one transformation of an arc $\Gamma$. Then,

$$
\left.\begin{array}{rl}
\left.\delta_{j k}^{(0)} \equiv[S(j, h) \circ \phi(z)]\right|_{z=z_{k}}= \begin{cases}0, & \text { if } k \neq j, \\
1, & \text { if } k=j,\end{cases} \\
\delta_{j k}^{(1)} & \left.\equiv h \frac{d}{d \phi}[S(j, h) \circ \phi(z)]\right|_{z=z_{k}}
\end{array}, \begin{array}{ll}
\frac{(-1)^{k-j}}{k-j}, & \text { if } k \neq j, \\
0, & \text { if } k=j,
\end{array}\right\} \begin{aligned}
\delta_{j k}^{(2)} & \left.\equiv h^{2} \frac{d^{2}}{d \phi^{2}}[S(j, h) \circ \phi(z)]\right|_{z=z_{k}}
\end{aligned}, \begin{array}{ll}
\frac{-2(-1)^{k-j}}{(k-j)^{2}}, & \text { if } k \neq j,  \tag{A.20}\\
-\frac{\pi^{2}}{3}, & \text { if } k=j .
\end{array}
$$

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## Research Article

# Mathematical Modeling of the Propagation of Democratic Support of Extreme Ideologies in Spain: Causes, Effects, and Recommendations for Its Stop 

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#### Abstract

This paper deals with the construction of a discrete population mathematical model for the short-term forecast until January 2016 of the electoral support of extreme ideology parties in Spain. Firstly, the nontrivial concept of extreme ideology is stated. Then, the electoral register is split in three subpopulations: supporters of extremist parties, abstentions/blank voters, and supporters of establishment parties. The model takes into account the following variables: economy measured throughout the Spanish unemployment rate; demography quantified in terms of birth and death rates and emigration; sociopolitical situation measured by the Spanish poverty indicator, trust on the Government labor indicator (GLI), and the indicator of political trust. By considering the dynamic subpopulations transits built throughout data obtained from public and private prestigious institutions and sociopolitical analysis, a system of difference equations models the electoral population behavior in Spain allowing us to compute the expected electoral support in the time horizon of January 2016. Sensitivity analysis versus uncertain parameters is performed in order to improve the reliability of the model results.


## 1. Introduction

It is difficult to believe that Europe could perform its economic recovery if previously it does not achieve confidence in its political system. The depth and length of the European economic crisis with high levels of unemployment (mainly youth), public debt, increasing taxation, and welfare deterioration combined with a lack of solutions by Government parties as well as the so called "cartelization" of the political parties are producing the emergence of political offers breaking the limits of traditional European trends. Thus, the populist and extremist political offers are not infrequent at the present times [1], and the Government parties are losing their power by each election period since 2008.

Literature is abundant in studies focused on the use of surveys or previous election's results applying purely statistic analysis and techniques [2]. However, these approaches are not reliable in the current situation previously described
due to the radical changes produced in the behavior of the European citizens and governments by the hard economic crisis.

In this paper, we propose a population discrete mathematical model represented by a system of difference equations to forecast in the short-term horizon of 2015 the electoral support of establishment and extremist parties as well as the level of abstention and blank voters. In fact the present situation in Spain is very special due to the combination of the economic and institutional crisis [3-6].

The study of the propagation of the extremist vote may be useful to recommend measures to prevent the risk of political instability by the implementation of political and economical alternatives focused on curbing the electoral propagation of extreme ideologies.

For the sake of accuracy and reliability, the period of study is split into quarters of year. Our approach splits the Spanish register of voters into three subpopulations.

The aim of this paper is the construction of a mathematical model for short-term (next national elections) forecast of the electoral support of Spanish citizenship up to 2015. Individual behavior may be erratic, but aggregate behavior is often quite predictable [7]. Our approach is epidemiological dealing with populations instead of individuals. The foundations of such approach are based on the mimetic human behavior [8], human herding [9], and social contagion [10].

The register of voters is split into three categories or subpopulations. The three relevant subpopulations are explained below.
(1) The extremist voters (EX) are identified as followers of any of these three alternatives.
(i) Breakers of the constitutional order (separatists).
(ii) Breakers of the capitalist system.
(iii) Those who are disrespectful of the human rights (racists of gender, race, minorities. . .).
(2) The second subpopulation is called abstentions (AB) composed of those potential electors who do not vote or plan to vote [11].
(3) In addition, the establishment voters (ES) composed of those citizens planning of vote for a nonextremist party that also achieved representation on the Spanish parliament for at least the last three national elections.

It is important to point out that these definitions depend on the type of election we are managing. In fact, nowadays there are political parties in Spain like Convergència i Unió (CiU) that play the role of extremists (EX) in the national elections, while in the Autonomous Community elections CiU and others they play the role of establishment parties (ES). This occurs because of the recent strategy of separatism developed by CiU . However, in this study we only focus on national elections.

It is worthy to remark that unlike what happens in Spain, in many European countries the extremist parties are represented by racist ideology parties [1].

It is important to point out that we use electoral data coming from the last three Spanish general elections (2004, 2008, and 2011) in order to know the initial level of each subpopulation. Then, we will study the dynamic subpopulations' transits taking into account economic (Spanish unemployment rate), demographic (birth rate, death rate, and emigration), and socio-political (corruption indicator, poverty index, and the trust Government's labor indicator) factors. Thus, solving the difference equation system, the amount of supporters of each subpopulation is quantified.

This type of population approach has been recently employed in the study of sociological problems such as anxiolytic consumption dependency [12], workaholism [13], and compulsive shopping behavior [14].

The paper is organized as follows. Section 2 deals with mathematical model construction with an explicit computation of the subpopulations coefficients transits. Results and simulations are included in Section 3. Section 4 shows
the robustness of the model throughout changes in the Government labor indicator (GLI) and also the political trust indicator. Finally, conclusions and recommendations section is included.

## 2. Model Construction

This section embraces the description of the population of study, then the sources of information and hypotheses employed and the modeling of the dynamic subpopulations transits.

The target population is composed of the Spanish citizens older than 18 years old. The total amount of the population is affected by demographic factors such as emigration, immigration, and birth and death rates.

We are building a discrete dynamic epidemiological model whose subpopulations are quarterly estimated. This way, we start forecasting the citizenship vote intention in Spanish national elections. Our period of study starts in January 2012 and ends at January 2016, splitted it into quarters.

Our starting hypothesis is that subpopulations change dynamically and these variations depend mainly of demographic factors (emigration, birth and death rates), economic ones (unemployment rate, poverty indicator), and sociological trends (Government and political trust indicators). The transit between subpopulations (ES, AB , and EX) is modeled by dynamic coefficients built on data provided by direct sources of information such as the Spanish Institute of Statistics (INE) [15], Center of Sociological Research (CIS) [16], the Organization for Economic Cooperation and Development (OECD) [17], Societé Générale Bank (SGB) [18], and International Monetary Fund (IMF) [19] combined with our hypotheses and analysis.

The financial and economic crisis started at the end of 2007 resulted in social and demographic changes. Due to the breakdown of the economic trend, we only take into account the national data of 2004, 2008 and 2011, since previous data does not provide evidence of this trend change.

The Spanish economic framework is characterized by high level of stable unemployment rates (above 20\%). Although we should distinguish between ideologized and non ideologized voters, in the present economic recession we assume that nonideologized ES voters transit to AB after becoming unemployed. This means that if the unemployment increases in $1 \%$ then $0.01 \mathrm{ES}(n)$ people will transit to the subpopulation $\mathrm{AB}(n)$. By the same reason if the unemployment decreases in $1 \%$, then $0.01 \mathrm{AB}(n)$ would transit to $\mathrm{ES}(n)$. Thus, we are dealing with a type of citizens whose vote support is mainly dependent on the economy measured in terms of unemployment.

This approach is conservative in the sense that the new unemployed people could impact the electoral behavioral of his/her close neighborhood, for instance the family.

Thus, the transit between subpopulations $\mathrm{ES}(n)$ and $\mathrm{AB}(n)$ follows the rule:

$$
\alpha_{1}(n)=\left\{\begin{array}{cl}
0.01(\gamma(n)-\gamma(n-1)), & \gamma(n) \neq \gamma(n-1) \\
0.01, & \gamma(n)=\gamma(n-1)
\end{array}\right\}
$$

$$
\alpha_{1}(n)=\text { unemployment economic transit }
$$

effect between quarters $n$ and $n+1$,
$\gamma(n)=$ unemployment rate at quarter $n(\%)$.

A proportion of ideologized voters who used to vote for the establishment parties in the recent past change their vote due to their lack of political trust due to the continuous emergence of corruption cases combined with the increase of taxes and the deterioration of the welfare system. The Spanish political trust indicator performed quarterly by [16] calibrates this lack of political trust. These voters punish the ES parties by moving from $\operatorname{ES}(n)$ to $\operatorname{EX}(n+1)$ at $n+1$ quarter. Thus, we have the transit coefficient:

$$
\begin{equation*}
\alpha 2(n)=C(\beta(n)-\beta(n-1)), \tag{2}
\end{equation*}
$$

where $\beta(n)$ is the value of the political trust indicator at quarter $n$.

The approximation of this proportion $C$ that is regarded as a sociological constant is adjusted by matching the electoral data of the last three Spanish general elections. After matching the data it turns out that $C=0.00299$.

As we commented previously, Spanish population is suffering from the consequences of long-term unemployment rate. A proportion of long-term unemployed citizens orient their vote intention towards extremist parties looking for possible solutions outside the ES parties. The non-ideologized citizen is affected by the principle of the continuity of transit which means moving from AB to EX instead of from ES to EX.

The proportion of population transiting from $A B$ to EX is measured throughout the Spanish poverty indicator [20]. Analyzing the behavior of this indicator for the last 4 years (2008-2011) and assuming that this remains stable until 2015 (because the Spanish economy is not going to improve in an important degree), we found that this quarter rate is about $0.5 \%$ of the AB subpopulation.

Thus, the rate of radicalization due to Spanish poverty indicator is denoted by $\alpha_{3}(n)=\alpha_{3}$.

Finally, we consider the supporters of the Government parties to become disappointed with the labor of the Government changing their vote intention from ES to AB . This may occurs due to two main reasons: the lack of Government's commitment with the electoral program and the weakness of Government's reaction to corruption scandals [21-23].

The coefficient $\alpha_{4}$ is expressed as follows:

$$
\alpha_{4}=0.329(f(n-1)-f(n)),
$$

$$
\begin{equation*}
f(n)=\text { is defined as the Government's } \tag{3}
\end{equation*}
$$

labor indicator at quarter $n$.
This indicator is estimated quarterly by [16].
The constant factor 0.329 ( $32.9 \%$ ) means the proportion of susceptible ES voters that can move to other subpopulations due to the lack of confidence in Government. That is, about $67.1 \%$ of ES voters remain loyal to the previous ES
voting intention. This percentage includes a proportion of the retired people among the age interval of $(60-75)$ who are afraid of losing their pensions, also a proportion of public servants, clergy, military, the socialist labor union (UGT members), and ES political parties' members.

This factor 0.329 has been adjusted by using electoral data from the general elections of 2004, 2008, and 2011. The sociological factor 0.329 is assumed constant in the short period of study (January 2016) reflecting the level of loyalty from different population segments to the ES parties. The data of $f(n)$ are provided by [16] until July 2013. From this time until the end of our period of study (January 2016), we assume that $f(n)$ goes slightly down due to the continuous diffusion of corruption cases of the two main Spanish ES parties (PSOE and PP), also a persistent application of new taxes combined with the welfare system deterioration, and finally, the demographic factors composed of the birth and death rates and the new effect of Spanish emigrating abroad seeking for a job.
$\alpha_{b}(n-72)$ is the birth rate of the Spanish population at quarter ( $n-72$ ). This parameter is estimated according to the data coming from [15]. We assume that all Spanish citizens becoming 18 -year-old entry into the system are distributed equally among the three subpopulations. Thus,

$$
\begin{equation*}
\frac{B(n-72)}{3}=\frac{\alpha_{b}(n-72) P(n-72)}{3} \tag{4}
\end{equation*}
$$

where $B(n-72)$ is the number of births at quarter $(n-72)$ and $P(n-72)$ is the Spanish population at quarter $(n-72)$.

Then, the exit from the system could happen due to two reasons, the biological death measured throughout the death rate $\left(\alpha_{d}(n)\right)$ at quarter $n$, uniformly quarterly distributed. This parameter is estimated throughout data provided by [15]. We assume that this rate remains constant for the period from 2013 until 2015. Consider the following:

$$
\begin{equation*}
\frac{D(n)}{3}=\alpha_{d}(n) \frac{P(n)}{3} \tag{5}
\end{equation*}
$$

The second reason for exiting the model is the Spanish emigration effect due to the lack of jobs in the Spanish market. The Spanish emigration number is about 200,000 people per year. We assume that the amount of Spanish people leaving Spain ( $E=150,000$ ) remains constant due to the economic crisis until January 2016.

We assume that this exit amount occurs equally from ES and AB subpopulations, and even when they have the right to vote (remotely) we assume they do not.

In further section, we will study the robustness of the model under the change of values of indicator $f(n)$ for the period of 2013-2016.

The above analysis is mathematically written by the following equations and also illustrated in Figure 1:

$$
\begin{aligned}
\operatorname{ES}(n)= & \mathrm{ES}(n-1)-\alpha_{1}(n) \mathrm{ES}(n-1)-\alpha_{4}(n) \operatorname{ES}(n-1) \\
& -\alpha_{2}(n) \operatorname{ES}(n-1)+\frac{1}{3} B(n-72)-\frac{1}{3} D(n)-\frac{1}{2} E,
\end{aligned}
$$

$$
\begin{align*}
\mathrm{AB}(n)= & \mathrm{AB}(n-1)+\alpha_{1}(n) \mathrm{ES}(n-1)+\alpha_{4}(n) \mathrm{ES}(n-1) \\
& -\alpha_{3} \mathrm{AB}(n-1)+\frac{1}{3} B(n-72)-\frac{1}{3} D(n)-\frac{1}{2} E \\
\mathrm{EX}(n)= & \mathrm{EX}(n-1)+\alpha_{2}(n) \mathrm{ES}(n-1)+\alpha_{3} \mathrm{AB}(n-1) \\
& +\frac{1}{3} B(n-72)-\frac{1}{3} D(n) . \tag{6}
\end{align*}
$$

## 3. Results and Simulations

This section deals with quantification and simulation of the subpopulation sizes throughout the period of study.

As we mentioned above, we are considering the Spanish unemployment rate and the trust on the Government labor indicator. These indicators have relevance in the construction of the dynamic subpopulations' transits coefficients. It is clear that for the short-term future period (2013-2016), we should assume an estimation of these indicators.

Related to the economic one, there are some prestigious institutions forecasting these values. Thus, we built one pessimistic scenario based on [17] (for 2013, 2014) and on [18] for 2015, While the optimistic scenario is based on [19] forecast embracing the whole period. As our study is split into quarters of a year periods, we distribute uniformly by quarters the annual variation of the Spanish unemployment rate due to the annually of the available information (see Table 1).

On the other hand, the trust on the GLI is based on [16] for the periods of (2008 and 2013), but also at our own estimations for (2014 and 2016). We assume that due to the continuous publications of corruption events connected with the two main parties (PSOE and PP), this indicator evolves worsening until the end of the period of study (January 2016). The information about this indicator is provided quarterly (see Table 2).

Following, the values of the transits' coefficients of the model are expressed quarterly (see Table 3).

Computing the subpopulations $\mathrm{ES}(n), \mathrm{AB}(n)$, and $\mathrm{EX}(n)$, starting from the initial subpopulations at January 2012, using the difference system (6) one can estimate the values collected in Table 4.

We assume that the vote intention for January 2012 is the electoral results at November 2011.

As Table 4 shows and Figure 2 illustrates, at the end of the period of study, January 2016, the expected electoral support of ES parties achieves $47.71 \%$ in the pessimistic case, while in the optimistic one the ES electoral support amounts to $49.78 \%$. With respect to the extremist support (EX) it is $16.95 \%$ for the optimistic scenario and $17.62 \%$ for the pessimistic one. Finally with respect to the abstentions (AB) it is $33.27 \%$ in the optimistic scenario while $34.67 \%$ in the pessimistic one.

## 4. Robustness of the Model

As the main goal of the paper is the short-term forecast of the electoral support of the three involved subpopulations, it

TABLE 1: Spanish unemployment rate (\%).

|  | Pessimistic | Optimistic |
| :--- | :---: | :---: |
| 2011 | 21.7 | 21.7 |
| 2012 | 25.0 | 25.0 |
| 2013 | 27 | 27.0 |
| 2014 | 28.1 | 26.0 |
| 2015 | 30.0 | 24.7 |

Table 2: Trust on the Spanish Government labor indicator (GLI), $\beta(n)$, (\%).

|  | $\beta(n)$ Optimist | $\beta(n)$ Pessimist |
| :--- | :---: | :---: |
| Oct-11 | 33.50 | 33.50 |
| Jan-12 | 42.70 | 42.70 |
| Apr-12 | 37.50 | 37.50 |
| Jul-12 | 32.90 | 32.90 |
| Oct-12 | 27.70 | 27.70 |
| Jan-13 | 29.30 | 26.25 |
| Apr-13 | 27.20 | 25.75 |
| Jul-13 | 26.55 | 25.25 |
| Oct-13 | 25.90 | 24.75 |
| Jan-14 | 25.25 | 24.25 |
| Apr-14 | 25.00 | 23.75 |
| Jul-14 | 24.75 | 23.25 |
| Oct-14 | 24.50 | 22.75 |
| Jan-15 | 24.25 | 22.25 |
| Apr-15 | 24.13 | 22.00 |
| Jul-15 | 24.00 | 21.75 |
| Oct-15 | 23.88 | 21.50 |
| Jan-16 | 23.75 | 21.25 |

is necessary to assume the values of the trust on the GLI and the political trust indicator. Thus, it is convenient to analyze the sensitivity of the model results under changes in assumed values of these indicators.
4.1. Robustness of the Model versus the Trust on the Government Indicator. Due to the constant political cases of corruption affecting the two main parties, the trust on the Government indicator is assumed to decrease. However for the time horizon of our study we analyzed the sensitivity of our model under changes in this indicator values among the interval $[-0.5,0.25]$. As a result the subpopulations at year 2016 are estimated for the different values of the trust on the Government labor indicator (see Figure 3).
4.2. Robustness of the Model versus Political Trust Indicator. The coefficient that explains the transit from ES to EX due to the political trust is built by the expression $\alpha_{2}(n)=C(\beta(n)-$ $\beta(n-1)$ ), where $\beta(n)$ is the value of the political trust indicator at quarter $n$ and $C$ is a sociological constant that is adjusted taking into account the electoral data of the last three Spanish general elections.

In order to improve the reliability of the results of the model and as the estimations of this constant $C$ are


Figure 1: Block diagram system.

Table 3: Subpopulations' transits.

| Quarter | Optimistic scenario |  |  |  | $\alpha_{3}(n)$ | $\alpha_{1}(n)$ | $\alpha_{2}(n)$ | $\alpha_{3}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |

approximated, it is convenient to study the sensitivity of the model results under changes in this parameter.

As a result $C$ was estimated in the value interval [1/2C,3/2C], what allow us to compute the subpopulations results at 2016 (see Figure 4).

## 5. Conclusions and Recommendations

From the mathematical point of view, the proposed model could be easily replicated to study the electoral behavior of any other country with the obvious considerations of appropriate electoral data, socio-political-economic indicators, electoral laws, [24-27] and demographic factors.

The model predicts that the support of abstentions and blank voters, that is, the subpopulation AB , remains approximately constant since the last election in November 2011 until the expected time (if general elections are not held in
advance) in spite of the coming reduction of the electoral register of about 800,000 Spanish citizens who emigrate looking for a job (we assume that they do not vote).

On the other hand, the predicted electoral support of the two main parties (PP and PSOE) scarcely will achieve 40\% in the next general elections, while in the previous general elections they got a support of $49 \%$ of the register. This scenario includes an important increase of minor ES parties like UPyD, Ciutadants. This situation is combined with a highly important growth of support of extremist parties, moving from $11 \%$ to $17.5 \%$ that in absolute terms means to be an increase of more than 2 millions of supporters.

The previous results imply that probably it is not going to be possible to get sufficient electoral support to constitute a Government without extremist parties. It looks like the only possible imagined Government coalition would involve at least three parties, including between them one extremist

Table 4: Forecasted subpopulations by quarters.

| Quarter | Optimist | ES | Pessimist | Optimist | AB | Pessimist |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $19,238,081$ | $19,238,081$ | $11,764,115$ | $11,764,115$ | Optimist | Pessimist |
| Jan-12 | $19,191,070$ | $19,191,070$ | $11,418,760$ | $11,418,760$ | $4,328,902$ | $3,975,684$ |
| Apr-2012 | $18,575,355$ | $18,575,355$ | $11,678,532$ | $11,678,532$ | $4,645,003$ | $4,328,902$ |
| Jul-12 | $17,768,050$ | $17,768,050$ | $12,103,273$ | $12,103,273$ | $4,987,024$ | $4,645,003$ |
| Oct-12 | $17,805,227$ | $17,623,338$ | $12,005,133$ | $12,026,023$ | $4,956,742$ | $4,987,024$ |
| Jan 2013 | $17,595,761$ | $17,489,724$ | $11,993,195$ | $12,023,580$ | $5,122,528$ | $5,117,742$ |
| Apr-13 | $17,423,610$ | $17,320,809$ | $12,020,360$ | $12,055,407$ | $5,210,029$ | $5,277,983$ |
| Jul-13 | $17,361,189$ | $17,248,098$ | $11,936,752$ | $11,989,883$ | $5,296,712$ | $5,356,672$ |
| Oct-13 | $17,298,280$ | $17,174,964$ | $11,852,936$ | $11,923,981$ | $5,382,237$ | $5,434,508$ |
| Jan-14 | $17,248,253$ | $17,119,914$ | $11,778,890$ | $11,841,857$ | $5,447,243$ | $5,512,615$ |
| Apr-14 | $17,199,008$ | $17,065,668$ | $11,705,893$ | $11,760,845$ | $5,512,554$ | $5,590,941$ |
| Jul-14 | $17,150,543$ | $17,012,227$ | $11,633,942$ | $11,680,938$ | $5,578,174$ | $5,669,494$ |
| Oct-14 | $17,102,858$ | $16,959,588$ | $11,563,031$ | $11,602,133$ | $5,644,111$ | $5,748,279$ |
| Jan-15 | $17,069,824$ | $16,783,681$ | $11,483,534$ | $11,659,030$ | $5,702,907$ | $5,813,555$ |
| Apr-15 | $17,036,449$ | $16,608,988$ | $11,404,072$ | $11,713,840$ | $5,760,934$ | $5,878,626$ |
| Jul-15 | $17,002,733$ | $16,435,501$ | $11,373,139$ | $11,766,587$ | $5,769,698$ | $5,943,482$ |
| Oct-15 | $16,968,678$ | $16,263,210$ | $11,341,868$ | $11,817,290$ | $5,778,069$ | $6,008,115$ |
| Jan-16 |  |  |  |  |  |  |

Op: Optimistic scenario; Pe: Pessimistic scenario.


Figure 2: Trend of the subpopulations (expressed in millions).
party. In some way this paper advances the end of the hegemonic two-party system in Spain although is well known that the Spanish electoral system favors the two party-system [24, 26].

As results of the model show, the economic crisis favors the shift of a part of ES voters to abstentions (AB) and from $A B$ to EX. Thus, all types of measures addressed to improve the economic scenario of the country will help to stop this trend. In our opinion, which agrees with [5], unlike the


Figure 3: Sensitivity analysis of trust on the GLI (Millions of predicted voters).
current measures of the Government, they are focused on a growth of taxes and also an irrelevant reduction of the public spending.

We mean that the cuts of the public spending should not be only addressed to reduce the size of public servants staff but mainly consultants are directly chosen without any technical or professional filter by local, regional, and national politicians (about 1,500,000 people in 2011, [5]). The current Government is not aware that Spanish citizens are waiting for an important reduction of the public spending related to the job of political administrators instead of a continuous growth of taxes and cuts of the welfare system.


Figure 4: Sensitivity analysis of the political trust indicator (Millions of predicted voters).

Apart from these issues, the lack of compliance of the electoral programs has an influence on the shift of ES voters (supporting the Government) to the abstentions (AB), and also the ES voters may punish the Government by voting for extremist parties.

In order to reduce the progressive political and institutional disappointment of Spanish citizens, it is clear that measures preventing the political corruption, political privileges, and excesses as well as an improvement of the transparency of public activity are necessary and urgent. Just to quote some of them, we have a reform of the constitution, stabilizing the political territorial administration (against the separatist stresses), the elimination of the public financial support to labor unions and firm associations, new formulas of management for expensive public services. What is also urgent is a reform of the current political parties law to improve the transparency of the political system as well as a change of the electoral law that punishes the appearance of new emergent political offers [4, 6, 28].

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## Research Article

# Turing Patterns in a Predator-Prey System with Self-Diffusion 

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For a predator-prey system, cross-diffusion has been confirmed to emerge Turing patterns. However, in the real world, the tendency for prey and predators moving along the direction of lower density of their own species, called self-diffusion, should be considered. For this, we investigate Turing instability for a predator-prey system with nonlinear diffusion terms including the normal diffusion, cross-diffusion, and self-diffusion. A sufficient condition of Turing instability for this system is obtained by analyzing the linear stability of spatial homogeneous equilibrium state of this model. A series of numerical simulations reveal Turing parameter regions of the interaction of diffusion parameters. According to these regions, we further demonstrate dispersion relations and spatial patterns. Our results indicate that self-diffusion plays an important role in the spatial patterns.

## 1. Introduction

In ecological systems, the interactions of different species indicate abundant dynamical features. It is informative to use mathematical model to study the interactions of species in these systems. Among these models, predator-prey systems, which were based on the pioneering works of Volterra [1], have been important in ecological problems. However, since we live in a spatial world, the predator-prey systems should include spatial factors. Thus, these systems should be described by using reaction-diffusion equations. As a result, it is an open problem to understand spatiotemporal behaviors of the temporal-spatial predator-prey systems [211]. Thereinto, the formation of spatial patterns of predatorprey systems is a very active research area [12-15], which is based on the pioneering work of Turing [16] in 1952.

In recent years, there are a lot of bodies of literature to study the predator-prey system by taking into account the normal diffusion as well as cross-diffusion [17-20]. Normal diffusion is a natural phenomenon of the movement of the prey or the predators from higher-density regions to lowerdensity ones. Cross-diffusion of the prey expresses a flux of the prey because of the presence of the predators and vice versa. Furthermore, in predator-prey systems, cross-diffusion can induce Turing instability to produce spatial patterns
even though spatial homogeneous equilibrium states for the corresponding system in the absence of cross-diffusion are stable [8, 11, 21-28].

Besides the normal diffusion and the cross-diffusion of the predator and prey in ecological systems, there exists, in fact, another diffusion form-self-diffusion for the pressure of their own species. It can describe the tendency to move along the direction of lower density of the predator's and prey's own species [29]. Unfortunately, most of the studies mainly focused on well-posedness of solutions for predatorprey systems with self-diffusion [30, 31]. Little attention was paid to examine Turing patterns of these systems. Based on the above discussion, in this paper we mainly concentrate on Turing instability of a predator-prey system that includes a normal diffusion, cross-diffusion, and self-diffusion terms. To this end, we find a sufficient condition to generate Turing patterns. By using numerical simulation, for this system we examine parameter regions of forming patterns and show snapshots of spatial patterns.

The paper is organized as follows. In Section 2, we build the predator-prey model with nonlinear diffusion terms including normal diffusion, cross-diffusion, and selfdiffusion terms and the biological meaning of these parameters are interpreted. In Section 3, we find the sufficient condition of Turing instability. By performing a series of
numerical simulations, we locate the Turing parameter spaces when parts of the parameters are fixed in Section 4. In Section 5, by choosing values of some parameters from the Turing parameter spaces, we illustrate Turing patterns. Finally, some conclusions and discussions are given.

## 2. A Predator-Prey Model with Nonlinear Diffusion

In this paper, we are interested in the spatiotemporal patterns of the following predator-prey system with nonlineardiffusion terms. Mathematical properties for this system have been investigated in $[30,31]$ :

$$
\begin{align*}
& \frac{\partial u_{1}}{\partial t}-\Delta\left[\left(d_{1}+a_{11} u_{1}+a_{12} u_{2}\right) u_{1}\right] \\
& \quad=u_{1}\left(a_{1}-b_{1} u_{1}-c_{1} u_{2}\right), \quad \text { in } \Omega \times[0, \infty), \\
& \frac{\partial u_{2}}{\partial t}-\Delta\left[\left(d_{2}+a_{21} u_{1}+a_{22} u_{2}\right) u_{2}\right] \\
& =u_{2}\left(a_{2}+b_{2} u_{1}-c_{2} u_{2}\right), \quad \text { in } \Omega \times[0, \infty),  \tag{1}\\
& \frac{\partial u_{1}}{\partial n}=\frac{\partial u_{2}}{\partial n}=0, \quad \text { on } \partial \Omega \times[0, \infty), \\
& \quad u_{1}(x, 0)=u_{1}^{0}(x) \geq 0, \\
& u_{2}(x, 0)=u_{2}^{0}(x) \geq 0, \quad \text { in } \Omega
\end{align*}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{2}$ with smooth boundary $\partial \Omega$ and represents the domain that these two species inhabit. The vector $n$ is the outward unit normal vector of $\Omega$. In this model, $u_{1}$ and $u_{2}$ are the prey and the predator densities, respectively; $a_{i}, b_{i}$, and $c_{i}(i=1,2)$ are positive constants. $a_{1}, a_{2}$ are rates of the prey and predator proliferations for food source; $a_{1} / b_{1}$ and $a_{2} / c_{2}$ are environmental carrying capacities for prey and predator, respectively. $c_{1}$ is a consumption rate; $b_{2}$ is a conversion rate. In the diffusion terms, the constant $d_{i}(i=1,2)$, which is usually termed as a normaldiffusion coefficient, represents the natural dispersive force of movement of a species. The positive constants $a_{12}$ and $a_{21}$ are referred to as cross-diffusion coefficients, which describe that the prey tends to avoid higher density of the predators and vice versa by diffusing away. In addition, for the predators and the prey, the positive constants $a_{11}$ and $a_{22}$ are self-diffusion rates due to pressure within their own species.

Next, we want to look for the condition on the parameter values such that a positive homogeneous equilibrium state is linearly stable in the absence of the cross-diffusion and the self-diffusion (i.e., a normal reaction-diffusion system) but unstable in the present of the cross-diffusion and the selfdiffusion.

For simplicity, we set up the following notation as in [22, 24].

Notation 1. Let $0=\mu_{1}<\mu_{2}<\cdots \rightarrow \infty$ be the eigenvalues of $-\Delta$ on $\Omega$ under no-flux boundary condition, and let $E\left(\mu_{i}\right)$
be the space of eigenfunction corresponding to $\mu_{i}$. We define the following space decomposition:
(i) $\mathbf{X}_{i j}:=\left\{\varphi_{i j} \mathbf{c}: \mathbf{c} \in \mathbb{R}^{2}\right\}$, where $\varphi_{i j}$ are orthonormal bases of $E\left(\mu_{i}\right)$ for $j=1, \ldots, \operatorname{dim} E\left(\mu_{i}\right)$;
(ii) $\mathbf{X}:=\left\{\mathbf{u} \in\left[\mathscr{C}^{1}(\bar{\Omega})\right]^{2}: \partial u_{1} / \partial n=\partial u_{2} / \partial n=0\right.$ on $\left.\partial \Omega\right\}$, and thus $\mathbf{X}=\oplus_{i=1}^{\mathbf{\infty}} \mathbf{X}_{i}$, where $\mathbf{X}_{i}=\oplus_{j=1}^{\operatorname{dim} E\left(\mu_{i}\right)} \mathbf{X}_{i j}$.

