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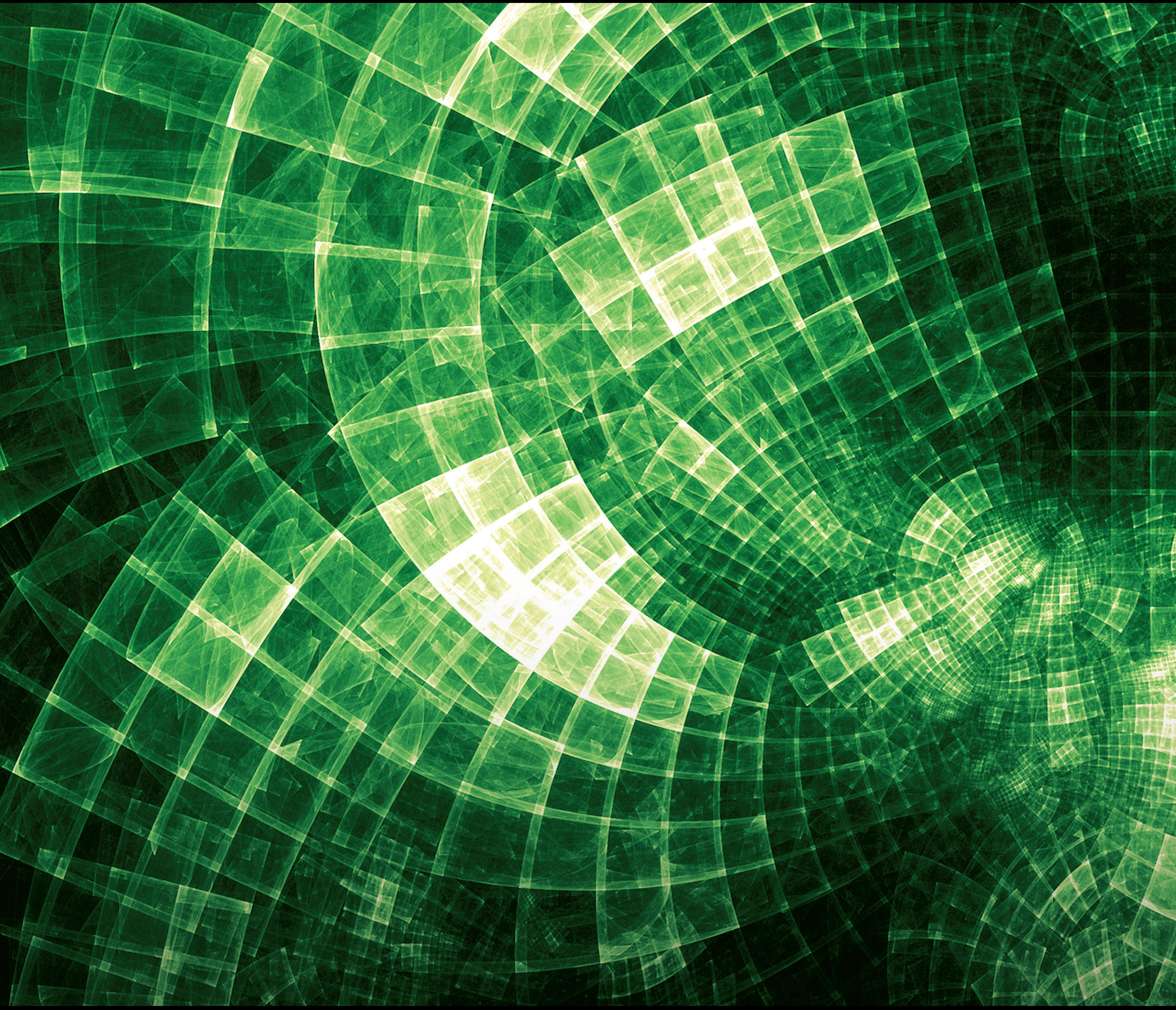
Guest Editors: Jamshaid Ahmad, Aftab Hussain, Jianhua Chen, and Ahmad S. Al-Rawashdeh



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Research Article

New Iteration Scheme for Approximating a Common Fixed Point of a Finite Family of Mappings

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Guest Editor: Aftab Hussain

We introduce a new algorithm (horizontal algorithm) in a real Hilbert space, for approximating a common fixed point of a finite family of mappings, without imposing on the finite family of the control sequences $\{\{\varsigma_n^i\}_{n=1}^{\infty}\}_{i=1}^{N}$, the condition that $\sum_{i=1}^{N}\varsigma_n^i = 1$, for each $n \geq 1$. Furthermore, under appropriate conditions, the horizontal algorithm converges both weakly and strongly to a common fixed point of a finite family of type-one demicontractive mappings. It is also applied to obtain some new algorithms for approximating a common solution of an equilibrium problem and the fixed point problem for a finite family of mappings. Our work is a contribution to ongoing research on iteration schemes for approximating a common solution of fixed point problems of a finite family of mappings and equilibrium problems.

1. Introduction

Let Y be a nonempty set and $S\colon Y \longrightarrow Y$ be a mapping. A point $y \in Y$ is called a fixed point of S if $y = Sy$. If $S\colon Y \longrightarrow 2^Y$ is a multivalued mapping, then y is a fixed point of S if $y \in Sy$. y is called a strict fixed point of S if $Sy = \{y\}$. The set $F(S) = \{y \in D(S)\colon y \in Sy\}$ (respectively, $F(S) = \{y \in D(S)\colon y = Sy\}$) is called the set of fixed points of the multivalued (respectively, single-valued) mapping S, while the set $F_s(S) = \{y \in D(S)\colon Sy = \{y\}\}$ is called the set of strict fixed points of S.

Let Y be a normed space. A subset K of Y is called proximinal if for each $y \in Y$, there exists $k \in K$ such that

$$\|y - k\| = \inf\{\|y - w\|\colon w \in K\} = d(y, K). \tag{1}$$

It is known that every convex closed subset of a uniformly convex Banach space is proximinal. We shall denote the family of all nonempty closed and bounded subsets of Y by $CB(Y)$, the family of all nonempty subsets of Y by 2^Y, and the family of all proximinal subsets of Y by $P(Y)$, for a nonempty set Y.

Let $\mathscr{D}$ denote the Hausdorff metric induced by the metric d on Y, that is, for every $A, B \in CB(Y)$,

$$\mathscr{D}(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}. \tag{2}$$

Let Y be a normed space and $S\colon D(S) \subseteq Y \longrightarrow 2^Y$ be a multivalued mapping on Y. S is called $L-$Lipschitzian if there exists $L \geq 0$ such that, for all $x, y \in D(S)$,

$$\mathscr{D}(Sx, Sy) \leq L\|x - y\|. \tag{3}$$

In (3), if $L \in [0, 1)$, then S is a contraction, while S is nonexpansive if $L = 1$. S is called quasi-nonexpansive if $F(S) \neq \emptyset$ and for all $p \in F(S)$,

$$\mathscr{D}(Sx, Sp) \leq \|x - p\|. \tag{4}$$

Clearly, every nonexpansive mapping with the nonempty fixed point set is quasi-nonexpansive. The multivalued mapping S is k-strictly pseudo-contractive-type of Isiogugu [1] using the terminology of Browder and Petryshen [2] for single-valued pseudo-contractive mapping and Markin [3] for the monotone operator if there exists $k \in [0, 1)$ such that given any pair $x, y \in D(S)$ and $u \in Sx$, there exists $v \in Sy$ satisfying $\|u - v\| \leq \mathscr{D}(Sx, Sy)$ and

$$\mathscr{D}^2(Sx, Sy) \leq \|x - y\|^2 + k\|x - u - (y - v)\|^2. \tag{5}$$

If $k = 1$ in (5), then S is pseudo-contractive-type, while S is nonexpansive-type if $k = 0$. Every multivalued nonexpansive mapping $S\colon D(S) \subseteq Y \longrightarrow P(Y)$ is nonexpansive-type. S is of type-one in the sense of Isiogugu et al. [4] if given any pair $x, y \in D(S)$, then

$$\|u - v\| \leq \mathscr{D}(Sx, Sy), \quad \text{for all } u \in P_S x,\ v \in P_S y, \tag{6}$$

where $P_S x\colon = \{u \in Sx\colon \|u - x\| = d(x, Sx)\}$. S is called a multivalued demicontractive in the sense of Isiogugu and Osilike [5] using the terminology of Hicks and Kubicek [6] for single-valued demicontractive if $F(S) \neq \emptyset$ and for all $p \in F(S)$ and $x \in D(S)$, there exists $k \in [0, 1)$ such that

$$\mathscr{D}^2(Sx, Sp) \leq \|x - p\|^2 + kd^2(x, Sx). \tag{7}$$

If $k = 1$ in (7), S is hemicontractive in the terminology of Naimpally and Singh [7] for single-valued hemicontractive, while S is quasi-nonexpansive if $k = 0$.

Furthermore, every multivalued k–strictly pseudo-contractive-type in the sense of [1] with the nonempty set of strict fixed points is demicontractive with respect to its set of strict fixed points.

A single-valued mapping $S\colon D(S) \subseteq H \longrightarrow H$ is called nonspreading in the sense of Kohsaka and Takahashi [8, 9] if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\langle x - Sx, y - Sy\rangle, \quad \forall x,\ y \in C. \tag{8}$$

Observe that if S is nonspreading and $F(S) \neq \emptyset$, then S is quasi-nonexpansive. S is k-strictly pseudo-nonspreading in the sense of Osilike and Isiogugu [10] if there exists $k \in [0, 1)$ such that

$$\begin{aligned} \|Sx - Sy\|^2 \leq{}& \|x - y\|^2 + k\|x - Sx - (y - Sy)\|^2 \\ &+ 2\langle x - Sx, y - Sy\rangle, \end{aligned} \tag{9}$$

for all $x, y \in D(S)$. Clearly, every nonspreading mapping is k-strictly pseudo-nonspreading. If S is k–strictly pseudo-nonspreading and $F(S) \neq \emptyset$, then S is demicontractive in the sense of [6] (see also [11]).

Several algorithms have been introduced by different authors for the approximation of common fixed points of finite family of mappings $\{S_i\}_{i=1}^N$, where $N \in \mathbb{N}$ (the set of nonnegative integers) (see, for example, [12–18] and references therein). One of the motivations for this aspect of research is the well-known convex feasibility problem which is reducible to the problem of finding a point in the intersection of the set of fixed points of a family of nonexpansive mappings (see, for example, [19, 20]). The earliest of such algorithms was the cyclic algorithm introduced by Bauschke [12] using a Halpern-type iterative process considered in [21] for the approximation of a common fixed point of a finite family of nonexpansive self-mappings. He proved the following theorem.

Theorem 1 (see [12], Theorem 3.1). *Let K be a nonempty convex closed subset of a real Hilbert space H and $S_1, S_2, \ldots, S_N$ be a finite family of nonexpansive mappings of K into itself with $F\colon = \cap_{i=1}^N F(S_i) \neq \emptyset$ with $F = F(S_N S_{N-1} \ldots S_1) = F(S_1 S_N \ldots S_2) = F(S_{N-1} S_{N-2} \ldots S_1 S_N)$. Given points $u, x_0 \in K$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \varsigma_n x_n + (1 - \varsigma_n) S_{n+1} x_{n+1}, \quad n \geq 0, \tag{10}$$

where $S_n \coloneqq S_{n(\mathrm{mod} N)}$ and $\varsigma_n \subset (0, 1)$ satisfies $\sum_{n\geq 1} |\varsigma_{n+N} - \varsigma_n| < \infty$. Then, $\{x_n\}$ converges strongly to $P_F u$, where $P_F\colon H \longrightarrow F$ is the metric projection.

The above algorithm of Bauschke was extended to approximate the family of more general class of strictly pseudo-contractive mappings (see, for example, [22, 23]). Suantai et al. also considered similar algorithms (see, for example, [24]) and references therein.

In 2008, Zhang and Guo [25] considered a parallel iteration for approximating the common fixed points of a finite family of strictly pseudo-contractive mapping. They obtained the following theorem.

Theorem 2 (see [25], Theorem 4.3). *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and K be a nonempty convex closed subset of E. Let $N \geq 1$ be an integer, and for each $1 \leq i \leq N$, let $S_i\colon K \longrightarrow K$ be a k_i-strictly pseudo-contractive mapping for some $0 \leq k_i < 1$. Let $k = \min\{k_i\colon 1 \leq i \leq N\}$. Assume the common fixed point set $\cap_{i-1}^N F(S_i)$ is nonempty. Assume also for each n, $\{\lambda_i^n\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{1=1}^N \lambda_i^n = 1$ for all n and $\inf_{n\geq 1} \lambda_i^n > 0$ for all $1 \leq i \leq N$. Given $x_0 \in K$, let $\{x_n\}_{n=1}^\infty$ be the sequence generated by the algorithm:*

$$x_{n+1} = \varsigma_n x_n + (1 - \varsigma_n) \sum_{i=1}^N \lambda_i^n S_i x_n, \quad n \geq 0. \tag{11}$$

Let $\{\varsigma_n\}_{n=1}^\infty$ be a real sequence satisfying the conditions

$$\begin{aligned} \sum_{n=0}^\infty \sum_{i=1}^N \left|\lambda_i^{n+1} - \lambda_i^n\right| &< \infty; \\ \sum_{n=0}^\infty (1 - \varsigma_n) qk - \left[Cq(1 - \varsigma_n)^{q-1}\right] &= \infty. \end{aligned} \tag{12}$$

Then, $\{x_n\}$ converges weakly to a common fixed point of $\{S_i\}_{i=1}^N$.

Motivated by the parallel algorithm above, many authors have considered in a real Hilbert space, another form of parallel algorithm for a finite family $\{S_i\}_{i=1}^N$ of k_i-strictly pseudo-contractive mappings defined by

$$x_{n+1} = \varsigma_n^0 x_n + \sum_{i=1}^N \varsigma_n^i S_i x_n, \quad n \geq 1, \tag{13}$$

where $\{\varsigma_n^i\}_{n=1}^\infty \subseteq (0, 1)$ for each i and $\sum_{i=0}^N \varsigma_n^i = 1$ for each n (see, for example, [13] and references therein).

In [14], Iemoto and Takahashi studied the approximation of common fixed points of a nonexpansive self-mapping T and a nonspreading self-mapping S in a real Hilbert space. If $T, S\colon C \longrightarrow C$ are, respectively, nonexpansive and nonspreading mappings, they considered the iterative scheme $\{x_n\}_{n=1}^\infty$ generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1-\varsigma_n)x_n + \varsigma_n[\beta_n S x_n + (1-\beta_n)T x_n], \quad n \geq 1, \tag{14}$$

where $\{\varsigma_n\}$ and $\{\beta_n\}$ are suitable sequences in $[0,1]$. They proved the following main theorem:

Theorem 3 (see [14], Theorem 4.1). *Let H be a real Hilbert space. Let C be a nonempty convex and closed subset of H. Let S be a nonspreading mapping of C into itself and T a nonexpansive mapping of C into itself such that $F(T) \cap F(S) \neq \emptyset$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in C as follows:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\varsigma_n)x_n + \varsigma_n[\beta_n S x_n + (1-\beta_n)T x_n], \end{cases} \tag{15}$$

for all $n \in \mathbb{N}$, where $\{\varsigma_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \subset [0,1]$.

Then, the following hold:

(i) If $\liminf_{n\longrightarrow\infty}\varsigma_n(1-\varsigma_n) > 0$, $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(S)$.

(ii) If $\sum_{n=1}^{\infty}\varsigma_n(1-\varsigma_n) = \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(T)$.

(iii) If $\liminf_{n\longrightarrow\infty}\varsigma_n(1-\varsigma_n) > 0$ and $\liminf_{n\longrightarrow\infty}\beta_n(1-\beta_n) > 0$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(T) \cap F(S)$.

Motivated by the above result, Osilike and Isiogugu obtained the following result.

Theorem 4 (see [10], Theorem 3.1.1). *Let C be a nonempty convex closed subset of a real Hilbert space, and let $T\colon C \longrightarrow C$ be a k-strictly pseudo-nonspreading mapping with a nonempty fixed point set $F(T)$. Let $\beta \in [k,1)$, and let $\{\varsigma_n\}_{n=1}^{\infty}$ be a real sequence in $[0,1)$ such that $\lim_{n\longrightarrow\infty}\varsigma_n = 0$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated for arbitrary $x_1 \in C$ by*

$$\begin{aligned} x_{n+1} &= \varsigma_n x_n + (1-\varsigma_n)[\beta x_n + (1-\beta)T x_n], \quad n \geq 1, \\ z_n &= \frac{1}{n}\sum_{k=1}^{n} x_k, \quad n \geq 1. \end{aligned} \tag{16}$$

Then, $\{z_n\}_{n=1}^{\infty}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\longrightarrow\infty}P_{F(T)}x_n$.

We observed that all the existing iteration schemes for the approximation of a common fixed point of a finite family $\{T_1, T_2, \ldots, T_N\}$ of mappings for $N > 2$, which are related to the parallel algorithm, require the condition that, for each n, $\sum_{i=1}^{N}\varsigma_n^i = 1$ on the control sequences $\{\{\varsigma_n^i\}_{n=1}^{\infty}\}_{i=1}^{N}$. However, in real-life applications, if N is very large, it is very difficult or almost impossible to generate a family of such control sequences. Moreover, the computational cost of generating such a family of control sequences is very high and also takes a very long process. On the contrary, the algorithms of Iemoto and Takahashi [14] and Osilike and Isiogugu [10] do not require the imposition $\sum_{i=1}^{N}\varsigma_n^i = 1$ on the control sequences for $N = 2$. Consequently, there is a need to extend the iteration schemes in [10, 14] for $N > 2$.

Motivated by the above observations and the algorithms of Iemoto and Takahashi [14] and Osilike and Isiogugu [10], which do not require the imposition $\sum_{i=1}^{N}\varsigma_n^i = 1$ on the control sequences for $N = 2$ and the need to extend the iteration schemes for $N > 2$, the aim of this work is first to study some possible linear combinations of the products of the elements of a family of sequence of real numbers $\{\{\varsigma_n^i\}_{n=1}^{\infty}\}_{i=1}^{N}$ whose sum is unity. Second, to apply the result to construct a new (horizontal) algorithm which does not require the condition $\sum_{i=1}^{N}\varsigma_n^i = 1$ on the finite family of the control sequences $\{\{\varsigma_n^i\}_{n=1}^{\infty}\}_{i=1}^{N}$. Third, to prove that the new algorithm converges weakly and strongly to an element in the intersection of the set of fixed points of a countable finite family of multivalued type-one demicontractive mappings. We also show that our algorithm is an extension of the algorithm of Osilike and Isiogugu [10] when $N = 2$. Furthermore, the algorithm is applied to establish some new algorithms for the approximation of the common solution of an equilibrium problem and a fixed point problem for a finite family of type demicontractive mappings. The numerical examples and computations of the horizontal algorithm were also presented. The obtained results complement, extend, and improve many results on the iteration schemes for the approximation of common fixed points for a finite family of single-valued and multivalued mappings.

2. Preliminaries

In the sequel, we shall need the following definitions and lemmas.

Definition 1 (see, e.g., [26–27]). Let Y be a Banach space and $S\colon D(S) \subseteq Y \longrightarrow 2^Y$ be a multivalued mapping. $I - S$ is weakly demiclosed at zero if for any sequence, $\{x_n\}_{n=1}^{\infty} \subseteq D(S)$ such that $\{x_n\}$ converges weakly to p and a sequence $\{y_n\}$ with $y_n \in Sx_n$ for all $n \in \mathbb{N}$ such that $\{x_n - y_n\}$ strongly converges to zero. Then, $p \in Sp$ (i.e., $0 \in (I - S)p$).

Definition 2. A Banach space Y is said to satisfy Opial's condition [28] if whenever a sequence $\{x_n\}$ weakly converges to $x \in Y$, then it is the case that

$$\liminf\|x_n - x\| < \liminf\|x_n - y\|, \tag{17}$$

for all $y \in Y$, $y \neq x$.

Definition 3 (see [29]). A multivalued mapping $S\colon C \longrightarrow P(C)$ is said to satisfy condition (1) (see, for example, [29]) if there exists a nondecreasing function $f\colon [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$d(x, Sx) \geq f(d(x, F(S))), \quad \forall x \in C. \tag{18}$$

Definition 4 (see [4]). Let Y be a normed space and $S\colon D(S) \subseteq Y \longrightarrow 2^Y$ be a multivalued map. S is of type-one if given any pair $x, y \in D(S)$, then

$$\|u - v\| \leq \mathscr{D}(Sx, Sy), \quad \text{for all}\, u \in P_S x,\ v \in P_S y. \tag{19}$$

Lemma 1 (see [30]). *Let $\{a_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq a_n + \gamma_n, \quad \forall n \in N. \tag{20}$$

If $\sum \gamma_n < \infty$, then $\lim_{n\longrightarrow\infty} a_n$ exists.

3. Main Results

Let K be a nonempty convex and closed subset of a real Hilbert space H. Suppose that $\{S_i\}_{i=1}^N$, $N \geq 2$ is a countable finite family of mappings S_i: $K \longrightarrow K$, and we consider the horizontal iteration process generated from arbitrary x_1 for the finite family of mappings $\{S^i\}_{i=1}^N$, using a finite family of the control sequences $\{\{\varsigma_n^i\}_{n=1}^\infty\}_{i=1}^N$ as follows:

For $N = 2$,

$$x_{n+1} = \varsigma_n^1 x_n + \left(1 - \varsigma_n^1\right)\left[\varsigma_n^2 S_1 x_n + \left(1 - \varsigma_n^2\right) S_2 x_n\right]. \tag{21}$$

For $N = 3$,

$$\begin{aligned} x_{n+1} = \varsigma_n^1 x_n &+ \left(1 - \varsigma_n^1\right)\left[\varsigma_n^2 S_1 x_n + \left(1 - \varsigma_n^2\right)\right. \\ &\left.\cdot\left[\varsigma_n^3 S_2 x_n + \left(1 - \varsigma_n^3\right) S_3 x_n\right]\right]. \end{aligned} \tag{22}$$

For arbitrary but finite $N \geq 2$,

$$\begin{aligned} x_{n+1} &= \varsigma_n^1 x_n + \left(1 - \varsigma_n^1\right)\left[\varsigma_n^2 S_1 x_n + \left(1 - \varsigma_n^2\right)\left[\varsigma_n^3 S_2 x_n\right.\right. \\ &\quad \left.\left. + \left(1 - \varsigma_n^3\right)\left[\ldots\left[\varsigma_N S_{N-1} x_n + (1 - \varsigma_N) S_N x_n\right]\ldots\right]\right]\right] \\ &= \varsigma_n^1 x_n + \sum_{i=2}^{N} \varsigma_n^i \prod_{j=1}^{i-1}\left(1 - \varsigma_n^j\right) S_{i-1} x_n \\ &\quad + \prod_{j=1}^{N}\left(1 - \varsigma_n^j\right) S_N x_n, \ n \geq 1. \end{aligned} \tag{23}$$

We now present the following results which are very useful in establishing our convergence theorems.

