Journal of Applied Mathematics

Variational Inequalities and Vector Optimization

Guest Editors: Jian-Wen Peng, Nan-Jing Huang, Xue-Xiang Huang, and Jen-Chih Yao

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Editorial Board

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Editorial

Variational Inequalities and Vector Optimization

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As very powerful and important tools in the study of nonlinear sciences, variational inequalities and vector optimization have attracted so much attention. Over the last decades, variational inequality and vector optimization techniques have been applied extensively in such diverse fields as biology, chemistry, economics, engineering, game theory, management science, physics, and so on. The thorough study of both theory and methods about variational inequalities and vector optimization contained in the literature will help us to find new variational inequalities and vector optimization techniques for solving the practical problems.

The aim of this special issue is to present new approaches and theories for variational inequalities and vector optimization problems arising in mathematics and applied sciences.

This special issue includes 24 high-quality peer-reviewed papers that deal with different aspects of variational inequalities and vector optimization problems. These papers contain some new, novel, and innovative techniques and ideas. We hope that all the papers published in this special issue can motivate and foster further scientific works and development of the research in the area of the theory, algorithms, and applications of variational inequalities and vector optimization problems.

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Research Article

New Optimality Conditions for a Nondifferentiable Fractional Semipreinvex Programming Problem

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We study a nondifferentiable fractional programming problem as follows: $(P)\min_{x\in K} f(x)/g(x)$ subject to $x \in K \subseteq X$, $h_i(x) \leq 0$, $i = 1, 2, \ldots, m$, where K is a semiconnected subset in a locally convex topological vector space X, $f : K \to \mathbb{R}$, $g : K \to \mathbb{R}_+$ and $h_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, m$. If f, $-g$, and h_i, $i = 1, 2, \ldots, m$, are arc-directionally differentiable, semipreinvex maps with respect to a continuous map $\gamma : [0, 1] \to K \subseteq X$ satisfying $\gamma(0) = 0$ and $\gamma'(0^+) \in K$, then the necessary and sufficient conditions for optimality of (P) are established.

1. Introduction

In recent years, there has been an increasing interest in studying the develpoment of optimality conditions for nondifferentiable multiobjective programming problems. Many authors established and employed some different Kuhn and Tucker type necessary conditions or other type necessary conditions to research optimal solutions; see [1–27] and references therein. In [7], Lai and Ho used the Pareto optimality condition to investigate multiobjective programming problems for semipreinvex functions. Lai [6] had obtained the necessary and sufficient conditions for optimality programming problems with semipreinvex assumptions. Some Pareto optimality conditions are established by Lai and Lin in [8]. Lai and Szilágyi [9] studied the programming with convex set functions and proved that the alternative theorem is valid for convex set functions defined on convex subfamily S of measurable subsets in X and showed that if the system

$$\begin{aligned} f(\Omega) &\ll \theta, \\ g(\Omega) &< \theta \end{aligned} \tag{1}$$

has on solution, where θ stands for zero vector in a topological vector space, then there exists a nonzero continuous linear function $(y^*, z^*) \in C^* \times D^*$ such that

$$\langle f(\Omega), y^* \rangle + \langle g(\Omega), z^* \rangle \geq 0 \quad \forall \Omega \in S. \tag{2}$$

In this paper, we study the following optimization problem:

$$\begin{aligned} &\min_{x\in K} \frac{f(x)}{g(x)} \\ &\text{subject to} \quad x \in K \subseteq X, \quad h_i(x) \leq 0, \\ &\qquad\qquad\qquad i = 1, 2, \ldots, m, \end{aligned} \tag{P}$$

where K is a semiconnected subset in a locally convex topological vector space X, $f : K \to \mathbb{R}$, $g : K \to \mathbb{R}_+$ and $h_i : K \to (-\infty, 0]$, $i = 1, 2, \ldots, m$, are functions satisfying some suitable conditions. The purpose of this study is dealt with such constrained fractional semipreinvex programming problem. Finally, we established the Fritz John type necessary and sufficient conditions for the optimality of a fractional semipreinvex programming problem.

2. Preliminaries

Throughout this paper, we let X be a locally convex topological vector space over the real field $\mathbb{R}$. Denote $L^1(X)$ by the space of all linear operators from X into $\mathbb{R}$.

Let W be a nonempty convex subset of X. Let $f : W \to \mathbb{R}$ be differentiable at $x_0 \in K$. Then there is a linear operator $A = f'(x_0) \in L^1(X)$, such that

$$\lim_{\alpha \to 0} \frac{f((1-\alpha)x_0 + \alpha x) - f(x_0)}{\alpha} = f'(x_0)(x - x_0). \quad (3)$$

Recall that a function $f : W \to \mathbb{R}$ is called convex on W, if

$$f((1-\alpha)x_0 + \alpha x) \le (1-\alpha) f(x_0) + \alpha f(x) \quad (4)$$

or

$$\frac{f((1-\alpha)x_0 + \alpha x) - f(x_0)}{\alpha} \le f(x) - f(x_0). \quad (5)$$

If $f : W \to \mathbb{R}$ is convex and differentiable at $x_0 \in K$, then by (3) and (5), we have

$$f'(x_0)(x - x_0) \le f(x) - f(x_0). \quad (6)$$

In 1981, Hanson [13, 14] introduced a generalized convexity on X, so-called invexity; that is, $x - x_0$ is replaced by a vector $\tau(x_0, x) \in X$ in (6), or

$$f'(x_0)\tau(x_0, x) \le f(x) - f(x_0). \quad (7)$$

So an invex function is indeed a generalization of a convex differentiable function.

Definition 1 (see [6]). (1) A set $K \subseteq X$ is said to be semiconnected with respect to a given $\tau : X \times X \to \mathbb{R}$ if

$$x, y \in K,\ 0 \le \alpha \le 1 \Longrightarrow y + \alpha\tau(x, y, \alpha) \in K. \quad (8)$$

(2) A map $f : X \to \mathbb{R}$ is said to be semipreinvex on a semiconnected subset $K \subset X$ if each $(x, y, \alpha) \in K \times K \times [0, 1]$ corresponds a vector $\tau(x, y, \alpha) \in X$ such that

$$\begin{gathered} f(x + \alpha\tau(x, y, \alpha)) \le (1-\alpha) f(x) + \alpha f(y), \\ \lim_{\alpha \downarrow 0} \alpha\tau(x, y, \alpha) = \theta, \end{gathered} \quad (9)$$

where θ stands for the zero vector of X.

The following is an example of a bounded semiconnected set in $\mathbb{R}$, which is semiconnected with respect to a nontrivial τ.

Example 2. Let $A := [4, 8]$, $B := [-8, -4]$ and $K := A \cup B$ be bounded sets. Let $\tau : K \times K \times [0, 1] \to \mathbb{R}$ be defined by

$$\begin{aligned}
\tau(x, y, \alpha) &= \frac{x - y}{1 - \alpha}, \quad \text{for } (x, y, \alpha) \in A \times A \times \left[0, \frac{1}{2}\right], \\
\tau(x, y, \alpha) &= \frac{x - y}{1 - \alpha}, \quad \text{for } (x, y, \alpha) \in B \times B \times \left[0, \frac{1}{2}\right], \\
\tau(x, y, \alpha) &= \frac{-8 - y}{1 - \alpha}, \quad \text{for } (x, y, \alpha) \in A \times B \times \left[0, \frac{1}{2}\right], \\
\tau(x, y, \alpha) &= \frac{4 - y}{1 - \alpha}, \quad \text{for } (x, y, \alpha) \in B \times A \times \left[0, \frac{1}{2}\right], \\
\tau(x, y, \alpha) &= \frac{x - y}{\alpha}, \quad \text{for } (x, y, \alpha) \in A \times A \times \left[\frac{1}{2}, 1\right], \\
\tau(x, y, \alpha) &= \frac{x - y}{\alpha}, \quad \text{for } (x, y, \alpha) \in B \times B \times \left[\frac{1}{2}, 1\right], \\
\tau(x, y, \alpha) &= \frac{-8 - y}{\alpha}, \quad \text{for } (x, y, \alpha) \in A \times B \times \left[\frac{1}{2}, 1\right], \\
\tau(x, y, \alpha) &= \frac{4 - y}{\alpha}, \quad \text{for } (x, y, \alpha) \in B \times A \times \left[\frac{1}{2}, 1\right].
\end{aligned} \quad (10)$$

Then K is a bound semiconnected set with respect to τ.

Theorem 3 (see [6, Theorem 2.2]). *Let $K \subset X$ be a semiconnected subset and $f : K \to \mathbb{R}$ a semipreinvex map. Then any local minimum of f is also a global minimum of f over K.*

From the assumption in problem 9, there exists a positive number λ such that

$$\begin{gathered} \frac{f(y)}{g(y)} \ge \lambda \quad \forall y \in X, \\ f(y) - \lambda g(y) \ge 0. \end{gathered} \quad (11)$$

Consequently, we can reduce the problem 9 to an equivalent nonfractional parametric problem:

$$v(\lambda) := \min_{y \in X} (f(y) - \lambda g(y)) \ge 0, \quad (P_\lambda)$$

where $\lambda \in [0, \infty)$ is a parameter.

We will prove that the problem (P) is equivalent to the problem (P_{λ^*}) for the optimal value λ^*. The following result is our main technique to derive the necessary and sufficient optimality conditions for problem (P).

Theorem 4. *Problem (P) has an optimal solution y_0 with optimal value λ^* if and only if $v(\lambda^*) = 0$ and y_0 is an optimal solution of (P_{λ^*}).*

Proof. If y_0 is an optimal solution of (P) with optimal value λ^*, that is,

$$\lambda^* := \frac{f(y_0)}{g(y_0)} = \min_{z \in X} \frac{f(z)}{g(z)} \le \frac{f(z)}{g(z)} \quad \forall z \in X. \quad (12)$$

It follows from (12) that

$$f(z)-\lambda^* g(z)\geq 0 \quad \forall z\in X, \qquad f(y_0)-\lambda^* g(y_0)=0. \tag{13}$$

Thus, we have

$$0\leq \min_{z\in X}\left(f(z)-\lambda^* g(z)\right)\leq f(y_0)-\lambda^* g(y_0)=0. \tag{14}$$

Then, by (14), we get

$$v(\lambda^*)=\min_{z\in X}\left(f(z)-\lambda^* g(z)\right)=f(y_0)-\lambda^* g(y_0)=0. \tag{15}$$

Therefore, y_0 is an optimal solution of (P_{λ^*}) and $v(\lambda^*)=0$.

Conversely, if y_0 is an optimal solution of (P_{λ^*}) with optimal value $v(\lambda^*)=0$, then

$$f(y_0)-\lambda^* g(y_0)=\min_{z\in X}\left(f(z)-\lambda^* g(z)\right)=0. \tag{16}$$

So

$$f(z)-\lambda^* g(z)\geq 0=f(y_0)-\lambda^* g(y_0) \quad \forall z\in X. \tag{17}$$

It follows from (17) that

$$\frac{f(z)}{g(z)}\geq \lambda^* \quad \forall z\in X, \qquad \frac{f(y_0)}{g(y_0)}=\lambda^*, \tag{18}$$

and hence

$$\min_{z\in X}\frac{f(z)}{g(z)}\geq \lambda^*, \qquad \min_{z\in X}\frac{f(z)}{g(z)}\leq \frac{f(y_0)}{g(y_0)}=\lambda^*. \tag{19}$$

Therefore,

$$\min_{z\in X}\frac{f(z)}{g(z)}=\lambda^*=\frac{f(y_0)}{g(y_0)} \tag{20}$$

and we know y_0 is an optimal solution of (P) with optimal value λ^*. □

3. The Existence of the Necessary and Sufficient Conditions for Semipreinvex Functions

Definition 5 (see [6]). A mapping $f: K\subset X\to\mathbb{R}$ is said to be arcwise directionally (in short, arc-directionally) differentiable at $x_0\in K$ with respect to a continuous arc $\beta:[0,1]\to K\subset X$ if $x_0+\beta(t)\in K$ for $t\in[0,1]$ with

$$\beta(0)=\theta, \quad \beta'(0^+)=u \quad (\text{in } X), \tag{21}$$

that is, the continuous function β is differentiable from right at 0, and the limit

$$\lim_{t\downarrow 0}\frac{f(x_0+\beta(t))-f(x_0)}{t}\cong f'(x_0;u) \text{ exists}. \tag{22}$$

Note that the arc directional derivative $f'(x_0;\cdot)$ is a mapping from X into $\mathbb{R}$. Moreover, how can we make K to be a semiconnected set? Indeed, we can construct a function τ concerned with β defined as follows.

For any $x,y\in K$ and $t\in[0,1]$, we choose a vector

$$\tau(x,y,t):=\frac{\beta(t)}{t}=\frac{\beta(t)-\beta(0)}{t-0}, \tag{23}$$

then

$$\lim_{t\downarrow 0}\tau(x,y,t)=\beta'(0^+)=u, \qquad \frac{d}{dt}\left[t\tau(x,y,t)\right]\Big|_{t=0^+}=\beta'(0^+)=u. \tag{24}$$

Let $f: X\to\mathbb{R}$, $-g: X\to\mathbb{R}_-$ and $h_i: X\to\mathbb{R}_-$, $i=1,2,\ldots,m$, be semipreinvex maps on a semiconnected subset K in X. Consider a constrained programming problem as (P).

The following Fritz John type theorem is essential in this section for programming problem (P).

Theorem 6 (Necessary Optimality Condition). *Suppose that $f,-g$ and h_i, $i=1,2,\ldots,m$ are arc-directionally differentiable at $x_0\in K$ and semipreinvex on K with respect to a continuous arc β defined as in Definition 5. If x_0 minimizes locally for the semipreinvex programming problem (P), then there exist $\lambda^*\in(0,\infty)$ and $\{\gamma_i\}_{i=1}^m\subseteq[0,\infty)$ such that*

$$f'(x_0;u)-\lambda^* g'(x_0;u)+\sum_{i=1}^m\gamma_i h_i'(x_0;u)\geq 0, \tag{25}$$

where $u=\beta'(0^+)$ and

$$\sum_{i=1}^m\gamma_i h_i(x_0)=0. \tag{26}$$

Proof. By Theorem 4, the minimum solution to (P) is also a minimum to (P_{λ^*}). Then x_0 is the local minimal solution to (P_{λ^*}). By Theorem 3, we have x_0 is the global minimal solution to (P_λ). It follows that the system

$$\left[f(x)-\lambda^* g(x)\right]-\left[f(x_0)-\lambda^* g(x_0)\right]<0, \qquad h_i(x)\leq 0, \quad i=1,2,\ldots,m \tag{27}$$

has no solution in K, then we have

$$\left[f(x)-\lambda^* g(x)\right]-\left[f(x_0)-\lambda^* g(x_0)\right]+\sum_{i=1}^m\gamma_i h_i(x)<0 \tag{28}$$

has no solution in K for any $\{\gamma_i\}_{i=1}^m\subseteq[0,\infty)$. Thus for any $x\in K$,

$$\left[f(x)-\lambda^* g(x)\right]-\left[f(x_0)-\lambda^* g(x_0)\right]+\sum_{i=1}^m\gamma_i h_i(x)\geq 0 \tag{29}$$

for some $\{\gamma_i\}_{i=1}^m \subseteq [0,\infty)$. Putting $x = x_0$ in (29), we get

$$\sum_{i=1}^{m} \gamma_i h_i(x_0) \geq 0. \tag{30}$$

Since $\gamma_i \geq 0$ and $h_i(x_0) \leq 0$, it follows that

$$\sum_{i=1}^{m} \gamma_i h_i(x_0) = 0. \tag{31}$$

So (26) is proved.

As K is a semiconnected set, for any $x \in K$ and $t \in [0,1]$, we have

$$x_0 + t\tau(x_0, x, t) \in K. \tag{32}$$

For $t \neq 0$, the point $\tilde{x} = x_0 + t\tau(x_0, x, t) \neq x_0$ does not solve the system (27). So substituting $\tilde{x}$ in (29) and using the result (26), we obtain

$$\begin{aligned}
&[f(x_0 + t\tau(x_0, x, t)) - f(x_0)] \\
&\quad - \lambda^* [g(x_0 + t\tau(x_0, x, t)) - g(x_0)] \\
&\quad + \sum_{i=1}^{m} \gamma_i (h_i(x_0 + t\tau(x_0, x, t)) - h_i(x_0)) \geq 0.
\end{aligned} \tag{33}$$

Since f and g are arc-directionally differentiable with respect to β, choose a vector $\tau(x_0, x, t)$ as (23), so that (24) hold. It follows that if we divide (33) by $t \neq 0$ and take the limit as $t \downarrow 0$, then we have

$$f'(x_0; u) - \lambda^* g'(x_0; u) + \sum_{i=1}^{m} \gamma_i h_i'(x_0; u) \geq 0, \tag{34}$$

which proves (25) and the proof of theorem is completed. □

Theorem 7 (Sufficient Optimality Condition). *Let f, $-g$ and h_i, $i = 1, 2, \ldots, m$ be arc-directionally differentiable at $x_0 \in K$ and semipreinvex on K with respect to a continuous arc β defined as in Definition 5. If there exist $\lambda \in (0,\infty)$ and $\{\gamma_i\}_{i=1}^m \subseteq [0,\infty)$ satisfying*

$$f'(x_0; u) - \lambda g'(x_0; u) + \sum_{i=1}^{m} \gamma_i h_i'(x_0; u) \geq 0, \tag{35}$$

with $u = \beta'(0^+)$ and

$$\sum_{i=1}^{m} \gamma_i h_i(x_0) = 0, \tag{36}$$

then x_0 is an optimal solution for problem (P).

Proof. Suppose to the contrary that x_0 is not optimal for problem (P) and $\lambda = f(x_0)/g(x_0)$. Then $f(x_0) - \lambda g(x_0) = 0$. Therefore,

$$0 \leq \min_{x \in X} (f(x) - \lambda g(x)) \leq f(x_0) - \lambda g(x_0) = 0, \tag{37}$$

thus $v(\lambda) = \min_{x \in X}(f(x) - \lambda g(x)) = 0$.

By Theorem 4, x_0 was not optimal for problem (P_λ). Then there is an $x \in X$ such that

$$\begin{gathered}
f(x) - \lambda g(x) < f(x_0) - \lambda g(x_0), \\
h_i(x) \leq 0
\end{gathered} \tag{38}$$

for $i = 1, 2, \ldots, m$. Moreover, we have

$$[f(x) - \lambda g(x)] - [f(x_0) - \lambda g(x_0)] < 0, \tag{39}$$

$$\sum_{i=1}^{m} \gamma_i [h_i(x) - h_i(x_0)] \leq 0 \qquad \left(\text{since} \sum_{i=1}^{m} \gamma_i h_i(x_0) = 0\right) \tag{40}$$

for any $\{\gamma_i\}_{i=1}^m \subseteq [0,\infty)$. Thus

$$\begin{aligned}
&[f(x) - \lambda g(x)] - [f(x_0) - \lambda g(x_0)] \\
&\quad + \sum_{i=1}^{m} \gamma_i [h_i(x) - h_i(x_0)] < 0.
\end{aligned} \tag{41}$$

Since the semi-preinvex maps f, $-g$ and h_i, $i = 1, 2, \ldots, m$ are arc-directionally differentiable, it follows that for $(x, x_0, t) \in K \times K \times [0,1]$ there corresponds a vector $\tau(x, x_0, t) \in X$ such that

$$\begin{gathered}
f(x_0 + t\tau(x, x_0, t)) \leq (1-t) f(x_0) + tf(x), \\
-g(x_0 + t\tau(x, x_0, t)) \leq (1-t)(-g)(x_0) + t(-g)(x), \\
h_i(x_0 + t\tau(x, x_0, t)) \leq (1-t) h_i(x_0) + th_i(x),
\end{gathered} \tag{42}$$

and so

$$\begin{aligned}
\frac{f(x_0 + t\tau(x, x_0, t)) - f(x_0)}{t} &\leq f(x) - f(x_0), \\
\frac{(-g)(x_0 + t\tau(x, x_0, t)) + g(x_0)}{t} &\leq (-g)(x) + g(x_0), \\
\frac{h_i(x_0 + t\tau(x, x_0, t)) - h_i(x_0)}{t} &\leq h_i(x) - h_i(x_0).
\end{aligned} \tag{43}$$

Letting $t \downarrow 0$, we have $\lim_{t\downarrow 0}\tau(x, x_0, t) = \beta'(0^+) = u$ and the last inequalities imply

$$\begin{gathered}
f'(x_0, u) \leq f(x) - f(x_0), \\
-g'(x_0, u) \leq -[g(x) - g(x_0)], \\
h_i'(x_0, u) \leq h_i(x) - h_i(x_0).
\end{gathered} \tag{44}$$

Consequently, from (41) and (44), we obtain

$$f'(x_0; u) - \lambda g'(x_0; u) + \sum_{i=1}^{m} \gamma_i h_i'(x_0; u) < 0, \tag{45}$$

which contradicts the fact of (35). Therefore x_0 is an optimal solution of problem (P). □

Since any global minimal is a local minimal, applying Theorems 6 and 7, we can obtain the necessary and sufficient conditions for problem (P).

Theorem 8. *Suppose that $f, -g$ and h_i, $i = 1, 2, \ldots, m$ are arc-directionally differentiable at at $x_0 \in K$ and semi-preinvex on K with respect to a continuous arc β defined as in Definition 5. If x_0 minimizes globally for the semi-preinvex programming problem (P) if and only if there exists $(\lambda, \gamma_i) \in \mathbb{R}^+ \times (\mathbb{R}^+ \cup \{0\})$, $i = 1, 2, \ldots, m$, such that*

$$f'(x_0; u) - \lambda g'(x_0; u) + \sum_{i=1}^{m} \gamma_i h_i'(x_0; u) \geq 0, \tag{46}$$

where $u = \beta'(0^+)$ and

$$\sum_{i=1}^{m} \gamma_i h_i(x_0) = 0. \tag{47}$$

Remark 9. Our results also hold for preinvex functions.

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References

[1] B. D. Craven, "Invex functions and constrained local minima," *Bulletin of the Australian Mathematical Society*, vol. 24, no. 3, pp. 357–366, 1981.

[2] B. D. Craven and B. M. Glover, "Invex functions and duality," *Australian Mathematical Society*, vol. 39, no. 1, pp. 1–20, 1985.

[3] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, NY, USA, 1983.

[4] F. H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, NY, USA, 2nd edition, 1983.

[5] H. Dietrich, "On the convexification procedure for nonconvex and nonsmooth infinite-dimensional optimization problems," *Journal of Mathematical Analysis and Applications*, vol. 161, no. 1, pp. 28–34, 1991.

[6] H.-C. Lai, "Optimality conditions for semi-preinvex programming," *Taiwanese Journal of Mathematics*, vol. 1, no. 4, pp. 389–404, 1997.

[7] H. C. Lai and C. P. Ho, "Duality theorem of nondifferentiable convex multiobjective programming," *Journal of Optimization Theory and Applications*, vol. 50, no. 3, pp. 407–420, 1986.

[8] H.-C. Lai and L.-J. Lin, "Moreau-Rockafellar type theorem for convex set functions," *Journal of Mathematical Analysis and Applications*, vol. 132, no. 2, pp. 558–571, 1988.

[9] H. C. Lai and P. Szilágyi, "Alternative theorems and saddlepoint results for convex programming problems of set functions with values in ordered vector spaces," *Acta Mathematica Hungarica*, vol. 63, no. 3, pp. 231–241, 1994.

[10] H. C. Lai and J. C. Liu, "Minimax fractional programming involving generalised invex functions," *The ANZIAM Journal*, vol. 44, no. 3, pp. 339–354, 2003.

[11] J.-C. Chen and H.-C. Lai, "Fractional programming for variational problem with F, ρ, θ-invexity," *Journal of Nonlinear and Convex Analysis*, vol. 4, no. 1, pp. 25–41, 2003.

[12] J. C. Liu, "Optimality and duality for generalized fractional programming involving nonsmooth (F, ρ)-convex functions," *Computers and Mathematics with Applications*, vol. 42, pp. 437–446, 1990.

[13] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

[14] M. A. Hanson and B. Mond, "Necessary and sufficient conditions in constrained optimization," *Mathematical Programming*, vol. 37, no. 1, pp. 51–58, 1987.

[15] S. K. Mishra and G. Giorgi, *Invexity and Optimization*, vol. 88 of *Nonconvex Optimization and Its Applications*, Springer, Berlin, Germnay, 2008.

[16] S. K. Mishra, S. Wang, and K. K. Lai, *V-Invex Functions and Vector Optimization*, vol. 14 of *Springer Optimization and Its Applications*, Springer, New York, NY, USA, 2008.

[17] S. K. Mishra, S.-Y. Wang, and K. K. Lai, *Generalized Convexity and Vector Optimization*, vol. 90 of *Nonconvex Optimization and Its Applications*, Springer, Berlin, Germnay, 2009.

[18] S. K. Mishra and V. Laha, "On approximately star-shaped functions and approximate vector variational inequalities," *Journal of Optimization Theory and Applications*. In press.

[19] S. K. Mishra and V. Laha, "On V-r-invexity and vector variational Inequalities," *Filomat*, vol. 26, no. 5, pp. 1065–1073, 2012.

[20] S. K. Mishra and B. B. Upadhyay, "Nonsmooth minimax fractional programming involving η-pseudolinear functions," *Optimization*. In press.

[21] S. K. Mishra and M. Jaiswal, "Optimality conditions and duality for nondifferentiable multiobjective semi-infinite programming," *Vietnam Journal of Mathematics*, vol. 40, pp. 331–343, 2012.

[22] N. G. Rueda and M. A. Hanson, "Optimality criteria in mathematical programming involving generalized invexity," *Journal of Mathematical Analysis and Applications*, vol. 130, no. 2, pp. 375–385, 1988.

[23] T. W. Reiland, "Nonsmooth invexity," *Bulletin of the Australian Mathematical Society*, vol. 12, pp. 437–446, 1990.

[24] T. Weir, "Programming with semilocally convex functions," *Journal of Mathematical Analysis and Applications*, vol. 168, no. 1, pp. 1–12, 1992.

[25] T. Weir and V. Jeyakumar, "A class of nonconvex functions and mathematical programming," *Bulletin of the Australian Mathematical Society*, vol. 38, no. 2, pp. 177–189, 1988.

[26] T. Antczak, "(p, r)-invex sets and functions," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 355–379, 2001.

[27] Y. L. Ye, "D-invexity and optimality conditions," *Journal of Mathematical Analysis and Applications*, vol. 162, no. 1, pp. 242–249, 1991.

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Research Article

Weighted Wilcoxon-Type Rank Test for Interval Censored Data

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Interval censored (IC) failure time data are often observed in medical follow-up studies and clinical trials where subjects can only be followed periodically, and the failure time can only be known to lie in an interval. In this paper, we propose a weighted Wilcoxon-type rank test for the problem of comparing two IC samples. Under a very general sampling technique developed by Fay (1999), the mean and variance of the test statistics under the null hypothesis can be derived. Through simulation studies, we find that the performance of the proposed test is better than that of the two existing Wilcoxon-type rank tests proposed by Mantel (1967) and R. Peto and J. Peto (1972). The proposed test is illustrated by means of an example involving patients in AIDS cohort studies.

1. Introduction

Interval censored (IC) failure time data often arise from medical studies such as AIDS cohort studies and leukemic blood cancer follow-up studies. In these studies, patients were divided into two groups according to different treatments. For example, in leukemic cancer studies, one group of the patients was treated with radiotherapy alone, and the other group of patients was treated with initial radiotherapy along with adjuvant chemotherapy. The two groups of patients were examined every month, and the failure time of interest is the time until the appearance of leukemia retraction; the object is to test the difference of the failure times between the two treatments. Some of the patients missed some successive scheduled examinations and came back later with a changed clinical status, and they contributed IC observations. For our convenience, we assume that in such a medical study, the underlying survival function can be either discrete or continuous, and there are only finitely many scheduled examination times. IC data only provide partial information about the lifetime of the subject, and the data is one kind of incomplete data. To deal with such incomplete data, Turnbull [1] introduced a self-consistent algorithm to compute the maximum likelihood estimate of the survival function for arbitrarily censored and truncated data. For IC data, there have been some related studies in the literature as well. For example, Mantel [2] extends Gehan's [3, 4] generalized Wilcoxon [5] test to interval censored data, and R. Peto and J. Peto [6] also develop a different version. Sun [7] applied Turnbull's algorithm to estimate the number of failures and risks of IC data and then propose a log-rank type test.

Fay [8], Sun [7], Zhao and Sun [9], Sun et al. [10], and Huang et al. [11] extend the log-rank test to interval censored data. Petroni and Wolfe [12] and Lim and Sun [13] generalize Pepe and Fleming's [14] weighted Kaplan-Meier (WKM) [15] test to interval censored data.

For the purpose of comparing the power of the test statistics, Fay [8] proposed a model for generating interval censored observation. A similar selection scheme can also be seen in the Urn model of Lee [16] and mixed cased model of Schick and Yu [17]. In this paper, we propose a Wilcoxon-type weighted rank test to compare with the existing two Wilcoxon-type rank tests proposed by Mantel [2] and R. Peto and J. Peto [6]. We restrict ourselves to the Wilcoxon-type rank tests because these tests are simple to use and have the robustness property that their powers are fairly stable under different lifetime distributions.

This paper is organized as follows. In Section 2, we review the Turnbull's [1] algorithm and introduce Fay's [8] selection model for generating interval censored data. This

Table 1: The probability of selected interval.

True value of X	Selected interval	Probability
	(0,1]	$p_1 a_1$
1	(0,2]	$p_1(1-a_1)a_2$
	(0,3]	$p_1(1-a_1)(1-a_2)$
	(1,2]	$p_2 a_1 a_2$
2	(0,2]	$p_2(1-a_1)a_2$
	(1,3]	$p_2 a_1(1-a_2)$
	(0,3]	$p_2(1-a_1)(1-a_2)$
	(2,3]	$p_3 a_2$
3	(1,3]	$p_3 a_1(1-a_2)$
	(0,3]	$p_3 a_1(1-a_1)(1-a_2)$

selection model can be extended to a more general one, and the consistency property can be found in Schick and Yu [17]. In Section 3, we introduce Mantel's [2] and R. Peto and J. Peto's [6] generalized Wilcoxon-type rank tests and propose our weighted rank test. In Section 4, a simulation study is conducted to compare the performance of the three tests under different configurations. Finally, an application to AIDS cohort study is presented in Section 5.

2. Data Treatment

Assume that X is the lifetime random variable of a survival study, measured in discrete units and taking values $0 = x_0 < x_1 < x_2 < \cdots < x_m$. Let $U = \{(x_i, x_j], 0 \le i < j \le m\}$ be the collection of all $m(m+1)/2$ admissible intervals, and define $p_j = P(X = x_j)$, where $\sum_{j=1}^{m} p_j = 1$, so that $F(x) = \sum_{x_j \le x} p_j$, and $S(x) = \sum_{x_j > x} p_j$. Note that the observed failure time data in a clinical trial can be discretized if the underlying variable is continuous.

2.1. Turnbull's Algorithm. Suppose that there is a sample of n i.i.d. observations $(X_L^i, X_R^i]$ of X, $i = 1, 2, \ldots, n$. Here, $(X_L^i, X_R^i]$ is the IC observation of the ith individual in the sample, where $X_L^i, X_R^i \in \{x_0, x_1, x_2, \ldots, x_m\}$, and $X_L^i < X_R^i$. The case $X_R^i = x_m$ is to denote that the failure time of the ith subject occurs after the last examination time x_{m-1}. Turnbull [1] proposed an algorithm to estimate the unknown probabilities $p = (p_1, p_2, \ldots, p_m)$. The algorithm can be described by the following four steps.

Step 1. Start with initial values $p^{(0)} = (p_1^{(0)}, p_2^{(0)}, \ldots, p_m^{(0)})$.

Step 2. Obtain improved estimates $p_j^{(1)}$ by setting

$$p_j^{(1)} = \frac{1}{n} \sum_{i=1}^{n} \frac{\alpha_j^i p_j^{(0)}}{\sum_{l=1}^{m} \alpha_l^i p_l^{(0)}}, \quad j = 1, 2, \ldots, m, \tag{1}$$

$$\text{where } \alpha_j^i = I\left\{x_j \in \left(X_L^i, X_R^i\right]\right\}.$$

Step 3. Return to Step 1 with $p^{(1)}$ replacing $p^{(0)}$.

Step 4. Stop when the required accuracy has been achieved.

Table 2: Selection probability $Q(I)$ for all admissible intervals.

Interval I	Probability $Q(I)$
(0,1]	$p_1 a_1$
(1,2]	$p_2 a_1 a_2$
(2,3]	$p_3 a_2$
(0,2]	$(p_1 + p_2)(1-a_1)a_2$
(1,3]	$(p_2 + p_3)a_1(1-a_2)$
(0,3]	$(p_1 + p_2 + p_3)(1-a_1)(1-a_2)$

The algorithm is simple and converges fairly rapidly. The estimate $\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m)$ yielded from the iteration is in fact the unique maximum likelihood estimate of $p = (p_1, p_2, \ldots, p_m)$ and is a self-consistent estimate.

2.2. Return Probability Model. To comply with the periodical clinical inspection, Fay [8] proposed a simulation model for generating IC data. He assumed that the probability for a patient to return to the clinic for inspection at time points $x_1, x_2, \ldots, x_{m-1}$ are i.i.d. Bernulli random variables $A_1, A_2, \ldots, A_{m-1}$; that is, $P(A_i = 1) = q$, $P(A_i = 0) = 1 - q$, $0 < q < 1$, $i = 1, 2, \ldots, m-1$. $A_i = 1$ means that the patient returned to the clinic at the inspection time x_i, and $A_i = 0$ means that the patient missed the inspection. In our model, we always assume that $A_m = 1$. The failure time X is independent of $(A_1, A_2, \ldots, A_{m-1})$, and the observable random interval is

$$\begin{aligned} (X_L, X_R] = \left(x_{s_j}, x_{t_j}\right], \quad \text{where } & s_j = \max_l \{0 \le l < j : A_l = 1\}, \\ & t_j = \min_l \{j \le l \le m : A_l = 1\}, \\ & x_{s_j} < X \le x_{t_j}. \end{aligned} \tag{2}$$

2.2.1. Model Consistency. Under Fay's [8] selection model, the consistency property has been proved. This selection model can be generalized to the case that the return probability at each examination time point may be different; say that $P(A_i = 1) = a_i$, $i = 1, 2, \ldots, m$. To demonstrate the generalized return model, we set $m = 3$ and $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. The selection probabilities for all admissible intervals are shown in Tables 1 and 2.

It is not difficult to see that the selection probability of the interval $I = (x_u, x_v]$ is

$$\begin{gathered} Q\{I\} = P(I)\left[a_u \times \prod_{i=u+1}^{v-1} (1 - a_i) \times a_v\right], \\ u = 0, 1, 2, \ldots, v-1, \quad v = 1, 2, \ldots, m-1, \end{gathered} \tag{3}$$

$$\begin{aligned} & Q\{(x_u, x_m]\} \\ & = P((x_u, x_m])\left[a_u \times \prod_{i=u+1}^{m-1} (1 - a_i)\right], \quad 0 \le u \le m-1, \end{aligned} \tag{4}$$

where $P(I) = (p_{u+1} + p_{u+2} + \cdots + p_v)$, $x_0 = 0$, and $a_0 = 1$. For instance, the interval $(0, 2]$ may be selected under two possibilities. First, the true value of X is $X = 1$, and the patient who missed the inspection at $x_1 = 1$ then goes to inspection at $x_2 = 2$; in this case, the interval is selected with probability $p_1(1-a_1)a_2$. Second, the true value of X is $X = 2$, and the patient missed the inspection at $x_1 = 1$ then goes to inspection at $x_2 = 2$; in this case, the interval is selected with probability $p_2(1-a_1)a_2$, and therefore $Q\{(0,2]\} = (p_1+p_2)(1-a_1)a_2$.

The generalized return probability model can be viewed as a special case of the mixed case model in Schick and Yu [17]; under very mild conditions, the estimate of $p = (p_1, p_2, \ldots, p_m)$ computed by Turnbull's algorithm is still consistent.

3. Wilcoxon-Type Rank Tests for Interval Censored Data

Two-sample Wilcoxon rank test is a well-known method to test whether two samples of exact data come from the same population. The method is constructed by ranking the pooled samples and giving an appropriate rank to each observation. However, this ranking technique is in general not admissible for intervals. In this section, we will discuss how to generalize the ranking technique and then propose a Wilcoxon-type rank test for IC data to compare with two existing rank tests proposed by Mantel [2] and R. Peto and J. Peto [6]. Suppose that two samples of IC data for X and Y are, respectively, $(X_L^i, X_R^i]$, $i = 1, 2, \ldots, n_1$ and $(Y_L^i, Y_R^i]$, $i = 1, 2, \ldots, n_2$. To test whether these two samples come from the same population is equivalent to testing the equality of survival functions $S_X(t)$ and $S_Y(t)$, for all $t \geq 0$; that is,

$$H_0 : S_X(t) = S_Y(t), \quad \forall t \geq 0. \tag{5}$$

3.1. Mantel's Test. Mantel [2] extended Gehan's [3, 4] generalized Wilcoxon test to interval censored data by defining the score of the kth observation as the number of observations that are definitely greater than the kth observation minus the number of observations that are definitely less than the kth observation. He proposed the test statistic

$$\begin{gathered} W = \sum_{k=1}^{n_1} V_k, \quad \text{where } V_k = \sum_{h=1}^{n_1+n_2} V_{kh}, \\ V_{kh} = \begin{cases} 1 & \text{if we know for sure obs-}k > \text{obs-}h, \\ -1 & \text{if we know for sure obs-}k < \text{obs-}h, \\ 0 & \text{if not sure.} \end{cases} \end{gathered} \tag{6}$$

Under H_0, the test statistic is approximately normal distributed with mean 0 and variance

$$\operatorname{Var}(W) = n_1 n_2 \sum_{k=1}^{n_1+n_2} \frac{V_k^2}{(n_1+n_2)(n_1+n_2-1)}. \tag{7}$$

3.2. R. Peto and J. Peto's Test. Different from the Mantel's generalized version, R. Peto and J. Peto [6] defined the score of the ith observation as

$$U_i = \frac{f(\widehat{S}(X_L^i)) - f(\widehat{S}(X_R^i))}{\widehat{S}(X_L^i) - \widehat{S}(X_R^i)}, \tag{8}$$

where $\widehat{S}$ is the estimated survival function, $f(y) = y^2 - y$; hence, $U_i = \widehat{S}(X_L^i) + \widehat{S}(X_R^i) - 1$. They proposed the test statistic

$$\begin{aligned} Z^2 &= \frac{(Y_1^2/n_1 + Y_2^2/n_2)}{s^2}, \quad \text{where } Y_1 = \sum_{i=1}^{n_1} U_i, \\ Y_2 &= \sum_{i=n_1+1}^{n_1+n_2} U_i, \\ s^2 &= \frac{\sum_{i=1}^{n_1+n_2} U_i^2}{(n_1+n_2-1)}. \end{aligned} \tag{9}$$

Under H_0, the test statistic Z^2 is approximately distributed as χ_1^2.

3.3. Our Proposed Wilcoxon-Type Weighted Rank Test. To transform an IC data to exact, we first assign each inspection time x_i a primary rank R_i; for instance, $R_i = i$. Rewrite any observation, say $(X_L^j, X_R^j]$, as $(X_L^j, X_R^j] = (x_{u^{(j)}}, x_{v^{(j)}}]$, where $x_{u^{(j)}}, x_{v^{(j)}} \in \{0, x_1, x_2, \ldots, x_m\}$, and $x_{u^{(j)}} < x_{v^{(j)}}$. Then, we associate the observation $(X_L^j, X_R^j]$ with the weighted rank

$$\operatorname{rank}((X_L^j, X_R^j]) = \sum_{l=u^{(j)}+1}^{v^{(j)}} \frac{p_l}{p_{u^{(j)}+1} + \cdots + p_{v^{(j)}}} R_l. \tag{10}$$

Let W_1, W_2 be, respectively, the average weighted rank of the X and Y samples, so that

$$\begin{aligned} W_1 &= \frac{1}{n_1} \sum_{i=1}^{n_1} \operatorname{rank}((X_L^i, X_R^i]) \\ &= \frac{1}{n_1} \sum_{i=1}^{n_1} \left(\sum_{l=u^{(i)}+1}^{v^{(i)}} \frac{p_l}{p_{u^{(i)}+1} + \cdots + p_{v^{(i)}}} R_l \right), \\ W_2 &= \frac{1}{n_2} \sum_{j=1}^{n_2} \operatorname{rank}((Y_L^j, Y_R^j]) \\ &= \frac{1}{n_2} \sum_{j=1}^{n_2} \left(\sum_{l=u^{(j)}+1}^{v^{(j)}} \frac{p_l}{p_{u^{(j)}+1} + \cdots + p_{v^{(j)}}} R_l \right). \end{aligned} \tag{11}$$

To test whether two IC samples come from the same population, we propose the test statistic

$$\text{W.R.T} = \frac{W_1 - W_2}{\sqrt{\operatorname{Var}(W_1) + \operatorname{Var}(W_2)}}. \tag{12}$$

Under H_0, the central limit theorem implies that W.R.T is approximately distributed as a standard normal random variable. However, the mean and variance of W_1 and W_2 may depend on the probability space where they are defined; it means, different selection probability for IC intervals in (4) leads to different mean and variance of W_1 and W_2. We therefore only consider the selection model of Fay defined in Section 2.2. In this model, the selection probability of an IC interval is in one of the following categories:

$$\text{(i)}\quad Q\{(0, x_r]\} = \sum_{j=1}^{r} p_j q(1-q)^{r-1}, \quad 1 \le r < m, \tag{13}$$

$$\text{(ii)}\quad Q\{(0, x_m]\} = \sum_{j=1}^{m} p_j (1-q)^{m-1}, \tag{14}$$

$$\text{(iii)}\quad Q\{(x_u, x_v]\} = \sum_{j=u+1}^{v} p_j q^2 (1-q)^{r-1}, \quad 1 \le u < v < m, \tag{15}$$

$$\text{(iv)}\quad Q\{(x_u, x_m]\} = \sum_{j=u+1}^{m} p_j q (1-q)^{m-u-1}, \quad 1 \le u < m. \tag{16}$$

Consider the probability space $(U, 2^U, Q)$, where the probability measure Q is defined in Section 2. To compute the variance of W_1 and W_2, we define a random variable Z on this space by assigning value $Z\{(x_u, x_v]\}$ to the interval $(x_u, x_v]$ in U, where

$$Z\{(x_u, x_v]\} = \sum_{l=u+1}^{v} \frac{p_l}{p_{u+1} + \cdots + p_v} R_l, \quad 0 \le u < v \le m. \tag{17}$$

The value $Z\{(x_u, x_v]\}$ can be viewed as the weighted rank of $(x_u, x_v]$. If R_l, $l = 1, 2, \ldots, m$ are chosen as in the Wilcoxon test for exact data, then our proposed test statistic W.R.T is a Wilcoxon-type weighted rank test. Under this probability space, the expectation $E(Z)$ can be simplified as in the following theorem.

Theorem 1. *Suppose that Z is the random variable defined on the probability space $(U, 2^U, Q)$ according to (17). Then, the expectation of Z, $E(Z)$, can be simplified as*

$$E(Z) = \sum_{l=1}^{m} p_l R_l, \tag{18}$$

which is independent of the choice of q.

Proof. It is obvious that $E(Z)$ can be written as $E(Z) = \sum_{l=1}^{m} b_l p_l R_l$, where the coefficients b_l, $l = 1, 2, \ldots, m$ are to be determined. The theorem is, hence, proved if we can show that all the coefficients b_l are ones.

Consider b_1 first. An interval $(x_u, x_v]$ contributes $p_1 R_1$ in $E(Z)$ if and only if it contains the point x_1. Therefore, it must be of the form $(0, x_v]$, $v = 1, 2, \ldots, m$. For intervals $(0, x_v]$, $1 \le v \le m-1$, the probabilities $Q\{(0, x_v]\}$ are defined in (13). For interval $(0, x_m]$, the probability $Q\{(0, x_m]\}$ is defined in (14). Therefore, the coefficient b_1 is

$$\begin{aligned} b_1 &= \sum_{v=1}^{m-1} q(1-q)^{v-1} + (1-q)^{m-1} \\ &= q\frac{1-(1-q)^{m-1}}{1-(1-q)} + (1-q)^{m-1} = 1. \end{aligned} \tag{19}$$

Next, consider the coefficient b_j for $1 < j \le m-1$. An interval contributes $p_j R_j$ if and only if it contains the point x_j. Therefore, it must be of the form $(x_u, x_v]$, where $0 \le u < j \le v \le m$. It is necessary to study the contribution of the interval $(x_u, x_v]$ to b_j in four different categories.

(i) $u = 0, v \le m-1$.

By (13), this category contributes $\sum_{v=j}^{m-1} q(1-q)^{v-1}$.

(ii) $u = 0, v = m$.

By (14), the interval $(0, x_m]$ contributes $(1-q)^{m-1}$.

(iii) $1 \le u < v \le m-1$.

By (15), this category contributes $\sum_{u=1}^{j-1}\sum_{v=j}^{m-1} q^2(1-q)^{v-u-1}$.

(iv) $u \ge 1, v = m$.

By (16), this category contributes $\sum_{u=1}^{j-1} q(1-q)^{m-u-1}$. Consequently, the coefficient of b_j is

$$\begin{aligned} b_j &= \sum_{v=j}^{m-1} q(1-q)^{v-1} + (1-q)^{m-1} \\ &\quad + \sum_{u=1}^{j-1}\sum_{v=j}^{m-1} q^2(1-q)^{v-u-1} + \sum_{u=1}^{j-1} q(1-q)^{m-u-1} \\ &= q\frac{(1-q)^{j-1}\left[1-(1-q)^{m-j}\right]}{1-(1-q)} + (1-q)^{m-1} \\ &\quad + \sum_{u=1}^{j-1} q\left(q\frac{(1-q)^{j-u-1}\left[1-(1-q)^{m-j}\right]}{1-(1-q)}\right) \\ &\quad + \sum_{u=1}^{j-1} q(1-q)^{m-u-1} \\ &= (1-q)^{j-1} - (1-q)^{m-1} + (1-q)^{m-1} \\ &\quad + \sum_{u=1}^{j-1} q(1-q)^{j-u-1} \\ &\quad - \sum_{u=1}^{j-1} q(1-q)^{m-u-1} + \sum_{u=1}^{j-1} q(1-q)^{m-u-1} \end{aligned}$$

Table 3: The mean, sample variance, and sample deviation of $\widehat{q}$.

n	q	0.8	0.5	0.3
	Estimate	0.8001	0.5029	0.3024
50	Variance	0.0020	0.0021	0.0012
	Std.	0.0448	0.0461	0.0341
	Estimate	0.8039	0.5036	0.3012
100	Variance	0.0010	0.001	0.0005
	Std.	0.0320	0.0312	0.0233
	Estimate	0.8009	0.4977	0.3033
150	Variance	0.0005	0.0008	0.0004
	Std.	0.0225	0.0277	0.0207

$$\begin{aligned} &= (1-q)^{j-1} + \left(1 - (1-q)^{j-1}\right) \\ &= 1. \end{aligned} \tag{20}$$

Finally, the proof for the case $j = m$ is

$$\begin{aligned} b_m &= \sum_{u=1}^{m-1} q(1-q)^{m-u-1} + (1-q)^{m-1} \\ &= 1 - (1-q)^{m-1} + (1-q)^{m-1} = 1. \end{aligned} \tag{21}$$

□

The variance of Z, $\mathrm{Var}(Z)$, is

$$\begin{aligned} \mathrm{Var}(Z) &= E\left(Z^2\right) - E^2(Z) \\ &= \sum_{i=1}^{m(m+1)/2} Q(I_i) R^2(I_i) - E^2(Z), \end{aligned} \tag{22}$$

where $Q(I_i)$ and $R(I_i)$ are the selected probability and the weighted rank of the ith admissible interval of I_i, respectively, $I_i \in U$.

Consider the formulas (13)–(16), the selection probability $Q(I)$ depends on $p = (p_1, p_2, \dots, p_m)$ and q; therefore, the likelihood function can be written as

$$L(p_1, p_2, \dots, p_m, q) = P(p_1, p_2, \dots, p_m) G(q), \tag{23}$$

where $G(q) = q^{k_1}(1-q)^{k_2}$, k_1 and k_2 are positive integers determined by the sample. Since the probability $p = (p_1, p_2, \dots, p_m)$ can be estimated by Turnbull's [1] algorithm discussed in Section 2.2, and q can also be estimated by $k_1/(k_1 + k_2)$ trivially.

For demonstration, we set $m = 6$, inspection times $x_i = i$, $i = 1, 2, \dots, 6$, and the true lifetime X is exponentially distributed with $\lambda = 1/3$. For different sample sizes $n = 50$, 100, and 150, different return probabilities of inspection $q = 0.8$, 0.5, and 0.3, and simulation with 100 replications, Table 3 presents the estimates of q and sample variance and sample deviation of $\widehat{q}$. To show the normality of W.R.T, we assume that the two populations (sample size $n_1 = n_2 = 30$) are coming from the same distribution exponential (1/5).

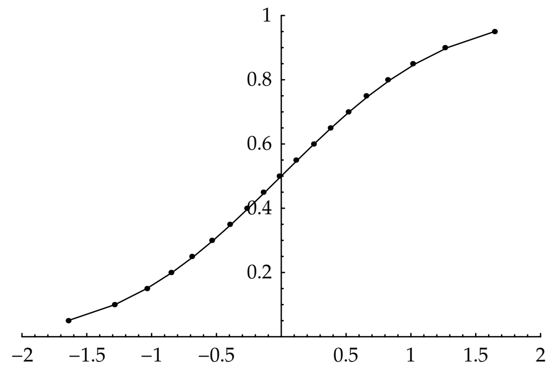

Figure 1: CDF of standard normal and simulation result of W.R.T. Line: standard normal. Point: simulation result of W.R.T ($q = 0.5$).

By simulation with 10000 replications and different return probabilities of inspection $q = 0.8$, 0.5, and 0.3, Table 4 presents the quantiles of W.R.T and $N(0, 1)$. Figure 1 shows the CDF plots of $N(0, 1)$ and W.R.T with $q = 0.5$.

4. Simulation Study

In this section, we carry out simulation studies to compare the performance of W.R.T test with Mantel's [2] and Peto's [6] tests. In the study, we assume that the failure time random variable is distributed as exponential, total sample sizes are $n = 100$ and 200, and each sample has $(n/2)$ subjects. The interval censored data are generated by the following four steps.

Step 1. Generate a failure time t_j from some distribution.

Step 2. Create a 0, 1 sequence $A = \{A_0, A_1, A_2, \dots, A_m\}$ with probabilities $P(A_i = 1) = q$, $i = 1, 2, \dots, m-1$, and $P(A_0 = 1) = P(A_m = 1) = 1$.

Step 3. The observation is $(a, b]$, if $a < t_j \le b$, $A_a = A_b = 1$, and $A_{a+1} = A_{a+2} = \cdots = A_{b-1} = 0$.

Step 4. Repeat Step 1 to Step 3 for n times.

Table 4: The quantiles of W.R.T and $N(0, 1)$.

Quantile	Normal (0,1)	W.R.T 0.8	W.R.T 0.5	W.R.T 0.3
0.05	−1.6449	−1.6757	−1.6421	−1.6786
0.10	−1.2816	−1.3083	−1.2855	−1.3064
0.15	−1.0364	−1.0543	−1.0354	−1.0700
0.20	−0.8416	−0.8647	−0.8494	−0.8649
0.25	−0.6745	−0.6874	−0.6892	−0.6877
0.30	−0.5244	−0.5326	−0.5351	−0.5338
0.35	−0.3853	−0.3883	−0.3966	−0.3946
0.40	−0.2533	−0.2623	−0.2651	−0.2663
0.45	−0.1257	−0.1247	−0.1379	−0.1314
0.50	0	−0.0007	−0.0152	0.0002
0.55	0.1257	0.1296	0.1136	0.1306
0.60	0.2533	0.2582	0.2503	0.2604
0.65	0.3853	0.3879	0.3789	0.4012
0.70	0.5244	0.5336	0.5176	0.5501
0.75	0.6745	0.6814	0.6549	0.6954
0.80	0.8416	0.8535	0.8215	0.8611
0.85	1.0364	1.0508	1.0146	1.0734
0.90	1.2816	1.2758	1.2628	1.3346
0.95	1.6449	1.6368	1.6458	1.6747

We consider three return probabilities, $q = 0.8, 0.5$, and 0.3, two sets of inspection time points, $m = 6, 10$, and 1000 replications at significance level 0.05.

In the case of $m = 6$, 6 return points, we set the hazards 1/3 for population 1 and $1/3e^{\beta}$ for population 2. Figure 2 shows the density plot of exponential distribution with $\beta = -0.4$, −0.2, 0, 0.2, 0.4. In the case of $m = 10$, 10 return points, we set the hazards 1/4 for population 1 and $1/4e^{\beta}$ for population 2. Figure 3 shows the density plot of exponential distribution with $\beta = -0.6, -0.3, 0, 0.3, 0.6$. Tables 5 and 6 present the powers of the three tests with sample size $n = 100$ and 200. Simulation result shows that when the failure times come from the exponential distribution, our proposed test W.R.T is the most powerful.

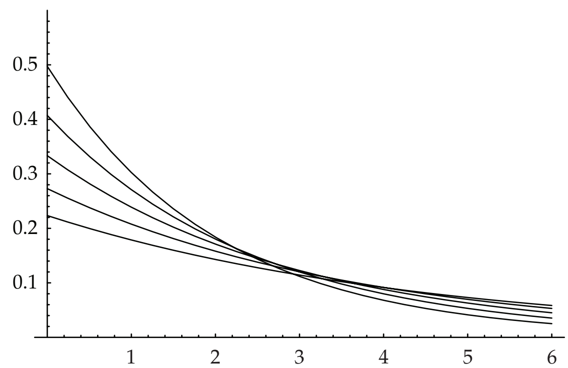

Figure 2: Density plot of exponential distribution with hazards $1/3e^{\beta}$.

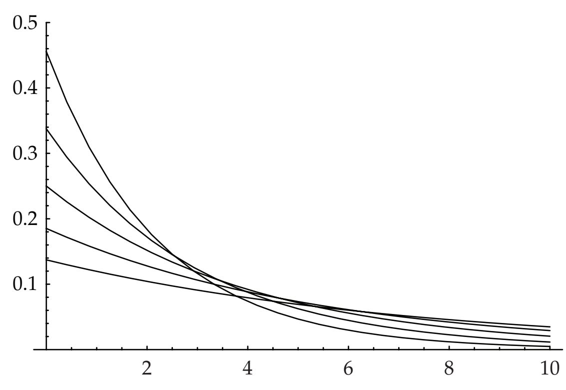

Figure 3: Density plot of exponential distribution with hazards $1/4e^{\beta}$.

5. An Application to AIDS Cohort Study

Consider the data of 262 hemophilia patients in De Gruttola and Lagakos [18], among them, 105 patients received at least 1,000 μg/kg of blood factor for at least one year between 1982 and 1985, and the other 157 patients received less than 1,000 μg/kg in each year. In this medical study, patients were treated between 1978 and 1988, the observations $(X_L, X_R]$ for the 262 patients, based on a discretization of the time axis into 6-month intervals. The failure time of interest is the time of HIV seroconversion. The object is to test the difference of the failure times between the two treatments. Applying our proposed test, namely, W.R.T, Mantel's [2] and Peto's [6] tests to this data set, the values of the three test statistics are −7.815, −7.352, and 56.476, respectively. All the three P values are less than 0.001 and have the same conclusion that the HIV seroconversion appeared in the two groups of patients being significantly different.

Table 5: Power comparison of tests under exponential distribution with sample $n = 100$.

m	q	Test	β −0.4	−0.2	0	0.2	0.4
6	0.8	W.R.T	0.419	0.131	0.050	0.150	0.371
		Mantel	0.391	0.120	0.047	0.143	0.362
		Peto	0.385	0.122	0.050	0.140	0.361
	0.5	W.R.T	0.383	0.123	0.045	0.132	0.345
		Mantel	0.360	0.121	0.041	0.124	0.344
		Peto	0.345	0.109	0.045	0.124	0.336
	0.3	W.R.T	0.313	0.102	0.042	0.103	0.254
		Mantel	0.307	0.101	0.040	0.096	0.255
		Peto	0.294	0.099	0.040	0.101	0.248
m	q	Test	β −0.6	−0.3	0	0.3	0.6
10	0.8	W.R.T	0.801	0.289	0.047	0.264	0.779
		Mantel	0.736	0.246	0.051	0.236	0.737
		Peto	0.717	0.242	0.050	0.237	0.740
	0.5	W.R.T	0.793	0.278	0.048	0.275	0.712
		Mantel	0.754	0.247	0.045	0.262	0.678
		Peto	0.718	0.240	0.052	0.256	0.663
	0.3	W.R.T	0.680	0.238	0.052	0.239	0.662
		Mantel	0.667	0.215	0.048	0.223	0.640
		Peto	0.624	0.216	0.049	0.224	0.632

Table 6: Power comparison of tests under exponential distribution with sample $n = 200$.

m	q	Test	β −0.4	−0.2	0	0.2	0.4
6	0.8	W.R.T	0.710	0.268	0.049	0.196	0.642
		Mantel	0.678	0.251	0.053	0.192	0.632
		Peto	0.667	0.253	0.054	0.190	0.630
	0.5	W.R.T	0.656	0.201	0.050	0.193	0.573
		Mantel	0.636	0.193	0.046	0.184	0.561
		Peto	0.621	0.188	0.047	0.188	0.558
	0.3	W.R.T	0.549	0.182	0.058	0.171	0.523
		Mantel	0.537	0.182	0.057	0.168	0.506
		Peto	0.523	0.181	0.052	0.164	0.501
m	q	Test	β −0.6	−0.3	0	0.3	0.6
10	0.8	W.R.T	0.984	0.520	0.049	0.473	0.945
		Mantel	0.964	0.472	0.050	0.441	0.930
		Peto	0.957	0.460	0.050	0.439	0.927
	0.5	W.R.T	0.971	0.484	0.046	0.448	0.957
		Mantel	0.961	0.458	0.045	0.424	0.946
		Peto	0.948	0.434	0.039	0.415	0.944
	0.3	W.R.T	0.942	0.429	0.053	0.402	0.901
		Mantel	0.927	0.413	0.050	0.387	0.892
		Peto	0.908	0.385	0.060	0.368	0.889

References

[1] B. W. Turnbull, "The empirical distribution function with arbitrarily grouped, censored and truncated data," *Journal of the Royal Statistical Society B*, vol. 38, no. 3, pp. 290–295, 1976.

[2] N. Mantel, "Ranking procedures for arbitrarily restricted observation," *Biometrics*, vol. 23, no. 1, pp. 65–78, 1967.

[3] E. A. Gehan, "A generalized Wilcoxon test for comparing arbitrarily singly-censored samples," *Biometrika*, vol. 52, no. 1-2, pp. 203–223, 1965.

[4] E. A. Gehan, "A generalized two-sample Wilcoxon test for doubly censored data," *Biometrika*, vol. 62, no. 3-4, pp. 650–653, 1965.

[5] F. Wilcoxon, "Individual comparisons by ranking methods," *Biometrika*, vol. 1, no. 6, pp. 80–83, 1945.

[6] R. Peto and J. Peto, "Asymptotically effcient rank invariant test procedures," *Journal of the Royal Statistical Society A*, vol. 135, no. 2, pp. 185–206, 1972.

[7] J. Sun, "A non-parametric test for interval-censored failure time data with application to AIDS studies," *Statistics in Medicine*, vol. 15, no. 13, pp. 1378–1395, 1996.

[8] M. P. Fay, "Comparing several score tests for interval censored data," *Statistics in Medicine*, vol. 18, no. 3, pp. 273–285, 1999.

[9] Q. Zhao and J. Sun, "Generalized log-rank test for mixed interval-censored failure time data," *Statistics in Medicine*, vol. 23, no. 10, pp. 1621–1629, 2004.

[10] J. Sun, Q. Zhao, and X. Zhao, "Generalized log-rank tests for interval-censored failure time data," *Scandinavian Journal of Statistics*, vol. 32, no. 1, pp. 49–57, 2005.

[11] J. Huang, C. Lee, and Q. Yu, "A generalized log-rank test for interval-censored failure time data via multiple imputation," *Statistics in Medicine*, vol. 27, no. 17, pp. 3217–3226, 2008.

[12] G. R. Petroni and R. A. Wolfe, "A two-sample test for stochastic ordering with interval-censored data," *Biometrics*, vol. 50, no. 1, pp. 77–87, 1994.

[13] H.-J. Lim and J. Sun, "Nonparametric tests for interval-censored failure time data," *Biometrical Journal*, vol. 45, no. 3, pp. 263–276, 2003.

[14] M. S. Pepe and T. R. Fleming, "Weighted Kaplan-Meier statistics: a class of distance tests for censored survival data," *Biometrics*, vol. 45, no. 2, pp. 497–507, 1989.

[15] E. L. Kaplan and P. Meier, "Nonparametric estimation from incomplete observations," *Journal of the American Statistical Association*, vol. 53, no. 282, pp. 457–481, 1958.

[16] C. Lee, "An urn model in the simulation of interval censored failure time data," *Statistics & Probability Letters*, vol. 45, no. 2, pp. 131–139, 1999.

[17] A. Schick and Q. Yu, "Consistency of the GMLE with mixed case interval-censored data," *Scandinavian Journal of Statistics*, vol. 27, no. 1, pp. 45–55, 2000.

[18] V. De Gruttola and S. W. Lagakos, "Analysis of doubly-censored survival data, with application to AIDS," *Biometrics*, vol. 45, no. 1, pp. 1–11, 1989.

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Research Article

Hybrid Extragradient Iterative Algorithms for Variational Inequalities, Variational Inclusions, and Fixed-Point Problems

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We investigate the problem of finding a common solution of a general system of variational inequalities, a variational inclusion, and a fixed-point problem of a strictly pseudocontractive mapping in a real Hilbert space. Motivated by Nadezhkina and Takahashi's hybrid-extragradient method, we propose and analyze new hybrid-extragradient iterative algorithm for finding a common solution. It is proven that three sequences generated by this algorithm converge strongly to the same common solution under very mild conditions. Based on this result, we also construct an iterative algorithm for finding a common fixed point of three mappings, such that one of these mappings is nonexpansive, and the other two mappings are strictly pseudocontractive mappings.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, and let P_C be the metric projection from H onto C. Let $S : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $A : C \to H$ is called monotone if

$$\left\langle Ax - Ay, x - y\right\rangle \geq 0, \quad \forall x, y \in C. \tag{1.1}$$

A mapping $A : C \to H$ is called L-Lipschitz continuous if there exists a constant $L > 0$, such that

$$\left\|Ax - Ay\right\| \leq L\left\|x - y\right\|, \quad \forall x, y \in C. \tag{1.2}$$

For a given mapping $A : C \to H$, we consider the following variational inequality (VI) of finding $x^* \in C$, such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

The solution set of the VI (1.3) is denoted by $\mathrm{VI}(C, A)$. The variational inequality was first discussed by Lions [1] and now is well known. Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, and equilibrium problems; see, for example, [2–4]. To construct a mathematical model which is as close as possible to a real complex problem, we often have to use more than one constraint. Solving such problems, we have to obtain some solution which is simultaneously the solution of two or more subproblem or the solution of one subproblem on the solution set of another subproblem. Actually, these subproblems can be given by problems of different types. For example, Antipin considered a finite-dimensional variant of the variational inequality, where the solution should satisfy some related constraint in inequality form [5] or some system of constraints in inequality and equality form [6]. Yamada [7] considered an infinite-dimensional variant of the solution of the variational inequality on the fixed-point set of some mapping.

A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a constant $\alpha > 0$, such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C; \tag{1.4}$$

see [8, 9]. It is obvious that an α-inverse strongly monotone mapping A is monotone and Lipschitz continuous. A self-mapping $S : C \to C$ is called k-strictly pseudocontractive if there exists a constant $k \in [0, 1)$, such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x,\ y \in C; \tag{1.5}$$

see [10]. In particular, if $k = 0$, then S is called a nonexpansive mapping; see [11].

A set-valued mapping M with domain $D(M)$ and range $R(M)$ in H is called monotone if its graph $G(M) = \{(x, f) \in H \times H : x \in D(M), f \in Mx\}$ is a monotone set in $H \times H$; that is, M is monotone if and only if

$$(x, f), (y, g) \in G(M) \Longrightarrow \langle x - y, f - g \rangle \geq 0. \tag{1.6}$$

A monotone set-valued mapping M is called maximal if its graph $G(M)$ is not properly contained in the graph of any other monotone mapping in H.

Let Φ be a single-valued mapping of C into H, and let M be a multivalued mapping with $D(M) = C$. Consider the following variational inclusion: find $x^* \in C$, such that

$$0 \in \Phi(x^*) + Mx^*. \tag{1.7}$$

We denote by Ω the solution set of the variational inclusion (1.7). In particular, if $\Phi = M = 0$, then $\Omega = C$.

In 1998, Huang [12] studied problem (1.7) in the case where M is maximal monotone, and Φ is strongly monotone and Lipschitz continuous with $D(M) = C = H$. Subsequently, Zeng et al. [13] further studied problem (1.7) in the case which is more general than Huang's one [12]. Moreover, the authors [13] obtained the same strong convergence conclusion as in Huang's result [12]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions.

In 2003, for finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ when $C \subset H$ is nonempty, closed, and convex, $S : C \to C$ is nonexpansive, and $A : C \to H$ is α-inverse strongly monotone. Takahashi and Toyoda [14] introduced the following iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n), \quad \forall n \geq 0, \tag{1.8}$$

where $x_0 \in C$ chosen arbitrarily, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to some $z \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$. In 2006, to solve this problem (i.e., to find an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$), Nadezhkina and Takahashi [15] introduced an iterative algorithm by a hybrid method. Generally speaking, the suggested algorithm is based on two well-known types of methods, that is, on the extragradient-type method due to Korpelevich [16] for solving variational inequality and so-called hybrid or outer-approximation method due to Haugazeau (see [15]) for solving fixed point problem. It is worth emphasizing that the idea of "hybrid" or "outer-approximation" types of methods was successfully generalized and extended in many papers; see, for example, [17–23]. In addition, the idea of the extragradient iterative algorithm introduced by Korpelevich [16] was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, for example, [24–29].

Theorem NT (see [15, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and k-Lipschitz-continuous mapping, and let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned} y_n &= P_C (x_n - \lambda_n A x_n), \\ z_n &= \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A y_n), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \tag{1.9}$$

where $x_0 \in C$ is chosen arbitrarily, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$, and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $P_{\mathrm{Fix}(S) \cap \mathrm{VI}(C,A)} x_0$.

It is easy to see that the class of α-inverse strongly monotone mappings in the above mentioned problem of Takahashi and Toyoda [14] is the quite important class of mappings in various classes of well-known mappings. It is also easy to see that while α-inverse strongly monotone mappings are tightly connected with the important class of nonexpansive mappings, α-inverse strongly monotone mappings are also tightly connected with the more general and also quite important class of strictly pseudocontractive mappings. That is, if a mapping $S : C \to C$ is nonexpansive, then the mapping $I - S$ is $(1/2\text{-})$ inverse strongly monotone; moreover, $\mathrm{Fix}(S) = \mathrm{VI}(C, I - S)$ (see, e.g., [14]). The construction of fixed points of nonexpansive mappings via Mann's algorithm has extensively been investigated in the literature (see, e.g., [30, 31] and references therein). At the same time, if a mapping $S : C \to C$ is k-strictly pseudocontractive, then the mapping $I - S$ is $(1-k)/2$-inverse strongly monotone and $2/(1-k)$-Lipschitz continuous.

Let $B_1, B_2 : C \to H$ be two mappings. Recently, Ceng et al. [32] introduced and considered the following problem of finding $(x^*, y^*) \in C \times C$, such that

$$\begin{gathered}\langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \ge 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \ge 0, \quad \forall x \in C,\end{gathered} \tag{1.10}$$

which is called a general system of variational inequalities (GSVI), where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. The set of solutions of problem (1.10) is denoted by $\mathrm{GSVI}(C, B_1, B_2)$. In particular, if $B_1 = B_2 = A$, then problem (1.10) reduces to the new system of variational inequalities (NSVI), introduced and studied by Verma [33]. Further, if $x^* = y^*$ additionally, then the NSVI reduces to the VI (1.3).

In particular, if $B_1 = A$ and $B_2 = 0$, then the GSVI (1.10) is equivalent to the VI (1.3).

Indeed, in this case, the GSVI (1.10) is equivalent to the following problem of finding $(x^*, y^*) \in C \times C$, such that

$$\begin{gathered}\langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \ge 0, \quad \forall x \in C, \\ \langle y^* - x^*, x - y^* \rangle \ge 0, \quad \forall x \in C.\end{gathered} \tag{1.11}$$

Thus we must have $x^* = y^*$. As a matter of fact, if $x^* \ne y^*$, then by setting $x = x^*$ we have

$$0 > -\|x^* - y^*\|^2 = \langle y^* - x^*, x^* - y^* \rangle \ge 0, \tag{1.12}$$

which hence leads to a contradiction. Therefore, the GSVI (1.10) coincides with the VI (1.3).

Recently, Ceng at al. [32] transformed problem (1.10) into a fixed-point problem in the following way.

Lemma 1.1 (see [32]). *For given $\overline{x}, \overline{y} \in C$, $(\overline{x}, \overline{y})$ is a solution of problem* (1.10) *if and only if $\overline{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)], \quad \forall x \in C, \tag{1.13}$$

where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.

In particular, if the mapping $B_i : C \to H$ is β_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\beta_i]$ for $i = 1, 2$.

Utilizing Lemma 1.1, they introduced and studied a relaxed extragradient method for solving the GSVI (1.10). Throughout this paper, the set of fixed points of the mapping G is denoted by Ξ. Based on the relaxed extragradient method and viscosity approximation method, Yao et al. [34] proposed and analyzed an iterative algorithm for finding a common solution of the GSVI (1.10) and the fixed point problem of a strictly pseudocontractive mapping $S : C \to C$.

Subsequently, Ceng et al. [35] further presented and analyzed an iterative scheme for finding a common element of the solution set of the VI (1.3), the solution set of the GSVI (1.10), and the fixed point set of a strictly pseudo-contractive mapping $S : C \to C$.

Theorem CGY (see [35, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone, and let $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that* $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A) \neq \emptyset$. *Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, 1/2)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= P_C(x_n - \lambda_n A x_n), \\ y_n &= \alpha_n Q x_n + (1 - \alpha_n) P_C\left[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)\right], \\ x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \tag{1.14}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\lambda_n\} \subset (0, 2\alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$, such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$, *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty}(\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$;

(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ generated by (1.14) converges strongly to $\overline{x} = P_{\mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)} Q\overline{x}$, and $(\overline{x}, \overline{y})$ is a solution of the GSVI (1.10), where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.

On the other hand, let $A : C \to H$ be a monotone, and let L-Lipschitz-continuous mapping, $\Phi : C \to H$ be an α-inverse strongly monotone mapping. Let M be a maximal monotone mapping with $D(M) = C$, and let $S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{VI}(C, A) \neq \emptyset$. Motivated Nadezhkina and Takahashi's hybrid-extragradient algorithm (1.9), Ceng et al. [36, Theorem 3.1] introduced another modified hybrid-extragradient algorithm

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n A x_n), \\ t_n &= P_C(x_n - \lambda_n A y_n), \\ \widehat{t}_n &= J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ z_n &= (1 - \alpha_n - \widehat{\alpha}_n) x_n + \alpha_n \widehat{t}_n + \widehat{\alpha}_n S \widehat{t}_n, \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \tag{1.15}$$

where $J_{M,\mu_n} = (I + \mu_n M)^{-1}$, $x_0 \in C$ chosen arbitrarily, $\{\lambda_n\} \subset (0, 1/L), \{\mu_n\} \subset (0, 2\alpha]$, and $\{\alpha_n\}, \{\widehat{\alpha}_n\} \subset (0, 1]$ such that $\alpha_n + \widehat{\alpha}_n \le 1$. It was proven in [36] that under very mild conditions three sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by (1.15) converge strongly to the same point $P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{VI}(C,A)} x_0$.

Inspired by the research going on this area, we propose and analyze the following hybrid extragradient iterative algorithm for finding a common element of the solution set Ξ of the GSVI (1.10), the solution set Ω of the variational inclusion (1.7), and the fixed point set $\mathrm{Fix}(S)$ of a strictly pseudo-contractive mapping $S : C \to C$.

Algorithm 1.2. Assume that $\mathrm{Fix}(S) \cap \Omega \cap \Xi \neq \emptyset$. Let $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\mu_n\} \subset (0, 2\alpha]$, and $\{\sigma_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\beta_n + \gamma_n + \delta_n = 1$, for all $n \ge 0$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequences generated by the hybrid extragradient iterative scheme

$$
\begin{aligned}
y_n &= P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right], \\
t_n &= P_C\left[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\right], \\
\widehat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\
z_n &= \beta_n x_n + \gamma_n \widehat{t}_n + \delta_n S \widehat{t}_n, \\
C_n &= \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0,
\end{aligned}
\tag{1.16}
$$

where $J_{M,\mu_n} = (I + \mu_n M)^{-1}$, for all $n \ge 0$.

Under very appropriate assumptions, it is proven that all the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\overline{x} = P_{\mathrm{Fix}(S)\cap\Omega\cap\Xi} x_0$. Furthermore, $(\overline{x}, \overline{y})$ is a solution of the GSVI (1.10), where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.

Let $T : C \to C$ be a k-strictly pseudocontractive mapping, let $\Gamma : C \to C$ be a κ-strictly pseudocontractive mapping, and let $S : C \to C$ be a nonexpansive mapping. Putting $B_1 = I - T, B_2 = 0$, $\Phi = I - \Gamma$, $M = 0$, and $\sigma_n = 0$, for all $n \ge 0$ in Algorithm 1.2, we consider and analyze the following hybrid extragradient iterative algorithm for finding a common fixed point of three mappings S, Γ, and T.

Algorithm 1.3. Assume that $\mathrm{Fix}(S) \cap \mathrm{Fix}(\Gamma) \cap \mathrm{Fix}(T) \neq \emptyset$. Let $\mu_1 \in (0, 1 - k)$, $\{\mu_n\} \subset (0, 1 - \kappa]$, and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\beta_n + \gamma_n + \delta_n = 1$, for all $n \ge 0$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequences generated by the hybrid extragradient iterative scheme

$$
\begin{aligned}
y_n &= x_n - \mu_1(x_n - T x_n), \\
t_n &= y_n - \mu_1(y_n - T y_n), \\
\widehat{t}_n &= t_n - \mu_n(t_n - \Gamma t_n),
\end{aligned}
$$

$$
\begin{aligned}
z_n &= \beta_n x_n + \gamma_n \widehat{t}_n + \delta_n S\widehat{t}_n, \\
C_n &= \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0.
\end{aligned}
\tag{1.17}
$$

Under quite mild conditions, it is shown that all the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the same point $P_{\mathrm{Fix}(S)\cap\mathrm{Fix}(\Gamma)\cap\mathrm{Fix}(T)} x_0$.

Observe that Ceng et al. [36, Theorem 3.1] considered the problem of finding an element of $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{VI}(C, A)$ where $S : C \to C$ is nonexpansive, Nadezhkina and Takahashi [15, Theorem 3.1] studied the problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ where $S : C \to C$ is nonexpansive, and Ceng et al. [35, Theorem 3.1] investigated the problem of finding an element of $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A)$ where $S : C \to C$ is strictly pseudocontractive. It is clear that every one of these three problems is very different from our problem of finding an element of $\mathrm{Fix}(S) \cap \Omega \cap \Xi$ where $S : C \to C$ is strictly pseudocontractive. Hence there is no doubt that the strong convergence results for solving our problem are very interesting and quite valuable. Because our hybrid extragradient iterative algorithms involve two inverse strongly monotone mappings B_1 and B_2, a strictly pseudo-contractive self-mapping S, and several parameter sequences, they are more flexible and more subtle than the corresponding ones in [36, Theorem 3.1] and [15, Theorem 3.1], respectively. Furthermore, the relaxed extragradient iterative scheme in Yao et al. [34, Theorem 3.2] is extended to develop our hybrid extragradient iterative algorithms. In our results, the hybrid extragradient iterative algorithms drop the requirements that $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ and $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$ in [34, Theorem 3.2] and [35, Theorem 3.1]. Therefore, our results represent the modification, supplementation, extension, and improvement of [36, Theorem 3.1], [15, Theorem 3.1], [34, Theorem 3.2], and [35, Theorem 3.1] to a great extent.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $\to$ to indicate that the sequence $\{x_n\}$ converges strongly to x and $\rightharpoonup$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$
\omega_w(x_n) := \left\{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\right\}. \tag{2.1}
$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$
\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.2}
$$

P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C; that is, there holds the following relation

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.3}$$

Consequently, P_C is nonexpansive and monotone. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{2.4}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.5}$$

for all $x \in H, y \in C$; see [11, 37] for more details. Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality, this implies that

$$x \in \mathrm{VI}(C, A) \Longleftrightarrow x = P_C(x - \lambda Ax), \quad \forall \lambda > 0. \tag{2.6}$$

It is also known that the norm of every Hilbert space H satisfies the weak lower semicontinuity [4]. That is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n\| \geq \|x\| \tag{2.7}$$

holds.

Recall that a set-valued mapping $M : D(M) \subset H \to 2^H$ is called maximal monotone if M is monotone and $(I + \lambda M)D(M) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(M)$ the graph of M. It is known that a monotone mapping M is maximal if and only if, for $(x, f) \in H \times H, \langle f - g, x - y\rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$. Here the following example illustrates the concept of maximal monotone mappings in the setting of Hilbert spaces.

Let $A : C \to H$ be a monotone, L-Lipschitz-continuous mapping, and let $N_C v$ be the normal cone to C at $v \in C$, that is,

$$N_C v = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{2.8}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [38].

Assume that $M : D(M) \subset H \to 2^H$ is a maximal monotone mapping. Then, for $\lambda > 0$, associated with M, the resolvent operator $J_{M,\lambda}$ can be defined as

$$J_{M,\lambda}x = (I + \lambda M)^{-1}x, \quad \forall x \in H. \tag{2.9}$$

In terms of Huang [12] (see also [13]), there holds the following property for the resolvent operator $J_{M,\lambda} : H \to H$.

Lemma 2.1. *$J_{M,\lambda}$ is single valued and firmly nonexpansive, that is,*

$$\langle J_{M,\lambda}x - J_{M,\lambda}y, x - y\rangle \geq \|J_{M,\lambda}x - J_{M,\lambda}y\|^2, \quad \forall x, y \in H. \tag{2.10}$$

Consequently, $J_{M,\lambda}$ is nonexpansive and monotone.

Lemma 2.2 (see [39]). *There holds the relation:*

$$\|\lambda x + \mu y + \nu z\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 + \nu\|z\|^2 - \lambda\mu\|x - y\|^2 - \mu\nu\|y - z\|^2 - \lambda\nu\|x - z\|^2, \tag{2.11}$$

for all $x, y, z \in H$ and λ, μ, $\nu \in [0,1]$ with $\lambda + \mu + \nu = 1$.

Lemma 2.3 (see [36]). *Let M be a maximal monotone mapping with $D(M) = C$. Then for any given $\lambda > 0$, $x^* \in C$ is a solution of problem* (1.7) *if and only if $x^* \in C$ satisfies*

$$x^* = J_{M,\lambda}(x^* - \lambda\Phi(x^*)). \tag{2.12}$$

Lemma 2.4 (see [13]). *Let M be a maximal monotone mapping with $D(M) = C$, and let $V : C \to H$ be a strong monotone, continuous, and single-valued mapping. Then for each $z \in H$, the equation $z \in Vx + \lambda Mx$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.5 (see [36]). *Let M be a maximal monotone mapping with $D(M) = C$, and let $A : C \to H$ be a monotone, continuous, and single-valued mapping. Then $(I + \lambda(M + A))C = H$ for each $\lambda > 0$. In this case, $M + A$ is maximal monotone.*

It is clear that, in a real Hilbert space H, $S : C \to C$ is k-strictly pseudo-contractive if and only if there holds the following inequality:

$$\langle Sx - Sy, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.13}$$

This immediately implies that if S is a k-strictly pseudocontractive mapping, then $I - S$ is $(1-k)/2$-inverse strongly monotone; for further detail, we refer to [10] and the references therein. It is well known that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings.

Lemma 2.6 (see [10, Proposition 2.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S : C \to C$ be a mapping.*

(i) *If S is a k-strict pseudo-contractive mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C. \tag{2.14}$$

(ii) *If S is a k-strict pseudo-contractive mapping, then the mapping $I - S$ is semiclosed at 0; that is, if $\{x_n\}$ is a sequence in C such that $x_n \to \tilde{x}$ weakly and $(I - S)x_n \to 0$ strongly, then $(I - S)\tilde{x} = 0$.*

(iii) *If S is k-quasistrict pseudo-contraction, then the fixed point set* $\mathrm{Fix}(S)$ *of S is closed and convex, so that the projection $P_{\mathrm{Fix}(S)}$ is well defined.*

Lemma 2.7 (see [34]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a k-strictly pseudo-contractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma+\delta)k \leq \gamma$. Then*

$$\|\gamma(x-y)+\delta(Sx-Sy)\| \leq (\gamma+\delta)\|x-y\|, \quad \forall x, y \in C. \tag{2.15}$$

The following lemma is well known to us.

Lemma 2.8 (see [11]). *Every Hilbert space H has the Kadec-Klee property; that is, for given $x \in H$ and $\{x_n\} \subset H$, we have*

$$\left.\begin{array}{c} x_n \rightharpoonup x \\ \|x_n\| \longrightarrow \|x\| \end{array}\right\} \Longrightarrow x_n \longrightarrow x. \tag{2.16}$$

3. Main Results

In this section, we first prove the strong convergence of the sequences generated by our hybrid extragradient iterative algorithm for finding a common solution of a general system of variational inequalities, a variational inclusion, and a fixed problem of a strictly pseudocontractive self-mapping.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$, let $\Phi : C \to H$ be an α-inverse strongly monotone mapping, let M be a maximal monotone mapping with $D(M) = C$, and let $S : C \to C$ be a k-strictly pseudocontractive mapping such that* $\mathrm{Fix}(S) \cap \Omega \cap \Xi \neq \emptyset$. *For given $x_0 \in C$ arbitrarily, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned} y_n &= P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right], \\ t_n &= P_C\left[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\right], \\ \hat{t}_n &= \sigma_n t_n + (1-\sigma_n) J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S\hat{t}_n, \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \tag{3.1}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$, and $\{\sigma_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ such that $\{\sigma_n\} \subset [0, c]$ for some $c \in [0,1)$, $\{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$, $\beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)k \leq \gamma_n$, for all $n \geq 0$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\overline{x} = P_{\mathrm{Fix}(S)\cap\Omega\cap\Xi} x_0$ if and only if $\|S\hat{t}_n - \hat{t}_n\| \to 0$. Furthermore, $(\overline{x}, \overline{y})$ is a solution of the GSVI (1.10), *where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.*

Proof. It is obvious that C_n is closed and Q_n is closed and convex for every $n = 0, 1, 2, \ldots$. As

$$C_n = \left\{ z \in C : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z \rangle \leq 0 \right\}, \tag{3.2}$$

we also know that C_n is convex for every $n = 0, 1, 2, \ldots$. As

$$Q_n = \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \tag{3.3}$$

we have $\langle x_n - z, x - x_n \rangle \geq 0$, for all $z \in Q_n$, and hence $x_n = P_{Q_n} x_0$ by (2.4).