Notation 2. For the sake of simplicity, we denote reaction terms for systems (1) by

$$
\begin{equation*}
\mathbf{G}(\mathbf{u})=\binom{G_{1}(\mathbf{u})}{G_{2}(\mathbf{u})}=\binom{u_{1} g_{1}(\mathbf{u}):=u_{1}\left(a_{1}-b_{1} u_{1}-c_{1} u_{2}\right)}{u_{2} g_{2}(\mathbf{u}):=u_{2}\left(a_{2}+b_{2} u_{1}-c_{2} u_{2}\right)}, \tag{2}
\end{equation*}
$$

and its Jacobian matrix at the point $\mathbf{u}^{*}$ is

$$
\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)=\left(\begin{array}{ll}
G_{11} & G_{12}  \tag{3}\\
G_{21} & G_{22}
\end{array}\right):=\left(\begin{array}{cc}
-b_{1} u_{1}^{*} & -c_{1} u_{1}^{*} \\
b_{2} u_{2}^{*} & -c_{2} u_{2}^{*}
\end{array}\right) .
$$

The diffusion term is denoted as

$$
\begin{align*}
\Phi(\mathbf{u}) & =\binom{\Phi_{1}(\mathbf{u})}{\Phi_{2}(\mathbf{u})} \\
& :=\binom{\left(d_{1}+a_{11} u_{1}+a_{12} u_{2}\right) u_{1}}{\left(d_{2}+a_{21} u_{1}+a_{22} u_{2}\right) u_{2}} \tag{4}
\end{align*}
$$

and its corresponding Jacobian matrix at the point $\mathbf{u}^{*}$ is

$$
\begin{align*}
\Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right) & =\left(\begin{array}{cc}
\Phi_{11} & \Phi_{12} \\
\Phi_{21} & \Phi_{22}
\end{array}\right) \\
& :=\left(\begin{array}{cc}
d_{1}+2 a_{11} u_{1}^{*}+a_{12} u_{2}^{*} & a_{12} u_{1}^{*} \\
a_{21} u_{2}^{*} & d_{2}+a_{21} u_{1}^{*}+2 a_{22} u_{2}^{*}
\end{array}\right) \tag{5}
\end{align*}
$$

## 3. Linear Stability Analysis of System (1) with a Normal Diffusion

Let $a_{i j}=0$ for $i, j=1,2$; then the system (1) degenerates into standard reaction-diffusion equations:

$$
\begin{gather*}
\frac{\partial u_{1}}{\partial t}-d_{1} \Delta u_{1}=u_{1}\left(a_{1}-b_{1} u_{1}-c_{1} u_{2}\right), \quad \text { in } \Omega \times[0, \infty), \\
\frac{\partial u_{2}}{\partial t}-d_{2} \Delta u_{2}=u_{2}\left(a_{2}+b_{2} u_{1}-c_{2} u_{2}\right), \quad \text { in } \Omega \times[0, \infty), \\
\frac{\partial u_{1}}{\partial n}=\frac{\partial u_{2}}{\partial n}=0, \quad \text { on } \partial \Omega \times[0, \infty), \\
u_{1}(x, 0)=u_{1}^{0}(x) \geq 0 \\
u_{2}(x, 0)=u_{2}^{0}(x) \geq 0, \quad \text { in } \Omega . \tag{6}
\end{gather*}
$$

Through this paper, we assume the following conditions:

$$
\begin{equation*}
[C 1] \frac{c_{2}}{c_{1}}>\frac{a_{2}}{a_{1}}, \quad[C 2] \quad \frac{b_{2}}{b_{1}}>\frac{c_{2}}{c_{1}}-\frac{2 a_{2}}{a_{1}} \tag{7}
\end{equation*}
$$

Then, there exists a unique positive equilibrium state for system (6), denoted by $\mathbf{u}^{*}=\left(u_{1}^{*}, u_{2}^{*}\right)$, where

$$
\begin{equation*}
u_{1}^{*}=\frac{a_{1} c_{2}-a_{2} c_{1}}{b_{1} c_{2}+b_{2} c_{1}}, \quad u_{2}^{*}=\frac{a_{1} b_{2}+a_{2} b_{1}}{b_{1} c_{2}+b_{2} c_{1}} . \tag{8}
\end{equation*}
$$

Theorem 1. If there are no cross-diffusion and self-diffusion, the positive equilibrium state $\mathbf{u}^{*}$ of the system (6) is locally asymptotically stable when condition [C1] holds.

Proof. The linearization of (6) around the steady state $\mathbf{u}^{*}$ can be therefore expressed by

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial t}=\left(D \Delta+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right) \mathbf{u} \tag{9}
\end{equation*}
$$

where $D=\operatorname{diag}\left(d_{1}, d_{2}\right)$. Obviously, the operator $D \Delta+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$ is invariant in the subspace $\mathbf{X}_{i}$, and $\lambda_{i}$ is an eigenvalue of this operator on $\mathbf{X}_{i}$, if and only if it is an eigenvalue of $-\mu_{i} D+$ $\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$. Direct calculation shows the characteristic equation

$$
\begin{equation*}
\lambda_{i}^{2}-m_{i} \lambda_{i}+n_{i}=0 \tag{10}
\end{equation*}
$$

where $m_{i}=-u_{1}^{*} b_{1}-u_{2}^{*} c_{2}-\left(d_{1}+d_{2}\right) \mu_{i}$ and $n_{i}=\left(u_{1}^{*} b_{1}+\right.$ $\left.d_{1} \mu_{i}\right)\left(u_{2}^{*} c_{2}+d_{2} \mu_{i}\right)+u_{1}^{*} u_{2}^{*} b_{2} c_{1}$. It is easy to be verified that $m_{i}$ is negative and $n_{i}$ positive. Thus, for each $i>1$, the two roots of (10) have negative real parts. Consequently, we complete this proof.

Remark 2. According to the proof of Theorem 1, we can also calculate out the eigenvector corresponding to the eigenvalue $\lambda_{i}$ for the operator $-\mu_{i} D+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$, which satisfies

$$
\begin{equation*}
\left(\lambda_{i}+\mu_{i} D-\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right) \mathbf{c}=0 \tag{11}
\end{equation*}
$$

Furthermore, we can yield corresponding eigenspace $\left\{\varphi_{i j} \mathbf{c}\right\}$ in $\mathrm{X}_{i}$.

By Theorem 1, under condition [C1] system (6) cannot destabilize $\mathbf{u}^{*}$. Next, the cross-diffusion and self-diffusion are taken into account.

Theorem 3. Assume that conditions [C1] and [C2] hold; then there exists a sufficiently large positive constant $a_{21}^{*}$ such that the homogeneous state $\mathbf{u}^{*}$ of the system (1) is unstable provided that $a_{21}>a_{21}^{*}$.

Proof. We first linearize the system (1) around $\left(u_{1}, u_{2}\right)=$ $\left(u_{1}^{*}, u_{2}^{*}\right)$ and arrive at

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial t}=\left(\Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right) \Delta+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right) \mathbf{u} \tag{12}
\end{equation*}
$$

By the same method as the proof of Theorem 1, we consider the operator $\Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right) \Delta+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$ on the subspace $\mathbf{X}_{i}$. The eigenvalue of this operator on $\mathbf{X}_{i}$ is denoted as $\lambda$, and then it is also an eigenvalue of the matrix $-\mu_{i} \Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right)+\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$. Thus, $\lambda$ satisfies the following equation:

$$
\begin{align*}
\lambda^{2}-\left[G_{11}\right. & \left.+G_{22}-\mu_{i}\left(\Phi_{11}+\Phi_{22}\right)\right] \lambda  \tag{13}\\
& +\operatorname{det}\left(\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mu_{i} \Phi_{\mathbf{u}}\right)=0
\end{align*}
$$

According to Notation 2, we can calculate and get

$$
\begin{align*}
G_{11}+ & G_{22}-\mu_{i}\left(\Phi_{11}+\Phi_{22}\right) \\
= & -\mu_{i}\left[d_{1}+d_{2}+\left(2 a_{11}+a_{21}\right) u_{1}^{*}+\left(a_{12}+2 a_{12}\right) u_{2}^{*}\right] \\
& -b_{1} u_{1}^{*}-c_{2} u_{2}^{*}<0 \tag{14}
\end{align*}
$$

$$
\begin{equation*}
\operatorname{det}\left(\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mu_{i} \Phi_{\mathbf{u}}\right)=a \mu_{i}^{2}+b \mu_{i}+c \tag{15}
\end{equation*}
$$

where

$$
\begin{align*}
a= & d_{1} d_{2}+d_{1}\left(a_{21} u_{1}^{*}+2 a_{22} u_{2}^{*}\right) \\
& +d_{2}\left(2 a_{11} u_{1}^{*}+a_{12} u_{2}^{*}\right)+2 a_{11} a_{21}\left(u_{1}^{*}\right)^{2} \\
& +4 a_{11} a_{22} u_{1}^{*} u_{2}^{*}+2 a_{12} a_{22}\left(u_{2}^{*}\right)^{2}>0, \\
b= & c_{2} u_{2}^{*}\left(d_{1}+2 a_{11} u_{1}^{*}+a_{12} u_{2}^{*}\right) \\
& +b_{1} u_{1}^{*}\left(d_{2}+a_{21} u_{1}^{*}+2 a_{22} u_{2}^{*}\right) \\
& +a_{12} b_{2} u_{1}^{*} u_{2}^{*}-a_{21} c_{1} u_{1}^{*} u_{2}^{*}, \\
c= & b_{1} c_{2} u_{1}^{*} u_{2}^{*}+b_{2} c_{1} u_{1}^{*} u_{2}^{*}>0 . \tag{16}
\end{align*}
$$

Let $\lambda_{1}\left(\mu_{i}\right)$ and $\lambda_{2}\left(\mu_{i}\right)$ be the two roots of (13); then we have

$$
\begin{gather*}
\lambda_{1}\left(\mu_{i}\right)+\lambda_{2}\left(\mu_{i}\right)=G_{11}+G_{22}-\mu_{i}\left(\Phi_{11}+\Phi_{22}\right), \\
\lambda_{1}\left(\mu_{i}\right) \lambda_{2}\left(\mu_{i}\right)=\operatorname{det}\left(\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mu_{i} \Phi_{\mathbf{u}}\right) \tag{17}
\end{gather*}
$$

In order to ensure that $\operatorname{Re} \lambda_{1}\left(\mu_{i}\right)<0$ and $\operatorname{Re} \lambda_{2}\left(\mu_{i}\right)>0$, a sufficient condition of Turing instability about homogeneous equilibrium $\mathbf{u}^{*}$ is

$$
\begin{equation*}
\operatorname{det}\left(\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mu_{i} \Phi_{\mathbf{u}}\right)<0 \tag{18}
\end{equation*}
$$

Next, we look for the diffusion conditions such that $\operatorname{det}\left(\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mu_{i} \Phi_{\mathbf{u}}\right)<0$ holds. Furthermore, we have

$$
\begin{gather*}
\lim _{a_{21} \rightarrow \infty} \frac{a}{a_{21}}=\left(d_{1}+2 a_{11} u_{1}^{*}\right) u_{1}^{*}, \\
\lim _{a_{21} \rightarrow \infty} \frac{b}{a_{21}}=\left(b_{1} u_{1}^{*}-c_{1} u_{2}^{*}\right) u_{1}^{*}, \\
\lim _{a_{21} \rightarrow \infty} \frac{c}{a_{21}}=0 . \tag{19}
\end{gather*}
$$

Taking condition [C2] in consideration, we can obtain that

$$
\begin{align*}
& \lim _{a_{21} \rightarrow \infty} \frac{\operatorname{det}\left(\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mu_{i} \Phi_{\mathbf{u}}\right)}{a_{21}}=\left(d_{1}+2 a_{11} u_{1}^{*}\right) u_{1}^{*} \mu_{i}^{2}  \tag{20}\\
&+\left(b_{1} u_{1}^{*}-c_{1} u_{2}^{*}\right) u_{1}^{*} \mu_{i}:=P\left(\mu_{i}\right) .
\end{align*}
$$

Then, $P\left(\mu_{i}\right)=0$ has two real roots, one being 0 and the other being positive. A continuity argument shows that there exists a positive constant $a_{21}^{*}$ such that when $a_{21}>a_{21}^{*}$, (15) and $\mu_{i}$ axis intersect on two real and positive points, denoted by $\mu_{i 1}$, $\mu_{i 2}\left(\mu_{i 1}<\mu_{i 2}\right)$. Hence, there exists a $\mu_{i} \in\left(\mu_{i 1}, \mu_{i 2}\right)$ such that $\operatorname{det}\left(\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-\mu_{i} \Phi_{\mathbf{u}}\right)<0$.


Figure 1: Turing parameter spaces for the system (1). Inside the gray zones, $\mathbf{u}^{*}$ is unstable, while, for parameters outside these zones, $\mathbf{u}^{*}$ remains stable. These graphics are obtained by fixing parameters in (21); besides $a_{12}=a_{22}=0.01(\mathrm{a}), a_{11}=a_{22}=0.01(\mathrm{~b})$, and $a_{11}=a_{12}=0.01$ (c).

Remark 4. Theorem 3 is available for a case of the system (1) equipped with cross-diffusion and self-diffusion; that is, $a_{11} \neq 0, a_{12} \neq 0$ and $a_{22} \neq 0$. When $a_{11}=a_{12}=a_{22}=0$ and $a_{21} \neq 0$, the system (1) possesses a diffusion term the same as in [22].

Corollary 5. If $a_{21}=0$, then the homogeneous steady state $\mathbf{u}^{*}$ of the system (1) is always stable.

## 4. Turing Parameter Space

In this section, we will find some parameter regions of nonlinear diffusion coefficients where the equilibrium state
$\mathbf{u}^{*}$ is unstable. For this, according to (15) the sufficient condition of Turing instability is $b<0$ and $b^{2}-4 a c>0$ besides [C1] and [C2]. In this paper, the parameter values satisfying conditions [C1] and [C2] are taken as follows:

$$
\begin{array}{ccc}
a_{1}=1, & a_{2}=2, & b_{1}=0.08 \\
b_{2}=0.05, & c_{1}=0.05, & c_{2}=0.15  \tag{21}\\
& d_{1}=d_{2}=0.1 . &
\end{array}
$$

Then, for these fixed parameters the homogeneous steady state $\mathbf{u}^{*}$ is given by $\left(u_{1}^{*}, u_{2}^{*}\right)=(3.4483,14.4828)$. In Figure 1, we examine the parameter regions where


Figure 2: Dispersion relations for different parameters: in (a), curves of the real part of the eigenvalue are derived by taking $a_{21}=10$, $a_{12}=a_{22}=0.01$, and $a_{11}=0.05,0.1,0.15,0.18,0.201,0.25$ from top to bottom; in (b) $a_{21}=10, a_{11}=a_{22}=0.01$, and $a_{12}=$ $0.15,0.2,0.25,0.3,0.353,0.4$ from top to bottom; and in (c) $a_{21}=5.06, a_{11}=a_{12}=0.01$, and $a_{22}=0.35,0.4,0.45,0.508,0.558,0.608$ from top to bottom. Other parameters are fixed as in (21).
the homogeneous steady state $\mathbf{u}^{*}$ is expected to be unstable. These charts are obtained by fixed parameters in (21) as well as $a_{12}=a_{22}=0.01$ for Figure 1(a), $a_{11}=a_{22}=0.01$ for Figure $1(\mathrm{~b})$, and $a_{11}=a_{12}=0.01$ for Figure 1(c).

From the mathematical viewpoint, the Turing bifurcation occurs when for the characteristic root of (13), $\operatorname{Im}(\lambda(\mu))=0$ and $\operatorname{Re}(\lambda(\mu))=0$ at $\mu=\mu_{c} \neq 0$. Next, we will look for the critical wave of spatial patterns and note the relationship of $\mu$
and the wave number $k$; that is, $\mu=k^{2}$ [26]. Thus, we only need to confirm that

$$
\begin{equation*}
\min _{k} \operatorname{det}\left(\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)-k^{2} \Phi_{\mathbf{u}}\right)=0 \tag{22}
\end{equation*}
$$

Then, the Turing bifurcation thresholds of parameters satisfy the following equation:

$$
\begin{align*}
& 4 \operatorname{det}\left(\mathbf{G}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right) \operatorname{det}\left(\Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right) \\
& \quad=\left(\Phi_{11} \mathbf{G}_{22}+\Phi_{22} \mathbf{G}_{11}-\Phi_{21} \mathbf{G}_{12}-\Phi_{12} \mathbf{G}_{21}\right)^{2} \tag{23}
\end{align*}
$$



Figure 3: Snapshots of contour pictures of time evolution of $u_{1}$ and $u_{2}$. From top to bottom: patterns of $u_{1}, u_{2}$ for $a_{11}=0.05$ and of $u_{1}$, $u_{2}$ for $a_{11}=0.1$. From left to right: $2 \times 10^{3}$ iterations, $4 \times 10^{4}$ iterations, and $1 \times 10^{6}$ iterations. Other diffusion parameters: $a_{12}=a_{22}=0.01$ and $a_{21}=10$.
and the critical wavenumber $k_{c}$ satisfies

$$
\begin{equation*}
k_{c}^{2}=\frac{\left(\Phi_{11} \mathbf{G}_{22}+\Phi_{22} \mathbf{G}_{11}-\Phi_{21} \mathbf{G}_{12}-\Phi_{12} \mathbf{G}_{21}\right)}{\left(2 \operatorname{det}\left(\Phi_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right)\right)} \tag{24}
\end{equation*}
$$

To well see the effect of the nonlinear diffusion, according to Turing parameter regions in Figure 1 we plot the dispersion relations in Figure 2. The critical parameter values in Figure 2(a) correspond to $a_{11}=0.201, a_{12}=0.353$ in Figure 2(b), and $a_{22}=0.558$ in Figure 2(c). In addition, we find that the lowest limit of wavenumber $k$ corresponding to the available

Turing modes $\operatorname{Re}(\lambda)>0$ turns small with $a_{11}$ increasing in Figure 2(a), with $a_{12}$ increasing in Figure 2(b), and with $a_{22}$ increasing in Figure 2(c).

## 5. Pattern Formation

In this section, using numerical methods, we perform numerical simulations of the system (1) in a two-dimensional space and illustrate that cross-diffusion and self-diffusion induce spatial patterns. Throughout this section, we assume that


FIGURE 4: Snapshots of contour pictures of time evolution of $u_{1}$ and $u_{2}$. From top to bottom: patterns of $u_{1}, u_{2}$ for $a_{12}=0.15$ and of $u_{1}$, $u_{2}$ for $a_{12}=0.3$. From left to right: $2 \times 10^{3}$ iterations, $4 \times 10^{4}$ iterations, and $1 \times 10^{6}$ iterations. Other diffusion parameters: $a_{11}=a_{22}=0.01$ and $a_{21}=10$.
the region of the system (1) is $[0, L] \times[0, L] \subset \mathbb{R}^{2}$. Hence, according to the definition of the eigenvalue $\mu_{i}$ in Notation 2, we can obtain $\mu_{i}=((i-1) \pi / L)^{2}$. To numerically solve partial differential equations, we first have to discretize the spacetime of the system (1). The region of $[0, L] \times[0, L]$ is solved in a discrete domain with $M \times N$ lattice sites. The length of the lattices is defined by a constant $h$. The time is also discrete by a constant step $\tau$. All our numerical simulations employ the Neumann boundary condition. Here, we use the standard five-point approximation for the two-dimensional Laplacian derivative and the time evolution is solved by using
the Euler method. More precisely, the value $\left(u_{1, i, j}^{n+1}, u_{2, i, j}^{n+1}\right)$ at the time $(n+1) \tau$ at the mesh position $\left(x_{i}, y_{i}\right)$ is obtained by

$$
\begin{align*}
& u_{1, i, j}^{n+1}=u_{1, i, j}^{n}+\tau \Delta \Phi_{1}+\tau G_{1}\left(u_{1, i, j}^{n}, u_{2, i, j}^{n}\right)  \tag{25}\\
& u_{2, i, j}^{n+1}=u_{2, i, j}^{n}+\tau \Delta \Phi_{2}+\tau G_{2}\left(u_{1, i, j}^{n}, u_{2, i, j}^{n}\right)
\end{align*}
$$

with Laplacian defined by

$$
\Delta \Phi_{1}=\frac{\Phi_{1}\left(u_{1, i+1, j}, u_{2, i+1, j}\right)+\Phi_{1}\left(u_{1, i-1, j}, u_{2, i-1, j}\right)}{h^{2}}
$$



Figure 5: Snapshots of contour pictures of time evolution of $u_{1}$ and $u_{2}$. From top to bottom: patterns of $u_{1}, u_{2}$ for $a_{22}=0.35$ and of $u_{1}$, $u_{2}$ for $a_{22}=0.508$. From left to right: $2 \times 10^{3}$ iterations, $4 \times 10^{4}$ iterations, and $1 \times 10^{6}$ iterations. Other diffusion parameters: $a_{11}=a_{12}=0.01$ and $a_{21}=5.06$.

$$
\begin{align*}
& +\frac{\Phi_{1}\left(u_{1, i, j+1}, u_{2, i, j+1}\right)+\Phi_{1}\left(u_{1, i, j-1}, u_{2, i, j-1}\right)}{h^{2}} \\
& -\frac{4 \Phi_{1}\left(u_{1, i, j}, u_{2, i, j}\right)}{h^{2}} . \tag{26}
\end{align*}
$$

In this paper, we set $h=1, \tau=0.001$, and $M=N=100$. The initial data of the system (1) is taken as a uniformly
distributed random perturbation in order of $1 \times 10^{-4}$ around the homogeneous equilibrium state $\mathbf{u}^{*}$. More precisely,

$$
\begin{equation*}
u_{1}(x, 0)=u_{1}^{*}+\eta_{1}(x), \quad u_{2}(x, 0)=u_{2}^{*}+\eta_{2}(x) \tag{27}
\end{equation*}
$$

where $\eta_{1}, \eta_{2} \in\left[-1 \times 10^{-4}, 1 \times 10^{-4}\right]$. We simulate different patterns according to the dispersion relations in Figure 2.

In Figure 3, we show the evolution of the spatial patterns of the prey and the predators at $2 \times 10^{3}, 4 \times 10^{4}$, and $1 \times 10^{6}$
iterations for $a_{11}=0.05,0.1$ when we set $a_{21}=10, a_{12}=0.01$, and $a_{22}=0.01$. One can see that the patterns arise from random initial conditions. After the cold spot patterns for the prey and the hot spot patterns for the predator arise, they turn steadily with time until these patterns are temporally independent. In addition, for $a_{11}=0.1$ the cold spot patterns for the prey and the hot spot patterns for the predator are looser compared with those for $a_{11}=0.05$.

In Figure 4, we fix $a_{11}=0.01, a_{22}=0.01$, and $a_{21}=$ 10 and obtain the spatial patterns of species $u_{1}, u_{2}$ of time evolution for $a_{12}=0.15$ and $a_{12}=0.3$. For the case of $a_{12}=0.15$, the random initial distribution leads to the formation of irregular patterns. After a long time evolution, we find that the cold spot-strip patterns emerge for the prey $u_{1}$ and that the hot spot-strip patterns for the predator $u_{2}$ arise. However, in the case of $a_{12}=0.3$, the steady patterns of the prey $u_{1}$ consist of hot spots in a bigger size, while the steady patterns of the predator $u_{2}$ are in the formation of bigger cold spots.

In Figure 5, diffusion parameters are set as $a_{11}=a_{12}=0$ and $a_{21}=5.06$. We plot the patterns for $a_{22}=0.35$ and $a_{22}=$ 0.508 , respectively. For both cases, one can see that as time goes on, the cold spot patterns of the prey $u_{1}$ and the hot spot patterns of the predator $u_{2}$ ultimately form.

## 6. Conclusion and Discussion

In this paper, we have studied the prey-predator model with the nonlinear diffusions including normal diffusion, crossdiffusion, and self-diffusion. By applying the mathematical analysis and suitable numerical simulations, we obtain the sufficient conditions of the formation of Turing patterns for this nonlinear diffusion and illustrate Turing parameter regions and Turing patterns when some parameters in system (1) are set.

In our results, we have provided Theorems 1 and 3 to demonstrate that for the nonlinear diffusion including self-diffusion and the cross-diffusion of the predator, the parameter $a_{21}$ plays an important role to induce Turing instability. Furthermore, if $a_{21}=0$, then the homogeneous equilibrium state $\mathbf{u}^{*}$ is always stable; that is, the system (1) has no Turing patterns. By performing numerical simulations, we find the Turing parameter regions of the interaction between cross-diffusion $a_{21}$ and other diffusion terms including cross-diffusion of the other species and self-diffusions. Besides, according to these parameter regions, we show the corresponding dispersion relations and the corresponding patterns. These results indicate that Turing patterns can emerge through the interaction between the cross-diffusion $a_{21}$ and self-diffusion as well as other cross-diffusions in the system (1).

It is well known that for a prey-predator system, the formation of patterns can occur by introducing the crossdiffusion. However, our results further show that under condition (see Theorem 3), self-diffusion can produce Turing patterns.

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## Research Article

# Mathematical Modeling of the Consumption of Low Invasive Plastic Surgery Practices: The Case of Spain 

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#### Abstract

Plastic surgery practice grows continuously among the women in Western countries due to their body image dissatisfaction, aging anxiety, and an ideal body image propagated by the media. The consumption growth is so important that plastic surgery is becoming a normal practice among women, like any other cosmetic product, with the risk of suffering psychopathology disorders in the sense that plastic surgery could be employed as an instrument to recover personal self-esteem or even happiness. Plastic surgery practice depends on economic, demographic, and social contagion factors. In this paper, a mathematical epidemiological model to forecast female plastic surgery consumption in Spain is fully constructed. Overconsumer subpopulation is predicted and simulated. Robustness of the model versus uncertain parameters is studied throughout a sensitivity analysis.


## 1. Introduction

Society pressures women continuously to be eternally young and beautiful. Not only botox or botulinum toxin and fillers (hyaluronic acid), but also others that involve low invasive procedures such as restylane, juvederm, chemical peelings, prevelle, rosacea treatments, and vampire facelift have shown a fast growth during last decades; these concern the antiaging treatments such as face wrinkle and eyelid and are nonsurgical procedures. The consumption of nonsurgical plastic procedures is not only related to psychological drivers, but also to economic factors explained by their low costs in comparison with other procedures like augmentation of breasts or butt. The fast growth of the consumption of these procedures makes these practices become affordable and not luxury, as consequence of the bandwagon [1], and therefore closely related to the economic situation measured by the Spanish unemployment rate. Furthermore, apart from the economy, people change because of emotional waves and social influences (media, mimetic behaviour, and social contagion).

Human beings could be understood as a combination of animal and spirit. When people suffer from low self-esteem levels they try to improve their wellbeing by overpracticing
those activities that make them feel better. These psychological factors are one of the motivations that explain the growth of social addictions like plastic surgery.

The attractiveness of physical appearance and stereotyped female body image pressure makes procedures such as augmentation or reduction of breasts, antiaging surgeries, rejuvenation, and/or look for an ideal body pattern become natural practices as any other consumption good in Western societies [2, 3]; for instance, gym practices combined with diet foods and drinks are usually correlated with the consumption of aesthetic surgical practices. Indeed, it is well known that a good image eases the professional promotion and social recognition [4-7].

Apart from these sociobiological factors (drivers) of plastic surgeries, we should also consider the social contagion effect propagated by marketing [6, 7], human herding, and mimetic behaviour [8-10].

After surgery practice a proportion of women could be dissatisfied again with their body image by comparison with other females from their neighborhood or proceeding from the media. This phenomenon produces a contagion effect of undergoing other surgical operations [8-11].

Thus, plastic surgery has become a normal good of consumption, and it is affected by the fluctuations of the economy
as any other economic sector. It is important to distinguish between invasive and expensive procedures that follow the pattern of luxury goods [12, 13] and are mainly affordable by high-income population while surface treatments like botox fillers are accessible to wide spread population [1416]. Moreover explicitly, during expansive economic periods expensive surgical procedures become more accessible to general population due to an improvement of credit liquidity, while during economic recessions those expensive surgery treatments are almost exclusive for rich people.

However, even when these procedures are becoming regular goods of consumption of western societies when average economic behaviour applies ( $11 \%$ unemployment rate, in Spain), for the particular case of Spain, deeply affected by the economic crisis ( $27 \%$ unemployment rate in 2013), the emergence of these products slows down.

Forecasting the growth of plastic surgery in the female population is interesting from the economic point of view as an emergent business in the western societies but is also relevant from the point of view of public health due to the potential abuse of these practices as artificial tools to enhance personal self-esteem and confidence [17].

In this paper, we construct an epidemiological discrete mathematical model to forecast the population of female consumers of nonsurgical plastic surgery in the coming years in Spain. We deal with populations more than individuals. Individual human behaviour may be erratic, but aggregated behaviour is often quite predictable [18]. The population model approach is also founded on the mimetic human behaviour [8], human herding [10], and social contagion [9].

Epidemiological mathematical models have been employed during last years to describe the propagation of several social disorders such as the consumption of substances like alcohol, cannabis, tobacco, and so forth $[19,20]$ as well as the development of habits like compulsive shopping, workaholism, bigorexia, sex addiction, and eating disorders [21-24].

The population object of study is the women population aged from 16 to 60 years old. This population is split in three subpopulations according to their level of practice of low invasive plastic surgery (LIPS) procedures. To our knowledge there is no questionnaire in the literature measuring the level of practice of LIPS consumption. The first part of the paper is devoted to the construction of such a questionnaire that will allow us to classify the collected data into subpopulations.

Thus, we distinguish between $R$ (rational consumers), $O$ (overconsumers), and $D$ (dependent consumers).

The intersubpopulations transits are modeled according to economic, demographic, and sociologic factors that will allow the construction of a system of difference equations whose solution will permit forecast of the population of consumers in the next years.

## 2. Methods

2.1. A Questionnaire Measuring the Level of Surgery Practice. Typically there are two main ways to produce questionnaires to measure the level of consumption of a good or service,
questionnaires focused on the amount of activity performed [25] and others focused on the psychological dependence of the consumer [22,26]. Hybrid questionnaires combine both parameters to produce an appropriated questionnaire [27].

In this section, we design a questionnaire for measuring the level of consumption of plastic surgery procedures. Although our questionnaire does not distinguish surgical and nonsurgical treatments, we assume that fearless women to practicing surgical treatments are also potential consumers of non-surgical practices.

In the questionnaire we include several demographic questions apart from those related to the aesthetic surgery practice, which will play an important role in the sociological study of the subpopulations.

The contents of the questionnaire are listed below.
(1) Your marital status
(a) Single.
(b) Married/Relationship.
(c) Divorced/Widowed.
(2) Age
(a) $16-20$
(b) 20-30
(c) $30-40$
(d) 40-50
(e) 50-60
(f) 60-70
(g) $70+$
(3) Do you have a paid activity?
(a) Yes.
(b) No.
(4) Level of income
(a) Low.
(b) Medium.
(c) High.
(5) Level of education
(a) Primary school.
(b) High school.
(c) University degree.
(6) Has any relative or close friend undergone any aesthetic surgery procedures/treatment?
(a) Yes, how many people? --
(b) No.
(7) Have you had any experiences with aesthetic surgery during last year?
(a) Yes, how many times? --
(b) No.
(8) If you answered YES to the previous question, would you undergo any other aesthetic surgery, if you had economic resources?
(a) Yes.
(b) No.
2.2. Data Collection and Sampling. The population of the study is composed of the Spanish women who undergo plastic surgery aged among the interval $[16,60]$.

We passed the questionnaire once, (March and April 2012) at different locations such as a private gym a private franchised gym multilocated, and a public beach of Valencia. Participants were asked to complete a questionnaire composed of 8 questions. Each question had two and/or three possible answers; questions from 1 to 5 were related to sociodemographic data while questions from 6 to 8 were related to the practice. The classification of the individuals is based on the answer of question number 7 .
(i) $R$ : defined as rational women when their score at question 7 was 0 or 1 at year $n$.
(ii) $O$ : overconsumer women whose answer at question 7 was equal to 2 or 3 at year $n$.
(iii) $D$ : women who scored higher than 3 at question 7 are considered dependent consumers at year $n$.

Question number 6 is related to the social contagion, which will be relevant in the model construction. Question number 8 focuses on the willingness to practice plastic surgery.

Table 1 shows data collected in our sample classifying the Spanish women by subpopulations after applying stratified sampling using data from the Spanish Statistical Institute [28].
2.3. Constructing the Mathematical Model. The purpose of this section is to develop a discrete mathematical model, which allows us to forecast the Spanish women's low invasive plastic surgery procedures for the interval 2012-2018.

As it was commented in the previous section, the population object of study was divided into three categories according to their answer to question number 7 that measures their amount of consumption.

