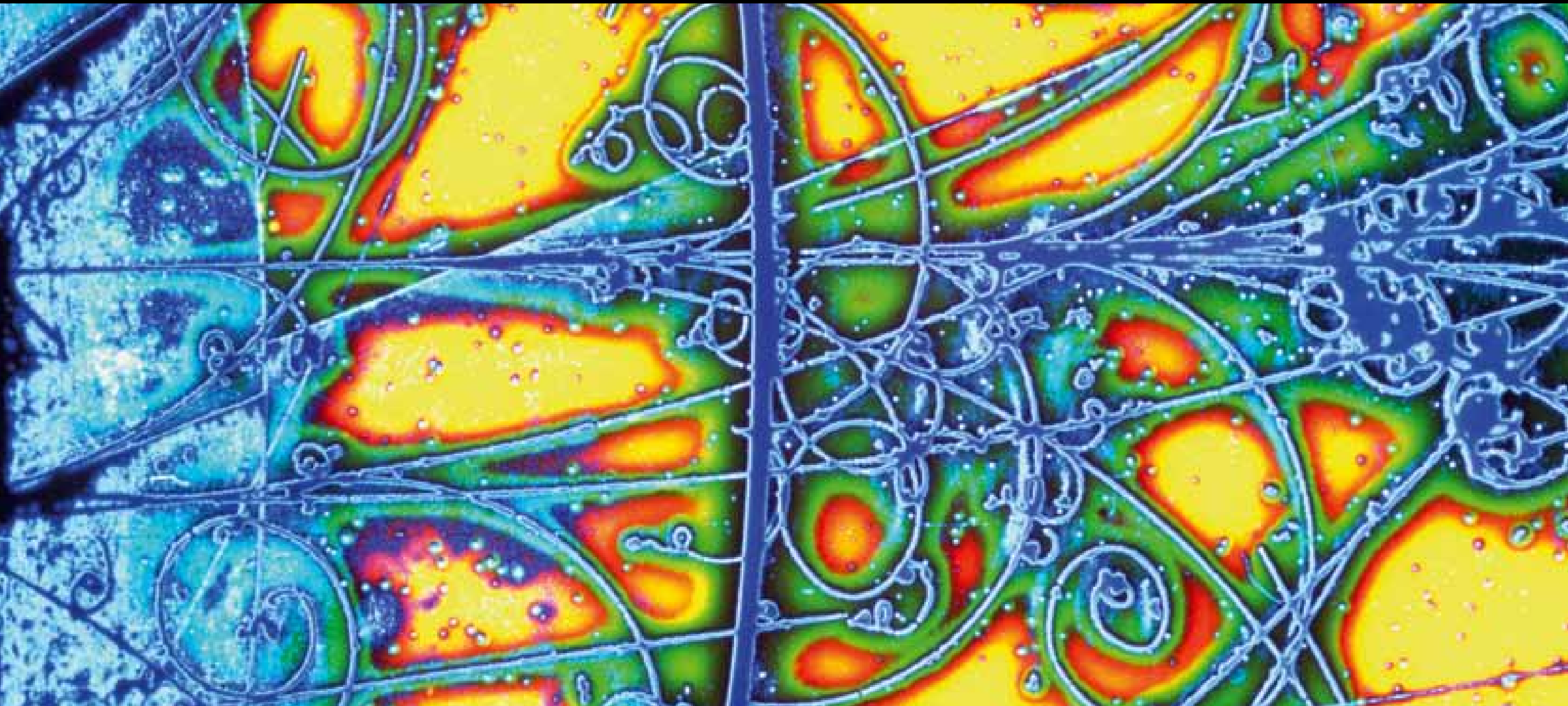
Advances in High Energy Physics

Information Geometry: From Black Holes to Condensed Matter Systems

Guest Editors: Tapobrata Sarkar, Hernando Quevedo, and Rong-Gen Cai



Information Geometry: From Black Holes to Condensed Matter Systems

Information Geometry: From Black Holes to Condensed Matter Systems

Guest Editors: Tapobrata Sarkar, Hernando Quevedo, and Rong-Gen Cai

Editorial Board

Contents

Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2013, Article ID 465957, 2 pages
http://dx.doi.org/10.1155/2013/465957

Editorial

Information Geometry: From Black Holes to Condensed Matter Systems

Tapobrata Sarkar,[1] Hernando Quevedo,[2] and Rong-Gen Cai[3]

[1] *Department of Physics, Indian Institute of Technology, Kanpur 208016, India*
[2] *Instituto de Ciencias Nucleares, Universidad Nacional Autonoma de Mexico, Apartado Postal 70-543, 04510 Mexico City, DF, Mexico*
[3] *Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

Correspondence should be addressed to Tapobrata Sarkar; tapo@iitk.ac.in

Received 8 September 2013; Accepted 8 September 2013

The role of geometry in understanding physical phenomena, from large to small scales, has become an exciting arena of research in the recent past. Novel geometric insights into the phases of matter at different length scales indicate a deep connection between properties of dissimilar physical systems. The present issue contains original research articles aimed towards furthering our understanding of these aspects.

The paper "*A cosmological scaling relation for describing the late time dynamics*" explores the relation between quantum fluctuations and Cosmological dynamics. This is obtained via a proposed scaling that arises from black hole physics, between the mass and minimal information of a given system. The emergence of dark energy is shown to follow as a consequence of this scaling relation.

In the paper "*On thermodynamics of charged and rotating asymptotically AdS black strings*," the authors study the thermodynamics of charged, rotating black strings. This is done by treating the Cosmological constant as pressure and using its conjugate volume. Issues of stability of these systems is analyzed, and it is established that they are stable and that they do not undergo second order phase transitions.

In the paper "*Two-dimensional Einstein manifolds in geometrothermodynamics*," a Riemannian geometric analysis is carried out for a class of thermodynamic systems, for which the associated curvature is a constant. For a particular class of such systems, a differential equation scheme is set up, and a description of a polytropic fluid is obtained.

In "*Legendre invariance and geometrothermodynamics description of the 3D charged-dilaton black hole*," the authors explore a Legendre invariant thermodynamic formalism for charged dilatonic black holes. This is used to obtain the phase properties of such black holes. Issues of stability are also studied, and a comparative analysis of various geometric approaches is made.

In the paper "*Geometrothermodynamics of Myers-Perry black holes*," thermodynamics and its Legendre invariant formulation is studied for various cases in black holes of five dimensions. A three-dimensional parameter manifold arising out of such a construction is also discussed, which gives rise to divergences in addition to the ones appearing due to the phase structure of the theory, and a theoretical explanation for these is offered.

The paper "*Geometric curvatures of plane symmetry black hole*" studies the geometric structure of a class of black holes, in different formalisms. First- and second-order phase transitions are investigated, and the scalar curvatures calculated in these cases. Local thermodynamic stability is also discussed via the heat capacity.

In "*Geometric description of the thermodynamics of the noncommutative Schwarzschild black hole*," a Legendre invariant thermodynamic formulation of a class of noncommutative black holes is discussed. Via the geometric structure of the equilibrium state space, the authors analyze the phase structure of the system, by treating the noncommutativity parameter at par with other thermodynamic variables.

This issue brings together a collection of research papers on the application of geometric methods to a wide variety of physical systems. We hope this will be a useful volume for researchers working in related areas.

Tapobrata Sarkar
Hernando Quevedo
Rong-Gen Cai

Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2013, Article ID 536832, 5 pages
http://dx.doi.org/10.1155/2013/536832

Research Article

A Cosmological Scaling Relation for Describing the Late Time Dynamics

Gerardo Cristofano[1,2] and Orlando Luongo[1,2,3]

[1] *Dipartimento di Fisica, Università di Napoli "Federico II", Via Cinthia, I-80126 Napoli, Italy*
[2] *INFN Sez. di Napoli, Compl. Univ. Monte S. Angelo Ed. N Via Cinthia, I-80126 Napoli, Italy*
[3] *Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, AP 70543, 04510 México, DF, Mexico*

Correspondence should be addressed to Orlando Luongo; luongo@na.infn.it

Received 24 March 2013; Revised 16 June 2013; Accepted 26 June 2013

Academic Editor: Rong-Gen Cai

A scaling relation between mass and minimal information of a given system is inferred from primordial black holes. Extending its validity, it is possible to describe different stages of the universe evolution. Particularly, the broad interest on matching the scaling law, from early to late redshift regimes, may suggest the mechanism to relate quantum and classical aspects of gravity. Under this hypothesis, the scaling relation is interpreted as a thermodynamic modification able to describe the cosmological dynamics at late times. In this scheme, dark energy emerges, as a consequence of assuming the validity of our scale relation. The corresponding equation of state reduces to a cosmological constant at early times and evolves in terms of the apparent horizon at late times.

1. Introduction

The challenge of reproducing the observed scales, from small to high structures, has been extensively investigated in the last decades [1]. It follows from the particular idea that self-gravitating aggregated structures may be predicted from prime principles [2]. This leads to intrinsic connections between quantum mechanics and gravity, providing a unified scheme for quantum gravity. A similar approach has been already employed several decades ago. In particular, it was found a possible correspondence between proton and higher sizes structure masses [3–5]. Even though this fact may represent a first step for obtaining a self-comprehensive theory, including quantum effects in gravitation, the strange correspondence between proton and heavier masses is not yet clarified [6]. The corresponding problem is known in the literature as *coincidence problem* [7–10]. All the suggestions and attempts spent to describe the latter coincidence appeared to be nonconclusive in the framework of astrophysics. However, more recently, an accurate explanation has been proposed, dealing with a possible quantization relation for physical systems [11, 12]. If one assumes black holes as physical systems, it follows that the main interest of the coincidence problem lies on the possible connection of macroscopic scales to the Planck constant $\hbar$ [13]. In doing so, under the hypothesis that the early time phases of the universe evolution could be characterized by primordial black holes, one may infer the observed structures as determined from a self comprehensive common origin, depending on quantum fluctuations [14]. The importance of investigating the consequences of a new quantization rule is to frame the large scale structures into a unifying theory. To this end, it has been shown [15, 16] that it is possible to get a quantization relation which has the meaning of scaling relation between mass and the quantization number.

In this work, by supposing its validity, we describe the consequences of it, at higher and smaller sizes, showing the role played by quantum fluctuations and the corresponding consequences at late time epochs. The observed dynamics today could be influenced by such conditions. We therefore show that it is possible to get a characteristic length of gravitational interaction, showing dark energy effects from prime principles [17]. Moreover, we describe in detail some thermodynamic consequences of our quantization rule, showing that both entropy and temperature become functions of the quantizing number, hereafter n. In doing so, it becomes evident that our scaling relation, although derived at Planck scales, can be used for all self-gravitating astrophysical systems, including the whole universe.

The paper is organized as follows. In Section 2 we describe in detail the main features of the quantization relation. In Section 3 we extend the quantization law from minimal physical regions to higher ones and we give emphasis to the form of entropy and temperature. In Section 4 we highlight a corresponding cosmological application of our framework, which is able to predict the dark energy effects at late times, without the need of introducing any further cosmological fluid. Finally, Section 5 is devoted to conclusions and perspectives of our work.

2. The Quantization Relation

In this section, we describe the main steps leading to the quantization relation, derived from a black hole effective potential, of the early phases of the universe evolution. In this picture, one may assume that primordial black holes dominated the whole universe dynamics [18, 19]. In doing so, the corresponding scalar potential is

$$V_{\mathrm{BH}} = e^{2\phi}\left(Q_e - aQ_m\right)^2 + e^{-2\phi}Q_m^2, \tag{1}$$

where Q_e and Q_m represent the electric and magnetic charges, respectively. The so-called dilaton field is defined by the function ϕ, while the axion field is here represented by a.

This framework is restricted to the so-called extremal black holes, in which the event horizons degenerate. It is worth noticing to focus on this case, following the work of [11, 12]. Let us consider

$$V_{\mathrm{eff}}^{\mathrm{CFT}} = R_c^2\left(Q_e - \frac{\theta}{2\pi}Q_m\right)^2 + \frac{1}{R_c^2}Q_m^2, \tag{2}$$

where R_c is the compactification radius of the so-called *Fubini scalar field*. Once the effective potential is defined, it is easy to get the following identifications:

$$R_c^2 = e^{2\phi}, \qquad \frac{\theta}{2\pi} = a. \tag{3}$$

Having considered one component charge in (1), stability requirements lead to $a = \theta/2\pi = 0$, which describes the case in which only the dilaton field is present, giving as the black hole potential the following simplified expression [20]:

$$V_{\mathrm{BH}} = e^{2\phi}Q_e^2 + e^{-2\phi}Q_m^2. \tag{4}$$

Furthermore, by considering ϕ as a constant field the following relation between black hole mass and effective potential gets implemented [21]:

$$GM^2 = V_{\mathrm{BH}}. \tag{5}$$

By imposing the stability $\partial V_{\mathrm{BH}}/\partial\phi = 0$, one gets

$$e^{2\phi_H} = \frac{Q_m}{Q_e} = R_H^2, \tag{6}$$

with ϕ_H and R_H referring to as their values at the horizon of the black hole. By recasting the mass in terms of the product of the two charges, that is, $GM^2 = 2Q_eQ_m$, and assuming the Dirac quantization condition $2Q_eQ_m = n\hbar c$, with n a positive integer, we finally get

$$GM^2 = n\hbar c. \tag{7}$$

For $n = 1$, we obtain the lowest mass allowed for a quantum black hole (primordial black hole) $M_{\mathrm{BH}} \equiv M_{\mathrm{Planck}}$.

Here, our main purpose is to relate thermodynamics to (7). In doing so, we show that the physical meaning of n, if it becomes a function of the size of the system, that is, $\mathscr{R} = \mathscr{R}(t)$, becomes that of an information parameter. In other words, by employing $n = n(\mathscr{R})$, as the minimal information of a given space time region, it is possible to recover the holographic principle and the Verlinde's recipe [22–26]. This will be highlighted in the next sections.

3. Extending the Quantization Law to Minimal Physical Regions

By following the pioneering work of [11, 12], we extend the quantizing relation, proposed for primordial black hole dominated era, to different epochs along the universe evolution. In particular, we find that one possibility to relate our formula to higher radii, is to consider a continuous n, function of the Hubble radius $\mathscr{R} = \mathscr{R}(t)$, with t the cosmic time. After cumbersome algebra, one is able to depict a model which shows a redshift evolution that predicts, at late time, an accelerating universe under some conditions. The holographic principle could be recovered if n is reviewed as the functional term which describes the minimal information of the space time region under exam.

3.1. The Entropy Representation. The concept of entropy turns out to be very useful to describe the thermodynamical processes which imply an increasing disorder. In astrophysics, all the processes are expected to occur only if the entropy increases as the universe expands. However, that is mainly true for all astrophysical systems. It follows that the use of entropy leads to numerous applications, spanning from virialized systems, to black hole physics. It is commonly believed that the black hole entropy depends on its area, rather than its volume, as in standard thermodynamics. In the context of black hole physics, this turns out to be a natural consequence of the macroscopic horizon, associated to every black hole. The net energy content is assumed to be confined within the horizon itself, and then the corresponding first and second laws of thermodynamics can be easily inferred if one considers the area as $A_{\mathrm{bh}} = 4\pi R_{\mathrm{bh}}^2$, with R_{bh} the black hole radius (In this section, we omit the units $\hbar$, c, and G for the sake of clearness. In the incoming sections we will restore their use.).

It is easy to show that the total mass contained into a black hole is $dM = (\kappa/8\pi)dA_{\mathrm{bh}}$, where κ is the surface gravity of the black hole which can be computed in a model independent way, showing that it is not necessary to fix *a priori* the black hole mass through a ruler constant. For our purposes, the first law of thermodynamics reads $dE = TdS$, which naturally defines a corresponding black hole temperature

$$T_{\rm bh} = \frac{1}{8\pi M}\left(\frac{\hbar c^3}{Gk_B}\right) \tag{8}$$

that can be recasted as

$$T_{\rm bh} = \frac{\kappa}{2\pi}, \tag{9}$$

showing a simple connection between the black hole surface gravity and its temperature. After cumbersome algebra, it is possible to define a net entropy of the form $S_{\rm bh} = (1/4)A_{\rm bh}$, which provides a second law

$$dS_{\rm bh} \geq -dS_{\rm matter}, \tag{10}$$

commonly called the generalized second law. In the next subsection, we adopt the validity of (9) and (10) to relate n to current observable universe, with particular attention to the validity of (10), studying the consequences of introducing an incoming particle into the physical system under interest.

3.2. The Generalized Scaling Relation. To get a scaling independent relation, applicable in modern cosmology, we must fulfill the conditions on entropy, emphasized in the above sections. Particularly, we can assume

$$GM^2 = n(\mathscr{R})\hbar c, \tag{11}$$

where a functional dependence of n on the radius of black holes has been introduced. Under this hypothesis, we get

$$2GMdM = dn\hbar c, \tag{12}$$

so, recalling (7), we infer the density $\rho \equiv dM/dV$, given by

$$\rho^2 = \frac{\hbar c}{G}\frac{dn}{dV^2}, \tag{13}$$

where we made use of the fact that $\rho \approx M/V$. Equation (13) differs from the approaches proposed in [27, 28], since our relation has been obtained in terms of a generic black hole radius, $\mathscr{R}$. In other words, since the validity of (11) is general, as one can see in [11, 12], our relation corresponds to a quantization rule valid for gravitational quantum systems. For these reasons, we want to demonstrate, in what follows, that the use of our quantization relation, in observational cosmology, may lead to accelerate the universe today, without invoking *a priori* a dark energy term, as responsible for the cosmic speed up.

Together with (13) we are able to assume that

$$dM = \frac{\lambda_c^M}{R_S^M}Mdn, \tag{14}$$

in which we used the definitions of Compton length and Schwarzschild radius, that is, λ_c^M and R_S^M, respectively. Equation (14) can be rewritten as

$$k_BT = \frac{1}{2\pi}\frac{\hbar}{c}\frac{GM}{R_S^2}, \tag{15}$$

and together with the quantization relation, we have

$$k_BT = \frac{1}{8\pi}\frac{Mc^2}{n}, \tag{16}$$

and then one infers

$$\frac{S}{k_B} = 4\pi n. \tag{17}$$

The entropy of a given region of space time is therefore associated to $n(\mathscr{R})$. By combining $N = 4\pi n$ and (17) with the expression for the degrees of freedom, one gets

$$Nk_BT = \frac{1}{2}Mc^2, \tag{18}$$

which relates the temperature to the mass of the universe. Since the Unruh temperature is recovered under our picture, it is easy to notice that our quantized rule appears to satisfy the basic demands of thermodynamics.

3.3. Information of Incoming Particles. From another point of view, one can wonder whether the information due to the introduction of $n(\mathscr{R})$ allows us to recover the first principle of thermodynamics, if one adds energy to the system. As a toy model, let us assume the simplest case of one black hole. By assuming that n scales with respect to the radius of our black hole we get

$$TdS = \frac{1}{8\pi}\frac{\hbar c^3}{GMk_B}4\pi dnk_B = \frac{1}{2}\frac{Mc^2}{n}dn, \tag{19}$$

which represents the first principle in terms of the entropy S. Easily we have $dn = 2(GMdM/\hbar c)$. So that, by substituting the definition of n into (19)

$$TdS = 2F_NR_S, \tag{20}$$

we define F_N as the standard Newtonian law. Equation (20) is compatible with the Newtonian dependence on the radius $\mathscr{R}$, because it reproduces the Uhruh law without corrections; that is,

$$TdS = 2(dM)R_S, \tag{21}$$

giving

$$dS = 4\pi k_BdM \cdot \frac{2\pi c}{\hbar}R_S = 4\pi k_B\frac{R_S^M}{\lambda_c^M}, \tag{22}$$

which is compatible with (17) for the definition, $R_S^M = dn\lambda_c^M$. On the other side, for an incoming particle, assuming that the volume of a BH is fixed,

$$dU = dMc^2, \tag{23}$$

therefore, noticing that

$$dU = c^2\frac{\hbar c}{2GM} = \frac{dn}{2}\frac{\hbar c}{GM^2}Mc^2 = \frac{1}{2}Mc^2\frac{dn}{n} \tag{24}$$

that reproduces the first principle $dU = TdS$ when one takes the fixed volume evolution. Moreover, the identity holds $F_N R_S = U/2 = (1/2)Mc^2$. The origin of the factor $1/2$ arises because we are considering a fixed mass M.

In other words, by evaluating the integral $-\int_{R_S}^{0}(GM(\mathscr{R})dM/\mathscr{R}^2)d\mathscr{R}$, with $M = \rho\cdot(4/3)\pi\mathscr{R}^3$, we cannot reproduce the exact expression for the Newtonian law, which is instead recovered if

$$M(\mathscr{R}) = \rho\cdot 4\pi\mathscr{R}^2, \tag{25}$$

showing that the internal density scales as $\mathscr{R}^{-2}$. Relating $M(\mathscr{R})$ to the n term, it is easy to show that n contains the physical information of a certain system, in analogy with the minimal size identified in the holographic principle.

In other words, by considering the quantizing relation and the fact that one adds physical information to the system, it is possible to reobtain the Newtonian law, if n is associated to the total energy budget.

4. Connection with Cosmology

An important consequence arises by assuming that the scale relation could be associated to the net energy budget of the whole universe. In other words, one may consider the universe, as the physical system under exam. So that, by keeping in mind the validity of the anthropic principle, in a Friedmann-Robertson-Walker metric, we consider the apparent horizon, that is, $\mathscr{R} \propto \mathscr{H}^{-1}$ [29, 30], as physical radius. Hence, for reproducing the corresponding expression for the entropy S, we notice that

$$n \propto \mathscr{R}^2, \tag{26}$$

which reproduces the form of the holographic principle in the limiting case in which the apparent horizon corresponds to the universe size, and n to the minimal energy. This represents an important result of our model, since holography seems to be recovered in a simple and concise way, by only postulating the validity of (13), for the whole universe, in which we considered the temperature as given by the standard Hawking radiation, in terms of the apparent horizon. Thus, by assuming that the volume of the universe scales as $V \propto \mathscr{R}^3$ [31, 32], we obtain

$$n \propto \frac{1}{\mathscr{H}^2}, \tag{27}$$

in which we considered the fact that the apparent horizon is proportional to the inverse square of the Hubble rate. By considering the Friedmann equations,

$$\mathscr{H}^2 = \frac{8\pi G}{3}\rho_t, \qquad \dot{\mathscr{H}} + \mathscr{H}^2 = -\frac{4\pi G}{3}\left(3P + \rho_t\right), \tag{28}$$

where ρ_t and P are, respectively, the total energy budget and pressure of the universe, through the use of the continuity equation: that is,

$$\frac{d\rho}{dz} = \frac{3\left(P+\rho\right)}{1+z}. \tag{29}$$

We are able to infer the conditions that relate ρ with n and V. If $\rho_t = \rho_{m,0}(1+z)^3 + \rho$, with $\rho_{m,0}(1+z)^3$ the standard pressureless matter term, by postulating that ρ represents the dark energy counterpart and assuming the standard definition of energy in thermodynamics [33], that is, $\rho \equiv n/V$, we find

$$\rho \propto n^{-1/2}, \tag{30}$$

which is equivalent to require $\rho \propto \mathscr{H}$. The corresponding equation of state of the dark species associated to n reads

$$\omega \equiv \frac{P}{\rho}, \tag{31}$$

and we can rewrite it, as follows:

$$\omega = -\left[1 + \frac{(1+z)}{6}\frac{d\ln n}{dz}\right], \tag{32}$$

which is actually negative, mimicking the dark energy effects, when the first derivative of n with respect to the redshift z is positive. If $dn/dz > 0$, the n parameter should increase as the universe expands, in agreement with the hypothesis that at early times a significative contribution due to n is significative, and the universe is black hole dominated. Moreover, (32) provides a dark energy term, reproducing a late time acceleration, which can be matched with current observations; that is, $-1 \le \omega < 0$, when $1+\omega \propto d\ln V/dz$. In addition, if the volume is negligibly small, at early times, we have $\omega \approx -1$. This turns out to give us an early time cosmological constant contribution. Nevertheless, as $z \to \infty$, a cosmological constant term does not influence the early pressure perturbations, that is, $\delta P \approx 0$, since matter dominates over dark energy. However, at late times, an evolving dark energy term is expected, since ω strongly depends on the form of V in terms of $\mathscr{H}$. Hence, our model reduces to a cosmological constant dark energy at early times, and to a late time evolving dark energy.

5. Conclusions and Perspectives

In this work, we propose a quantum scaling relation, derived from black hole physics. We postulate that our scaling relation is able to describe the universe dynamics, by considering prime principles only. In particular, it is possible to show that, under the hypothesis that n is not a integer number, but a function of the apparent horizon of the universe, one infers the Newtonian law, in agreement with the first principle of thermodynamics. This is analogous to the Verlinde's recipe in which gravity appears as a derived effect. So that, by extending this result to cosmological scales, one finds the interesting fact that dark energy arises as an emerging effect due to our scaling relation. In other words, from our basic demands, it is easy to show that volume, force, and thermodynamics functions can be reobtained in a simple and compact picture. Our goal is to recover the holographic principle, by postulating that $n \propto \mathscr{R}^2$, where $\mathscr{R}$ represents the apparent horizon of the universe. The corresponding dark energy model predicts an evolving equation of state at late times, reducing to a cosmological constant at early times.

Future efforts will be devoted to constrain the cosmological model with current data and to extend the validity of our scaling relation to different cosmological scales.

References

[1] A. D. Sakharov, "The initial stage of an expanding universe and the appearance of a nonuniform distribution of matter," *Journal of Experimental and Theoretical Physics*, vol. 22, no. 1, p. 241, 1966.

[2] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, "Theory of cosmological perturbations," *Physics Report*, vol. 215, no. 5-6, pp. 203–333, 1992.

[3] F. Calogero, "Cosmic origin of quantization," *Physics Letters A*, vol. 228, no. 6, pp. 335–346, 1997.

[4] S. Capozziello and S. Funkhouser, "A scaling law for the cosmological constant from a stochastic model for cosmic structures," *Modern Physics Letters A* , vol. 24, no. 14, p. 1121, 2009.

[5] S. Capozziello and S. Funkhouser, "Fractal large-scale structure from a stochastic scaling law model," *Modern Physics Letters A*, vol. 24, no. 22, p. 1743, 2009.

[6] P. M. A. Dirac, "Quantised singularities in the electromagnetic field," *Proceedings of the Royal Society A*, vol. 133, p. 60, 1931.

[7] P. A. M. Dirac, "A new basis for cosmology," *Proceedings of the Royal Society A*, vol. 165, p. 199, 1938.

[8] P. A. M. Dirac, "The cosmological constants," *Nature*, vol. 139, no. 3512, p. 323, 1937.

[9] P. A. M. Dirac, "Cosmological models and the large numbers hypothesis," *Proceedings of the Royal Society A*, vol. 338, p. 439, 1974.

[10] A. S. Eddington, "Preliminary note on the masses of the electron, the proton, and the universe," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 27, p. 15, 1931.

[11] S. Capozziello, G. Cristofano, and M. De Laurentis, "Primordial black holes, astrophysical systems and the eddington-weinberg relation," *Modern Physics Letters A* , vol. 26, no. 34, pp. 2549–2558, 2011.

[12] S. Capozziello, G. Cristofano, and M. de Laurentis, "Astrophysical structures from primordial quantum black holes," *European Physical Journal C*, vol. 69, no. 1, pp. 293–303, 2010.

[13] S. Capozziello, S. De Martino, S. De Siena, and F. Illuminati, "Phenomenological scaling laws relating the observed galactic dimensions to the microscopic fundamental scales," *Modern Physics Letters A*, vol. 15, no. 16, pp. 1063–1070, 2000.

[14] J. J. Halliwell, "Introductory lectures on quantum cosmology," in *Proceedings of the 1990 Jerusalem Winter School on Quantum Cosmology and Baby Universes*, S. Coleman, J. B. Hartle, T. Piran, and S. Weinberg, Eds., World Scientinc, Singapore, 1991.

[15] R. Kallosh, A. Linde, T. Ortín, A. Peet, and A. Van Proeyen, "Supersymmetry as a cosmic censor," *Physical Review D*, vol. 46, no. 12, pp. 5278–5302, 1992.

[16] S. Fubini, "Vertex operators and quantum Hall effect," *Modern Physics Letters A*, vol. 6, no. 4, pp. 347–358, 1991.

[17] D. A. Demir, "Vacuum energy as the origin of the gravitational constant," *Foundations of Physics*, vol. 39, no. 12, pp. 1407–1425, 2009.

[18] C. T. Byrnes, E. J. Copeland, and A. M. Green, "Primordial black holes as a tool for constraining non-gaussianity," *Physical Review D*, vol. 86, no. 4, Article ID 043512, 9 pages, 2012.

[19] A. G. Polnarev, T. Nakama, and J. Yokoyama, "Self-consistent initial conditions for primordial black hole formation," *Journal of Cosmology and Astroparticle Physics*, vol. 09, p. 027, 2012.

[20] R. Kallosh, A. Linde, T. Ortín, A. Peet, and A. Van Proeyen, "Supersymmetry as a cosmic censor," *Physical Review D*, vol. 46, no. 12, pp. 5278–5302, 1992.

[21] S. Ferrara, K. Hayakawa, and A. Marrani, "Lectures on attractors and black holes," *Fortschritte der Physik. Progress of Physics*, vol. 56, no. 10, pp. 993–1046, 2008.

[22] S. Roy, *Statistical Geometry and Applications to Micro-Physics and Cosmology*, Kluwer, 1998.

[23] R. Bousso and R. Harnik, "Entropic landscape," *Physical Review D*, vol. 82, no. 12, Article ID 123523, 19 pages, 2010.

[24] R. Bousso, "Holographic probabilities in eternal inflation," *Physical Review Letters*, vol. 97, no. 19, Article ID 191302, 4 pages, 2006.

[25] E. P. Verlinde, "On the origin of gravity and the laws of newton," *Journal of High Energy Physics* , vol. 2011, article 29, 2011.

[26] R. Bousso, "The holographic principle," *Reviews of Modern Physics*, vol. 74, no. 3, pp. 825–874, 2002.

[27] S. Hod, "Bohr's correspondence principle and the area spectrum of quantum black holes," *Physical Review Letters*, vol. 81, no. 20, pp. 4293–4296, 1998.

[28] L. Liu and S. Y. Pei, "Sommerfeld's quantum condition of action and the spectra of quantum schwarzschild black hole," *Chinese Physics Letters*, vol. 21, no. 10, pp. 1887–1889, 2004.

[29] K. Bamba, S. Capozziello, S. Nojiri, and S. D. Odintsov, "Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests," *Astrophysics and Space Science* , vol. 342, no. 1, pp. 155–228, 2012.

[30] S. Capozziello and M. de Laurentis, "Extended theories of gravity," *Physics Reports*, vol. 509, no. 4-5, pp. 167–321, 2011.

[31] O. Luongo and H. Quevedo, "Cosmographic study of the universe's specific heat: a landscape for Cosmology?" In press, http://arxiv.org/abs/1211.0626.

[32] R.-G. Cai and S. P. Kim, "First law of thermodynamics and Friedmann equations of Friedmann-Robertson-Walker universe," *Journal of High Energy Physics*, no. 02, article 050, 2005.

[33] H. B. Callen, *Thermodynamics and an Introduction to Thermostatics*, John Wiley and Sons, New York, NY, USA, 1985.

Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2013, Article ID 371084, 7 pages
http://dx.doi.org/10.1155/2013/371084

Research Article

On Thermodynamics of Charged and Rotating Asymptotically AdS Black Strings

Ren Zhao,[1,2] Mengsen Ma,[1,2] Huaifan Li,[1,2] and Lichun Zhang[1,2]

[1] *Institute of Theoretical Physics, Shanxi Datong University, Datong 037009, China*
[2] *Department of Physics, Shanxi Datong University, Datong 037009, China*

Correspondence should be addressed to Ren Zhao; zhaoren2969@yahoo.com.cn

Received 14 March 2013; Revised 17 June 2013; Accepted 4 July 2013

Academic Editor: Tapobrata Sarkar

In this paper, we study thermodynamics of cylindrically symmetric black holes and calculate the equation of states and heat capacity of charged and rotating black strings. In the process, we treat the cosmological constant as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume. It is shown that, when taking the equivalence between the thermodynamic quantities of black strings and the ones of general thermodynamic system, the isothermal compressibility and heat capacity of black strings satisfy the stability conditions of thermodynamic equilibrium and no divergence points exist for heat capacity. Thus, we obtain the conclusion that the thermodynamic system relevant to black strings is stable and there is no second-order phase transition for AdS black holes in the cylindrically symmetric spacetime.

1. Introduction

Black hole physics, especially black hole thermodynamics, refers to many fields such as theories of gravitation, statistical physics, particle physics, and field theory, which makes the profound and fundamental connection between the theories, and much attention has been paid to the subject. It can be said that black hole physics has become the laboratory of many relevant theories. The pioneering works of Bekenstein and Hawking have opened many interesting aspects of unification of quantum mechanics, gravity, and thermodynamics. These are known for the last forty years [1–5]. The black hole thermodynamics has the similar forms to the general thermodynamics, which attracted great attention. In particular the case with negative cosmological constant (AdS case) has concerned many physicists [6–22]. Asymptotically, AdS black hole spacetimes admit a gauge duality description and are described by dual conformal field theory. Correspondingly, one has a microscopic description of the underlying degrees of freedom at hand. This duality has been recently exploited to study the behavior of quark-gluon plasmas and for the qualitative description of various condensed matter phenomena [12].

Recently, the studies on black hole thermodynamics in spherically symmetric spacetime by considering cosmological constant as the variable have got many attentions [12–16, 23–25]. In the previous works on the AdS black hole, cosmological constant corresponds to pressure in general thermodynamic system, the relation is [12, 13, 15]

$$P = -\frac{1}{8\pi}\Lambda = \frac{3}{8\pi}\frac{1}{l^2}, \tag{1}$$

and the corresponding thermodynamic volume is

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q_i,J_k}. \tag{2}$$

In [16], the relation between cosmological constant and pressure is given in the higher dimensional AdS spherically symmetric spacetime, which supplies the basis for the study on the black hole thermodynamics in AdS spherically symmetric spacetime.

Theoretically, if we consider black holes in AdS spacetime as a thermodynamic system, the critical behaviors and phase transitions should also exist. Until now the statistical origin of black hole thermodynamics is still unclear. Therefore, the search for the connection between kinds of thermodynamic quantities in AdS spacetime is meaningful, which may help to understand the entropy, temperature, and heat capacity of black holes and to build the consistent theory for black hole thermodynamics.

In this paper, we generalize the works of [12–19] and research the charged and rotating cylindrically symmetric spacetime. According to (1), we analyzed the thermodynamic properties of charged and rotating black string, calculated the heat capacity, and discussed the critical behaviors and phase transition of black string.

2. Rotating Charged Black Strings

The asymptotically AdS solution of the Einstein-Maxwell equations with cylindrical symmetry can be written as [26–28]

$$\begin{aligned} ds^2 &= -\Xi^2\left(f(r) - \frac{a^2r^2}{\Xi^2 l^4}\right)dt^2 + \frac{dr^2}{f(r)} \\ &\quad - 2\frac{a\Xi l}{r}\left(b - \frac{l}{r}\lambda^2\right)dt\,d\phi \\ &\quad + \left[\Xi^2 r^2 - a^2 f(r)\right]d\phi^2 + \frac{r^2}{l^2}dz^2, \\ A_\mu &= -\Xi\frac{l\lambda}{r}\left(\delta^0_\mu - \frac{a}{\Xi}\delta^2_\mu\right), \end{aligned} \tag{3}$$

where

$$f(r) = \frac{r^2}{l^2} - \frac{bl}{r} + \frac{\lambda^2 l^2}{r^2}, \qquad \Xi^2 = 1 + \frac{a^2}{l^2}. \tag{4}$$

a, b, and λ are the constant parameters of the metric. The entropy, mass, electric charge, and angular momentum per unit length of black string are

$$\begin{aligned} S &= \frac{\pi\Xi r_+^2}{2l}, \qquad M = \frac{1}{8}\left(3\Xi^2 - 1\right)b, \\ Q &= \frac{\Xi\lambda}{2}, \qquad J = \frac{3}{8\Xi}ba, \end{aligned} \tag{5}$$

where r_+ is the location of the event horizon of black hole, which satisfies $f(r_+) = 0$. The Hawking temperature, angular velocity, and electric potential of black string are

$$T = \frac{3r_+^4 - \lambda^2 l^4}{4\pi\Xi l^2 r_+^3}, \qquad \Omega_+ = \frac{a}{\Xi l^2}, \qquad \Phi = \frac{\lambda l}{\Xi r_+}. \tag{6}$$

Expressing the mass per unit length of black string as the function of entropy S, angular momentum J, electric charge Q, and pressure P (cosmological constant l), from (4) and (5), we have

$$\begin{aligned} Q^2 &= \frac{l^2S^2}{\pi^2 r_+^4}\left(\frac{br_+}{l} - \frac{r_+^4}{l^4}\right), \qquad \frac{Q^2\pi^2}{lS^2} + \frac{1}{l^3} = \frac{b}{r_+^3}, \\ b &= \frac{r_+^3\left(S^2 + Q^2\pi^2 l^2\right)}{l^3 S^2}, \\ \frac{4J\pi r_+^2}{3} &= \frac{\left(Q^2\pi^2 l^2 + S^2\right)}{l^2 S}r_+^3 a, \end{aligned} \tag{7}$$

$$\begin{aligned} r_+^2 &= \frac{2lS}{9\pi\left(S^2 + Q^2\pi^2 l^2\right)^2} \\ &\quad \times\left[\sqrt{16J^4S^2l^2\pi^6 + 81\left(S^2 + Q^2\pi^2 l^2\right)^4} - 4J^2Sl\pi^3\right] \\ &= \frac{2lS}{9\pi\left(S^2 + Q^2\pi^2 l^2\right)^2}Y, \end{aligned} \tag{8}$$

where $Y = \sqrt{16J^4S^2l^2\pi^6 + 81(S^2 + Q^2\pi^2l^2)^4} - 4J^2Sl\pi^3$.

From this we can get

$$\begin{aligned} M &= \frac{1}{8}\left[\frac{12l^2S^2}{\pi^2 r_+^4} - 1\right]b \\ &= \frac{1}{8}\left[\frac{12l^2S^2 - \pi^2 r_+^4}{\pi^2 r_+^4}\right]\frac{\left(S^2 + Q^2\pi^2 l^2\right)}{l^3 S^2}r_+^3 \\ &= \frac{1}{8\pi^2 S^2 l^3}\left(S^2 + Q^2\pi^2 l^2\right)\left[\frac{12l^2S^2 - \pi^2 r_+^4}{r_+}\right] \\ &= \frac{3}{\sqrt{8\pi^3 l^3 S}}\left[\frac{3\times 81\left(S^2 + Q^2\pi^2 l^2\right)^4 - Y^2}{81\left(S^2 + Q^2\pi^2 l^2\right)^2\sqrt{Y}}\right] \\ &= \frac{3}{\sqrt{8\pi^3 l^3 S}}\left[\frac{2\times 81\left(S^2 + Q^2\pi^2 l^2\right)^4 + 8J^2Sl\pi^3 Y}{81\left(S^2 + Q^2\pi^2 l^2\right)^2\sqrt{Y}}\right], \end{aligned} \tag{9}$$

where $Y^2 = -8J^2Sl\pi^3Y + 81(S^2 + Q^2\pi^2l^2)^4$. From (9), we can find that the thermodynamic quantities of black string satisfy the first law of thermodynamics as

$$dM = TdS + \Phi dQ + \Omega dJ + VdP. \tag{10}$$

From (10), one can deduce

$$\begin{aligned} T &= \left(\frac{\partial M}{\partial S}\right)_{J,Q,P}, \qquad \Omega = \left(\frac{\partial M}{\partial J}\right)_{S,Q,P}, \\ \Phi &= \left(\frac{\partial M}{\partial Q}\right)_{J,S,P}, \qquad V = \left(\frac{\partial M}{\partial P}\right)_{J,Q,S}, \end{aligned}$$

$$\left(\frac{\partial M}{\partial l}\right)_{S,Q,J} = -\frac{9}{2l\sqrt{8\pi^3 l^3 S}} \times \left[\frac{2\times 81\left(S^2+Q^2\pi^2l^2\right)^4 + 8J^2Sl\pi^3 Y}{81\left(S^2+Q^2\pi^2l^2\right)^2\sqrt{Y}}\right] + \frac{3\left(S^2+Q^2\pi^2l^2\right)}{\sqrt{Y}\sqrt{8\pi^3l^3S}} \times \left[8Q^2\pi^2 l - \frac{\left(S^2+Q^2\pi^2l^2\right)}{Y}\frac{\partial Y}{\partial l} + \frac{4J^2S\pi^3}{81\left(S^2+Q^2\pi^2l^2\right)^3} \times \left(2Y + l\frac{\partial Y}{\partial l}\right) - \frac{32J^2Sl^2\pi^5Q^2Y}{81\left(S^2+Q^2\pi^2l^2\right)^4}\right], \tag{11}$$

where $\partial Y/\partial l = (81\times 4Q^2\pi^2 l(S^2+Q^2\pi^2l^2)^3 - 4J^2S\pi^3Y)/\sqrt{Y}$. From (1), one can derive the corresponding "thermodynamic" volume of black string as

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,J} = \frac{b\pi}{2}l^2\Xi^2 - \frac{2\pi\lambda^2 l^3}{3r_+}. \tag{12}$$

From (12), one can get

$$\Xi^2 = \frac{lV}{\pi r_+^3}\left(V - \frac{2\pi l^3Q^2}{r_+}\right) \pm \frac{l}{r_+^2}\sqrt{\frac{1}{\pi^2 r_+^2}\left(V - \frac{2\pi l^3 Q^2}{r_+}\right)^2 + \frac{16l^2Q^2}{3}}. \tag{13}$$

Because of $Q = 0$, only the plus sign is kept, namely,

$$\Xi^2 = \frac{lV}{\pi r_+^3}\left(V - \frac{2\pi l^3Q^2}{r_+}\right) + \frac{l}{r_+^2}\sqrt{\frac{1}{\pi^2 r_+^2}\left(V - \frac{2\pi l^3 Q^2}{r_+}\right)^2 + \frac{16l^2Q^2}{3}}. \tag{14}$$

From $\Xi^2 = 1 + a^2/l^2$, we can derive

$$a^2 = \left[\frac{lV}{\pi r_+^3}\left(V - \frac{2\pi l^3Q^2}{r_+}\right) + \frac{l}{r_+^2}\sqrt{\frac{1}{\pi^2 r_+^2}\left(V - \frac{2\pi l^3 Q^2}{r_+}\right)^2 + \frac{16l^2Q^2}{3}} - 1\right] l^2. \tag{15}$$

From $J = (3/8)\Xi b a$, we get

$$J^2 = \frac{9}{64}\Xi^2\left(\frac{r_+^3}{l^3} + \frac{4Q^2 l}{r_+\Xi^2}\right)^2 a^2. \tag{16}$$

3. Thermodynamics of Charged Black String

In this section, we discuss thermodynamics of static charged black string. When $a = 0$, $\Xi = 1$. From (12), one can get

$$V = \frac{\pi r_+^3}{2l} - \frac{2\pi Q^2 l^3}{3r_+}, \qquad T = \frac{3r_+}{4\pi l^2} - \frac{l^2Q^2}{\pi r_+^3}. \tag{17}$$

From this, we obtain

$$\left(\frac{\partial P}{\partial V}\right)_{T,Q} = \left(\frac{3}{4\pi l^2} + \frac{3l^2Q^2}{\pi r_+^4}\right) \times \left(\left(\frac{\pi r_+^3}{2l^2} + \frac{2\pi Q^2 l^2}{r_+}\right) \times \left(\frac{3}{4\pi l^2} + \frac{3l^2Q^2}{\pi r_+^4}\right) - \left(\frac{3\pi r_+^2}{2l} + \frac{2\pi Q^2 l^3}{3r_+^2}\right) \times \left(\frac{3r_+}{2\pi l^3} + \frac{2Q^2 l}{r_+^3}\right)\right)^{-1} \times \frac{3}{4\pi l^3} = \frac{3/4\pi l^2 + 3l^2Q^2/\pi r_+^4}{14Q^4l^4/3r_+^5 - \left(15r_+^3/8l^4 + Q^2/r_+\right)}\frac{3}{4\pi l^3}. \tag{18}$$

From (18), when $14Q^4l^4/3r_+^5 > 15r_+^3/l^4 + Q^2/r_+$, $(\partial P/\partial V)_{T,Q} > 0$, the thermodynamic system is unstable. When

$$\frac{14Q^4l^4}{3r_+^5} < \frac{15r_+^3}{l^4} + \frac{Q^2}{r_+}, \tag{19}$$

$(\partial P/\partial V)_{T,Q} < 0$, the thermodynamic system is stable. When $Q \to 0$, $(\partial P/\partial V)_{T,Q} < 0$, the thermodynamic system is stable. Heat capacity at constant pressure is

$$C_{P,Q} = T\left(\frac{\partial S}{\partial T}\right)_{P,Q} = T\left[\frac{\pi r_+}{3/4\pi l + 3l^3Q^2/\pi r_+^4}\right]. \tag{20}$$

According to (20), if $T > 0$, namely, $3r_+/4l^2 > Q^2l^2/r_+^3$, $C_{P,Q}$ will be greater than zero, which fulfills the stable condition

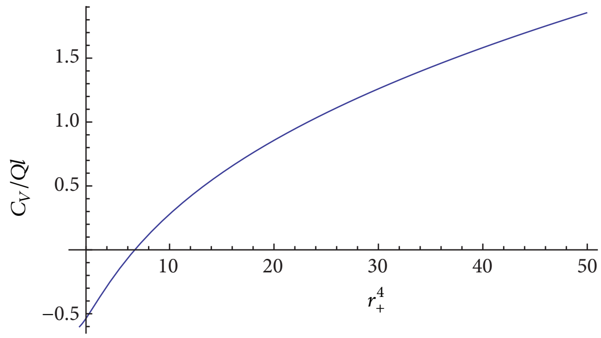


Figure 1: Plots of the heat capacity at constant V and Q versus the r_+^4. The horizontal axis started from 4/3, because of $T > 0$. The intersection point of $C_{V,Q}$ with the horizontal axis is 20/3, which shows that when $r_+^4 > (20/3)Q^2l^4$ holds up, $C_{V,Q} > 0$.

of thermodynamic equilibrium. The heat capacity at constant volume is

$$\begin{aligned}
C_{V,Q} &= T\left(\frac{\partial S}{\partial T}\right)_{V,Q} \\
&= T\left(\left(\frac{\pi r_+}{l}\left(\frac{3r_+^5+12Q^2l^4r_+}{9lr_+^4+4Q^2l^5}\right)-\frac{\pi r_+^2}{2l^2}\right)\right. \\
&\quad\times\left(\left(\frac{3}{4\pi l^2}+\frac{3Q^2l^2}{\pi r_+^4}\right)\times\left(\frac{3r_+^5+12Q^2l^4r_+}{9lr_+^4+4Q^2l^5}\right)\right. \\
&\quad\left.\left.-\left(\frac{3r_+}{2\pi l^3}+\frac{2Q^2l}{\pi r_+^3}\right)\right)^{-1}\right) \\
&= T\frac{10\pi Q^2l^3r_+^2-3\pi r_+^6/2l}{28Q^4l^6/\pi r_+^3-45r_+^5/4\pi l^2-6Q^2r_+l^2/\pi} \\
&= \left(\frac{3r_+}{4l^2}-\frac{Q^2l^2}{r_+^3}\right)\frac{10\pi Q^2l^3r_+^2-3\pi r_+^6/2l}{28Q^4l^6/r_+^3-45r_+^5/4l^2-6Q^2r_+l^2}.
\end{aligned} \tag{21}$$

We can plot the curve of $C_{V,Q}$, which shows that only when the condition $r_+^4 > (20/3)Q^2l^4$ holds up, $C_{V,Q} > 0$ will work. See Figure 1.

From (20) and $T > 0$, when $r_+^4 > (4/3)Q^2l^4$, $C_{P,Q}$ will be greater than zero, which fulfills the stable condition of thermodynamic equilibrium. When $r_+^4 = (4/3)Q^2l^4$, the Hawking temperature of heat capacity is zero which corresponds to the extreme case. However, for the $C_{V,Q}$ only the condition $T > 0$ is not enough. One needs more strict condition $r_+^4 > (20/3)Q^2l^4$, under which the $C_{V,Q}$ is greater than zero. This suggests that the thermodynamic system of charged black strings does not have the first-order phase transition only when $r_+^4 > (20/3)Q^2l^4$. On the other hand, the second-order phase transition points of the thermodynamic system of charged black strings turn up when heat capacities diverge. In this charged black string spacetime, under the given condition $r_+^4 > (20/3)Q^2l^4$, $C_{V,Q}$ and $C_{P,Q}$ are always greater than zero, which suggests that the second-order phase transition of black string will not happen. Whether the phase transition exists when the condition breaks out will be discussed later.

4. Thermodynamics of Rotating Black String

In this section, we discuss thermodynamics of stationary rotating black string. When $Q = 0$, from (12) and (14), we have

$$\Xi^2 = \frac{2lV}{\pi r_+^3}, \qquad V\left(\frac{2lV}{\pi r_+^3}-1\right)=\frac{32\pi J^2l^3}{9r_+^3},$$
$$V = \frac{\pi r_+^3}{4l}+\frac{\pi}{2}\sqrt{\frac{r_+^6}{4l^2}+\frac{64J^2l^2}{9}}, \tag{22}$$

$$T = \frac{3r_+^2\sqrt{r_+}}{4l^2\sqrt{2\pi lV}}. \tag{23}$$

From (22) and (23), we deduce

$$\left(\frac{\partial l}{\partial V}\right)_{T,J} = \frac{3\left(20lV-8\pi r_+^3\right)l}{5\pi\left(9Vr_+^3+32\pi J^2l^3\right)-30lV^2} = \frac{5lV-2\pi r_+^3}{5V}. \tag{24}$$

Thus,

$$\left(\frac{\partial P}{\partial V}\right)_{T,J} = \frac{3\left(2\pi r_+^3-5lV\right)}{20\pi l^3V}. \tag{25}$$

From (22), we have $2lV > \pi r_+^3$, so $(\partial P/\partial V)_{T,J} < 0$, which satisfies the condition of thermodynamic equilibrium. We can derive the heat capacities of rotating string at constant pressure and constant volume as follows:

$$\begin{aligned}
C_{V,J} &= T\left(\frac{\partial S}{\partial T}\right)_{V,J} \\
&= T\left(\left(\frac{1}{6r_+^2V}\left(\frac{\pi V}{2lr_+}\right)^{1/2}\left(\frac{2V^2}{\pi}-\frac{32J^2l^2}{3}\right)\right.\right. \\
&\quad\left.-\frac{1}{2l}\left(\frac{\pi Vr_+}{2l}\right)^{1/2}\right) \\
&\quad\times\left(\frac{5}{8l^2V\sqrt{2\pi lVr_+}}\left(\frac{2V^2}{\pi}-\frac{32J^2l^2}{3}\right)\right. \\
&\quad\left.\left.-\frac{15r_+^{5/2}}{8l^3\sqrt{2\pi lV}}\right)^{-1}\right) \\
&= T\frac{2\pi lV\sqrt{2\pi lVr_+}}{15r_+^3} = \frac{\pi}{10}\frac{V}{l},
\end{aligned}$$

$$\begin{aligned}C_{P,J} &= T\left(\frac{\partial S}{\partial T}\right)_{P,J} \\ &= T\left(\left(\frac{1}{6r_+^2V}\left(\frac{\pi V}{2lr_+}\right)^{1/2}\left(\frac{4lV}{\pi}-r_+^3\right)+\frac{1}{2}\left(\frac{\pi r_+}{2lV}\right)^{1/2}\right)\right. \\ &\quad\times\left(\frac{5}{8l^2V\sqrt{2\pi lVr_+}}\left(\frac{4lV}{\pi}-r_+^3\right)\right. \\ &\quad\left.\left.-\frac{3r_+^{5/2}}{8l^2V\sqrt{2\pi lV}}\right)^{-1}\right) \\ &= \frac{1}{2}\left(\frac{\pi r_+ V}{2l}\right)^{1/2}\left(\frac{2lV-\pi r_+^3}{5lV-2\pi r_+^3}\right).\end{aligned} \tag{26}$$

From (22), we have $lV > \pi r_+^3/2$, so $C_{P,J} > 0$, which satisfies the condition of thermodynamic equilibrium. The second-order phase transition points of thermodynamic systems will appear when heat capacities diverge. According to (26), the heat capacities do not have divergent points; therefore, the second-order phase transition of rotating black string also cannot happen.

5. Thermodynamics of Charged and Rotating Black String

In this section, we discuss thermodynamics of static charged and rotating black string. The location r_+ of event horizon satisfies

$$\frac{r_+^2}{l^2}-\frac{bl}{r_+}+\frac{\lambda^2 l^2}{r_+^2}=0, \tag{27}$$

where

$$\begin{aligned}r_+ &= \frac{1}{2}\left\{\gamma^{1/2}+\left[-\gamma+2\left(\gamma^2-4\lambda^2 l^4\right)^{1/2}\right]^{1/2}\right\}, \\ \gamma &= \left\{\frac{b^2l^6}{2}+\left[\left(\frac{b^2l^6}{2}\right)^2+\left(\frac{4\lambda^2l^4}{3}\right)^2\right]^{1/2}\right\}^{1/3} \\ &\quad+\left\{\frac{b^2l^8}{2}-\left[\left(\frac{b^2l^8}{2}\right)^2+\left(\frac{4\lambda^2l^4}{3}\right)^2\right]^{1/2}\right\}^{1/3}.\end{aligned} \tag{28}$$

For discussion purpose and without loss of generality, we take $a(J)$ and Q to be small quantities relative to M or l and r_+, namely, $a^2/l^2 \ll 1$, $r_+^2 \gg a^2$. From (16), we have

$$J \approx \frac{3}{8}\left(\frac{r_+^3}{l^3}+\frac{4Q^2l}{r_+}\right)a, \tag{29}$$

and when Q is small

$$a \approx \frac{8l^3}{3r_+^3}J. \tag{30}$$

From (12), we can get the approximate value of volume

$$V \approx \frac{\pi r_+^3}{2l}-\frac{2\pi Q^2 l^3}{3r_+}+\frac{32\pi l^6}{9r_+^6}J^2. \tag{31}$$

According to (6), we can obtain the approximate Hawking temperature

$$T \approx \frac{3r_+}{4\pi l^2}\left(1-\frac{32l^4}{r_+^6}J^2\right)-\frac{Q^2l^2}{\pi r_+^3}. \tag{32}$$

From this, we can deduce

$$\left(\frac{\partial P}{\partial V}\right)_{T,J,Q} \approx \frac{3/4\pi l^2+120l^2J^2/\pi r_+^6+3Q^2l^2/\pi r_+^4}{16l^3J^2/r_+^6-\left(15r_+^3/8l^4+Q^2/r_+ +12J^2/r_+^3\right)} \times\frac{3}{4\pi l^3}. \tag{33}$$

From (32), when requiring $T > 0$, the following equation should be satisfied:

$$\frac{3r_+}{4l^2} > \frac{24l^2J^2}{r_+^5}+\frac{Q^2l^2}{r_+^3}. \tag{34}$$

From (33), when

$$\frac{15r_+^3}{8l^4} > \frac{16l^3J^2}{r_+^6}-\left(\frac{12J^2}{r_+^3}+\frac{Q^2}{r_+}\right), \tag{35}$$

we have $(\partial P/\partial V)_{T,J,Q} < 0$, which satisfies the condition of thermodynamic equilibrium. Substituting (34) into (33), we can get $15r_+^3/8l^4 > 16l^3J^2/r_+^6 - 3r_+^3/4l^4$, or

$$\frac{21r_+^3}{8l^4} > \frac{16l^3J^2}{r_+^6} \approx \frac{9a^2}{4l^3}. \tag{36}$$

From (28), one can deduce $r_+/l \propto (4M)^{1/3} > a$, $r_+^2 \gg a^2$; thus, (36) is satisfied.

In order to show the relation between P and V clearly, we plot the V-P curve. According to (1), (31), and (32), we can depict the V-P curve of charged and rotating black strings (Figure 2).

From this figure, we know that the V-P curves of charged and rotating black strings are smooth and continuous; therefore, under the condition of isothermality the first-order and second-order phase transitions caused by the variation of pressure or volume do not exist.

The approximate expression of entropy is

$$S = \frac{\pi\Xi r_+^2}{2l} \approx \frac{\pi r_+^2}{2l}\left(1+\frac{a^2}{2l^2}\right) \approx \frac{\pi r_+^2}{2l}\left(1+\frac{32l^4}{9r_+^6}J^2\right). \tag{37}$$

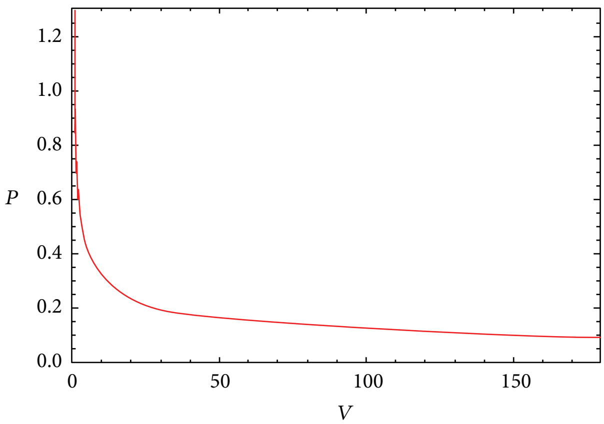


FIGURE 2: Plots of the pressure P versus the volume V. The curves correspond to the parameters $T = 1$, $J = 0.1$, and $Q = 0.0, 0.5, 1.0$. The three curves roughly coincide.

The heat capacity of charged and rotating black strings at constant pressure and constant volume is

$$\begin{aligned}C_{V,J,Q} &= T\left(\frac{\partial S}{\partial T}\right)_{V,J,Q}\\ &\approx T\left(\left(\frac{5\pi^2Q^2l}{3}-\frac{\pi^2r_+^4}{l^3}-\frac{22\pi^2l^4J^2}{3r_+^5}+\frac{40\pi^2lJ^2}{9r_+^2}\right)\right.\\ &\quad\left.\times\left(-\frac{33r_+^3}{8l^4}-\frac{Q^2}{r_+}+\frac{16l^3J^2}{r_+^6}-\frac{12J^2}{r_+^3}\right)^{-1}\right),\\ C_{P,J,Q} &= T\left(\frac{\partial S}{\partial T}\right)_{P,J,Q}\\ &= T\frac{\pi r_+/l-\left(64\pi l^3/9r_+^5\right)J^2}{3/4\pi l^2+120l^2J^2/\pi r_+^6+3Q^2l^2/\pi r_+^4}.\end{aligned} \tag{38}$$

From (35), one can deduce $\pi^2r_+^4/l^3 > 22l^4J^2/3r_+^5-(40lJ^2/9r_+^2+5Q^2l/3)$, $33r_+^3/8l^4 > +16l^3J^2/r_+^6-(12J^2/r_+^3+Q^2/r_+)$; therefore,

$$C_{V,J,Q} > 0, \qquad C_{P,J,Q} > 0. \tag{39}$$

Thus we can consider the charged and rotating black strings as a thermodynamic system and the system can satisfy the stable conditions of equilibrium under the assumption of small a and Q, because the second order phase transition points of the thermodynamic system turn up when heat capacities diverge. According to (39), the heat capacities are always greater than zero, which suggests that the second-order phase transition of black string will not happen when a and Q are small quantities.

6. Conclusion

In this paper, we study the thermodynamic properties of charged and rotating black strings in cylindrically symmetric AdS spacetime. Like the spherically symmetric case for the charged and rotating black strings we take the cosmological constant to correspond to the pressure in general thermodynamic system. The relation is (1). We consider the identification, because when solving Einstein equations the cosmological constant l is independent of the symmetry of spacetime under consideration and the pressure in thermodynamic system also has nothing to do with the surface morphology. Thus the relation (1) should also be appropriate to the charged and rotating cylindrically symmetric spacetime.

On the basis of (1), we analyze the corresponding thermodynamic quantities for charged and rotating black strings. We find that, under some conditions, the heat capacities are greater than zero and $(\partial P/\partial V)_{T,J,Q} < 0$, which satisfy the stable condition of thermodynamic equilibrium. Thus, when the system is perturbed slightly and deviates from equilibrium, some process will appear automatically and makes the system restore equilibrium.

Compared with the works of [12–14], it is found that the thermodynamic properties of black holes in spherically symmetric spacetime are different from the ones of black holes in cylindrically symmetric spacetime, specially that the heat capacities of black holes in cylindrically symmetric spacetime do not have divergent points; thus, no second-order phase transition occurs and no critical phenomena similar to Van der Waals gas occur. At present, the problem cannot be explained logically and it deserves further discussion.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China (Grant nos. 11075098 and 11175109), the Young Scientists Fund of the National Natural Science Foundation of China (Grant no. 11205097), the Natural Science Foundation for Young Scientists of Shanxi Province, China (Grant no. 2012021003-4), and the Shanxi Datong University Doctoral Sustentation Fund (nos. 2008-B-06 and 2011-B-04), China.

References

[1] J. D. Bekenstein, "Black holes and the second law," *Lettere Al Nuovo Cimento Series 2*, vol. 4, no. 15, pp. 737–740, 1972.

[2] J. D. Bekenstein, "Generalized second law of thermodynamics in black-hole physics," *Physical Review D*, vol. 9, no. 12, pp. 3292–3300, 1974.

[3] J. M. Bardeen, B. Carter, and S. W. Hawking, "The four laws of black hole mechanics," *Communications in Mathematical Physics*, vol. 31, no. 2, pp. 161–170, 1973.

[4] S. W. Hawking, "Black hole explosions?" *Nature*, vol. 248, no. 5443, pp. 30–31, 1974.

[5] S. W. Hawking, "Particle creation by black holes," *Communications in Mathematical Physics*, vol. 43, no. 3, pp. 199–220, 1975.

[6] R. Banerjee, S. K. Modak, and S. Samanta, "Second order phase transition and thermodynamic geometry in Kerr-AdS black holes," *Physical Review D*, vol. 84, no. 6, Article ID 064024, 8 pages, 2011.

[7] R. Banerjee and D. Roychowdhury, "Critical behavior of Born-Infeld AdS black holes in higher dimensions," *Physical Review D*, vol. 85, no. 10, Article ID 104043, 14 pages, 2012.

[8] R. Banerjee and D. Roychowdhury, “Critical phenomena in Born-Infeld AdS black holes,” *Physical Review D*, vol. 85, no. 4, Article ID 044040, 10 pages, 2012.

[9] R. Banerjee and D. Roychowdhury, “Thermodynamics of phase transition in higher dimensional AdS black holes,” *Journal of High Energy Physics*, vol. 2011, article 4, 2011.

[10] R. Banerjee, S. Ghosh, and D. Roychowdhury, “New type of phase transition in Reissner Nordström-AdS black hole and its thermodynamic geometry,” *Physics Letters B*, vol. 696, no. 1-2, pp. 156–162, 2011.

[11] R. Banerjee, S. K. Modak, and S. Samanta, “Glassy phase transition and stability in black holes,” *European Physical Journal C*, vol. 70, no. 1, pp. 317–328, 2010.

[12] D. Kubiznak and R. B. Mann, “$P-V$ criticality of charged AdS black holes,” *Journal of High Energy Physics*, vol. 2012, no. 7, article 33, 2012.