First of all, assume that the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\overline{x} = P_{\mathrm{Fix}(S) \cap \Omega \cap \Xi} x_0$. Then it is clear that $\|x_n - y_n\| \to 0$ and $\|x_n - z_n\| \to 0$. Observe that from the nonexpansiveness of the mappings $P_C(I - \mu_1 B_1)$ and $P_C(I - \mu_2 B_2)$ (due to $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$), we have

$$\begin{aligned}
\|y_n - t_n\| &= \big\| P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n) \right] \\
&\qquad - P_C\left[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n) \right] \big\| \\
&= \big\| P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2) x_n - P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2) y_n \big\| \\
&\leq \big\| P_C(I - \mu_2 B_2) x_n - P_C(I - \mu_2 B_2) y_n \big\| \\
&\leq \|x_n - y_n\|.
\end{aligned} \tag{3.4}$$

Hence, we conclude that $\|y_n - t_n\| \to 0$ and $t_n \to \overline{x}$. Since $\overline{x} \in \mathrm{Fix}(S) \cap \Omega \cap \Xi$, we obtain that $S\overline{x} = \overline{x}$ and $\overline{x} = J_{M,\mu_n}(\overline{x} - \mu_n \Phi(\overline{x}))$. Thus, from the nonexpansiveness of the mapping $J_{M,\mu_n}(I - \mu_n \Phi)$, we have

$$\begin{aligned}
\left\| \hat{t}_n - t_n \right\| &= \|\sigma_n t_n + (1 - \sigma_n) J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) - t_n\| \\
&= (1 - \sigma_n) \big\| J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) - t_n \big\| \\
&\leq \big\| J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) - \overline{x} \big\| + \|\overline{x} - t_n\| \\
&= \big\| J_{M,\mu_n}(I - \mu_n \Phi) t_n - J_{M,\mu_n}(I - \mu_n \Phi) \overline{x} \big\| + \|t_n - \overline{x}\| \\
&\leq \|t_n - \overline{x}\| + \|t_n - \overline{x}\| = 2\|t_n - \overline{x}\|.
\end{aligned} \tag{3.5}$$

So, we deduce that $\|\hat{t}_n - t_n\| \to 0$ and $\hat{t}_n \to \overline{x}$. Note that

$$\begin{aligned}
\left\| S\hat{t}_n - \hat{t}_n \right\| &\leq \left\| S\hat{t}_n - \overline{x} \right\| + \left\| \overline{x} - \hat{t}_n \right\| \\
&= \left\| S\hat{t}_n - S\overline{x} \right\| + \left\| \overline{x} - \hat{t}_n \right\| \\
&\leq \left(\frac{1 + k}{1 - k} + 1 \right) \left\| \hat{t}_n - \overline{x} \right\| = \frac{2}{1 - k} \left\| \hat{t}_n - \overline{x} \right\|.
\end{aligned} \tag{3.6}$$

This implies that $\|S\hat{t}_n - \hat{t}_n\| \to 0$ as $n \to \infty$.

For the remainder of the proof, we divide it into several steps.

Step 1. We claim that $\mathrm{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n$ for every $n = 0, 1, 2, \ldots$.

Indeed, take a fixed $p \in \mathrm{Fix}(S) \cap \Omega \cap \Xi$ arbitrarily. Then $Sp = p$, $J_{M,\mu_n}(p - \mu_n \Phi(p)) = p$, for all $n \geq 0$, and

$$p = P_C\left[P_C(p - \mu_2 B_2 p) - \mu_1 B_1 P_C(p - \mu_2 B_2 p)\right]. \tag{3.7}$$

For simplicity, we write $q = P_C(p - \mu_2 B_2 p)$, $\widetilde{x}_n = P_C(x_n - \mu_2 B_2 x_n)$, and $\widetilde{y}_n = P_C(y_n - \mu_2 B_2 y_n)$,

$$\begin{aligned} y_n &= P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right] = P_C(\widetilde{x}_n - \mu_1 B_1 \widetilde{x}_n), \\ t_n &= P_C\left[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\right] = P_C(\widetilde{y}_n - \mu_1 B_1 \widetilde{y}_n), \end{aligned} \tag{3.8}$$

for each $n \geq 0$. Since $B_i : C \to H$ is β_i-inverse strongly monotone, and $0 < \mu_i < 2\beta_i$ for $i = 1, 2$, we know that for all $n \geq 0$,

$$\begin{aligned} &\|y_n - p\|^2 \\ &\quad = \left\|P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right] - p\right\|^2 \\ &\quad = \left\|P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right]\right. \\ &\qquad \left. - P_C\left[P_C(p - \mu_2 B_2 p) - \mu_1 B_1 P_C(p - \mu_2 B_2 p)\right]\right\|^2 \\ &\quad \leq \left\|\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right]\right. \\ &\qquad \left. - \left[P_C(p - \mu_2 B_2 p) - \mu_1 B_1 P_C(p - \mu_2 B_2 p)\right]\right\|^2 \\ &\quad = \left\|\left[P_C(x_n - \mu_2 B_2 x_n) - P_C(p - \mu_2 B_2 p)\right] - \mu_1\left[B_1 P_C(x_n - \mu_2 B_2 x_n) - B_1 P_C(p - \mu_2 B_2 p)\right]\right\|^2 \\ &\quad \leq \left\|P_C(x_n - \mu_2 B_2 x_n) - P_C(p - \mu_2 B_2 p)\right\|^2 \\ &\qquad - \mu_1(2\beta_1 - \mu_1)\left\|B_1 P_C(x_n - \mu_2 B_2 x_n) - B_1 P_C(p - \mu_2 B_2 p)\right\|^2 \\ &\quad \leq \left\|(x_n - \mu_2 B_2 x_n) - (p - \mu_2 B_2 p)\right\|^2 - \mu_1(2\beta_1 - \mu_1)\left\|B_1 \widetilde{x}_n - B_1 q\right\|^2 \\ &\quad = \left\|(x_n - p) - \mu_2(B_2 x_n - B_2 p)\right\|^2 - \mu_1(2\beta_1 - \mu_1)\left\|B_1 \widetilde{x}_n - B_1 q\right\|^2 \\ &\quad \leq \|x_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2 x_n - B_2 p\|^2 - \mu_1(2\beta_1 - \mu_1)\left\|B_1 \widetilde{x}_n - B_1 q\right\|^2 \leq \|x_n - p\|^2. \end{aligned} \tag{3.9}$$

Repeating the same argument, we can obtain that for all $n \geq 0$,

$$\begin{aligned} \|t_n - p\|^2 &\leq \|y_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2 y_n - B_2 p\|^2 \\ &\quad - \mu_1(2\beta_1 - \mu_1)\left\|B_1 \widetilde{y}_n - B_1 q\right\|^2 \leq \|y_n - p\|^2. \end{aligned} \tag{3.10}$$

Furthermore, by Lemma 2.1 we derive from (3.9) and (3.10)

$$\begin{aligned}
\left\|\hat{t}_n - p\right\|^2 &= \left\|\sigma_n(t_n - p) + (1-\sigma_n)\left(J_{M,\mu_n}(t_n - \mu_n\Phi(t_n)) - p\right)\right\|^2 \\
&\le \sigma_n\|t_n - p\|^2 + (1-\sigma_n)\left\|J_{M,\mu_n}(t_n - \mu_n\Phi(t_n)) - p\right\|^2 \\
&= \sigma_n\|t_n - p\|^2 + (1-\sigma_n)\left\|J_{M,\mu_n}(t_n - \mu_n\Phi(t_n)) - J_{M,\mu_n}(p - \mu_n\Phi(p))\right\|^2 \\
&\le \sigma_n\|t_n - p\|^2 + (1-\sigma_n)\left\|(t_n - \mu_n\Phi(t_n)) - (p - \mu_n\Phi(p))\right\|^2 \\
&\le \sigma_n\|t_n - p\|^2 + (1-\sigma_n)\left[\|t_n - p\|^2 + \mu_n(\mu_n - 2\alpha)\|\Phi(t_n) - \Phi(p)\|^2\right] \\
&\le \|t_n - p\|^2 \\
&\le \|y_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2y_n - B_2p\|^2 - \mu_1(2\beta_1 - \mu_1)\|B_1\tilde{y}_n - B_1q\|^2 \\
&\le \|x_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2x_n - B_2p\|^2 - \mu_1(2\beta_1 - \mu_1)\|B_1\tilde{x}_n - B_1q\|^2 \\
&\quad - \mu_2(2\beta_2 - \mu_2)\|B_2y_n - B_2p\|^2 - \mu_1(2\beta_1 - \mu_1)\|B_1\tilde{y}_n - B_1q\|^2 \\
&= \|x_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\left(\|B_2x_n - B_2p\|^2 + \|B_2y_n - B_2p\|^2\right) \\
&\quad - \mu_1(2\beta_1 - \mu_1)\left(\|B_1\tilde{x}_n - B_1q\|^2 + \|B_1\tilde{y}_n - B_1q\|^2\right).
\end{aligned} \tag{3.11}$$

Since $(\gamma_n + \delta_n)k \le \gamma_n$, for all $n \ge 0$, utilizing Lemmas 2.2 and 2.7, we get from (3.11)

$$\begin{aligned}
&\|z_n - p\|^2 \\
&\quad = \left\|\beta_n(x_n - p) + \gamma_n\left(\hat{t}_n - p\right) + \delta_n\left(S\hat{t}_n - p\right)\right\|^2 \\
&\quad = \left\|\beta_n(x_n - p) + (\gamma_n + \delta_n)\frac{1}{\gamma_n + \delta_n}\left[\gamma_n\left(\hat{t}_n - p\right) + \delta_n\left(S\hat{t}_n - p\right)\right]\right\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\left\|\frac{1}{\gamma_n + \delta_n}\left[\gamma_n\left(\hat{t}_n - p\right) + \delta_n\left(S\hat{t}_n - p\right)\right]\right\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\left\|\hat{t}_n - p\right\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\Big\{\|x_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\left(\|B_2x_n - B_2p\|^2 + \|B_2y_n - B_2p\|^2\right) \\
&\qquad\qquad - \mu_1(2\beta_1 - \mu_1)\left(\|B_1\tilde{x}_n - B_1q\|^2 + \|B_1\tilde{y}_n - B_1q\|^2\right)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - p\|^2 - (\gamma_n + \delta_n)\Big\{\mu_2(2\beta_2 - \mu_2)\Big(\|B_2x_n - B_2p\|^2 + \|B_2y_n - B_2p\|^2\Big) \\
&\qquad\qquad + \mu_1(2\beta_1 - \mu_1)\Big(\|B_1\tilde{x}_n - B_1q\|^2 + \|B_1\tilde{y}_n - B_1q\|^2\Big)\Big\} \\
&\leq \|x_n - p\|^2,
\end{aligned} \tag{3.12}
$$

for every $n = 0,1,2,\ldots$, and hence $p \in C_n$. So, $\mathrm{Fix}(S) \cap \Omega \cap \Xi \subset C_n$ for every $n = 0,1,2,\ldots$. Next, let us show by mathematical induction that $\{x_n\}$ is well defined and $\mathrm{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n$ for every $n = 0,1,2,\ldots$. For $n = 0$, we have $Q_0 = C$. Hence we obtain $\mathrm{Fix}(S) \cap \Omega \cap \Xi \subset C_0 \cap Q_0$. Suppose that x_n is given and $\mathrm{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n$ for some integer $n \geq 0$. Since $\mathrm{Fix}(S) \cap \Omega \cap \Xi$ is nonempty, $C_n \cap Q_n$ is a nonempty closed convex subset of C. So, there exists a unique element $x_{n+1} \in C_n \cap Q_n$ such that $x_{n+1} = P_{C_n \cap Q_n} x_0$. It is also obvious that there holds $\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0$ for $z \in \mathrm{Fix}(S) \cap \Omega \cap \Xi$, and hence $\mathrm{Fix}(S) \cap \Omega \cap \Xi \subset Q_{n+1}$. Therefore, we derive $\mathrm{Fix}(S) \cap \Omega \cap \Xi \cap C_{n+1} \cap Q_{n+1}$.

Step 2. We claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.13}$$

Indeed, let $l_0 = P_{\mathrm{Fix}(S)\cap\Omega\cap\Xi} x_0$. From $x_{n+1} = P_{C_n \cap Q_n} x_0$, and $l_0 \in \mathrm{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - x_0\| \tag{3.14}$$

for every $n = 0,1,2,\ldots$. Therefore, $\{x_n\}$ is bounded. From (3.9)–(3.12), we also obtain that $\{\tilde{x}_n\}$, $\{y_n\}$, $\{\tilde{y}_n\}$, $\{t_n\}$, $\{\hat{t}_n\}$, and $\{z_n\}$ all are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n} x_0$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \tag{3.15}$$

for every $n = 0,1,2,\ldots$. Therefore, there exists $\lim_{n\to\infty}\|x_n - x_0\|$. Since $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$, utilizing (2.5), we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \tag{3.16}$$

for every $n = 0,1,2,\ldots$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.17}$$

Since $x_{n+1} \in C_n$, we have $\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$, and hence

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2\|x_{n+1} - x_n\| \tag{3.18}$$

for every $n = 0, 1, 2, \ldots$. From $\|x_{n+1} - x_n\| \to 0$ it follows that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.19}$$

Step 3. We claim that

$$\lim_{n\to\infty} \left\|x_n - y_n\right\| = \lim_{n\to\infty} \|x_n - t_n\| = \lim_{n\to\infty} \left\|\hat{t}_n - t_n\right\| = 0. \tag{3.20}$$

Indeed, for $p \in \mathrm{Fix}(S) \cap \Omega \cap \Xi$, we obtain from (3.12)

$$\begin{aligned} \left\|z_n - p\right\|^2 \leq{} & \left\|x_n - p\right\|^2 \\ & - (\gamma_n + \delta_n)\Big\{\mu_2(2\beta_2 - \mu_2)\Big(\left\|B_2x_n - B_2p\right\|^2 + \left\|B_2y_n - B_2p\right\|^2\Big) \\ & \quad + \mu_1(2\beta_1 - \mu_1)\Big(\left\|B_1\tilde{x}_n - B_1q\right\|^2 + \left\|B_1\tilde{y}_n - B_1q\right\|^2\Big)\Big\}. \end{aligned} \tag{3.21}$$

Therefore, we have

$$\begin{aligned} & (\gamma_n + \delta_n)\Big\{\mu_2(2\beta_2 - \mu_2)\Big(\left\|B_2x_n - B_2p\right\|^2 + \left\|B_2y_n - B_2p\right\|^2\Big) \\ & \qquad + \mu_1(2\beta_1 - \mu_1)\Big(\left\|B_1\tilde{x}_n - B_1q\right\|^2 + \left\|B_1\tilde{y}_n - B_1q\right\|^2\Big)\Big\} \\ & \quad \leq \left\|x_n - p\right\|^2 - \left\|z_n - p\right\|^2 \\ & \quad = \left(\left\|x_n - p\right\| - \left\|z_n - p\right\|\right)\left(\left\|x_n - p\right\| + \left\|z_n - p\right\|\right) \\ & \quad \leq \|x_n - z_n\|\left(\left\|x_n - p\right\| + \left\|z_n - p\right\|\right). \end{aligned} \tag{3.22}$$

Since $\{\delta_n\} \subset [d,1]$ for some $d \in (0,1]$, $\|x_n - z_n\| \to 0$, and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we deduce that

$$\begin{aligned}\lim_{n\to\infty}\|B_2x_n - B_2p\| &= \lim_{n\to\infty}\|B_2y_n - B_2p\| = \lim_{n\to\infty}\|B_1\tilde{x}_n - B_1q\| \\ &= \lim_{n\to\infty}\|B_1\tilde{y}_n - B_1q\| = 0.\end{aligned} \tag{3.23}$$

On the other hand, by firm nonexpansiveness of P_C, we have

$$\begin{aligned}\|\tilde{x}_n - q\|^2 &= \|P_C(x_n - \mu_2B_2x_n) - P_C(p - \mu_2B_2p)\|^2 \\ &\leq \langle (x_n - \mu_2B_2x_n) - (p - \mu_2B_2p), \tilde{x}_n - q\rangle \\ &= \frac{1}{2}\Big[\|x_n - p - \mu_2(B_2x_n - B_2p)\|^2 + \|\tilde{x}_n - q\|^2 \\ &\qquad -\|(x_n - p) - \mu_2(B_2x_n - B_2p) - (\tilde{x}_n - q)\|^2\Big] \\ &\leq \frac{1}{2}\Big[\|x_n - p\|^2 + \|\tilde{x}_n - q\|^2 - \|(x_n - \tilde{x}_n) - \mu_2(B_2x_n - B_2p) - (p - q)\|^2\Big] \\ &= \frac{1}{2}\Big[\|x_n - p\|^2 + \|\tilde{x}_n - q\|^2 - \|x_n - \tilde{x}_n - (p - q)\|^2 \\ &\qquad +2\mu_2\langle x_n - \tilde{x}_n - (p - q), B_2x_n - B_2p\rangle - \mu_2^2\|B_2x_n - B_2p\|^2\Big] \\ &\leq \frac{1}{2}\Big[\|x_n - p\|^2 + \|\tilde{x}_n - q\|^2 - \|x_n - \tilde{x}_n - (p - q)\|^2 \\ &\qquad +2\mu_2\|x_n - \tilde{x}_n - (p - q)\|\|B_2x_n - B_2p\|\Big];\end{aligned} \tag{3.24}$$

that is,

$$\|\tilde{x}_n - q\|^2 \leq \|x_n - p\|^2 - \|x_n - \tilde{x}_n - (p - q)\|^2 + 2\mu_2\|x_n - \tilde{x}_n - (p - q)\|\|B_2x_n - B_2p\|. \tag{3.25}$$

Repeating the same argument, we can also obtain

$$\|\tilde{y}_n - q\|^2 \leq \|y_n - p\|^2 - \|y_n - \tilde{y}_n - (p - q)\|^2 + 2\mu_2\|y_n - \tilde{y}_n - (p - q)\|\|B_2y_n - B_2p\|. \tag{3.26}$$

Moreover, using the argument technique similar to the above one, we derive

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(\tilde{x}_n - \mu_1 B_1 \tilde{x}_n) - P_C(q - \mu_1 B_1 q)\|^2 \\
&\le \langle (\tilde{x}_n - \mu_1 B_1 \tilde{x}_n) - (q - \mu_1 B_1 q), y_n - p \rangle \\
&= \frac{1}{2}\Big[\|\tilde{x}_n - q - \mu_1(B_1\tilde{x}_n - B_1 q)\|^2 + \|y_n - p\|^2 \\
&\qquad -\|(\tilde{x}_n - q) - \mu_1(B_1\tilde{x}_n - B_1 q) - (y_n - p)\|^2\Big] \\
&\le \frac{1}{2}\Big[\|\tilde{x}_n - q\|^2 + \|y_n - p\|^2 - \|(\tilde{x}_n - y_n) - \mu_1(B_1\tilde{x}_n - B_1 q) + (p - q)\|^2\Big] \\
&= \frac{1}{2}\Big[\|\tilde{x}_n - q\|^2 + \|y_n - p\|^2 - \|\tilde{x}_n - y_n + (p - q)\|^2 \\
&\qquad +2\mu_1\langle \tilde{x}_n - y_n + (p - q), B_1\tilde{x}_n - B_1 q\rangle - \mu_1^2\|B_1\tilde{x}_n - B_1 q\|^2\Big] \\
&\le \frac{1}{2}\Big[\|\tilde{x}_n - q\|^2 + \|y_n - p\|^2 - \|\tilde{x}_n - y_n + (p - q)\|^2 \\
&\qquad +2\mu_1\|\tilde{x}_n - y_n + (p - q)\|\|B_1\tilde{x}_n - B_1 q\|\Big];
\end{aligned} \tag{3.27}$$

that is,

$$\|y_n - p\|^2 \le \|\tilde{x}_n - q\|^2 - \|\tilde{x}_n - y_n + (p - q)\|^2 + 2\mu_1\|\tilde{x}_n - y_n + (p - q)\|\|B_1\tilde{x}_n - B_1 q\|. \tag{3.28}$$

Repeating the same argument, we can also obtain

$$\|t_n - p\|^2 \le \|\tilde{y}_n - q\|^2 - \|\tilde{y}_n - t_n + (p - q)\|^2 + 2\mu_1\|\tilde{y}_n - t_n + (p - q)\|\|B_1\tilde{y}_n - B_1 q\|. \tag{3.29}$$

Utilizing (3.11), (3.25)–(3.29), we have

$$\begin{aligned}
&\|z_n - p\|^2 \\
&\quad = \left\|\beta_n(x_n - p) + \gamma_n\left(\hat{t}_n - p\right) + \delta_n\left(S\hat{t}_n - p\right)\right\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\left\|\hat{t}_n - p\right\|^2 \\
&\quad = \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\|t_n - p\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 \\
&\qquad + (\gamma_n + \delta_n)\Big[\|\tilde{y}_n - q\|^2 - \|\tilde{y}_n - t_n + (p - q)\|^2 \\
&\qquad\qquad +2\mu_1\|\tilde{y}_n - t_n + (p - q)\|\|B_1\tilde{y}_n - B_1 q\|\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n \|x_n - p\|^2 \\
&\quad + (\gamma_n + \delta_n)\Big[\|y_n - p\|^2 - \|y_n - \tilde{y}_n - (p-q)\|^2 \\
&\qquad\qquad + 2\mu_2 \|y_n - \tilde{y}_n - (p-q)\| \|B_2 y_n - B_2 p\| \\
&\qquad\qquad - \|\tilde{y}_n - t_n + (p-q)\|^2 + 2\mu_1 \|\tilde{y}_n - t_n + (p-q)\| \|B_1 \tilde{y}_n - B_1 q\|\Big] \\
&\leq \beta_n \|x_n - p\|^2 \\
&\quad + (\gamma_n + \delta_n)\Big\{\|\tilde{x}_n - q\|^2 - \|\tilde{x}_n - y_n + (p-q)\|^2 \\
&\qquad\qquad + 2\mu_1 \|\tilde{x}_n - y_n + (p-q)\| \|B_1 \tilde{x}_n - B_1 q\| \\
&\qquad\qquad - \|y_n - \tilde{y}_n - (p-q)\|^2 + 2\mu_2 \|y_n - \tilde{y}_n - (p-q)\| \|B_2 y_n - B_2 p\| \\
&\qquad\qquad - \|\tilde{y}_n - t_n + (p-q)\|^2 + 2\mu_1 \|\tilde{y}_n - t_n + (p-q)\| \|B_1 \tilde{y}_n - B_1 q\|\Big\} \\
&\leq \beta_n \|x_n - p\|^2 \\
&\quad + (\gamma_n + \delta_n)\Big\{\|x_n - p\|^2 - \|x_n - \tilde{x}_n - (p-q)\|^2 \\
&\qquad\qquad + 2\mu_2 \|x_n - \tilde{x}_n - (p-q)\| \|B_2 x_n - B_2 p\| \\
&\qquad\qquad - \|\tilde{x}_n - y_n + (p-q)\|^2 + 2\mu_1 \|\tilde{x}_n - y_n + (p-q)\| \|B_1 \tilde{x}_n - B_1 q\| \\
&\qquad\qquad - \|y_n - \tilde{y}_n - (p-q)\|^2 + 2\mu_2 \|y_n - \tilde{y}_n - (p-q)\| \|B_2 y_n - B_2 p\| \\
&\qquad\qquad - \|\tilde{y}_n - t_n + (p-q)\|^2 + 2\mu_1 \|\tilde{y}_n - t_n + (p-q)\| \|B_1 \tilde{y}_n - B_1 q\|\Big\} \\
&\leq \|x_n - p\|^2 + 2\mu_2 \|x_n - \tilde{x}_n - (p-q)\| \|B_2 x_n - B_2 p\| \\
&\quad + 2\mu_1 \|\tilde{x}_n - y_n + (p-q)\| \|B_1 \tilde{x}_n - B_1 q\| \\
&\quad + 2\mu_2 \|y_n - \tilde{y}_n - (p-q)\| \|B_2 y_n - B_2 p\| + 2\mu_1 \|\tilde{y}_n - t_n + (p-q)\| \|B_1 \tilde{y}_n - B_1 q\| \\
&\quad - (\gamma_n + \delta_n)\Big[\|x_n - \tilde{x}_n - (p-q)\|^2 + \|\tilde{x}_n - y_n + (p-q)\|^2 \\
&\qquad\qquad + \|y_n - \tilde{y}_n - (p-q)\|^2 + \|\tilde{y}_n - t_n + (p-q)\|^2\Big],
\end{aligned}
\tag{3.30}
$$

which hence implies that

$$
\begin{aligned}
(\gamma_n + \delta_n)&\Big[\|x_n - \tilde{x}_n - (p-q)\|^2 + \|\tilde{x}_n - y_n + (p-q)\|^2 \\
&\quad + \|y_n - \tilde{y}_n - (p-q)\|^2 + \|\tilde{y}_n - t_n + (p-q)\|^2\Big] \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 + 2\mu_2 \|x_n - \tilde{x}_n - (p-q)\| \|B_2 x_n - B_2 p\|
\end{aligned}
$$

$$\begin{aligned}
&+ 2\mu_1 \left\| \tilde{x}_n - y_n + (p-q) \right\| \left\| B_1 \tilde{x}_n - B_1 q \right\| \\
&+ 2\mu_2 \left\| y_n - \tilde{y}_n - (p-q) \right\| \left\| B_2 y_n - B_2 p \right\| \\
&+ 2\mu_1 \left\| \tilde{y}_n - t_n + (p-q) \right\| \left\| B_1 \tilde{y}_n - B_1 q \right\| \\
\leq{}& \|x_n - z_n\| \left(\|x_n - p\| + \|z_n - p\| \right) \\
&+ 2\mu_2 \left\| x_n - \tilde{x}_n - (p-q) \right\| \left\| B_2 x_n - B_2 p \right\| \\
&+ 2\mu_1 \left\| \tilde{x}_n - y_n + (p-q) \right\| \left\| B_1 \tilde{x}_n - B_1 q \right\| \\
&+ 2\mu_2 \left\| y_n - \tilde{y}_n - (p-q) \right\| \left\| B_2 y_n - B_2 p \right\| \\
&+ 2\mu_1 \left\| \tilde{y}_n - t_n + (p-q) \right\| \left\| B_1 \tilde{y}_n - B_1 q \right\|.
\end{aligned} \tag{3.31}$$

Since $\{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$, $\|x_n - z_n\| \to 0$, and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$, and $\{t_n\}$ all are bounded, it follows from (3.23) that

$$\begin{aligned}
\lim_{n\to\infty} \left\| x_n - \tilde{x}_n - (p-q) \right\| &= \lim_{n\to\infty} \left\| \tilde{x}_n - y_n + (p-q) \right\| = 0, \\
\lim_{n\to\infty} \left\| y_n - \tilde{y}_n - (p-q) \right\| &= \lim_{n\to\infty} \left\| \tilde{y}_n - t_n + (p-q) \right\| = 0.
\end{aligned} \tag{3.32}$$

Consequently, it immediately follows that

$$\lim_{n\to\infty} \left\| x_n - y_n \right\| = 0, \qquad \lim_{n\to\infty} \left\| y_n - t_n \right\| = 0. \tag{3.33}$$

This shows that

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{3.34}$$

Also, note that

$$\begin{aligned}
\|z_n - x_n\|^2 &= \left\| \gamma_n \left(\hat{t}_n - x_n \right) + \delta_n \left(S\hat{t}_n - x_n \right) \right\|^2 \\
&= \left\| (\gamma_n + \delta_n) \frac{1}{\gamma_n + \delta_n} \left[\gamma_n \left(\hat{t}_n - x_n \right) + \delta_n \left(S\hat{t}_n - x_n \right) \right] \right\|^2 \\
&= (\gamma_n + \delta_n)^2 \left[\frac{\gamma_n}{\gamma_n + \delta_n} \left\| \hat{t}_n - x_n \right\|^2 + \frac{\delta_n}{\gamma_n + \delta_n} \left\| S\hat{t}_n - x_n \right\|^2 \right. \\
&\qquad \left. - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} \left\| \hat{t}_n - S\hat{t}_n \right\|^2 \right] \\
&= (\gamma_n + \delta_n) \left[\gamma_n \left\| \hat{t}_n - x_n \right\|^2 + \delta_n \left\| S\hat{t}_n - x_n \right\|^2 \right] - \gamma_n \delta_n \left\| \hat{t}_n - S\hat{t}_n \right\|^2.
\end{aligned} \tag{3.35}$$

Thus we have

$$\begin{aligned}
d^2\left\|S\widehat{t}_n - x_n\right\|^2\Big] &\le (\gamma_n + \delta_n)\left[\gamma_n\left\|\widehat{t}_n - x_n\right\|^2 + \delta_n\left\|S\widehat{t}_n - x_n\right\|^2\right] \\
&= \|z_n - x_n\|^2 + \gamma_n\delta_n\left\|\widehat{t}_n - S\widehat{t}_n\right\|^2 \\
&\le \|z_n - x_n\|^2 + \left\|\widehat{t}_n - S\widehat{t}_n\right\|^2.
\end{aligned} \tag{3.36}$$

This together with $\|z_n - x_n\| \to 0$ and $\|\widehat{t}_n - S\widehat{t}_n\| \to 0$ implies that

$$\lim_{n\to\infty}\left\|S\widehat{t}_n - x_n\right\| = 0, \qquad \lim_{n\to\infty}\left\|\widehat{t}_n - x_n\right\| = 0. \tag{3.37}$$

Consequently, from (3.34) we immediately derive

$$\lim_{n\to\infty}\left\|\widehat{t}_n - t_n\right\| = 0. \tag{3.38}$$

Step 4. We claim that $\omega_w(x_n) \subset \operatorname{Fix}(S) \cap \Omega \cap \Xi$.

Indeed, as $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some $u \in \omega_w(x_n)$. We can obtain that $u \in \operatorname{Fix}(S) \cap \Omega \cap \Xi$. First, it is clear from Lemma 2.6(ii) that $u \in \operatorname{Fix}(S)$. Now let us show that $u \in \Xi$. We note that

$$\begin{aligned}
\|x_n - G(x_n)\| &= \left\|x_n - P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right]\right\| \\
&= \left\|x_n - y_n\right\| \longrightarrow 0 \ (n \longrightarrow \infty),
\end{aligned} \tag{3.39}$$

where $G : C \to C$ is defined as that in Lemma 1.1. According to Lemma 2.6(ii) we obtain $u \in \Xi$. Further, let us show that $u \in \Omega$. As a matter of fact, since Φ is α-inverse strongly monotone, and M is maximal monotone, by Lemma 2.5 we know that $M + \Phi$ is maximal monotone. Take a fixed $(y, g) \in G(M + \Phi)$ arbitrarily. Then we have $g \in My + \Phi(y)$. So, we have $g - \Phi(y) \in My$. Since

$$\widehat{t}_{n_i} = \sigma_{n_i} t_{n_i} + (1 - \sigma_{n_i}) J_{M,\mu_{n_i}}\left(t_{n_i} - \mu_{n_i}\Phi(t_{n_i})\right) \tag{3.40}$$

implies

$$\frac{1}{\mu_{n_i}}\left(t_{n_i} - s_{n_i} - \mu_{n_i}\Phi(t_{n_i})\right) \in Ms_{n_i}, \tag{3.41}$$

where $s_{n_i} = t_{n_i} + (\widehat{t}_{n_i} - t_{n_i})/(1 - \sigma_{n_i})$, we have

$$\left\langle y - s_{n_i}, g - \Phi(y) - \frac{1}{\mu_{n_i}}\left(t_{n_i} - s_{n_i} - \mu_{n_i}\Phi(t_{n_i})\right)\right\rangle \ge 0, \tag{3.42}$$

which hence yields

$$\begin{aligned}
\langle y - s_{n_i}, g\rangle &\geq \left\langle y - s_{n_i}, \Phi(y) + \frac{1}{\mu_{n_i}}\left(t_{n_i} - s_{n_i} - \mu_{n_i}\Phi(t_{n_i})\right)\right\rangle \\
&= \langle y - s_{n_i}, \Phi(y) - \Phi(t_{n_i})\rangle + \left\langle y - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i})\right\rangle \\
&\geq \alpha\left\|\Phi(y) - \Phi(s_{n_i})\right\|^2 + \langle y - s_{n_i}, \Phi(s_{n_i}) - \Phi(t_{n_i})\rangle + \left\langle y - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i})\right\rangle \\
&\leq \langle y - s_{n_i}, \Phi(s_{n_i}) - \Phi(t_{n_i})\rangle + \left\langle y - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i})\right\rangle.
\end{aligned} \tag{3.43}$$

Observe that

$$\begin{aligned}
&\left|\langle y - s_{n_i}, \Phi(s_{n_i}) - \Phi(t_{n_i})\rangle + \left\langle y - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i})\right\rangle\right| \\
&\quad\leq \left\|y - s_{n_i}\right\|\|\Phi(s_{n_i}) - \Phi(t_{n_i})\| + \left\|y - s_{n_i}\right\|\left\|\frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i})\right\| \\
&\quad\leq \frac{1}{\alpha}\left\|y - s_{n_i}\right\|\|s_{n_i} - t_{n_i}\| + \frac{1}{\epsilon}\left\|y - s_{n_i}\right\|\|t_{n_i} - s_{n_i}\| \\
&\quad= \left(\frac{1}{\alpha} + \frac{1}{\epsilon}\right)\left\|y - s_{n_i}\right\|\|t_{n_i} - s_{n_i}\|.
\end{aligned} \tag{3.44}$$

From $\|s_{n_i} - t_{n_i}\| = (1/(1 - \sigma_{n_i}))\|\hat{t}_{n_i} - t_{n_i}\| \leq (1/(1 - c))\|\hat{t}_{n_i} - t_{n_i}\| \to 0$, it follows that

$$\lim_{i\to\infty}\left|\langle y - s_{n_i}, \Phi(s_{n_i}) - \Phi(t_{n_i})\rangle + \left\langle y - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i})\right\rangle\right| = 0. \tag{3.45}$$

Since $\|x_n - t_n\| \to 0$, $\|\hat{t}_n - t_n\| \to 0$, and $x_{n_i} \rightharpoonup u$, we derive $s_{n_i} = t_{n_i} + ((\hat{t}_{n_i} - t_{n_i})/(1 - \sigma_{n_i})) \rightharpoonup u$, and hence by letting $i \to \infty$ we get from (3.43)

$$\langle y - u, g\rangle \geq 0. \tag{3.46}$$

This shows that $0 \in \Phi(u) + Mu$. Thus, $u \in \Omega$. Therefore, $u \in \mathrm{Fix}(S) \cap \Omega \cap \Xi$.

Step 5. We claim that

$$\lim_{n\to\infty}\|x_n - l_0\| = \lim_{n\to\infty}\left\|y_n - l_0\right\| = \lim_{n\to\infty}\|z_n - l_0\| = 0, \tag{3.47}$$

where $l_0 = P_{\mathrm{Fix}(S)\cap\Omega\cap\Xi}x_0$.

Indeed, Since $l_0 = P_{\mathrm{Fix}(S)\cap\Omega\cap\Xi}x_0$, and $u \in \mathrm{Fix}(S) \cap \Omega \cap \Xi$, from (3.14) we have

$$\|l_0 - x_0\| \leq \|u - x_0\| \leq \liminf_{i\to\infty}\|x_{n_i} - x_0\| \leq \limsup_{i\to\infty}\|x_{n_i} - x_0\| \leq \|l_0 - x_0\|. \tag{3.48}$$

So, we obtain

$$\lim_{i \to \infty} \|x_{n_i} - x_0\| = \|u - x_0\|. \tag{3.49}$$

From $x_{n_i} - x_0 \rightharpoonup u - x_0$, we have $x_{n_i} - x_0 \to u - x_0$ (due to the Kadec-Klee property of Hilbert spaces [37]), and hence $x_{n_i} \to u$. Since $x_n = P_{Q_n} x_0$, and $l_0 \in \operatorname{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|l_0 - x_{n_i}\|^2 = \langle l_0 - x_{n_i}, x_{n_i} - x_0 \rangle + \langle l_0 - x_{n_i}, x_0 - l_0 \rangle \geq \langle l_0 - x_{n_i}, x_0 - l_0 \rangle. \tag{3.50}$$

As $i \to \infty$, we obtain $-\|l_0 - x_{n_i}\|^2 \geq \langle l_0 - u, x_0 - l_0 \rangle \geq 0$ by $l_0 = P_{\operatorname{Fix}(S) \cap \Omega \cap \Xi} x_0$, and $u \in \operatorname{Fix}(S) \cap \Omega \cap \Xi$. Hence we have $u = l_0$. This implies that $x_n \to l_0$. It is easy to see that $y_n \to l_0$ and $z_n \to l_0$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$, let $\Phi : C \to H$ be an α-inverse strongly monotone mapping, let M be a maximal monotone mapping with $D(M) = C$, and let $S : C \to C$ be a nonexpansive mapping such that* $\operatorname{Fix}(S) \cap \Omega \cap \Xi \neq \emptyset$. *For given $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned}
y_n &= P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right], \\
t_n &= P_C\left[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\right], \\
\widehat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\
z_n &= \beta_n x_n + \gamma_n \widehat{t}_n + \delta_n S \widehat{t}_n, \\
C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{3.51}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$, and $\{\sigma_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\} \subset [0, 1]$ such that $\{\sigma_n\} \subset [0, c]$ for some $c \in [0, 1)$, $\{\gamma_n\}, \{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$, and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\overline{x} = P_{\operatorname{Fix}(S) \cap \Omega \cap \Xi} x_0$. Furthermore, $(\overline{x}, \overline{y})$ is a solution of the GSVI (1.10), where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.

Proof. Since S is a nonexpansive mapping, S must be a k-strictly pseudocontractive mapping with $k = 0$. Take a fixed $p \in \operatorname{Fix}(S) \cap \Omega \cap \Xi$ arbitrarily. Note that in Step 1 for the proof of Theorem 3.1, we have obtained that $\{x_n\}$ is bounded and the relation holds

$$\left\|\widehat{t}_n - p\right\| \leq \|x_n - p\|, \quad \forall n \geq 0 \tag{3.52}$$

(due to (3.11)). Moreover, in Step 2 for the proof of Theorem 3.1, we have proven that

$$\lim_{n \to \infty} \|z_n - x_n\| = 0. \tag{3.53}$$

Now, utilizing Lemma 2.2, from the nonexpansiveness of S we deduce that

$$\begin{aligned}
\|z_n - p\|^2 &= \left\|\beta_n(x_n - p) + \gamma_n\left(\hat{t}_n - p\right) + \delta_n\left(S\hat{t}_n - p\right)\right\|^2 \\
&\le \beta_n\|x_n - p\|^2 + \gamma_n\left\|\hat{t}_n - p\right\|^2 + \delta_n\left\|S\hat{t}_n - p\right\|^2 - \gamma_n\delta_n\left\|S\hat{t}_n - \hat{t}_n\right\|^2 \\
&\le \beta_n\|x_n - p\|^2 + \gamma_n\left\|\hat{t}_n - p\right\|^2 + \delta_n\left\|\hat{t}_n - p\right\|^2 - \gamma_n\delta_n\left\|S\hat{t}_n - \hat{t}_n\right\|^2 \\
&= \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\left\|\hat{t}_n - p\right\|^2 - \gamma_n\delta_n\left\|S\hat{t}_n - \hat{t}_n\right\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\|x_n - p\|^2 - \gamma_n\delta_n\left\|S\hat{t}_n - \hat{t}_n\right\|^2 \\
&= \|x_n - p\|^2 - \gamma_n\delta_n\left\|S\hat{t}_n - \hat{t}_n\right\|^2.
\end{aligned} \tag{3.54}$$

This together with $\{\gamma_n\}, \{\delta_n\} \subset [d, 1]$ implies that

$$\begin{aligned}
d^2\left\|S\hat{t}_n - \hat{t}_n\right\|^2 \le \gamma_n\delta_n\left\|S\hat{t}_n - \hat{t}_n\right\|^2 &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned} \tag{3.55}$$

So, we immediately derive

$$\lim_{n\to\infty}\left\|S\hat{t}_n - \hat{t}_n\right\| = 0. \tag{3.56}$$

It is easy to see that all the conditions of Theorem 3.1 are satisfied. Therefore, in terms of Theorem 3.1 we obtain the desired result. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$, and let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Xi \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned}
y_n &= P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\
z_n &= \beta_n x_n + \gamma_n P_C[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)] \\
&\quad + \delta_n S P_C[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\
C_n &= \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0,
\end{aligned} \tag{3.57}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$, $\beta_n + \gamma_n + \delta_n = 1$, and $(\gamma_n + \delta_n)k \le \gamma_n$ for all $n \ge 0$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge *strongly to the same point* $\overline{x} = P_{\mathrm{Fix}(S)\cap\Xi}x_0$. *Furthermore,* $(\overline{x}, \overline{y})$ *is a solution of the GSVI* (1.10), *where* $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.

Proof. Putting $\Phi = M = 0$ in Theorem 3.1, we have $\Omega = C$ and $\mathrm{Fix}(S)\cap\Omega\cap\Xi = \mathrm{Fix}(S)\cap\Xi$. Let α be any positive number in the interval $(0,\infty)$, and take any sequence $\{\sigma_n\} \subset [0,c]$ for some $c \in [0,1)$ and any sequence $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$. Then Φ is α-inverse strongly monotone, and we have

$$\begin{aligned}
y_n &= P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right],\\
t_n &= P_C\left[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\right],\\
\widehat{t}_n &= \sigma_n t_n + (1-\sigma_n) J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) = \sigma_n t_n + (1-\sigma_n)(I + \mu_n M)^{-1} t_n = t_n,\\
z_n &= \beta_n x_n + \gamma_n \widehat{t}_n + \delta_n S\widehat{t}_n,\\
C_n &= \{z \in C : \|z_n - z\| \le \|x_n - z\|\},\\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{C_n\cap Q_n} x_0, \quad \forall n \ge 0,
\end{aligned} \tag{3.58}$$

which is just equivalent to (3.57). In this case, we have

$$z_n = \beta_n x_n + \gamma_n \widehat{t}_n + \delta_n S\widehat{t}_n = \beta_n x_n + \gamma_n t_n + \delta_n S t_n. \tag{3.59}$$

Note that in Steps 2 and 3 for the proof of Theorem 3.1, we have proven that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0, \qquad \lim_{n\to\infty}\|t_n - x_n\| = 0, \tag{3.60}$$

respectively. Thus, we have

$$\|\delta_n(St_n - x_n)\| \le \|z_n - x_n\| + \gamma_n\|t_n - x_n\| \longrightarrow 0. \tag{3.61}$$

Consequently, it follows from $\{\delta_n\} \subset [d,1]$ that $\|St_n - x_n\| \to 0$, and hence $\|St_n - t_n\| \to 0$. This shows that $\|S\widehat{t}_n - \widehat{t}_n\| \to 0$. Utilizing Theorem 3.1, we obtain the desired result. □

Remark 3.4. Our Theorems 3.1 improves, extends, and develops [36, Theorem 3.1], [15, Theorem 3.1], [34, Theorem 3.2], and [35, Theorem 3.1] in the following aspects.

(i) Compared with the relaxed extragradient iterative algorithm in [34, Theorem 3.2] and the hybrid extragradient iterative algorithm in [35, Theorem 3.1], our hybrid extragradient iterative algorithms remove the requirements that $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ and $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$.

(ii) The problem of finding an element of $\mathrm{Fix}(S) \cap \Omega \cap \Xi$ in our Theorem 3.1 is more general than the corresponding ones in [36, Theorem 3.1], [15, Theorem 3.1], and [34, Theorem 3.2] to a great extent. Thus, beyond question our results are very interesting and quite valuable.

(iii) The relaxed extragradient method for finding an element of $\mathrm{Fix}(S) \cap \Xi$ in [34, Theorem 3.2] is extended to develop our hybrid extragradient iterative algorithms for finding an element of $\mathrm{Fix}(S) \cap \Omega \cap \Xi$.

(iv) The proof of our results are very different from that of [15, Theorem 3.1] because our argument technique depends on two inverse strongly monotone mappings B_1 and B_2, the property of strict pseudocontractions (see Lemmas 2.6 and 2.7), and the properties of the resolvent $J_{M,\lambda}$ to a great extent.

(v) Because our iterative algorithms involve two inverse strongly monotone mappings B_1 and B_2, a k-strictly pseudocontractive self-mapping S, and several parameter sequences, they are more flexible and more subtle than the corresponding ones in [36, Theorem 3.1], [15, Theorem 3.1], and [34, Theorem 3.2], respectively.

4. Applications

Utilizing Theorem 3.1, we prove some strong convergence theorems in a real Hilbert space.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1,2$, let $\Phi : C \to H$ be an α-inverse strongly monotone mapping, and let M be a maximal monotone mapping with $D(M) = C$ such that $\Omega \cap \Xi \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned}
y_n &= P_C\left[P_C\left(x_n - \mu_2 B_2 x_n\right) - \mu_1 B_1 P_C\left(x_n - \mu_2 B_2 x_n\right)\right], \\
t_n &= P_C\left[P_C\left(y_n - \mu_2 B_2 y_n\right) - \mu_1 B_1 P_C\left(y_n - \mu_2 B_2 y_n\right)\right], \\
z_n &= \beta_n x_n + \left(1 - \beta_n\right)\left[\sigma_n t_n + \left(1 - \sigma_n\right) J_{M,\mu_n}\left(t_n - \mu_n \Phi(t_n)\right)\right], \\
C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{4.1}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1,2$, $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$, and $\{\sigma_n\}, \{\beta_n\} \subset [0,1]$ such that $\{\sigma_n\} \subset [0, c]$ for some $c \in [0,1)$, and $\{\beta_n\} \subset [0, d]$ for some $d \in [0,1)$. Then the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\overline{x} = P_{\Omega \cap \Xi} x_0$. Furthermore, $(\overline{x}, \overline{y})$ is a solution of the GSVI (1.10), *where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.*

Proof. In Corollary 3.2, putting $S = I$, we have

$$\begin{aligned}
y_n &= P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right],\\
t_n &= P_C\left[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\right],\\
\hat{t}_n &= \sigma_n t_n + (1-\sigma_n) J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)),\\
z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S\hat{t}_n = \beta_n x_n + (1-\beta_n)\hat{t}_n,\\
C_n &= \{z \in C : \|z_n - z\| \le \|x_n - z\|\},\\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0,
\end{aligned} \tag{4.2}$$

which is just equivalent to (4.1). In this case, we know that $\mathrm{Fix}(S) \cap \Omega \cap \Xi = \Omega \cap \Xi$. Therefore, by Corollary 3.2 we obtain desired result. □

Theorem 4.2 (see [15, Theorem 4.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S)$ *is nonempty. For given $x_0 \in C$ arbitrarily, let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned}
z_n &= (1-\delta_n)x_n + \delta_n S x_n,\\
C_n &= \{z \in C : \|z_n - z\| \le \|x_n - z\|\},\\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0,
\end{aligned} \tag{4.3}$$

where $\{\delta_n\} \subset [d,1]$ for some $d \in (0,1]$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{\mathrm{Fix}(S)} x_0$.

Proof. Putting $B_1 = B_2 = \Phi = M = 0$ in Corollary 3.2, we let β_1, β_2, and α be any positive numbers in the interval $(0,\infty)$, and take any numbers $\mu_i \in (0, 2\beta_i)$ for $i = 1,2$ and any sequence $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$. Then $B_i : C \to H$ is β_i-inverse strongly monotone for $i = 1,2$, and $\Phi : C \to H$ is α-inverse strongly monotone. In this case, we know that $\mathrm{Fix}(S) \cap \Omega \cap \Xi = \mathrm{Fix}(S)$ and

$$\begin{aligned}
y_n &= P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right] = x_n,\\
t_n &= P_C\left[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\right] = y_n,
\end{aligned}$$

$$
\begin{aligned}
\widehat{t}_n &= \sigma_n t_n + (1-\sigma_n) J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) = t_n, \\
z_n &= \beta_n x_n + \gamma_n \widehat{t}_n + \delta_n S \widehat{t}_n = (1-\delta_n) x_n + \delta_n S x_n, \\
C_n &= \{ z \in C : \|z_n - z\| \le \|x_n - z\| \}, \\
Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0,
\end{aligned}
\tag{4.4}
$$

which is just equivalent to (4.3). Therefore, by Corollary 3.2 we obtain the desired result. □

Remark 4.3. Originally Theorem 4.2 is the result of Nakajo and Takahashi [22].

Theorem 4.4. *Let H be a real Hilbert space. Let $A : H \to H$ be a λ-inverse strongly monotone mapping, let $\Phi : H \to H$ be an α-inverse strongly monotone mapping, let $M : H \to 2^H$ be a maximal monotone mapping, and let $S : H \to H$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Omega \cap A^{-1}0 \neq \emptyset$*. For given $x_0 \in H$ arbitrarily, let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by*

$$
\begin{aligned}
t_n &= x_n - \mu [A x_n + A(x_n - \mu A x_n)], \\
z_n &= \beta_n x_n + \gamma_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) + \delta_n S J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\
C_n &= \{ z \in C : \|z_n - z\| \le \|x_n - z\| \}, \\
Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0,
\end{aligned}
\tag{4.5}
$$

where $\mu \in (0, 2\lambda)$, $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$, and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ such that $\{\gamma_n\}, \{\delta_n\} \subset [d, 1]$ for some $d \in (0,1]$, and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \ge 0$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{\mathrm{Fix}(S) \cap \Omega \cap A^{-1}0} x_0$.

Proof. Putting $C = H$, $B_1 = A$, $B_2 = 0$, $\mu_1 = \mu$, and $\sigma_n = 0$, for all $n \ge 0$ in Corollary 3.2, we know that $P_C = P_H = I$ and the GSVI (1.10) coincides with the VI (1.3). Hence we have $A^{-1}0 = \mathrm{VI}(H, A) = \Xi$. In this case, we conclude that $\mathrm{Fix}(S) \cap \Omega \cap \Xi = \mathrm{Fix}(S) \cap \Omega \cap A^{-1}0$ and

$$
\begin{aligned}
y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] = x_n - \mu A x_n, \\
t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)] = x_n - \mu A x_n - \mu A(x_n - \mu A x_n), \\
\widehat{t}_n &= \sigma_n t_n + (1-\sigma_n) J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) = J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\
z_n &= \beta_n x_n + \gamma_n \widehat{t}_n + \delta_n S \widehat{t}_n, \\
C_n &= \{ z \in C : \|z_n - z\| \le \|x_n - z\| \}, \\
Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0.
\end{aligned}
\tag{4.6}
$$

Therefore, by Corollary 3.2 we obtain the desired result. □

Let $B : H \to 2^H$ be a maximal monotone mapping. Then, for any $x \in H$ and $r > 0$, consider $J_{B,r}x = (I + rB)^{-1}x$. It is known that such a $J_{B,r}$ is the resolvent of B.

Theorem 4.5. *Let H be a real Hilbert space. Let $A : H \to H$ be a λ-inverse strongly monotone mapping, let $\Phi : H \to H$ be an α-inverse strongly monotone mapping, and let $B, M : H \to 2^H$ be two maximal monotone mappings such that $A^{-1}0 \cap B^{-1}0 \cap \Omega \neq \emptyset$. Let $J_{B,r}$ be the resolvent of B for each $r > 0$. For given $x_0 \in H$ arbitrarily, let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned} t_n &= x_n - \mu\left[Ax_n + A(x_n - \mu Ax_n)\right], \\ z_n &= \beta_n x_n + \gamma_n J_{M,\mu_n}(t_n - \mu_n\Phi(t_n)) + \delta_n J_{B,r} J_{M,\mu_n}(t_n - \mu_n\Phi(t_n)), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \tag{4.7}$$

where $\mu \in (0, 2\lambda)$, $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$, and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\{\gamma_n\}, \{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$, and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{A^{-1}0 \cap B^{-1}0 \cap \Omega} x_0$.

Proof. Putting $S = J_{B,r}$ in Theorem 4.4, we know that $\mathrm{Fix}(S) = \mathrm{Fix}(J_{B,r}) = B^{-1}0$. In this case, (4.5) is coincident with (4.7). Therefore, by Theorem 4.4 we obtain the desired result. □

It is well known that a mapping $T : C \to C$ is called pseudocontractive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$, for all $x, y \in C$. It is easy to see that this definition is equivalent to the one that a mapping $T : C \to C$ is called pseudocontractive if $\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2$, for all $x, y \in C$; see [8]. In the meantime, we also know one more definition of a k-strictly pseudocontractive mapping, which is equivalent to the definition given in the introduction. A mapping $T : C \to C$ is called k-strictly pseudocontractive if there exists a constant $k \in [0, 1)$, such that

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - k}{2}\|(I - T)x - (I - T)y\|^2, \tag{4.8}$$

for all $x, y \in C$. It is clear that in this case the mapping $I - T$ is $(1 - k)/2$-inverse strongly monotone. From [10], we know that if T is a k-strictly pseudocontractive mapping, then T is Lipschitz continuous with constant $(1+k)/(1-k)$, such that $\mathrm{Fix}(T) = \mathrm{VI}(C, I-T)$ (see, e.g., the proof of Theorem 4.6). It is obvious that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings and the class of pseudocontractions strictly includes the class of strict pseudocontractions.

In the following theorem we introduce an iterative algorithm that converges strongly to a common fixed point of three mappings: one of which is nonexpansive, and the other two ones are strictly pseudocontractive mappings.

Theorem 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping, let $\Gamma : C \to C$ be a κ-strictly pseudocontractive mapping,* *and let* $S : C \to C$ *be a nonexpansive mapping such that* $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(\Gamma) \neq \emptyset$*. For given* $x_0 \in C$ *arbitrarily, let* $\{x_n\}$*,* $\{y_n\}$*, and* $\{z_n\}$ *be the sequences generated by*

$$\begin{aligned}
y_n &= x_n - \mu_1 (x_n - T x_n), \\
t_n &= y_n - \mu_1 (y_n - T y_n), \\
\hat{t}_n &= t_n - \mu_n (t_n - \Gamma t_n), \\
z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\
C_n &= \{ z \in C : \|z_n - z\| \leq \|x_n - z\| \}, \\
Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{4.9}$$

where $\mu_1 \in (0, 1-k)$*,* $\{\mu_n\} \subset [\epsilon, 1-\kappa]$ *for some* $\epsilon \in (0, 1-\kappa]$*, and* $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ *such that* $\{\gamma_n\}, \{\delta_n\} \subset [d, 1]$ *for some* $d \in (0,1]$*, and* $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 0$*. Then the sequences* $\{x_n\}, \{y_n\}$*, and* $\{z_n\}$ *converge strongly to* $P_{\mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(\Gamma)} x_0$*.*

Proof. Putting $B_1 = I - T$, $B_2 = 0$, $\Phi = I - \Gamma$, $M = 0$, and $\sigma_n = 0$, for all $n \geq 0$ in Corollary 3.2, we know that B_1 is β_1-inverse strongly monotone with $\beta_1 = (1-k)/2$ and Φ is α-inverse strongly monotone with $\alpha = (1-\kappa)/2$. Moreover, we have $\Xi = \mathrm{VI}(C, B_1) = \mathrm{VI}(C, I - T)$. Noticing $\mu_1 \in (0, 1-k)$ and $k \in [0,1)$, we know that $\mu_1 \in (0,1)$, and hence $(1-\mu_1)x_n + \mu_1 T x_n \in C$. Also, noticing $\{\mu_n\} \subset [\epsilon, 1-\kappa] \subset (0, 1-\kappa]$, we know that $\{\mu_n\} \subset (0,1]$, and hence $(1-\mu_n)t_n + \mu_n \Gamma t_n \in C$. This implies that

$$\begin{aligned}
y_n &= P_C \left[P_C (x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C (x_n - \mu_2 B_2 x_n) \right] \\
&= P_C \left((1-\mu_1) x_n + \mu_1 T x_n \right) = x_n - \mu_1 (x_n - T x_n), \\
t_n &= P_C \left[P_C (y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C (y_n - \mu_2 B_2 y_n) \right] \\
&= P_C \left((1-\mu_1) y_n + \mu_1 T y_n \right) = y_n - \mu_1 (y_n - T y_n), \\
\hat{t}_n &= \sigma_n t_n + (1-\sigma_n) J_{M,\mu_n} (t_n - \mu_n \Phi(t_n)) = t_n - \mu_n (t_n - \Gamma t_n), \\
z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\
C_n &= \{ z \in C : \|z_n - z\| \leq \|x_n - z\| \}, \\
Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0.
\end{aligned} \tag{4.10}$$

Now let us show $\mathrm{Fix}(T) = \mathrm{VI}(C, B_1)$. In fact, we have, for $\lambda > 0$,

$$\begin{aligned}
u \in \mathrm{VI}(C, B_1) &\Longleftrightarrow \langle B_1 u, y - u \rangle \geq 0, \quad \forall y \in C \\
&\Longleftrightarrow \langle u - \lambda B_1 u - u, u - y \rangle \geq 0, \quad \forall y \in C \\
&\Longleftrightarrow u = P_C(u - \lambda B_1 u) \\
&\Longleftrightarrow u = P_C(u - \lambda u + \lambda T u) \\
&\Longleftrightarrow \langle u - \lambda u + \lambda T u - u, u - y \rangle \geq 0, \quad \forall y \in C \\
&\Longleftrightarrow \langle u - T u, u - y \rangle \leq 0, \quad \forall y \in C \\
&\Longleftrightarrow u = T u \\
&\Longleftrightarrow u \in \mathrm{Fix}(T).
\end{aligned} \tag{4.11}$$

Next let us show $\Omega = \mathrm{Fix}(\Gamma)$. In fact, noticing that $M = 0$ and $\Phi = I - \Gamma$, we have

$$u \in \Omega \Longleftrightarrow 0 \in \Phi(u) + Mu \Longleftrightarrow 0 = \Phi(u) = u - \Gamma u \Longleftrightarrow u \in \mathrm{Fix}(\Gamma). \tag{4.12}$$

Consequently,

$$\mathrm{Fix}(S) \cap \Omega \cap \Xi = \mathrm{Fix}(S) \cap \mathrm{Fix}(\Gamma) \cap \mathrm{VI}(C, B_1) = \mathrm{Fix}(S) \cap \mathrm{Fix}(\Gamma) \cap \mathrm{Fix}(T). \tag{4.13}$$

Therefore, by Theorem 3.1 we obtain the desired result. □

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References

[1] J. L. Lions, *Quelques Methodes de Resolution des Problems aux Limites Non Lineaires*, Dunod, Paris, France, 1969.
[2] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer Series in Computational Physics, Springer, New York, NY, USA, 1984.
[3] J. T. Oden, *Quantitative Methods on Nonlinear Mechanics*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1986.
[4] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, Springer, New York, NY, USA, 1985.
[5] A. S. Antipin, "Methods for solving variational inequalities with related constraints," *Computational Mathematics and Mathematical Physics*, vol. 40, no. 9, pp. 239–1254, 2000.
[6] A. S. Antipin and F. P. Vasil'ev, "A regularized prediction method for solving variational inequalities with an inexactly specified set," *Computational Mathematics and Mathematical Physics*, vol. 44, no. 5, pp. 796–804, 2004.

[7] I. Yamada, "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa, 2000)*, D. Butnariu, Y. Censor, and S. Reich, Eds., vol. 8 of *Studies in Computational Mathematics*, pp. 473–504, Kluwer Academic, Dordrecht, The Netherlands, 2001.

[8] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[9] F. Liu and M. Z. Nashed, "Regularization of nonlinear ill-posed variational inequalities and convergence rates," *Set-Valued Analysis*, vol. 6, no. 4, pp. 313–344, 1998.

[10] G. Marino and H. K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.

[11] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

[12] N. J. Huang, "A new completely general class of variational inclusions with noncompact valued mappings," *Computers & Mathematics with Applications*, vol. 35, no. 10, pp. 9–14, 1998.

[13] L. C. Zeng, S. M. Guu, and J. C. Yao, "Characterization of H-monotone operators with applications to variational inclusions," *Computers & Mathematics with Applications*, vol. 50, no. 3-4, pp. 329–337, 2005.

[14] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.

[15] N. Nadezhkina and W. Takahashi, "Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings," *SIAM Journal on Optimization*, vol. 16, no. 4, pp. 1230–1241, 2006.

[16] G. M. Korpelevich, "An extragradient method for finding saddle points and for other problems," *Èkonomika i Matematicheskie Metody*, vol. 12, no. 4, pp. 747–756, 1976.

[17] H. H. Bauschke and P. L. Combettes, "A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces," *Mathematics of Operations Research*, vol. 26, no. 2, pp. 248–264, 2001.

[18] H. H. Bauschke and P. L. Combettes, "Construction of best Bregman approximations in reflexive Banach spaces," *Proceedings of the American Mathematical Society*, vol. 131, no. 12, pp. 3757–3766, 2003.

[19] R. S. Burachik, J. O. Lopes, and B. F. Svaiter, "An outer approximation method for the variational inequality problem," *SIAM Journal on Control and Optimization*, vol. 43, no. 6, pp. 2071–2088, 2005.

[20] P. L. Combettes, "Strong convergence of block-iterative outer approximation methods for convex optimization," *SIAM Journal on Control and Optimization*, vol. 38, no. 2, pp. 538–565, 2000.

[21] H. Iiduka and W. Takahashi, "Strong convergence theorem by a hybrid method for nonlinear mappings of nonexpansive and monotone type and applications," *Advances in Nonlinear Variational Inequalities*, vol. 9, no. 1, pp. 1–10, 2006.

[22] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[23] M. V. Solodov and B. F. Svaiter, "Forcing strong convergence of proximal point iterations in a Hilbert space," *Mathematical Programming*, vol. 87, no. 1, pp. 189–202, 2000.

[24] R. Gárciga Otero and A. Iusem, "Proximal methods with penalization effects in Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 25, no. 1-2, pp. 69–91, 2004.

[25] B. S. He, Z. H. Yang, and X. M. Yuan, "An approximate proximal-extragradient type method for monotone variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 2, pp. 362–374, 2004.

[26] M. A. Noor, "New extragradient-type methods for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 379–394, 2003.

[27] M. V. Solodov, "Convergence rate analysis of iteractive algorithms for solving variational inquality problems," *Mathematical Programming*, vol. 96, no. 3, pp. 513–528, 2003.

[28] M. V. Solodov and B. F. Svaiter, "An inexact hybrid generalized proximal point algorithm and some new results on the theory of Bregman functions," *Mathematics of Operations Research*, vol. 25, no. 2, pp. 214–230, 2000.

[29] P. Katchang and P. Kumam, "Hybrid-extragradient type methods for a generalized equilibrium problem and variational inequality problems of nonexpansive semigroups," *Fixed Point Theory*, vol. 13, no. 1, pp. 107–120, 2012.

[30] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[31] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[32] L. C. Ceng, C. Y. Wang, and J. C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[33] R. U. Verma, "On a new system of nonlinear variational inequalities and associated iterative algorithms," *Mathematical Sciences Research Hot-Line*, vol. 3, no. 8, pp. 65–68, 1999.

[34] Y. Yao, Y. C. Liou, and S. M. Kang, "Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3472–3480, 2010.

[35] L. C. Ceng, S. M. Guu, and J. C. Yao, "Finding common solutions of a variational inequality, a general system of variational inequalities, and a fixed-point problem via a hybrid extragradient method," *Fixed Point Theory and Applications*, vol. 2011, Article ID 626159, 22 pages, 2011.

[36] L. C. Ceng, Q. H. Ansari, M. M. Wong, and J. C. Yao, "Mann type hybrid extragradient method for variational inequalities, variational inclusions and fixed point problems," *Fixed Point Theory*, vol. 13, no. 2, pp. 1–20, 2012.

[37] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

[38] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, pp. 75–88, 1970.

[39] M. O. Osilike and D. I. Igbokwe, "Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations," *Computers & Mathematics with Applications*, vol. 40, no. 4-5, pp. 559–567, 2000.

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Research Article

A Proximal Analytic Center Cutting Plane Algorithm for Solving Variational Inequality Problems

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Under the condition that the values of mapping F are evaluated approximately, we propose a proximal analytic center cutting plane algorithm for solving variational inequalities. It can be considered as an approximation of the earlier cutting plane method, and the conditions we impose on the corresponding mappings are more relaxed. The convergence analysis for the proposed algorithm is also given at the end of this paper.

1. Introduction

According to [1], the history of algorithms for solving finite-dimensional variational inequalities is relatively short. A recent development of such methods is the analytic center method based on cutting plane methods. It combines the feature of the newly developed interior point methods with the classical cutting plane scheme to achieve polynomial complexity in theory and quick convergence in practice. More details can be found in [2, 3]. Specifically, Goffin et al. [4] developed a convergent framework for finding a solution x^* of the variational inequality $\mathrm{VI}_P(F, X)$ associated with the continuous mapping F from X to R^n and the polyhedron $X = \{x \in R^n \mid Ax \leq b\}$ under an assumption slightly stronger than pseudomonotonicity. Again, Marcotte and Zhu [5] extended this algorithm to quasimonotone variational inequalities that satisfy a weak additional assumption. Such methods are effective in practice.

Note that the facts in optimization problems, see [6–8], some functions from R^n to R are themselves defined through other minimization problems. For example, consider the Lagrangian relaxation, see [9–12], the primal problem is

$$\max\{q(\xi) \mid \xi \in P, h(\xi) = 0\}, \tag{1.1}$$

where P is a compact subset of R^m and $q : R^m \to R$, $h : R^m \to R^n$ are two functions. Lagrangian relaxation in this problem leads to the problem $\min\{f(x) \mid x \in R^n\}$, where

$$f(x) = \max_{\xi\in P}\{q(\xi) + \langle x, h(\xi)\rangle\} \tag{1.2}$$

is the dual function. Trying to solve problem (1.1) by means of solving its dual problem $\min\{f(x) \mid x \in R^n\}$ becomes more difficult since in this case evaluating the function value $f(x)$ requires solving exactly another optimization problem (1.2). Let us see another example: consider the problem

$$\min\{f(x) \mid x \in C\}, \tag{1.3}$$

where f is convex (not necessarily differentiable), $C \subset R^n$ is a nonempty closed convex set, F is called the Moreau-Yosida regularization of f on C, that is,

$$F(x) = \min_{z\in C}\left\{f(z) + \frac{1}{2\alpha}\|z - x\|^2\right\}, \tag{1.4}$$

where α is a positive parameter. A point $x \in C$ is a solution to (1.3) if and only if it is a solution to the problem:

$$\min_{x\in R^n} F(x). \tag{1.5}$$

The problem (1.5) is easier to deal with than (1.3), see [13]. But in this case, computing the exact function value of F at an arbitrary point x is difficult or even impossible since F itself is defined through a minimization problem involving another function f. Intuitively, we consider the approximate computation of function F.

The above-mentioned phenomenon also exists for mappings from X to Y, where X and Y are two subspaces of any two finite-dimensional spaces, respectively. Once a mapping, and more specifically, a continuous mapping is defined implicitly rather than explicitly, the approximation of the mapping becomes inevitable, see [14]. In this paper we try to solve $\mathrm{VI}_P(F, X)$ by assuming the values of the mapping F from X to R^n can be only computed approximately. Under the assumption, we construct an algorithm for solving the approximate variational inequality problem $\mathrm{AVI}_P(F, X)$ and we also prove that there exists a cluster point of the iteration points generated by the proposed algorithm, it is a solution to the original problem $\mathrm{VI}_P(F, X)$.

This paper is organized as follows. Some basic concepts and results are introduced in Section 2. In Section 3, a proximal analytic center cutting plane algorithm for solving the variational inequality problems is given. The convergence analysis of the proposed algorithm is addressed in Section 4. In the last section, we give some conclusions.

2. Basic Concepts and Results

Let $X = \{x \in R^n \mid Ax \leq b\}$ be a polyhedron and F a continuous mapping from X to R^n. A vector x^* is a solution to the variational inequality $\mathrm{VI}_P(F, X)$ if and only if it satisfies the system of nonlinear inequalities:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in X. \tag{2.1}$$

The vector x^* is a solution to the dual variational inequality $\mathrm{VI}_D(F, X)$ of $\mathrm{VI}_P(F, X)$ if and only if it satisfies

$$\langle F(x), x - x^* \rangle \geq 0, \quad \forall x \in X. \tag{2.2}$$

We denote by X_P^* the solution set of $\mathrm{VI}_P(F, X)$, and X_D^* the solution set of $\mathrm{VI}_D(F, X)$, respectively. Whenever F is continuous, we have $X_D^* \subseteq X_P^*$, see [15]. If F is pseudomonotone on X, then $X_D^* = X_P^*$, see [15]. If F is quasimonotone at $x^* \in X_P^*$ and $F(x^*)$ is not normal to X at x^*, then X_D^* is nonempty, see Proposition 1 in [5]. For the definitions of monotone, pseudomonotone and quasimonotone, see [5, 15].

Definition 2.1. The gap functions $g_P(x)$ and $g_D(x)$ of $\mathrm{VI}_P(F, X)$ and $\mathrm{VI}_D(F, X)$ are, respectively, defined by

$$\begin{aligned} g_P(x) &= \max_{y \in X} \langle F(x), x - y \rangle, \\ g_D(x) &= \max_{y \in X} \langle F(y), x - y \rangle. \end{aligned} \tag{2.3}$$

Note that $g_P(x) \geq 0$, $g_D(x) \geq 0$, and $g_P(x^*) = 0$ if and only if x^* is a solution to $\mathrm{VI}_P(F, X)$, $g_D(x^*) = 0$ if and only if x^* is a solution to $\mathrm{VI}_D(F, X)$. Thus, $X_P^* = \{x \in X \mid g_P(x) = 0\}$, $X_D^* = \{x \in X \mid g_D(x) = 0\}$.

Definition 2.2. A point $\tilde{x} \in X$ is called an ε-solution to $\mathrm{VI}_P(F, X)$ if $g_P(\tilde{x}) \leq \varepsilon$ for given $\varepsilon > 0$.

Definition 2.3. For $x, y \in X$, we say $F(x) \preceq F(y)$ if and only if $F_i(x) \leq F_i(y)$, for $i = 1, 2, \ldots, n$, where $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^T$.

Assumptions 2.4. Throughout this paper, we make the following assumptions: for each $x, y \in X$, given any $\overline{\varepsilon} = (\varepsilon, \varepsilon, \ldots, \varepsilon)$, $\overline{\delta} = (\delta, \delta, \ldots, \delta)$, where $\varepsilon, \delta \in (0, 1)$, we can always find a $\overline{F}_x \in R^n$ and a $\overline{F}_y \in R^n$ such that

$$\begin{aligned} &\text{(a)} \quad F(x) \preceq \overline{F}_x \preceq F(x) + \overline{\varepsilon}, \ F(y) \preceq \overline{F}_y \preceq F(y) + \overline{\delta}, \\ &\qquad \text{that is, we can compute the approximate value of } F \text{ with any precision;} \\ &\text{(b)} \quad \overline{F}_y \longrightarrow \overline{F}_x \text{ if } y \longrightarrow x, \text{ no matter the relationship between } \overline{\varepsilon} \text{ and } \overline{\delta}; \\ &\text{(c)} \quad \left\| \overline{F}_y - \overline{F}_x \right\| \leq L \| y - x \|, \text{ where } L > 0 \text{ is a constant.} \end{aligned} \tag{2.4}$$

These assumptions are realistic in practice, see [16, 17]. By using the given architecture in [16, 17], we can approximate the mapping F arbitrary well since neural networks are capable of approximating any function from one finite-dimensional real vector space to another one arbitrary well, see [18]. Specifically, let us consider the case of univariate function. If f is a min-type function of the form

$$f(x) = \inf\{N_z(x) \mid z \in Z\}, \tag{2.5}$$

where each $N_z(x)$ is convex and Z is an infinite set, then it may be impossible to calculate $f(x)$. However, we may still consider two cases. In the first case of controllable accuracy, for each positive $\varepsilon > 0$ one can find an ε-minimizer of (2.5), that is, an element $z_x \in Z$ satisfying $N_{z_x}(x) \le f(x) + \varepsilon$; in the second case, this may be possible only for some fixed (any possibly unknown) $\varepsilon < \infty$. In both cases, we may set $\overline{f}_x = N_{z_x}(x) \le f(x) + \varepsilon$. A special case of (2.5) arises from Lagrangian relaxation [15], where the problem $\max\{f(x) \mid x \in S\}$ with $S = R^n_+$ is the Lagrangian dual of the primal problem

$$\inf \ \psi_0(x) \quad \text{s.t. } \psi_j(x) \ge 0, \quad j = 1, 2, \dots, n, \ x \in X, \tag{2.6}$$

with $N_z(x) = \psi_0(z) + \langle x, \psi(z)\rangle$ for $\psi = (\psi_1, \psi_2, \dots, \psi_n)$. Then, for each multiplier $x \ge 0$, we need only to find $z_x \in Z$ such that $\overline{f}_x = N_{z_x}(x) \le f(x) + \varepsilon$, see [19].

Under the above assumptions (2.4), we introduce an approximate problem $\mathrm{AVI}_P(F, X)$ associated with $\mathrm{VI}_P(F, X)$: finding $x^* \in X$ such that

$$\left\langle \overline{F}_{x^*}, x - x^* \right\rangle \ge 0, \quad \forall x \in X, \tag{2.7}$$

where $\overline{F}_{x^*}$ satisfies $F(x^*) \preceq \overline{F}_{x^*} \preceq F(x^*) + \overline{\varepsilon}$ for arbitrary $\overline{\varepsilon} \succ 0$. Its dual problem $\mathrm{AVI}_D(F, X)$ is to find $x^* \in X$ such that

$$\left\langle \overline{F}_{x}, x - x^* \right\rangle \ge 0, \quad \forall x \in X, \tag{2.8}$$

where $\overline{F}_x$ satisfies $F(x) \preceq \overline{F}_x \preceq F(x) + \overline{\varepsilon}$ for arbitrary $\overline{\varepsilon} \succ 0$.

Definition 2.5. The gap function of $\mathrm{AVI}_P(F, X)$ is defined by $\overline{g}_P(x) = \max_{y \in X}\langle \overline{F}_x, x - y\rangle$.

Definition 2.6. A point $\tilde{x} \in X$ is called an ε-solution to $\mathrm{AVI}_P(F, X)$ if $\overline{g}_P(\tilde{x}) \le \varepsilon$ for given $\varepsilon > 0$.

The optimal solution sets of $\mathrm{AVI}_P(F, X)$ and $\mathrm{AVI}_D(F, X)$ are denoted by AX_P^* and AX_D^*, respectively. The following proposition ensures that AX_D^* is nonempty.

Proposition 2.7. *If there exists a point $x^* \in AX_P^*$ such that*

$$\left\langle \overline{F}_{x^*}, y - x^* \right\rangle > 0 \Longrightarrow \left\langle \overline{F}_{y}, y - x^* \right\rangle \ge 0, \quad \forall y \in X, \tag{2.9}$$

and $\overline{F}_{x^*}$ is not normal to X at x^*, then AX_D^* is nonempty.

Proof. Since $\overline{F}_{x^*}$ is not normal to X at x^*, there exists a point $x_0 \in X$ such that $\langle \overline{F}_{x^*}, x_0 - x^* \rangle > 0$. $\forall\, x \in X$, set $x_t = tx_0 + (1-t)x$ for $t \in (0,1]$, then we have $\langle \overline{F}_{x^*}, x - x^* \rangle \geq 0$ and $\langle \overline{F}_{x^*}, x_t - x^* \rangle > 0$. Note the condition (2.9), we obtain $\langle \overline{F}_{x_t}, x_t - x^* \rangle \geq 0$. Letting $t \to 0$, it follows from the condition (b) in (2.4) that $\langle \overline{F}_x, x - x^* \rangle \geq 0$, that is, $x^* \in AX_D^*$. □

In the following part, we focus our attention on solving $\mathrm{AVI}_P(F, X)$. Let $\Gamma(y, x) : R^n \times R^n \to R^n$ denote an auxiliary mapping, continuous in x and y, strongly monotone in y, that is,

$$\langle \Gamma(y,x) - \Gamma(z,x), y - z \rangle \geq \beta \left\| y - z \right\|^2, \quad \forall y, z \in X, \tag{2.10}$$

for some $\beta > 0$. We consider the auxiliary variational inequality associated with Γ, whose solution $w(x)$ satisfies

$$\left\langle \Gamma(w(x), x) - \Gamma(x,x) + \overline{F}_x, y - w(x) \right\rangle \geq 0, \quad \forall y \in X. \tag{2.11}$$

In view of the strong monotonicity of $\Gamma(y, x)$ with respect to y, this auxiliary variational inequality has a unique solution $w(x)$.

Proposition 2.8. *The mapping $w : X \to X$ is continuous on X. Furthermore, x is a solution to $AVI_P(F, X)$ if and only if $x = w(x)$.*

Proof. The first part of the proposition follows from Theorem 5.4 in [1]. To prove the second part, we first suppose that $x = w(x)$. This yields $\langle \overline{F}_x, y - x \rangle \geq 0$, $\forall y \in X$, that is, x solves $\mathrm{AVI}_P(F, X)$. Conversely, suppose that x solves $\mathrm{AVI}_P(F, X)$, then

$$\left\langle \overline{F}_x, w(x) - x \right\rangle \geq 0, \tag{2.12}$$

and from (2.11), we have

$$\left\langle \Gamma(w(x), x) - \Gamma(x,x) + \overline{F}_x, x - w(x) \right\rangle \geq 0. \tag{2.13}$$

Adding the two preceding inequalities, one obtains

$$\langle \Gamma(w(x), x) - \Gamma(x,x), x - w(x) \rangle \geq 0, \tag{2.14}$$

and we conclude, from the strong monotonicity of Γ with respect to y, that $x = w(x)$. □

Let $\rho < 1$, $\alpha < \beta$ be two positive numbers. Let l be the smallest nonnegative integer satisfying

$$\left\langle \overline{F}_{x+\rho^l(w(x)-x)}, x - w(x) \right\rangle \geq \alpha \| w(x) - x \|^2, \tag{2.15}$$

where $\overline{F}_{x+\rho^l(w(x)-x)}$ satisfies $F(x+\rho^l(w(x)-x)) \preceq \overline{F}_{x+\rho^l(w(x)-x)} \preceq F(x+\rho^l(w(x)-x)) + \overline{\varepsilon}$ for arbitrary $\overline{\varepsilon} \succ 0$. The existence of a finite l will be proved in Proposition 2.9. The composite mapping G is defined, for every $x \in X$, by

$$G(x) = \overline{G}_x = \overline{F}_{x+\rho^l(w(x)-x)}. \tag{2.16}$$

If x^* is a solution to $\mathrm{AVI}_P(F, X)$, then we have $w(x^*) = x^*$, $l = 0$ and $\overline{G}_{x^*} = \overline{F}_{x^*}$.

Proposition 2.9. *The operator G is well defined for every $x \in X$. Moreover, we have*

$$l \leq \frac{\ln((\beta-\alpha)/L)}{\ln \rho}, \tag{2.17}$$

where L is the number given in (2.4)-*(c).*

Proof. From the definition of $w(x)$, we have

$$\langle \overline{F}_x, x - w(x) \rangle \geq \langle \Gamma(w(x), x) - \Gamma(x, x), w(x) - x \rangle \geq \beta \|x - w(x)\|^2. \tag{2.18}$$

Suppose (2.15) does not hold for any finite integer l, that is,

$$\langle \overline{F}_{x+\rho^l(w(x)-x)}, x - w(x) \rangle < \alpha \|x - w(x)\|^2. \tag{2.19}$$

Note the assumption (2.4)-(b), we obtain

$$\overline{F}_{x+\rho^l(w(x)-x)} \longrightarrow \overline{F}_x \quad \text{as } l \longrightarrow +\infty, \tag{2.20}$$

therefore,

$$\left\langle \overline{F}_x, x - w(x) \right\rangle \leq \alpha \|x - w(x)\|^2. \tag{2.21}$$

Since $\alpha < \beta$, (2.21) is in contradiction with (2.18). To prove the second part, we notice that

$$\begin{aligned} \left\langle \overline{F}_{x+\rho^l(w(x)-x)}, x - w(x) \right\rangle &= \left\langle \overline{F}_x, x - w(x) \right\rangle + \left\langle \overline{F}_{x+\rho^l(w(x)-x)} - \overline{F}_x, x - w(x) \right\rangle \\ &\geq \beta \|w(x) - x\|^2 - L\rho^l \|w(x) - x\|^2 \\ &\geq \alpha \|w(x) - x\|^2 \end{aligned} \tag{2.22}$$

if $\alpha \leq \beta - L\rho^l$, which means the second conclusion of Proposition 2.9 holds. □

Proposition 2.10. *If $x \notin AX_P^*$, then $\forall y^* \in AX_D^*$, we have*

$$\left\langle \overline{G}_x, x - y^* \right\rangle > 0. \tag{2.23}$$

Proof. Let $y(x) = x + \rho^l(w(x) - x)$, then $\overline{G}_x = \overline{F}_{y(x)}$ and

$$\left\langle \overline{F}_{y(x)}, w(x) - x \right\rangle \leq -\alpha \|w(x) - x\|^2. \tag{2.24}$$

Since $x \notin AX_P^*$, so $w(x) \neq x$. Therefore,

$$\left\langle \overline{F}_{y(x)}, y(x) - x \right\rangle = \rho^l \left\langle \overline{F}_{y(x)}, w(x) - x \right\rangle < 0. \tag{2.25}$$

For all $y^* \in AX_D^*$, there holds

$$\left\langle \overline{F}_{y(x)}, y(x) - y^* \right\rangle \geq 0. \tag{2.26}$$

By combining (2.25) with (2.26), we obtain $\langle \overline{F}_{y(x)}, x - y^* \rangle > 0$, that is, $\langle \overline{G}_x, x - y^* \rangle > 0$. □

3. A Proximal Analytic Center Cutting Plane Algorithm

Algorithm 3.1 offered in this section is a modification of the algorithm in [5]. Algorithm 3.1 is described as follows.

Algorithm 3.1. Let β be the strong monotonicity constant of $\Gamma(x, y)$ with respect to y, and let $\alpha \in (0, \beta)$, $\varepsilon \in (0, 1)$ be two constants. Set $k = 0$, $A^k = A \in R^{m \times n}$, $b^k = b$, and $\varepsilon^k = \varepsilon$.

Step 1 (computation of the center). Find an approximate analytic center x^k of $X^k = \{x \in R^n \mid A^k x \leq b^k\}$.

Step 2 (stopping criterion). If $g_P(x^k) \leq \varepsilon$, stop.

Step 3 (solving the approximate auxiliary variational inequality problem). Find $w(x^k)$, such that

$$\left\langle \overline{F}_{x^k} + \Gamma\left(w\left(x^k\right), x^k\right) - \Gamma\left(x^k, x^k\right), y - w\left(x^k\right) \right\rangle \geq 0, \quad \forall y \in X, \tag{3.1}$$

where $\overline{F}_{x^k}$ satisfies $F(x^k) \leq \overline{F}_{x^k} \leq F(x^k) + \overline{\varepsilon}_{x^k}$, $\overline{\varepsilon}_{x^k} = (\varepsilon^k, \varepsilon^k, \ldots, \varepsilon^k)^T$.

Step 4 (construction of the approximate cutting plane). Let $y^k = x^k + \rho^{l_k}(w(x^k) - x^k)$ and $\overline{G}_{x^k} = \overline{F}_{y^k}$, where l_k is the smallest integer that satisfies

$$\left\langle \overline{F}_{x^k + \rho^{l_k}(w(x^k) - x^k)}, x^k - w\left(x^k\right) \right\rangle \geq \alpha \left\| w(x^k) - x^k \right\|^2, \tag{3.2}$$

where $\overline{F}_{x^k + \rho^{l_k}(w(x^k) - x^k)}$ satisfies $F(x^k + \rho^{l_k}(w(x^k) - x^k)) \leq \overline{F}_{x^k + \rho^{l_k}(w(x^k) - x^k)} \leq F(x^k + \rho^{l_k}(w(x^k) - x^k)) + \overline{\varepsilon}_{x^k + \rho^{l_k}(w(x^k) - x^k)}$, $\overline{\varepsilon}_{x^k + \rho^{l_k}(w(x^k) - x^k)} = (\varepsilon^k, \varepsilon^k, \ldots, \varepsilon^k)^T$.