Proposition 1. *Let $\{\varsigma_i\}_{i=1}^N \subseteq \mathbb{R}$ be a countable subset of the set of real numbers $\mathbb{R}$, where $N \geq 2$ is an arbitrary integer. Then, the following holds:*

$$\varsigma_1 + \sum_{i=2}^{N} \varsigma_i \prod_{j=1}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=1}^{N}\left(1 - \varsigma_j\right) = 1. \tag{24}$$

Proof. For $N = 2$,

$$\begin{aligned} \varsigma_1 &+ \sum_{i=2}^{2} \varsigma_i \prod_{j=1}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=1}^{2}\left(1 - \varsigma_j\right) \\ &= \varsigma_1 + \varsigma_2\left(1 - \varsigma_1\right) + \left(1 - \varsigma_1\right)\left(1 - \varsigma_2\right) \\ &= \varsigma_1 + \left(1 - \varsigma_1\right)\left[\varsigma_2 + \left(1 - \varsigma_2\right)\right] \\ &= \varsigma_1 + \left(1 - \varsigma_1\right) = 1. \end{aligned} \tag{25}$$

We assume it is true for N and prove for N+1.

$$\begin{aligned} \varsigma_1 &+ \sum_{i=2}^{N+1} \varsigma_i \prod_{j=1}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=1}^{N+1}\left(1 - \varsigma_j\right) \\ &= \varsigma_1 + \sum_{i=2}^{N} \varsigma_i \prod_{j=1}^{i-1}\left(1 - \varsigma_j\right) + \varsigma_{N+1} \prod_{j=1}^{N}\left(1 - \varsigma_j\right) + \prod_{j=1}^{N+1}\left(1 - \varsigma_j\right) \\ &= \varsigma_1 + \sum_{i=2}^{N} \varsigma_i \prod_{j=1}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=1}^{N}\left(1 - \varsigma_j\right)\left[\varsigma_{N+1} + \left(1 - \varsigma_{N+1}\right)\right] \\ &= \varsigma_1 + \sum_{i=2}^{N} \varsigma_i \prod_{j=1}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=1}^{N}\left(1 - \varsigma_j\right) = 1. \end{aligned} \tag{26}$$

□

Remark 1. Proposition 1 holds if $\{\varsigma_i\}_{i=1}^N$ is replaced with $\{\varsigma_i\}_{i=0}^N$.

Proposition 2. *Let $\{\varsigma_i\}_{i=k}^N \subseteq \mathbb{R}$ be a countable subset of the set of real numbers $\mathbb{R}$, where k is a fixed nonnegative integer and $N \in \mathbb{N}$ is any integer with $k + 1 \leq N$. Then, the following holds:*

$$\varsigma_k + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=k}^{N}\left(1 - \varsigma_j\right) = 1. \tag{27}$$

Proof. For $k = 0$, $N = 1$, and $k = 1$, $N = 2$, the proofs follow from Remark 1 and Proposition 1, respectively.

We assume it is true for k and N. Now, for k and $N + 1$,

$$\begin{aligned} \varsigma_k &+ \sum_{i=k+1}^{N+1} \varsigma_i \prod_{j=k}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=k}^{N+1}\left(1 - \varsigma_j\right) \\ &= \varsigma_k + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1 - \varsigma_j\right) + \varsigma_{N+1} \prod_{j=k}^{N}\left(1 - \varsigma_j\right) + \prod_{j=k}^{N+1}\left(1 - \varsigma_j\right) \\ &= \varsigma_k + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=k}^{N}\left(1 - \varsigma_j\right)\left[\varsigma_{N+1} + \left(1 - \varsigma_{N+1}\right)\right] \\ &= \varsigma_k + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=k}^{N}\left(1 - \varsigma_j\right) \\ &= \varsigma_k + \sum_{i=k}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1 - \varsigma_j\right) + \prod_{j=k}^{N}\left(1 - \varsigma_j\right) = 1. \end{aligned} \tag{28}$$

□

Proposition 3. *Let t, u, and v be arbitrary elements of a real Hilbert space H. Let k be a fixed nonnegative integer and $N \in \mathbb{N}$ be such that $k + 1 \leq N$. Let $\{v_i\}_{i=k}^{N-1} \subseteq H$ and $\{\varsigma_i\}_{i=k}^N \subseteq [0, 1]$ be a countable finite subset of H and $\mathbb{R}$, respectively. Define*

$$y = \varsigma_k t + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1 - \varsigma_j\right) v_{i-1} + \prod_{j=k}^{N}\left(1 - \varsigma_j\right) v. \tag{29}$$

Then,

$$\begin{aligned}\|y-u\|^2 = {} & \varsigma_k\|t-u\|^2 + \sum_{i=k+1}^{N}\varsigma_i\prod_{j=k}^{i-1}\left(1-\varsigma_j\right)\left\|v_{i-1}-u\right\|^2 + \prod_{j=k}^{N}\left(1-\varsigma_j\right)\|v-u\|^2 \\ & - \varsigma_k\left[\sum_{i=k+1}^{N}\varsigma_i\prod_{j=k}^{i-1}\left(1-\varsigma_j\right)\left\|t-v_{i-1}\right\|^2 + \prod_{j=k}^{N}\left(1-\varsigma_j\right)\|t-v\|^2\right] \\ & - \left(1-\varsigma_k\right)\left[\sum_{i=k+1}^{N-1}\varsigma_i\prod_{j=k}^{i}\left(1-\varsigma_j\right)\left\|v_{i-1}-\left[\varsigma_{i+1}v_i+w_{i+1}\right]\right\|^2 \right. \\ & \left. +\varsigma_N\prod_{j=k}^{N}\left(1-\varsigma_j\right)\left\|v-v_{N-1}\right\|^2\right],\end{aligned} \tag{30}$$

where $w_k = \sum_{i=k+1}^{N}\varsigma_i\prod_{j=k}^{i-1}(1-\varsigma_j)v_{i-1} + \prod_{j=k}^{N}(1-\varsigma_j)v$, $k = 1,2,\ldots,N-1$, and $w_N = (1-\varsigma_N)v$.

Proof. Using the well-known identity,

$$\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \tag{31}$$

which holds for all $x, y \in H$ and for all $t \in [0,1]$, we prove by (i) direct computation and (ii) induction.

Observe that, for $k \le N-1$, $w_k = (1-\varsigma_k)[\varsigma_{k+1}v_k + w_{k+1}]$. Consequently, by the direct computation, we have

$$\begin{aligned}\|y-u\|^2 & = \left\|\varsigma_k t + \sum_{i=k+1}^{N}\varsigma_i\prod_{j=k}^{i-1}\left(1-\varsigma_j\right)v_{i-1} + \prod_{j=k}^{N}\left(1-\varsigma_j\right)v-u\right\|^2 \\ & = \left\|\varsigma_k t + w_k - u\right\|^2 \\ & = \left\|\varsigma_k t + \left(1-\varsigma_k\right)\left[\varsigma_{k+1}v_k + w_{k+1}\right]-u\right]\right\|^2 \\ & = \varsigma_k\|t-u\|^2 + \left(1-\varsigma_k\right)\left\|\varsigma_{k+1}v_k + w_{k+1}-u\right\|^2 - \varsigma_k\left(1-\varsigma_k\right)\left\|t-\left[\varsigma_{k+1}v_k+w_{k+1}\right]\right\|^2 \\ & = \varsigma_k\|t-u\|^2 + \left(1-\varsigma_k\right)\left[\varsigma_{k+1}\left\|v_k-u\right\|^2 + \left(1-\varsigma_{k+1}\right)\left\|\varsigma_{k+2}v_{k+1}+w_{k+2}-u\right\|^2\right. \\ & \quad \left. -\varsigma_{k+1}\left(1-\varsigma_{k+1}\right)\left\|v_k-\left[\varsigma_{k+2}v_{k+1}+w_{k+2}\right]\right\|^2\right] \\ & \quad - \varsigma_k\left(1-\varsigma_k\right)\left[\varsigma_{k+1}\left\|t-v_k\right\|^2 + \left(1-\varsigma_{k+1}\right)\left\|t-\left[\varsigma_{k+2}v_{k+1}+w_{k+2}\right]\right\|^2\right. \\ & \quad \left. -\varsigma_{k+1}\left(1-\varsigma_{k+1}\right)\left\|v_k-\left[\varsigma_{k+2}v_{k+1}+w_{k+2}\right]\right\|^2\right] \\ & = \varsigma_k\|t-u\|^2 + \left(1-\varsigma_k\right)\varsigma_{k+1}\left\|v_k-u\right\|^2 \\ & \quad + \left(1-\varsigma_k\right)\left(1-\varsigma_{k+1}\right)\left\|\varsigma_{k+2}v_{k+1}+w_{k+2}-u\right\|^2 \\ & \quad - \left(1-\varsigma_k\right)\varsigma_{k+1}\left(1-\varsigma_{k+1}\right)\left\|v_k-\left[\varsigma_{k+2}v_{k+1}+w_{k+2}\right]\right\|^2 \\ & \quad - \varsigma_k\left(1-\varsigma_k\right)\varsigma_{k+1}\left\|t-v_k\right\|^2 - \varsigma_k\left(1-\varsigma_k\right)\left(1-\varsigma_{k+1}\right)\left\|t-\left[\varsigma_{k+2}v_{k+1}+w_{k+2}\right]\right\|^2 \\ & \quad \left.+\varsigma_k\left(1-\varsigma_k\right)\varsigma_{k+1}\left(1-\varsigma_{k+1}\right)\left\|v_k-\left[\varsigma_{k+2}v_{k+1}+w_{k+2}\right]\right\|^2\right] \\ & = \varsigma_k\|t-u\|^2 + \left(1-\varsigma_k\right)\varsigma_{k+1}\left\|v_k-u\right\|^2 - \varsigma_k\left(1-\varsigma_k\right)\varsigma_{k+1}\left\|t-v_k\right\|^2 \\ & \quad + \left(1-\varsigma_k\right)\left(1-\varsigma_{k+1}\right)\left\|\varsigma_{k+2}v_{k+1}+w_{k+2}-u\right\|^2 \\ & \quad - \varsigma_k\left(1-\varsigma_k\right)\left(1-\varsigma_{k+1}\right)\left\|t-\left[\varsigma_{k+2}v_{k+1}+w_{k+2}\right]\right\|^2 \\ & \quad - \varsigma_{k+1}\left(1-\varsigma_k\right)\left(1-\varsigma_{k+1}\right)\left(1-\varsigma_k\right)\left\|v_k-\left[\varsigma_{k+2}v_{k+1}+w_{k+2}\right]\right\|^2\end{aligned}$$

$$
\begin{aligned}
&= \varsigma_k\|t-u\|^2 + (1-\varsigma_k)\varsigma_{k+1}\|v_k-u\|^2 - \varsigma_k(1-\varsigma_k)\varsigma_{k+1}\|t-v_k\|^2 \\
&\quad - \varsigma_{k+1}(1-\varsigma_k)(1-\varsigma_{k+1})(1-\varsigma_k)\|v_k-[\varsigma_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&\quad + (1-\varsigma_k)(1-\varsigma_{k+1})\|\varsigma_{k+2}v_{k+1}+w_{k+2}-u\|^2 \\
&\quad - \varsigma_k(1-\varsigma_k)(1-\varsigma_{k+1})\|t-[\varsigma_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&= \varsigma_k\|t-u\|^2 + (1-\varsigma_k)\varsigma_{k+1}\|v_k-u\|^2 - \varsigma_k(1-\varsigma_k)\varsigma_{k+1}\|t-v_k\|^2 \\
&\quad - \varsigma_{k+1}(1-\varsigma_k)(1-\varsigma_{k+1})(1-\varsigma_k)\|v_k-[\varsigma_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&\quad + (1-\varsigma_k)(1-\varsigma_{k+1})\|\varsigma_{k+2}v_{k+1}+(1-\varsigma_{k+2})[\varsigma_{k+3}v_{k+2}+w_{k+3}]-u\|^2 \\
&\quad - \varsigma_k(1-\varsigma_k)(1-\varsigma_{k+1})\|\varsigma_{k+2}v_{k+1}+(1-\varsigma_{k+2})[\varsigma_{k+3}v_{k+2}+w_{k+3}]-t\|^2 \\
&= \varsigma_k\|t-u\|^2 + \left(1-\varsigma_n^1\right)\varsigma_{k+1}\|v_k-u\|^2 - \varsigma_n^1\left(1-\varsigma_n^1\right)\varsigma_{k+1}\|t-v_k\|^2 \\
&\quad + (1-\varsigma_k)(1-\varsigma_{k+1})\varsigma_{k+2}\|v_{k+1}-u\|^2 \\
&\quad + (1-\varsigma_k)(1-\varsigma_{k+1})(1-\varsigma_{k+2})\|\varsigma_{k+3}v_{k+2}+w_{k+3}-u\|^2 \\
&\quad - (1-\varsigma_k)(1-\varsigma_{k+1})\varsigma_{k+2}(1-\varsigma_{k+2})\|v_{k+1}-[\varsigma_{k+3}v_{k+2}+w_{k+3}]\|^2 \\
&\quad - \varsigma_k(1-\varsigma_k)(1-\varsigma_{k+1})\varsigma_{k+2}\|v_{k+1}-t\|^2 \\
&\quad - \varsigma_k(1-\varsigma_k)(1-\varsigma_{k+1})(1-\varsigma_{k+2})\|\varsigma_{k+3}v_{k+2}+w_{k+3}-t\|^2 \\
&\quad + \varsigma_k(1-\varsigma_k)(1-\varsigma_{k+1})\varsigma_{k+2}\Big(1-\varsigma_{k+2}\|v_{k+1}-\varsigma_{k+3}v_{k+2}+w_{k+3}\|^2 \\
&\quad - \varsigma_{k+1}(1-\varsigma_k)(1-\varsigma_{k+1})(1-\varsigma_k)\|v_k-[\varsigma_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&= \varsigma_k\|t-u\|^2 + (1-\varsigma_k)\varsigma_{k+1}\|v_k-u\|^2 - \varsigma_k(1-\varsigma_k)\varsigma_{k+1}\|t-v_k\|^2 \\
&\quad + (1-\varsigma_k)(1-\varsigma_{k+1})\varsigma_{k+2}\|v_{k+1}-u\|^2 \\
&\quad + (1-\varsigma_k)(1-\varsigma_{k+1})(1-\varsigma_{k+2})\|\varsigma_{k+3}v_{k+2}+w_{k+3}-u\|^2 \\
&\quad - \varsigma_k(1-\varsigma_k)(1-\varsigma_{k+1})\varsigma_{k+2}\|v_{k+1}-t\|^2 \\
&\quad - \varsigma_k(1-\varsigma_k)(1-\varsigma_{k+1})(1-\varsigma_{k+2})\|\varsigma_{k+3}v_{k+2}+w_{k+3}-t\|^2 \\
&\quad - \varsigma_{k+1}(1-\varsigma_k)(1-\varsigma_{k+1})(1-\varsigma_k)\|v_k-[\varsigma_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&\quad - (1-\varsigma_k)^2(1-\varsigma_{k+1})\varsigma_{k+2}(1-\varsigma_{k+2})\|v_{k+1}-[\varsigma_{k+3}v_{k+2}+w_{k+3}]\|^2, \\
&= \varsigma_k\|t-u\|^2 + \sum_{i=k+1}^{k+2}\varsigma_i\prod_{j=k}^{i-1}\left(1-\varsigma_j\right)\|v_{i-1}-u\|^2 \\
&\quad - \varsigma_k\left[\sum_{i=k+1}^{k+2}\varsigma_i\prod_{j=k}^{i-1}\left(1-\varsigma_j\right)\|t-v_{i-1}\|^2\right] \\
&\quad - (1-\varsigma_k)\left[\sum_{i=k+1}^{k+2}\varsigma_i\prod_{j=k}^{i}\left(1-\varsigma_j\right)\|v_{i-1}-[\varsigma_{i+1}v_i+w_{i+1}]\|^2\right. \\
&\quad + \prod_{j=k}^{k+2}\left(1-\varsigma_j\right)\|[\varsigma_{k+3}v_{k+2}+w_{k+3}]-u\|^2 \\
&\quad - \varsigma_k\prod_{j=k}^{k+2}\left(1-\varsigma_j\right)\|t-[\varsigma_{k+3}v_{k+2}+w_{k+3}]\|^2 \\
&= \\
&\quad \vdots
\end{aligned}
$$

$$
\begin{aligned}
&= \varsigma_k\|t-u\|^2 + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1-\varsigma_j\right)\left\|v_{i-1}-u\right\|^2 + \prod_{j=k}^{N}\left(1-\varsigma_j\right)\|v-u\|^2 \\
&\quad - \varsigma_k\left[\sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1-\varsigma_j\right)\left\|t-v_{i-1}\right\|^2 + \prod_{j=k}^{N}\left(1-\varsigma_j\right)\|t-v\|^2\right] \\
&\quad - \left(1-\varsigma_k\right)\left[\sum_{i=k+1}^{N-1} \varsigma_i \prod_{j=k}^{i}\left(1-\varsigma_j\right)\left\|v_{i-1}-\left[\varsigma_{i+1}v_i + w_{i+1}\right]\right\|^2\right. \\
&\quad \left.+\varsigma_N \prod_{j=k}^{N}\left(1-\varsigma_j\right)\left\|v-v_{N-1}\right\|^2\right].
\end{aligned} \tag{32}
$$

Therefore, it holds for k, N from direct computation.

Since induction holds for a fixed k and each N from direct computation, then it is true for $k,\ N = 1, 2, 3$. Thus, to prove by induction, we then assume that it is true for $k,\ N$ and prove for k and $N+1$. From

$$
y = \varsigma_k t + \sum_{i=k+1}^{N+1} \varsigma_i \prod_{j=k}^{i-1}\left(1-\varsigma_j\right)v_{i-1} + \prod_{j=k}^{N+1}\left(1-\varsigma_j\right)v, \tag{33}
$$

we have that

$$
\begin{aligned}
\|y-u\|^2 &= \left\|\varsigma_k t + \sum_{i=k+1}^{N+1} \varsigma_i \prod_{j=k}^{i-1}\left(1-\varsigma_j\right)v_{i-1} + \prod_{j=k}^{N+1}\left(1-\varsigma_j\right)v - u\right\|^2 \\
&= \left\|\varsigma_k t + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1-\varsigma_j\right)v_{i-1} + \varsigma_{N+1}\prod_{j=k}^{N}\left(1-\varsigma_j j\right)v_N + \prod_{j=k}^{N+1}\left(1-\varsigma_j\right)v - u\right\|^2 \\
&= \left\|\varsigma_k t + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1-\varsigma_j\right)v_{i-1} + \prod_{j=k}^{N}\left(1-\varsigma_j\right)\left[\varsigma_{N+1}v_N + \left(1-\varsigma_{N+1}\right)v\right] - u\right\|^2 \\
&= \left\|\varsigma_k t + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1-\varsigma_j\right)v_{i-1} + \prod_{j=k}^{N}\left(1-\varsigma_j\right)v^* - u\right\|^2 \\
&= \varsigma_k\|t-u\|^2 + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1}\left(1-\varsigma_j\right)\left\|v_{i-1}-u\right\|^2 + \prod_{j=k}^{N}\left(1-\varsigma_j\right)\left\|v^*-u\right\|^2 \\
&\quad - \varsigma_k\left[\sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-k}\left(1-\varsigma_j\right)\left\|t-v_{i-1}\right\|^2 + \prod_{j=k}^{N}\left(1-\varsigma_j\right)\left\|t-v^*\right\|^2\right] \\
&\quad - \left(1-\varsigma_k\right)\left[\sum_{i=k+1}^{N-1} \varsigma_i \prod_{j=k}^{i}\left(1-\varsigma_j\right)\left\|v_{i-1}-\left[\varsigma_{i+1}v_i + w_{i+1}\right]\right\|^2\right. \\
&\quad \left.+\varsigma_N \prod_{j=k}^{N}\left(1-\varsigma_j\right)\left\|v^*-v_{N-1}\right\|^2\right].
\end{aligned} \tag{34}
$$

Observe that

$$
\begin{aligned}
\left\|t-v^*\right\|^2 &= \left\|\varsigma_{N+1}v_N + \left(1-\varsigma_{N+1}\right)v - t\right\|^2 \\
&= \varsigma_{N+1}\left\|t-v_N\right\|^2 + \left(1-\varsigma_{N+1}\right)\|t-v\|^2 \\
&\quad - \varsigma_{N+1}\left(1-\varsigma_{N+1}\right)\left\|v_N-v\right\|^2.
\end{aligned} \tag{35}
$$

Also,

$$
\begin{aligned}
\left\|v^*-v_{N-1}\right\|^2 &= \left\|\varsigma_{N+1}v_N + \left(1-\varsigma_{N+1}\right)v - v_{N-1}\right\|^2 \\
&= \left\|v_{N-1} - \left[\varsigma_{N+1}v_N + \left(1-\varsigma_{N+1}\right)v\right]\right\|^2 \\
&= \left\|v_{N-1} - \left[\varsigma_{N+1}v_N + w_{N+1}\right]\right\|^2.
\end{aligned} \tag{36}
$$

Furthermore,

$$\begin{aligned}\left\|v^{*}-u\right\|^{2}&=\left\|\varsigma_{N+1}v_{N}+\left(1-\varsigma_{N+1}\right)v-u\right\|^{2}\\&=\varsigma_{N+1}\left\|u-v_{N}\right\|^{2}+\left(1-\varsigma_{N+1}\right)\|v-u\|^{2}\\&\quad-\varsigma_{N+1}\left(1-\varsigma_{N+1}\right)\left\|v_{N}-v\right\|^{2}.\end{aligned}\tag{37}$$

It then follows from (34–37) that

$$\begin{aligned}\|y-u\|^{2}&=\varsigma_{k}\|t-u\|^{2}+\sum_{i=k+1}^{N+1}\varsigma_{i}\prod_{j=k}^{i-1}\left(1-\varsigma_{j}\right)\left\|v_{i-1}-u\right\|^{2}\\&\quad+\prod_{j=k}^{N+1}\left(1-\varsigma_{j}\right)\|v-u\|^{2}\\&\quad-\varsigma_{k}\left[\sum_{i=k+1}^{N+1}\varsigma_{i}\prod_{j=k}^{i-i}\left(1-\varsigma_{j}\right)\left\|t-v_{i-1}\right\|^{2}+\prod_{j=k}^{N+1}\left(1-\varsigma_{j}\right)\|t-v\|^{2}\right]\\&\quad-\left(1-\varsigma_{k}\right)\left[\sum_{i=k+1}^{N}\varsigma_{i}\prod_{j=k}^{i}\left(1-\varsigma_{j}\right)\left\|v_{i-1}-\left[\varsigma_{i+1}v_{i}+w_{i+1}\right]\right\|^{2}\right.\\&\quad\left.+\varsigma_{N+1}\prod_{j=k}^{N+1}\left(1-\varsigma_{j}\right)\left\|v-v_{N}\right\|^{2}\right].\end{aligned}\tag{38}$$

We now apply Propositions 2 and 3 to prove the following weak and strong convergence theorems for type-one demicontractive mappings.