[13] B. P. Dolan, D. Kastor, D. Kubiznak, R. B. Mann, and J. Traschen, “Thermodynamic volumes and isoperimetric inequalities for de sitter black holes,” *Physical Review D*, vol. 87, no. 10, Article ID 104017, 14 pages, 2013.

[14] S. Gunasekaran, D. Kubiznak, and R. B. Mann, “Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization,” *Journal of High Energy Physics*, vol. 2012, article 110, 2012.

[15] M. Cvetic, G. W. Gibbons, D. Kubiznak, and C. N. Pope, “Black hole enthalpy and an entropy inequality for the thermodynamic volume,” *Physical Review D*, vol. 84, no. 2, Article ID 024037, 17 pages, 2011.

[16] S. W. Wei and Y. X. Liu, “Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes,” *Physical Review D*, vol. 87, no. 4, Article ID 044014, 14 pages, 2013.

[17] A. Fatima and K. Saifullah, “Thermodynamics of charged and rotating black strings,” *Astrophysics and Space Science*, vol. 341, no. 2, pp. 437–443, 2012.

[18] M. Akbar, H. Quevedo, K. Saifullah, A. Sánchez, and S. Taj, “Thermodynamic geometry of charged rotating BTZ black holes,” *Physical Review D*, vol. 83, no. 8, Article ID 084031, 10 pages, 2011.

[19] A. Belhaj, M. Chabab, H. El Moumni, and M. B. Sedra, “On thermodynamics of AdS black holes in arbitrary dimensions,” *Chinese Physics Letters*, vol. 29, no. 10, Article ID 100401, 2012.

[20] M. H. Dehghani and S. Asnafi, “Thermodynamics of rotating Lovelock-Lifshitz black branes,” *Physical Review D*, vol. 84, no. 6, Article ID 064038, 8 pages, 2011.

[21] S. Bellucci and B. N. Tiwari, “Thermodynamic geometry and topological Einstein-Yang-Mills black holes,” *Entropy*, vol. 14, no. 6, pp. 1045–1078, 2012.

[22] J. Shen, R. Cai, B. Wang, and R. Su, “Thermodynamic geometry and critical behavior of black holes,” *International Journal of Modern Physics A*, vol. 22, no. 1, pp. 11–27, 2007.

[23] B. Dolan, “The cosmological constant and the black hole equation of state,” *Classical and Quantum Gravity*, vol. 28, no. 12, Article ID 125020, 2011.

[24] B. P. Dolan, “Pressure and volume in the first law of black hole thermodynamics,” *Classical and Quantum Gravity*, vol. 28, no. 23, Article ID 235017, 2011.

[25] B. P. Dolan, “Compressibility of rotating black holes,” *Physical Review D*, vol. 84, no. 12, Article ID 127503, 3 pages, 2011.

[26] J. P. S. Lemos and V. T. Zanchin, “Rotating charged black strings and three-dimensional black holes,” *Physical Review D*, vol. 54, no. 6, pp. 3840–3853, 1996.

[27] M. H. Dehghani, “Thermodynamics of rotating charged black strings and (A)dS/CFT correspondence,” *Physical Review D*, vol. 66, no. 4, Article ID 044006, 6 pages, 2002.

[28] R. G. Cai and Y. Z. Zhang, “Black plane solutions in four-dimensional spacetimes,” *Physical Review D*, vol. 54, no. 8, pp. 4891–4898, 1996.

Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2013, Article ID 967618, 6 pages
http://dx.doi.org/10.1155/2013/967618

Research Article

Two-Dimensional Einstein Manifolds in Geometrothermodynamics

Antonio C. Gutiérrez-Piñeres,[1,2] Cesar S. López-Monsalvo,[1] and Francisco Nettel[3]

[1] *Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A.P. 70-543, 04510 Mexico, DF, Mexico*
[2] *Facultad de Ciencias Básicas, Universidad Tecnológica de Bolívar, Cartagena 13001, Colombia*
[3] *Departamento de Física, Facultad de Ciencias, Universidad Nacional Autónoma de México, A.P. 50-542, 04510 México, DF, Mexico*

Correspondence should be addressed to Cesar S. López-Monsalvo; cesar.slm@correo.nucleares.unam.mx

Received 16 March 2013; Accepted 15 June 2013

Academic Editor: Hernando Quevedo

We present a class of thermodynamic systems with constant thermodynamic curvature which, within the context of geometric approaches of thermodynamics, can be interpreted as constant thermodynamic interaction among their components. In particular, for systems constrained by the vanishing of the Hessian curvature we write down the systems of partial differential equations. In such a case it is possible to find a subset of solutions lying on a circumference in an abstract space constructed from the first derivatives of the isothermal coordinates. We conjecture that solutions on the characteristic circumference are of physical relevance, separating them from those of pure mathematical interest. We present the case of a one-parameter family of fundamental relations that—when lying in the circumference—describe a polytropic fluid.

1. Introduction

The study of physical systems that admit a geometric description in terms of Riemannian manifolds is an interesting and timely subject. Over the last few years, there have been a number of efforts towards an integrated description of thermodynamics in terms of Legendre invariant quantities. In particular, analogously to the case of field theories, it has been argued that the curvature of the appropriate manifold should be linked to the notion of thermodynamic interaction [1]. There have been numerous proposals in this direction. On the one hand, there are the conformally related metric theories of Ruppeiner and Weinhold, where the metric takes the form of a Hessian of the extensive parameters in the entropy and energy representations, respectively [2, 3]. However, both fail to comply with the spirit of the geometric construction of field theories; that is, those are *not* invariant under the natural set of transformations in thermodynamics. On the other hand, the Geometrothermodynamics programme (GTD) has successfully managed to provide us with a set of metrics which are independent of the potential used [1] and the fundamental representation one uses [4, 5]. Once the metric parameters are fully specified the Riemannian manifold is uniquely defined.

In the GTD programme one posits that the physical information about a thermodynamic system cannot depend on the potential used to describe it and that such information is encoded in the curvature of the maximal integral manifold of the Pfaffian system defining the first law of thermodynamics (cf. (1)). We call such a manifold the *space of equilibrium states*. The curvature of such manifold is obtained from the first fundamental form, induced from a Legendre invariant metric for the thermodynamic phase space [4]. In Section 2 we present a brief review of the programme; in particular, we centre our attention on a metric whose curvature does not depend on the fundamental representation.

Thus far, the GTD formalism has been applied to a number of thermodynamic systems in order to test the consistency of the programme (e.g., ordinary thermodynamic systems such as ideal gasses, van der Waals, Ising model [5], and black hole thermodynamics [6]) In this work, we take a step forward and set ourselves to the task of finding those systems which exhibit constant *thermodynamic* interaction. That is, we will find the class of fundamental functions producing a manifold of constant curvature. We will restrict ourselves to the two-dimensional case; that is, we consider only systems with two degrees of freedom. These systems are interesting from the mathematical point of view since any two-dimensional metric can be cast into a conformally flat form.

The paper is organised as follows. In Section 2 we present a brief account of the Geometrothermodynamics programme and two-dimensional thermodynamic systems of constant curvature. In Section 3 we analyse the system of partial differential equations to find the set of isothermal coordinates for metrics with vanishing Hessian curvature. There we propose a criterion to single out physical fundamental relations based on a circumference-like equation in an abstract space related to the system of differential equations for the isothermal coordinates. To close this section we present an example illustrating these matters. Finally, in Section 4 we present a summary of the results and address future work on the subject.

2. Two-Dimensional Thermodynamic Systems of Constant Curvature

The GTD programme promotes the natural formalism of thermodynamics in terms of contact manifolds to a Legendre invariant Riemannian structure. Let us begin with a brief review of the programme by considering the case of two thermodynamic degrees of freedom. In this case, we need a five-dimensional manifold which admits a set of local coordinates corresponding to the collection of extensive and intensive variables—denoted by q^i and p_i, respectively—together with the thermodynamic potential, Φ, such that the kernel of the 1-form

$$\Theta = \mathrm{d}\Phi - p_1 \mathrm{d}q_1 - p_2 \mathrm{d}q_2 \tag{1}$$

generates a maximally nonintegrable set of hyperplanes, $\xi \subset T\mathcal{T}$. A manifold $\mathcal{T}$ together with the 1-form Θ is called a contact manifold. In the present case we refer to it as the *thermodynamic phasespace*. Of special interest is the maximal integral submanifold, $\mathcal{E} \subset \mathcal{T}$, that is, the largest sub-manifold which can be embedded in $\mathcal{T}$ such that $T\mathcal{E} \subset \xi$. It is easy to see that this is a two-dimensional manifold defined by the first law of thermodynamics

$$\begin{gathered} \mathrm{d}\Phi = p_1 \mathrm{d}q_1 + p_2 \mathrm{d}q_2, \\ \text{where } \Phi = \Phi(q_1, q_2), \qquad p_i = \frac{\partial \Phi}{\partial q_i} \equiv \Phi_{,i}. \end{gathered} \tag{2}$$

Thus, we see that if we know the fundamental function $\Phi = \Phi(q_1, q_2)$, then we know how $\mathcal{E}$ is embedded in $\mathcal{T}$. We call the sub-manifold $\mathcal{E}$ the *space of equilibrium states*.

In addition, the GTD programme introduces a metric structure for the thermodynamic phase space. Such a structure is constructed in order to satisfy the criterion of Legendre invariance; that is, Legendre transformations correspond to isometries. Within the GTD programme there have been two distinct classes of metrics which have been studied according to their invariance properties, those which are invariant under every possible Legendre transformation and those which are only invariant under *total* Legendre transformations. The metric structure of $\mathcal{T}$ induces a Riemannian metric on $\mathcal{E}$, its first fundamental form, whose intrinsic curvature is associated with the thermodynamic interaction of the system. In our two-dimensional scenario, this whole information is contained in the curvature scalar of $\mathcal{E}$.

If the curvature of the space of equilibrium states is to give a faithful account of the thermodynamic interaction, it should not depend on the choice of fundamental representation; that is, one is free to work in the energy or entropy representation indistinctly. It has been shown that the metric compatible with both Legendre and representation invariance is

$$G^{\natural} = \Theta \otimes \Theta + \frac{1}{q_2 p_2} \left(\mathrm{d}q_1 \otimes \mathrm{d}p_1 + \mathrm{d}q_2 \otimes \mathrm{d}p_2 \right). \tag{3}$$

Thus, the induced metric on $\mathcal{E}$ is simply given by

$$g^{\natural} = \Omega(q_1, q_2)\, h. \tag{4}$$

Here h is the Hessian metric

$$\begin{aligned} h = {} & \Phi_{,11} \mathrm{d}q_1 \otimes \mathrm{d}q_1 \\ & + \left(\Phi_{,12} + \Phi_{,21} \right) \mathrm{d}q_1 \otimes \mathrm{d}q_2 + \Phi_{,22} \mathrm{d}q_2 \otimes \mathrm{d}q_2, \end{aligned} \tag{5}$$

where we have used a coma to denote partial differentiation with respect to the corresponding coordinate function of $\mathcal{E}$ and the conformal factor is given by

$$\Omega(q_1, q_2) = \frac{1}{q_2 \Phi_{,2}}. \tag{6}$$

The interested reader in the derivation of the metric (4) is referred to [4] and to [5] for applications to ordinary thermodynamic systems.

Note that the components of the metric (4) depend on the second derivatives of the fundamental function Φ but are otherwise unspecified. It is an interesting exercise to find a class of fundamental functions for which the space of equilibrium states $\mathcal{E}$ becomes an Einstein manifold for the metric (4). That is, we look for solutions of the system

$$R^{\natural}_{ab} = K g^{\natural}_{ab}, \tag{7}$$

where $R^{\natural}_{ab}$ is the Ricci tensor associated with $g^{\natural}$ and K is a constant, which in the present case corresponds to the Gaussian curvature of $\mathcal{E}$.

It is worth noting that (7) represents a system of three, *third* order, nonlinear partial differential equations for the thermodynamic potential Φ. Indeed, it is straightforward to show that in two dimensions, the fourth order terms in the curvature exactly cancel whenever the metric is the Hessian of a scalar function.

We can reduce the system (7) by raising one of the indexes to obtain

$$R^{\natural\,b}_{\;\;a} = K\delta^b_a. \tag{8}$$

Thus, the system reduces to the single PDE

$$F\left(\Phi_{,i},\Phi_{,ij},\Phi_{,ijj},\Phi_{,iii}\right) = 4K\frac{\rho^2}{\Omega^5\left(q_1,q_2\right)} \quad \text{with } i,j=1,2. \tag{9}$$

Here ρ is the determinant of the metric (4) given by the expression

$$\rho = \Omega^2\left(q_1,q_2\right)\left(\Phi_{,11}\Phi_{,22}-\Phi_{,12}^2\right), \tag{10}$$

and the lhs of (9) is

$$\begin{aligned} F&\left(\Phi_{,i},\Phi_{,ij},\Phi_{,ijj},\Phi_{,iii}\right) \\ &= \Phi_{,2}^2\left(A\Phi_{,11}^2 + B\Phi_{,11} - 2q_2\Phi_{,211}\Phi_{,12}^2 + C\Phi_{,12} + D\Phi_{,22}\right) \\ &\quad + \Phi_{,2}\left(q_2^2\Phi_{,22}\Phi_{,222}\Phi_{,11}^2 + E\Phi_{,11} + 2q_2^2\Phi_{,12}^3\Phi_{,221}\right. \\ &\qquad\qquad \left.-q_2^2\Phi_{,12}^2\Phi_{,22}\Phi_{,211}\right) \\ &\quad - 2q_2^2\rho\Omega^{-2}\left(\Phi_{,22}^2\Phi_{,11} - \Phi_{,12}^2\Phi_{,22}\right), \end{aligned} \tag{11}$$

where

$$\begin{aligned} A &= -q_2\Phi_{,222} - 2\Phi_{,22}, \\ B &= 2\Phi_{,12}^2 + 3q_2\Phi_{,12}\Phi_{,221} \\ &\quad - q_2\Phi_{,211}\Phi_{,22} + q_2^2\Phi_{,221}^2 - q_2^2\Phi_{,222}\Phi_{,211}, \\ C &= -q_2^2\Phi_{,221}\Phi_{,211} + q_2^2\Phi_{,111}\Phi_{,222} + q_2\Phi_{,22}\Phi_{,111}, \\ D &= -q_2^2\Phi_{,111}\Phi_{,221} + q_2^2\Phi_{211}^2, \\ E &= -2q_2^2\Phi_{,22}\Phi_{,12}\Phi_{,221} - q_2^2\Phi_{,12}^2\Phi_{,222} + q_2^2\Phi_{,22}^2\Phi_{,211}. \end{aligned} \tag{12}$$

Motivated by the results of a previous work by the authors (c.f. Section III.D in [5, 7]), we know that a solution to (8) is given by the fundamental relation

$$\Phi = \Phi_0 \log\left(q_1^\alpha + cq_2^\alpha\right). \tag{13}$$

Since we are working in two dimensions, the Gaussian and scalar curvature are proportional and we see that the constancy of K is satisfied and has the value

$$K = -\frac{1}{4}\frac{\alpha^2}{\alpha-1}. \tag{14}$$

Thus, we can propose a general solution of the form

$$\Phi = f\left(\xi q_1 + \chi q_2\right). \tag{15}$$

Here f is a sufficiently differentiable function of the sum of the extensive parameters, where ξ and χ are constants. This type of ansatz does solve (11). However, a quick inspection to the metric determinant reveals the degeneracy of this case (c.f. (17), below). Therefore, let us propose the more general solution

$$\Phi = f\left(\xi q_1^\alpha + \chi q_2^\alpha\right), \tag{16}$$

where α is a constant different from one hence the case of the dark fluid cannot be analysed with this metric. Now, the metric determinant is in general different from zero and has the form

$$\rho = \frac{a-1}{\chi q_2^{2+a} q_1^2 f'}\left[f' + \xi a q_1^a\left(\xi q_1^\alpha + \chi q_2^\alpha\right)f''\right], \tag{17}$$

where f' and f'' are the first and second total derivatives of the fundamental relation (16) evaluated at $(\xi q_1^\alpha + \chi q_2^\alpha)$. Now we can clearly see the degeneracy for $\alpha = 1$.

Substituting our ansatz (16) into (8) we obtain again the same result as in the case of (13), that is, the Gaussian curvature is the same constant, (14); regardless of the particular form of the function f as long as the argument is $(\xi q_1^\alpha + \chi q_2^\alpha)$. Therefore, the generalised Chaplygin gas, (13), belongs to a class of thermodynamic systems with the same type of interaction given by (16). Moreover, the Hessian metric for the logarithmic form of this type of fundamental relation has vanishing curvature. In this case it becomes a simpler problem to find the set of isothermal coordinates for the space of equilibrium states.

3. Isothermal Coordinates

It is a well-known result that every two-dimensional Riemannian manifold is conformally flat. That is, we can always find a set of coordinates for which the metric takes the form

$$g = \widetilde{\Omega}^2\left(x,y\right)g^\flat, \quad \text{where } g^\flat = dx\otimes dx + dy\otimes dy. \tag{18}$$

Such a coordinate system is called *isothermal*. In this section we find the isothermal coordinates for the space of equilibrium states $(\mathscr{E}, g^\natural)$ under the assumption of the Hessian flatness, that is, by demanding that the curvature scalar of the Hessian part of the metric (4) vanishes.

Let us consider the diffeomorphism $\varphi : \mathscr{E} \to \mathscr{E}$ accounting for the change of coordinates $x = x(q_1, q_2)$ and $y = y(q_1, q_2)$. Then we can pull back the metric $g^\flat$ (c.f. (18)) and solve the equation

$$\varphi^* g^\flat - h = 0 \tag{19}$$

for the coordinate functions x, y and the thermodynamic potential Φ. This will provide us thermodynamic fundamental relations with zero Hessian curvature together with their isothermal coordinates. Equation (19) above corresponds to the system of equations

$$\begin{aligned} x_{,1}^2 + y_{,1}^2 &= \Phi_{,11}, \\ x_{,1}x_{,2} + y_{,1}y_{,2} &= \Phi_{,12}, \\ x_{,2}^2 + y_{,2}^2 &= \Phi_{,22}. \end{aligned} \tag{20}$$

There is a large family of solutions for such a system. In particular, for separable fundamental relations,

$$\Phi = S(q_1) + T(q_2), \tag{21}$$

we have that the change of coordinates is given by

$$\begin{aligned} x &= \int \sqrt{(1-c^2)S''}\,\mathrm{d}q_1 + c\int \sqrt{\ddot{T}}\,\mathrm{d}q_2, \\ y &= \int \sqrt{(1-c^2)\ddot{T}}\,\mathrm{d}q_2 - c\int \sqrt{S''}\,\mathrm{d}q_1, \end{aligned} \tag{22}$$

where c is a constant and the primes and dots denote differentiation with respect to q_1 and q_2, respectively.

A more general solution is given by those fundamental functions satisfying the third order system of PDE's

$$\Phi_{,211} = \frac{\Phi_{,222}\Phi_{,12}^2}{\Phi_{,22}^2}, \qquad \Phi_{,1,22} = \frac{\Phi_{,222}\Phi_{1,2}}{\Phi_{22}}. \tag{23}$$

In this case, the isothermal coordinates must satisfy the system

$$x_{,1}^2 = \frac{1}{\Phi_{,22}}\left(2\Phi_{,12}x_{,1}x_{,2} + \Phi_{,11}\Phi_{,22} - x_{,2}^2\Phi_{,11} - \Phi_{,12}^2\right), \tag{24}$$

$$x_{,22} = \frac{1}{2}\frac{\Phi_{,222}x_{,2}}{\Phi_{,22}}, \tag{25}$$

$$\begin{aligned} y = \int \sqrt{\Phi_{,22} - x_{,2}^2}\,\mathrm{d}q_2 + \frac{1}{2}\int \frac{1}{\sqrt{\Phi_{,22} - x_{,2}^2}} \\ \times \left[\sqrt{\Phi_{,22} - x_{,2}^2}\int \frac{2x_{,2}x_{,12} - \Phi_{,221}}{\sqrt{\Phi_{,22} - x_{,2}^2}}\,\mathrm{d}q_2 \right. \\ \left. -2x_{,1}x_{,2} + 2\Phi_{,12}\right]\mathrm{d}q_1. \end{aligned} \tag{26}$$

One can verify that a fundamental relation of the form (15) is a solution of (23). We have seen that this type of functions generates degenerate Hessian metrics. However, we can use them to learn some properties about the space of solutions of the system (20). For example, consider the fundamental relations given by

$$\Phi = \log(\xi q_1 + \chi q_2). \tag{27}$$

In this case we can solve the pair of equations for x, that is, (24), and (25) to obtain

$$x = c\log\left(q_2 + \frac{\xi}{\chi}q_1\right), \tag{28}$$

and substitution in (26) yields

$$y = \sqrt{-(1 + c^2)}\log(\xi q_1 + \chi q_2). \tag{29}$$

Thus we see that, indeed, this type of fundamental relation fails to produce a real change of coordinates satisfying (19). Moreover, note that we can find particular solutions to the system (20) if we restrict ourselves to a circumference in an abstract XY plane. Thus, we have

$$X^2 + Y^2 = R^2, \tag{30}$$

where

$$\begin{aligned} X^2 &= (x_{,1} + x_{,2})^2, \\ Y^2 &= (y_{,1} + y_{,2})^2, \\ R^2 &= \Phi_{,11} + 2\Phi_{,12} + \Phi_{,22}. \end{aligned} \tag{31}$$

From this point of view, we observe that the fundamental relation (27) corresponds to an "imaginary" radius of the circumference (30); that is

$$R^2 = -\frac{(\xi + \chi)^2}{(\xi q_1 + \chi q_2)^2}. \tag{32}$$

This is not surprising since we knew that the Hessian corresponding to this fundamental relation is degenerate and thus the system is not well posed except for the case $\xi = -\chi = 1$, for which $R = 0$.

We can use this geometric construction to probe the space of solutions for a *fixed* fundamental relation by noting that a solution to the system (20) must lie on the circumference associated with the particular fundamental relation we use (c.f. (32)), but not every solution lying on the circumference solves the system we are probing.

Example 1. To see how this construction works, let us choose a family of fundamental relations in the form of (13) parametrised by the exponent α. We work in the entropy representation using molar quantities. Thus we set by $q_1 = u$ the specific energy and $q_2 = v$ is the specific volume. The fundamental relation is written as

$$s_\alpha = \log(u^\alpha + v^\alpha). \tag{33}$$

Each of these functions defines a Hessian metric of zero curvature and a natural metric of constant thermodynamic interaction (c.f. (14)). The change to isothermal coordinates for this type of functions cannot be expressed analytically. However, we can use the circumference to classify the various types of differential equations obtained for each value of α.

The squared radii of the circles associated with each function are given by

$$\begin{aligned} R_\alpha^2 = &-\frac{\alpha}{u^2v^2(u^\alpha + v^\alpha)} \\ &\times \left[u^{2\alpha}v^2 - v^\alpha u^\alpha \right. \\ &\left. \times \left((\alpha-1)u^2 - 2\alpha uv + v^2(\alpha-1)\right) + v^{2\alpha}u^2\right]. \end{aligned} \tag{34}$$

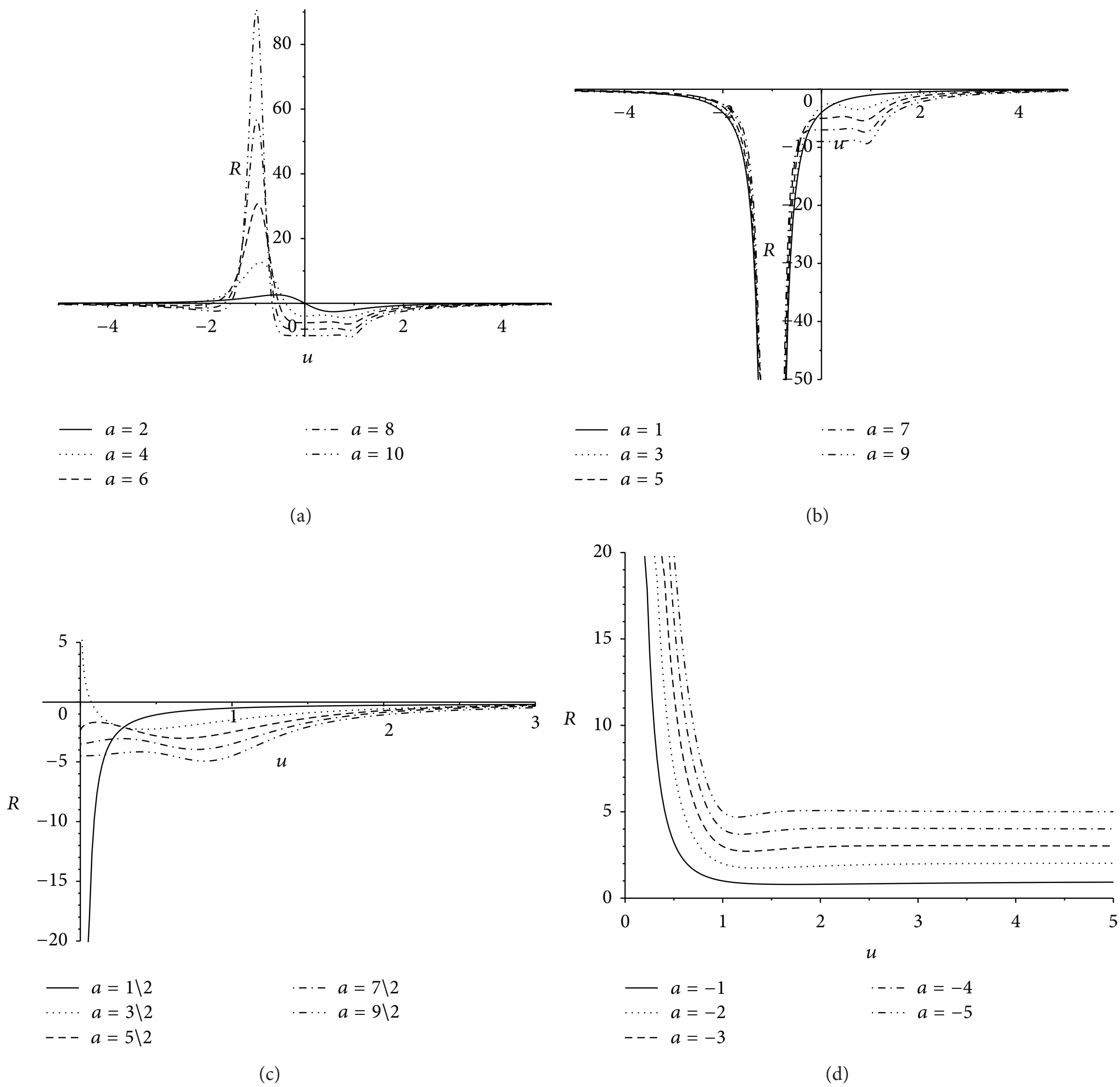


FIGURE 1: The three qualitatively different types of radial functions. The horizontal axis corresponds to the energy, while the vertical represents the radius of the circumference. We observe that only the curves corresponding to negative values of the exponent α are positive definite in the full physical domain, that is, positive values of energy and volume.

Thus, fixing the volume of the system, we find the subset of fundamental functions for which the PDE represented by the circumference (30) is well defined. The three qualitatively different types of behaviour are depicted in Figure 1. It is easy to observe that for $\alpha > 0$, even in the case when $\alpha < 1$ (c.f. Figure 2), the PDE is ill defined; thus, only fundamental relations for which $\alpha < 0$ are meaningful. This corresponds to a positive thermodynamic curvature as can be seen from (14). Furthermore, noticing the symmetry of u and v in the expression for the circumference radius (34), it is easy to observe that the PDE also restrict the domain of the thermodynamic variables to $u, v > 0$. The fundamental relation with $\alpha < 0$ describes a polytropic fluid with equation of state given by

$$P = \rho^{1-\alpha}, \tag{35}$$

where $\rho = u/v$ is the energy density of the fluid. It is a simple task to obtain the heat capacity at constant volume for these systems

$$c_v = \frac{a u^{\alpha}}{u^{\alpha} + (1-\alpha) v^{\alpha}}. \tag{36}$$

From this expression we observe that the heat capacity remains finite for any value of the thermodynamic variables and is always negative whenever $\alpha < 0$.

4. Closing Remarks

In this paper we studied two-dimensional Einstein manifolds for the Geometrothermodynamics programme. We found the differential equation that must be satisfied by the

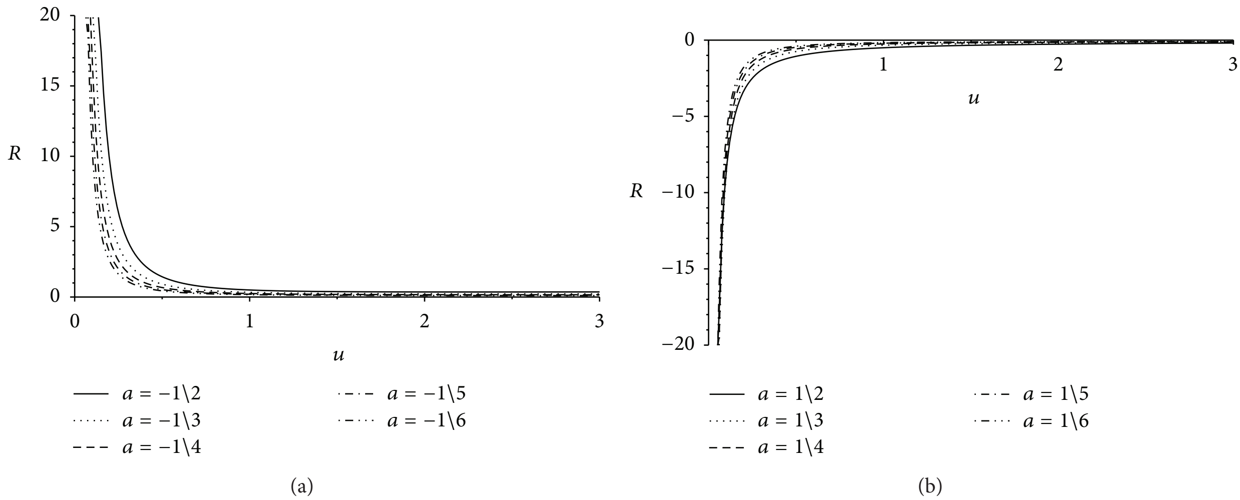


Figure 2: These plots correspond to the radial function for $-1 \leq \alpha \leq 1$. We observe that only those values of $\alpha < 0$ can be associated with physical systems.

fundamental relation in order to describe a system with constant thermodynamic interaction, that is, a fundamental relation producing a representation invariant metric whose associated curvature is constant. In particular, building on previous work (c.f. [5]) we analysed the one-parameter family of fundamental relations given by (16).

Noting the conformal structure of $g^{\natural}$ (see (4)), we centre our study in the class of functions whose associated Hessian metric has vanishing curvature. With this assumption, we set up the system of differential equations defining the set of isothermal coordinates. As expected, we found an analytic expression for the change of coordinates for the case in which the fundamental function is separable. An interesting exercise allowed us to explore some properties of the space of solutions of (19), that is, the set of isothermal coordinates with their corresponding fundamental relation of vanishing Hessian curvature.

Observing the algebraic structure of the system (20), we note that there will be a class of fundamental relations satisfying the Hessian curvature constraint for which we can build a characteristic circumference on which the solutions of the PDE system lie. This can be done whenever the derivatives of the fundamental relation define a positive squared radial function. Moreover, we conjecture that only the set of fundamental relations for which such a construction is possible can describe physical systems of constant thermodynamic interaction. In particular, we work out the example given by (13). Here, we work in specific thermodynamic variables in the entropy representation of a system characterised by the exponent α. Indeed, only those systems for which $\alpha < 0$ correspond to a polytropic fundamental relation.

In sum, we have analysed a particular class of fundamental relations of constant thermodynamic curvature. It remains to explore the larger class of functions within the set of solutions of (19) and study their thermodynamic implications. This will be done in a forthcoming article.