Let $H^k = \{x \mid \langle \overline{G}_{x^k}, x - x^k \rangle = 0\}$,

$$A^{k+1} = \begin{pmatrix} A^k \\ \overline{G}_{x^k} \end{pmatrix}, \qquad b^{k+1} = \begin{pmatrix} b^k \\ \overline{G}_{x^k}^T x^k \end{pmatrix}. \tag{3.3}$$

Increase k by one and go to Step 1.
End of Algorithm 3.1

4. Convergence Analysis

In [20], the authors proposed a column generation scheme to generate the polytope X^k, and they proved if k satisfies the following inequality

$$\frac{\varepsilon^2}{m} \geq \frac{1/2 + 2m\ln\left(1 + (k+1)/8m^2\right)}{2m + k + 1} e^{-2\alpha((k+1)/(2m+k+1))}, \tag{4.1}$$

where $\varepsilon < 1/2$ is a constant, the scheme will stop with a feasible solution, that is, they can find a vector a_{k+1} such that $\{y \mid a_{k+1}^T y \leq a_{k+1}^T y^k\} \supset \Gamma$ with $\|a_{k+1}\| = 1$, Γ contains a full-dimensional closed ball with $\varepsilon < 1/2$ radius. In other words, there exists the smallest $k(\varepsilon)$ such that $X^{k(\varepsilon)}$ generated by the column generation scheme does not contain the ball with $\varepsilon < 1/2$ radius, and it is known as the finite cut property. It is easy to know that the result of Theorem 6.6 in [20] also holds without much change for our Algorithm 3.1 using the approximate centers. That is, by using the row generation scheme, there exists the smallest $k(\rho)$ such that $X^{k(\rho)}$ generated in Step 4 in Algorithm 3.1 does not contain the ball with ρ radius lying inside the polytope X. This result plays an important role in proving the convergence of the described Algorithm 3.1 in Section 3.

Theorem 4.1. *Let the polyhedron X have nonempty interior, and let AX_D^* be nonempty. Assumption* (2.4) *holds. Then either Algorithm 3.1 stops with a solution to $AVI_P(F, X)$ after a finite number of iterations, or there exists a subsequence of the infinite sequence $\{x^k\}$ that converges to a point in AX_P^*.*

Proof. Assume that $x^k \notin AX_P^*$ for every iteration k, and let $y^* \in AX_D^*$. From Proposition 2.10, we know that $y^* \in X^k$ never lies on H^k for any k. Let $\{\overline{y}_i\}_{i\in N}$ be an arbitrary sequence of point in the interior of X converging to y^* and δ_i a sequence of positive numbers such that $\lim_{i\to+\infty}\delta_i = 0$ and that the sequence of closed balls $\{B(\overline{y}_i, \delta_i)\}_{i\in N}$ lies in the interior of X. Note that $\lim_{i\to+\infty} B(\overline{y}_i, \delta_i) = \{y^*\}$. From finite cut property, we know that there exists the smallest index $k(i)$ and a point $\widetilde{y}_i \in B(\overline{y}_i, \delta_i)$ such that $\widetilde{y}_i$ satisfies

$$\langle \overline{G}_{x^{k(i)}}, x^{k(i)} - \widetilde{y}_i \rangle < 0. \tag{4.2}$$

As $\langle \overline{G}_{x^{k(i)}}, x^{k(i)} - y^* \rangle > 0$, there exists a point $\widehat{y}_i$ on the segment $[\widetilde{y}_i, y^*]$ such that $\langle \overline{G}_{x^{k(i)}}, x^{k(i)} - \widehat{y}_i \rangle = 0$. Since X is compact, we can extract from $\{x^{k(i)}\}_{i\in N}$ a convergent subsequence $\{x^{k(i)}\}_{i\in S}$. Denote by $\widehat{x}$ its limit point, we have

$$\langle \overline{G}_{x^{k(i)}}, \widehat{y}_i - x^{k(i)} \rangle = 0, \quad \forall i \in S. \tag{4.3}$$

From Proposition 2.9, we know that $l_{k(i)}$ is bounded. Consequently, we can extract form $\{l_{k(i)}\}_{i\in S}$ a constant subsequence $l_{k^*(i)} = k^*$. Now from the continuity of $w(x)$ for fixed k and the relations (2.15) and (4.3), it follows by taking the limit in (4.3) that

$$\langle \overline{G}_{\widehat{x}}, y^* - \widehat{x} \rangle = 0. \tag{4.4}$$

By Proposition 2.10, we conclude that $\widehat{x} \in AX_P^*$. □

Theorem 4.2. *Under the conditions of Theorem 4.1, either Algorithm 3.1 stops with a solution to $AVI_P(F, X)$ after a finite number of iterations, or there exists a subsequence of the infinite sequence $\{x^k\}$ that converges to a point in X_P^*.*

Proof. Since $\varepsilon \in (0,1)$, $\varepsilon^k \in (0,1)$. At the end of Step 4 in Algorithm 3.1 we increase k by one, so we have $\varepsilon^{k+1} < \varepsilon^k$, $\varepsilon^k \to 0$ as $k \to \infty$. Moreover, $\overline{\varepsilon}_{x^k} \to \overline{0}$ as $k \to \infty$ in Algorithm 3.1, where $\overline{0}$ denotes the zero vector. This means $\overline{F}_{\widehat{x}} = F(\widehat{x})$ as $k \to \infty$. Therefore, from the second result of Theorem 4.1, we know $\widehat{x}$ is the solution to the problem $\mathrm{VI}_P(F, X)$. □

5. Conclusions

In [5], the authors proposed a cutting plane method for solving the quasimonotone variational inequalities, but throughout the paper they employed the exact information of the mapping F from X to R^n. Just like the discussion in the first part of our paper, sometimes, it is not so easy or even impossible to compute the exact values of the mapping F. Motivated by this fact, we consider constructing an approximate problem $\mathrm{AVI}_P(F, X)$ of $\mathrm{VI}_P(F, X)$, and try out a proximal analytic center cutting plane algorithm for solving $\mathrm{AVI}_P(F, X)$. In contrast to [5], our algorithm can be viewed as an approximation algorithm, and it is easier to implement than [5] since it only requires the inexact information of the corresponding mapping. At the same time, the conditions we impose on the corresponding mappings are more relaxed, for example, [5] requires the mapping F satisfies the Lipschitz condition, but we only require that the so-called approximate Lipschitz condition (2.4)-(c) holds.

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References

[1] P. T. Harker and J.-S. Pang, “Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications,” *Mathematical Programming B*, vol. 48, no. 2, pp. 161–220, 1990.

[2] J.-L. Goffin and J.-P. Vial, “Convex nondifferentiable optimization: a survey focused on the analytic center cutting plane method,” *Optimization Methods & Software*, vol. 17, no. 5, pp. 805–867, 2002.

[3] Z. Q. Luo and J. Sun, “An analytic center based column generation algorithm for convex quadratic feasibility problems,” *SIAM Journal on Optimization*, vol. 9, no. 1, pp. 217–235, 1999.

[4] J. L. Goffin, P. Marcotte, and D. Zhu, “An analytic center cutting plane method for pseudomonotone variational inequalities,” *Operations Research Letters*, vol. 20, no. 1, pp. 1–6, 1997.

[5] P. Marcotte and D. L. Zhu, "A cutting plane method for solving quasimonotone variational inequalities," *Computational Optimization and Applications*, vol. 20, no. 3, pp. 317–324, 2001.
[6] M. Hintermüller, "A proximal bundle method based on approximate subgradients," *Computational Optimization and Applications*, vol. 20, no. 3, pp. 245–266, 2001.
[7] K. C. Kiwiel, "An algorithm for nonsmooth convex minimization with errors," *Mathematics of Computation*, vol. 45, no. 171, pp. 173–180, 1985.
[8] M. V. Solodov, "On approximations with finite precision in bundle methods for nonsmooth optimization," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 151–165, 2003.
[9] D. P. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, Mass, USA, 1995.
[10] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1989.
[11] J. F. Bonnans, J. C. Gilbert, C. Lemaréchal, and C. Sagastizábal, *Optimization Numerique: Aspects Theoriques et Pratiques*, Springer, Berlin, Germany, 1997.
[12] C. Lemaréchal, "Lagrangian decomposition and nonsmooth optimization: bundle algorithm, prox iteration, augmented Lagrangian," in *Nonsmooth optimization: Methods and Applications*, pp. 201–216, Gordon and Breach, Philadelphia, Pa, USA, 1992.
[13] Y. R. He, "Minimizing and stationary sequences of convex constrained minimization problems," *Journal of Optimization Theory and Applications*, vol. 111, no. 1, pp. 137–153, 2001.
[14] A. Pinkus, "Density in approximation theory," *Surveys in Approximation Theory*, vol. 1, pp. 1–45, 2005.
[15] A. Auslender, *Optimization, Méthodes*, Numériques, Masson, Paris, 1976.
[16] B. Hammer, "Universal approximation of mappings on structured objects using the folding architecture," Technical Report. Reihd P, Heft 183, Universitat Osnabruck, Fachbereich Informatic, 1996.
[17] B. Hammer and V. Sperschneider, "Neural networks can approximate mappings on structured objects," in *Proceedings of the International Conference on Computational Inteliigence and Neural Network*, P. P. Wang, Ed., pp. 211–214, 1997.
[18] K. Hornik, M. Stinchcombe, and H. White, "Multilayer feedforward networks are universal approximators," *Neural Networks*, vol. 2, no. 5, pp. 359–366, 1989.
[19] K. C. Kiwiel, "A proximal bundle method with approximate subgradient linearizations," *SIAM Journal on Optimization*, vol. 16, no. 4, pp. 1007–1023, 2006.
[20] J. L. Goffin, Z. Q. Luo, and Y. Ye, "Complexity analysis of an interior cutting plane method for convex feasibility problems," *SIAM Journal on Optimization*, vol. 6, no. 3, pp. 638–652, 1996.

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Research Article

Metric Subregularity for Subsmooth Generalized Constraint Equations in Banach Spaces

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This paper is devoted to metric subregularity of a kind of generalized constraint equations. In particular, in terms of coderivatives and normal cones, we provide some necessary and sufficient conditions for subsmooth generalized constraint equations to be metrically subregular and strongly metrically subregular in general Banach spaces and Asplund spaces, respectively.

1. Introduction

Let X be a Banach space and $f : X \to \overline{R}$ be a function. Consider the following inequality:

$$f(x) \leq 0. \tag{1.1}$$

Let $S := \{x \in X : f(x) \leq 0\}$. Recall that (1.1) has a local error bound at $a \in S$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, S) \leq \tau \left[f(x)\right]_+ \quad \forall x \in B(a, \delta), \tag{1.2}$$

where $[f(x)]_+ := \max\{f(x), 0\}$ and $B(a, \delta)$ denotes the open ball of center a and radius δ. The error bound has been studied by many authors (see [1–3] and the references therein).

Let Y be another Banach space, $b \in Y$, and let $F : X \rightrightarrows Y$ be a closed multifunction. The following generalized equation:

$$b \in F(x) \tag{GE}$$

concludes most of systems in optimization and was investigated by many researchers (see [4–9] and the references therein). Let $x \in X$ and $b \in F(a)$. Recall that (GE) is metrically subregular at (a, b) if there exist $\tau, \delta \in (0, \infty)$ such that

$$d\left(x, F^{-1}(b)\right) \leq \tau d(b, F(x)) \quad \forall x \in B(a, \delta) \tag{1.3}$$

(see [4–6] and the references therein). This property provides an estimate how far for an element x near a can be from the solution set of (GE). A stronger notion is the metric regularity: a multifunction F is metrically regular at (a, b) if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d\left(x, F^{-1}(y)\right) \leq \tau d(y, F(x)) \quad \forall (x, y) \in B((a, b), \delta). \tag{1.4}$$

There exists a wide literature on this topic. We refer the interested readers to [3, 7–11] and to the references contained therein. Let A be a closed subset of X. Consider the generalized constraint equation as follows:

$$b \in F(x) \quad \text{subject to } x \in A. \tag{GCE}$$

Let S denote the solution set of (GCE), that is, $S = \{x \in A : b \in F(x)\}$. We say that (GCE) is metrically subregular at $a \in S$ if there exist $\tau, \delta \in (0, \infty)$ such that

$$d(x, S) \leq \tau(d(b, F(x)) + d(x, A)) \quad \forall x \in B(a, \delta). \tag{1.5}$$

When $A = X$, (GCE) reduces (GE) and (1.5) means that (GE) is metrically subregular at (a, b). When $F(x) = [f(x), +\infty), b = 0$ and $A = X$, (GCE) reduces the inequality (1.1) and (1.5) means that this inequality has a local error bound at a. Error bounds, metric subregularity and regularity have important applications in mathematical programming and have been extensively studied (see [1–12] and the references therein). The Authors [13] introduced the notions of primal smoothness and investigated the properties of primal smooth functions. Under proper conditions, the distance function is primal smooth. Differentiability of the distance function was discussed in [14]. As extension of primal smoothness and convexity, the notion of subsmoothness was introduced and some functional characterizations were provided in [15]. Recently, by variational analysis techniques (for more details, see [16–19]), Zheng and Ng [6] investigated metric subregularity of (GE) under the subsmooth assumption. In this paper, in terms of normal cones and coderivatives, we devote to metric subregularity of generalized constraint equation (GCE) under the subsmooth assumption. We will build some new necessary and sufficient conditions for (GCE) to be metrically subregular and strongly metrically subregular.

2. Preliminaries

Let X be a Banach space. We denote by B_X and X^* the closed unit ball and the dual space of X, respectively. Let A be a nonempty subset of X, $\text{int}(A)$ and $\text{bd}(A)$, respectively, denote the interior and the boundary of A. For $a \in X$ and $\delta > 0$, let $B(a, \delta)$ denote the open ball with center a and radius δ.

We introduce some notions of variations and derivatives needed to state our results.

For a closed subset A of X and $a \in A$, let $T_c(A, a)$ and $T(A, a)$, respectively, denote the Clarke tangent cone and contingent (Bouligand) cone of A at a defined by

$$T_c(A, a) := \liminf_{x \xrightarrow{A} a, t \to 0^+} \frac{A - x}{t}, \qquad T(A, a) := \limsup_{t \to 0^+} \frac{A - a}{t}, \tag{2.1}$$

where $x \xrightarrow{A} a$ means that $x \to a$ with $x \in A$. It is easy to verify that $v \in T_c(A, a)$ if and only if for each sequence $\{a_n\}$ in A converging to a and each sequence $\{t_n\}$ in $(0, \infty)$ decreasing to 0, there exists a sequence $\{v_n\}$ in X converging to v such that $a_n + t_n v_n \in A$ for each natural number n; while $v \in T(A, a)$ if and only if there exists a sequence $\{v_n\}$ in X converging to v and a sequence $\{t_n\}$ in $(0, \infty)$ decreasing to 0 such that $a + t_n v_n \in A$ for all n.

We denote by $N_c(A, a)$ the Clarke normal cone of A at a, that is,

$$N_c(A, a) := \{x^* \in X^* : \langle x^*, h \rangle \leq 0 \ \forall h \in T_c(A, a)\}. \tag{2.2}$$

For $\varepsilon \geq 0$ and $a \in A$, the nonempty set

$$\widehat{N}_\varepsilon(A, a) := \left\{ x^* \in X^* : \limsup_{x \xrightarrow{A} a} \frac{\langle x^*, x - a \rangle}{\|x - a\|} \leq \varepsilon \right\} \tag{2.3}$$

is called the set of *Fréchet* ε-normals of A at a. When $\varepsilon = 0$, $\widehat{N}_\varepsilon(A, a)$ is a convex cone which is called the *Fréchet* normal cone of A at a and is denoted by $\widehat{N}(A, a)$. Let $N(A, a)$ denote the Mordukhovich limiting or basic normal cone of A at a, that is,

$$N(A, a) := \limsup_{x \xrightarrow{A} a, \varepsilon \to 0^+} \widehat{N}_\varepsilon(A, x), \tag{2.4}$$

that is, $x^* \in N(A, a)$ if and only if there exist sequences $\{(x_n, \varepsilon_n, x_n^*)\}$ in $A \times R_+ \times X^*$ such that $(x_n, \varepsilon_n) \to (a, 0)$, $x_n^* \xrightarrow{w^*} x^*$ and $x_n^* \in \widehat{N}_{\varepsilon_n}(A, x_n)$ for each natural number n. It is known that

$$\widehat{N}(A, a) \subseteq N(A, a) \subseteq N_c(A, a), \tag{2.5}$$

(see [4, 9, 16, 18, 19] and the references contained therein). If A is convex, then

$$\begin{gathered} T(A, a) = T_c(A, a), \\ \widehat{N}(A, a) = N(A, a) = N_c(A, a) = \{x^* \in X^* : \langle x^*, x - a \rangle \leq 0 \ \forall x \in A\}. \end{gathered} \tag{2.6}$$

Recall that a Banach space X is called an Asplund space if every continuous convex function on X is *Fréchet* differentiable at each point of a dense subset of X (for other definitions and their equivalents, see [19]). It is well known that X is an Asplund space if and only if every separable subspace of X has a separable dual space. In particular, every reflexive Banach space is an Asplund space. When X is an Asplund space, it is well known that

$$N_c(A,a) = \mathrm{cl}^*(\mathrm{co}(N(A,a))), \qquad N(A,a) = \limsup_{x \xrightarrow{A} a} \widehat{N}(A,x), \tag{2.7}$$

where $\mathrm{cl}^*(\cdot)$ denotes the closure with respect to the weak* topology, see [9, 19]. Recently, Zheng and Ng [5] established an approximate projection result for a closed subset of X, which will play a key role in the proofs of our main results.

Lemma 2.1. *Let A be a closed nonempty subset of a Banach space X and let $\beta \in (0,1)$. Then for any $x \notin A$ there exist $a \in \mathrm{bd}(A)$ and $a^* \in N_c(A,a)$ with $\|a^*\| = 1$ such that*

$$\beta\|x-a\| < \min\{d(x,A), \langle a^*, x-a\rangle\}. \tag{2.8}$$

If X is an Asplund space, then $N_c(A,a)$ can be replaced by $\widehat{N}(A,a)$.

Let $F : X \rightrightarrows Y$ be a multifunction and let $\mathrm{Gr}(F)$ denote the graph of F, that is,

$$\mathrm{Gr}(F) := \{(x,y) \in X \times Y : y \in F(x)\}. \tag{2.9}$$

As usual, F is said to be closed (resp., convex) if $\mathrm{Gr}(F)$ is a closed (resp., convex) subset of $X \times Y$. Let $(x,y) \in \mathrm{Gr}(F)$. The Clarke tangent and contingent derivatives $D_cF(x,y), DF(x,y)$ of F at (x,y) are defined by

$$\begin{aligned} D_cF(x,y)(u) &:= \{v \in Y : (u,v) \in T_c(\mathrm{Gr}(F),(x,y))\} \quad \forall u \in X, \\ DF(x,y)(u) &:= \{v \in Y : (u,v) \in T(\mathrm{Gr}(F),(x,y))\} \quad \forall u \in X, \end{aligned} \tag{2.10}$$

respectively. Let $\widehat{D}^*F(x,y), D^*F(x,y)$, and $D_c^*F(x,y)$ denote the coderivatives of F at (x,y) associated with the *Fréchet*, Mordukhovich, and Clarke normal structures, respectively. They are defined by the following:

$$\begin{aligned} \widehat{D}^*F(x,y)(y^*) &:= \left\{x^* \in X^* : (x^*,-y^*) \in \widehat{N}(\mathrm{Gr}(F),(x,y))\right\} \quad \forall y^* \in Y^*, \\ D^*F(x,y)(y^*) &:= \{x^* \in X^* : (x^*,-y^*) \in N(\mathrm{Gr}(F),(x,y))\} \quad \forall y^* \in Y^*, \\ D_c^*F(x,y)(y^*) &:= \{x^* \in X^* : (x^*,-y^*) \in N_c(\mathrm{Gr}(F),(x,y))\} \quad \forall y^* \in Y^*. \end{aligned} \tag{2.11}$$

The more details of the coderivatives can be found in [9, 18, 19] and the references therein.

3. Subsmooth Generalized Constraint Equation

Let A be a closed subset of X. Recall (see [13, 14]) that A is said to be prox-regular at $a \in A$ if there exist $\tau, \delta > 0$ such that

$$\langle x^* - u^*, x - u \rangle \geq -\tau \|x - u\|^2 \tag{3.1}$$

whenever $x, u \in A \cap B(a, \delta), x^* \in N_c(A, x) \cap B_{X^*}$, and $u^* \in N_c(A, u) \cap B_{X^*}$.

As a generalization of the prox-regularity, Aussel et al. [15] introduced and studied the subsmoothness. A is said to be subsmooth at $a \in A$ if for any $\varepsilon > 0$ there exist $\tau, \delta > 0$ such that

$$\langle x^* - u^*, x - u \rangle \geq -\varepsilon \|x - u\|, \tag{3.2}$$

whenever $x, u \in A \cap B(a, \delta), x^* \in N_c(A, x) \cap B_{X^*}$, and $u^* \in N_c(A, u) \cap B_{X^*}$.

It is easy to verify that A is subsmooth at $a \in A$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle u^*, x - u \rangle \leq \varepsilon \|x - u\| \tag{3.3}$$

whenever $x, u \in A \cap B(a, \delta)$ and $u^* \in N_c(A, u) \cap B_{X^*}$.

Let $F : X \rightrightarrows Y$ be a closed multifunction, $b \in Y$ and $a \in F^{-1}(b)$. Zheng and Ng [6] introduce the concept of the L-subsmoothness of F at a for b: F is called to be L-subsmooth at a for b if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle u^*, x - a \rangle + \langle v^*, y - v \rangle \leq \varepsilon \left(\|x - a\| + \|y - v\| \right), \tag{3.4}$$

whenever $v \in F(a) \cap B(b, \delta), (u^*, v^*) \in N_c(\mathrm{Gr}(F), (a, v)) \cap B_{X^* \times Y^*}$ and $(x, y) \in \mathrm{Gr}(F)$ with $\|x - a\| + \|y - b\| < \delta$. Next, we introduce the concept of the subsmoothness of generalized constraint equation (GCE) which will be useful in our discussion.

Definition 3.1. Generalized equation (GCE) is subsmooth at $a \in S$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle u^*, x - u \rangle - \langle v^*, y - b \rangle \leq \varepsilon \left(\|x - u\| + \|y - b\| \right), \qquad \langle w^*, x' - u \rangle \leq \varepsilon \|x' - u\|, \tag{3.5}$$

whenever $x \in B(a, \delta), x' \in A \cap B(a, \delta), u \in S \cap B(a, \delta), y \in F(x) \cap B(b, \delta), v^* \in B_{Y^*}, u^* \in D_c^* F(u, b)(v^*) \cap B_{X^*}$, and $w^* \in N_c(A, u) \cap B_{X^*}$.

Remark 3.2. The subsmoothness of (GCE) at a means the subsmoothness of F at a for b when $A = X$, while the subsmoothness of (GCE) at a means the subsmoothness of A at a when $F(x) = b$ for all $x \in X$. If $A = X$ and $\mathrm{Gr}(F)$ is prox-regular at (a, b), then generalized equation (GCE) is subsmooth at a. If A and $\mathrm{Gr}(F)$ are convex, then F is also subsmooth at a. Finally when A is prox-regular and F is single-valued and smooth, (GCE) is subsmooth at a, too. Hence, Definition 3.1 extends notions of smoothness, convexity and prox-regularity.

Proposition 3.3. *Suppose that Generalized equation* (GCE) *is subsmooth at $a \in S$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\langle u^*, x-u\rangle \leq (2+\varepsilon)d(b, F(x)) + \varepsilon\|x-u\|, \tag{3.6}$$

$$\langle w^*, x-u\rangle \leq d(x, A) + \varepsilon\|x-u\|, \tag{3.7}$$

whenever $x \in B(a,\delta), u \in S \cap B(a,\delta)$, $u^* \in D_c^* F(u,b)(B_{Y^*}) \cap B_{X^*}$, and $w^* \in N_c(A,u) \cap B_{X^*}$.

Proof. Suppose that (GCE) is subsmooth at $a \in S$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle u^*, x-u\rangle - \langle v^*, y-b\rangle \leq \frac{\varepsilon}{2}\left(\|x-u\| + \|y-b\|\right), \qquad \langle w^*, x'-u\rangle \leq \frac{\varepsilon}{2}\|x'-u\|, \tag{3.8}$$

whenever $x \in B(a,2\delta), x' \in A \cap B(a,2\delta), u \in S \cap B(a,2\delta), y \in F(x) \cap B(b,2\delta), v^* \in B_{Y^*}, u^* \in D_c^* F(u,b)(v^*) \cap B_{X^*}$, and $w^* \in N_c(A,u) \cap B_{X^*}$.

Let $x \in B(a,\delta), u \in S \cap B(a,\delta), v^* \in B_{Y^*}, u^* \in D_c^* F(u,b)(v^*) \cap B_{X^*}$, and $w^* \in N_c(A,u) \cap B_{X^*}$. If $F(x) \cap B(b,\delta) = \emptyset$, then

$$\langle u^*, x-u\rangle \leq \|x-u\| \leq \|x-a\| + \|a-u\| \leq 2\delta, \qquad d(b, F(x)) \geq \delta. \tag{3.9}$$

Thus, (3.6) holds. Otherwise, one has

$$\langle u^*, x-u\rangle \leq \langle v^*, y-b\rangle + \frac{\varepsilon}{2}\left(\|x-u\| + \|y-b\|\right) \leq (1+\varepsilon)\|y-b\| + \varepsilon\|x-u\|, \tag{3.10}$$

whenever $y \in F(x) \cap B(b,\delta)$. Noting that $d(b, F(x)) = d(b, F(x) \cap B(b,\delta))$ (sine $F(x) \cap B(b,\delta) \neq \emptyset$), it follows that

$$\langle u^*, x-u\rangle \leq (1+\varepsilon)d(b, F(x)) + \varepsilon\|x-u\|. \tag{3.11}$$

It remains to show that (3.7) holds. Since

$$\begin{aligned} \langle w^*, x-u\rangle &= \langle w^*, x-x'\rangle + \langle w^*, x'-u\rangle \leq \|x-x'\| + \frac{\varepsilon}{2}\|x'-u\| \\ &\leq \left(1+\frac{\varepsilon}{2}\right)\|x-x'\| + \frac{\varepsilon}{2}\|x-u\|, \end{aligned} \tag{3.12}$$

whenever $x' \in A \cap B(a,\delta)$. One has

$$\langle w^*, x-u\rangle \leq d(x, A) + \frac{\varepsilon}{2}\left(d(x, A) + \|x-u\|\right). \tag{3.13}$$

Noting that $u \in A$, it follows that $d(x, A) \leq \|x-u\|$ which implies that (3.7) holds and completes the proof. □

4. Main Results

This section is devoted to metric subregularity of generalized equation (GCE). We divide our discussion into two subsections addressing the necessary conditions and the sufficient conditions for metric subregularity.

4.1. Necessary Conditions for Metric Subregularity

There are two results in this subsection: one is on the Banach space setting and the other on the Asplund space setting.

Theorem 4.1. *Suppose that X, Y are Banach spaces and that generalized equation (GCE) is metrically subregular at $a \in S$. Then there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\widehat{N}(S,u) \cap B_{X^*} \subseteq \tau(D_c^* F(u,b)(B_{Y^*}) + N_c(A,u) \cap B_{X^*}) \quad \forall u \in S \cap B(a,\delta). \tag{4.1}$$

Proof. Let $\delta_{\mathrm{Gr}(F)}$ denote the indicator function of $\mathrm{Gr}(F)$ and $\delta > 0$ such that (1.5) holds. Then (1.5) can be rewritten as

$$d(x,S) \leq \delta_{\mathrm{Gr}(F)}(x,y) + \tau(\|y-b\| + d(x,A)) \quad \forall (x,y) \in B(a,\delta) \times Y. \tag{4.2}$$

Let $u \in S \cap B(a,\delta)$ and $u^* \in \widehat{N}(S,u) \cap B_{X^*}$. Noting (cf. [9, Corollary 1.96]) that $\widehat{N}(S,u) \cap B_{X^*} = \widehat{\partial} d(\cdot, S)(u)$, one gets that for any natural number n, there exists $r \in (0,\delta)$ such that $B(u,r) \subseteq B(a,\delta)$ and

$$\langle u^*, x-u \rangle \leq d(x,S) + \frac{1}{n}\|x-u\| \quad \forall x \in B(u,r). \tag{4.3}$$

Hence, by (4.2), it follows that

$$\langle u^*, x-u \rangle \leq \delta_{\mathrm{Gr}(F)}(x,y) + \tau\|y-b\| + \tau d(x,A) + \frac{1}{n}\|x-u\| \quad \forall (x,y) \in B(u,r) \times Y, \tag{4.4}$$

that is, (u,b) is a local minimizer of ϕ defined by

$$\phi(x,y) := -\langle u^*, x-u \rangle + \delta_{\mathrm{Gr}(F)}(x,y) + \tau\|y-b\| + \tau d(x,A) + \frac{1}{n}\|x-u\| \quad \forall (x,y) \in X \times Y. \tag{4.5}$$

Hence, $(0,0) \in \partial_c \phi(u,b)$. It follows from [16] that

$$(0,0) \in (-u^*, 0) + N_c(\mathrm{Gr}(F), (u,b)) + \{0\} \times \tau B_{Y^*} + \tau \partial_c d(\cdot, A)(u) \times \{0\} + \frac{1}{n} B_{X^*} \times \{0\}, \tag{4.6}$$

that is,

$$\left(\frac{1}{\tau}u^* + \frac{1}{\tau n}x_n^*, -y_n^*\right) \in N_c(\mathrm{Gr}(F), (u, b)) + \partial_c d(\cdot, A)(u) \times \{0\}, \tag{4.7}$$

for some $x_n^* \in B_{X^*}$ and $y_n^* \in B_{Y^*}$. Since B_{X^*} and B_{Y^*} are weak* compact, without loss of generality (otherwise take a generalized subsequence), we can assume $x_n^* \xrightarrow{w^*} x^*, y_n^* \xrightarrow{w^*} y^*$ for some $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$ as $n \to \infty$. Noting that

$$N_c(\mathrm{Gr}(F), (u, b)) + \partial_c d(\cdot, A)(u) \times \{0\} \tag{4.8}$$

is weak* closed (since $N_c(\mathrm{Gr}(F), (u, b))$ is weak* closed and $\partial_c d(\cdot, A)(u) \times \{0\}$ is weak* compact), one has

$$\left(\frac{u^*}{\tau}, -y^*\right) \in N_c(\mathrm{Gr}(F), (u, b)) + \partial_c d(\cdot, A)(u) \times \{0\}. \tag{4.9}$$

This implies that

$$u^* \in \tau\left(D_c^* F(u, b)(B_{Y^*}) + N_c(A, u) \cap B_{X^*}\right). \tag{4.10}$$

This shows that (4.1) holds true. The proof is completed. □

When X and Y are Asplund spaces, the conclusion in Theorem 4.1 can be strengthened with $D_c^* F(u, b)(B_{Y^*})$ and $N_c(A, u) \cap B_{X^*})$ replaced by $D^* F(u, b)(B_{Y^*})$ and $N(A, u) \cap B_{X^*})$, respectively. Its proof is similar to that of Theorem 4.1.

Theorem 4.2. *Suppose that X and Y are Asplund spaces and that generalized equation (GCE) is metrically subregular at $a \in S$. Then there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\widehat{N}(S, u) \cap B_{X^*} \subseteq \tau\left(D^* F(u, b)(B_{Y^*}) + N(A, u) \cap B_{X^*}\right) \quad \forall u \in S \cap B(a, \delta). \tag{4.11}$$

4.2. Sufficient Conditions for Metric Subregularity

Under the subsmooth assumption, we will show in the next result that some conditions similar to (4.1) turns out to be sufficient conditions for metric subregularity.

Theorem 4.3. *Let X and Y be Banach spaces. Suppose that generalized constraint equation (GCE) is subsmooth at a and that there exist $\tau, \delta \in (0, +\infty)$ such that*

$$N_c(S, u) \cap B_{X^*} \subseteq \tau\left(D_c^* F(u, b)(B_{Y^*}) + N_c(A, u) \cap B_{X^*}\right), \tag{4.12}$$

whenever $u \in \mathrm{bd}(S) \cap B(a, \delta)$. *Then* (GCE) *is metrically subregular at* a *and, more precisely, for any* $\varepsilon \in (0, 1/(2(1+\tau)))$ *there exists* $\delta_\varepsilon \in (0, \delta/2)$ *such that*

$$d(x, S) \leq \frac{(1+\tau)(2+\varepsilon)}{1-2(1+\tau)\varepsilon}(d(b, F(x)) + d(x, A)) \quad \forall x \in B(a, \delta_\varepsilon). \tag{4.13}$$

Proof. Let $\varepsilon \in (0, 1/(2(1+\tau)))$. Then, by subsmooth assmption of (GCE) at a and Proposition 3.3, there exists $\delta' \in (0, \delta/2)$ such that

$$\begin{gathered} \langle u_1^*, x-u \rangle \leq (2+\varepsilon) d(b, F(x)) + \varepsilon\|x-u\|, \\ \langle u_2^*, x-u \rangle \leq d(x, A) + \varepsilon\|x-u\|, \end{gathered} \tag{4.14}$$

whenever $x \in B(a, \delta'), u \in S \cap B(a, \delta'), u_1^* \in D_c^* F(u, b)(B_{Y^*}) \cap B_{X^*}$ and $u_2^* \in N_c(A, u) \cap B_{X^*}$.

Let $\delta_\varepsilon \in (0, \delta'/2)$ and $x \in B(a, \delta_\varepsilon) \setminus S$. Now we need only show (4.13).

(i) If $F(x) \cap B(b, \delta') = \emptyset$, then $d(x, S) \leq \|x-a\| < \delta_\varepsilon$, $d(b, F(x)) \geq \delta'$. Hence (4.13) holds.

(ii) Suppose $F(x) \cap B(b, \delta') \neq \emptyset$ and let

$$\beta \in \left(\max\left\{\frac{d(x, S)}{\delta_\varepsilon}, 2(1+\tau)\varepsilon, \frac{1}{2}\right\}, 1\right). \tag{4.15}$$

By Lemma 2.1 there exist $u_0 \in \mathrm{bd}(S)$ and $u^* \in N_c(S, u_0)$ with $\|u^*\| = 1$ such that

$$\beta\|x-u_0\| \leq \min\{\langle u^*, x-u_0 \rangle, d(x, S)\}. \tag{4.16}$$

Thus, $\|x-u_0\| \leq (d(x, S)/\beta) < \delta_\varepsilon$. Hence,

$$\|u_0 - a\| \leq \|u_0 - x\| + \|x-a\| < 2\delta_\varepsilon < \delta' < \delta. \tag{4.17}$$

By (4.12) there exist $y_1^* \in B_{Y^*}, x_1^* \in D_c^* F(u_0, b)(y_1^*)$, and $x_2^* \in N_c(A, u_0) \cap B_{X^*}$ such that $u^* = \tau(x_1^* + x_2^*)$. Applying (4.14) with $((\tau/(1+\tau))x_1^*, (\tau/(1+\tau))x_2^*, u_0)$ in place of (u_1^*, u_2^*, u), it follows that

$$\begin{aligned} \langle u^*, x-u_0 \rangle &= \tau(\langle x_1^*, x-u_0 \rangle + \langle x_2^*, x-u_0 \rangle) \\ &\leq (1+\tau)(2+\varepsilon)(d(b, F(x)) + d(x, A)) + 2(1+\tau)\varepsilon\|x-u_0\|. \end{aligned} \tag{4.18}$$

This and (4.16) imply that

$$d(x, S) \leq \|x-u_0\| \leq \frac{(1+\tau)(2+\varepsilon)}{\beta - 2(1+\tau)\varepsilon}(d(b, F(x)) + d(x, A)). \tag{4.19}$$

Letting $\beta \to 1$, it follows that (4.13) holds. The proof is completed. □

When X and Y are Asplund spaces, the assumption in Theorem 4.3 can be weakened with $N_c(S, u)$ replaced by $\widehat{N}(S, u)$.

Theorem 4.4. *Suppose X and Y are Asplund spaces. Suppose that generalized constraint equation (GCE) is subsmooth at a and that there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\widehat{N}(S,u) \cap B_{X^*} \subseteq \tau(D_c^* F(u,b)(B_{Y^*}) + N_c(A,u) \cap B_{X^*}), \tag{4.20}$$

whenever $u \in \mathrm{bd}(S) \cap B(a,\delta)$. Then for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that (4.13) *holds.*

With the Asplund space version of Lemma 2.1 applied in place of the Banach space version, similar to the proof of Theorem 4.3, it is easy to verify Theorem 4.4.

In general, (GCE) is not necessarily metrically subregular at a if (GCE) only has that $N_c(S,a) \cap B_{X^*} \subseteq \tau(D_c^* F(a,b)(B_{Y^*}) + N_c(A,a) \cap B_{X^*})$.

Finally, we end this subsection with a sufficient and necessary condition for the Clarke tangent derivative mapping $D_c F(a,b)$ to be metrically subregular at 0 for 0 over the Clerke tangent cone $T_c(A,a)$.

Let

$$\tau(F,a,b;A) := \inf\{\tau > 0 : \text{ there exists } \delta > 0 \text{ such that (1.5) holds}\}. \tag{4.21}$$

For $u \in S$, let

$$\gamma(F,u,b;A) := \inf\{\tau > 0 : N_c(S,u) \cap B_{X^*} \subseteq \tau(D_c^* F(a,b)(B_{Y^*}) + N_c(A,u) \cap B_{X^*})\}. \tag{4.22}$$

The following lemma is known ([5, Theorem 3.2]) and useful for us in the sequel.

Lemma 4.5. *Assume that $F : X \rightrightarrows Y$ is a closed convex multifunction, A is a closed convex subset of X, and $a \in S$. And suppose that there exist a cone C and a neighborhood V of a such that $S \cap V = (a + C) \cap V$. Then,*

$$\tau(F,a,b;A) = \gamma(F,a,b;A). \tag{4.23}$$

Consequently, (GCE) *is metrically subregular at a if and only if $\gamma(F,a,b;A) < +\infty$.*

Theorem 4.6. *Let $a \in S$ and*

$$\tau := \inf\{\tau > 0 : d(h, T_c(S,a)) \le \tau(d(0, D_c F(a,b)(h)) + d(h, T_c(A,a)))\ \forall h \in X\}. \tag{4.24}$$

Suppose that

$$T_c(S,a) \subseteq T_c(A,a) \cap D_c F(a,b)^{-1}(0). \tag{4.25}$$

Then,

$$\tau = \gamma(F,a,b;A). \tag{4.26}$$

If, in addition, $\tau < +\infty$, *then*

$$T_c(S,a) = T_c(A,a) \cap D_cF(a,b)^{-1}(0). \tag{4.27}$$

Consequently, $D_cF(a,b)$ *is metrically subregular at* $(0,0)$ *over* $T_c(A,a)$ *if and only if* $\gamma(F,a,b;A) < +\infty$.

Proof. First, we assume that $\tau < +\infty$. By the definition of τ, we have

$$d(x, T_c(S,a)) \leq \tau(d(0, D_cF(a,b)(x)) + d(x, T_c(A,a))) \quad \forall x \in X. \tag{4.28}$$

This implies that

$$T_c(A,a) \cap D_cF(a,b)^{-1}(0) \subseteq T_c(S,a). \tag{4.29}$$

This and (4.25) imply that (4.27) holds.

We consider the following constraint equation:

$$0 \in D_cF(a,b)(x) \quad \text{subject to } x \in T_c(A,a). \tag{GCE$'$}$$

Let S' denote the solution set of (GCE'). Then,

$$S' = T_c(A,a) \cap D_cF(a,b)^{-1}(0). \tag{4.30}$$

Noting that

$$\begin{gathered} N_c(S',0) = N_c(T_c(S,a),0) = N_c(S,a), \\ N_c(\text{Gr}(F),(a,b)) = N_c(\text{Gr}(D_cF(a,b)),(0,0)), \end{gathered} \tag{4.31}$$

it is straightforward to verify that

$$\tau = \tau(D_cF(a,b),0,0;T_c(A,a)), \qquad \gamma(F,a,b;A) = \gamma(D_cF(a,b),0,0;T_c(A,a)). \tag{4.32}$$

On the other hand, since $D_cF(a,b)$ is a closed convex multifunction from X to Y and $T_c(A,a)$ is a closed convex cone, Lemma 4.5 implies that

$$\tau(D_cF(a,b),0,0;T_c(A,a)) = \gamma(D_cF(a,b),0,0;T_c(A,a)). \tag{4.33}$$

This gives us $\tau = \gamma(F,a,b;A)$.

It remains to show that $\gamma(F,a,b;A) = +\infty$ when $\tau = +\infty$. Suppose that

$$\gamma(F,a,b;A) < +\infty. \tag{4.34}$$

We need only show that $\tau < +\infty$. Let $x \in X \setminus T_c(S,a)$ and $\beta \in (0,1)$. By Lemma 2.1 there exist $u \in T_c(S,a)$ and $x^* \in N_c(T_c(S,a),u)$ with $\|x^*\| = 1$ such that

$$\beta\|x-u\| \le \langle x^*, x-u\rangle. \tag{4.35}$$

Noting that $T_c(S,a)$ is a convex cone, it is easy to verify that

$$x^* \in N_c(T_c(S,a),0) = N_c(S,a), \qquad \langle x^*, u\rangle = 0. \tag{4.36}$$

Take a fixed r in $(\gamma(F,a,b;A),+\infty)$. Then there exist $y^* \in rB_{Y^*}, x_1^* \in D_c^*F(a,b)(y^*)$ and $x_2^* \in N_c(A,a) \cap rB_{X^*}$ such that

$$x^* = x_1^* + x_2^*. \tag{4.37}$$

We equip the product space $X \times Y$ with norm

$$\|(x,y)\|_r := \frac{r}{1+r}\|x\| + \|y\| \quad \forall (x,y) \in X \times Y. \tag{4.38}$$

Noting that the unit ball of the dual space of $(X \times Y, \|\cdot\|_r)$ is $(((1+r)/r)B_{X^*}) \times B_{Y^*}$, it follows from the convexity of $D_cF(x,y)$ and $T_c(A,a)$ that

$$\begin{aligned}
\frac{1}{r}(x_1^*, -y^*) &\in N_c(\mathrm{Gr}(F),(a,b)) \cap \left(\left(\frac{1+r}{r}B_{X^*}\right) \times B_{Y^*}\right) \\
&= N_c(\mathrm{Gr}(D_cF(a,b)),(0,0)) \cap \left(\left(\frac{1+r}{r}B_{X^*}\right) \times B_{Y^*}\right) \\
&= \partial_c d_{\|\cdot\|_r}(\cdot, \mathrm{Gr}(D_cF(a,b)))(0,0), \\
\frac{1}{r}x_2^* \in N_c(A,a) \cap B_{X^*} &= N_c(T_c(A,a),0) \cap B_{X^*} = \partial_c d(\cdot, T_c(A,a))(0).
\end{aligned} \tag{4.39}$$

Hence,

$$\begin{aligned}
\frac{1}{r}\langle x_1^*, x\rangle &\le d_{\|\cdot\|_r}((x,0),\mathrm{Gr}(D_cF(a,b))) \le d(0, D_cF(a,b)(x)), \\
\frac{1}{r}\langle x_2^*, x\rangle &\le d(x, T_c(A,a)),
\end{aligned} \tag{4.40}$$

whenever $x \in X$. Noting that $\langle x^*, u\rangle = 0$, it follows from (4.35) that

$$\frac{\beta\|x-u\|}{r} \le d(0, D_cF(a,b)(x)) + d(x, T_c(A,a)). \tag{4.41}$$

Therefore,

$$\frac{\beta d(x, T_c(S,a))}{r} \leq d(0, D_c F(a,b)(x)) + d(x, T_c(A,a)). \tag{4.42}$$

Letting $\beta \to 1$, one has

$$d(x, T_c(S,a)) \leq r(d(0, D_c F(a,b)(x)) + d(x, T_c(A,a))). \tag{4.43}$$

This contradicts with $\tau = +\infty$. The proof is completed. □

4.3. Strongly Metric Subregularity

Let $F : X \rightrightarrows Y$ be a multifunction and $b \in F(a)$. Recall that F is strongly subregular at a if there exist $\tau \in (0, +\infty)$, neighborhoods U of a, and V of b such that

$$\|x - a\| \leq \tau d(b, F(x) \cap V) \quad \forall x \in U. \tag{4.44}$$

It is clear that this definition is equivalent to the next one when $A = \mathrm{X}$.

Definition 4.7. One says that generalized constraint equation (GCE) is strongly metrically subregular at a if there exists $\tau, \delta \in (0, \infty)$ such that

$$\|x - a\| \leq \tau(d(b, F(x)) + d(x, A)) \quad \forall x \in B(a, \delta). \tag{4.45}$$

It is clear that (GCE) is strongly metrically subregular at a if and only if a is an isolated point of S (i.e., $S \cap B(a,r) = \{a\}$ for some $r > 0$) and it is metrically subregular at a. Thus, if (GCE) is strongly metrically subregular at a, Then $N_c(S,a) = X^*$. We immediately have the following Corollary 4.8 from Theorem 4.1.

Corollary 4.8. *Suppose that there exists $\tau, \delta \in (0, \infty)$ such that* (4.45) *holds. Then,*

$$B_{X^*} \subseteq \tau(D_c^* F(a,b)(B_{Y^*}) + N_c(A,a) \cap B_{X^*}). \tag{4.46}$$

Applying Theorem 4.3, one obtains a sufficient condition for (GCE) to be strongly metrically subregular at a.

Corollary 4.9. *Let X, Y be Banach spaces. Suppose that generalized constraint equation* (*GCE*) *is subsmooth at a and that there exists $\tau \in (0, +\infty)$ such that*

$$B_{X^*} \subseteq \tau(D_c^* F(a,b)(B_{Y^*}) + N_c(A,a) \cap B_{X^*}). \tag{4.47}$$

Then (*GCE*) *is strongly metrically subregular at* a *and, more precisely, for any* $\varepsilon \in (0, 1/(1+2\tau))$ *there exists* $\delta_\varepsilon > 0$ *such that*

$$\|x - a\| \leq \frac{(1+\tau)(2+\varepsilon)}{1 - 2(1+\tau)\varepsilon}(d(b, F(x)) + d(x, A)) \quad \forall x \in B(a, \delta_\varepsilon). \tag{4.48}$$

Proof. From Theorem 4.3, we need only show that $S \cap B(a, \delta) = \{a\}$ for some $\delta > 0$. Since the assumption that (GCE) is subsmooth at a, by Proposition 3.3, for any $\varepsilon \in (0, 1/2(1+\tau))$, there exists $\delta > 0$ such that

$$\langle a_1^*, x - a \rangle \leq (2+\varepsilon) d(b, F(x)) + \varepsilon \|x - u\|, \tag{4.49}$$

$$\langle a_2^*, x - a \rangle \leq d(x, A) + \varepsilon \|x - u\|, \tag{4.50}$$

whenever $x \in B(a, \delta)$, $a_1^* \in D_c^* F(a, b)(B_{Y^*}) \cap B_{X^*}$ and $a_2^* \in N_c(A, a) \cap B_{X^*}$.

Take an arbitrary $x^* \in B_{X^*}$. By (4.47), there exist $x_1^* \in D_c^* F(a, b)(B_{Y^*})$, $x_2^* \in N_c(A, a) \cap B_{X^*}$ such that $x^* = \tau(x_1^* + x_2^*)$. Let $x \in S \cap B(a, \delta)$. Applying (4.49) with $(\tau/(1+\tau))x_1^*$ in place of a_1^*, it follows from this and (4.50) that we have

$$\tau \langle x_1^*, x - a \rangle \leq (1+\tau)\varepsilon \|x - a\|, \qquad \langle x_2^*, x - a \rangle \leq \varepsilon \|x - a\|. \tag{4.51}$$

Then,

$$\begin{aligned} \langle x^*, x - a \rangle &= \tau\left(\langle x_1^*, x - a \rangle + \langle x_2^*, x - a \rangle\right) \\ &\leq (1+2\tau)\varepsilon \|x - a\|. \end{aligned} \tag{4.52}$$

And so,

$$\|x - a\| = \sup_{x^* \in B_{X^*}} \langle x^*, x - a \rangle \leq (1+2\tau)\varepsilon \|x - a\|. \tag{4.53}$$

This shows that $S \cap B(a, \delta) = \{a\}$. The proof is completed. □

From Corollaries 4.8 and 4.9, we also have the following equivalent results.

Corollary 4.10. *Suppose that generalized constraint equation* (*GCE*) *is subsmooth at* a. *Then the following statements are equivalent:*

(i) (*GCE*) *is strongly metrically subregular at* a;

(ii) *there exists* $\tau \in (0, \infty)$ *such that* $B_{X^*} \subseteq \tau(D_c^* F(a, b)(B_{Y^*}) + N_c(A, a) \cap B_{X^*}))$;

(iii) $0 \in \operatorname{int}(D_c^* F(a, b)(Y^*) + N_c(A, a))$;

(iv) $X^* = D_c^* F(a, b)(Y^*) + N_c(A, a)$;

(v) $D_c F(a, b)$ *is strongly metrically subregular at* 0 *for* 0 *over* $T_c(A, a)$.

Proof. First, by Corollaries 4.8 and 4.9, it is clear that (i)$\Leftrightarrow$(ii). Noting that $D_c^*(D_cF(a,b))(0,0)(B_{Y^*}) = D_c^*F(a,b)(B_{Y^*})$, (ii)$\Leftrightarrow$(v) is immediate from (i)$\Leftrightarrow$(ii).

It is clear that (ii)$\Leftrightarrow$(iii). Noting that $N_c(A,a)$ and $D_c^*F(a,b)(Y^*)$ are cones, hence,

$$D_c^*F(a,b)(Y^*) + N_c(A,a) = \bigcup_{n=1}^{\infty}(D_c^*F(a,b)(nB_{Y^*}) + N_c(A,a) \cap nB_{X^*}). \tag{4.54}$$

This shows that (ii)$\Rightarrow$(iv).

It remains to show that (iv)$\Rightarrow$(ii). Suppose that (iv) holds, by the Alaogu theorem, for each n, the set $D_c^*F(a,b)(nB_{Y^*}) + N_c(A,a) \cap nB_{X^*}$ is weakly star-closed, it follows from the well-known Baire category theorem and (iv) that

$$0 \in \operatorname{int}(D_c^*F(a,b)(B_{Y^*}) + N_c(A,a) \cap B_{X^*}). \tag{4.55}$$

Hence, (ii) holds. The proof is completed. □

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References

[1] W. Li and I. Singer, "Global error bounds for convex multifunctions and applications," *Mathematics of Operations Research*, vol. 23, no. 2, pp. 443–462, 1998.
[2] X. Y. Zheng and K. F. Ng, "Perturbation analysis of error bounds for systems of conic linear inequalities in Banach spaces," *SIAM Journal on Optimization*, vol. 15, no. 4, pp. 1026–1041, 2005.
[3] W. Li, "Abadie's constraint qualification, metric regularity, and error bounds for differentiable convex inequalities," *SIAM Journal on Optimization*, vol. 7, no. 4, pp. 966–978, 1997.
[4] B. S. Mordukhovich and Y. Shao, "Nonconvex differential calculus for infinite-dimensional multifunctions," *Set-Valued Analysis*, vol. 4, no. 3, pp. 205–236, 1996.
[5] X. Y. Zheng and K. F. Ng, "Metric subregularity and constraint qualifications for convex generalized equations in Banach spaces," *SIAM Journal on Optimization*, vol. 18, no. 2, pp. 437–460, 2007.
[6] X. Y. Zheng and K. F. Ng, "Calmness for L-subsmooth multifunctions in Banach spaces," *SIAM Journal on Optimization*, vol. 19, no. 4, pp. 1648–1673, 2008.
[7] D. Klatte and B. Kummer, *Nonsmooth Equations in Optimization*, vol. 60, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.
[8] B. S. Mordukhovich, "Complete characterization of openness, metric regularity, and Lipschitzian properties of multifunctions," *Transactions of the American Mathematical Society*, vol. 340, no. 1, pp. 1–35, 1993.
[9] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. I, II*, Springer, Berlin, Germany, 2006.
[10] R. Henrion, A. Jourani, and J. Outrata, "On the calmness of a class of multifunctions," *SIAM Journal on Optimization*, vol. 13, no. 2, pp. 603–618, 2002.
[11] X. Y. Zheng and K. F. Ng, "Metric regularity and constraint qualifications for convex inequalities on Banach spaces," *SIAM Journal on Optimization*, vol. 14, no. 3, pp. 757–772, 2003.
[12] X. Y. Zheng and K. F. Ng, "Linear regularity for a collection of subsmooth sets in Banach spaces," *SIAM Journal on Optimization*, vol. 19, no. 1, pp. 62–76, 2008.
[13] F. H. Clarke, R. J. Stern, and P. R. Wolenski, "Proximal smoothness and the lower-C^2 property," *Journal of Convex Analysis*, vol. 2, no. 1-2, pp. 117–144, 1995.

[14] R. A. Poliquin, R. T. Rockafellar, and L. Thibault, "Local differentiability of distance functions," *Transactions of the American Mathematical Society*, vol. 352, no. 11, pp. 5231–5249, 2000.
[15] D. Aussel, A. Daniilidis, and L. Thibault, "Subsmooth sets: functional characterizations and related concepts," *Transactions of the American Mathematical Society*, vol. 357, no. 4, pp. 1275–1301, 2005.
[16] F. H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, NY, USA, 1983.
[17] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, vol. 1364 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1989.
[18] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, vol. 317, Springer, Berlin, Germany, 1998.
[19] B. S. Mordukhovich and Y. H. Shao, "Nonsmooth sequential analysis in Asplund spaces," *Transactions of the American Mathematical Society*, vol. 348, no. 4, pp. 1235–1280, 1996.

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Research Article

Travel Demand-Based Assignment Model for Multimodal and Multiuser Transportation System

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In this paper, the structural characteristic of urban multimodal transport system is fully analyzed and then a two-tier network structure is proposed to describe such a system, in which the first-tier network is used to depict the traveller's mode choice behaviour and the second-tier network is used to depict the vehicle routing when a certain mode has been selected. Subsequently, the generalized travel cost is formulated considering the properties of both traveller and transport mode. A new link impedance function is proposed, in which the interferences between different vehicle flows are taken into account. Simultaneously, the bi-equilibrium patterns for multimodal transport network are proposed by extending Wardrop principle. Correspondingly, a bi-level programming model is then presented to describe the bi-equilibrium based assignment for multi-class multimodal transport network. The solution algorithm is also given. Finally, a numerical example is provided to illustrate the model and algorithm.

1. Introduction

With the rapid development of economy, the transportation infrastructures have been improved significantly in the most cities of China, especially in some metropolitan cities like Beijing and Shanghai, where the integrated urban transportation systems have been established gradually. Synchronously, the modal share for passenger travel has been dramatically changed. Table 1 lists statistics of the trip intensity and mode split of Beijing in 1986, 2000, 2005, and 2010, respectively [1].

It shows that Beijing's transportation development mode is a typical multimodal transportation system. The system consists of different transportation subsystems or subnets for passenger cars, buses, trains, bicycles, and so forth, in which the multimodal traffic flows are interdependent and interactive. Obviously, the equilibriums between the various

Table 1: Trip intensity and mode split characteristics of Beijing by year.

Year	Total trips/day	Average trip distance (km)	Mode split (%)				
			Bus	Subway	Taxi	Car	Bicycle
1986	1.61	—	29.31		0.36	5.24	65.09
2000	2.77	8.0	27.33		9.03	23.96	39.68
2005	2.64	9.3	26.60	5.70	7.60	29.80	30.30
2010	2.82	10.6	28.20	11.50	6.70	34.20	16.40

subsystems and within each subsystem are much more complicated than that for a pure private vehicle system. Thus, the following critical issues should be carefully considered for resolving the multimodal network equilibrium problem.

(i) The multimodal transportation network is a superposition or compound of various physical subnets for different transportation modes.

(ii) The performance of each mode depends on both self-demand and the demands in other modes, which means that there are interactions between different modes.

(iii) The traffic flow pattern in a multimodal network involves traveler's combined choice behaviors in which the travelers not only choose trip modes through the whole multimodal network but also select routes within each subnet.

(iv) The criteria of mode choice and route choice during a trip are usually different. In the mode choice stage, the travelers' decisions are generally influenced by a combination of travel time, potential expense, and other factors. Once the trip mode has been selected, the travelers only care about how to minimize their travel time through route choice within the specific subnet. Therefore, different types of travelers have different psychological preferences for mode choice while the characteristics of travelers have no impact on their route choices.

(v) There are feedbacks between these two choices. Firstly, the traveler's mode choice results in the total demands for different transportation modes, which determines the traffic flows through the multimodal network. Secondly, the traffic assignment patterns corresponding to the route choice within subnets determine the travel time in the respective modes, which conversely affect the mode choice.

The user equilibrium (UE) assignment problem for the private vehicle traffic network has been formulated by Beckmann et al. [2], Sheffi [3], Patriksson [4], and so forth. For multimodal networks, the earlier models [5–9] were developed for modal choice using logit type functions to split travel demand for each travel mode. However, these models cannot reflect a multimodal network's configuration and how the traffic flows are distributed in the network [10]. In order to overcome this, the combined or integrated models have been developed [11–13], in which modal split and flow assignment are incorporated together. Based on the assumption that travel cost structures are either separable or symmetric, the above models were formulated as convex optimization programs. However, the assumption for the cost structure may be not realistic in certain situations [14, 15]. In order to model the asymmetric interactions, some general combined travel demand models were formulated as a variational inequality problem [10, 16–20] and a fixed point problem [21]. Although the above studies have combined the traveler's modal choice and traffic flow assignment, they usually focus on user equilibrium for path flow assignment within single mode traffic network. However, the multimodal equilibrium issue associated with mode split has rarely been explored, and few studies investigate the relationship between these two equilibriums.

From the viewpoint of economics, the transportation service can be measured by the generalized travel cost in addition to the expense charged. The generalized cost is not fixed given a specific trip but is dependent on travel demand. If many individuals choose to use a certain mode, it will get congested and its travel time will increase. In response, some travelers may take alternative modes. Consequently, the alternative modes can also be congested, which will push travelers back to the original mode. Therefore, the UE principle in a multimodal system includes two categorizes. One is the user equilibrium between different modes, and the other is the traditional user equilibrium among different routes in respective subnets. The first equilibrium derives from the monotonically increasing of the generalized cost of each mode with travel demand while the later one derives from travel time in a specific subnet also with the monotonically increasing nature of the link impedance functions. There exists a two-way influence between the two types of equilibriums. If the travel demands of transportation modes are all given, the demands will be assigned in each subnet based on traditional user equilibrium, under which the travel time of various transportation modes will be obtained. Subsequently, such travel times will lead to the changes in the generalized costs of different modes, and then the travelers will reselect transportation modes. Eventually, both user equilibrium between different modes and user equilibrium between different routes in each mode are achieved.

The generalized travel cost is associated with the properties of transportation mode, such as travel time, travel expense, and convenience. Meanwhile, the traveler's psychological preference is another important factor leading to different mode-choice behaviors. Different types of travelers may perceive different values of the properties stated above. For example, the high-income travelers will concern more about the factor of time while the low-income travelers will care more about the factor of expense. To account for multiple user classes that can be distinguished by the value of travel time, the multiclass, multicriteria traffic network equilibrium models were developed, in which each class of travelers perceives the travel disutility associated with a route as a subjective weighting for travel time and travel cost [22]. The models allow both travel time and travel cost of a link to depend on the entire link load pattern, rather than on the particular link flow only [23–25]. The multiclass, multicriteria models were further applied for dynamic traffic assignment [26] and multimodal network issues, such as using aggregate hierarchical logit structures for mode choice [17, 27] and extending the fixed point theory to the multimodal network equilibrium model [21]. However, the previous studies considered the factors of travel time and expense in the link impedance function, and the same criteria are adopted for the travelers' mode choice and route choice. So, the travelers have different preferences not only in the stage of mode choice but also for route choice. In fact, the influence factors of traveler's mode choice in the multimodal network and route choice in a single modal network are different. The generalized travel cost involving travel time, expense, or other factors should be addressed as user preference or multiclass problems in the stage of mode choice. Once travelers decide which mode they would take, travel time is the only factor for their route choices unless there is an imposed charge for selecting the shortest travel time, for example, toll for urban freeways. Such a special issue in the multimodal transportation system is beyond the scope of this study. Nevertheless, few studies clearly indicate the criteria difference between mode choice and route choice, neither is the issue formalised in previous multimodal network models.

In addition, the structure of multimodal system is generally more complicated than that of private car roadway system. Hierarchical structure is an efficient way to model the multimodal system with multiple subnet levels. Mainguenaud [28] presented a data model to manage multimodal networks with a Geographical Information System (GIS), which allows definition of a node and link as an abstraction of a subnet. Jing et al. [29] proposed the Hierarchical Encoded Path View (HEPV) model that partitions large graph into smaller subgraphs and organizes them in a hierarchical fashion. Van Nes [30] introduced a strategy for hierarchical multimodal network levels utilizing specific journey functions according to travel distance as well as quality in terms of travel speed and comfort. Jung and Pramanik [31] developed a graph model, called hierarchical multilevel graph, for very large topographical road maps. This graph model provides a tool to structure and abstract a topographic road map in a hierarchical fashion. These studies mainly focused on topology of multimodal transportation network and discussed how to deal with the problem of a very large volume of data, but the travelers' choice behaviors in the multimodal network were rarely investigated. Not until recent years, researchers have started to integrate traveler's mode choice and route choice with complex network structure in the multimodal network models. Lo et al. [32] transformed a multimodal network to a so-called State Augmented Multimodal (SAM) network, by which the network equilibrium problem can be resolved directly. Wu and Lam [33] used a multilayer network to represent the multimodal network with combined modes that can facilitate generating feasible routes. García and Marín [34] explored the network equilibrium model in the space of hyperroute flows, which contributes to considering asymmetric costs and modeling multimodal network in a more flexible way. Si et al. [35–37] presented an augmented network model for urban transit system. The route choice in the augmented transit network was defined according to the passengers' behaviors, and the corresponding network equilibrium model with an improved shortest path algorithm was developed for the urban transit assignment problem.

The objective of this study is to address the aforementioned concerns of the urban multimodal network equilibrium issue, including (1) assigning traffic based on both user equilibrium between different modes and user equilibrium between different routes; (2) adopting different criteria for travelers' mode choice and route choice behaviors, namely, using multiclass-related general travel cost in the stage of mode choice and traditional link impedance for route choice within each single mode subnet; (3) constructing a hierarchical network to describe the multimodal transportation system, in which the first-tier network is used to depict the travelers' mode choice behaviors and the second-tier network is used to describe travelers' route choice behaviors within the single mode subnets. In this paper, the biequilibrium patterns for multimodal transportation network are proposed by extending Wardrop principle. Correspondingly, a bilevel programming model with its solution algorithm is applied for the biequilibrium traffic assignment in the multimodal transportation network. Finally, a numerical example is provided to illustrate the model and algorithm.

2. Hierarchical Network for Multimodal Transportation System

In this paper, the multimodal transportation system is expressed as $G = (N, A, K)$, where A is the set of roads, N is the set of nodes that usually represent the intersections or zones, and K is the set of transportation modes. Clearly, there are K subnets in the multimodal transportation system, and each subnet, represented by $G_k = (A_k, N_k)$, corresponds to transport mode $k(k \in K)$.

Figure 1 illustrates a physical network example for the proposed multimodal network system, which consists of one O-D pair (r-s), nine nodes, twelve roads, and three transportation modes (car, bus, and bike). It shows that the different modes have different network

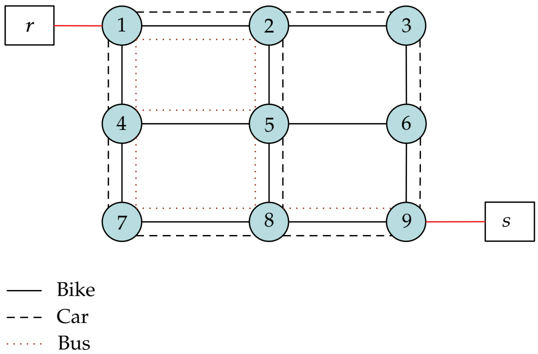


Figure 1: The multimodal transportation system.

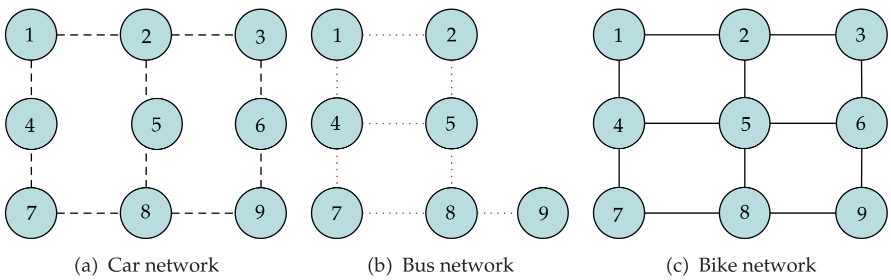


Figure 2: The subnets for different modes.

structures. Figure 2 shows the subnets for different modes separately, in which the traveler can choose the different routes from node 1 to node 9.

Generally, the traveler during a trip from origin to destination should make two successive decisions in the multimodal system. The first one is the mode choice in the whole network, and the second is the route choice in the corresponding subnet once a mode is selected. At the first stage, the multimodal system can be represented as a simple network by the different connections, as shown in Figure 3.

According to the structural features of urban multimodal transportation system demonstrated above, a hierarchical network model can be used to describe such a system. In the model, each node is described by two variables (n, k), where $n(n \in N)$ denotes the location in the physical network and $k(k \in K)$ denotes the transportation mode. Note that the notations of origin and destination nodes require special attention. An origin node is denoted as single variable r and a destination is denoted as s, where r and s designate their physical locations. The set of links connecting the different nodes is divided into two categories. One category includes loading link and unloading link, the end of which is either origin or destination; the other category only includes in-vehicle link that indicates connectivity in each subnet. Both categories are all described by two variables (a, k), where $a(a \in A)$ denotes the physical road and $k(k \in K)$ denotes the transportation mode. The hierarchical multimodal transport system is described in Figure 4.

In such a hierarchical network, the origin is connected with different subnets by the loading links. Similarly, the destination is connected with different subnets by the unloading links. If all travelers are assumed to complete their trips through only one mode, it implies that there should be no connectivity between subnets in the hierarchical network. Based on

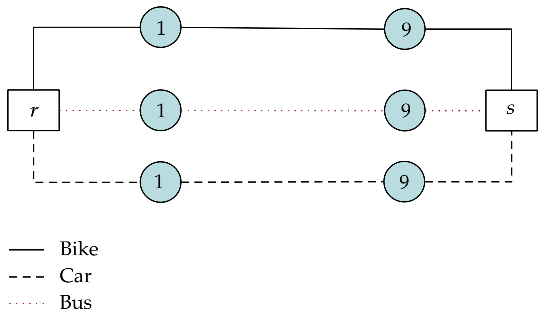


Figure 3: The simplified multimodal transport network.

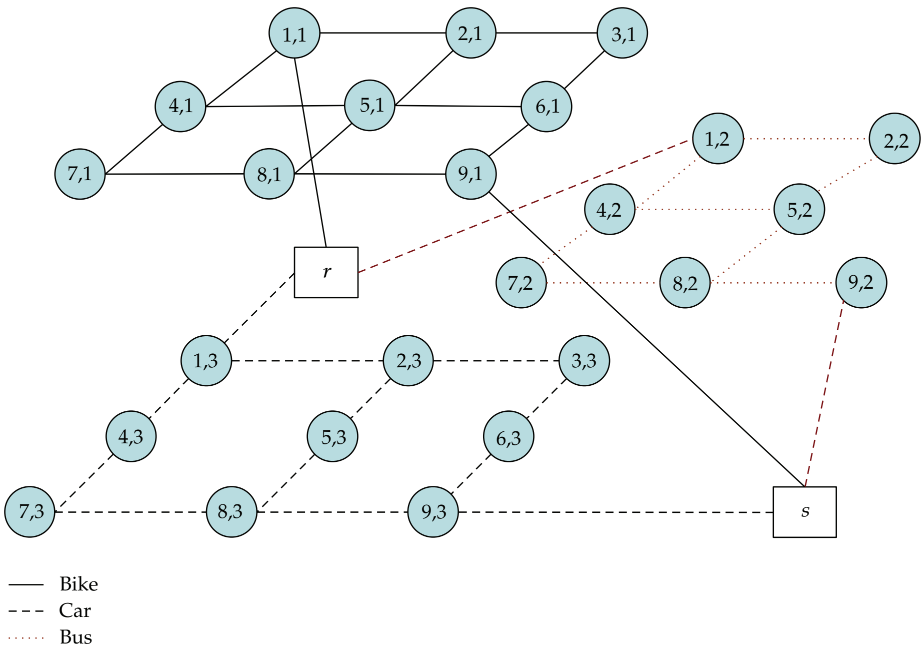


Figure 4: The hierarchical network for multimodal transport system.

the hierarchical network, the multimodal transportation system can be used as a generalized network for traffic assignment or network analysis.

3. Travel Costs Based on Traveler's Characteristics

In this paper, all travelers are divided into I classes by socioeconomic attributes, assuming that the mode-choice decision is homogeneous within each class, but differs among classes. Moreover, the travel time of each mode depends on the travel demands for the mode, and the potential expense of each mode is included in the generalized travel cost for different travelers. The generalized travel costs of different modes for different traveler classes can be expressed as follows:

$$c_w^{i,k} = \alpha^i \mu_w^k(\mathbf{q}) + \beta^i \tau_w^k, \quad \forall w, k, i, \tag{3.1}$$

where $c_w^{i,k}$ is the generalized cost of mode k for class i between O-D pair w; $\mu_w^k(\mathbf{q})$ represents the equilibrium travel time for transportation mode k between O-D pair w, which is decided by the travel demands (represented by $\mathbf{q}$); τ_w^k denotes the potential expense of transportation mode k between O-D pair w; α^i and β^i are parameters related to socioeconomic attributes of class i.

Similar to the general traffic network, the travel time of class i on route r in subnet k between the O-D pair w, denoted by $t_{w,r}^{i,k}$, can be obtained by the travel time on the link, that can be expressed as follows:

$$t_{w,r}^{i,k} = \sum_a t_a^{i,k} \delta_{a,r}^{k,w}, \quad \forall w, k, i, r, \tag{3.2}$$

where $t_a^{i,k}$ denotes the travel time of class i selecting mode k on road a; $\delta_{a,r}^{k,w}$ is route and road incidence variable in the subnet k between O-D pair w; if road a is on the route r, then $\delta_{a,r}^{k,w} = 1$, otherwise, $\delta_{a,r}^{k,w} = 0$.

Generally, no matter what class of travelers, as long as the transportation mode is selected, the travel time in the corresponding subnet is not relevant to the personal properties. In other words, the travel time on the road network is only related to the characteristics of transportation modes, but not related to the traveler's personal properties. Let $t_{w,r}^k$ and t_a^k denote the travel time of mode k on route r between the O-D pair w and the travel time of transportation mode k on road a, respectively. Then,

$$t_{w,r}^{i,k} = t_{w,r}^{j,k} = \cdots = t_{w,r}^k, \quad \forall w, k, r, \ \forall i \neq j, \tag{3.3a}$$

$$t_a^{i,k} = t_a^{j,k} = \cdots = t_a^k, \quad \forall k, a, \ \forall i \neq j. \tag{3.3b}$$

Obviously, (3.2) can be rewritten as

$$t_{w,r}^k = \sum_a t_a^k \delta_{a,r}^{k,w}, \quad \forall w, k, r. \tag{3.4}$$

In the traffic network, the link impedance function mainly describes the relationship between travel time and link flow. It should be noted that the interferences among different modes will occur in the multimodal traffic network if there are no physical barriers between different flows on the road. Therefore, the link impedance function in the multimodal traffic network is very different from that in a single-mode traffic network. The travel time of different modes is decided by not only the road flow of its own mode but also the road flows of the other modes. Accordingly, the link impedance function in multimodal traffic network can be formulated as

$$t_a^k = f\left(t_a^{k(0)}, v_a^1, \ldots, v_a^k, C_a^k\right), \quad \forall k, a, \tag{3.5}$$

where $t_a^{k(0)}$ is the free-flow travel time of mode k on road a; C_a^k is the practical capacity on road a; v_a^k is the vehicle flow of mode k on road a. Generally, $t_a^{k(0)}$ and C_a^k can be assumed as constants.

In the multimodal traffic network, the link flow is defined as the number of vehicles including cars, buses, and bikes that have traveled over the road sections during a time unit, which is a congregative result by all travelers' mode choice and route choice behaviors. Therefore, the number of travelers can be looked upon as a variable in the link impedance function, by which the link flows and corresponding travel time can be calculated. Accordingly, the road flow can be represented by the travel demand as follows:

$$v_a^k = x_a^k \cdot \left(\frac{U_k}{A_k}\right), \quad \forall k, a, \tag{3.6}$$

where x_a^k is the travel demand of mode k on road a; U_k is the PCU conversion coefficient of mode k; A_k is the occupancy rate of mode k, which indicates the average number of travelers within each vehicle of mode k.

As stated above, the road flows of different modes on road a can be expressed by the travel demand of corresponding mode on road a. Consequently, (3.5) can be rewritten as follows:

$$t_a^k = \hat{f}_a^k\left(x_a^1, \ldots, x_a^k\right), \quad \forall k, a. \tag{3.7}$$

4. Conservations of Demand in Multimodal Transportation Network

Assuming that the total demands of different travelers between each O-D pair are given and fixed, for a certain class, the sum of demands of different modes equals the total demand between O-D pair, which can be represented as

$$\sum_k q_w^{i,k} = q_w^i, \quad \forall w, i, \tag{4.1}$$

where q_w^i is the total demand of class i between O-D pair w; $q_w^{i,k}$ is the demand of class i selecting mode k between O-D pair w.

Secondly, for a certain class selecting a certain mode, the sum of demands on different routes in each subnet equals the demand of the corresponding mode between O-D pair, that is:

$$\sum_r h_{w,r}^{i,k} = q_w^{i,k}, \quad \forall w, k, i, \tag{4.2}$$

where $h_{w,r}^{i,k}$ is the demand of class i on the route r in subnet k between O-D pair w.

Obviously, the following formulation can be obtained according to (4.2):

$$\sum_i \sum_r h_{w,r}^{i,k} = \sum_i q_w^{i,k}, \quad \forall w, k. \tag{4.3}$$

Let q_w^k and $h_{w,r}^k$ denote the total demand of mode k between O-D pair w and the demand on the route r in subnet k between O-D pair w, respectively. Then the following two equations can be obtained easily:

$$\sum_i q_w^{i,k} = q_w^k, \quad \forall w, k, \tag{4.4}$$

$$\sum_i h_{w,r}^{i,k} = h_{w,r}^k, \quad \forall w, k, r. \tag{4.5}$$

Then, (4.3) can be rewritten as

$$\sum_r h_{w,r}^k = \sum_i q_w^{i,k} = q_w^k, \quad \forall w, k. \tag{4.6}$$

In addition, for class i in subnet k between O-D pair w, the demand on road a can be represented by the demand on the routes passing though the road, that is:

$$x_a^{i,k} = \sum_w \sum_r h_{w,r}^{i,k} \delta_{a,r}^{k,w}, \quad \forall a, i, k, \tag{4.7}$$

where $x_a^{i,k}$ is the demand of class i selecting mode k on road a.

Similarly, the following formulation can be obtained according to (4.7):

$$\sum_i x_a^{i,k} = \sum_i \sum_w \sum_r h_{w,r}^{i,k} \delta_{a,r}^{k,w} = \sum_w \sum_r \sum_i h_{w,r}^{i,k} \delta_{a,r}^{k,w}, \quad \forall a, i, k. \tag{4.8}$$

Thus, the total demand of mode k on road a is the sum of demand of different classes selecting mode k on road a, that is:

$$x_a^k = \sum_i x_a^{i,k}, \quad \forall a, k. \tag{4.9}$$

The following formulation can be gotten easily according to (4.5) and (4.9):

$$x_a^k = \sum_w \sum_r h_{w,r}^k \delta_{a,r}^{k,w}, \quad \forall a, k. \tag{4.10}$$

5. Biequilibrium Model for Multimodal Transport Network

Equilibrium is a central concept in numerous disciplines from economics and regional science to operational research/management science [38]. The example in transportation science is the famous Wardrop equilibrium. In the conventional equilibrium of transportation, the single-mode traffic network with purely automobile flow is considered, and only the motorists' route choices are examined, while the traveler's mode and route combined choices and the resulting complicated equilibrium in the multimodal traffic network have not been explored substantially.

As aforementioned, the UE principle in multimodal transportation system can be divided into two categories in order to be consistent with the travelers' combined choice behaviors. One category of equilibrium exists between different modes, where the generalized travel cost for a certain class selecting a certain mode is the same and the minimum generalized travel costs of unselected transportation modes must not be less than the minimum cost between O-D pair. The other category is the traditional equilibrium among different routes in each single-mode subnet between O-D pair. The biequilibriums in the multimodal transportation system can be described as

$$c_w^{i,k} \begin{cases} = \eta_w^i, & \text{if} \quad q_w^{i,k} \geq 0, \\ \geq \eta_w^i, & \phantom{\text{if}} \quad q_w^{i,k} = 0, \end{cases} \qquad \forall w, k, i, \tag{5.1}$$

$$t_{w,r}^k \begin{cases} = \mu_w^k, & \text{if} \quad h_{w,r}^k \geq 0, \\ \geq \mu_w^k, & \phantom{\text{if}} \quad h_{w,r}^k = 0, \end{cases} \qquad \forall w, k, r, \tag{5.2}$$

where η_w^i and μ_w^k are the generalized travel cost for class i and the travel time for mode k between O-D pair w at equilibrium.

In this paper, the following bilevel programming model is proposed to describe the combined equilibrium assignment through the multimodal transportation network.

The upper-level problem is to find $\tilde{\mathbf{q}} \in \Omega = \{\mathbf{q} \mid \sum_k q_w^{i,k} = q_w^i,\ q_w^{i,k} \geq 0,\ \forall w, i, k\}$ such that

$$\sum_w \sum_i \sum_k \left\{\alpha^i \cdot \mu_w^k(\tilde{\mathbf{q}}) + \beta^i \cdot \tau_w^k\right\} \times \left(q_w^{i,k} - \tilde{q}_w^{i,k}\right) \geq 0, \tag{5.3}$$

where $\mathbf{q}$ is the vector of $q_w^{i,k}$; the function $\mu_w^k(\mathbf{q})$ is decided by the following lower-level problem.

The lower-level problem is to find

$$\tilde{\mathbf{x}}(\mathbf{q}) \in \Psi = \left\{\mathbf{x} \mid \sum_r h_{w,r}^k = \sum_i q_w^{i,k},\ x_a^k = \sum_w \sum_r h_{w,r}^k \delta_{a,r}^{k,w},\ h_{w,r}^k \geq 0,\ \forall w, k, r, a\right\} \tag{5.4}$$

such that

$$\sum_a \sum_k \hat{f}_a^k\{\tilde{\mathbf{x}}(\mathbf{q})\} \times \left(x_a^k - \tilde{x}_a^k(\mathbf{q})\right) \geq 0, \tag{5.5}$$

where $\mathbf{x}$ is the vector of x_a^k.

It can been seen that the variational inequality (VI) model for upper-level problem is to find equilibrium demand of class i selecting mode k between O-D pair w, that is, $\tilde{q}_w^{i,k}$, to meet the first equilibrium principle in (5.1). The travelers' generalized costs are partially decided by the equilibrium flow patterns and the corresponding travel time through the different subnets. The relationship between them is described by the lower-level VI model with parameters in (5.5). The lower-level problem represents the equilibrium assignment reflecting travelers' route choice behaviors within each subnet, and the goal is to find the equilibrium flows and corresponding travel time under the condition that the demands of different classes and selected different modes are all given. The variables $\mathbf{q}$ and $\mathbf{x}$ can be regarded as decision variables for the bilevel problem. The biequilibrium for the urban multimodal network can be achieved by solving the bilevel problem.

The equivalence between the solution to the previous model and the equilibrium conditions for multimodal transportation network is given as follows.

Assuming that $\tilde{\mathbf{q}} \in \Omega$ is a solution to VI problem in (5.3), then $\tilde{\mathbf{q}}$ is bound to meet the following conditions:

$$\tilde{q}_w^{i,k}\left(c_w^{i,k} - \eta_w^i\right) = 0, \quad \forall w, k, i, \tag{5.6a}$$

$$c_w^{i,k} - \eta_w^i \geq 0, \quad \forall w, k, i, \tag{5.6b}$$

where η_w^i is the dual multiplier of the constraint condition (4.1).

Similarly, assuming that $\tilde{\mathbf{x}} \in \Psi$ is a solution to VI problem in (5.5), then $\tilde{\mathbf{x}}$ is bound to meet the following conditions:

$$h_{w,r}^k\left(t_{w,r}^k - \mu_w^k\right) = 0, \quad \forall w, k, r, \tag{5.7a}$$

$$t_{w,r}^k - \mu_w^k \geq 0, \quad \forall w, k, r, \tag{5.7b}$$

where μ_w^k is the dual multiplier of the constraint condition in (4.6).

Obviously, the first equilibrium condition in (5.1) can be gotten from (5.6a) and (5.6b), and the second equilibrium condition in (5.2) can be gotten by (5.7a) and (5.7b).

6. Solution Algorithm

Due to the intrinsic complexity of model formulation, the bilevel programming problem has been recognized as one of the most difficult, yet challenging, problems for global optimality in transportation system. In the past decades, researchers [35, 36, 39–42] developed alternative solution algorithms for this problem. The sensitivity analysis-based method proposed by Tobin and Friesz [43] is used to solve the bilevel programming model proposed in this paper.

It is necessary to derive the derivatives of the decision variables with respect to the parameters for the lower-level problem in the sensitivity analysis approach. In our proposed problem, we need to calculate the derivatives of the optimal dual multiplier of the constraint condition in (4.6), that is, the equilibrium of O-D travel time (represented by $\boldsymbol{\mu}$), with respect to the travel demand (represented by $\mathbf{q}$). By assuming that the initial $\mathbf{q}^{(0)}$ is given and other conditions are fixed, the equilibrium O-D travel time matrix for a multimodal traffic network, $\tilde{\boldsymbol{\mu}}(\mathbf{q}^{(0)})$ can be obtained by solving the lower level of the model. Through conducting a sensitivity analysis of VI model in (5.5) (see appendix), the approximate differential coefficient, $\nabla_{\mathbf{q}}\boldsymbol{\mu}$, can be obtained. Then the response function can be approximated by the Taylor expansions. That is,

$$\boldsymbol{\mu}(\mathbf{q}) \approx \tilde{\boldsymbol{\mu}}\left(\mathbf{q}^{(0)}\right) + \left(\nabla_{\mathbf{q}}\boldsymbol{\mu}\right)^T\left(\mathbf{q} - \mathbf{q}^{(0)}\right). \tag{6.1}$$

By substituting (6.1) into the upper-level problem, the whole optimization model can be simplified as one-level optimization problem. The solution of this one-level optimization will then be input into the lower level of the model to run the next iteration. By repeating the iteration process, it is possible to obtain an optimum solution for the above bilevel programming model. This process can be summarized as the following steps.