□

Theorem 5. *Let K be a nonempty convex and closed subset of a real Hilbert space H. Suppose that $\{S_i\}_{i=1}^{N}$, $N\geq 2$ is a countable finite family of type-one demicontractive mappings S_i: $K\longrightarrow P(K)$ from K into the family of all proximinal subsets of K with contractive coefficients $\lambda_i\in[0,1)$ for each i. Suppose that $\cap_{i=1}^{N}F(S_i)\neq\varnothing$ and for each i, $(I-S_i)$ is weakly demiclosed at zero; then, the sequence of the horizontal algorithm defined by*

$$\begin{aligned}x_{n+1}&=\varsigma_{n,1}x_{n}+\sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)y_{n,i-1}\\&\quad+\prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)y_{n,N},\quad n\geq 1,\end{aligned}\tag{39}$$

converges weakly to $q\in\cap_{i=1}^{N}F(S_i)$, where $y_{n,i}\in S_i x_n$ for each i and $\left\{\{\varsigma_{n,i}\}_{n=1}^{\infty}\right\}_{i=1}^{N}$ is a countable finite family of real sequences in [0, 1] satisfying the following:

(i) $\varsigma_{n,1}>\lambda>\max\{\lambda_i\}_{i=1}^{N}$; $\varsigma_{n,i}<\varsigma<1$, for each i.
(ii) $\liminf_{n\longrightarrow\infty}\varsigma_{n,i}\prod_{j=1}^{i-1}(1-\varsigma_{n,j})(\varsigma_{n,1}-\lambda_{i-1})>0$, $i=2,3,\ldots,N$.
(iii) $\liminf_{n\longrightarrow\infty}\prod_{j=1}^{N}(1-\varsigma_{n,j})(\varsigma_{n,1}-\lambda_N)>0$.

Also, if, in addition, S_i is L-Lipschitzian and satisfies condition (1) for each i, then $\{x_n\}$ converges strongly to $q\in\cap_{i=1}^{N}F(S_i)$.

Proof. Setting $x_{n+1}=y$, $x_n=t$, $p=u$, $k=1$, and $y_{n,N}\in S_N x_n=v$ in Proposition 3, we obtain

$$\begin{aligned}\left\|x_{n+1}-p\right\|^{2}&\leq\varsigma_{n,1}\left\|x_{n}-p\right\|^{2}+\sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)\left\|y_{n,(i-1)}-p\right\|^{2}\\&\quad+\prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)\left\|y_{n,N}-p\right\|^{2}\\&\quad-\varsigma_{n,1}\left[\sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)\left\|x_{n}-y_{n,i-1}\right\|^{2}\right.\\&\quad\left.+\prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)\left\|x_{n}-y_{n,N}\right\|^{2}\right].\end{aligned}\tag{40}$$

Applying type-one demicontractive condition on each S_i, we obtain

$$\begin{aligned}\left\|x_{n+1}-p\right\|^{2}&\leq\varsigma_{n,1}\left\|x_{n}-p\right\|^{2}+\sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)\left[\left\|x_{n}-p\right\|^{2}+\lambda_{i-1}\left\|x_{n}-y_{n,(i-1)}\right\|^{2}\right]\\&\quad+\prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)\left[\left\|x_{n}-p\right\|^{2}+\lambda_{N}\left\|x_{n}-y_{n,N}\right\|^{2}\right]\\&\quad-\varsigma_{n,1}\left[\sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)\left\|x_{n}-y_{n,i-1}\right\|^{2}+\prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)\left\|x_{n}-y_{n,N}\right\|^{2}\right]\\&=\left[\varsigma_{n,1}+\sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)+\prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)\right]\left\|x_{n}-p\right\|^{2}\\&\quad-\left(\varsigma_{n,1}-\lambda_{i-1}\right)\sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)\left\|x_{n}-y_{n,i-1}\right\|^{2}\\&\quad-\left(\varsigma_{n,1}-\lambda_{N}\right)\prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)\left\|x_{n}-y_{n,N}\right\|^{2}.\end{aligned}\tag{41}$$

Consequently, if we set $k = 1$ in Proposition 2, we obtain

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 - \left[\sum_{i=2}^{N} \varsigma_{n,i} \prod_{j=1}^{i-1}\left(1 - \varsigma_{n,j}\right)\left(\varsigma_{n,1} - \lambda_{i-1}\right) \cdot \|x_n - y_{n,i-1}\|^2 + \prod_{j=1}^{N}\left(1 - \varsigma_{n,j}\right)\left(\varsigma_{n,1} - \lambda_N\right)\|x_n - y_{n,N}\|^2\right]. \quad (42)$$

Furthermore, condition (i) on the control sequences implies that $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists; hence, $\{x_n\}$ is bounded. Similarly, conditions (ii) and (iii) imply that $\lim_{n\longrightarrow\infty}\|x_n - y_{n,i}\|0$, $i = 1, 2, \ldots, N$. Finally, the demiclosedness property of each $(I - S_i)$, boundedness of $\{x_n\}$, uniqueness of the limit of a weakly convergent sequence, and Opial property of a real Hilbert space guarantee the weak convergence of $\{x_n\}$ to $q \in \cap_{i=1}^{N} F(S_i)$. Also, since S_i is L-Lipschitzian and satisfies condition (1) for each i, it then follows from standard argument that $\{x_n\}$ converges strongly to $q \in \cap_{i=1}^{N} F(S_i)$.

□

Remark 2. If $N = 2$ and we set $\varsigma_{n,1} = \varsigma_n$ and $\varsigma_{n,2} = \beta$, for all n, I $(the\ identity\ mapping) = S_1$, and $S = S_2$, we obtain

$$\begin{aligned} x_{n+1} &= \varsigma_{n,1}x_n + \sum_{i=2}^{2} \varsigma_{n,i} \prod_{j=1}^{i-1}\left(1 - \varsigma_{n,j}\right)S_{n,i-1} + \prod_{j=1}^{2}\left(1 - \varsigma_{n,j}\right)S_{n,2} \\ &= \varsigma_n x_n + (1 - \varsigma_n)\beta_n I x_n + (1 - \varsigma_n)(1 - \beta_n)Sx_n \\ &= \varsigma_n x_n + (1 - \varsigma_n)[\beta x_n + (1 - \beta)Sx_n], \end{aligned} \quad (43)$$

which was considered by Osilike and Isiogugu [10].

4. Applications

We now present the application of Theorem 5 in the construction of algorithms for approximating a common solution of an equilibrium problem and fixed point problem.

For solving the equilibrium problems for a bifunction $F: C \times C \longrightarrow \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1): $F(g, g) = 0$ for all $g \in C$

(A2): F is monotone, that is, $F(g, h) + F(h, g) \le 0$, for all $g, h \in C$

(A3): for each $g, h, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)g, h) \le F(g, h)$

(A4): for each $g \in C$, $h \mapsto F(g, h)$ is convex and lower semicontinuous

Lemma 2 (see [31]). *Let C be a nonempty convex closed subset of a real Hilbert space H and $F: C \times C \longrightarrow \mathbb{R}$, a bifunction satisfying (A1)–(A4). Let $r > 0$ and $g \in H$. Then, there exists $z \in C$ such that*

$$F(z, h) + \frac{1}{r}\langle h - z, z - g\rangle \ge 0, \quad \forall h \in C. \quad (44)$$

Lemma 3 (see [32]). *Let C be a nonempty convex closed subset of a real Hilbert space H. Assume that $F: C \times C \longrightarrow \mathbb{R}$ satisfies (A1)–(A4). Let $r > 0$ and $g \in H$. Define $T_r: H \longrightarrow 2^C$ by*

$$T_r(g) = \left\{z \in C: F(z, h) + \frac{1}{r}\langle h - z, z - g\rangle \ge 0\right\}, \quad \forall h \in C. \quad (45)$$

Then, the following hold:

(1) T_r is single valued.

(2) T_r is firmly nonexpansive, that is, for any $g, h \in H$, $\|T_r g - T_r h\|^2 \le \langle T_r g - T_r h, g - h\rangle$.

(3) $F(T_r) = EP(F)$.

(4) $EP(F)$ is convex and closed.

Lemma 4 (see [33]). *Let C be a nonempty convex closed subset of a real Hilbert space H and $F: C \times C \longrightarrow \mathbb{R}$, a bifunction satisfying (A1)–(A4). Let $r > 0$ and $g \in H$. Then, for all $g \in H$ and $p \in F(T_r)$,*

$$\|p - T_r g\|^2 + \|T_r g - g\|^2 \le \|p - g\|^2. \quad (46)$$

Lemma 5. *Let H be a real Hilbert space, and let C be a nonempty convex closed subset of H. Let P_C be the convex projection onto C. Then, convex projection is characterized by the following relations:*

(i) $g^* = P_C(g) \Leftrightarrow \langle g - g^*, h - g^*\rangle \le 0$, for all $h \in C$.

(ii) $\|g - P_C g\|^2 \le \|g - h\|^2 - \|h - P_C g\|^2$.

(iii) $\|g - P_C h\|^2 \le \|g - h\|^2 - \|P_C h - h\|^2$.

Motivated by Algorithm 19 of Isiogugu et al. [34], we obtain the following result using a selection of Algorithm 4.2 above in the sense of [34].

Theorem 6. *Let C be a nonempty convex closed subset of a real Hilbert space H, $f: C \times C \longrightarrow \mathbb{R}$, a bifunction satisfying (A1)–(A4) and $\{T_i\}_{i=1}^{N}$ be such that $T_i: C \longrightarrow P(C)$ is type-one λ_i-strictly pseudo-contractive-type mappings, and $(I - T_i)$ is weakly demiclosed at zero for each $i = 1, 2, \ldots, N$. Suppose that $\mathbb{F} = \cap_{i=1}^{N} F_s(T_i) \cap EP(f) \neq \varnothing$. Let $\{x_n\}$ be a sequence generated from arbitrary $x_0 \in C$ as follows:*

Algorithm 1.

$$
\begin{cases}
y_n = \varsigma_{n,1}x_n + \sum_{i=2}^{N} \varsigma_{n,i} \prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right) y_{n,i-1} + \prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right) y_{n,N},\\
u_n \in K \text{ such that } F\left(u_n, y\right) + \frac{1}{r_n}\langle y-u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in K,\\
x_{n+1} = \frac{1}{2}\left(u_n + x_n\right),
\end{cases}
\tag{47}
$$

where $y_{n,i} \in T_i x_n$ for each i and $\left\{\left\{\varsigma_{n,i}\right\}_{n=1}^{\infty}\right\}_{i=1}^{N}$ is a finite family of real sequences in [0, 1] for each i satisfying

(i) $\varsigma_{n,1} > \lambda > \max\{\lambda_i\}_{i=1}^{N}$; $\varsigma_{n,i} < \varsigma < 1$, for each i.

(ii) $\liminf_{n\longrightarrow\infty} \varsigma_{n,i}\prod_{j=1}^{i-1}(1-\varsigma_{n,j})(\varsigma_{n,1}-\lambda_{i-1}) > 0$, $i = 2, 3, \ldots, N$.

(iii) $\liminf_{n\longrightarrow\infty} \prod_{j=1}^{N}(1-\varsigma_{n,j})(\varsigma_{n,1}-\lambda_N) > 0$.

Also, if, in addition, T_i satisfies condition (1) for each i,

(iv) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then, $\{x_n\}$ converges strongly to $p \in \mathbb{F}$.

Proof

$$
\begin{aligned}
\left\|x_{n+1}-p\right\|^2 &= \left\|\frac{1}{2}\left(x_n+u_n\right)-p\right\|^2\\
&= \frac{1}{2}\left\|x_n-p\right\|^2 + \frac{1}{2}\left\|u_n-p\right\|^2 - \frac{1}{4}\left\|x_n-u_n\right\|^2\\
&\leq \frac{1}{2}\left\|x_n-p\right\|^2 + \frac{1}{2}\left\|y_n-p\right\|^2 - \frac{1}{4}\left\|x_n-u_n\right\|^2\\
&= \frac{1}{2}\left\|x_n-p\right\|^2 - \frac{1}{4}\left\|x_n-u_n\right\|^2 + \frac{1}{2}\left\|\varsigma_{n,1}x_n + \sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)y_{n,i-1} + \prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)y_{n,N} - p\right\|^2\\
&\leq \frac{1}{2}\left\|x_n-p\right\|^2 - \frac{1}{4}\left\|x_n-u_n\right\|^2 + \frac{1}{2}\left[\left\|x_n-p\right\|^2 - \left[\sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)\left(\varsigma_{n,1}-\lambda_{i-1}\right)\left\|x_n-y_{n,i-1}\right\|^2\right.\right.\\
&\quad \left.\left. + \prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)\left(\varsigma_{n,1}-\lambda_N\right)\left\|x_n-y_{n,N}\right\|^2\right]\right]\\
&= -\frac{1}{4}\left\|x_n-u_n\right\|^2 + \left\|x_n-p\right\|^2 - \frac{1}{2}\left[\sum_{i=2}^{N}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)\left(\varsigma_{n,1}-\lambda_{i-1}\right)\left\|x_n-y_{n,i-1}\right\|^2\right.\\
&\quad \left. + \prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)\left(\varsigma_{n,1}-\lambda_N\right)\left\|x_n-y_{n,N}\right\|^2\right].
\end{aligned}
\tag{48}
$$

It then follows that $\lim_{n\longrightarrow\infty}\|x_n-p\|$ exists; hence, $\{x_n\}$ is bounded. Also,

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{2}\varsigma_{n,i}\prod_{j=1}^{i-1}\left(1-\varsigma_{n,j}\right)\left(\varsigma_{n,1}-\lambda_{i-1}\right)\left\|x_n-y_{n,i-1}\right\|^2\\
&\quad \leq \left\|x_0-p\right\|^2 < \infty, \quad i = 1, 2, \ldots, N-1.\\
&\sum_{n=1}^{\infty}\frac{1}{2}\prod_{j=1}^{N}\left(1-\varsigma_{n,j}\right)\left(\varsigma_{n,1}-\lambda_N\right)\left\|x_n-y_{n,N}\right\|^2 \leq \left\|x_0-p\right\|^2 < \infty.
\end{aligned}
\tag{49}
$$

Thus, from (i), (ii), and (iii), we have that $\lim_{n\longrightarrow\infty}\|x_n-y_{n,i}\| = 0$, for all $i = 1, 2, \ldots, N$. Furthermore, $\lim_{n\longrightarrow\infty}\|x_n-u_n\| = 0$. Consequently, $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|^2 = \lim_{n\longrightarrow\infty}\|1/2(x_n-u_n)\|^2 = 0$ which implies that $\{x_n\}$ is a Cauchy sequence in K. Also, since K is convex and closed, $\{x_n\}$ converges strongly to some $q \in K$. From the Opial condition of H and the demiclosedness property of T_i, we have that $q \in T_i q$, for all $i - 1, 2, \ldots, N$.

The remaining part of the proof is similar to the method of [34], Theorem 20. Therefore, it is omitted.

□

Table 1: Sequences of coordinates $\left\{\overline{x}_{t,n}\right\}_{n=1}^{\infty}$ of $\{\overline{x}_n\}_{n=1}^{\infty} = \left\{(x_{1,n}, x_{2,n}, \ldots, x_{t,n}, \ldots x_{m,n})\right\}_{n=1}^{\infty}$ for each $t = 1, 2, \ldots, m$.

$N = 5,\ x_{t,1} = 10$		$N = 5,\ x_{t,1} = -10$		$N = 10,\ x_{t,1} = -10$		$N = 10,\ x_{t,1} = 10$	
n	$x_{t,n}$	N	$x_{t,n}$	n	$x_{t,n}$	n	$x_{t,n}$
1	−10	1	10	1	10	1	−10
2	−4.863391744	2	4.863391744	2	4.86333914	2	−4.86333914
3	−2.426008398	3	2.426008398	3	2.42595782	3	−2.42595782
4	−1.220221064	4	1.220221064	4	1.22018365	4	−1.22018365
5	−0.615820431	5	0.615820431	5	0.61579556	5	−0.61579556
6	−0.311310767	6	0.311310767	6	0.31129518	6	−0.31129518
7	−0.157521707	7	0.157521707	7	0.1575123	7	−0.1575123
8	−0.079750993	8	0.079750993	8	0.07974546	8	−0.07974546
9	−0.040392014	9	0.040392014	9	0.04038883	9	−0.04038883
10	−0.020462938	10	0.020462938	10	0.02046113	10	−0.02046113
11	−0.01036864	11	0.01036864	11	0.01036763	11	−0.01036763
12	−0.005254554	12	0.005254554	12	0.00525399	12	−0.00525399
13	−0.002663152	13	0.002663152	13	0.00266284	13	−0.00266284
14	−0.00134987	14	0.00134987	14	0.0013497	14	−0.0013497
15	−0.000684252	15	0.000684252	15	0.00068416	15	−0.00068416
16	−0.000346866	16	0.000346866	16	0.00034682	16	−0.00034682
17	−0.000175843	17	0.000175843	17	0.00017582	17	−0.00017582
18	−0.000089144	18	0.000089144	18	0.00008913	18	−0.00008913
19	−0.000045193	19	0.000045193	19	0.00004519	19	−0.00004519
20	−0.000022912	20	0.000022912	20	0.00002291	20	−0.00002291
21	−0.000011615	21	0.000011615	21	0.00001161	21	−0.00001161
22	−0.000005887	22	0.000005887	22	0.00000588	22	−0.00000588
23	−0.000002983	23	0.000002983	23	0.00000298	23	−0.00000298
24	−0.000001509	24	0.000001509	24	0.00000151	24	−0.00000151
25	−0.000000762	25	0.000000762	25	0.00000076	25	−0.00000076
26	−0.000000384	26	0.000000384	26	0.00000038	26	−0.00000038
27	−0.000000191	27	0.000000191	27	0.00000019	27	−0.00000019
28	−0.000000095	28	0.000000095	28	0.00000009	28	−0.00000009
29	−0.000000046	29	0.000000046	29	0.00000004	29	−0.00000004
30	−0.00000002	30	0.00000002	30	0.00000002	30	−0.00000002
31	−0.000000007	31	0.000000007	31	0.00000001	31	−0.00000001
32	0	32	0				

5. Examples

We present the numerical computation of the iteration scheme of Theorem 5.

Let $H = (\mathbb{R}^m, \|.\|, \leq) R^m$ with the usual norm "$\|.\|$" on R^m and partial order "$\leq$" on $\mathbb{R}$, $C = \{\overline{x} = (x_1, x_2, \ldots, x_t, \ldots, x_m) \in \mathbb{R}^m\colon x_1 = x_2 = \ldots = x_t = \ldots = x_m\}$. Observe that $(C, \|.\|, \leq)$ is a convex closed linear total ordered subset of R^n with $\overline{a} \leq \overline{b}$ if and only if $a_t \leq b_t$ for all $t = 1, 2, 3, \ldots, m$. Denote the order interval $\overline{a} \leq \overline{x} \leq \overline{b}$ by $[\overline{a}, \overline{b}]$, and let $\{S_i\}_{i=1}^{\infty}$ be a countable infinite family of mappings and $S_i\colon C \longrightarrow CB(C)$ define for each i and $\overline{x} \in C$ by

$$S_i\overline{x} = \begin{cases} \left[-\left(\dfrac{4i}{2i+1}\right)\overline{x},\ -\left(\dfrac{3i}{2i+1}\right)\overline{x}\right], & \overline{x} \geq \overline{0}, \\ \left[-\left(\dfrac{3i}{2i+1}\right)\overline{x}, -\left(\dfrac{4i}{2i+1}\right)\overline{x}\right], & \overline{x} < \overline{0}. \end{cases} \tag{50}$$

Clearly, for each i,

(I) $F(S_i) = \{\overline{0}\}$.

(II) $P_{S_i}\overline{x} = \{-(3i/(2i+1))\overline{x}\}$.

$$\mathscr{D}(S_i\overline{x}, S_i\overline{y}) = \begin{cases} \dfrac{4i}{2i+1}\|\overline{x} - \overline{y}\|, & \overline{x}, \overline{y} \geq \overline{0}, \\ \dfrac{4i}{2i+1}\|\overline{x} - \overline{y}\|, & \overline{x}, \overline{y} < \overline{0}, \\ \left\|\dfrac{3i}{2i+1}\overline{x} - \dfrac{4i}{2i+1}\overline{y}\right\| \geq \left\|\dfrac{3i}{2i+1}\overline{x} - \dfrac{3i}{2i+1}\overline{y}\right\|, & \overline{x} \geq \overline{0}, \overline{y} < \overline{0}. \end{cases} \tag{51}$$

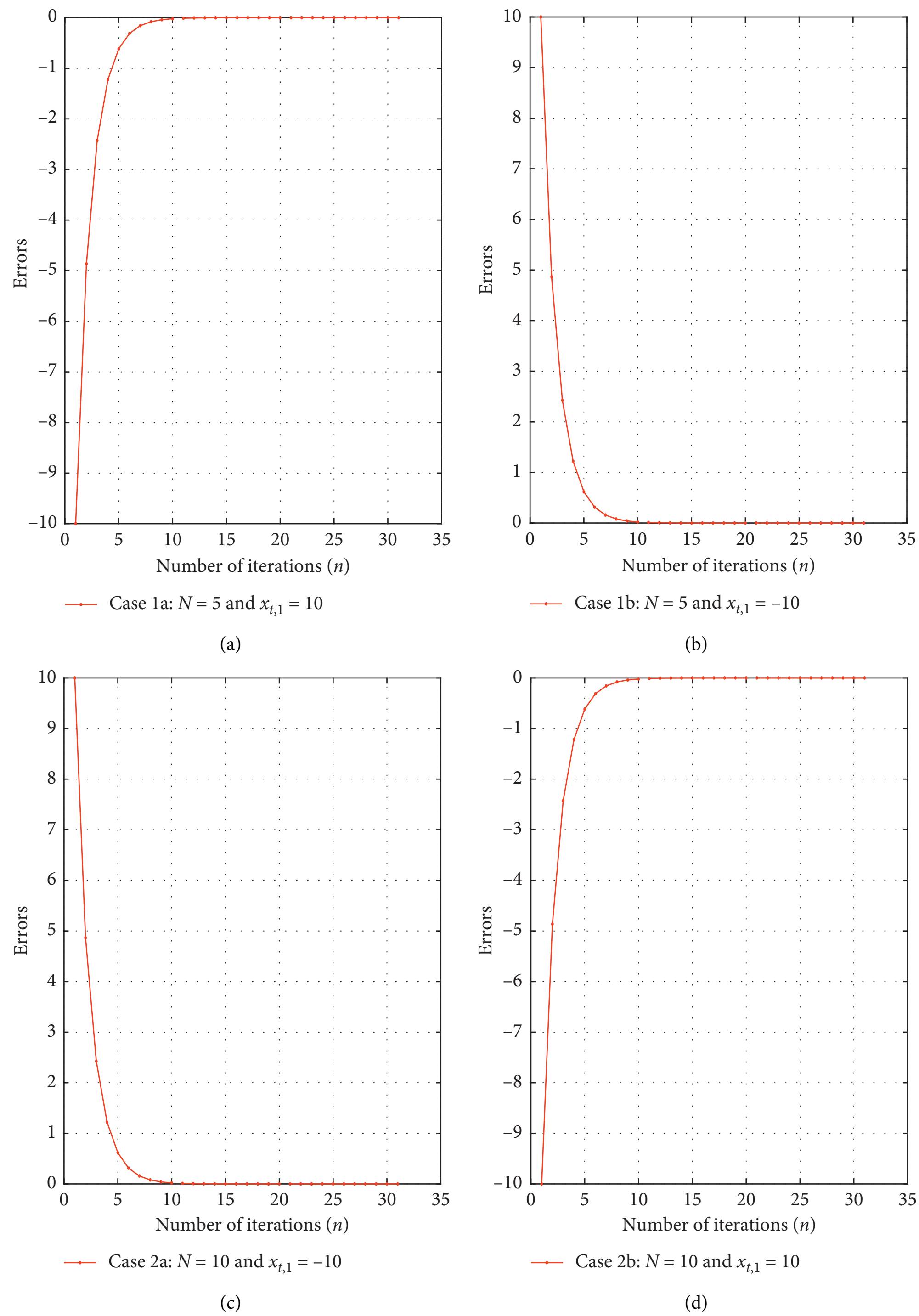

Figure 1: Errors vs. iteration numbers (n): case 1a (a); case 1b (b); case 2a (c); case 2b (d).