Acknowledgments

A. C. Gutiérrez-Pińeres is funded by a TWAS-CONACYT postdoctoral grant. C. S. López-Monsalvo is thankful to CONACYT, postdoctoral Grant no. 290679_UNAM. F. Nettel acknowledges support from DGAPA-UNAM (postdoctoral fellowship).

References

[1] H. Quevedo, "Geometrothermodynamics," *Journal of Mathematical Physics*, vol. 48, no. 1, Article ID 13506, 2007.

[2] G. Ruppeiner, "Thermodynamics: a Riemannian geometric model," *Physical Review A*, vol. 20, no. 4, pp. 1608–1613, 1979.

[3] F. Weinhold, "Metric geometry of equilibrium thermodynamics," *The Journal of Chemical Physics*, vol. 63, no. 6, pp. 2479–2483, 1975.

[4] A. Bravetti, C. S. Lopez-Monsalvo, F. Nettel, and H. Quevedo, "The conformal metric structure of geometrothermodynamics," *Journal of Mathematical Physics*, vol. 54, Article ID 033513, 11 pages, 2013.

[5] H. Quevedo, F. Nettel, C. S. Lopez-Monsalvo, and A. Bravetti, "Representation invariant Geometrothermodynamics: applications to ordinary thermodynamic systems," http://arxiv.org/abs/1303.1428.

[6] A. Bravetti, D. Momeni, R. Myrzakulov, and H. Quevedo, "Geometrothermodynamics of higher dimensional black holes," *General Relativity and Gravitation*, 2013.

[7] A. Aviles, A. Basterrechea-Almodovar, L. Campuzano, and H. Quevedo, "Extending the generalized Chaplygin gas model by using geometrothermodynamics," *Physical Review D*, vol. 86, Article ID 063508, 2012.

Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2013, Article ID 865354, 5 pages
http://dx.doi.org/10.1155/2013/865354

Research Article

Legendre Invariance and Geometrothermodynamics Description of the 3D Charged-Dilaton Black Hole

Yiwen Han[1] and XiaoXiong Zeng[2]

[1] *College of Computer Science, Chongqing Technology and Business University, Chongqing 400067, China*
[2] *School of Science, Chongqing Jiaotong University, Chongqing 400074, China*

Correspondence should be addressed to XiaoXiong Zeng; xxzengphysics@163.com

Received 13 February 2013; Revised 9 June 2013; Accepted 15 June 2013

Academic Editor: Hernando Quevedo

We first review Weinhold information geometry and Ruppeiner information geometry of 3D charged-dilaton black hole. Then, we use the Legendre invariant to introduce a 2-dimensional thermodynamic metric in the space of equilibrium states, which becomes singular at those points. According to the analysis of the heat capacities, these points are the places where phase transitions occur. This result is valid for the black hole, therefore, provides a geometrothermodynamics description of black hole phase transitions in terms of curvature singularities.

1. Introduction

Since Ferrara et al. [1] investigated the critical points of moduli space by using Weinhold metric and Ruppeiner metric; the black hole thermodynamic in geometry framework becomes a hot spot of theoretical physics.

It is well known that an equilibrium thermodynamic system poses interesting geometric features. An interesting inner product on the equilibrium thermodynamic state space in the energy representation was provided by Weinhold as the Hessian matrix of the internal energy U with respect to the extensive thermodynamic variables N^a, namely, [2] $g_{ij}^W = \partial_i\partial_j M(U, N^a)$. However, there was no physical interpretation associated with this metric structure. As a modification, Ruppeiner introduced Riemannian metric into thermodynamic system once more and defended it as the second derivative of entropy S (here, entropy is a function of internal energy U and its extensive variables N^a) [3] $g_{ij}^R = -\partial_i\partial_j S(U, N^a)$. An interesting phenomenon is that these two metrics are conformally related, that is, $g_{ij}^W = Tg_{ij}^R$, and the conformal factor is the temperature, $T = \partial M/\partial S$. It has been applied to all kinds of thermodynamic models, for example, the ideal gas, the van der Waals gas, and the two-dimensional Fermi gas et al. Studies showed that Ruppeiner geometry can overcome the covariant and self-consistent problem of general thermodynamics. Based on the Ruppeiner and Weinhold metrics, consideration of different black hole families under various assumptions has led to numerous puzzling results for both metrics [4–13]. So it is then natural to try to describe the phase transitions of black holes in terms of curvature singularities in the space of equilibrium states. Unfortunately, the obtained results, at least some, are contradictory. For instance, for Reissner-Nordström black hole, the Ruppeiner metric is flat [14], whereas the Weinhold metric presents a curvature singularity. Similarly, the 3D charged-dilaton black hole also showed similar result [15]. Nevertheless, a simple change of the thermodynamic potential [16] affects Ruppeiner's geometry in such a way that the resulting curvature singularity now corresponds to a phase transition. A dimensional reduction of Ruppeiner's curvature seems to affect its properties too [17]. However, it is well known that ordinary thermodynamics does not depend on the thermodynamic potential.

Recently, Quevedo [18] proposed a formalism of geometrothermodynamics (GTDs) as a geometric approach that incorporates Legendre invariance in a natural way and allows us to derive Legendre invariant metrics in the space of equilibrium states. Since Weinhold and Ruppeiner metrics are not Legendre invariant, one of the first results in the context of GTD was the derivation of simple Legendre invariant generalizations of these metrics and their application to black hole thermodynamics [19–26]. These results end the controversy regarding the application of geometric structures in black hole thermodynamics. The phase transition structure contained in the heat capacity of black holes becomes completely integrated in the scalar curvature of the Legendre invariant metric so that a curvature singularity corresponds to a phase transition.

In this paper, we first review Weinhold information geometry and Ruppeiner information geometry of 3D charged-dilaton black hole. Then, based on our previous work, the thermodynamics scalar curvature of the black hole is described. We explored the geometrothermodynamics of 3D charged-dilaton black hole.

The organization of this paper is outlined as follows. In Section 2, we review Weinhold information geometry and Ruppeiner information geometry of the 3D charged-dilaton black hole. Then, in Section 3, geometrothermodynamics of the charged black hole ise described. Finally, some discussions and conclusions are given in Section 4. Throughout the paper, the units ($G = c = \hbar = 1$) are used.

2. Information Geometry Description of 3D Charged-Dilaton Black Hole

Let us first review Weinhold information geometry and Ruppeiner information geometry of the 3D charged-dilaton black hole.

The starting action with the dilaton field ϕ is given by [27]

$$S = \int d^3x\sqrt{-g}\left[R + 2e^{4\phi}\Lambda + 4(\nabla\phi)^2 - e^{-4\phi}F_{\mu\nu}F^{\mu\nu}\right]. \quad (1)$$

The cosmology constant $\Lambda > 0$ for anti-de Sitter space-time. This action is conformably related to the low-energy string action black hole that is given by

$$\begin{gathered} ds^2 = -f(r)\,dt^2 + \frac{4r^2}{\gamma^4 f(r)}dr^2 + r^2 d^2\varphi, \\ \phi = \frac{1}{4}\ln\left[\frac{r}{\gamma^2}\right]; \qquad F_{rt} = \frac{Q}{r^2}, \end{gathered} \quad (2)$$

where the metric function $f(r) = -2Mr + 8\Lambda r^2 + 8Q^2$ and an integration constant γ with dimension $[L]^{1/2}$ is necessary to have correct dimensions.

In our previous work, the mass and electric charge of the 3D charged-dilaton black hole have been expressed in terms of the inner and outer horizons as [15]

$$M = 4\Lambda(r_+ + r_-), \qquad Q^2 = \Lambda r_+ r_-, \quad (3)$$

and, the electric potential is given by

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S = \frac{8Q}{r_+}. \quad (4)$$

According to the energy conservation law,

$$dM = TdS + \Phi dQ. \quad (5)$$

The temperature is

$$T = \left(\frac{\partial M}{\partial S}\right)_Q = \frac{2\Lambda}{r_+^2}(r_+ - r_-). \quad (6)$$

By using the area law, the entropy of the black hole is given by

$$S = \frac{k_B}{4}A = k_B\pi r_+^2 = r_+^2, \quad (7)$$

with k_B Boltzmann's constant, and $k_B = 1/\pi$. In its natural coordinates, the Weinhold metric can be obtained as follows:

$$ds_W^2 = \frac{1}{r_+^3}\left(\frac{3Q^2}{r_+^2} - \Lambda\right)dS^2 - 8QdS\,dQ + r_+^2 dQ^2, \quad (8)$$

where the index W denotes the Weinhold information geometry. Here, we have made the choice that the mass M corresponds to the thermodynamic potential; entropy S and charge Q correspond to the extensive variables, from which it can be shown that the Weinhold scalar curvature in the entropy representation becomes

$$\mathfrak{R}_W = -\frac{r_+}{4\Lambda(r_+ - r_-)^2}. \quad (9)$$

We see that the curvature $\mathfrak{R}_W$ naively diverges at the extreme limit of the black hole, where $r_+ = r_-$, which is of less interest physically since at the extreme limit, the Hawking temperature vanishes, and the thermodynamics description breaks down as mentioned above. We interpret this result as an indication of the limit of applicability of geometric thermodynamics as a geometric model for equilibrium thermodynamics.

By using the coordinate transformation $u = Q/r_+$ [14, 15], we obtain the diagonalized Ruppeiner metric for the 3D charged-dilaton black hole as follows:

$$ds_R^2 = \frac{1}{T}ds_W^2 = -\frac{1}{2S}dS^2 + \frac{4S}{\Lambda - u}du^2. \quad (10)$$

Let us do a new transformation as follows:

$$\tau = \sqrt{2}S, \qquad \sqrt{\Lambda}\sin\left(\frac{\sigma}{\sqrt{2}}\right) = u. \quad (11)$$

Then, the Ruppeiner metric can be written in the above Rindler coordinates as

$$ds_R^2 = -d\tau + \tau^2 d\sigma^2. \quad (12)$$

Obviously, this is a flat metric; its curvature is zero. The vanished thermodynamic curvature implies that no phase transition points exist and no thermodynamic interactions appear. This result implies that the Ruppeiner curvature cannot describe the phase transitions of the black hole either.

3. Geometrothermodynamics Description of 3D Charged-Dilaton Black Hole

In this section, we turn to use the recent geometric formulation of extended thermodynamic behavior of the 3D charged-dilaton black hole.

The formulation of GTD is based on the use of contact geometry as a framework for thermodynamics [18]. Consider the $(2n+1)$-dimensional thermodynamic phase space $\mathfrak{T}$ coordinated by the thermodynamic potential Φ, extensive variables E^a, and intensive variables I^a $(a = 1, \ldots, n)$. Consider on $\mathfrak{T}$ a nondegenerate metric $G = G(Z^A)$, with $Z^A = \{\Phi, E^a, I^a\}$, and the Gibbs1-form $\Theta = d\Phi - \delta_{ab} I^a dE^b$, with $\delta_{ab} = \operatorname{diag}(1, 1, \ldots, 1)$. The set $(\mathfrak{T}, \Theta, G)$ defines a contact Riemannian manifold if the condition $\Theta \wedge (d\Theta)^n \neq 0$ is satisfied. Moreover, the metric G is Legendre invariant if its functional dependence on Z^A does not change under a Legendre transformation. The Gibbs 1g-form Θ is also invariant with respect to Legendre transformations. Legendre invariance guarantees that the geometric properties of G do not depend on the thermodynamic potential used in its construction.

The thermodynamic phase space $\mathfrak{T}$ which in the case of the 3D charged-dilaton black hole can be defined as a 4-dimensional space with coordinates $Z^A = \{M, S, T, Q\}$, $A = 0, \ldots, 4$. Equation (3) represents the fundamental relationship $M = (S, Q)$ from which all the thermodynamic information can be obtained; therefore, we would like to consider a 5-dimensional phase space $\mathfrak{T}$ with coordinates (M, S, T, Q, Φ), a contact one-form

$$\Theta = dM - TdS - \Phi dQ, \tag{13}$$

and an invariant metric

$$G = (dM - TdS - \Phi dQ)^2 + (TS + \Phi Q)(-dT\, dS + d\Phi\, dQ). \tag{14}$$

The triplet $(\mathfrak{T}, \Theta, G)$ defines a contact Riemannian manifold that plays an auxiliary role in GTD. It is used to properly handle the invariance with respect to Legendre transformations. In fact, for the charged black hole, a Legendre transformation involves in general all the thermodynamic variables M, S, Q, T, and Φ so that they must be independent from each other as they are in the phase space. We introduce also the geometric structure of the space of equilibrium states ε in the following manner: ε is a 2-dimensional submanifold of $\mathfrak{T}$ that is defined by the smooth embedding map $\varphi : \varepsilon \mapsto \mathfrak{T}$, satisfying the condition that the "projection" of the contact form Θ on ε vanishes, namely, $\varphi^*(\Theta) = 0$, where φ^* is the pullback of φ, and that G induces a Legendre invariant metric g on ε by means of ε. In principle, any 2-dimensional subset of the set of coordinates of $\mathfrak{T}$ can be used to label ε. For the sake of simplicity, we will use the set of extensive variables S and Q which in ordinary thermodynamics corresponds to the energy representation. Then, the embedding map for this specific choice is

$$\varphi : \{S, Q\} \longmapsto \left\{M(S,Q), S, Q, \frac{\partial M}{\partial S}, \frac{\partial M}{\partial Q}\right\}. \tag{15}$$

The condition $\varphi^*(\Theta) = 0$ is equivalent to (5) (the first law of thermodynamics) and (4), (6) (the conditions of thermodynamic equilibrium); the induced metric is obtained as follows:

$$g = \left(S\frac{\partial M}{\partial S} + Q\frac{\partial M}{\partial Q}\right)\left(-\frac{\partial^2 M}{\partial S^2} dS^2 + \frac{\partial^2 M}{\partial Q^2} dQ^2\right). \tag{16}$$

This metric determines all the geometric properties of the equilibrium space ε. We see that in order to obtain the explicit form of the metric, it is only necessary to specify the thermodynamic potential M as a function of S and Q. In ordinary thermodynamics, this function is usually referred to as the fundamental equation from which all the equations of state can be derived. From (3), the fundamental equation $M = M(S, Q)$ is given by

$$M(S,Q) = 4\Lambda\sqrt{S}\left(1 + \frac{Q^2}{\Lambda S}\right). \tag{17}$$

The first-order and the second-order partial differentials can be expressed, respectively, as

$$\begin{aligned} \frac{\partial M}{\partial S} &= \frac{2}{r_+}\left(\Lambda - \frac{Q^2}{r_+^2}\right), \qquad \frac{\partial M}{\partial Q} = \frac{8Q}{r_+}, \\ \frac{\partial^2 M}{\partial S^2} &= -\frac{1}{r_+^3}\left(\Lambda - \frac{3Q^2}{r_+^2}\right), \qquad \frac{\partial^2 M}{\partial Q^2} = \frac{8}{r_+}. \end{aligned} \tag{18}$$

Substituting (18) into (16), the lines of GTD for the 3D charged-dilaton black hole are written as

$$dS_G^2 = \frac{2}{r_+^6}\left(\Lambda r_+^4 - 9Q^4\right) dS^2 + \frac{16}{r_+^2}\left(\Lambda r_+^2 + 3Q^2\right) dQ^2, \tag{19}$$

where the index G denotes the geometrothermodynamics. Thus, the curvature scalar can be obtained by

$$\mathfrak{R}_G = \frac{9r_+ r_- (r_+ - r_-)}{2\Lambda^4 (3r_- - r_+)^2 (r_+ + 3r_-)^3}. \tag{20}$$

We see in our setup that the scalar curvature $\mathfrak{R}_G$ vanishes only at the extremal limit to where $r_+ = r_-$. In a general case, the scalar curvature $\mathfrak{R}_G$ does not vanish and it goes positive infinity when $r_+ = 3r_-$, which stands for a kind of phase transition or long rang correlation of the system according to the Ruppeiner theory [28]. It is interesting to note that the divergence point of the scalar curvature is just the transition point of Davies [29]. In the fact, it is easy to check this by calculating the heat capacity with a fixed charge as follows:

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = \frac{2r_+^2 (r_+ - r_-)}{3r_- - r_+}, \tag{21}$$

which is singular at $r_+ = 3r_-$ corresponding to $M^2 = 266Q^2/3$ and indicates that the black hole has a second-order phase transition. Moreover, we see that all thermodynamic variables are well behaved, except perhaps in the extremal limit $r_+ = r_-$, at this point, changes sign and the scalar curvature diverge. Therefore, there will be a phase transition $\mathfrak{R}_G$.

4. Discussion and Conclusions

In this work, we investigated the Weinhold metric and the Ruppeiner metric as well as the geometrothermodynamics of a 3D Charged-Dilaton Black Hole. In all these cases, our results showed that the thermodynamic curvature is in general different, indicating the presence of thermodynamic interaction. For instance, the scalar curvature $\mathfrak{R}_W$ indicates the presence of second-order transition points in $r_+ = r_-$. Nevertheless, the scalar curvature $\mathfrak{R}_R$ is zero, indicating that no phase transitions can occur, and the scalar curvature $\mathfrak{R}_R$ lost the information about charge Q. Moreover, the scalar curvature $\mathfrak{R}_G$ indicates more physical interest since the first-order phase transition point and the second-order transition both occur in the extremal limit at $r_+ = r_-$ and $r_+ = 3r_-$.

In addition, the thermodynamic metric proposed in this work has been applied to the case of black hole configurations in four dimensions with and without the cosmological constant. It has been shown that this thermodynamic metric correctly describes the thermodynamic behavior of the corresponding black hole configurations. One additional advantage of this thermodynamic metric is its invariance with respect to total Legendre transformations. This means that the results are independent of the thermodynamic potential used to generate the thermodynamic metric. A very interesting result is that it can recreate the lost information in Ruppeiner metric by using Legendre transformation. In summary, all of the above thermodynamic geometries leading to different results indicate that it is still unresolved to introduce geometrical concepts into all kinds of black holes; we also expect that this unified geometry description may give more information about a thermodynamic system.

Acknowledgment

This work is supported by the Scientific and Technological Foundation of Chongqing Municipal Education Commission (Grant no. KJ100706).

References

[1] S. Ferrara, G. W. Gibbons, and R. Kallosh, "Black holes and critical points in moduli space," *Nuclear Physics B*, vol. 500, no. 1–3, pp. 75–93, 1997.

[2] F. Weinhold, "Metric geometry of equilibrium thermodynamics," *The Journal of Chemical Physics*, vol. 63, no. 6, pp. 2479–2483, 1975.

[3] G. Ruppeiner, "Thermodynamics: a Riemannian geometric model," *Physical Review A*, vol. 20, no. 4, pp. 1608–1613, 1979.

[4] G. Ruppeiner, "Stability and fluctuations in black hole thermodynamics," *Physical Review D*, vol. 75, no. 2, Article ID 024037, 11 pages, 2007.

[5] G. Ruppeiner, "Thermodynamic curvature and phase transitions in Kerr-Newman black holes," *Physical Review D*, vol. 78, no. 2, Article ID 024016, 13 pages, 2008.

[6] J. E. Åman and N. Pidokrajt, "Geometry of higher-dimensional black hole thermodynamics," *Physical Review D*, vol. 73, no. 2, Article ID 024017, 7 pages, 2006.

[7] J. E. Åman, I. Bengtsson, and N. Pidokrajt, "Flat information geometries in black hole thermodynamics," *General Relativity and Gravitation*, vol. 38, no. 8, pp. 1305–1315, 2006.

[8] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, "New attractor mechanism for spherically symmetric extremal black holes," *Physical Review D*, vol. 76, no. 10, Article ID 104045, 9 pages, 2007.

[9] A. J. M. Medved, "A commentary on Ruppeiner metrics for black holes," *Modern Physics Letters A*, vol. 23, no. 26, pp. 2149–2161, 2008.

[10] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, "Thermodynamics and phase transitions in the Born-Infeld-anti-de Sitter black holes," *Physical Review D*, vol. 78, no. 8, Article ID 084002, 9 pages, 2008.

[11] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, "Ruppeiner geometry and 2D dilaton gravity in the thermodynamics of black holes," *Physics Letters B*, vol. 663, no. 4, pp. 342–350, 2008.

[12] R.-G. Cai, L.-M. Cao, and N. Ohta, "Thermodynamics of Hořava-Lifshitz black holes," *Physics Letters B*, vol. 679, no. 5, pp. 504–509, 2009.

[13] W. Janke, D. A. Johnston, and R. Kenna, "Geometrothermodynamics of the Kehagias-Sfetsos black hole," *Journal of Physics A*, vol. 43, no. 42, Article ID 425206, 2010.

[14] J. E. Åman, I. Bengtsson, and N. Pidokrajt, "Geometry of black hole thermodynamics," *General Relativity and Gravitation*, vol. 35, no. 10, pp. 1733–1743, 2003.

[15] Y. Han and J. Zhang, "Hawking temperature and thermodynamics geometry of the 3D charged-dilaton black holes," *Physics Letters B*, vol. 692, no. 2, pp. 74–77, 2010.

[16] J. Y. Shen, R. G. Cai, B. Wang, and R. K. Su, "Thermodynamic geometry and critical behavior of black holes," *International Journal of Modern Physics A*, vol. 22, no. 1, p. 11, 2007.

[17] B. Mirza and M. Zamaninasab, "Ruppeiner geometry of RN black holes: flat or curved?" *Journal of High Energy Physics*, vol. 2007, no. 6, article 059, 2007.

[18] H. Quevedo, "Geometrothermodynamics," *Journal of Mathematical Physics*, vol. 48, no. 1, Article ID 013506, 14 pages, 2007.

[19] H. Quevedo, "Geometrothermodynamics of black holes," *General Relativity and Gravitation*, vol. 40, no. 5, pp. 971–984, 2008.

[20] H. Quevedo and A. Vazquez, "The geometry of thermodynamics," *AIP Conference Proceedings*, vol. 977, pp. 165–172, 2008.

[21] J. L. Álvarez, H. Quevedo, and A. Sánchez, "Unified geometric description of black hole thermodynamics," *Physical Review D*, vol. 77, no. 8, Article ID 084004, 6 pages, 2008.

[22] H. Quevedo and A. Sánchez, "Geometrothermodynamics of asymptotically anti-de Sitter black holes," *Journal of High Energy Physics*, vol. 2008, no. 9, article 034, 2008.

[23] H. Quevedo and A. Sánchez, "Geometric description of BTZ black hole thermodynamics," *Physical Review D*, vol. 79, no. 2, Article ID 024012, 9 pages, 2009.

[24] H. Quevedo and A. Sánchez, "Geometrothermodynamics of black holes in two dimensions," *Physical Review D*, vol. 79, no. 8, Article ID 087504, 4 pages, 2009.

[25] Y. W. Han and G. Chen, "Thermodynamic, geometric representation and the critical behaviors of the $(2+1)$-dimensional black holes in the Legendre transformations," *Physics Letters B*, vol. 714, no. 6, pp. 127–130, 2012.

[26] Y.-W. Han, Z.-Q. Bao, and Y. Hong, "Thermodynamic curvature and phase transitions from black hole with a Coulomb-like field," *Communications in Theoretical Physics*, vol. 55, no. 4, pp. 599–601, 2011 (Chinese).

[27] K. C. K. Chan and R. B. Mann, "Static charged black holes in $(2 + 1)$-dimensional dilaton gravity," *Physical Review D*, vol. 50, no. 10, pp. 6385–6393, 1994.

[28] G. Ruppeiner, "Riemannian geometry in thermodynamic fluctuation theory," *Reviews of Modern Physics*, vol. 67, no. 3, pp. 605–659, 1995.

[29] P. C. W. Davies, "Thermodynamics of black holes," *Reports on Progress in Physics*, vol. 41, no. 8, p. 1313, 1978.

Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2013, Article ID 549808, 11 pages
http://dx.doi.org/10.1155/2013/549808

Research Article

Geometrothermodynamics of Myers-Perry Black Holes

Alessandro Bravetti,[1,2] Davood Momeni,[3] Ratbay Myrzakulov,[3] and Aziza Altaibayeva[3]

[1] *Dipartimento di Fisica and ICRA, "Sapienza" Università di Roma, Piazzale Aldo Moro 5, 00185 Rome, Italy*
[2] *Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, AP 70543, 04510 México, DF, Mexico*
[3] *Eurasian International Center for Theoretical Physics, L.N. Gumilyov Eurasian National University, Astana 010008, Kazakhstan*

Correspondence should be addressed to Alessandro Bravetti; bravetti@icranet.org

Received 13 March 2013; Revised 4 June 2013; Accepted 5 June 2013

Academic Editor: Rong-Gen Cai

We consider the thermodynamics and geometrothermodynamics of the Myers-Perry black holes in five dimensions for three different cases, depending on the values of the angular momenta. We follow Davies approach to study the thermodynamics of black holes and find a nontrivial thermodynamic structure in all cases, which is fully reproduced by the analysis performed with the techniques of Geometrothermodynamics. Moreover, we observe that in the cases when only one angular momentum is present or the two angular momenta are fixed to be equal, that is, when the thermodynamic system is two dimensional, there is a complete agreement between the divergences of the generalized susceptibilities and the singularities of the equilibrium manifold, whereas when the two angular momenta are fully independent, that is, when the thermodynamic system is three dimensional, additional singularities in the curvature appear. However, we prove that such singularities are due to the changing from a stable phase to an unstable one.

1. Introduction

Black holes are very special thermodynamic systems. They are thermodynamic system since they have a temperature, the celebrated Hawking temperature [1], and a definition of entropy via the Bekenstein area law [2], from which one can prove that the laws of thermodynamics apply to black holes [3]. On the other side, they are very special thermodynamic systems, and since, for instance, the entropy is not extensive, they cannot be separated into small subsystems, and perhaps the worst fact, their thermodynamics does not possess a microscopic description yet (see, e.g., [4] for a clear description of these problems).

In this puzzling situation, one of the most successful and at the same time discussed approach to the study of black holes phase transitions is the work of Davies [5]. According to Davies, black holes can be regarded as ordinary systems, showing phase transitions right at those points where the generalized susceptibilities, that is, second-order derivatives of the potential, change sign most notably through an infinite discontinuity. Since there is no statistical mechanical description of black holes as thermodynamic systems, it is hard to verify Davies approach with the usual technique of calculating the corresponding critical exponents (although very interesting works on this subject exist, see, e.g., [4, 6–18]). In fact, the main drawback of this approach is that one has to choose arbitrarily the order parameter for black holes.

A possible resolution to this situation can then come from the use of thermodynamic geometry. Since the pioneering works of Gibbs [19] and Carathéodory [20], techniques from geometry have been introduced into the analysis of thermodynamics. In particular, Fisher [21] and Radhakrishna Rao [22] proposed a metric structure in the space of probability distributions which has been extensively used both in statistical physics and in economics (for a recent review see [23]). Later, Weinhold [24] introduced an inner product in the equilibrium space of thermodynamics based on the stability conditions of the internal energy, taken as the thermodynamic potential. The work of Weinhold was then developed by Ruppeiner [25] from a different perspective. Ruppeiner moved from the analysis of fluctuations around equilibrium and from the gaussian approximation of the probability of fluctuations and found a thermodynamic metric which is defined as (minus) the hessian of the entropy of the system. Remarkably, Ruppeiner geometry was found to be conformally related to the one proposed by Weinhold. Moreover, Ruppeiner metric is intrinsically related to the underlying statistical model, and in fact the scalar curvature of the Riemannian manifold representing the system using Ruppeiner metric happens to have exactly the same exponent as the correlation volume for ordinary systems (see, e.g., [26] for a review).

All these approaches have been widely used to study ordinary systems, and in particular Ruppeiner metric has been also used to study many black holes configurations (see [27] and refernces therein). This is because one can argue that being Ruppeiner metric defined only from thermodynamic quantities and on the other side giving information about the statistical model, then it can provide some hints towards the resolution of the long standing problem of understanding the microscopic properties of black holes (see, e.g., [27]).

On the other side, the problem with the use of thermodynamic geometries to study black holes thermodynamics is that black holes are not ordinary systems, as we argued previous. For instance, Ruppeiner metric in many cases gives exactly the same results of Davies approach (which is based upon ordinary thermodynamics), while in some other important cases it does not converge to the same results, as it happens for example, in the Reissner-Nordström and Kerr cases (see, e.g., [27–29]). One can argue either that Davies approach is inaccurate or that the application of Ruppeiner metric to black holes may be imperfect, due to the strange nature of black holes as thermodynamic systems. In fact, there is still an open debate on this topic (see, e.g., the discussion in [27, 30]).

Furthermore, in the recent years, a new approach in the context of thermodynamic geometry has been proposed by Quevedo [31], known as geometrothermodynamics (GTD). According to this approach, the Riemannian structure to be introduced in the equilibrium space should be consistent with the property of Legendre invariance, a property which is at the core of ordinary thermodynamics. In GTD some families of metrics have been found that share the Legendre invariance property, and they have also been proven to interpret in a consistent manner the thermodynamic properties of ordinary systems, chemical interactions, black holes configurations, and cosmological solutions (see [31–39]). In particular, the correspondence between the divergences of the scalar curvature of the equilibrium manifold of GTD and the phase transition points signaled by the divergences of the heat capacity (i.e., phase transitions *à la Davies*) seems to be a general fact, according to the variety of systems analyzed so far and to the general expressions given in [40]. In addition, a recent study [41] of the thermodynamics of the Reissner-Nordström and Kerr black holes in any dimensions suggested that the GTD approach can detect not only the points of phase transitions due to singularities of the heat capacities but also divergences of the full spectrum of the generalized susceptibilities.

On the other side, the thermodynamic properties of the Myers-Perry black holes in five dimensions have been extensively studied in the literature from completely different points of view (see, e.g., [4, 29, 42–46]). In this work, we give special emphasis on the relation between divergences of the generalized susceptibilities and curvature singularities of the metric from GTD. For example, we do not consider here possible phase transitions related to change in the topology of the event horizon, an intriguing question which was addressed, for example, in [4]. We find out that the GTD thermodynamic geometry is always curved for the considered cases, showing the presence of thermodynamic interaction and that its singularities always correspond to divergences of the susceptibilities or to points where there is a change from a stable to an unstable phase. This will allow us to infer new results on the physical meaning of the equilibrium manifold of GTD.

The work is organized as follows. In Section 2, we present the basic aspects of GTD and introduce all the mathematical concepts that are needed. In Section 3, we perform the parallel between the thermodynamic quantities and the Geometrothermodynamic description of the five-dimensional Myers-Perry black holes for three different cases, depending on the values of the angular momenta. Finally, in Section 4, we comment on the results and discuss possible developments.

2. Basics of Geometrothermodynamics

Geometrothermodynamics (GTD) is a geometric theory of thermodynamics recently proposed by Quevedo [31]. It makes use of contact geometry in order to describe the phase space $\mathcal{T}$ of thermodynamic systems and express the first law of thermodynamics in a coordinate-free fashion. Furthermore, GTD adds a Riemannian structure G to the phase space and requests G to be invariant under Legendre transformations, in order to give it the same properties which one expects for ordinary thermodynamics. Moreover, GTD introduces the manifold of the equilibrium space $\mathcal{E}$ as the maximum integral submanifold of the contact structure of $\mathcal{T}$, characterized by the validity of the first law of thermodynamics [31]. At the same time, GTD prescribes also to pull back the Riemannian structure G to the equilibrium space. This results in a naturally induced Riemannian structure g in $\mathcal{E}$, which is supposed to be the geometric counterpart of the thermodynamic system. Such a description has been proposed in order to give thermodynamic geometry a new symmetry which was not present in previous approaches, that is, the Legendre invariance.