Our attention is focused on forecasting the number of rational plastic surgery consumers $(R)$, over-consumers of plastic surgery $(O)$, and those dependent $(D)$ to low invasive plastic surgery procedures.

The low invasive plastic surgery behaviour is dynamic which means that it changes over time and embraces interactions among subpopulations. For this, a mathematical model is going to be constructed to describe this dynamic system as a system of difference equations whose unknowns are the following subpopulations.
$R_{n}$ : number of rational consumers who undergo plastic surgery procedures, aged among the interval $[16,60]$ at year $n$.

Table 1: Data collection and Spanish population distribution for 2012.

|  | 2012 |  |
| :--- | :---: | :---: |
|  | Sample | Spanish women by population |
| $R$ | 213 | $13,730,450$ |
| $O$ | 11 | 709,084 |
| $D$ | 2 | 128,924 |
| Total | 226 | $14,568,458$ |

$O_{n}$ : number of over-consumers of plastic surgery practice aged among the interval $[16,60]$ at year $n$.
$D_{n}$ : number of dependent consumers of plastic surgery practices aged among the interval $[16,60]$ at year $n$.

The total population $(P)$ in the time $n$ is given by

$$
\begin{equation*}
P(n)=R(n)+O(n)+D(n) . \tag{1}
\end{equation*}
$$

As at any other western developed country the Spanish birth and mortality rates are quietly balanced, which makes the total population not increase in a significant degree [28]. However, for the particular case of Spain, due to the ferocious economic crisis (actually, 27\% unemployment rate; $57 \%$ youth unemployment rate) we have exceptional emigration events. Firstly, jobless immigrants are returning to their countries of origin but also unemployed Spanish citizens are emigrating looking for a job abroad.

In particular, the impact of the crisis over the Spanish economy, 200,000 foreign immigrants are coming back to their countries yearly. In addition, about 100,000 Spanish citizens yearly leave Spain seeking for a job overseas [29]. Both facts involve a net decrease of the Spanish population of about 350,000 people per year and approximately 175,000 women. Disregarding people under 16 years old, we estimate that about 300,000 people older than 16 years old leave Spain every year since 2012 until 2018. Thus, the amount of approximately $E=150,000$ women leaves Spain yearly during the considered period of study.

The dynamic behaviour of the low invasive plastic surgery practices is based on subpopulations' transits explained by coefficients that need to be modeled according to economic, sociodemographic, and contagion hypotheses, and also collected data analysis.

The period of study started in 2012, when the first sample of the study was obtained, and finishes in 2018. People behaviour changes along the time, not only because of biology but also because of external influences (media, contagion, mimetic behaviour, human herding, and economy) and psychological drivers (low self-esteem, poor wellbeing and self-confidence). This motivates the consideration of subpopulation transits that are computed throughout sociological analysis and assuming behavioural hypotheses.

Then, the dynamics between subpopulations $R, O$, and $D$ are estimated for the interval $[n, n+1]$.
(a) The influence of an economic change could cause the transit of women between categories $R, O$, and $D$ following the economic trend.
(b) The contagion effect based on low levels of selfesteem combined with the mimetic behaviour, social contagion, and bandwagon [ $1,8-10$ ] leading to the transit from $R$ to $O$ and also from $O$ to $D$ due to the contact between women who underwent LIPS procedures.
(c) We estimated from the annual survey on aesthetic procedures provided by [30] the relationship between the economic behaviour throughout the Spanish unemployment rate and women who underwent plastic surgery.

The dynamics of the model are illustrated in Figure 1 and described by the following equations:

$$
\begin{gather*}
R(n+1)=\left(1+\alpha_{b}-\frac{d_{i}+\alpha_{f}}{3}\right) R(n)-\alpha_{r} R(n) \\
\\
\quad-\alpha_{e}(n) R(n)-\gamma_{1} R(n)-E, \\
O(n+1)=\left(1-\frac{d_{i}+\alpha_{f}}{3}\right) O(n)+\alpha_{e}(n) R(n)+\alpha_{r} R(n) \\
 \tag{2}\\
\\
-\beta_{e}(n) O(n)+\gamma_{1}(R(n)-O(n)), \\
D(n+1)= \\
\left(1-\frac{d_{i}+\alpha_{f}}{3}\right) D(n)+\beta_{e}(n) O(n)+\gamma_{1} O(n),
\end{gather*}
$$

or in the following simplified form:

$$
\begin{gather*}
R(n+1)=\left(1+\alpha_{b}-\frac{d_{i}+\alpha_{f}}{3}\right) R(n) \\
-\left(\alpha_{r}-\alpha_{e}(n)-\gamma_{1}\right) R(n)-E, \\
O(n+1)=\left(1-\frac{d_{i}+\alpha_{f}}{3}\right) O(n)+\left(\alpha_{e}(n)+\alpha_{r}\right) R(n) \\
\\
\quad-\beta_{e}(n) O(n)+\gamma_{1}(R(n)-O(n)),  \tag{3}\\
D(n+1)=\left(1-\frac{d_{i}+\alpha_{f}}{3}\right) D(n)+\left(\beta_{e}(n)+\gamma_{1}\right) O(n),
\end{gather*}
$$

The values of all parameters were estimated from different sources of information and hypotheses with the exception of the transit $\alpha_{r}$ and $\gamma_{1}$ obtained from [28] and the survey designed and implemented in this study.

In the following, we describe the construction of the parameters of the model.
(i) $\alpha_{b}(n-16)$ is the birth rate of the Spanish population at year $(n-16)$. This parameter is estimated according to the data coming from [28]. We assume that all women becoming 16 years old in Spain enter into the system as rational consumers.
(ii) Then, the exit from the system could happen due to two reasons: the biological deaths measured throughout the rate of death $\left(d_{i}\right)$ of women up to 60 years old as well as those of women aging 61 years old. $\alpha_{f}$ are women who become out of our age period interval. Both coefficients are assumed to


Figure 1: Block diagram of the system's dynamics.
remain constant for the period of study. Due to the lack of reliability of the available information, we assume that the system death probability is the same for all three categories.
(iii) $\alpha_{r}$ is the recovery rate of self-esteem. The coefficient is split into two components: $\alpha_{r 1}$ and $\alpha_{r 2}$.
(a) $\alpha_{r 1}$ : we estimated that from the rate of rational Spanish consumers of botox ( $C=0.49 \%$ ) [31], $8 \%$ of the rational Spanish women aged in the interval [30, 45] were treated with botox looking for a better physical image [32], and $1 \%$ did it in order to overcome some emotional drawback (such as a relative passing away, a divorce, or a loss of job). We weighted $\alpha_{r 1}$ as $2 / 3$ of $\alpha_{r}$. In general, women between the age interval [30, 45] are more interested in taking care of their physical appearance, which justifies the weight $2 / 3$ :

$$
\begin{align*}
\alpha_{r 1} & =0.42 \times 0.004924787 \times(0.08+0.01) \times \frac{2}{3}  \tag{4}\\
& =0.000124105 .
\end{align*}
$$

(b) $\alpha_{r 2}$ : again, we estimated that from the proportion of Spanish women that underwent botox treatments ( $0.49 \%$ ), those who were among the age interval [46, 60] were treated looking for a better body image and for improving their wellbeing after suffering from emotional slowdowns; about one-third of the $35 \%$ of the divorced Spanish women underwent these plastic surgery procedures:

$$
\begin{align*}
& \alpha_{r 2}=0.32 \times 0.004924787 \times(0.35) \times \frac{1}{3} \\
&=0.000183859  \tag{5}\\
& \alpha_{r}=\alpha_{r 1}+\alpha_{r 2} .
\end{align*}
$$

Then,

$$
\begin{align*}
\alpha_{r} & =\alpha_{r 1}+\alpha_{r 2} \\
& =0.000124105+0.000183859  \tag{6}\\
& =0.000307963 .
\end{align*}
$$

The distribution of weights, two-thirds versus one-third, between both coefficients $\alpha_{r 1}, \alpha_{r 2}$ is based on the hypothesis that in western countries like Spain, women aged in the interval $[30,45]$ used to be more concerned about their physical appearance.
(iv) $\alpha_{e}(n)$ : the impact of the economy measured in terms of the annual Spanish unemployment rate $\delta(n)$, allowing the transit between subpopulations bidirectionally according to the economic trend as it occurs with most of consumption goods.

Observing the behaviour of both the Spanish economy the Spanish LIPS data consumption [30] and the approximated distribution of the consumers among the three categories $(R, O$, and $D)$, during the full economic crisis period 2009-2011, we suggest an approximated correlation of both variables quantified as follows.

If the unemployment rate increases by $1 \%$, the consumption of LIPS decreases by about $3 \%$. In addition, we should consider which are the transits in periods of economic recovery. However, after a long period of economic crisis in Spain, the recovery of consumption habits slows down in accordance with the economic improvement mostly for sectors such as products and services related to the body care.

We assume that if the unemployment rate $(\delta(n))$ improves by $1 \%$, the LIPS consumption recovery is $0.5 \%$ until the economic scenario achieves a stable admissible ( $11 \%$ unemployment rate) situation. This transit only affects the categories $R$ and $O$.

The coefficient is modeled as follows:

$$
\alpha_{e}(n)\left\{\begin{array}{cl}
3(\delta(n-1)-\delta(n)) C, & \delta(n)>\delta(n-1)  \tag{7}\\
-1.5 C, & \delta(n)=\delta(n-1) \\
0.5(\delta(n-1)-\delta(n)) C, & \delta(n)<\delta(n-1)
\end{array}\right\}
$$

The consumption of this kind of product is differently affected by the category of consumer. The dependent consumer behavior is influenced by psychological factors. Indeed, the $D$ consumer reacts differently than an $R$ or $O$ consumer to any change in the economic environment. Thus, we assume that the transit coefficient $\beta_{e}(n)$ is as follows:

$$
\beta_{e}(n)\left\{\begin{array}{cl}
4(\delta(n-1)-\delta(n)) C, & \delta(n)>\delta(n-1)  \tag{8}\\
-2 C, & \delta(n)=\delta(n-1) \\
2(\delta(n-1)-\delta(n)) C, & \delta(n)<\delta(n-1)
\end{array}\right\}
$$

In the present economic context of crisis, when Spanish unemployment rate is higher than $20 \%$, the propagation of the consumption of LIPS only affects the segment of women with secured incomes or wealthy people. According to our survey, women interact on a daily bases with 30 other females, from those, 1 belongs to the category $O$ or $D(3.3 \%)[9,10,33]$. This fact suggests a contagion rate of $\gamma_{1}=0.033 C$. We assume that this parameter is constant all the time, but also this is a continuous jump contagion; people only jump from one category to the next level of consumption $(R \rightarrow O ; O \rightarrow D)$.
(v) $E=150,000$ is the constant approximated value of women that leave Spain looking for a job abroad due to the economic crisis. We assume that all of them are rational potential consumers of plastic surgery procedures.

## 3. Results and Simulations

The mathematical model allows us to predict the subpopulations $R(n), O(n)$, and $D(n)$ at any year $n$, in the study interval [2012, 2018]. In order to compute the subpopulations it was

Table 2: Economic forecast of the annual Spanish unemployment rate expressed in percentage.

|  | Pessimistic | Optimistic |
| :--- | :---: | :---: |
| 2011 | $\mathbf{2 1 . 6}$ | $\mathbf{2 1 . 7}$ |
| 2012 | $\mathbf{2 5 . 0}$ | $\mathbf{2 5 . 1}$ |
| 2013 | 26.9 | $\mathbf{2 7 . 0}$ |
| 2014 | $\mathbf{2 8 . 1}$ | $\mathbf{2 6 . 0}$ |
| 2015 | $\mathbf{3 0 . 0}$ | $\mathbf{2 4 . 7}$ |
| 2016 | 29.0 | $\mathbf{2 3 . 2}$ |
| 2017 | 28.5 | $\mathbf{2 1 . 7}$ |
| 2018 | 28.0 | $\mathbf{2 0 . 1}$ |

Bold: forecasts provided by institutions (OECD, Societé Générale, IMF). Italic: own forecasts.
necessary to estimate the coefficients of the system for the next years according to two economic scenarios, one more optimistic and another one more pessimistic, reflecting the Spanish economic trend.

Thus, we based our pessimistic scenario on the economic forecast of the Spanish unemployment rate according to the Organization of Economic Cooperation and Development [34] for the years 2011 and 2014 and for the optimistic scenario the International Monetary Fund for the period [2011, 2018] [35]. For 2015 we applied the rate of unemployment predicted by Societé Générale [36] for the pessimistic scenario, while we forecasted by ourselves the economic trend for the period [2016, 2018] (see Table 2).

Since organizations only forecast the Spanish unemployment rate until 2015 (Societé Générale, 30\%), we continue the predictions of the Spanish economic scenario following the $O E C D /$ Societé Générale forecast with a more pessimistic scenario while the IMF one agrees with a more optimistic approach.

Once the economic scenarios were built, we estimated the annual value of each subpopulation $R(n), O(n)$, and $D(n)$. Table 3 collects the results performed by the computation of the system expressed in absolute terms (number of women at each subpopulation at year $n$ ).

Even when the consumption of these products shows a growing trend due to the hard Spanish economic crisis, the level of activity of these services slows down between the middle class. Figure 2 illustrates this fact.

Then, we performed a sensitivity analysis of both models (pessimistic and optimistic) versus the contagious parameter $C$, involved in the coefficient $\gamma_{1}=0.033 C$. Results showed the same effect for both economic scenarios. In fact, regarding a variation interval for $C$ ranging in the interval [0.0049, 0.1], where the extreme value 0.0049 corresponds to the current consumption rate of LIPS procedures by Spanish women whereas the highest value 0.1 would correspond to a situation where middle class sums up to the imitation effect and the consumption of this type of goods. Figure 3 illustrates how the subpopulations vary continuously with the contagion parameter measured in terms of $C$ with robustness.

Table 3: Subpopulation forecast in number of women who practice LIPS by economic scenarios.

|  | Pessimistic |  |  |  | $D$ | $R$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $R$ | $O$ | $D$ | $R$ | $O$ | $D$ |
| 2011 | $13,778,952$ | 711,589 | 129,379 | $13,778,952$ | 711,589 | 129,379 |
| 2012 | $13,669,723$ | 707,090 | 128,118 | $13,669,743$ | 707,071 | 128,211 |
| 2013 | $13,558,991$ | 705,473 | 127,079 | $13,559,067$ | 705,403 | 127,261 |
| 2014 | $13,444,811$ | 705,099 | 126,139 | $13,442,149$ | 707,531 | 126,651 |
| 2015 | $13,336,065$ | 703,400 | 125,109 | $13,329,129$ | 709,665 | 126,067 |
| 2016 | $13,227,621$ | 705,431 | 124,418 | $13,220,562$ | 711,780 | 125,500 |
| 2017 | $13,120,360$ | 707,266 | 123,697 | $13,112,952$ | 713,827 | 124,938 |
| 2018 | $13,014,809$ | 709,038 | 122,980 | $13,007,344$ | 715,578 | 124,317 |



Figure 2: Simulated subpopulations $R, O$, and $D$ during the period 2011-2018 by economic scenario.

## 4. Conclusions

In this paper, a complete mathematical model for predicting the low invasive consumption of plastic surgery practices in Spain during the period 2012-2018 has been constructed.

To our knowledge, this is the first time the consumption of these services has been modeled from a behavioral perspective, although some statistics about the level of consumption may be found in the literature. However, previous studies focused on the willingness to consume LIPS rather than forecasting real consumption of LIPS. As consequences, our results are conservative, since we predict LIPS activity; we are taking into account economic environment, individual contagion, and emotional drives.

The LIPS consumption improves the possible professional success and the wellbeing of the person, improving her body
image and social skills. However, it is important to point out that an overconsumption of these treatments can lead the person to develop disorders of dependence and identification of happiness by care of the body.

Thus, from the point of view of business and marketing it is interesting to study the propagation of these services while from the public health point of view it is interesting to study the evolution of the over-consumers and addicts.

Our discrete mathematical model is of epidemiological nature, where consumers are classified by subpopulations in accordance with a survey designed by the authors in this paper.

The study of the consumption of this product in Spain is particularly interesting due to the emergence of its consumption by middle classes in western countries as any other service employed to body care during the last decade.


Figure 3: Sensitivity analysis of subpopulations (expressed in millions) versus the contagion parameter for the optimistic scenario. (Abscises represents values of the parameter C.)

However, the impact of the ferocious Spanish economic crisis slowed down its trend suddenly until the point where these services are just affordable by high-income women.

The model allows the possibility of simulating artificial scenarios simply by changing the values of the parameters and computing the subpopulations with the model. Furthermore, the model is applicable to any other region where economic and sociologic data are available.

In addition, the model has the potential advantage that the period of study can be modified assuming a loss of accuracy due to the uncertainty of the economic future.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Dealing with Dependent Uncertainty in Modelling: A Comparative Study Case through the Airy Equation 

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#### Abstract

The consideration of uncertainty in differential equations leads to the emergent area of random differential equations. Under this approach, inputs become random variables and/or stochastic processes. Often one assumes that inputs are independent, a hypothesis that simplifies the mathematical treatment although it could not be met in applications. In this paper, we analyse, through the Airy equation, the influence of statistical dependence of inputs on the output, computing its expectation and standard deviation by Fröbenius and Polynomial Chaos methods. The results are compared with Monte Carlo sampling. The analysis is conducted by the Airy equation since, as in the deterministic scenario its solutions are highly oscillatory, it is expected that differences will be better highlighted. To illustrate our study, and motivated by the ubiquity of Gaussian random variables in numerous practical problems, we assume that inputs follow a multivariate Gaussian distribution throughout the paper. The application of Fröbenius method to solve Airy equation is based on an extension of the method to the case where inputs are dependent. The numerical results show that the existence of statistical dependence among the inputs and its magnitude entails changes on the variability of the output.


## 1. Introduction and Motivation

Deterministic differential equations (ddes) have demonstrated to be powerful tools for modelling numerous problems appearing in different areas including physics, chemistry, economy, engineering, and epidemiology. Their practical application requires knowing their inputs (coefficients, forcing terms, initial/boundary conditions, etc.). This task can only be done after accurate measurements that usually contain uncertainty due to measuring errors or the inherent complexity or ignorance of the phenomena under study. This approach leads us to consider the inputs of such models as random variables (rvs) or stochastic processes (sps) rather than deterministic constants or functions, respectively. Differential equations containing in their formulation randomness are usually referred to as random differential equations (rdes) or stochastic differential equations (sdes) depending on the kind of uncertainty therein. When randomness is just considered through the white noise process (i.e., the generalized derivative of the Wiener process), they are usually
called sdes Then, Itô calculus is required in order to conduct the study. Otherwise, the term rde is used (see, [1, page 66], [2]).

Rdes constitute a natural extension of ddes. Generalized Polynomial Chaos (usually denoted by gPC) and Monte Carlo sampling (MCs) are probably the most popular techniques to deal with rdes (see for instance [3, 4], resp.). In addition to these approaches, the extension of some deterministic techniques to the random scenario, based on the so-called $p$-stochastic calculus, also constitutes useful tools to solve rdes (see $[5,6]$ and the references therein). In particular, a random power series Fröbenius method has been recently proposed by some of the authors to study some significant rdes by assuming that random inputs are independent $[7,8]$. Although independence assumption simplifies the mathematical treatment of the models, it should not be assumed in many practical situations. Apart from few contributions such as $[9,10]$ where the authors study the dependent scenario by taking advantage of gPC approach, most methods developed to study rdes rely on independence of random inputs. In
particular, to the best of our knowledge, applications of the random Fröbenius method considering dependent rvs have not been studied yet. As a consequence, the study of rdes with dependent inputs is currently an active research area, mainly stimulated by the necessity of providing more realistic approaches and more accurate answers in mathematical modelling.

In this paper, we present a comparative study about the capability of previous approaches to deal with rdes containing dependent random inputs. To conduct the study, we will consider the Airy random differential equation:

$$
\begin{align*}
& \ddot{X}(t)+t A X(t)=0, \quad t>0,  \tag{1}\\
& X(0)=Y_{0}, \quad \dot{X}(0)=Y_{1},
\end{align*}
$$

where $A, Y_{0}, Y_{1}$ are assumed to be Gaussian dependent rvs on a probability space $(\Omega, \mathfrak{F}, P)$. We point out that the Airy equation has been selected since, as it is well-known, in the deterministic scenario its solutions are highly oscillatory [11]; therefore, it is expected that differences among gPC, Fröbenius method, and MCs will be better highlighted.

Specifically, we will compare, by means of several illustrative examples, the quality of the numerical approximations provided by the three approaches to compute the average and standard deviation of the solution sp to the initial value problem (ivp) (1). These examples will allow us to elucidate, through the random Airy differential equation, whether the statistical independence between the random inputs (initial conditions and coefficients), usually assumed in many applications, has a significant influence on the output.

The paper is organized as follows. Section 2 is divided in two parts. The first one is devoted to construct a mean square convergent random power series solution to the ivp (1) using the random Fröbenius method including approximations of the average and the variance of the solution sp In the second part, we summarize the main features of the gPC method to study rdes, and we apply gPC to the particular case where inputs are Gaussian dependent rvs. In Section 3 we will present several illustrative examples. The aim of these examples is twofold. First, to highlight the similarities and differences between the three approaches in dealing with both dependent/independent (Gaussian) rvs; second, to reveal, through the Airy rde, the importance of setting appropriately the statistical dependence of random inputs in dealing with mathematical models. Conclusions are drawn in the last section.

## 2. Development

In this section, we present the main results required to construct using Fröbenius and gPC methods the approximations of the mean and standard deviation of the solution sp to the ivp (1) when inputs $A, Y_{0}, Y_{1}$ are assumed to be dependent rvs. We point out that the description of Fröbenius method in this scenario is presented briefly deliberately since it follows in broad outline that of independent case, which has been developed by some of the authors previously [7]. Foundations
and further details about gPC method can be found in [3], for instance.
2.1. Tackling Random Dependence Using Fröbenius Method. Random Fröbenius method consists of constructing a power series solution to the ivp (1), say, $X(t)=\sum_{n \geq 0} X_{n} t^{n}$ which is mean square convergent on a certain $t$-domain. The rigorous construction of such a mean square convergent random infinite series requires $L^{p}=L^{p}(\Omega, \mathfrak{F}, P)$-calculus with $p=$ 2,4 , where convergence is defined in the $p$-norm: $\|X\|_{p}=$ $\left(\mathrm{E}\left[|X|^{p}\right]\right)^{1 / p}[5,7]$. Convergence with $p=2 / p=4$ norm is usually referred to as mean square (ms)/fourth (mf) convergence. By the Schwarz inequality, it is straightforward to proof that mean fourth convergence implies mean square convergence. Based on the ideas developed in [7], now we assume that the following joint absolute moments with respect to the origin increase at most exponentially; that is, there is a nonnegative integer $n_{0}$ and positive constants $H_{i}$ and $M_{i}, i \in\{0,1\}$ such that

$$
\begin{equation*}
E\left[|A|^{n}\left|Y_{i}\right|^{4}\right] \leq H_{i}\left(M_{i}\right)^{n}<\infty, \quad i=0,1, \quad \forall n \geq n_{0} \tag{2}
\end{equation*}
$$

In the case of ivp (1), the random power series solution is given by [7]

$$
\begin{align*}
X(t)= & Y_{0}+\sum_{n \geq 1} \frac{(-1)^{n}(3 n-2)!!!A^{n} Y_{0}}{(3 n)!} t^{3 n} \\
& +Y_{1} t+\sum_{n \geq 1} \frac{(-1)^{n}(3 n-1)!!!A^{n} Y_{1}}{(3 n+1)!} t^{3 n+1} \tag{3}
\end{align*}
$$

Under hypothesis (2) mf convergence (and hence ms convergence) of the first series appearing in (3) follows straightforwardly:

$$
\begin{align*}
\left\|\frac{(-1)^{n}(3 n-2)!!!A^{n} Y_{0}}{(3 n)!} t^{3 n}\right\|_{4}= & \left\|A^{n} Y_{0}\right\|_{4} \frac{(3 n-2)!!!}{(3 n)!}|t|^{3 n} \\
= & \left(E\left[|A|^{4 n}\left|Y_{0}\right|^{4}\right]\right)^{1 / 4} \\
& \times \frac{(3 n-2)!!!}{(3 n)!}|t|^{3 n}  \tag{4}\\
\leq & L_{0}\left(M_{0}\right)^{n} \frac{(3 n-2)!!!}{(3 n)!}|t|^{3 n}
\end{align*}
$$

where $L_{0}=\sqrt[4]{H_{0}}>0$. As a consequence, for $n \geq n_{0}$ we have obtained as a majorant series

$$
\begin{equation*}
\sum_{n \geq n_{0}} \alpha_{n}, \quad \text { where } \alpha_{n}=L_{0}\left(M_{0}\right)^{n} \frac{(3 n-2)!!!}{(3 n)!}|t|^{3 n} \tag{5}
\end{equation*}
$$

that is convergent for all $t \in \mathbb{R}$ as it can be directly checked by D'Alembert test:

$$
\begin{equation*}
\frac{\alpha_{n+1}}{\alpha_{n}}=M_{0} \frac{(3 n+1)}{(3 n+3)(3 n+2)(3 n+1)}|t|^{3} \underset{n \rightarrow \infty}{\longrightarrow} 0, \quad t \in \mathbb{R} . \tag{6}
\end{equation*}
$$

The mf convergence for the second series in (3) follows analogously.

Remark 1. By assuming that there are positive constants $H$ and $M>0$ such that $E\left[|A|^{n}\right] \leq H M^{n}<\infty$ for every $n \geq n_{0}$ and $Y_{i} \in L^{8}$ for $i \in\{0,1\}$, m.s. convergence of series (3) can also be established analogously as we have shown previously. In fact, this follows immediately from the Schwarz inequality

$$
\begin{equation*}
E\left[|A|^{n}\left|Y_{i}\right|^{4}\right] \leq\left(E\left[|A|^{2 n}\right]\right)^{1 / 2}\left(E\left[\left|Y_{i}\right|^{8}\right]\right)^{1 / 2} \leq H_{i} M^{n} \tag{7}
\end{equation*}
$$

where $H_{i}=\sqrt{H K_{i}}$, being $K_{i}=E\left[\left|Y_{i}\right|^{8}\right]<\infty, i \in\{0,1\}$. However, notice that this condition is stronger than (2).

Taking advantage of m.s. convergence of series appearing in right-hand side of (3) together with the following property ([2, page 88$]$ ):

$$
\begin{align*}
\left\{X_{n}: n \geq 0\right\} \in L^{2}: X_{n} & \xrightarrow[n \rightarrow \infty]{\mathrm{m} . \mathrm{s} .}
\end{align*} X, \begin{array}{ll}
E\left[X_{n}\right] & \xrightarrow[n \rightarrow \infty]{\longrightarrow} E[X], \\
\operatorname{Var}\left[X_{n}\right] & \underset{n \rightarrow \infty}{\longrightarrow} \operatorname{Var}[X], \tag{8}
\end{array}
$$

we can obtain approximations for the average, $E[X(t)]$, and variance, $\operatorname{Var}[X(t)]$, (or equivalently standard deviation) by truncating the random power series of $X(t)$ given by (3). For the approximation of the average one gets

$$
\begin{align*}
E\left[X_{N}(t)\right]= & E\left[Y_{0}\right]+\sum_{n=1}^{N}(-1)^{n} E\left[A^{n} Y_{0}\right] \frac{(3 n-2)!!!}{(3 n)!} t^{3 n} \\
& +E\left[Y_{1}\right] t+\sum_{n=1}^{N}(-1)^{n} E\left[A^{n} Y_{1}\right] \frac{(3 n-1)!!!}{(3 n+1)!} t^{3 n+1} \tag{9}
\end{align*}
$$

In order to obtain approximations of the standard deviation of $X(t)$, we will take into account the well-known representation of the variance in terms of the two first moments, $\operatorname{Var}[X(t)]=E\left[(X(t))^{2}\right]-(E[X(t)])^{2}$. Therefore, it is enough to approximate the second moment:

$$
\begin{aligned}
E\left[\left(X_{N}(t)\right)^{2}\right]= & E\left[\left(Y_{0}\right)^{2}\right] \\
& +2 \sum_{n=1}^{N}(-1)^{n} E\left[A^{n}\left(Y_{0}\right)^{2}\right] \frac{(3 n-2)!!!}{(3 n)!} t^{3 n} \\
& +\sum_{n=1}^{N} E\left[A^{2 n}\left(Y_{0}\right)^{2}\right]\left(\frac{(3 n-2)!!!}{(3 n)!}\right)^{2} t^{6 n} \\
& +2 \sum_{n=2}^{N} \sum_{m=1}^{n-1}(-1)^{n+m} E\left[A^{n+m}\left(Y_{0}\right)^{2}\right] \\
& \times \frac{(3 n-2)!!!(3 m-2)!!!}{(3 n)!(3 m)!} t^{3(n+m)} \\
& +E\left[\left(Y_{1}\right)^{2}\right] t^{2}
\end{aligned}
$$

$$
\begin{align*}
& +2 \sum_{n=1}^{N}(-1)^{n} E\left[A^{n}\left(Y_{1}\right)^{2}\right] \frac{(3 n-1)!!!}{(3 n+1)!} t^{3 n+2} \\
& +\sum_{n=1}^{N} E\left[A^{2 n}\left(Y_{1}\right)^{2}\right]\left(\frac{(3 n-1)!!!}{(3 n+1)!}\right)^{2} t^{6 n+2} \\
& +2 \sum_{n=2}^{N} \sum_{m=1}^{n-1}(-1)^{n+m}+E\left[A^{n+m}\left(Y_{1}\right)^{2}\right] \\
& \quad \times \frac{(3 n-1)!!!(3 m-1)!!!}{(3 n+1)!(3 m+1)!} t^{3(n+m)+2} \\
& \quad+2 E\left[Y_{0} Y_{1}\right] t \\
& +2 \sum_{n=1}^{N}(-1)^{n} E\left[A^{n} Y_{0} Y_{1}\right] \frac{(3 n-1)!!!}{(3 n+1)!} t^{3 n+1} \\
& +2 \sum_{n=1}^{N}(-1)^{n} E\left[A^{n} Y_{0} Y_{1}\right] \frac{(3 n-2)!!!}{(3 n)!} t^{3 n+1} \\
& +2 \sum_{n=1}^{N} \sum_{m=1}^{N}(-1)^{n+m} E\left[A^{n+m} Y_{0} Y_{1}\right] \\
& \quad \times \frac{(3 n-2)!!!(3 m-1)!!!!}{(3 n)!(3 m+1)!} t^{3(n+m)+1} . \tag{10}
\end{align*}
$$

Remark 2. In order to legitimate the use of the previous approximations to the average and the standard deviation, condition (2) must be checked in practice. However, there is a lack of explicit formulae for the absolute moments with respect to the origin of some rvs. This aims us to look for a general approach to deal with a wide range of random inputs taking advantage of the so-called censuring method (see [12, chapter V]). Let us assume that rvs $A, Y_{i}, i \in\{0,1\}$ satisfy

$$
\begin{equation*}
a_{1} \leq a=A(\omega) \leq a_{2}, \quad y_{i, 1} \leq y_{i}=Y_{i}(\omega) \leq y_{i, 2}, \quad \forall \omega \in \Omega \tag{11}
\end{equation*}
$$

Then

$$
\begin{equation*}
E\left[|A|^{n}\left|Y_{i}\right|^{4}\right]=\int_{a_{1}}^{a_{2}} \int_{y_{i, 1}}^{y_{i, 2}}|a|^{n}\left|y_{i}\right|^{4} f_{A Y_{i}}\left(a, y_{i}\right) \mathrm{d} y_{i} \mathrm{~d} a \leq H_{i} M^{n} \tag{12}
\end{equation*}
$$

where $f_{A Y_{i}}\left(a, y_{i}\right)$ denotes the joint probability density function (p.d.f.) of rvs $A, Y_{i}$ and $M=\max \left(\left|a_{1}\right|,\left|a_{2}\right|\right), H_{i}=\left(h_{i}\right)^{4}$, being $h_{i}=\max \left(\left|y_{i, 1}\right|,\left|y_{i, 2}\right|\right), i \in\{0,1\}$. Indeed, in the case that $H_{i}, M>1, i \in\{0,1\}$ one gets

$$
\begin{align*}
& \int_{a_{1}}^{a_{2}} \int_{y_{i, 1}}^{y_{i, 2}}|a|^{n}\left|y_{i}\right|^{4} f_{A Y_{i}}\left(a, y_{i}\right) \mathrm{d} y_{i} \mathrm{~d} a \\
& \quad \leq H_{i} M^{n} \int_{a_{1}}^{a_{2}} \int_{y_{i, 1}}^{y_{i, 2}} f_{A Y_{i}}\left(a, y_{i}\right) \mathrm{d} y_{i} \mathrm{~d} a=H_{i} M^{n} \tag{13}
\end{align*}
$$

Notice that in the last step the double integral is just 1 , since $f_{A Y_{i}}\left(a, y_{i}\right)$ is a pdf The other cases can be analyzed
analogously. Substituting the integral by a sum in (12), previous reasoning remains true when $A$ and/or $Y_{i}, i \in\{0,1\}$ are discrete rvs. As a consequence, important rvs such as binomial, hypergeometric, uniform or beta satisfy condition (2), which is related to joint absolute moments of $A, Y_{i}, i \in$ $\{0,1\}$. It is worthwhile to point out that there are significant rvs that do not satisfy condition (2) such as the exponential rv, for instance. In fact, taking $Y_{i} \equiv 1, i=0,1$ in (2), if $A \sim \operatorname{Exp}(\lambda), \lambda>0$, then $E\left[|A|^{n}\right]=n!/ \lambda^{n}$. As a consequence, in this case condition (2) does not fulfill. Although other unbounded rvs can also verify condition (2), we do not need to check it each time, since if we censure its codomain suitably, we are legitimated to compute approximations to the mean and standard deviation according to formulae (9)(10), respectively. The larger the censured interval, the better the approximations. However, in practice, intervals relatively short provide very good approximations. For instance, as an illustrative example notice that the truncated interval [ $\mu-$ $3 \sigma, \mu+3 \sigma]$ contains the $99.7 \%$ of the probability mass of a Gaussian rv with mean $\mu$ and standard deviation $\sigma>0$.