(III) $\|\overline{u} - \overline{v}\| = (3i/(2i+1))\|\overline{x} - \overline{y}\| \leq \mathcal{D}(S_i\overline{x}, S_i\overline{y})$, for all $\overline{u} \in P_{S_i}\overline{x}$, $\overline{v} \in P_{S_i}\overline{y}$.

(IV) $\cap_{i=1}^{N} F(S_i) = \{\overline{0}\}$.

(V) $d^2(\overline{x}, S_i\overline{x}) = \|\overline{x} - (-(3i/(2i+1))\overline{x})\|^2 = \|\overline{x} + (3i/(2i+1))\overline{x}\|^2 = ((5i+1)/(2i+1))^2\|\overline{x}\|^2$.

(VI) $d(\overline{x}, F(S_i)) = d(\overline{x}, \{\overline{0}\}) = \|\overline{x}\|$.

(VII) $H^2(S_i\overline{x}, S\overline{0}) = \|((4i/2i+1))\overline{x}\|^2 = \|\overline{x}\|^2 + [((4i/2i+1))^2 - 1]\|\overline{x}\|^2 = \|\overline{x}\|^2 + (12(i)^2 - 4i - 1/(2i+1)^2)\|\overline{x}\|^2$.

It then follows from (V) and (VII) that

(VIII) $H^2(S_i\overline{x}, S\overline{0}) = \|\overline{x} - \overline{0}\|^2 + (12(i)^2 - 4i - 1/25(i)^2 + 10i + 1)d^2(\overline{x}, S_i\overline{x})$.

Also, from (V) and (VI), we obtain that

(IX) $d(\overline{x}, S_i\overline{x}) \geq f(d(\overline{x}, F(S_i)))$, where $f\colon [0,\infty) \longrightarrow [0,\infty)$ is defined by $f(r) = r$.

In summary, for each i, we have from (III), (VIII), and (IX) that S_i is type-one demicontractive mapping with contraction coefficient $\lambda_i = (12(i)^2 - 4i - 1/25(i)^2 + 10i + 1)$ and satisfies condition (1).

Observe that $\sup\{\lambda_i\}_{i=1}^{\infty} = (12/25) = \lim_{i\longrightarrow\infty}\lambda_i$. Therefore, if we set $-(3i/(2i+1))\overline{x}_n = y_{n,i} \in S_i\overline{x}_n$ and define $\left\{\{\varsigma_{n,i}\}_{n=1}^{\infty}\right\}_{i=1}^{N} \subseteq 2^{[0,1]}$ by

$$\varsigma_{n,i} = \frac{38(ni)^2 + 37}{50\left[(ni)^2 + 1\right]}, \tag{52}$$

then

(i) $\varsigma_{n,1} > (37/50) > (12/25) = \sup\{\lambda_i\}_{i=1}^{\infty}$; $\varsigma_{n,i} < (38/50) < 1$.

(ii) $\liminf_{n\longrightarrow\infty}\varsigma_{n,i}\prod_{j=1}^{i-1}(1-\varsigma_{n,j})(\varsigma_{n,1} - \lambda_{i-1}) = \lim_{n\longrightarrow\infty}\varsigma_{n,i}\prod_{j=1}^{i-1}(1-\varsigma_{n,j})(\varsigma_{n,1} - \lambda_{i-1}) = (38/50)(1-(38/50))^{i-1}((38/50) - \lambda_{i-1}) > 0$, $2 \leq i < N-1$.

(iii) $\liminf_{n\longrightarrow\infty}\prod_{j=1}^{N}(1-\varsigma_{n,j})(\varsigma_{n,1} - \lambda_{i_N}) = \lim_{n\longrightarrow\infty}\prod_{j=1}^{N}(1-\varsigma_{n,j})(\varsigma_{n,1} - \lambda_{i_N}) = (1-(38/50))^{N}((38/50) - \lambda_{i_N}) > 0$.

Table 1 and Figure 1 show the sequences for $N = 5$ and $N = 10$. The values are rounded up to 9 decimal places.

6. Conclusion

A horizontal iteration scheme for the approximation of a common fixed point of a finite family of mappings is introduced in a real Hilbert space. This algorithm does not require the imposition of sum = 1 on the control sequences. Its applicability in developing other algorithms is demonstrated in Algorithm 1. Furthermore, its computability is also exhibited in our numerical computations presented in Section 5.

Data Availability

All data generated or analyzed during this study are included in this published article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Research Article

On Mixed Equilibrium Problems in Hadamard Spaces

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The main purpose of this paper is to study mixed equilibrium problems in Hadamard spaces. First, we establish the existence of solution of the mixed equilibrium problem and the unique existence of the resolvent operator for the problem. We then prove a strong convergence of the resolvent and a Δ-convergence of the proximal point algorithm to a solution of the mixed equilibrium problem under some suitable conditions. Furthermore, we study the asymptotic behavior of the sequence generated by a Halpern-type PPA. Finally, we give a numerical example in a nonlinear space setting to illustrate the applicability of our results. Our results extend and unify some related results in the literature.

1. Introduction

Let C be a nonempty set and Ψ be any real-valued function defined on C. The minimization problem (MP) is defined as

$$\text{find } x^* \in C \text{ such that } \Psi(x^*) \leq \Psi(y), \quad \forall y \in C. \tag{1}$$

In this case, x^* is called a minimizer of Ψ and $\operatorname{argmin}_{y\in C}\Psi(y)$ denotes the set of minimizers of Ψ. MPs are very useful in optimization theory and convex and nonlinear analysis. One of the most popular and effective methods for solving MPs is the proximal point algorithm (PPA) which was introduced in Hilbert space by Martinet [1] in 1970 and was further extensively studied in the same space by Rockafellar [2] in 1976. The PPA and its generalizations have also been studied extensively for solving MP (1) and related optimization problems in Banach spaces and Hadamard manifolds (see [3–7] and the references therein), as well as in Hadamard and p-uniformly convex metric spaces (see [8–13] and the references therein).

An important generalization of Problem (1) is the following equilibrium problem (EP), defined as

$$\text{find } x^* \in C \text{ such that } F(x^*, y) \geq 0, \quad \forall y \in C. \tag{2}$$

The point x^* for which (2) is satisfied is called an equilibrium point of F. The solution set of problem (2) is denoted by $EP(C, F)$. The EP is one of the most important problems in optimization theory that has received a lot of attention in recent time since it includes many other optimization and mathematical problems as special cases, namely, MPs, variational inequality problems, complementarity problems, fixed point problems, and convex feasibility problems, among others (see, for example, [5, 14–18]). Thus, EPs are of central importance in optimization theory as well as in nonlinear and convex analysis. As a result of this, numerous authors have studied EPs in Hilbert, Banach, and topological vector spaces (see [19, 20] and the references therein), as well as in Hadamard manifolds (see [3, 21]).

Very recently, Kumam and Chaipunya [5] extended these studies to Hadamard spaces. First, they established the existence of an equilibrium point of a bifunction satisfying some convexity, continuity, and coercivity assumptions, and they also established some fundamental properties of the resolvent of the bifunction. Furthermore, they proved that the PPA Δ-converges to an equilibrium point of a monotone bifunction in a Hadamard space. More precisely, they proved the following theorem.

Theorem 1. *Let C be a nonempty closed and convex subset of an Hadamard space X and $F : C \times C \longrightarrow \mathbb{R}$ be monotone and Δ-upper semicontinuous in the first variable such that $D(J_\lambda^F) \supset C$ for all $\lambda > 0$ (where $D(J_\lambda^F)$ means the domain of J_λ^F). Suppose that $\mathrm{EP}(C, F) \neq \varnothing$ and for an initial guess $x_0 \in C$, the sequence $\{x_n\} \subset C$ is generated by*

$$x_n := J_{\lambda_n}^F (x_{n-1}), \quad n \in \mathbb{N}, \tag{3}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers bounded away from 0. Then, $\{x_n\}$Δ-converges to an element of $\mathrm{EP}(C, F)$.

Other authors have also studied EPs in Hadamard spaces (see, for example, [14, 15]).

In the linear settings (for example, in Hilbert spaces), EPs have been generalized into what is called the mixed equilibrium problem (MEP), defined as

$$\text{find } x^* \in C \text{ such that } F(x^*, y) + \Psi(y) - \Psi(x^*) \geq 0, \quad \forall y \in C. \tag{4}$$

The MEP is an important class of optimization problems since it contains many other optimization problems as special cases. For instance, if $F \equiv 0$ in (3), then the MEP (4) reduces to MP (1). Also, if $\Psi \equiv 0$ in (3), then the MEP (4) reduces to the EP (2). The existence of solutions of the MEP (4) was established in Hilbert spaces by Peng and Yao [22] (see also [23]). More so, different iterative algorithms have been developed by numerous authors for approximating solutions of MEP (4) in real Hilbert spaces (see, for example, [22–24] and the references therein).

Since MEPs contain both MPs and EPs as special cases in Hilbert spaces, it is important to extend their study to Hadamard spaces, so as to unify other optimization problems (in particular, MPs and EPs) in Hadamard spaces. Moreover, Hadamard spaces are more suitable frameworks for the study of optimization problems and other related mathematical problems since many recent results in these spaces have already found applications in diverse fields than they do in Hilbert spaces. For instance, the minimizers of the energy functional (which is an example of a convex and lower semicontinuous functional in a Hadamard space), called harmonic maps, are very useful in geometry and analysis (see [9]). Also, the gradient flow theorem in Hadamard spaces was employed to investigate the asymptotic behavior of the Calabi flow in Kahler geometry (see [25]). Furthermore, the study of the PPA for optimization problems has successfully been applied in Hadamard spaces, for computing medians and means, which are very important in computational phylogenetics, diffusion tensor imaging, consensus algorithms, and modeling of airway systems in human lungs and blood vessels (see [26, 27], for details). It is also worthy to note that many nonconvex problems in the linear settings can be viewed as convex problems in Hadamard spaces (see Section 4 of this paper).

Therefore, it is our interest in this paper to extend the study of the MEP (4) to Hadamard spaces. First, we establish the existence of solution of the MEP (4) and the unique existence of the resolvent operator associated with F and Ψ. We then prove a strong convergence of the resolvent and a Δ-convergence of the PPA to a solution of MEP (4) under some suitable conditions on F and Ψ. Furthermore, we study the asymptotic behavior of the sequence generated by the Halpern-type PPA. Finally, we give a numerical example in a nonlinear space setting to illustrate the applicability of our results. Our results extend and unify the results of Kumam and Chaipunya [5] and Peng and Yao [22].

The rest of this paper is organized as follows: In Section 2, we recall the geometry of geodesic spaces and some useful definitions and lemmas. In Section 3, we establish the existence of solution for MEP (4) and the unique existence of the resolvent operator associated with F and Ψ. Some fundamental properties of the resolvent operator are also established in this section. In Section 4, we prove a strong convergence of the resolvent and a Δ-convergence of the PPA to a solution of MEP (4) under some suitable conditions on F and Ψ. In Section 5, we study the asymptotic behavior of the sequence generated by the Halpern-type PPA. In Section 6, we generate some numerical results in nonlinear setting for the PPA and the Halpern-type PPA, to show the applicability of our results.

2. Preliminaries

2.1. Geometry of Geodesic Spaces

Definition 1. Let (X, d) be a metric space, $x, y \in X$ and $I = [0, d(x, y)]$ be an interval. A curve c (or simply a geodesic path) joining x to y is an isometry $c : I \longrightarrow X$ such that $c(0) = x$, $c(d(x, y)) = y$, and $d(c(t), c(t') = |t - t'|)$ for all $t, t' \in I$. The image of a geodesic path is called a geodesic segment, which is denoted by $[x, y]$ whenever it is unique.

Definition 2 (see [28]). A metric space (X, d) is called a geodesic space if every two points of X are joined by a geodesic path, and X is said to be uniquely geodesic if every two points of X are joined by exactly one geodesic path. A subset C of X is said to be convex if C includes every geodesic segments joining two of its points. Let $x, y \in X$ and $t \in [0, 1]$, and we write $tx \oplus (1 - t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(z, y) = td(x, y). \tag{5}$$

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three vertices (points in X) with unparameterized geodesic segment between each pair of vertices. For any geodesic triangle, there is comparison (Alexandrov) triangle $\overline{\Delta} \subset \mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j)$ for $i, j \in \{1, 2, 3\}$. Let Δ be a geodesic triangle in X and $\overline{\Delta}$ be a comparison triangle for $\overline{\Delta}$, then Δ is said to satisfy the CAT(0) inequality if for all points $x, y \in \Delta$ and $\overline{x}, \overline{y} \in \overline{\Delta}$:

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{6}$$

Let x, y, and z be points in X and y_0 be the midpoint of the segment $[y, z]$; then, the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{7}$$

Inequality (7) is known as the CN inequality of Bruhat and Titis [29].

Definition 3. A geodesic space X is said to be a CAT(0) space if all geodesic triangles satisfy the CAT(0) inequality. Equivalently, X is called a CAT(0) space if and only if it satisfies the CN inequality.

CAT(0) spaces are examples of uniquely geodesic spaces, and complete CAT(0) spaces are called Hadamard spaces.

Definition 4. Let C be a nonempty closed and convex subset of a CAT(0) space X. The metric projection is a mapping $P_C : X \longrightarrow C$ which assigns to each $x \in X$, the unique point $P_C x$ in C such that

$$d(x, P_C x) = \inf\{d(x, y) : y \in C\}. \tag{8}$$

Definition 5 (see [30]). Let X be a CAT(0) space. Denote the pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, a mapping $\langle .,. \rangle : (X \times X) \times (X \times X) \longrightarrow \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad \forall a, b, c, d \in X, \tag{9}$$

is called a quasilinearization mapping.

It is easy to check that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all a, b, c, d, $e \in X$. A geodesic space X is said to satisfy the Cauchy–Swartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)\, \forall a, b, c, d \in X$. It has been established in [30] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy–Schwartz inequality. Examples of CAT(0) spaces include Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature [31], $\mathbb{R}$-trees, and Hilbert ball [32], among others.

We end this section with the following important lemmas which characterize CAT(0) spaces.

Lemma 1. *Let X be a CAT(0) space, $x, y, z \in X$, and $t, s \in [0, 1]$. Then,*

(i) $d(tx \oplus (1 - t)y, z) \leq td(x, z) + (1 - t)d(y, z)$ (see [28])

(ii) $d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y)$ (see [28])

2.2. Notion of Δ-Convergence

Definition 6. Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. Then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \bar{v} \in X : \limsup_{n \longrightarrow \infty} d(\bar{v}, x_n) = \inf_{v \in X} \limsup_{n \longrightarrow \infty} d(v, x_n). \tag{10}$$

A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\bar{v} \in X$ if $A(\{x_{n_k}\}) = \{\bar{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \longrightarrow \infty} x_n = \bar{v}$ (see [33]). The concept of Δ-convergence in metric spaces was first introduced and studied by Lim [34]. Kirk and Panyanak [35] later introduced and studied this concept in CAT(0) spaces and proved that it is very similar to the weak convergence in Banach space setting.

We now end this section with the following important lemmas which are concerned with Δ-convergence.

Lemma 2 (see [28, 36]). *Let X be an Hadamard space. Then,*

(i) Every bounded sequence in X has a Δ-convergent subsequence

(ii) Every bounded sequence in X has a unique asymptotic center

Lemma 3 ([37], Opial's Lemma). *Let X be an Hadamard space and $\{x_n\}$ be a sequence in X. If there exists a nonempty subset F in which*

(i) $\lim_{n \longrightarrow \infty} d(x_n, z)$ exists for every $z \in F$

(ii) if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ which is Δ-convergent to x, then $x \in F$

Then, there is a $p \in F$ such that $\{x_n\}$ is Δ-convergent to p.

Lemma 4 ([14], Proposition 4.3). *Suppose that $\{x_n\}$ is Δ-convergent to q and there exists $y \in X$ such that $\limsup d(x_n, y) \leq d(q, y)$, then $\{x_n\}$ converges strongly to q.*

3. Existence and Uniqueness of Solution

In this section, we establish the existence of solution for MEP (4). We also establish the unique existence of the resolvent operator associated with the bifunction F and the convex functional Ψ. In addition, we study some fundamental properties of this resolvent operator. We begin with the following known results.

Definition 7. Let X be a CAT(0) space. A function $\Psi : D(\Psi) \subseteq X \longrightarrow \mathbb{R}$ (where $D(\Psi)$ means the domain of Ψ) is said to be convex, if

$$\Psi(tx \oplus (1 - t)y) \leq t\Psi(x) + (1 - t)\Psi(y), \quad \forall x, y \in X, \quad t \in (0, 1). \tag{11}$$

Ψ is lower semicontinuous (or upper semicontinuous) at a point $x \in D(\Psi)$, if

$$\Psi(x) \leq \liminf_{n \longrightarrow \infty} \Psi(x_n) \left(\text{or } \Psi(x) \geq \limsup_{n \longrightarrow \infty} \Psi(x_n) \right), \quad (12)$$

for each sequence $\{x_n\}$ in $D(\Psi)$ such that $\lim_{n \longrightarrow \infty} x_n = x$. We say that Ψ is lower semicontinuous (or upper semicontinuous) on $D(\Psi)$, if it is lower semicontinuous (or upper semicontinuous) at any point in $D(\Psi)$.

Lemma 5 (See [9]). *Let X be a Hadamard space and $\Psi : C \longrightarrow \mathbb{R}$be a convex and lower semicontinuous function. Then, $\Psi$is Δ-lower semicontinuous.*

For a nonempty subset C of X, we denote by $\text{conv}(C)$, the convex hull of C. That is, the smallest convex subset of X containing C. Recall that the convex hull of a finite set is the set of all convex combinations of its points.

Theorem 2 (the KKM principle) (see [5], Theorem 3.3; see also [14], Lemma 1.8). *Let C be a nonempty, closed, and convex subset of an Hadamard space X and $G : C \longrightarrow 2^C$ be a set-valued mapping with closed values. Suppose that for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of C,*

$$\text{conv}(\{x_1, x_2, \ldots, x_m\}) \subset \% \cup_{i=1}^m G(x_i). \quad (13)$$

Then, the family $\{G(x)\}_{x \in C}$ has the finite intersection property. Moreover, if $G(x_0)$ is compact for some $x_0 \in C$, then $\cap_{x \in C} G(x) \neq \varnothing$.

3.1. Existence of Solution for Mixed Equilibrium Problem

Theorem 3. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let $\Psi : C \longrightarrow \mathbb{R}$be a real-valued function and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction such that the following assumptions hold:*

(A1) $F(x, x) = 0$, $\forall x \in C$

(A2) For every $x \in C$, the set $\{y \in C : F(x, y) + \Psi(y) - \Psi(x) < 0\}$is convex

(A3) There exists a compact subset $D \subset C$ containing a point $y_0 \in D$ such that $F(x, y_0) + \Psi(y_0) - \Psi(x) < 0$whenever $x \in C/D$

Then, the MEP *(4) has a solution.*

Proof. For each $y \in C$, define the set-valued mapping $G : C \longrightarrow 2^C$ by

$$G(y) \coloneqq \{x \in C : F(x, y) + \Psi(y) - \Psi(x) \geq 0\}. \quad (14)$$

By (A1), we obtain that, for each $y \in C$, $G(y) \neq \varnothing$ since $y \in G(y)$. Also, we obtain from (A2) that $G(y)$ is a closed subset of C for all $y \in C$.

We claim that G satisfies the inclusion (13). Suppose for contradiction that this is not true, then there exist a finite subset $\{y_1, y_2, \ldots, y_m\}$ of C and $\alpha_i \geq 0$, $\forall i = 1, 2, \ldots, m$ with $\sum_{i=1}^m \alpha_i = 1$ such that $y^* = \sum_{i=1}^m \alpha_i y_1 \notin G(y_i)$ for each $i = 1, 2, \ldots, m$. That is, there exists $y^* \in \text{conv}(\{y_1, y_2, \ldots, y_m\})$ such that $y^* \notin G(y_i)$, for each $1, 2, \ldots, m$. By (14), we obtain for each $i = 1, 2, \ldots, m$ that

$$F(y^*, y_i) + \Psi(y_i) - \Psi(y^*) < 0. \quad (15)$$

Thus, for each $i = 1, 2, \ldots, m$, $y_i \in \{y \in C : F(y^*, y) + \Psi(y) - \Psi(y^*) < 0\}$, which is convex by (A2). Since $\text{conv}(\{y_1, y_2, \ldots, y_m\})$ is the smallest convex set containing $y_1, y_2, \ldots, y_m$, we have that $\text{conv}(\{y_1, y_2, \ldots, y_m\}) \subset \{y \in C : F(y^*, y) + \Psi(y) - \Psi(y^*) < 0\}$, which implies that $y^* \in \{y \in C : F(y^*, y) + \Psi(y) - \Psi(y^*) < 0\}$. That is, $0 = F(y^*, y^*) + \Psi(y^*) - \Psi(y^*) < 0$, which is a contradiction. Therefore, G satisfies the inclusion (13).

Now, observe that (A3) implies that there exists a compact subset D of C containing $y_0 \in D$ such that for any $x \in C/D$, we have

$$F(x, y_0) + \Psi(y_0) - \Psi(x) < 0, \quad (16)$$

which further implies that

$$G(y_0) = \{x \in C : F(x, y_0) + \Psi(y_0) - \Psi(x) \geq 0\} \subset D. \quad (17)$$

Thus, $G(y_0)$ is compact. It then follows from Theorem 2 that $\cap_{y \in C} G(y) \neq \varnothing$. This implies that there exists $x^* \in C$ such that

$$F(x^*, y) + \Psi(y) - \Psi(x^*) \geq 0, \quad \forall y \in C. \quad (18)$$

That is, MEP (4) has a solution. □

3.2. Existence and Uniqueness of Resolvent Operator

Definition 8. Let X be an Hadamard space and C be a nonempty subset of X. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction, $\Psi : C \longrightarrow \mathbb{R}$ be a real-valued function, $\overline{x} \in X$, and $\lambda > 0$; then, we define the perturbation $\widetilde{F}_{\overline{x}} : C \times C \longrightarrow \mathbb{R}$ of F and Ψ, by

$$\widetilde{F}_{\overline{x}}(x, y) \coloneqq F(x, y) + \Psi(y) - \Psi(x) + \frac{1}{\lambda} \langle \overrightarrow{\overline{x}y}, \overrightarrow{\overline{x}x} \rangle, \quad \forall x, y \in C. \quad (19)$$

In the next theorem, we shall prove the existence and uniqueness of solution of the following auxiliary problem: find $x^* \in C$ such that

$$\widetilde{F}_{\overline{x}}(x^*, y) \geq 0, \quad \forall y \in C, \quad (20)$$

where $\widetilde{F}_{\overline{x}}$ is as defined in (19). The proof for existence is similar to the proof of Theorem 3. But for completeness, we shall give the proof here.

Theorem 4. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let $\Psi : C \longrightarrow \mathbb{R}$be a convex function and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction such that the following assumptions hold:*

(A1) $F(x, x) = 0$, $\forall x \in C$

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$, $\forall x, y, \in C$

(A3) $F(x,.) : C \longrightarrow \mathbb{R}$ *is convex* $\forall\, x \in C$

(A4) *For each* $\overline{x} \in X$ *and* $\lambda > 0$, *there exists a compact subset* $D_{\overline{x}} \subset C$ *containing a point* $y_{\overline{x}} \in D_{\overline{x}}$ such that $F(x, y_{\overline{x}}) + \Psi(y_{\overline{x}}) - \Psi(x) + (1/\lambda)\langle \overrightarrow{xy_{\overline{x}}}, \overrightarrow{\overline{x}x}\rangle <$ 0*whenever* $x \in C/D_{\overline{x}}$.