Let us see now the mathematical definitions of the GTD objects that we shall use in this work. If we are given a system with n thermodynamic degrees of freedom, we introduce first a $(2n+1)$-dimensional space $\mathcal{T}$ with coordinates $Z^A = \{\Phi, E^a, I^a\}$, with $A = 0, \ldots, 2n$ and $a = 1, \ldots, n$, which is known as *the thermodynamic phase space* [31]. We make use of the phase space $\mathcal{T}$ in order to correctly handle both the Legendre transformations and the first law of thermodynamics. In fact, in classical thermodynamics, we can change the thermodynamic potential using *a Legendre transformation*, which is defined in $\mathcal{T}$ as the change of coordinates given by [47]:

$$\left\{\Phi, E^a, I^a\right\} \longrightarrow \left\{\widetilde{\Phi}, \widetilde{E}^a, \widetilde{I}^a\right\},$$

$$\Phi = \widetilde{\Phi} - \delta_{kl}\widetilde{E}^k\widetilde{I}^l, \qquad E^i = -\widetilde{I}^i, \qquad E^j = \widetilde{E}^j,$$
$$I^i = \widetilde{E}^i, \qquad I^j = \widetilde{I}^j, \tag{1}$$

where $i \cup j$ can be any disjoint decomposition of the set of indices $\{1,\dots,n\}$ and $k,l = 1,\dots,i$. We remark that Legendre transformations are change of coordinates in $\mathcal{T}$ and that they are not defined in the equilibrium space. Moreover, the phase space $\mathcal{T}$ is equipped with a canonical contact structure called *the Gibbs* 1*-form* defined as

$$\Theta = d\Phi - \delta_{ab}I^b dE^a = d\Phi - I_a dE^a, \tag{2}$$

which extremely resembles the first law of thermodynamics and hence will be the starting point to define the equilibrium space.

Furthermore, *the equilibrium space* $\mathcal{E}$ is the n-dimensional submanifold of $\mathcal{T}$ defined by the embedding $\varphi : \mathcal{E} \to \mathcal{T}$ under the condition

$$\varphi^*(\Theta) = 0, \quad \text{that is, } d\Phi = I_a dE^a, \quad I_a = \frac{\partial\Phi}{\partial E^a}, \tag{3}$$

where φ^* is the pullback of φ. It follows immediately from (2) that (3) represents both the first law and the equilibrium conditions for the thermodynamic system under analysis, so that $\mathcal{E}$ results to be (by definition) the submanifold of points where the first law and the equilibrium conditions hold, that is, the geometric counterpart of the thermodynamic system. It also follows that the coordinates $\{Z^A\}$ of $\mathcal{T}$ assume a physical meaning in $\mathcal{E}$. In fact, the set $\{E^a\}$, with $a = 1,\dots,n$, can be identified with the extensive thermodynamic variables, while $\Phi = \Phi(E^a)$ with the fundamental equation for the thermodynamic potential, and finally the coordinates $\{I^a\} = \{I^a(E^a)\} \equiv \{\partial_{E^a}\Phi\}$, $a = 1,\dots,n$ represent the intensive quantities corresponding to the extensive set $\{E^a\}$ (see, e.g., [48] for these definitions).

Now, let us add the Riemannian structure. Since we want the Riemannian structure to share the same properties of the first law and since the first law is invariant under Legendre transformations, we introduce in the phase space $\mathcal{T}$ a metric G which is invariant under Legendre transformations. In GTD, there are several families of metrics which have this property (for a recent work on this topic see [49]). Among them, one has been proven particularly successful to describe systems with second-order phase transitions, as black holes are supposed to be. Thus, the candidate metric we shall use in this work is

$$G = \left(d\Phi - I_a dE^a\right)^2 + \Lambda\left(\xi_{ab}E^a I^b\right)\left(\chi_{cd}dE^c dI^d\right), \tag{4}$$

where ξ_{ab} and χ_{ab} are diagonal constant tensors and Λ is an arbitrary Legendre invariant function of the coordinates $\{Z^A\}$. In particular, we choose to fix $\Lambda = 1, \xi_{ab} = \delta_{ab} \equiv \mathrm{diag}(1,\dots,1)$ and $\chi_{ab} = \eta_{ab} \equiv \mathrm{diag}(-1,1,\dots,1)$ in order to get the exact expression for the metric describing black holes phase transitions (see also [33]).

On the other side, we are not interested in the geometric representation of the phase space, while we care about the geometric properties of the thermodynamic system, which is paralleled by the equilibrium space $\mathcal{E}$. Thus, we pull back the metric G onto $\mathcal{E}$ and obtain a Riemannian structure for the equilibrium space which reads

$$g^{II} \equiv \varphi^*(G) = \left(E^a\frac{\partial\Phi}{\partial E^a}\right)\left(\eta_b^c\frac{\partial^2\Phi}{\partial E^c\partial E^d}dE^b dE^d\right), \tag{5}$$

where φ^* is the pullback of φ as in (3) and $\eta_b^c = \mathrm{diag}(-1,1,\dots,1)$. We remark that g^{II} is (by definition) invariant under (total) Legendre transformations (see, e.g., [33]). Moreover, we also note that g^{II} can be calculated explicitly once the fundamental equation $\Phi = \Phi(E^a)$ is known.

The main thesis of GTD is that the thermodynamic properties of a system described by a fundamental equation $\Phi(E^a)$ can be translated into geometrical features of the equilibrium manifold $\mathcal{E}$, which in our case is described by the metric g^{II}. For example, the scalar curvature of $\mathcal{E}$ should give information about the thermodynamic interaction. This means that systems without interaction shall correspond to flat geometries and systems showing interaction and phase transitions should correspond to curved equilibrium manifolds having curvature singularities. It has been tested in a number of works (see, e.g. [31–35]) that indeed such correspondence works. Furthermore, a previous work [41] studying the thermodynamics and GTD of the Reissner-Nordström and of the Kerr black holes in any dimensions, highlighted that curvature singularities of g^{II} are exactly at the same points where the generalized susceptibilities diverge.

In this work, we extend the work in [41] to the case of Myers-Perry black holes in five dimensions, with the aim of both to analyze their thermodynamic geometry from a new perspective and to focus on the idea of checking what happens with a change of the potential from $\Phi = M$ to $\Phi = S$ in the GTD analysis and when the equilibrium manifold is 3 dimensional. The investigation of the phase structure of Myers-Perry black holes in five dimensions is thus a matter which is interesting by itself and that will provide us with the necessary ground for a new test of the correspondence between the thermodynamic geometry g^{II} of GTD and black holes thermodynamics.

3. Myers-Perry Black Holes

The Kerr black hole can be generalized to the case of arbitrary dimensions and arbitrary number of spins. It turns out that, provided, d is the number of spacetime dimensions, that the maximum number of possible independent spins is $(d-1)/2$ if d is odd and $(d-2)/2$ if d is even [50]. Such general configurations are called Myers-Perry black holes. They deserve a special interest because they are the natural generalization of the well-known Kerr black hole to higher number of spins and because they are shown to coexist with the Emparan-Reall black ring solution for some values of the parameters, thus providing the first explicit example of a violation in a dimension higher than four of the uniqueness theorem (see, e.g., [51] for more details). The line element of the Myers-Perry black hole with an arbitrary number of independent angular momenta in Boyer-Lindquist coordinates for $d = 2n + 1$ (i.e., odd d) reads [50]

$$ds^2 = -dt^2 + \frac{\mu r^2}{\Pi F}\left(dt + \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i\right)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 + \sum_{i=1}^{n}\left(r^2 + a_i^2\right)\left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right), \tag{6}$$

with

$$F \equiv 1 - \sum_{i=1}^{n} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}, \qquad \Pi \equiv \prod_{i=1}^{n}\left(r^2 + a_i^2\right), \qquad \mu \equiv \frac{16\pi G M}{(d-2)\,\Omega_{(d-2)}}, \qquad a_i \equiv \frac{(d-2)}{2}\frac{J_i}{M}, \tag{7}$$

where $\Omega_{(d-2)} = 2\pi^n/\Gamma(n)$, M is the mass of the black hole, $J_i = J_1, \ldots, J_n$ are the $(d-1)/2$ independent angular momenta, and the constraint $\sum_{i=1}^{n} \mu_i^2 = 1$ holds. Solving the equation $g^{rr} = 1/g_{rr} = 0$, one finds the radius of the event horizon (in any dimensions) and thus derives the area and the corresponding entropy, using Bekenstein area law [29].

In particular, in this work, we are interested in the five dimensional case, that is, when $d = 5$. Myers-Perry black holes in five-dimensions can have up to 2 independent angular momenta, and the general equation for the area reads [29]

$$A = \frac{2\pi^2}{r_+}\left(r_+^2 + a_1^2\right)\left(r_+^2 + a_2^2\right), \tag{8}$$

where r_+ is the radius of the event horizon. From the previous expression the entropy can be calculated, being

$$S = \frac{k_B A}{4G} = \frac{1}{r_+}\left(r_+^2 + a_1^2\right)\left(r_+^2 + a_2^2\right), \tag{9}$$

where we choose k_B and G such that S simplifies as in the second equality in (9).

Since it is rather complicated to calculate explicitly the previous expression for the entropy, we will use the M representation throughout the paper. This is possible since the mass can be written in terms of S, J_1, and J_2 as [29]

$$M(S, J_1, J_2) = \frac{3}{4} S^{2/3}\left(1 + 4\frac{J_1^2}{S^2}\right)^{1/3}\left(1 + 4\frac{J_2^2}{S^2}\right)^{1/3}. \tag{10}$$

Equation (10) thus represents the fundamental equation for the Myers-Perry black hole in five dimensions as a thermodynamic system. Starting from (10), we can analyze both the thermodynamic properties and their geometrothermodynamic counterparts. We will split the work in order to consider the three most interesting cases, that is, when one of the two angular momenta is zero, when they are both nonzero but equal, and finally when they are both nonzero and different among each other.

3.1. The Case $J_2 = 0$. If either $J_1 = 0$ or $J_2 = 0$, we obtain the Kerr black hole in 5 dimensions, which has been analyzed in [41]. We briefly review here some of the results presented there and improve the analysis, including the investigation of the response functions defined in the total Legendre transformation of the mass M, which we will call the Gibbsian response functions, in analogy with standard thermodynamics [48]. Therefore, let us suppose that $J_2 = 0$. According to our previous results [41], we know that the response functions defined in the mass representation read

$$C_{J_1} = -\frac{3S\left(S^2 + 4J_1^2\right)}{S^2 - 12J_1^2}, \qquad \kappa_S = \frac{3\left(S^2 + 4J_1^2\right)^{5/3}}{2J_1\left(3S^2 - 4J_1^2\right)}, \qquad \alpha_S = -\frac{3}{8}\frac{\left(S^2 + 4J_1^2\right)^{5/3}}{J_1^2 S}, \tag{11}$$

where we make use of the notation $M_{E^a} \equiv \partial M/\partial E^a$ and $M_{E^a E^b} \equiv \partial^2 M/\partial E^a \partial E^b$, for $E^i = S, J_1$. It follows that α_S does not show any singularity (apart from the extremal limit $S = 0$), while C_{J_1} diverges at the Davies point $S^2 = 12J_1^2$ and κ_S shows an additional possible phase transition at $3S^2 = 4J_1^2$. As it was pointed out in [41], both singularities of the heat capacity and of the compressibility are in the black hole region and hence are physically relevant. It was also shown that the GTD geometry (5) with fundamental equation (10) (with $J_2 = 0$) is curved, indicating the presence of thermodynamic interaction, and that the singularities of the scalar curvature are situated exactly at the same points where the response functions C_{J_1}, and κ_S diverge, both in the mass and in the entropy representations. Furthermore, it was also commented that Weinhold geometry is flat in this case and Ruppeiner thermodynamic metric diverges only in the extremal limit $S = 0$ (see, e.g., [29] for a complete analysis using these metrics).

Moreover, since the thermodynamics of black holes can depend on the chosen ensemble (see, e.g., [52, 53]), we now proceed to calculate the Gibbsian response functions, which can possibly give new information about the phase structure. Using the relations between thermodynamic derivatives (see, [48]), we find out that the expressions for such response functions in the coordinates (S, J_1) used here are

$$C_{\Omega_1} = -\frac{S\left(3S^2 - 4J_1^2\right)}{S^2 + 4J_1^2}, \qquad \kappa_T = -\frac{S^2 - 12J_1^2}{2J_1\left(S^2 + 4J_1^2\right)^{1/3}}, \qquad \alpha_{\Omega_1} = -\frac{8S}{\left(S^2 + 4J_1^2\right)^{1/3}}. \tag{12}$$

It is immediate to see that C_{Ω_1} never diverges and it vanishes exactly at the same points where κ_S diverges. On the other side, κ_T is never divergent and it vanishes exactly where C_{J_1} diverges, while α_{Ω_1} is always finite. It follows that the Gibbsian response functions do not add any information to the knowledge of the phase structure of this configuration, as they change the sign exactly at the points that we have already analyzed; therefore, we conclude that the divergences of the scalar curvature of the metric (5) match exactly the points of second-order phase transitions.

Let us now add a second spin parameter and show that there is still a concrete correlation between the geometric description performed with g^{II} and the thermodynamic properties. To do so, we first focus on the special case of (10) in which $J_1 = J_2 = J$, and afterwards we will consider the completely general case, that is, with J_1 and J_2 both different from zero and from each other. In particular, in the latter case, we will get a 3-dimensional thermodynamic manifold labelled by ($E^1 = S, E^2 = J_1, E^3 = J_2$), and hence we will consider the 3-dimensional version of the metric (5).

3.2. The Case $J_1 = J_2 \equiv J$. Another special case in (10) which is of interest is the case in which the two angular momenta are fixed to be equal, that is, $J_1 = J_2 \equiv J$. This is interesting from the mathematical and physical point of view since it is the only case in which the angular momenta are both different from zero, and at the same time the thermodynamic manifold is 2-dimensional. In fact, the mass fundamental equation (10) in this case is given by

$$M(S,J) = \frac{3}{4}\, S^{2/3}\left(1 + 4\frac{J^2}{S^2}\right)^{2/3}, \tag{13}$$

and the response functions can then be accordingly calculated to give

$$C_J = -\frac{3S\left(S^4 - 16J^4\right)}{S^4 - 32J^2S^2 - 80J^4}, \qquad \kappa_S = \frac{3S^{2/3}\left(S^2 + 4J^2\right)^{4/3}}{4J\left(3S^2 + 4J^2\right)},$$
$$\alpha_S = -\frac{3}{16}\frac{S^{5/3}\left(S^2 + 4J^2\right)^{4/3}}{J^2\left(S^2 + 2J^2\right)}. \tag{14}$$

From (14), it follows that in this case α_S and κ_S do not show any singularity, while C_J diverges at the roots of the denominator $\mathscr{D}_C = S^4 - 32J^2S^2 - 80J^4$. We also observe that the temperature of this black hole is given by

$$T \equiv \left(\frac{\partial M}{\partial S}\right)_J = \frac{1}{2}\frac{S^2 - 4J^2}{S^{5/3}\left(S^2 + 4J^2\right)^{1/3}}, \tag{15}$$

therefore, the extremal limit $T = 0$ is reached when $J^2/S^2 = 1/4$.

Solving $\mathscr{D}_C = 0$, we find that the singularities of the heat capacity are situated at a value S_{critical} for the entropy such that

$$\left.\frac{J^2}{S^2}\right|_{S=S_{\text{critical}}} = \frac{\sqrt{21} - 4}{20}, \tag{16}$$

which is less than the extremal limit. Therefore Davies point of phase transition belongs to the black hole region and we shall investigate it.

It is convenient also in this case to write the full set of thermodynamic response functions, including the Gibbsian ones. Again, making use of the relations between thermodynamic derivatives, we find out that they read

$$C_\Omega = -\frac{S\left(S^2 - 4J^2\right)\left(3S^2 + 4J^2\right)}{\left(S^2 + 4J^2\right)^2},$$
$$\kappa_T = -\frac{S^{2/3}}{4J}\frac{\mathscr{D}_C}{\left(S^2 + 4J^2\right)^{5/3}}, \tag{17}$$
$$\alpha_\Omega = \frac{8S^{5/3}\left(S^2 + 2J^2\right)}{\left(S^2 + 4J^2\right)^{5/3}}.$$

In this case, we observe that the only divergence of the response functions in (14), that is, the divergence of C_J, is again controlled by the vanishing of κ_T. Furthermore, both C_J and C_Ω vanish at the extremal limit, but this does not correspond to any divergence of κ_S, and hence we expect the curvature of the thermodynamic metric to diverge only at the points where $\mathscr{D}_C = 0$.

From the point of view of Geometrothermodynamics, given the fundamental equation (13) and the general metric (5), we can calculate the particular metric and the scalar curvature for the equilibrium manifold of the MP black hole with two equal angular momenta, both in the mass and in the entropy representations.

The metric in the M representation reads

$$g_M^{II} = \frac{1}{S^{4/3}\left(S^2 + 4J^2\right)^{2/3}} \times \left\{\frac{1}{12}\frac{\mathscr{D}_C}{S^2}dS^2 + \frac{2\left(3S^2 + 4J^2\right)}{3}dJ^2\right\}. \tag{18}$$

Therefore, its scalar curvature is

$$R_M = 24S^{10/3}\left(S^2 + 4J^2\right)^{2/3} \times \left(5S^6 + 48J^2S^4 - 368J^4S^2 - 896J^6\right) \times \left(\mathscr{D}_C^2\left(3S^2 + 4J^2\right)^2\right)^{-1}. \tag{19}$$

The numerator is a not very illuminating function that never vanishes when the denominator is zero, and $\mathscr{D}_C$ is exactly the denominator of the heat capacity C_J. Therefore, the singularities of R_M correspond exactly to those of C_J (resp., to the vanishing of κ_T). We remark that the factor $3S^2+4J^2$ in the denominator of R_M, despite being always different from 0 (thus not indicating any phase transition in this case), is exactly the denominator of the compressibility κ_S (resp., a factor in the numerator of C_Ω).

To continue with the analysis, in [49], a general relation was presented (see (34) therein) to express $g^{II}$with $\Phi = S$ (i.e., in the S representation) in the coordinates of the M

representation (i.e., $\{E^a\} = (S, J)$). Such relation in the present case reads

$$g_S^{II} = \frac{M - J\Omega}{T^3} \times \left[TM_{SS} dS^2 + 2\Omega M_{SS} dS\, dJ + \left(2\Omega M_{SJ} - TM_{JJ}\right) dJ^2 \right], \tag{20}$$

where $T \equiv \partial M / \partial S$ is the temperature, $\Omega \equiv \partial M / \partial J$ is the angular velocity at the horizon and $M_{E^a E^b} \equiv \partial^2 M / \partial E^a \partial E^b$, for $E^i = S, J$. Using (20) and (13) for the mass in terms of S and J, we can calculate the expression for metric g_S^{II} in the coordinates (S, J), which reads

$$\begin{aligned} g_S^{II} = {} & \frac{1}{3\left(S^2 + 4J^2\right)(S+2J)^2 (S-2J)^2} \times \left\{ -\frac{3S^2 - 4J^2}{2} \mathscr{D}_C dS^2 + \right. \\ & + \frac{8SJ\left(3S^2 - 4J^2\right)}{(S+2J)(S-2J)} \mathscr{D}_C dS dJ - 4S^2 \\ & \left. \times \frac{9S^6 + 156S^4 J^2 + 112 S^2 J^4 - 448 J^6}{(S+2J)(S-2J)} dJ^2 \right\}. \end{aligned} \tag{21}$$

Consequently, the scalar curvature is

$$R_S = \frac{\mathscr{N}_S}{\left(3S^2 - 4J^2\right)^3 \left(S^2 + 4J^2\right)^2 \mathscr{D}_C^2}, \tag{22}$$

where $\mathscr{N}_S$ is again a function which never vanishes at the points where the denominator is zero. From (22), we see that the denominator of C_J is present in the denominator of R_S. Furthermore, the factor $S^2 + 4J^2$ is never zero, and hence it does not give any additional singularity. On the other hand, the factor $3S^2 - 4J^2$ is clearly vanishing when $J^2/S^2 = 3/4$, which is readily greater than the extremal limit $J^2/S^2 = 1/4$, and hence it has no physical relevance in our analysis.

We thus conclude that also in this case the GTD geometry g^{II} exactly reproduces the phase transition structure of the Myers-Perry black holes both in the mass and in the entropy representation. We comment that in the entropy representation there is an additional singularity which does not correspond to any singularity of the response functions. However, such singularity is situated out of the black hole region, and thus it is not to be considered here. We also remark that Ruppeiner curvature in this case reads $R = -S(S^2 + 12J^2)/(S^4 - 16J^4)$, and hence it diverges only in the extremal limit, while Weinhold metric is flat.

In the next subsection, we will analyze the general case of the Myers-Perry black hole in five dimensions, that is, when the two angular momenta are allowed to vary freely.

3.3. The General Case in Which $J_1 \neq J_2 \neq 0$. Perhaps the most interesting case is the most general one, in which the two angular momenta are allowed to vary freely. In this case, the thermodynamic manifold is 3 dimensional and the mass fundamental equation is given by (10).

The generalized susceptibilities can then be accordingly calculated. The heat capacity at constant angular momenta J_1 and J_2 reads

$$C_{J_1,J_2} = \frac{M_S}{M_{SS}} = -\frac{3S\left(S^2 + 4J_1^2\right)\left(S^2 + 4J_2^2\right)\left(S^4 - 16J_1^2 J_2^2\right)}{\mathscr{D}_C}, \tag{23}$$

where

$$\mathscr{D}_C = S^8 - 12\left(J_1^2 + J_2^2\right) S^6 - 320 J_1^2 J_2^2 S^4 - 576 J_1^2 J_2^2 \left(J_1^2 + J_2^2\right) S^2 - 1280 J_1^4 J_2^4. \tag{24}$$

Furthermore, one can define the 3 analogues of the adiabatic compressibility as

$$\begin{aligned} (\kappa_S)_{11} &\equiv \left(\frac{\partial J_1}{\partial \Omega_1}\right)_S = \frac{3S^{2/3}\left(S^2 + 4J_1^2\right)^{5/3}}{2\left(S^2 + 4J_2^2\right)^{1/3}\left(3S^2 - 4J_1^2\right)}, \\ (\kappa_S)_{22} &\equiv \left(\frac{\partial J_2}{\partial \Omega_2}\right)_S = \frac{3S^{2/3}\left(S^2 + 4J_2^2\right)^{5/3}}{2\left(S^2 + 4J_1^2\right)^{1/3}\left(3S^2 - 4J_2^2\right)}, \\ (\kappa_S)_{12} &\equiv \left(\frac{\partial J_1}{\partial \Omega_2}\right)_S = \frac{3}{16}\frac{S^{2/3}\left(S^2 + 4J_1^2\right)^{2/3}\left(S^2 + 4J_2^2\right)^{2/3}}{J_1 J_2}. \end{aligned} \tag{25}$$

Finally, the analogues of the expansion are given by

$$\begin{aligned} \alpha_{S,J_2} &\equiv \left(\frac{\partial J_1}{\partial T}\right)_S = -\frac{3}{8}\frac{S^{5/3}\left(S^2 + 4J_1^2\right)^{5/3}\left(S^2 + 4J_2^2\right)^{2/3}}{J_1\left(S^4 + 6S^2 J_2^2 + 8J_1^2 J_2^2\right)}, \\ \alpha_{S,J_1} &\equiv \left(\frac{\partial J_1}{\partial T}\right)_S = -\frac{3}{8}\frac{S^{5/3}\left(S^2 + 4J_2^2\right)^{5/3}\left(S^2 + 4J_1^2\right)^{2/3}}{J_2\left(S^4 + 6S^2 J_1^2 + 8J_1^2 J_2^2\right)}. \end{aligned} \tag{26}$$

In this case, neither $(\kappa_S)_{12}$ nor the expansions show any singularity, while C_{J_1,J_2} diverges when $\mathscr{D}_C = 0$ and the compressibilities $(\kappa_S)_{11}$ and $(\kappa_S)_{22}$ diverge when $3S^2 - 4J_1^2 = 0$

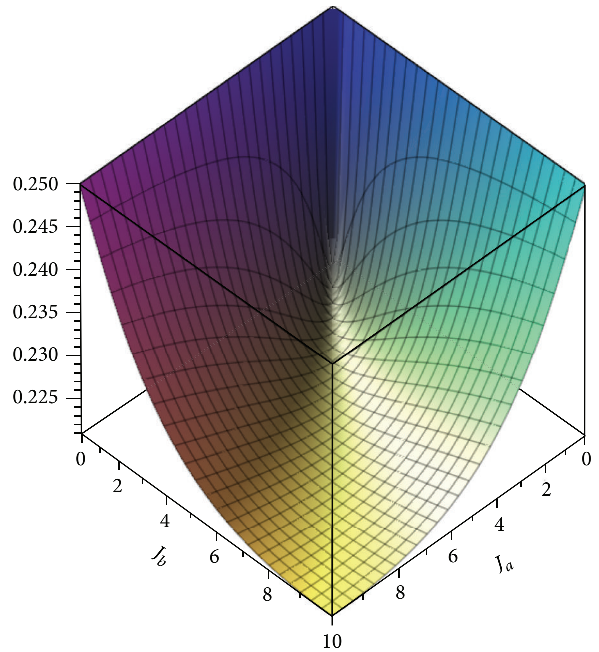


FIGURE 1: The difference between the extremal limit ($J_1 J_2/S^2 = 1/4$) and the value of $J_1 J_2/S^2$ at the critical point of the heat capacity, plotted for values of J_1 and J_2 in the interval $[0, 10]$.

and $3S^2 - 4J_2^2 = 0$, respectively. Furthermore, the temperature reads

$$T = \frac{1}{2S^{5/3}} \frac{S^4 - 16J_1^2 J_2^2}{\left(S^2 + 4J_1^2\right)^{2/3} \left(S^2 + 4J_2^2\right)^{2/3}}. \tag{27}$$

Hence; the extremal limit is reached for $J_1 J_2/S^2 = 1/4$. The heat capacity diverges when $\mathscr{D}_C = 0$, which is an algebraic equation of degree 8 in S. We can solve numerically such equation and obtain the critical value $S = S_{\text{critical}}$ in terms of J_1 and J_2. Taking only the roots which are real and positive, we can compare them with the extremal limit by doing

$$\left.\frac{J_1 J_2}{S^2}\right|_{S=\text{Sextremal}} - \left.\frac{J_1 J_2}{S^2}\right|_{S=\text{Scritical}} = \frac{1}{4} - \left.\frac{J_1 J_2}{S^2}\right|_{S=\text{Scritical}}. \tag{28}$$

The plot of the result is given in Figure 1 for some values of J_1 and J_2. As we can see from Figure 1, the difference in (28) is always positive, and hence the point of phase transition signaled by the divergence of the heat capacity is always in the black hole region.

In the same way, we can solve $3S^2 - 4J_1^2 = 0$ and see whether the divergence of $(\kappa_S)_{11}$ lies in the black hole region or not. It turns out that the denominator of $(\kappa_S)_{11}$ vanishes for values of S such that $J_1^2/S^2 = 3/4$, which means that $J_1 J_2/S^2 = (3/4)(J_2/J_1)$. Therefore, we have that $(1/4) - (3/4)(J_2/J_1)$ is positive provided that $J_1 > 3J_2$ for $J_1 > 0$ or $J_1 < 3J_2$ for $J_1 < 0$. Summing up, the divergences of $(\kappa_S)_{11}$ can be in the black hole region for appropriate values of J_1 and J_2. Analogously, the divergences of $(\kappa_S)_{22}$ can also be in the black hole region.

As in the preceding sections, we will now focus on the Gibbsian response functions, in order to make the analysis complete. The heat capacity at constant angular velocities read

$$\begin{aligned} C_{\Omega_1,\Omega_2} &\equiv T\left(\frac{\partial S}{\partial T}\right)_{\Omega_1,\Omega_2} \\ &= -3S\left(S^4 - 16J_1^2 J_2^2\right)\left(3S^2 - 4J_1^2\right) \\ &\quad \times \left(3S^2 - 4J_2^2\right)\left(S^2 + 4J_1^2\right)\left(S^2 + 4J_2^2\right) \\ &\quad \times \left(\mathscr{D}\left(S, J_1, J_2\right)\right)^{-1}, \end{aligned} \tag{29}$$

where the denominator is given by

$$\begin{aligned} \mathscr{D}\left(S, J_1, J_2\right) = {} & 9S^{12} + 72\left(J_1^2 + J_2^2\right)S^{10} \\ & + 16\left(9J_1^4 + 95J_1^2 J_2^2 + 9J_2^4\right)S^8 \\ & + 5376J_1^2 J_2^2\left(J_1^2 + J_2^2\right)S^6 \\ & - 256J_1^2 J_2^2\left(9J_1^4 - 101J_1^2 J_2^2 + 9J_2^4\right)S^4 \\ & - 6144J_1^4 J_2^4\left(J_1^2 + J_2^2\right)S^2 - 53248J_1^6 J_2^6. \end{aligned} \tag{30}$$

Furthermore, one can define three generalized susceptibilities, analogous to the isothermal compressibility, as

$$\begin{gathered} \left(\kappa_T\right)_{11} \equiv \left(\frac{\partial J_1}{\partial \Omega_1}\right)_T, \qquad \left(\kappa_T\right)_{12} \equiv \left(\frac{\partial J_1}{\partial \Omega_2}\right)_T, \\ \left(\kappa_T\right)_{22} \equiv \left(\frac{\partial J_2}{\partial \Omega_2}\right)_T. \end{gathered} \tag{31}$$

For the Myers-Perry black hole it can be written as

$$\begin{aligned} &\left(\kappa_T\right)_{11} \\ &\quad = -\frac{S^{2/3}}{2}\left(\mathscr{D}_C\left(\left(S^2 + 4J_1^2\right)^{1/3}\left(S^2 + 4J_2^2\right)^{1/3}\right.\right. \\ &\qquad \left.\left. \times \left(S^6 - 12J_2^2 S^4 + 48J_1^2 J_2^2 S^2 + 192J_2^2 J_2^4\right)\right)^{-1}\right), \\ &\left(\kappa_T\right)_{22} \\ &\quad = -\frac{S^{2/3}}{2}\left(\mathscr{D}_C\left(\left(S^2 + 4J_1^2\right)^{1/3}\left(S^2 + 4J_2^2\right)^{1/3}\right.\right. \\ &\qquad \left.\left. \times \left(S^6 - 12J_1^2 S^4 + 48J_1^2 J_2^2 S^2 + 192J_2^2 J_1^4\right)\right)^{-1}\right), \end{aligned} \tag{32}$$

while $(\kappa_T)_{12}$ has a more cumbersome expression and we will not write it here, since it has the same properties of $(\kappa_T)_{11}$ and $(\kappa_T)_{22}$ as regards to our analysis; that is, it is proportional to the denominator of C_{J_1,J_2} defined in (24) and it has a nontrivial denominator (one can also introduce the two analogues of the thermal expansion, but for the sake of simplicity, we are not going to write them here, since they do not show any singularities, and hence they do not play any role in our analysis).

Therefore, from the thermodynamic point of view, we remark that the divergences of C_{J_1,J_2} are matched by the vanishing of the three quantities $(\kappa_T)_{ij}$, while the divergences of $(\kappa_S)_{11}$ and $(\kappa_S)_{22}$ are reproduced as zeroes of the heat capacity C_{Ω_1,Ω_2}. This behavior is in agreement with the analysis of the preceding sections. Furthermore, in this case, the heat capacity C_{Ω_1,Ω_2} and the generalized compressibilities $(\kappa_T)_{ij}$ possibly show additional phase transitions, which is a further indication of the fact that black holes exhibit different thermodynamic behavior in different potentials.