### 2.2. Tackling Random Dependence by Generalized Polynomial

 Chaos Method. As it has been underlined in Section 1, gPC constitutes a powerful method to deal with randomness in differential equations, say$$
\begin{equation*}
\mathfrak{D}(t, \eta(\omega) ; X)=f(t, \eta(\omega)), \tag{14}
\end{equation*}
$$

where $\mathfrak{D}$ denotes a differential operator; $X=X(t, \eta(\omega))$ is the solution sp to be determined and $f(t, \eta(\omega))$ is a forcing term. Notice that in the rde (14) uncertainty is represented by $\eta$, and it just enters through its coefficients and forcing term, although in practice it could also be considered via initial and/or boundary conditions. For the sake of clarity, in the following each scalar random input will be denoted by $\eta$.
gPC permits to represent spectrally each $\eta$ in the random dimension, and the solution sp, $X(t)$, in $L^{2}=L^{2}(\Omega, \mathfrak{F}, P)$. These representations are given by infinite random series defined in terms of certain orthogonal polynomial expansions $\left\{\Phi_{i}\right\}$ which depend on a number of rvs $\zeta(\omega)=$ $\left(\zeta_{1}(\omega), \zeta_{2}(\omega), \ldots\right), \omega \in \Omega$,

$$
\begin{equation*}
\eta=\sum_{i=0}^{\infty} \eta_{i} \Phi_{i}(\zeta(\omega)), \quad X(t)=\sum_{i=0}^{\infty} X_{i}(t) \Phi_{i}(\zeta(\omega)) \tag{15}
\end{equation*}
$$

This set $\left\{\Phi_{i}\right\}$ constitutes a complete orthogonal basis in $L^{2}(\Omega$, $\mathfrak{F}, P$ ) with the inner product

$$
\begin{equation*}
\left\langle\Phi_{i}, \Phi_{j}\right\rangle=\left\langle\Phi_{i}^{2}\right\rangle \delta_{i j}, \tag{16}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker delta and $\langle\cdot\rangle$ denotes the ensemble average defined as follows:

$$
\begin{equation*}
\langle g(\zeta), h(\zeta)\rangle=\int_{\operatorname{supp}(\zeta)} g(\zeta) h(\zeta) f_{\zeta}(\zeta) \mathrm{d} \zeta, \tag{17}
\end{equation*}
$$

being $f_{\zeta}(\zeta)$ the joint $\operatorname{pdf}$ of $\zeta$ and $\operatorname{supp}(\zeta)$ its support.
The choice of the trial basis $\left\{\Phi_{i}\right\}$ is crucial in dealing with rdes. In [3], authors provide a comprehensive way to choose
the trial basis according to the statistical distribution of the random input $\eta$ in order to achieve optimal convergence in (15). For instance, if rv $\eta$ follows a binomial, negative binomial, hypergeometric, Poisson, beta, or gamma distribution, then $\left\{\Phi_{i}\right\}$ should be taken as Krawtchouk, Meixner, Hahn, Charlier, Jacobi, Laguerre orthogonal polynomials belonging to the Wiener-Askey scheme, respectively. In the significant case that $\eta$ is a Gaussian rv, Hermite polynomials are required. This particular case is referred to as Polynomial Chaos rather than gPC. Throughout this paper only PC will be used. The key connection to do an adequate selection of the trial basis lies in the close relationship between the pdf $f_{\zeta}(\zeta)$ of some standard rvs and the weight function that defines the inner product (17) with respect to which some classical polynomials are orthogonal.

In order to keep the computations affordable in dealing with rdes, each random model parameter $\eta$ as well as the solution sp $X(t)$ is represented by truncated series of the form (15),

$$
\begin{gather*}
\eta \approx \sum_{i=0}^{P} \eta_{i} \Phi_{i}(\zeta(\omega)),  \tag{18}\\
X(t) \approx \sum_{i=0}^{P} X_{i}(t) \Phi_{i}(\zeta(\omega)),
\end{gather*}
$$

where the number of components of random vector $\zeta(\omega)$ also needs to be truncated at a number $s$ called the order of chaos, $\zeta(\omega)=\left(\zeta_{1}(\omega), \zeta_{2}(\omega), \ldots, \zeta_{s}(\omega)\right)$. The truncation order $P$ is made so that all expansion polynomials up to a certain maximum degree, $p$, are included. This entails the following relationship between the number of terms $P+1$ in the series expansions, the maximum degree $p$, and the order of chaos $s$ :

$$
\begin{equation*}
P=\frac{(p+s)!}{p!s!}-1 \tag{19}
\end{equation*}
$$

In this context, solving the rde (14) consists of computing coefficients $X_{i}(t)$ appearing in (18) which also allows to compute approximations of the expectation and the standard deviation to the solution sp $X(t)$ as follows:

$$
\begin{gather*}
E[X(t)] \approx \mu_{X(t)}^{P}=X_{0}(t) E\left[\Phi_{0}(\zeta)\right] \\
\sqrt{\operatorname{Var}[X(t)]} \approx \sigma_{X(t)}^{P}=\sqrt{\sum_{i=1}^{P}\left(X_{i}(t)\right)^{2} E\left[\left(\Phi_{i}(\zeta)\right)^{2}\right]} \tag{20}
\end{gather*}
$$

To achieve this goal, first the expansion of $X(t)$ given by (18) is substituted into the rde (14). Second, a Galerkin projection is done by multiplying the rde by every polynomial of the expansion basis $\left\{\Phi_{i}\right\}$, and then, the ensemble average is taken. This leads to

$$
\begin{array}{r}
\left\langle\mathfrak{D}\left(t, \zeta ; \sum_{i=0}^{P} X_{i}(t) \Phi_{i}(\zeta)\right), \Phi_{j}(\zeta)\right\rangle=\left\langle f(t, \zeta), \Phi_{j}(\zeta)\right\rangle, \\
j=0,1, \ldots, P \tag{21}
\end{array}
$$

that corresponds to a deterministic system of $P+1$ coupled differential equations whose unknowns are the node functions $X_{i}(t)$. These unknowns can be computed by standard numerical techniques such as Runge-Kutta scheme.

Most of the contributions based on gPC assume that rvs $\zeta_{i}(\omega), 1 \leq i \leq s$ are independent which facilitates the study. The case in which random parameters are assumed to be dependent is currently a topic under study. In $[9,10]$, authors present methods based on gPC to tackle dependence in differential equations. Both contributions provide general techniques that can be applied whenever the joint pdf of the random inputs is known. However, in practice the availability of this joint pdf can be very difficult even impossible. In the particular case where the inputs are dependent Gaussian rvs, an alternative method can be applied to conduct the corresponding study taking advantage that uncorrelation and independence are equivalent notions for Gaussian rvs together with Cholesky matrix decomposition. To exhibit how the method is going to be applied in our case, let us remember the following basic result.

Proposition 3. Let $\zeta=\left(\zeta_{1}, \ldots, \zeta_{n}\right)^{\top}$ be a Gaussian vector with mean $\mu_{\zeta}$ and variance-covariance matrix $\Sigma_{\zeta}: \zeta$ ~ $N\left(\boldsymbol{\mu}_{\zeta} ; \boldsymbol{\Sigma}_{\zeta}\right)$ (the symbol $\top$ denotes the usual matrix transpose operator). For each deterministic vector $\boldsymbol{\nu} \in \mathbb{R}^{n}$ and deterministic matrix $\boldsymbol{\Lambda} \in M_{n \times n}(\mathbb{R})$, the random vector $\boldsymbol{\xi}=\left(\xi_{1}, \ldots, \xi_{n}\right)^{\top}$ defined by the linear transformation: $\boldsymbol{\xi}=\boldsymbol{\nu}+\boldsymbol{\Lambda}$ follows the Gaussian distribution: $\boldsymbol{\xi} \sim N\left(\boldsymbol{\nu}+\boldsymbol{\Lambda} \boldsymbol{\mu}_{\zeta} ; \boldsymbol{\Lambda} \boldsymbol{\Sigma}_{\zeta} \boldsymbol{\Lambda}^{\top}\right)$.

On the other hand, let $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ be a Gaussian random vector. As $\boldsymbol{\Sigma}_{\eta}$ is a variance-covariance matrix, it is Hermitian and positive definite. Hence, there is a matrix, say $\mathbf{H}$, such that $\boldsymbol{\Sigma}_{\eta}=\mathbf{H H}^{\top}$. For instance, Cholesky decomposition provides a well-known procedure to compute matrix H [13].

Keeping the notation of the previous context, we apply Proposition 3 to the particular case where $\zeta=\mathbf{Z} \sim N\left(\mathbf{0} ; \mathbf{I}_{n}\right)$ (so, $\boldsymbol{\mu}_{\zeta}=\mathbf{0}$ and $\boldsymbol{\Sigma}_{\zeta}=\mathbf{I}_{n}$, being $\mathbf{I}_{n}$ the identity matrix of size $n) ; \boldsymbol{\nu}=\boldsymbol{\mu}_{\eta}$ and $\boldsymbol{\Lambda}=\mathbf{H}$. Then

$$
\begin{equation*}
\boldsymbol{\mu}_{\eta}+\mathbf{H Z} \sim N\left(\boldsymbol{\mu}_{\eta} ; \mathbf{H H}^{\top}\right)=N\left(\boldsymbol{\mu}_{\eta} ; \Sigma_{\eta}\right) \tag{22}
\end{equation*}
$$

As $\mathbf{Z}=\left(Z_{1}, \ldots, Z_{n}\right)^{\top} \sim N\left(0 ; \mathbf{I}_{n}\right)$, then $Z_{j}, 1 \leq j \leq n$ are Gaussian and uncorrelated r.v.s and, therefore, independent. As a consequence, expression (22) provides a direct way to represent a Gaussian random vector $\eta$ with components statistically dependent by means of a linear transformation of a Gaussian vector $\mathbf{Z}$ whose components are independent: $\eta=\mu_{\eta}+\mathrm{HZ}$.

Now we detail how the previous development can be applied to transform the ivp (1), where random inputs $A, Y_{0}, Y_{1}$ are assumed to be Gaussian dependent into another one with Gaussian independent random inputs. Let $\eta=$ $\left(A, Y_{0}, Y_{1}\right)^{\top} \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ be the multivariate Gaussian distribution of the random data, and let us denote by $\mathbf{H}=\left(h_{i j}\right)$ the Cholesky decomposition of variance-covariance matrix $\boldsymbol{\Sigma}_{\eta}$ of $\eta$. According to (22), we define the linear transformation:

$$
\begin{align*}
\eta & =\left(\begin{array}{l}
A \\
Y_{0} \\
Y_{1}
\end{array}\right)=\boldsymbol{\mu}_{\eta}+\mathbf{H Z} \\
& =\left(\begin{array}{l}
\mu_{A} \\
\mu_{Y_{0}} \\
\mu_{Y_{1}}
\end{array}\right)+\left(\begin{array}{lll}
h_{11} & h_{12} & h_{13} \\
h_{21} & h_{22} & h_{23} \\
h_{31} & h_{32} & h_{33}
\end{array}\right)\left(\begin{array}{l}
Z_{1} \\
Z_{2} \\
Z_{3}
\end{array}\right), \tag{23}
\end{align*}
$$

where $\left(\mu_{A}, \mu_{Y_{0}}, \mu_{Y_{1}}\right)^{\top}$ denotes the mean of vector $\eta=\left(A, Y_{0}\right.$, $\left.Y_{1}\right)^{\top}$ and $\mathbf{Z}=\left(Z_{1}, Z_{2}, Z_{3}\right)^{\top}$, being $Z_{j}$ independent and identically distributed standard Gaussian rvs, that is, $Z_{j} \sim$ $N(0 ; 1), 1 \leq j \leq 3$. By (23), ivp (1) can be recast as follows:

$$
\begin{gather*}
\ddot{X}(t)+t\left(\mu_{A}+\sum_{j=1}^{3} h_{1 j} Z_{j}\right) X(t)=0, \quad t>0, \\
X(0)=\mu_{Y_{0}}+\sum_{j=1}^{3} h_{2 j} Z_{j},  \tag{24}\\
\dot{X}(0)=\mu_{Y_{1}}+\sum_{j=1}^{3} h_{3 j} Z_{j},
\end{gather*}
$$

where random inputs $Z_{j}, 1 \leq j \leq 3$, are independent standard Gaussian rvs. This allows us to compute approximations of the expectation and standard deviation functions by PC according to (20).

## 3. Examples

In this section we will present several illustrative examples based on ivp (1) in order to compare the approximations provided by Polynomial Chaos, Fröbenius, and Monte Carlo simulation. The comparison is performed by computing the average and standard deviation functions of the solution of ivp (1). As we pointed out in the Introduction section, the study is conducted through Airy differential equation since, as in the deterministic case its solutions are highly oscillatory, it is expected that differences among the three previous approaches will be better highlighted in the random framework. Computations have been carried out with Mathematica package [14]. In particular, the coupled systems of differential equations obtained after applying gPC in each example are numerically solved with this software.

The examples have been designed to explore both the marginal influence of randomness on the output when it is assumed that only some inputs of ivp (1) are rvs (see Examples 1 and 2 , where $\left(Y_{0}, Y_{1}\right)$ and $\left(Y_{0}, A\right)$ are assumed to follow a bivariate Gaussian rvs, resp.), and the cases presented in Examples 3 and 4 where all the random inputs $\left(Y_{0}, Y_{1}, A\right)$ are assumed to follow a multivariate Gaussian distribution. In the two first examples, we also investigate the influence of the numerical value of the correlation coefficient of the twodimensional random input changes. Examples 3 and 4 seek to illustrate the different qualitative behaviour of the solution sp of ivp (1) depending on rv $A$.

Example 1. This first example has been devised to investigate whether statistical dependence between the initial conditions
entails a substantial change in the output with respect to independence assumption. In addition, it permits to highlight some significant advantages of Fröbenius and PC methods in comparison with MCs. Let us consider the ivp (1) where $A=1$ and the initial conditions $\eta=\left(Y_{0}, Y_{1}\right)^{\top}$ are assumed to be dependent on a Gaussian r.v.s: $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$, where

$$
\boldsymbol{\mu}_{\eta}=\binom{\mu_{Y_{0}}}{\mu_{Y_{1}}}, \quad \boldsymbol{\Sigma}_{\eta}=\left(\begin{array}{cc}
\left(\sigma_{Y_{0}}\right)^{2} & \rho_{Y_{0} Y_{1}} \sigma_{Y_{0}} \sigma_{Y_{1}}  \tag{25}\\
\rho_{Y_{0} Y_{1}} \sigma_{Y_{0}} \sigma_{Y_{1}} & \left(\sigma_{Y_{1}}\right)^{2}
\end{array}\right) .
$$

In this case, we can directly monitor the influence of the statistical dependence between $Y_{0}$ and $Y_{1}$, measured through its correlation coefficient $\rho_{Y_{0} Y_{1}}$, in the computations of the average and the standard deviation when these moments are calculated using anyone of the three methods. For the Fröbenius method, taking into account that $A=1$ and expression (10), we observe that dependence only contributes through the term: $E\left[Y_{0} Y_{1}\right]=\rho_{Y_{0} Y_{1}} \sigma_{Y_{0}} \sigma_{Y_{1}}+\mu_{Y_{0}} \mu_{Y_{1}}$. In case of PC method, monitoring dependence is made directly over the resulting ivp:

$$
\begin{gather*}
\ddot{X}(t)+t X(t)=0, \quad t>0, \\
X(0)=\mu_{Y_{0}}+\sum_{j=1}^{2} h_{1 j} Z_{j},  \tag{26}\\
\dot{X}(0)=\mu_{Y_{1}}+\sum_{j=1}^{2} h_{2 j} Z_{j},
\end{gather*}
$$

since some coefficients $h_{i j}, 1 \leq i, j \leq 2$ depend on $\rho_{Y_{0} Y_{1}}$. Specifically, $h_{i j}$ are the entries of the Cholesky decomposition of variance-covariance matrix $\boldsymbol{\Sigma}_{\eta}=\mathbf{H} \mathbf{H}^{\top}$. In this case one gets

$$
\mathbf{H}=\left(\begin{array}{cc}
\sigma_{Y_{0}} & 0  \tag{27}\\
\rho_{Y_{0} Y_{1}} \sigma_{Y_{1}} & \sigma_{Y_{1}} \sqrt{1-\left(\rho_{Y_{0} Y_{1}}\right)^{2}}
\end{array}\right) .
$$

Notice that in (26), $Z_{1}$ and $Z_{2}$ are independent standard Gaussian rvs: $Z_{1}, Z_{2} \sim N(0 ; 1)$. With respect to MC simulation, it is obvious that dependence is monitored directly through the bivariate pdf which depends on $\rho_{Y_{0} Y_{1}}$ :

$$
\begin{aligned}
f\left(Y_{0}, Y_{1}\right)= & \frac{1}{\sqrt{2 \pi\left(1-\left(\rho_{Y_{0} Y_{1}}\right)^{2}\right)} \sigma_{Y_{0}} \sigma_{Y_{1}}} \\
& \times \exp \left\{-\frac{1}{2\left(1-\left(\rho_{Y_{0} Y_{1}}\right)^{2}\right)}\right. \\
& \times\left[\left(\frac{Y_{0}-\mu_{Y_{0}}}{\sigma_{Y_{0}}}\right)^{2}-2 \rho_{Y_{0} Y_{1}}\left(\frac{Y_{0}-\mu_{Y_{0}}}{\sigma_{Y_{0}}}\right)\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.\left.\times\left(\frac{Y_{1}-\mu_{Y_{1}}}{\sigma_{Y_{1}}}\right)+\left(\frac{Y_{1}-\mu_{Y_{1}}}{\sigma_{Y_{1}}}\right)^{2}\right]\right\} . \tag{28}
\end{equation*}
$$

The random samples needed to apply MC method have been sampled with the Mathematica instruction: "RandomVariate[MultinormalDistribution[,]]."

Notice that Fröbenius and PC approaches turn out much more fruitful than MCs in this particular example. In the case of Fröbenius method, the series representation of the solution sp $X(t)$ given by (3) permits not only to compute reliable approximations of the average and standard deviation but also to determine its full statistical distribution, which in general is a major challenge. Firstly, notice that under condition (2), we have proven that the infinite series (3) converges in the mean square sense for each $t$; therefore, it also converges in distribution for every $t$. In accordance with (3), the truncated approximation of $X(t)$ is given by

$$
\begin{gather*}
X_{N}(t)=Y_{0} f_{N}(t)+Y_{1} g_{N}(t), \\
f_{N}(t)=\sum_{n=0}^{N} \frac{(-1)^{n}(3 n-2)!!!}{(3 n)!} t^{3 n},  \tag{29}\\
g_{N}(t)=\sum_{n=0}^{N} \frac{(-1)^{n}(3 n-1)!!!}{(3 n+1)!} t^{3 n+1},
\end{gather*}
$$

where we adopt the convention $k!!!=1$ for $k<0$. Note that

$$
\begin{align*}
& \lim _{N \rightarrow \infty} f_{N}(t)={ }_{0} F_{1}\left(; \frac{2}{3} ;-\frac{t^{3}}{9}\right), \\
& \lim _{N \rightarrow \infty} g_{N}(t)={ }_{0} F_{1}\left(; \frac{4}{3},-\frac{t^{3}}{9}\right), \tag{30}
\end{align*}
$$

where ${ }_{p} F_{q}\left[a_{1}, \ldots, a_{p} ; b_{1}, \ldots, b_{q} ; z\right]$ denotes the confluent hypergeometric function of order $p$ and $q$ evaluated at the point $z$.

Since the vector $\left(Y_{0}, Y_{1}\right)^{\top}$ has a bivariate Gaussian distribution, $X_{N}(t)$ follows a univariate Gaussian distribution whose average and variance are given by

$$
\begin{align*}
& E\left[X_{N}(t)\right]=E\left[Y_{0}\right] f_{N}(t)+E\left[Y_{1}\right] g_{N}(t),  \tag{31}\\
& \operatorname{Var}\left[X_{N}(t)\right]= \operatorname{Var}\left[Y_{0}\right]\left(f_{N}(t)\right)^{2} \\
&+\operatorname{Var}\left[Y_{1}\right]\left(g_{N}(t)\right)^{2} \\
&+2 \operatorname{Cov}\left[Y_{0} f_{N}(t), Y_{1} g_{N}(t)\right]  \tag{32}\\
&= \operatorname{Var}\left[Y_{0}\right]\left(f_{N}(t)\right)^{2}+\operatorname{Var}\left[Y_{1}\right]\left(g_{N}(t)\right)^{2} \\
&+2 f_{N}(t) g_{N}(t) \rho_{Y_{0} Y_{1}} \sigma_{Y_{0}} \sigma_{Y_{1}},
\end{align*}
$$

respectively. By property (8), these expressions converge as $N \rightarrow \infty$ to the exact average, $E[X(t)]$, and variance $\operatorname{Var}[X(t)]$, respectively. This determines completely the statistical distribution of $X(t)$ for every $t$.

On the other hand, PC method also provides, in this example, a useful series representation of the solution sp $X(t)$ that permits to obtain its statistical distribution. It is straightforward to check that the PC series representation obtained from ivp (26) has the following finite linear form:

$$
\begin{equation*}
X(t)=X_{0}(t) \Phi_{0}(\boldsymbol{\xi})+X_{1}(t) \Phi_{1}(\boldsymbol{\xi})+X_{2}(t) \Phi_{2}(\boldsymbol{\xi}) \tag{33}
\end{equation*}
$$

where $\boldsymbol{\xi}=\left(Z_{1}, Z_{2}\right)^{\top}, Z_{i} \sim N(0 ; 1)$ independent, $i=1,2$ and $\Phi_{0}(\boldsymbol{\xi})=\left(Z_{1}\right)^{0}\left(Z_{2}\right)^{0}=1, \Phi_{1}(\boldsymbol{\xi})=\left(Z_{1}\right)^{1}\left(Z_{2}\right)^{0}=$ $Z_{1}$ and $\Phi_{2}(\boldsymbol{\xi})=\left(Z_{1}\right)^{0}\left(Z_{2}\right)^{1}=Z_{2}$. Hence, following an analogous reasoning as in the previous case, we deduce the full distribution of $X(t)$. With respect to MC approach, it only provides a set of numerical values to the solution at some selected time instants from which only rough approximations of the statistical distribution of the solution $X(t)$ can be given.

In Tables 1 and 2, we compare the approximations of the expectation and standard deviation at different time instants $t$ and correlation values: $\rho_{Y_{0} Y_{1}}=-0.5,0,0.5,0.9$, respectively. Notice that these values correspond to average negative dependence; independence; average positive dependence; and strong positive dependence, respectively. We assume $\mu_{Y_{0}}=1, \mu_{Y_{1}}=1, \sigma_{Y_{0}}=0.2, \sigma_{Y_{1}}=0.2$. $E\left[X_{N}(t)\right]\left(\sigma\left[X_{N}(t)\right]\right), \mu_{X}^{P}(t)\left(\sigma_{X}^{P}(t)\right)$ and $\tilde{\mu}_{X}^{m}(t)\left(\widetilde{\sigma}_{X}^{m}(t)\right)$ denote the approximate expectation (and standard deviation) at time $t$ obtained by Fröbenius (see expressions (9) and (10), or equivalently, (31) and (32)) with truncation orden $N$, PC (see expression (20)) of order $P$ and MCs using $m$ simulations, respectively. The values of $N$ and $P$ show in Tables 1 and 2 are those obtained when the numerical stabilization at six significant digits is achieved.

We observe the numerical results provided by Fröbenius and PC approaches match while MCs captures about three significant digits. From Tables 1 and 2, we realize that the correlation value $\rho_{Y_{0}, Y_{1}}$ between $Y_{0}$ and $Y_{1}$ does not influence the average, but it does decisively influuence the standard deviation of the solution. This indicates to us that it is crucial to know not only the existence of statistical dependence between initial conditions but also quantifying, as accurate as possible, its value by the correlation coefficient $\rho_{Y_{0}, Y_{1}}$. Notice that these numerical conclusions agree with formulae (31)(32).

Example 2. In the previous example, uncertainty entered in the equation just through initial conditions $Y_{0}$ and $Y_{1}$. Although both rvs are dependent, we have showed that it does not influence the expectation of the solution but its standard deviation. Does this answer change in case that randomness is considered through coefficient $A$ and an initial condition? In order to answer this question let us consider the ivp (1) where $\eta=\left(A, Y_{0}\right)^{\top}$ are assumed to be dependent Gaussian rvs: $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$, where

$$
\begin{gather*}
\boldsymbol{\mu}_{\eta}=\binom{\mu_{A}}{\mu_{Y_{0}}}, \\
\boldsymbol{\Sigma}_{\eta}=\left(\begin{array}{cc}
\left(\sigma_{A}\right)^{2} & \rho_{A Y_{0}} \sigma_{A} \sigma_{Y_{0}} \\
\rho_{A Y_{0}} \sigma_{A} \sigma_{Y_{0}} & \left(\sigma_{Y_{0}}\right)^{2}
\end{array}\right) . \tag{34}
\end{gather*}
$$



Figure 1: Comparison of the approximations for the expectation $\mu[X(t)]$ on the time-interval $t \in[0,5]$ for different correlation values $\rho=\rho_{A, Y_{0}} \in\{-0.5,0,0.5,0.9\}$ in Example 2 by using Fröbenius, Polynomial Chaos, and Monte Carlo simulations. Since the numerical values are indistinguishable graphically, for the sake of clarity in the presentation, nonlabel has been introduced. We assume that $Y_{1}=1$ and $\eta=\left(A, Y_{0}\right)^{\top}$ follows a bivariate Gaussian distribution: $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ where, according to (34), $\mu_{A}=1$, $\mu_{Y_{0}}=1, \sigma_{A}=0.2, \sigma_{Y_{0}}=0.2$.

To perform the computations of the average $(\mu[X(t)])$ and standard deviation $(\sigma[X(t)])$ shown in Figures 1 and 2, respectively, we have taken $\mu_{A}=\mu_{Y_{0}}=1, \sigma_{A}=\sigma_{Y_{0}}=0.2$, and $\rho_{A Y_{0}} \in\{-0.5,0,0.5,0.9\}$. The initial condition $Y_{1}$ is assumed to be deterministic: $Y_{1}=1$. Computations for Fröbenius and PC have been carried out until the numerical stabilization at six significant digits is achieved. This stabilization is got for Fröbenius method with $N=15$ for the average and $N=16$ for the standard deviation. The corresponding values for PC are $P=5$ and $P=7$. These results for $N$ and $P$ do not depend on the correlation coefficient $\rho_{A Y_{0}}$. Results obtained by MCs have been carried out with $m=500000$ simulations. As a reference of the computational burden required by each one of the three methods, we indicate the CPU seconds to carry out the standard deviation plotted in Figure 2 in an Intel Core $i 7$ with 2.7 GHz : Fröbenius ( 1.3 s ), PC ( 0.8 s ), and MC ( 385 s ).

Although similar numerical differences between MCs and the other techniques (Fröbenius and PC) could be reported in a table as we did in the foregoing example, for both the average and the standard deviation, now we present the numerical approximations in Figures 1 and 2 without labelling each method since they are indistinguishable graphically. We underline that numerical results for Fröbenius and PC methods match with the six significant digits. While, MCs captures three of these digits.