Then, (20) has a unique solution.

Proof. Let $\overline{x}$ be a point in X. For each $y \in C$, define the set-valued mapping $G : C \longrightarrow 2^C$ by

$$G(y) = \left\{x \in C : F(x, y) + \Psi(y) - \Psi(x) + \frac{1}{\lambda}\langle \overrightarrow{xy}, \overrightarrow{\overline{x}x}\rangle \geq 0\right\}. \tag{21}$$

Then, it is easy to see that $G(y)$ is a nonempty closed subset of C. As in the proof of Theorem 3, we claim that G satisfies the inclusion (13). Suppose for contradiction that this is not true, then there exists $y^* = \sum_{i=1}^m \alpha_i y_i \in \operatorname{conv}(\{y_1, y_2, \ldots, y_m\})$ such that

$$F(y^*, y_i) + \Psi(y_i) - \Psi(y^*) + \frac{1}{\lambda}\langle \overrightarrow{y^* y_i}, \overrightarrow{\overline{x}y^*}\rangle < 0, \qquad i = 1, 2, \ldots, m. \tag{22}$$

By (A3) and the convexity of Ψ, we obtain that

$$\begin{aligned} 0 &= F(y^*, y^*) + \Psi(y^*) - \Psi(y^*) + \frac{1}{\lambda}\langle \overrightarrow{y^* y^*}, \overrightarrow{\overline{x}y^*}\rangle \\ &\leq \textstyle\sum_{i=1}^m \alpha_i (F(y^*, y_i) + \Psi(y_i) - \Psi(y^*)) \\ &\quad + \frac{1}{\lambda}\left(\textstyle\sum_{i=1}^m \alpha_i \langle \overrightarrow{y^* y_i}, \overrightarrow{\overline{x}y^*}\rangle\right) < 0, \end{aligned} \tag{23}$$

which is a contradiction. Therefore, G satisfies the inclusion (13). By (A4), we obtain that $G(y_{\overline{x}}) \subset D_{\overline{x}}$. Thus, $G(y_{\overline{x}})$ is compact and by Theorem 2, we get that $\cap_{y\in C} G(y) \neq \varnothing$. Therefore, (20) has a solution.

Next, we show that this solution is unique. Suppose that x and x^* solve (20). Then,

$$\begin{aligned} 0 &\leq \widetilde{F}_{\overline{x}}(x, x^*) = F(x, x^*) + \Psi(x^*) - \Psi(x) + \frac{1}{\lambda}\langle \overrightarrow{\overline{x}x}, \overrightarrow{xx^*}\rangle, \\ 0 &\leq \widetilde{F}_{\overline{x}}(x^*, x) = F(x^*, x) + \Psi(x) - \Psi(x^*) + \frac{1}{\lambda}\langle \overrightarrow{\overline{x}x^*}, \overrightarrow{x^*x}\rangle. \end{aligned} \tag{24}$$

Adding both inequalities and noting that F is monotone, we obtain that

$$\begin{aligned} 0 &\leq -\frac{1}{\lambda}\left(\langle \overrightarrow{\overline{x}x}, \overrightarrow{xx^*}\rangle + \langle \overrightarrow{\overline{x}x^*}, \overrightarrow{xx^*}\rangle\right) \\ &= -\frac{1}{\lambda} d(x, x^*)^2, \end{aligned} \tag{25}$$

which implies that $x = x^*$. □

Definition 9. Let X be an Hadamard space and C be a nonempty closed and convex subset of X. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction and $\Psi : C \longrightarrow \mathbb{R}$ be a convex function. Assume that (20) has a unique solution for each $\lambda > 0$ and $x \in X$. This unique solution is denoted by $J_{\lambda F}^{\Psi} x$, and it is called the resolvent operator associated with F and Ψ of order $\lambda > 0$ and at $x \in X$. In other words, the resolvent operator associated with F and Ψ is the set-valued mapping $J_{\lambda F}^{\Psi} : X \longrightarrow 2^C$ defined by

$$\begin{aligned} J_{\lambda F}^{\Psi}(x) &:= EP(C, \widetilde{F}_x) = \Big\{z \in C : F(z, y) + \Psi(y) - \Psi(z) \\ &\quad + \frac{1}{\lambda}\langle \overrightarrow{zy}, \overrightarrow{xz}\rangle \geq 0, \forall y \in C\Big\}, \quad \text{for all } x \text{ in } X. \end{aligned} \tag{26}$$

Under the assumptions of Theorem 4, we have the unique existence of $J_{\lambda F}^{\Psi}(x)$. Therefore, $J_{\lambda F}^{\Psi}$ is well defined.

3.3. Fundamental Properties of the Resolvent Operator. In the following theorem, we shall study some fundamental properties of the resolvent operator. First, we recall the following definitions which will be needed for our study.

Definition 10. Let X be a CAT(0) space. A point $x \in X$ is called a fixed point of a nonlinear mapping $T : X \longrightarrow X$, if $Tx = x$. We denote the set of fixed points of T by $\operatorname{Fix}(T)$. The mapping T is said to be

(i) Firmly nonexpansive, if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle, \quad \forall x, y \in X. \tag{27}$$

(ii) Nonexpansive, if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x,\ y \in X. \tag{28}$$

Theorem 5. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let $\Psi : C \longrightarrow \mathbb{R}$be a convex function and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying assumptions (A1)–(A4) of Theorem 4. For $\lambda > 0$, we have that $J_{\lambda F}^{\Psi}$is single valued. Moreover, if $C \subset D(J_{\lambda F}^{\Psi})$,then*

(i) $J_{\lambda F}^{\Psi}$*is firmly nonexpansive restricted to* C

(ii) *For* $F(J_{\lambda F}^{\Psi}) \neq \varnothing$,*we have*

$$\begin{aligned} d^2\left(J_{\lambda F}^{\Psi} x, x\right) &\leq d^2(x, v) - d^2\left(J_{\lambda F}^{\Psi} x, v\right), \\ &\quad \forall x \in C,\ \forall v \in \operatorname{fix}\left(J_{\lambda F}^{\Psi}\right), \end{aligned} \tag{29}$$

(iii) *For* $0 < \lambda \leq \mu$, *we have* $d(J_{\mu F}^{\Psi} x, J_{\lambda F}^{\Psi} x) \leq \sqrt{1 - (\lambda/\mu)} d(x, J_{\mu F}^{\Psi} x)$,*which implies that* $d(x, J_{\lambda F}^{\Psi} x) \leq 2d(x, J_{\mu F}^{\Psi} x)$, $\forall x \in C$

(iv) $\operatorname{Fix}(J_{\lambda F}^{\Psi}) = \operatorname{MEP}(C, F, \Psi)$

Proof. For each $x \in D(J_{\lambda F}^{\Psi})$ and $\lambda > 0$, let $z_1, z_2 \in J_{\lambda F}^{\Psi}x$. Then from (26), we have

$$F(z_1, z_2) + \Psi(z_2) - \Psi(z_1) + \frac{1}{\lambda}\langle \overrightarrow{z_1z_2}, \overrightarrow{xz_1}\rangle \geq 0,$$
$$F(z_2, z_1) + \Psi(z_1) - \Psi(z_2) + \frac{1}{\lambda}\langle \overrightarrow{z_2z_1}, \overrightarrow{xz_2}\rangle \geq 0. \tag{30}$$

Adding both inequalities and using assumption (A2), we obtain that

$$\langle \overrightarrow{z_2z_1}, \overrightarrow{z_1z_2}\rangle \geq 0, \tag{31}$$

which implies that $d^2(z_1, z_2) \leq 0$. This further implies that $z_1 = z_2$. Therefore, $J_{\lambda F}^{\Psi}$ is single valued.

(i) Let $x, y \in C$, then

$$\begin{aligned} &F\left(J_{\lambda F}^{\Psi}x, J_{\lambda F}^{\Psi}y\right) + \Psi\left(J_{\lambda F}^{\Psi}y\right) - \Psi\left(J_{\lambda F}^{\Psi}x\right) \\ &\quad + \frac{1}{\lambda}\langle \overrightarrow{J_{\lambda F}^{\Psi}xJ_{\lambda F}^{\Psi}y}, \overrightarrow{xJ_{\lambda F}^{\Psi}x}\rangle \geq 0, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &F\left(J_{\lambda F}^{\Psi}y, J_{\lambda F}^{\Psi}x\right) + \Psi\left(J_{\lambda F}^{\Psi}x\right) - \Psi\left(J_{\lambda F}^{\Psi}y\right) \\ &\quad + \frac{1}{\lambda}\langle \overrightarrow{J_{\lambda F}^{\Psi}yJ_{\lambda F}^{\Psi}x}, \overrightarrow{yJ_{\lambda F}^{\Psi}y}\rangle \geq 0. \end{aligned} \tag{33}$$

Adding (32) and (33), and noting that F is monotone, we obtain

$$\frac{1}{\lambda}\left(\langle \overrightarrow{xJ_{\lambda F}^{\Psi}x}, \overrightarrow{J_{\lambda F}^{\Psi}xJ_{\lambda F}^{\Psi}y}\rangle + \langle \overrightarrow{yJ_{\lambda F}^{\Psi}y}, \overrightarrow{J_{\lambda F}^{\Psi}yJ_{\lambda F}^{\Psi}x}\rangle\right) \geq 0, \tag{34}$$

which implies that

$$\langle \overrightarrow{xy}, \overrightarrow{J_{\lambda F}^{\Psi}xJ_{\lambda F}^{\Psi}y}\rangle \geq \langle \overrightarrow{J_{\lambda F}^{\Psi}xJ_{\lambda F}^{\Psi}y}, \overrightarrow{J_{\lambda F}^{\Psi}xJ_{\lambda F}^{\Psi}y}\rangle. \tag{35}$$

That is,

$$\langle \overrightarrow{xy}, \overrightarrow{J_{\lambda F}^{\Psi}xJ_{\lambda F}^{\Psi}y}\rangle \geq d^2\left(J_{\lambda F}^{\Psi}x, J_{\lambda F}^{\Psi}y\right). \tag{36}$$

(ii) It follows from (36) and the definition of quasilinearization that

$$\begin{aligned} &d^2\left(x, J_{\lambda F}^{\Psi}x\right) \leq d^2(x, v) - d^2\left(v, J_{\lambda F}^{\Psi}x\right), \\ &\quad \forall x \in C,\ v \in \operatorname{fix}\left(J_{\lambda F}^{\Psi}\right). \end{aligned} \tag{37}$$

(iii) Let $x \in C$ and $0 < \lambda \leq \mu$, then we have that

$$\begin{aligned} &F\left(J_{\lambda F}^{\Psi}x, J_{\mu F}^{\Psi}x\right) + \Psi\left(J_{\mu F}^{\Psi}x\right) - \Psi\left(J_{\lambda F}^{\Psi}x\right) \\ &\quad + \frac{1}{\lambda}\langle \overrightarrow{xJ_{\lambda F}^{\Psi}x}, \overrightarrow{J_{\lambda F}^{\Psi}xJ_{\mu F}^{\Psi}x}\rangle \geq 0, \end{aligned} \tag{38}$$

and

$$\begin{aligned} &F\left(J_{\mu F}^{\Psi}x, J_{\lambda F}^{\Psi}x\right) + \Psi\left(J_{\lambda F}^{\Psi}x\right) - \Psi\left(J_{\mu F}^{\Psi}x\right) \\ &\quad + \frac{1}{\mu}\langle \overrightarrow{xJ_{\mu F}^{\Psi}x}, \overrightarrow{J_{\mu F}^{\Psi}xJ_{\lambda F}^{\Psi}x}\rangle \geq 0. \end{aligned} \tag{39}$$

Adding (38) and (39), and using the monotonicity of F, we obtain that

$$\langle \overrightarrow{J_{\lambda F}^{\Psi}xx}, \overrightarrow{J_{\mu F}^{\Psi}xJ_{\lambda F}^{\Psi}x}\rangle \geq \frac{\lambda}{\mu}\langle \overrightarrow{J_{\mu F}^{\Psi}xx}, \overrightarrow{J_{\mu F}^{\Psi}xJ_{\lambda F}^{\Psi}x}\rangle. \tag{40}$$

By quasilinearization, we obtain that

$$\begin{aligned} &\left(\frac{\lambda}{\mu} + 1\right)d^2\left(J_{\mu F}^{\Psi}x, J_{\lambda F}^{\Psi}x\right) \leq \left(1 - \frac{\lambda}{\mu}\right)d^2\left(x, J_{\mu F}^{\Psi}x\right) \\ &\quad + \left(\frac{\lambda}{\mu} - 1\right)d^2\left(x, J_{\lambda F}^{\Psi}x\right). \end{aligned} \tag{41}$$

Since $(\lambda/\mu) \leq 1$, we obtain that

$$\left(\frac{\lambda}{\mu} + 1\right)d^2\left(J_{\mu F}^{\Psi}x, J_{\lambda F}^{\Psi}x\right) \leq \left(1 - \frac{\lambda}{\mu}\right)d^2\left(x, J_{\mu F}^{\Psi}x\right), \tag{42}$$

which implies that

$$d\left(J_{\mu F}^{\Psi}x, J_{\lambda F}^{\Psi}x\right) \leq \sqrt{1 - \frac{\lambda}{\mu}}d\left(x, J_{\mu F}^{\Psi}x\right). \tag{43}$$

Moreover, we obtain by triangle inequality and (43) that

$$d\left(x, J_{\lambda F}^{\Psi}x\right) \leq 2d\left(x, J_{\mu F}^{\Psi}x\right). \tag{44}$$

(iv) Observe that

$$\begin{aligned} x \in \operatorname{fix}\left(J_{\lambda F}^{\Psi}\right) &\Longleftrightarrow F(x, y) + \Psi(y) - \Psi(x) + \frac{1}{\lambda}\langle \overrightarrow{xx}, \overrightarrow{xy}\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow F(x, y) + \Psi(y) - \Psi(x) \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow x \in \operatorname{MEP}(C, F, \Psi). \end{aligned} \tag{45}$$

□

Remark 1. It follows from Cauchy–Schwartz inequality that firmly nonexpansive mappings are nonexpansive, and it is well known that the set of fixed points of nonexpansive mappings is closed and convex. Thus, by (i) and (iv) of Theorem 5, we have that $\mathrm{MEP}(C, F, \Psi)$ is closed and convex.

4. Convergence Results

For the rest of this paper, we shall assume that C is a nonempty closed and convex subset of an Hadamard space X and that $D(J^{\Psi}_{\lambda F}) \supset C$.

4.1. Convergence of Resolvent. In the following theorem, we shall prove that $\{J^{\Psi}_{\lambda F}x\}$ converges strongly to a solution of MEP (4) as $\lambda \longrightarrow 0$.

Theorem 6. *Let $\Psi : C \longrightarrow \mathbb{R}$be a convex and lower semicontinuous function and $F : C \times C \longrightarrow \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)–(A4) of Theorem 4. If $\mathrm{MEP}(C, F, \Psi) \neq \varnothing$, then $\{J^{\Psi}_{\lambda_n F}x\}$converges strongly to $q \in \mathrm{MEP}(C, F, \Psi)$,which is the nearest point of $\mathrm{MEP}(C, F, \Psi)$to x as $\lambda \longrightarrow 0$.*

Proof. Let $v \in \mathrm{MEP}(C, F, \Psi)$, since $J^{\Psi}_{\lambda F}$ is nonexpansive (by Remark 1), we obtain that $\{J^{\Psi}_{\lambda F}x\}$ is bounded. Let $\{\lambda_n\}$ be a sequence that converges to 0 as $n \longrightarrow \infty$. Then, $\{J^{\Psi}_{\lambda_n F}x\}$ is bounded. Thus, by Lemma 2(i), there exists a subsequence $\{J^{\Psi}_{\lambda_{nk} F}x\}$ of $\{J^{\Psi}_{\lambda_n F}x\}$ that Δ-converges to $q \in C$.

Now, observe that, by the definition of $J^{\Psi}_{\lambda F}$, the Δ-upper semicontinuity of F, lower semicontinuous of Ψ, and Lemma 5, we obtain that

$$F(q, y) + \Psi(y) - \Psi(q) \geq 0. \tag{46}$$

Therefore, $q \in \mathrm{MEP}(C, F, \Psi)$. Hence, we obtain from Theorem 5(ii) that

$$d^2\left(J^{\Psi}_{\lambda_{nk} F}x, x\right) \leq d^2(x, v), \quad \forall v \in \mathrm{MEP}(C, F, \Psi). \tag{47}$$

Since $d^2(., x)$ is Δ-lower semicontinuous, we obtain that

$$d^2(q, x) \leq \liminf_{k \longrightarrow \infty} d^2\left(J^{\Psi}_{\lambda_{nk} F}x, x\right) \leq d^2(x, v), \quad \forall v \in \mathrm{MEP}(C, F, \Psi), \tag{48}$$

which implies that

$$d(q, x) \leq d(x, v), \quad \forall v \in \mathrm{MEP}(C, F, \Psi). \tag{49}$$

Thus, $q = P_\Gamma x$, where P_Γ is the metric projection of X onto Γ, and $\Gamma = \mathrm{MEP}(C, F, \Psi)$. Therefore, by taking $\lambda_{nk} = \lambda$, we have that $\{J^{\Psi}_{\lambda F}x\}$$\Delta$-converges to $q = P_\Gamma x$ as $\lambda \longrightarrow 0$.

Now, observe also that Theorem 5(ii) implies that

$$d\left(J^{\Psi}_{\lambda F}x, x\right) \leq d(q, x). \tag{50}$$

It then follows from Lemma 4 that $\{J^{\Psi}_{\lambda F}x\}$ converges strongly to $q = P_\Gamma x$ as $\lambda \longrightarrow 0$.

By setting $\Psi \equiv 0$ in Theorem 6, we obtain the following result which is similar to ([14], Theorem 4.4). □

Corollary 1. *Let $F : C \times C \longrightarrow \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)–(A4) of Theorem 4. If $\mathrm{MEP}(C, F) \neq \varnothing$, then $\{J_{\lambda F}x\}$converges strongly to $q \in \mathrm{MEP}(C, F)$, which is the nearest point of $\mathrm{MEP}(C, F)$ to x as $\lambda \longrightarrow 0$.*

4.2. Proximal Point Algorithm. In this section, we study the Δ-convergence of the sequence generated by the following PPA for approximating solutions of MEP (4): For an initial starting point x_1 in C, define the sequence $\{x_n\}$ in C by

$$x_{n+1} = J^{\Psi}_{\lambda_n F}x_n, \quad n \geq 1, \tag{51}$$

where $\{\lambda_n\}$ is a sequence in $(0, \infty)$, $F : C \times C \longrightarrow \mathbb{R}$ is a bifunction, and $\Psi : C \longrightarrow \mathbb{R}$ is a convex function.

Recall that the PPA does not converge strongly in general without additional assumptions even for the case where $F \equiv 0$. See for example, the question of interest raised by Rockafella as to whether the PPA can be improved from weak convergence (an analogue of Δ-convergence) to strong convergence in Hilbert space settings. Several counterexamples have been constructed to resolve this question in the negative (see [38, 39]). Therefore, only weak convergence of the PPA is expected without additional assumptions. For this reason, we propose the following Δ-convergence theorem for the PPA (51).

Theorem 7. *Let $\Psi : C \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous function and $F : C \times C \longrightarrow \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)–(A4) of Theorem 4. Let $\{\lambda_n\}$ be a sequence in $(0, \infty)$ such that $0 < \lambda \leq \lambda_n$, $\forall n \geq 1$. Suppose that $\mathrm{MEP}(C, F, \Psi) \neq \varnothing$, then, the sequence given by (51) Δ-converges to an element of $\mathrm{MEP}(C, F, \Psi)$.*

Proof. Let $v \in \mathrm{MEP}(C, F, \Psi)$. Then, by Remark 1 and Theorem 5(iv), we obtain that

$$d(v, x_{n+1}) = d\left(v, J^{\Psi}_{\lambda_n F}x_n\right) \leq d(v, x_n), \tag{52}$$

which implies that $\lim_{n \longrightarrow \infty} d(x_n, v)$ exists for all $v \in \mathrm{MEP}(C, F, \Psi)$. Hence $\{x_n\}$ is bounded. It then follows from Theorem 5(ii) that

$$d^2(x_{n+1}, x_n) \leq d^2(x_n, v) - d^2(x_{n+1}, v) \longrightarrow 0, \quad \text{as}\, n \longrightarrow \infty. \tag{53}$$

That is,

$$\lim_{n \longrightarrow \infty} d(x_{n+1}, x_n) = 0. \tag{54}$$

Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ that Δ-converges to a point, say $q \in C$. From (51) and (26), we obtain that

$$\begin{aligned} F(x_{nk+1}, y) + \Psi(y) - \Psi(x_{nk+1}) &\geq -\frac{1}{\lambda_{nk}}\langle \overrightarrow{x_{nk}x_{nk+1}}, \overrightarrow{x_{nk+1}y}\rangle \\ &\geq -\frac{1}{\lambda_{nk}} d(x_{nk+1}, x_{nk}) d(x_{nk+1}, y). \end{aligned} \tag{55}$$

Since $0<\lambda\le\lambda_{nk}$, $\{x_n\}$ is bounded, F is Δ-upper semicontinuous in the first argument and Ψ is lower semicontinuous, we obtained from (54) and (55) that

$$\begin{aligned} F(q,y)+\Psi(y)-\Psi(q) &\ge \limsup_{k\longrightarrow\infty}\left(F\left(x_{nk+1},y\right)+\Psi(y)\right) \\ &\quad - \liminf_{k\longrightarrow\infty}\Psi\left(x_{nk+1}\right) \\ &\ge -\frac{M}{\lambda}\limsup_{k\longrightarrow\infty} d\left(x_{nk+1},x_{nk}\right)=0, \end{aligned} \tag{56}$$

for some $M>0$ and for all $y\in C$. This implies that $q\in \mathrm{MEP}(C,F,\Psi)$.

It then follows from Lemma 3 that $\{x_n\}$$\Delta$-converges to an element of $\mathrm{MEP}(C,F,\Psi)$.

By setting $\Psi\equiv 0$ in Theorem 7, we obtain the following result which coincides with ([5], Theorem 7.3). □

Corollary 2. *Let $F: C\times C\longrightarrow\mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)–(A4) of Theorem 4 and $\{\lambda_n\}$ be a sequence in $(0,\infty)$ such that $0<\lambda\le\lambda_n\,\forall n\ge 1$. Suppose that $\mathrm{EP}(C,F)\ne\varnothing$; then, the sequence given for $x_1\in C$ by*

$$x_{n+1}=J_{\lambda_n F}x_n,\quad n\ge 1. \tag{57}$$

Δ-converges to an element of $\mathrm{EP}(C,F)$.

By setting $F\equiv 0$ in Theorem 7, we obtain the following corollary which is similar to ([9], Theorem 1.4).

Corollary 3. *Let $\Psi: C\longrightarrow\mathbb{R}$ be a convex and lower semicontinuous function and $\{\lambda_n\}$ be a sequence in $(0,\infty)$ such that $0<\lambda\le\lambda_n$, $\forall n\ge 1$. Suppose that* $\operatorname{argmin}_{y\in C}\Psi(y)\ne\varnothing$; *then, the sequence given for $x_1\in C$ by*

$$x_{n+1}=J^{\Psi}_{\lambda_n}x_n,\quad n\ge 1. \tag{58}$$

Δ-converges to an element of $\operatorname{argmin}_{y\in C}\Psi(y)$.