Now, let us turn to the GTD analysis. Given the fundamental equation (10) and the general metric (5), we can calculate the particular metric and the scalar curvature for the MP black hole with two free angular momenta, both in the mass and in the entropy representations. The metric in the M representation reads

$$\begin{aligned} g_M^{II} = & \frac{1}{3S^{4/3}\left(S^2+4J_1^2\right)^{1/3}\left(S^2+4J_2^2\right)^{1/3}} \\ & \times \left\{ -\frac{1}{4}\frac{\mathscr{D}_C}{S^2\left(S^2+4J_1^2\right)\left(S^2+4J_2^2\right)}dS^2 \right. \\ & + \frac{\left(3S^2-4J_1^2\right)\left(S^2+4J_2^2\right)}{S^2+4J_1^2}dJ_1^2 \\ & + \frac{\left(3S^2-4J_2^2\right)\left(S^2+4J_1^2\right)}{S^2+4J_2^2}dJ_2^2 \\ & \left. +16J_1J_2dJ_1\ dJ_2 \right\}. \end{aligned} \tag{33}$$

Hence, its scalar curvature is

$$\begin{aligned} R_M = & \mathscr{N}_M \\ & \times \left(\mathscr{D}_C^2\left[3S^4-4S^2\left(J_1^2+J_2^2\right)-16J_1^2J_2^2\right]^2 \right. \\ & \left. \times \left(S^2+4J_1^2\right)^{2/3}\left(S^2+4J_2^2\right)^{2/3} \right)^{-1}, \end{aligned} \tag{34}$$

where $\mathscr{D}_C$ is as usual the denominator of C_{J_1,J_2} defined in (24). Since there is no term in the numerator $\mathscr{N}_M$ which cancels out the divergences that happen when $\mathscr{D}_C = 0$, we can conclude that every phase transition related to the heat capacity C_{J_1,J_2} is properly reproduced by the scalar curvature R_M. In addition, in this case, the factor $3S^4 - 4S^2(J_1^2 + J_2^2) - 16J_1^2J_2^2$ can also vanish, possibly giving an additional singularity which does not correspond to the ones shown by the response functions. It is easy to calculate that $3S^4-4S^2(J_1^2+J_2^2) - 16J_1^2J_2^2 = 0$ for values of S such that

$$\frac{J_1J_2}{S^2} = \frac{1}{8}\frac{\left(-J_1^2-J_2^2+\sqrt{J_1^4+14J_1^2J_2^2+J_2^4}\right)}{J_1J_2}. \tag{35}$$

We can thus calculate the difference between the extremal limit $J_1J_2/S^2 = 1/4$ and the critical value (35). The result is

$$\begin{aligned} & \frac{1}{4} - \frac{1}{8}\frac{\left(-J_1^2-J_2^2+\sqrt{J_1^4+14J_1^2J_2^2+J_2^4}\right)}{J_1J_2} \\ & \quad = -\frac{1}{48}\left(\left(\left(J_1^2+J_2^2-6J_1J_2+\sqrt{J_1^4+14J_1^2J_2^2+J_2^4}\right)\right.\right. \\ & \qquad \left. \times \left(J_1^2+J_2^2-\sqrt{J_1^4+14J_1^2J_2^2+J_2^4}\right)\right) \\ & \qquad \left. \times \left(J_1^2J_2^2\right)^{-1}\right), \end{aligned} \tag{36}$$

which can be positive for appropriate values of J_1 and J_2. Therefore, such points of divergence of R_M are in the black hole region for some values of the parameters. Hence, we conclude that the behavior of R_M perfectly matches the behavior of C_{J_1,J_2}, but in this case, it does not reproduce the additional possible phase transitions indicated by the singularities of the compressibilities $(\kappa_S)_{11}$ and $(\kappa_S)_{22}$ and possibly shows some additional unexpected singularities.

However, we can give a precise physical meaning to such additional singularities. In fact, if we evaluate the determinant of the Hessian of the mass with respect to the angular momenta J_1 and J_2, we get

$$\begin{aligned} \det\left(\mathrm{Hess}(M)_{J_1J_2}\right) & \equiv M_{J_1J_1}M_{J_2J_2} - M_{J_1J_2}^2 \\ & = \frac{4}{3}\frac{3S^4-4S^2\left(J_1^2+J_2^2\right)-16J_1^2J_2^2}{S^{4/3}\left(S^2+4J_1^2\right)^{4/3}\left(S^2+4J_2^2\right)^{4/3}}, \end{aligned} \tag{37}$$

from which we can see that the numerator is exactly the factor in the denominator of R_M whose roots give the additional singularities. Since the Hessians of the energy in thermodynamics are related to the stability conditions, we suggest that the physical meaning of such additional divergences of R_M is to be found in a change of stability of the system, for example, from a stable phase to an unstable one.

On the other side, using the relation (20) for g^{II} between the M and the S representations, naturally extended to the 3-dimensional case with coordinates (S, J_1, J_2), that is,

$$\begin{aligned} g_S^{II} = & \frac{M-J_1\Omega_1-J_2\Omega_2}{T^3} \\ & \times \left[TM_{SS}dS^2 + 2\Omega_1M_{SS}dS\ dJ_1 + 2\Omega_2M_{SS}dS\ dJ_2 \right. \\ & + \left(2\Omega_1M_{SJ_1} - TM_{J_1J_1}\right)dJ_1^2 \\ & + \left(2\Omega_2M_{SJ_2} - TM_{J_2J_2}\right)dJ_2^2 \\ & \left. -2\left(TM_{J_1J_2} - \Omega_1M_{SJ_2} - \Omega_2M_{SJ_1}\right)dJ_1\ dJ_2\right], \end{aligned} \tag{38}$$

we can now calculate the metric in the S representation, which reads

$$
\begin{aligned}
g_S^{II} = {} & \frac{\left[3S^4 + 4S^2\left(J_1^2 + J_2^2\right) - 16J_1^2J_2^2\right]\mathscr{D}_C}{3\left(S^2 + 4J_1^2\right)\left(S^2 + 4J_2^2\right)\left(S^2 - 4J_1J_2\right)^2\left(S^2 + 4J_1J_2\right)^2} \\
& \times \left\{ \frac{1}{2}dS^2 + 4\frac{SJ_1\left(S^2 + 4J_2^2\right)}{\left(S^2 - 4J_1J_2\right)\left(S^2 + 4J_1J_2\right)}\, dSdJ_1 \right. \\
& + 4\frac{SJ_2\left(S^2 + 4J_1^2\right)}{\left(S^2 - 4J_1J_2\right)\left(S^2 + 4J_1J_2\right)}dSdJ_2 \\
& - 2\left(S^2\left(S^2 + 4J_2^2\right)^2\right. \\
& \quad \times \left[3S^6 + 26S^4J_1^2 + 144S^2J_1^2J_2^2 + 320J_1^4J_2^2\right] \\
& \quad \left. \times \left(\mathscr{D}_C\left(S^2 - 4J_1J_2\right)\left(S^2 + 4J_1J_2\right)\right)^{-1}\right)\, dJ_1^2 \\
& - 2\left(S^2\left(S^2 + 4J_1^2\right)^2\right. \\
& \quad \times \left[3S^6 + 26S^4J_2^2 + 144S^2J_1^2J_2^2 + 320J_2^4J_1^2\right] \\
& \quad \left. \times \left(\mathscr{D}_C\left(S^2 - 4J_1J_2\right)\left(S^2 + 4J_1J_2\right)\right)^{-1}\right)dJ_2^2 \\
& - 32\left(S^2J_1J_2\left(S^2 + 4J_1^2\right)\left(S^2 + 4J_2^2\right)\right. \\
& \quad \times \left[5S^4 + 12S^2\left(J_1^2 + J_2^2\right) + 16J_1^2J_2^2\right] \\
& \quad \times \left(\mathscr{D}_C\left(S^2 - 4J_1J_2\right)\right. \\
& \quad \left.\left.\left. \times \left(S^2 + 4J_1J_2\right)\right)^{-1}\right)\right)dJ_1\, dJ_2 \Bigg\}.
\end{aligned} \tag{39}
$$

The scalar curvature can thus be calculated to obtain

$$
\begin{aligned}
R_S = \mathscr{N}_S \times \Big(& \mathscr{D}_C^2\left[3S^4 + 4S^2\left(J_1^2 + J_2^2\right) - 16J_1^2J_2^2\right]^3 \\
& \times S^2\left(S^2 + 4J_1^2\right)\left(S^2 + 4J_2^2\right)\Big)^{-1}.
\end{aligned} \tag{40}
$$

In this case, we see again that the denominator of the heat capacity $\mathscr{D}_C$ is present in the denominator of R_S. Furthermore, the second factor, which is slightly different from the factor in the denominator of R_M, vanishes for values of S such that

$$
\frac{J_1J_2}{S^2} = \frac{1}{8}\frac{J_1^2 + J_2^2 + \sqrt{J_1^4 + 14J_1^2J_2^2 + J_2^4}}{J_1J_2}. \tag{41}
$$

The earlier discussion for the additional singularity of R_M does not apply in this case, since one can easily show that the points described by (41) do not belong to the black hole region for any values of J_1 and J_2. However, we comment in passing that such additional singularities are still related to the vanishing of the determinant of the Hessian of the entropy S with respect to the angular momenta J_1 and J_2. Therefore, they still indicate the points where the Hessian vanishes, although they are not situated in the black hole region in this case. We infer from these results that the physical meaning of the divergences of the scalar curvature of the metric g^{II} for such a 3-dimensional equilibrium manifold is related to the divergences of the heat capacity at constant angular momenta and to the zeroes of the Hessian of the potential with respect to those momenta, both in the mass and in the entropy representations. On the other side, from the full analysis of the divergences of the generalized response functions, we see that there are other possible points of phase transitions related to divergences of the compressibilities, which appear to be not enclosed by the analysis given with g^{II}. We also comment that we could have used the potential $\Phi = G \equiv M - TS - J_1\Omega_1 - J_2\Omega_2$ in writing the metric (5) to study the GTD analysis in the G representation, but such investigation would have led to exactly the same results, as it has to be, since the metric (5) is invariant under total Legendre transformations.

To conclude, we observe that in [29] the case of the full Myers-Perry black hole thermodynamics has been investigated using Weinhold and Ruppeiner thermodynamic geometries. The authors proved that both Weinhold and Ruppeiner scalar curvatures only diverge in the extremal limit.

4. Conclusions

In this work, we have analyzed the thermodynamics and thermodynamic geometry of different Myers-Perry black holes configurations in five dimensions, classifying them according to the values of the two possible independent angular momenta.

To this end, we followed the approach of Davies for the standard analysis of the thermodynamic properties in different potentials and used the approach of GTD for the thermodynamic geometric investigation. The present work has been carried out with the twofold aim of understanding the phase structure of Myers-Perry black holes in five dimensions and inferring new conclusions on the physical meaning of the metric g^{II}, both in the mass and in the entropy representations.

Our results indicate that the Myers-Perry black holes in five dimensions have a nontrivial phase structure in the sense of Davies. In particular, the analysis of the response functions indicates that both the heat capacities and the compressibilities defined in the M potential diverge at some points, which is usually interpreted as the hallmark of a phase transition. Interestingly, such a behavior is matched by the vanishing of the corresponding Gibbsian response functions in all the cases studied here. Moreover, in the most general case when the two angular momenta vary freely, we have shown that the Gibbsian response functions provide some additional singularities, indicating that the analysis in the M potential is different from that performed in the G potential.

In all the cases studied in this work, the phase transitions are well reproduced by the GTD analysis, while they are not reproduced by the thermodynamic geometries of Weinhold and Ruppeiner, whose analysis has been observed to correspond to other approaches (see e.g., [4]). We have also found that the scalar curvature of the metric g^{II} shows a very similar behavior in the M representation to that of the S representation. In particular, for the cases in which we have only two degrees of freedom we argue that no physical difference has been detected and we have shown that not only the phase transitions indicated by C_J are reproduced, but also the ones indicated by divergences of κ_S. Moreover, a detailed analysis of the Gibbsian response functions showed that such divergences correspond to points where κ_T and C_Ω vanish and change their character. We therefore conclude that for such cases the divergences of the scalar curvature of g^{II} reproduce the full set of second order phase transitions considered here.

On the other side, it seems that analyzing the general case in which both angular momenta are switched on, that is, a thermodynamic system with three degrees of freedom, some differences might appear. In fact, the phase transitions signaled by C_{J_1,J_2} are still obtained as curvature singularities in both representations. Nevertheless, the scalar curvature has some additional divergences, which for the case of the M representation can be in the black hole region for appropriate values of the angular momenta and that apparently are not directly related to the response functions of the system. However, we claim that such additional divergences are linked to the vanishing of the Hessian determinant of the potential M with respect to the two angular momenta, and therefore they mark the transition from a stable phase to an unstable one. In our opinion, this means that the physical meaning of the scalar curvature of the metric g^{II} for thermodynamic systems with three degrees of freedom goes beyond the well-established correspondence with the generalized susceptibilities, that is, second-order derivatives of the potential, encompassing also questions of stability related to their mutual relation, that is, determinants of the Hessian of the potential. This is also supported by the analysis of the scalar curvature in the S representation, which again shows singularities exactly at those points where the Hessian of the entropy with respect to the two angular momenta vanishes, so from the mathematical point of view the situation is basically the same. It is interesting, however, to note that in the S representation such points are not in the black hole region, a direct evidence of the fact that black hole thermodynamics strictly depends on the potential being used. Moreover, in the completely general case, some additional divergences appear when considering the Gibbsian response functions, which are not present in the thermodynamic analysis in the M potential nor are indicated as curvature singularities of g^{II}. The study of such additional singularities goes beyond the scope of this work and may be the matter of further investigation. We also expect to extend this work in the nearest future and find a number of further examples which support (or discard) the interpretation of the thermodynamic metric g^{II} for thermodynamic systems with 3 degrees of freedom given here.

Acknowledgments

The authors want to thank Professor H. Quevedo for insightful suggestions. Alessandro Bravetti wants to thank ICRA for financial support.

References

[1] S. W. Hawking, "Particle creation by black holes," *Communications in Mathematical Physics*, vol. 43, no. 3, pp. 199–220, 1975.

[2] J. D. Bekenstein, "Black holes and entropy," *Physical Review D*, vol. 7, pp. 2333–2346, 1973.

[3] J. M. Bardeen, B. Carter, and S. W. Hawking, "The four laws of black hole mechanics," *Communications in Mathematical Physics*, vol. 31, pp. 161–170, 1973.

[4] G. Arcioni and E. Lozano-Tellechea, "Stability and critical phenomena of black holes and black rings," *Physical Review D*, vol. 72, no. 10, Article ID 104021, 22 pages, 2005.

[5] P. C. W. Davies, "Thermodynamics of black holes," *Reports on Progress in Physics*, vol. 41, no. 8, p. 1313, 1978.

[6] C. O. Loustó, "The fourth law of black-hole thermodynamics," *Nuclear Physics B*, vol. 410, no. 1, pp. 155–172, 1993.

[7] C. O. Lousto, "Erratum," *NuclearPhysics B*, vol. 449, 433 pages, 1995.

[8] C. O. Lousto, "Some thermodynamic aspects of black holes and singularities," *International Journal of Modern Physics D*, vol. 6, pp. 575–590, 1997.

[9] R.-G. Cai and J.-H. Cho, "Thermodynamic curvature of the BTZ black hole," *Physical Review D*, vol. 60, no. 6, Article ID 067502, 4 pages, 1999.

[10] J. Shen, R.-G. Cai, B. Wang, and R.-K. Su, "Thermodynamic geometry and critical behavior of black holes," *International Journal of Modern Physics A*, vol. 22, no. 1, pp. 11–27, 2007.

[11] R. Banerjee, S. K. Modak, and S. Samanta, "Glassy phase transition and stability in black holes," *European Physical Journal C*, vol. 70, no. 1, pp. 317–328, 2010.

[12] R. Banerjee, S. K. Modak, and S. Samanta, "Second order phase transition and thermodynamic geometry in Kerr-AdS black holes," *Physical Review D*, vol. 84, Article ID 064024, 8 pages, 2011.

[13] R. Banerjee and D. Roychowdhury, "Thermodynamics of phase transition in higher dimensional AdS black holes," *Journal of High Energy Physics*, vol. 2011, article 4, 2011.

[14] R. Banerjee, S. K. Modak, and D. Roychowdhury, "A unified picture of phase transition: from liquid-vapour systems to AdS black holes," *Journal of High Energy Physics*, vol. 2012, article 125, 2012.

[15] R. Banerjee, S. Ghosh, and D. Roychowdhury, "New type of phase transition in Reissner Nordström–AdS black hole and its thermodynamic geometry," *Physics Letters B*, vol. 696, pp. 156–162, 2011.

[16] R. Banerjee and D. Roychowdhury, "Critical behavior of Born-Infeld AdS black holes in higher dimensions," *Physical Review D*, vol. 85, Article ID 104043, 14 pages, 2012.

[17] R. Banerjee and D. Roychowdhury, "Critical phenomena in Born-Infeld AdS black holes," *Physical Review D*, vol. 85, no. 4, Article ID 044040, 10 pages, 2012.

[18] A. Lala and D. Roychowdhury, "Ehrenfest's scheme and thermodynamic geometry in Born-Infeld AdS black holes," *Physical Review D*, vol. 86, Article ID 084027, 8 pages, 2012.

[19] J. W. Gibbs, *The Collected Works*, vol. 1 of *Thermodynamics*, Yale University Press, 1948.

[20] C. Carathéodory, "Untersuchungen über die Grundlagen der Thermodynamik," *Mathematische Annalen*, vol. 67, no. 3, pp. 355–386, 1909.

[21] R. A. Fisher, "Theory of statistical estimation," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 22, no. 5, pp. 700–725, 1925.

[22] C. Radhakrishna Rao, "Information and the accuracy attainable in the estimation of statistical parameters," *Bulletin of the Calcutta Mathematical Society*, vol. 37, pp. 81–91, 1945.

[23] D. C. Brody and D. W. Hook, "Information geometry in vapour-liquid equilibrium," *Journal of Physics A*, vol. 42, no. 2, Article ID 023001, 33 pages, 2009.

[24] F. Weinhold, "Metric geometry of equilibrium thermodynamics," *The Journal of Chemical Physics*, vol. 63, no. 6, pp. 2479–2483, 1975.

[25] G. Ruppeiner, "Thermodynamics: a Riemannian geometric model," *Physical Review A*, vol. 20, no. 4, pp. 1608–1613, 1979.

[26] G. Ruppeiner, "Riemannian geometry in thermodynamic fluctuation theory," *Reviews of Modern Physics*, vol. 67, no. 3, pp. 605–659, 1995.

[27] G. Ruppeiner, "Thermodynamic curvature and phase transitions in Kerr-Newman black holes," *Physical Review D*, vol. 78, no. 2, Article ID 024016, 13 pages, 2008.

[28] J. E. Åman, I. Bengtsson, and N. Pidokrajt, "Flat information geometries in black hole thermodynamics," *General Relativity and Gravitation*, vol. 38, no. 8, pp. 1305–1315, 2006.

[29] J. E. Åman and N. Pidokrajt, "Geometry of higher-dimensional black hole thermodynamics," *Physical Review D*, vol. 73, no. 2, Article ID 024017, 7 pages, 2006.

[30] L. Á. Gergely, N. Pidokrajt, and S. Winitzki, "Geometrothermodynamics of tidal charged black holes," *European Physical Journal C*, vol. 71, no. 3, pp. 1–11, 2011.

[31] H. Quevedo, "Geometrothermodynamics," *Journal of Mathematical Physics*, vol. 48, no. 1, Article ID 013506, 14 pages, 2007.

[32] H. Quevedo, "Geometrothermodynamics of black holes," *General Relativity and Gravitation*, vol. 40, no. 5, pp. 971–984, 2008.

[33] H. Quevedo, "Exterior and interior metrics with quadrupole moment," *General Relativity and Gravitation*, vol. 43, no. 4, pp. 1141–1152, 2011.

[34] H. Quevedo and A. Sanchez, "Geometrothermodynamics of asymptotically Anti-de Sitter black holes," *Journal of High Energy Physics*, vol. 9, p. 34, 2008.

[35] W. Janke, D. A. Johnston, and R. Kenna, "Geometrothermodynamics of the Kehagias-Sfetsos black hole," *Journal of Physics A*, vol. 43, no. 42, Article ID 425206, 11 pages, 2010.

[36] H. Quevedo and D. Tapias, "Geometric description of chemicalreactions," submitted, http://arxiv.org/abs/1301.0262.

[37] Y. Han and G. Chen, "Thermodynamics, geometrothermodynamics and critical behavior of $(2+1)$-dimensional black holes," *Physics Letters B*, vol. 714, no. 2-5, pp. 127–130, 2012.

[38] A. Aviles, A. Bastarrachea-Almodovar, L. Campuzano, and H. Quevedo, "Extending the generalized Chaplygin gas model by using geometrothermodynamics," *Physical Review D*, vol. 86, Article ID 063508, 10 pages, 2012.

[39] M. E. Rodrigues and Z. A. A. Oporto, "Thermodynamics of phantom black holes in Einstein-Maxwell-dilaton theory," *Physical Review D*, vol. 85, no. 10, Article ID 104022, 12 pages, 2012.

[40] A. Bravetti and F. Nettel, "Second order phase transitions and thermodynamic geometry: a general approach," submitted, http://arxiv.org/abs/1208.0399.

[41] A. Bravetti, D. Momeni, R. Myrzakulov, and H. Quevedo, "Geometrothermodynamics of higher dimensional black holes," *General Relativity and Gravitation*, 2013.

[42] J. E. Aman and N. Pidokrajt, "On explicit thermodynamic functions and extremal limits of Myers-Perry black holes," submitted, http://arxiv.org/abs/1004.5550.

[43] R. Emparan and R. C. Myers, "Instability of ultra-spinning black holes," *Journal of High Energy Physics*, vol. 309, p. 25, 2003.

[44] R. Monteiro, M. J. Perry, and J. E. Santos, "Thermodynamic instability of rotating black holes," *Physical Review D*, vol. 80, no. 2, Article ID 024041, 18 pages, 2009.

[45] D. Astefanesei, R. B. Mann, M. J. Rodriguez, and C. Stelea, "Quasilocal formalism and thermodynamics of aymptotically flat black objects," *Classical and Quantum Gravity*, vol. 27, no. 16, Article ID 165004, 22 pages, 2010.

[46] D. Astefanesei, M. J. Rodriguez, and S. Theisen, "Thermodynamic instability of doubly spinning black objects," *Journal of High Energy Physics*, vol. 1008, article 46, 2010.

[47] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York, NY, USA, 1989.

[48] H. B. Callen, *Thermodynamics and an Introduction to Thermostatics*, John Wiley & Sons, New York, NY, USA, 1985.

[49] A. Bravetti, C. S. Lopez-Monsalvo, F. Nettel, and H. Quevedo, "The conformal metric structure of Geometrothermodynamics," *Journal of Mathematical Physics*, vol. 54, Article ID 033513, 11 pages, 2013.

[50] R. C. Myers and M. J. Perry, "Black holes in higher-dimensional space-times," *Annals of Physics*, vol. 172, no. 2, pp. 304–347, 1986.

[51] R. Emparan and H. S. Reall, "Black holes in higher dimensions," *Living Reviews in Relativity*, vol. 11, p. 6, 2008.

[52] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, "Charged AdS black holes and catastrophic holography," *Physical Review D*, vol. 60, no. 6, Article ID 064018, 17 pages, 1999.

[53] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, "Holography, thermodynamics, and fluctuations of charged AdS black holes," *Physical Review D*, vol. 60, no. 10, Article ID 104026, 20 pages, 1999.

Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2013, Article ID 734138, 8 pages
http://dx.doi.org/10.1155/2013/734138

Research Article

Geometric Curvatures of Plane Symmetry Black Hole

Shao-Wen Wei, Yu-Xiao Liu, Chun-E. Fu, and Hai-Tao Li

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, China

Correspondence should be addressed to Yu-Xiao Liu; liuyx@lzu.edu.cn

Received 16 February 2013; Accepted 23 May 2013

Academic Editor: Hernando Quevedo

We study the properties and thermodynamic stability of the plane symmetry black hole from the viewpoint of geometry. We find that the Weinhold curvature gives the first-order phase transition at $N = 1$, where N is a parameter of the plane symmetry black hole while the Ruppeiner one shows first-order phase transition points for arbitrary $N \neq 1$. Considering the Legendre invariant proposed by Quevedo et al., we obtain a unified geometry metric, which contains the information of the second-order phase transition. So, the first-order and second-order phase transitions can be both reproduced from the geometry curvatures. The geometry is also found to be curved, and the scalar curvature goes to negative infinity at the Davie phase transition points beyond semiclassical approximation.

1. Introduction

Several decades ago, the original work of Bekenstein and Hawking showed that the black hole is indeed a thermodynamics system [1, 2]. It was also found that the black hole satisfies four laws of the elementary thermodynamics with regarding the surface gravity and the outer horizon area as its temperature and entropy, respectively [3]. Although, it is widely believed that a black hole is a thermodynamic system, the statistical origin of the black hole entropy is still one of the most fascinating and controversial subjects today.

The investigation of thermodynamic properties of black holes is also a fascinating subject. Much of work had been carried out on the stability and phase transitions of black holes. It is generally thought that the local stability of a black hole is mainly determined by its heat capacity. Negative heat capacity usually gives a thermodynamically unstable system, and the positive one implies a local stable one. The divergent points of the heat capacity are usually consistent with the Davies points, where the second-order phase transition takes place [4–6].

The properties and phase transitions of a thermodynamic system can also be studied with the idea of geometry. Weinhold [7] first introduced the geometrical concept into the thermodynamics. He suggested that a Riemannian metric can be defined as the second derivatives of internal energy U with respect to the entropy and other extensive quantities of a thermodynamic system. However, it seems that the Weinhold geometry has not much physical meanings. Few years later, Ruppeiner [8] introduced another metric, which is analogous to the Weinhold one. The thermodynamic potential of the Ruppeiner geometry is the entropy S of the thermodynamic system rather than the internal energy U. In fact, the two metrics are conformally related to each other:

$$ds_R^2 = \frac{1}{T} ds_W^2, \tag{1}$$

with the temperature T as the conformal factor. The Ruppeiner geometry had been used to study the ideal gas and the van der Waals gas. It was shown that the curvature vanishes for the ideal gas, whereas for the van der Waals gas, the curvature is nonzero and diverges only at those points where the phase transitions take place (for details see the review paper [9]). The black hole, as a thermodynamics system, has been extensively investigated. The Weinhold geometry and the Ruppeiner geometry were obtained for various black holes and black branes [4–30]. In particular, it was found that the Ruppeiner geometry carries the information of phase structure of a thermodynamic system. In general, its curvature is singular at the points, where the phase transition takes place. However, for the Banados-Teitelboim-Zanelli (BTZ) and Reissner-Nordström (RN) black holes, the cases are quite different. The Ruppeiner geometries give a vanished curvature, which means there exist no thermodynamic interactions and no phase transition points. There exist phase transition points for the two kinds of black hole. For the contradiction, much research has been carried out to explain it. The main focus is on the thermodynamic potential, which is generally believed to be the internal energy U rather than the mass M. For the Reissner-Nordström black hole, it was argued in [18] that the thermodynamic curvature should be reproduced from the Kerr-Newmann anti-de sitter black hole with the angular momentum $J \rightarrow 0$ and cosmological constant $\Lambda \rightarrow 0$. Another explanation of the contradiction was presented by Queved et al. few years ago [31, 32]. They pointed out that the origin of the contradiction is that the Weinhold metric and Ruppeiner metric are not Legendre invariants. A Legendre invariant metric was introduced by them, which could reproduce correctly the behavior of the thermodynamic interactions and second-order phase transitions for the BTZ and RN black holes [33, 34] and other black hole configurations and models [35–38]. Inspired by the thermodynamic geometry, Liu et al. recently proposed a free energy metric [39], which can give a better description on the phase transition for a black hole. The authors also showed that, for a system with n-pairs of intensive/extensive variables, different thermodynamic metrics can be embedded into a flat (n, n)-dimensional space. The method has been extended to different black holes [40–44].

Another interesting and important question of this field is how the geometry behaves beyond semiclassical approximation. It is generally believed that there will be a logarithmic corrected term to the entropy when the semiclassical black hole extends to its quantum level [45, 46]. Considering the correction term, the geometry structure was studied in [33, 47] for the BTZ black hole. Especially, its Ruppeiner curvature will be nonzero beyond semiclassical approximation. The aim of this paper is to study the phase transitions and geometry structure of the plane symmetry black hole. Firstly, we study the thermodynamic stability of the plane symmetric black hole. It is shown that there always exist locally thermodynamically stable phases and unstable phases for the plane symmetric black hole due to suitable parameter regimes. Then, three different geometry structures are obtained. The Weinhold curvature gives phase transition points, which correspond to that of the first-order phase transition only at $N = 1$, while the Ruppeiner one shows first-order phase transition points for arbitrary $N \neq 1$. Considering the Legendre invariant, we obtain a unified geometry metric, which gives a correct behavior of the thermodynamic interactions and phase transitions. It is found that the curvature constructed from the unified metric goes to negative infinity at the Davies' points, where the second-order phase transition takes place. The geometry structure is also studied as the logarithmic correction is included. The results show that the unified geometry behaviors differ when logarithmic correction term is included. However, it has no effect on the unified geometry depicting the phase transitions of the plane symmetric black hole.

The paper is organized as follows. In Section 2, we first review some thermodynamic quantities of the plane symmetric black hole. The thermodynamic stability is also studied. In Section 3, both the Weinhold and Ruppeiner geometry structures are obtained. However, they fail to give the information about the second-order phase transition points. For the reason, we give a detailed analysis and obtain a new Legendre invariant metric structure which could give a good description of the thermodynamic interactions and phase transitions in Section 4. Unified geometry structure beyond semiclassical approximation is also considered in Section 5. Finally, the paper ends with a brief conclusion.

2. Thermodynamic Quantities and Thermodynamic Stability of the Plane Symmetric Black Hole

In this section, we will present the thermodynamic quantities and other properties of the plane symmetric black hole. The local thermodynamic stability of it is also discussed. The action depicting the plane symmetric black hole is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\mathscr{R} - 2(\nabla\varphi)^2 - 2\Lambda e^{2b\varphi} - e^{-2a\varphi} F^2 \right), \tag{2}$$

where φ is a dilaton field and a, b are constants. The negative cosmological constant is $\Lambda = -3\alpha^2$. Static plane symmetric black hole solutions in this theory were first given in [48] (some detail work for the black hole can also be found in [49–53]). Consider the following:

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + g(r)\left(dx^2 + dy^2\right). \tag{3}$$

The metric functions are given by, respectively,

$$\begin{gathered} f(r) = -\frac{4\pi M}{N\alpha^N} r^{1-N} + \frac{6\alpha^2}{N(2N-1)} r^N + \frac{2Q^2}{N\alpha^{2N}} r^{-N}, \\ g(r) = (r\alpha)^N. \end{gathered} \tag{4}$$

The dilaton field φ reads

$$\varphi(r) = -\frac{\sqrt{2N - N^2}}{2} \ln r. \tag{5}$$

And the constant $a = b = \sqrt{2N - N^2}/N$. From (5), one easily finds the parameter $N \in (1/2, 2)$. The scalar curvature of this spacetime can be calculated as

$$R = 12\alpha^2 r^{N-2} + \frac{\left(2N - N^2\right)}{2r^2} f(r). \tag{6}$$

Obviously, R diverges at $r = 0$ for any value of N, which implies that $r = 0$ plane is a singularity plane. From Figure 1, we see that the scalar curvature R is a monotonically decreasing function of r for different N. It is also clear that R has a large value for small value of N near $r = 0$.

The parameters M and Q are the mass and charge of the black hole. The event horizon is located at $f(r_h) = 0$ and the radius r_h satisfies

$$\frac{3\alpha^2}{2N-1} r_h^{2N} - \frac{2\pi M}{\alpha^N} r_h + \frac{Q^2}{\alpha^{2N}} = 0. \tag{7}$$

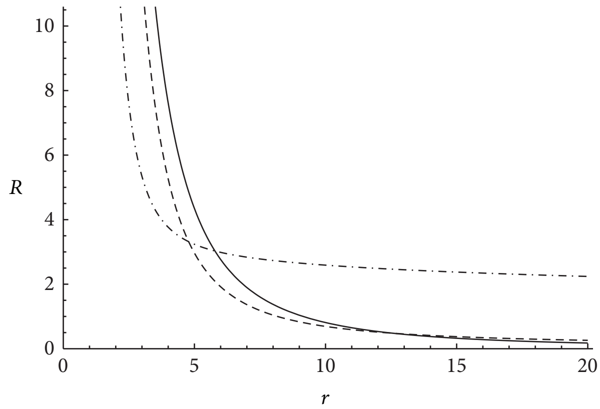


FIGURE 1: The behavior of scalar curvature R with $\alpha = \sqrt{3}/3$, $Q = 10$, and $M = 1$. The full line, dashed line, and dashed-dotted line are for $N = 0.6$, 1, and 1.8, respectively.