In contrast to what happened in Example 1, we now observe that average changes, but slightly, when $\rho_{A Y_{0}}$ does (see Figure 1). These changes are greater when computing standard deviation (see Figure 2). Both conclusions agree with formulae (9)-(10). Again, these results indicate that independence between random parameters (in this case, between coefficient $A$ and the initial condition $Y_{0}$ ) must be checked previously since the existence of statistical dependence influences significantly on the output.
TABLE 1: Comparison of the approximations for the expectations at different time instants $t$ and correlation values $\rho_{Y_{0}, Y_{1}}$ in Example 1 by using Fröbenius method with a truncated series with $N+1$ terms $\left(E\left[X_{N}(t)\right]\right)$, Polynomial Chaos of order $P\left(\mu_{X}^{P}(t)\right)$, and Monte Carlo with $m$ simulations $\left(\tilde{\mu}_{X}^{m}(t)\right)$. We assume that $A=1$ and the initial condition: $\eta=\left(Y_{0}, Y_{1}\right)^{\top}$ follows a bivariate Gaussian distribution: $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ where, according to (25), $\mu_{Y_{0}}=1, \mu_{Y_{1}}=1, \sigma_{Y_{0}}=0.2, \sigma_{Y_{1}}=0.2$.

| $t$ | $\rho_{Y_{0}, Y_{1}}=-0.5$ |  |  | $\rho_{Y_{0}, Y_{1}}=0$ |  |  | $\rho_{Y_{0}, Y_{1}}=0.5$ |  |  | $\rho_{Y_{0}, Y_{1}}=0.9$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{gathered} E\left[X_{N}(t)\right] \\ N=1 \\ \hline \end{gathered}$ | $\begin{aligned} & \mu_{X}^{P}(t) \\ & P=1 \end{aligned}$ | $\begin{gathered} \tilde{\mu}_{\times}^{m}(t) \\ m=500000 \end{gathered}$ | $\begin{gathered} E\left[X_{N}(t)\right] \\ N=14 \end{gathered}$ | $\begin{aligned} & \mu_{P}^{P}(t) \\ & P=1 \end{aligned}$ | $\begin{gathered} \substack{\tilde{\mu}_{3}^{m}(t) \\ m=50000 \\ \hline} \end{gathered}$ | $\begin{gathered} \left.E\left[X_{N}(t)\right]\right] \\ N=14 \end{gathered}$ | $\begin{aligned} & \mu_{X}^{P}(t) \\ & P=1 \end{aligned}$ | $\begin{gathered} \tilde{\mu}_{\Omega}^{m}(t) \\ m=500000 \end{gathered}$ | $\begin{gathered} E\left[X_{N}(t)\right] \\ N=14 \end{gathered}$ | $\begin{aligned} & \mu_{X}^{P}(t) \\ & P=1 \end{aligned}$ | $\begin{gathered} \tilde{\mu}_{\times}^{m}(t) \\ m=500000 \end{gathered}$ |
| 0.00 | 1. | 1. | 0.999945 | 1. | 1. | 0.999723 | 1. | 1. | 0.999838 | 1. | 1. | 0.999615 |
| 0.25 | 1.24707 | 1.24707 | 1.24705 | 1.24707 | 1.24707 | 1.24689 | 1.24707 | 1.24707 | 1.24681 | 1.24707 | 1.24707 | 1.24661 |
| 0.50 | 1.47406 | 1.47406 | 1.47408 | 1.47406 | 1.47406 | 1.47397 | 1.47406 | 1.47406 | 1.47371 | 1.47406 | 1.47406 | 1.47353 |
| 0.75 | 1.65457 | 1.65457 | 1.65462 | 1.65457 | 1.65457 | 1.65457 | 1.65457 | 1.65457 | 1.65413 | 1.65457 | 1.65457 | 1.65398 |
| 1.00 | 1.75744 | 1.75744 | 1.75752 | 1.75744 | 1.75744 | 1.75754 | 1.75744 | 1.75744 | 1.75694 | 1.75744 | 1.75744 | 1.75683 |
| 2.00 | 0.884201 | 0.884201 | 0.884327 | 0.884201 | 0.884201 | 0.884535 | 0.884201 | 0.884201 | 0.88385 | 0.884201 | 0.884201 | 0.883924 |
| 3.00 | -1.20538 | -1.20538 | -1.20541 | -1.20538 | -1.20538 | -1.20537 | -1.20538 | -1.20538 | -1.20506 | -1.20538 | -1.20538 | -1.20495 |
| 4.00 | -0.353252 | -0.353252 | -0.353344 | -0.353252 | -0.353252 | -0.353523 | -0.353252 | -0.353252 | -0.353062 | -0.353252 | -0.353252 | -0.353156 |
| 5.00 | 1.21343 | 1.21343 | 1.21352 | 1.21343 | 1.21343 | 1.21363 | 1.21343 | 1.21343 | 1.21304 | 1.21343 | 1.21343 | 1.21302 |

Table 2: Comparison of the approximations for the standard deviation at different time instants $t$ and correlation values $\rho_{Y_{0}, Y_{1}}$ in Example 1 by using Fröbenius method with a truncated series with $N+1$ terms $\left(\sigma\left[X_{N}(t)\right]\right)$, Polynomial Chaos of order $P\left(\sigma_{X}^{P}(t)\right)$, and Monte Carlo with $m$ simulations $\left(\widetilde{\sigma}_{X}^{m}(t)\right)$. We assume that $A=1$ and the initial condition: $\eta=\left(Y_{0}, Y_{1}\right)^{\top}$ follows a bivariate Gaussian distribution: $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ where, according to (25), $\mu_{Y_{0}}=1, \mu_{Y_{1}}=1, \sigma_{Y_{0}}=0.2, \sigma_{Y_{1}}=0.2$.

| $t$ | $\rho_{Y_{0}, Y_{1}}=-0.5$ |  |  | $\rho_{Y_{0}, Y_{1}}=0$ |  |  | $\rho_{Y_{0}, Y_{1}}=0.5$ |  |  | $\rho_{Y_{0}, Y_{1}}=0.9$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{gathered} \sigma_{\left[1 X_{N}(t)\right]}^{N=14} \end{gathered}$ | $\begin{aligned} & \sigma_{X}^{P}(t) \\ & P=1 \\ & \hline \end{aligned}$ | $\begin{gathered} \tilde{\sigma}_{X}^{m}(t) \\ m=500000 \end{gathered}$ | $\begin{gathered} \sigma_{[ }\left(X_{N}(t)\right] \\ N=14 \end{gathered}$ | $\begin{aligned} & \sigma_{X}^{p}(t) \\ & P=1 \end{aligned}$ | $\begin{gathered} \tilde{\sigma}_{\pi}^{n}(t) \\ m=500000 \end{gathered}$ | $\begin{gathered} \sigma_{\left[X_{N}(t)\right]}^{N=15} \end{gathered}$ | $\begin{aligned} & \sigma_{X}^{p}(t) \\ & P=1 \end{aligned}$ | $\begin{gathered} \tilde{\sigma}_{X}^{n}(t) \\ m=500000 \end{gathered}$ | $\begin{gathered} \sigma_{\left[\mid X_{N}(t)\right]}^{N=14} \end{gathered}$ | $\begin{gathered} \sigma_{X}^{p}(t) \\ P=1 \end{gathered}$ | $\begin{gathered} \tilde{\sigma}_{n}^{m}(t) \\ m=50000 \end{gathered}$ |
| 0.00 | 0.2 | 0.2 | 0.199788 | 0.2 | 0.2 | 0.20002 | 0.2 | 0.2 | 0.199918 | 0.2 | 0.2 | 0.199962 |
| 0.25 | 0.17979 | 0.17979 | 0.179672 | 0.205634 | 0.205634 | 0.205696 | 0.228575 | 0.228575 | 0.228578 | 0.245388 | 0.245388 | 0.245399 |
| 0.50 | 0.169615 | 0.169615 | 0.169576 | 0.219433 | 0.219433 | 0.219502 | 0.25987 | 0.25987 | 0.259943 | 0.288163 | 0.288163 | 0.288216 |
| 0.75 | 0.169288 | 0.169288 | 0.169293 | 0.235811 | 0.235811 | 0.235858 | 0.287324 | 0.287324 | 0.287452 | 0.322667 | 0.322667 | 0.322755 |
| 1.00 | 0.176287 | 0.176287 | 0.176287 | 0.248796 | 0.248796 | 0.248799 | 0.304502 | 0.304502 | 0.304667 | 0.342607 | 0.342607 | 0.342723 |
| 2.00 | 0.181352 | 0.181352 | 0.181106 | 0.179861 | 0.179861 | 0.17966 | 0.178357 | 0.178357 | 0.178418 | 0.177145 | 0.177145 | 0.177237 |
| 3.00 | 0.124683 | 0.124683 | 0.124684 | 0.172443 | 0.172443 | 0.172481 | 0.209588 | 0.209588 | 0.209677 | 0.235116 | 0.235116 | 0.235177 |
| 4.00 | 0.141854 | 0.141854 | 0.141629 | 0.122796 | 0.122796 | 0.122633 | 0.100176 | 0.100176 | 0.100135 | 0.0774611 | 0.0774611 | 0.0774863 |
| 5.00 | 0.144263 | 0.144263 | 0.144177 | 0.183048 | 0.183048 | 0.182964 | 0.214945 | 0.214945 | 0.215076 | 0.237397 | 0.237397 | 0.237501 |



Figure 2: Comparison of the approximations for the standard deviation $\sigma[X(t)]$ on the time-interval $t \in[0,5]$ for different correlation values $\rho=\rho_{A, Y_{0}} \in\{-0.5,0,0.5,0.9\}$ in Example 2 by using Fröbenius, Polynomial Chaos, and Monte Carlo simulations. Since the numerical values are indistinguishable graphically, for the sake of clarity in the presentation, nonlabel has been introduced. We assume that $Y_{1}=1$ and $\eta=\left(A, Y_{0}\right)^{\top}$ follows a bivariate Gaussian distribution: $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ where, according to (34), $\mu_{A}=1$, $\mu_{Y_{0}}=1, \sigma_{A}=0.2, \sigma_{Y_{0}}=0.2$.

Example 3. Now, we consider the case where all the data involved in ivp (1) are rvs. We will assume that the random vector $\eta=\left(A, Y_{0}, Y_{1}\right)^{\top}$ follows a multivariate Gaussian distribution. The aim of this example is twofold: first, to confirm the conclusions drawn in the two previous examples and, second, to show and compare the capabilities of the three methods under analysis to tackling satisfactorily full randomness in the ivp (1). Computations have been performed taking as average: $\boldsymbol{\mu}_{\eta}=(4,3,6)^{\top}$ and, $\boldsymbol{\Sigma}_{\eta}^{1}$ and $\boldsymbol{\Sigma}_{\eta}^{2}$ as the variance-covariance matrices given by (35) that correspond to statistical independence and dependence, respectively,

$$
\begin{gather*}
\Sigma_{\eta}^{1}=\left(\begin{array}{ccc}
0.4 & 0 & 0 \\
0 & 0.3 & 0 \\
0 & 0 & 0.25
\end{array}\right), \\
\Sigma_{\eta}^{2}=\left(\begin{array}{ccc}
0.4 & -0.2 & 0.15 \\
-0.2 & 0.3 & 0.1 \\
0.15 & 0.1 & 0.25
\end{array}\right) . \tag{35}
\end{gather*}
$$

Computations for Fröbenius and PC have been carried out until the numerical stabilization at six significant digits is achieved. This stabilization is got for Fröbenius method with $N=28$ for the average and $N=33$ for the standard deviation. The corresponding values for PC are $P=7$ and $P=12$. As it happened in the foregoing example, the obtained results for $N$ and $P$ do not depend on the variance-covariance matrix. Numerical values obtained by MCs have been carried out with $m=500000$ simulations. Again, as in the previous example the numerical results provided by Fröbenius and PC methods coincide with the six significant digits, while MCs only captures three of these digits.

Figures 3 and 4 show the results for the average and the standard deviation on the time interval $0 \leq t \leq 5$,


Figure 3: Comparison of the approximations for the expectation $\mu[X(t)]$ on the time-interval $t \in[0,5]$ in Example 3 by using Fröbenius, Polynomial Chaos, and Monte Carlo simulations. Since the numerical values are indistinguishable graphically, for the sake of clarity in the presentation, nonlabel has been introduced. We assume that $\eta=\left(A, Y_{0}, Y_{1}\right)^{\top}$ follows a multivariate Gaussian distribution: $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ where $\boldsymbol{\mu}_{\eta}=(4,3,6)^{\top}$ and $\boldsymbol{\Sigma}_{\eta}^{1}$ and $\boldsymbol{\Sigma}_{\eta}^{2}$ are given by (35) corresponding to statistical independence and dependence, respectively.


Figure 4: Comparison of the approximations for the standard deviation $\sigma[X(t)]$ on the time-interval $t \in[0,5]$ in Example 3 by using Fröbenius, Polynomial Chaos, and Monte Carlo simulations. Since the numerical values are indistinguishable graphically, for the sake of clarity in the presentation, nonlabel has been introduced. We assume that $\eta=\left(A, Y_{0}, Y_{1}\right)^{\top}$ follows a multivariate Gaussian distribution: $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ where $\boldsymbol{\mu}_{\eta}=(4,3,6)^{\top}$ and $\boldsymbol{\Sigma}_{\eta}^{1}$ and $\Sigma_{\eta}^{2}$ are given by (35) corresponding to statistical independence and dependence, respectively.
respectively. Analogous comments as we did in Example 2 can be done: one presents changes for both the expectation and standard deviation of the solution depending on the statistical dependence of the input, being these changes greater on the standard deviation.

Example 4. The aim of this example is to show that the qualitative behaviour of the solution sp of the random Airy differential (1) is different depending on the random input $A$.


Figure 5: Comparison of the approximations for the expectation $\mu[X(t)]$ on the time-interval $t \in[0,2]$ in Example 4 by using Fröbenius and Polynomial Chaos. The numerical values obtained by both techniques are indistinguishable graphically. We assume that $\eta=\left(A, Y_{0}, Y_{1}\right)^{\top}$ follows a multivariate Gaussian distribution: $\eta \sim N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ where $\boldsymbol{\mu}_{\eta}=(-2,3,6)^{\top}$ and $\boldsymbol{\Sigma}_{\eta}^{1}$ and $\boldsymbol{\Sigma}_{\eta}^{2}$ are given by (35) corresponding to statistical independence and dependence, respectively.

To illustrate this fact, we will assume that the random vector $\eta=\left(A, Y_{0}, Y_{1}\right)^{\top}$ follows a multivariate Gaussian distribution with average: $\boldsymbol{\mu}_{\eta}=(-2,3,6)^{\top}$ and the same variancecovariance matrices as the ones considered in Example 3 (see expression (35)). Note that we are assuming that most part of the probability mass of $\mathrm{rv} A$ is on the negative real line which implies a different behaviour of the solution with respect to previous examples. Figure 5 shows the mean of the solution in both cases, that is, when the random inputs are independent and dependent. Although both plots are quite similar, in Figure 6 we see in contrast to what happens in previous examples that now standard deviations as well as their difference increase over time.

## 4. Conclusions

The consideration of uncertainty in models based on differential equations leads to random differential equations. Over the last few decades, these types of random continuous models have demonstrated to be powerful tools in dealing with mathematical modelling. However, for simplicity, most of these contributions rely on the assumption that random inputs are statistically independent, a hypothesis that could not be met in many applications. The study of differential equations whose inputs are statistically dependent constitutes currently a topic under development.

In this paper we have studied two methods for solving differential equations whose parameters are assumed to be Gaussian dependent, namely, Fröbenius and Polynomial Chaos. The numerical results, for the average and standard deviation of the solution, provided by both methods have been compared with those computed by Monte Carlo simulations, which can be considered the most common


Figure 6: (a) Comparison of the approximations for the standard deviation $\sigma[X(t)]$ on the time-interval $t \in[0,2]$ in Example 4 by using Fröbenius and Polynomial Chaos. We assume that $\eta=$ $\left(A, Y_{0}, Y_{1}\right)^{\top}$ follows a multivariate Gaussian distribution: $\eta \sim$ $N\left(\boldsymbol{\mu}_{\eta} ; \boldsymbol{\Sigma}_{\eta}\right)$ where $\boldsymbol{\mu}_{\eta}=(-2,3,6)^{\top}$ and $\boldsymbol{\Sigma}_{\eta}^{1}$ and $\boldsymbol{\Sigma}_{\eta}^{2}$ are given by (35) corresponding to statistical independence and dependence, respectively. (b) Difference between the approximations showed in the figure on (a).
approach to deal with random differential equations. The study has been performed through the Airy equation, which is expected to be an excellent test model to highlight differences among previous approaches, due to the highly oscillatory behaviour of its solutions in the deterministic case. The examples reveal that Fröbenius and Polynomial Chaos perform better than Monte Carlo simulations since both are more accurate. A major conclusion drawn from the study case performed through the examples is the significant influence of statistical dependence among the random inputs on the variability of the output. As a consequence, the usual hypothesis of statistical independence for the random parameters should be checked carefully in modelling. Furthermore, when dependence is assumed, its numerical value (measured, for example, by the correlation coefficient) must be determined as accurately as possible, since it has been shown that it also influences both the average and the variability of the output.

Notice that in order to conduct the study, we have had to extend the Fröbenius method presented by some of the authors in the previous contribution [7] to the case where random inputs are dependent. Although we have focused the study on the case in which random variables are just Gaussian, notice that Fröbenius approach does not depend on the statistical type of the involved random variables, while tackling statistical dependence with Polynomial Chaos has been carried out through a direct approach based on nice properties of Gaussian random variables. This approach has allowed us to transform the random initial value problem (1) into another having independent Gaussian random variables, which has facilitated the study.

Solving random differential equations mainly consists of computing the average and standard deviation of the solution stochastic process. A major challenge is to determine its statistical distribution. Example 1 shows, by means of a simple but still illustrative scenario, the potentiality of both, Fröbenius and gPC methods, to deal with this issue when other random differential equations appear in modelling. We think that the combined application of the novel theory of copulas [15] and previous methods constitutes a promising approach that will be considered in the forthcoming works.

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## Research Article

# Epidemic Random Network Simulations in a Distributed Computing Environment 

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#### Abstract

We discuss a computational system following the paradigm of distributed computing, which will allow us to simulate the epidemic propagation in random networks with large number of nodes up to one million. This paradigm consists of a server that delivers tasks to be carried out by client computers. When the task is finished, the client sends the obtained results to the server to be stored until all tasks are finished and then ready to be analysed. Finally, we show that this technique allows us to disclose the emergence of seasonal patterns in the respiratory syncytial virus transmission dynamics which do not appear neither in smaller systems nor in continuous systems.


## 1. Introduction

Networks have become a paradigm of paramount importance in the analysis of many complex systems. Recent applications include disciplines as varied as evolutionary biology [1], the structure of on-line social networks [2], transportation and economics networks [3], or neural networks for the storage and retrieval of memories [4]. Social networks have been ascertained from real data and used to study the social pandemics of smoking [5] and obesity [6]. In the field of epidemiology, the debate about targeted or mass vaccination in the control of smallpox has also been addressed within the context of network models [7, 8]. Following this idea, an outburst of interest in cellular automata models [9] for epidemic spread has been generalised among researchers in epidemiology. Among many others, we have seen in the literature some applications to hepatitis B transmission [10], immune system [11], plagues which devastate some crops [12], and HIV pandemic [13]. Most cellular automata models are defined in euclidean lattice substrates with the individuals occupying the sites of the lattice. Nevertheless, in many situations, only the topology of the network is relevant for the epidemic spreading. More refined models using random
networks as the basic substrate have also been considered [1416], where the authors describe algorithms for simulating the spread of diseases in large networks.

Networks also provide an alternative to the traditional differential equations approach inaugurated by the seminal work of Kermack and McKendrick [17], Edelstein-Keshet [18], and Murray [19]. Differential equations are a powerful and well-known mathematical tool for studying the dynamics of any system, and, consequently, it is not surprising that they have dominated the research in epidemiology for many years. Typically, in these models, we consider the fraction of susceptible (S), infected (I), and recovered (R) individuals and propose a compartmental model for the transitions between these states. The resulting model has been widely studied [19, 20] but, albeit it is a good approximation in some cases, it is clear that it cannot be the final word in the epidemiology of any real disease. The continuous approach cannot, by its own nature, distinguish among individuals and, consequently, the effects of age, sex, previous illnesses, and any other parameters influencing the propagation of the epidemic under study are difficult to implement.

In order to avoid some of the fundamental problems of continuous models, network models emerged. A network is
a set of nodes representing individuals. Labels or properties may be assigned to each node, such as age, sex, and state respect to the disease (susceptibility, infection, recovery, latency, etc.). Nodes are connected by ties that represent disease transmission paths. Once the network model and the disease evolution rules are stated, it is possible to simulate the evolution of the disease on the network nodes over time and to study its spread on the population.

The science of networks provides several standard alternatives for implementing the network substrate. The most traditional one is based upon the pioneering work of Bollobas [21], the so-called "random graphs," where connections among the pairs of subjects are created with the same probability. Alternative models are the scale-free networks [22] or the small-world networks of Watts [23]. The small-world phenomenon (i.e., every pair of nodes is connected through a path which crosses a small amount of neighbours) is found in many social networks linked by friendship, collaboration, or other social binds.

Nevertheless, the studies of real social networks are restricted to a relatively small number of individuals usually not larger than 10,000 individuals [5], but pandemics involve large number of people in the range of millions. So, the development of a distributed computing solution for simulating pandemics in very large networks is a necessary challenge to be dealt with in epidemiology.

In this paper, we describe how we tackled the problem by using two computational systems which follow the paradigm of distributed computing that allowed us to estimate the parameters in random network epidemic models, depending on the amount of tasks to be carried out: one of them, Sisifo, designed by us to work in intranets, is simpler, uses less resources, and has a quicker development, implementation, and deployment; the other is the well-known Berkeley Open Architecture for Network Computing (BOINC) platform [24]. The main difference among Sisifo and BOINC concerns the security issues and the possibility of widespread distributed computing in personal computers of clients connected to Internet, in the case of BOINC, in contrast with the limitation of Sisifo to a computer intranet where security issues are not so important. In particular, BOINC implements public-key encryption against virus attack in which we did not consider in Sisifo.

Both computational paradigms consist of a client/server structure where the server delivers tasks to be carried out by client computers, and when the task is finished, the client sends the obtained results to the server to be stored until all tasks are finished and ready to be analysed. In our case, every task consists of a network model simulation with a set of parameters and the results are the number of persons of each age in each disease state at any time instant.

We must also remark that in this problem, as in many similar problems arisen in statistical physics, it is integral to the computational process to evaluate averages over many realizations over a single substrate and also over many substrates. This is particularly interesting in the simulation of nonequilibrium processes such as diffusion and reactions as well as epidemic propagation [25]. In our case, the first average refers to the evolution of the disease in a given
configuration of the random network. However, as we must also consider different network configurations, in order to take into account the variability of the social structure, a second average over these configurations is also necessary. This is the so-called second average.

In order to achieve this, these double average simulations are usually carried out sequentially in a single computer. In this paper, we show two methods on how to parallelize this calculation and perform the averages directly on the server.

The paper is organized as follows. In Section 2, we describe our two proposals for a distributed computing system: the first one is based on intranets, while the second one uses the BOINC platform. Both of them are capable of managing the simulation of random networks up to one million nodes. An application to the simulation of the seasonal respiratory syncytial virus (RSV) pandemic is given in Section 3. In Section 4 we show that seasonality emerges spontaneously as an amplification of intrinsic fluctuations in large networks without any external forcing. The paper ends with some conclusions and a plan for further study in Section 5.

## 2. Distributed Computing

Distributed computing is the use of distributed systems to solve complex computational problems by dividing them into independent tasks (each of them can be done separately in one computer). The partial solutions are returned to the server and joined (somehow) to obtain the complete solution. In most cases, the more computers we have, the faster problems are solved. The only exception could be the situation in which the communications are slow or the number of communications per second necessary to deal with the problem could increase above the computing time in a single processor.

In 1993-94, the message-passing interface standard was developed to tap on the incipient multiple processor capabilities available at the time [26]. In recent years, a different form of distributed computing has become increasingly popular, the so-called desktop grid, that allows to take advantage of the resources of a set of computers (e.g., part-time idle computer rooms in the universities) joining them together into a computing network. One of the most known projects using this platform is SETI, dedicated to search for extraterrestrial life by analysing radio signals [27]. The software that makes possible to develop this type of distributed computing projects is BOINC [24]. However, the BOINC setting up is complex, demands advanced knowledge of system administration, software, and web development in a deployment process that may last several months [28]. BOINC is a solution that does not require a supercomputer, with many processors on the same motherboard, as MPI because it relies on the distributed computing over many independent personal computers connected throught the Internet.
2.1. Sisifo: Our Design Proposal. In the context described in the previous section, we have developed Sisifo [29]. Sisifo


Figure 1: Operation scheme of Sisifo.
is our proposal for a distributed computing system, quite simpler than BOINC, requiring less resources, and providing a significantly quicker development, implementation, and deployment. Although our first objective is to use Sisifo to estimate parameters in epidemic models, of course it can also be used to solve other kinds of problems if they match the requirements for distributed computing. Taking into account that we have previous experience in the development of client/server systems with centralized databases and TCP/IP communication [30], we have used that background to build Sisifo. The operation scheme of Sisifo is shown in Figure 1.

To implement the Sisifo system, we need the following strategy: the problem we want to solve (e.g., simulation of the propagation of an epidemic in different conditions) should be divided into independent tasks, in the sense that they can be computed in a client separately from the others. Tasks are coded in text files and they indicate all necessary data to start the running of a simulation once received by the clients. An example of task for the epidemic model to be discussed in the next section is given in Table 1.

In this file, we detail the number of days of evolution for the epidemic we are simulating, the number of nodes of the network, and the average connectivity degree for the random network as well as the transition probabilities for the susceptible-infected-recovered-susceptible (SIRS) model. Initial conditions for the percentage of susceptibles and infected individuals are also provided.

Each client computer has a Sisifo client installed which requests tasks to the server. The server provides to the client a package containing the solver (the computer program which runs the simulation) and the task to be solved. The task is then marked as pending by the server. Once the client has received the package, it runs the solver with the data provided by the task and waits for the solver to finish. Then, the client takes the obtained results and transmits them back to the server. The

Table 1: An example of task file for the Sisifo project.

| Task_ID | Prueba10372 |
| :--- | :---: |
| RAM_needed | 223672 |
| Simulation_days | 2555 |
| No_realisations | 1 |
| No_nodes | 1000000 |
| Node_average_degree_( $k$ ) | 29 |
| Average_from_infected_to_recovered_(days) | 10 |
| Average_from_recovered_to_susceptible_(days) | 198 |
| Parameter_b | 0.00311 |
| Initial_percentage_of_susceptibles | 0.99 |
| Initial_percentage_of_infected | 0.01 |
| Initial_percentage_of_recovered | 0 |

results are received by the server and stored and the related task is marked as done. Then, the client requests a new task and the cycle starts again until all tasks have been solved.

Taking into account that we use client computers under our supervision, it is not necessary to include in Sisifo the management of users, statistics, and security control which BOINC does.

The minimum server requirements are as follows:
(i) storage of tasks and their state (undone, pending, and done);
(ii) storage of the solver. In fact, two solvers: one for MSWindows and one for Linux;
(iii) storage of the client updates (if required, also for MSWindows and Linux);
(iv) admission of client requests via TCP/IP in a certain (configurable) port. Several Sisifo servers may be installed in the server computer, each one listening to its own port for requests from the clients;
(v) processing clients requests. The client identifies itself providing its version and platform (MS-Windows or Linux) and demands a new task to the server or returns the results to the server;
(vi) answer the clients by, at least,
(a) providing the pair solver+task, taking into account the operating system of the client,
(b) providing an updated version of the client software when the current client version is older than the one stored in the server,
(c) accepting the results at the end of a task execution and informing if the transmission has been completed successfully in order to retransmit in case of errors,
(d) managing the time when the processing task expires. The server marks the transmission date of each package (solver + task), and in case of an excessive delay (configurable) occurs in the resolution reception; this leads the server to mark the task as undone and resend it to another client,
(e) creating a record with solved tasks and time used,
(f) showing in real-time basic information about connected clients, served tasks, and solved tasks,
(g) checking the validity of the results, because otherwise the results should be rejected and sent to be computed again,
(h) verifying data communication integrity, that is, via CRC.

Moreover, the minimum client requirements are as follows:
(i) use the minimum resources as possible,
(ii) create a TCP/IP connection to the server to do requests,
(iii) process the server messages, including
(a) the solver preparation and its execution with the corresponding task,
(b) the verification of a proper execution of the solver and the correct transmission of the results to the server.

Furthermore, the solver does not have special requirements to be executed by Sisifo apart from some conventions with input/output data format.

Note that emphasis has been put in the simplicity and development time because the network is under our supervision. Of course, this development may be not suitable for a wide network of unreliable clients.
2.2. Distributed Computing throughout Internet. Every task corresponds to the evolution of the disease for a given set of parameters for the epidemic model as shown in Table 1 and a given network configuration.

Although the first experiment with Sisifo gave us satisfactory results in a very short period of time, during the computations, a lot of tasks were unsuccessful because with our first set of tasks, the total number of infected individuals became zero, and therefore, nobody could be infected. We needed a deeper search into the solution space to find what parameters were able to produce a network where the disease did not die out. This required processing a much higher number of tasks, but the intranet where Sisifo was running did not have enough power to cope with all computations in a reasonable time and we wondered if we could boost this.

For this reason, we decided to try out BOINC. BOINC is an open source software actively maintained and used by the scientific community to manage projects of distributed computing as the abovementioned SETI@home [31, 32], ROSETA [33], or Climate Prediction [34]. BOINC protects against several types of attacks and the distribution of viruses using digital signatures based on public-key encryption; its server architecture is highly scalable and the core client is available for most common platforms. These are some of its main features that assure the correct transmission and reception of the task results [35].

To do that, we requested the help of the Falua project [36, 37], an initiative supported by the CES Felipe II of Aranjuez (campus of the Universidad Complutense de Madrid) which provides ad hoc BOINC deployment and computing power for small to medium computation problems. They adapted our solver to the BOINC platform and opened the possibility of foreign collaboration from the BOINC community.

With the increased computation power of the many volunteers of the BOINC community, we were able to simulate more than 145000 tasks in a short time. With this number of tasks processed, a reliable picture of the phase diagram of the model in the plane $k-b$, where $k$ is the average degree of connectivity of the network and $b$ is the transmission probability, was obtained. The processing of this large number of tasks is attainable because we have sustained peak performance over 370 Gigaflops in contrast with the 225 Gigaflops (at best and during short periods of time) of the Sisifo project, where only 80 computers were dedicated to the project.

## 3. Epidemic Transmission Dynamics in a Random Network Model

Seasonal pandemics are a serious public health problem. In particular, respiratory infections typically peak in their incidence once a year during the autumn-winter period. These epidemics' effects are particularly acute in young children and the elderly whose immune systems are either underdeveloped or deteriorated.

Among them, the respiratory syncytial virus (RSV) is one of the most severe causes of respiratory disease epidemics in many countries. RSV is a single-stranded RNA virus discovered more than fifty years ago in a child with bronchiolitis [38]. This virus is the cause of a seasonal epidemic in many countries all around the world. Only in Spain, there are around $15,000-20,000$ visits to primary care due to RSV every year. Also, up to $18 \%$ of the pneumonia hospitalisations of individuals older than 65 are due to RSV [38]. This epidemic is also a major concern in immunocompromised patients at any age [38-40].

Its coincidence with other seasonal epidemics such as influenza and rotavirus produces a large number of hospitalisations year after year, saturating the National Health System. In particular, the cost of pediatric hospitalisation for the Health System of the Community of Valencia (Spain) [41] has been estimated to be 3.5 million euros per year [42] without taking into account indirect costs [43], with a cohort of newborns of around 45,000 children.

Epidemiological continuous models dealing with RSV have been proposed in [43-45], for instance. To our knowledge, only a network model of RSV has been studied in [15, 46].

Alternatively, other network substrates could be used as small-world or scale-free networks. However, respiratory infectious diseases are transmitted by random encounters among people in their social environment including public transportation and schools. For these reasons, a random network model seems reasonable and we have found that
it simulates the epidemic propagation at least as efficiently as traditional continuous differential models. On the other hand, scale-free networks only seem realistic for the propagation of sexually transmitted diseases [47, 48], where a few individuals with a large number of connections are the main hubs for the transmission of the disease. Transmission of respiratory infectious diseases is mainly mediated by random close spatial encounters among people and they do not rely heavily on stable social links (family, friends, work, etc.); for this reason, the random network model seems suitable as a first approach.
3.1. Population Model. For a realistic simulation of a disease which affects, with different degrees of severity, all age groups of society, we need to incorporate a reasonable population model. We have retrieved mortality rates for Valencia from the Valencian Institute of Statistics [49] and simulated a Forster-McKendrick dynamic model with constant population [20]. The contribution of the disease itself to death rates is not taken into account because it is marginal in comparison with other causes [50]. This is implemented in the random network by the following algorithm.
(i) Every individual in the network has an age assigned to him/her following the Forster-McKendrick model. The age is increased in every time-step (our unit of time is 1 day).
(ii) Every time step we check whether the subject dies or survives until the next time-step. This is calculated by generating a pseudorandom number and comparing it with the mortality rate per day corresponding to the age of the subject. This is performed in a daily basis because the natural scale of the problem, that is, the average duration of the disease, is 10 days [15].
(iii) If a subject dies, he or she is replaced by a susceptible newborn. This way, the population remains constant by definition.

A warming-up period is allowed for the population pyramid to stabilize and the epidemic propagation is simulated afterwards. The results for the Autonomous Region of Valencia are shown in Figure 2.
3.2. Random Network. Random networks are characterised by the number of individuals or nodes $N$, usually large (in our case a million nodes), and the average number of contacts of every individual, $k$ (called the degree of this node). Consequently, the number of ties in the network is given by $N \times k / 2$. These links are randomly assigned to pairs of individuals with the obvious rule that, at most, only a tie can connect two individuals. The degrees of the nodes in the resulting random network follow a Poisson distribution with mean $k$. We must take into account that random networks do not capture some structures found in real networks such as degree heterogeneity (different social groups could be connected with different average degrees) or communities (some social groups form connections mainly within the same group). However, a random network provides a first


Figure 2: Percentage of persons with a given age in years obtained from the Foerster-McKendrick model after the warming-up period is performed.
approach to the modelling of epidemic transmission in complex systems and it constitutes a good starting point for testing simulation techniques.

An initial state in which $1 \%$ of the individuals are infected is chosen as the initial state ( $99 \%$ of the individuals are susceptible and no recovered individuals are found at the beginning). The age of the nodes is chosen from the warmedup population pyramid in Figure 2. Then, the evolution algorithm of the RSV model proceeds as follows.
(1) Infected individuals recover following an exponential distribution of mean lâĄĎ10, because 10 days is the average time to recover from the illness.
(2) Recovered individuals become susceptible again following an exponential distribution of mean $\gamma$ per time step, where $1 / \gamma$ days is the average time an individual remains immune against re-infection. Weber et al. proposed a model in which $\gamma=1 / 200$ [44], however we will find a slightly different value after fitting hospitalization data for RSV in Valencia. In order to perform this fitting we will use the BOINC implementation of the model.
(3) The main difference with respect to the standard continuous model is found in the infection procedure: every susceptible individual can only be infected by infected individuals connected through existing ties with him or her. This occurs with a probability $b$ per time step (one day in our case) in each contact, that is, every infected individual can infect a neighbouring susceptible individual with probability $b$ per unit time. To simulate this process we select a random number, $x$, for every link connecting an infected with a susceptible individual. If $x<b$ the susceptible individual becomes infected. The same node cannot be infected twice, so if it becomes infected we stop the checking algorithm. The network is stored by means of the efficient algorithm of adjacent lists typically
used in graph theory [51]. By using this algorithm the links are stored in a list of $(k+2) N / 2$ integers.
(4) The age of the individuals is increased in one week on every time-step, that is, our time-step is one week because the duration of the infectious disease (the average time before recovering from the infection) is 10 days and we need a sufficiently small time-step for the model to follow the recovery process.