5. Asymptotic Behavior of Halpern's Algorithm

To obtain strong convergence result, we modify the PPA into the following Halpern-type PPA and study the asymptotic behavior of the sequence generated by it: For $x_1, u\in C$, define the sequence $\{x_n\}\subset C$ by

$$x_{n+1}=\alpha_n u\oplus(1-\alpha_n)J^{\Psi}_{\lambda_n F}x_n, \tag{59}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$, F and Ψ are as defined in (51).

We begin by establishing the following lemmas which will be very useful to our study.

Lemma 6. *Let $\Psi: C\longrightarrow\mathbb{R}$ be a convex and lower semicontinuous function and $F: C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying (A1)–(A4) of Theorem 4. If $\lambda,\mu>0$ and $x,y\in C$, then the following inequalities hold:*

$$\begin{aligned} d^2\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right) &\le 2\lambda F\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right)+2\lambda\left(\Psi\left(J^{\Psi}_{\mu F}y\right)-\Psi\left(J^{\Psi}_{\lambda F}x\right)\right)+d^2\left(x,J^{\Psi}_{\mu F}y\right)-d^2\left(x,J^{\Psi}_{\lambda F}x\right),\\ (\lambda+\mu)d^2\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right)+\mu d^2\left(J^{\Psi}_{\lambda F}x,x\right)+\lambda d^2\left(J^{\Psi}_{\mu F}y,y\right) &\le \lambda d^2\left(J^{\Psi}_{\lambda F}x,y\right)+\mu d^2\left(J^{\Psi}_{\lambda F}y,x\right). \end{aligned} \tag{60}$$

Proof. We first prove (60). Let $\lambda,\mu>0$ and $x,y\in C$. Then, by (26), we obtain that

$$F\left(J^{\Psi}_{\lambda F}x,z\right)+\Psi(z)-\Psi\left(J^{\Psi}_{\lambda F}x\right)+\frac{1}{\lambda}\langle\overrightarrow{xJ^{\Psi}_{\lambda F}x},\overrightarrow{J^{\Psi}_{\lambda F}xz}\rangle\ge 0,\quad \forall z\in C, \tag{61}$$

which implies that

$$\begin{aligned} 2\lambda\Psi\left(J^{\Psi}_{\lambda F}x\right) &\le 2\lambda F\left(J^{\Psi}_{\lambda F}x,z\right)+2\lambda\Psi(z)+2\langle\overrightarrow{xJ^{\Psi}_{\lambda F}x},\overrightarrow{J^{\Psi}_{\lambda F}xz}\rangle\\ &=2\lambda F\left(J^{\Psi}_{\lambda F}x,z\right)+2\lambda\Psi(z)+d^2(x,z)-d^2\left(x,J^{\Psi}_{\lambda F}\right)\\ &\quad -d^2\left(J^{\Psi}_{\lambda F}x,z\right)\\ &\le 2\lambda F\left(J^{\Psi}_{\lambda F}x,z\right)+2\lambda\Psi(z)+d^2(x,z)\\ &\quad -d^2\left(x,J^{\Psi}_{\lambda F}x\right). \end{aligned} \tag{62}$$

Now, set $z=tJ^{\Psi}_{\mu F}y\oplus(1-t)J^{\Psi}_{\lambda F}x$ for all $t\in(0,1)$ in (5). Since Ψ is convex and F satisfies conditions (A1) and (A3) of Theorem 4, we obtain that

$$\begin{aligned} 2\lambda\Psi\left(J^{\Psi}_{\lambda F}x\right)+d^2\left(x,J^{\Psi}_{\lambda F}x\right) &\le 2\lambda\Big(tF\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right)\\ &\quad +(1-t)F\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\lambda F}x\right)\Big)\\ &\quad +2\lambda\left(t\Psi\left(J^{\Psi}_{\mu F}y\right)+(1-t)\Psi\left(J^{\Psi}_{\lambda F}x\right)\right)\\ &\quad +td^2\left(x,J^{\Psi}_{\mu F}y\right)+(1-t)d^2\left(x,J^{\Psi}_{\lambda F}x\right)\\ &\quad -t(1-t)d^2\left(J^{\Psi}_{\mu F}y,J^{\Psi}_{\lambda F}x\right)\\ &=2\lambda tF\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right)\\ &\quad +2\lambda\left(t\Psi\left(J^{\Psi}_{\mu F}y\right)+(1-t)\Psi\left(J^{\Psi}_{\lambda F}x\right)\right)\\ &\quad +td^2\left(x,J^{\Psi}_{\mu F}y\right)+(1-t)d^2\left(x,J^{\Psi}_{\lambda F}x\right)\\ &\quad -t(1-t)d^2\left(J^{\Psi}_{\mu F}y,J^{\Psi}_{\lambda F}x\right), \end{aligned} \tag{63}$$

which implies that

$$2\lambda\Psi\left(J_{\lambda F}^{\Psi}x\right)+d^{2}\left(x,J_{\lambda F}^{\Psi}x\right)\leq 2\lambda F\left(J_{\lambda F}^{\Psi}x,J_{\mu F}^{\Psi}y\right)+2\lambda\Psi\left(J_{\mu F}^{\Psi}y\right)+d^{2}\left(x,J_{\mu F}^{\Psi}y\right)-(1-t)d^{2}\left(J_{\mu F}^{\Psi}y,J_{\lambda F}^{\Psi}x\right). \tag{64}$$

As $t\longrightarrow 0$ in (64), we obtain (60).

Next, we prove (60). From (60), we obtain that

$$\mu d^{2}\left(J_{\lambda F}^{\Psi}x,J_{\mu F}^{\Psi}y\right)\leq 2\lambda\mu\left[F\left(J_{\lambda F}^{\Psi}x,J_{\mu F}^{\Psi}y\right)+\Psi\left(J_{\mu F}^{\Psi}y\right)-\Psi\left(J_{\lambda F}^{\Psi}x\right)\right]+\mu d^{2}\left(x,J_{\mu F}^{\Psi}y\right)-\mu d^{2}\left(x,J_{\lambda F}^{\Psi}x\right). \tag{65}$$

Similarly, we have

$$\lambda d^{2}\left(J_{\mu F}^{\Psi}y,J_{\lambda F}^{\Psi}x\right)\leq 2\mu\lambda\left[F\left(J_{\mu F}^{\Psi}y,J_{\lambda F}^{\Psi}x\right)+\Psi\left(J_{\lambda F}^{\Psi}x\right)-\Psi\left(J_{\mu F}^{\Psi}y\right)\right]+\lambda d^{2}\left(y,J_{\lambda F}^{\Psi}x\right)-\lambda d^{2}\left(y,J_{\mu F}^{\Psi}y\right). \tag{66}$$

Adding both inequalities and noting that F is monotone, we get

$$(\lambda+\mu)d^{2}\left(J_{\lambda F}^{\Psi}x,J_{\mu F}^{\Psi}y\right)+\mu d^{2}\left(x,J_{\lambda F}^{\Psi}x\right)+\lambda d^{2}\left(y,J_{\mu F}^{\Psi}y\right)\leq \mu d^{2}\left(x,J_{\mu F}^{\Psi}y\right)+\lambda d^{2}\left(y,J_{\lambda F}^{\Psi}x\right). \tag{67}$$

□

Lemma 7. *Let $\Psi:C\longrightarrow\mathbb{R}$ be a convex and lower semicontinuous function and $F:C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying (A1)–(A4) of Theorem 4. Let $\{\lambda_n\}$ be a sequence in $(0,\infty)$ and $\overline{v}$ be an element of C. Suppose that $\lim_{n\longrightarrow\infty}\lambda_n=\infty$ and $A(\{J_{\lambda_n}^{\Psi}x_n\})=\{\overline{v}\}$ for some bounded sequence $\{x_n\}$ in X, then $\overline{v}\in\mathrm{MEP}(C,F,\Psi)$.*

Proof. From (60), we obtain that

$$(\lambda_n+1)d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,J_{F}^{\Psi}\overline{v}\right)+d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,x_n\right)+\lambda_n d^{2}\left(J_{F}^{\Psi}\overline{v},\overline{v}\right)\leq d^{2}\left(J_{F}^{\Psi}\overline{v},x_n\right)+\lambda_n d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,\overline{v}\right), \tag{68}$$

which implies that

$$d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,J_{F}^{\Psi}\overline{v}\right)\leq\frac{1}{\lambda_n}d^{2}\left(J_{F}^{\Psi}\overline{v},x_n\right)+d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,\overline{v}\right)^{2}. \tag{69}$$

Since $\lim_{n\longrightarrow\infty}\lambda_n=\infty$, $\{x_n\}$ is bounded and $A(\{J_{\lambda_n}^{\Psi}x_n\})=\{\overline{v}\}$, we obtain that

$$\limsup_{n\longrightarrow\infty}d\left(J_{\lambda_n F}^{\Psi}x_n,J_{F}^{\Psi}\overline{v}\right)\leq\limsup_{n\longrightarrow\infty}d\left(J_{\lambda_n F}^{\Psi}x_n,\overline{v}\right)=\inf_{y\in X}\limsup_{n\longrightarrow\infty}d\left(J_{\lambda_n F}^{\Psi}x_n,y\right), \tag{70}$$

which by Lemma 2(ii) and Theorem 5(iv) implies that $\overline{v}\in\mathrm{fix}(J_{F}^{\Psi})=\mathrm{MEP}(C,F,\Psi)$.

Lemma 8 (Xu, [40]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1}\leq(1-\alpha_n)a_n+\alpha_n\sigma_n+\gamma_n,\quad n\geq 0, \tag{71}$$

where (i) $\{\alpha_n\}\subset[0,1]$, $\sum\alpha_n=\infty$; (ii) $\limsup\sigma_n\leq 0$; (iii) $\gamma_n\geq 0$; $(n\geq 0)$, $\sum\gamma_n<\infty$. Then, $a_n\longrightarrow 0$ as $n\longrightarrow\infty$.

Theorem 8. *Let $\Psi:C\longrightarrow\mathbb{R}$ be a convex and lower semicontinuous function and $F:C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying (A1–A4) of Theorem 4. Let $\{x_n\}$ be a sequence defined by (59), where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\longrightarrow\infty}\lambda_n=\infty$. Then, we have the following:*

(i) The sequence $\{J_{\lambda_n F}^{\Psi}x_n\}$ is bounded if and only if $\mathrm{MEP}(C,F,\Psi)\neq\varnothing$

(ii) If $\lim_{n\longrightarrow\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\Gamma\coloneqq\mathrm{MEP}(C,F,\Psi)\neq\varnothing$, then $\{x_n\}$ and $\{J_{\lambda_n F}^{\Psi}x_n\}$ converge to $\overline{v}=P_{\Gamma}u$, where P_{Γ} is the metric projection of X onto Γ

Proof. (i) Suppose that $\{J_{\lambda_n}^{f}x_n\}$ is bounded. Then by Lemma 2(ii), there exists $\overline{v}\in X$ such that $A(\{J_{\lambda_n}^{f}x_n\})=\{\overline{v}\}$. From (59) and Lemma 1(i), we obtain that

$$d(x_{n+1},\overline{v})\leq\alpha_n d(u,\overline{v})+(1-\alpha_n)d\left(J_{\lambda_n F}^{\Psi}x_n,\overline{v}\right), \tag{72}$$

which implies that $\{x_n\}$ is bounded. Also, since $\lim_{n\longrightarrow\infty}\lambda_n=\infty$ and $A(\{J_{\lambda_n F}^{\Psi}x_n\})=\{\overline{v}\}$, we obtain by Lemma 7 that $\mathrm{MEP}(C,F,\Psi)\neq\varnothing$.

Conversely, let $\mathrm{MEP}(C,F,\Psi)\neq\varnothing$. Then, we may assume that $\overline{v}\in\mathrm{MEP}(C,F,\Psi)\neq\varnothing$. Thus, by (59) and Lemma 1, we obtain that

$$\begin{aligned}d(x_{n+1},\overline{v})&\leq\alpha_n d(u,\overline{v})+(1-\alpha_n)d\left(J_{\lambda_n F}^{\Psi}x_n,\overline{v}\right)\\&\leq\alpha_n d(u,\overline{v})+(1-\alpha_n)d(x_n,\overline{v})\\&\leq\max\{d(u,\overline{v}),d(x_n,\overline{v})\},\end{aligned} \tag{73}$$

which implies by induction that

$$d(x_n,\overline{v})\leq\max\{d(u,\overline{v}),d(x_1,\overline{v})\},\quad\forall n\geq 1. \tag{74}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{J_{\lambda_n F}^{\Psi}x_n\}$ is also bounded.

(ii) Since $\Gamma\coloneqq\mathrm{MEP}(C,F,\Psi)\neq\varnothing$, we obtain from (74) that $\{x_n\}$ and $\{J_{\lambda_n F}^{\Psi}x_n\}$ are bounded. Furthermore, we obtain from Lemma 1(ii) that

$$\begin{aligned}d^{2}(x_{n+1},\overline{v})&\leq\alpha_n d^{2}(u,\overline{v})+(1-\alpha_n)d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,\overline{v}\right)-\alpha_n(1-\alpha_n)d^{2}\left(u,J_{\lambda_n F}^{\Psi}x_n\right)\\&\leq\alpha_n d^{2}(u,\overline{v})+(1-\alpha_n)d^{2}(x_n,\overline{v})-\alpha_n(1-\alpha_n)d^{2}\left(u,J_{\lambda_n F}^{\Psi}x_n\right)\\&=(1-\alpha_n)d^{2}(x_n,\overline{v})+\alpha_n\delta_n,\quad\forall n\geq 1,\end{aligned} \tag{75}$$

where $\delta_n=d^{2}(u,\overline{v})+(\alpha_n-1)d^{2}(u,J_{\lambda_n F}^{\Psi}x_n)$. Now, set $v_n=J_{\lambda_n F}^{\Psi}x_n$, $\forall n\geq 1$. Then, by the boundedness of $\{v_n\}$ and Lemma 2(i), we obtain that there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$

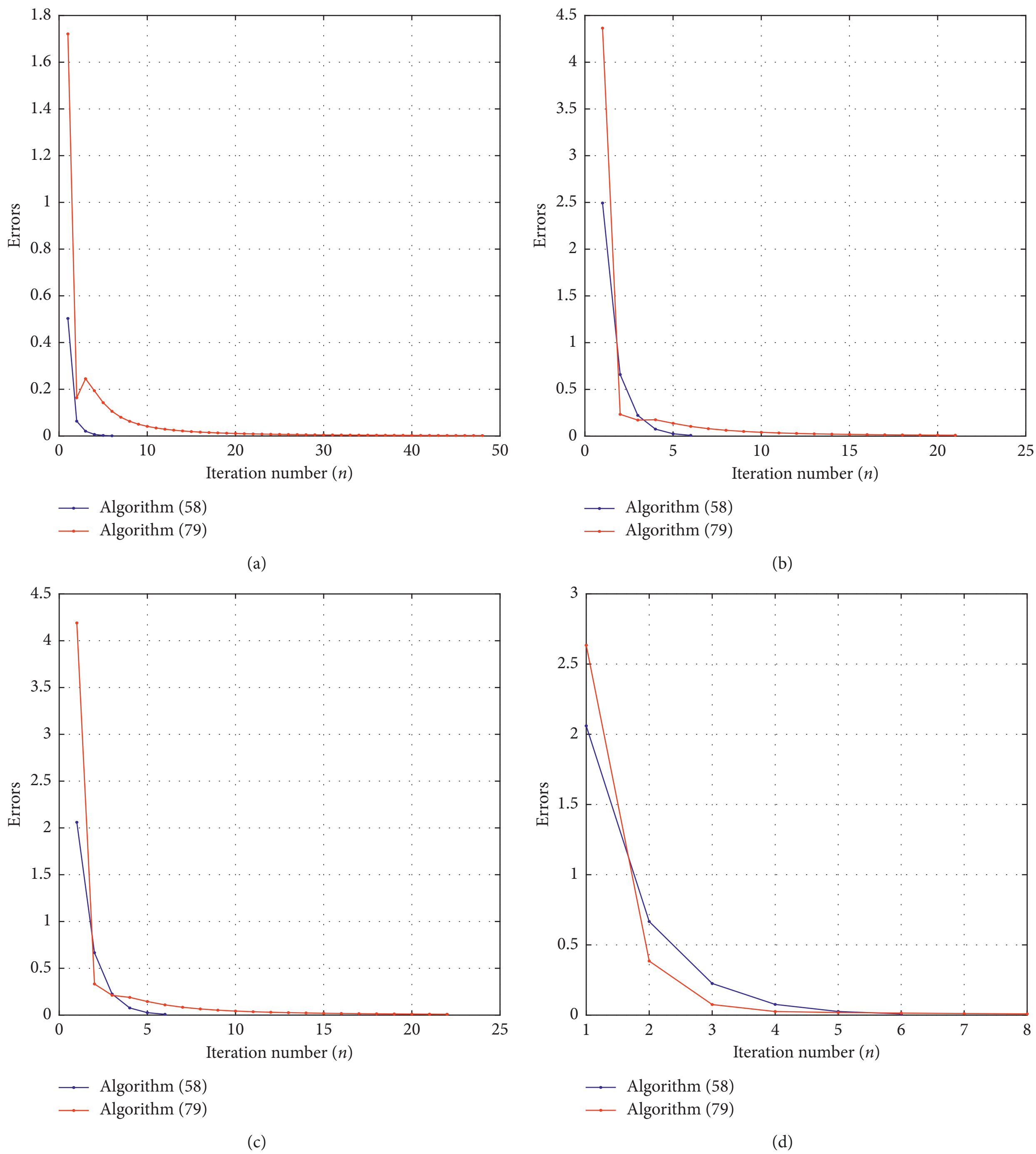

Figure 1: Errors vs iteration numbers n: Case 1 (a); Case 2 (b); Case 3 (c); Case 4 (d).

that Δ-converges to some $\widehat{v} \in C$. Thus, by Lemma 2(ii), we obtain that $A(\{v_{n_k}\}) = \{\widehat{v}\}$. Moreover, $\lim_{k\longrightarrow\infty}\lambda_{n_k} = \infty$ and $\{x_{n_k}\}$ is bounded. Hence, by Lemma 7, we obtain that $\widehat{v} \in \mathrm{MEP}(C, F, \Psi)$.

Next, we show that $\{x_n\}$ converges to $\widehat{v}$. By the Δ-lower semicontinuity of $d^2(u, .)$, we obtain that

$$\begin{aligned} d^2(u,\widehat{v}) &\leq \liminf_{k\longrightarrow\infty} d^2\left(u, v_{n_k}\right) = \lim_{k\longrightarrow\infty} d^2\left(u, v_{n_k}\right) \\ &= \liminf_{n\longrightarrow\infty} d^2(u, v_n). \end{aligned} \tag{76}$$

Since $\delta_n = d^2(u,\overline{v}) + (\alpha_n - 1)d^2(u, v_n)$, $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\overline{v} = P_\Gamma u$, and $\widehat{v} \in \Gamma$, we obtain from the definition of P_Γ and (76) that

$$\limsup_{n\to\infty}\delta_n \le d^2(u,\bar{v}) - \liminf_{n\to\infty} d^2(u,v_n) \le d^2(u,\hat{v}) - \liminf_{n\to\infty} d^2(u,v_n) \le 0. \tag{77}$$

Thus, applying Lemma 8 to (75) gives that $\{x_n\}$ converges to $\bar{v} = P_\Gamma u$. It then follows that $\{J^\Psi_{\lambda_n F} x_n\}$ is convergent to $\bar{v} = P_\Gamma u$.

By setting $\Psi \equiv 0$ in Theorem 8, we obtain the following new result for equilibrium problem in an Hadamard space. □

Corollary 4. *Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1–A3) of Theorem 4 and $\{x_n\}$ be a sequence defined for u, $x_1 \in C$, by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J_{\lambda_n F} x_n, \tag{78}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\longrightarrow\infty}\lambda_n = \infty$. Then, we have the following:

(i) *The sequence $\{J_{\lambda_n F} x_n\}$ is bounded if and only if* $\mathrm{EP}(C,F) \neq \varnothing$

(ii) *If $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\Gamma \coloneqq \mathrm{EP}(C,F) \neq \varnothing$*, then $\{x_n\}$ and $\{J_{\lambda_n F} x_n\}$ *converge to $\bar{v} = P_\Gamma u$,where P_Γ is the metric projection of X onto Γ*

By setting $F \equiv 0$ in Theorem 8, we obtain the following result which coincides with ([41], Theorem 5.1).

Corollary 5. *Let $\Psi : C \longrightarrow C$ be a proper convex and lower semicontinuous function and $\{x_n\}$ be a sequence defined for u, $x_1 \in C$, by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J^\Psi_{\lambda_n} x_n, \tag{79}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\longrightarrow\infty}\lambda_n = \infty$. Then, we have the following:

(i) *The sequence $\{J^\Psi_{\lambda_n} x_n\}$ is bounded if and only if* $\mathrm{argmin}_{y\in C}\Psi(y) \neq \varnothing$

(ii) *If $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\Gamma \coloneqq$* $\mathrm{argmin}_{y\in C}\Psi(y) \neq \varnothing$,*then $\{x_n\}$ and $\{J^\Psi_{\lambda_n} x_n\}$ converge to $\bar{v} = P_\Gamma u$, where $P_\Gamma$is the metric projection of X onto Γ*

6. Numerical Results

In this section, we generate some numerical results in nonlinear setting for Algorithms (58) and (79).

Let $X = \mathbb{R}^2$ be endowed with a metric $d_X : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [0,\infty)$ defined by

$$d_X(x,y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2}, \quad \forall x,\ y \in \mathbb{R}^2. \tag{80}$$

Then, $(\mathbb{R}^2, d_X)$ is an Hadamard space (see ([42], Example 5.2)) with the geodesic joining x to y given by

$$(1-t)x \oplus ty = \Big((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)\big(x_1^2 - x_2\big) - t\big(y_1^2 - y_2\big)\Big). \tag{81}$$

Now, define $\Psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$\Psi(x_1,x_2) = 100\big((x_2 - 2) - (x_1 - 2)^2\big)^2 + (x_1 - 3)^2. \tag{82}$$

Then, it follows from ([42], Example 5.2) that Ψ is a proper convex and lower semicontinuous function in $(\mathbb{R}^2, d_X)$ but not convex in the classical sense (Figure1).

Now, take $\alpha_n = 1/(n+1)$ and $\lambda_n = n+1$ for all $n \ge 1$, then all the conditions of Corollaries 4.5 and 5.6 are satisfied. Hence, by considering the following initial vectors, we obtain the numerical results for Algorithms (58) and (79) as shown by the graphs as follows:

Case 1: $x_1 = (0.5, -0.25)^T$ and $u = (0.5, 3)^T$
Case 2: $x_1 = (-1.5, -3)^T$ and $u = (0.5, 3)^T$
Case 3: $x_1 = (0.5, 3)^T$ and $u = (-1.5, -3)^T$
Case 4: $x_1 = (0.5, 3)^T$ and $u = (0.5, -0.25)^T$

Data Availability

No data were used to support this study.