In general, there exist two horizons, the inner horizon and the outer horizon. Under the extreme case, the two horizons will merge into one. Here, we have denoted r_h as the radius of outer horizon. The surface area of the outer horizon corresponds to unit x-y plane is [52]. One may consider the following:

$$\mathscr{A} = \left(\alpha r_h\right)^{2N}. \quad (8)$$

From (7), the mass can be expressed in the form

$$M = \frac{3\alpha^{N+2}}{2\pi\left(2N-1\right)} r_h^{2N-1} + \frac{Q^2}{2\pi\alpha^N} r_h^{-1}. \quad (9)$$

With the relation between area and entropy, that is, $S = \mathscr{A}/4$, we can obtain

$$r_h = \frac{1}{\alpha}(4S)^{1/2N}. \quad (10)$$

Substituting (10) into (9), the mass can be obtained as a function of entropy S and charge Q in the form

$$M = \frac{3\alpha^{3-N}}{2\pi\left(2N-1\right)}(4S)^{(2N-1)/2N} + \frac{Q^2}{2\pi\alpha^{N-1}}(4S)^{-1/2N}. \quad (11)$$

From the energy conservation law of the black hole

$$dM = TdS + \phi dQ, \quad (12)$$

the relevant thermodynamic variables, the temperature, and electric potential are obtained

$$T = \left(\frac{\partial M}{\partial S}\right)_Q = \frac{\left(12\alpha^2 S - Q^2\right)\alpha^{1-N}}{2^{2+1/N}\pi N S^{1+1/2N}},$$
$$\phi = \left(\frac{\partial M}{\partial Q}\right)_S = \frac{\alpha^{1-N} Q}{2^{1/N}\pi S^{1/2N}}. \quad (13)$$

For a given charge Q, the heat capacity has the expression

$$C_Q = -\frac{2NS\left(12\alpha^2 S - Q^2\right)}{12S\alpha^2 - (1+2N)Q^2}, \quad (14)$$

with the zeropoints and singular points

$$Q^2 = 12\alpha^2 S, \quad \text{(zero points)}, \quad (15)$$

$$Q^2 = \frac{12\alpha^2 S}{2N+1}, \quad \text{(singular points)}, \quad (16)$$

respectively. The heat capacity C_Q goes to zero at $Q^2 = 12\alpha^2 S$ continuously, which is considered to be the first-order phase transition point. On other hand, it is generally believed that the Davies' points where the second-order phase transition takes place correspond to the diverge points of heat capacity. So the heat capacity C_Q may indicate that the second-order phase transition takes place at $Q^2 = 12\alpha^2 S/(2N+1)$. The heat capacity also contains the information of the local stability of the black hole thermodynamics. The negative heat capacity always implies an unstable thermodynamics system, and the positive one shows a stable system. Here, we would like to give a brief discussion about the local stability of the plane symmetric black hole. For $Q^2 > 12\alpha^2 S$, the numerator of the heat capacity (14) is negative, while the denominator is positive, which gives a negative heat capacity. For $Q^2 < 12\alpha^2 S/(2N+1)$, the numerator is positive, but the denominator turns to be negative, which also shows a negative heat capacity. So, in both cases, the heat capacity C_Q implies an unstable black hole thermodynamics. However, when $|Q| \in (2\sqrt{3}\alpha\sqrt{S/(2N+1)}, 2\sqrt{3}\alpha\sqrt{S})$, both the numerator and denominator are positive, which implies a stable black hole thermodynamics. The behavior of the heat capacity C_Q can be directly found from Figure 2. For larger and smaller values of $|Q|$, the heat capacity C_Q is negative. While in the middle zone, it is positive, which means that the black hole can be in stable thermal equilibrium with an arbitrary volume heat bath. In summary, we have found that there always exist locally thermodynamically stable phases and unstable phases for the plane symmetric black hole due to suitable parameter regimes.

3. Weinhold Geometry and Ruppeiner Geometry of the Plane Symmetric Black Hole

In this section, we would like to study the Weinhold and Ruppeiner geometries of the plane symmetric black hole. In the first step, we will calculate the Weinhold geometry. Then, using the conformal relation, we could obtain the Ruppeiner geometry naturally. The Weinhold geometry is charactered by the metric

$$ds_W^2 = \frac{\partial^2 M}{\partial S^2}dS^2 + 2\frac{\partial^2 M}{\partial S \partial Q}dSdQ + \frac{\partial^2 M}{\partial Q^2}dQ^2, \quad (17)$$

where the index W denotes the Weinhold geometry. Here, the thermodynamic potential is the mass M, and the entropy S and charge Q are the extensive variables.

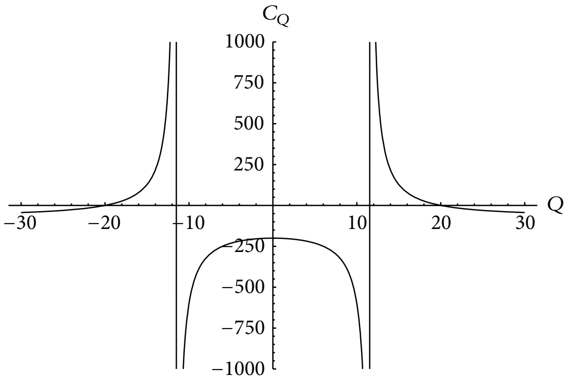


FIGURE 2: The behavior of the heat capacity C_Q, with $N = 1$, $\alpha = \sqrt{3}/3$, and $S = 100$. C_Q is singular at $Q = \pm 11.5470$ and vanishes at $Q = \pm 20$.

Using (11), the Weinhold metric can be obtained in the form

$$g_W = -\frac{\alpha^{1-N}}{2^{1/N}S^{1/2N}}\begin{pmatrix} \dfrac{12\alpha^2 S - (2N+1)Q^2}{8\pi N^2 S^2} & \dfrac{Q}{2\pi NS} \\ \dfrac{Q}{2\pi NS} & -\dfrac{1}{\pi} \end{pmatrix}. \quad (18)$$

Its determinant is $\det(g) = -\alpha^{2-2N}[12\alpha^2 S - (2N-1)Q^2]/2^{3+2/N}S^{2+1/N}\pi^2 N^2$. Note that the determinant disappears as the heat capacity vanishes only at $N = 1$. A simple calculation shows that the Christoffel symbols are

$$\begin{aligned}
\Gamma^S_{SS} &= -\frac{(2N+1)\left[12\alpha^2 S - (4N-1)Q^2\right]}{4NS\left[12\alpha^2 S - (2N-1)Q^2\right]}, \\
\Gamma^S_{QS} = \Gamma^S_{SQ} &= -\frac{2NQ}{12\alpha^2 S - (2N-1)Q^2}, \\
\Gamma^S_{QQ} &= -\frac{2NS}{12\alpha^2 S - (2N-1)Q^2}, \\
\Gamma^Q_{SS} &= -\frac{(2N+1)Q^3}{4NS^2\left[12\alpha^2 S - (2N-1)Q^2\right]}, \\
\Gamma^Q_{SQ} = \Gamma^Q_{QS} &= -\frac{12\alpha^2 S + (2N+1)Q^2}{4S\left[12N\alpha^2 S - N(2N-1)Q^2\right]}, \\
\Gamma^Q_{QQ} &= \frac{Q}{12\alpha^2 S - (2N-1)Q^2},
\end{aligned} \quad (19)$$

where the Christoffel symbols are calculated with

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\tau}\left(g_{\nu\tau,\mu} + g_{\mu\tau,\nu} - g_{\mu\nu,\tau}\right). \quad (20)$$

The Riemannian curvature tensor, Ricc curvature, and scalar curvature are given, respectively,

$$\begin{aligned}
R^\mu_{\sigma\nu\tau} &= \Gamma^\mu_{\sigma\nu,\tau} - \Gamma^\mu_{\sigma\tau,\nu} + \Gamma^\mu_{\lambda,\tau}\Gamma^\lambda_{\sigma,\nu} - \Gamma^\mu_{\lambda,\nu}\Gamma^\lambda_{\sigma,\tau}, \\
R_{\mu\nu} &= R^\lambda_{\mu\lambda\nu}, \\
R &= g^{\mu\nu}R_{\mu\nu}.
\end{aligned} \quad (21)$$

With (21), we get the scalar curvature

$$\mathscr{R}_W = -\frac{24\ 2^{1/N}N\pi S^{1+1/2N}\alpha^{1+N}}{\left[12\alpha^2 S - (2N-1)Q^2\right]^2}. \quad (22)$$

This curvature is always negative for any values of charge Q and positive entropy S. It also diverges at $Q^2 = 12\alpha^2 S/(2N+1)$, which consists with the first-order transition points (15) reproduced from the capacity C_Q only at $N = 1$. Its behavior can be seen in Figure 3. However, it implies no information about the second-order phase transition. So, it is natural to ask how the Ruppeiner curvature behaves. Could it give the proper phase transition points?

With that question, we now turn to the Ruppeiner geometry of the plane symmetric black hole. Recalling the conformal relation (1) between the Ruppeiner geometry and the Weinhold geometry, we obtain the Ruppeiner metric

$$g_R = \frac{1}{T}g_W = \begin{pmatrix} -\dfrac{12\alpha^2 S - (2N+1)Q^2}{2NS\left(12\alpha^2 S - Q^2\right)} & -\dfrac{2Q}{12\alpha^2 S - Q^2} \\ -\dfrac{2Q}{12\alpha^2 S - Q^2} & -\dfrac{4NS}{12\alpha^2 S - Q^2} \end{pmatrix}, \quad (23)$$

where the index R denotes the Ruppeiner geometry. After some calculations, we obtain the Ruppeiner curvature

$$\mathscr{R}_R = -\frac{12\alpha^2 Q^2 (N-1)\left[36S\alpha^2 - (4N-1)Q^2\right]}{\left(12\alpha^2 S - Q^2\right)\left[12S\alpha^2 - (2N-1)Q^2\right]^2}. \quad (24)$$

It is obvious that the curvature will be zero at $N = 1$. The vanished thermodynamic curvature $\mathscr{R}_R$ implies that there exist no phase transition points and no thermodynamic interactions. So, the Ruppeiner curvature is not proper to describe the phase transitions of the plane symmetric black hole at $N = 1$. The divergence of the Ruppeiner curvature is at $Q^2 = 12\alpha^2 S/(2N+1)$ and $Q^2 = 12\alpha^2 S$, which can be seen from Figure 4. The points $Q^2 = 12\alpha^2 S$ consist with the zero-points (15) of the heat capacity C_Q. This means that the Ruppeiner curvature always implies the first-order phase transition points. Like the Weinhold curvature, the Ruppeiner curvature also implies no any information about the second-order phase transition.

4. Unified Geometry of the Plane Symmetric Black Hole

In the previous section, we show that the Weinhold curvature implies the first-order phase transition points only at $N = 1$, while the Ruppeiner curvature implies the first-order phase transition points except $N = 1$. Both of the geometry structures gave no information about the second-order phase transition points of the plane symmetric black hole. Quevedo pointed out that the two geometries are not Legendre invariants, which makes them inappropriate to describe the geometry of thermodynamic systems [34]. Considering the Legendre invariant, a unified geometry was presented in [35], where the metric structure can give a good description

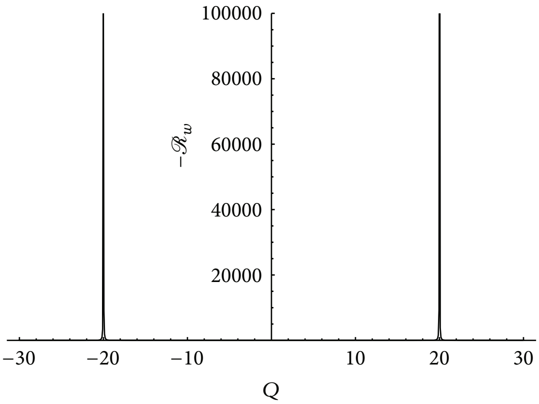


FIGURE 3: The negative Weinhold curvature $\mathscr{R}_w$ versus the charge Q with $N = 1$, $\alpha = \sqrt{3}/3$, and $S = 100$. The divergence points are at $Q = \pm 20$.

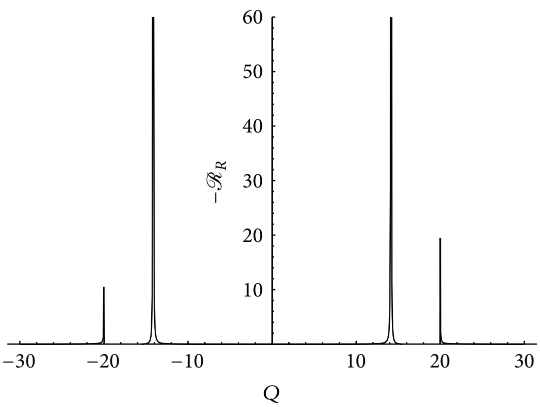


FIGURE 4: The negative Ruppeiner curvature $\mathscr{R}_R$ versus the charge Q with $N = 3/2$, $\alpha = \sqrt{3}/3$, and $S = 100$. The divergence points are at $Q = \pm 20, \pm 14.1421$.

of various types of black hole thermodynamics. So, in this section, we would like to discuss the unified geometry of the plane symmetric black hole, and we want to know whether it works.

Here, we still take the mass M as the thermodynamic potential. Then the unified geometry metric can be expressed as

$$\begin{aligned} ds_L^2 &= \left(S\frac{\partial M}{\partial S} + Q\frac{\partial M}{\partial Q}\right)\begin{pmatrix} -\dfrac{\partial^2 M}{\partial S^2} & 0 \\ 0 & \dfrac{\partial^2 M}{\partial Q^2}\end{pmatrix}\begin{pmatrix} dS^2 \\ dQ^2 \end{pmatrix} \\ &= \frac{\alpha^{2-2N}\left[12\alpha^2 S - (2N+1)Q^2\right]\left[12\alpha^2 S + (4N-1)Q^2\right]}{2^{5+2/N}\pi^2 N^3 S^{2+1/N}} \\ &\quad \times dS^2 + \frac{\alpha^{2-2N}\left(12\alpha^2 S + (4N-1)Q^2\right)}{2^{2(N+1)/N}\pi^2 N S^{1/N}} dQ^2. \end{aligned} \tag{25}$$

The index L denotes the curvature reproduced from the Legendre invariant metric.

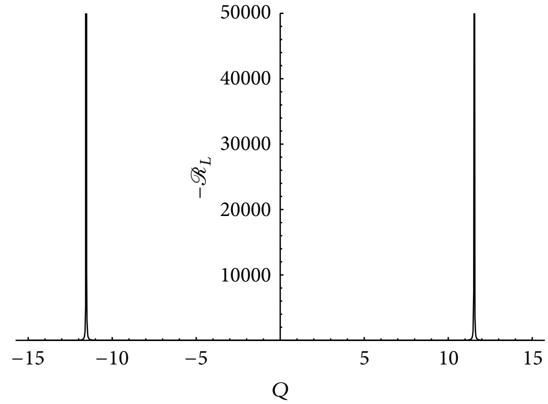


FIGURE 5: The negative unified geometric curvature $\mathscr{R}_L$ versus the charge Q with $N = 1$, $\alpha = \sqrt{3}/3$, and $S = 100$. The divergence points are at $Q = \pm 11.5470$, which are consistent with the divergence points of the heat capacity C_Q. The positive curvature region is at $|Q| \geq 20$ and it is not shown in this figure.

This diagonal metric reproduces the thermodynamic curvature $\mathscr{R}_L$, which turns out to be non-zero and the scalar curvature is

$$\begin{aligned} \mathscr{R}_L = {} & \frac{192 \quad 4^{1/N}\pi^2\alpha^{2N} N S^{1+1/N}}{\left[12\alpha^2 S - (2N+1)Q^2\right]^2\left[12\alpha^2 S + (4N-1)Q^2\right]^3} \\ & \cdot \left\{(4N-1)Q^2\left[\left(N\left(4N^2 - 6N - 3\right) - 1\right)Q^2 \right.\right. \\ & \qquad \left. - 12\left((N-5)N - 2\right)\alpha^2 S\right] \\ & \left. + 144(N-1)^2\alpha^4 S^2\right\}. \end{aligned} \tag{26}$$

The thermodynamic curvature vanishes at $Q^2 = 12S\alpha^2$ when $N = 1$, which are just the points of the first-order phase transition. It is shown that the diverge points are at $Q^2 = 12\alpha^2 S/(2N+1)$, which implies that there exist second-order phase transitions at these points. This result exactly consists with that of the heat capacity (14). The detail behavior of $\mathscr{R}_L$ can be found in Figure 5, where the singularities are just the divergence points of the heat capacity C_Q. Now, we can see that the thermodynamic curvature $\mathscr{R}_L$ reproduced from the Legendre invariant metric (25) could give an exact description of the second-order phase transitions of a thermodynamics system. Beside this, we also expect that this unified geometry description may give more information about a thermodynamics system.

5. Unified Geometry beyond Semiclassical Approximation

In this section, we will discuss the unified geometry of the plane symmetric black hole beyond semiclassical approximation. With the idea, each quantity of the black hole will

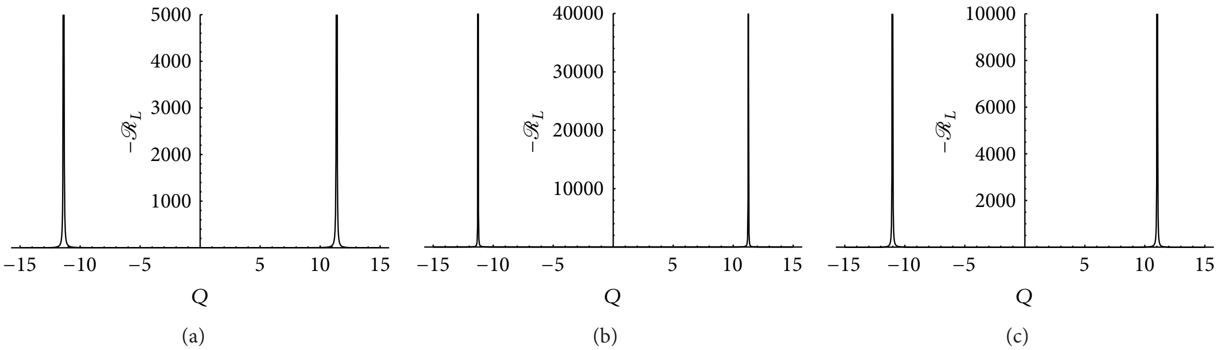


FIGURE 6: The negative unified geometric curvature $\mathscr{R}_L$ versus the charge Q, including the logarithmic correction with $N = 1$, $\alpha = \sqrt{3}/3$, and $S = 100$. The parameter γ is set to 1/2 (left), 5/6 (middle), and 3/2 (right), respectively. The divergence points are at $Q = \pm 11.3756, \pm 11.2613$, and ± 11.0326.

be modified accordingly. For general, we suppose that the corrected entropy is of the form

$$S' = S - \gamma \ln S. \tag{27}$$

The parameter γ is a constant. In fact, the origin of the logarithmic correction term can be accounted by the uncertainty principle or the tunneling method. With (27), the heat capacity (14) is also modified to

$$C'_Q = \frac{2NS(S-\gamma)(S-\gamma\ln S)\left(12S\alpha^2 - Q^2 - 12\alpha^2\gamma\ln S\right)}{A_2Q^2 - A_1}, \tag{28}$$

with

$$A_1 = 12\alpha^2(S-\gamma\ln S)\left[(1+2N\ln S)\gamma^2 - 2(N+1)S\gamma + S^2\right],$$
$$A_2 = [2N(1+\ln S)+1]\gamma^2 - 2(3N+1)S\gamma + (2N+1)S^2. \tag{29}$$

The singular points of the heat capacity are determined by $A_2Q^2 - A_1 = 0$ and are given by

$$Q^2 = \frac{A_1}{A_2}. \tag{30}$$

If $\gamma = 0$, the singular points of the heat capacity will reduce to (14). Following Section 4, we obtain the curvature $\mathscr{R}'_L$:

$$\mathscr{R}'_L = \frac{h(S,Q,N,\gamma)}{K^3(A_2Q^2 - A_1)^2}, \tag{31}$$

where $K = [4N(S-\gamma\ln S) + \gamma - S]Q^2 + 12\alpha^2(S-\gamma)(S-\gamma\ln S)$ and $h(S,Q,N,\gamma)$ is a complex function and we do not write it here. It is found that the divergence points for the heat capacity C'_Q and the curvature $\mathscr{R}'_L$ consist with each other, which means that the curvature gives proper points, where second-order phase transitions take place. So, it is easy to summarize that the logarithmic correction term does not affect the unified geometry to depict the plane symmetry black hole's phase transitions.

Now, we would like to discuss how the geometry behaved as the parameter γ takes different values. For simplicity, we turn back to the case $N = 1$. The Legendre invariant metric for this case is

$$g_L = \begin{pmatrix} -\dfrac{BC}{128\pi^2S^2(S-\gamma\ln S)^4} & 0 \\ 0 & \dfrac{B}{16\pi^2S^2(S-\gamma\ln S)^2} \end{pmatrix}, \tag{32}$$

where $B = K|_{N=1}$ and $C = (A_2Q^2 - A_1)|_{N=1}$. After some tedious calculations, we can obtain the curvature. The numerator of the curvature is a cumbersome expression and can not be written in a compact form. While the denominator of it is proportional to the determinant of the metric (32) and is given by

$$D = B^3 \cdot C^2. \tag{33}$$

Fixing the parameters α and entropy S, the characteristic behavior of the curvature is depicted in Figure 6, where the parameter γ is set to 1/2, 5/6 and 3/2, respectively. The values of the charge Q at the divergence points of the curvature $\mathscr{R}'_L$ are given as

$$Q = \pm 2\sqrt{3}\alpha\sqrt{\frac{(S-\gamma\ln S)\left[(1+2\ln S)\gamma^2 - 4S\gamma + S^2\right]}{[2(1+\ln S)+1]\gamma^2 - 8S\gamma + 3S^2}}. \tag{34}$$

When $\Delta = 3 - 2\ln S \geq 0$, there are three points of γ for the vanished charge Q:

$$\gamma_1 = \frac{S}{\ln S}, \qquad \gamma_\pm = \frac{2 \pm \sqrt{3 - 2\ln S}}{1 + 2\ln S}S. \tag{35}$$

In general, we consider $S \gg 1$, which leads to $\Delta < 0$. So, the vanished charge Q is only at $\gamma = \gamma_1$.

6. Conclusion

In this paper, we study the phase transitions and geometry structure of the plane symmetry black hole. The local thermodynamic stability of it is also discussed through the heat capacity C_Q. It is shown that there always exist locally thermodynamically stable phases and unstable phases for plane symmetric black hole due to suitable parameter regimes. The Weinhold geometry and the Ruppeiner geometry are obtained. The Weinhold curvature gives phase transition points, which correspond to that of the first-order phase transition only at $N = 1$, while the Ruppeiner one shows first-order phase transition points for arbitrary $N \neq 1$. Both of which give no information about the second-order phase transition. Quevedo et al. first pointed out that the two geometry metrics are not Legendre invariant and they introduced a Legendre invariant metric, which can give a good description of various types of black hole thermodynamics. Considering the Legendre invariant, we obtain a unified geometry metric, which gives a correct behavior of the thermodynamic interaction and second-order phase transition. Including the logarithmic corrected term, we study the geometry structure of the plane symmetry black hole. The result shows that the logarithmic correction term does not affect the unified geometry to depict the phase transitions. In this paper, we show that the unified geometry description gives a good description of the second-order phase transitions of the plane symmetry black hole. We also expect that this unified geometry description may give more information about a thermodynamic system.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 11205074 and Grant no. 11075065) and the Huo Ying-Dong Education Foundation of the Chinese Ministry of Education (Grant no. 121106), and the Fundamental Research Funds for the Central Universities (no. lzujbky-2013-18 and no. lzujbky-2013-21).

References

[1] J. D. Bekenstein, "Black holes and entropy," *Physical Review D*, vol. 7, pp. 2333–2346, 1973.

[2] S. W. Hawking, "Particle creation by black holes," *Communications in Mathematical Physics*, vol. 43, no. 3, pp. 199–220, 1975.

[3] J. M. Bardeen, B. Carter, and S. W. Hawking, "The four laws of black hole mechanics," *Communications in Mathematical Physics*, vol. 31, pp. 161–170, 1973.

[4] P. C. W. Davies, "The thermodynamic theory of black holes," *Proceedings of the Royal Society A*, vol. 353, pp. 499–521, 1977.

[5] P. C. W. Davies, "Thermodynamics of black holes," *Reports on Progress in Physics*, vol. 41, p. 1313, 1977.

[6] P. C. W. Davies, "Thermodynamic phase transitions of Kerr-Newman black holes in de Sitter space," *Classical and Quantum Gravity*, vol. 6, no. 12, p. 1909, 1989.

[7] F. Weinhold, "Metric geometry of equilibrium thermodynamics," *The Journal of Chemical Physics*, vol. 63, no. 6, pp. 2479–2483, 1975.

[8] G. Ruppeiner, "Thermodynamics: a Riemannian geometric model," *Physical Review A*, vol. 20, no. 4, pp. 1608–1613, 1979.

[9] G. Ruppeiner, "Riemannian geometry in thermodynamic fluctuation theory," *Reviews of Modern Physics*, vol. 67, pp. 605–659, 1995, Erratum: *Reviews of Modern Physics*, vol. 68, p. 313, 1996.

[10] S. Ferrara, G. W. Gibbons, and R. Kallosh, "Black holes and critical points in moduli space," *Nuclear Physics B*, vol. 500, no. 1–3, pp. 75–93, 1997.

[11] R.-G. Cai and J.-H. Cho, "Thermodynamic curvature of the BTZ black hole," *Physical Review D*, vol. 60, no. 6, Article ID 067502, 1999.

[12] J. E. Åman, I. Bengtsson, and N. Pidokrajt, "Geometry of black hole thermodynamics," *General Relativity and Gravitation*, vol. 35, no. 10, pp. 1733–1743, 2003.

[13] D. A. Johnston, W. Janke, and R. Kenna, "Information geometry, one, two, three (and four)," *Acta Physica Polonica B*, vol. 34, no. 10, pp. 4923–4937, 2003.

[14] G. Arcioni and E. Lozano-Tellechea, "Stability and critical phenomena of black holes and black rings," *Physical Review D*, vol. 72, no. 10, Article ID 104021, 2005.

[15] J. Shen, R.-G. Cai, B. Wang, and R.-K. Su, "Thermodynamic geometry and critical behavior of black holes," *International Journal of Modern Physics A*, vol. 22, no. 1, pp. 11–27, 2007.

[16] J. E. Åman, I. Bengtsson, and N. Pidokrajt, "Flat information geometries in black hole thermodynamics," *General Relativity and Gravitation*, vol. 38, no. 8, pp. 1305–1315, 2006.

[17] J. E. Åman and N. Pidokrajt, "Geometry of higher-dimensional black hole thermodynamics," *Physical Review D*, vol. 73, no. 2, Article ID 024017, 2006.

[18] B. Mirza and M. Zamaninasab, "Ruppeiner geometry of RN black holes: flat or curved?" *Journal of High Energy Physics*, no. 6, 2007.

[19] J. E. Aman, N. Pidokrajt, and J. Ward, "On geometrothermodynamics of Dilaton black holes," *EAS Publications Series*, vol. 30, p. 279, 2008.

[20] J. E. Åman and N. Pidokrajt, "Ruppeiner geometry of black hole thermodynamics," *EAS Publications Series*, vol. 30, pp. 269–273, 2008.

[21] A. J. M. Medved, "A commentary on Ruppeiner metrics for black holes," *Modern Physics Letters A*, vol. 23, no. 26, pp. 2149–2161, 2008.

[22] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, "Ruppeiner geometry and 2D dilaton gravity in the thermodynamics of black holes," *Physics Letters B*, vol. 663, no. 4, pp. 342–350, 2008.

[23] L. A. Gergely, N. Pidokrajt, and S. Winitzki, "Geometrothermodynamics of tidal charged black holes," *European Physical Journal C*, vol. 71, p. 1569, 2011.

[24] Y.-H. Wei, "Thermodynamic critical and geometrical properties of charged BTZ black hole," *Physical Review D*, vol. 80, no. 2, Article ID 024029, 2009.

[25] R. Biswas and S. Chakraborty, "The geometry of the higher dimensional black hole thermodynamics in Einstein-Gauss-Bonnet theory," *General Relativity and Gravitation*, vol. 42, no. 5, pp. 1311–1322, 2010.

[26] T. Sarkar, G. Sengupta, and B. N. Tiwari, "Thermodynamic geometry and extremal black holes in string theory," *Journal of High Energy Physics*, no. 10, 2008.

[27] S. Bellucci and B. N. Tiwari, "On the microscopic perspective of black branes thermodynamic geometry," *Entropy*, vol. 12, no. 10, pp. 2097–2143, 2010.

[28] J. E. Aman, J. Bedford, D. Grumiller, N. Pidokrajt, and J. Ward, "Ruppeiner theory of black hole thermodynamics," *Journal of Physics: Conference Series*, vol. 66, Article ID 012007, 2007.

[29] G. Ruppeiner, "Thermodynamic curvature and phase transitions in Kerr-Newman black holes," *Physical Review D*, vol. 78, no. 2, Article ID 024016, 2008.

[30] S. I. Vacaru, "Thermodynamic geometry and locally anisotropic black holes," http://arxiv.org/abs/gr-qc/9905053.

[31] H. Quevedo, "Geometrothermodynamics," *Journal of Mathematical Physics*, vol. 48, no. 1, Article ID 013506, 2007.

[32] H. Quevedo and A. Vazquez, "The geometry of thermodynamics," *AIP Conference Proceedings*, vol. 977, p. 165, 2008.

[33] H. Quevedo and A. Sánchez, "Geometric description of BTZ black hole thermodynamics," *Physical Review D*, vol. 79, no. 2, Article ID 024012, 2009.

[34] H. Quevedo, "Geometrothermodynamics of black holes," *General Relativity and Gravitation*, vol. 40, no. 5, pp. 971–984, 2008.

[35] J. L. Álvarez, H. Quevedo, and A. Sánchez, "Unified geometric description of black hole thermodynamics," *Physical Review D*, vol. 77, no. 8, Article ID 084004, 2008.

[36] H. Quevedo and A. Sánchez, "Geometrothermodynamics of asymptotically anti-de Sitter black holes," *Journal of High Energy Physics*, no. 9, 2008.

[37] H. Quevedo, A. Sanchez, and A. Vazquez, "Invariant geometry of the ideal gas," http://arxiv.org/abs/0811.0222.

[38] H. Quevedo and A. Sánchez, "Geometrothermodynamics of black holes in two dimensions," *Physical Review D*, vol. 79, no. 8, Article ID 087504, 2009.

[39] H. Liu, H. Lu, M. Luo, and K.-N. Shao, "Thermodynamical metrics and black hole phase transitions," *Journal of High Energy Physics*, no. 54, 2010.

[40] Q. J. Cao, Y. X. Chen, and K. N. Shao, "Black hole phase transitions in Hořava-Lifshitz gravity," *Physical Review D*, vol. 83, Article ID 064015, 2011.

[41] M. Akbar, H. Quevedo, K. Saifullah, A. Sanchez, and S. Taj, "Thermodynamic geometry of charged rotating BTZ black holes," *Physical Review D*, vol. 83, Article ID 084031, 2011.

[42] P. Chen, "Thermodynamic Geometry of the Born-Infeld-anti-de Sitter black holes," *International Journal of Modern Physics A*, vol. 26, p. 3091, 2011.