Then, for each node, for every day, we simulate the disease evolution following the above rules, obtaining the number of susceptible, infected, and recovered for all days at any age. By drawing these data, we can visualise the behaviour of the disease spread over time.

The third step of the algorithm, corresponding to the propagation of the infection, is extremely time consuming: the computer program must check the evolution of the state of every individual on a very large set by analysing the propagation of the infection through the ties with every infected individual in its neighbourhood. The use of adjacent lists allows us to efficiently store and identify the neighbourhood of a site.

Note that parameters $k$ and $b$ should be estimated by fitting the random network model with real data. The above description of the model leads to the fitting of two parameters, but note that $b$ may depend on age, gender, or other situations, in which case the number of parameters will increase.

In order to estimate the unknown parameters, we should choose a feasible parameter space and build a mesh, where, for each mesh point $(k, b)$, a simulation, following the disease rules described above, should be carried out and the results have to be compared to determine its closeness to the real data in the least square sense. The number of simulations depends on the mesh-point size, but in order to explore sufficiently the chosen parameter space, this number must be necessarily high, and here is where the computing distributed environment can help us to obtain the best $k$ and $b$.
3.3. An Experiment with Sisifo. Two part-time people have developed, implemented, and tested the client and the server in less than three weeks using open source tools. The server has been installed in a personal computer Pentium IV with 1 GB of RAM.

The solver has to be developed specifically for each experiment. For this one in particular, we considered an SIRS random network model applied to the transmission dynamics RSV. The solver implementation for RSV in order to estimate the parameters $b$ (transmission rate) and $k$ (average node degree) has taken ten weeks. Then, we prepared 60120 tasks for
(i) a million nodes, divided into age groups following a Foster-McKendrick constant demographic model as described in Section 3.1. In this model individuals within the same year of age constitute an age-group,
(ii) each node that is labelled by age and state respect to RSV,
(iii) a Poisson nodes distribution with mean $k=5,6, \ldots$, 124,
(iv) a transmission rate $b$ from 0 to $1 / 200$ with steps of $10^{-5}$,
(v) an average time of infection of 10 days,
(vi) an average time of immunity after the infection of 200 days according to Weber et al. [44],
(vii) an initial situation where all the population is susceptible, but for $1 \%$ of infected individuals.

After that, we started the computing process, installing Sisifo server with all the tasks and solver in the server PC. The number of simultaneous clients has been varied from 10 (usually) up to 150 (during week-ends). Performance achieved the equivalent of more than three years of computing time in just five weeks. To calculate this performance, we must notice that the 60120 tasks lasted an average of 25 minutes each on every Dual Core 3 GHz computer.

## 4. BOINC: Continuing the Experiment

The 145099 tasks were generated by combining $k$ (in the range 25-60), the transmission probability in a person-toperson contact ( $0.00216 \leq b \leq 0.005208$ with $10^{-5}$ jumps), and the average immunity time after infection, 80 days $\leq$ $1 / \gamma \leq 200$ days, with jumps of 3 days. We wanted to consider an interval range for $\gamma$ because there is no consensus on what is the good value for the recovering rate [43, 44]. These important data for the epidemiology of RSV can be derived from our model fitting to the number of hospitalisations due to RSV infection [42, 43].

In this case, we restricted the intervals for $b$ and $k$ because our main goal is to find the best solution fitting hospitalisation data, varying also the average immunity time after infection, $\gamma$. From the experiments with Sisifo, we know that the best solutions always lie in this region. In order to achieve a more detailed exploration of this region, we also increased the number of tasks. Each task corresponds to a different set of parameters of the model as explained in Section 2.1. For each set of parameters, we perform a simulation over 10 different realizations of the random network.

Each task needed an average of 80 minutes for completion and this would have needed more than twenty years of computing time in a single computer dedicated to the task, but using the BOINC distributed computing, we only needed 3 weeks. When the BOINC project was made available to the international community, we got an average of 800 computers connected on peak performance. This made 8500 successful tasks completed each day.

In Figure 3, we have plotted the number of tasks that were carried out successfully each day in the first phase of the BOINC experiment.

The best fit was obtained for the following values: $b=$ 0.00267 (a mean of 267 per 100000 successful contacts per day), $k=54$ (a mean of 54 contacts, successful or not, per individual per day), and $s=4.979$ (4.979\% of the weekly infected children under one year old are hospitalized). The immunity period was found to be $1 / \gamma=188$ days (very close to the prediction of continuous models [44]); this value was obtained as a consequence of the best fit of hospitalization


Figure 3: The number of successful tasks completed versus time (in days) for the first phase of the BOINC experiment.


Figure 4: The number of hospitalisations of children under 1 year of age in the Spanish region of Valencia: real data (dotted line) and fitting corresponding to the random network model (solid line). The period of time goes from January 2001 to December 2004.
data in contrast to previous works, where $\gamma$ was a fixed parameter. The mean quadratic error ( $\mathrm{MSE}=11.38$ ) is even better than the one obtained by fitting the continuous models [43]. The fitting is depicted in Figure 4. Notice that the parameter $s$, the fraction of infected children under one year old who become hospitalised as a consequence of RSV infection, is also a fitting parameter. We considered a fraction $s$ of the children aged one year old or less and compared it with the real data to find the optimum result.

The interesting fact about the results depicted in Figure 4 is that the oscillatory behaviour was obtained without resorting to external forcing as usually considered in continuous models [43-45]. Further analysis of the phase diagram of the random network model has already been given [46]. However, it is difficult to attribute the origin of the seasonal behaviour to purely intrinsic dynamics. This is an open problem and many authors believe that the seasonal outbreaks are


Figure 5: The probability for the emergence of a seasonal pattern in a network with $N$ nodes and the parameters for RSV propagation fitted to the Spanish region of Valencia data. Circles are simulation data; the continuous line is an exponential fit.
provoked by a slight change in the transmission rate correlated with atmospheric factors [52, 53], although Dushoff et al. have also suggested that dynamical resonance can account for seasonal behaviour of respiratory pandemics (including influenza) without any detectable change in the transmission rate [54]. We have shown that without a seasonal external forcing the model can fit the data for hospitalisations of children under 1 year of age.

Our distributed computing solutions allow us to address another important epidemiological problem. It is useful to determine the size of the population capable of sustaining the seasonal behaviour for a long period of time. By using the parameters corresponding to the disease as fitted in Figure 4, we have reduced the population size and calculated the probability that the seasonal behaviour emerges. Otherwise, the epidemic becomes extinct. We consider several population sizes: $100000,250000,400000,550000,700000$, 850000 , and 1000000 . For every population size, we build 100 random networks with the parameters corresponding to the fitting of the seasonal epidemic and observe the emergence of oscillatory behaviour in any of these realizations. The fraction of realizations in which a seasonal pattern is observed gives us the probability $P(N)$. Results are plotted in Figure 5.

A good fitting is found by the simple exponential function which we propose heuristically:

$$
\begin{equation*}
P(N)=1-e^{-N / M} \tag{1}
\end{equation*}
$$

where $M$ depends on the parameters of the model, but we have not considered this dependence because we are interested only in a particular set of parameters, that is, those corresponding to the fitting of the seasonal epidemic. In this particular case $M=138480$, so we can conclude that a population similar to 1000000 nodes is necessary to sustain the oscillatory behaviour. This result presupposes an ideally isolated population and does not take into account the fact that the virus can be reintroduced after extinction by migratory displacement of nodes.

## 5. Conclusion

In this paper, we have discussed two computing distributed techniques in order to accelerate the simulation of epidemic models in very large random networks. The first one is an intranet solution (Sisifo) which is simpler to implement and to operate, while the second one is based on the BOINC software and has the advantage of being distributed throughout the whole Internet with the help of volunteers.

Both of them have their advantages and disadvantages and should be chosen depending on the problem and the availability of resources. Sisifo is an intranet solution for distributed computing independently developed for this project, where its main advantages are its fast deployment and easy management. Being an intranet solution, we can control every computer in the project and avoid the complications of security issues. However, we must have a large number of sufficiently efficient computers at our disposal. BOINC allows us to distribute the computation throughout the whole internet, but its deployment is far more complicated and the resources availability is highly variable. For this reason, Sisifo could be a useful solution in many problems which do not require so large computational capabilities as those attained with BOINC. We have shown this by studying the fitting problem for our epidemiological model entirely with Sisifo, using only BOINC for a more accurate calculation involving an additional fitting parameter, $\gamma$, the probability, per unit time, for a recovered individual to become susceptible again.

By using this distributed architecture, we have been able to tackle the problem of double statistical average in networks: we consider a set of different random networks characterized by the connectivity, $k$, and for each of them, a simulation of the propagation of the epidemic is performed. Then, an average over the propagation of the disease in a random network with a given $k$ can be given. This is a fundamental statistical problem in many models arisen in statistical physics and our solution provides an efficient way to implement a computational solution even for very large systems. In order to test its efficiency, we have considered epidemic propagation in random networks as an example.

Random networks provide an alternative to continuous differential models in the simulation of disease transmission dynamics. However, they demand a high amount of computations in parameter estimation, simulations, and so forth.

To test the feasibility of this approach, we performed an experiment in a random network of a million nodes in order to simulate the spread of RSV in the region of Valencia (Spain). Then, computations were carried out to estimate the RSV parameters $b$, to determine the probability of successful transmission contacts, $k$, and to determine the average number of individual daily contacts, and $\gamma$, the immunity period after infection, as well as the fraction of infected children under one year of age who became hospitalised, $s$, in a matter of a few weeks by using a distributed computing network, instead of requiring several years for a single PC.

In these simulations, we have shown the emergence of seasonal patterns in the incidence of infections which do not require any external forcing. This could provide a social network explanation to seasonal patterns without the need
of invoking weather effects. Moreover, we have shown that a population around one million persons is required for the RSV pandemic not to become extinct in the long run. This result was obtained by considering 100 realizations of the random network for the epidemic model parameters corresponding to the fitting solution of the real hospitalisation data. The stationary state achieved for an initial $1 \%$ of infected individuals could be extinction or seasonal behaviour, the fraction $P(N)$ of realizations in which seasonal behaviour is observed to rise progressively from a $50 \%$ around $N=$ 150000 to $100 \%$ at $N=1000000$.

Now, once the model has been parametrised, we are ready to design specific or targeted vaccination or prophylactic strategies and evaluate their effectiveness.

The next step in our research will consist of the inclusion of an optimization system using genetic algorithms (optimized for the distributed computing environment), for instance, in order to reduce the amount of computations so as to get the optimum and then be capable of facing the fitting of network models with more than two or three parameters to be estimated, as well as to increase the number of nodes. The simulation method discussed in this paper can also be applied to many other similar problems. Currently, we are considering meningococcal C disease spread and vaccination strategies as well as a network model of the brain [55]. Migration and diffusion can also be considered to take into account spatial dynamics effects [56]. Results for these projects will be published elsewhere.

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## Research Article

# Analytic-Numerical Solution of Random Boundary Value Heat Problems in a Semi-Infinite Bar 

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#### Abstract

This paper deals with the analytic-numerical solution of random heat problems for the temperature distribution in a semi-infinite bar with different boundary value conditions. We apply a random Fourier sine and cosine transform mean square approach. Random operational mean square calculus is developed for the introduced transforms. Using previous results about random ordinary differential equations, a closed form solution stochastic process is firstly obtained. Then, expectation and variance are computed. Illustrative numerical examples are included.


## 1. Introduction

Temperature and heat flow are two important quantities in the problem of heat conduction. Temperature at any point of a solid is completely determined by its numerical value because it is a scalar quantity, whereas heat flow is defined by its value and direction, and it depends on the properties of the material that are neither uniform nor pure [1]. These uncertainties about material apart from measurement error deserve a random approach. Random heat transfer in a finite medium has been treated in [2] by developing a random perturbation method, in [3] by using finite element method, and in $[4,5]$ using finite difference methods.

It is well known that the integral transform technique to solve deterministic heat problems is very powerful and efficient because, by the combined use of the direct and inverse transform, the problem is simplified, by transforming partial differential equations into ordinary differential ones. In this paper, we introduce the random Fourier sine and cosine transforms and their mean square operational calculus to solve random temperature distribution in a semi-infinite bar with random temperature or random heat flow at one end. Using results about random ordinary differential equations of [6] allows us to find closed-form solutions stochastic process (s.p.) of random heat problems as in the deterministic case.

Although some authors deal with the uncertainty by using the Brownian motion and the Itô calculus [7, 8], we
use a random mean square approach (m.s.) for two main reasons. The first one is that the m.s. solution coincides with the deterministic solution when data are deterministic. The second one is that if $V(x, t)$ is an m.s. approximation of $u(x, t)$, then the expectation and the variance of $V(x, t)$ converge to the expectation and variance of $u(x, t)$, respectively; see [ 9 , chapter 4].

This paper is organized as follows. In Section 2, some preliminaries about m.s. calculus definitions, properties, and results are included. In Section 3, the random Fourier sine and cosine transforms are introduced and important m.s. operational rules related to the random Fourier sine (cosine) transform of a s.p. and of its m.s. derivatives, that is, the key to solve partial differential equations in terms of ordinary differential equations, are given. Sections 4 and 5 deal with random boundary value problems associated with the random heat equation

$$
\begin{equation*}
u_{t}(x, t)=L u_{x x}(x, t), \quad x>0, t>0, \tag{1}
\end{equation*}
$$

where $L$ is a positive random variable (r.v.) whose properties will be specified later. For the deterministic case, the boundary conditions of these problems are called of third and second kind, respectively [1, page 52]. In Section 6, some illustrative examples are studied and in Section 7 a set of conclusions are given.

## 2. Preliminaries about Random Mean Square Calculus

In this section, we review some important concepts, definitions, and results related to the random $L_{p}$ calculus, mainly focusing on the mean square (m.s.) and mean fourth (m.f.) calculus, which correspond to $p=2$ and $p=4$, respectively (see [10] for further details). After that, a relevant class of r.v.'s that will play an important role in the development of next sections is studied.

Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probabilistic space. Let $p \geq 1$ be a real number. A real r.v. $U$ defined on $(\Omega, \mathscr{F}, \mathscr{P})$ is called of order $p$, if

$$
\begin{equation*}
\mathrm{E}\left[|U|^{p}\right]<+\infty, \tag{2}
\end{equation*}
$$

where E[•] denotes the expectation operator. The space $L_{p}$ of all the real r.v.s of order $p$, endowed with the norm

$$
\begin{equation*}
\|U\|_{p}=\left(\mathrm{E}\left[|U|^{p}\right]\right)^{1 / p} \tag{3}
\end{equation*}
$$

is a Banach space, [11, page 9].
Let $\left\{U_{n}: n \geq 0\right\}$ be a sequence of r.v.s of order $p$. We say that it is convergent in the $p$ th mean to the real r.v. $U \in L_{p}$, if

$$
\begin{equation*}
\lim _{n \rightarrow+\infty}\left\|U_{n}-U\right\|_{p}=0 \tag{4}
\end{equation*}
$$

This type of convergence is often expressed by $\operatorname{li.im}_{n \rightarrow \infty} U_{n}=$ $U$. The symbol l.i.m. denotes the limit in the $p$ th mean. If $p_{2} \geq p_{1}$, then $L_{p_{2}} \subseteq L_{p_{1}}$. In addition, if $\left\{U_{n}: n \geq 0\right\}$ is $p_{2}$ th mean convergent to $U \in L_{p_{2}}$, then $\left\{U_{n}: n \geq 0\right\}$ is also $p_{1}$ th mean convergent to $U \in L_{p_{1}}$ [11, page 13]. Convergences in $L_{2}$ and $L_{4}$ are usually referred to as m.s. and m.f. convergence, respectively. If $\left\{U_{n}: n \geq 0\right\}$ is a sequence of 2-r.v.s in $L_{2}$ m.s. convergent to $U \in L_{2}$, then from Theorem 4.3.1 of [9, page 88] one gets

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathrm{E}\left[U_{n}\right]=\mathrm{E}[U], \quad \lim _{n \rightarrow \infty} \operatorname{Var}\left[U_{n}\right]=\operatorname{Var}[U] \tag{5}
\end{equation*}
$$

where $\operatorname{Var}[\cdot]$ denotes the variance operator.
Let $T$ be a subset of the real line. A family $\{U(t): t \in T\}$ of real r.v.s of order $p$ is said to be a s.p. of order $p$ or, in short, a $p$-s.p. if

$$
\begin{equation*}
\mathrm{E}\left[|U(t)|^{p}\right]<+\infty, \quad \forall t \in T \tag{6}
\end{equation*}
$$

We say that $\{U(t): t \in T\}$ is $p$ th mean continuous at $t \in T$, if

$$
\begin{equation*}
\|U(t+h)-U(t)\|_{p} \longrightarrow 0 \quad \text { as } h \longrightarrow 0, t, t+h \in T \tag{7}
\end{equation*}
$$

Furthermore, if there exists a s.p. $U^{\prime}(t)$ of order $p$, such that

$$
\begin{array}{r}
\left\|\frac{U(t+h)-U(t)}{h}-U^{\prime}(t)\right\|_{p} \longrightarrow 0  \tag{8}\\
\text { as } h \longrightarrow 0, t, t+h \in T
\end{array}
$$

then we say that $\{U(t): t \in T\}$ is $p$ th mean differentiable at $t \in T$ and $U^{\prime}(t)$ is the $p$-derivative of $U(t)$.

In the particular cases that $p=2,4$, aforementioned definitions lead to the corresponding concepts of mean square (m.s.) and mean fourth (m.f.) continuity and differentiability. Furthermore, it is easy to establish by the Schwarz's inequality that $\|U V\|_{2} \leq\|U\|_{4}\|V\|_{4}$ (see [10]), which prove that m.f. continuity and differentiability entail m.s. continuity and differentiability, respectively.

In accordance with [9, page 99], [12], we say that a s.p. $\{V(x): x \in \mathbb{R}\}$ with $V(x) \in L_{p}$ for all $x$, is locally integrable in $\mathbb{R}$ if, for all finite interval $[a, b] \subset \mathbb{R}$, the integral

$$
\begin{equation*}
\int_{a}^{b} V(x) d x \tag{9}
\end{equation*}
$$

exits in $L_{p}$. We say that $\{V(x): x \in \mathbb{R}\}$ is absolutely integrable in $L_{p}$, if

$$
\begin{equation*}
\int_{-\infty}^{+\infty}\|V(x)\|_{p} d x<+\infty \tag{10}
\end{equation*}
$$

Now, we introduce an important type of r.v.s, $L$, that have played a significant role in the m.s. solution of random ordinary differential equations (see [13] and references therein), and which will be used later. We will assume that such r.v.'s $L$ have absolute moments with respect to the origin that increases at the most exponentially; that is, there exist a nonnegative integer $n_{0}$ and positive constants $M$ and $H$ such that

$$
\begin{equation*}
\mathrm{E}\left[|L|^{n}\right] \leq M H^{n}, \quad \forall n \geq n_{0}, \quad \text { that is, } \mathrm{E}\left[|L|^{n}\right]=\mathcal{O}\left(H^{n}\right) . \tag{11}
\end{equation*}
$$

From (11) and definition (3) for $p=2$, for each $x \in \mathbb{R}$, one gets

$$
\begin{align*}
& \left(\left\|e^{-L x^{2}}\right\|_{2}\right)^{2}=\mathrm{E}\left[e^{-2 L x^{2}}\right]=\mathrm{E}\left[\sum_{n \geq 0} \frac{(-2)^{n} L^{n} x^{2 n}}{n!}\right] \\
& =\sum_{n \geq 0} \frac{(-2)^{n} x^{2 n} \mathrm{E}\left[L^{n}\right]}{n!} \\
& \leq \sum_{n \geq 0} \frac{(-2)^{n} x^{2 n} \mathrm{E}\left[|L|^{n}\right]}{n!} \\
& \leq M \sum_{n \geq 0} \frac{(-2)^{n} x^{2 n} H^{n}}{n!}=M e^{-2 H x^{2}}, \quad M>0 . \tag{12}
\end{align*}
$$

Thus,

$$
\begin{equation*}
\left\|e^{-L x^{2}}\right\|_{2} \leq \widetilde{M} e^{-H x^{2}}, \quad x \in \mathbb{R}, \widetilde{M}=\sqrt{M}>0 \tag{13}
\end{equation*}
$$

Remark 1. The lack of explicit formulae for the absolute moments with respect to the origin of some standard r.v.s as well as the aim of looking for a general approach to deal with the widest range of random inputs, we are going to take advantage of censuring method (see [14, chapter 5]) to show that truncated r.v.'s satisfy condition (11). Let us assume a r.v. $L$ that satisfies

$$
\begin{equation*}
l_{1} \leq l=L(\omega) \leq l_{2}, \quad \forall \omega \in \Omega \tag{14}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\mathrm{E}\left[|L|^{n}\right]=\int_{l_{1}}^{l_{2}}|l|^{n} f_{L}(l) d l \leq H^{n} \tag{15}
\end{equation*}
$$

where $f_{L}(l)$ denotes the probability density function (p.d.f.) of r.v. $L$ and $H=\max \left(\left|l_{1}\right|,\left|l_{2}\right|\right)$. Indeed, in the case that $H>1$, one gets

$$
\begin{equation*}
\int_{l_{1}}^{l_{2}}|l|^{n} f_{L}(l) d l \leq H^{n} \int_{l_{1}}^{l_{2}} f_{L}(l) d l=H^{n} \tag{16}
\end{equation*}
$$

Notice that, in the last step, we have applied that the integral of the right-hand side is just 1 because $f_{L}(l)$ is a p.d.f. The other cases can be analyzed analogously. Substituting the integral by a sum in (15), previous reasoning remains true when $L$ is a discrete r.v. As a consequence, important r.v.s such as binomial, hypergeometric, uniform, or beta satisfy condition (11) related to the absolute moments of $L$. Although many other unbounded r.v.'s can also verify condition (11), we do not need to check it each case; since censuring their codomain suitably, we are legitimated to approximate them. Hence, truncations of r.v.'s such as exponential or Gaussian satisfy condition (11). The larger the censured interval is, the better the approximations are. However, in practice, intervals relatively short provide very good approximations. For instance, as an illustrative example, notice that the truncated interval $[\mu-3 \sigma, \mu+3 \sigma$ ] contains the $99.7 \%$ of the probability mass of a Gaussian r.v. with mean $\mu$ and standard deviation $\sigma>0$.

For the sake of clarity in the presentation, we state the m.s. differentiation of integrals whose proof is a direct consequence of the deterministic case [12, page 99] and the $\mathrm{m} . \mathrm{s}$. differentiation theorem for a sequence of 2-s.p. [15].

Lemma 2 (m.s. differentiation of infinite integrals). Let $g(x, t)$ be a 2-s.p. m.s. continuous with m.s. continuous partial derivative $\partial g(x, t) / \partial t$. Assume the following hypotheses.
(i) $G(t)=\int_{a}^{+\infty} g(x, t) d x$ is m.s. pointwise convergent for each $t>0$.
(ii) $\int_{a}^{+\infty}(\partial g(x, t) / \partial t) d x$ is m.s. uniformly convergent in $[t-\delta, t+\delta], \delta>0$ for each $t>0$.

Then, the process $G(t)$ is m.s. differentiable, and

$$
\begin{equation*}
G^{\prime}(t)=\int_{a}^{+\infty} \frac{\partial g(x, t)}{\partial t} d x \tag{17}
\end{equation*}
$$

## 3. Random Fourier Sine and Cosine Transforms' Operational Calculus

We begin this section by introducing the definition of the random Fourier sine and cosine transforms of a 2-s.p. $\{u(x)$ : $x>0\}$ m.s. locally integrable, and m.s. absolutely integrable; that is,

$$
\begin{equation*}
\int_{0}^{\infty}\|u(x)\|_{2} d x<+\infty \tag{18}
\end{equation*}
$$

as the s.p.'s

$$
\begin{array}{ll}
\mathfrak{F}_{\mathfrak{s}}[u(x)](\xi)=\int_{0}^{\infty} u(x) \sin (\xi x) d x, & \xi>0, \\
\mathfrak{F}_{\mathfrak{c}}[u(x)](\xi)=\int_{0}^{\infty} u(x) \cos (\xi x) d x, & \xi>0, \tag{20}
\end{array}
$$

respectively. Note that from (18) both integrals appearing in (19) and (20) are convergent in $L_{2}$ and thus they are 2-s.p.s well defined.

Following the ideas of the deterministic inverse Fourier sine and cosine transforms, [16, chapter 2], we define the random inverse Fourier sine and cosine transforms of a 2s.p. $F(\xi)$ m.s. locally and m.s. absolutely integrable by the formulae

$$
\begin{array}{ll}
\mathfrak{F}_{\mathfrak{B}}^{-1}[F(\xi)](x)=\frac{2}{\pi} \int_{0}^{\infty} F(\xi) \sin (\xi x) d \xi, & x>0 \\
\mathfrak{F}_{\mathfrak{c}}^{-1}[F(\xi)](x)=\frac{2}{\pi} \int_{0}^{\infty} F(\xi) \cos (\xi x) d \xi, & x>0 \tag{22}
\end{array}
$$

respectively.
The following result contains some m.s. operational rules that will be used in Sections 4 and 5 to solve random heat problems in a semi-infinite medium.

Theorem 3. Let $\{u(x): x>0\}$ be a 2-s.p. twice m.s. differentiable with $u^{\prime \prime}(x)$ m.s. locally integrable, and with $u(x)$, $u^{\prime}(x)$, and $u^{\prime \prime}(x)$ m.s. absolutely integrable in $[0, \infty[$. Then,
(i) $\mathfrak{F}_{\mathfrak{z}}\left[u^{\prime}(x)\right](\xi)=-\xi \mathfrak{F}_{\mathfrak{c}}[u(x)](\xi)$,
(ii) $\mathfrak{F}_{\mathfrak{c}}\left[u^{\prime}(x)\right](\xi)=-u(0)+\xi \mathfrak{F}_{\mathfrak{\xi}}[u(x)](\xi)$,
(iii) $\mathfrak{F}_{\mathfrak{3}}\left[u^{\prime \prime}(x)\right](\xi)=u(0) \xi-\xi^{2} \mathfrak{F}_{\mathfrak{\xi}}[u(x)](\xi)$,

$$
\begin{equation*}
\xi>0 \tag{25}
\end{equation*}
$$

(iv) $\mathfrak{F}_{c}\left[u^{\prime \prime}(x)\right](\xi)=-u^{\prime}(0)-\xi^{2} \mathfrak{F}_{c}[u(x)](\xi)$,

$$
\begin{equation*}
\xi>0 \tag{26}
\end{equation*}
$$

Proof. We present the proof of each formula separately.
(i) By the rule for the m.s. derivative of a product of a 2s.p. by a deterministic function, [9, page 96], it follows that

$$
\begin{equation*}
(u(x) \sin (\xi x))^{\prime}=u^{\prime}(x) \sin (\xi x)+\xi u(x) \cos (\xi x) \tag{27}
\end{equation*}
$$

or
$u^{\prime}(x) \sin (\xi x)=(u(x) \sin (\xi x))^{\prime}-\xi u(x) \cos (\xi x)$.

From definitions (19) and (28), one gets

$$
\begin{align*}
\mathfrak{F}_{\mathfrak{3}}\left[u^{\prime}(x)\right](\xi)= & \int_{0}^{\infty} u^{\prime}(x) \sin (\xi x) d x \\
= & \int_{0}^{\infty}(u(x) \sin (\xi x))^{\prime} d x  \tag{29}\\
& -\xi \int_{0}^{\infty} u(x) \cos (\xi x) d x .
\end{align*}
$$

From the fundamental theorem of the m.s. calculus [9, page 104], we have

$$
\begin{equation*}
\int_{0}^{R} u^{\prime}(x) d x=u(R)-u(0) \tag{30}
\end{equation*}
$$

From (30), as $u^{\prime}(x)$ is m.s. absolutely integrable, it follows that

$$
\begin{align*}
\int_{0}^{\infty} u^{\prime}(x) d x & =\underset{R \rightarrow \infty}{\operatorname{li.m.}} \int_{0}^{R} u^{\prime}(x) d x  \tag{31}\\
& =(\underset{R \rightarrow \infty}{\operatorname{li.m.} .} u(R))-u(0) .
\end{align*}
$$

Hence, the limit $\ell=$ l.i.m. ${ }_{R \rightarrow \infty} u(R)$ is finite. Furthermore, as $u(x)$ is m.s. absolutely integrable, by the Cauchy condition of the integral

$$
\begin{equation*}
\int_{0}^{\infty}\|u(x)\|_{2} d x \tag{32}
\end{equation*}
$$

one gets that

$$
\begin{equation*}
\underset{R \rightarrow \infty}{\operatorname{li.m.m.~}_{\infty}} u(R)=0 \tag{33}
\end{equation*}
$$

By the fundamental theorem of the m.s. calculus [9, page 104], we also have

$$
\begin{equation*}
\int_{0}^{R}(u(x) \sin (\xi x))^{\prime} d x=u(R) \sin (\xi R), \quad \xi>0 \tag{34}
\end{equation*}
$$

From (33) and (34) one gets that

$$
\begin{align*}
\int_{0}^{\infty} & (u(x) \sin (\xi x))^{\prime} d x \\
\quad & \operatorname{li.i.m.~}_{R \rightarrow \infty} \int_{0}^{R}(u(x) \sin (\xi x))^{\prime} d x=0 \tag{35}
\end{align*}
$$

Finally, from (29) and (35) and definition (20), one gets the proof of (i).
(ii) By the rule for the m.s. derivative of a product of a 2s.p. by a deterministic function [9, page 96] we have

$$
\begin{equation*}
(u(x) \cos (\xi x))^{\prime}=u^{\prime}(x) \cos (\xi x)-\xi u(x) \sin (\xi x) \tag{36}
\end{equation*}
$$

or
$u^{\prime}(x) \cos (\xi x)=(u(x) \cos (\xi x))^{\prime}+\xi u(x) \sin (\xi x)$.

Now, by definition (20) applied to $u^{\prime}(x)$ and (37), it follows that

$$
\begin{align*}
\mathfrak{F}_{\mathrm{c}}\left[u^{\prime}(x)\right](\xi)= & \int_{0}^{\infty} u^{\prime}(x) \cos (\xi x) d x \\
= & \int_{0}^{\infty}(u(x) \cos (\xi x))^{\prime} d x  \tag{38}\\
& +\xi \int_{0}^{\infty} u(x) \sin (\xi x) d x
\end{align*}
$$

By the fundamental theorem of the m.s. calculus [9, page 104],
$\int_{0}^{\infty}(u(x) \cos (\xi x))^{\prime} d x=\underset{R \rightarrow \infty}{\operatorname{li.m.} .}\{u(R) \cos (\xi R)-u(0)\}$.

As in the proof of part (i), l.i.m. ${ }_{R \rightarrow \infty} u(R)=0$, and thus from (39) one gets

$$
\begin{equation*}
\int_{0}^{\infty}(u(x) \cos (\xi x))^{\prime} d x=-u(0) \tag{40}
\end{equation*}
$$

From (38) and (40), one gets (24).
(iii) By applying part (i) to $u^{\prime}(x)$, it follows directly (25).
(iv) It is a direct consequence of the application of part (ii) to $u^{\prime}(x)$.