Disclosure

Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS and NRF.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Halpern-Type Iteration Method for Bregman Nonspreading Mapping and Monotone Operators in Reflexive Banach Spaces

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In this paper, we introduce an iterative method for approximating a common solution of monotone inclusion problem and fixed point of Bregman nonspreading mappings in a reflexive Banach space. Using the Bregman distance function, we study the composition of the resolvent of a maximal monotone operator and the antiresolvent of a Bregman inverse strongly monotone operator and introduce a Halpern-type iteration for approximating a common zero of a maximal monotone operator and a Bregman inverse strongly monotone operator which is also a fixed point of a Bregman nonspreading mapping. We further state and prove a strong convergence result using the iterative algorithm introduced. This result extends many works on finding a common solution of the monotone inclusion problem and fixed-point problem for nonlinear mappings in a real Hilbert space to a reflexive Banach space.

1. Introduction

Let E be a real reflexive Banach space with a norm $\|\cdot\|$ and E^* be the dual space of E. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x^*, x\rangle$. A mapping A is called a monotone mapping if for any $x, y \in \operatorname{dom} A$, we have

$$\begin{aligned} &\mu \in Ax, \\ &\nu \in Ay \Longrightarrow \langle \mu - \nu, x - y\rangle \geq 0. \end{aligned} \tag{1}$$

A monotone mapping $A : E \longrightarrow 2^{E^*}$ is said to be maximal monotone if its graph, $G(A) := \{(x, u) \in E \times E^* : u \in Ax\}$, is not properly contained in the graph of any other monotone operator. A basic problem that arises in several branches of applied mathematics [1–7] is to find $x \in E$ such that

$$0 \in Ax. \tag{2}$$

One of the methods for solving this problem is the well-known proximal point algorithm (PPA) introduced by Martinet [8]. Let H be a Hilbert space and let I denote the identity operator on H. The PPA generates for any starting point $x_0 = x \in H$, a sequence $\{x_n\}$ in H by

$$x_{n+1} = (I + \lambda_n A)^{-1} x_n, \quad n = 1, 2, \ldots, \tag{3}$$

where A is a maximal monotone mapping and $\{\lambda_n\}$ is a given sequence of positive real numbers. It has been observed that (3) is equivalent to

$$0 \in Ax_{n+1} + \frac{1}{\lambda_n}(x_{n+1} - x_n), \quad n = 1, 2, \ldots. \tag{4}$$

This algorithm was further developed by Rockafellar [5], who proved that the sequence generated by (3) converges weakly to an element of $A^{-1}(0)$ when $A^{-1}(0)$ is nonempty and $\liminf_{n\longrightarrow\infty}\lambda_n > 0$. Furthermore, Rockafellar [5] asked if the sequence generated by (3) converges strongly in general. This question was answered in the negative by Güler [9] who presented an example of a subdifferential for which the sequence generated by (3) converges weakly but not strongly. Also, the works of Bruck and Reich [10] and

Bauschke et al. [11] are very important in this direction. For more recent results on PPA, see [12–14].

The problem of finding the zeros of the sum of two monotone mappings A and B, is to find a point $x^* \in E$ such that

$$0 \in (A+B)x^*, \quad (5)$$

has recently received attention due to its significant importance in many physical problems. One classical method for solving problem (5) is the forward-backward splitting method [15], which is as follows: for $x_1 \in E$,

$$x_{n+1} = (I+rB)^{-1}(x_n - rAx_n), \quad n \geq 1, \quad (6)$$

where $r > 0$. This method combines the proximal point algorithm and the gradient projection algorithm. In [16], Lions and Mercier introduced the following splitting iterative methods in a real Hilbert space H:

$$\begin{aligned} x_{n+1} &= \left(2J_r^A - I\right)\left(2J_r^B - I\right)x_n, \quad n \geq 1, \\ x_{n+1} &= J_r^A\left(2J_r^B - I\right)x_n + \left(I - J_r^B\right)x_n, \quad n \geq 1, \end{aligned} \quad (7)$$

where $J_r^T = (I+rT)^{-1}$. The first one is called Peaceman–Rachford algorithm and the second one is called Douglas–Rachford algorithm [15]. It was noted that both algorithms converge weakly in general [16, 17].

Many authors have studied the approximation of zero of the sum of two monotone operators (in Hilbert space) and accretive operators (in Banach spaces), but the approximation of the sum of two monotone operators in more general Banach spaces other the Hilbert spaces has not enjoyed such popularity.

Throughout this paper, $f : E \longrightarrow (-\infty, +\infty]$ is a proper lower semicontinuous and convex function, and the Fenchel conjugate of f is the function $f^* : E^* \longrightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}. \quad (8)$$

We denote by dom$f$ the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$. For any $x \in \operatorname{intdom} f$ and $y \in E$, the right-hand derivative of f at x in the direction of t is defined by

$$f^o(x,y) := \lim_{t \longrightarrow 0^+} \frac{f(x+ty) - f(x)}{t}. \quad (9)$$

The function f is said to be Gâteaux differentiable at x if the limit as $t \longrightarrow 0^+$ in (9) exists for any y. In this case, $f^o(x,y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{intdom} f$. The function f is Fréchet differentiable at x if the limit is attained with $\|y\| = 1$ and uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

The function f is said to be Legendre if it satisfies the following two conditions:

(L1) $\operatorname{intdom} f \neq \emptyset$ and the subdifferential ∂f is single-valued in its domain

(L2) $\operatorname{tdom} f^* \neq \emptyset$ and ∂f^* is single-valued on its domain

The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke et al. in [18]. Their definition is equivalent to conditions (L1) and (L2) because the space E is assumed to be reflexive (see [18], Theorems 5.4 and 5.6, p. 634). It is well known that in reflexive Banach spaces, $\nabla f = (\nabla f^*)^{-1}$ (see [19], p. 83). When this fact is combined with conditions (L1) and (L2), we obtain

$$\begin{aligned} \operatorname{ran}\nabla f &= \operatorname{dom}\nabla f^* = \operatorname{int}(\operatorname{dom} f)^*, \\ \operatorname{ran}\nabla f^* &= \operatorname{dom}\nabla f = \operatorname{int}(\operatorname{dom} f). \end{aligned} \quad (10)$$

It also follows that f is Legendre if and only if f^* is Legendre (see [18], Corollary 5.5, p. 634) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains.

Several interesting examples of the Legendre functions are presented in [18, 20, 21]. A very important example of Legendre function is the function $1/s\|\cdot\|^s$ with $s \in (1,\infty)$, where the Banach space E is smooth and strictly convex, and in particular, a Hilbert space. Throughout this article, we assume that the convex function $f : E \longrightarrow (-\infty, +\infty]$ is Legendre.

Definition 1. Let $f : E \longrightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function, the function $D_f : \operatorname{dom} f \times \operatorname{intdom} f \longrightarrow [0,\infty)$ which is defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle, \quad (11)$$

is called the Bregman distance [22–24].

The Bregman distance does not satisfy the well-known metric properties, but it does have the following important property, which is called the three-point identity: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{intdom} f$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \quad (12)$$

Let C be a nonempty subset of a Banach space E and $T : C \longrightarrow C$ be a mapping, then a point x is called fixed point of T if $Tx = x$. The set of fixed point of T is denoted by $F(T)$. Also, a point $x^* \in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to x^* and $\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$ [25]. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Definition 2 [26, 27]. Let C be a nonempty, closed, and convex subset of E. A mapping $T : C \longrightarrow \operatorname{int}(\operatorname{dom} f)$ is called

(i) Bregman firmly nonexpansive (BFNE for short) if

$$\begin{aligned} &\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \\ &\leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle, \quad \forall x, y \in C. \end{aligned} \quad (13)$$

(ii) Bregman strongly nonexpansive (BSNE) with respect to a nonempty $\widehat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x), \tag{14}$$

for all $p \in \widehat{F}(T)$ and $x \in C$ and if whenever $\{x_n\}_{n=1}^{\infty} \subset C$ is bounded, $p \in \widehat{F}(T)$ and

$$\lim_{n \longrightarrow \infty} \left(D_f(p, x_n) - D_f(p, Tx_n) \right) = 0, \tag{15}$$

it follows that

$$\lim_{n \longrightarrow \infty} D_f(Tx_n, x_n) = 0. \tag{16}$$

(iii) Bregman quasi-nonexpansive if $F(T) \neq \varnothing$ and

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C \text{ and } p \in F(T). \tag{17}$$

(iv) Bregman skew quasi-nonexpansive if $F(T) \neq \varnothing$ and

$$D_f(Tx, p) \leq D_f(x, p), \quad \forall x \in C \text{ and } p \in F(T). \tag{18}$$

(v) Bregman nonspreading if

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) &\leq D_f(Tx, y) \\ &+ D_f(Ty, x), \quad \forall x, y \in C. \end{aligned} \tag{19}$$

It is easy to see that every Bregman nonspreading mapping T with $F(T) \neq \varnothing$ is Bregman quasi-nonexpansive. Also Bregman nonspreading mappings include, in particular, the class of nonspreading functions studied by Takahashi et al. in [28, 29]. For more information on Bregman nonspreading mappings, see [30].

In a real Hilbert space H, the nonlinear mapping $T : C \longrightarrow C$ is said to be

(i) Nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{20}$$

(ii) Quasi-nonexpansive if $F(T) \neq \varnothing$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C \text{ and } p \in F(T). \tag{21}$$

(iii) Nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C. \tag{22}$$

Clearly, every nonspreading mapping T with $F(T) \neq \varnothing$ is also quasi-nonexpansive mapping. The class of nonspreading mappings is very important due to its relation with maximal monotone operators (see, e.g., [28]).

Let $B : E \longrightarrow 2^{E^*}$ be a maximal monotone operator. The resolvent of B, $\mathrm{Res}_B^f : E \longrightarrow 2^E$, is defined by (see [26])

$$\mathrm{Res}_B^f := (\nabla f + B)^{-1} \circ \nabla f. \tag{23}$$

It is known that Res_B^f is a BFNE operator, single-valued, and $F(\mathrm{Res}_B^f) = B^{-1}(0^*)$ (see [26]). If $f : E \longrightarrow \mathbb{R}$ is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded subsets of E, then Res_B^f is BSNE and $\widehat{F}(\mathrm{Res}_B^f) = F(\mathrm{Res}_B^f)$ (see [31]).

Assume that the Legendre function f satisfies the following range condition:

$$\mathrm{ran}(\nabla f - A) \subseteq \mathrm{ran} \nabla f. \tag{24}$$

An operator $A : E \longrightarrow 2^{E^*}$ is called Bregman inverse strongly monotone (BISM) if $(\mathrm{dom} A)$ $(\mathrm{dom} f)$, and for any $x, y \in \mathrm{int}\,\mathrm{dom} f$ and each $u \in Ax$ and $v \in Ay$, we have

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v) \rangle \geq 0. \tag{25}$$

The class of BISM mappings is a generalization of the class of firmly nonexpansive mappings in Hilbert spaces. Indeed, if $f = 1/2\|\cdot\|^2$, then $\nabla f = \nabla f^* = I$, where I is the identity operator and (25) becomes

$$\langle u - v, x - u - (y - v) \rangle \geq 0, \tag{26}$$

which means

$$\|u - v\|^2 \leq \langle x - y, u - v \rangle. \tag{27}$$

Observe that

$$\begin{aligned} \mathrm{dom} A^f &= (\mathrm{dom} A) \cap (\mathrm{int}\,\mathrm{dom} f), \\ \mathrm{ran} A^f &\subset \mathrm{int}\,\mathrm{dom} f. \end{aligned} \tag{28}$$

In other words, T is a (single-valued) firmly nonexpansive operator.

For any operator $A : E \longrightarrow 2^{E^*}$, the antiresolvent operator $A^f : E \longrightarrow 2^E$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A). \tag{29}$$

It is known that the operator A is BISM if and only if the antiresolvent A^f is a single-valued BFNE (see [32], Lemma 3.2(c) and (d), p. 2109) and $F(A^f) = A^{-1}(0^*)$. For examples and further information on BISM, see [32].

Since the monotone inclusion problems have very close connections with both the fixed-point problems and the equilibrium problems, finding the common solutions of these problems has drawn many people's attention and has become one of the hot topics in the related fields in the past few years [33, 34]. Furthermore, interest in finding the common solution of these problems has also grown because of the possible application of these problems to mathematical models whose constraints can be present as fixed points of mappings and/or monotone inclusion problems and/or equilibrium problems. Such a problem occurs, in particular, in the practical problems as signal processing, network resource allocation, and image recovery (see [35, 36]).

In this paper, we introduce an iterative method for approximating a common solution of monotone inclusion problem and fixed point of Bregman nonspreading mapping in a reflexive Banach space and prove a strong convergence of the sequence generated by our iterative algorithm. This result extends many works on finding common solution of monotone inclusion problem and fixed problem of nonlinear mapping in a real Hilbert space to a reflexive Banach space.

2. Preliminaries

The Bregman projection [22] of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty, closed, and convex subset $C \subset \operatorname{int}(\operatorname{dom} f)$ is defined as the necessarily unique vector $\operatorname{Proj}_C^f(x) \in C$ satisfying

$$D_f\left(\operatorname{Proj}_C^f(x), x\right) = \inf\left\{D_f(y, x) : y \in C\right\}. \tag{30}$$

It is known from [37] that $z = \operatorname{Proj}_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \quad \text{for all } y \in C. \tag{31}$$

We also have

$$\begin{aligned} &D_f\left(y, \operatorname{Proj}_C^f(x)\right) + D_f\left(\operatorname{Proj}_C^f(x), x\right) \\ &\leq D_f(y, x), \quad \text{for all } x \in E, y \in C. \end{aligned} \tag{32}$$

Note that if E is a Hilbert space and $f(x) = 1/2\|x\|^2$, then the Bregman projection of x onto C, i.e., $\arg\min\{\|y - x\| : y \in C\}$, is the metric projection P_C.

Lemma 1 [37]. *Let f be totally convex on int (domf). Let C be a nonempty, closed, and convex subset of int (domf) and* $x \in \operatorname{int}(\operatorname{dom} f)$*; if* $z \in C$*, then the following conditions are equivalent:*

(i) $z = Proj_C^f(x)$

(ii) $\langle \nabla f(x) - \nabla f(z), z - y\rangle \geq 0$ *for all* $y \in C$

(iii) $D_f(y, z) + D_f(z, x) \leq D_f(y, x)$ *for all* $y \in C$

Let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The function f is said to be totally convex at $x \in \operatorname{intdom} f$ if its modulus of totally convexity at x, that is, the function $v_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty)$ defined by

$$v_f(x, t) \coloneqq \inf\left\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\right\}, \tag{33}$$

is positive for any $t > 0$. The function f is said to be totally convex when it is totally convex at every point $x \in \operatorname{int}(\operatorname{dom} f)$. In addition, the function f is said to be totally convex on bounded set if $v_f(B, t)$ is positive for any nonempty bounded subset B, where the modulus of total convexity of the function f on the set B is the function v_f : $\operatorname{int}(\operatorname{dom} f) \times [0, +\infty)$ defined by

$$v_f(B, t) \coloneqq \inf\left\{v_f(x, t) : x \in B \cap \operatorname{dom} f\right\}. \tag{34}$$

For further details and examples on totally convex functions, see [37–39].

Let $f : E \longrightarrow \mathbb{R}$ be a convex, Legendre, and Gâteaux differentiable function and let the function $V_f : E \times E^* \longrightarrow [0, \infty)$ associated with f (see [23, 40]) be defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*. \tag{35}$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$, $\forall x \in E, x^* \in E^*$. Furthermore, by the subdifferential inequality, we have (see [41])

$$\begin{aligned} &V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \\ &\leq V_f(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*. \end{aligned} \tag{36}$$

In addition, if $f : E \longrightarrow (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : E^* \longrightarrow (-\infty, +\infty]$ is a proper *weak** lower semicontinuous and convex function (see [42]). Hence, V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\left(z, \nabla f^* \sum_{i=1}^{N} t_i \nabla f(x_i)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{37}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2 (see [43]). *Let* $r > 0$ *be a constant and let* $f : E \longrightarrow \mathbb{R}$ *be a continuous uniformly convex function on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^{\infty} \alpha_k x_k\right) \leq \sum_{k=0}^{\infty} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r\left(\left\|x_i - x_j\right\|\right), \tag{38}$$

for all $i, j \in \mathbb{N} \cup 0$, $x_k \in B_r$, $\alpha_k \in (0, 1)$, *and* $k \in \mathbb{N} \cup 0$ *with* $\sum_{k=0}^{\infty} \alpha_k = 1$, *where* ρ_r *is the gauge of uniform convexity of f.*

Recall that a function f is said to be sequentially consisted (see [37]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n \longrightarrow \infty} D_f(y_n, x_n) = 0 \Longrightarrow \lim_{n \longrightarrow \infty} \|y_n - x_n\| = 0. \tag{39}$$

The following lemma follows from [44].

Lemma 3. *If domf contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 4 (see [45]). *Let* $f : E \longrightarrow (-\infty, +\infty]$ *be a Legendre function and let* $A : E \longrightarrow 2^{E^*}$ *be a BISM operator such that* $A^{-1}(0^*) \neq \emptyset$*. Then the following statements hold:*

(i) $A^{-1}(0^*) = F(A^f)$

(ii) For any $w \in A^{-1}(0^*)$ and $x \in \operatorname{dom} A^f$, we have

$$D_f\left(w, A^f x\right) + D_f\left(A^f x, x\right) \leq D_f(w, x). \tag{40}$$

Remark 1. If the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the antiresolvent A^f is a single-valued BSNE operator which satisfies $F(A^f) = \widehat{F}(A^f)$ (cf. [31]).

Lemma 5 (see [46]). *If $f : E \longrightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 6 (see [44]). *Let $f : E \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_1 \in E$ and the sequence $\{D_f(x_n, x_1)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 7 (see [45]). *Assume that $f : E \longrightarrow \mathbb{R}$ is a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subset of E. Let C be a nonempty, closed, and convex subset of E. Let $\{T_i : 1 \leq i \leq N\}$ be BSNE operators which satisfy $\widehat{F}(T_i) = F(T_i)$ for each $1 \leq i \leq N$ and let $T := w_n T_{N-1} \ldots T_1$. If*

$$\cap\{F(T_i) : 1 \leq i \leq N\}, \tag{41}$$

and $F(T)$ are nonempty, then T is also BSNE with $F(T) = \widehat{F}(T)$.

Lemma 8 (Demiclosedness principle [30]). *Let C be a nonempty subset of a reflexive Banach space. Let $g : E \longrightarrow \mathbb{R}$ be a strict convex, Gâteaux differentiable, and locally bounded function. Let $T : C \longrightarrow E$ be a Bregman nonspreading mapping. If $x_n \rightharpoonup p$ in C and $\lim_{n\longrightarrow\infty} \|Tx_n - x_n\| = 0$, then $p \in F(T)$.*

Lemma 9 (see [47]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n\delta_n \forall n \geq 0, \tag{42}$$

where $\{t_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=o}^{\infty} t_n = \infty$

(ii) $\limsup_{n\longrightarrow\infty} \delta_n \leq 0$

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 10 [48]. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a nondecreasing subsequence $\{n_i\}$ of $\{n\}$, that is, $a_{n_i} \leq a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \longrightarrow \infty$, and the following properties are satisfied for all (sufficiently large number $k \in \mathbb{N}$): $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.*

3. Main Results

Theorem 1. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \longrightarrow \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $A : E \longrightarrow 2^{E^*}$ be a Bregman inverse strongly monotone operator, $B : E \longrightarrow 2^{E^*}$ be a maximal monotone operator, and $T : C \longrightarrow C$ be a Bregman nonspreading mapping. Suppose $\Gamma := F(\mathrm{Res}_A^f \circ A^f) \cap F(T) \neq \emptyset$. Let $\{\gamma_n\} \subset (0, 1)$ and $\{\alpha_n\}, \beta_n$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \delta_n = 1$. Given $u \in E$ and $x_1 \in C$ arbitrarily, let $\{x_n\}$ and $\{y_n\}$ be sequences in E generated by*

$$\begin{cases} y_n = \nabla f^*(\gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n)), \\ x_{n+1} = \mathrm{Proj}_C^f\Big(\nabla f^*\Big(\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \delta_n \nabla f\Big(\mathrm{Res}_B^f \circ A^f(y_n)\Big)\Big)\Big), \quad n \geq 1. \end{cases} \tag{43}$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$

(ii) $(1 - \alpha_n)a < \delta_n$, $\alpha_n \leq b < 1$, $a \in (0, 1/2)$

(iii) $0 \leq c < \liminf_{n\longrightarrow\infty} \gamma_n \leq \limsup_{n\longrightarrow\infty} \gamma_n < 1$

Then $\{x_n\}$ converges strongly to $\mathrm{Proj}_\Gamma^f u$, where Proj_Γ^f is the Bregman projection of E onto Γ.

Proof. First we observe that $F(\mathrm{Res}_B^f \circ A^f) = (A + B)^{-1}0$ and $F(\mathrm{Res}_B^f \circ A^f) = F(\mathrm{Res}_B^f) \cap F(A^f)$. Thus, since Res_B^f and A^f are BSNE operators and $F(\mathrm{Res}_B^f) \cap F(A^f) = (A + B)^{-1}0 \neq \emptyset$, it then follows from Lemma 7 that $\mathrm{Res}_B^f \circ A^f$ is BSNE and $F(\mathrm{Res}_B^f \circ A^f) = \widehat{F}(\mathrm{Res}_B^f \circ A^f)$.

We next show that $\{x_n\}$ and $\{y_n\}$ are bounded.

Let $p \in \Gamma$, then from (43), we have

$$\begin{aligned} D_f(p, y_n) &= D_f(p, \nabla f^*(\gamma_n \nabla f(x_n)) + (1-\gamma_n)\nabla f(Tx_n)) \\ &\leq \gamma_n D_f(p, x_n) + (1-\gamma_n) D_f(p, Tx_n) \\ &\leq \gamma_n D_f(p, x_n) + (1-\gamma_n) D_f(p, x_n) \\ &= D_f(p, x_n). \end{aligned} \tag{44}$$

Also

$$\begin{aligned} D_f(p, x_{n+1}) &\leq D_f\Big(p, \nabla f^*\Big[\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) \\ &\qquad + \delta_n \nabla f\Big(\mathrm{Res}_B^f \circ A^f(y_n)\Big)\Big]\Big) \\ &\leq \alpha_n D_f(p, u) + \beta_n D_f(p, y_n) \\ &\qquad + \delta_n D_f\Big(p, \mathrm{Res}_B^f \circ A^f y_n\Big) \\ &\leq \alpha_n D_f(p, u) + \beta_n D_f(p, y_n) + \delta_n D_f(p, y_n) \\ &= \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, y_n) \\ &= \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) \\ &\leq \max\{D_f(p, u), D_f(p, x_n)\} \\ &\vdots \\ &\leq \max\{D_f(p, u), D_f(p, x_1)\}. \end{aligned} \tag{45}$$

Hence $\{D_f(p, x_n)\}$ is bounded. Therefore, by Lemma 6, $\{x_n\}$ is also bounded, and consequently, $\{y_n\}$ is also bounded.

We now show that x_n converges strongly to $\overline{x} = \mathrm{Proj}_{\Gamma}^{f}(u)$. To do this, we first show that if there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q \in C$, then $q \in \Gamma$.