[43] S. W. Wei, Y. X. Liu, Y. Q. Wang, and H. Guo, "Thermodynamic geometry of black hole in the deformed Hořava-Lifshitz gravity," *Europhysics Letters*, vol. 99, Article ID 20004, 2012.

[44] G. Ruppeiner, "Thermodynamic curvature: pure fluids to black holes," *Journal of Physics: Conference Series*, vol. 410, Article ID 012138, 2013.

[45] K. Huang, *Statistical Mechanics*, John Wiley & Sons, New York, NY, USA, 1963.

[46] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Pergamon, 1969.

[47] T. Sarkar, G. Sengupta, and B. N. Tiwari, "On the thermodynamic geometry of BTZ black holes," *Journal of High Energy Physics*, no. 11, 2006.

[48] R.-G. Cai and Y.-Z. Zhang, "Black plane solutions in four-dimensional spacetimes," *Physical Review D*, vol. 54, no. 8, pp. 4891–4898, 1996.

[49] A. S. Miranda, J. Morgan, and V. T. Zanchin, "Quasinormal modes of plane-symmetric black holes according to the AdS/CFT correspondence," *Journal of High Energy Physics*, no. 11, 2008.

[50] A. S. Miranda and V. T. Zanchin, "Quasinormal modes of plane-symmetric anti-de Sitter black holes: a complete analysis of the gravitational perturbations," *Physical Review D*, vol. 73, Article ID 064034, 2006.

[51] X.-X. Zeng, Y.-W. Han, and Y. S. Zheng, "Hawking radiation from plane symmetric black hole covariant anomaly," *Communications in Theoretical Physics*, vol. 51, no. 1, pp. 187–189, 2009.

[52] R. Zhao, L. C. Zhang, and Y. Q. Wu, "Calculating entropy of plane symmetry black hole via generalized uncertainty relation," *International Journal of Theoretical Physics*, vol. 46, p. 3128, 2007.

[53] J. P. S. Lemos and F. S. N. Lobo, "Plane symmetric traversable wormholes in an anti-de Sitter background," *Physical Review D*, vol. 69, no. 10, Article ID 104007, 2004.

Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2013, Article ID 641273, 6 pages
http://dx.doi.org/10.1155/2013/641273

Research Article

Geometric Description of the Thermodynamics of the Noncommutative Schwarzschild Black Hole

Alexis Larrañaga,[1] Natalia Herrera,[2] and Juliana Garcia[2]

[1] *National Astronomical Observatory, National University of Colombia, Bogotá 11001000, Colombia*
[2] *Department of Physics, National University of Colombia, Bogotá 11001000, Colombia*

Correspondence should be addressed to Alexis Larrañaga; ealarranaga@unal.edu.co

Received 13 March 2013; Accepted 8 May 2013

Academic Editor: Rong-Gen Cai

The thermodynamics of the noncommutative Schwarzschild black hole is reformulated within the context of the recently developed formalism of geometrothermodynamics (GTD). Using a thermodynamic metric which is invariant with respect to Legendre transformations, we determine the geometry of the space of equilibrium states and show that phase transitions, which correspond to divergencies of the heat capacity, are represented geometrically as singularities of the curvature scalar. This further indicates that the curvature of the thermodynamic metric is a measure of thermodynamic interaction.

1. Introduction

The work of Hawking [1] gives rise to an extensive study of the thermodynamics of black holes. One of the most interesting aspects of such study is the notion of critical behaviour that has arisen in several contexts, for example, Hawking and Page's [2] phase transition in AdS space and the idea that the extremal limit of different black hole families might themselves be regarded as genuine critical points [3–6]. A complete explanation for the final state of the black hole after the evaporation is important, but it has not been achieved, presumably because there is not yet a full quantum gravity theory. Today, one of the strongest candidates for quantum gravity is string theory, in which coordinates of the target spacetime become noncommuting operators on a D-brane as

$$[x^{\mu}, x^{\nu}] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is an antisymmetric matrix which determines the fundamental cell discretization of spacetime in the same way as the Planck constant discretizes the phase space. It has also been shown that Lorentz invariance and unitary can be achieved by assuming $\theta^{\mu\nu} = \theta\,\mathrm{diag}(\epsilon_1, \ldots, \epsilon_{n/2})$, where n is the dimension of spacetime, and $\sqrt{\theta}$ is a constant that provides a minimum scale. A noncommutative static and spherically symmetric black hole solution whose commutative limit is the Schwarzschild metric has been found in [7–11]. The thermodynamics and evaporation process of this black hole have been studied in [12], while the entropy issue is discussed in [13, 14] and its Hawking radiation in [15].

On the other hand, the use of geometry in statistical mechanics was pioneered by Ruppeiner [16, 17] and Weinhold [18, 19], who suggested that the curvature of a metric defined on the space of parameters of a statistical mechanical theory could provide information about the phase structure. However, when these methods are applied to the study of black hole thermodynamics, some puzzling anomalies appear. A possible solution to these issues was suggested by Quevedo's geometrothermodynamics (GTD), whose starting point [20] was the observation that standard thermodynamics was invariant with respect to Legendre transformations, and an interesting aspect in this formalism is that it indicates that phase transitions occur at those points where the thermodynamic curvature is singular.

In this paper, we apply the GTD formalism to the noncommutative Schwarzschild black hole to investigate the behaviour of the thermodynamical curvature in the search of phase transitions. This noncommutative black hole has two horizons and an evaporation process that ends up in a extremal zero-temperature configuration. Thus, similar evaporation process and thermodynamical properties when compared with the behaviour of Reissner-Nordström black hole are expected as stressed in [21, 22]. Since the noncommutative Schwarzschild black hole is described by the thermodynamical variables mass, temperature, and entropy, the space of equilibrium thermodynamic states has just two dimensions, making it impossible to analyse the phase transitions structure from the curvature scalar. Therefore, we need to treat the noncommutative parameter θ as a thermodynamical variable, playing a similar role to that of the electric charge in Reissner-Nordström solution. This consideration can be justified from the point of view of the quantum fluctuations of geometry which are naturally expected to arise from the noncommutativity of spacetime.

2. Geometrothermodynamics in Brief

Let $\mathcal{T}$ be the $(2n+1)$-dimensional *thermodynamic phase space* with coordinates given by the thermodynamic potential Φ, the extensive variables E^a, and the intensive variables I^a. These coordinates will be noted as $Z^A = \{\Phi, E^a, I^a\}$ with $a = 1, \dots, n$. We define on $\mathcal{T}$ a nondegenerate metric $G = G(Z^A)$ and Gibbs 1-form $\Theta = d\Phi - \delta_{ab} I^a dE^b$. If the condition $\Theta \wedge (d\Theta)^n \neq 0$ is satisfied, the set $(\mathcal{T}, \Theta, G)$ is called a contact Riemannian manifold. Gibbs 1-form is invariant with respect to Legendre transformations, while the metric G is Legendre invariant if its functional dependence on Z^A does not change under a Legendre transformation. Following the GTD formalism, we will impose this invariance in order to guarantee that the geometric properties of G do not depend on the thermodynamic potential used for its construction. We introduce the n-dimensional subspace $\mathcal{E} \subset \mathcal{T}$ called the *space of equilibrium thermodynamic states* through the following smooth mapping:

$$\begin{gathered} \varphi : \mathcal{E} \longrightarrow \mathcal{T}, \\ (E^a) \longrightarrow (\Phi, E^a, I^a), \end{gathered} \tag{2}$$

with $\Phi = \Phi(E^a)$ and subject to the condition $\varphi^*(\Theta) = 0$ which gives the following relations:

$$d\Phi = \delta_{ab} I^a dE^b, \tag{3}$$

$$\frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b. \tag{4}$$

Equation (3) corresponds to the first law of thermodynamics, whereas (4) is usually known as the condition for thermodynamic equilibrium, that is, the intensive thermodynamic variables are dual to the extensive ones. The second law of thermodynamics is equivalent to the convexity condition on the thermodynamic potential,

$$\frac{\partial^2 \Phi}{\partial E^a \partial E^b} \geq 0. \tag{5}$$

The mapping φ implies that the equation $\Phi = \Phi(E^a)$, known as the fundamental equation, must be explicitly given and from this relation all the thermodynamical information can be derived. The potential satisfies the homogeneity condition $\Phi(\lambda E^a) = \lambda^\beta \Phi(E^a)$, with λ and β constant parameters, so it also satisfies Euler's identity as follows:

$$\beta \Phi (E^a) = \delta_{ab} I^b E^a. \tag{6}$$

Using the first law of thermodynamics, this equation becomes Gibbs-Duhem relation:

$$(1-\beta)\, \delta_{ab} I^a dE^b + \delta_{ab} E^a dI^b = 0. \tag{7}$$

A thermodynamic system is described by a thermodynamic metric G if it is invariant with respect to transformations which do not modify the contact structure of $\mathcal{T}$. In particular, G must be invariant with respect to Legendre transformations in order for GTD to describe thermodynamic properties in terms of geometric concepts. A Legendre invariant metric G induces a Legendre invariant, nondegenerate metric structure g on $\mathcal{E}$ through the pullback φ^* as $g = \varphi^*(G)$ [20].

The results of Quevedo et al. [23–25] showed that if the curvature of the thermodynamic metric is to be considered as a measure of the thermodynamic interaction, this metric should be flat only for systems with no thermodynamic interaction. Hence, phase transitions must occur at those points where the thermodynamic curvature is singular. There is a vast number of metrics on $\mathcal{T}$ that satisfy the Legendre invariance condition, and some results seem to show that the metric structure of the phase manifold determines the type of systems that can be described by a specific thermodynamic metric. For instance, a pseudo-Euclidean structure of the form

$$G = \Theta^2 + \left(\delta_{ab} E^a I^b\right) \left(\eta_{cd} dE^c dI^d\right) \tag{8}$$

with $\eta_{cd} = \mathrm{diag}(-1, 1, 1, \dots, 1)$ is Legendre invariant and induces the following metric:

$$g = \left(E^f \frac{\partial \Phi}{\partial E^f}\right) \left(\eta_{ab} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d\right), \tag{9}$$

which appears to describe thermodynamical systems characterized with second-order phase transitions.

3. The Noncommutative Schwarzschild Black Hole

In a commutative space, the mass density of a point particle is expressed as a product of its mass with the Dirac delta function, but in a noncommutative space, it is expected that such a description of point mass is not possible due to the fuzziness of space which arises as a consequence of position-position uncertainty relation. To introduce the noncommutative correction in the expression of mass density, we replace the Dirac delta function by a Gaussian distribution of minimal width $\sqrt{\theta}$,

$$\rho_\theta = \frac{M}{(4\pi\theta)^{3/2}} e^{-r^2/4\theta}, \tag{10}$$

where the noncommutativity parameter θ, which defines the minimum scale, is considered to be a small positive number. Consequently, the mass of the black hole can be determined by integrating (10) over a volume of radius r,

$$m_\theta(r) = \int_0^r 4\pi r'^2 \rho_\theta(r')\, dr' = \frac{2M}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right), \quad (11)$$

where $\gamma(3/2, r^2/4\theta)$ is the lower incomplete gamma function,

$$\gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt. \quad (12)$$

In the commutative limit, $\theta \to 0$, $\gamma(3/2, r^2/4\theta)$ becomes the usual gamma function $\Gamma(3/2)$ and $m_\theta(r) \to M$. Substituting this result in the mass term of the Schwarzschild's solution, we obtain the noncommutative Schwarzschild metric,

$$\begin{aligned} ds^2 &= -\left(1 - \frac{4M}{r\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)\right) dt^2 \\ &+ \left(1 - \frac{4M}{r\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)\right)^{-1} dr^2 + r^2 d\Omega^2. \end{aligned} \quad (13)$$

The classical Schwarzschild's metric is obtained from (13) in the limit $r/\sqrt{\theta} \to \infty$, and event horizon(s) can be found at points where $g_{00}(r_H) = 0$, which corresponds to the following condition:

$$r_H = \frac{4M}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r_H^2}{4\theta}\right). \quad (14)$$

The analysis of (14) determines that, instead of a single-event horizon, there are three different possibilities depending on a critical mass $M_0 = 1.9\sqrt{\theta}$, [9]:

(1) two distinct horizons for $M > M_0$,

(2) one degenerate horizon at $r_0 = 3.0\sqrt{\theta}$, when $M = M_0$ (corresponds to the extremal black hole),

(3) no horizon for $M < M_0$.

Equation (14) can be conveniently rewritten in terms of the gamma function as

$$r_H = 2M\left[1 - \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}, \frac{M^2}{\theta}\right)\right], \quad (15)$$

where the first term in the right hand side is the Schwarzschild radius, while the second term brings in noncommutative corrections. The area of the event horizon can be written as

$$A = 4\pi r_H^2 = 16\pi M^2\left[1 - \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}, \frac{M^2}{\theta}\right)\right]^2 \quad (16)$$

and, hence, the entropy associated with the black hole is simply

$$S = \frac{A}{4} = 4\pi M^2\left[1 - \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}, \frac{M^2}{\theta}\right)\right]^2. \quad (17)$$

The Hawking temperature may be calculated as usual, giving the temperature of Schwarzschild's black hole plus a correction term,

$$T = \frac{1}{4\pi}\left.\frac{dg_{00}}{dr}\right|_{r=r_H} = \frac{1}{4\pi r_H}\left[1 - \frac{r_H^3}{4\theta^{3/2}}\frac{e^{-r_H^2/4\theta}}{\gamma(3/2, r_H^2/4\theta)}\right]. \quad (18)$$

As is well known, in the commutative case, the temperature diverges, putting a limit on the validity of the conventional description of Hawking radiation. However, the temperature obtained in (18) includes noncommutative effects which are relevant at small distances, making T to deviate from the standard hyperbola and, instead of diverge, it reaches a maximum value at the radius $r_c \simeq 4.7\sqrt{\theta}$ or correspondingly at the mass $M_c \simeq 2.4\sqrt{\theta}$ and then quickly drops to zero temperature for $r_H = r_0$, which corresponds to the radius of the extremal black hole. This behaviour of the evaporation precession is typical of black holes with two horizons as, for example, Reissner-Nordström's solution.

From (14), we may write

$$M = \frac{\sqrt{S}}{4\gamma(3/2, S/4\pi\theta)}, \quad (19)$$

which may be considered as the fundamental thermodynamical equation $M = M(S, \theta)$ which relates the total energy of the black hole, M, with the extensive variables, entropy, and noncommutativity parameter, and from which all the thermodynamical information can be derived. The inclusion of the parameter θ as a thermodynamical variable is justified from the analogy between the noncommutative Schwarzschild's black hole and the Reissner-Nordström solution which has been commented in [21] and studied extensively in [22], where the authors showed that the noncommutativity parameter plays a similar role with the electric charge. Even more, the authors showed that the thermodynamical properties and the evaporation process are similar in both solutions.

In the geometric formulation of thermodynamics, we will choose the extensive variables as $E^a = \{S, \theta\}$ and the corresponding intensive variables as $I^a = \{T, \Psi\}$, where T is the temperature, and Ψ is the generalised variable conjugate to the state parameter θ. Therefore, the coordinates that we will use in the 5-dimensional thermodynamical space $\mathcal{T}$ are $Z^A = \{M, S, \theta, T, \Psi\}$. The contact structure of $\mathcal{T}$ is generated by the 1-form,

$$\Theta = dM - TdS - \Psi d\theta. \quad (20)$$

To obtain the induced metric in the space of equilibrium states $\mathcal{E}$, we will introduce the following smooth mapping:

$$\varphi : \{S, \theta\} \longmapsto \{M(S, \theta), S, \theta, T(S, \theta), \Psi(S, \theta)\} \quad (21)$$

along with the condition $\varphi^*(\Theta) = 0$ that corresponds to the first law $dM = TdS + \Psi d\theta$. The conjugate variables to S and θ are evaluated as

$$\begin{aligned} T &= \frac{\partial M}{\partial S} \\ &= \frac{1}{8\sqrt{S}\gamma(3/2, S/4\pi\theta)}\left[1 - \frac{1}{4}\sqrt{\left(\frac{S}{\pi\theta}\right)^3}\frac{e^{-S/4\pi\theta}}{\gamma(3/2, S/4\pi\theta)}\right], \\ \Psi &= \frac{\partial M}{\partial\theta} = \frac{S^2}{32\sqrt{\pi^3\theta^5}}\frac{e^{-S/4\pi\theta}}{\gamma^2(3/2, S/4\pi\theta)}. \end{aligned} \tag{22}$$

Using (8), $\mathcal{T}$ becomes a Riemannian manifold by defining

$$G = (dM - TdS - \Psi d\theta)^2 + (ST + \Psi\theta)(-dSdT + d\Lambda d\theta). \tag{23}$$

This metric has nonzero curvature, and its determinant is $\det[G] = (ST + \Psi\theta)^4/16$. To obtain the induced metric in the space of equilibrium states $\mathcal{E}$, we use (9), obtaining

$$g = (SM_S + \theta M_\theta)\begin{pmatrix} -M_{SS} & 0 \\ 0 & M_{\theta\theta} \end{pmatrix}, \tag{24}$$

where subscripts represent partial derivative with respect to the corresponding coordinate. Clearly, the determinant of this metric is

$$\det[g] = -M_{SS}M_{\theta\theta}(SM_S + \theta M_\theta). \tag{25}$$

4. Phase Transitions and the Curvature Scalar

Phase transitions are an interesting subject in the study of black holes thermodynamics because there is no unanimity in their definition. As is well known, ordinary thermodynamics defines phase transitions by looking for singular points in the behaviour of thermodynamical variables. For example, Davies [26, 27] showed that divergences in the heat capacity

$$C = T\frac{\partial S}{\partial T} = \frac{M_S}{M_{SS}} \tag{26}$$

indicate phase transitions. From these arguments one can expect that phase transitions occur at $M_{SS} = 0$. In geometrothermodynamics, the apparition of phase transitions is related with the divergences of the curvature scalar R in the space of equilibrium states $\mathcal{E}$. This can be understood by remembering that R always contains the determinant of the metric g in the denominator, so that zeros of $\det[g]$ could lead to curvature singularities if those zeros are not canceled by the zeros of the numerator.

The metric given by (24) has the determinant (25) which is proportional to S_{MM} and $S_{\theta\theta}$. This fact makes clear the coincidence with the divergence of the heat capacity. Even more, the curvature scalar R for the metric g has the denominator

$$D = (SM_S + \theta M_\theta)^3 M_{SS}^2 M_{\theta\theta}^2, \tag{27}$$

which makes R diverge when $S_{MM} = 0$ or $S_{\theta\theta} = 0$, whereas the numerator is a rather cumbersome expression that cannot be written in a compact form. From (19), we have

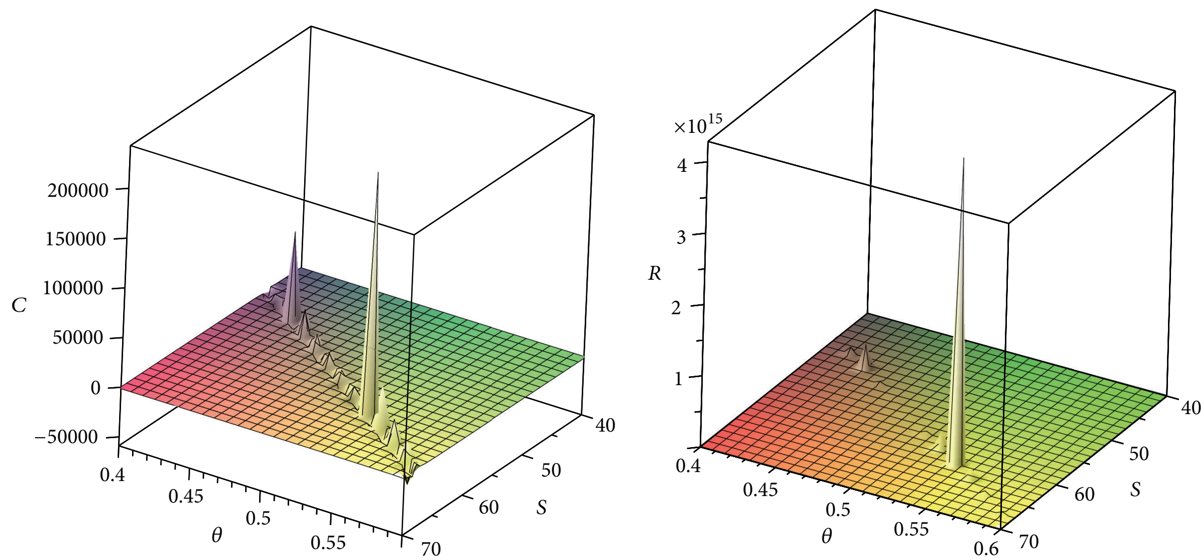
$$\begin{aligned} M_{SS} = &-\frac{1}{16\sqrt{S^3}\gamma(3/2, S/4\pi\theta)} \\ &\times\left[1 + \frac{1}{6}\sqrt{\left(\frac{S}{\pi\theta}\right)^3}\frac{e^{-S/4\pi\theta}}{\gamma(3/2, S/4\pi\theta)}\right. \\ &\quad - \frac{1}{8}\sqrt{\left(\frac{S}{\pi\theta}\right)^5}\frac{e^{-S/4\pi\theta}}{\gamma(3/2, S/4\pi\theta)} \\ &\quad \left. -\frac{1}{8}\left(\frac{S}{\pi\theta}\right)^3\frac{e^{-S/2\pi\theta}}{\gamma^2(3/2, S/4\pi\theta)}\right], \\ M_{\theta\theta} = &\frac{5S^2}{64\sqrt{\pi^3\theta^7}}\left[1 - \frac{S}{10\sqrt{\pi^2}\theta} - \frac{S\sqrt{S}}{10\sqrt{\pi^3\theta^3}}\frac{e^{-S/4\pi\theta}}{\gamma(3/2, S/4\pi\theta)}\right] \\ &\times\frac{e^{-S/4\pi\theta}}{\gamma^2(3/2, S/4\pi\theta)}. \end{aligned} \tag{28}$$

A numerical analysis shows that singularities in the curvature scalar R appear exactly at the same points where the behaviour of the heat capacity indicates the presence of phase transitions (see Figure 1). In Figure 2, we show the particular phase transition for $\theta = 0.5$ and located at the approximate value $S = 57$. For all analysed values of the noncommutativity parameter θ, it was obtained a similar behaviour, showing that, in fact, the points where phase transitions occur are characterised by curvature singularities of the thermodynamic metric.

5. Conclusion

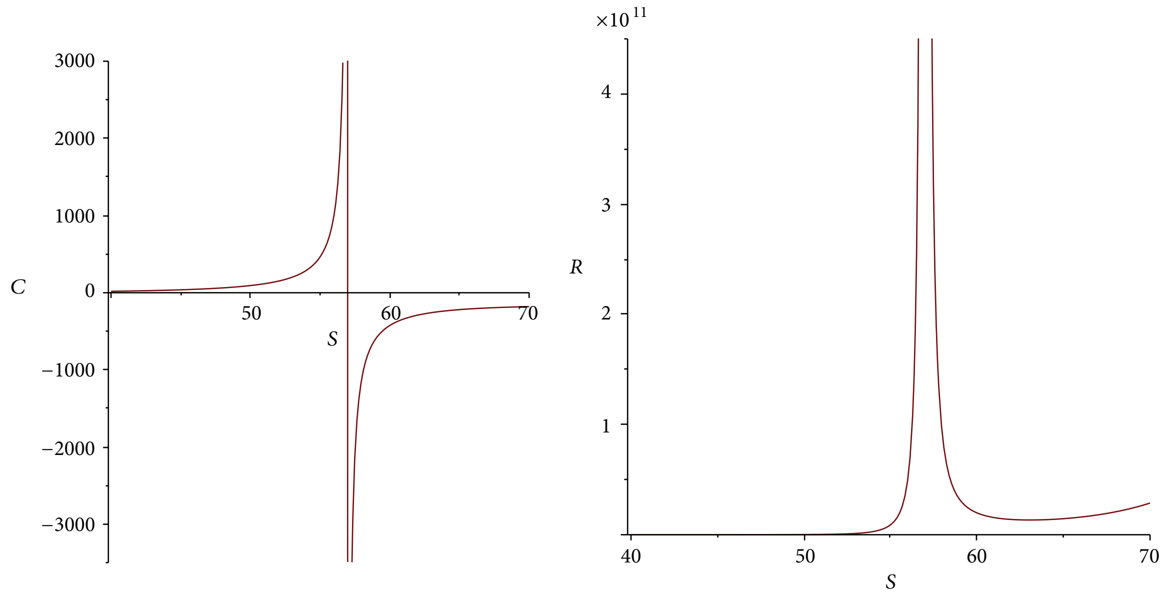
Geometrothermodynamics is a differential geometry formalism whose objective is to describe in an invariant manner the properties of thermodynamic systems using geometric concepts. In this work, we reformulated the thermodynamics of the noncommutative Schwarzschild's black hole under the GTD formalism and considered the noncommutativity parameter θ as a new thermodynamical state variable. The total mass of the black hole is interpreted as its total energy, and the formalism gives a curvature scalar that diverges exactly at the point at which the heat capacity indicates the presence of a phase transition. Thus, we conclude that the curvature of the space of thermodynamic equilibrium states can be interpreted as a measure of the thermodynamic interaction.

These results clearly confirm that the phase manifold contains information about thermodynamic systems; however, a further exploration is necessary in order to understand how the thermodynamic information is encoded in the geometrical properties. For example, it is really interesting to address the problem of describing the black hole in an isolated cavity which is bigger than its Schwarzschild



(a) (b)

Figure 1: (a) Characteristic behaviour of the heat capacity as a function of S and θ. (b) Characteristic behaviour of the thermodynamic curvature scalar R as a function of S and θ. Note that the singularities follow the same pattern in both functions, indicating the presence of phase transitions.



(a) (b)

Figure 2: (a) Characteristic behaviour of the heat capacity as a function of the entropy S. The divergence indicates the point of a phase transition. (b) Behaviour of the thermodynamic curvature scalar R as a function of the entropy S. The singularity is located at the point of the phase transition. In both figures, the noncommutativity parameter is set to $\theta = 0.5$.

radius. As is known, the asymptotically flat solution has an unstable behaviour, because of the negative heat capacity; but when introducing the cavity, the temperature is fixed at a finite spatial boundary, and it is expected that the black hole reaches the thermodynamical equilibrium with the surrounding radiation if the total energy E of the system is greater than some critical value E_c, depending on the volume of the cavity and the number of fields in the radiation [28]. It is well known that asymptotic flatness is not satisfied in reality and therefore, it is important to consider this situation in the specific case of the noncommutative black hole for which the heat capacity becomes positive for certain ranges of the horizon radius, making stable small and large black holes [22]. In a forthcoming paper, we will analyse how the GTD formalism describes this kind of system and how to implement the cavity in the geometric model.

Acknowledgment

This work was supported by the Universidad Nacional de Colombia, Hermes Project Code 13038.

References

[1] S. W. Hawking, "Particle creation by black holes," *Communications in Mathematical Physics*, vol. 43, no. 3, pp. 199–220, 1975.

[2] S. W. Hawking and D. N. Page, "Thermodynamics of black holes in anti-de Sitter space," *Communications in Mathematical Physics*, vol. 87, no. 4, pp. 577–588, 1983.

[3] J. Louko and S. N. Winters-Hilt, "Hamiltonian thermodynamics of the Reissner-Nordström-anti-de Sitter black hole," *Physical Review D*, vol. 54, no. 4, pp. 2647–2663, 1996.

[4] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, "Holography, thermodynamics, and fluctuations of charged AdS black holes," *Physical Review D*, vol. 60, no. 10, Article ID 104026, 1999.

[5] R.-G. Cai, "Critical Behavior in Black Hole Thermodynamics," *Journal of the Korean Physical Society*, vol. 33, p. S477, 1998.

[6] R.-G. Cai and J.-H. Cho, "Thermodynamic curvature of the BTZ black hole," *Physical Review D*, vol. 60, Article ID 067502, 1999.

[7] A. Smailagic and E. Spallucci, "Feynman path integral on the non-commutative plane," *Journal of Physics A*, vol. 36, no. 33, pp. L467–L471, 2003.

[8] T. G. Rizzo, "Noncommutative inspired black holes in extra dimensions," *Journal of High Energy Physics*, no. 9, 2006.

[9] P. Nicolini, A. Smailagic, and E. Spallucci, "Noncommutative geometry inspired Schwarzschild black hole," *Physics Letters B*, vol. 632, no. 4, pp. 547–551, 2006.

[10] E. Spallucci, A. Smailagic, and P. Nicolini, "Non-commutative geometry inspired higher-dimensional charged black holes," *Physics Letters B*, vol. 670, no. 4-5, pp. 449–454, 2009.

[11] P. Nicolini, "Noncommutative black holes, the final appeal to quantum gravity: a review," *International Journal of Modern Physics A*, vol. 24, pp. 1229–1308, 2009.

[12] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, "Thermodynamics and evaporation of the noncommutative black hole," *Journal of High Energy Physics*, no. 2, 2007.

[13] R. Banerjee, B. R. Majhi, and S. Samanta, "Noncommutative black hole thermodynamics," *Physical Review D*, vol. 77, no. 12, Article ID 124035, 2008.

[14] R. Banerjee, B. R. Majhi, and S. K. Modak, "Noncommutative Schwarzschild black hole and area law," *Classical and Quantum Gravity*, vol. 26, no. 8, 2009.

[15] K. Nozari and S. H. Mehdipour, "Hawking radiation as quantum tunneling from a noncommutative Schwarzschild black hole," *Classical and Quantum Gravity*, vol. 25, no. 17, Article ID 175015, 2008.

[16] G. Ruppeiner, "Thermodynamics: a Riemannian geometric model," *Physical Review A*, vol. 20, p. 1608, 1979.

[17] G. Ruppeiner, "Stability and fluctuations in black hole thermodynamics," *Physical Review D*, vol. 75, Article ID 024037, 2007.

[18] F. Weinhold, "Metric geometry of equilibrium thermodynamics," *Journal of Chemical Physics*, vol. 63, p. 2479, 1975.

[19] F. Weinhold, "Metric geometry of equilibrium thermodynamics. II. Scaling, homogeneity, and generalized Gibbs-Duhem relations," *Journal of Chemical Physics*, vol. 63, p. 2484, 1975.

[20] H. Quevedo, "Geometrothermodynamics," *Journal of Mathematical Physics*, vol. 48, no. 1, Article ID 013506, 2007.

[21] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, "Quantum cooling evaporation process in regular black holes," *Physics Letters B*, vol. 656, no. 4-5, pp. 221–225, 2007.

[22] W. Kim, E. J. Son, and M. Yoon, "Thermodynamic similarity between the noncommutative Schwarzschild black hole and the Reissner-Nordström black hole," *Journal of High Energy Physics*, no. 4, 2008.

[23] H. Quevedo, "Geometrothermodynamics of black holes," *General Relativity and Gravitation*, vol. 40, no. 5, pp. 971–984, 2008.

[24] H. Quevedo and A. Sánchez, "Geometric description of BTZ black hole thermodynamics," *Physical Review D*, vol. 79, no. 2, Article ID 024012, 2009.

[25] H. Quevedo, A. Sánchez, S. Taj, and A. Vázquez, "Phase transitions in geometrothermodynamics," *General Relativity and Gravitation*, vol. 43, no. 4, pp. 1153–1165, 2011.

[26] P. C. W. Davies, "Thermodynamic theory of black holes," *Proceedings of the Royal Society A*, vol. 353, no. 1975, pp. 499–521, 1977.

[27] P. C. W. Davies, "Thermodynamics of black holes," *Reports on Progress in Physics*, vol. 41, p. 1313, 1978.

[28] J. D. Brown, J. Creighton, and R. B. Mann, "Temperature, energy, and heat capacity of asymptotically anti-de Sitter black holes," *Physical Review D*, vol. 50, no. 10, pp. 6394–6403, 1994.