Step 1. Set the initial value $\mathbf{q}^{(0)}$, and set the number of iterations to $i = 1$.

Step 2. Find the solution of the lower-level model, $\tilde{\boldsymbol{\mu}}^{(i)}$.

Step 3. Find the linear equation of the matrix, $\boldsymbol{\mu}(\mathbf{q})$, through sensitivity analysis and Taylor expansion.

Step 4. Put the linear equation of the matrix into the upper-level model to update the value of $\mathbf{q}^{(i)}$ by solving upper-level problem.

Step 5. Examine the convergence. If $\mathbf{q}^{(i)} \approx \mathbf{q}^{(i-1)}$ or $i = N$, then iteration stops, where N is the maximum number of iterations. Otherwise, set $i = i + 1$ and start a new iteration.

Note that both Steps 2 and 4 solve different VI models. The approach most commonly used to solve VI model is the popularly known "diagonalization" method, which mimics the Jacobi (resp., Gauss-Seidel) decomposition approach used for solving systems of equations [44]. The idea behind the method is to fix flows for all but one group of variables and to iteratively solve a sequence of separable subproblems which can be described as mathematical programs. As for VI model in (5.3), the vector function $\mu_w^k(\mathbf{q})$ is "diagonalized" by the current solution in nth iteration, yielding a symmetric assignment problem, which can be represented by the following mathematical program:

$$\min_{\mathbf{q}\in\Omega} Z(\mathbf{q}) = \sum_w \sum_k \sum_i \int_0^{q_w^{i,k}} c_w^{i,k}\left(q_{w(n-1)}^{1,1}, \ldots, q_{w(n-1)}^{1,k}, \ldots, q_{w(n-1)}^{i,1}, \ldots, \omega\right) \mathrm{d}\omega. \tag{6.2}$$

Similarly, as for VI model in (5.5), the vector function $\widehat{f}_a^k(\mathbf{x})$ is "diagonalized" at the current solution, yielding the following mathematical program:

$$\min_{\mathbf{x}\in\Psi} F(\mathbf{x}) = \sum_a \sum_k \int_0^{x_a^k} \widehat{f}_a^k\left(x_{a(n-1)}^1, x_{a(n-1)}^2, \ldots, \omega\right) \mathrm{d}\omega. \tag{6.3}$$

The Frank-Wolfe method or MSA method can be employed to solve the diagonalization problem (6.2) and (6.3). Due to the limited space, the detailed process MSA method for (6.3) is given as follows here.

Step 1. Initialization: set $x_a^k = 0$ and compute $t_a^{k(0)}$ for any k and a. Find the shortest route in subnet k between O-D pair w. Then perform all-or-nothing assignment to load q_w^k for subnet k and obtain $x_a^{k(1)}$ for any k and a. Set iteration $n = 1$.

Step 2. Compute $t_a^{k(n)}$ based on $x_a^{k(n)}$.

Step 3. Find the shortest route in subnet k between O-D pair w. Perform all-or-nothing assignment to load q_w^k and obtain $y_a^{k(n)}$ for any k and a.

Step 4. Compute

$$x_a^{k(n+1)} = x_a^{k(n)} + \frac{1}{n}\left(y_a^{k(n)} - x_a^{k(n)}\right), \quad \forall k, a. \tag{6.4}$$

Step 5. Convergence test: if a convergence criterion is met, stop. The current solutions, $\{x_a^{k(n+1)}\}$, are the sets of equilibrium solutions; otherwise, set $n = n + 1$ and go to Step 2.

7. Numerical Example

A simple numerical example is used to illustrate the effectiveness of the proposed model and algorithm. The multimodal transportation system and the corresponding hierarchical network structure are, respectively, given by Figures 1 and 4.

The following impedance functions are used in this example [18, 19]:

$$t_a^k = t_a^{k(0)} \prod_m \left[1 + \gamma\left(\frac{U_m \cdot x_a^m}{A_m \cdot C_a^m}\right)^{\lambda}\right], \quad \forall k, a. \tag{7.1}$$

The relevant data of different roads are given in Table 2, where the PCU conversion coefficient, the average occupancy rate, and potential expense, which are pertinent to different modes, are illustrated in Table 3.

The values of $\gamma = 0.15$, $\lambda = 4$ are set for the parameters in (7.1). In this example, the travelers are divided into two classes: (i) for the first class, $\alpha^1 = 2.5$ and $\beta^1 = 0.5$ indicate that this class is sensitive to the travel time; (ii) for the second class, $\alpha^2 = 1.5$ and $\beta^2 = 0.5$ indicate that this class is sensitive to the potential expense. The demands of these two classes are all assumed as $5000/\mathrm{Ph}^{-1}$.

The convergences of the diagonalization method for the lower-level problem and the upper-level problem are, respectively, analyzed using the gap measure proposed by Boyce et al. [45]. The gaps at iteration n for the assignment models can be defined as

$$\begin{aligned} \text{gap}(n) &= -\sum_a \sum_k t_a^{k(n)} \cdot \left(y_a^{k(n)} - x_a^{k(n)}\right), \\ \text{gap}(n) &= -\sum_w \sum_i \sum_k c_w^{i,k(n)} \cdot \left(v_w^{i,k(n)} - q_w^{i,k(n)}\right), \end{aligned} \tag{7.2}$$

where $y_a^{k(n)}$ is the auxiliary flow of mode k on link a at iteration n given by an all-or-nothing assignment based on link travel time, $t_a^{k(n)}$, and $v_w^{i,k(n)}$ is the auxiliary demand of class i selecting mode k between O-D pair w at iteration n given by an all-or-nothing assignment based on the generalized costs, $c_w^{i,k(n)}$.

Figure 5 shows the gaps against the iteration number for the lower-level problem and upper-level problem, respectively. It can be seen that the solution algorithm has a good convergence especially for the upper-level problem. It can be explained that the network structure of the traveler's mode choice in the upper-level problem is simpler than that of the traveler's route choice in the lower-level problem.

Table 2: The relevant data of different roads.

Road	$t_a^{1(0)}/(\mathrm{h})$	$t_a^{2(0)}/(\mathrm{h})$	$t_a^{3(0)}/(\mathrm{h})$	$C_a^1/(\mathrm{Ph}^{-1})$	$C_a^2/(\mathrm{Ph}^{-1})$	$C_a^3/(\mathrm{Ph}^{-1})$
(1,2)	0.111	0.178	0.361	1000	1000	600
(2,3)	0.128	—	0.378	700	—	400
(1,4)	0.100	0.167	0.350	1500	1500	800
(2,5)	0.106	0.172	0.356	700	700	400
(3,6)	0.089	—	0.339	700	—	400
(4,5)	—	0.144	0.328	—	1000	600
(5,6)	—	—	0.344	—	—	600
(4,7)	0.133	0.200	0.383	900	900	500
(5,8)	0.111	0.178	0.361	700	700	400
(6,9)	0.144	—	0.394	700	—	400
(7,8)	0.094	0.161	0.344	900	900	500
(8,9)	0.100	0.167	0.350	900	900	500

Table 3: The relevant data of different modes.

Mode	U_k	A_k	τ_w^k
Car	1	4	10
Bus	1.5	20	4
Bike	0.25	1	0

Table 4 shows the equilibrium results of mode demand and the corresponding generalized costs of different classes. Table 5 shows the equilibrium results of road demand and the corresponding travel time of different modes.

Next, we analyze the impacts of the pertinent parameters in this example on the modal share and the performance of the whole network. Here, the shares of different modes, denoted by P_w^k, can be computed by

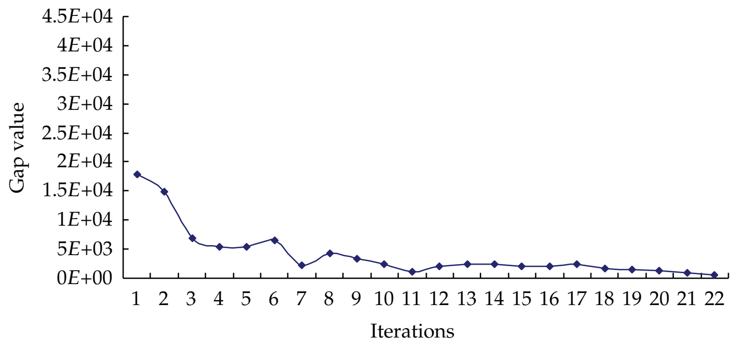
$$P_w^k = \frac{q_w^k}{\sum_i q_w^i}, \quad \forall w, k. \tag{7.3}$$

The total travel time of the network, denoted by T, is used to represent the performance of the whole network, that is:

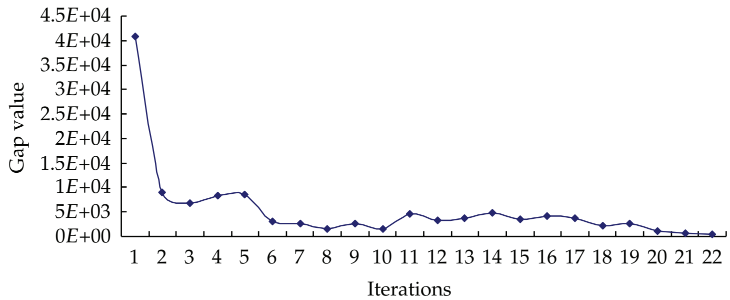
$$T = \sum_a \sum_k \frac{U_k \cdot x_a^k}{A_k} \cdot t_a^k. \tag{7.4}$$

Figures 6(a)–6(d), respectively, show the changes in modal share and the total travel time of the whole network with the changes of the parameters (α^1, α^2), which indicate the travelers' sensitivity to the factor of travel time (or congestion). It shows that the share of bike and the total travel time of whole network will decrease while the shares of bus and car will increase symmetrically with the increasing of α^1 or α^2. The travelers who select the bike mode with longer travel time will shift into the car or bus mode when such travelers become more sensitive to travel time.

Figures 7(a)–7(d), respectively, display the changes in modal share and the total travel time with the changes of the parameters (β^1, β^2), which indicate the travelers' sensitivity to the potential travel expense. It can be found that the share of bike and the total travel time



(a)



(b)

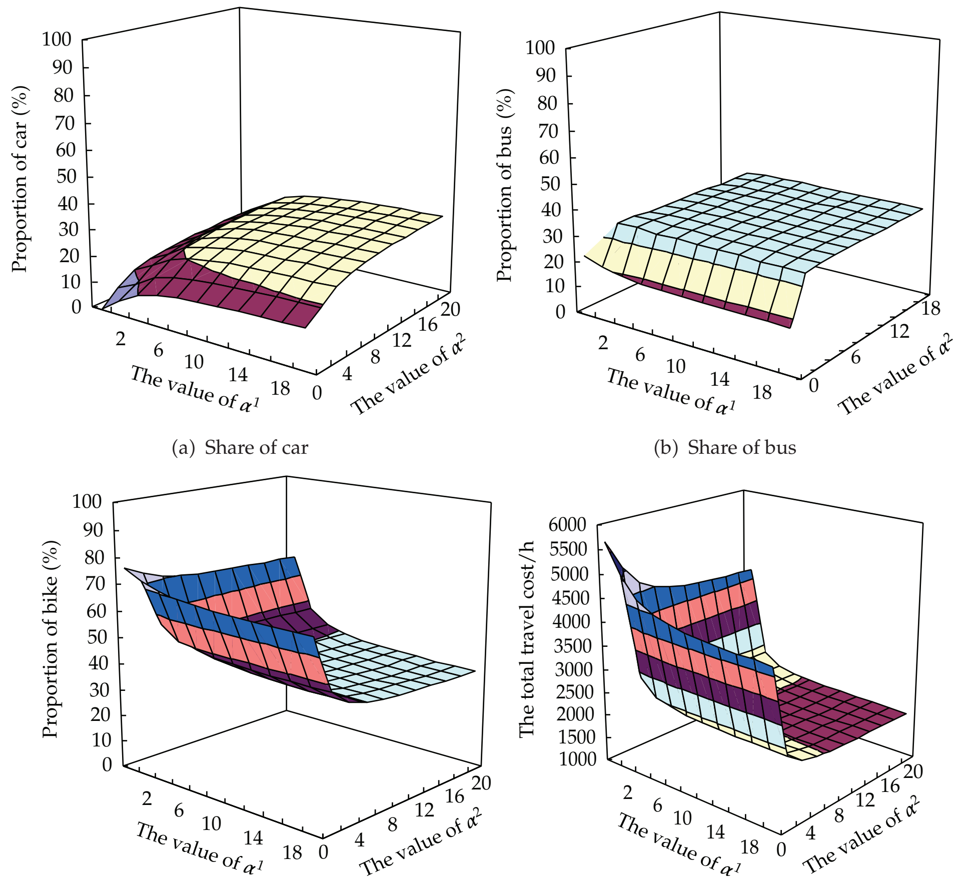
Figure 5: The convergences of the algorithms for upper problem (a) and lower problem (b).

Table 4: The equilibrium results of demand and the corresponding costs of different classes.

	Mode 1 (car)		Mode 2 (bus)		Mode 3 (bike)	
	Class 1	Class 2	Class 1	Class 2	Class 1	Class 2
Demand/Ph^{-1}	609	284	1809	1684	2582	3032
Generalized cost/h	21.0199	13.0646	21.0195	13.0635	21.0196	13.0644

Table 5: The equilibrium results of road demands and its travel time of different modes.

Roads	$x_a^1/(\mathrm{Ph}^{-1})$	$x_a^2/(\mathrm{Ph}^{-1})$	$x_a^3/(\mathrm{Ph}^{-1})$	$t_a^1/(\mathrm{h})$	$t_a^2/(\mathrm{h})$	$t_a^3/(\mathrm{h})$
(1,2)	166.72	0	2148.55	0.1147	0.1835	0.3960
(2,3)	0	—	900.00	0.1284	—	0.3835
(1,4)	726.28	3493.00	3465.45	0.1070	0.1783	0.4225
(2,5)	166.72	0	1248.55	0.1075	0.1754	0.3754
(3,6)	0	—	900.00	0.0893	—	0.3440
(4,5)	—	3443.00	1940.48	—	0.1476	0.3489
(5,6)	—	—	1572.41	—	—	0.3540
(4,7)	726.28	50.00	1524.97	0.1356	0.2034	0.4028
(5,8)	166.72	3443.00	1616.62	0.1175	0.1880	0.4183
(6,9)	0	—	2472.41	0.1858	—	0.7331
(7,8)	726.28	50.00	1524.97	0.0961	0.1639	0.3619
(8,9)	893.00	3493.00	3141.59	0.1321	0.2202	0.6725



(a) Share of car

(b) Share of bus

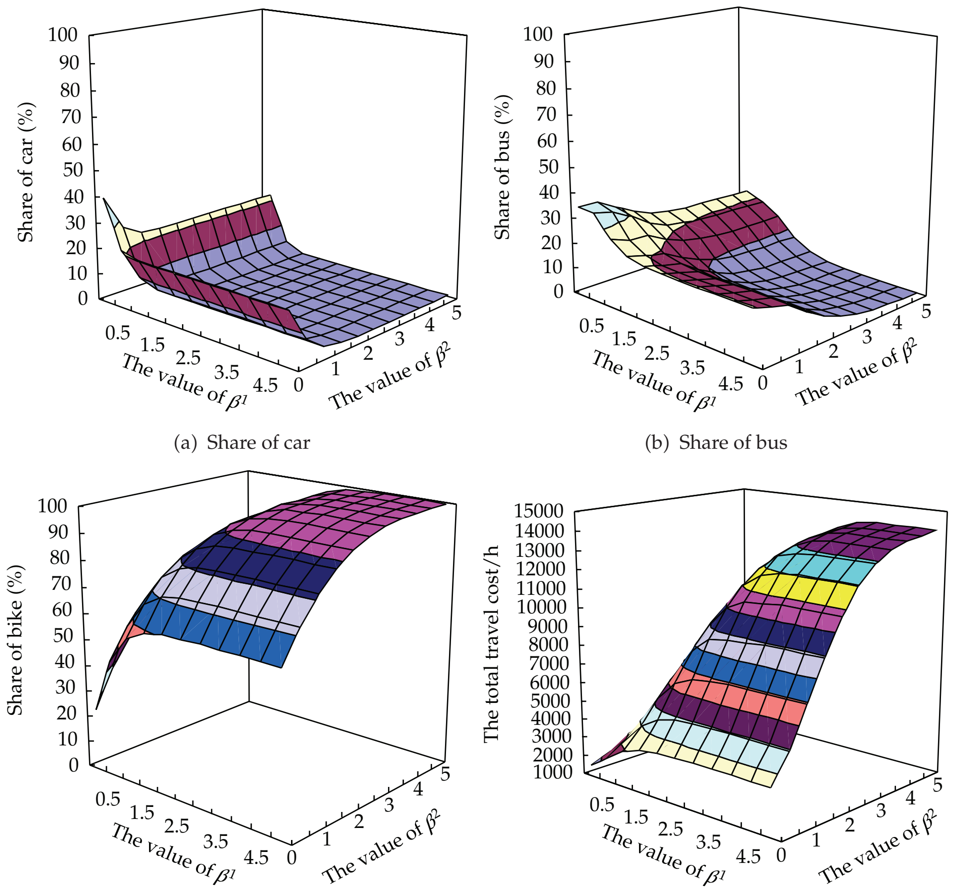
(c) Share of bike

(d) The total travel time

Figure 6: The changes of shares of different modes and the total travel time with the changes of (α^1, α^2).

will increase while the shares of bus and car will decrease with the increment of β^1 or β^2. The travelers who select the bus or car mode will shift into the bike mode without any potential expense when such travelers become more sensitive to the potential expense. The previous results imply that the performance of the whole network in terms of total travel time would be better when the travelers are more sensitive to the travel time and less sensitive to potential expense.

The shares of various modes and the total travel time can be dramatically changed with the changes of (α^1, α^2) or (β^1, β^2) in a certain range. However, these values will not change significantly when these parameters reach a certain value. When the travelers are all excessive time-sensitive or cost-sensitive, the other factor can affect their mode choice behaviors to a very small extent. For example, when the travelers are very sensitive to the potential expense, they would not consider the factor of travel time. In such a condition, most of the travelers would choose bike as their traffic tools since they do not bear any costs. As the speed of bike is slowest, the total cost of network will reach the maximum. On the contrary, when the traveler is very sensitive to the travel time, they will not consider the factor of money. Therefore, the travelers always tend to choose the mode with the shortest travel time (such as car). Simultaneously, the travel time of such mode will become longer



(a) Share of car

(b) Share of bus

(c) Share of bike

(d) The total travel time

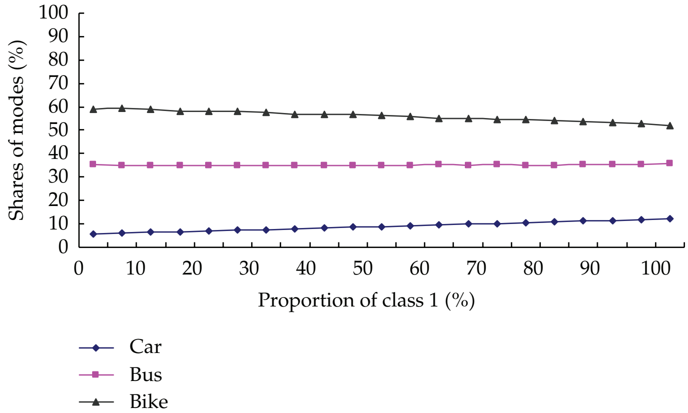
Figure 7: The changes of shares of different modes and the total travel time with the changes of (β^1, β^2).

and longer with its increasing demand. The equilibrium between different modes will be achieved ultimately, and the shares and the total travel time of the whole network will not be changed at such equilibrium.

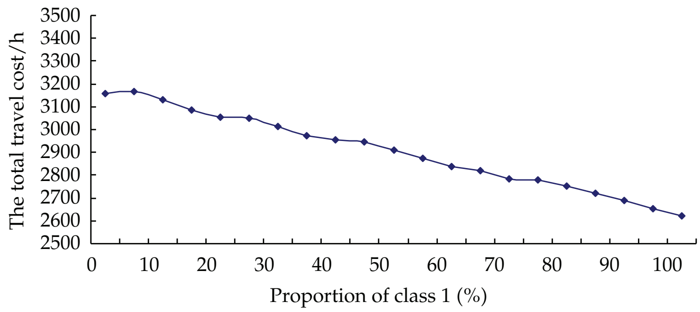
Assuming that the total number of travelers between O-D remains unchanged (take the value of 10000 persons each hour), Figures 8(a) and 8(b), respectively, show the change trends of the shares and the total travel time with the proportion of class I which is sensitive to the travel time. It can be shown that the share of bike mode will decrease and the share of car will go up slightly, while the share of bus remained unchanged. Meanwhile, the total travel time of whole network will increase with the increment of the proportion of class I. The results also imply that in the multiclass multimodal transportation network, the more travelers who focus on the factor of travel time, the lower the total travel time of network is.

8. Conclusions

This paper presents a biequilibrium traffic assignment model for multimodal transportation networks using the bilevel programming method. The model development is based on several important concepts that are not explored by the previous studies.



(a)



(b)

Figure 8: The changes of shares of different modes and the total travel time with the proportions of class 1.

First, a two-tier hierarchical multimodal network is proposed for the model, in which the first-tier network is used for mode choice and the second-tier network is used to for route choice in the single mode subnets.

Second, the model distinguishes the criteria between mode choice and route choice. The mode choice behavior is based on the multiclass generalized travel cost while the route choice behavior is based on the travel time only. The generalized cost functions of different modes and the link impendence functions are formulated while the interferences between different modes are considered. The approach can better reflect traveler's preference and decision-making process in a multimodal transportation system.

Third, the biequilibrium pattern of traffic assignment is firstly proposed for multimodal traffic network modeling. Its major advantage is integrating the separated two steps of mode split and traffic assignment in the traditional transportation planning method into a unified process.

The solution algorithm for the bilevel programming model is illustrated by a simple numerical example. The sensitivity analysis shows that as travelers are more sensitive to travel time, they are more likely to choose the mode with less travel time, which will mitigate the congestion of whole network. In contrast, as travelers are more sensitive to travel expense, they are more likely to choose the mode without expense, such as bike, which will aggravate the congestion of whole network. As for the travelers who are more sensitive to travel time, the changes of their choice behaviors will impact on the performance of whole network significantly. Additionally, with the increment of the proportion of traveler class that is more sensitive to travel time, the network congestion will be mitigated gradually.

It should be noted that there are some limitations in this paper. For example, all travelers are assumed to complete their trips through only one mode; in other words, they are assumed not to change modes during their journey. Such reasonable assumption will preclude the possibility of park-and-ride or similar mode-change mechanisms. In addition, the case of fixed demand is considered, while the case of elastic demand or demand uncertainty is not taken into account. So, the promising future work would be extension to reliability analysis for these situations.

Appendix

Sensitivity Analysis of VI Model (5.5)

Assume that the solutions to $\tilde{\mathbf{x}}(\mathbf{q}^{(0)})$ and $\boldsymbol{\mu}(\mathbf{q}^{(0)})$ of the variational inequality problem in (5.5) at $\mathbf{q} = \mathbf{q}^{(0)}$ have been obtained and that $\hat{f}_a^k(\mathbf{x})$ is strongly monotone in $\mathbf{x}$, so that the solutions are unique. According to Tobin and Friesz [43], the necessary conditions excluding the nonbinding constraints for solution at $\mathbf{q} = \mathbf{q}^{(0)}$ for VI problem in (5.5) can be expressed as follows:

$$\begin{gathered}\sum_a \hat{f}_a^k(\tilde{\mathbf{x}}) \cdot \delta_{a,r}^{k,w} - \tilde{\mu}_w^k = 0, \quad \forall w, k, r, \\ \sum_r \tilde{h}_{w,r}^k - \sum_i q_w^{i,k} = 0, \quad \forall w, k.\end{gathered} \tag{A.1}$$

Let $\mathbf{y} = [\mathbf{h}, \boldsymbol{\mu}]^T$, where $\mathbf{h}$ is the vector of $h_{w,r}^k$. Let $\mathbf{J}_\mathbf{y}$ and $\mathbf{J}_\mathbf{q}$ denote the Jacobian matrixes of (A.1) with respect to $\mathbf{y}$ and $\mathbf{q}$ at the point $\mathbf{q} = \mathbf{q}^{(0)}$, respectively:

$$\mathbf{J}_\mathbf{y} = \begin{bmatrix} \nabla_\mathbf{h}\mathbf{t} & \boldsymbol{\Lambda}^T \\ \boldsymbol{\Lambda} & \mathbf{0} \end{bmatrix}, \tag{A.2}$$

where $\mathbf{t}$ is the vector of $t_{w,r}^k$, and $\boldsymbol{\Lambda}$ is the O-D and path incidence matrix. Suppose

$$[\mathbf{J}_\mathbf{y}]^{-1} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}. \tag{A.3}$$

The following results can be obtained:

$$\begin{aligned}
\mathbf{B}_{22} &= \left[\boldsymbol{\Lambda} \cdot \nabla_\mathbf{h}\mathbf{t}^{-1} \cdot \boldsymbol{\Lambda}^T\right]^{-1}, \\
\mathbf{B}_{12} &= \nabla_\mathbf{h}\mathbf{t}^{-1} \cdot \boldsymbol{\Lambda}^T \cdot \mathbf{B}_{22} = \nabla_\mathbf{h}\mathbf{t}^{-1} \cdot \boldsymbol{\Lambda}^T\left[\boldsymbol{\Lambda} \cdot \nabla_\mathbf{h}\mathbf{t}^{-1} \cdot \boldsymbol{\Lambda}^T\right]^{-1}, \\
\mathbf{B}_{21} &= -\mathbf{B}_{22} \cdot \boldsymbol{\Lambda} \cdot \nabla_\mathbf{h}\mathbf{t}^{-1} = -\left[\boldsymbol{\Lambda} \cdot \nabla_\mathbf{h}\mathbf{t}^{-1} \cdot \boldsymbol{\Lambda}^T\right]^{-1} \cdot \boldsymbol{\Lambda} \cdot \nabla_\mathbf{h}\mathbf{t}^{-1}, \\
\mathbf{B}_{11} &= \nabla_\mathbf{h}\mathbf{t}^{-1} \cdot \left[\mathbf{I} + \boldsymbol{\Lambda}^T \cdot \mathbf{B}_{21}\right] = \nabla_\mathbf{h}\mathbf{t}^{-1} \cdot \left\{\mathbf{I} - \boldsymbol{\Lambda}^T \cdot \left[\boldsymbol{\Lambda} \cdot \nabla_\mathbf{h}\mathbf{t}^{-1} \cdot \Lambda^T\right]^{-1} \cdot \boldsymbol{\Lambda} \cdot \nabla_\mathbf{h}\mathbf{t}^{-1}\right\},
\end{aligned} \tag{A.4}$$

where $\mathbf{I}$ is unit matrix:

$$\mathbf{J}_{\mathbf{q}} = \begin{bmatrix} \nabla_{\mathbf{q}}\mathbf{t} \\ -\mathbf{I} \end{bmatrix}. \tag{A.5}$$

From theorems in Tobin and Friesz [43], the following result can be obtained:

$$\begin{bmatrix} \nabla_{\mathbf{q}}\mathbf{h} \\ \nabla_{\mathbf{q}}\boldsymbol{\mu} \end{bmatrix} = \left[\mathbf{J}_{\mathrm{y}}\right]^{-1} \cdot \left[-\mathbf{J}_{\mathbf{q}}\right] = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} -\nabla_{\mathbf{q}}\mathbf{t} \\ \mathbf{I} \end{bmatrix}. \tag{A.6}$$

Thus, the approximate differential coefficient, $\nabla_{\mathbf{q}}\boldsymbol{\mu}$, can be obtained:

$$\nabla_{\mathbf{q}}\boldsymbol{\mu} = -\mathbf{B}_{21} \cdot \nabla_{\mathbf{q}}\mathbf{t} + \mathbf{B}_{22} = \left[\boldsymbol{\Lambda} \cdot \nabla_{\mathbf{h}}\mathbf{t}^{-1} \cdot \boldsymbol{\Lambda}^{T}\right]^{-1} \cdot \boldsymbol{\Lambda} \cdot \nabla_{\mathbf{h}}\mathbf{t}^{-1} \cdot \nabla_{\mathbf{q}}\mathbf{t} + \left[\boldsymbol{\Lambda} \cdot \nabla_{\mathbf{h}}\mathbf{t}^{-1} \cdot \boldsymbol{\Lambda}^{T}\right]^{-1}. \tag{A.7}$$

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References

[1] Beijing Transportation Research Center (BTRC), "Investigation and analysis of Beijing residents' travel behavior in 2010," Research Report, Beijing, China, 2011.
[2] A. B. Beckmann, C. B. McGuire, and C. B. Winsten, *Studies in the Economics of Transportation*, Yale University Press, New Haven, Conn, USA, 1956.
[3] Y. Sheffi, *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1985.
[4] M. Patriksson, *The Traffic Assignment Problem: Models and Methods*, VSP, Utrecht, The Netherlands, 1994.
[5] M. Florian, "A traffic equilibrium model of travel by car and public transit modes," *Transportation Science*, vol. 11, no. 2, pp. 166–179, 1977.
[6] M. Florian and H. Spiess, "On binary mode choice/assignment models," *Transportation Science*, vol. 17, no. 1, pp. 32–47, 1983.
[7] A. B. Nagurney, "Comparative tests of multimodal traffic equilibrium methods," *Transportation Research Part B*, vol. 18, no. 6, pp. 469–485, 1984.
[8] S. C. Wong, "Multi-commodity traffic assignment by continuum approximation of network flow with variable demand," *Transportation Research Part B*, vol. 32B, no. 8, pp. 567–581, 1998.
[9] P. Ferrari, "A model of urban transport management," *Transportation Research Part B*, vol. 33B, no. 1, pp. 43–61, 1999.
[10] S. Li, W. Deng, and Y. Lv, "Combined modal split and assignment model for the multimodal transportation network of the economic circle in China," *Transport*, vol. 24, no. 3, pp. 241–248, 2009.
[11] M. Florian and S. Nguyen, "A combined trip distribution modal split and trip assignment model," *Transportation Research*, vol. 12, no. 4, pp. 241–246, 1978.
[12] K. N. A. Safwat and T. L. Magnanti, "A combined trip generation, trip distribution, modal split, and trip assignment model," *Transportation Science*, vol. 22, no. 1, pp. 14–30, 1988.
[13] T. Abrahamsson and L. Lundqvist, "Formulation and estimation of combined network equilibrium models with applications to Stockholm," *Transportation Science*, vol. 33, no. 1, pp. 80–100, 1999.
[14] D. Boyce, "Network equilibrium models of urban location and travel choices: a new research agenda," in *New Frontiers in Regional Science*, M. Chatterji and R. E. Kuenne, Eds., pp. 238–256, Macmillan, New York, NY, USA, 1990.

[15] D. Boyce, "Long-term advances in the state of the art of travel forecasting methods," in *Equilibrium and Advanced Transportation ModelingEdited By*, P. Marcotte and S. Nguyen, Eds., pp. 73–86, Kluwer, Dordrecht, The Nertherlands, 1998.
[16] C. S. Fisk and D. E. Boyce, "Alternative variationa1 inequality formulations of the network equilibrium travel choice problem," *Transportation Science*, vol. 17, no. 4, pp. 454–463, 1983.
[17] M. Florian, J. H. Wu, and S. He, "A multi-class multi-mode variable demand network equilibrium model with hierarchical logit structures," in *Transportation and Network Analysis: Current Trends*, vol. 63, pp. 119–133, Kluwer Academic Publishers, London, UK, 2002.
[18] B. Si, J. Long, and Z. Gao, "Optimization model and algorithm for mixed traffic of urban road network with flow interference," *Science in China. Series E*, vol. 51, no. 12, pp. 2223–2232, 2008.
[19] B. F. Si, M. Zhong, and Z. Y. Gao, "A link resistance function of urban mixed traffic network," *Journal of Transportation Systems Engineering and Information Technology*, vol. 8, no. 1, pp. 68–73, 2008.
[20] Z. Zhou, A. Chen, and S. C. Wong, "Alternative formulations of a combined trip generation, trip distribution, modal split, and trip assignment model," *European Journal of Operational Research*, vol. 198, no. 1, pp. 129–138, 2009.
[21] M. Xu and Z. Y. Gao, "Multi-class multi-modal network equilibrium with regular choice behaviors: a general fixed point approach," in *Transportation and Traffic Theory*, W. H. K. Lam, S. C. Wong, and H. K. Lo, Eds., pp. 301–325, Springer, 2009.
[22] A. Nagurney, "A multiclass, multicriteria traffic network equilibrium model," *Mathematical and Computer Modelling*, vol. 32, no. 3-4, pp. 393–411, 2000.
[23] R. B. Dial, "A model and algorithm for multicriteria route-mode choice," *Transportation Research Part B*, vol. 13, no. 4, pp. 311–316, 1979.
[24] S. Dafermos, *A Multicriteria Route-Mode Choice Traffic Equilibrium Model*, Lefschetz Center for Dynamical Systems, Brown University, Providence, RI, USA, 1981.
[25] A. Nagurney and J. Dong, "A multiclass, multicriteria traffic network equilibrium model with elastic demand," *Transportation Research Part B*, vol. 36, no. 5, pp. 445–469, 2002.
[26] M. C. J. Bliemer and P. H. L. Bovy, "Quasi-variational inequality formulation of the multiclass dynamic traffic assignment problem," *Transportation Research Part B*, vol. 37, no. 6, pp. 501–519, 2003.
[27] K. I. Wong, S. C. Wong, J. H. Wu, H. Yang, and W. H. K. Lam, "A combined distribution, hierarchical mode choice, and assignment network model with multiple user and mode classes," in *Urban and Regional Transportation Modeling*, Essays in Honor of David Boyce, pp. 25–42, 2004.
[28] M. Mainguenaud, "Modelling the network component of geographical information systems," *International Journal of Geographical Information Systems*, vol. 9, no. 6, pp. 575–593, 1995.
[29] N. Jing, Y. W. Huang, and E. A. Rundensteiner, "Hierarchical encoded path views for path query processing: an optimal model and its performance evaluation," *IEEE Transactions on Knowledge and Data Engineering*, vol. 10, no. 3, pp. 409–432, 1998.
[30] R. Van Nes, "Hierarchical levels networks in the design of multimodal transport networks," in *Proceedings of the Nectar Conference*, Delft, The Netherlands, 1999.
[31] S. Jung and S. Pramanik, "An efficient path computation model for hierarchically structured topographical road maps," *IEEE Transactions on Knowledge and Data Engineering*, vol. 14, no. 5, pp. 1029–1046, 2002.
[32] H. K. Lo, C. W. Yip, and K. H. Wan, "Modeling transfer and non-linear fare structure in multi-modal network," *Transportation Research B*, vol. 37, pp. 149–170, 2003.
[33] Z. X. Wu and W. H. K. Lam, "Network equilibrium for congested multi-mode networks with elastic demand," *Journal of Advanced Transportation*, vol. 37, no. 3, pp. 295–318, 2003.
[34] R. García and A. Marín, "Network equilibrium with combined modes: models and solution algorithms," *Transportation Research Part B*, vol. 39, no. 3, pp. 223–254, 2005.
[35] B. Si, M. Zhong, and Z. Gao, "Bilevel programming for evaluating revenue strategy of railway passenger transport under multimodal market competition," *Transportation Research Record*, no. 2117, pp. 1–6, 2009.
[36] B. Si, M. Zhong, H. Sun, and Z. Gao, "Equilibrium model and algorithm of urban transit assignment based on augmented network," *Science in China. Series E*, vol. 52, no. 11, pp. 3158–3167, 2009.
[37] B. Si, M. Zhong, X. Yang, and Z. Gao, "Urban transit assignment model based on augmented network with in-vehicle congestion and transfer congestion," *Journal of Systems Science and Systems Engineering*, vol. 20, no. 2, pp. 155–172, 2011.
[38] A. Nagurney, *Network Economics: A Variational Inequality Approach*, vol. 1 of *Advances in Computational Economics*, Kluwer Academic Publishers, Dordrecht, The Netherrlands, 1993.

[39] L. J. LeBlanc and D. E. Boyce, "A bilevel programming algorithm for exact solution of the network design problem with user-optimal flows," *Transportation Research B*, vol. 20, no. 3, pp. 259–265, 1986.

[40] C. Suwansirikul, T. L. Friesz, and R. L. Tobin, "Equilibrium decomposed optimization: a heuristic for the continuous equilibrium network design problem," *Transportation Science*, vol. 21, no. 4, pp. 254–263, 1987.

[41] J. Clegg, M. Smith, Y. Xiang, and R. Yarrow, "Bilevel programming applied to optimising urban transportation," *Transportation Research Part B*, vol. 35, no. 1, pp. 41–70, 2001.

[42] M. Patriksson, "On the applicability and solution of bilevel optimization models in transportation science: a study on the existence, stability and computation of optimal solutions to stochastic mathematical programs with equilibrium constraints," *Transportation Research Part B*, vol. 42, no. 10, pp. 843–860, 2008.

[43] R. L. Tobin and T. L. Friesz, "Sensitivity analysis for equilibrium network flow," *Transportation Science*, vol. 22, no. 4, pp. 242–250, 1988.

[44] P. T. Harker, "Accelerating the convergence of the diagonalization and projection algorithms for finite-dimensional variational inequalities," *Mathematical Programming*, vol. 41, no. 1, pp. 29–59, 1988.

[45] D. Boyce, B. Ralevic-Dekic, and H. Bar-Gera, "Convergence of traffic assignments: how much is enough?" *Journal of Transportation Engineering*, vol. 130, no. 1, pp. 49–55, 2004.

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Research Article

The Solution Set Characterization and Error Bound for the Extended Mixed Linear Complementarity Problem

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For the extended mixed linear complementarity problem (EML CP), we first present the characterization of the solution set for the EMLCP. Based on this, its global error bound is also established under milder conditions. The results obtained in this paper can be taken as an extension for the classical linear complementarity problems.

1. Introduction

We consider that the extended mixed linear complementarity problem, abbreviated as EMLCP, is to find vector $(x^*; y^*) \in R^{2n}$ such that

$$\begin{gathered} F(x^*) \geq 0, \qquad G(x^*, y^*) \geq 0, \qquad F(x^*)^\top G(x^*, y^*) = 0, \\ Ax^* + By^* + b \geq 0, \qquad Cx^* + Dy^* + d = 0, \end{gathered} \tag{1.1}$$

where $F(x) = Mx + p$, $G(x) = Nx + Qy + q$, $M, N, Q \in R^{m\times n}$, $p, q \in R^m$, $A, B \in R^{s\times n}$, C, $D \in R^{t\times n}$, $b \in R^s$, $d \in R^t$. We assume that the solution set of the EMLCP is nonempty throughout this paper.

The EMLCP is a direct generalization of the classical linear complementarity problem and a special case of the generalized nonlinear complementarity problem which was discussed in the literature ([1, 2]). The extended complementarity problem plays a significant role in economics, engineering, and operation research, and so forth [3]. For example, the balance of supply and demand is central to all economic systems; mathematically, this fundamental equation in economics is often described by a complementarity relation between two sets of decision variables. Furthermore, the classical Walrasian law of competitive equilibria of exchange economies can be formulated as a generalized nonlinear complementarity problem in the price and excess demand variables [4].

Up to now, the issues of the solution set characterization and numerical methods for the classical linear complementarity problem or the classical nonlinear complementarity problem were fully discussed in the literature (e.g., [5–8]). On the other hand, the global error bound is also an important tool in the theoretical analysis and numerical treatment for variational inequalities, nonlinear complementarity problems, and other related optimization problems [9]. The error bound estimation for the classical linear complementarity problems (LCP) was fully analyzed (e.g., [7–12]).

Obviously, the EMLCP is an extension of the LCP, and this motivates us to extend the solution set characterization and error bound estimation results of the LCP to the EMLCP. To this end, we first detect the solution set characterization of the EMLCP under milder conditions in Section 2. Based on these, we establish the global error bound estimation for the EMLCP in Section 3. These constitute what can be taken as an extension of those for linear complementarity problems.

We end this section with some notations used in this paper. Vectors considered in this paper are all taken in Euclidean space equipped with the standard inner product. The Euclidean norm of vector in the space is denoted by $\|\cdot\|$. We use R^n_+ to denote the nonnegative orthant in R^n and use x_+ and x_- to denote the vectors composed by elements $(x_+)_i := \max\{x_i, 0\}$ and $(x_-)_i := \max\{-x_i, 0\}, 1 \le i \le n$, respectively. For simplicity, we use $(x; y)$ for column vector $(x^\top, y^\top)^\top$. We also use $x \ge 0$ to denote a nonnegative vector $x \in R^n$ if there is no confusion.

2. The Solution Set Characterization for EMLCP

In this section, we will characterize the solution set of the EMLCP. First, we can give the needed assumptions for our analysis.

Assumption 2.1. For the matrices M, N, Q involved in the EMLCP, we assume that the matrix $\begin{pmatrix} M^\top N + N^\top M & M^\top Q \\ Q^\top M & 0 \end{pmatrix}$ is positive semidefinite.

Theorem 2.2. *Suppose that Assumption 2.1 holds; the following conclusions hold.*

(i) *If $(x_0; y_0)$ is a solution of the EMLCP, then*

$$X^* = \Big\{ (x; y) \in X \mid \Big\{ (M, 0_{m\times n})^\top (N, Q) + (N, Q)^\top (M, 0_{m\times n}) \Big\} \{ (x; y) - (x_0; y_0) \} = 0, \quad \Big\{ (M, 0_{m\times n})^\top q + (N, Q)^\top p \Big\}^\top \{ (x; y) - (x_0; y_0) \} = 0 \Big\}, \tag{2.1}$$

where $X = \{(x; y) \in R^{2n} \mid Mx + p \ge 0,\ Nx + Qy + q \ge 0,\ Ax + By + b \ge 0,\ Cx + Dy + d = 0\}$, and X^* denotes the solution set of EMLCP.

(ii) *If* $(x_1; y_1)$ *and* $(x_2; y_2)$ *are two solutions of the EMLCP, then*

$$(Mx_1 + p)^\top (Nx_2 + Qy_2 + q) = (Mx_2 + p)^\top (Nx_1 + Qy_1 + q) = 0. \tag{2.2}$$

(iii) *The solution set of EMLCP is convex.*

Proof. (i) Set

$$\begin{aligned} W = \Big\{ (x; y) \in X \mid & \Big\{ (M,0)^\top (N,Q) + (N,Q)^\top (M,0) \Big\} \{ (x;y) - (x_0; y_0) \} = 0, \\ & \Big\{ (M,0)^\top q + (N,Q)^\top p \Big\}^\top \{ (x;y) - (x_0;y_0) \} = 0 \Big\}. \end{aligned} \tag{2.3}$$

For any $(\tilde{x}; \tilde{y}) \in X^*$, since $(x_0; y_0) \in X$, we have

$$\begin{aligned} &((x_0; y_0) - (\tilde{x}; \tilde{y}))^\top \Big((M,0)^\top (N,Q)(\tilde{x};\tilde{y}) + (M,0)^\top q \Big) \\ &\quad = \big[(Mx_0 + p) - (M\tilde{x} + p) \big]^\top (N\tilde{x} + Q\tilde{y} + q) \\ &\quad = \big[(Mx_0 + p) \big]^\top (N\tilde{x} + Q\tilde{y} + q) - (M\tilde{x} + p)^\top (N\tilde{x} + Q\tilde{y} + q) \\ &\quad = \big[(Mx_0 + p) \big]^\top (N\tilde{x} + Q\tilde{y} + q) \geq 0. \end{aligned} \tag{2.4}$$

Since $(\tilde{x}; \tilde{y}) \in X$, $(x_0; y_0) \in X^*$, using the similar arguments to that in (2.4), we have

$$((\tilde{x}; \tilde{y}) - (x_0; y_0))^\top \Big((M,0)^\top (N,Q)(x_0; y_0) + (M,0)^\top q \Big) \geq 0. \tag{2.5}$$

Combining (2.4) with (2.5), one has

$$((\tilde{x}; \tilde{y}) - (x_0; y_0))^\top (M,0)^\top (N,Q) ((\tilde{x}; \tilde{y}) - (x_0; y_0)) \leq 0. \tag{2.6}$$

By (2.6), we have

$$((\tilde{x}; \tilde{y}) - (x_0; y_0))^\top \Big((M,0)^\top (N,Q) + (N,Q)^\top (M,0) \Big) ((\tilde{x}; \tilde{y}) - (x_0; y_0)) \leq 0. \tag{2.7}$$

By Assumption 2.1, one has

$$\begin{aligned} &((\tilde{x}; \tilde{y}) - (x_0; y_0))^\top \Big((M,0)^\top (N,Q) + (N,Q)^\top (M,0) \Big) ((\tilde{x}; \tilde{y}) - (x_0; y_0)) \\ &\quad = ((\tilde{x}; \tilde{y}) - (x_0; y_0))^\top \begin{pmatrix} M^\top N + N^\top M & M^\top Q \\ Q^\top M & 0 \end{pmatrix} ((\tilde{x}; \tilde{y}) - (x_0; y_0)) \geq 0. \end{aligned} \tag{2.8}$$

Combining (2.7) with (2.8), we have

$$((\tilde{x};\tilde{y})-(x_0;y_0))^\top\Big((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\Big)((\tilde{x};\tilde{y})-(x_0;y_0))=0. \tag{2.9}$$

That is,

$$\Big((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\Big)((\tilde{x};\tilde{y})-(x_0;y_0))=0. \tag{2.10}$$

Using $(x_0;y_0)\in X, (\tilde{x};\tilde{y})\in X^*$ again, we have

$$\begin{aligned}
&((x_0;y_0)-(\tilde{x};\tilde{y}))^\top\Big((N,Q)^\top(M,0)(\tilde{x};\tilde{y})+(N,Q)^\top p\Big)\\
&\quad=\left[(Nx_0+Qy_0+q)-(N\tilde{x}+Q\tilde{y}+q)\right]^\top(M\tilde{x}+p)\\
&\quad=(Nx_0+Qy_0+q)^\top(M\tilde{x}+p)-(N\tilde{x}+Q\tilde{y}+q)^\top(M\tilde{x}+p)\\
&\quad=(Nx_0+Qy_0+q)^\top(M\tilde{x}+p)\geq 0.
\end{aligned} \tag{2.11}$$

Using $(\tilde{x};\tilde{y})\in X, (x_0;y_0)\in X^*$ again, using the similar arguments to that in (2.11), we have

$$((\tilde{x};\tilde{y})-(x_0;y_0))^\top\Big((N,Q)^\top(M,0)(x_0;y_0)+(N,Q)^\top p\Big)\geq 0. \tag{2.12}$$

From (2.9), (2.4), and (2.11), one has

$$\begin{aligned}
&((x_0;y_0)-(\tilde{x};\tilde{y}))^\top\Big\{\Big((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\Big)(x_0;y_0)+(M,0)^\top q+(N,Q)^\top p\Big\}\\
&\quad=((x_0;y_0)-(\tilde{x};\tilde{y}))^\top\Big((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\Big)((x_0;y_0)-(\tilde{x};\tilde{y}))\\
&\qquad+((x_0;y_0)-(\tilde{x};\tilde{y}))^\top\Big\{\Big((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\Big)(\tilde{x};\tilde{y})\\
&\qquad\qquad+(M,0)^\top q+(N,Q)^\top p\Big\}\\
&\quad=((x_0;y_0)-(\tilde{x};\tilde{y}))^\top\Big\{(M,0)^\top(N,Q)(\tilde{x};\tilde{y})+(M,0)^\top q\Big\}\\
&\qquad+((x_0;y_0)-(\tilde{x};\tilde{y}))^\top\Big\{(N,Q)^\top(M,0)(\tilde{x};\tilde{y})+(N,Q)^\top p\Big\}\geq 0.
\end{aligned} \tag{2.13}$$

Combining (2.5) with (2.12) yields

$$((\tilde{x};\tilde{y})-(x_0;y_0))^\top\Big(\Big((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\Big)(x_0;y_0)+(M,0)^\top q+(N,Q)^\top p\Big)\geq 0. \tag{2.14}$$

Combining this with (2.13) yields

$$((\widetilde{x};\widetilde{y})-(x_0;y_0))^{\top}\Big(\big((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\big)(x_0;y_0)+(M,0)^{\top}q+(N,Q)^{\top}p\Big)=0. \tag{2.15}$$

From (2.10) and (2.15), one has

$$\Big((M,0)^{\top}q+(N,Q)^{\top}p\Big)^{\top}((\widetilde{x};\widetilde{y})-(x_0;y_0))=0. \tag{2.16}$$

By (2.10) and (2.16), we obtain that $(\widetilde{x};\widetilde{y})\in W$ follows.

On the other hand, for any $(\widehat{x};\widehat{y})\in W$, then $(\widehat{x};\widehat{y})\in X$, and

$$\begin{gathered}\Big((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\Big)((\widehat{x};\widehat{y})-(x_0;y_0))=0,\\ \Big((M,0)^{\top}q+(N,Q)^{\top}p\Big)((\widehat{x};\widehat{y})-(x_0;y_0))=0,\end{gathered} \tag{2.17}$$

and one has

$$\begin{aligned}0&=((\widehat{x};\widehat{y})-(x_0;y_0))^{\top}\Big[\Big((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\Big)(x_0;y_0)\\&\qquad+(M,0)^{\top}q+(N,Q)^{\top}p\Big]\\&=((\widehat{x};\widehat{y})-(x_0;y_0))^{\top}\Big((M,0)^{\top}(N,Q)(x_0;y_0)+(M,0)^{\top}q\Big)\\&\quad+((\widehat{x};\widehat{y})-(x_0;y_0))^{\top}\Big((N,Q)^{\top}(M,0)(x_0;y_0)+(N,Q)^{\top}p\Big)\\&=\big[(M\widehat{x}+p)-(Mx_0+p)\big]^{\top}(Nx_0+Qy_0+q)\\&\quad+\big[(N\widehat{x}+Q\widehat{y}+q)-(Nx_0+Qy_0)+q)\big]^{\top}(Mx_0+p)\\&=(M\widehat{x}+p)^{\top}(Nx_0+Qy_0+q)+(N\widehat{x}+Q\widehat{y}+q)^{\top}(Mx_0+p).\end{aligned} \tag{2.18}$$

Using (2.18), one has

$$\begin{aligned}0&=((\widehat{x};\widehat{y})-(x_0;y_0))^{\top}\Big((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\Big)((\widehat{x};\widehat{y})-(x_0;y_0))\\&=2((\widehat{x};\widehat{y})-(x_0;y_0))^{\top}(M,0)^{\top}(N,Q)((\widehat{x};\widehat{y})-(x_0;y_0))\\&=2\big[(M\widehat{x}+p)-(Mx_0+p)\big]^{\top}\big[(N\widehat{x}+Q\widehat{y}+q)-(Nx_0+Qy_0+q)\big]\\&=2\Big[(M\widehat{x}+p)^{\top}(N\widehat{x}+Q\widehat{y}+q)-(M\widehat{x}+p)^{\top}(Nx_0+Qy_0+q)\\&\qquad-(Mx_0+p)^{\top}(N\widehat{x}+Q\widehat{y}+q)+(Mx_0+p)^{\top}(Nx_0+Qy_0+q)\Big]\\&=2(M\widehat{x}+p)^{\top}(N\widehat{x}+Q\widehat{y}+q).\end{aligned} \tag{2.19}$$

Thus, we have that $(\widehat{x};\widehat{y})\in X^{*}$.

(ii) Since $(x_1; y_1)$ and $(x_2; y_2)$ are two solutions of the EMLCP, by Theorem 2.2 (i), we have

$$\begin{aligned}
&\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)((x_1;y_1)-(x_2;y_2))\\
&\quad=\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)((x_1;y_1)-(x_0;y_0))\\
&\qquad-\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)((x_2;y_2)-(x_0;y_0))=0.
\end{aligned}\tag{2.20}$$

Combining this with $(Mx_1+p)^{\top}(Nx_1+Qy_1+q)=(Mx_2+p)^{\top}(Nx_2+Qy_2+q)=0$, one has

$$\begin{aligned}
0&=((x_1;y_1)-(x_2;y_2))^{\top}\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)((x_1;y_1)-(x_2;y_2))\\
&=2((x_1;y_1)-(x_2;y_2))^{\top}(M,0)^{\top}(N,Q)((x_1;y_1)-(x_2;y_2))\\
&=2[(Mx_1+p)-(Mx_2+p)]^{\top}[(Nx_1+Qy_1+q)-(Nx_2+Qy_2+q)]\\
&=-2\left[(Mx_1+p)^{\top}(Nx_2+Qy_2+q)+(Mx_2+p)^{\top}(Nx_1+Qy_1+q)\right].
\end{aligned}\tag{2.21}$$

On the other hand, from $Mx_i+p\geq 0$, $Nx_i+Qy_i+q\geq 0$, $i=1,2$, we can deduce

$$(Mx_1+p)^{\top}(Nx_2+Qy_2+q)\geq 0,\qquad (Mx_2+p)^{\top}(Nx_1+Qy_1+q)\geq 0.\tag{2.22}$$

From (2.21) and (2.22), thus, we have that Theorem 2.2 (ii) holds.

(iii) If solution set of the EMLCP is single point set, then it is obviously convex. In this following, we suppose that $(x^1; y^1)$ and $(x^2; y^2)$ are two solutions of the EMLCP. By Theorem 2.2 (i), we have

$$\begin{aligned}
\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)\left(\left(x^1;y^1\right)-(x_0;y_0)\right)&=0,\\
\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)\left(\left(x^2;y^2\right)-(x_0;y_0)\right)&=0,\\
\left((M,0)^{\top}q+(N,Q)^{\top}p\right)^{\top}\left(\left(x^1;y^1\right)-(x_0;y_0)\right)&=0,\\
\left((M,0)^{\top}q+(N,Q)^{\top}p\right)^{\top}\left(\left(x^2;y^2\right)-(x_0;y_0)\right)&=0.
\end{aligned}\tag{2.23}$$

For the vector $(x; y)=\tau(x^1; y^1)+(1-\tau)(x^2; y^2)$, for all $\tau\in[0,1]$, by (2.23), we have

$$\begin{aligned}
&\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)((x;y)-(x_0;y_0))\\
&\quad=\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)\left(\tau\left(x^1;y^1\right)-\tau(x_0;y_0)\right)\\
&\qquad+\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)\left((1-\tau)\left(x^2;y^2\right)-(1-\tau)(x_0;y_0)\right)=0.
\end{aligned}\tag{2.24}$$

Using the similar arguments to that in (2.24), we can also obtain

$$\left((M,0)^{\top}q+(N,Q)^{\top}p\right)^{\top}((x;y)-(x_0;y_0))=0. \tag{2.25}$$

Combining (2.24) and (2.25) with the conclusion of Theorem 2.2 (i), we obtain the desired result. □

Corollary 2.3. *Suppose that Assumption 2.1 holds. Then, the solution set for EMLCP has the following characterization:*

$$\begin{aligned} X^{*}=\Big\{(x;y)\in X \mid &\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)((x;y)-(x_0;y_0))=0,\\ &((x;y)-(x_0;y_0))^{\top}\Big[\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)(x_0;y_0)\\ &+(M,0)^{\top}q+(N,Q)^{\top}p\Big]\le 0.\Big\}. \end{aligned} \tag{2.26}$$

Proof. Set

$$\begin{aligned} \widetilde{W}=\Big\{(x;y)\in X \mid &\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)((x;y)-(x_0;y_0))=0\,,\\ &((x;y)-(x_0;y_0))^{\top}\Big[\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)(x_0;y_0)\\ &+(M,0)^{\top}q+(N,Q)^{\top}p\Big]\le 0\Big\}. \end{aligned} \tag{2.27}$$

For any $(\hat{x};\hat{y})\in\widetilde{W}$, then $(\hat{x};\hat{y})\in X$, combining this with $(x_0;y_0)\in X^{*}$. Using the similar arguments to that in (2.5) and (2.12), we have

$$((\hat{x};\hat{y})-(x_0;y_0))^{\top}\Big[\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)(x_0;y_0)+(M,0)^{\top}q+(N,Q)^{\top}p\Big]\ge 0. \tag{2.28}$$

Combining this with $(\hat{x};\hat{y})\in\widetilde{W}$, one has

$$((\hat{x};\hat{y})-(x_0;y_0))^{\top}\Big[\left((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0)\right)(x_0;y_0)+(M,0)^{\top}q+(N,Q)^{\top}p\Big]=0. \tag{2.29}$$

From $((M,0)^{\top}(N,Q)+(N,Q)^{\top}(M,0))((\hat{x};\hat{y})-(x_0;y_0))=0$, we have

$$\left((M,0)^{\top}q+(N,Q)^{\top}p\right)^{\top}((\hat{x};\hat{y})-(x_0;y_0))=0. \tag{2.30}$$

Thus, by Theorem 2.2 (i), one has $(\hat{x};\hat{y})\in X^{*}$.

On the other hand, for any $(\widehat{x};\widehat{y}) \in X^*$, by Theorem 2.2 (i), we have $(\widehat{x};\widehat{y}) \in X$, $((M,0)^\top(N,Q) + (N,Q)^\top(M,0))((\widehat{x};\widehat{y}) - (x_0;y_0)) = 0$, and $((M,0)^\top q + (N,Q)^\top p)^\top((\widehat{x};\widehat{y}) - (x_0;y_0)) = 0$, that is,

$$((\widehat{x};\widehat{y}) - (x_0;y_0))^\top\left[\left((M,0)^\top(N,Q) + (N,Q)^\top(M,0)\right)(x_0;y_0) + (M,0)^\top q + (N,Q)^\top p\right] = 0. \tag{2.31}$$

Thus, $(\widehat{x};\widehat{y}) \in \widetilde{W}$. □

Using the following definition developed from EMLCP, we can further detect the solution structure of the EMLCP.

Definition 2.4. A solution $(\overline{x};\overline{y})$ of the EMLCP is said to be nondegenerate if it satisfies

$$\left(M\overline{x} + p\right) + \left(N\overline{x} + Q\overline{y} + q\right) > 0. \tag{2.32}$$

Theorem 2.5. *Suppose that Assumption 2.1 holds, and the EMLCP has a nondegenerate solution, say $(x_0;y_0)$. Then, the following conclusions hold.*

(i) *The solution set of EMLCP*

$$X^* = \Big\{(x;y) \in X \mid ((x;y) - (x_0;y_0))^\top\Big[\Big((M,0)^\top(N,Q) + (N,Q)^\top(M,0)\Big)(x_0;y_0) + (M,0)^\top q + (N,Q)^\top p\Big] \le 0\Big\}. \tag{2.33}$$

(ii) *If the matrices $M_{\overline{\alpha}}$ and Q_α are the full-column rank, where $\alpha = \{i \mid (Mx_0 + p)_i > 0, i = 1,2,\ldots,m\}$, $\overline{\alpha} = \{i \mid i = 1,2,\ldots,m, i \notin \alpha\}$, then $(x_0;y_0)$ is the unique nondegenerate solution of EMLCP.*

Proof. (i) Set

$$\overline{W} = \Big\{(x;y) \in X \mid ((x;y) - (x_0;y_0))^\top\Big[\Big((M,0)^\top(N,Q) + (N,Q)^\top(M,0)\Big)(x_0;y_0) + (M,0)^\top q + (N,Q)^\top p\Big] \le 0\Big\}. \tag{2.34}$$

From Corollary 2.3, one has $X^* \subseteq \overline{W}$. In this following, we will show that $\overline{W} \subseteq X^*$. For any $(x;y) \in \overline{W}$, then $(x;y) \in X$, combining this with $(x_0;y_0) \in X^*$. Using the similar arguments to that in (2.14), we have

$$((x;y) - (x_0;y_0))^\top\left[\left((M,0)^\top(N,Q) + (N,Q)^\top(M,0)\right)(x_0;y_0) + (M,0)^\top q + (N,Q)^\top p\right] \ge 0. \tag{2.35}$$

Combining this with $(x; y) \in \overline{W}$, one has

$$\begin{aligned}
0 &= ((x;y) - (x_0;y_0))^{\top}\Big[\Big((M,0)^{\top}(N,Q) + (N,Q)^{\top}(M,0)\Big)(x_0;y_0) + (M,0)^{\top}q + (N,Q)^{\top}p\Big] \\
&= ((x;y) - (x_0;y_0))^{\top}\Big[(M,0)^{\top}(N,Q)(x_0;y_0) + (M,0)^{\top}q\Big] \\
&\quad + ((x;y) - (x_0;y_0))^{\top}\Big[(N,Q)^{\top}(M,0)(x_0;y_0) + (N,Q)^{\top}p\Big] \\
&= \left[(Mx+p) - (Mx_0+p)\right]^{\top}(Nx_0 + Qy_0 + q) \\
&\quad + \left[(Nx+Qy+q) - (Nx_0+Qy_0+q)\right]^{\top}(Mx_0+p) \\
&= (Mx+p)^{\top}(Nx_0+Qy_0+q) + (Nx+Qy+q)^{\top}(Mx_0+p).
\end{aligned} \tag{2.36}$$

Combining $Mx + p \geq 0, Nx + Qy + q \geq 0$ with (2.36), one has

$$(Mx+p)^{\top}(Nx_0+Qy_0+q) = (Mx_0+p)^{\top}(Nx+Qy+q) = 0. \tag{2.37}$$

Since $(x_0; y_0)$ is a nondegenerate solution, combining this with (2.37), we have $(Mx + p)^{\top}(Nx + Qy + q) = 0$. That is, $(x; y) \in X^*$.

(ii) Let $(\hat{x}; \hat{y})$ be any nondegenerate solution. Since $(x_0; y_0)$ is a nondegenerate solution, then we have

$$(Mx_0+p)^{\top}(Nx_0+Qy_0+q) = 0, \tag{2.38}$$

$$(Mx_0+p) + (Nx_0+Qy_0+q) > 0. \tag{2.39}$$

Combining (2.38) with (2.39), we have

$$(Nx_0+Qy_0+q)_i = 0, \quad \forall i \in \alpha. \tag{2.40}$$

If $i \notin \alpha$, then $(Nx_0 + Qy_0 + q)_i > 0$ by (2.39). By (2.38) again, we can deduce that

$$(Mx_0+p)_i = 0, \quad \forall i \notin \alpha. \tag{2.41}$$

On the other hand, for the $(x_0; y_0)$ and $(\hat{x}; \hat{y})$ which are solutions of EMLCP, and combining Theorem 2.2 (ii), we have $(M\hat{x}+p)^{\top}(Nx_0+Qy_0+q) = 0$. Using $(Nx_0+Qy_0+q)_i > 0$, for all $i \notin \alpha$, we can deduce that

$$(M\hat{x}+p)_i = 0, \quad \forall i \notin \alpha. \tag{2.42}$$

Combining Theorem 2.2 (ii) again, we also have

$$((Mx_0+p))^{\top}(N\hat{x}+Q\hat{y}+q) = 0. \tag{2.43}$$

For any $i \in \alpha$, that is, $(Mx_0 + p)_i > 0$, and combining (2.43), we obtain

$$(N\widehat{x} + Q\widehat{y} + q)_i = 0, \quad \forall i \in \alpha. \tag{2.44}$$

Combining this with the fact that $(M\widehat{x} + p) + (N\widehat{x} + Q\widehat{y} + q) > 0$, we can deduce that

$$(M\widehat{x} + p)_i > 0, \quad \forall i \in \alpha. \tag{2.45}$$

From (2.41) and (2.42), we obtain

$$M_{\overline{\alpha}}(\widehat{x} - x_0) = 0. \tag{2.46}$$

Thus, $\widehat{x} = x_0$ by the full-column rank assumption on $M_{\overline{\alpha}}$. Using $\widehat{x} = x_0$, combining (2.40) with (2.44), we can deduce that

$$Q_\alpha \widehat{y} = -N_\alpha \widehat{x} - q = -N_\alpha x_0 - q = Q_\alpha y_0. \tag{2.47}$$

That is, $\widehat{y} = y_0$ by the full-column rank assumption on Q_α. Thus, the desired result follows. □

The solution set characterization obtained in Theorem 2.2 (i) coincides with that of Lemma 2.1 in [7], and the solution set characterization obtained in Theorem 2.5 (i) coincides with that of Lemma 2.2 in [8] for the linear complementarity problem.

3. Global Error Bound for the EMLCP

In this following, we will present a global error bound for the EMLCP based on the results obtained in Corollary 2.3 and Theorem 2.5 (i). Firstly, we can give the needed error bound for a polyhedral cone from [13] and following technical lemmas to reach our claims.

Lemma 3.1. *For polyhedral cone $P = \{x \in R^n \mid D_1 x = d_1, B_1 x \le b_1\}$ with $D_1 \in R^{l\times n}$, $B_1 \in R^{m\times n}$, $d_1 \in R^l$ and $b_1 \in R^m$, there exists a constant $c_1 > 0$ such that*

$$\operatorname{dist}(x, P) \le c_1 [\|D_1 x - d_1\| + \|(B_1 x - b_1)_+\|] \quad \forall x \in R^n; \tag{3.1}$$

Lemma 3.2. *Suppose that $(x_0; y_0)$ is a solution of EMLCP, and let*

$$\omega = \left[\left((M,0)^\top (N,Q) + (N,Q)^\top (M,0)\right)(x_0; y_0) + (M,0)^\top q + (N,Q)^\top p\right], \tag{3.2}$$

then, there exists a constant $\tau > 0$, such that for any $(x; y) \in R^{2n}$, one has

$$\begin{aligned} &\left[\omega^\top ((x;y) - (x_0;y_0))\right]_- \\ &\quad \le \tau\left(\|(Mx+p)_-\| + \|(Nx+Qy+q)_-\| + \|(Ax+By+b)_-\| + \|Cx+Dy+d\|\right). \end{aligned} \tag{3.3}$$

Proof. Similar to the proof of (2.14), we can obtain

$$\omega^\top\left((x;y)-(x_0;y_0)\right)\geq 0,\quad \forall(x;y)\in X. \tag{3.4}$$

We consider the following linear programming problems

$$\begin{aligned}\min\quad & \omega^\top(x;y)\\ \text{s.t.}\quad & Mx+p\geq 0,\\ & Nx+Qy+q\geq 0,\\ & Ax+By+b\geq 0,\\ & Cx+Dy+d=0.\end{aligned} \tag{3.5}$$

From the assumption, we know that (x_0,y_0) is an optimal point of the linear programming problem. Thus, there exist optimal Lagrange multipliers $\lambda_1,\lambda_2\in R^m_+$, $\lambda_3\in R^s_+$, and $\lambda_4\in R^t$ such that

$$\begin{gathered}\omega=(M,0)^\top\lambda_1+(N,Q)^\top\lambda_2+(A,B)^\top\lambda_3+(C,D)^\top\lambda_4,\\ Mx_0+p\geq 0,\qquad Nx_0+Qy_0+q\geq 0,\\ Ax_0+By_0+b\geq 0,\qquad Cx_0+Dy_0+d=0,\\ \left((M,0)(x_0;y_0)+p\right)^\top\lambda_1=0,\\ \left(Nx_0+Qy_0+q\right)^\top\lambda_2=0,\\ \left(Ax_0+By_0+b\right)^\top\lambda_3=0.\end{gathered} \tag{3.6}$$

From (3.6), we can easily deduce that

$$\begin{aligned}\omega^\top(x_0;y_0)&=\left\{(M,0)^\top\lambda_1+(N,Q)^\top\lambda_2+(A,B)^\top\lambda_3+(C,D)^\top\lambda_4\right\}^\top(x_0;y_0)\\ &=\lambda_1^\top(M,0)(x_0;y_0)+\lambda_2^\top(N,Q)(x_0;y_0)\\ &\quad+\lambda_3^\top(A,B)(x_0;y_0)+\lambda_4^\top(C,D)(x_0;y_0)\\ &=-\lambda_1^\top p-\lambda_2^\top q-\lambda_3^\top b-\lambda_4^\top d.\end{aligned} \tag{3.7}$$

Thus, for any $(x;y)\in R^{2n}$, from the first equation in (3.6), we have

$$\begin{aligned}
\left[\omega^\top((x;y)-(x_0;y_0))\right]_- &= \left\{\lambda_1^\top((M,0)(x;y)+p)+\lambda_2^\top((N,Q)(x;y)+q)\right.\\
&\quad \left.+\lambda_3^\top((A,B)(x;y)+b)+\lambda_4^\top((C,D)(x;y)+d)\right\}_-\\
&\le \left\{\lambda_1^\top((M,0)(x;y)+p)\right\}_- + \left\{\lambda_2^\top((N,Q)(x;y)+q)\right\}_-\\
&\quad + \left\{\lambda_3^\top((A,B)(x;y)+b)\right\}_- + \left\{\lambda_4^\top((C,D)(x;y)+d)\right\}_-\\
&\le \lambda_1^\top\{(M,0)(x;y)+p\}_- + \lambda_2^\top\{(N,Q)(x;y)+q\}_-\\
&\quad + \lambda_3^\top\{(A,B)(x;y)+b\}_-\\
&\quad + \{\lambda_4\}_-^\top\{(C,D)(x;y)+d\}_+ + \{\lambda_4\}_+^\top\{(C,D)(x;y)+d\}_-\\
&\le \|\lambda_1\|\|\{(M,0)(x;y)+p\}_-\| + \|\lambda_2\|\|\{(N,Q)(x;y)+q\}_-\|\\
&\quad + \|\lambda_3\|\|\{(A,B)(x;y)+b\}_-\| + \nu\|(C,D)(x;y)+d\|,
\end{aligned} \tag{3.8}$$

Where $\nu \ge 0$ is a constant. Let $\tau = \max\{\|\lambda_1\|, \|\lambda_2\|, \|\lambda_3\|, \nu\}$, then the desired result follows. □

Now, we are at the position to state our results.

Theorem 3.3. *Suppose that Assumption 2.1 holds. Then, there exists a constant $\eta > 0$ such that for any $(x;y) \in R^{2n}$, there exists $(x^*;y^*) \in X^*$ such that*

$$\|(x;y)-(x^*;y^*)\| \le \eta\left\{s(x,y)+s(x,y)^{1/2}\right\}, \tag{3.9}$$

where

$$\begin{aligned}
s(x,y) = {} & \|(Mx+p)_-\| + \|(Nx+Qy+q)_-\|\\
& + \|(Ax+By+b)_-\| + \|Cx+Dy+d\| + \left[(Mx+p)^\top(Nx+Qy+q)\right]_+.
\end{aligned} \tag{3.10}$$

Proof. Using Corollary 2.3 and Lemma 3.1, there exists a constant $\mu_1 > 0$, for any $(x;y) \in R^{2n}$, and there exists $(x^*;y^*) \in X^*$ such that

$$\begin{aligned}
\|(x;y)-(x^*;y^*)\| \le \mu_1\Big\{ & \|(Mx+p)_-\| + \|(Nx+Qy+q)_-\|\\
& + \|(Ax+By+b)_-\| + \|Cx+Dy+d\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\left[\left(\left((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\right)(x_0;y_0)\right.\right.\right.\\
&\qquad\left.\left.\left.+\ (M,0)^\top q+(N,Q)^\top p\right)^\top((x;y)-(x_0;y_0))\right]_+\right\|\\
&+\left\|\left((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\right)((x;y)-(x_0;y_0))\right\|\Big\},
\end{aligned}
\tag{3.11}
$$

Where $(x_0;y_0)$ is a solution of EMLCP. Now, we consider the right-hand-side of expression (3.11).

Firstly, by Assumption 2.1, we obtain that

$$
H(x,y)=(Mx+p)^\top(Nx+Qy+q) \tag{3.12}
$$

is a convex function. For any $(x;y)\in R^{2n}$, we have

$$
\begin{aligned}
H(x,y)-H(x_0;y_0)\geq{}&\left[\left((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\right)(x_0;y_0)\right.\\
&\left.+\ (M,0)^\top q+(N,Q)^\top p\right]^\top((x;y)-(x_0;y_0)).
\end{aligned}
\tag{3.13}
$$

Combining this with $H(x_0;y_0)=0$, we can deduce that

$$
\begin{aligned}
&\left\{\left[\left((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\right)(x_0;y_0)+(M,0)^\top q+(N,Q)^\top p\right]^\top((x;y)-(x_0;y_0))\right\}_+\\
&\quad\leq\left[(Mx+p)^\top(Nx+Qy+q)\right]_+.
\end{aligned}
\tag{3.14}
$$

Secondly, we consider the last item in (3.11). By Assumption 2.1, there exists a constant $\mu_2>0$ such that for any $(x;y)\in R^{2n}$,

$$
\begin{aligned}
&\left\|\left((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\right)((x;y)-(x_0;y_0))\right\|^2\\
&\quad\leq\mu_2((x;y)-(x_0;y_0))^\top\left((M,0)^\top(N,Q)+(N,Q)^\top(M,0)\right)((x;y)-(x_0;y_0))\\
&\quad=2\mu_2\Big\{(Mx+p)^\top(Nx+Qy+q)-(Mx_0+p)^\top(Nx_0+Qy_0+q)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \Big[\Big((M,0)^{\top}(N,Q) + (N,Q)^{\top}(M,0)\Big)(x_0;y_0) + (M,0)^{\top}q + (N,Q)^{\top}p\Big]^{\top} \\
&\quad \times \big((x;y) - (x_0;y_0)\big)\Big\} \\
&\le \mu_2\Big[(Mx+p)^{\top}(Nx+Qy+q)\Big]_+ + 2\mu_2\Big\{\Big[\Big((M,0)^{\top}(N,Q) + (N,Q)^{\top}(M,0)\Big)(x_0;y_0) \\
&\qquad + (M,0)^{\top}q + (N,Q)^{\top}p\Big]^{\top}\big((x;y) - (x_0;y_0)\big)\Big\}_- \\
&\le 2\mu_2\Big[(Mx+p)^{\top}(Nx+Qy+q)\Big]_+ + 2\mu_2\tau\big(\|(Mx+p)_-\| + \|(Nx+Qy+q)_-\| \\
&\qquad + \|(Ax+By+b)_-\| + \|Cx+Dy+d\|\big),
\end{aligned}
\tag{3.15}
$$

where the first equality is based on the Taylor expansion of function $H(x,y)$ on $(x_0;y_0)$ point, the second inequality follows from the fact that $(x_0;y_0)$ is a solution of EMLCP and the fact that $a+b \le a_+ + b_+$ for any $a,b \in R$, and the last inequality is based on Lemma 3.2. By (3.11)–(3.15), we have that (3.9) holds. □

The error bound obtained in Theorem 3.3 coincides with that of Theorem 2.4 in [11] for the linear complementarity problem, and it is also an extension of Theorem 2.7 in [7] and Corollary 2 in [14].

Theorem 3.4. *Suppose that the assumption of Theorem 2.5 holds. Then, there exists a constant $\eta_1 > 0$, such that for any $(x;y) \in R^{2n}$, there exists a solution $(x^*;y^*) \in X^*$ such that*

$$
\|(x;y) - (x^*;y^*)\| \le \eta_1 s(x,y), \tag{3.16}
$$

where $s(x,y)$ is defined in Theorem 3.3.

Proof. From Theorem 2.5, using the proof technique is similar to that of Theorem 3.3. For any $(x;y) \in R^{2n}$, there exist $(x^*;y^*) \in X^*$ and a constant $\mu_4 > 0$ such that

$$
\begin{aligned}
\|(x;y) - (x^*;y^*)\| \le \mu_4\Big\{ & \|(Mx+p)_-\| + \|(Nx+Qy+q)_-\| \\
& + \|(Ax+By+b)_-\| + \|Cx+Dy+d\| \\
& + \Big\|\Big[\Big(\Big((M,0)^{\top}(N,Q) + (N,Q)^{\top}(M,0)\Big)(x_0;y_0) \\
& \quad + (M,0)^{\top}q + (N,Q)^{\top}p\Big)^{\top}\big((x;y) - (x_0;y_0)\big)\Big]_+\Big\|\Big\}.
\end{aligned}
\tag{3.17}
$$

Combining this with (3.14), we can deduce that (3.16) holds. □

4. Conclusion

In this paper, we presented the solution Characterization, and also established global error bounds on the extended mixed linear complementarity problems which are the extensions of those for the classical linear complementarity problems. Surely, we may use the error bound estimation to establish quick convergence rate of the noninterior path following method for solving the EMLCP just as was done in [14], and this is a topic for future research.

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References

[1] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*, Academic Press, New York, NY, USA, 1992.
[2] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequality and Complementarity Problems*, Springer, New York, NY, USA, 2003.
[3] M. C. Ferris and J. S. Pang, "Engineering and economic applications of complementarity problems," *Society for Industrial and Applied Mathematics*, vol. 39, no. 4, pp. 669–713, 1997.
[4] L. Walras, *Elements of Pure Economics*, George Allen and Unwin, London, UK, 1954.
[5] S. Karamardian, "Generalized complementarity problem," *Journal of Optimization Theory and Applications*, vol. 8, pp. 161–168, 1971.
[6] G. J. Habetler and A. L. Price, "Existence theory for generalized nonlinear complementarity problems," *Journal of Optimization Theory and Applications*, vol. 7, pp. 223–239, 1971.
[7] O. L. Mangasarian and T. H. Shiau, "Error bounds for monotone linear complementarity problems," *Mathematical Programming*, vol. 36, no. 1, pp. 81–89, 1986.
[8] O. L. Mangasarian, "Error bounds for nondegenerate monotone linear complementarity problems," *Mathematical Programming*, vol. 48, no. 3, pp. 437–445, 1990.
[9] J.-S. Pang, "Error bounds in mathematical programming," *Mathematical Programming*, vol. 79, no. 1–3, pp. 299–332, 1997.
[10] Z.-Q. Luo, O. L. Mangasarian, J. Ren, and M. V. Solodov, "New error bounds for the linear complementarity problem," *Mathematics of Operations Research*, vol. 19, no. 4, pp. 880–892, 1994.
[11] O. L. Mangasarian and J. Ren, "New improved error bounds for the linear complementarity problem," *Mathematical Programming*, vol. 66, no. 2, pp. 241–255, 1994.
[12] R. Mathias and J.-S. Pang, "Error bounds for the linear complementarity problem with a P-matrix," *Linear Algebra and its Applications*, vol. 132, pp. 123–136, 1990.
[13] A. J. Hoffman, "On approximate solutions of systems of linear inequalities," *Journal of Research of the National Bureau of Standards*, vol. 49, pp. 263–265, 1952.
[14] J. Zhang and N. Xiu, "Global s-type error bound for the extended linear complementarity problem and applications," *Mathematical Programming B*, vol. 88, no. 2, pp. 391–410, 2000.

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Research Article

Robust Local Regularity and Controllability of Uncertain TS Fuzzy Descriptor Systems

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The robust local regularity and controllability problem for the Takagi-Sugeno (TS) fuzzy descriptor systems is studied in this paper. Under the assumptions that the nominal TS fuzzy descriptor systems are locally regular and controllable, a sufficient criterion is proposed to preserve the assumed properties when the structured parameter uncertainties are added into the nominal TS fuzzy descriptor systems. The proposed sufficient criterion can provide the explicit relationship of the bounds on parameter uncertainties for preserving the assumed properties. An example is given to illustrate the application of the proposed sufficient condition.

1. Introduction

Recently, it has been shown that the fuzzy-model-based representation proposed by Takagi and Sugeno [1], known as the TS fuzzy model, is a successful approach for dealing with the nonlinear control systems, and there are many successful applications of the TS-fuzzy-model-based approach to the nonlinear control systems (e.g., [2–19] and references therein). Descriptor systems represent a much wider class of systems than the standard systems [20]. In recent years, some researchers (e.g., [4–6, 8, 21–28] and references therein) have studied the design issue of the fuzzy parallel-distributed-compensation (PDC) controllers for each fuzzy rule of the TS fuzzy descriptor systems. Both regularity and controllability are actually two very important properties of descriptor systems with control inputs [29]. So, before the design of the fuzzy PDC controllers in the corresponding rule of the TS fuzzy descriptor systems, it is necessary to consider both properties of local regularity and controllability for each fuzzy rule [23]. However, both regularity and controllability of the TS fuzzy systems are not considered by those mentioned-above researchers before the fuzzy PDC controllers are designed. Therefore, it is meaningful to further study the criterion that the local regularity and controllability for each fuzzy rule of the TS fuzzy descriptor systems hold [30].

On the other hand, in fact, in many cases it is very difficult, if not impossible, to obtain the accurate values of some system parameters. This is due to the inaccurate measurement, inaccessibility to the system parameters, or variation of the parameters. These parametric uncertainties may destroy the local regularity and controllability properties of the TS fuzzy descriptor systems. But, to the authors' best knowledge, there is no literature to study the issue of robust local regularity and controllability for the uncertain TS fuzzy descriptor systems.

The purpose of this paper is to present an approach for investigating the robust local regularity and controllability problem of the TS fuzzy descriptor systems with structured parameter uncertainties. Under the assumptions that the nominal TS fuzzy descriptor systems are locally regular and controllable, a sufficient criterion is proposed to preserve the assumed properties when the structured parameter uncertainties are added into the nominal TS fuzzy descriptor systems. The proposed sufficient criterion can provide the explicit relationship of the bounds on structured parameter uncertainties for preserving the assumed properties. A numerical example is given in this paper to illustrate the application of the proposed sufficient criterion.

2. Robust Local Regularity and Controllability Analysis

Based on the approach of using the sector nonlinearity in the fuzzy model construction, both the fuzzy set of premise part and the linear dynamic model with parametric uncertainties of consequent part in the exact TS fuzzy control model with parametric uncertainties can be derived from the given nonlinear control model with parametric uncertainties [5]. The TS continuous-time fuzzy descriptor system with parametric uncertainties for the nonlinear control system with structured parametric uncertainties can be obtained as the following form:

$$\begin{aligned} \widetilde{R}^i:\ & \text{IF } z_1 \text{ is } M_{i1} \text{ and } \dots \text{ and } z_g \text{ is } M_{ig}, \\ & \text{then } E_i \dot{x}(t) = (A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t), \end{aligned} \tag{2.1}$$

or the uncertain discrete-time TS fuzzy descriptor system can be described by

$$\begin{aligned} \widetilde{R}^i:\ & \text{IF } z_1 \text{ is } M_{i1} \text{ and } \dots \text{ and } z_g \text{ is } M_{ig}, \\ & \text{then } E_i x(k+1) = (A_i + \Delta A_i)x(k) + (B_i + \Delta B_i)u(k), \end{aligned} \tag{2.2}$$

with the initial state vector $x(0)$, where $\widetilde{R}^i$ $(i = 1, 2, \dots, N)$ denotes the ith implication, N is the number of fuzzy rules, $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ and $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]^T$ denote the n-dimensional state vectors, $u(t) = [u_1(t), u_2(t), \dots, u_p(t)]^T$ and $u(k) = [u_1(k), u_2(k), \dots, u_p(k)]^T$ denote the p-dimensional input vectors, z_i $(i = 1, 2, \dots, g)$ are the premise variables, E_i, A_i, and B_i $(i = 1, 2, \dots, N)$ are, respectively, the $n \times n$, $n \times n$ and

$n \times p$ consequent constant matrices, ΔA_i and ΔB_i $(i = 1, 2, \ldots, N)$ are, respectively, the parametric uncertain matrices existing in the system matrices A_i and the input matrices B_i of the consequent part of the ith rule due to the inaccurate measurement, inaccessibility to the system parameters, or variation of the parameters, and M_{ij} ($i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, g$) are the fuzzy sets. Here the matrices E_i $(i = 1, 2, \ldots, N)$ may be singular matrices with $\operatorname{rank}(E_i) \leq n$ $(i = 1, 2, \ldots, N)$. In many applications, the matrices E_i $(i = 1, 2, \ldots, N)$ are the structure information matrices; rather than parameter matrices, that is, the elements of E_i $(i = 1, 2, \ldots, N)$ contain only structure information regarding the problem considered.