## 4. Random Heat Problem with Third Kind Boundary Condition

In this section, we deal with the random heat problem for the temperature distribution $u(x, t)$ in a semi-infinite bar with random temperature at the end $x=0$ and zero initial temperature

$$
\begin{gather*}
u_{t}(x, t)=L u_{x x}(x, t), \quad x>0, t>0,  \tag{41}\\
u(0, t)=A, \quad t>0  \tag{42}\\
u(x, 0)=0, \quad x>0 \tag{43}
\end{gather*}
$$

where $L$ and $A$ both are independent positive 4-r.v.s, satisfying properties to be specified later. Assume that problem (41)-(43) admits a solution 2-s.p. $u(x, t)$ m.s. locally and m.s. absolutely integrable, and let us denote

$$
\begin{equation*}
\mathfrak{F}_{\mathfrak{z}}[u(\cdot, t)](\xi)=\mathscr{U}(t)(\xi), \quad \xi>0, \tag{44}
\end{equation*}
$$

what means that $u(x, t)$ is regarded as a process of the active variable $x$, for fixed $t>0$. By applying the random Fourier sine transform to both members of (41) and properties of Theorem 3, it follows that

$$
\begin{align*}
\mathfrak{F}_{\mathfrak{3}}\left[u_{x x}(\cdot, t)\right](\xi) & =\xi u(0, t)-\xi^{2} \mathfrak{F}_{\mathfrak{3}}[u(\cdot, t)](\xi)  \tag{45}\\
& =A \xi-\xi^{2} \mathscr{U}(t)(\xi),
\end{align*}
$$

and from Lemma 2,

$$
\begin{equation*}
\mathfrak{F}_{\mathfrak{3}}\left[u_{t}(\cdot, t)\right](\xi)=\frac{d}{d t}(\mathscr{U}(t)(\xi)) . \tag{46}
\end{equation*}
$$

From condition (43), it follows that

$$
\begin{equation*}
\mathfrak{F}_{\mathfrak{z}}[u(\cdot, 0)](\xi)=\mathscr{U}(0)(\xi)=0 . \tag{47}
\end{equation*}
$$

Hence, the transformed problem becomes the following random initial value problem for the variable $t$ :

$$
\begin{gather*}
\frac{d}{d t}(\mathscr{U}(t)(\xi))+L \xi^{2} \mathscr{U}(t)(\xi)=\xi A L, \quad t>0,  \tag{48}\\
\mathscr{U}(0)(\xi)=0
\end{gather*}
$$

Let us assume that

$$
\begin{equation*}
\Phi_{L}(t)=\mathrm{E}\left[e^{t L}\right] \text { is locally bounded about } t=0 \tag{49}
\end{equation*}
$$

being $\Phi_{L}(t)$ the moment generating function of r.v. $L$. Then, by Theorem 8 of [6], the solution s.p. of problem (48) is given by

$$
\begin{equation*}
\mathscr{U}(t)(\xi)=\frac{A}{\xi}-\frac{A}{\xi} e^{-t \xi^{2} L} . \tag{50}
\end{equation*}
$$

By using the random inverse Fourier sine transform (21), one gets

$$
\begin{equation*}
u(x, t)=\mathfrak{F}_{\mathfrak{B}}^{-1}[\mathscr{U}(t)(\xi)]=\frac{2 A}{\pi}\left\{\mathrm{I}_{1}(x)-\mathrm{I}_{2}(x, t)\right\} \tag{51}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathrm{I}_{1}(x)=\int_{0}^{\infty} \frac{\sin (\xi x)}{\xi} d \xi  \tag{52}\\
\mathrm{I}_{2}(x, t)=\int_{0}^{\infty} e^{-t \xi^{2} L} \frac{\sin (\xi x)}{\xi} d \xi .
\end{gather*}
$$

Putting the substitution $\xi x=\tau$, we have

$$
\begin{equation*}
\mathrm{I}_{1}(x)=\int_{0}^{\infty} \frac{\sin (\tau)}{\tau} d \tau=\frac{\pi}{2} \tag{53}
\end{equation*}
$$

and dealing with $\mathrm{I}_{2}(x, t)$, we have

$$
\begin{equation*}
\mathrm{I}_{2}(x, t)=\int_{0}^{\infty} e^{-t\left(\tau^{2} / x^{2}\right) L} \frac{\sin (\tau)}{\tau} d \tau \tag{54}
\end{equation*}
$$

We start paying attention to the derivative with respect to the variable $x$ of the s.p. $\mathrm{I}_{2}(x, t)$. Note that, under condition (11), the property (13) holds:

$$
\begin{equation*}
\left\|e^{-t L \xi^{2}}\right\|_{2} \leq \widetilde{M} e^{-H t \xi^{2}}, \quad t>0 \tag{55}
\end{equation*}
$$

for some positive constants $\widetilde{M}$ and $H$. Hence, it is easy to check that $\mathrm{I}_{2}(x, t)$ is m.s. uniformly convergent in a neighbourhood $\left[x_{0}-\delta, x_{0}+\delta\right], \delta>0$, of each $x_{0}>0, t>0$.

Note also that, from (55), the integral of the derivative with respect to $x$ of the integrand of $\mathrm{I}_{2}(x, t)$ (see (52))

$$
\begin{equation*}
J(x, t)=\int_{0}^{\infty} e^{-t \xi^{2} L} \cos (\xi x) d \xi \tag{56}
\end{equation*}
$$

is m.s. absolutely uniformly convergent in a neighbourhood $\left[x_{0}-\delta, x_{0}+\delta\right], \delta>0$, of each $x_{0}>0, t>0$. Hence, for each $x>0, t>0$, (56) defines a 2-r.v. By Lemma 2, it follows that

$$
\begin{equation*}
\frac{\partial \mathrm{I}_{2}(x, t)}{\partial x}=J(x, t)=\int_{0}^{\infty} e^{-t \xi^{2} L} \cos (\xi x) d \xi \tag{57}
\end{equation*}
$$

By [12, page 61], fixed $x>0, t>0$, each realization of (56), denoted by $J(x, t)(\omega)$, can be computed as follows:

$$
\begin{align*}
J(x, t)(\omega)= & \frac{1}{2} \sqrt{\frac{\pi}{t L(\omega)}} e^{-x^{2} / 4 t L(\omega)}  \tag{58}\\
& x>0, t>0, \omega \in \Omega
\end{align*}
$$

Thus,

$$
\begin{equation*}
J(x, t)=\frac{1}{2} \sqrt{\frac{\pi}{t L}} e^{-x^{2} / 4 t L}, \quad x>0, t>0 \tag{59}
\end{equation*}
$$

By (52) $\mathrm{I}_{2}(0, t)=0$, the fundamental theorem of the m.s. calculus [9, page 104] yields

$$
\begin{align*}
\mathrm{I}_{2}(x, t) & =\mathrm{I}_{2}(0, t)+\int_{0}^{x} \frac{\partial \mathrm{I}_{2}(s, t)}{\partial s} d s \\
& =\frac{1}{2} \sqrt{\frac{\pi}{t L}} \int_{0}^{x} e^{-s^{2} / 4 t L} d s \tag{60}
\end{align*}
$$

From (51), (53), and (60), one gets

$$
\begin{equation*}
u(x, t)=A\left(1-\frac{1}{\sqrt{\pi t L}} \int_{0}^{x} e^{-s^{2} / 4 t L} d s\right) \tag{61}
\end{equation*}
$$

Using the independence of r.v.'s $A$ and $L$, one gets that the expectation and the variance function of the solution s.p. are, respectively,

$$
\mathrm{E}[u(x, t)]=\mathrm{E}[A]\left(1-\frac{1}{\sqrt{\pi t}} \int_{0}^{x} \mathrm{E}\left[\frac{1}{\sqrt{L}} e^{-s^{2} / 4 t L}\right] d s\right),
$$

$$
\begin{equation*}
x>0, t>0, \tag{62}
\end{equation*}
$$

$$
\begin{equation*}
\operatorname{Var}[u(x, t)]=\mathrm{E}\left[(u(x, t))^{2}\right]-(\mathrm{E}[u(x, t)])^{2} \tag{63}
\end{equation*}
$$

where

$$
\begin{align*}
\mathrm{E}\left[(u(x, t))^{2}\right]= & \mathrm{E}\left[A^{2}\right] \\
& \times\left(1-\frac{2}{\sqrt{\pi t}} \int_{0}^{x} \mathrm{E}\left[\frac{1}{\sqrt{L}} e^{-s^{2} / 4 t L}\right] d s+\frac{1}{\pi t}\right. \\
& \left.\times \iint_{0}^{x} \mathrm{E}\left[\frac{1}{L} e^{-\left(s_{1}^{2}+s_{2}^{2}\right) / 4 t L}\right] d s_{1} d s_{2}\right) \tag{64}
\end{align*}
$$

Summarizing, the following result has been established.
Theorem 4. Let us consider the random heat problem given by (41)-(43) where $L$ and A are independent positive 4-r.v.'s. Assume that L also satisfies conditions (11) and (49). Then, the solution stochastic process $u(x, t)$ of this problem is given by (61). In addition, (62)-(64) are closed expressions for its expectation and variance.

## 5. Random Heat Problem with Second Kind Boundary Condition

This section deals with the random heat problem for the temperature distribution $u(x, t)$ in a semi-infinite bar with
zero initial temperature and where the heat flow $u_{x}(0, t)$ at the end $x=0$ is given by s.p. $g(t ; B)$ :

$$
\begin{gather*}
u_{t}(x, t)=L u_{x x}(x, t), \quad x>0, t>0,  \tag{65}\\
u_{x}(0, t)=g(t ; B), \quad t>0,  \tag{66}\\
u(x, 0)=0, \quad x \geq 0 \tag{67}
\end{gather*}
$$

where $L$ and $B$ both are independent 4-r.v.'s and $L$ is positive and satisfies properties (11) and (49). We also assume that $g(t ; B)$ is m.f. continuous. Assuming that problem (65)-(67) admits a solution 2 -s.p. $u(x, t)$ locally and absolutely m.s. integrable, let us denote

$$
\begin{equation*}
\mathfrak{F}_{\mathfrak{c}}[u(\cdot, t)](\xi)=\mathscr{U}(t)(\xi), \quad \xi>0, \tag{68}
\end{equation*}
$$

which means that $u(x, t)$ is regarded as a process of the active variable $x$, for fixed $t>0$. By applying the random Fourier cosine transform to both members of (65) and properties of Theorem 3, one gets

$$
\begin{align*}
\mathfrak{F}_{\mathfrak{c}}\left[u_{x x}(\cdot, t)\right](\xi) & =-u_{x}(0, t)-\xi^{2} \mathfrak{F}_{\mathfrak{c}}[u(\cdot, t)](\xi) \\
& =-g(t ; B)-\xi^{2} \mathscr{U}(t)(\xi) \tag{69}
\end{align*}
$$

In addition, from (68) and (67), one gets

$$
\begin{equation*}
\mathfrak{F}_{\mathfrak{c}}[u(\cdot, 0)](\xi)=\mathscr{U}(0)(\xi)=0 \tag{70}
\end{equation*}
$$

By Lemma 2, we also have

$$
\begin{equation*}
\mathfrak{F}_{\mathrm{c}}\left[u_{t}(\cdot, t)\right](\xi)=\frac{d}{d t}(\mathscr{U}(t)(\xi)) . \tag{71}
\end{equation*}
$$

Thus, the transformed random ordinary initial value problem in the variable $t$ becomes

$$
\begin{gather*}
\frac{d}{d t}(\mathscr{U}(t)(\xi))+L \xi^{2} \mathscr{U}(t)(\xi) \\
=-L g(t ; B), \quad t>0  \tag{72}\\
\mathscr{U}(0)(\xi)=0
\end{gather*}
$$

By using Theorem 8 of [6], the solution of (72) is given by

$$
\begin{equation*}
\mathscr{U}(t)(\xi)=-L e^{-t \xi^{2} L} \int_{0}^{t} g(s ; B) e^{s \xi^{2} L} d s, \quad t>0 \tag{73}
\end{equation*}
$$

Taking in (73) the random inverse Fourier cosine transform (22) and using Fubini theorem in normed spaces [17, page 175], it follows that

$$
\begin{align*}
u(x, t) & =\mathfrak{F}_{\mathrm{c}}^{-1}[\mathscr{U}(t)(\xi)] \\
& =-\frac{2 L}{\pi} \int_{0}^{\infty} e^{-t \xi^{2} L} \int_{0}^{t} g(s ; B) e^{s \xi^{2} L} \cos (\xi x) d s d \xi \\
& =-\frac{2 L}{\pi} \int_{0}^{t} g(s ; B)\left\{\int_{0}^{\infty} e^{-\xi^{2}(t-s) L} \cos (\xi x) d \xi\right\} d s \tag{74}
\end{align*}
$$

With the notation of (56), we have

$$
\begin{equation*}
u(x, t)=-\frac{2 L}{\pi} \int_{0}^{t} g(s ; B) J(x, t-s) d s, \quad x>0, t>0 \tag{75}
\end{equation*}
$$

For the sake of convenience, let us write

$$
\begin{align*}
J(x, t-s) & =\int_{0}^{\infty} e^{-\xi^{2}(t-s) L} \cos (\xi x) d \xi \\
& =[\xi \sqrt{t-s}=\nu]  \tag{76}\\
& =\frac{1}{\sqrt{t-s}} \int_{0}^{\infty} e^{-\nu^{2} L} \cos \left(\frac{v x}{\sqrt{t-s}}\right) d \nu
\end{align*}
$$

Note that under condition (11), the random integral appearing in (76) is m.s. absolutely uniformly convergent in a neighbourhood $\left[x_{0}-\delta, x_{0}+\delta\right], \delta>0$, of each $x_{0}>0, t>0$. Furthermore, from [12, page 61] each realization

$$
\begin{align*}
& \int_{0}^{\infty} e^{-\nu^{2} L(\omega)} \cos \left(\frac{v x}{\sqrt{t-s}}\right) d \nu \\
& \quad=\frac{1}{2} \sqrt{\frac{\pi}{(t-s) L(\omega)}} e^{-x^{2} / 4(t-s) L(\omega)}, \quad \omega \in \Omega . \tag{77}
\end{align*}
$$

Hence, from (76)-(77), one gets

$$
\begin{equation*}
J(x, t-s)=\frac{1}{2} \sqrt{\frac{\pi}{(t-s) L}} e^{-x^{2} / 4(t-s) L} \tag{78}
\end{equation*}
$$

and from (75) it follows that

$$
\begin{align*}
u(x, t) & =-\sqrt{\frac{L}{\pi}} \int_{0}^{t} \frac{g(s ; B)}{\sqrt{t-s}} e^{-x^{2} / 4(t-s) L} d s \\
& =[\sqrt{t-s}=v]  \tag{79}\\
& =2 \sqrt{\frac{L}{\pi}} \int_{0}^{\sqrt{t}} g\left(t-v^{2} ; B\right) e^{-(x / 2 v \sqrt{L})^{2}} d v \\
& x>0, t>0
\end{align*}
$$

Using the independence of r.v.s $B$ and $L$, one computes the expectation of the solution s.p. $u(x, t)$ :

$$
\begin{align*}
\mathrm{E}[u(x, t)]=\frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{t}} & \mathrm{E}\left[g\left(t-v^{2} ; B\right)\right]  \tag{80}\\
& \times \mathrm{E}\left[\sqrt{L} e^{-(x / 2 v \sqrt{L})^{2}}\right] d v
\end{align*}
$$

Taking into account (63), the variance of $u(x, t)$ is computed from (80) and

$$
\begin{align*}
& \mathrm{E}\left[(u(x, t))^{2}\right] \\
& \left.\left.\begin{array}{rl}
=\frac{4}{\pi} \iint_{0}^{\sqrt{t}} & \mathrm{E}
\end{array}\right] g\left(t-v_{1}^{2} ; B\right) g\left(t-v_{2}^{2} ; B\right)\right]  \tag{81}\\
& \\
& \quad \times \mathrm{E}\left[L e^{-\left(x / 2 v_{1} v_{2} \sqrt{L}\right)^{2}\left(v_{1}^{2}+v_{2}^{2}\right)}\right] d v_{1} d v_{2} .
\end{align*}
$$

Summarizing, the following result has been established.


FIgURE 1: Approximations for the expectation $\mathrm{E}[u(x, t)]$ (a), and the standard deviation $\sqrt{\operatorname{Var}[u(x, t)]}$ (b), on the spatial domain $0 \leq x \leq 50$ for some selected values in the time interval $0<t \leq 100$ in the context of Example 1.

Theorem 5. Let one consider the random heat problem given by (65)-(67) where $L$ is a positive $4-r . v$. satisfying conditions (11) and (49). Let one assume that $g(t ; B)$ is a mean fourth continuous process depending on r.v. B such that, for each $t>0$, the 4-r.v. $g(t ; B)$ is independent of $L$. Then, the solution stochastic process $u(x, t)$ of this problem is given by (79). In addition, (80) together with (63) and (81) are closed expressions for its expectation and variance.

## 6. Numerical Examples

Example 1. Let us consider problem (41)-(43) where the positive 4-r.v.s $L$ and $A$ are assumed to follow a beta distribution of parameters $\alpha=3$ and $\beta=1: L \sim B e(3 ; 1)$ and a gamma distribution of parameters $\alpha=4$ and $\beta=8$ : $A \sim$ Gamma $(4 ; 8)$, respectively. We will assume that both r.v.s are independent. Note that $L$ satisfies condition (55) since it is bounded (see Remark 1 and (11)-(13)). Furthermore, it is easy to check that the moment generating function of r.v. $L, \Phi_{L}(t)$, satisfies

$$
\begin{equation*}
\Phi_{L}(t)=\mathrm{E}\left[e^{t L}\right]=3 \frac{\left(-2+2 e^{t}-2 e^{t} t+e^{t} t^{2}\right)}{t^{3}} \xrightarrow{t \rightarrow 0} 1 ; \tag{82}
\end{equation*}
$$

hence, it is locally bounded about $t=0$. Therefore, by Theorem 4 expression (61) is a solution s.p. of problem (41)(43). In Figure 1, we have plotted the expectation, $\mathrm{E}[u(x, t)]$, and the standard deviation, $\sqrt{\operatorname{Var}[u(x, t)]}$, of the solution s.p. on the spatial domain $0 \leq x \leq 50$ for some selected values in the time interval $0<t \leq 100$. One observes the average of temperature pulls out of zero as time increases and, as a consequence, its variability, measured through standard deviation, behaves analogously.

Example 2. Let us consider problem (65)-(67) where $L$ is assumed to follow a beta distribution of parameters $\alpha=6$ and $\beta=8: L \sim B e(6 ; 8)$. Notice that $L$ is a positive 4-r.v. and satisfies condition (11) because it is bounded. In addition, it is straightforward to check that its moment generating function is given by

$$
\begin{align*}
\Phi_{L}(t)=\frac{1235520}{t^{13}} & \left(3991680-3991680 e^{t}\right. \\
& +2328480 t+1663200 e^{t} t \\
& +635040 e^{2}-302400 e^{t} t^{2}+105840 t^{3} \\
& +30240 e^{t} t^{3}+11760 t^{4}-1680 e^{t} t^{4} \\
& \left.+882 t^{5}+42 e^{t} t^{5}+42 t^{6}+t^{7}\right) \tag{83}
\end{align*}
$$

and satisfies $\Phi_{L}(t) \xrightarrow{t \rightarrow 0} 1$; hence, it is locally bounded about $t=0$. Let us consider the boundary condition $g(t ; B)=t B$, where $B$ is a Gaussian 4-r.v. of mean $\mu=4$ and standard deviation $\sigma=0.5$, that is, $B \sim N(4 ; 0.5)$ independent of r.v. $L$. Since $\mathrm{E}\left[B^{4}\right]=3(0.5)^{4}<\infty$ (see [9, page 26]), $\|g(t ; B)-g(s ; B)\|_{4}=\|B\|_{4}|t-s| \xrightarrow{t \rightarrow s} 0, g(t ; B)$ is m.f. continuous. Hence, the hypotheses of Theorem 5 are satisfied and expression given by (79) is a solution s.p. of problem (65)-(67). In Figure 2, we show, by means of a surface, approximations on the spatial domain $0<x \leq 8$ to the expectations and standard deviation according to (80), (63), and (81) at different instants in the time interval $1 \leq t \leq 10$.


Figure 2: Approximations for the expectation $\mathrm{E}[u(x, t)]$ (a), and the standard deviation $\sqrt{\operatorname{Var}[u(x, t)]}$ (b), on the spatial domain $0<x \leq 8$ for some values of time in the interval $1 \leq t \leq 10$ for the data of Example 2.

## 7. Conclusions

In this paper, we show that the well-known Fourier sine and cosine transforms' technique used in the deterministic case can also be used to solve random heat problems with the same quality answer. This fact requires the proof of the m.s. operational rules for the random Fourier sine and cosine transforms, as well as results about random ordinary differential equations obtained by the authors.

Thus, the paper opens a fruitful research activity in the management of random partial differential problems not only with these random Fourier transforms but also with other random transforms.

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# A Structural Equation Model for Analysis of Factors Associated with the Choice of Engineering Degrees in a Technical University 

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#### Abstract

Many different factors are taken into account by students when choosing a degree and university. Some of these are general considerations, such as the quality of the degree course (ratio of available places/places in first choice, cut-off mark, etc.), while others are subjective factors (e.g., friends doing the same course). This paper presents a partial multivariate model that considers the weight of the different variables linked to this decision, as identified in the bibliography. We analyzed four samples of first-year students (total $n=1790$ ) from different engineering degree courses at the Universitat Politècnica de València (UPV) in the 20102011 and 2011-2012 academic years. All the students involved in the study had chosen this university and their courses as their first option. The overall effect shows that the structural model adjusts reasonably well to the different engineering courses analyzed. Similarly, the individual models for each engineering degree manage to identify the different effects involved. In the case of the engineering degree based on new technologies (ICT), the statistical effects are much greater and more statistically significant than in the other three branches of engineering considered. Social and individual factors were seen to have more impact on the choice of ICT degrees at the UPV.


## 1. Introduction

The question of which factors determine a student's selection of university and degree course has been the subject of debate for some time. Many specialists have opined on the subject with more or less structured inputs and empirical contributions to justify their positions.

It is easy to obtain questionable but often-repeated evidence on the criteria used in selecting a university and study course. Arguments such as "my friends are also going," "It's close to home," or the well-known "the cutoff is low" are often cited. Studies such as those published by the Universidad Antonio de Nebrija (2001) (http:// www.nebrija.com/servicios/publicaciones.php) or the Universitat Oberta de Catalunya (2001) (http://elcrps.uoc.edu/ ojs/index.php/elcrps) show how most students use this type
of argument at the expense of those that academic specialists would prefer them to use. These arguments must be understood in the Spanish social context, in which the family relationship is very strong, and usually university student, over $76.6 \%$ at UPV, live with their parents, and the homes and apartments for rent are used by students whose families live far from the university.

This issue has also been raised in Europe, where the universities (and even more so the universities of our immediate environment) have found that new students are highly motivated but are somewhat lacking in the necessary capacities. The European Access Network (EAN) (http://www.ean-edu.org/), for example, has been very active in this field. In the USA the tradition is very similar with the "slight" difference that recruitment systems are supposedly competitive but are ultimately based on the financial
resources of prospective students, very different from our policy of scholarships and student grants. In our system grants play a social role, while in the USA there is a policy of recruiting talent, as has been shown by Murphy and McGarrity [1] in a descriptive study of 350 American colleges. Their conclusions have been verified by other authors, who also found this vision of college selection as the key to the future of students. Supporting evidence for this position can be found in the Hispanic Association of Colleges and Universities (http://www.hacu.net/), whose aim is to incorporate Hispanic minorities in the American university system, based on the quality of the system and the "trust" in the university. The situation described by Murphy and McGarrity is similar to the present situation in Spain and, as we have already said, can be clearly seen in some proven schemes, as in the case of Engineering [2] or that of women in certain degrees [3] or in exclusion-prone groups such as ethnic minorities [4].

Most authors agree on the relationship of the factors that determine the selection of a college or university and these are clearly identified in some of the works cited, as in [4]. In fact, they are consistent with many of those proposed some time ago in various forums and have even been included in the general documents of the European Students Union (http://www.esib.org/) to establish unsystematic models, somewhat more phenomenological than demonstrable. Many of the proposals are similar in type. First, it is considered that the demand for a degree and a university has a direct relationship with a factor generated from the subjective perception of this degree by students and their environment. The ratio between places and the demand for first choice is established from indirect indicators, as shown by the consolidation of the faculties and degrees most in demand. In more specific models, a certain longitudinal effect is admitted, so that the ratio between supply and demand in previous years is taken into account in the subjective perception of the present situation.

The Spanish public university admittance system is a relatively simple one. Prospective students range from those who have obtained the highest to those with the lowest acceptable grades. The condition is that there is a minimum requirement to access the system, so that only those who have successfully passed the preuniversity (high school or Baccalaureate) stage are eligible for consideration in the place allocation process. Those who have successfully passed high school and also a common university entrance exam, known as the Prueba de Acceso a la Universidad (PAU), can apply for a place at any university in the public system. The access grades of students considered for admittance consist of a weighted linear combination of the average grade of all high school subjects and the average marks of subjects taken in the PAU.

The system of access grades means that all students seeking a place at a public university can be ranked from the highest rating to the lowest and that places can be systematically assigned according to the preferences of the students. This ensures that no student gets a place in the public system with a lower entrance grade than another student who would not have been admitted to the same degree course. It sometimes happens that when the time comes to assign a place to a student and due to his/her
access grade there are no vacancies for the specified course or university. In such cases of students they are assigned their second option, or third if the second is not feasible due to lack of capacity. However, recent data from the Spanish universities admittance system shows that more than $92 \%$ are assigned their first choice and $98 \%$ one of their first three options. Finally, the access grade of the last student admitted to a given degree course establishes what is known as the cut-off mark, that is, the qualification below which a place cannot be obtained in the public system for that degree and university; fortunately, in Valencia, $80 \%$ of students are admitted to their first choice.

Obviously, the number of places available for the different degrees and universities and the applications received are the mechanisms that define the cut-off mark, so that a high demand coupled with limited places carries high cut-off marks, and likewise the degrees widely on offer with a low demand imply low cut-off marks. In no case does the level of the cut-off mark reflect the perceived difficulty of the degree or any other circumstance that is not strictly the relationship between supply and demand. Similarly, the number of first choice applications for a certain degree and university is a good indicator of these universities' capacity to attract students.

Other studies, such as Capilla [5], Ting [6], Huang and Fang [7], and Veenstra et al. [8], also see the demand for a university course as a very nonspecific assessment of the "social value" given to the degree and also to the university. This more qualitative or more subjective perception is constructed from certain parameters such as the absolute value of the cut-off mark in the application process, since it is used socially as an indicator of quality in sought-after degree courses, but not in degrees with a low demand and cut-off mark. No one doubts the difficulty of a college course in mathematics, but few consider its cut-off mark. Only when the available places are scarce the cut-off mark is used as an indicator of quality. The relationship between both factors is clear and some authors, like the aforementioned Ford, [4], attribute most of the variance to it, focusing on aspects such as the perceived utility of the degree. Such a proposal coincides with some contextual data. For example, the utility data offered by the Quality Agency of the University System in Catalonia (http://www.aqu.cat/) indicate that most students and parents attribute utility to many degrees that are not in fact in great demand by employers (such as computer science, e.g.). These subjective perceptions, in the opinion of these authors and also in ours, are decisive when selecting a university and course of studies. Similarly, some unsystematic data that has appeared in forums and seminars points to the existence of personal factors in this process, such as geographical issues, transport, gender, vocation, and interest in the subject. These personal aspects, as could be expected, sometimes coincide with aspects of the student's preuniversity academic history.

All these factors have certain similarities and links, occasionally spurious, which must be carefully analyzed, and which give rise to a complex network of effects which should be able to explain the variability observed in the university and degree course selection process. If we know
the weight of each factor, its effect, intensity, and direction, we can plan much more effectively university access guidance campaigns, generate strategies for the advancement of underrepresented groups, and, finally, make the transition between preuniversity and university much smoother. There are many who consider that the inadequate transition and lack of proper planning are largely responsible for drop-outs, absenteeism, and academic failure; however, this question is outside the scope of the present paper. Finally, we should point out that an analysis of the phenomenon dealt with here only acquires a realistic dimension in longitudinal terms and with a large sample of subjects that not only considers demographic variations due, for example, to immigration, but also takes into account the impact of new degrees in Spain and the perhaps excessively wide range of courses available. In view of the previously mentioned, the main objective of this work is to verify a multivariate model based on the principles of the structural equations model (SEM), which analyzes the impact of the different variables and factors identified by the bibliography as linked to the decision process on university degree courses, in the Universitat Politècnica de València (UPV) as paradigmatic institution in this environment.

The study was carried out on various cohorts of freshmen students enrolled for the first time in the Spanish university system in the academic years 2010-2011 and 2011-2012 and formed part of a larger project which analyzed the decision process in other areas of knowledge.

## 2. Path Diagram

The structural relationships used in the analysis were based on the structural model (SEM) proposed by Guàrdia et al. in [9], which obtained good fits of the structural model shown in Figure 1 as applied to a psychology degree course. A further development of the model was applied to several other degrees and universities [10].

As can be seen, this model involves the simultaneous use of directly observable variables free of error (represented by rectangles in Figure 1) and latent variables (represented by ovals). This created certain difficulties of notation on translating the proposal into statistical terms specific to structural models, as shown in Figure 2.

The structural equations that can be specified from Figure 2 are as follows:

$$
\begin{gather*}
Y_{1}=\beta_{11} X_{1}+\beta_{12} X_{2}+\zeta_{1}  \tag{1}\\
Y_{2}=\beta_{21} Y_{1}+\gamma_{21} \xi_{1}+\gamma_{22} \xi_{2}+\gamma_{23} \xi_{3}+\zeta_{2}
\end{gather*}
$$

Finally, to conform to the general precepts and assumptions of structural equation models, we considered the following statistical assumptions for quantitative variables $E\left(X_{i}\right)=$ $E\left(Y_{i}\right)=E\left(\xi_{i}\right)=0$ and $\operatorname{Var}\left(X_{i}\right)=\operatorname{Var}\left(Y_{i}\right)=\operatorname{Var}\left(\xi_{i}\right)=$ 1. Consequently, all quantitative variables were transformed by reduction and standardization, and similarly $E\left(\varepsilon_{i} \varepsilon_{j}\right)=$ $E\left(\delta_{i} \delta_{j}\right)=E(\xi \delta)=E(\eta \varepsilon)=E\left(\zeta_{i} \zeta_{j}\right)=0$, assuming initially that the errors of measurement were uncorrelated with each other, as in the case of the observable and latent variables. The categorical observable variables (type of baccalaureate and

Table 1: General characteristics of the UPV.

Universitat Politècnica de València (UPV)<br>Autonomous Community: Valencia

Located in Valencia, on the east coast of Spain, it is highly focused on technology transfer and specialized engineering degrees together with others in the field of social sciences and also fine arts. However, it has the characteristics of a medium-sized university as regards the number of students
Data 2012
Students: 36.855
Degrees: 74
Faculties and schools: 13
Lecturers: 2.764
Staff: 2.617
gender) were considered separately and subjected to an own estimation process described in the following.

For the sake of brevity, the structures of exogenous measurement models $\left(\Lambda_{x}\right)$ are not included here. We adopted the correlations between exogenous variables (both observable and latent) that had been shown to be significant in previous pilot studies. In all cases, the exogenous measurement models specified in the model comply with the conditions for applying the usual order conditions. In addition, the proposed model meets the identification condition, since it presents positive degrees of freedom (degree of freedom $\mathrm{df}=321$ ).

## 3. Method

3.1. Participants. Four accidental samples were obtained ( $n=$ 2244) composed of undergraduate students doing different degree courses at the Universitat Politécnica de València. The UPV is one of four polytechnic universities in Spain and offers a wide range of courses. Table 1 shows some of its main features.

The UPV sample consisted of 265 students enrolled in social science degree programs ( $12 \%$ ), 189 in experimental sciences and health ( $8 \%$ ), and the rest ( $80 \%$ ) were engaged in engineering studies, the fundamental faculty at the UPV. In engineering, the percentage of women is much lower than usual, ranging from $15 \%$ to $30 \%$. This paper is focused exclusively on engineering students ( $n=1790$ ), whose basic variables are shown in Table 2.

Some subjects, aeronautical engineering, for example, were not included in the sample due to insufficient numbers. Ages ranged from 18 to 21 in all samples, so that there was wide homogeneity in the distribution ( $\mathrm{M}=18.77$; $\mathrm{SD}=$ 0.38 ). The students tended to come from the technical areas of the Baccalaureate ( $98 \%$ from science and technology and the residual $2 \%$ from other branches). The cut-off mark ranged from 5 (minimum access) to 12.96 (maximum possible 14 points), reaching $\mathrm{M}=8.12$ and $\mathrm{SD}=1.52$, but with a very skewed distribution towards the tail on the right.
3.2. Questionnaire. Each student was administered the questionnaire proposed by Guàrdia et al. (2013), [10], which


Figure 1: Diagram of the proposed structural model.