Let $s = \sup\{\|\nabla f(x_n)\|, \|\nabla f(Tx_n)\|\}$ and $\rho_s^* : E^* \longrightarrow \mathbb{R}$ be the gauge of uniform convexity of the conjugate function f^*. From Lemma 2 and (9), we have

$$\begin{aligned}
D_f(p, y_n) &\le D_f(p, \nabla f^*(\gamma_n \nabla f(x_n)) + (1-\gamma_n)\nabla f(Tx_n)) \\
&= V_f(p, \gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n)) \\
&= f(p) - \langle p, \gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n)\rangle \\
&\quad + f^*(\gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n)) \\
&\le \gamma_n f(p) - \gamma_n \langle p, \nabla f(x_n)\rangle + \gamma_n f^*(\nabla f(x_n)) \\
&\quad + (1-\gamma_n) f(p) - (1-\gamma_n)\langle p, \nabla f(Tx_n)\rangle \\
&\quad + (1-\gamma_n) f^*(\nabla f(Tx_n)) - \gamma_n(1-\gamma_n)\rho_s^* \\
&\quad \cdot (\|\nabla f(x_n) - \nabla f(Tx_n)\|) \\
&= \gamma_n D_f(p, x_n) + (1-\gamma_n) D_f(p, Tx_n) \\
&\quad - \gamma_n(1-\gamma_n)\rho_s^*(\|\nabla f(x_n) - \nabla f(Tx_n)\|) \\
&\le D_f(p, x_n) - \gamma_n(1-\gamma_n)\rho_s^* \\
&\quad \cdot (\|\nabla f(x_n) - \nabla f(Tx_n)\|).
\end{aligned} \tag{46}$$

Thus, from (45), we have

$$\begin{aligned}
D_f(p, x_{n+1}) &\le \alpha_n D_f(p, u) + (1-\alpha_n) \\
&\quad \cdot \left[D_f(p, x_n) - \gamma_n(1-\gamma_n)\rho_s^*(\|\nabla f(x_n) - \nabla f(Tx_n)\|)\right].
\end{aligned} \tag{47}$$

We consider the following two cases for the rest of the proof. □

Case A. Suppose $\{D_f(p, x_n)\}$ is monotonically nonincreasing. Then, $\{D_f(p, x_n)\}$ converges and $D_f(p, x_n) - D_f(p, x_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, from (47), we have

$$\begin{aligned}
&(1-\alpha_n)(1-\gamma_n)\gamma_n \rho_s^*(\|\nabla f(x_n) - \nabla f(Tx_n)\|) \\
&\le \alpha_n\left(D_f(p, u) - D_f(p, x_n)\right) + D_f(p, x_n) - D_f(p, x_{n+1}).
\end{aligned} \tag{48}$$

Since $\alpha_n \longrightarrow 0$, $n \longrightarrow \infty$, then we have

$$\lim_{n\longrightarrow\infty} \gamma_n(1-\gamma_n)\rho_s^*(\|\nabla f(x_n) - \nabla f(Tx_n)\|) = 0, \tag{49}$$

and hence, by condition (iii) and the property of ρ_s^*, we have

$$\lim_{n\longrightarrow\infty} \|\nabla f(x_n) - \nabla f(Tx_n)\| = 0. \tag{50}$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subset of E^*, we have

$$\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0. \tag{51}$$

Again

$$\begin{aligned}
\|\nabla f(x_n) - \nabla f(y_n)\| &= \|\nabla f(x_n) - (\gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n))\| \\
&= (1-\gamma_n)\|\nabla f(x_n) - \nabla f(Tx_n)\| \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{52}$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we have that

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0. \tag{53}$$

Now, let $w_n = \nabla f^*(\beta_n/1-\alpha_n \nabla f(y_n) + \delta_n/1-\alpha_n \nabla f(\mathrm{Res}_B^f \circ A^f y_n))$, then

$$\begin{aligned}
D_f(p, w_n) &= D_f\left(p, \nabla f^*\left[\frac{\beta_n}{1-\alpha_n}\nabla f(y_n) + \frac{\delta_n}{1-\alpha_n}\nabla f\left(\mathrm{Res}_B^f \circ A^f y_n\right)\right]\right) \\
&\le \frac{\beta_n}{1-\alpha_n} D_f(p, y_n) + \frac{\delta_n}{1-\alpha_n} D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right) \\
&\le \frac{\beta_n + \delta_n}{1-\alpha_n} D_f(p, y_n) \\
&= D_f(p, y_n).
\end{aligned} \tag{54}$$

Therefore, we have

$$\begin{aligned}
0 &\le D_f(p, x_n) - D_f(p, w_n) \\
&= D_f(p, x_n) - D_f(p, x_{n+1}) + D_f(p, x_{n+1}) - D_f(p, w_n) \\
&\le D_f(p, x_n) - D_f(p, x_{n+1}) + \alpha_n D_f(p, u) \\
&\quad + (1-\alpha_n) D_f(p, w_n) - D_f(p, w_n) \\
&= D_f(p, x_n) - D_f(p, x_{n+1}) \\
&\quad + \alpha_n\left[D_f(p, u) - D_f(p, w_n)\right] \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{55}$$

More so

$$\begin{aligned}
D_f(p, w_n) &\le \frac{\beta_n}{1-\alpha_n} D_f(p, y_n) + \frac{\delta_n}{1-\alpha_n} D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right) \\
&= D_f(p, y_n) - \left(1 - \frac{\beta_n}{1-\alpha_n}\right) D_f(p, y_n) \\
&\quad + \frac{\delta_n}{1-\alpha_n} D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right) \\
&\le D_f(p, x_n) + \frac{\delta_n}{1-\alpha_n}\left[D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right)\right. \\
&\quad \left. - D_f(p, y_n)\right].
\end{aligned} \tag{56}$$

Since $(1-\alpha_n)a < \delta_n$ and $\alpha_n \le b < 1$, we have

$$a\left(D_f(p,y_n)-D_f\left(p,\mathrm{Res}_B^f\circ A^f y_n\right)\right) < \frac{\delta_n}{1-\alpha_n}\left[D_f(p,y_n) - D_f\left(p,\mathrm{Res}_B^f\circ A^f y_n\right)\right] \le D_f(p,x_n) - D_f(p,w_n)\longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \tag{57}$$

Thus,

$$D_f(p,y_n)-D_f\left(p,\mathrm{Res}_B^f\circ A^f y_n\right)\longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \tag{58}$$

Therefore, since $\mathrm{Res}_B^f\circ A^f$ is BSNE, we have that $\lim_{n\longrightarrow\infty}D_f(y_n,\mathrm{Res}_B^f\circ A^f y_n)=0$, which implies that

$$\lim_{n\longrightarrow\infty}\left\|y_n-\mathrm{Res}_B^f\circ A^f y_n\right\|=0. \tag{59}$$

Setting $u_n=\nabla f^*[\alpha_n\nabla f(u)+\beta_n\nabla f(y_n)+\delta_n\nabla f(\mathrm{Res}_B^f\circ A^f y_n)]$, for each $n\ge 1$, we have

$$\begin{aligned} D_f(y_n,u_n) &= D_f\left(y_n,\nabla f^*\left[\alpha_n\nabla f(u)+\beta_n\nabla f(y_n) + \delta_n\nabla f\left(\mathrm{Res}_B^f\circ A^f y_n\right)\right]\right) \\ &\le \alpha_n D_f(y_n,u)+\beta_n D_f(y_n,y_n)+\delta_n D_f \cdot \left(y_n,\mathrm{Res}_B^f\circ A^f y_n\right)\longrightarrow 0. \end{aligned} \tag{60}$$

Thus,

$$\lim_{n\longrightarrow\infty}\left\|y_n-u_n\right\|=0. \tag{61}$$

Therefore, from (47), we have

$$\left\|u_n-x_n\right\|\le\left\|u_n-y_n\right\|+\left\|y_n-x_n\right\|\longrightarrow 0, \quad n\longrightarrow\infty. \tag{62}$$

Moreover, since $x_{n+1}=\mathrm{Proj}_C^f u_n$, then

$$D_f(p,x_{n+1})+D_f(x_{n+1},u_n)\le D_f(p,u_n), \tag{63}$$

and therefore, we have that

$$\begin{aligned} D_f(x_{n+1},u_n) &\le D_f(p,u_n)-D_f(p,x_{n+1}) \\ &\le \alpha_n D_f(p,u)+\beta_n D_f(p,y_n)+\delta_n D_f\left(p,\mathrm{Res}_B^f\circ A^f y_n\right) - D_f(p,x_{n+1}) \\ &= \alpha_n D_f(p,u)+(1-\alpha_n)D_f(p,y_n)-D_f(p,x_{n+1}) \\ &\le \alpha_n\left(D_f(p,u)-D_f(p,x_n)\right)+D_f(p,x_n) - D_f(p,x_{n+1})\longrightarrow 0, \quad n\longrightarrow\infty, \end{aligned} \tag{64}$$

which implies

$$\left\|x_{n+1}-u_n\right\|\longrightarrow 0, \quad n\longrightarrow\infty. \tag{65}$$

Hence,

$$\left\|x_{n+1}-x_n\right\|\le\left\|x_{n+1}-u_n\right\|+\left\|u_n-x_n\right\|\longrightarrow 0, \quad n\longrightarrow\infty. \tag{66}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $q\in C$ as $n\longrightarrow\infty$. Since $\lim_{n\longrightarrow\infty}\|x_{n_i}-Tx_{n_i}\|=0$, it follows from Lemma 8 that $q\in F(T)$. Also, since $\|x_{n_i}-y_{n_i}\|\longrightarrow 0$, it implies that y_{n_i} also converges weakly to $q\in E$. Therefore, from (59), we have that $q\in F(\mathrm{Res}_B^f\circ A^f)$, and hence, $q\in\Gamma=F(T)\cap F(\mathrm{Res}_B^f\circ A^f)$.

Next, we show that $\{x_n\}$ converges strongly to $\overline{x}=\mathrm{Proj}_\Gamma^f(u)$.

Now from (43), we have

$$\begin{aligned} D_f(\overline{x},x_{n+1}) &\le D_f\left(\overline{x},\nabla f^*\left[\alpha_n\nabla f(u)+\beta_n\nabla f y_n + \delta_n\nabla f\mathrm{Res}_B^f\circ A^f y_n\right]\right) \\ &= V_f\left(\overline{x},\alpha_n\nabla f(u)+\beta_n\nabla f(y_n)+\delta_n\nabla f\left(\mathrm{Res}_B^f\circ A^f\right)y_n\right) \\ &\le V_f\left(\overline{x},\alpha_n\nabla f(u)+\beta_n\nabla f(y_n)+\delta_n\nabla f\left(\mathrm{Res}_B^f\circ A^f\right)y_n - \alpha_n(\nabla f(u)-\nabla f(\overline{x}))\right) \\ &\quad - \langle -\alpha_n(\nabla f(u)-\nabla f(\overline{x})),\nabla f^* \cdot \left[\alpha_n\nabla f(u)+\beta_n\nabla f(y_n)+\delta_n\nabla f\left(\mathrm{Res}_B^f\circ A^f\right)y_n\right]-\overline{x}\rangle \\ &= V_f\left(\overline{x},\alpha_n\nabla f(\overline{x})+\beta_n\nabla f(y_n)+\delta_n\nabla f\left(\mathrm{Res}_B^f\circ A^f\right)y_n\right) + \alpha_n\langle\nabla f(u)-\nabla f(w),u_n-\overline{x}\rangle \\ &= D_f\left(\overline{x},\nabla f^*\left[\alpha_n\nabla f(\overline{x})+\beta_n\nabla f(y_n) + \delta_n\nabla f\left(\mathrm{Res}_B^f\circ A^f\right)y_n\right]\right)+\alpha_n\langle\nabla f(u) - \nabla f(\overline{x}),u_n-\overline{x}\rangle \\ &= \alpha_n D_f(\overline{x},\overline{x})+\beta_n D_f(\overline{x},y_n)+\delta_n D_f\left(\overline{x},\mathrm{Res}_B^f\circ A^f y_n\right) + \alpha_n\langle\nabla f(u)-\nabla f(\overline{x}),u_n-\overline{x}\rangle \\ &\le \beta_n D_f(\overline{x},y_n)+\delta_n D_f(\overline{x},y_n) + \alpha_n\langle\nabla f(u)-\nabla f(\overline{x}),u_n-\overline{x}\rangle \\ &= (1-\alpha_n)D_f(\overline{x},y_n)+\alpha_n\langle\nabla f(u)-\nabla f(\overline{x}),u_n-\overline{x}\rangle \\ &\le (1-\alpha_n)D_f(\overline{x},x_n)+\alpha_n\langle\nabla f(u)-\nabla f(\overline{x}),u_n-\overline{x}\rangle. \end{aligned} \tag{67}$$

Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle\nabla f(u)-\nabla f(\overline{x}),x_n-\overline{x}\rangle = \lim_{j\longrightarrow\infty}\langle\nabla f(u) - \nabla f(\overline{x}),x_{n_j}-\overline{x}\rangle. \tag{68}$$

Since $x_{n_j}\rightharpoonup q$, it follows from Lemma 1(ii) that

$$\limsup_{n\to\infty}\langle\nabla f(u)-\nabla f(\bar{x}),x_n-\bar{x}\rangle=\lim_{j\to\infty}\langle\nabla f(u)-\nabla f(\bar{x}),x_{n_j}-\bar{x}\rangle=\langle\nabla f(u)-\nabla f(\bar{x}),q-\bar{x}\rangle\leq 0. \tag{69}$$

Since $\|u_n-x_n\|\longrightarrow 0$, $n\longrightarrow\infty$, then,

$$\limsup_{n\to\infty}\langle\nabla f(u)-\nabla f(\bar{x}),u_n-\bar{x}\rangle\leq 0. \tag{70}$$

Hence, by Lemma 9 and (67), we conclude that $D_f(\bar{x},x_n)\longrightarrow 0$, $n\longrightarrow\infty$. Therefore, $\{x_n\}$ converges strongly to $\bar{x}=\text{Proj}_\Gamma^f(u)$.

Case B. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$D_f\left(x_{n_j},w\right)<D_f\left(x_{n_j+1},w\right), \tag{71}$$

for all $j\in\mathbb{N}$. Then, by Lemma 10, there exists a non-decreasing sequence $\{m_k\}\subset\mathbb{N}$ with $m_k\longrightarrow\infty$ as $n\longrightarrow\infty$ such that

$$\begin{aligned}&D_f\left(p,x_{m_k}\right)\leq D_f\left(p,x_{m_k+1}\right),\\&D_f(p,x_k)\leq D_f\left(p,x_{m_k+1}\right),\end{aligned} \tag{72}$$

for all $k\in\mathbb{N}$. Following the same line of arguments as in Case I, we have that

$$\begin{aligned}&\lim_{k\to\infty}\left\|Tx_{m_k}-x_{m_k}\right\|=0,\\&\lim_{k\to\infty}\left\|\text{Res}_B^f A^f y_{m_k}-y_{m_k}\right\|=0,\\&\lim_{k\to\infty}\left\|w_{m_k}-x_{m_k}\right\|=0,\\&\limsup_{k\to\infty}\langle\nabla f(u)-\nabla f(p),w_{m_k}-p\rangle\leq 0.\end{aligned} \tag{73}$$

From (67), we have

$$\begin{aligned}D_f\left(p,x_{m_k+1}\right)\leq{}&\left(1-\alpha_{m_k}\right)D_f\left(p,x_{m_k}\right)\\&+\alpha_{m_k}\langle\nabla f(u)-\nabla f(p),w_{m_k}-p\rangle.\end{aligned} \tag{74}$$

Since $D_f(p,x_{m_k})\leq D_f(p,x_{m_k+1})$, it follows from (74) that

$$\begin{aligned}\alpha_{m_k}D_f\left(p,x_{m_k}\right)\leq{}&D_f\left(p,x_{m_k}\right)-D_f\left(p,x_{m_k+1}\right)\\&+\alpha_{m_k}\langle\nabla f(u)-\nabla f(x^*),w_{m_k}-p\rangle\\\leq{}&\alpha_{m_k}\langle\nabla f(u)-\nabla f(p),w_{m_k}-p\rangle.\end{aligned} \tag{75}$$

Since $\alpha_{m_k}>0$, we obtain

$$D_f\left(p,x_{m_k}\right)\leq\langle\nabla f(u)-\nabla f(p),w_{m_k}-p\rangle. \tag{76}$$

Then from (73), it follows that $D_f(p,x_{m_k})\longrightarrow 0$ as $k\longrightarrow\infty$. Combining $D_f(p,x_{m_k})\longrightarrow 0$ with (74), we obtain $D_f(p,x_{m_k+1})\longrightarrow 0$ as $k\longrightarrow\infty$. Since $D_f(p,x_k)\leq D_f(p,x_{m_k+1})$ for all $k\in\mathbb{N}$, we have $x_k\longrightarrow p$ as $k\longrightarrow\infty$, which implies that $x_n\longrightarrow p$ as $n\longrightarrow\infty$.

Therefore, from the above two cases, we conclude that $\{x_n\}$ converges strongly to $\bar{x}=\text{Proj}_\Gamma^f u$.

This completes the proof.

Corollary 1. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f:E\longrightarrow\mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $A:E\longrightarrow 2^{E^*}$ be a Bregman inverse strongly monotone operator, $B:E\longrightarrow 2^{E^*}$ be a maximal monotone operator, and $T:C\longrightarrow C$ be a Bregman firmly nonexpansive mapping. Suppose $\Gamma:=F(Res_A^f\circ A^f)\cap F(T)\neq\varnothing$. Let $\{\gamma_n\}\subset(0,1)$ and $\{\alpha_n\},\beta_n$, and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\delta_n=1$. Given $u\in E$ and $x_1\in C$ arbitrarily, let $\{x_n\}$ and $\{y_n\}$ be sequences in E generated by*

$$\begin{cases}y_n=\nabla f^*(\gamma_n\nabla f(x_n)+(1-\gamma_n)\nabla f(Tx_n)),\\x_{n+1}=\text{Proj}_C^f\left(\nabla f^*\left(\alpha_n\nabla f(u)+\beta_n\nabla f(y_n)+\delta_n\nabla f\left(\text{Res}_B^f\circ A^f(y_n)\right)\right)\right),\quad n\geq 1.\end{cases} \tag{77}$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$
(ii) $(1-\alpha_n)a<\delta_n$, $\alpha_n\leq b<1$, $a\in(0,1/2)$
(iii) $0\leq c<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n<1$

Then, $\{x_n\}$ converges strongly to $\text{Proj}_\Gamma^f u$, where Proj_Γ^f is the Bregman projection of E onto Γ.

Corollary 2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A:H\longrightarrow H$ be a single-valued 1-inverse strongly monotone operator, $B:E\longrightarrow 2^{E^*}$ be a maximal monotone operator, and $T:C\longrightarrow C$ be a firmly nonexpansive mapping. Suppose $\Gamma:=F((I+B)^{-1}(I-A))\cap F(T)\neq\varnothing$. Let $\{\gamma_n\}\subset(0,1)$ and $\{\alpha_n\},\beta_n$, and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\delta_n=1$. Given $u\in E$ and $x_1\in C$ arbitrarily, let $\{x_n\}$ and $\{y_n\}$ be sequences in E generated by*

$$\begin{cases}y_n=\gamma_n x_n+(1-\gamma_n)Tx_n,\\x_{n+1}=P_C\left(\alpha_n u+\beta_n y_n+\delta_n(I+B)^{-1}(I-A)y_n\right),\quad n\geq 1.\end{cases} \tag{78}$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$
(ii) $(1-\alpha_n)a<\delta_n$, $\alpha_n\leq b<1$, $a\in(0,1/2)$
(iii) $0\leq c<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n<1$

Then, $\{x_n\}$ converges strongly to $P_\Gamma u$, where P_Γ is the metric projection of H onto Γ.

4. Application

In this section, we apply our result to obtain a common solution of variational inequality problem (VIP) and equilibrium problem (EP) in real reflexive Banach spaces.

Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E. Suppose $g : C \times C \longrightarrow \mathbb{R}$ is a bifunction that satisfies the following conditions:

A1 $g(x, x) = 0, \quad \forall x \in C$

A2 $g(x, y) + g(y, x) \leq 0, \quad \forall x, y \in C$

A3 $\limsup_{t \downarrow 0} g(tz + (1-t)x, y) \leq g(x, y), \quad \forall x, y, z \in C$

A4 $g(x, .)$ is convex and lower semicontinuous, for each $x \in C$.

The equilibrium problem with respect to g is to find $\overline{x} \in C$ such that

$$g(\overline{x}, y) \geq 0, \quad \forall y \in C. \tag{79}$$

We denote the set of solutions of (79) by EP(g). The resolvent of a bifunction $g : C \times C \longrightarrow \mathbb{R}$ that satisfies $A1 - A4$ (see [49]) is the operator $T_g^f : E \longrightarrow 2^C$ defined by

$$T_g^f(x) := \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \quad \forall y \in C\}. \tag{80}$$

Lemma 11 ([27], Lemma 1, 2). *Let $f : E \longrightarrow (-\infty, \infty)$ be a coercive Legendre function and let C be a nonempty, closed, and convex subset of E. Suppose the bifunction $g : C \times C \longrightarrow \mathbb{R}$ satisfies $A1 - A4$, then*

(1) $\operatorname{dom}(T_g^f) = E$.

(2) T_g^f is single valued

(3) T_g^f is Bregman firmly nonexpansive

(4) $F(T_g^f) = \mathrm{EP}(g)$

(5) $\mathrm{EP}(g)$ is a closed and convex subset of C

(6) $D_f(u, T_g^f(x)) + D_f(T_g^f(x), x) \leq D_f(u, x)$, for all $x \in E$ and for all $u \in F(T_g^f)$

Let $A : E \longrightarrow E^*$ be a Bregman inverse strongly monotone mapping and let C be a nonempty, closed, and convex subset of dom$A$. The variational inequality problem corresponding to A is to find $x \in C$, such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{81}$$

The set of solutions of (81) is denoted by $\mathrm{VI}(C, A)$.

Lemma 12 (see [25, 46]). *Let $A : E \longrightarrow E^*$ be a Bregman inverse strongly monotone mapping and $f : E \longrightarrow (-\infty, \infty]$ be a Legendre and totally convex function that satisfies the range condition. If C is a nonempty, closed, and convex subset of $\operatorname{dom}A \cap \operatorname{int}(\operatorname{dom} f)$, then*

(1) $P_C^f \circ A^f$ is Bregman relatively nonexpansive mapping

(2) $F(P_C^f \circ A^f) = \mathrm{VI}(C, A)$

Now let i_C be the indicator function of a closed convex subset C of E, defined by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & \text{otherwise}. \end{cases} \tag{82}$$

The subdifferential of the indicator function $\partial i_C(\overline{x}) = N_C(\overline{x})$, where C is a closed subset of a Banach space E and $N_C \subset E^*$ is the normal cone defined by

$$N_C(\overline{x}) = \begin{cases} \{v \in E^* : \langle v, x - \overline{x} \rangle \leq 0, & \text{for all } x \in C\}, \ \overline{x} \in C, \\ \varnothing, & x \notin C. \end{cases} \tag{83}$$

The normal cone N_C is maximal monotone and the resolvent of the normal cone corresponds to the Bregman projection (see [50], Example 4.4) that is $\mathrm{Res}_{N_C}^f = \mathrm{Proj}_C^f$.

Therefore, if we let $B = N_C$ and $T = T_g^f$, then the iterative algorithm (77) becomes

$$\begin{cases} y_n = \nabla f^*\left(\gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f\left(T_g^f x_n\right)\right), \\ x_{n+1} = \mathrm{Proj}_C^f\left(\nabla f^*\left(\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \delta_n \nabla f\left(\mathrm{Proj}_C^f \circ A^f(y_n)\right)\right)\right), \quad n \geq 1. \end{cases} \tag{84}$$

Thus, from Corollary 1, we obtain a strong convergence result for approximating a point $x \in \mathrm{VI}(C, A) \cap \mathrm{EP}(g)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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