In many interesting problems (e.g., plant uncertainties, constant output feedback with uncertainty in the gain matrix), we have only a small number of uncertain parameters, but these uncertain parameters may enter into many entries of the system and input matrices [31, 32]. Therefore, in this paper, we suppose that the parametric uncertain matrices ΔA_i and ΔB_i take the forms

$$\Delta A_i = \sum_{k=1}^{m} \varepsilon_{ik} A_{ik}, \qquad \Delta B_i = \sum_{k=1}^{m} \varepsilon_{ik} B_{ik}, \tag{2.3}$$

where ε_{ik} ($i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, m$) are the elemental parametric uncertainties, and A_{ik} and B_{ik} ($i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, m$) are, respectively, the given $n \times n$ and $n \times p$ constant matrices which are prescribed a priori to denote the linearly dependent information on the elemental parametric uncertainties ε_{ik}.

In this paper, for the uncertain TS fuzzy descriptor system in (2.1) (or (2.2)), each fuzzy-rule-nominal model $E_i\dot{x}(t) = A_i x(t) + B_i u(t)$ or $E_i x(k+1) = A_i x(k) + B_i u(k)$, which is denoted by $\{E_i, A_i, B_i\}$, is assumed to be regular and controllable. Due to inevitable uncertainties, each fuzzy-rule-nominal model $\{E_i, A_i, B_i\}$ is perturbed into the fuzzy-rule-uncertain model $\{E_i, A_i + \Delta A_i, B_i + \Delta B_i\}$. Our problem is to determine the conditions such that each fuzzy-uncertain model $\{E_i, A_i + \Delta A_i, B_i + \Delta B_i\}$ for the uncertain TS fuzzy descriptor system (2.1) (or (2.2)) is robustly locally regular and controllable. Before we investigate the robust properties of regularity and controllability for the uncertain TS fuzzy descriptor system (2.1) (or (2.2)), the following definitions and lemmas need to be introduced first.

Definition 2.1 (see [33]). The measure of a matrix $\overline{W} \in C^{n\times n}$ is defined as

$$\mu\left(\overline{W}\right) \equiv \lim_{\theta \to 0} \frac{\left(\left\|I + \theta \overline{W}\right\| - 1\right)}{\theta}, \tag{2.4}$$

where $\|\cdot\|$ is the induced matrix norm on $C^{n\times n}$.

Definition 2.2 (see [34]). The system $\{E_i, A_i, B_i\}$ is called controllable, if for any $t_1 > 0$ (or $k_1 > 0$), $x(0) \in R^n$, and $w \in R^n$, there exists a control input $u(t)$ (or $u(k)$) such that $x(t_1) = w$ (or $x(k_1) = w$).

Definition 2.3. The uncertain TS fuzzy descriptor system in (2.1) (or (2.2)) is locally regular, if each fuzzy-rule-uncertain model $\{E_i, A_i + \Delta A_i, B_i + \Delta B_i\}$ $(i = 1,\ 2, \dots, N)$ is regular.

Definition 2.4. The uncertain TS fuzzy descriptor system in (2.1) (or (2.2)) is locally controllable, if each fuzzy-rule-uncertain model $\{E_i, A_i + \Delta A_i, B_i + \Delta B_i\}$ $(i = 1, 2, \dots, N)$ is controllable.

Lemma 2.5 (see [34]). *The system $\{E_i, A_i, B_i\}$ is regular if and only if* $\operatorname{rank}[E_{ni} \quad B_{di}] = n^2$, *where $E_{ni} \in R^{n^2 \times n}$ and $E_{di} \in R^{n^2 \times n^2}$ are given by*

$$E_{ni} = \begin{bmatrix} E_i \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \qquad E_{di} = \begin{bmatrix} A_i & & & & \\ E_i & A_i & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & E_i & A_i \end{bmatrix}. \tag{2.5}$$

Lemma 2.6 (see [29, 35]). *Suppose that the system $\{E_i, A_i, B_i\}$ is regular. The system $\{E_i, A_i, B_i\}$ is controllable if and only if* $\operatorname{rank}[E_{di} \quad E_{bi}] = n^2$ *and* $\operatorname{rank}[E_i \quad B_i] = n$, *where $E_{di} \in R^{n^2 \times n^2}$ is given in* (2.5) *and* $E_{bi} = \operatorname{diag}\{B_i, B_i, \dots, B_i\} \in R^{n^2 \times np}$.

Lemma 2.7 (see [33]). *The matrix measures of the matrices $\overline{W}$ and $\overline{V}$, namely, $\mu(\overline{W})$ and $\mu(\overline{V})$, are well defined for any norm and have the following properties:*

(i) $\mu(\pm I) = \pm 1$, *for the identity matrix I;*

(ii) $-\|\overline{W}\| \le -\mu(-\overline{W}) \le \operatorname{Re}(\lambda(\overline{W})) \le \mu(\overline{W}) \le \|\overline{W}\|$, *for any* norm $\|\cdot\|$ *and any matrix $\overline{W} \in C^{n\times n}$;*

(iii) $\mu(\overline{W} + \overline{V}) \le \mu(\overline{W}) + \mu(\overline{V})$, *for any two matrices $\overline{W}, \overline{V} \in C^{n\times n}$;*

(iv) $\mu(\gamma\overline{W}) = \gamma\mu(\overline{W})$, *for any matrix $\overline{W} \in C^{n\times n}$ and any non-negative real number γ,*

where $\lambda(\overline{W})$ denotes any eigenvalue of $\overline{W}$, and $\operatorname{Re}(\lambda(\overline{W}))$ *denotes the real part of $\lambda(\overline{W})$.*

Lemma 2.8. *For any $\gamma < 0$ and any matrix $\overline{W} \in C^{n\times n}$, $\mu(\gamma\overline{W}) = -\gamma\mu(-\overline{W})$.*

Proof. This lemma can be immediately obtained from the property (iv) in Lemma 2.7. □

Lemma 2.9. *Let $\overline{N} \in C^{n\times n}$. If $\mu(-\overline{N}) < 1$, then* $\det(I + \overline{N}) \neq 0$.

Proof. From the property (ii) in Lemma 2.7 and since $\mu(-\overline{N}) < 1$, we can get that $\operatorname{Re}(\lambda(\overline{N})) \ge -\mu(-\overline{N}) > -1$. This implies that $\lambda(\overline{N}) \neq -1$. So, we have the stated result. □

Now, let the singular value decompositions of $R_i = [E_{ni} \quad E_{di}]$, $Q_i = [E_{di} \quad E_{bi}]$, and $P_i = [E_i \quad B_i]$ be, respectively,

$$R_i = U_i [S_i \ \ 0_{n^2 \times n}] V_i^H, \tag{2.6}$$

$$Q_i = U_{ri} [S_{ri} \ \ 0_{n^2 \times np}] V_{ri}^H, \tag{2.7}$$

$$P_i = U_{ci} [S_{ci} \ \ 0_{n^2 \times q}] V_{ci}^H, \tag{2.8}$$

where $U_i \in R^{n^2 \times n^2}$ and $V_i \in R^{(n^2+n)\times(n^2+n)}$ are the unitary matrices, $S_i = \operatorname{diag}\{\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{in^2}\}$, and $\sigma_{i1} \geq \sigma_{i2} \geq \cdots \geq \sigma_{in^2} > 0$ are the singular values of R_i; $U_{ri} \in R^{n^2 \times n^2}$ and $V_{ri} \in R^{(n^2+np)\times(n^2+np)}$ are the unitary matrices, $S_{ri} = \operatorname{diag}\{\sigma_{ri1}, \sigma_{ri2}, \ldots, \sigma_{rin^2}\}$ and $\sigma_{ri1} \geq \sigma_{ri2} \geq \cdots \geq \sigma_{rin^2} > 0$ are the singular values of Q_i; $U_{ci} \in R^{n \times n}$ and $V_{ci} \in R^{(n+p)\times(n+p)}$ are the unitary matrices, $S_{ci} = \operatorname{diag}\{\sigma_{ci1}, \sigma_{ci2}, \ldots, \sigma_{cin}\}$ and $\sigma_{ci1} \geq \sigma_{ci2} \geq \cdots \geq \sigma_{cin} > 0$ are the singular values of P_i; V_i^H, V_{ri}^H, and V_{ci}^H denote, respectively, the complex-conjugate transposes of the matrices V_i, V_{ri}, and V_{ci}.

In what follows, with the preceding definitions and lemmas, we present a sufficient criterion for ensuring that the uncertain TS fuzzy descriptor system in (2.1) or (2.2) remains locally regular and controllable.

Theorem 2.10. *Suppose that the each fuzzy-rule-nominal descriptor system $\{E_i, A_i, B_i\}$ is regular and controllable. The uncertain TS fuzzy descriptor system in* (2.1) *(or* (2.2)*) is still locally regular and controllable (i.e., each fuzzy-rule-uncertain descriptor system $\{E_i, A_i + \Delta A_i, B_i + \Delta B_i\}$ remains regular and controllable), if the following conditions simultaneously hold*

$$\sum_{k=1}^{m} \varepsilon_{ik} \varphi_{ik} < 1, \tag{2.9a}$$

$$\sum_{k=1}^{m} \varepsilon_{ik} \theta_{ik} < 1, \tag{2.9b}$$

$$\sum_{k=1}^{m} \varepsilon_{ik} \phi_{ik} < 1, \tag{2.9c}$$

where $i = 1, 2, \ldots, N$, and $k = 1, 2, \ldots, m$:

$$\varphi_{ik} = \begin{cases} \mu\left(-S_i^{-1} U_i^H R_{ik} V_i [I_{n^2}, 0_{n^2 \times n}]^T\right), & \text{for } \varepsilon_{ik} \geq 0, \\ -\mu\left(S_i^{-1} U_i^H R_{ik} V_i [I_{n^2}, 0_{n^2 \times n}]^T\right), & \text{for } \varepsilon_{ik} < 0, \end{cases}$$

$$R_{ik} = \left[0_{n^2 \times n} \quad \widetilde{R}_{ik}\right] \in R^{n^2 \times (n^2+n)},$$

$$\widetilde{R}_{ik} = \operatorname{diag}\{A_{ik}, \ldots, A_{ik}\} \in R^{n^2 \times n^2},$$

$$\theta_{ik} = \begin{cases} \mu\left(-S_{ri}^{-1} U_{ri}^H Q_{ik} V_{ri} [I_{n^2}, 0_{n^2 \times np}]^T\right), & \text{for } \varepsilon_{ik} \geq 0, \\ -\mu\left(S_{ri}^{-1} U_{ri}^H Q_{ik} V_{ri} [I_{n^2}, 0_{n^2 \times np}]^T\right), & \text{for } \varepsilon_{ik} < 0, \end{cases}$$

$$Q_{ik} = \begin{bmatrix} A_{ik} & & & & B_{ik} & & & \\ & A_{ik} & & & & B_{ik} & & \\ & & \ddots & & & & \ddots & \\ & & & A_{ik} & & & & B_{ik} \end{bmatrix} \in R^{n^2\times(n^2+np)},$$

$$\phi_{ik} = \begin{cases} \mu\left(-S_{ci}^{-1}U_{ci}^{H}P_{ik}V_{ci}\left[I_n, 0_{n\times p}\right]^T\right), & \text{for } \varepsilon_{ik} \geq 0, \\ -\mu\left(S_{ci}^{-1}U_{ci}^{H}P_{ik}V_{ci}\left[I_n, 0_{n\times p}\right]^T\right), & \text{for } \varepsilon_{ik} < 0, \end{cases}$$

$$P_{ik} = \left[0_{n\times n} \quad B_{ik}\right] \in R^{n\times(n+p)}, \tag{2.10}$$

the matrices S_i, U_i,V_i, S_{ri}, U_{ri},V_{ri}, S_{ci}, U_{ci}, *and* V_{ci} $(i = 1,2,\ldots,N)$ *are, respectively, defined in* (2.6)–(2.8), *and* I_{n^2} *denotes the* $n^2 \times n^2$ *identity matrix.*

Proof. Firstly, we show the regularity. Since each fuzzy-rule-nominal descriptor system $\{E_i, A_i, B_i\}$ $(i = 1,2,\ldots,N)$ is regular, then, from Lemma 2.5, we can get that the matrix $R_i = [E_{ni} \quad E_{di}] \in R^{n^2\times(n^2+n)}$ has full row rank (i.e., $\text{rank}(R_i) = n^2$). With the uncertain matrices $A_i + \Delta A_i$ and $B_i + \Delta B_i$, each fuzzy-rule-uncertain descriptor system $\{E_i, A_i + \Delta A_i, B_i + \Delta B_i\}$ is regular if and only if

$$\widetilde{R}_i = R_i + \sum_{k=1}^{m} \varepsilon_{ik} R_{ik} \tag{2.11}$$

has full row rank, where $R_{ik} = [0_{n^2\times n} \quad \widetilde{R}_{ik}] \in R^{n^2\times(n^2+n)}$ and $\widetilde{R}_{ik} = \text{diag}\{A_{ik}, \ldots, A_{ik}\} \in R^{n^2\times n^2}$.

$$\text{It is known that } \text{rank}\left(\widetilde{R}_i\right) = \text{rank}\left(S_i^{-1}U_i^{H}\widetilde{R}_i V_i\right). \tag{2.12}$$

Thus, instead of $\text{rank}(\widetilde{R}_i)$, we can discuss the rank of

$$[I_{n^2}, 0_{n^2\times n}] + \sum_{k=1}^{m} \varepsilon_{ik} \widehat{R}_{ik}, \tag{2.13}$$

where $\widehat{R}_{ik} = S_i^{-1}U_i^{H}R_{ik}V_i$, for $i = 1,2,\ldots,N$ and $k = 1,2,\ldots,m$. Since a matrix has at least rank n^2 if it has at least one nonsingular $n^2 \times n^2$ submatrix, a sufficient condition for the matrix in (2.13) to have rank n^2 is the nonsingularity of

$$L_i = I_{n^2} + \sum_{k=1}^{m} \varepsilon_{ik} \overline{R}_{ik}, \tag{2.14}$$

where $\overline{R}_{ik} = S_i^{-1}U_i^{H}R_{ik}V_i[I_{n^2}, 0_{n^2\times n}]^T$ (for $i = 1,2,\ldots,N$ and $k = 1,2,\ldots,m$).

Using the properties in Lemmas 2.7 and 2.8 and from (2.9a), we get

$$\begin{aligned}
\mu\left(-\sum_{k=1}^{m}\varepsilon_{ik}\overline{R}_{ik}\right) &= \mu\left(-\sum_{k=1}^{m}\varepsilon_{ik}S_i^{-1}U_i^H R_{ik}V_i[I_{n^2},0_{n^2\times n}]^T\right) \\
&\leq \sum_{k=1}^{m}\mu\left(-\varepsilon_{ik}S_i^{-1}U_i^H R_{ik}V_i[I_{n^2},0_{n^2\times n}]^T\right) \\
&= \sum_{k=1}^{m}\varepsilon_{ik}\varphi_{ik} < 1.
\end{aligned} \tag{2.15}$$

From Lemma 2.9, we have that

$$\det(L_i) = \det\left(I_{n^2}+\sum_{k=1}^{m}\varepsilon_{ik}\overline{R}_{ik}\right) \neq 0. \tag{2.16}$$

Hence, the matrix L_i in (2.14) is nonsingular. That is, the matrix $\widetilde{R}_i$ in (2.11) has full row rank n^2. Thus, from the Lemma 2.5, the regularity of each fuzzy-rule-uncertain descriptor system $\{E_i, A_i+\Delta A_i, B_i+\Delta B_i\}$ is ensured.

Next, we show the controllability. Since each fuzzy-rule-nominal descriptor system $\{E_i, A_i, B_i\}$ $(i = 1,2,\ldots,N)$ is controllable, then from Lemma 2.6, we have that the matrix $Q_i = [E_{di} \quad E_{bi}]$ has full row rank (i.e., $\operatorname{rank}(Q_i) = n^2$) and $P_i = [E_i \quad B_i]$ has full row rank (i.e., $\operatorname{rank}(P_i) = n$). With the uncertain matrices $A_i+\Delta A_i$ and $B_i+\Delta B_i$, each fuzzy-rule-uncertain descriptor system $\{E_i, A_i+\Delta A_i, B_i+\Delta B_i\}$ is controllable if and only if

$$\widetilde{Q}_i = Q_i + \sum_{k=1}^{m}\varepsilon_{ik}Q_{ik}, \tag{2.17}$$

$$\widetilde{P}_i = P_i + \sum_{k=1}^{m}\varepsilon_{ik}P_{ik} \tag{2.18}$$

have full row rank, where

$$Q_{ik} = \begin{bmatrix} A_{ik} & & & & B_{ik} & & & \\ & A_{ik} & & & & B_{ik} & & \\ & & \ddots & & & & \ddots & \\ & & & A_{ik} & & & & B_{ik} \end{bmatrix} \in R^{n^2\times(n^2+np)}, \tag{2.19}$$

and $P_{ik} = [0_{n\times n} \quad B_{ik}] \in R^{n\times(n+p)}$.

It is known that

$$\operatorname{rank}\left(\widetilde{Q}_i\right) = \operatorname{rank}\left(S_{ri}^{-1} U_{ri}^{H} \widetilde{Q}_i V_{ri}\right). \tag{2.20}$$

Thus, instead of $\operatorname{rank}(\widetilde{Q}_i)$, we can discuss the rank of

$$\left[I_{n^2}, 0_{n^2 \times np}\right] + \sum_{k=1}^{m} \varepsilon_{ik} \widehat{Q}_{ik}, \tag{2.21}$$

where $\widehat{Q}_{ik} = S_{ri}^{-1} U_{ri}^{H} Q_{ik} V_{ri}$, for $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, m$. Since a matrix has at least rank n^2 if it has at least one nonsingular $n^2 \times n^2$ submatrix, a sufficient condition for the matrix in (2.21) to have rank n^2 is the nonsingularity of

$$G_i = I_{n^2} + \sum_{k=1}^{m} \varepsilon_{ik} \overline{Q}_{ik}, \tag{2.22}$$

where $\overline{Q}_{ik} = S_{ri}^{-1} U_{ri}^{H} Q_{ik} V_{ri} [I_{n^2}, 0_{n^2 \times np}]^T$ (for $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, m$).

Applying the properties in Lemmas 2.7 and 2.8 and from (2.9b), we get

$$\begin{aligned} \mu\left(-\sum_{k=1}^{m} \varepsilon_{ik} \overline{Q}_{ik}\right) &= \mu\left(-\sum_{k=1}^{m} \varepsilon_{ik} S_{ri}^{-1} U_{ri}^{H} Q_{ik} V_{ri} \left[I_{n^2}, 0_{n^2 \times np}\right]^T\right) \\ &\le \sum_{k=1}^{m} \mu\left(-\varepsilon_{ik} S_{ri}^{-1} U_{ri}^{H} Q_{ik} V_{ri} \left[I_{n^2}, 0_{n^2 \times np}\right]^T\right) \\ &= \sum_{k=1}^{m} \varepsilon_{ik} \theta_{ik} < 1. \end{aligned} \tag{2.23}$$

From Lemma 2.9, we have that

$$\det(G_i) = \det\left(I_{n^2} + \sum_{k=1}^{m} \varepsilon_{ik} \overline{Q}_{ik}\right) \neq 0. \tag{2.24}$$

Hence, the matrix G_i in (2.22) is nonsingular. That is, the matrix $\widetilde{Q}_i$ in (2.17) has full row rank n^2.

And then, it is also known that

$$\operatorname{rank}\left(\widetilde{P}_i\right) = \operatorname{rank}\left(S_{ci}^{-1} U_{ci}^{H} \widetilde{P}_i V_{ci}\right). \tag{2.25}$$

Thus, instead of rank$(\widetilde{P}_i)$, we can discuss the rank of

$$[I_n, 0_{n\times p}] + \sum_{k=1}^{m} \varepsilon_{ik} \widehat{P}_{ik}, \tag{2.26}$$

where $\widehat{P}_{ik} = S_{ci}^{-1} U_{ci}^{H} P_{ik} V_{ci}$, for $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, m$. Since a matrix has at least rank n if it has at least one nonsingular $n \times n$ submatrix, a sufficient condition for the matrix in (2.26) to have rank n is the nonsingularity of

$$H_i = I_n + \sum_{k=1}^{m} \varepsilon_{ik} \overline{P}_{ik}, \tag{2.27}$$

where $\overline{P}_{ik} = S_{ci}^{-1} U_{ci}^{H} P_{ik} V_{ci} [I_n, 0_{n\times p}]^T$ (for $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, m$).

Adopting the properties in Lemmas 2.7 and 2.8 and from (2.9c), we obtain

$$\begin{aligned}
\mu\left(-\sum_{k=1}^{m} \varepsilon_{ik} \overline{P}_{ik} \right) &= \mu\left(-\sum_{k=1}^{m} \varepsilon_{ik} S_{ci}^{-1} U_{ci}^{H} P_{ik} V_{ci} \left[I_n,\, 0_{n\times p}\right]^T \right) \\
&\le \sum_{k=1}^{m} \mu\left(-\varepsilon_{ik} S_{ci}^{-1} U_{ci}^{H} P_{ik} V_{ci} \left[I_n,\, 0_{n\times p}\right]^T \right) \\
&= \sum_{k=1}^{m} \varepsilon_{ik} \phi_{ik} < 1.
\end{aligned} \tag{2.28}$$

From Lemma 2.9, we get that

$$\det(H_i) = \det\left(I_n + \sum_{k=1}^{m} \varepsilon_{ik} \overline{P}_{ik} \right) \neq 0. \tag{2.29}$$

Hence, the matrix H_i in (2.27) is nonsingular. That is, the matrix $\widetilde{P}_i$ in (2.18) has full row rank n. Thus, from the Lemma 2.6 and the results mentioned above, the controllability of each fuzzy-rule-uncertain descriptor system $\{E_i, A_i + \Delta A_i, B_i + \Delta B_i\}$ is ensured. Therefore, we can conclude that the uncertain TS fuzzy descriptor system in (2.1) (or (2.2)) is locally regular and controllable, if the inequalities (2.9a), (2.9b), and (2.9c) are simultaneously satisfied. Thus, the proof is completed. □

Remark 2.11. The proposed sufficient conditions in (2.9a)–(2.9c) can give the explicit relationship of the bounds on ε_{ik} ($i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, m$) for preserving both regularity and controllability. In addition, the bounds, that are obtained by using the proposed sufficient conditions, on ε_{ik} are not necessarily symmetric with respect to the origin of the parameter space regarding ε_{ik} ($i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, m$).

Remark 2.12. This paper studies the problem of robust local regularity and controllability analysis. If the proposed conditions in (2.9a)–(2.9c) are satisfied, each rule of the uncertain TS fuzzy descriptor system $\{E_i, A_i + \Delta A_i, B_i + \Delta B_i\}$ is guaranteed to be robustly locally regular and controllable. This implies that, in the fuzzy PDC controller design, if the proposed conditions in (2.9a)–(2.9c) are satisfied, the PDC controller of each fuzzy rule can control every state variable in the corresponding rule of the uncertain TS fuzzy descriptor system $\{E_i, A_i + \Delta A_i, B_i + \Delta B_i\}$. However, here, it should be noticed that although the PDC controller of each control rule can control every state variable in the corresponding rule under the presented conditions being held, the PDC controller gains should be determined using global design criteria that are needed to guarantee the global stability and control performance [5], where many useful global design criteria have been proposed by some researchers (e.g., [4-6, 8, and 21-28] and references therein).

3. Illustrative Example

Consider a two-rule fuzzy descriptor system as that considered by Wang et al. [21]. The TS fuzzy descriptor system with the elemental parametric uncertainties is described by

$$\begin{aligned}\widetilde{R}^1:\ & \text{IF } z_1 \text{ is } M_{11},\\ & \text{then } E_1 x(k+1) = (A_1 + \Delta A_1)x(k) + (B_1 + \Delta B_1)\ u(k);\end{aligned} \tag{3.1a}$$

$$\begin{aligned}\widetilde{R}^2:\ & \text{IF } z_1 \text{ is } M_{21},\\ & \text{then } E_2 x(k+1) = (A_2 + \Delta A_2)x(k) + (B_2 + \Delta B_2)\ u(k),\end{aligned} \tag{3.1b}$$

where

$$\begin{gathered} x(k) = \begin{bmatrix} x_1(k) & x_2(k) \end{bmatrix}^T, \qquad E_1 = E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0.848 & 0 \\ 0 & -0.315 \end{bmatrix}, \\ A_2 = \begin{bmatrix} -0.236 & 0 \\ 0 & 0.113 \end{bmatrix}, \qquad B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \Delta A_i = \sum_{k=1}^{2} \varepsilon_{ik} A_{ik}, \\ \Delta B_i = \sum_{k=1}^{2} \varepsilon_{ik} B_{ik}, \qquad A_{11} = \begin{bmatrix} 0 & 0 \\ -0.1 & 0 \end{bmatrix}, \qquad A_{21} = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \end{bmatrix}, \qquad A_{12} = A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ B_{i1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad B_{i2} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \qquad M_{11} = \frac{1}{1 + \exp(-0.5(z_1 - 0.3))}, \\ M_{21} = \frac{\exp(-0.5(z_1 - 0.3))}{1 + \exp(-0.5(z_1 - 0.3))}, \quad \varepsilon_{i1} \in [-1 \quad 1.1], \quad \varepsilon_{i2} \in [-1.2 \quad 10], \text{ in which } i = 1, 2. \end{gathered} \tag{3.2}$$

Now, applying the sufficient conditions in (2.9a)–(2.9c) with the two-norm-based matrix measure, we can get the following:

(I) for the fuzzy rule 1:

$$\text{(i)}\ \sum_{k-1}^{2} \varepsilon_{ik}\varphi_{ik} \leq 0.17460 < 1, \quad \text{for } \varepsilon_{i1} \in [0 \quad 1.1],\ \varepsilon_{i2} \in [-1.2 \quad 10], \tag{3.3a}$$

$$\text{(ii)}\ \sum_{k-1}^{2} \varepsilon_{ik}\varphi_{ik} \leq 0.15873 < 1, \quad \text{for } \varepsilon_{i1} \in [-1 \quad 0],\ \varepsilon_{i2} \in [-1.2 \quad 10], \tag{3.3b}$$

$$\text{(iii)}\ \sum_{k-1}^{2} \varepsilon_{ik}\theta_{ik} \leq 0.03297 < 1, \quad \text{for } \varepsilon_{i1} \in [0 \quad 1.1],\ \varepsilon_{i2} \in [0 \quad 10], \tag{3.3c}$$

$$\text{(iv)}\ \sum_{k-1}^{2} \varepsilon_{ik}\theta_{ik} \leq 0.13861 < 1, \quad \text{for } \varepsilon_{i1} \in [-1 \quad 0],\ \varepsilon_{i2} \in [0 \quad 10], \tag{3.3d}$$

$$\text{(v)}\ \sum_{k-1}^{2} \varepsilon_{ik}\theta_{ik} \leq 0.25078 < 1, \quad \text{for } \varepsilon_{i1} \in [-1 \quad 0],\ \varepsilon_{i2} \in [-1.2 \quad 0], \tag{3.3e}$$

$$\text{(vi)}\ \sum_{k-1}^{2} \varepsilon_{ik}\theta_{ik} \leq 0.14514 < 1, \quad \text{for } \varepsilon_{i1} \in [0 \quad 1.1],\ \varepsilon_{i2} \in [-1.2 \quad 0], \tag{3.3f}$$

$$\text{(vii)}\ \sum_{k-1}^{2} \varepsilon_{ik}\phi_{ik} = 0 < 1, \quad \text{for } \varepsilon_{i1} \in [-1 \quad 1.1],\ \varepsilon_{i2} \in [0 \quad 10], \tag{3.3g}$$

$$\text{(viii)}\ \sum_{k-1}^{2} \varepsilon_{ik}\phi_{ik} \leq 0.1200 < 1, \quad \text{for } \varepsilon_{i1} \in [-1 \quad 1.1],\ \varepsilon_{i2} \in [-1.2 \quad 0]; \tag{3.3h}$$

$$\text{(i)}\ \sum_{k-1}^{2} \varepsilon_{ik}\varphi_{ik} \leq 0.97345 < 1, \quad \text{for } \varepsilon_{i1} \in [0 \quad 1.1],\ \varepsilon_{i2} \in [-1.2 \quad 10], \tag{3.4a}$$

$$\text{(ii)}\ \sum_{k-1}^{2} \varepsilon_{ik}\varphi_{ik} \leq 0.88496 < 1, \quad \text{for } \varepsilon_{i1} \in [-1 \quad 0],\ \varepsilon_{i2} \in [-1.2 \quad 10], \tag{3.4b}$$

$$\text{(iii)}\ \sum_{k-1}^{2} \varepsilon_{ik}\theta_{ik} \leq 0.56719 < 1, \quad \text{for } \varepsilon_{i1} \in [0 \quad 1.1],\ \varepsilon_{i2} \in [0 \quad 10], \tag{3.4c}$$

$$\text{(iv)}\ \sum_{k-1}^{2} \varepsilon_{ik}\theta_{ik} \leq 0.87768 < 1, \quad \text{for } \varepsilon_{i1} \in [-1 \quad 0],\ \varepsilon_{i2} \in [0 \quad 10], \tag{3.4d}$$

$$\text{(v)}\ \sum_{k-1}^{2} \varepsilon_{ik}\theta_{ik} \leq 0.99740 < 1, \quad \text{for } \varepsilon_{i1} \in [-1 \quad 0],\ \varepsilon_{i2} \in [-1.2 \quad 0], \tag{3.4e}$$

$$\text{(vi)}\ \sum_{k-1}^{2} \varepsilon_{ik}\theta_{ik} \leq 0.168691 < 1, \quad \text{for } \varepsilon_{i1} \in [0 \quad 1.1],\ \varepsilon_{i2} \in [-1.2 \quad 0], \tag{3.4f}$$

$$\text{(vii)}\ \sum_{k-1}^{2} \varepsilon_{ik}\phi_{ik} = 0 < 1, \quad \text{for}\, \varepsilon_{i1} \in [-1 \quad 1.1],\ \varepsilon_{i2} \in [0 \quad 10], \tag{3.4g}$$

$$\text{(viii)}\ \sum_{k-1}^{2} \varepsilon_{ik}\phi_{ik} \leq 0.1200 < 1, \quad \text{for}\, \varepsilon_{i1} \in [-1 \quad 1.1],\ \varepsilon_{i2} \in [-1.2 \quad 0]. \tag{3.4h}$$

From the results in (3.3a)–(3.3h) and (3.4a)–(3.4h), we can conclude that the uncertain TS fuzzy descriptor system (3.1a) and (3.1b) is locally robustly regular and controllable.

4. Conclusions

The robust local regularity and controllability problem for the uncertain TS fuzzy descriptor systems has been investigated. The rank preservation problem for robust local regularity and controllability of the uncertain TS fuzzy descriptor systems is converted to the nonsingularity analysis problem. Under the assumption that each fuzzy rule of the nominal TS fuzzy descriptor system has the full row rank for its related regularity and controllability matrices, a sufficient criterion has been proposed to preserve the assumed properties when the elemental parameter uncertainties are added into the nominal TS fuzzy descriptor systems. The proposed sufficient conditions in (2.9a)–(2.9c) can provide the explicit relationship of the bounds on elemental parameter uncertainties for preserving the assumed properties. One example has been given to illustrate the application of the proposed sufficient conditions. On the other hand, the issue of robust global regularity and controllability with evolutionary computation [36] for the uncertain TS fuzzy descriptor systems will be an interesting and important topic for further research.

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References

[1] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control,” *IEEE Transactions on Systems, Man and Cybernetics*, vol. 15, no. 1, pp. 116–132, 1985.
[2] R. Babuska, *Fuzzy Modeling for Control*, Kluwer, Boston, Mass, USA, 1998.
[3] S. S. Farinwata, D. Filev, and R. Langari, *Fuzzy Control: Synthesis and Analysis*, John Wiley & Sons, Chichester, UK, 2000.
[4] T. Taniguchi, K. Tanaka, and H. O. Wang, “Fuzzy descriptor systems and nonlinear model following control,” *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 4, pp. 442–452, 2000.
[5] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*, John Wiley & Sons, New York, NY, USA, 2001.
[6] Y. Wang, Q. L. Zhang, and W. Q. Liu, “Stability analysis and design for T-S fuzzy descriptor systems,” in *Proceedings of the 40th IEEE Conference on Decision and Control (CDC ’01)*, pp. 3962–3967, Orlando, Fla, USA, December 2001.
[7] S. C. Tong, T. Wang, Y. P. Wang, and J. T. Tang, *Design and Stability Analysis of Fuzzy Control Systems*, Science Press, Beijing, China, 2004.
[8] Y. Wang, Z. Q. Sun, and F. C. Sun, “Robust fuzzy control of a class of nonlinear descriptor systems with time-varying delay,” *International Journal of Control, Automation and Systems*, vol. 2, no. 1, pp. 76–82, 2004.

[9] T.-S. Lee, Y.-H. Chen, and J. C.-H. Chuang, "Robust control design of fuzzy dynamical systems," *Applied Mathematics and Computation*, vol. 164, no. 2, pp. 555–572, 2005.

[10] J. S. Ren and Y. S. Yang, "Robust control of chaotic system based on T-S fuzzy model," *Systems Engineering and Electronics*, vol. 27, no. 3, pp. 474–478, 2005.

[11] K. Y. Lian, J. J. Liou, and C. Y. Huang, "LMI-based integral fuzzy control of DC-DC converters," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 1, pp. 71–80, 2006.

[12] Z. Li, *Fuzzy Chaotic Systems: Modeling, Control and Applications*, Springer, Berlin, Germany, 2006.

[13] W. H. Ho and J. H. Chou, "Design of optimal controllers for Takagi-Sugeno fuzzy-model-based systems," *IEEE Transactions on Systems, Man, and Cybernetics Part A*, vol. 37, no. 3, pp. 329–339, 2007.

[14] W. H. Ho, J. T. Tsai, and J. H. Chou, "Robust-stable and quadratic-optimal control for TS-fuzzy-model-based control systems with elemental parametric uncertainties," *IET Control Theory and Applications*, vol. 1, no. 3, pp. 731–742, 2007.

[15] M. R. Hsu, W. H. Ho, and J. H. Chou, "Stable and quadratic optimal control for TS fuzzy-model-based time-delay control systems," *IEEE Transactions on Systems, Man, and Cybernetics Part A*, vol. 38, no. 4, pp. 933–944, 2008.

[16] W. H. Ho, J. T. Tsai, and J. H. Chou, "Robust quadratic-optimal control of TS-fuzzy-model-based dynamic systems with both elemental parametric uncertainties and norm-bounded approximation error," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 3, pp. 518–531, 2009.

[17] Z. Hidayat, Z. Lendek, R. Babuška, and B. De Schutter, "Fuzzy observer for state estimation of the METANET traffic model," in *Proceedings of the 13th International IEEE Conference on Intelligent Transportation Systems (ITSC '10)*, pp. 19–24, Funchal, Portugal, September 2010.

[18] C. W. Chen, "Stability analysis and robustness design of nonlinear systems: an NN-based approach," *Applied Soft Computing Journal*, vol. 11, no. 2, pp. 2735–2742, 2011.

[19] W. H. Ho, "Takagi-Sugeno fuzzy model of nonlinear HIV dynamics: chebyshev-series approach integrated with genetic algorithm," *International Journal of Innovative Computing Information and Control*, vol. 8, pp. 1439–1452, 2012.

[20] G. C. Verghese, B. C. Lévy, and T. Kailath, "A generalized state-space for singular systems," *IEEE Transactions on Automatic Control*, vol. 26, no. 4, pp. 811–831, 1981.

[21] Y. Wang, Q. L. Zhang, and X. D. Liu, "Robustness design of uncertain discrete-time fuzzy descriptor systems with guaranteed admissibility," in *Proceedings of the American Control Conference*, pp. 1699–1704, Anchorage, Alaska, USA, May 2002.

[22] S.-W. Kau, Y.-S. Liu, C.-H. Lee, L. Hong, and C.-H. Fang, "Static output feedback fuzzy controller design for T-S fuzzy descriptor systems," *International Journal of Fuzzy Systems*, vol. 6, no. 4, pp. 187–193, 2004.

[23] B. Y. Zhu, Q. L. Zhang, and K. N. Da, "State feedback robust H_∞ control for T-S fuzzy descriptor systems," *Journal of Northeastern University. Natural Science*, vol. 25, no. 5, pp. 453–456, 2004.

[24] X. Shi and Z. W. Gao, "Stability analysis for fuzzy descriptor systems," *Systems Engineering and Electronics*, vol. 27, no. 6, pp. 1087–1089, 2005.

[25] B.-Y. Zhu, Q.-L. Zhang, and S.-C. Tong, "Passivity control for uncertain T-S fuzzy descriptor systems," *Acta Automatica Sinica*, vol. 32, no. 5, pp. 674–679, 2006.

[26] K. Tanaka, H. Ohtake, and H. O. Wang, "A descriptor system approach to fuzzy control system design via fuzzy Lyapunov functions," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 3, pp. 333–341, 2007.

[27] W. H. Tian and H. G. Zhang, "Optimal guaranteed cost control of uncertain fuzzy descriptor systems with time-varying delay," *Journal of Northeastern University*, vol. 29, no. 4, pp. 464–468, 2008.

[28] J. R. Wu, "The generalized quadratically stability for a fuzzy descriptor system," in *Proceedings of the International Conference on Machine Learning and Cybernetics*, pp. 855–859, Baoding, China, July 2009.

[29] L. Dai, *Singular Control Systems*, vol. 118, Springer, Berlin, Germany, 1989.

[30] Y. H. Yuan and G. M. Zhang, "Research on T-S fuzzy descriptor systems: a review," *Acta Automatica Sinica*, vol. 36, no. 7, pp. 901–911, 2010.

[31] J. Chen and Z. Ren, "A comparison of small gain versus Lyapunov type robust stability bounds," *International Journal of Robust and Nonlinear Control*, vol. 11, no. 15, pp. 1407–1414, 2001.

[32] C. H. Hsieh and J. H. Chou, "Robust stability analysis of TS-fuzzy-model-based control systems with both elemental parametric uncertainties and norm-bounded approximation error," *JSME International Journal, Series C*, vol. 47, no. 2, pp. 686–693, 2004.

[33] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, NY, USA, 1975.

[34] E. L. Yip and R. F. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Transactions on Automatic Control*, vol. 26, no. 3, pp. 702–707, 1981.

[35] D. Cobb, "Controllability, observability, and duality in singular systems," *IEEE Transactions on Automatic Control*, vol. 29, no. 12, pp. 1076–1082, 1984.

[36] W.-H. Ho, J.-H. Chou, and C.-Y. Guo, "Parameter identification of chaotic systems using improved differential evolution algorithm," *Nonlinear Dynamics*, vol. 61, no. 1-2, pp. 29–41, 2010.

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Research Article

Analysis on Dynamic Decision-Making Model of the Enterprise Technological Innovation Investment under Uncertain Environment

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Under the environment of fuzzy factors including the return of market, performance of product, and the demanding level of market, we use the method of dynamic programming and establish the model of investment decision, in technology innovation project of enterprise, based on the dynamic programming. Analysis of the influence caused by the changes of fuzzy uncertainty factors to technological innovation project investment of enterprise.

1. Introduction

The enterprise technological innovation is a creative process. The uncertainty risk mainly includes the environment, technology, market, and risk management. At the same time, the process of enterprise technological innovation is a dynamic process. In the initial stage of the technical innovation, the enterprise must evaluate and select the innovation project and also consider the social and economic benefits and the development of technology with the combination of their own development strategies; at the end select the most suitable for the development of innovative investment projects. Sarkar [1] had studied market uncertainty and corporate investment relationship in consideration of system risk conditions, and he thinks that increasing the uncertainty may increase the probability of investment of enterprises to some low growth and low risk of investment project. In fact, a technical innovation project can be regarded as embedded in a series of options chain, and each option gives investors investment rights, so every decision stage contains an "improvement option"; when the difficult technology problem definitely is solved, we can make further investment in product prototype development and innovative design and then continue investing to enter the testing phase. At the same time, in every decision stage there is an abandonment option, so the flexibility of project management not only increases the value of the project, but also reduces the investment risk of the technical innovation project. Weeds [2] described if research and development is successful as a Poisson process, and it is used to describe the uncertainty of technology and of Hershey's bad market opportunity arrival timing. He builds two stages' R&D investment pricing model and discusses the reality of "Sleeping Patent" rationality; that is, the enterprise is willing to develop a technology, but after the success of the technology research and development, will put it away unheeded and not put it into the market. Mitchell and Hamilton [3] point out that due to R&D plays an important role in creating competitive advantage and, therefore, should be treated from a strategic perspective, discussing in detail the multiple features of the R&D strategic options and then taking three steps: clear strategic aim, evaluation strategy options, and select influence strategic target to study R&D strategy option. Lint and Pennings [4, 5] mainly studied the innovation of real option in the process of marketization and pointed out that there were two choices, disposable rapid advancing, and slow advancing and related options opportunities and options value hid in slow advancing. Huchzermeier and Loch [6] proposed a decision model of multiple stages and considered in each phase of this model that managers had three solutions: continue to invest in the project, improve the project, and give up the project. From the market returns, assumed total return consists of two levels: a basic income can be relatively easy to be obtained; only in the project performance exceeding market demands becomes uncertain. They also identified several different sources of the flexibility and uncertainty and made analysis of the impact problem increased by uncertainty and flexibility. From the characteristics that the state variable of R&D project is a nonfinancial parameter, using an equivalent method, dynamic planning of the option evaluation to build the dynamic programming model of R&D project, without the need for asset replication, Sheng [7] had solved the flexible problem of R&D project well.

The aforementioned is the study conducted under random environment (some discrete environmental). In fact, the essence of fuzzy real option is that tolerance in the same information shows diversification before the rationality and that the complexity of the human mind is admitted, namely, the introduction of nonuniform rational in the value assessment. At the same time, there are often still some realities that we cannot accurate valuation or expect net cash flow situation and due to objective factors some variables cannot be estimated by the exact data, and some actual situation etc (Liu [8]). So evaluation results often deviate from the actual if we use the accurate values to determine model input parameters. In this paper, we mainly combine the dynamic programming method and option analysis method, in a fuzzy environment making an analysis of some flexible decision problems of enterprise technology innovation and innovation investment. Firstly the Huchzermeier and Loch [6] model is extended. Secondly a discussion of dynamic programming model of the second stage under fuzzy environment is made. Thirdly analysis is made about technological innovation project decision model under fuzzy environment. Then the elastic value of the project of technology innovation is discussed. It focuses on the analysis of changes of the fuzzy uncertainty factors (including market returns and demanding level).

2. The Dynamic Programming Decision Model of the Technological Innovation Investment under Fuzzy Environment

2.1. Extending of Huchzermeier and Loch Model

Under the fuzzy environment, we consider a technical innovation project; the success of the project depends mainly on the performance of the products during commercialization in the market, and the fuzzy uncertainty of market performance is caused by market and technology risk, denoted by (i, t) project at time t expected market performance (the market performance can be expected through the simulation test, get). Typically managers have difficulty to predict the distribution of i during the stage of outcome of the market commercialization, assuming i as fuzzy variables and i obeying credibility distribution; the expected profits of the maximum and minimum values are Q and q, respectively, actual process of project meets no aftereffect, and the state transfer of market performance obeys two distribution; if the products of the project development reach to the performance state i, the expected profit is

$$\pi_i = q + \Phi(i)\left(Q - q\right) \quad \left(\Phi(i) \text{ is credibility distribution function}\right). \tag{2.1}$$

By Liu [8], in the processing of fuzzy event, credibility measure plays a similar role in the probability measure on random events. So, modeled as stochastic events in the transition probability, in the evaluation model of two-fork tree option, assuming the state transfer of market performance obeys two distributions, namely, the condition of market upgraded by the credibility of $\mathrm{Cr}\{A\}$, and under adverse conditions turning for the worse with probability $(1 - \mathrm{Cr}\{A\})$, we generalize this process as the market's performance improvement and deterioration and then easily conclude that the transition probability P_{ij} is expressed as follows under fuzzy environment:

$$P_{ij} = \begin{cases} \dfrac{\mathrm{Cr}\{A\}}{N}, & j \in \left(i + \dfrac{1}{2}, \dots, i + \dfrac{N}{2}\right), \\ \dfrac{1 - \mathrm{Cr}\{A\}}{N}, & j \in \left(i - \dfrac{1}{2}, \dots, i - \dfrac{N}{2}\right), \\ 0, & \text{other}. \end{cases} \tag{2.2}$$

At the same time, while expanding the scale of investment, under fuzzy environment the transition probability is

$$P_{ij} = \begin{cases} \dfrac{\mathrm{Cr}\{A\}}{N}, & j \in \left(i + 1 + \dfrac{1}{2}, \dots, i + 1 + \dfrac{N}{2}\right), \\ \dfrac{1 - \mathrm{Cr}\{A\}}{N}, & j \in \left(i + 1 - \dfrac{1}{2}, \dots, i + 1 - \dfrac{N}{2}\right), \\ 0, & \text{other}. \end{cases} \tag{2.3}$$

Under fuzzy environment, the project management dynamic programming optimal value function, in the last stage of commercial stage project:

$$V_i(T) = \max \begin{cases} -C(T) + \dfrac{\sum_j^N \left[\mathrm{Cr}\{A\}\pi_{i+j/2} + (1 - \mathrm{Cr}\{A\})\pi_{i-j/2}\right]}{N(1+r)} & \text{continue}, \\ -C(T) - A(T) + \dfrac{\sum_j^N \left[\mathrm{Cr}\{A\}\pi_{i+1+j/2} + (1 - \mathrm{Cr}\{A\})\pi_{i+1-j/2}\right]}{N(1+r)} & \text{improvement}, \\ 0 & \text{give up}, \end{cases} \tag{2.4}$$

where $C(T)$ is the cost of project's continuation and $A(T)$ are the cost when the project is increased in size, including the net present value of option strategic: $V = V(0) - I$; theoretically speaking, fuzzy environment is closer to the reality of the technical innovation. So, this model has certain practical significance, of course, according to the difference of the actual investment situation; we also can extend the model to the fuzzy random environment, fuzzy environment, rough fuzzy environment, and so forth.

2.2. The Dynamic Programming Model of the Two Stages under Fuzzy Environment

Assuming I is sunk cost, the interest rate without the risk is $r > 0$, and u, d are parameters; hypothesize; the price of the product in stage 0 is P_0, from the beginning of stage 1, the feasibility of the prices $(1+u)P_0$ is $\mathrm{Cr}\{A\}$, the feasibility of the prices of $(1-d)P_0$ is $(1-\mathrm{Cr}\{A\})$, assuming that the investment opportunity lies only in stage 0; if the technical innovation enterprises do not invest at this stage, so in stage 1, it will not change the decision forever; we use V_0 to be the symbol for the expectation value obtained from its investment of technology innovation enterprise, then

$$\begin{aligned} V_0 &= P_0 + [\mathrm{Cr}\{A\}(1+u)P_0 + (1-\mathrm{Cr}\{A\})(1-d)P_0]\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots\right] \\ &= \frac{P_0[1+r+\mathrm{Cr}\{A\}(u+d)-d]}{r}. \end{aligned} \tag{2.5}$$

Now, we consider the reality of the situation; in any future stage, the investment opportunities still exist. So, at this time, in stage 0, we can choose to invest or wait to select until stage 1; from stage 1 forward conditions will not change; if in phase 1 waiting, the price becomes

$$P_1 = \begin{cases} (1+u)P_0, & \text{when the feasibility of } \mathrm{Cr}\{A\}, \\ (1-d)P_0, & \text{when the feasibility of } 1-\mathrm{Cr}\{A\}. \end{cases} \tag{2.6}$$

To either possibilities (price changing in stage 0 and stage 1), if $V_0 > I$, then enterprise invests, we can get the net return: $F_1 = \max\{V_1 - I, 0\}$.

The discounted value is $V_1 = P_1(1+r)/r$; from stage 0, the price is P_1 in stage 1, the value is V_1, F_1 is the random variables, and E_0 is the expected value calculated by feasible weighted average in stage 0, then

$$\begin{aligned} E_0(F_1) = \mathrm{Cr}\{A\}\max&\left[\frac{(1+u)P_0(1+r)}{r} - I, 0\right] \\ &+ (1-\mathrm{Cr}\{A\})\max\left[\frac{(1-D)P_0(1+r)}{r} - I, 0\right]. \end{aligned} \tag{2.7}$$

Back to stage 0, the enterprise has two kinds of choices. If it invests, the income is $V_0 - I$; if not, the enterprise has continuous value $E_0(F_1)$, but the value is obtained in stage 1, so it should use $1/(1+r)$ to discount; therefore, the whole investment opportunities are the net present value of the investment profit arranged optimally, credited as F_0

$$F_0 = \max\left\{ V_0 - I, \frac{1}{1+r} E(F_1) \right\}. \tag{2.8}$$

Previously we discussed the dynamic programming model in Sheng [7] under fuzzy environment. Theoretically speaking, the fuzzy environment is closer to the technical innovation in reality, so this model has certain practical significance. Below we start from Liu [8] model, combine Dixit and Pindyck [9], and make discussion about the multistage model.

2.3. The Decision Model of Technological Innovation Project during Multistages under Fuzzy Environment

2.3.1. Fuzzy Uncertainty of Technological Innovation Project during Development Stage

Investment management of technical innovation project is a decision process of multistages. Each stage has decision points of project evaluation. Each decision point includes the project evaluation at present, investment decision making of the current state, and future earnings evaluation based on each kind of decision-making choice. In this section, we use the fuzzy theory proposed by Liu [8] to handle uncertainty of technical innovation project's development phase. Firstly the decision model of technology innovation project during multistage is described [7].

Santiago and Vakili Model

Assuming totally there are T stages of decision making of technology innovation project, $t = 0, 1, 2, \dots, T-1$, in each stage t, decision makers will face three alternatives, "continue" "improvement," and "give up". The success of technical innovation project depends on the performance of the product put into market; we use the state variables of the project to show the product performance in the process of the development. Let the X_t project be the state variables in the initial stage t, and assume when $t = 0$, $X_0 = 0$; ξ_t is the fuzzy uncertainty of the project inside and outside during t stage in the process of innovation, and $\{\xi_0, \xi_1, \dots, \xi_{T-1}\}$ are independent from each other; u_t is the choice decision of the project in stage of starting time. So, when the t stage is completed, the state variable of project can be expressed as [10]

$$X_{t+1} = \begin{cases} X_t + k(u_t) + \xi_t, & \text{if } u_t \text{ choose "continue" or "improvement"}, \\ X_t, & \text{if } u_t \text{ choose to "give up"}, \end{cases} \tag{2.9}$$

where the feasibility of $k(\text{continue}) = 0$ and $k(\text{improvement}) = 1$ is $\text{Cr}\{A\}/N$, the feasibility of $\xi_t = -i/2$ is $(1 - \text{Cr}\{A\})/N$, $i = 1, 2, \dots, N$, N is regarded as a measure of the uncertainty and fuzziness. In other words, if the item "continues," then in the next stage, the expected performance will be present together with some fuzzy uncertainty; if the item "improves," the project of state variable will be plused one more improved unit and fuzzy uncertain effects; if the project "gives up", the project stops at current state variables and remains unchanged. According to the discussion of Sheng [7], we have the following

Nature 1

A decision-making problem of a technical innovation project is considered under fuzzy environment:

$$X_t = \begin{cases} X_t', & \text{if choose "continue" or "improvement"}, \\ X_t, & \text{if choose to "give up"}. \end{cases} \tag{2.10}$$

According to hypothesis at the initial time of t stage, the state variables of project in two cases are X_t and X_t', respectively, and X_t and X_t' are fuzzy variables on possibility space $(\Theta, P(\Theta), \mathrm{Pos})$, then, $X_t \geq X_t'$.

Proof. Assume in two cases that the initial state variables of project equal, that is, $X_0 = X_0' = 0$, because X_t and X_t' are fuzzy variables of $(\Theta, P(\Theta), \mathrm{Pos})$ on possibility space. According to hypothesis in $t-1$ phase, for all $\theta \in \Theta$, we have $X_{t-1}(\theta) \geq X_{t-1}'(\theta)$. Assume in $t-1$ phase situation 1, decision makers take u^* as the optimal decision, and then in the t stage by Santiago and Vakili [10] model, we can get

$$X_t(\theta) = X_{t-1}(\theta) + k(u^*) + \xi_{t-1}(\theta) \geq X_{t-1}'(\theta) + \xi_{t-1}(\theta) = X_t'(\theta), \quad \text{then } X_t \geq X_t'. \tag{2.11}$$

□

2.3.2. Fuzzy Expected Value, Variance

The fuzzy uncertainty of technical innovation project ξ_t $(t = 0, \ldots, T-1)$ includes the technical risk of the project development process inside and the fuzziness influenced by the external environment, though there is statistical data of the project of the same type for reference, because technical innovation project is unique and singular, which makes subjective judgment of decision makers essential. We describe this kind of fuzzy uncertainty as fuzzy variable, which is independent and identically distributed, and its expected value is 0. It indicates that the project performance may be improved due to favorable fuzzy uncertain events, and also may be worse due to the occurrence of adverse; by assumptions of Yi Chang sheng, the fuzzy variable ξ_t expectations of the technological innovation project are expressed as

$$E[\xi_t] = \int_0^{+\infty} \mathrm{Cr}\{\xi_t \geq r\} dr - \int_{-\infty}^{0} \mathrm{Cr}\{\xi_t \leq r\} dr = 0. \tag{2.12}$$

According to Liu [8] and $E(\xi) = 0$, then the variance is $V[\xi_t] = E[(\xi_t - E[\xi_t])^2]$. The variance of $V[\xi_t]$ can be used as a measure of fuzzy degree of uncertainty of technical innovation project. If $V[\xi_t]$ is smaller, then the fuzzy uncertainty is smaller. The solvent of fuzzy uncertainty can be reflected by the cumulative value of technical innovation project. Assumed in the of the initial state item variables is X_t, at the end of t phase and the initial time of $t+1$ stage, the fuzzy uncertain factors ξ_t of t phase are solved, and the state variables of project will change to $X_t + k(u_t) + \xi_t$. ξ_t shows the fuzzy uncertainty of the project in the phase under internal and external environment. It includes the technical risk during the process of project development, the evaluation of the project's profitability made by the project team, the external market information of project, and other aspects.

2.3.3. Development Costs and Market Returns

According to the model of Huchzermeier and Loch [6], because during the development process of each stage of the development cost and decision-making choices, we can assume the project in phase t development costs as $C_t(u_t)$; if the decision maker chooses "give up," then the cost of project development is 0. If the decision maker selects "continue," then the cost of project development is $c(t)$ (the "continue" cost of $c(t)$ for stage t); if the decision maker selects "improvement," then the cost of project development is $c(t) + a(t)$ ($a(t)$ is the additional cost invested when taking corrective action and does not need to extend the project schedule, such as processing engineers and the experimental equipment; it can make the project status variables improve a unit). If project's initial investment is I while $t = 0$, the revenue and cost of project discount according to the nonrisk free rate r. At the end of $t - 1$ phase, the project is completed and products are put into market, enterprises will get market gains closely related to product performance, and $R(X_T)$ expressed as follows:

$$R(X_T) = \begin{cases} m, & \text{if } X_T < \eta, \\ M, & \text{if } X_T \geq \eta, \end{cases} \tag{2.13}$$

where η is the market's demand level of the product and X_T is the state variable after $t - 1$ stage is over, that is, the final product performance obtained by enterprise. If the product's performance meets or exceeds η, then the business has more advantages than its competitors in product performance and will gain a perfect profit of M; conversely, the enterprise can only get a small profit of m (clearly, $m < M$); because the η is unknown before the product entering market, we postulate it is the fuzzy variable. For any $\theta \in \Theta$, assums the variable is $X_T(\theta)$ during project phase, we use $\mathrm{Cr}\{X_T(\theta) \geq \eta\}$ to show that project state variable $X_T(\theta)$ reaches or exceeds the credibility of η; then, when the state variable is $X_T(\theta)$, the expected value of fuzzy market returns is

$$\begin{aligned} E[\Pi(X_T)] &= \int_0^{+\infty} \mathrm{Cr}\{R(X_T(\theta)) \geq r\} dr \\ &= \int_0^{m} \mathrm{Cr}\{R(X_T(\theta)) \geq r\} dr + \int_m^{M} \mathrm{Cr}\{R(X_T(\theta)) \geq r\} dr \\ &= m + \int_m^{M} \mathrm{Cr}\{X_T(\theta) \geq \eta\} dr \\ &= m + \mathrm{Cr}\{X_T(\theta) \geq \eta\} \int_m^{M} dr \\ &= m + \mathrm{Cr}\{X_T(\theta) \geq \eta\}(M - m). \end{aligned} \tag{2.14}$$

We use $\phi(\cdot)$ to show credibility distribution function of the fuzzy variable η, then

$$E[\Pi(X_T)] = m + \phi(X_T) \cdot (M - m). \tag{2.15}$$

Huchzermeier and Loch [6] show the benefit function: if the performance level is X_T at the moment of T, it will generate the expected market return Π_i. According to the different actual investment situations, we can also extend the conclusion to fuzzy random environment, rough fuzzy environment, and so on. For example, under fuzzy random environment, when the variables of product state are $X_T(\theta)$, the expectation obtained by fuzzy stochastic market returns $\Pi(X_T)$ can also be presented in this way.

Definition 2.1. Let ξ be a fuzzy random variable, and then one has a finite expected value $E(\xi)$, $V[\xi] = E[(\xi - E[\xi])^2]$ that is called fuzzy variable ξ variance.

From the previous discussion, when the product state variable is $X_T(\theta)$, the expectation of fuzzy stochastic market returns $\Pi(X_T)$ is obtained by

$$\begin{aligned} E[\Pi(X_T)] &= \int_0^{+\infty} \Pr\{\xi \in \Omega \mid R(X_T(\theta)) \geq r\} dr \\ &= \int_0^{m} \Pr\{\xi \in \Omega \mid R(X_T(\theta)) \geq r\} dr + \int_m^{M} \Pr\{\xi \in \Omega \mid R(X_T(\theta)) \geq r\} dr \\ &= m + \int_m^{M} \Pr\{\xi \in \Omega \mid R(X_T(\theta)) \geq r\} dr \\ &= m + \Pr\{\xi \in \Omega \mid X_T(\theta) \geq r\} \int_m^{M} dr \\ &= m + \Pr\{\xi \in \Omega \mid X_T(\theta) \geq r\}(M - m). \end{aligned} \tag{2.16}$$

Therefore, a double-fuzzy environment and rough fuzzy environment are similar to be launched.

2.3.4. The Dynamic Programming Model of Technological Innovation Project

We consider the decision behavior of the enterprise technology innovation investment; the project current state variable is indicated by x, it will affect opportunities of the enterprise's decision-making and expansion, at any stage of t, and the value of variable x_t is known, let the future value $x_{t+1}, x_{t+2}, \ldots$ be random variables. $\{x_t\}$ is Markov process; we use the $V_t(x)$ to be the whole decision results of company from t; when selecting u_t as the control variables, its cash flow is $\Pi_t(x_t, u_t)$, at the phase of $t+1$, the state is x_{t+1}, the result of optimal decision result is $V_{t+1}(x_{t+1})$; at the phase of t, this result is a random variable, so, we take $E(V_{t+1}(x_{t+1}))$ as expected value, discounting to the stage of t; the plus of sight cash flow and the continuous value is

$$\Pi_t(x_t, u_t, \xi_t) + \frac{1}{1+\rho} E(V_{t+1}(x_{t+1})). \tag{2.17}$$

Enterprises will choose u_t to be maximum, and the result just is $V_t(x_t)$; then we have

$$V_t(x_t) = \max_{u_t} \left\{ \Pi_t(x_t, u_t) + \frac{1}{1+\rho} E(V_{t+1}(x_{t+1})) \right\}. \tag{2.18}$$

This equation is the optimal basic equation (see [9]).

If multiple phase problems have limited stage T, the ultimate returns of enterprises are $\Omega_T(x_t)$; then in the previous stage, we have

$$V_{T-1}\ (x_{T-1}) = \max_{u_{T-1}}\left\{\Pi_{T-1}(x_{T-1}, u_{T-1}) + \frac{1}{1+\rho}E(\Omega_{T-1}(x_{T-1}))\right\}, \tag{2.19}$$

which provides theoretical basis to simulate by using computer.

In fact, the x_t, x_{t+1} may be in any state, and it can be generally denoted as x, x'; then, for any x, we have Bellman equation of infinite duration dynamic programming:

$$V(x) = \max_{t}\left\{\Pi(x,u) + \frac{1}{1+\rho}E\big(V(x') \mid x,u\big)\right\}. \tag{2.20}$$

In the following we put this problem to further discussion under fuzzy environment.

Technical innovation project is on the stage of development spending is by the final market returns to compensate for an evaluation, according to the Santiago and Vakili model, since each choose temporary investment costs are known, therefore, management decision based primarily on the final market assessment. We use the value of the function $V_t(x)$ to be this evaluation, fuzzy uncertainty ξ_t $(t = 0, \dots, T-1)$ of technical innovation project includes the technical inside risk of the project during the development process, and the uncertainty influenced by external environment, and ξ_t is fuzzy variable independent and identically distributed, and its expected value is 0. Assuming in the stage that the development cost is $C_t(u_t)$, according to Santiago and Vakili model, it can be described by dynamic programming equation:

$$V_t(x_t) = \max_{u_t}\left\{-C_t(u_t) + \frac{1}{1+r}E[V_{t+1}(X_{t+1}(x_t, u_t, \xi_t))]\right\}. \tag{2.21}$$

Value of the function is $V_t(x)$ at t phase; initial project state variable is x. At the end of the project phase of $t = T$, we have

$$V_t(x) = E[\Pi(x)]. \tag{2.22}$$

At the same time, under fuzzy environment, if multiple phase problems have limited stage T, the ultimate returns obtained by the enterprises are $\Omega_T(x_t)$; at an earlier stage, we have

$$V_{T-1}(x_{T-1}) = \max_{u_{\,T-1}}\left\{-C_{T-1}(u_{T-1}) + \frac{1}{1+\rho}E[V_T(\Omega_T(x_{T-1}, u_{T-1}, \xi_{T-1}))]\right\}. \tag{2.23}$$

In fact, x_t, x_{t+1} may be any state, it can be generally written as x, x'; and then for any x, we have Bellman equation of infinite duration dynamic programming under fuzzy environment:

$$V(x) = \max_{t}\left\{-C(x,u,\xi) + \frac{1}{1+\rho}E\big(V(x') \mid x,u,\xi\big)\right\}. \tag{2.24}$$

The discussion of Santiago and Vakili [10] and Sheng [7] also gives a property of value function under the stochastic uncertainty environment; the following is the promotion under fuzzy environment.

Nature 2

Assuming the technology innovation project of state variables is indicated by x under fuzzy environment, if the expected market return function $E[\Pi(x)]$ is nondecreasing, then the value of the function $V_t(x)$ $(t = 0, 1, \ldots, T-1)$ of technical innovation project in any stage is non-decreasing too.

Proof. In reference Santiago and Vakili [10], if $V_{t+1}(x)$ is non-decreasing, we assume that $V_{t+1}(x)$ is nondecreasing; for two project state variables x_1 and x_2 of a given t phase, if $x_2 > x_1$, then we only need to prove $V_t(x_2) \geq V_t(x_1)$. Assuming that at t phase, the project of state variable is x_1, selecting the optimal decision u^*, makes the enterprise obtain the maximum $V_t(x_1)$. When the state variable of t phase is x_2, the taken decisions u^* make enterprises get the project value $V_t'(x_2)$; then consider the following;

(1) If u^* is "continue" or "improved," we have

$$\begin{aligned} V_t'(x_2) - V_t(x_1) &= \max_{u_t}\left\{-C_t(u_t) + \frac{1}{1+r}E[V_{t+1}(X_{t+1}(x_2, u_t, \xi_t))]\right\} \\ &\quad - \max_{u_t}\left\{-C_t(u_t) + \frac{1}{1+r}E[V_{t+1}(X_{t+1}(x_1, u_t, \xi_t))]\right\} \\ &= \frac{1}{1+r}E[V_{t+1}(x_2 + k(u^*) + \xi_t) - V_{t+1}(x_1 + k(u^*) + \xi_t)]. \end{aligned} \tag{2.25}$$

(2) If u^* is "gives up." $V_t'(x_2) - V_t(x_1) = 0$. Because ξ_t is a fuzzy variable, so, $x_1 + k(u^*) + \xi_t$ and $x_2 + k(u^*) + \xi_t$ are fuzzy variables. If $x_2 > x_1$, we have $x_1 + k(u^*) + \xi_t > x_2 + k(u^*) + \xi_t$. Because $V_{t+1}(x)$ is monotonicity, we have

$$V_{t+1}(x_2 + k(u^*) + \xi_t) - V_{t+1}(x_1 + k(u^*) + \xi_t) \geq 0. \tag{2.26}$$

From the properties that the fuzzy variable is nonnegative, and its expected value is also non-negative, we have

$$E[V_{t+1}(x_2 + k(u^*) + \xi_t) - V_{t+1}(x_1 + k(u^*) + \xi_t)] \geq 0. \tag{2.27}$$

Therefore, $V_t'(x_2) - V_t(x_1) \geq 0$. When the status of technical innovation project is x_2, the maximum value of technology innovation obtained, and the makers' optimal decision is $V_t(x_2)$, so we have $V_t(x_2) \geq V_t'(x_2)$.

$$\text{Therefore, } V_t(x_2) - V_t(x_1) \geq V_t'(x_2) - V_t(x_1) \geq 0, \text{ that is, } V_t(x_2) \geq V_t(x_1). \tag{2.28}$$

□

Nature 3

From the aforementioned, we also can get that under fuzzy environment, if the optimal decision selected by the decision maker is "give up," when the state variable is x during the state of t and when the variable is less than x, the optimal decision is "give up" too.

3. Conclusion

Combining the dynamic programming method and option analysis method, we make analysis of flexible decision problems of enterprise technological innovation investment, under fuzzy environment, mainly introducing fuzzy factors based on the model of Huchzermeier and Loch [6], Santiago and Vakili [10], Dixit and Pindyck [9], Sheng [7], and so forth. We establish the model, focus on the promotion of the Huchzermeier and Loch model under fuzzy environment, establish models of two-phase, multi-stage dynamic programming decision and make some analysis, and then draw valuable conclusions. But it only extends the models of Huchzermeier and Loch and Santiago and Vakili [10] to the fuzzy environment; in fact, this kind of promotion can also be extended to the fuzzy random environment and rough fuzzy environment. Although some attempt has been made, it is still not enough. This is what we should try our best in during the next step.

References

[1] S. Sarkar, "On the investment-uncertainty relationship in a real options model," *Journal of Economic Dynamics and Control*, vol. 24, no. 2, pp. 219–225, 2000.

[2] H. Weeds, "Sleeping patents and compulsory licensing: an options analysis," Working Paper, Fitzwilliam College, University of Cambridge, 1999.

[3] G. R. Mitchell and W. F. Hamilton, "Managing R&D as a strategic option," *Research Technology Management*, vol. 50, no. 2, pp. 41–50, 2007.

[4] O. Lint and E. Pennings, "R&D as an option on market introduction," *R&D Management*, vol. 28, no. 4, pp. 279–287, 1998.

[5] E. Pennings and O. Lint, "Market entry, phased rollout or abandonment? A real option approach," *European Journal of Operational Research*, vol. 124, no. 1, pp. 125–138, 2000.

[6] A. Huchzermeier and C. H. Loch, "Project management under risk: using the real options approach to evaluate flexibility in R&D," *Management Science*, vol. 47, no. 1, pp. 85–101, 2001.

[7] Y. C. Sheng, *Under the uncertain environment project decision analysis [Doctoral Dissertation]*, Tianjin University, 2006.

[8] B. Liu, *Uncertainty Theory*, vol. 154 of *Studies in Fuzziness and Soft Computing*, Springer, Berlin, 2nd edition, 2007.

[9] A. K. Dixit and R. S. Pindyck, *Investment under Uncertainty Princeton*, Princeton University, Princeton, NJ, USA, 1994.

[10] L. P. Santiago and P. Vakili, "On the value of flexibility in R&D projects," *Management Science*, vol. 51, no. 8, pp. 1206–1218, 2005.

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Research Article

Option-Game Approach to Analyze Technology Innovation Investment under Fuzzy Environment

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Based on the model of symmetric and asymmetric duopoly option game, this paper discusses the present value of profit flows and the sunk investment costs for the trapezoidal fuzzy number. It constructs the fuzzy expressions of the investment value and investment threshold of followers and leaders under fuzzy environment and conducts numerical analysis. This offers a kind of explanation to the investment strategies under fuzzy environment.

1. Introduction

Under the condition of uncertain symmetric duopoly game model, the model was firstly proposed by Smets [1]. For the research of asymmetric enterprises, Huisman [2] considered the initial investment cost and two asymmetry enterprise option-game models. Technology investment extends the already existing real option models by the introduction of the game theory. Under the environment of imperfect competition, the original idea of decision-making model of investment cost asymmetry investment came from the Grenadier's [3] duopoly model. Kong and Kwok [4] studied the real options problem of strategic investment game between two asymmetry enterprises. Pawlina and Kort [5] studied the influence to the investment decision caused by the difference of the enterprise and discussed the relationship of the value of enterprises and the different investment cost. Zmeškal [6] used the European call option of fuzzy random variables to assess the value of the enterprise. Yoshida [7] constructed symmetric triangular fuzzy numbers with the assumption that the stock price was fuzzy and stochastic; the fuzzy objective definition was introduced with fuzzy expectation with the assumption that the fuzzy degree and stock price were in proportion. It gave rise to the pricing formula of the European option and the fuzzy hedge strategy. Hui and Yong [8] studied the influence of the enterprise investment strategy given by the investment cost's variance and the time required by the success of technical innovation strategy. Carlsson et al. [9] considered the rates' fuzzy option formula of fuzzy relation and used the optimization theory to build R & D project's investment decision model. Liu [10] applied the fuzzy theory to build the model of currency option pricing, which converts the risk-free interest rate. The volatility and the original asset price to the fuzzy number under fuzzy environment, based on the model of equivalent martingale measuring and Black-Scholes.

The Problem of Trapezoidal Fuzzy Number

The concept of fuzzy set was initiated by Zadeh [11]. From the definition of Carlsson and Fullér [12], a fuzzy set A is a fuzzy set of the real line, (fuzzy) convex, and continuous membership function of bounded support; the family of fuzzy numbers will be denoted by F, for any for all $A \in F$; we will use the notation $[A]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$ as γ-sets of A; If $A \in F$ is a fuzzy number and $x \in \mathbb{R}$ is a real number, then $A(x)$ may be interpreted as the degree of possibility of the statement "x is A".

Definition 1.1 (See [7, 12]). A fuzzy set $A \in F$ is called a trapezoidal fuzzy number with core $[a, b]$, left width α, and right width β, if its membership function has the following form:

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \leq t < a \\ 1 & \text{if } a \leq t \geq b \\ 1 - \dfrac{b-t}{\beta} & \text{if } b \leq t \leq b + \beta \\ 0 & \text{Otherwise.} \end{cases} \tag{1.1}$$

And we use the notation $A = (a, b, \alpha, \beta)$, for all $\gamma \in [0, 1]$, then it can easily be shown that

$$[A]^{\gamma} = \left[a - (1-\gamma)\alpha, b - (1-\gamma)\beta\right]. \tag{1.2}$$

The support of A is $(a - \alpha, b + \beta)$.

Let $[A]^{\gamma} = (a_1(\gamma), b_1(\gamma), \alpha_1(\gamma), \beta_1(\gamma))$, $[B]^{\gamma} = (a_2(\gamma), b_2(\gamma), \alpha_2(\gamma), \beta_2(\gamma))$ be fuzzy numbers and let $\lambda \in \mathbb{R}$ be a real number; using the extension principle, we can verify the following rules for addition and scalar multiplication of fuzzy numbers

$$[A + B]^{\gamma} = \left[a_1(\gamma) + a_2(\gamma), b_1(\gamma) + b_2(\gamma), \alpha_1(\gamma) + \alpha_2(\gamma), \beta_1(\gamma) + \beta_2(\gamma)\right], \tag{1.3}$$

$$[A - B]^{\gamma} = \left[a_1(\gamma) - b_2(\gamma), b_1(\gamma) - a_2(\gamma), \alpha_1(\gamma) + \alpha_2(\gamma), \beta_1(\gamma) + \beta_2(\gamma)\right], \tag{1.4}$$

$$[\lambda A]^{\gamma} = \lambda [A]^{\gamma}. \tag{1.5}$$

From Carlsson and Fullér [12] and Yoshida [7], it is easy to see that the (crisp) possibility mean (or expected) value of A and the (possibility) variance of A:

$$E(A) = \int_0^1 \gamma\left(a - (1-\gamma)\alpha + b + (1-\gamma)\beta\right) = \frac{a+b}{2} + \frac{\beta-\alpha}{6},$$
$$\delta^2(A) = \frac{(b-a)^2}{4} + \frac{(b-a)(\alpha+\beta)}{6} + \frac{(\alpha+\beta)^2}{24}. \tag{1.6}$$

This paper reviews the symmetrical and asymmetrical enterprises technology innovation investment decision problem under fuzzy environment, discusses the model under fuzzy environment, and concludes followers' and leaders' fuzzy expression of the investment and the critical value under fuzzy environment. By the technology of fuzzy simulation and data, we can find that the symmetrical and asymmetrical enterprises have the optimal investment strategy under fuzzy environment.

2. Symmetric Model in Fuzzy Environment

2.1. Basic Assumptions

Assuming that there exist two technology innovation investment enterprises with neutral and rational risk in market and assuming that both of them have opportunities of technology innovation and irreversible investment, where such investment opportunity exists forever, their competition and strategies are both symmetrical. The sunk investment cost I is a fuzzy number, because the enterprise technology innovation investment sunk cost is often difficult to be expressed by a number; using fuzzy numbers to estimate is more objective and actual. Market is in the symmetrical duopoly, supposing that the two enterprises are of nonconstraint conspiracy. The present value of expected profit value of technology innovation investment project has uncertainty and fuzziness. In order to incorporate uncertainty, assuming the uncertainty factor $X(t)$ of market demanding follows a geometric Brown motion process

$$dX(t) = \mu X(t)dt + \sigma X(t)dz, \tag{2.1}$$

where $\mu \in (0, r)$ is drift, and σ is the rate of fluctuation. For the technical innovation of enterprises, assuming their anticipated profits of the present value's stream depends on $X(t)$, and the firms' investment behavior at the same time, the expected profit of the present value's stream is a trapezoidal fuzzy number and can be given by

$$\pi(t) = X(t)D\left(N_i, N_j\right), \tag{2.2}$$

where $D(N_i, N_j)$ is determined by the market demand parameters. Therefore, according to the trapezoidal fuzzy number (1.5), the nature of $X(t)$ is also for the trapezoidal fuzzy number. For any $k \in \{i, j\}$, we can conclude

$$N_k = \begin{cases} 0 & \text{The enterprise } k \text{ has not invest} \\ 1 & \text{The enterprise } k \text{ has to invest.} \end{cases} \tag{2.3}$$

If the investment can increase the expected income stream and have advantage of the first mover, the following restrictions on $D(N_i, N_j)$ are implied:

$$D(1,0) > D(1,1) > D(0,0) > D(0,1). \tag{2.4}$$

Further, we assume that there is a first mover advantage to investment

$$D(1,0) - D(0,0) > D(1,1) - D(0,1). \tag{2.5}$$

2.2. The Investment Value and Investment Threshold Value of Followers

According to the calculation rules of the trapezoidal fuzzy number, the solution $(X/X^f)^{\theta_1}$ is difficult, according to the literature [6, 7, 9, 10, 12] in solving similar problems by expectations to approximate estimation methods; we also can use $(E(X)/E(X^f))^{\theta_1}$ to estimate. We have taken the similar approach, just using $E(X) < E(X^f)$ instead of $X < X^f$. According to the equation of Itô's lemma and the Behrman equation, the option value $F(X)$ of followers can be shown by the following partial differential equation. According to the method of Dixit and Pindyck [13] and Huisman [2], we have the following investment value:

$$F^f(X) = \begin{cases} \left(\dfrac{X}{X^f}\right)^{\theta_1} \dfrac{X^f(D(1,1) - D(0,1))}{(r-\mu)\theta_1} & \\ \quad + \dfrac{(X_1, X_2, \alpha_1, \beta_1) D(0,1)}{r-\mu} & \text{if } E(X) < E(X^f) \\ \dfrac{(X_1, X_2, \alpha_1, \beta_1) D(1,1)}{r-\mu} - (I_1, I_2, \alpha_2, \beta_2) & \text{if } E(X) \geq E(X^f). \end{cases} \tag{2.6}$$

Here, it is optimal for the investment to invest when $E(X) \geq E(X^f)$. Equation (2.6) is derived by solving the optimal stopping problem with use of Itô's lemma. X^f expresses the followers' investment value and the investment threshold value under fuzzy environment, then

$$X^f = \frac{\theta_1}{\theta_1 - 1} \frac{(r-\mu)}{D(1,1) - D(0,1)} (I_1, I_2, \alpha_2, \beta_2), \tag{2.7}$$

where

$$\theta_1 = \frac{-(\mu - (1/2)\sigma^2) + \sqrt{(\mu - (1/2)\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \tag{2.8}$$

2.3. Investment Value and Investment Threshold Value of Leaders

Similar with the followers' discussion, we have the leading value function $F^l(X)$ and the optimal investment's threshold X^l, using X^m as the monopolist's critical value of investment

$$F^l(X) = \begin{cases} \left(\dfrac{X}{X^f}\right)^{\theta_1} \dfrac{X^f(D(1,1) - D(1,0))}{r-\mu} & \\ \quad + \dfrac{(X_1, X_2, \alpha_1, \beta_1) D(1,0)}{r-\mu} - (I_1, I_2, \alpha_2, \beta_2) & \text{if } E(X) < E(X^f) \\ \dfrac{(X_1, X_2, \alpha_1, \beta_1) D(1,1)}{r-\mu} - (I_1, I_2, \alpha_2, \beta_2) & \text{if } E(X) \geq E(X^f). \end{cases} \tag{2.9}$$

When there does not exist an initial investment, the critical value of the optimal investment strategy in a critical point is

$$X^m = \frac{\theta_1}{\theta_1 - 1} \frac{(r-\mu)}{D(1,0) - D(0,0)} I. \tag{2.10}$$

When there exists an initial investment, X^l is the leader's' critical value of investment, then the leader's optimal investment strategy is

$$X^l = \operatorname{Inf}\left\{0 < E\left(X^l\right) < E\left(X^f\right), F^l\left(X^l\right) = F^f\left(X^l\right)\right\}. \tag{2.11}$$

3. Establishments of the Asymmetric Model under Fuzzy Environment

3.1. Review of Model and Basic Assumptions

Under imperfect competition environment, the thought of the investment cost asymmetry investment decision-making's model is from the Grenadier's [3] duopoly model, according to Hui and Yong's [8] model, using standard backward induction of solving dynamic game. Promote the symmetry enterprises considered by Grenadiar to the enterprise of asymmetry. Suppose there are two technology innovation enterprises in the market, neutral risk and pursuing the largest expected value, expressed as i and j, they have opportunity to do a technical innovation investment to increase their profit flow and investment cost is asymmetry; that is, $(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_1 \neq (I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_2$, followers' profit flow begins to be immediately affected, and the net profit of leaders changes to zero in the process of implementation of technology innovation. From the beginning to the successful implementation of investment, it needs a fixed period of time, δ years; the inverse demanding curve faced by enterprises can be shown by market prices $P(t)$ of unit product of enterprise i:

$$P(t) = X(t) D_{N_i N_j}. \tag{3.1}$$

Here, $X(t)$ shows product market demanding uncertainty, if we assume that the two enterprises' fuzzy uncertainty is equal, and is $(X_1, X_2, \alpha_1, \beta_1)$; in order not to cause confusion of circumstances, we use $X(t)$ to express, assuming that it follows the geometric Brown motion

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW_t, \tag{3.2}$$

where α is the drift term, σ is a variable rate, dW_t is the increment of standard Wiener process, and $D_{N_iN_j}$ is the deterministic demand parameters of enterprises i, showing the effect of the strategic decision to profit flow between enterprises, which depends on the identity of enterprise $k \in \{i, j\}$, and the following inequality is established:

$$D_{10} > D_{11} > D_{00} > D_{01} > 0, \tag{3.3}$$

where $D_{10} > D_{00}$ shows that the profit of unsuccessful innovation enterprise decreases because of competitor's success; $D_{00} > D_{01}$ shows that the profits of enterprise's innovation success exceeds the failed; $D_{11} > D_{00}$ shows that when the competitor succeeds, the success of enterprise innovation can increase the profits; $D_{11} < D_{10}$ shows that if the competition has innovation success cases, successful innovation will improve their profit level; $D_{N_iN_j} > 0$ said corporate profits for non negative. In addition, also assumes the existence of investment first mover advantage over a competitor is enterprise innovation successful case of the comparative income greater than after competitor's innovation successful case of the comparative income, and have the following relations (refer to Grenadier [3]):

$$D_{10} - D_{00} > D_{11} - D_{01}. \tag{3.4}$$

3.2. *The Followers' Investment Value and Investment Threshold Value under Fuzzy Environment*

When the leader has invested, investment value of followers is the combination of the profit stream $XD(0, 1)$ and the investment option value. Inspired by the [2–5] and other relative literatures, according to Itô's lemma and the Behrman equation [13], we can obtain the option-game model of enterprise technology innovation under fuzzy environment (avoiding causing confusion circumstances, we still use the original symbol, $F(X)$, X_{iF} express the followers' investment value and investment threshold value under fuzzy environment)

$$F_i(X) = \begin{cases} \left(\dfrac{X}{X_{iF}}\right)^{\beta} \dfrac{(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i}{\beta - 1} + \dfrac{(X_1, X_2, \alpha_1, \beta_1)D_{01}}{(r-\alpha)}, & E(X) < E(X_{iF}) \\ \dfrac{(X_1, X_2, \alpha_1, \beta_1)D_{01}}{(r-\alpha)} - (I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i & \\ \quad + \dfrac{(X_1, X_2, \alpha_1, \beta_1)(D_{11} - D_{01})}{(r-\alpha)} e^{-(r-\alpha)\delta}, & E(X) \geq E(X_{iF}), \end{cases} \tag{3.5}$$

where $(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i$ $(i = 1, 2)$ is a follower of cost of the enterprise technological innovation investment. Assuming that $(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_1 \neq (I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_2$, the β in (3.5) is the positive root of the following quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha\beta - r = 0. \tag{3.6}$$

From Dixit and Pindyck [13], we can conclude the investment threshold of followers

$$X_{iF} = \left(\frac{\beta}{\beta - 1}\right)\frac{(r - \alpha)(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i}{D_{11} - D_{01}}e^{-(r-\alpha)\delta}. \tag{3.7}$$

According to the method from the literature [6, 7, 9, 10, 12], using the expectations of X to approximate estimation in solving similar problems, we can also use $(E(X)/E(X_{iF}))^{\beta}$ to estimate; in the comparison, we adopt a similar approach too, namely, using $E(X) < E(X_{iF})$ to be instead of $X < X_{iF}$.