Figure 2: Specification of proposed structural model with identification of the free parameters to be estimated.
showed good values for reliability and validity. The questionnaire variables related to access was divided into two secondorder factors (social and individual) defined by six primary factors: consideration of the university; perceived Utility and Social Considerations as social factors. The primary factors were vocational aspects; influence of geographical location for the individual factor, including the access grade, method of access, and gender. In the initial study, Cronbach's $\alpha$ values ranged between .84 and .95 for all factors. The factorial validity analyses carried out with confirmatory factor analysis also showed a good fit that confirmed the structure of the factors described ( $\chi^{2}=1234.74 ; P=.18$ ). The questionnaire consisted of a total of 25 items defined in a scale of ordinal response from 1 (maximum disagreement) to 7 (maximum agreement) leaving a complementary value for cases in which the proposed item did not apply, for example, if students had not attended the university's open day or did not have access to the services of an educational counselor, and so forth.

In addition to the variables included in the questionnaire, data were obtained from institutional variables, that is, cutoff marks, places offered in the academic years considered, numbers of students enrolled in each degree course, and so forth. While some of this information was gathered
from the students themselves, data was also obtained from official sources, that is, the universities involved and the official statistical data from the Education Ministry (http://www.mecd.gob.es/portada-mecd/). The data from the students themselves was not analyzed since it responded to secondary objectives of the overall project and was designed to estimate the real knowledge of the students accessing the university system.
3.3. Procedure. The questionnaire was administered by computer in such a way that the final total sample emerged from an accidental sampling. The results and data for each student were then processed according to the planned statistical analysis that was performed using SPSS v 21.0 and Amos v 19.0.
3.4. Statistical Analysis. To conform to the general precepts and assumptions of structural equation models, the following statistical assumptions were made for quantitative variables $E\left(X_{i}\right)=E\left(Y_{i}\right)=E\left(\xi_{i}\right)=0$ and $\operatorname{Var}\left(X_{i}\right)=\operatorname{Var}\left(Y_{i}\right)=$ $\operatorname{Var}\left(\xi_{i}\right)=1$. As a result, all quantitative variables were transformed through reduction and standardization and,

TABLE 2: Samples of UPV engineering students in 2010/2011 and 2011/2012.

|  | Agronomics | Construction and <br> civil engineering | Degree groups <br> Information and <br> communication <br> technology | Industrial <br> engineering | 2010-2011 | Cohorts |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Female |  |  |  |  |  |  |
| $376(21 \%)$ | 395 | 420 | 440 | 535 | 805 | 985 |
| Male | $22 \%$ | $25 \%$ | $37 \%$ | $45 \%$ | $55 \%$ |  |
| $1414(79 \%)$ | $22 \%$ |  |  |  |  |  |

likewise, $E\left(\varepsilon_{i} \varepsilon_{j}\right)=E\left(\delta_{i} \delta_{j}\right)=E(\xi \delta)=E(\eta \varepsilon)=E\left(\zeta_{i} \zeta_{j}\right)=$ 0 , assuming initially that the errors of measurement were uncorrelated to each other, as was the case of the observable and latent variables. The categorical observable variables (type of Baccalaureate and gender) were considered as such and subjected to an estimation process described in the following.

The structures of the exogenous measurement models $\left(\Lambda_{x}\right)$ are not described for reasons of space. We should just point out that we measured the correlations between exogenous variables (both observable and latent) that previous studies have shown to be relevant. The exogenous measurement models specified in the proposed model meet the usual application conditions of order. In addition, the proposed model verifies the identification condition since it presents positive degrees of freedom (degree of freedom $\mathrm{df}=$ 321). Further information on specific model data can be found in [9].

## 4. Results

First, parametric statistical tests were carried out to assess whether various samples considered showed any significant differences in relation to the gender and access marks of the students. No significant difference was obtained, so that the modifying effects of marginal distributions could be ruled out of the later analyses. The only exception was the distribution of the gender variable, in which the proportion of women compared to an equiprobable population distribution was significantly lower than that of men $\left(\chi^{2}=89.43\right.$; ) $(P<$ $.001, V=.82$ ), as the UPV students are mostly men. We analyzed the replies to the questionnaire according to the factors described, comparing results by degree courses and did not find any statistically significant differences between courses.

Thus, for each of the subsamples (engineering branches) the matrix of Pearson correlation between all the variables involved in the analysis was obtained, taking into account that in pairs in which the gender variable was considered it was estimated by biserial correlations and the type of Baccalaureate was estimated by polychoric correlations. The different parameters of the model were estimated using the AFD technique to estimate free distribution, since many of the variables involved presented high values of asymmetry. Although these did not affect the estimation of the correlation, given the large sample size, they could have affected the value of the estimates of the structural parameters. All

Table 3: Overall setting for structural model only for UPV engineering students, with an indication of reliability (estimation of $\alpha$ of Satorra-Bentler).

| Indicator | Estimation |
| :--- | :---: |
| Goodness of Fit Index (GFI) | .976 |
| Adjusted Goodness of Fit Index (AGFI) | .958 |
| Bentler Bonnet Normed Fit Index (BBNFI) | .961 |
| Bentler Bonnet Nonnormed Fit Index (BBNNFI) | .969 |
| Comparative Fit Index (CFI) | .952 |
| Coefficient of Determination (R2) | .491 |
| Root Mean Standard Errors (RMSE) | .006 |
| Standardized Root Mean Standard Errors (SRMSE) | .002 |
| $\chi^{2}$ with df = 321 | 915.67 |
| Ratio $\chi^{2} /$ df | $(P<.05)$ |
| Values of reliability for latent factors | 2.85 |
| Opinion of the university |  |
| Perceived utility of degree | $\alpha=.799$ |
| Social considerations | $\alpha=.801$ |
| Vocational aspects | $\alpha=.812$ |
| Influence of surroundings | $\alpha=.799$ |
| Geographical location | $\alpha=.791$ |

the analyses were carried out using SPSS and Amos software. Table 3 shows the overall fit in the UPV, assuming the four branches of engineering in a joint mode. Table 4 shows these parameters differentiated by study field. As can be seen in both tables, the goodness of fit indicators, show a good overall fit and also in the different fields. It should also be pointed out that in the construction engineering field the fit of the model is not so good.

The structural parameters derived from each of the four models described in Table 4 were also estimated to make it possible to analyze any differences from the effects in the general model, dividing the degree courses into different groups. Table 5 summarizes the standardized estimates of each structural parameter value.

## 5. Conclusions

The first issue to highlight has to do with the adjustment values of the models analyzed; while the fit of the $\chi^{2}$ statistic was not particularly good, we consider that the general model

TABLE 4: Overall settings of each specialty.

| Indicator | Universitat Politècnica de València (UPV) |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | Agronomics | Construction and civil engineering | Information and communications technology | Industrial engineering |
| GIF | . 900 | . 877 | . 944 | . 922 |
| AGIF | . 901 | . 878 | . 945 | . 924 |
| BBNFI | . 900 | . 871 | . 933 | . 931 |
| BBNNFI | . 902 | . 873 | . 928 | . 928 |
| CFI | . 899 | . 878 | . 919 | . 918 |
| $R^{2}$ | . 312 | . 302 | . 387 | . 318 |
| RMSE | . 010 | . 010 | . 007 | . 008 |
| SRMSE | . 004 | . 005 | . 002 | . 002 |
| $\chi^{2}(\mathrm{df}=321)$ | $\begin{gathered} 594.23 \\ P=.078 \end{gathered}$ | $\begin{gathered} 643.28 \\ P=0.069 \end{gathered}$ | $\begin{gathered} 699.54 \\ P=.09 \end{gathered}$ | $\begin{gathered} 745.12 \\ P=.11 \end{gathered}$ |
| $\chi^{2} / \mathrm{df}$ | 2.787 | 2.48 | 2.79 | 2.312 |

Table 5: Estimation of each standardized structural parameter for degrees at the UPV.

| Start of the effect according to the model in Figure 1 <br> Effect from | End of the effect according to the model in Figure 1 <br> Effect until | Parameter | Engineering fields |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | Agronomy | Construction | Information and communications technology | Industrial engineering |
| Perception offer/demand in access year | Demand in first option access year | $\Gamma_{23}$ | .501* | .592* | .676* | .423* |
| Previous course registration | Offer of places access year | $B_{11}$ | .198* | .335* | . $644^{*}$ | . 216 * |
| Offer of places last year | Offer of places access year | $B_{12}$ | .218* | . 321 * | . 612 * | .299* |
| Offer of places access year | First option year access demand | $B_{21}$ | . 256 * | .299* | . $618{ }^{*}$ | . $261{ }^{*}$ |
| Social factors | Demand for first option in access year | $\Gamma_{21}$ | . $399^{*}$ | . $441{ }^{*}$ | . $649 *$ | . 381 * |
| Individual factors | Demand for first option in access year | $\Gamma_{22}$ | . $643^{*}$ | .678* | .612* | . 551 * |
| Correlation between factors | social and individual | $\Phi_{21}$ | .512* | .571* | . $623^{*}$ | .493* |
| Correlation between and demand for pla the year prior to the individual factors | perception of offer es in first choice in year studied with | $\Phi_{32}$ | . $621^{*}$ | .699* | .679* | . 612 * |

proposed to explain the demand for the students' first choice in engineering studies at the UPV under the Spanish public system could be a suitable model. This conclusion is based on the fact both in the case of the overall adjustment of the UPV total and in the adjustment for the different engineering courses; the adjustment rates are good, since the values of GFI, AGFI, BBNFI, BBNNFI, or IFC indices are over . 90 and
in some cases more than .95. The SRMR values are lower than .011 and SRMSE values are below .005 (as proposed by Hu and Bentler [11]). These indices are acceptable when they are superior to .90 (GFI, AGFI, BBNFI, BBNNFI, and CFI), and for SRMR $\leq .05$; and RMSEA $\leq .06$, also the $\chi^{2} / \mathrm{df}$ ratio $<2$ indicates an excellent fit, $\chi^{2} / \mathrm{df}<3$ a good fit, and $\chi^{2} / \mathrm{df}<5$ an acceptable fit [11], and in our case for all the models fitted
these ratios are lower than 3. These results coincide with the model proposed by Guàrdia et al. [10] for the first version of the present model.

It can also be pointed out that in general the hypothesized parameters in the model are statistically significant in all cases, which is another argument in favor of the proposed model's ability to explain the demand for the first choice of university degree courses. All the estimated values are statistically significant with a confidence level of $95 \%$. However, there exist certain differences that should be pointed out. In Table 5 the value of the estimated parameter is very different for the specialties analyzed. The standardized estimates provide a simple descriptive analysis. Thus, in the case of ICT, the end endogenous variable to be explained (the demand for first choice $\left(Y_{2}\right)$ ) presents higher parameter values due to the impact of the exogenous variables $\left(\xi_{1}, \xi_{2}\right.$, and $\xi_{3}$ ), in most cases $\gamma_{i j}$ is of the order of .60 or higher. This could be explained by the fact that ICT is composed of a range of courses identified with the present time and thus is given preference in the demands of the students. Despite new offers, new UPV students see ICT as an attractive choice with a strong impact of individual and social factors for selection as the first option.

The cut-off mark also has a great influence on the perception of the offer/demand in the access year, especially in ICT $\left(\gamma_{23}=.676\right)$. This is consistent with the previously mentioned, since the UPV is particularly strong in engineering, which therefore reaffirms the predominant role of the ICT field as compared to the other three fields of construction, agricultural, and industrial engineering.

This work has certain limitations that should be considered and which mainly involve two issues. The first is the size of the asymmetrical samples, which means that some estimates are somewhat skewed by this effect. The use of standardized estimates facilitates presentation and study but does not solve all the problems of sampling. The second issue is that, in order to keep the results as simple as possible, we did not carry out strict statistical comparisons between the various parameters using the usual mechanisms (LM or Wald Test). The additional parameters and models would have involved extra complexity without providing special information. For the same reason we also decided not to incorporate the global adjustment values based on the Akaike (AIC) or Bayesian criteria (BIC).

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# Two-Dimension Hydrodynamic Dispersion Equation with Seepage Velocity and Dispersion Coefficient as Function of Space and Time 

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The contamination through the geological formation cannot move and disperse with the same speed and dispersion coefficient, respectively, due to the variability of the geological formation. This paper is therefore first devoted to the description of the hydrodynamic advection dispersion equation with the seepage velocity and dispersion coefficient as function of space and time. Secondly the equation is solved via two analytical techniques: the homotopy decomposition method and the differential transform method. The numerical simulations of the approximated solutions are presented.

## 1. Introduction

Most of the Earth's liquid freshwater is found, not in lakes and rivers, but is stored underground in conglomerations of voids, called aquifers, and present in the geological formations constituting the earth's mantle, where they act as reservoirs of water for rivers and streams, especially during periods of drought or low rainfall-a phenomenon commonly called baseflow. This resource, conventionally referred to as groundwater, therefore forms an essential, in fact major, component of the fresh water resources, as illustrated by the fact that that nearly two billion people depend directly on groundwater or their drinking water, while $40 \%$ of the world's food is produced on farmlands irrigated with groundwater [1].

One reason why groundwater so often constitutes the main source of drinking water in many cities and towns around the world is because it is frequently present in sufficient quantities at the point of demand. However, this seemingly advantage may sometimes be its greatest disadvantage, especially in situations where the groundwater occurs at shallow depths and the area overlying the aquifer is populated densely. This problem is particularly relevant
in the present technological age to its vast quantities of waste that is often disposed in an uncontrolled manner. Since space, time, and mass are measurable, and numbers form the basis of all mathematics, the question arises if it is not possible to combine the power of human reason with wellplanned observations and describe natural phenomena in abstract mathematical terms? This question leads us to the mathematical formulation of the flow of subsurface water in the geological formations. The development of a theory for natural phenomenon, can be briefly summarized in four steps [2,3].

Apply the rules of mathematical analysis to the observables and try to establish a framework, or hypothesis, able to predict the behaviour of the phenomenon, under different conditions from those used in establishing the hypothesis. The main effort is to establish a suitable mathematical relation between different observables, or what [4] calls a mathematical model for the observables. An ideal mathematical model should not only provide links between different observables, but also lead to a better understanding of the phenomenon. The attractiveness of this approach is that the mathematical model could, in principle, be used to investigate the future
behaviour of a given phenomenon and under various conditions. For simplicity, it is always assumed that, the seepage velocity retardation factor and the dispersion coefficient appearing are the mathematical equation describing the movement of contamination in groundwater as constants. However, in the real world observation, these coefficients depend on space and time. One therefore needs to include such behaviour in the mathematical formulation. This paper is therefore devoted to the discursion underpinning the effect of variation in time space of the seepage velocity and dispersion coefficient on one hand, and on the other hand a possible analytical solution of this equation.

## 2. Mathematical Formulation

The mathematical model, discussed below, has been used extensively in the studies of groundwater pollution problems, with satisfactory results. A relatively complete set of onedimensional analytical solutions for convective-dispersive solute equations has been recently published by [5]; here we will review a case having a practical application. Let us consider a two-dimensional model consisting of and infinitely ling homogenous isotropic porous media with steady-state uniform flow with seepage velocity $v$. We inject a particular chemical from one end of the model for a period of time $t_{0}$ such that the input concentration varies as an exponential function of time [5]. The value of that chemical concentration at any time $t$ and at a distance $x$ from the injection boundary, allowing for the decay and adsorption, may be obtained from the solution of the following set of equations [5]:

$$
\begin{align*}
D & {\left[\frac{\partial^{2} c(x, y, t)}{\partial x^{2}}+\frac{\partial^{2} c(x, y, t)}{\partial y^{2}}\right] } \\
& -v\left[\frac{\partial c(x, y, t)}{\partial x}+\frac{\partial c(x, y, t)}{\partial y}\right]  \tag{1}\\
& -\lambda R c(x, y, t) \\
& =R \frac{\partial c(x, y, t)}{\partial t}
\end{align*}
$$

where $D$ is the dispersion coefficient and $R$ is the retardation factor and subject to the initial condition:

$$
\begin{gather*}
c(x, y, t)=c_{1}, \quad t=0  \tag{2}\\
c(0,0, t)=c_{0} \exp (-\gamma t), \quad 0<t \leq t_{0}
\end{gather*}
$$

which means that the system is initially free of that chemical, $\gamma$ and $c_{0}$ are constants, and $\lambda$ is the radioactive decay constant. And boundaries conditions:

$$
\begin{equation*}
\frac{\partial c(x, y, t)}{\partial x}=\frac{\partial c(x, y, t)}{\partial y}=0, \quad x, y \longrightarrow \infty \tag{3}
\end{equation*}
$$

In order to include explicitly the variation of the seepage velocity and dispersion coefficient into the mathematical formulation, we replace the constants coefficients $R, D$, and
$v$ of equation (1) by the functions $R(x, y, t), D(x, y, t)$, and $v(x, y, t)$ to obtain

$$
\begin{align*}
D(x, y, t)[ & \left.\frac{\partial^{2} c(x, y, t)}{\partial x^{2}}+\frac{\partial^{2} c(x, y, t)}{\partial y^{2}}\right] \\
& -v(x, y, t)\left[\frac{\partial c(x, y, t)}{\partial x}+\frac{\partial c(x, y, t)}{\partial y}\right]  \tag{4}\\
& -\lambda R(x, y, t) c(x, y, t) \\
& =R(x, y, t) \frac{\partial c(x, y, t)}{\partial t}
\end{align*}
$$

To satisfy the physical behaviour of the aquifer, there exist positives constants $C_{1}, C_{2}$, and $C_{3}$ such that

$$
\begin{align*}
& 0<R(x, y, t) \leq C_{1}, \\
& 0<v(x, y, t) \leq C_{2}, \quad x, y, t \geq 0,  \tag{5}\\
& 0<D(x, y, t) \leq C_{3} .
\end{align*}
$$

## 3. Analytical Solutions

Environmental phenomena, such as groundwater pollution, are highly complex phenomena, which do not lend themselves readily to analysis of analytical models. The discussion presented in this section will therefore be devoted to the derivation of analytical solution to the advection dispersion equation (1). As V. M. Alexandrov wrote in the foreword of a popular science book "Asymtotology: ideas, methods, and applications [sic]", asymptotic methods belong to the, perhaps, the most romantic area of modern mathematics [6]. Though computer science is growing very fast and numerical simulation is applied everywhere, nonnumerical issues will still play a large role [7-11].

There exist some alternative analytical asymptotic approaches, such as the nonperturbative method [12], modified Lindstedt-Poincare method [13], variational iteration method [14], Adomian decomposition method [14], homotopy perturbation method [15], and book-keeping artificial parameter perturbation method [15]. Recently to solve the groundwater flow equation, Atangana and Botha proposed a modified decomposition method called homotopy decomposition [16]. In this section we make use of two different methods including the HDM and DTM to derive approximate solution of equation (1). We will start with HDM.
3.1. Useful Tools for the Homotopy Decomposition Method [1719]. To accommodate readers that are not acquainted with this modified form of decomposition method, we present in this section the basic methodology of the method. To illustrate the fundamental design of this technique, we consider
a universal nonlinear nonhomogeneous partial differential equation with initial conditions of the following form:

$$
\begin{align*}
& \frac{\partial^{m} R(x, t)}{\partial t^{m}} \\
& \quad=L(R(x, t))+N(R(x, t))+h(x, t), \quad m=1,2,3 \ldots \tag{6}
\end{align*}
$$

subject to the initial condition:

$$
\begin{gather*}
\frac{\partial^{i} R(x, 0)}{\partial t^{i}}=f_{m}(x), \\
\frac{\partial^{m-1} R(x, 0)}{\partial t^{m-1}}=0, \quad i=0,1,2, \ldots, m-2, \tag{7}
\end{gather*}
$$

where $h$ is a known function, $N$ is the general nonlinear differential operator, and $L$ represents a linear differential operator. The first step of the method here is to apply the inverse operator $\partial^{m} / \partial t^{m}$ of on both sides of (1) to obtain

$$
\begin{align*}
& R(x, t) \\
& \begin{array}{l}
=\sum_{k=0}^{m-1} \frac{t^{k}}{k!} \frac{d^{k} R(x, 0)}{d t^{k}} \\
\quad+\int_{0}^{t} \int_{0}^{t_{1}} \cdots \int_{0}^{t_{m-1}} L(R(x, \tau))+N(R(x, \tau)) \\
\\
\quad+h(x, \tau) d \tau \cdots d t
\end{array} \tag{8}
\end{align*}
$$

The multi-integral in (2) can be transformed to

$$
\begin{align*}
& \int_{0}^{t} \int_{0}^{t_{1}} \cdots \int_{0}^{t_{m-1}} L(R(x, \tau))+N(R(x, \tau))+h(x, \tau) d \tau \cdots d t_{1} \\
& \quad=\frac{1}{(m-1)!} \int_{0}^{t}(t-\tau)^{m-1} L(R(x, \tau))+N(R(x, \tau))+h(x, \tau) d \tau \tag{9}
\end{align*}
$$

so that (8) can be reformulated as

$$
\begin{align*}
R(x, t)=\sum_{k=0}^{m-1} & \frac{t^{k}}{k!}\left\{\frac{d^{k} R(x, 0)}{d t^{k}}\right\}+\frac{1}{(m-1)!} \\
& \times \int_{0}^{t}(t-\tau)^{m-1} L(R(x, \tau))  \tag{10}\\
& +N(R(x, \tau))+h(x, \tau) d \tau
\end{align*}
$$

Using the Homotopy scheme the solution of the above integral equation is given in series form as

$$
\begin{align*}
R(x, t, p) & =\sum_{n=0}^{\infty} \beta^{n} R_{n}(x, t)  \tag{11}\\
R(x, t) & =\lim _{p \rightarrow 1} R(x, t, p)
\end{align*}
$$

and the nonlinear term can be represented as

$$
\begin{equation*}
N R(r, t)=\sum_{n=1}^{\infty} \beta^{n} \mathrm{P}_{n}(R) \tag{12}
\end{equation*}
$$

where $\beta \in(0,1]$ is an embedding parameter. $\mathrm{P}_{n}(R)$ is the polynomials that can be engendered by

$$
\begin{align*}
\mathrm{P}_{n} & \left(R_{0}, \ldots, R_{\mathrm{n}}\right) \\
& =\frac{1}{n!} \frac{\partial^{n}}{\partial p^{n}}\left[N\left(\sum_{j=0}^{n} \beta^{j} R_{j}(x, t)\right)\right], \quad n=0,1,2 \ldots \tag{13}
\end{align*}
$$

The homotopy decomposition method is achieved via the polished combination of decomposition method with Abel's integral and is given by

$$
\begin{align*}
& \sum_{n=0}^{\infty} \beta^{n} R_{n}(x, t) \\
& \quad=T(x, t)+\beta \frac{1}{(m-1)!} \\
& \quad \times \int_{0}^{t}(t-\tau)^{m-1}\left[h(x, \tau)+L\left(\sum_{n=0}^{\infty} \beta^{n} R_{n}(x, \tau)\right)\right.  \tag{14}\\
& \left.\quad+\sum_{n=0}^{\infty} \beta^{n} \mathrm{P}_{n}(R)\right] d \tau
\end{align*}
$$

With

$$
\begin{equation*}
K(x, t)=\sum_{k=0}^{m-1} \frac{t^{k}}{k!}\left\{\frac{d^{k} R(x, 0)}{d t^{k}}\right\} \tag{15}
\end{equation*}
$$

matching up to the expressions of similar powers of $\beta$, we can get hold of solutions of assorted orders. The early estimate of the approximation is $K(x, t)$. It is significant to draw attention to the fact that the early estimate is the Taylor series of order $m$ of the exact solution.
3.2. Differential Transform Method (DTM) [17, 20-22]. The necessary definitions of the three-dimensional transform are shown. Given a $w$ function which has three components such as $x, y, t$, the three-dimensional differential transform function of the function $w(x, y, t)$ is defined as

$$
\begin{equation*}
W(k, h, m)=\frac{1}{k!h!m!}\left[\frac{\partial^{(k+h+m)} W(x, y, t)}{\partial x^{k} \partial y^{h} \partial t^{m}}\right]_{(0,0,0)} \tag{16}
\end{equation*}
$$

where $w(x, y, t)$ is the original function and $W(k, h, m)$ is the transform function.

The inverse differential transform of $w(k, h, m)$ is defined as

$$
\begin{equation*}
w(x, y, t)=\sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} W(k, h, k) x^{k} y^{k} t^{k} \tag{17}
\end{equation*}
$$

from (16) and (17) it can be concluded that

$$
\begin{align*}
& w(x, y, t) \\
& \quad=\sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k!h!m!}\left[\frac{\partial^{(k+h+m)} W(x, y, t)}{\partial x^{k} \partial y^{h} \partial t^{m}}\right]_{(0,0,0)} x^{k} y^{k} t^{k} . \tag{18}
\end{align*}
$$

Some fundamental mathematical operators performed by three-dimensional differential transform methods are listed below [20]:
(1) if $w(x, y, t)=W_{1}(x, y, t) \mp W_{2}(x, y, t)$, then $W(k, h, m)=W_{1}(k, h, m) \mp W_{2}(k, h, m)$,
(2) if $w(x, y, t)=c w_{1}(x, y, t)$, then $W(k, h, t)=$ $c W_{1}(k, h, t)$ where $c$ is a constant,
(3) if $w(x, y, t)=d^{1} w_{1}(x, y, t) / d x^{1}$, then $W(k, h, t)=$ $(k+1) W_{1}(k+1, h, m)$,
(4) if $w(x, y, t)=d^{1} w_{1}(x, y, t) / d y^{1}$, then $W(k, h, t)=$ $(h+1) W_{1}(h+1, k, m)$,
(5) if $w(x, y, t)=d^{1} w_{1}(x, y, t) / d t^{1}$, then $W(k, h, t)=$ $(m+1) W_{1}(m+1, k, h)$,
(6) if $w(x, y, t)=w_{1}(x, y, t) w_{2}(x, y, t)$, then $W(k, h, m)=$ $\sum_{r=0}^{k} \sum_{s=0}^{h} W_{1}(r, h-s, m-p) W_{2}(k-r, s, p)$.

## 4. Applications

In this section, we apply these methods for solving the twodimension hydrodynamic dispersion equation with seepage velocity and dispersion coefficient as function of space and time.

Since $0<R(x, y, t) \leq C_{1}$, we first divide (1) by $R(x, y, t)$ and let

$$
\begin{align*}
& f(x, y, t)=\frac{D(x, y, t)}{R(x, y, t)} \\
& h(x, y, t)=\frac{v(x, y, t)}{R(x, y, t)} \tag{19}
\end{align*}
$$

to obtain

$$
\begin{align*}
f(x, y, t)[ & \left.\frac{\partial^{2} c(x, y, t)}{\partial x^{2}}+\frac{\partial^{2} c(x, y, t)}{\partial y^{2}}\right] \\
& -h(x, y, t)\left[\frac{\partial c(x, y, t)}{\partial x}+\frac{\partial c(x, y, t)}{\partial y}\right]  \tag{20}\\
& -\lambda c(x, y, t) \\
& =\frac{\partial c(x, y, t)}{\partial t}
\end{align*}
$$

Applying the HDM to the above equation yields

$$
\begin{gather*}
c_{0}(x, y, t)=c(x, y, 0) \\
c_{n}(x, y, t)  \tag{21}\\
=\int_{0}^{t}\left(f(x, y, \tau)\left[\frac{\partial^{2} c_{n-1}(x, y, \tau)}{\partial x^{2}}+\frac{\partial^{2} c_{n-1}(x, y, \tau)}{\partial y^{2}}\right]\right. \\
-h(x, y, \tau)\left[\frac{\partial c_{n-1}(x, y, \tau)}{\partial x}+\frac{\partial c_{n-1}(x, y, \tau)}{\partial y}\right] \\
\left.-\lambda c_{n-1}(x, y, \tau)\right) d \tau, \quad n \geq 1
\end{gather*}
$$

Simulation of $c(x, y=2, t)$ in the aquifer


FIGURE 1: Simulation of the approximate solution obtained via HDM for a fixed $y=2$.

Following the DTM method we supposed that the solution of (1) is in the form of

$$
\begin{align*}
& c(x, y, t) \\
& \quad=\sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k!h!m!}\left[\frac{\partial^{(k+h+m)} C(x, y, t)}{\partial x^{k} \partial y^{h} \partial t^{m}}\right]_{(0,0,0)} x^{k} y^{k} t^{k} \tag{22}
\end{align*}
$$

with

$$
\begin{equation*}
c(x, y, t)=c_{1}(x, y, t)+c_{2}(x, y, t) \tag{23}
\end{equation*}
$$

so that

$$
\begin{equation*}
C(k, h, m)=C_{1}(k, h, m)+C_{2}(k, h, m) . \tag{24}
\end{equation*}
$$

Example 1. Let us consider in (1) the following coefficients:

$$
\begin{gather*}
f(x, y, t)=2+\sin (x) \sin (y) \cos (t) \\
h(x, y, t)=2-2 \sin (2 \pi x) \sin (y) \cos (\pi+t) \tag{25}
\end{gather*}
$$

The theoretical parameters used in this simulation are $c_{0}=$ $100, \gamma=0.86, \lambda=1$.

The following figure shows the numerical simulations of the approximated solutions obtained via HDM and DTM.

Figures 1 and 2 show the numerical simulation of the approximate solution via HDM and DTM, respectively. Here we fixed $y=2$ and we consider the concentration to be a function of $x$ and $t$. It can be observed that both methods give pretty much the same results for the first four components of the series solution.

Figures 3 and 4 show the numerical simulation of the approximate solution via HDM and DTM, respectively. Here we fixed $x=2$ and we consider the concentration to be a function of $x$ and $t$.

Simulation of $c(x, y=2, t)$ in the aquifer


Figure 2: Simulation of the approximate solution obtained via DTM for a fixed $y=2$.


Figure 3: Simulation of the approximate solution obtained via HDM for a fixed $x=2$.

Figures 5 and 6 are the contour plot obtained via Mathematica. These figures show the possible pathway taken by the pollution through the geological formation called aquifer.

Figure 7 shows the contour plot of the approximate solution as a function of $x$ and $y$ for a fixed time. In practice this figure shows the behaviour of the contamination through the geological formation for a fixed time $t=10$. The figure shows that since the seepage velocity and the dispersion coefficient are function of space, the pollution will not have the same intensities everywhere in the aquifer. There will be certain place in the aquifer with more pollution than other, depending on the velocity and the dispersion coefficients associated.

Simulation of $c(x=2, y, t)$ in the aquifer


Figure 4: Simulation of the approximate solution obtained via DTM for a fixed $x=2$.


Figure 5: Contour plot of the approximated solution in $x$-direction for fixed $y$.

## 5. Discussion and Conclusion

Natural geological deposits with highly contrasting permeability may form mobile and relatively immobile zones, where the potential mass exchange between mobile and immobile zones results in a wide time distribution for solute trapping. The transport process in groundwater is, by its very nature, always in contact with the matrix of an aquifer. There is thus a possibility that the solutes may interact with the rock matrix and one another. A true mathematical model for groundwater pollution must therefore be able to account for interactions between the dissolved solids and matrix of the aquifer. It will thus be advantageous to look at the nature of the interactions between dissolved solids and a porous medium that may be expected in groundwater pollution. Experimental evidence indicates that when a dissolved solid comes in contact with the matrix of a porous medium, it


Figure 6: Contour plot of the approximate solution in $y$-direction for fixed $x$.


Figure 7: Contour plot of the approximated solution for a fixed time $t=2$.
may (a) pass through the medium with no apparent effect, (b) be absorbed by the porous matrix, and (c) react with the porous matrix and other substances dissolved in the fluid. The dissolved solids encountered in porous flow are, for this reason, often classified as conservative, nonconservative, and reactive tracers [22]. This behaviour implies that the quantity of dissolved solids in a porous medium depends not only on the flow pattern, but also on the nature of the porous matrix and the solution. These situations (a), (b), and (c) can be characterized efficiently by the time-nonlocal model, including the hydrodynamic dispersion equation with seepage velocity and dispersion coefficients as function of space and time as described by Figure 5. If the high-permeable material tends to form preferential flow paths, such as the interconnected paleochannels observed in alluvial depositional systems, then the solute transport may show a heavy leading edge, which
can be described by the hydrodynamic dispersion equation with seepage velocity and dispersion coefficients as function of space and time.

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