Conclusion 1. The optimal investment strategy of followers to invest is at the time of $T_F = \{t \geq 0 : X(t) \geq X_{iF}\}$. The optimal investment threshold of followers is $X_F = (\beta/\beta - 1)((r - \alpha)(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i/D_{11} - D_{01})e^{-(r-\alpha)\delta}$. The coefficient $\beta/(\beta - 1) > 1$ is more expanded with the rule of the optimal investment than that with the rule of net present value; in the symmetric case, X_{iF} is influenced under the previous symmetric case $e^{-(r-\alpha)\delta}$; it is different from the discussion. Using the same method, we can establish the value function and the investment threshold of the enterprise as the leader under fuzzy environment

$$L_i(X) = \begin{cases} \left(\dfrac{X}{X_{jF}}\right)^{\beta}\dfrac{\beta}{\beta - 1}\dfrac{D_{11} - D_{10}}{D_{11} - D_{01}}(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i & \\ \quad + \dfrac{(X_1, X_2, \alpha_1, \beta_1)D_{10}}{(r - \alpha)}e^{-(r-\alpha)\delta} - (I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i, & E(X) < E(X_{jF}) \\ \dfrac{(X_1, X_2, \alpha_1, \beta_1)D_{11}}{(r - \alpha)}e^{-(r-\alpha)\delta} - (I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i, & E(X) \geq E(X_{jF}). \end{cases} \tag{3.8}$$

Under fuzzy environment, the value function and monopoly investment threshold of the enterprise i and its competitors are given by $S_i(X)$ when they invest at the same time

$$S_i(X) = \begin{cases} \dfrac{(X_1, X_2, \alpha_1, \beta_1)D_{00}}{(r - \alpha)} + \dfrac{(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i}{\beta - 1}\left(\dfrac{X}{X_{iS}}\right)^{\beta}, & E(X) < E(X_{iS}) \\ \dfrac{(X_1, X_2, \alpha_1, \beta_1)D_{00}}{(r - \alpha)} - (I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i & \\ \quad + \dfrac{(X_1, X_2, \alpha_1, \beta_1)(D_{11} - D_{00})}{(r - \alpha)}e^{-(r-\alpha)\delta} & E(X) \geq E(X_{iS}). \end{cases} \tag{3.9}$$

Conclusion 2. Under the situation of two companies investing at the same time, the value of each enterprise investment

$$S_i(X) = \frac{\beta}{\beta - 1} \frac{(r - \alpha)}{(D_{11} - D_{00})} \left(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i}\right)_i e^{-(r-\alpha)\delta}. \tag{3.10}$$

Conclusion 3. The optimal investment strategy of two businesses is to invest at the time of $T_S = \inf\{t \geq 0 : X \geq X_{iS}\}$.

4. Analysis with Comparison

(1) In most cases, under the condition of symmetry and asymmetry, the present value and investment sunk of expected profit flow uncertainty factor are not a definite number, but an estimate. This provides the basis for dealing with the above problem.

(2) From symmetry model, we can see the following.

When $E(X) < E(X^f)$, $F^f(X)$ and $(D(1,1) - D(0,1))$, X is in correlation positively, with $(r - \mu)$, I negative correlation.

At the same time, influenced by σ, θ_1, $(r - \mu)$, $D(1,1) - D(0,1)$ and the fuzzy of investment sunk cost according to the formula (2.6). From the model of asymmetry.

When $E(X) < E(X_{iF})$, $F(X)$ and $D_{11} - D_{01}$, $(I_{1i}, I_{2i}, \alpha_{2i}, \beta_{2i})_i$, and $(X_1, X_2, \alpha_1, \beta_1)$ are related positively. This is different according to the formula (3.5), from the symmetric case.

(3) To solve the symmetry model, firstly we calculate the θ_1, X^f, X^l, then we get the X range by fuzzy mathematics, we not only know the optimal strategy of two enterprise which can also be calculated through the simulation, but also can calculate the investment value of followers and leaders.

When $E(X) \geq E(X^l)$ and $E(X) \geq E(X^f)$ the investment value of followers is equal to the leader, that is, $F^f(X) = F^l(X)$. Due to the assumption that the two enterprises have symmetry, the leader and followers have no difference for any enterprise. But for the asymmetric model, this is different.

5. Conclusion

Due to space limitations, we do not use the fuzzy simulation method to compute. This method can be found in [10]. As long as we are given the estimation value of parameter, we can find the investment strategy and estimate the strategy of investment value according to the relative model. We also can make in-depth analysis according to the features of different districts caused by the different parameters.

Of course, when the uncertain factor of the enterprise technology innovation investment project cash flow, and the investment sunk cost of the enterprise technology innovation are trapezoidal fuzzy numbers, we consider the cost difference and balanced relationship types of operating costs of the symmetrical and asymmetrical enterprise, and obtain the value function and the corresponding threshold value of investment of the leaders and followers under fuzzy environment. Through numerical analysis, we find the symmetrical and asymmetrical enterprises under fuzzy environment still have the optimal investment strategies and make comparison. Of course, since in reality there exist the situation of symmetry and asymmetry of market demanding under fuzzy environment and technology, two or more factors, and the problems in process of dynamic stage, we will try our best to solve these problems in the future.

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References

[1] F. Smets, *Exporting FDI: The Effect of Uncertainty, Irreversibilities and Strategic Interactions*, Yale University, 1991.
[2] K. J. M. Huisman, *Technology Investment: A Game Theoretic Real Options Approach*, Kluwer Academic, Boston, Mass, USA, 2001.
[3] S. R. Grenadier, "The strategic exercise of options: development Cascades and overbuilding in real estate markets," *Journal of Finance*, vol. 51, no. 5, pp. 1653–1679, 1996.
[4] J. J. Kong and Y. K. Kwok, "Real options in strategic investment games between two asymmetric firms," *European Journal of Operational Research*, vol. 181, no. 2, pp. 967–985, 2007.
[5] G. Pawlina and P. M. Kort, "Real options in an asymmetric duopoly: who benefits from your competitive disadvantage," *Journal of Economics and Management Strategy*, vol. 15, no. 1, pp. 1–35, 2006.
[6] Z. Zmeškal, "Application of the fuzzy-stochastic methodology to appraising the firm value as a European call option," *European Journal of Operational Research*, vol. 135, no. 2, pp. 303–310, 2001.
[7] Y. Yoshida, "The valuation of European options in uncertain environment," *European Journal of Operational Research*, vol. 145, no. 1, pp. 221–229, 2003.
[8] X. Hui and Z. Yong, "Research on hybrid index hierarchy fuzzy decision-making method," *Journal of Management Sciences in China*, vol. 8, no. 1, pp. 31–41, 2005.
[9] C. Carlsson, R. Fullér, M. Heikkilä, and P. Majlender, "A fuzzy approach to R&D project portfolio selection," *International Journal of Approximate Reasoning*, vol. 44, no. 2, pp. 93–105, 2007.
[10] B. Liu, "Fuzzy process, hybrid process and uncertain process," *Journal of Uncertain Systems*, no. 1, pp. 3–16, 2008.
[11] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
[12] C. Carlsson and R. Fullér, "A fuzzy approach to real option valuation," *Fuzzy Sets and Systems*, vol. 139, no. 2, pp. 297–312, 2003.
[13] A. K. Dixit and R. S. Pindyck, *Investment under Uncertainty*, Princeton University, Princeton, NJ, USA, 1994.

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Research Article

Research on Regional Strategic Emerging Industry Selection Models Based on Fuzzy Optimization and Entropy Evaluation

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It is with a great significance to discuss the selection model of regional strategic emerging industry. First, the paper uses the fuzzy optimization theory to select the central industry of area as the regional strategic emerging industry and try to optimize the weight's calculation in the multistage fuzzy comprehensive evaluation to get more accurate results. Then it will do a strategic emerging industrial inspection about the advantage and ecological related index based on a multiobjective programming model and the maximum entropy.

1. Introduction

Strategic emerging industry includes both the characteristics of the huge development potential and the strong leading ability as strategic industry and the highly innovative characteristics as emerging industry. After the financial crisis period, countries all over the world especially some major developed countries take the development of new energy, new materials, energy conservation and environmental protection, and green economy emerging industry as a key of a new round of industry development. In September 2009, China first raised the concept of strategic emerging industry and selected seven industries as the national strategic emerging industries.

At present the provinces in our country also take a national strategic emerging industry planning as a guide and put forward the strategic emerging industry which will be developed in this area. Making a strategic arrangement in the development of national strategic emerging industry in 30 provinces of our country, we found that the choice of a regional strategic new industry convergence was serious that the choice of the new material area should achieve 29, new energy 26, and the choice of biological 25. But the regional industry structure characteristics and resources endowment difference are not taken into consideration, which will inevitably lead to three prominent problems that are serious industry homogenization phenomenon, excess capacity, and the waste of resources. Therefore, how to scientifically choose accurate selection and evaluation of regional strategic emerging industry is of a strong theoretical and practical significance.

2. Literature Review

The American economists Hector Seaman (A. O. Hirschman), the first person to put forward the concept of strategic industry, has called the put-output relationship between the most closely related economic system "strategic department" [1]. Cambridge scholars such as Heffeman have carried on a deep analysis of the emerging industry characteristics, development, evolution path system, and so on. From the perspective of commercial development, it emphasizes on the idea-to-product of the conversion process, and that emerging industry development is a dynamic evolution process [2]. Liu from strategic emerging industry concept holds that the strategic emerging industry has at least several characteristics such as strategy, innovation, growth, relevance, guidance, and risk [3]. Some researchers have a try on the choice of strategic emerging industry. Hu and Zhao of Liaoning province evaluated 18 scales above high and new technology industrial economic benefits in 2006, using factor analysis, and got development strategic emerging industry potential and comparative advantage of industry [4]. He et al. [5–7] established the evaluation model based on analytic hierarchy process (AHP) and integrated fuzzy evaluation method, and they knew about a region that was whether or not fit for the development of strategic new industry by the model. and helped the region to have comprehensive, accurate, and objective selection and evaluation of strategic emerging industry. Qing [8], with the aid of expert questionnaire marking method and analytic hierarchy process (AHP), calculated the weights of the six factors such as natural, economic, social and human, science and technology, industrial competitiveness, and the government which had influenced the emerging industry development in Henan province and put in order the seven big emerging industry developments according to quality. Qiao et al., [9] based on the gray theory analysis, established the evaluation index system and evaluation mode 1 which are suited to the characteristics of the strategic emerging industry in biological medicine. Hu et al., [10] based on the combination weights "AHP-IE-PCA", the selection model of regional strategic emerging industries was tentative proposed, which was applied to N county in the selection and evaluation of strategic emerging industries.

It is obvious that, at present, the empirical research of regional strategic emerging industry choice always uses the analytic hierarchy process and expert scoring method, and this kind of method subjectivity is too strong and lacks dynamics. In addition the choice of empirical research is more than an unidirectional choice, and the lack of the selected conclusion effectiveness and efficiency evaluation makes the industry choice lack research reliability analysis and policy persuasion.

Vector optimization ideas originated from utility theory research in economics from 1776. In 1896, French economist Pareto [11] first put forward the problem of the multi-objective programming for a limited number of evaluation index in economic balance study.

At that time, from the perspective of political economics, he summarized a lot of targets difficult to compare as multi-objective optimization and proposed the thought of the later called Pareto efficient solution, and this thought had an important and profound impact upon the forming of the vector optimization disciplines. In 1944, from the point of view of game theory, Neumann and Morgenstern [12] put forward the problem of multi-objective decision which contradicted each other and had several decision makers. In 1951, from the analysis of production and distribution activities, Koopmans [13] put forward the problem of multi-objective optimization, and for the first time he put forwards the concept of Pareto effective solution. In the same year, from the point of view of mathematical programming, Kuhn and Tucker [14] proposed the concept of Pareto efficient solution for the vector extremism and studied this solution's optimally sufficient and necessary conditions. The evaluation balanced research by Debreu [15] in 1954, the research of pushing multi-objective optimization problem to the general topological vector space by Harwicz [16] in 1958, and the vector optimization problems were concerned by people gradually. In 1968, Johnsen [17] published the first monograph about multi-objective decision-making model. In 1970 Bellman and Zadeh put forward "fuzzy optimization" concept and provided effective tool for linear programming in several fields which had more fuzzy factors, such as multi-objective optimization [18].

Therefore, with the aid of the industrial structure similarity coefficient, this paper will first measure the area of industrial isomorphism, thus give system dynamical judgment of the regional industrial structure convergence change characteristics, and determine the regional industrial structure from the macroscopic perspective preliminary. Secondly, in view of regional strategic emerging industry, selection is influenced by multiple factors and belongs to multistage comprehensive evaluation problem, the paper uses the method of fuzzy mathematics comprehensive evaluation. Since it is needed to optimize and improve multistage fuzzy comprehensive evaluation model, the evaluation results will be as accurate and objective as possible. Finally, the paper will evaluate the chosen results and calculate the quotient of location to find out whether the area industry has advantage in the same industry of country or not. It will calculate intersection to the two kinds of industry and confirm the area strategic emerging industry. In addition, it will be coupled to an evaluation about the traditional industries and the strategic emerging industries, including double coupling relevance and developmental evaluation.

3. The Selection Model of Strategic Emerging Industries

3.1. Regional Industrial Structure Convergence Degree Test

Firstly the research is to analyze the convergence of industrial structure, and carry on the measure. The convergence of industrial structure generally refers to a phenomenon that the areas of different geographical location, different resource abundance and different development path form a similar industrial structure, which refers in particular to the convergence within the industry and the internal structure of industry. The convergence of industrial structure mainly performances for the structure of the industry between the areas difference contractible, industrial categories and industrial systems between the areas resembling gradually, spatial distribution equalization of the major industries and product production. As to the specific measure of industrial isomorphism, this paper uses the similarity coefficient method put forward by the United Nations industrial development organization international industrial research center and Shingling Wang's related research; the similarity coefficient is defined as [19]:

$$S_{ij} = \frac{\sum_{k=1}^{n} X_{ik} \cdot \sum_{j=1}^{n} X_{jk}}{\sqrt{\sum_{k=1}^{n} X_{ik}^2 \cdot \sum_{k=1}^{n} X_{jk}^2}}. \tag{3.1}$$

The type of S_{ij} is similarily coefficient, I, j is two phase comparison areas, and X_{ik}, X_{jk} are industries k in the area I and j region as a proportion of the industrial structure. S_{ij} is a direct link between the number of 0 and 1. And if the value is 0, it means the two compared regional industrial structures are completely different; if the value is 1, it means the two compared regional manufacturing structures are exactly the same. Through the observation of the certain period S_{ij} value changes, it may give dynamic judgments to the changes in regional industrial structure. If S_{ij} value tends to rise, it is "structure convergence"; if S_{ij} value tends to decline, it is "structure divergence." In this way, it can judge on the whole the similar degree of different regional industrial structure so as to provide macro basis for a strategic choice of emerging industry.

3.2. Multistep Fuzzy Comprehensive Evaluation Model

Multistage fuzzy comprehensive evaluation model is the organic union of multistage fuzzy theory and classic comprehensive evaluation method, which is mainly used to solve the evaluation object affected by various uncertain factors and the various factors, and has different levels. The choice of strategic emerging industry involves many fuzzy factors, and various factors have obvious hierarchy; therefore, we will choose the multistage fuzzy comprehensive evaluation method to research it. In the fuzzy comprehensive evaluation model, establishing single factor evaluation matrix R and determining the weight distribution A are two key jobs. But at the same time there is no unified format which can be abided by. Typically it uses the expert evaluation method to work out, but in practical operation this kind of method has weakness in the long survey period, which may delay decision time and affect the real-time performance of the evaluation results to a certain extent. Here the paper will mainly discuss how to use the statistical method to determine the weights model.

The strategic emerging industry index system is as shown in Figure 1.

3.2.1. The Fuzzy Weighted Vector Sure [20]

Multistage fuzzy comprehensive evaluation method often uses Delphi method to determine the index weight, which affects the model practicality and objectivity of evaluation results. In order to overcome this shortcoming of this model, we will determine the weight of each index by the variation coefficient method. First, through the Figure 1 index system, determine the level one evaluation index set $U = \{u_1, u_2, \ldots, u_n\}$; u_i means the ith one class index. Second, we will further divide it into several evaluation levels according to the assessment and evaluation factors: $u_{ij} = \{u'_{j1}, u'_{j2}, \ldots, u'_{jk}\}$; k is the number of evaluation factors for the ith evaluation, the jth evaluation factor. Third, using variance method to determine the weight

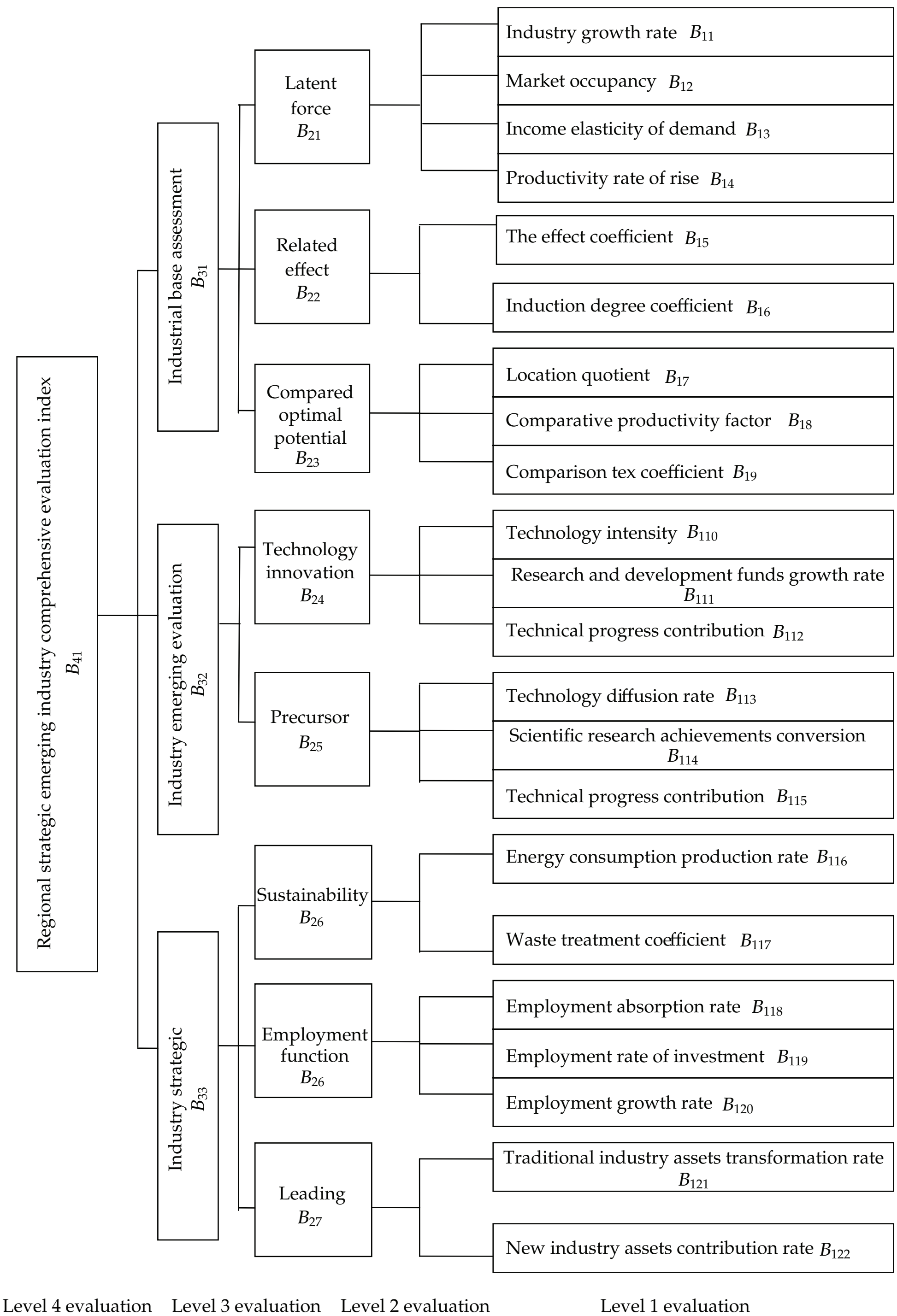

Figure 1: Index system of regional strategic emerging industry.

of each basic index A reflects the importance of the various factors in the whole. That is, each index is A group of survey data $z_1, z_2, \ldots, z_n$, written as

$$\overline{z} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$
$$t_k = \left(\frac{1}{n-1}\sum_{i=1}^{n}(z_i - \overline{z})^2\right)^{1/2}. \tag{3.2}$$

That is

$$\nu_k = \frac{t_k}{|\overline{z}|}. \tag{3.3}$$

ν_k is the coefficient of the variation of data $z_1, z_2, \ldots, z_n$, the weight a_j is:

$$a_j = \frac{\nu_j}{\sum_{i=1}^{m} \nu_i}. \tag{3.4}$$

3.2.2. Comment Set and the Determination of Membership Function

Be sure that comment set $U = \{$shall not choose, will consider, should select$\}$ and membership functions in order to determine the evaluation factors on the comments membership.

3.2.3. The Generation of Fuzzy Judgment Vector B [21]

Starting from the U_i, determining the evaluation object of evaluation element set ν and degrees of membership R_i is, namely, called single factor fuzzy evaluation. Factor U_i and evaluation results R_i are called single factor fuzzy evaluation set, which is the ν fuzzy subset, $R_i = \{r_{i1}, r_{i2}, \ldots, r_{im}\}$.

Put the single factor evaluation set as the line, then it can get deviation fuzzy matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{bmatrix}. \tag{3.5}$$

In type (3.4),

$$r_{ij} = \frac{|a_{ij} - \lambda_i|}{\max\{a_{ij}\} - \min\{a_{ij}\}}, \qquad \lambda_i = \begin{cases} \max\{a_{ij}\},\ a_{ij} \text{ is performance indicator,} \\ \min\{a_{ij}\},\ a_{ij} \text{ is cost indicator.} \end{cases} \tag{3.6}$$

Single factor fuzzy evaluation can only reflect a factor of evaluation object, but cannot reflect the comprehensive influence of all factors. Thus, all factors must be comprehensively considered. All single factor fuzzy evaluation can be expressed as

$$B = A \cdot R = (a_1, a_2, \ldots, a_n) \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{bmatrix} = (b_1, b_2, \ldots, b_n),$$
$$b_i = \bigvee_{i=1}^{n} (a_i \wedge r_{ij}), \quad j = 1, 2, \ldots, m. \tag{3.7}$$

In formula (3.7), "$\wedge$" and "$\vee$" are "take small" operator, "take big" operator.

3.2.4. Construct an Evaluation Function

In order to facilitate and get an accurate evaluation result, suppose the level of the value of a variable range is 0–60 (not to be chosen), 60–80 (to be considered), and 80–100 (should be chosen) and calculate the group data to get evaluation matrix P:

$$P = \begin{bmatrix} 50 \\ 70 \\ 90 \end{bmatrix}. \tag{3.8}$$

Then the comprehensive evaluation function is

$$S = B \cdot R. \tag{3.9}$$

Then according to the size of the S, Table 1 found out the corresponding level evaluation, which is the final evaluation result of whether the strategic emerging industry has been chosen or not.

3.3. Fuzzy Evaluation Model Optimization

3.3.1. Optimization of Multistage Fuzzy Comprehensive Evaluation Methods [20]

Because the evaluation vector is at the next higher level for the fuzzy evaluation vector, it can use the level evaluation of fuzzy evaluation vector to structure the evaluation fuzzy evaluation set again. In addition, another important problem is to determine the index weight above Level two. As the variation coefficient method applies only to statistical index, and the indicators above Level two have no direct statistics, the paper will adjust the above evaluation method so that researchers can use the statistical method to calculate the weight of the index above the second level.

After the calculation of the level of the index weight, we separately judge each questionnaire for the primary evaluation, get the fuzzy evaluation vector of secondary evaluation index $(y_1, y_2, \ldots, y_n)$, use the membership functions of inverse function to calculate secondary statistical indexes x_{2i}, put each index value of the questionnaire

Table 1: Comparison between the results of evaluation and the level of remark.

Level comments	Not chosen	To be considered	Should be chosen
The comprehensive evaluation value	$[0, 60]$	$[60, 80]$	$[80, 100]$

calculated as a secondary index of the statistics, and repeat the process of the different coefficient method to determine the secondary index weight. The index weights above the second level are calculated with this method. Then repeat the above steps so as to complete the fuzzy comprehensive evaluation of each layer index.

3.3.2. Optimization of the Weights in the Fuzzy Comprehensive Evaluation [22]

Objective weighting method is got according to current sample data statistics, related to weight and the current sample data, which has a stronger objectivity and avoids the deviation of the artificial factors. It also has the determination of the weight of the consideration on each factor, such as principal component analysis focused on the relationship between index data, the mean square error method, maximizing deviations method, the entropy weight method, and the variation coefficient method. All of them have more consideration of the influence of the data discrete sex to the real evaluation result. Therefore there is also likely the situation in disagreement with the importance of the determination between the weight of index and the index itself.

Therefore, this paper will optimize the method of the fuzzy comprehensive evaluation weights calculation, combine the principal component analysis method and the entropy weighting method of mutual confluence, so that it can keep both the principal component analysis and the linear combination of the original data, simplify the advantages of the index, and at the same time compensate the defect without the consideration of the principal component analysis and the determination of the weights for data features by means of entropy value method. The concrete methods are as follow.

(1) Make the original data of n industry p indexes matrix $X(i = 1, 2, \ldots, n; j = 1, 2, \ldots, p)$, which were then dimensionless or standardized processing and generally use Z-score method, which is dimensionless, and next get new M_{ij} matrix.

(2) The correlation coefficient matrix R_{jk} of the calculation index:

$$R_{jk} = \frac{1}{n} \cdot \sum_{i=1}^{n} \frac{\left(X_{ij} - \overline{X_j}\right)}{S_j} \cdot \frac{\left(X_{ik} - \overline{X_k}\right)}{S_k} = \frac{1}{n} \sum_{i=1}^{n} Z_{ij} \cdot Z_{ik}, \quad R_{jj} = 1,\ R_{jk} = R_{kj}. \tag{3.10}$$

(3) Work out the R_{jk} array value of characteristics λ_k $(k = 1, 2, \ldots, p)$ and characteristic vector L_k $(k = 1, 2, \ldots, p)$. According to the characteristic equation:

$$|R - \lambda I| = 0. \tag{3.11}$$

Calculate characteristic value λ_k, citing the characteristic value λ_k, and feature vector L_k.

(4) Calculation of contribution rate:

$$T_k = \frac{\lambda_k}{\sum_{j=1}^{p} \lambda_j} \tag{3.12}$$

and cumulative contribution:

$$D_k = \sum_{j=1}^{k} T_j. \tag{3.13}$$

The selection of $D_m \geq 85\%$ of the characteristic value λ_m $(m < p)$ corresponding several principal components.

(5) Use each principal component proportion of the principal components to explain reflects index righteousness.

(6) Calculate the principal component index weights. The first m a principal component characteristic value of the product contribution D_m is set 1. Calculate the $T_1, T_2, \ldots, T_m$ of the corresponding new $T_1', T_2', \ldots, T_m'$, that is, the main component of the index weights.

(7) Calculate the principal components and get part matrix F_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$. So far, we have achieved the purpose of simplified index number with the aid of the principal component analysis method. Next we will use entropy weight method to calculate the weight of each index factor.

(8) Data translation. Make

$$y_{ij} = y_{ij} + 1 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n). \tag{3.14}$$

Because using entropy value to work out weight should be made use of logarithmic calculation, in this way we can avoid taking logarithmic time nonsense.

(9) Calculating the proportion of the index in the j indexes and i the value of the industry:

$$P_{ij} = \frac{y_{ij}}{\sum_{i=1}^{m} y_{ij}} \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n). \tag{3.15}$$

(10) Computing the first j indexes entropy:

$$e_j = -k \sum_{i=1}^{m} P_{ij} \ln P_{ij}, \quad k > 0,\ k = \frac{1}{\ln(n)},\ e_j \geq 0. \tag{3.16}$$

(11) Compute the difference coefficient of the j indexes. As to the j indexes, the greater the differences of the Y_{ij} index value is, the greater effect the scheme evaluation Y_{ij}

is and the smaller the entropy value is. The definition of the difference coefficient is as follows:

$$g_i = \frac{1-e_j}{\sum_{j=1}^{n}(1-e_j)} = \frac{1-e_j}{n-E_e}, \quad E = \sum_{j=1}^{n} e_j, \; 0 \le g_i \le 1, \; \sum_{j=1}^{n} g_j = 1. \tag{3.17}$$

(12) Working out weight:

$$w_j = \frac{g_j}{\sum_{j=1}^{n} g_j} \quad (1 \le j \le n). \tag{3.18}$$

(13) Weighted comprehensive evaluation model according to many indexes to calculate the comprehensive evaluation value:

$$S_i = \sum_{j=1}^{m} w_j \cdot P_{ij} \quad (i = 1, 2 \ldots m; \; j = 1, 2 \ldots n). \tag{3.19}$$

4. The Inspection Model Strategic Emerging Industries

4.1. The Industry Advantage Index

4.1.1. The Escalating Rate of Productivity Index

As a strategic emerging industry, the high escalating rate of productivity has a high potential for growth and development advantages, and such ability will drive the development of the industry system. The escalating rate of productivity index,

$$v_i = \frac{a_i(t_n) - a_i(t_0)}{t_n - t_0}. \tag{4.1}$$

$a_i(t_n)$ is the productivity of the$i$th sector and the t_n year, $a_i(t_0)$ is the productivity of the$i$th sector and the t_0 year.

4.1.2. The Comparative Advantage Coefficient

The selection of strategic emerging industry should be beneficial in this area's development, and should have a certain comparative advantage in the output value, profit tax rate, and so forth. The relative advantage of ith sector will be shown η_i,

$$\eta_i = \frac{x_i/x}{X_i/X} \cdot \frac{o_i/o}{O_i/O} \cdot \frac{p_i/p}{P_i/P} \cdot \frac{t_i}{T_i} \cdot \frac{q_i}{Q_i}. \tag{4.2}$$

x_i, o_i, p_i and t_i are, respectively, the product value, product output, total factor productivity, and profit tax rate of the ith sector; q_i is the year-end staff's total of i industry departments; x, o, and p are, respectively, total product value, total value of product output, and average productivity of each sector; Q_i is the year-beginning staff's total of i industry departments; X_i, O_i, P_i, and T_i are, respectively, total product value, product output, total factor productivity, and profit tax rate of the i sector; X, O, and P are, respectively, total product value, total value of product output, total profit tax rate. Q is average productivity of all of sectors.

4.2. The Industry Lead Function Index

4.2.1. The Technology Index

Strategic emerging industry has a new technical support, such as the capital fusion, science and technology innovation, and talent cultivation that will focus on the industry. One has

$$\pi_i = \frac{v_i}{\left[y_i(t_n) - y_i(t_0)\right]/(t_n - t_0)} \cdot \frac{x_i}{X_i} \cdot \frac{\omega_i}{y_i}. \tag{4.3}$$

v_i is the improved rate of productivity, $y_i(t_n)$—is the total product value of the ith sector in the ith year, x_i—is total technology personnel of the ith sector, X_i—is total employment personnel of the ith sector, and ω_i is R&D funds.

4.2.2. The Location Entropy

Location entropy coefficient compares the specialized level of area industrial department with the average level of one country. It can evaluate the competitive level of an area strategic emerging industry in the country. One has

$$LQ = \frac{y_i/y}{Y_i/Y}. \tag{4.4}$$

4.3. The Industry Ecosystem Index

4.3.1. The Output Value Rate of Unit Energy Index

The output value rate of unit energy reflects the industry department energy dissipation capacity, and the strategic emerging industry should be the low consumption industry. One has

$$h_i = \frac{y_i}{\sum_{j=1}^{n_2} e_{ij} \times f_{2j}}. \tag{4.5}$$

e_{ij} is the total of j energy utilized by i sector; f_{2j} is the use of the fees of unit j energy; n_2 is the type of energy.

4.3.2. The Output Value Rate of Unit Three-Kinds-Waste Discharge Index

One has

$$g_i = \frac{y_i}{\sum_{j=1}^{3} w_{ij} \times f_{1j}}, \tag{4.6}$$

w_{ij} is the total of j waste outputted by i sector; f_{1j} is the administered fees of unit j waste.

4.4. The Selection Model of the Maximum Entropy

4.4.1. Establish Evaluation Matrix

Definition 4.1. Suppose there is n industry to participate in the selection, notes for $Q = \{Q_1, Q_2, Q_3, \ldots, Q_n\}$, the number of each industry evaluation index is m; notes for $P = \{P_1, P_2, P_3, \ldots, P_m\}$, x_{ij} is the evaluation value of the ith industry and the jth evaluation index. $A = [x_{ij}]_{n\times m}$ is the evaluation matrix of industry set Q for index set P:

$$A = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{vmatrix}. \tag{4.7}$$

Presume standard index:

$$x_{0j} = \frac{1}{n}\sum_{i=1}^{n} x_{ij}. \tag{4.8}$$

Let the jth index $(x_{1j}, x_{2j}, \ldots, x_{nj})$ compare with standard index, determine its corresponding equivalent value $(r_{1j}, r_{2j}, \ldots, r_{nj})$. We can get the corresponding equivalent matrix:

$$R = \begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{vmatrix}. \tag{4.9}$$

4.4.2. Establishment of Multi-Objective Programming (MP) Model

If m evaluation index weight vectors $W = (w_1, w_2, \ldots, w_m)^T$, and the final evaluation value of the ith industry for U_i,

$$U_i = \sum_{j=i}^{m} w_j r_{ij}. \tag{4.10}$$

According to the idea of the maximum entropy and the corresponding equivalent index, we can set up the goal programming equation:

$$\begin{gathered}
(\mathrm{MP}) \max \left\{ -\sum_{i=1}^{n} w_i \ln w_i \right\}, \\
(\mathrm{MP}) \min \sum_{i=1}^{n} f_i(w) = \sum_{i=1}^{n} \sum_{j=1}^{m} w_j \left(1 - r_{ij}\right)^2.
\end{gathered} \tag{4.11}$$

Constraint conditions for

$$\sum_{j=1}^{m} w_j = 1, \quad w_j \geq 0,\ j = 1, 2, \ldots, m, \tag{4.12}$$

$$w_i > w_k, \quad 1 \leq i,\ k \leq m,\ i \neq m. \tag{4.13}$$

For the solution of type (4.10)–(4.12) multi-objective programming function, researchers will structure the following mathematical model as (4.13)

$$\begin{aligned}
\min \quad & \delta \sum_{i=1}^{n} \sum_{j=1}^{m} w_j \left(1 - r_{ij}\right)^2 + (1 - \delta) + \sum_{j=1}^{m} w_j \ln w_j, \\
\text{s.t.} \quad & \sum_{j=1}^{m} w_j = 1,\ w_j \geq 0,\ j = 1, 2, \ldots, m, \\
& 0 \leq \delta \leq 1.
\end{aligned} \tag{4.14}$$

For the solution of type (4.13), we will structure Lagrange function:

$$F(w, \lambda) = \delta \sum_{i=1}^{n} \sum_{j=1}^{m} w_j \left(1 - r_{ij}\right)^2 + (1 - \delta) + \sum_{j=1}^{m} w_j \ln w_j - \lambda \left(\sum_{j=1}^{m} w_j - 1 \right). \tag{4.15}$$

According to a necessary condition for the extreme existence, we can get

$$\begin{gathered}
\frac{\partial F}{\partial w_j} = \delta \sum_{i=1}^{n} \left(1 - r_{ij}\right)^2 + (1 - \delta)\left(\ln w_j + 1\right) - \lambda = 0, \quad j = 1, 2, \ldots, m, \\
\frac{\partial F}{\partial \lambda} = \sum_{j=1}^{m} w_j - 1 = 0.
\end{gathered} \tag{4.16}$$

Next get

$$w_j = \frac{\exp\left\{-\left[1+\delta\sum_{i=1}^{n}\left(1-r_{ij}\right)^2/(1-\delta)\right]\right\}}{\sum_{j=1}^{m}\exp\left\{-\left[1+\delta\sum_{i=1}^{n}\left(1-r_{ij}\right)^2/(1-\delta)\right]\right\}}, \quad j=1,2,\ldots,m. \tag{4.17}$$

Through changing δ values for the many different results of empowerment choose a group result that most conforms to the formula (4.17) of the combination of empowerment, and get w_j, $j = 1,2,\ldots,m$ of the empowerment results of formula (4.12) and (4.13). Then based on the industry's index, find out the industrial appraisal value:

$$U_i = \sum_{j=1}^{m} w_j r_{ij}. \tag{4.18}$$

We will order and choose strategic emerging industry according to the size of the U_i.

5. Conclusions

It is a theoretical and practical significance to make the scientific and accurate selection and evaluation of regional strategic emerging industry. This paper tried to use the fuzzy optimization theory and the maximum entropy to select the central and sustainable development industry of area as the regional strategic emerging industry.

First, the paper depends on the aid of the industrial structure similarity coefficient to measure the area of industrial isomorphism, thus gives system dynamically the judgment of the regional industrial structure convergence change characteristics and determines the regional industrial structure from the macroscopic perspective preliminary.

Secondly, it establishes and optimizes fuzzy evaluation model. The problem in view of regional strategic emerging industry selection is influenced by multiple factors and belongs to multistage comprehensive evaluation problem. We can use the method of fuzzy mathematics comprehensive evaluation, but we also need to optimize and improve multistage fuzzy comprehensive evaluation model, so that the evaluation results will be as accurate and objective as possible.

Thirdly, it evaluates the chosen results. We calculate the quotient of location to find out whether the area industry has advantage in the same industry of country. We will calculate intersection to the two kinds of industry and confirm the area strategic emerging industry.

In the future, we will also do empirical analysis based on these models. In addition, we will be coupled to an evaluation about the traditional industries and the strategic emerging industries, including double coupling relevance and developmental evaluation.

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References

[1] P. R. Brown, V. E. Soybel, and C. P. Stickney, "Achieving comparability of US and Japanese financial statement data," *Japan and The World Economy*, vol. 5, no. 1, pp. 51–72, 1993.

[2] C. Perez, "The double bubble at the turn of the century: technological roots and structural implications," *Cambridge Journal of Economics*, vol. 33, no. 4, pp. 779–805, 2009.

[3] H. Liu, "China's emerging industry strategic principle of selecting and cultivating policy orientation study," *Public Understanding of Science*, vol. 3, pp. 87–92, 2011.

[4] Y. Hu and J. Zhao, "The application of factor analysis to the choice of strategic new industry," *Journal of Social Science Series*, vol. 6, pp. 127–129, 2010.

[5] Z. He, R. W. kurz, Geissler k, and Z. Zhou, "Strategic emerging industry choice and evaluation and empirical analysis," *Public Understanding of Science*, no. 12, pp. 62–67, 2010.

[6] H. Li, "Regional strategic emerging industry selection evaluation," *Journal of North China Water Conservancy and Hydropower College*, no. 4, pp. 91–93, 2011.

[7] D.-S. Gong, ". Strategic emerging industries, the establishment of the evaluation index system based on the analysis of the industrial characteristics and thinking," *Journal of City Economy*, vol. 1, pp. 7–9, 2012.

[8] S. Qing, "AHP method in Henan province strategic emerging industry evaluation and selection of decision analysis," *Journal of Henan College of Engineering*, no. 1, pp. 45–48, 2011.

[9] Z. Qiao, H. Chu, and Y. Wu, "Based on the grey relation analysis of strategic emerging industry evaluation—the biological medicine as an example," *Journal of Mathematical Economics*, no. 3, pp. 62–67, 2010.

[10] Z. Hu, C. Li, and Y. Xiong, "Based on the AHP-IE-PCA, 'combination weighting method of strategic emerging industry selection model'," *Public Understanding of Science*, no. 7, pp. 104–110, 2011.

[11] V. Pareto, *Manuale Di Economic Politiea*, Societ a Editrice Linraria, Milano, Italy, 1906.

[12] V. Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princetion University Press, NJ, USA, 1944.

[13] T. C. Koopmans, "Analysis of production as an efficient combination of activities," in *Activity Analysis of Production and Alloeation*, T. C. KooPmans, Ed., vol. 13, pp. 33–97, Cowles Commission Monograph, John Wiley & Sons, New York, NY, USA, 1951.

[14] H. W. Kuhn and A. W. Tucker, "Nonlinear analysis," in *Proceeding of the 2d Berley Symposium on Mathematical Statistics and Probability*, pp. 481–492, University of California Press, Calif, USA.

[15] G. Debreu, "Representation of a preference ordering by numerical function," in *Decision Processes*, R. M. Thrall, C. H. Coombs, and R. L. Davis, Eds., pp. 159–165, John Wiley & Sons, New York, NY, USA, 1954.

[16] L. Harwicz, "Programming in linear spaces," in *Studies in Linear and Nonlinear Programming*, K. J. Arrow, L. Harwicz, and H. Uzawa, Eds., pp. 38–102, Standford University Press, Calif, USA, 1958.

[17] Z. Johnsen, *Studies in Multi Objective Decision Models*, Economic Research Center in Lund, Lund, Sweden, 1968.

[18] R. E. Bellman and L. A. Zadeh, "Decision-making in a fuzzy environment," *Management Science*, vol. 17, pp. B141–B164, 1970.

[19] Z. Wang and H. Wang, "Interregional industry transfer the power of the formation," *Journal of Dalian University of Technology*, no. 1, pp. 22–26, 2007.

[20] T. Zhang and J. Li, "Application of multi-step fuzzy comprehensive evaluation," *Journal of Harbin Engineering University*, vol. 3, pp. 132–135, 2002.

[21] L. Shi, M. Ju, Z. Li, and M. Chen, "Application of regional environmental impact assessment with fuzzy mathematics method," http://www.paper.edu.cn/.

[22] Y. Li, Z. Gao, and Y. Han, "The determination of weight value and the choice of composite operators in fuzzy comprehensive evaluation," *Computer Engineering and Applications*, vol. 23, pp. 38–43, 2006.

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Research Article

Strong Convergence Theorems for Modifying Halpern Iterations for Quasi-ϕ-Asymptotically Nonexpansive Multivalued Mapping in Banach Spaces with Applications

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An iterative sequence for quasi-ϕ-asymptotically nonexpansive multivalued mapping for modifying Halpern's iterations is introduced. Under suitable conditions, some strong convergence theorems are proved. The results presented in the paper improve and extend the corresponding results in the work by Chang et al. 2011.

1. Introduction

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively. Let D be a nonempty closed subset of a real Banach space X. A mapping $T : D \to D$ is said to be *nonexpansive*, if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in D$. Let $N(D)$ and $\mathrm{CB}(D)$ denote the family of nonempty subsets and nonempty closed bounded subsets of D, respectively. The Hausdorff metric on $\mathrm{CB}(D)$ is defined by

$$H(A_1, A_2) = \max\left\{\sup_{x\in A_1} d(x, A_2), \sup_{y\in A_2} d(y, A_1)\right\} \tag{1.1}$$

for A_1, $A_2 \in \mathrm{CB}(D)$, where $d(x, A_1) = \inf\{\|x - y\|, y \in A_1\}$. The multivalued mapping $T : D \to \mathrm{CB}(D)$ is called *nonexpansive*, if $H(Tx, Ty) \le \|x - y\|$, for all $x, y \in D$. An element $p \in D$ is called a *fixed point* of $T : D \to N(D)$, if $p \in T(p)$. The set of fixed points of T is represented by $F(T)$.

Let X be a real Banach space with dual X^*. We denote by J the normalized duality mapping from X to 2^{X^*} which is defined by

$$J(x) = \left\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad x \in X, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A Banach space X is said to be *strictly convex*, if $\|(x+y)/2\| \leq 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space is said to be *uniformly convex*, if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\} \subset X$ with $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 0$.

The norm of Banach space X is said to be *Gâteaux differentiable*, if for each $x, y \in S(x)$, the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{1.3}$$

exists, where $S(x) = \{x : \|x\| = 1, x \in X\}$. In this case, X is said to be *smooth*. The norm of Banach space X is said to be *Fréchet differentiable*, if for each $x \in S(x)$, the limit (1.3) is attained *uniformly*, for $y \in S(x)$, and the norm is uniformly Fréchet differentiable if the limit (1.3) is attained uniformly for $x, y \in S(x)$. In this case, X is said to be *uniformly smooth*.

Remark 1.1. The following basic properties for Banach space X and for the normalized duality mapping J can be found in Cioranescu [1].

(1) X (X^*, resp.) is uniformly convex if and only if X^* (X, resp.) is uniformly smooth.

(2) If X is smooth, then J is single-valued and norm-to-weak* continuous.

(3) If X is reflexive, then J is onto.

(4) If X is strictly convex, then $Jx \bigcap Jy \neq \emptyset$, for all $x, y \in X$.

(5) If X has a Fréchet differentiable norm, then J is norm-to-norm continuous.

(6) If X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X.

(7) Each uniformly convex Banach space X has the Kadec-Klee property, that is, for any sequence $\{x_n\} \subset X$, if $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x \in X$.

(8) If X is a reflexive and strictly convex Banach space with a strictly convex dual X^* and $J^* : X^* \to X$ is the normalized duality mapping in X^*, then $J^{-1} = J^*$, $JJ^* = I_{X^*}$ and $J^*J = I_X$.

Next, we assume that X is a smooth, strictly convex, and reflexive Banach space and D is a nonempty, closed and convex subset of X. In the sequel, we always use $\phi : X \times X \to R^+$ to denote the *Lyapunov functional* defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad x, y \in X. \tag{1.4}$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2,$$
$$\phi(y,x) = \phi(y,z) + \phi(z,x) + 2\langle z-y, Jx - Jz\rangle, \quad x,y,z \in X, \tag{1.5}$$

$$\phi\left(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\right) \le \lambda\phi(x,y) + (1-\lambda)\phi(x,z), \tag{1.6}$$

for all $\lambda \in [0,1]$ and $x,y,z \in X$.

Following Alber [2], the generalized projection $\Pi_D : X \to D$ is defined by

$$\Pi_D(x) = \arg\inf_{y\in D}\phi(y,x), \quad \forall x \in X. \tag{1.7}$$

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping.

Example 1.2 (see [3]). Let Π_D be the generalized projection from a smooth, reflexive and strictly convex Banach space X onto a nonempty closed convex subset D of X, then Π_D is a closed and quasi-ϕ-nonexpansive from X onto D.

In 1953, Mann [4] introduced the following iterative sequence $\{x_n\}$:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \tag{1.8}$$

where the initial guess $x_1 \in D$ is arbitrary, and $\{\alpha_n\}$ is a real sequence in $[0,1]$. It is known that under appropriate settings the sequence $\{x_n\}$ converges weakly to a fixed point of T. However, even in a Hilbert space, Mann iteration may fail to converge strongly [5]. Some attempts to construct iteration method guaranteeing the strong convergence have been made. For example, Halpern [6] proposed the following so-called Halpern iteration:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \tag{1.9}$$

where u, $x_1 \in D$ are arbitrary given and $\{\alpha_n\}$ is a real sequence in $[0,1]$. Another approach was proposed by Nakajo and Takahashi [7]. They generated a sequence as follows:

$$\begin{aligned} & x_1 \in X \text{ is arbitrary}, \\ & y_n = \alpha_n u + (1-\alpha_n)Tx_n, \\ & C_n = \{z \in D : \|y_n - z\| \le \|x_n - z\|\}, \\ & Q_n = \{z \in D : \langle x_n - z, x_1 - x_n\rangle \ge 0\}, \\ & x_{n+1} = P_{C_n \cap Q_n} x_1 \quad (n = 1,2,\ldots), \end{aligned} \tag{1.10}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ and P_K denotes the metric projection from a Hilbert space H onto a closed and convex subset K of H. It should be noted here that the iteration previous works only in Hilbert space setting. To extend this iteration to a Banach space, the concept of relatively nonexpansive mappings are introduced (see [8–12]).

Inspired by Matsushita and Takahashi, in this paper, we introduce modifying Halpern-Mann iterations sequence for finding a fixed point of multivalued mapping $T : D \to \mathrm{CB}(D)$.

2. Preliminaries

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Lemma 2.1 (see [2]). *Let X be a smooth, strictly convex, and reflexive Banach space, and let D be a nonempty closed and convex subset of X. Then the following conclusions hold*

(a) $\phi(x,y) = 0$ *if and only if* $x = y$, *for all* $x, y \in X$;

(b) $\phi(x,\Pi_D y) + \phi(\Pi_D y, y) \leq \phi(x,y)$, *for all* $x \in D$, *for all* $y \in X$;

(c) *if* $x \in X$ *and* $z \in D$, *then* $z = \Pi_D x \Leftrightarrow \langle z - y, Jx - Jz\rangle \geq 0$, *for all* $y \in D$.

Remark 2.2. If H is a real Hilbert space, then $\phi(x,y) = \|x-y\|^2$ and Π_D is the metric projection P_D of H onto D.

Definition 2.3. A point $p \in D$ is said to be an asymptotic fixed point of $T : D \to \mathrm{CB}(D)$, if there exists a sequence $\{x_n\} \subset D$ such that $x_n \rightharpoonup x \in X$ and $d(x_n, T(x_n)) \to 0$. Denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$.

Definition 2.4. (1) A multivalued mapping $T : D \to \mathrm{CB}(D)$ is said to be relatively nonexpansive, if $F(T) \neq \emptyset$, $\widehat{F}(T) = F(T)$, and $\phi(p,z) \leq \phi(p,x)$, for all $x \in D$, $p \in F(T)$, $z \in T(x)$.

(2) A multivalued mapping $T : D \to \mathrm{CB}(D)$ is said to be closed, if for any sequence $\{x_n\} \subset D$ with $x_n \to x \in D$ and $d(y, T(x_n)) \to 0$, then $d(y, T(x)) = 0$.

Next, we present an example of relatively nonexpansive multivalued mapping.

Example 2.5 (see [13]). Let X be a smooth, strictly convex, and reflexive Banach space, let D be a nonempty closed and convex subset of X, and let $f : D \times D \to R$ be a bifunction satisfying the conditions: (A1) $f(x,x) = 0$, for all $x \in D$; (A2) $f(x,y) + f(y,x) \leq 0$, for all $x, y \in D$; (A3) $\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x,y)$, for each $x, y, z \in D$; (A4) the function $y \mapsto f(x,y)$ is convex and lower semicontinuous, for each $x \in D$. The "so-called" equilibrium problem for f is to find a $x^* \in D$ such that $f(x^*, y) \geq 0$, for all $y \in D$. The set of its solutions is denoted by $\mathrm{EP}(f)$.

Let $r > 0$, $x \in X$ and define mapping $T_r : X \to D$ as follows:

$$T_r(x) = \left\{ x \in D, f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in X, \tag{2.1}$$

then (1) T_r is single-valued, and so $\{z\} = T_r(x)$; (2) T_r is a relatively nonexpansive mapping, therefore it is a closed and quasi-ϕ-nonexpansive mapping; (3) $F(T_r) = \mathrm{EP}(f)$.

Definition 2.6. (1) A multivalued mapping $T : D \to \mathrm{CB}(D)$ is said to be quasi-ϕ-nonexpansive, if $F(T) \neq \emptyset$, and $\phi(p,z) \leq \phi(p,x)$, for all $x \in D$, $p \in F(T)$, $z \in T(x)$.

(2) A multivalued mapping $T : D \to \mathrm{CB}(D)$ is said to be quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \emptyset$, and there exists a real sequence $k_n \subset [1, +\infty)$, $k_n \to 1$ such that

$$\phi(p, z_n) \leq k_n \phi(p, x), \quad \forall x \in D,\ p \in F(T),\ z_n \in T^n(x). \tag{2.2}$$

(3) A multivalued mapping $T : D \to \mathrm{CB}(D)$ is said to be totally quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \emptyset$, and there exist nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ with v_n, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that

$$\phi(p, z_n) \leq \phi(p, x) + v_n \zeta[\phi(p, x)] + \mu_n, \quad \forall x \in D,\ \forall n \geq 1,\ p \in F(T),\ z_n \in T^n(x). \tag{2.3}$$

Remark 2.7. From the definitions, it is obvious that a relatively nonexpansive multivalued mapping is a quasi-ϕ-nonexpansive multivalued mapping, and a quasi-ϕ-nonexpansive multivalued mapping is a quasi-ϕ-asymptotically nonexpansive multivalued mapping, but the converse is not true.

Lemma 2.8. *Let X be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and let D be a nonempty closed and convex subset of X. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in D such that $x_n \to p$ and $\phi(x_n, y_n) \to 0$, where ϕ is the function defined by* (1.4)*, then $y_n \to p$.*

Proof. For $\phi(x_n, y_n) \to 0$, we have $(\|x_n\| - \|y_n\|)^2 \to 0$. This implies that $\|y_n\| \to \|p\|$ and so $\|Jy_n\| \to \|Jp\|$. Since D is uniformly smooth, X^* is reflexive and $JX = X^*$, therefore, there exist a subsequence $\{Jy_{n_i}\} \subset \{Jy_n\}$ and a point $x \in X$ such that $Jy_{n_i} \rightharpoonup Jx$. Because the norm $\|\cdot\|$ is weakly lower semi continuous, we have

$$\begin{aligned} 0 &= \lim_{n_i \to \infty} \phi(x_{n_i}, y_{n_i}) = \lim_{n_i \to \infty} \left\{ \|x_{n_i}\|^2 - 2\langle x_{n_i}, Jy_{n_i} \rangle + \left\|Jy_{n_i}\right\|^2 \right\} \\ &\geq \left\|p\right\|^2 - 2\langle p, Jx \rangle + \|Jx\|^2 = \phi(p, x). \end{aligned} \tag{2.4}$$

By Lemma 2.1(a), we have $p = x$. Hence we have $Jy_{n_i} \rightharpoonup Jp$. Since $\|Jy_{n_i}\| \to \|Jp\|$ and X^* has the Kadec-Klee property, we have $Jy_{n_i} \to Jp$. By Remark 1.1, it follows that $y_{n_i} \rightharpoonup p$. Since $\|Jy_{n_i}\| \to \|Jp\|$, by using the Kadec-Klee property of X, we get $y_{n_i} \to p$. If there exists another subsequence $\{Jy_{n_j}\} \subset \{Jy_n\}$ such that $y_{n_j} \to q$, then we have

$$\begin{aligned} 0 &= \lim_{n_j \to \infty} \phi\left(x_{n_j}, y_{n_j}\right) = \lim_{n_j \to \infty} \left\{ \left\|x_{n_j}\right\|^2 - 2\left\langle x_{n_j}, Jy_{n_j} \right\rangle + \left\|Jy_{n_j}\right\|^2 \right\} \\ &= \left\|p\right\|^2 - 2\langle p, Jq \rangle + \left\|q\right\|^2 = \phi(p, q). \end{aligned} \tag{2.5}$$

This implies that $p = q$. So $y_n \to p$. The conclusion of Lemma 2.8 is proved. □

Lemma 2.9. *Let X and D be as in Lemma 2.8. Let $T : D \to CB(D)$ be a closed and quasi-ϕ-asymptotically nonexpansive multivalued mapping with nonnegative real sequences $\{k_n\} \subset [1, +\infty)$, if $k_n \to 1$, then the fixed point set $F(T)$ of T is a closed and convex subset of D.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$, such that $x_n \to x^*$. Since T is quasi-ϕ-asymptotically nonexpansive multivalued mapping, we have

$$\phi(x_n, z) \leq k_1 \phi(x_n, x^*) \tag{2.6}$$

for all $z \in Tx^*$ and for all $n \in N$. Therefore,

$$\phi(x^*, z) = \lim_{n \to \infty} \phi(x_n, z) \leq \lim_{n \to \infty} k_1 \phi(x_n, x^*) = k_1 \phi(x^*, x^*) = 0. \tag{2.7}$$

By Lemma 2.1, we obtain $z = x^*$, Hence, $Tx^* = \{x^*\}$. So, we have $x^* \in F(T)$. This implies that $F(T)$ is closed.

Let $p, q \in F(T)$ and $t \in (0,1)$, and put $w = tp + (1-t)q$. we prove that $w \in F(T)$. Indeed, in view of the definition of ϕ, let $z_n \in T^n w$, we have

$$\begin{aligned}
\phi(w, z_n) &= \|w\|^2 - 2\langle w, Jz_n \rangle + \|z_n\|^2 \\
&= \|w\|^2 - 2\langle tp + (1-t)q, Jz_n \rangle + \|z_n\|^2 \\
&= \|w\|^2 + t\phi(p, z_n) + (1-t)\phi(q, z_n) - t\|p\|^2 - (1-t)\|q\|^2.
\end{aligned} \tag{2.8}$$

Since

$$\begin{aligned}
&t\phi(p, z_n) + (1-t)\phi(q, z_n) \\
&\quad \leq tk_n \phi(p, w) + (1-t)k_n \phi(q, w) \\
&\quad = t\left\{ \|p\|^2 - 2\langle p, Jw \rangle + \|w\|^2 + (k_n - 1)\phi(p, w) \right\} \\
&\qquad + (1-t)\left\{ \|q\|^2 - 2\langle q, Jw \rangle + \|w\|^2 + (k_n - 1)\phi(q, w) \right\} \\
&\quad = t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2 + t(k_n - 1)\phi(p, w) + (1-t)(k_n - 1)\phi(q, w).
\end{aligned} \tag{2.9}$$

Substituting (2.8) into (2.9) and simplifying it, we have

$$\phi(w, z_n) \leq t(k_n - 1)\phi(p, w) + (1-t)(k_n - 1)\phi(q, w) \longrightarrow 0, \quad (\text{as } n \longrightarrow \infty). \tag{2.10}$$

Hence, we have $z_n \to w$. This implies that $z_{n+1} (\in TT^n w) \to w$. Since T is closed, we have $Tw = \{w\}$, that is, $w \in F(T)$. This completes the proof of Lemma 2.9. □

Definition 2.10. A mapping $T : D \to \mathrm{CB}(D)$ is said to be uniformly L-Lipschitz continuous, if there exists a constant $L > 0$ such that $\|x_n - y_n\| \leq L\|x - y\|$, where $x, y \in D$, $x_n \in T^n x$, $y_n \in T^n y$.

3. Main Results

Theorem 3.1. *Let X be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, let D be a nonempty, closed and convex subset of X, and let $T : D \to CB(D)$ be a closed and* *uniformly L-Lipschitz continuous quasi-ϕ-asymptotically nonexpansive multivalued mapping with nonnegative real sequences $\{k_n\} \subset [1, +\infty)$ and $k_n \to 1$ satisfying condition* (2.2). *Let $\{\alpha_n\}$ be a sequence in* $(0, 1)$. *If $\{x_n\}$ is the sequence generated by*

$$\begin{gathered} x_1 \in X \quad \text{is arbitrary;} \quad D_1 = D, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J z_n], \quad z_n \in T^n x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{gathered} \tag{3.1}$$

where $\xi_n = (k_n - 1)\sup_{p \in F(T)} \phi(p, x_n)$, $F(T)$ is the fixed point set of T, and $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.

Proof. (I) *First, we prove that D_n are closed and convex subsets in D.* By the assumption that $D_1 = D$ is closed and convex. Suppose that D_n is closed and convex for some $n \ge 1$. In view of the definition of ϕ, we have

$$\begin{aligned} D_{n+1} &= \{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\} \\ &= \{z \in D : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\} \cap D_n \\ &= \Big\{z \in D : 2\alpha_n \langle z, J x_1\rangle + 2(1-\alpha_n)\langle z, J x_n\rangle - 2\langle z, J z_n\rangle \\ &\qquad \le \alpha_n \|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|z_n\|^2 \Big\} \cap D_n. \end{aligned} \tag{3.2}$$

This shows that D_{n+1} is closed and convex. The conclusions are proved.

(II) *Next, we prove that $F(T) \subset D_n$, for all $n \ge 1$.* In fact, it is obvious that $F(T) \subset D_1$. Suppose $F(T) \subset D_n$, for some $n \ge 1$. Hence, for any $u \in F(T) \subset D_n$, by (1.6), we have

$$\begin{aligned} \phi(u, y_n) &= \phi\Big(u, J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J z_n)\Big) \\ &\le \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, z_n) \\ &\le \alpha_n \phi(u, x_1) + (1-\alpha_n) k_n \phi(u, x_n) \\ &= \alpha_n \phi(u, x_1) + (1-\alpha_n)\{\phi(u, x_n) + (k_n - 1)\phi(u, x_n)\} \\ &\le \alpha_n \phi(u, x_1) + (1-\alpha_n)\Big\{\phi(u, x_n) + (k_n - 1)\sup_{u \in F(T)} \phi(u, x_n)\Big\} \\ &= \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, x_n) + \xi_n. \end{aligned} \tag{3.3}$$

This shows that $u \in F(T) \subset D_{n+1}$ and so $F(T) \subset D_n$.

(III) *Now, we prove that $\{x_n\}$ converges strongly to some point p^*.* In fact, since $x_n = \Pi_{D_n} x_1$, from Lemma 2.1(c), we have

$$\langle x_n - y, J x_1 - J x_n\rangle \ge 0, \quad \forall y \in D_n. \tag{3.4}$$

Again since $F(T) \subset D_n$, we have

$$\langle x_n - u, Jx_1 - Jx_n \rangle \geq 0, \quad \forall u \in F(T). \tag{3.5}$$

It follows from Lemma 2.1(b) that for each $u \in F(T)$ and for each $n \geq 1$,

$$\phi(x_n, x_1) = \phi(\Pi_{D_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1). \tag{3.6}$$

Therefore, $\{\phi(x_n, x_1)\}$ is bounded, and so is $\{x_n\}$. Since $x_n = \Pi_{D_n} x_1$ and $x_{n+1} = \Pi_{D_{n+1}} x_1 \in D_{n+1} \subset D_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. Since X is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^*$ (some point in $D = D_1$). Since D_n is closed and convex and $D_{n+1} \subset D_n$. This implies that D_n is weakly closed and $p^* \in D_n$ for each $n \geq 1$. In view of $x_{n_i} = \Pi_{D_{n_i}} x_1$, we have

$$\phi(x_{n_i}, x_1) \leq \phi(p^*, x_1), \quad \forall n_i \geq 1. \tag{3.7}$$

Since the norm $\|\cdot\|$ is weakly lower semicontinuous, we have

$$\begin{aligned}
\liminf_{n_i\to\infty} \phi(x_n, x_1) &= \liminf_{n_i\to\infty} \left(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_1\rangle + \|x_1\|^2 \right) \\
&\geq \|p^*\|^2 - 2\langle p^*, Jx_1\rangle + \|x_1\|^2 \\
&= \phi(p^*, x_1),
\end{aligned} \tag{3.8}$$

and so

$$\phi(p^*, x_1) \leq \liminf_{n_i\to\infty} \phi(x_n, x_1) \leq \limsup_{n_i\to\infty} \phi(x_n, x_1) = \phi(p^*, x_1). \tag{3.9}$$

This shows that $\lim_{n_i\to\infty} \phi(x_{n_i}, x_1) = \phi(p^*, x_1)$, and we have $\|x_{n_i}\| \to \|p^*\|$. Since $x_{n_i} \rightharpoonup p^*$, by the virtue of Kadec-Klee property of X, we obtain that $x_{n_i} \to p^*$. Since $\{\phi(x_n, x_1)\}$ is convergent, this together with $\lim_{n_i\to\infty} \phi(x_{n_i}, x_1) = \phi(p^*, x_1)$ shows that $\lim_{n_i\to\infty} \phi(x_n, x_1) = \phi(p^*, x_1)$. If there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_n \to q$, then from Lemma 2.1, we have

$$\begin{aligned}
\phi(p^*, q) &= \lim_{n_i, n_j\to\infty} \phi\left(x_{n_i}, x_{n_j}\right) = \lim_{n_i, n_j\to\infty} \phi\left(x_{n_i}, \Pi_{D_{n_j}} x_1\right) \\
&\leq \lim_{n_i, n_j\to\infty} \left[\phi(x_{n_i}, x_1) - \phi\left(\Pi_{D_{n_j}} x_1, x_1\right)\right] = \lim_{n_i, n_j\to\infty} \left[\phi(x_{n_i}, x_1) - \phi\left(x_{n_j}, x_1\right)\right] \\
&= \phi(p^*, x_1) - \phi(p^*, x_1) = 0,
\end{aligned} \tag{3.10}$$

that is, $p^* = q$ and hence

$$x_n \longrightarrow p^*. \tag{3.11}$$

By the way, from (3.11), it is easy to see that

$$\xi_n = (k_n - 1) \sup_{p \in F(T)} \phi(p, x_n) \longrightarrow 0. \tag{3.12}$$

(IV) *Now, we prove that* $p^* \in F(T)$. In fact, since $x_{n+1} \in D_{n+1}$, from (3.1), (3.11), and (3.12), we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n)\phi(x_{n+1}, x_n) + \xi_n \longrightarrow 0. \tag{3.13}$$

Since $x_n \to p^*$, it follows from (3.13) and Lemma 2.8 that

$$y_n \longrightarrow p^*. \tag{3.14}$$

Since $\{x_n\}$ is bounded and T is quasi-ϕ-asymptotically nonexpansive multivalued mapping, $T^n x_n$ is bounded. In view of $\alpha_n \to 0$. Hence from (3.1), we have that

$$\lim_{n \to \infty} \| J y_n - J z_n \| = \lim_{n \to \infty} \| J x_1 - J z_n \| = 0. \tag{3.15}$$

Since $J y_n \to J p^*$, this implies $J z_n \to J p^*$. From Remark 1.1, it yields that

$$z_n \rightharpoonup p^*. \tag{3.16}$$

Again since

$$\|z_n\| - \|p^*\| = \|J z_n\| - \|J p^*\| \leq \|J z_n - J p^*\| \longrightarrow 0, \tag{3.17}$$

this together with (3.16) and the Kadec-Klee-property of X shows that

$$z_n \longrightarrow p^*. \tag{3.18}$$

On the other hand, by the assumptions that T is L-Lipschitz continuous, thus we have

$$\begin{aligned} d(T z_n, z_n) &\leq d(T z_n, z_{n+1}) + \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \\ &\leq (L+1)\|x_{n+1} - x_n\| + \|z_{n+1} - x_{n+1}\| + \|x_n - z_n\|. \end{aligned} \tag{3.19}$$

From (3.18) and $x_n \to p^*$, we have that $d(T z_n, z_n) \to 0$. In view of the closeness of T, it yields that $T(p^*) = \{p^*\}$, this implies that $p^* \in F(T)$.

(V) *Finally, we prove that* $p^* = \Pi_{F(T)} x_1$ *and so* $x_n \to \Pi_{F(T)} x_1$. Let $w = \Pi_{F(T)} x_1$. Since $w \in F(T) \subset D_n$, we have $\phi(p^*, x_1) \leq \phi(w, x_1)$. This implies that

$$\phi(p^*, x_1) = \lim_{n \to \infty} \phi(x_n, x_1) \leq \phi(w, x_1), \tag{3.20}$$

which yields that $p^* = w = \Pi_{F(T)}x_1$. Therefore, $x_n \to \Pi_{F(T)}x_1$. This completes the proof of Theorem 3.1. □

By Remark 2.7, the following corollaries are obtained.

Corollary 3.2. *Let X and D be as in Theorem 3.1, and let $T : D \to CB(D)$ be a closed and uniformly L-Lipschitz continuous a relatively nonexpansive multivalued mapping. Let $\{\alpha_n\}$ in $(0,1)$ with $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered} x_1 \in X \quad \text{is arbitrary;} \quad D_1 = D, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J z_n], \quad z_n \in T x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{gathered} \tag{3.21}$$

where $F(T)$ is the set of fixed points of T, and $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.

Corollary 3.3. *Let X and D be as in Theorem 3.1, and let $T : D \to CB(D)$ be a closed and uniformly L-Lipschitz continuous quasi-ϕ-nonexpansive multivalued mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $\alpha_n \in (0,1)$ for all $n \in N$, and satisfying: $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be the sequence generated by* (3.21). *Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.*

4. Application

We utilize Corollary 3.3 to study a modified Halpern iterative algorithm for a system of equilibrium problems.

Theorem 4.1. *Let D, X, and $\{\alpha_n\}$ be the same as in Theorem 3.1. Let $f : D \times D \to R$ be a bifunction satisfying conditions (A1)–(A4) as given in Example 2.5. Let $T_r : X \to D$ be a mapping defined by* (2.1), *that is,*

$$T_r(x) = \left\{x \in D, f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \forall y \in D\right\}, \quad \forall x \in X. \tag{4.1}$$

Let $\{x_n\}$ be the sequence generated by

$$\begin{gathered} x_1 \in X \quad \text{is arbitrary;} \quad D_1 = D, \\ f(u_n, y) + \frac{1}{r}\langle y - u_n, Ju_n - Jx_n\rangle \ge 0, \quad \forall y \in D,\ r > 0,\ u_n \in T_r x_n, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J u_n], \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots). \end{gathered} \tag{4.2}$$

If $F(T_r) \neq \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$ which is a common solution of the system of equilibrium problems for f.

Proof. In Example 2.5, we have pointed out that $u_n = T_r(x_n)$, $F(T_r) = \mathrm{EP}(f)$, and T_r is a closed and quasi-ϕ-nonexpansive mapping. Hence (4.2) can be rewritten as follows:

$$\begin{gathered} x_1 \in X \quad \text{is arbitrary;} \quad D_1 = D, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J u_n], \quad u_n \in T_r x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots). \end{gathered} \tag{4.3}$$

Therefore the conclusion of Theorem 4.1 can be obtained from Corollary 3.3. □

References

[1] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[2] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. G. Kartosator, Ed., vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Marcel Dekker, New York, NY, USA, 1996.

[3] S. S. Chang, C. K. Chan, and H. W. J. Lee, "Modified block iterative algorithm for quasi-ϕ-asymptotically nonexpansive mappings and equilibrium problem in Banach spaces," *Applied Mathematics and Computation*, vol. 217, no. 18, pp. 7520–7530, 2011.

[4] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[5] A. Genel and J. Lindenstrauss, "An example concerning fixed points," *Israel Journal of Mathematics*, vol. 22, no. 1, pp. 81–86, 1975.

[6] B. Halpern, "Fixed points of nonexpansive maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.

[7] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[8] S. Y. Matsushita and W. Takahashi, "Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2004, no. 1, pp. 37–47, 2004.

[9] S. Matsushita and W. Takahashi, "An iterative algorithm for relatively nonexpansive mappings by hybrid method and applications," in *Proceedings of the 3rd International Conference on Nonlinear Analysis and Convex Analysis*, pp. 305–313, 2004.

[10] S.-y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.

[11] X. Qin, Y. J. Cho, S. M. Kang, and H. Zhou, "Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings," *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 22, no. 7, pp. 1051–1055, 2009.

[12] Z. Wang, Y. Su, D. Wang, and Y. Dong, "A modified Halpern-type iteration algorithm for a family of hemi-relatively nonexpansive mappings and systems of equilibrium problems in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2364–2371, 2011.

[13] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

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Research Article

A Class of G-Semipreinvex Functions and Optimality

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A class of G-semipreinvex functions, which are some generalizations of the semipreinvex functions, and the G-convex functions, is introduced. Examples are given to show their relations among G-semipreinvex functions, semipreinvex functions and G-convex functions. Some characterizations of G-semipreinvex functions are also obtained, and some optimality results are given for a class of G-semipreinvex functions. Ours results improve and generalize some known results.

1. Introduction

Generalized convexity has been playing a central role in mathematical programming and optimization theory. The research on characterizations of generalized convexity is one of most important parts in mathematical programming and optimization theory. Many papers have been published to study the problems of how to weaken the convex condition to guarantee the optimality results. Schaible and Ziemba [1] introduced G-convex function which is a generalization of convex function and studied some characterizations of G-convex functions. Hanson [2] introduced invexity which is an extension of differentiable convex function. Ben-Israel and Mond [3] considered the functions for which there exists $\eta : R^n \times R^n \rightarrow R^n$ such that, for any $x, y \in R^n$, $\lambda \in [0,1]$,

$$f(y + \lambda\eta(x,y)) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1.1}$$

Weir et al. [4, 5] named such kinds of functions which satisfied the condition (1.1) as preinvex functions with respect to η. Further study on characterizations and generalizations of convexity and preinvexity, including their applications in mathematical programming, has been done by many authors (see [6–18]). As a generalization of preinvexity, Yang and Chen [15] introduced semipreinvex functions and discussed the applications in prevariational inequality. Yang et al. [16] investigated some properties of semipreinvex functions. As a generalization of G-convex functions and preinvex functions, Antczak [17] introduced G-preinvex functions and obtained some optimality results for a class of constrained optimization problems. As a generalization of B-vexity and semipreinvexity, Long and Peng [18] introduced the concept of semi-B-preinvex functions. Zhao et al. [19] introduced r-semipreinvex functions and established some optimality results for a class of nonlinear programming problems.

Motivated by the results in [17–19], in this paper, we propose the concept of G-semipreinvex functions and obtain some important characterizations of G-semipreinvexity. At the same time, we study some optimality results under G-semipreinvexity. Our results unify the concepts of G-convexity, preinvexity, G-preinvexity, semipreinvexity, and r-semipreinvexity.

2. Preliminaries and Definitions

Definition 2.1 (see [1]). Let G be a continuous real-valued strictly monotonic function defined on $D \subset R$. A real-valued function f defined on a convex set $X \subset R^n$ is said to be G-convex if for any $x, y \in X$, $\lambda \in [0,1]$,

$$f(y + \lambda(x - y)) \leq G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))), \tag{2.1}$$

where G^{-1} is the inverse of G, $f(X) \subset D$.

Remark 2.2. Every convex functions is G-convex, but the converse is not necessarily true.

Example 2.3. Let $X = [-1,1]$, $f : X \to R$, $I_f(X)$ be the range of real-valued function f, and let $G : I_f(X) \to R$ be defined by

$$f(x) = \arctan(|x| + 1), \qquad G(t) = \tan(t). \tag{2.2}$$

Then, we can verify that f is a G-convex function. But f is not a convex function because the following inequality

$$f(y + \lambda(x - y)) > \lambda f(x) + (1 - \lambda) f(y) \tag{2.3}$$

holds for $x = 1/4$, $y = 3/4$, and $\lambda = 1/2$.

Weir et al. [4, 5] presented the concepts of invex sets and preinvex functions as follows.

Definition 2.4 (see [4, 5]). A set $X \subseteq R^n$ is said to be invex if there exists a vector-valued function $\eta : X \times X \to R^n$ such that for any $x, y \in X$, $\lambda \in [0,1]$,

$$y + \lambda\eta(x, y) \in X. \tag{2.4}$$

Definition 2.5 (see [4, 5]). Let $X \subseteq R^n$ be invex with respect to vector-valued function $\eta : X \times X \to R^n$. Function $f(x)$ is said to be preinvex with respect to η if for any $x, y \in X$, $\lambda \in [0,1]$,

$$f(y + \lambda\eta(x,y)) \le \lambda f(x) + (1-\lambda) f(y). \tag{2.5}$$

Remark 2.6. Every convex function is a preinvex function with respect to $\eta = x - y$, but the converse is not necessarily true.

Example 2.7. Let $X = [-1,1]$. $f : X \to R$ be defined by

$$f(x) = \arctan(|x| + 1). \tag{2.6}$$

Then, we can verify that f is a preinvex function with respect to η, where

$$\eta(x,y) = \begin{cases} -y - x^2 + 2x, & 0 \le x \le 1,\ 0 \le y \le 1, \\ -y - x, & -1 \le x < 0,\ 0 \le y \le 1, \\ -y - x, & 0 \le x \le 1,\ -1 \le y < 0, \\ -y + x, & -1 \le x < 0,\ -1 \le y < 0. \end{cases} \tag{2.7}$$

But f is not convex a function in Example 2.3.

Antczak [17] introduced the concept of G-preinvex functions as follows.

Definition 2.8 (see [17]). Let X be a nonempty invex (with respect to η) subset of R^n. A function $f : X \to R$ is said to be (strictly) G-preinvex at y with respect to η if there exists a continuous real-valued increasing function $G : I_f(X) \to R$ such that for all $x \in X$ $(x \neq y)$, $\lambda \in [0,1]$,

$$\begin{gathered} f(y + \lambda\eta(x,y)) \le G^{-1}(\lambda(G(f(x))) + (1-\lambda)G(f(y))), \\ \Big(f(y + \lambda\eta(x,y)) < G^{-1}(\lambda(G(f(x))) + (1-\lambda)G(f(y)))\Big). \end{gathered} \tag{2.8}$$

If (2.8) is satisfied for any $y \in X$, then f is said to be (strictly) a G-preinvex function on X with respect to η.

Remark 2.9. Every preinvex function with respect to η is G-preinvex function with respect to the same η, where $G(x) = x$. Every G-convex function is G-preinvex function with respect to $\eta(x,y,\lambda) = x - y$. However, the converse is not necessarily true.

Example 2.10. Let $X = [-1,1]$. $f : X \to R$, $G : I_f(X) \to R$ be defined by

$$f(x) = \arctan(2 - |x|), \qquad G(t) = \tan t. \tag{2.9}$$

Then, we can verify that f is a G-preinvex function with respect to η, where

$$\eta(x,y) = \begin{cases} -y - x^2 + 2x, & 0 \le x \le 1,\ 0 \le y \le 1, \\ -y - x^2 - 2x, & -1 \le x < 0,\ 0 \le y \le 1, \\ -y - x, & 0 \le x \le 1,\ -1 \le y < 0, \\ -y + x, & -1 \le x < 0,\ -1 \le y < 0. \end{cases} \tag{2.10}$$

But f is not a preinvex function because the following inequality

$$f(y + \lambda\eta(x,y)) > \lambda f(x) + (1-\lambda) f(y) \tag{2.11}$$

holds for $x = 0$, $y = 1$, and $\lambda = 1/2$.

And $f(x)$ is not a G-convex function because the following inequality

$$f(y + \lambda(x - y)) > G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))) \tag{2.12}$$

holds for $x = 1$, $y = -1$, and $\lambda = 1/2$.

Definition 2.11 (see [15]). A set $X \subseteq R^n$ is said to be a semiconnected set if there exists a vector-valued function $\eta : X \times X \times [0,1] \to R^n$ such that for any $x, y \in X$, $\lambda \in [0,1]$,

$$y + \lambda\eta(x,y,\lambda) \in X. \tag{2.13}$$

Definition 2.12 (see [15]). Let $X \subseteq R^n$ be a semiconnected set with respect to a vector-valued function $\eta : X \times X \times [0,1] \to R^n$. Function f is said to be semipreinvex with respect to η if for any $x, y \in X$, $\lambda \in [0,1]$, $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda) = 0$,

$$f(y + \lambda\eta(x,y,\lambda)) \le \lambda f(x) + (1-\lambda) f(y). \tag{2.14}$$

Next we present the definition of G-semipreinvex functions as follows.

Definition 2.13. Let $X \subseteq R^n$ be semiconnected set with respect to vector-valued function $\eta : X \times X \times [0,1] \to R^n$. A function $f : X \to R$ is said to be (strictly) G-semipreinvex at y with respect to η if there exists a continuous real-valued strictly increasing function $G : I_f(X) \to R$ such that for all $x \in X$ $(x \ne y)$, $\lambda \in [0,1]$, $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda) = 0$,

$$\begin{gathered} f(y + \lambda\eta(x,y,\lambda)) \le G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))), \\ \Big(f(y + \lambda\eta(x,y,\lambda)) < G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y)))\Big). \end{gathered} \tag{2.15}$$

If (2.15) is satisfied for any $y \in X$, then f is said to be (strictly) G-semipreinvex on X with respect to η.

Remark 2.14. Every semipreinvex function with respect to η is a G-semipreinvex function with respect to the same η, where $G(x) = x$. However, the converse is not true.

Example 2.15. Let $X = [-6,6]$. Then X is a semiconnected set with respect to $\eta(x,y,\lambda)$ and $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda) = 0$, where

$$\eta(x,y,\lambda) = \begin{cases} \dfrac{x-y}{\sqrt[3]{\lambda}}, & -6 \le x < 0,\ -6 \le y < 0,\ x > y,\ 0 < \lambda \le 1, \\ \lambda^2(x-y), & 0 \le x \le 6,\ 0 \le y \le 6,\ x \ge y, \\ \lambda^2(x-y), & -6 \le x < 0,\ -6 \le y < 0,\ x \le y, \\ x-y, & 0 \le x \le 6,\ 0 \le y \le 6,\ x < y, \\ x-y, & 0 \le x \le 6,\ -6 \le y < 0,\ x < -y, \\ x-y, & -6 \le x < 0,\ 0 \le y \le 6,\ x > -y, \\ 0, & 0 \le x \le 6,\ -6 \le y < 0,\ x \ge -y, \\ 0, & -6 \le x < 0,\ 0 \le y \le 6,\ x \le -y. \end{cases} \tag{2.16}$$

Let $f : X \to R$, $G : I_f(X) \to R$ be defined by

$$f(x) = \arctan\left(x^2+2\right), \qquad G(t) = \tan t. \tag{2.17}$$

Then, we can verify that f is a G-semipreinvex function with respect to η. But f is not a semipreinvex function with respect to η because the following inequality

$$f(y+\lambda\eta(x,y,\lambda)) > \lambda f(x) + (1-\lambda)f(y) \tag{2.18}$$

holds for $x = 2$, and $y = 4$, $\lambda = 1/2$.

Example 2.16. Let $X = [-6,6]$. Then X is a semiconnected set with respect to $\eta(x,y,\lambda)$ and $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda) = 0$, where

$$\eta(x,y,\lambda) = \begin{cases} \dfrac{x-y}{\varphi(\lambda)}, & -6 \le x < 0,\ -6 \le y < 0,\ x > y,\ 0 < \lambda \le 1, \\ \varphi(\lambda)(x-y), & 0 \le x \le 6,\ 0 \le y \le 6,\ x \ge y, \\ \varphi(\lambda)(x-y), & -6 \le x < 0,\ -6 \le y < 0,\ x \le y, \\ x-y, & 0 \le x \le 6,\ 0 \le y \le 6,\ x < y, \\ x-y, & 0 \le x \le 6,\ -6 \le y < 0,\ x < -y,\ \lambda < \varphi(\lambda) < 1 \\ x-y, & -6 \le x < 0,\ 0 \le y \le 6,\ x > -y, \\ 0, & 0 \le x \le 6,\ -6 \le y < 0,\ x \ge -y, \\ 0, & -6 \le x < 0,\ 0 \le y \le 6,\ x \le -y. \end{cases} \tag{2.19}$$

Let $f : X \to R$, $G : I_f(X) \to R$ be defined by

$$f(x) = \arctan\left(x^2+k\right), \quad G(t) = \tan t, \quad \forall k \in R. \tag{2.20}$$

Then, we can verify that $f(x)$ is a G-semipreinvex function with respect to classes of functions η. But $f(x)$ is not semipreinvex function with respect to η because the following inequality

$$f(y+\lambda\eta(x,y,\lambda)) > \lambda f(x) + (1-\lambda)f(y) \tag{2.21}$$

holds for $x = 2$, $y = 4$, and $\lambda = 1/2$.

Remark 2.17. Every a G-convex function is G-semipreinvex function with respect to $\eta(x,y,\lambda) = x - y$. But the converse is not true.

Example 2.18. Let $X = (-6,6)$, it is easy to check that X is a semiconnected set with respect to $\eta(x,y,\lambda)$ and $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda) = 0$, where

$$\eta(x,y,\lambda) = \begin{cases} \lambda(x-y), & 0 \le x < 6,\ 0 \le y < 6,\ x < y, \\ \lambda(x-y), & -6 < x < 0,\ -6 < y < 0,\ x > y, \\ \dfrac{x-y}{\sqrt{\lambda}}, & 0 \le x < 6,\ 0 \le y < 6,\ x \ge y,\ 0 < \lambda \le 1, \\ \dfrac{x-y}{\sqrt{\lambda}}, & -6 < x < 0,\ -6 < y < 0,\ x \le y,\ 0 < \lambda \le 1, \\ -x-y, & 0 \le x < 6,\ -6 < y < 0,\ x \ge -y, \\ -x-y, & -6 < x < 0,\ 0 \le y < 6,\ x \le -y, \\ 0, & 0 \le x < 6,\ -6 < y < 0,\ x < -y, \\ 0, & -6 < x < 0,\ 0 \le y < 6,\ x > -y. \end{cases} \tag{2.22}$$

Let $f : X \to R$, $G : I_f(X) \to R$ be defined by

$$f(x) = \arctan(6 - |x|), \qquad G(t) = \tan t. \tag{2.23}$$

Then, we can verify that f is a G-semipreinvex function with respect to η. But f is not a G-convex function, because the following inequality

$$f(y+\lambda(x-y)) > G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))) \tag{2.24}$$

holds for $x = 1$, $y = -1$, and $\lambda = 1/2$.

3. Some Properties of G-Semipreinvex Functions

In this section, we give some basic characterizations of G-semipreinvex functions.

Theorem 3.1. *Let f be a G_1-semipreinvex function with respect to η on a nonempty semiconnected set $X \subset R^n$ with respect to η, and let G_2 be a continuous strictly increasing function on $I_f(X)$. If the function $g(t) = G_2G_1^{-1}(t)$ is convex on the image under G_1 of the range of f, then f is also G_2-semipreinvex function on X with respect to the same function η.*

Proof. Let X be a nonempty semiconnected subset of R^n with respect to η, and we assume that f is G_1-semipreinvex with respect to η. Then, for any $x, y \in X$, $\lambda \in [0,1]$,

$$f(y + \lambda\eta(x,y,\lambda)) \le G_1^{-1}(\lambda G_1(f(x)) + (1-\lambda)G_1(f(y))). \tag{3.1}$$

Let G_2 be a continuous strictly increasing function on $I_f(X)$. Then,

$$G_2(f(y + \lambda\eta(x,y,\lambda))) \le G_2G^{-1}(\lambda G_1(f(x)) + (1-\lambda)G_1(f(y))). \tag{3.2}$$

By the convexity of $g(t) = G_2G_1^{-1}$, it follows the following inequality

$$\begin{aligned} G_2G^{-1}(\lambda G_1(f(x)) + (1-\lambda)G_1(f(y))) &\le \lambda G_2G_1^{-1}\Big(G_1(f(x)) + (1-\lambda)G_2G_1^{-1}(G_1 f(y))\Big) \\ &= \lambda G_2(f(x)) + (1-\lambda)G_2(f(y)) \end{aligned} \tag{3.3}$$

for all $x, y \in X$, $\lambda \in [0,1]$. Therefore,

$$\begin{aligned} &G_1^{-1}[\lambda(G_1(f(x))) + (1-\lambda)G_1(f(y))] \\ &\quad \le G_2^{-1}[\lambda(G_2(f(x))) + (1-\lambda)G_2(f(y))]. \end{aligned} \tag{3.4}$$

Thus, we have

$$f(y + \lambda\eta(x,y,\lambda)) \le G_2^{-1}(\lambda G_2(f(x)) + (1-\lambda)G_2(f(y))). \tag{3.5}$$

□

Theorem 3.2. *Let f be a G-semipreinvex function with respect to η on a nonempty semiconnected set $X \subset R^n$ with respect to η. If the function G is concave on $I_f(X)$, then f is semipreinvex function with respect to the same function η.*

Proof. Let $y, z \in I_f(X)$, from the assumption G is concave on $I_f(X)$, we have

$$G(z + \lambda(y - z)) \ge \lambda G(y) + (1-\lambda)G(z), \quad \lambda \in [0,1]. \tag{3.6}$$

Let

$$G(y) = x, \qquad G(z) = u, \qquad y = G^{-1}(x), \qquad z = G^{-1}(u), \tag{3.7}$$

then

$$\begin{aligned} G\Big(G^{-1}(u) + \lambda\Big(G^{-1}(x) - G^{-1}(u)\Big)\Big) &\ge \lambda G\Big(G^{-1}(x)\Big) + (1-\lambda)G\Big(G^{-1}(u)\Big) \\ &= \lambda x + (1-\lambda)u. \end{aligned} \tag{3.8}$$

It follows that

$$G^{-1}G\Big(\lambda G^{-1}(x)+(1-\lambda)G^{-1}(u)\Big)\geq G^{-1}(\lambda x+(1-\lambda)u). \tag{3.9}$$

Then,

$$\lambda G^{-1}(x)+(1-\lambda)G^{-1}(u)\geq G^{-1}(\lambda x+(1-\lambda)u). \tag{3.10}$$

This means that G^{-1} is convex. Let $G_1=G$, $G_2=t$, then $g(t)=G_2G_1^{-1}(t)$ is convex. Hence by Theorem 3.1, f is G_2-semipreinvex with respect to η. But G_2 is the identity function; hence, f is a semipreinvex function with respect to the same function η. □

Theorem 3.3. *Let X be a nonempty semiconnected set with respect to η subset of R^n and let $f_i: X\to R$, $i\in I$, be finite collection of G-semipreinvex function with respect to the same η and G on X. Define $f(x)=\sup(f_i(x): i\in I)$, for every $x\in X$. Further, assume that for every $x\in X$, there exists $i^*=i(x)\in I$, such that $f(x)=f_{i^*}(x)$. Then f is G-semipreinvex function with respect to the same function η.*

Proof. Suppose that the result is not true, that is, f is not G-semipreinvex function with respect to η on X. Then, there exists $x,y\in X$, $\lambda\in[0,1]$ such that

$$f(y+\lambda\eta(x,y,\lambda))>G^{-1}(\lambda G(f(x))+(1-\lambda)G(f(y))). \tag{3.11}$$

We denote $z=y+\lambda\eta(x,y,\lambda)$ there exist $i(z):=i_z\in I$, $i(x):=i_x\in I$, and $i(y):=i_y\in I$, satisfying

$$f(z)=f_{i_z}(z),\qquad f(x)=f_{i_x}(x),\qquad f(y)=f_{i_y}(y). \tag{3.12}$$

Therefore, by (3.11),

$$f_{i_z}(z)>G^{-1}\Big(\lambda G(f_{i_x}(x))+(1-\lambda)G\Big(f_{i_y}(y)\Big)\Big). \tag{3.13}$$

By the condition, we obtain

$$f_{i_z}(z)\leq G^{-1}(\lambda G(f_{i_z}(x))+(1-\lambda)G(f_{i_z}(y))). \tag{3.14}$$

From the definition of G-semipreinvexity, G is an increasing function on its domain. Then, G^{-1} is increasing. Since $f_{i_z}(x)\leq f_{i_x}(x)$, $f_{i_z}(y)\leq f_{i_y}(y)$, then (3.14) gives

$$f_{i_z}(z)\leq G^{-1}\Big(\lambda G(f_{i_x}(x))+(1-\lambda)G\Big(f_{i_y}(y)\Big)\Big). \tag{3.15}$$

The inequality (3.15) above contradicts (3.13). □

Theorem 3.4. *Let f be a G-semipreinvex function with respect to η on a nonempty semiconnected set $X \subset R^n$ with respect to η. Then, the level set $S_\alpha = \{x \in X : f(x) \leq \alpha\}$ is a semiconnected set with respect to η, for each $\alpha \in R$.*

Proof. Let $x, y \in S_\alpha$, for any arbitrary real number α. Then, $f(x) \leq \alpha$, $f(y) \leq \alpha$. Hence, it follows that

$$f(y + \lambda\eta(x, y, \lambda)) \leq G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))) \leq G^{-1}(G(\alpha)) = \alpha. \tag{3.16}$$

Then, by the definition of level set we conclude that $y + \lambda\eta(x, y, \lambda) \in S_\alpha$, for any $\lambda \in [0, 1]$, we conclude that S_α is a semiconnected set with respect to η.

Let f is a G-semipreinvex function with respect to η, its epigraph $E_f = \{(x, \alpha) : x \in X, \alpha \in R, f(x) \leq \alpha\}$ is said to be G-semiconnected set with respect to η if for any $(x, \alpha) \in E_f$, $(y, \beta) \in E_f$, $\lambda \in [0, 1]$,

$$\left(y + \lambda\eta(x, y, \lambda), G^{-1}(\lambda G(\alpha) + (1 - \lambda)G(\beta))\right) \in E_f. \tag{3.17}$$

□

Theorem 3.5. *Let $X \subset R^n$ with respect to η be a nonempty semiconnected set, and let f be a real-valued function defined on X. Then, f is a G-semipreinvex function with respect to η if and only if its epigraph $E_f = \{(x, \alpha) : x \in X, \alpha \in R, f(x) \leq \alpha\}$ is a G-semiconnected set with respect to η.*

Proof. Let $(x, \alpha) \in E_f$, $(y, \beta) \in E_f$, then $f(x) \leq \alpha, f(y) \leq \beta$. Thus, for any $\lambda \in [0, 1]$,

$$\begin{aligned} f(y + \lambda\eta(x, y, \lambda)) &\leq G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))) \\ &\leq G^{-1}(\lambda G(\alpha) + (1 - \lambda)G(\beta)). \end{aligned} \tag{3.18}$$

By the definition of an epigraph of f, this means that

$$\left(y + \lambda\eta(x, y, \lambda), G^{-1}(\lambda G(\alpha) + (1 - \lambda)G(\beta))\right) \in E_f. \tag{3.19}$$

Thus, we conclude that E_f is a G semiconnected set with respect to η.

Conversely, let E_f be a G semiconnected set. Then, for any $x, y \in X$, we have $(x, f(x)) \in E_f$, $(y, f(y)) \in E_f$. By the definition of an epigraph of f, the following inequality

$$f(y + \lambda\eta(x, y, \lambda)) \leq G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))) \tag{3.20}$$

holds for any $\lambda \in [0, 1]$. This implies that f is a G-semipreinvex function on X with respect to η. □

The following results characterize the class of G-semipreinvex functions.

Theorem 3.6. *Let $X \subseteq R^n$ be a semiconnected set with respect to $\eta : X \times X \times [0,1] \to R^n$; $f : X \to R$ is a G-semipreinvex function with respect to the same η if and only if for all $x, y \in X$, $\lambda \in [0,1]$, and $u, v \in R$,*

$$f(x) \le u, \qquad f(y) \le v \Longrightarrow f(y + \lambda\eta(x,y,\lambda)) \le G^{-1}(\lambda G(u) + (1-\lambda)G(v)). \tag{3.21}$$

Proof. Let f be G-semipreinvex functions with respect to η, and let $f(x) \le u$, $f(y) \le v$, $0 < \lambda < 1$. From the definition of G-semipreinvexity, we have

$$\begin{aligned} f(y + \lambda\eta(x,y,\lambda)) &\le G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))) \\ &\le G^{-1}(\lambda G(u) + (1-\lambda)G(v)). \end{aligned} \tag{3.22}$$

Conversely, let $x, y \in X$, $\lambda \in [0,1]$. For any $\delta > 0$,

$$\begin{aligned} f(x) &< f(x) + \delta, \\ f(y) &< f(y) + \delta. \end{aligned} \tag{3.23}$$

By the assumption of theorem, we can get that for $0 < \lambda < 1$,

$$\begin{aligned} f(y + \lambda\eta(x,y,\lambda)) &\le G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))) \\ &\le G^{-1}(\lambda G(f(x)+\delta) + (1-\lambda)G(f(y)+\delta)). \end{aligned} \tag{3.24}$$

Since G is a continuous real-valued increasing function, and $\delta > 0$ can be arbitrarily small, let $\delta \to 0$, it follows that

$$f(y + \lambda\eta(x,y,\lambda)) \le G^{-1}(\lambda G(u) + (1-\lambda)G(v)). \tag{3.25}$$

□

4. G-Semipreinvexity and Optimality

In this section, we will give some optimality results for a class of G-semipreinvex functions.

Theorem 4.1. *Let $f : X \to R$ be a G-semipreinvex function with respect to η, and we assume that η satisfies the following condition: $\eta(x,y,\lambda) \ne 0$, when $x \ne y$. Then, each local minimum point of the function f is its point of global minimum.*

Proof. Assume that $\overline{y} \in X$ is a local minimum point of f which is not a global minimum point. Hence, there exists a point $\overline{x} \in X$ such that $f(\overline{x}) < f(\overline{y})$. By the G-semipreinvexity of f with respect to η, we have

$$f(\overline{y} + \lambda\eta(\overline{x},\overline{y},\lambda)) \le G^{-1}(\lambda G(f(\overline{x})) + (1-\lambda)G(f(\overline{y}))), \quad \lambda \in [0,1]. \tag{4.1}$$

Then, for $\lambda \in [0,1]$,

$$\begin{aligned} f(\overline{y} + \lambda\eta(\overline{x},\overline{y},\lambda)) &< G^{-1}(\lambda G(f(\overline{y})) + (1-\lambda)G(f(\overline{y}))) \\ &= G^{-1}(G(f(\overline{y}))) \\ &= f(\overline{y}). \end{aligned} \tag{4.2}$$

Thus, we have

$$f(\overline{y} + \lambda\eta(\overline{x},\overline{y},\lambda)) < f(\overline{y}). \tag{4.3}$$

This is a contradiction with the assumption. □

Theorem 4.2. *The set of points which are global minimum of f is a semiconnected set with respect to η.*

Proof. Denote by A the set of points of global minimum of f, and let $x, y \in A$. Since f is G-semipreinvex with respect to η, then

$$f(y + \lambda\eta(x,y,\lambda)) \leq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))), \quad \lambda \in [0,1] \tag{4.4}$$

is satisfied. Since $f(x) = f(y)$, we have

$$f(y + \lambda\eta(x,y,\lambda)) \leq G^{-1}(\lambda G(f(y)) + (1-\lambda)G(f(y))). \tag{4.5}$$

So, for any $\lambda \in [0,1]$,

$$f(y + \lambda\eta(x,y,\lambda)) \leq G^{-1}(G(f(y))) = f(y) = f(x). \tag{4.6}$$

Since $x, y \in A$ are points of a global minimum of f, it follows that, for any $\lambda \in [0,1]$, the following relation:

$$y + \lambda\eta(x,y,\lambda) \in A \tag{4.7}$$

is satisfied. Then, A is a semiconnected set with respect to η. □

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References

[1] S. Schaible and W. T. Ziemba, *Generalized Concavity in Optimization and Economics*, Academic Press, London, UK, 1981.

[2] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

[3] A. Ben-Israel and B. Mond, "What is invexity?" *Bulletin of Australian Mathematical Society B*, vol. 28, no. 1, pp. 1–9, 1986.

[4] T. Weir and B. Mond, "Pre-invex functions in multiple objective optimization," *Journal of Mathematical Analysis and Applications*, vol. 136, no. 1, pp. 29–38, 1988.

[5] T. Weir and V. Jeyakumar, "A class of nonconvex functions and mathematical programming," *Bulletin of the Australian Mathematical Society*, vol. 38, no. 2, pp. 177–189, 1988.

[6] R. Pini, "Invexity and generalized convexity," *Optimization*, vol. 22, no. 4, pp. 513–525, 1991.

[7] D. Bhatia and P. Kumar, "Multiobjective control problem with generalized invexity," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 3, pp. 676–692, 1995.

[8] S. R. Mohan and S. K. Neogy, "On invex sets and preinvex functions," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 3, pp. 901–908, 1995.

[9] L. V. Reddy and R. N. Mukherjee, "Some results on mathematical programming with generalized ratio invexity," *Journal of Mathematical Analysis and Applications*, vol. 240, no. 2, pp. 299–310, 1999.

[10] X. M. Yang and D. Li, "On properties of preinvex functions," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 1, pp. 229–241, 2001.

[11] D. S. Kim and S. Schaible, "Optimality and duality for invex nonsmooth multiobjective programming problems," *Optimization*, vol. 53, no. 2, pp. 165–176, 2004.

[12] T. Antczak, "r-preinvexity and r-invexity in mathematical programming," *Computers & Mathematics with Applications*, vol. 50, no. 3-4, pp. 551–566, 2005.

[13] T. Antczak, "(p, r)-invex sets and functions," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 355–379, 2001.

[14] T. Antczak, "A class of B-(p, r)-invex functions and mathematical programming," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 1, pp. 187–206, 2003.

[15] X. Q. Yang and G. Y. Chen, "A class of nonconvex functions and pre-variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 169, no. 2, pp. 359–373, 1992.

[16] X. M. Yang, X. Q. Yang, and K. L. Teo, "On properties of semipreinvex functions," *Bulletin of the Australian Mathematical Society*, vol. 68, no. 3, pp. 449–459, 2003.

[17] T. Antczak, "G-pre-invex functions in mathematical programming," *Journal of Computational and Applied Mathematics*, vol. 10, pp. 1–15, 2007.

[18] X. J. Long and J. W. Peng, "Semi-B-preinvex functions," *Journal of Optimization Theory and Applications*, vol. 131, no. 2, pp. 301–305, 2006.

[19] K. Q. Zhao, X. W. Liu, and Z. Chen, "A class of r-semipreinvex functions and optimality in nonlinear programming," *Journal of Global Optimization*, vol. 49, no. 1, pp. 37–47, 2011.

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Research Article

The Optimal Dispatch of Traffic and Patrol Police Service Platforms

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The main goal of this paper is to present a minmax programming model for the optimal dispatch of Traffic and Patrol Police Service Platforms with single traffic congestion. The objective is to minimize the longest time of the dispatch for Traffic and Patrol Police Service Platforms. Some numerical experiments are carried out, and the optimal project is given.

1. Introduction

Traffic and Patrol Police Service Platforms (in short, TPPSP) in the city have been playing an important role in dealing with emergency and traffic administration. The national college mathematical modeling contest of China in 2011 proposed the problem related to the optimal dispatch of TPPSP. However, only the case without any traffic congestion is considered for the problem. It is well known that the optimal dispatch and design of TPPSP is very complicated and it is affected by many real factors, such as

(i) the influence of traffic congestion on the optimal dispatch;

(ii) the influence of police resources allocation for each platform;

(iii) the influence of the uncertainty of road weights.

The shortest path between any two nodes in urban traffic network is usually solved by Floyd shortest path algorithm in traffic computing and path search. Also the shortest path algorithms are widely applied to computer science, operational research, geographic information systems and traffic guidance, navigation systems, and so forth [1–5]. Especially, given a detailed GIS mapping and image display program, Liao and Zhong [4] proved that the Floyd shortest path algorithm can quickly and easily retrieve the shortest path between two locations, saving computing time and overhead.

The minmax programming model has received more attentions in operations research and optimization fields in the literatures [6–9]. Averbakh and Berman [8] considered the location minmax p-TSP problem, where only optimal locations of the servers must be found, without the corresponding tours and without the optimal value of the objective function. Exact linear time algorithms for the cases $p = 2$ and $p = 3$ are presented.

In recent years, the research on optimal dispatch of TPPSP has also received some attentions in the literatures [10–12]. However, we noted that these works only focused on the case without any traffic congestion for the optimal dispatch of TPPSP.

In this paper, we first consider the optimal dispatch with single traffic congestion when the emergent event and establish a minmax programming model (model II) which objective is to minimize the longest time of the dispatch for TPPSP. Furthermore, some numerical experiments are carried out, and the optimal project is presented.

2. Notations

m: The number of the TPPSP

n: The number of intersections that should be blockaded

d'_{ij}: $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ The shortest distance from the ith TPPSP to the jth intersection without traffic congestion

t_{ij}: $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ The shortest time from the ith TPPSP to the jth intersection

v: The speed of police vehicles

x_{ij}: $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ The ith TPPSP is dispatched to the jth intersection or not.

Assume that every TPPSP has almost the same police force, TPPSP have been settled at some traffic centers and key parts of an urban area of a city. The average of the police car is 60 km/h.

3. Mathematical Models

In this section, we first introduce a minimax programming model for the optimal dispatch of TPPSP without any traffic congestion in the literatures. Then, we present our main model for the optimal dispatch of TPPSP with single traffic congestion.

3.1. *The Case without Any Traffic Congestion*

When the road section has no traffic congestion, some authors presented the following minimax programming model, which is also a 0-1 integer programming model, see literatures [10, 12], and so forth.

(model I)

$$\min\max\{t_{ij}x_{ij}\} \tag{3.1}$$

$$\text{s.t.}\quad \sum_{i=1}^{m}x_{ij}=1,\quad j=1,2,\ldots,n; \tag{3.2}$$

$$\sum_{j=1}^{n}x_{ij}\le 1,\quad i=1,2,\ldots,m; \tag{3.3}$$

$$x_{ij}(1-x_{ij})=0,\quad i=1,2,\ldots,m,\ j=1,2,\ldots,n; \tag{3.4}$$

$$t_{ij}=\frac{100d'_{ij}}{60v},\quad i=1,2,\ldots,m,\ j=1,2,\ldots,n. \tag{3.5}$$

The objective function (3.1) requires that the maximum time from the ith TPPSP to the jth node which is minimum. Besides, constraint (3.2) requires that every intersection should be blockaded by only one TPPSP. Constraint (3.3) ensures that one TPPSP can only blockade one intersection. Constraint (3.4) requires that x_{ij} is 0-1 variable. Constraint (3.5) shows that the relation between the time and the distance from the ith TPPSP to the jth intersection. Furthermore, in Constraint (3.5), the unite of t_{ij} is minutes, v is meters per second and the symbol d'_{ij} (the unite is millimeters) is the distance of map, $100d'_{ij}$ (the unite is meters) is the real distance.

3.2. The Case with Single Traffic Congestion

In the real life, traffic congestion may occur in urban traffic network. Therefore, the research on the optimal dispatch of TPPSP with Traffic congestions is important and meaningful. Considering that the emergency may occur at any time and place, and the road section may have some traffic congestions, in this subsection, we present one minmax programming model for the optimal dispatch of TPPSP with single traffic congestion. The optimal dispatch of TPPSP with Traffic congestions model is more effective than model I. Moreover, the optimal dispatch of TPPSP with Traffic congestions model's results has immediate practical applications.

We assume that the traffic congestion occurs on the road section from the node p to the node q, where the node p is adjacent to the node q. And T_{pq} denotes the average time of blocking. Besides, d''_{ij} denotes the shortest distance from the TPPSP i to the jth intersection without the road section from the node p to the node q, P_{ij} denotes the shortest path from the ith TPPSP to the jth node. R denotes the set of P_{ij} which go through p and q nodes. $t'_{ij}(i=1,2,\ldots,m,\ j=1,2,\ldots,n)$ denotes the shortest time from the ith TPPSP to the jth intersection without traffic congestion, and t''_{ij} denotes the shortest time from the ith TPPSP to the jth intersection without the road section from the node p to the node q.

We establish the following minmax programming model II for the dispatch of TPPSP with single traffic congestion:

(model II)

$$\min \max\{t_{ij}x_{ij}\}, \tag{3.6}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} x_{ij} = 1, \quad j = 1, 2, \ldots, n; \tag{3.7}$$

$$\sum_{j=1}^{n} x_{ij} \le 1, \quad i = 1, 2, \ldots, m; \tag{3.8}$$

$$x_{ij}(1 - x_{ij}) = 0, \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n, \tag{3.9}$$

where

$$\begin{gathered} t_{ij} = \begin{cases} t'_{ij}, & P_{ij} \in A_1; \\ t''_{ij}, & P_{ij} \in A_2; \\ t'_{ij} + T_{pq} - \min\left\{t'_{ip}, t'_{iq}\right\}, & P_{ij} \in A_3; \end{cases} \\ t'_{ij} = \frac{100 d'_{ij}}{60 v} \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n; \\ t''_{ij} = \frac{100 d''_{ij}}{60 v} \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n; \\ A_1 = \left\{P_{ij} \mid P_{ij} \notin R \text{ or } P_{ij} \in R, \; \min\left\{t'_{ip}, t'_{iq}\right\} \ge T_{pq}\right\}; \\ A_2 = \left\{P_{ij} \mid P_{ij} \in R, \; \min\left\{t'_{ip}, t'_{iq}\right\} < T_{pq}, \; t'_{ij} + T_{pq} - \min\left\{t'_{ip}, t'_{iq}\right\} > t''_{ij}\right\}; \\ A_3 = \left\{P_{ij} \mid P_{ij} \in R, \; \min\left\{t'_{ip}, t'_{iq}\right\} < T_{pq}, \; t'_{ij} + T_{pq} - \min\left\{t'_{ip}, t'_{iq}\right\} \le t''_{ij}\right\}. \end{gathered} \tag{3.10}$$

The objective function (3.6) requires that the maximum times from the ith TPPSP to the jth node which is minimum. The analysis of constraints (3.6)–(3.9) is the same as constraints (3.2)–(3.4). However, the value of t_{ij} is different from the time of the model I. In the model II, the function of t_{ij} is divided into three segments.

4. Numerical Experiments

In this paper, we take $m = 20$, $n = |I| = 13$, where

$$I = \{12, 14, 16, 21, 22, 23, 24, 28, 29, 30, 38, 48, 62\}, \tag{4.1}$$

to do specific analysis for our model. The data is based on (http://www.mcm.edu.cn/). We use *Floyd* Shortest Path Algorithm to figure out d'_{ij} and d''_{ij} by Matlab software. We can

Table 1: The optimal dispatch project of TPPSP without any traffic congestion.

Dispatching project	Path of choosing project	Time from the ith TPPSP to the jth intersection
1–38	1-69-70-2-40-39-38	5.880900
2–16	2-40-39-38-16	7.388100
4–48	4-57-58-59-51-50-5-47-48	7.395900
7–29	7-30-29	8.015500
9–30	9-34-33-32-7-30	3.492300
10–12	10-26-27-12	7.586600
11–22	11-22	3.269600
12–23	12-25-24-13-23	6.477000
13–24	13-24	2.385400
14–21	14-21	3.265000
15–28	15-28	4.751800
16–14	16-14	6.741700
20–62	20-85-62	6.448900

Table 2: The optimal dispatch project of TPPSP with $T_{pq} = 5\,\text{min}$.

Dispatching project	Path of choosing project	Time from the ith TPPSP to the jth intersection
1–38	1-69-70-2-40-39-38	5.880900
2–16	2-40-39-38-16	7.388100
4–48	4-57-58-59-51-50-5-47-48	7.395900
5–30	5-47-48-30	3.182900
7–29	7-30-29	8.015500
10–12	10-26-27-12	7.586600
11–24	11-25-24	3.805300
12–22	12-25-24-13-22	6.882500
13–23	13-23	0.500000
14–21	14-21	3.265000
15–28	15-28	4.751800
16–14	16-14	6.741700
18–62	18-80-79-19-77-76-64-63-4-62	6.734400

obtain the dispatch project of TPPSP without any traffic congestion when the emergent event happens in the city as Table 1.

Considering the case with single traffic congestion for the optimal dispatch of TPPSP in urban traffic network, we do the numerical experiments for the minmax programming model II by using Matlab software. Here, we take $p = 36$, $q = 16$, $T_{pq} = 5\,\text{min}$. The dispatching project of TPPSP with one road section having single traffic congestion when the emergent event happens is shown in Table 2.

Table 3: The optimal dispatch project of TPPSP with $T_{pq} = 10$ min.

Dispatching project	Path of choosing project	Time from the ith TPPSP to the jth intersection
1–38	1-69-70-2-40-39-38	5.880900
2–16	2-40-39-38-16	7.388100
5–62	5-50-51-59-58-57-60-62	5.255100
6–48	6-47-48	2.506400
7–29	7-30-29	8.015500
8–30	8-33-32-7-30	3.060800
10–12	10-26-27-12	7.586600
11–23	11-22-13-23	4.675100
12–22	12-25-24-13-22	6.882500
13–24	13-24	2.385400
14–21	14-21	3.265000
15–28	15-28	4.751800
16–14	16-14	6.741700

Table 4: The optimal dispatch project of TPPSP with $T_{pq} = 30$ min.

Dispatching project	Path of choosing project	Time from the ith TPPSP to the jth intersection
1–62	1-75-76-64-63-4-62	4.885200
2–16	2-40-39-38-16	7.388100
6–30	6-47-48-30	3.213500
7–29	7-30-29	8.015500
8–48	8-47-48	3.099500
9–38	9-35-36-39-38	4.725700
10–22	10-26-11-22	7.707900
11–23	11-22-13-23	4.675100
12–12	12-12	0.000000
13–24	13-24	2.385400
14–21	14-21	3.265000
15–28	15-28	4.751800
16–14	16-14	6.741700

From Tables 1 and 2, we can clearly see that the maximum time of the optimal dispatch for TPPSP is the same for the case without any traffic congestion and the case with single traffic congestion in the given urban traffic network. However, the optimal dispatch project of TPPSP is different each other. Consequently, this shows that traffic congestion between the nodes in urban traffic network system will influence the optimal dispatch project of TPPSP in a certain degree when the emergent event happens.

Table 5: The optimal dispatch project of TPPSP with $T_{pq} = 60$ min.

Dispatching project	Path of choosing project	Time from the ith TPPSP to the jth intersection
2–16	2-40-39-38-16	7.388100
4–48	4-57-58-59-51-50-5-47-48	7.395900
7–29	7-30-29	8.015500
9–30	9-34-33-32-7-30	3.492300
10–22	10-26-11-22	7.707900
11–24	11-25-24	3.805300
12–23	12-25-24-13-23	6.477000
13–12	13-24-25-12	5.977000
14–21	14-21	3.265000
15–28	15-28	4.751800
16–14	16-14	6.741700
19–38	19-79-78-1-69-70-2-40-39-38	7.639300
20–62	20-85-62	6.448900

From Tables 2, 3, 4, and 5, we can gain the different dispatch project when the time of a traffic congestions is different. We can know the traffic congestion can influence the dispatching project. The influence degree is different when the time of a traffic congestion is different. However, for the node p and node q, the maximum time from the TPPSP to the intersection is 8.015500 min when the time of a traffic congestion is different.

In order to avoid the data that we use may be too special, we further take $p = 7$, $q = 30$. Still take $m = 20$, $n = 13$, and $T_{pq} = 5$ or 10 or 30 or 60 min, respectively, the dispatching project about the TPPSP with one road section having a traffic congestion when the emergent event happens as Table 6, where M_1 means blocking time, M_2 means dispatching project, M_3 means path of choosing project, and M_4 means time from the ith TPPSP to the jth intersection.

In Table 6, where $p = 7$, $q = 30$, we can also gain the different dispatching project when the time of a traffic congestion is different. However, the maximum time from the TPPSP to the intersection is 8.570200 min when the time of a traffic congestion is different.

Road section with having a traffic congestion is different, the maximum time from the TPPSP to the intersection is different. The influence degree of the time of a traffic congestion is not too large to the maximum time from the TPPSP to the intersection, but is large to the dispatching project.

5. Concluding Remarks

In this paper, we present a minmax programming models for the optimal dispatch of TPPSP with single traffic congestion. Some numerical experiments are carried out by using Matlab software and the optimal dispatch projects are given. However, in this paper, we only consider the case with single traffic congestion in model II. Hence, it is possible and meaningful to study the optimal dispatch project of TPPSP with several traffic congestions. This will be the future topics that we study.

Table 6: The optimal dispatch project of TPPSP with T_{pq} = 5, 10, 30, 60 min.

M_1	5 min	10 min	30 min	60 min
M_2	$1 \rightarrow 38$	$1 \rightarrow 38$	$1 \rightarrow 38$	$3 \rightarrow 48$
	$4 \rightarrow 48$	$5 \rightarrow 48$	$2 \rightarrow 16$	$7 \rightarrow 28$
	$5 \rightarrow 16$	$7 \rightarrow 28$	$4 \rightarrow 48$	$8 \rightarrow 30$
	$6 \rightarrow 62$	$9 \rightarrow 14$	$7 \rightarrow 28$	$10 \rightarrow 24$
	$7 \rightarrow 28$	$10 \rightarrow 12$	$9 \rightarrow 14$	$11 \rightarrow 21$
	$9 \rightarrow 14$	$11 \rightarrow 21$	$10 \rightarrow 22$	$12 \rightarrow 21$
	$10 \rightarrow 22$	$12 \rightarrow 22$	$11 \rightarrow 12$	$13 \rightarrow 22$
	$11 \rightarrow 23$	$13 \rightarrow 24$	$12 \rightarrow 24$	$14 \rightarrow 23$
	$12 \rightarrow 24$	$14 \rightarrow 23$	$13 \rightarrow 23$	$15 \rightarrow 29$
	$13 \rightarrow 12$	$15 \rightarrow 29$	$14 \rightarrow 21$	$16 \rightarrow 14$
	$14 \rightarrow 21$	$16 \rightarrow 30$	$15 \rightarrow 29$	$17 \rightarrow 16$
	$15 \rightarrow 29$	$17 \rightarrow 16$	$16 \rightarrow 30$	$19 \rightarrow 38$
	$16 \rightarrow 30$	$20 \rightarrow 62$	$20 \rightarrow 62$	$20 \rightarrow 62$
M_3	1-69-70-2-40-39-38	1-69-70-2-40-39-38	1-69-70-2-40-39-38	3-55-54-53-49-5-47-48
	4-57-58-59-51-50-5-47-48	5-47-48	2-40-39-38-16	7-15-28
	5-47-8-9-35-36-16	7-15-28	4-57-58-59-51-50-5-47-48	8-47-48-30
	6-59-58-57-60-62	9-35-36-16-14	7-15-28	10-26-11-25-24
	7-15-28	10-26-27-12	9-35-36-16-14	11-22-21
	9-35-36-16-14	11-22-21	10-26-11-22	12-12
	10-26-11-22	12-25-24-13-22	11-25-12	13-22
	11-22-13-23	13-24	12-25-24	14-21-22-13-23
	12-25-24	14-21-22-13-23	13-23	15-28-29
	13-24-25-12	15-28-29	14-21	16-14
	14-21	16-36-35-9-8-47-48-30	15-28-29	17-40-39-38-16
	15-28-29	17-40-39-38-16	16-36-35-9-8-47-48-30	19-79-78-1-69-70-2-40-39-38
	16-36-37-7-30	20-85-62	20-85-62	20-85-62
M_4	5.880900	5.880900	5.880900	8.197900
	7.395900	2.475800	7.388100	8.570200
	6.228000	8.570200	7.395900	3.806600
	5.337300	8.274200	8.570200	8.243600
	8.570200	7.586600	8.274200	5.072300
	8.274200	5.072300	7.707900	0.000000
	7.707900	6.882500	3.791400	0.905540
	4.675100	2.385400	3.591600	6.473300
	3.591600	6.473300	0.500000	5.700500
	5.977000	5.700500	3.265000	6.741700
	3.265000	6.498900	5.700500	8.161600
	5.700500	8.161600	6.498900	7.639300
	5.583100	6.448900	6.448900	6.448900

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References

[1] T. Dudás and R. Rudolf, "Spanning trees and shortest paths in monge graphs," *Computing*, vol. 60, no. 2, pp. 109–119, 1998.
[2] S. Bistarelli, U. Montanari, and F. Rossi, "Soft constraint logic programming and generalized shortest path problems," *Journal of Heuristics*, vol. 8, no. 1, pp. 25–41, 2002.
[3] B. Huang, W. M. Wu, and Y. N. Zheng, "The research and improvement of path optimization in vehicle navigation," *Advances in Intelligent and Soft Computing*, vol. 129, pp. 203–207, 2012.
[4] Y. M. Liao and J. Zhong, "FLOYD Algorithm Based on the Shortest Path in GIS," *Communications in Computerand Information Science*, vol. 267, no. 4, pp. 574–579, 2012.
[5] H. X. Li, Q. J. Xu, and D. Q. Zhang, "An efficient constrained shortest path algorithm for traffic navigation," *Advanced Materials Research*, vol. 356–360, pp. 2880–2885, 2012.
[6] K. Fan, "Minimax theorems," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 39, pp. 42–47, 1953.
[7] G. Mosheiov and M. Shadmon, "Minmax earliness-tardiness costs with unit processing time jobs," *European Journal of Operational Research*, vol. 130, no. 3, pp. 638–652, 2001.
[8] I. Averbakh and O. Berman, "Minmax p-traveling salesmen location problems on a tree," *Annals of Operations Research*, vol. 110, no. 1–4, pp. 55–68, 2002.
[9] G. Mosheiov and D. Oron, "Minmax scheduling with job-classes and earliness-tardiness costs," *European Journal of Operational Research*, vol. 177, no. 1, pp. 612–622, 2007.
[10] J. X. Yu, D. C. Song, X. Y. Zhao, and J. Q. Li, "Research on the reasonable dispatch of service platform of trafficand patrol police," *Science Technology and Engineering*, vol. 12, pp. 126–128, 2012.
[11] S. L. Fu, L. Fang, C. L. Wang, and X. W. Yang, "The establishment and scheduling of traffic and patrol police service platform," *Journal of Logistical Engineering Universtiy*, vol. 28, pp. 79–84, 2012.
[12] X. Wei, X. X. Dong, and D. Tang, "The setting and dispatch model of the traffic and cruising police platform," *Mathematical Modeling and Its Applications*, vol. 1, pp. 73–76, 2012.

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Research Article

Traffic Network Equilibrium Problems with Capacity Constraints of Arcs and Linear Scalarization Methods

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Traffic network equilibrium problems with capacity constraints of arcs are studied. A (weak) vector equilibrium principle with vector-valued cost functions, which are different from the ones in the work of Lin (2010), and three kinds of parametric equilibrium flows are introduced. Some necessary and sufficient conditions for a (weak) vector equilibrium flow to be a parametric equilibrium flow are derived. Relationships between a parametric equilibrium flow and a solution of a scalar variational inequality problem are also discussed. Some examples are given to illustrate our results.

1. Introduction

The earliest traffic network equilibrium model was proposed by Wardrop [1] for a transportation network. After getting Wardrop's equilibrium principle, many scholars have studied variant kinds of network equilibrium models, see, for example, [2–5]. However, most of these equilibrium models are based on a single criterion. The assumption that the network users choose their paths based on a single criterion may not be reasonable. It is more reasonable to assume that no user will choose a path that incurs both a higher cost and a longer delay than some other paths. In other words, a vector equilibrium should be sought based on the principle that the flow of traffic along a path joining an *O-D* pair is positive only if the vector cost of this path is the minimum possible among all the paths joining the same *O-D* pair. Recently, equilibrium models based on multiple criteria or on a vector cost function have been proposed. In [6], Chen and Yen first introduced a vector equilibrium principle for vector traffic network without capacity constraints. In [7, 8], Khanh and Luu extended vector equilibrium principle to the case of capacity constraints of paths. For other results of vector equilibrium principle with capacity constraints of paths, we refer to [9–17].

Very recently, in [18, 19], Lin extended traffic network equilibrium principle to the case of capacity constraints of arcs and obtained a sufficient condition and stability results of vector traffic network equilibrium flows with capacity constraints of arcs. In [20], Xu et al. also considered that vector network equilibrium problems with capacity constraints of arcs. By virtue of a Δ function, which was introduced by Zaffaroni [21], the authors introduced a Δ-equilibrium flow and a weak Δ-equilibrium flow, respectively, and obtained sufficient and necessary conditions for a weak vector equilibrium flow to be a (weak) Δ-equilibrium flow.

In this paper, our aim is to further investigate traffic network equilibrium problems with capacity constraints for arcs. We introduce a (weak) vector equilibrium principle with vector-valued cost functions, which are more reasonable from practical point of view than the ones in [18, 19]. In order to obtain necessary and sufficient conditions for a (weak) vector equilibrium, we introduce three kinds of parametric equilibrium flows. Simultaneously, we also discuss relationships between a parametric equilibrium flow and a solution of a scalar variational inequality problem.

The outline of the paper is as follows. In Section 2, a (weak) equilibrium principle with capacity constraints of arcs is introduced. In Section 3, three kinds of parametric equilibrium flows are introduced. Some sufficient and necessary conditions for a (weak) vector equilibrium flow are obtained. Relationships between a parametric equilibrium flow and a solution of a scalar variational inequality problem are also discussed.

2. Preliminaries

For a traffic network, let N and E denote the set of nodes and directed arcs, respectively, and let $C = (c_e)_{e\in E}$ denote the capacity vector, where c_e (>0) denotes the capacity of arc $e \in E$. Let W denote the set of origin-destination (O-D) pairs and let $D = (d_w)_{w\in W}$ denote the demand vector, where d_w (>0) denotes the demand of traffic flow on O-D pair w. A traffic network with capacity constraints of arcs is usually denoted by $G = (N, E, C, W, D)$. For each arc $e \in E$, the arc flow needs to satisfy the capacity constraints: $c_e \geq v_e \geq 0$, for each $e \in E$. For each $w \in W$, let P_w denote the set of available paths joining O-D pair w. Let $m = \sum_{w\in W} |P_w|$. For a given path $k \in P_w$, let h_k denote the traffic flow on this path and $h = (h_1, h_2, \ldots, h_m) \in R^m$ is called a path flow. The path flow vector h induces an arc flow v_e on each arc $e \in E$ given by

$$v_e = \sum_{w\in W} \sum_{k\in P_w} \delta_{ek} h_k, \tag{2.1}$$

where $\delta_{ek} = 1$ if the arc e is contained in path k and 0, otherwise. Suppose that the demand of network flow is fixed for each O-D pair w. We say that a path flow h satisfies demand constraints

$$\sum_{k\in P_w} h_k = d_w, \quad \forall w \in W. \tag{2.2}$$

A path flow h satisfying the demand constraints and capacity constraints is called a feasible path flow. Let $H = \{h \in R_+^m : \text{for all } w \in W,\ \sum_{k\in P_w} h_w = d_w \text{ and for all } e \in E, c_e \geq v_e \geq 0\} = \{h \in R_+^m : \text{for all } w \in W,\ \sum_{k\in P_w} h_w = d_w \text{ and for all } e \in E,\ c_e \geq \sum_{w\in W} \sum_{k\in P_w} \delta_{ek} h_k \geq 0\}$ and let $H \neq \emptyset$. Clearly, H is convex and compact. Let $t_e(h_k) : R_+ \to R^r$ be a vector-valued cost function for the path k on the arc e. Let $T_k(h) : R_+^m \to R^r$ be a vector-valued cost function along the path k. Then the vector-valued cost on the path k is equal to the sum of the all costs of the flow h_k through arcs, which belong to the path k, that is,

$$T_k(h) = \sum_{e \in E} \delta_{ek} t_e(h_k). \tag{2.3}$$

Let $T(h) = (T_1(h), T_2(h), \ldots, T_m(h)) \in R^{r\times m}$.

Remark 2.1. In [18, 19], Lin defined the vector cost function along the path k as follows:

$$\overline{T}_k(h) = \sum_{e \in E} \delta_{ek} \bar{t}_e(h), \tag{2.4}$$

where $\bar{t}_e(h) : R^m \to R^r$ be a vector-valued cost function for arc e. If the paths have common arcs, then the definition is unreasonable. The following example can illustrate the case.

Example 2.2. Consider the network problem depicted in Figure 1. $V = \{1, 2, 3, 4\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, $C = (3, 2, 2, 4, 3)$, $W = \{(1, 4), (3, 4)\}$, $D = (3, 4)$. The cost functions of arcs from R to R are, respectively, as follows:

$$\begin{gathered}
\bar{t}_{e_1}(h) = \bar{t}_{e_1}(v_{e_1}) = 50v_{e_1} + 100, \qquad \bar{t}_{e_2}(h) = \bar{t}_{e_2}(v_{e_2}) = 20v_{e_2} + 500, \\
\bar{t}_{e_3}(h) = \bar{t}_{e_3}(v_{e_3}) = 60v_{e_3} + 100, \qquad \bar{t}_{e_4}(h) = \bar{t}_{e_4}(v_{e_4}) = 30v_{e_4} + 200, \\
\bar{t}_{e_5}(h) = \bar{t}_{e_5}(v_{e_5}) = 70v_{e_5} + 300.
\end{gathered} \tag{2.5}$$

For O-D pair (1, 4): $P_{(1,4)}$ includes path $1 = (1, 2, 4)$ and path $2 = (1, 4)$, for O-D pair $(3, 4) : P_{(3,4)}$ includes path $3 = (3, 2, 4)$ and path $4 = (3, 4)$. And by (2.4), we have

$$\begin{aligned}
\overline{T}_1(h) &= \bar{t}_{e_1}(h) + \bar{t}_{e_5}(h) = 50v_{e_1} + 70v_{e_5} + 400, \qquad \overline{T}_2(h) = \bar{t}_{e_2}(h) = 20v_{e_2} + 500, \\
\overline{T}_3(h) &= \bar{t}_{e_3}(h) + \bar{t}_{e_5}(h) = 60v_{e_3} + 70v_{e_5} + 400, \qquad \overline{T}_4(h) = \bar{t}_{e_4}(h) = 30v_{e_4} + 200.
\end{aligned} \tag{2.6}$$

Then, for flow $h = (h_1, h_2, h_3, h_4) = (2, 1, 1, 3)$, we have that arc flows

$$v = (v_{e_1}, v_{e_2}, v_{e_3}, v_{e_4}, v_{e_5}) = (2, 1, 1, 3, 3). \tag{2.7}$$

It follows from (2.4) that

$$\begin{aligned}
\overline{T}_1(h) &= \bar{t}_{e_1}(v_{e_1}) + \bar{t}_{e_5}(v_{e_5}) = 50 \times 2 + 100 + 70 \times 3 + 300 = 710, \\
\overline{T}_3(h) &= \bar{t}_{e_3}(v_{e_3}) + \bar{t}_{e_5}(v_{e_5}) = 60 \times 1 + 100 + 70 \times 3 + 300 = 670.
\end{aligned} \tag{2.8}$$

However, from the practical point of view, the cost values of the path 1 and path 3 with respect to h are, respectively, as follows:

$$\begin{aligned}
T_1(h) &= \bar{t}_{e_1}(h_1) + \bar{t}_{e_5}(h_1) = 50 \times 2 + 100 + 70 \times 2 + 300 = 640, \\
T_3(h) &= \bar{t}_{e_3}(h_3) + \bar{t}_{e_5}(h_3) = 60 \times 1 + 100 + 70 \times 1 + 300 = 530.
\end{aligned} \tag{2.9}$$

So, in this paper, we define the vector-valued cost function on a path as (2.3).

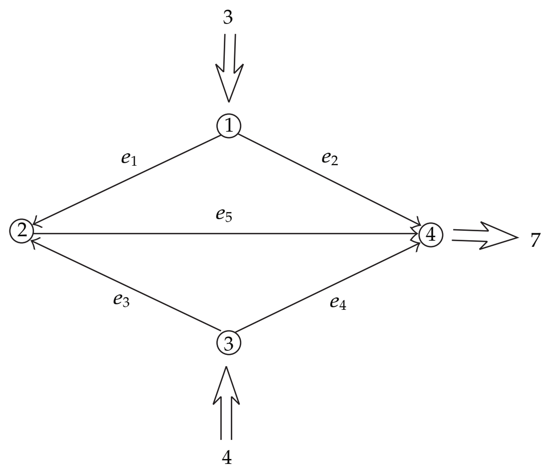

Figure 1: Network topology for an example.

In this paper, the cost space is an r-dimensional Euclidean space R^r, with the ordering cone $S = R^r_+$, a pointed, closed, and convex cone with nonempty interior $\operatorname{int} S$. We define the ordering relation as follows:

$$x \leq_S y, \quad \text{iff } y - x \in S;$$
$$x <_S y, \quad \text{iff } y - x \in \operatorname{int} S. \tag{2.10}$$

The orderings $\geq_S$ and $>_S$ are defined similarly. In the sequel, we let the set $S^+ := \{\varphi \in R^r : \varphi(s) \geq 0, \text{ for all } s \in S\}$ be the dual cone of S. Denote the interior of S^+ by

$$\operatorname{int} S^+ := \left\{\varphi \in R^r : \varphi(s) > 0, \ \forall s \in S \setminus \{0\}\right\}. \tag{2.11}$$

Lemma 2.3 (see [22]). *Consider*

$$S \setminus \{0\} := \left\{x \in R^r : \varphi(x) > 0, \ \forall \varphi \in \operatorname{int}\ S^+\right\},$$
$$\operatorname{int} S := \left\{x \in R^r : \varphi(x) > 0, \ \forall \varphi \in S^+ \setminus \{0\}\right\}. \tag{2.12}$$

Definition 2.4 (see [18, 19]). Assume that a flow $h \in H$,

(i) for $e \in E$, if $v_e = c_e$, then arc e is said to be a saturated arc of flow h, otherwise a nonsaturated arc of flow h.

(ii) for $k \in \bigcup_{w \in W} P_w$, if there exists a saturated arc e of flow h such that e belongs to path k, then path k is said to be a saturated path of flow h, otherwise a nonsaturated path of flow h.

We introduced the following vector equilibrium principle and weak vector equilibrium principle.

Definition 2.5 (vector equilibrium principle). A flow $h \in H$ is said to be a vector equilibrium flow if for all $w \in W$, for all $k, j \in P_w$, we have

$$T_k(h) - T_j(h) \in S \setminus \{0\} \Longrightarrow h_k = 0 \text{ or path } j \text{ is a saturated path of flow } h. \tag{2.13}$$

Definition 2.6 (weak vector equilibrium principle). A flow $h \in H$ is said to be a weak vector equilibrium flow if for all $w \in W$, for all $k, j \in P_w$, we have

$$T_k(h) - T_j(h) \in \operatorname{int} S \Longrightarrow h_k = 0 \text{ or path } j \text{ is a saturated path of flow } h. \tag{2.14}$$

If for all $e \in E$, $c_e = c \geq \sum_{w \in W} d_w$, then the capacity constraints of arcs are invalid, in this case, the traffic equilibrium problem with capacity constraints of arcs reduces to the traffic equilibrium problem without capacity constraints of arcs.

3. Sufficient and Necessary Conditions for a (Weak) Vector Equilibrium Flow

In this section, we introduce an $\operatorname{int} S^+$-parametric equilibrium flow, a $S^+ \setminus \{0\}$-parametric equilibrium flow and a φ-parametric equilibrium flow, respectively. By using the three new concepts, we can obtain some sufficient and necessary conditions of a vector equilibrium flow and a weak vector equilibrium flow, respectively.

Definition 3.1. A flow $h \in H$ is said to be in $\operatorname{int} S^+$-parametric equilibrium if for all $w \in W$, for all $k, j \in P_w$ and for all $\varphi \in \operatorname{int} S^+$, we have

$$\varphi\big(T_k(h) - T_j(h)\big) > 0 \Longrightarrow h_k = 0 \text{ or path } j \text{ is a saturated path of flow } h. \tag{3.1}$$

Definition 3.2. A flow $h \in H$ is said to be in $S^+ \setminus \{0\}$-parametric equilibrium if for all $w \in W$, for all $k, j \in P_w$ and for all $\varphi \in S^+ \setminus \{0\}$, we have

$$\varphi\big(T_k(h) - T_j(h)\big) > 0 \Longrightarrow h_k = 0 \text{ or path } j \text{ is a saturated path of flow } h. \tag{3.2}$$

Definition 3.3. Let a $\varphi \in S^+ \setminus \{0\}$ be given. A flow $h \in H$ is said to be in φ-parametric equilibrium flow if for all $w \in W$ and for all $k, j \in P_w$, we have

$$\varphi\big(T_k(h) - T_j(h)\big) > 0 \Longrightarrow h_k = 0 \text{ or path } j \text{ is a saturated path of flow } h. \tag{3.3}$$

The $\operatorname{int} S^+$-equilibrium flow and φ-parametric equilibrium flow for some $\varphi \in \operatorname{int} S^+$ are defined in Definitions 3.1 and 3.2, respectively. They can be used to characterize vector equilibrium flow in the following theorems.

Theorem 3.4. *A flow $h \in H$ is in vector equilibrium if and only if the flow h is in* int *S^+-parametric equilibrium.*

Proof. It can get immediately the above conclusion by Lemma 2.3. Thus the proof is omitted here. □

Theorem 3.5. *If there exists $\varphi \in \operatorname{int} S^+$ such that a flow $h \in H$ is in φ-parametric equilibrium, then the flow h is in vector equilibrium.*

Proof. Suppose that for any *O-D* pair $w \in W$, for all $k, j \in P_w$, we have

$$T_k(h) - T_j(h) \in S \setminus \{0\}. \tag{3.4}$$

By $\varphi \in \operatorname{int} S^+$ and Lemma 2.3, we get immediately

$$\varphi\left[T_k(h) - T_j(h)\right] > 0. \tag{3.5}$$

Since h is in φ-parametric equilibrium, we have

$$h_k = 0 \text{ or path } j \text{ is a saturated path of flow } h. \tag{3.6}$$

Thus, the flow $h \in H$ is in vector equilibrium. □

Now, we give the following example to illustrate Theorem 3.5.

Example 3.6. Consider the network problem depicted in Figure 2. $N = \{1,2,3,4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, $C = (3,3,3,2,3,4)^T$, $W = \{(1,4),(3,4)\}$, $D = (6,4)$. The cost functions of arcs from R to R^2 are defined as follows:

$$t_{e_1}(h_1) = \begin{pmatrix} h_1^2 + 1 \\ 2h_1 \end{pmatrix}, \qquad t_{e_2}(h_2) = \begin{pmatrix} 5h_2 \\ 3h_2^2 \end{pmatrix}, \qquad t_{e_3}(h_3) = \begin{pmatrix} h_3^2 + 7 \\ 5h_3 \end{pmatrix},$$
$$t_{e_4}(h_4) = \begin{pmatrix} 2h_4 + 1 \\ 3h_4 \end{pmatrix}, \qquad t_{e_5}(h_5) = \begin{pmatrix} 3h_5^2 \\ 6h_5 \end{pmatrix}, \qquad t_{e_6}(h_1) = \begin{pmatrix} h_1^2 \\ 2h_1 \end{pmatrix}, \qquad t_{e_6}(h_4) = \begin{pmatrix} h_4^2 \\ 2h_4 \end{pmatrix}. \tag{3.7}$$

Then, we have

$$T_1(h) = t_{e_1}(h_1) + t_{e_6}(h_1) = \begin{pmatrix} 2h_1^2 + 1 \\ 4h_1 \end{pmatrix}, \qquad T_4(h) = t_{e_4}(h_4) + t_{e_6}(h_4) = \begin{pmatrix} h_4^2 + 2h_4 + 1 \\ 5h_4 \end{pmatrix},$$
$$T_2(h) = t_{e_2}(h_2) = \begin{pmatrix} 5h_2 \\ 3h_2^2 \end{pmatrix}, \qquad T_3(h) = t_{e_3}(h_3) = \begin{pmatrix} h_3^2 + 7 \\ 5h_3 \end{pmatrix}, \qquad T_5(h) = t_{e_5}(h_5) = \begin{pmatrix} 3h_5^2 \\ 6h_5 \end{pmatrix}. \tag{3.8}$$

Taking $h^* = (2,2,2,2,2)' \in H$, then there exists $\overline{\varphi} = (1,1) \in \operatorname{int} R_+^2$ such that the flow h^* is in $\overline{\varphi}$-parametric equilibrium. Thus, by Theorem 3.5, we have that the flow h^* is in vector equilibrium.

For weak vector equilibrium flows, we have following similar results.

Theorem 3.7. *A path flow $h \in H$ is in weak vector equilibrium if and only if the flow h is in $S^+ \setminus \{0\}$-parametric equilibrium.*

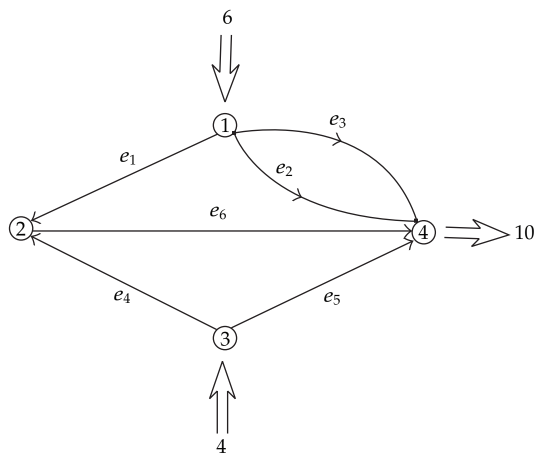

Figure 2: Network topology for an example.

Theorem 3.8. *If there exists $\varphi \in S^+ \setminus \{0\}$ such that a path flow $h \in H$ is in φ-parametric equilibrium, then the flow h is in weak vector equilibrium.*

From Theorems 3.4–3.8, we can get immediately the following corollaries.

Corollary 3.9. *If there exists $\varphi \in \operatorname{int} S^+$ such that a flow $h \in H$ is in φ-parametric equilibrium, then the flow h is in $\operatorname{int} S^+$-parametric equilibrium.*

Corollary 3.10. *If there exists $\varphi \in S^+ \setminus \{0\}$ such that a flow $h \in H$ is in φ-parametric equilibrium, then the flow h is in $S^+ \setminus \{0\}$-parametric equilibrium.*

Remark 3.11. When a flow $h \in H$ is in $\operatorname{int} S^+$-parametric equilibrium, then, the flow h may not be in φ-parametric equilibrium for some $\varphi \in \operatorname{int} S^+$. Of course, when a flow $h \in H$ is in $S^+ \setminus \{0\}$-parametric equilibrium, then, the flow h may not be in φ-parametric equilibrium for some $\varphi \in S^+ \setminus \{0\}$. The following example can explain these cases.

Example 3.12. Consider the network problem depicted in Figure 1. $N = \{1,2,3,4\}$, $E = \{e_1, e_2, e_3, e_4, e_5,\}$, $C = (3,3,2,4,3,)$, $W = \{\{1,4\},\{3,4\}\}$, $D = \{3,4\}$. Let the cost functions of arcs are defined as follows:

$$t_{e_1}(h_1) = \begin{pmatrix} h_1^2 + 2 \\ h_1^2 + 3 \end{pmatrix}, \qquad t_{e_2}(h_2) = \begin{pmatrix} h_2^2 + h_2 + 2 \\ h_2 + 2 \end{pmatrix}, \qquad t_{e_3}(h_3) = \begin{pmatrix} 3h_3^2 + 2 \\ 2h_3 + 2 \end{pmatrix},$$
$$t_{e_4}(h_4) = \begin{pmatrix} 2h_4 + 4 \\ h_4 + 1 \end{pmatrix}, \qquad t_{e_5}(h_1) = \begin{pmatrix} h_1^2 + 2 \\ 2h_1 \end{pmatrix}, \qquad t_{e_5}(h_3) = \begin{pmatrix} h_3^2 + 2 \\ 2h_3 \end{pmatrix}. \tag{3.9}$$

Then, we have

$$T_1(h) = t_{e_1}(h_1) + t_{e_6}(h_1) = \begin{pmatrix} 2h_1^2 + 4 \\ h_1^2 + 2h_1 + 3 \end{pmatrix}, \qquad T_2(h) = t_{e_2}(h_2) = \begin{pmatrix} h_2^2 + h_2 + 2 \\ h_2 + 2 \end{pmatrix},$$
$$T_3(h) = t_{e_3}(h_3) = t_{e_3}(h_3) + t_{e_5}(h_3) = \begin{pmatrix} 4h_3^2 + 4 \\ 4h_3 + 2 \end{pmatrix}, \qquad T_4(h) = t_{e_4}(h_4) = \begin{pmatrix} 2h_4 + 4 \\ h_4 + 1 \end{pmatrix}. \tag{3.10}$$

Taking

$$h^* = (1, 2, 1, 3)', \tag{3.11}$$

we have

$$T_1(h^*) = \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \qquad T_2(h^*) = \begin{pmatrix} 8 \\ 4 \end{pmatrix}, \qquad T_3(h^*) = \begin{pmatrix} 8 \\ 6 \end{pmatrix}, \qquad T_4(h^*) = \begin{pmatrix} 10 \\ 4 \end{pmatrix}. \tag{3.12}$$

Thus, by Definitions 3.1 and 3.2, we know that the flow h^* is a $\operatorname{int} S^+$-parametric equilibrium flow and is a $S^+ \setminus \{0\}$-parametric equilibrium flow as well. On the other hand, for $\varphi = (1, 1/2)' \in \operatorname{int} S^+ \subset S^+ \setminus \{0\}$, there exists $w = \{1, 4\}$ and path $1, 2 \in P_w$, we have

$$\varphi[T_2(h^*) - T_1(h^*)] = 1 > 0. \tag{3.13}$$

But, $h_2 = 2 > 0$ and path 1 is nonsaturated path of h^*. Thus, it follows from Definition 3.3 that the flow h^* is not in φ-parametric equilibrium.

Theorem 3.13. *Let $\varphi \in S^+ \setminus \{0\}$ be given. A flow $h \in H$ is in φ-parametric equilibrium if the flow h solves the following scalar variational inequality:*

$$\sum_{w \in W} \sum_{p \in P_w} \varphi(T_p(h))(f_p - h_p) \geq 0, \quad \forall f \in H. \tag{3.14}$$

Proof. Assume that $h \in H$ solves above scalar variational inequality problem. For all $w \in W$, for all $k, j \in P_w$, if $\varphi[T_k(h) - T_j(h)] = \varphi[T_k(h)] - \varphi[T_j(h)] > 0$ and path j is nonsaturated path of flow h, we need to prove that $h_k = 0$. Denote that $p_j = \{e \in E \mid \operatorname{arc} e \text{ belongs to path } j\}$. If the conclusion is false, then

$$\epsilon = \min\left\{ \min_{e \in p_j}(c_e - v_e), h_k \right\} > 0. \tag{3.15}$$

Construct a flow f as follows:

$$f = (f_l) = \begin{cases} h_l, & \text{if } l \neq k \text{ or } j, \\ (h_k - \epsilon), & \text{if } l = k, \\ (h_j + \epsilon), & \text{if } l = j. \end{cases} \tag{3.16}$$

It is easy to verify that

$$f \in H. \tag{3.17}$$

It follows readily that

$$\begin{aligned}\sum_{w\in W}\sum_{p\in P_w}\varphi(T_p(h))(f_p-h_p) &= \varphi(T_k(h))(f_k-h_k)+\varphi(T_j(h))(f_j-h_j)\\ &= \epsilon(\varphi[T_j(h)]-\varphi[T_k(h)])\\ &< 0,\end{aligned} \tag{3.18}$$

which contradicts (3.14). Thus, h is in φ-parametric equilibrium and the proof is complete. □

From Theorems 3.4–3.13, we can get the following corollary.

Corollary 3.14. *If there exists $\varphi \in \operatorname{int} S^+$ ($\varphi \in S^+ \setminus \{0\}$) such that a flow $h \in H$ is a solution of the following scalar variational inequality:*

$$\sum_{w\in W}\sum_{p\in P_w}\varphi(T_p(h))(f_p-h_p)\geq 0, \quad \forall f\in H, \tag{3.19}$$

then the flow h is in (weak) vector equilibrium.

Remark 3.15. We can prove that the the converse of Theorem 3.13 is valid when the traffic network equilibrium problem without capacity constraints of arcs, such as traffic network equilibrium problems without capacity constraints or with capacity constraints of paths. The result will be showed on Theorem 3.18. But, if the traffic network equilibrium problem with capacity constraints of arcs, then the converse of Theorem 3.13 may not hold. The following example is given to illustrate the case.

Example 3.16. Consider the network problem depicted in Figure 1. $N = \{1,2,3,4\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, $C = (3,2,2,4,3)$, $W = \{\{1,4\},\{3,4\}\}$, $D = \{3,4\}$. Let the cost functions of arcs are defined as follows:

$$\begin{gathered} t_{e_1}(h_1)=\begin{pmatrix}h_1\\ h_1^2\end{pmatrix},\qquad t_{e_2}(h_2)=\begin{pmatrix}h_2^2+3h_2+5\\ h_2^3+4h_2+3\end{pmatrix},\qquad t_{e_3}(h_3)=\begin{pmatrix}h_3^3+3\\ h_3^2+4\end{pmatrix},\\ t_{e_4}(h_4)=\begin{pmatrix}h_4+4\\ h_4+4\end{pmatrix},\qquad t_{e_5}(h_1)=\begin{pmatrix}h_1^2+1\\ h_1\end{pmatrix},\qquad t_{e_5}(h_3)=\begin{pmatrix}h_3^2+1\\ h_3\end{pmatrix}. \end{gathered} \tag{3.20}$$

Then, we have

$$\begin{gathered} T_1(h)=t_{e_1}(h_1)+t_{e_5}(h_1)=\begin{pmatrix}h_1^2+h_1+1\\ h_1^2+h_1\end{pmatrix},\qquad T_2(h)=t_{e_2}(h_2)=\begin{pmatrix}h_2^2+3h_2+5\\ h_2^3+4h_2+3\end{pmatrix},\\ T_3(h)=t_{e_3}(h_3)+t_{e_5}(h_3)=\begin{pmatrix}h_3^3+h_3^2+4\\ h_3^2+h_3+4\end{pmatrix},\qquad T_4(h)=t_{e_4}(h_4)=\begin{pmatrix}h_4+4\\ h_4+4\end{pmatrix}. \end{gathered} \tag{3.21}$$

Taking

$$h^* = (2, 1, 1, 3)', \tag{3.22}$$

we have

$$T_1(h^*) = \begin{pmatrix} 7 \\ 6 \end{pmatrix}, \qquad T_2(h^*) = \begin{pmatrix} 9 \\ 8 \end{pmatrix}, \qquad T_3(h^*) = \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \qquad T_4(h^*) = \begin{pmatrix} 7 \\ 7 \end{pmatrix}. \tag{3.23}$$

Then for any $\varphi \in \operatorname{int} S^+ (\varphi \in S^+ \setminus \{0\})$, we have

$$\begin{aligned} &\varphi[T_2(h^*) - T_1(h^*)] > 0, \\ &\varphi[T_4(h^*) - T_3(h^*)] > 0, \end{aligned} \tag{3.24}$$

and path 1 is a saturated arc path of h^*, and path 3 is a saturated arc path of h^* as well. Thus, the flow h^* is a φ-parametric equilibrium flow by Definition 3.3. However, taking $f = (3, 0, 0, 4)' \in H$, we have

$$\sum_{w \in W} \sum_{p \in P_w} T_p(h^*) \left(f_p - h_p^* \right) = (-1, -1)'. \tag{3.25}$$

Thus, for any $\varphi \in \operatorname{int} S^+ (\varphi \in S^+ \setminus \{0\})$, we can always get

$$\sum_{w \in W} \sum_{p \in P_w} \varphi(T_p(h^*)) \left(f_p - h_p^* \right) < 0. \tag{3.26}$$

Therefore, the converse of Theorem 3.13 is not valid.

The following theorem shows that the converse of Theorem 3.13 is valid when the traffic equilibrium problem with capacity constraints of paths. The proof is similar when the traffic network equilibrium problem without capacity constraints. Let

$$K := \left\{ h \mid \lambda \le h \le \mu, \sum_{p \in P_w} h_p = d_w, \ \forall w \in W \right\}, \tag{3.27}$$

be the feasible set of traffic network equilibrium problem with capacity constraints of paths, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ are lower and upper capacity constraints of paths, respectively. The φ-parametric equilibrium principle of traffic equilibrium problem with capacity constraints of paths is as follows.

Definition 3.17. Let a $\varphi \in S^+ \setminus \{0\}$ be given. A flow $h \in H$ is said to be in φ-parametric equilibrium flow if for all $w \in W$ and for all $k, j \in P_w$, we have

$$\varphi(T_k(h) - T_j(h)) > 0 \Longrightarrow h_j = \mu_j \text{ or } h_k = \lambda_k. \tag{3.28}$$

Theorem 3.18. *Let $\varphi \in S^+ \setminus \{0\}$ be given. A path $h \in K$ is in φ-parametric equilibrium if and only if the flow h solves the following scalar variational inequality:*

$$\sum_{w\in W}\sum_{p\in P_w}\varphi\left(T_p(h)\right)\left(f_p - h_p\right) \geq 0, \quad \forall f \in K. \tag{3.29}$$

Proof. From Theorem 3.13, we only prove necessity. So, we set

$$A_w := \{v \in P_w \mid h_v > \lambda_v\}, \qquad B_w := \{u \in P_w \mid h_u < \mu_u\}. \tag{3.30}$$

It follows from the definition of the φ-parametric equilibrium flow that

$$\varphi[T_u(h)] \geq \varphi[T_v(h)], \quad \forall u \in B_w,\ v \in A_w. \tag{3.31}$$

Thus, there exists a $\gamma_w \in R$ such that

$$\min_{u\in B_w}\varphi[T_u(h)] \geq \gamma_w \geq \max_{v\in A_w}\varphi[T_v(h)]. \tag{3.32}$$

Let $f \in K$ be arbitrary. Then, for every $r \in P_w$, we consider three cases.

Case 1. If $\varphi[T_r(h)] < \gamma_w$, then $r \notin B_w$. Hence, $h_r = \mu_r$, $f_r - h_r \leq 0$ and

$$\left[\varphi(T_k(h)) - \gamma_w\right]\left(f_r - h_r\right) \geq 0. \tag{3.33}$$

Case 2. If $\varphi[T_r(h)] > \gamma_w$, then $r \notin A_w$. Hence, $h_r = \lambda_r$, $f_r - h_r \geq 0$ and

$$\left[\varphi(T_k(h)) - \gamma_w\right]\left(f_r - h_r\right) \geq 0. \tag{3.34}$$

Case 3. If $\varphi[T_r(h)] = \gamma_w$, then we have

$$\left[\varphi(T_k(h)) - \gamma_w\right]\left(f_r - h_r\right) \geq 0. \tag{3.35}$$

From (3.33), (3.34), and (3.35), we have

$$\sum_{w\in W}\sum_{p\in P_w}\varphi\left(T_p(h)\right)\left(f_p - h_p\right) \geq \sum_{w\in W}\sum_{p\in P_w}\gamma_w\left(d_w - d_w\right) = 0. \tag{3.36}$$

Thus, the proof is complete. □

4. Conclusions

In this paper, we have studied traffic network equilibrium problems with capacity constraints of arcs. We have introduced some new parametric equilibrium flows, such as: $S^+ \setminus \{0\}$-parametric equilibrium flows, int S^+-parametric equilibrium flows, and φ-parametric equilibrium flows. By using these new concepts, we have characterized vector equilibrium problems on networks and derived some necessary and sufficient conditions for a (weak) vector equilibrium flow.

References

[1] J. Wardrop, "Some theoretical aspects of road traffic research," *Proceedings of the Institute of Civil Engineers*, vol. 1, pp. 325–378, 1952.

[2] Q. Meng, Y. K. Huang, and R. L. Cheu, "A note on supply chain network equilibrium models," *Transportation Research E*, vol. 43, no. 1, pp. 60–71, 2007.

[3] A. Nagurney and J. Dong, "A multiclass, multicriteria traffic network equilibrium model with elastic demand," *Transportation Research B*, vol. 36, no. 5, pp. 445–469, 2002.

[4] A. Nagurney, J. Dong, and D. Zhang, "A supply chain network equilibrium model," *Transportation Research E*, vol. 38, no. 5, pp. 281–303, 2002.

[5] A. Nagurney, "On the relationship between supply chain and transportation network equilibria: a supernetwork equivalence with computations," *Transportation Research E*, vol. 42, no. 4, pp. 293–316, 2006.

[6] G. Y. Chen and N. D. Yen, *On the Variational Inequality Model For Network Equilibrium*, vol. 724, Department of Mathematics, University of Pisa, 1993, Internal Report 3.

[7] P. Q. Khanh and L. M. Luu, "On the existence of solutions to vector quasivariational inequalities and quasicomplementarity problems with applications to traffic network equilibria," *Journal of Optimization Theory and Applications*, vol. 123, no. 3, pp. 533–548, 2004.

[8] P. Q. Khanh and L. M. Luu, "Some existence results for vector quasivariational inequalities involving multifunctions and applications to traffic equilibrium problems," *Journal of Global Optimization*, vol. 32, no. 4, pp. 551–568, 2005.

[9] G. Y. Chen, C. J. Goh, and X. Q. Yang, "Vector network equilibrium problems and nonlinear scalarization methods," *Mathematical Methods of Operations Research*, vol. 49, no. 2, pp. 239–253, 1999.

[10] T. C. E. Cheng and Y. N. Wu, "A multiproduct, multicriterion supply-demand network equilibrium model," *Operations Research*, vol. 54, no. 3, pp. 544–554, 2006.

[11] S. J. Li, X. Q. Yang, and G. Y. Chen, "A note on vector network equilibrium principles," *Mathematical Methods of Operations Research*, vol. 64, no. 2, pp. 327–334, 2006.

[12] S. J. Li, K. L. Teo, and X. Q. Yang, "A remark on a standard and linear vector network equilibrium problem with capacity constraints," *European Journal of Operational Research*, vol. 184, no. 1, pp. 13–23, 2008.

[13] S. J. Li, K. L. Teo, and X. Q. Yang, "Vector equilibrium problems with elastic demands and capacity constraints," *Journal of Global Optimization*, vol. 37, no. 4, pp. 647–660, 2007.

[14] C. J. Goh and X. Q. Yang, "Vector equilibrium problem and vector optimization," *European Journal of Operational Research*, vol. 116, no. 3, pp. 615–628, 1999.

[15] F. Raciti, "Equilibrium conditions and vector variational inequalities: a complex relation," *Journal of Global Optimization*, vol. 40, no. 1–3, pp. 353–360, 2008.

[16] X. Q. Yang and C. J. Goh, "On vector variational inequalities: application to vector equilibria," *Journal of Optimization Theory and Applications*, vol. 95, no. 2, pp. 431–443, 1997.

[17] D. T. Luc, M. Rocca, and M. Papalia, "Equilibrium in a vector supply-demand network with capacity constraints," *Applicable Analysis*, vol. 90, no. 6, pp. 1029–1045, 2011.

[18] Z. Lin, "The study of traffic equilibrium problems with capacity constraints of arcs," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 4, pp. 2280–2284, 2010.

[19] Z. Lin, "On existence of vector equilibrium flows with capacity constraints of arcs," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 72, no. 3-4, pp. 2076–2079, 2010.

[20] Y. D. Xu, S. J. Li, and K. L. Teo, "Vector network equilibrium problems with capacity constraints of arcs," *Transportation Research E*, vol. 48, pp. 567–577, 2012.

[21] A. Zaffaroni, "Degrees of efficiency and degrees of minimality," *SIAM Journal on Control and Optimization*, vol. 42, no. 3, pp. 1071–1086, 2003.

[22] J. Jahn, *Vector Optimization. Theory, Applications, and Extensions*, Springer, Berlin, Germany, 2004.

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Research Article

An Optimization Model of the Single-Leg Air Cargo Space Control Based on Markov Decision Process

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Based on the single-leg air cargo issues, we establish a dynamic programming model to consider the overbooking and space inventory control problem. We analyze the structure of optimal booking policy for every kind of booking requests and show that the optimal booking decision is of threshold type (known as booking limit policy). Our research provides a theoretical support for the air cargo space control.

1. Introduction

For the air cargo carrier, the main purpose of implementing seat inventory control in the actual operation is to avoid cargo space being occupied by too many low-value goods, causing the lack of timely transportation of high-value goods and resulting in the loss of some potential gains. In fact, some customers (FITs or agents) who order the space through telephone or network temporarily cancel the booking or directly do not appear while the aircraft is taking off. It brings a lot of losses to the air cargo carrier. In order to reduce the losses caused by the empty cabin, decision makers tend to overbook. However, overbooking too much may lead to greater economic losses. For this reason, during the air cargo overbooking, it is necessary to consider freight revenue, but at the same time decision makers have to consider the cost of overbooking as well. Above all, the decision makers' aim is to obtain the maximum profit of freight. Usually, most of the goods are transported by single legs (single routes). For some relatively tight (the market demand is greater than supply) segments (routes), it is more suitable for scientific space control and proper overbooking. The problem can be described in detail as follows: in some tight legs, if there are some booking requests, decision makers will decide whether to accept or not. If it is accepted, decision makers will have to set aside proper space to its customers in accordance with the cargo information. Simply put, in the case of implementing overbooking, the goal is to maximize air cargo revenue by properly accepting customers' booking requests.

2. Related Literature Review

Based on the analysis of the characteristics of the products, Kasilingam [1] establishes the air cargo overbooking decision-making model on the case of space capacity random, discusses the production capacity in discrete case and continuous case, and identifies the optimal overbooking for each situation. Kleywegt and Papastavrou [2] use dynamic random backpack model to discuss air cargo revenue management problem. The above-mentioned method can be used to describe the multidimensional problems, but the model is too difficult to solve easily. Luo et al. [3] establish a two-dimensional air cargo overbooking model and then determine the approximate optimal overbooking level. In this model, both the weight attributes and volume attributes are taken into account, and the objective is to minimize freight costs. Moussawi and Cakanyildrima [4] further study the two-dimensional air cargo overbooking model. The objective is to maximize freight revenue. What they do is different from Luo et al. [3]. Considering the benefits of passengers and cargo, Sandhu and Klabjan [5] establish space inventory control model with the static method. On the assumption of fixed volume, using the method of reducing dimension, they get the model's approximate solution and gain the maximum revenue. Considering the freight forwarders and the delay of cargo transportation, Chew et al. [6] establish a stochastic dynamic programming model of short-term space inventory control, and the objective is to minimize transportation costs. Amaruchkul et al. [7] consider a single-leg air cargo space control problem. Under the random cargo volume and weight, they build a Markov decision process model and take a heuristic algorithm to analyze the model. Considering protocol sale customers and free sale customers, Levin and Nediak [8] build a space inventory control model by using dynamic programming methods, and the purpose is to make maximum total income rooted in the receipt of the goods. Under uncertain environment, Wang and Kao [9] establish an air cargo overbooking model and solve it by the fuzzy systems approach. Taking into account the two different demands of customers, Modarres and Sharifyazdi [10] build a random capacity space inventory control model and get the optimal decision. By analyzing the expected revenue function in dynamic programming model, combining the randomness of cargo volume and weight, Huang and Chang [11] establish a more efficient algorithm in air cargo revenue management problem. Supposing the cargo volume, weight and the yields of the air cargo are random and the cargo space booking process has no aftereffect, Han et al. [12] set up a single-leg air cargo revenue management space allocation model and take the bid control strategy to determine the goods receiving. Amaruchkul and Lorchirachoonkul [13] study the allocation of air transport capacity for the number of freight forwarders, get the probability distribution of the actual use of the transport capacity through discrete Markov chain, and solve the model by dynamic programming methods.

By contrast, our work is to consider a single-leg air cargo overbooking and space inventory control problem by dynamic programming. We analyze the structure of optimal booking policy for every kind of booking requests and show that the optimal booking decision is of booking limit policy.

3. Model Description

Next we will study the single-leg air cargo space inventory control and overbooking under the conditions of shipping season (the demand is greater than supply). Because of the complexity of the actual situation, in order to abstract practical problems to theoretical issues, we need some basic assumptions.

(i) Aircraft total capacity is fixed (cargo size is unchanged);

(ii) customers' booking requests are sufficient, namely, the supply is adequate;

(iii) the booking requests are divided into multiple classes;

(iv) each class's arrival is independent;

(v) there is at most one booking request at any time;

(vi) on the condition that the aircraft is due to take off, whether or not to show up for each booking request is independent;

(vii) each booking request's weight (volume) is identically independently distributed;

(viii) during whole booking period, accepted booking requests will not be free to cancel.

Based on the above assumptions, we will establish mathematical model and make decisions on air cargo. In our model, the main parameters and variables are as follows:

t: decision time, $0 \leq t \leq T$, where T is a booking period;

j: jth class booking request, $j = 1, 2, \ldots, n$, where n is the number of booking classes;

p_{jt}: the arrival probability of jth class booking request in time t;

$p_{0t} = 1 - \sum_{j=1}^{n} p_{jt}$: probability of no booking request in time t;

r_j: unit tariff of jth class booking request;

ρ: penalty cost because of overbooking resulting in denying a booking request;

y_j: cumulative number of the accepted jth class booking request in time t;

$y = \sum_{j=1}^{n} y_j$: cumulative number of all accepted booking requests in time t;

θ: the show-up rate of each accepted booking request near the takeoff;

C_w: the maximum available load of the aircraft;

C_v: the maximum available volume of the aircraft;

W: weight of each booking request in time t;

V: volume of each booking request in time t;

$\overline{w}$: expected weight of each booking request in time t, namely, $E(W)$;

$\overline{v}$: expected volume of each booking request in time t, namely, $E(V)$;

λ: IATA required standard density, value of $0.006\,\mathrm{m}^3/\mathrm{kg}$.

In order to meet up only one request in a period, we will divide booking lead time into a number of small discrete booking periods. $t = 0$ means that air cargo carrier begins to accept the booking; $t = T$ marks the end of booking, then the air cargo carrier should consider the DB (denying booking) problem.

When $0 < t < T$, $U_t(y)$ is the maximum total expected profit given state y in time t. Given t and y, the dynamic programming optimality equation can be written as

$$U_t(y) = \sum_{j=1}^{n} p_{jt} \max\left\{U_{t+1}(y+1) + r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\}, U_{t+1}(y)\right\} + p_{0t}U_{t+1}(y) \tag{3.1}$$

$$= U_{t+1}(y) + \sum_{j=1}^{n} p_{jt} \max\left\{r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y), 0\right\}, \tag{3.1$'$}$$

where $\Delta U_t(y) = U_{t+1}(y) - U_{t+1}(y+1)$ is the opportunity cost of accepting some class request in time t. From (3.1)$'$, it is optimal to accept jth class booking request in time t if $r_j \max\{\overline{w}, \overline{v}/\lambda\} - \Delta U_t(y) \geq 0$.

Obviously, $U_0(0)$ is the maximum total expected profit from the beginning of booking to the end of booking. In $t = T$, the penalty cost of overbooking is

$$U_T(y) = -\rho E\left[(y\theta W - C_w)^+ + \left(\frac{y\theta V - C_v}{\lambda}\right)^+\right] \quad \forall y. \tag{3.2}$$

The boundary condition (3.2) shows that because of overbooking and relatively fixed capacity, air cargo carrier needs to deny some accepted requests at the end of booking.

4. Structural Properties

In (3.2), there exist continuous random variables W and V. The cumulative distribution function and probability density function are $F_W(\cdot)$, $F_V(\cdot)$ and $f_W(\cdot)$, $f_V(\cdot)$, respectively.

Next, we will prove that $U_t(y)$ satisfies the following first-order and second-order properties as follows.

Property 1. $\Delta U_t(y) \geq 0$, namely, $U_t(y)$ is decreasing in y.

Proof. When $t = T$,

$$\begin{aligned}
U_T(y) &= -\rho E\left[(y\theta W - C_w)^+ + \left(\frac{y\theta V - C_v}{\lambda}\right)^+\right] \\
&= -\rho\left[\int_{C_w/y\theta}^{+\infty} (y\theta w - C_w) f_W(w)dw + \int_{C_v/y\theta}^{+\infty} \left(\frac{y\theta v - C_v}{\lambda}\right) f_V(v)dv\right] \\
&= -\rho\left[y\theta \int_{C_w/y\theta}^{+\infty} w f_W(w)dw - C_w \int_{C_w/y\theta}^{+\infty} f_W(w)dw \right. \\
&\qquad \left. + \frac{y\theta}{\lambda} \int_{C_v/y\theta}^{+\infty} v f_V(v)dv - \frac{C_v}{\lambda} \int_{C_v/y\theta}^{+\infty} f_V(v)dv\right],
\end{aligned}$$

$$\therefore \frac{dU_T(y)}{dy} = -\rho\theta\left[\int_{C_w/y\theta}^{+\infty} w f_W(w)dw + \frac{1}{\lambda} \int_{C_v/y\theta}^{+\infty} v f_V(v)dv\right] \leq 0. \tag{4.1}$$

Then, $U_T(y)$ is decreasing in y.

Suppose that $U_{t+1}(y)$ is decreasing in y. Next, we will show that $U_t(y)$ is decreasing in y.

From (3.1), we can get the conclusion easily. This completes the proof. □

Property 2. $U_t(y)$ is concave in y, that is, for each given t, $\Delta U_t(y)$ is increasing in y.

Proof. When $t = T$,

$$\frac{d^2U_T(y)}{dy^2} = -\rho\theta\left[\frac{C_w}{y\theta} f_W\left(\frac{C_w}{y\theta}\right)\frac{C_w}{\theta}\frac{1}{y^2} + \frac{1}{\lambda}\frac{C_v}{y\theta} f_W\left(\frac{C_v}{y\theta}\right)\frac{C_v}{\theta}\frac{1}{y^2}\right] \le 0. \tag{4.2}$$

Then $\Delta U_T(y)$ is increasing in y, namely, $U_T(y)$ is concave in y.

Suppose that $\Delta U_t(y)$ is increasing in y. Next, we will show that $\Delta U_{t-1}(y)$ is increasing in y.

By (3.1)′, we have

$$\begin{aligned}\Delta U_{t-1}(y) = \Delta U_t(y) &+ \sum_{j=1}^{n} p_{jt} \max\left\{ r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y), 0\right\} \\ &- \sum_{j=1}^{n} p_{jt} \max\left\{ r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y+1), 0\right\}.\end{aligned} \tag{4.3}$$

Let

$$\begin{aligned} A &= \Delta U_t(y) + \sum_{j=1}^{n} p_{jt} \max\left\{ r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y), 0\right\}, \\ B &= -\sum_{j=1}^{n} p_{jt} \max\left\{ r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y+1), 0\right\}, \\ &\therefore\ \Delta U_{t-1}(y) = A + B. \end{aligned} \tag{4.4}$$

For B, by the above assumption, $\Delta U_t(y+1)$ is increasing in y, then B is increasing in y.

For A, suppose that there are $m = 0, 1, \ldots, n$ classes booking requests satisfying $r_j \max\{\overline{w}, \overline{v}/\lambda\} \ge \Delta U_t(y)$, $n-m$ classes booking requests meeting $r_j \max\{\overline{w}, \overline{v}/\lambda\} \le \Delta U_t(y)$. We may assume that the first m classes in total n satisfy $r_j \max\{\overline{w}, \overline{v}/\lambda\} \ge \Delta U_t(y)$, and the last $n-m$ classes satisfy $r_j \max\{\overline{w}, \overline{v}/\lambda\} \le \Delta U_t(y)$. Then,

$$\begin{aligned} A &= \Delta U_t(y) + \sum_{j=1}^{m} p_{jt}\left(r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y)\right) \\ &= \Delta U_t(y) + \sum_{j=1}^{m} p_{jt} r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y)\sum_{j=1}^{m} p_{jt} \\ &= \left(1 - \sum_{j=1}^{m} p_{jt}\right)\Delta U_t(y) + \sum_{j=1}^{m} p_{jt} r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\}. \end{aligned} \tag{4.5}$$

By above hypothesis, we know that $\Delta U_t(y)$ is increasing in y, then A is increasing in y.

From above, we can get: $\Delta U_{t-1}(y)$ is increasing in y. This completes the proof. □

Property 3. $\Delta U_t(y)$ is decreasing in t, that is, $\Delta U_{t-1}(y) \geq \Delta U_t(y)$.

Proof. From (4.3), we have

$$\begin{aligned}\Delta U_{t-1}(y) - \Delta U_t(y) &= \sum_{j=1}^{n} p_{jt} \max\left\{r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y), 0\right\} \\ &\quad - \sum_{j=1}^{n} p_{jt} \max\left\{r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y+1), 0\right\} \\ &= \sum_{j=1}^{n} p_{jt} \left\{\max\left\{r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y), 0\right\}\right. \\ &\quad \left. - \max\left\{r_j \max\left\{\overline{w}, \frac{\overline{v}}{\lambda}\right\} - \Delta U_t(y+1), 0\right\}\right\}.\end{aligned} \tag{4.6}$$

By Property 2, we have $\Delta U_t(y) \leq \Delta U_t(y+1)$, then $\Delta U_{t-1}(y) \geq \Delta U_t(y)$. This completes the proof. □

Define $n_{jt}^* = \max\{y | r_j \max\{\overline{w}, \overline{v}/\lambda\} \geq \Delta U_t(y)\}$, combining Property 3, we have at any time t, as long as $y \leq n_{jt}^*$, air cargo carrier will accept jth class booking request. And then, we have the following optimal control policy.

Theorem 4.1. *The optimal control policy of the n classes booking requests is a booking limit policy: it is optimal to accept jth class booking request in time t if $y \leq n_{jt}^*$, or reject it. Furthermore, the threshold n_{jt}^* has the following properties:*

(1) *If $r_1 \geq r_2 \geq \cdots \geq r_n$, n_{jt}^* is decreasing in j, that is, $n_{1t}^* \geq n_{2t}^* \geq \cdots \geq n_{nt}^*$;*

(2) *n_{jt}^* is increasing in t, that is, $n_{j1}^* \leq n_{j2}^* \leq \cdots \leq n_{jn}^*$.*

Proof. From Properties 2 and 3, we can have the conclusions easily. This theorem is verified. □

The theorem shows that the optimal booking limits are time-dependent and nested in classes. Furthermore, the unit tariff is higher, the optimal booking control policy is more relaxed; the optimal booking limit policy of each class is increasing in time.

5. Conclusions

In this paper, we consider a single-leg air cargo overbooking and space inventory control problem. Based on actual problem, we discrete the booking time into small pieces and establish the dynamic space inventory control model of considering overbooking. After some powerful proofs, we get the optimal booking-limit policy for each class of goods.

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References

[1] R. G. Kasilingam, "An economic model for air cargo overbooking under stochastic capacity," *Computers and Industrial Engineering*, vol. 32, no. 1, pp. 221–226, 1997.

[2] A. J. Kleywegt and J. D. Papastavrou, "The dynamic and stochastic knapsack problem," *Operations Research*, vol. 46, no. 1, pp. 17–35, 1998.

[3] S. Luo, M. Çakanyıldırım, and R. G. Kasilingam, "Two-dimensional cargo overbooking models," *European Journal of Operational Research*, vol. 197, no. 3, pp. 862–883, 2009.

[4] L. Moussawi and M. Cakanyildrima, "Profit maximization in air cargo overbooking," Working Paper, 2005.

[5] R. Sandhu and D. Klabjan, "Fleeting with passenger and cargo origin-destination booking control," *Transportation Science*, vol. 40, no. 4, pp. 517–528, 2006.

[6] E. P. Chew, H. C. Huang, E. L. Johnson, G. L. Nemhauser, J. S. Sokol, and C. H. Leong, "Short-term booking of air cargo space," *European Journal of Operational Research*, vol. 174, no. 3, pp. 1979–1990, 2006.

[7] K. Amaruchkul, W. L. Cooper, and D. Gupta, "Single-leg air-cargo revenue management," *Transportation Science*, vol. 41, no. 4, pp. 457–469, 2007.

[8] Y. Levin and M. Nediak, "Cargo capacity management with allotments and spot market demand," *Operations Research*, vol. 60, no. 2, pp. 351–365, 2012.

[9] Y. J. Wang and C. S. Kao, "An application of a fuzzy knowledge system for air cargo overbooking under uncertain capacity," *Computers and Mathematics with Applications*, vol. 56, no. 10, pp. 2666–2675, 2008.

[10] M. Modarres and M. Sharifyazdi, "Revenue management approach to stochastic capacity allocation problem," *European Journal of Operational Research*, vol. 192, no. 2, pp. 442–459, 2009.

[11] K. Huang and K. C. Chang, "An approximate algorithm for the two-dimensional air cargo revenue management problem," *Transportation Research Part E*, vol. 46, no. 3, pp. 426–435, 2010.

[12] D. L. Han, L. C. Tang, and H. C. Huang, "A Markov model for single-leg air cargo revenue management under a bid-price policy," *European Journal of Operational Research*, vol. 200, no. 3, pp. 800–811, 2010.

[13] K. Amaruchkul and V. Lorchirachoonkul, "Air-cargo capacity allocation for multiple freight forwarders," *Transportation Research Part E*, vol. 47, no. 1, pp. 30–40, 2011.

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Research Article

The Convergent Behavior for Parametric Generalized Vector Equilibrium Problems

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We study some properties for parametric generalized vector equilibrium problems and the convergent behavior for the correspondent solution sets of this problem under some suitable conditions. Several existence results and the topological structures of the efficient solutions set are established. Some new results of existence for weak solutions and strong solutions are derived. Finally, we give some examples to illustrate our theory including the example studied by Fang (1992), who established the perturbed nonlinear program (P_μ) and described successfully that the optimal solution of (P_μ) will approach the optimal solution of linear program (P).

1. Introduction and Preliminaries

In recent years, the topological structures of the set of efficient solutions for vector equilibrium problems or generalized systems or variational inequality problems have been discussed in several aspects, as we show in [1–29]. More precisely, we divide this subject into several topics as following. First, the closedness of the set of efficient solutions are studied in [1, 4, 6, 13–16, 27]. Second, the lower semicontinuity of the set of efficient solutions are studied in [1, 9, 10, 19, 21, 23–26, 30]. Third, the upper semicontinuity of the set of efficient solutions are studied in [1, 4, 7, 8, 16, 21, 23–26, 30]. Fourth, the connectedness of the set of efficient solutions are studied in [2, 3, 17, 20, 27, 29]. Fifth, the existence of efficient solutions are studied in [5, 6, 8–12, 16–18, 22, 27, 29, 31].

Gong and Yao [19] establish the lower semicontinuity of the set of efficient solutions for parametric generalized systems with monotone bifunctions in real locally convex

Hausdorff topological vector spaces. They also discuss the connectedness of the efficient solutions for generalized systems, we refer to [20]. Luc [27, Chapter 6] investigates the structures of efficient point sets of linear, convex, and quasiconvex problems and also points out that the closedness and connectedness of the efficient solutions sets are important in mathematical programming. Huang et al. [8] discuss a class of parametric implicit vector equilibrium problems in Hausdorff topological vector spaces, where the mappings f and K are perturbed by parameters, say η and μ, respectively. They establish the upper semicontinuity and lower semicontinuity of the solution mapping for such problems and derive the closedness of the set of efficient solutions. Li et al. [1] discuss the generalized vector quasivariational inequality problem and obtain both upper semicontinuous and lower semicontinuous properties of the set of efficient solutions for parametric generalized vector quasivariational inequality problems. The closedness of the set of efficient solutions is also derived. Cheng [2] discusses the connectedness of the set of weakly efficient solutions for vector variational inequalities in $\mathbb{R}^n$. In 1992, Fang [32] established the perturbed nonlinear program (P_μ) and described successfully that the optimal solution of (P_μ) will approach the optimal solution of linear program (P). We will state the result in Example 3.7 below. We further point out that, in some suitable conditions, such convergent behavior will display continuity. Furthermore, the correspondent solution sets will preserve some kinds of topological properties under the convergent process. These results will show the convergent behavior about the sets of solutions by two kinds of parameters. As mentioned in [20], for the connectedness, "there are few papers which deal with this subject." But from above descriptions, we can understand and the topological structures of the sets of efficient solutions for some problems are more and more popular and interesting subjects. On the other hand, for our recent result [15], we study the generalized vector equilibrium problems in real Hausdorff topological vector space settings. The concepts of weak solutions and strong solutions are introduced. Several new results of existence for weak solutions and strong solutions of the generalized vector equilibrium problems are derived. These inspired us to discuss the parametric generalized vector equilibrium problems (PGVEPs). Let us introduce some notations as follows. We will use these notations through all this paper.

Let X, Y, and Z be arbitrary real Hausdorff topological vector spaces, where X and Z are finite dimensional. Let Δ_1, and Δ_2 be two parametric sets, $K : \Delta_2 \to 2^X$ be a mapping with nonempty values, $\mathcal{K} = \cup_{\eta\in\Delta_1} K(\eta)$, $C : \mathcal{K} \to 2^Y$ a set-valued mapping such that for each $x \in \mathcal{K}$, $C(x)$ is a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} C(x) \neq \emptyset$. For each $x \in \mathcal{K}$, we can define relations "$\leq_{C(x)}$" and "$\nleq_{C(x)}$" as follows: (1) $z \leq_{C(x)} y \Leftrightarrow y - z \in C(x)$ and (2) $z \nleq_{C(x)} y \Leftrightarrow y - z \notin C(x)$. Furthermore, we use the following notations:

$$y \geq_{C(x)} z \Longleftrightarrow z \leq_{C(x)} y, \qquad y \ngeq_{C(x)} z \Longleftrightarrow z \nleq_{C(x)} y. \tag{1.1}$$

Similarly, we can define the relations "$\leq_{\operatorname{int} C(x)}$" and "$\nleq_{\operatorname{int} C(x)}$" if we replace the set $C(x)$ by $\operatorname{int} C(x)$. If the mapping $C(x)$ is constant, then we denote it by C. The mappings $f : \Delta_1 \times Z \times \mathcal{K} \times \mathcal{K} \to Y$ and $T : \mathcal{K} \to 2^Z$ are given. The parametric generalized vector equilibrium problem (PGVEP, for short) is as follows: For every $(\xi, \eta) \in \Delta_1 \times \Delta_2$, we will like to find an $\overline{x} \in K(\eta)$ such that

$$f(\xi, \overline{s}, \overline{x}, y) \notin -\operatorname{int} C(\overline{x}), \tag{1.2}$$

for all $y \in K(\eta)$ and for some $\overline{s} \in T(\overline{x})$. Such set of weak efficient solutions for (PGVEP) is denoted by $\Gamma_w(\xi,\eta)$. If we find $\overline{x} \in K(\eta)$ and some $\overline{s} \in T(\overline{x})$ such that

$$f(\xi,\overline{s},\overline{x},y) \notin -\operatorname{int} C(\overline{x}), \tag{1.3}$$

for all $y \in K(\eta)$. Such set of efficient solutions for (PGVEP) is denoted by $\Gamma(\xi,\eta)$. Our main purpose is to find some topological structures for these two sets, $\Gamma_w(\xi,\eta)$ and $\Gamma(\xi,\eta)$, of efficient solutions of the parametric generalized vector equilibrium problem. Furthermore, we try to find some sufficient conditions lead them to be nonempty or closed or connected or even compact sets.

2. Some Properties for $\Gamma_w(\xi,\eta)$

Theorem 2.1. *Let $X, Y, Z, C, K, \mathcal{K}, T$, and f be given as in Section 1, the parametric spaces Δ_1, Δ_2 be two Hausdorff topological vector spaces. Let the mapping $f : \Delta_1 \times Z \times \mathcal{K} \times \mathcal{K} \to Y$ be such that $(\xi,s,x,y) \to f(\xi,s,x,y)$ is continuous and $y \to f(\xi,s,x,y)$ is $C(x)$-convex for every $(\xi,s,x) \in \Delta_1 \times Z \times \mathcal{K}$, the mapping $T : \mathcal{K} \to 2^Z$ be an upper semicontinuous with nonempty compact values, and the mapping $K : \Delta_2 \to 2^X$ is continuous with nonempty compact and convex values. Suppose that the following conditions hold the following:*

(a) *for any $\xi \in \Delta_1$, $x \in \mathcal{K}$, there is an $s \in Tx$, such that $f(\xi,s,x,x) \notin (-\operatorname{int} C(x))$;*

(b) *the mapping $x \to Y \setminus (-\operatorname{int} C(x))$ is closed [33] on $\mathcal{K}$.*

Then, we have

(1) *for every $(\xi,\eta) \in \Delta_1 \times \Delta_2$, the weak efficient solutions for (PGVEP) exist, that is, the set $\Gamma_w(\xi,\eta)$ is nonempty, where $\Gamma_w(\xi,\eta) = \{\overline{x} \in K(\eta) : f(\xi,\overline{s},\overline{x},y) \notin -\operatorname{int} C(\overline{x})$ for some $\overline{s} \in T(\overline{x})$ for all $y \in K(\eta)\}$.*

(2) *$\Gamma_w : \Delta_1 \times \Delta_2 \to 2^X$ is upper semicontinuous on $\Delta_1 \times \Delta_2$ with nonempty compact values.*

Proof. (1) For any fixed $(\xi,\eta) \in \Delta_1 \times \Delta_2$, we can easy check that the mappings $(s,x) \to f(\xi,s,x,y)$, $y \to f(\xi,s,x,y)$ satisfy all conditions of Corollary 2.2 in [15] with $K = \mathcal{K}$ and $D = \operatorname{conv}(\mathcal{K})$. Hence, from this corollary, we know that $\Gamma_w(\xi,\eta)$ is nonempty.

(2) For any fixed $(\xi,\eta) \in \Delta_1 \times \Delta_2$, we first claim that $\Gamma_w(\xi,\eta)$ is closed in $K(\eta)$, hence it is compact. Indeed, let a net $\{x_\alpha\} \subset \Gamma_w(\xi,\eta)$ and $x_\alpha \to p$ for some $p \in X$. Then, $x_\alpha \in K(\eta)$ and $f(\xi,s_{\alpha y},x_{\alpha y},y) \notin -\operatorname{int} C(x_\alpha)$ for all $y \in K(\eta)$ and for some $s_{\alpha y} \in T(x_\alpha)$. Since $K(\eta)$ is compact, $p \in K(\eta)$. For each α and for each $y \in K(\eta)$, there exists an $s_{\alpha y} \in T(x_\alpha)$ such that $f(\xi,s_{\alpha y},x_\alpha,y) \in Y \setminus (-\operatorname{int} C(x_\alpha))$. Since T is upper semicontinuous with nonempty compact values, and the set $\{x_\alpha\} \cup \{p\}$ is compact, $T(\{x_\alpha\} \cup \{p\})$ is compact. Therefore, without loss of generality, we may assume that the net $\{s_{\alpha y}\}$ converges to some s_y. Then $s_y \in T(p)$. Since the mapping $(s,x) \to f(\xi,s,x,y)$ is continuous, we have

$$\lim_{\alpha} f(\xi,s_{\alpha y},x_\alpha,y) = f(\xi,s_y,p,y). \tag{2.1}$$

Since $f(\xi,s_{\alpha y},x_\alpha,y) \in Y \setminus (-\operatorname{int} C(x_\alpha))$, $x_\alpha \to p$ and the mapping $x \to Y \setminus (-\operatorname{int} C(x))$ is closed, we have

$$f(\xi,s_y,p,y) \in Y \setminus (-\operatorname{int} C(p)). \tag{2.2}$$

This proves that $p \in \Gamma_w(\xi,\eta)$, and hence $\Gamma_w(\xi,\eta)$ is closed. Since $K(\eta)$ is compact, so is $\Gamma_w(\xi,\eta)$.

We next prove that the mapping $\Gamma_w : \Delta_1 \times \Delta_2 \to 2^{K(\eta)}$ is upper semicontinuous. That is, for any $(\xi,\eta) \in \Delta_1 \times \Delta_2$, if there is a net $\{(\xi_\beta,\eta_\beta)\}$ converges to (ξ,η) and some $x_\beta \in \Gamma_w(\xi_\beta,\eta_\beta)$, we need to claim that there is a $p \in \Gamma_w(\xi,\eta)$ and a subnet $\{x_{\beta_\nu}\}$ of $\{x_\beta\}$ such that $x_{\beta_\nu} \to p$. Indeed, since $x_\beta \in K(\eta_\beta)$ and $K : \Delta_2 \to 2^X$ are upper semicontinuous with nonempty compact values, there is a $p \in K(\eta)$ and a subnet $\{x_{\beta_\nu}\}$ of $\{x_\beta\}$ such that $x_{\beta_\nu} \to p$.

If we can claim that $p \in \Gamma_w(\xi,\eta)$, then we can see that $\Gamma_w : \Delta_1 \times \Delta_2 \to 2^X$ is upper semicontinuous on $\Delta_1 \times \Delta_2$, and complete our proof. Indeed, if not, there is a $y \in K(\eta)$ such that for every $s \in T(p)$ we have

$$f(\xi,s,x,y) \in -\operatorname{int} C(p). \tag{2.3}$$

Since K is lower semicontinuous, there is a net $\{y_{\beta_\nu}\}$ with $y_{\beta_\nu} \in K(\eta_{\beta_\nu})$ and $y_{\beta_\nu} \to y$. Since $x_{\beta_\nu} \in \Gamma_w(\xi_{\beta_\nu},\eta_{\beta_\nu})$, we have $x_{\beta_\nu} \in K(\eta_{\beta_\nu})$ and, for each y_{β_ν},

$$f(\xi_{\beta_\nu},s_{\beta_\nu},x_{\beta_\nu},y_{\beta_\nu}) \in Y \setminus \left(-\operatorname{int} C(x_{\beta_\nu})\right), \tag{2.4}$$

for some $s_{\beta_\nu} \in T(x_{\beta_\nu})$.

Since T is upper semicontinuous and the net $x_{\beta_\nu} \to x$, without loss of generality, we may assume that $s_{\beta_\nu} \to s$ for some $s \in T(x)$. Since the mapping $(\xi,s,x,y) \to f(\xi,s,x,y)$ is continuous, we have

$$\lim_{\beta_\nu} f(\xi_{\beta_\nu},s_{\beta_\nu},x_{\beta_\nu},y_{\beta_\nu}) = f(\xi,s,p,y). \tag{2.5}$$

From (2.4) and the closedness of the mapping $x \to Y \setminus (-\operatorname{int} C(x))$, we have

$$f(\xi,s,p,y) \in Y \setminus \left(-\operatorname{int} C(p)\right), \tag{2.6}$$

which contradicts (2.3). Hence, we have $p \in \Gamma_w(\xi,\eta)$. □

3. Some Properties for $\Gamma(\xi,\eta)$

In the section, we discuss the set $\Gamma(\xi,\eta)$ of the efficient solutions for (PGVEP), where $\Gamma(\xi,\eta) = \{\overline{x} \in K(\eta) :$ there is an $\overline{s} \in T(\overline{x})$, such that $f(\xi,\overline{s},\overline{x},y) \notin -\operatorname{int} C(\overline{x})$ for all $y \in K(\eta)\}$. The sets of minimal points, maximum points, weak minimal points, and weak maximum points for some set A with respect to the cone $C(\overline{x})$ are denoted by $\mathrm{Min}^{C(\overline{x})}A$, $\mathrm{Max}^{C(\overline{x})}A$, $\mathrm{Min}_w^{C(\overline{x})}A$, and $\mathrm{Max}_w^{C(\overline{x})}A$, respectively. For more detail, we refer the reader to Definition 1.2 of [28].

Theorem 3.1. *Under the framework of Theorem 2.1, for each $(\xi,\eta) \in \Delta_1 \times \Delta_2$, there is an $\overline{x} \in \Gamma_w(\xi,\eta)$ with $\overline{s} \in T(\overline{x})$. In addition, if $T(\overline{x})$ is convex, the mapping $s \to -f(\xi,s,\overline{x},y)$ is properly quasi $C(\overline{x})$-convex (Definition 1.1 of [28]) on $T(\overline{x})$ for each $(\xi,y) \in \Delta_1 \times K(\eta)$. Assume that the mapping $(s,y) \to f(\xi,s,\overline{x},y)$ satisfies the following conditions:*

(i)

$$\mathrm{Max}^{C(\overline{x})} \bigcup_{s \in T(\overline{x})} \mathrm{Min}_w^{C(\overline{x})} \bigcup_{y \in K(\eta)} \{f(\xi, s, \overline{x}, y)\} \subset \mathrm{Min}_w^{C(\overline{x})} \bigcup_{y \in K(\eta)} \{f(\xi, s, \overline{x}, y)\} + C(\overline{x}) \tag{3.1}$$

for every $s \in T(\overline{x})$;

(ii) *for any fixed* $x \in K(\eta)$, *if* $\delta \in \mathrm{Max}^{C(\overline{x})} \bigcup_{s \in T(\overline{x})} \{f(\xi, s, \overline{x}, y)\}$ *and* δ *cannot be comparable with* $f(\xi, \overline{s}, \overline{x}, y)$ *which does not equal to* δ, *then* $\delta \nleq_{\mathrm{int}\, C(\overline{x})} 0$;

(iii) *if* $\mathrm{Max}^{C(\overline{x})} \bigcup_{s \in T(\overline{x})} \mathrm{Min}_w^{C(\overline{x})} \bigcup_{y \in K(\eta)} \{f(\xi, s, \overline{x}, y)\} \subset Y \setminus (-\mathrm{int}\, C(\overline{x}))$, *there exists an* $s \in T(\overline{x})$ *such that* $\mathrm{Min}_w^{C(\overline{x})} \bigcup_{y \in K(\eta)} \{f(\xi, s, \overline{x}, y)\} \subset Y \setminus (-\mathrm{int}\, C(\overline{x}))$.

Then, we have

(a) *for every* $(\xi, \eta) \in \Delta_1 \times \Delta_2$, *the efficient solutions exists, that is, the set* $\Gamma(\xi, \eta)$ *is nonempty, furthermore, it is compact;*

(b) *the mapping* $\Gamma : \Delta_1 \times \Delta_2 \to 2^X$ *is upper semicontinuous on* $\Delta_1 \times \Delta_2$ *with nonempty compact values;*

(c) *for each* $(\xi, \eta) \in \Delta_1 \times \Delta_2$, *the set* $\Gamma(\xi, \eta)$ *is connected if* $C : K(\eta) \to 2^Y$ *is constant, and for any* $(\xi, \eta) \in \Delta_1 \times \Delta_2$, $x \in K(\eta)$ *and* $s \in T(K(\eta))$, $f(\xi, s, x, K(\eta)) + C$ *is convex.*

Proof. (a) Fixed any $(\xi, \eta) \in \Delta_1 \times \Delta_2$, we can easy see that all conditions of Theorem 2.3 of [15] hold, hence from Theorem 2.3 of [15], we know that $\Gamma(\xi, \eta)$ is nonempty and compact.

(b) Let $\{(\xi_\alpha, \eta_\alpha)\} \subset \Delta_1 \times \Delta_2$ be a net such that $(\xi_\alpha, \eta_\alpha) \to (\xi, \eta)$ and $\{x_\alpha\}$ be a net with $x_\alpha \in \Gamma(\xi_\alpha, \eta_\alpha)$. Since $x_\alpha \in K(\eta_\alpha)$ and $K : \Delta_2 \to 2^X$ are upper semicontinuous with nonempty compact values, there are an $x \in K(\eta)$ and a subnet $\{x_{\alpha_\iota}\}$ of $\{x_\alpha\}$ such that $x_{\alpha_\iota} \to x$. Since $T : K \to 2^Z$ is upper semicontinuous with nonempty compact values, $T(\{x_{\alpha_\iota}\} \cup \{x\})$ is compact. Since $s_{\alpha_\iota} \in T(x_{\alpha_\iota})$, there is an $s \in T(x)$ such that a subnet of $\{s_{\alpha_\iota}\}$ converges to s. Without loss of generality, we still denote the subnet by $\{s_{\alpha_\iota}\}$, and hence $s_{\alpha_\iota} \to s$.

If $x \notin \Gamma(\xi, \eta)$, then there is a $y \in K(\eta)$ such that

$$f(\xi, s, x, y) \in -\mathrm{int}\, C(x). \tag{3.2}$$

Since $K(\eta)$ is compact, there is a net, say $\{y_{\alpha_\iota}\}$, in $K(\eta)$ converges to y. Since the mapping $(\xi, s, x, y) \to f(\xi, s, x, y)$ is continuous, and the mapping $x \to Y \setminus (-\mathrm{int}\, C(x))$ is closed, we have

$$f(\xi, s, x, y) = \lim_{\alpha_\iota} f(\xi_{\alpha_\iota}, s_{\alpha_\iota}, x_{\alpha_\iota}, y_{\alpha_\iota}) \in Y \setminus (-\mathrm{int}\, C(x)), \tag{3.3}$$

which contracts (3.2). Thus, $x \in \Gamma(\xi, \eta)$.

In order to prove (c), we introduce Lemmas 3.2–3.4 as follows.

Let $Y^\star$ be the topological dual space of Y. For each $x \in \mathcal{K}$,

$$C^\star(x) = \{g \in Y^\star : g(y) \geq 0 \ \forall y \in C(x)\}. \tag{3.4}$$

Let $C^\star = \cap_{x\in\mathcal{K}} C^\star(x)$, then $C^\star$ is nonempty and connected. If $C^\star : \mathcal{K} \to 2^{Y^\star}$ is a constant mapping, then $C^\star(x) = C^\star$ for all $x \in \mathcal{K}$. In the sequel, we suppose that $C^\star$ is not a singleton. That is, $C^\star \setminus \{0\} \neq \emptyset$, and hence it is connected. For each $g \in C^\star \setminus \{0\}$, let us denote the set of g-efficient solutions to (PGVEP) by

$$S^{\xi,\eta}(g) = \left\{ x \in K(\eta) : \sup_{s\in T(x)} g(f(\xi, s, x, y)) \geq 0 \text{ for every } y \in K(\eta) \right\}. \tag{3.5}$$

Lemma 3.2. *Under the framework of Theorem 3.1,*

$$S^{\xi,\eta}(g) \neq \emptyset, \tag{3.6}$$

for every $g \in C^\star \setminus \{0\}$.

Proof. From (a) of Theorem 3.1, we know that, for each $(\xi, \eta) \in \Delta_1 \times \Delta_2$, there is an $\overline{x} \in K(\eta)$ with $\overline{s} \in T(\overline{x})$ such that

$$g(f(\xi, \overline{s}, \overline{x}, y)) \geq 0, \tag{3.7}$$

for all $y \in K(\eta)$ and for all $g \in C^\star \setminus \{0\}$. Thus, $\overline{x} \in S^{\xi,\eta}(g)$ for every $g \in C^\star \setminus \{0\}$. Hence, $S^{\xi,\eta}(g) \neq \emptyset$ for every $g \in C^\star \setminus \{0\}$.

Lemma 3.3. *Suppose that for any $(\xi, \eta) \in \Delta_1 \times \Delta_2$ and $y \in K(\eta)$, $f(\xi, T(K(\eta)), K(\eta), y)$ are bounded. Then, the mapping $S^{\xi,\eta} : C^\star \setminus \{0\} \to 2^{K(\eta)}$ is upper semicontinuous with compact values.*

Proof. Fixed any $(\xi, \eta) \in \Delta_1 \times \Delta_2$. We first claim that the mapping $S^{\xi,\eta} : C^\star \setminus \{0\} \to 2^{K(\eta)}$ is closed. Let $x_\nu \in S^{\xi,\eta}(g_\nu)$, $x_\nu \to x$ and $g_\nu \to g$ with respect to the strong topology $\sigma(Y^\star, Y)$ in $Y^\star$.

Since $x_\nu \in S^{\xi,\eta}(g_\nu)$, there is an $s_\nu \in T(x_\nu)$ such that $g(f(\xi, s_\nu, x_\nu, y)) \geq 0$ for all $y \in K(\eta)$. Since T is upper semicontinuous with nonempty compact values, by a similar argument in the proof of Theorem 3.1(b), there is an $s \in T(x)$ such that a subnet of $\{s_\nu\}$ converges to s. Without loss of generality, we still denote the subnet by $\{s_\nu\}$.

For each $y \in K(\eta)$, we define $P_{f(\xi,T(K(\eta)),K(\eta),y)}(g) = \sup_{z\in f(\xi,T(K(\eta)),K(\eta),y)} |g(z)|$ for all $g \in Y^\star$. We note that the set $f(\xi, T(K(\eta)), K(\eta), y)$ is bounded by assumption, hence $P_{f(\xi,T(K(\eta)),K(\eta),y)}(g)$ is well defined and is a seminorm of $Y^\star$. For any $\varepsilon > 0$, let $\mathcal{U}_\varepsilon = \{g \in Y^\star : P_{f(\xi,T(K(\eta)),K(\eta),y)}(g) < \varepsilon\}$ be a neighborhood of 0 with respect to $\sigma(Y^\star, Y)$. Since $g_\nu \to g$, there is a $\alpha_0 \in \Lambda$ such that $g_\nu - g \in \mathcal{U}_\varepsilon$ for every $\nu \geq \nu_0$. That is, $P_{f(\xi,T(K(\eta)),K(\eta),y)}(g_\nu - g) = \sup_{z\in f(\xi,T(K(\eta)),K(\eta),y)} |(g_\nu - g)(z)| < \varepsilon/2$ for every $\nu \geq \nu_0$. This implies that

$$\left|(g_\nu - g)(f(\xi, s_\nu, x_\nu, y))\right| < \frac{\varepsilon}{2}, \tag{3.8}$$

for all $\nu \geq \nu_0$. Since the mapping $(s, x) \to f(\xi, s, x, y)$ is continuous and $(s_\nu, x_\nu) \to (s, x)$, we have

$$f(\xi, s_\nu, x_\nu, y) \to f(\xi, s, x, y). \tag{3.9}$$

By the continuity of g, we have

$$\left|g(f(\xi, s_\nu, x_\nu, y)) - g(f(\xi, s, x, y))\right| < \frac{\varepsilon}{2}, \tag{3.10}$$

for some ν_1 and all $\nu \geq \nu_1$. Let us choose $\nu_2 = \max\{\nu_0, \nu_1\}$. Combining (3.8) and (3.10), we know that, for all $\nu \geq \nu_2$,

$$\begin{aligned}
&\left|g_\nu(f(\xi, s_\nu, x_\nu, y)) - g(f(\xi, s, x, y))\right| \\
&\quad \leq \left|g_\nu(f(\xi, s_\nu, x_\nu, y)) - g(f(\xi, s_\nu, x_\nu, y))\right| \\
&\qquad + \left|g(f(\xi, s_\nu, x_\nu, y)) - g(f(\xi, s, x, y))\right| \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\quad = \varepsilon.
\end{aligned} \tag{3.11}$$

That is $g_\nu(f(\xi, s_\nu, x_\nu, y)) \to g(f(\xi, s, x, y))$. Since $g_\nu(f(\xi, s_\nu, x_\nu, y)) \geq 0$, there is an $s \in T(x)$ such that $g(f(\xi, s, x, y)) \geq 0$, which proves that $x \in S^{\xi,\eta}(g)$. Therefore, the mapping $S^{\xi,\eta} : C^\star \setminus \{0\} \to 2^{K(\eta)}$ is closed. By the compactness and Corollary 7 in [33, page 112], the mapping $S^{\xi,\eta}$ is upper semicontinuous with compact values.

Lemma 3.4. *Suppose that for any $(\xi, \eta) \in \Delta_1 \times \Delta_2$, $x \in K(\eta)$ and $s \in T(K(\eta))$, $f(\xi, s, x, K(\eta)) + C(x)$ is convex. Then*

$$\Gamma(\xi, \eta) \supseteq \bigcup_{g \in C^\star \setminus \{0\}} S^{\xi,\eta}(g). \tag{3.12}$$

Furthermore, if $C : K(\eta) \to 2^Y$ is constant, then we have

$$\Gamma(\xi, \eta) = \bigcup_{g \in C^\star \setminus \{0\}} S^{\xi,\eta}(g). \tag{3.13}$$

Proof. We first claim that $\Gamma(\xi, \eta) \supseteq \cup_{g \in C^\star \setminus \{0\}} S^{\xi,\eta}(g)$.

If $x \in \cup_{g \in C^\star \setminus \{0\}} S^{\xi,\eta}(g)$, there is a $g \in C^\star \setminus \{0\}$ such that $x \in S^{\xi,\eta}(g)$. Then, there is a $g \in C^\star \setminus \{0\}$ such that

$$g(f(\xi, s, x, y)) \geq 0 \tag{3.14}$$

for all $y \in K(\eta)$. This implies that $f(\xi, s, x, y) \notin -\operatorname{int} C(x)$ for all $y \in K(\eta)$. Indeed, if there is a $\overline{y} \in K(\eta)$ such that $f(\xi, s, x, \overline{y}) \in -\operatorname{int} C(x)$. Since $g \in C^\star \setminus \{0\}$, we have $g(f(\xi, s, x, y)) < 0$ which contracts (3.14). Thus, $x \in \Gamma(\xi, \eta)$. This proves (3.12) holds.

Second, if $C : K(\eta) \to 2^Y$ is constant, we claim that $\Gamma(\xi, \eta) \subseteq \cup_{g \in C^\star \setminus \{0\}} S^{\xi,\eta}(g)$.

If $x \in \Gamma(\xi,\eta)$, then $x \in K(\eta)$ with $s \in T(x)$ and $f(\xi,s,x,y) \notin -\operatorname{int} C$ for all $y \in K(\eta)$, that is, $f(\xi,s,x,K(\eta)) \cap (-\operatorname{int} C) = \emptyset$. Hence,

$$(f(\xi,s,x,K(\eta)) + C) \cap (-\operatorname{int} C) = \emptyset. \tag{3.15}$$

Since $f(\xi,s,x,K(\eta)) + C$ is convex, by Eidelheit separation theorem, there is a $g \in Y^\star \setminus \{0\}$ and $\rho \in \mathbb{R}$ such that

$$g(w') < \rho \leq g(f(\xi,s,x,y) + w), \tag{3.16}$$

for all $y \in K(\eta)$, $w \in C$, $w' \in -\operatorname{int} C$. Then,

$$(g-\rho)(w') < 0 \leq (g-\rho)(f(\xi,s,x,y) + w), \tag{3.17}$$

for all $y \in K(\eta)$, $w \in C$, $w\prime \in -\operatorname{int} C$.

Without loss of generality, we denote $g-\rho$ by g, then

$$g(w') < 0 \leq g(f(\xi,s,x,y) + w), \tag{3.18}$$

for all $y \in K(\eta)$, $w \in C$, $w' \in -\operatorname{int} C$. By the left-hand side inequality of (3.18) and the linearity of g, we have $g(m) > 0$ for all $m \in \operatorname{int} C$. Since C is closed, for any m in the boundary of C, there is a net $\{m_\nu\} \subset \operatorname{int} C$ such that $m_\nu \to m$. By the continuity of g, $g(m) = g(\lim_\nu m_\nu) = \lim_\nu g(m_\nu) \geq 0$. Hence, for all $w \in C$, $g(w) \geq 0$, that is $g \in C^\star \setminus \{0\}$.

By the right-hand side inequality of (3.18), for all $w \in C$, there is an $s \in T(x)$ such that $g(f(\xi,s,x,y) + w) \geq 0$ for all $y \in K(\eta)$. This implies that $g(f(\xi,s,x,y)) \geq 0$ for all $y \in K(\eta)$ if we choose $w = 0$. Hence, $\sup_{s\in T(x)} g(f(\xi,s,x,y)) \geq 0$ for all $y \in K(\eta)$. Thus, $x \in S^{\xi,\eta}(g)$. Therefore, $x \in \cup_{g\in C^\star\setminus\{0\}} S^{\xi,\eta}(g)$, and hence

$$\Gamma(\xi,\eta) \subseteq \bigcup_{g\in C^\star\setminus\{0\}} S^{\xi,\eta}(g). \tag{3.19}$$

Combining this with (3.12), we have

$$\Gamma(\xi,\eta) = \bigcup_{g\in C^\star\setminus\{0\}} S^{\xi,\eta}(g). \tag{3.20}$$

□

Now, we go back to prove Theorem 3.1(c).

Proof of Theorem 3.1(c). From Lemmas 3.2 and 3.3, the mapping $S^{\xi,\eta} : C^\star\setminus\{0\} \to 2^{K(\eta)}$ is upper semicontinuous with nonempty compact values. From Lemma 3.4 and Theorem 3.1 [29], we know that for each $(\xi,\eta) \in \Delta_1 \times \Delta_2$, the set $\Gamma(\xi,\eta)$ is connected. □

Modifying the Example 3.1 [8], we give the following examples to illustrate Theorems 2.1 and 3.1 as follows.

Example 3.5. Let $\Delta_1 = \Delta_2 = X = Y = Z = \mathbb{R}$, $K(\eta) = [0,1]$ for all $\eta \in \Delta_2$, $\mathcal{K} = \cup_{\eta\in\Delta_2} K(\eta) = [0,1]$, $C(x) = [0,\infty)$ for all $x \in \mathcal{K}$. Choose $T : \mathcal{K} \to 2^Z$ by $T(x) = \{x, x/2\}$ for all $x \in \mathcal{K}$. Define $f(\xi, s, x, y) = s - y + \xi^2$ for all $(\xi, x, y) \in \Delta_1 \times X \times Y$. Then, all the conditions of Theorem 2.1 hold, and $\Gamma_w(\xi,\eta) = [1-\xi^2, 1] \cap [0,1]$ for all $(\xi,\eta) \in \Delta_1 \times \Delta_2$. Indeed, since there are two choices for s, one is x, and the other is $x/2$. If the nonnegative number ξ^2 is less than 1, for any y in $[0,1]$, and we always choose $s = x/2$, then for this case, the set $\Gamma_w(\xi,\eta)$ will contain all elements of the set $[2(1-\xi^2), 1]$. Furthermore, if we always choose $s = x$, then the set $\Gamma_w(\xi,\eta)$ will contain all elements of the set $[(1-\xi^2), 1]$. If the nonnegative number ξ^2 is greater than or equal to 1, then the set $\Gamma_w(\xi,\eta)$ will contain all elements of the set $[0,1]$. Hence,

$$\begin{aligned}\Gamma_w(\xi,\eta) &= \left(\left[2\left(1-\xi^2\right),1\right] \cap [0,1]\right) \bigcup \left(\left[\left(1-\xi^2\right),1\right] \cap [0,1]\right) \bigcup [0,1] \\ &= \left[\left(1-\xi^2\right),1\right] \cap [0,1].\end{aligned} \tag{3.21}$$

Here, we note that we cannot apply Theorem 3.1 since $T(x)$ is not convex.

Example 3.6. Following Example 3.5, let $T(x) = [x/2, x]$ for all $x \in \mathcal{K} = [0,1]$. By Theorem 2.1, the set $\Gamma_w(\xi,\eta) \neq \emptyset$. We choose any $\overline{x} \in \Gamma_w(\xi,\eta)$, and we can see the mapping $s \to -f(\xi, s, \overline{x}, y)$ is properly quasi $C(\overline{x})$-convex on $T(\overline{x})$ for any $(\xi, y) \in \Delta_1 \times K(\eta)$. Since $\mathrm{Max}^{C(\overline{x})} \cup_{s\in[\overline{x}/2,\overline{x}]} \mathrm{Min}_w^{C(\overline{x})} \cup_{y\in[0,1]} \{s - y + \xi^2\} = \{\overline{x} - 1 + \xi^2\} \subset \{s - 1 + \xi^2\} + [0,\infty) = \mathrm{Min}_w^{C(\overline{x})} \cup_{y\in[0,1]} \{s - y + \xi^2\} + C(\overline{x})$ for all $s \in [\overline{x}/2, \overline{x}] = T(\overline{x})$. So, condition (i) of Theorem 3.1 holds. Obviously, the condition (ii) also holds, since no such δ exists in this example. Now, we can see condition (iii) holds. Indeed, from the facts

$$\begin{aligned}\mathrm{Min}_w^{C(\overline{x})} \bigcup_{y\in[0,1]} \left\{s - y + \xi^2\right\} &= \left\{s - 1 + \xi^2\right\}, \\ \mathrm{Max}^{C(\overline{x})} \bigcup_{s\in T(\overline{x})} \mathrm{Min}_w^{C(\overline{x})} \bigcup_{y\in[0,1]} \left\{s - y + \xi^2\right\} &= \left\{\overline{x} - 1 + \xi^2\right\},\end{aligned} \tag{3.22}$$

we know that if $\overline{x} - 1 + \xi^2 \geq 0$, then we can choose $s = \overline{x} \in T(\overline{x})$ such that $s - 1 + \xi^2 \geq 0$. Hence, we can apply Theorem 3.1, and we know that $\Gamma(\xi,\eta)$ is nonempty compact and connected. Let us compute the set $\Gamma(\xi,\eta)$ for any $(\xi,\eta) \in \Delta_1 \times \Delta_2$. If we choose any $\overline{s} = t\overline{x}$ for some $t \in [1/2, 1]$, we can see that all the points in the set $[(1-\xi^2)/t, 1]$ are efficient solutions for (PGVEP). Hence,

$$\begin{aligned}\Gamma(\xi,\eta) &= \bigcup_{t\in[1/2,1]} \left[\left(1-\xi^2\right)/t, 1\right] \cap [0,1] \\ &= \left[\left(1-\xi^2\right),1\right] \cap [0,1],\end{aligned} \tag{3.23}$$

for any $(\xi,\eta) \in \Delta_1 \times \Delta_2$.

Example 3.7 (see [32]). The perturbed nonlinear program (P_μ) described successfully that the optimal solutions set $\Gamma(\mu)$ of (P_μ) will approach the optimal solutions set Γ of linear program (P), where (P) and (P_μ) are as follows

$$\begin{aligned} \text{Minimise} \quad & c^T x, \\ \text{subject to} \quad & Ax = 0, \\ & e^T x = 1, \\ & x \geq 0, \end{aligned} \tag{P}$$

$$\begin{aligned} \text{minimise} \quad & \mathbf{c}^T \mathbf{x} + \mu \sum_{j=1}^{n} x_j \log x_j, \\ \text{subject to} \quad & Ax = 0, \\ & e^T x = 1, \\ & x \geq 0, \end{aligned} \tag{P_μ}$$

where $\mu > 0$.

We further note that, such convergent behavior will be described by upper semicontinuity by Theorems 2.1 and 3.1. That is,

$$\Gamma(\mu) \xrightarrow{\text{u.s.c.}} \Gamma \quad \text{as } \mu \to 0^+. \tag{3.24}$$

Furthermore, the correspondent solution sets will preserve some kinds of topological properties, such as compactness and connectedness, under the convergent process.

We would like to point out an open question that naturally raises from Theorems 2.1 and 3.1. Under what conditions the mappings Γ_w and Γ will be lower semicontinuous?

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References

[1] S. J. Li, G. Y. Chen, and K. L. Teo, "On the stability of generalized vector quasivariational inequality problems," *Journal of Optimization Theory and Applications*, vol. 113, no. 2, pp. 283–295, 2002.

[2] Y. H. Cheng, "On the connectedness of the solution set for the weak vector variational inequality," *Journal of Mathematical Analysis and Applications*, vol. 260, no. 1, pp. 1–5, 2001.

[3] X. H. Gong, "Connectedness of the solution sets and scalarization for vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 133, no. 2, pp. 151–161, 2007.

[4] L. Gong, "Global stability result for the generalized quasivariational inequality problem," *Journal of Optimization Theory and Applications*, vol. 70, no. 2, pp. 365–375, 1991.

[5] R. L. Tobin, "Sensitivity analysis for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 48, no. 1, pp. 191–204, 1986.

[6] S. J. Yu and J. C. Yao, "On vector variational inequalities," *Journal of Optimization Theory and Applications*, vol. 89, no. 3, pp. 749–769, 1996.

[7] Y. H. Cheng and D. L. Zhu, "Global stability results for the weak vector variational inequality," *Journal of Global Optimization*, vol. 32, no. 4, pp. 543–550, 2005.

[8] N. J. Huang, J. Li, and H. B. Thompson, "Stability for parametric implicit vector equilibrium problems," *Mathematical and Computer Modelling*, vol. 43, no. 11-12, pp. 1267–1274, 2006.

[9] O. Chadli, N. C. Wong, and J. C. Yao, "Equilibrium problems with applications to eigenvalue problems," *Journal of Optimization Theory and Applications*, vol. 117, no. 2, pp. 245–266, 2003.

[10] Q. H. Ansari, S. Schaible, and J.-C. Yao, "The system of generalized vector equilibrium problems with applications," *Journal of Global Optimization*, vol. 22, no. 1–4, pp. 3–16, 2002.

[11] X.-P. Ding, J.-C. Yao, and L.-J. Lin, "Solutions of system of generalized vector quasi-equilibrium problems in locally G-convex uniform spaces," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 2, pp. 398–410, 2004.

[12] L. J. Lin, Z. T. Yu, and G. Kassay, "Existence of equilibria for multivalued mappings and its application to vectorial equilibria," *Journal of Optimization Theory and Applications*, vol. 114, no. 1, pp. 189–208, 2002.

[13] M. Bianchi and R. Pini, "A note on stability for parametric equilibrium problems," *Operations Research Letters*, vol. 31, no. 6, pp. 445–450, 2003.

[14] F. Ferro, "A minimax theorem for vector-valued functions," *Journal of Optimization Theory and Applications*, vol. 60, no. 1, pp. 19–31, 1989.

[15] Y.-C. Lin, "On generalized vector equilibrium problems," *Nonlinear Analysis*, vol. 70, no. 2, pp. 1040–1048, 2009.

[16] P. Q. Khanh and L. M. Luu, "Upper semicontinuity of the solution set to parametric vector quasivariational inequalities," *Journal of Global Optimization*, vol. 32, no. 4, pp. 569–580, 2005.

[17] X. H. Gong, "Efficiency and Henig efficiency for vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 108, no. 1, pp. 139–154, 2001.

[18] R. N. Mukherjee and H. L. Verma, "Sensitivity analysis of generalized variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 167, no. 2, pp. 299–304, 1992.

[19] X. H. Gong and J. C. Yao, "Lower semicontinuity of the set of efficient solutions for generalized systems," *Journal of Optimization Theory and Applications*, vol. 138, no. 2, pp. 197–205, 2008.

[20] X. H. Gong and J. C. Yao, "Connectedness of the set of efficient solutions for generalized systems," *Journal of Optimization Theory and Applications*, vol. 138, no. 2, pp. 189–196, 2008.

[21] X. H. Gong, "Continuity of the solution set to parametric weak vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 139, no. 1, pp. 35–46, 2008.

[22] I. V. Konnov and J. C. Yao, "On the generalized vector variational inequality problem," *Journal of Mathematical Analysis and Applications*, vol. 206, no. 1, pp. 42–58, 1997.

[23] L. Q. Anh and P. Q. Khanh, "Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 2, pp. 699–711, 2004.

[24] L. Q. Anh and P. Q. Khanh, "On the Hölder continuity of solutions to parametric multivalued vector equilibrium problems," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 1, pp. 308–315, 2006.

[25] L. Q. Anh and P. Q. Khanh, "Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces," *Journal of Global Optimization*, vol. 37, no. 3, pp. 449–465, 2007.

[26] P. Q. Khanh and L. M. Luu, "Lower semicontinuity and upper semicontinuity of the solution sets and approximate solution sets of parametric multivalued quasivariational inequalities," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 329–339, 2007.

[27] D. T. Luc, *Theory of Vector Optimization*, vol. 319 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, Germany, 1989.

[28] Y.-C. Lin and M.-M. Wong, "Note on F-implicit generalized vector variational inequalities," *Taiwanese Journal of Mathematics*, vol. 14, no. 2, pp. 707–718, 2010.

[29] A. R. Warburton, “Quasiconcave vector maximization: connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives,” *Journal of Optimization Theory and Applications*, vol. 40, no. 4, pp. 537–557, 1983.

[30] K. Kimura and J.-C. Yao, “Sensitivity analysis of vector equilibrium problems,” *Taiwanese Journal of Mathematics*, vol. 12, no. 3, pp. 649–669, 2008.

[31] Q. H. Ansari, S. Schaible, and J.-C. Yao, “System of vector equilibrium problems and its applications,” *Journal of Optimization Theory and Applications*, vol. 107, no. 3, pp. 547–557, 2000.

[32] S.-C. Fang, “An unconstrained convex programming view of linear programming,” *Zeitschrift für Operations Research*, vol. 36, no. 2, pp. 149–161, 1992.

[33] C. Berge, *Espaces Topologiques*, Dunod, Paris, France, 1959.

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Research Article

Expected Residual Minimization Method for a Class of Stochastic Quasivariational Inequality Problems

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We consider the expected residual minimization method for a class of stochastic quasivariational inequality problems (SQVIP). The regularized gap function for quasivariational inequality problem (QVIP) is in general not differentiable. We first show that the regularized gap function is differentiable and convex for a class of QVIPs under some suitable conditions. Then, we reformulate SQVIP as a deterministic minimization problem that minimizes the expected residual of the regularized gap function and solve it by sample average approximation (SAA) method. Finally, we investigate the limiting behavior of the optimal solutions and stationary points.

1. Introduction

The quasivariational inequality problem is a very important and powerful tool for the study of generalized equilibrium problems. It has been used to study and formulate generalized Nash equilibrium problem in which a strategy set of each player depends on the other players' strategies (see, for more details, [1–3]).

QVIP is to find a vector $x^* \in S(x^*)$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in S(x^*), \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping, the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$, and $S : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued mapping of which $S(x)$ is a closed convex set in $\mathbb{R}^n$ for each x.

In particular, if S is a closed convex set and $S(x) \equiv S$ for each x, then QVIP (1.1) becomes the classical variational inequality problem (VIP): find a vector $x^* \in S$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in S. \tag{1.2}$$

In most important practical applications, the function F always involves some random factors or uncertainties. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Taking the randomness into account, we get stochastic quasivariational inequality problem (SQVIP): find an $x^* \in S(x^*)$ such that

$$P\{\omega \in \Omega : \langle F(x^*, \omega), x - x^* \rangle \geq 0, \ \forall x \in S(x^*)\} = 1, \tag{1.3}$$

or equivalently,

$$\langle F(x^*, \omega), x - x^* \rangle \geq 0, \quad \forall x \in S(x^*), \ \omega \in \Omega \text{ a.s.}, \tag{1.4}$$

where $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a mapping and a.s. is abbreviation for "almost surely" under the given probability measure P.

Due to the introduction of randomness, SQVIP (1.4) becomes more practical and also evokes more and more attentions in the recent literature [4–16]. However, to our best knowledge, most publications in the existing literature discuss the stochastic complementarity problems and the stochastic variational inequality problems, which are two special cases of (1.4). It is well known that quasivariational inequalities are more complicated than variational inequalities and complementarity problems and that they have widely applications. Therefore, it is meaningful and interesting to study the general problem (1.4).

Because of the existence of a random element ω, we cannot generally find a vector $x^* \in S(x^*)$ such that (1.4) holds almost surely. That is, (1.4) is not well defined if we think of solving (1.4) before knowing the realization ω. Therefore, in order to get a reasonable resolution, an appropriate deterministic reformulation for SQVIP becomes an important issue in the study of the considered problem.

Recently, one of the mainstreaming research methods on the stochastic variational inequality problem is expected residual minimization method (see [4, 5, 7, 11–13, 16] and the references therein). Chen and Fukushima [5] formulated the stochastic linear complementarity problem (SLCP) as a minimization problem which minimizes the expectation of gap function (also called residual function) for SLCP. They regarded the optimal solution of this minimization problem as a solution to SLCP. This method is the so-called expected residual minimization method (ERM). Following the ideas of Chen and Fukushima [5], Zhang and Chen [16] considered the stochastic nonlinear complementary problems. Luo and Lin [12, 13] generalized the expected residual minimization method to solve stochastic variational inequality problem.

In this paper, we focus on ERM method for SQVIP. We first show that the regularized gap function for QVIP is differentiable and convex under some suitable conditions. Then, we formulate SQVIP (1.4) as an optimization problem and solve this problem by SAA method.

The rest of this paper is organized as follows. In Section 2, some preliminaries and the reformulation for SQVIP are given. In Section 3, we give some suitable conditions under which the regularized gap function for QVIP is differentiable and convex. In Section 4, we show that the objective function of the reformulation problem is convex and differentiable under some suitable conditions. Finally, the convergence results of optimal solutions and stationary points are given in Section 5.

2. Preliminaries

Throughout this paper, we use the following notations. $\|\cdot\|$ denotes the Euclidean norm of a vector. For an $n \times n$ symmetric positive-definite matrix G, $\|\cdot\|_G$ denotes the G-norm defined by $\|x\|_G = \sqrt{\langle x, Gx\rangle}$ for $x \in \mathbb{R}^n$ and $\mathrm{Proj}_{S,G}(x)$ denotes the projection of the point x onto the closed convex set S with respect to the norm $\|\cdot\|_G$. For a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, $\nabla_x F(x)$ denotes the usual gradient of $F(x)$ in x. It is easy to verify that

$$\sqrt{\lambda_{\min}}\|x\| \leq \|x\|_G \leq \sqrt{\lambda_{\max}}\|x\|, \tag{2.1}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of G, respectively.

The regularized gap function for the QVIP (1.1) is given as follows:

$$f_\alpha(x) := \max_{y \in S(x)} \left\{ -\langle F(x), y - x\rangle - \frac{\alpha}{2}\|y - x\|_G^2 \right\}, \tag{2.2}$$

where α is a positive parameter. Let $X \subseteq \mathbb{R}^n$ be defined by $X = \{x \in \mathbb{R}^n : x \in S(x)\}$. This is called a feasible set of QVIP (1.1). For the relationship between the regularized gap function (2.2) and QVIP (1.1), the following result has been shown in [17, 18].

Lemma 2.1. *Let $f_\alpha(x)$ be defined by* (2.2)*. Then $f_\alpha(x) \geq 0$ for all $x \in X$. Furthermore, $f_\alpha(x^*) = 0$ and $x^* \in X$ if and only if x^* is a solution to QVIP* (1.1)*. Hence, problem* (1.1) *is equivalent to finding a global optimal solution to the problem:*

$$\min_{x \in X} f_\alpha(x). \tag{2.3}$$

Though the regularized gap function $f_\alpha(x)$ is directional differentiable under some suitable conditions (see, [17, 18]), it is in general nondifferentiable.

The regularized gap function (or residual function) for SQVIP (1.4) is as follows:

$$f_\alpha(x, \omega) := \max_{y \in S(x)} \left\{ -\langle F(x, \omega), y - x\rangle - \frac{\alpha}{2}\|y - x\|_G^2 \right\}, \tag{2.4}$$

and the deterministic reformulation for SQVIP is

$$\min_{x \in X} \Theta(x) := \mathbb{E} f_\alpha(x, \omega), \tag{2.5}$$

where $\mathbb{E}$ denotes the expectation operator.

Note that the objective function $\Theta(x)$ contains mathematical expectation. Throughout this paper, we assume that $\mathbb{E} f_\alpha(x, \omega)$ cannot be calculated in a closed form so that we will have to approximate it through discretization. One of the most well-known discretization approaches is sample average approximation method. In general, for an integrable function $\phi : \Omega \to \mathbb{R}$, we approximate the expected value $\mathbb{E}[\phi(\omega)]$ with sample average $(1/N_k)\sum_{\omega_i\in\Omega_k}\phi(\omega_i)$, where $\omega_1,\ldots,\omega_{N_k}$ are independently and identically distributed random samples of ω and $\Omega_k := \{\omega_1,\ldots,\omega_{N_k}\}$. By the strong law of large numbers, we get the following lemma.

Lemma 2.2. *If $\phi(\omega)$ is integrable, then*

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\phi(\omega_i) = \mathbb{E}[\phi(\omega)] \tag{2.6}$$

holds with probability one.

Let

$$\Theta_k(x) := \frac{1}{N_k}\sum_{\omega_i\in\Omega_k} f_\alpha(x,\omega_i). \tag{2.7}$$

Applying the above techniques, we can get the following approximation of (2.5):

$$\min_{x\in X}\Theta_k(x). \tag{2.8}$$

3. Convexity and Differentiability of $f_\alpha(x)$

In the remainder of this paper, we restrict ourself to a special case, where $S(x) = S + m(x)$. Here, S is a closed convex set in $\mathbb{R}^n$ and $m(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping. In this case, we can show that $f_\alpha(x)$ is continuously differentiable whenever so are the functions $F(x)$ and $m(x)$. In order to get this result, we need the following lemma (see [19, Chapter 4, Theorem 1.7]).

Lemma 3.1. *Let $S \in \mathbb{R}^n$ be a nonempty closed set and $U \in \mathbb{R}^m$ be an open set. Assume that $f : \mathbb{R}^n \times U \to \mathbb{R}$ be continuous and the gradient $\nabla_u f(\cdot,\cdot)$ is also continuous. If the problem $\min_{x\in S} f(x,u)$ is uniquely attained at $x(u)$ for any fixed $u \in U$, then the function $\phi(u) := \min_{x\in S} f(x,u)$ is continuously differentiable and $\nabla_u\phi(u)$ is given by $\nabla_u\phi(u) = \nabla_u f(x(u),u)$.*

For any $y \in S(x) = S + m(x)$, we can find a vector $z \in S$ such that $y = z + m(x)$. Thus, we can rewrite (2.2) as follows:

$$\begin{aligned} f_\alpha(x) &= \max_{z\in S}\left\{-\langle F(x), z + m(x) - x\rangle - \frac{\alpha}{2}\|z - (x - m(x))\|_G^2\right\} \\ &= -\min_{z\in S}\left\{\langle F(x), z - (x - m(x))\rangle + \frac{\alpha}{2}\|z - (x - m(x))\|_G^2\right\}. \end{aligned} \tag{3.1}$$

The minimization problem in (3.1) is essentially equivalent to the following problem:

$$\min_{z\in S}\left\| z - \left[x - m(x) - \alpha^{-1}G^{-1}F(x)\right]\right\|_G^2. \tag{3.2}$$

It is easy to know that problem (3.2) has a unique optimal solution $\mathrm{Proj}_{S,G}(x - m(x) - \alpha^{-1}G^{-1}F(x))$. Thus, $\mathrm{Proj}_{S,G}(x - m(x) - \alpha^{-1}G^{-1}F(x))$ is also a unique solution of problem (3.1). The following result is a natural extension of [20, Theorem 3.2].

Theorem 3.2. *If S is a closed convex set in $\mathbb{R}^n$ and $m(x)$ and $F(x)$ are continuously differentiable, then the regularized gap function $f_\alpha(x)$ given by* (2.2) *is also continuously differentiable and its gradient is given by*

$$\nabla f_\alpha(x) = [I - \nabla m(x)]F(x) - [\nabla F(x) - \alpha(I - \nabla m(x))G][z_\alpha(x) - (x - m(x))], \tag{3.3}$$

where $z_\alpha(x) = Proj_{S,G}(x - m(x) - \alpha^{-1}G^{-1}F(x))$ and I denotes the $n \times n$ identity matrix.

Proof. Let us define the function $h : \mathbb{R}^n \times S \to \mathbb{R}$ by

$$h(x, z) = \langle F(x), z - (x - m(x))\rangle + \frac{\alpha}{2}\|z - (x - m(x))\|_G^2. \tag{3.4}$$

It is obviously that if $F(x)$ and $m(x)$ are continuous, then $h(x, z)$ is continuous in (x, z). If $F(x)$ and $m(x)$ are continuously differentiable, then

$$\nabla_x h(x, z) = -[I - \nabla m(x)]F(x) + [\nabla F(x) - \alpha(I - \nabla m(x))G][z - (x - m(x))] \tag{3.5}$$

is continuous in (x, z). By (3.1), we have

$$f_\alpha(x) = -\min_{z\in S} h(x, z). \tag{3.6}$$

Since the minimum on the right-hand side of (3.6) is uniquely attained at $z = z_\alpha(x)$, it follows from Lemma 3.1 that $f_\alpha(x)$ is differentiable and its gradient is given by

$$\begin{aligned} \nabla f_\alpha(x) &= -\nabla_x h(x, z_\alpha(x)) \\ &= [I - \nabla m(x)]F(x) - [\nabla F(x) - \alpha(I - \nabla m(x))G][z_\alpha(x) - (x - m(x))]. \end{aligned} \tag{3.7}$$

This completes the proof. □

Remark 3.3. When $m(x) \equiv 0$, we have $S(x) \equiv S$ and so QVIP (1.1) reduces to VIP (1.2). In this case

$$\nabla f_\alpha(x) = F(x) - [\nabla F(x) - \alpha G][z_\alpha(x) - x], \tag{3.8}$$

where

$$z_\alpha(x) = \mathrm{Proj}_{S,G}\left(x - \alpha^{-1}G^{-1}F(x)\right). \tag{3.9}$$

Moreover, when $\alpha = 1$, we have

$$\begin{aligned}
\nabla f_\alpha(x) &= F(x) - [\nabla F(x) - G][z_\alpha(x) - x], \\
z_\alpha(x) &= \mathrm{Proj}_{S,G}\left(x - G^{-1}F(x)\right),
\end{aligned} \tag{3.10}$$

which is the same as [20, Theorem 3.2].

Now we investigate the conditions under which $f_\alpha(x)$ is convex.

Theorem 3.4. *Suppose that $F(x) = Mx + q$ and $m(x) = Nx$, where M and N are $n \times n$ matrices and $q \in \mathbb{R}^n$ is a vector. Denote $\beta_{\min}$ and $\mu_{\max}$ by the smallest and largest eigenvalues of $M^T(I - N) + (I - N)^T M$ and $(N - I)^T G(N - I)$, respectively. We have the following statements.*

(i) *If $\mu_{\max} > 0$, $\beta_{\min} \geq 0$ and $\alpha \leq (\beta_{\min}/\mu_{\max})$, then the function $f_\alpha(x)$ is convex. Moreover, if there exists a constant $\beta > 0$ such that $\alpha \leq (\beta_{\min}/\mu_{\max}(1 + \beta))$, then $f_\alpha(x)$ is strongly convex with modulus $\alpha\beta\mu_{\max}$.*

(ii) *If $\mu_{\max} = 0$ and $\beta_{\min} \geq 0$, then the function $f_\alpha(x)$ is convex. Moreover, if $\beta_{\min} > 0$, then $f_\alpha(x)$ is strongly convex with modulus $\beta_{\min}$.*

Proof. Substituting $F(x) = Mx + q$ and $m(x) = Nx$ into (3.1), we have

$$f_\alpha(x) = \max_{z \in S}\left\{-\langle Mx + q, z + (N - I)x\rangle - \frac{\alpha}{2}\|z - (I - N)x\|_G^2\right\}. \tag{3.11}$$

Define

$$H(x, z) = -\langle Mx + q, z + (N - I)x\rangle - \frac{\alpha}{2}\|z - (I - N)x\|_G^2. \tag{3.12}$$

Noting that

$$\nabla_x^2 H(x, z) = M^T(I - N) + (I - N)^T M - \alpha(N - I)^T G(N - I), \tag{3.13}$$

we have, for any $y \in \mathbb{R}^n$,

$$\begin{aligned}
y^T \nabla_x^2 H(x, z) y &= y^T\left[M^T(I - N) + (I - N)^T M\right]y - \alpha y^T(N - I)^T G(N - I)y \\
&\geq (\beta_{\min} - \alpha\mu_{\max})\|y\|^2.
\end{aligned} \tag{3.14}$$

If $\mu_{\max} > 0$, $\beta_{\min} \geq 0$ and $\alpha \leq (\beta_{\min}/\mu_{\max})$, we have

$$y^T \nabla_x^2 H(x,z) y \geq (\beta_{\min} - \alpha\mu_{\max}) \|y\|^2 \geq 0. \tag{3.15}$$

This implies that the Hessen matrix $\nabla_x^2 H(x,z)$ is positive semidefinite and hence $H(x,z)$ is convex in x for any $z \in S$. In consequence, by (3.11), the regularized gap function $f_\alpha(x)$ is convex. Moreover, if $\alpha \leq (\beta_{\min}/\mu_{\max}(1+\beta))$, then

$$y^T \nabla_x^2 H(x,z) y \geq (\beta_{\min} - \alpha\mu_{\max}) \|y\|^2 \geq \alpha\beta\mu_{\max} \|y\|^2, \tag{3.16}$$

which means that $H(x,z)$ is strongly convex with modulus $\alpha\beta\mu_{\max}$ in x for any $z \in S$. From (3.11), we know that the regularized gap function $f_\alpha(x)$ is strongly convex.

If $\mu_{\max} = 0$ and $\beta_{\min} \geq 0$, we have

$$y^T \nabla_x^2 H(x,z) y \geq \beta_{\min} \|y\|^2 \geq 0. \tag{3.17}$$

Thus, the regularized gap function $f_\alpha(x)$ is convex. Moreover, if $\beta_{\min} > 0$, then the regularized gap function $f_\alpha(x)$ is strongly convex with modulus $\beta_{\min}$. This completes the proof. □

Remark 3.5. When $N = 0$, QVIP (1.1) reduces to VIP (1.2). Denote $\overline{\beta}_{\min}$ and $\overline{\mu}_{\max}$ by the smallest and largest eigenvalues of $M^T + M$ and G, respectively. In this case, the function

$$\overline{f}_\alpha(x) = \max_{z\in S}\left\{ -\langle F(x), z-x\rangle - \frac{\alpha}{2}\|z-x\|_G^2 \right\} \tag{3.18}$$

is convex when $\overline{\mu}_{\max} > 0$, $\overline{\beta}_{\min} \geq 0$ and $\alpha \leq (\overline{\beta}_{\min}/\overline{\mu}_{\max})$.

Remark 3.6. When $N = 0$ and $G = I$, we have that $\overline{\mu}_{\max} = 1$. In this case, the function

$$\widehat{f}_\alpha(x) = \max_{z\in S}\left\{ -\langle F(x), z-x\rangle - \frac{\alpha}{2}\|z-x\|^2 \right\} \tag{3.19}$$

is convex when $\overline{\beta}_{\min} \geq 0$ and $\alpha \leq \overline{\beta}_{\min}$. This is consistent with [4, Theorem 2.1].

4. Properties of Function Θ

In this section, we consider the properties of the objective function $\Theta(x)$ of problem (2.5). In what follows we show that $\Theta(x)$ is differentiable under some suitable conditions.

Theorem 4.1. *Suppose that $F(x,\omega) := M(\omega)x + Q(\omega)$, where $M : \Omega \to \mathbb{R}^{n\times n}$ and $Q : \Omega \to \mathbb{R}^n$ with*

$$\mathbb{E}\left[\|M(\omega)\|^2 + \|Q(\omega)\|^2\right] < +\infty. \tag{4.1}$$

Let $S(x) = S + Nx$. Then the function $\Theta(x)$ is differentiable and

$$\nabla_x \Theta(x) = \mathbb{E}\nabla_x f_\alpha(x,\omega). \tag{4.2}$$

Proof. Since $S(x) = S + Nx$, it is easy to know that

$$f_\alpha(x,\omega) = -\langle F(x,\omega), y_\alpha(x,\omega) - (x - Nx)\rangle - \frac{\alpha}{2}\left\|y_\alpha(x,\omega) - (x - Nx)\right\|_G^2, \tag{4.3}$$

where

$$y_\alpha(x,\omega) = \mathrm{Proj}_{S,G}\left(x - Nx - \alpha^{-1}G^{-1}F(x,\omega)\right). \tag{4.4}$$

It follows from Lemma 2.1 that $f_\alpha(x,\omega) \geq 0$ and so

$$\begin{aligned} \frac{\alpha}{2}\left\|y_\alpha(x,\omega) - x + Nx\right\|_G^2 &\leq -\langle F(x,\omega), y_\alpha(x,\omega) - x + Nx\rangle \\ &\leq \|F(x,\omega)\|\left\|y_\alpha(x,\omega) - x + Nx\right\| \\ &\leq \frac{1}{\sqrt{\lambda_{\min}}}\|F(x,\omega)\|\left\|y_\alpha(x,\omega) - x + Nx\right\|_G. \end{aligned} \tag{4.5}$$

Thus,

$$\begin{gathered} \left\|y_\alpha(x,\omega) - x + Nx\right\|_G \leq \frac{2}{\alpha\sqrt{\lambda_{\min}}}\|F(x,\omega)\|, \\ \left\|y_\alpha(x,\omega) - x + Nx\right\| \leq \frac{1}{\sqrt{\lambda_{\min}}}\left\|y_\alpha(x,\omega) - x + Nx\right\|_G \leq \frac{2}{\alpha\lambda_{\min}}\|F(x,\omega)\|. \end{gathered} \tag{4.6}$$

In a similar way to Theorem 3.2, we can show that $f_\alpha(x,\omega)$ is differentiable with respect to x and

$$\nabla_x f_\alpha(x,\omega) = (I - N)F(x,\omega) - [M(\omega) - \alpha(I - N)G]\left[y_\alpha(x,\omega) - (I - N)x\right]. \tag{4.7}$$

It follows that

$$\begin{aligned}
\left\|\nabla_x f_\alpha(x,\omega)\right\| &\le \|I-N\|\|F(x,\omega)\| - \|M(\omega)-\alpha(I-N)G\|\left\|y_\alpha(x,\omega)-(I-N)x\right\| \\
&\le \left\{\|I-N\| + \frac{2}{\alpha\lambda_{\min}}\|M(\omega)-\alpha(I-N)G\|\right\}\|F(x,\omega)\| \\
&\le \left(1+\frac{2\|G\|}{\lambda_{\min}}\right)\|I-N\|\|F(x,\omega)\| + \frac{2}{\alpha\lambda_{\min}}\|M(\omega)\|\|F(x,\omega)\| \\
&\le \left(1+\frac{2\|G\|}{\lambda_{\min}}\right)\|I-N\|(1+\|x\|)(\|M(\omega)\|+\|Q(\omega)\|) \\
&\quad + \frac{2}{\alpha\lambda_{\min}}(1+\|x\|)(\|M(\omega)\|+\|Q(\omega)\|)^2 \\
&\le \left(1+\frac{2\|G\|}{\lambda_{\min}}\right)\|I-N\|(1+\|x\|)(\|M(\omega)\|+\|Q(\omega)\|) \\
&\quad + \frac{4}{\alpha\lambda_{\min}}(1+\|x\|)\left(\|M(\omega)\|^2+\|Q(\omega)\|^2\right).
\end{aligned} \tag{4.8}$$

By [21, Theorem 16.8], the function Θ is differentiable and $\nabla_x\Theta(x) = \mathbb{E}\nabla_x f_\alpha(x,\omega)$. This completes the proof. □

The following theorem gives some conditions under which $\Theta(x)$ is convex.

Theorem 4.2. *Suppose that the assumption of Theorem 4.1 holds. Let*

$$\beta_0 := \inf_{\omega\in\Omega\setminus\Omega_0} \lambda_{\min}\left(M(\omega)^T(I-N)+(I-N)^T M(\omega)\right), \tag{4.9}$$

where Ω_0 is a null subset of Ω and $\lambda_{\min}(G)$ denotes the smallest eigenvalue of G. We have the following statements.

(i) *If $\mu_{\max} > 0$, $\beta_0 > 0$ and $\alpha \le (\beta_0/\mu_{\max})$, then the function $\Theta(x)$ is convex. Moreover, if $\alpha \le (\beta_0/\mu_{\max}(1+\beta))$ with $\beta > 0$, then $\Theta(x)$ is strongly convex with modulus $\alpha\beta\mu_{\max}$.*

(ii) *If $\mu_{\max} = 0$ and $\beta_0 \ge 0$, then the function $\Theta(x)$ is convex. Moreover, if $\beta_0 > 0$, then $\Theta\alpha(x)$ is strongly convex with modulus β_0.*

Proof. Define

$$H(x,z,\omega) = -\langle M(\omega)x + Q(\omega), z + (N-I)x\rangle - \frac{\alpha}{2}\|z-(I-N)x\|_G^2. \tag{4.10}$$

Noting that

$$\nabla_x^2 H(x,z,\omega) = M(\omega)^T(I-N) + (I-N)^T M(\omega) - \alpha(N-I)^T G(N-I), \tag{4.11}$$

we have, for any $y \in \mathbb{R}^n$,

$$y^T \nabla_x^2 H(x,z,\omega) y = y^T \left[M(\omega)^T (I-N) + (I-N)^T M(\omega) \right] y - \alpha y^T (N-I)^T G (N-I) y \\ \geq (\beta_0 - \alpha \mu_{\max}) \, \|y\|^2, \tag{4.12}$$

where the inequality holds almost surely.

If $\mu_{\max} > 0$, $\beta_0 > 0$ and $\alpha \leq (\beta_0/\mu_{\max})$, then

$$y^T \nabla_x^2 H(x,z,\omega) y \geq 0. \tag{4.13}$$

This implies that the Hessen matrix $\nabla_x^2 H(x,z,\omega)$ is positive semidefinite and hence $H(x,z,\omega)$ is convex in x for any $z \in S$. Since

$$f_\alpha(x,\omega) = \max_{y \in S(x)} \left\{ -\langle F(x,\omega), y - x \rangle - \frac{\alpha}{2} \|y - x\|_G^2 \right\} = \max_{z \in S} H(x,z,\omega), \tag{4.14}$$

the regularized gap function $f_\alpha(x,\omega)$ is convex and so is $\Theta(x)$. Moreover, if $\alpha \leq (\beta_0/\mu_{\max}(1+\beta))$, then

$$y^T \nabla_x^2 H(x,z,\omega) y \geq \alpha \beta \mu_{\max} \|y\|^2, \tag{4.15}$$

which means that $H(x,z,\omega)$ is strongly convex in x for any $z \in S$. From the definitions of $H(x,z,\omega)$ and $f_\alpha(x,\omega)$, we know that $f_\alpha(x,\omega)$ is strongly convex with modulus $\alpha\beta\mu_{\max}$ and so is $\Theta(x)$.

If $\mu_{\max} = 0$ and $\beta_0 \geq 0$, then

$$y^T \nabla_x^2 H(x,z,\omega) y \geq \beta_0 \|y\|^2 \geq 0, \tag{4.16}$$

which implies that the regularized gap function $f_\alpha(x,\omega)$ is convex and so is $\Theta(x)$. Moreover, if $\beta_0 > 0$, then $\Theta(x)$ is strongly convex with modulus β_0. This completes the proof. □

It is easy to verify that $X = \{x \in \mathbb{R}^n : x \in S(x)\}$ is a convex subset when $S(x) = S + Nx$. Thus, Theorem 4.2 indicates that problem (2.5) is a convex program. From the proof details of Theorem 4.2, we can also get that problem (2.8) is a convex program. Hence we can obtain a global optimal solution using existing solution methods.

5. Convergence of Solutions and Stationary Points

In this section, we will investigate the limiting behavior of the optimal solutions and stationary points of (2.8).

Note that if the conditions of Theorem 4.1 are satisfied, then the set X is closed, and

$$\mathbb{E}\|M(\omega)\| < \infty, \qquad \mathbb{E}[\|M(\omega)\| + \|Q(\omega)\| + c]^2 < \infty, \tag{5.1}$$

where c is a constant.

Theorem 5.1. *Suppose that the conditions of Theorem 4.1 are satisfied. Let x^k be an optimal solution of problem* (2.8) *for each k. If x^* is an accumulation point of $\{x^k\}$, then it is an optimal solution of problem* (2.5).

Proof. Without loss of generality, we assume that x^k itself converges to x^* as k tends to infinity. It is obvious that $x^* \in X$.

We first show that

$$\lim_{k\to\infty}\left|\Theta_k\left(x^k\right) - \Theta_k(x^*)\right| = 0. \tag{5.2}$$

It follows from mean-value theorem that

$$\begin{aligned}
\left|\Theta_k\left(x^k\right) - \Theta_k(x^*)\right| &= \left|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left[f_\alpha\left(x^k,\omega_i\right) - f_\alpha(x^*,\omega_i)\right]\right| \\
&\le \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left|f_\alpha\left(x^k,\omega_i\right) - f_\alpha(x^*,\omega_i)\right| \\
&\le \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left\|\nabla_x f_\alpha\left(y_i^k,\omega_i\right)\right\|\left\|x^k - x^*\right\|,
\end{aligned} \tag{5.3}$$

where $y_i^k = \gamma_i^k x^k + (1-\gamma_i^k)x^*$ and $\gamma_i^k \in [0,1]$. From the proof details of Theorem 4.1, we have

$$\begin{aligned}
\left\|\nabla_x f_\alpha\left(y_i^k,\omega_i\right)\right\| \le{}& \left(1 + \frac{2\|G\|}{\lambda_{\min}}\right)\left(1 + \left\|y_i^k\right\|\right)(\|M(\omega_i)\| + \|Q(\omega_i)\|)\|I - N\| \\
&+ \frac{2}{\alpha\lambda_{\min}}\left(1 + \left\|y_i^k\right\|\right)(\|M(\omega_i)\| + \|Q(\omega_i)\|)^2.
\end{aligned} \tag{5.4}$$

Since $\lim_{k\to+\infty} x^k = x^*$, there exists a constant C such that $\|x^k\| \le C$ for each k. By the definition of y_i^k, we know that $\|y_i^k\| \le C$. Hence,

$$\begin{aligned}
\left\|\nabla_x f_\alpha\left(y_i^k,\omega_i\right)\right\| \le{}& \left(1 + \frac{2\|G\|}{\lambda_{\min}}\right)(1 + C)(\|M(\omega_i)\| + \|Q(\omega_i)\|)\|I - N\| \\
&+ \frac{2}{\alpha\lambda_{\min}}(1 + C)(\|M(\omega_i)\| + \|Q(\omega_i)\|)^2 \\
\le{}& C'(\|M(\omega_i)\| + \|Q(\omega_i)\| + 1)^2,
\end{aligned} \tag{5.5}$$

where

$$C' = \max\left\{\left(1 + \frac{2\|G\|}{\lambda_{\min}}\right)(1 + C)\|I - N\|,\ \frac{2}{\alpha\lambda_{\min}}(1 + C)\right\}. \tag{5.6}$$

It follows that

$$\left|\Theta_k\left(x^k\right) - \Theta_k(x^*)\right| \le C' \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} (\|M(\omega_i)\| + \|Q(\omega_i)\| + 1)^2 \left\|x^k - x^*\right\| \longrightarrow 0, \tag{5.7}$$

which means that (5.2) holds.

Now, we show that x^* is an optimal solution of problem (2.5). It follows from (5.2) and

$$\left|\Theta_k\left(x^k\right) - \Theta(x^*)\right| \le \left|\Theta_k\left(x^k\right) - \Theta_k(x^*)\right| + |\Theta_k(x^*) - \Theta(x^*)|, \tag{5.8}$$

that $\lim_{k\to+\infty}\Theta_k(x^k) = \Theta(x^*)$. Since x^k is an optimal solution of problem (2.8) for each k, we have that, for any $x \in X$,

$$\Theta_k\left(x^k\right) \le \Theta_k(x). \tag{5.9}$$

Letting $k \to \infty$ above, we get from (5.2) and Lemma 2.2 that

$$\Theta(x^*) \le \Theta(x), \tag{5.10}$$

which means x^* is an optimal solution of problem (2.5). This completes the proof. □

In general, it is difficult to obtain a global optimal solution of problem (2.8), whereas computation of stationary points is relatively easy. Therefore, it is important to study the limiting behavior of stationary points of problem (2.8).

Definition 5.2. x^k is said to be stationary to problem (2.8) if

$$\left\langle \nabla_x \Theta_k\left(x^k\right), y - x^k \right\rangle \ge 0, \quad \forall y \in X, \tag{5.11}$$

and x^* is said to be stationary to problem (2.5) if

$$\langle \nabla_x \Theta(x^*), y - x^* \rangle \ge 0, \quad \forall y \in X. \tag{5.12}$$

Theorem 5.3. *Let x^k be stationary to problem* (2.8) *for each k. If the conditions of Theorem 4.1 are satisfied, then any accumulation point x^* of $\{x^k\}$ is a stationary point of problem* (2.5).

Proof. Without loss of generality, we assume that $\{x^k\}$ itself converges to x^*.

At first, we show that

$$\lim_{k\to\infty} \left\| \nabla_x \Theta_k\left(x^k\right) - \nabla_x \Theta_k(x^*) \right\| = 0. \tag{5.13}$$

It follows from (2.1) and the nonexpansivity of the projection operator that

$$\begin{aligned}
&\left\|y_\alpha\left(x^k,\omega\right)-y_\alpha(x^*,\omega)\right\| \\
&\quad\le \frac{1}{\sqrt{\lambda_{\min}}}\left\|y_\alpha(x^k,\omega)-y_\alpha(x^*,\omega)\right\|_G \\
&\quad= \frac{1}{\sqrt{\lambda_{\min}}}\Big\|\mathrm{Proj}_{S,G}\left(x^k-Nx^k-\alpha^{-1}G^{-1}F\left(x^k,\omega\right)\right) \\
&\qquad\qquad -\mathrm{Proj}_{S,G}\left(x^*-Nx^*-\alpha^{-1}G^{-1}F(x^*,\omega)\right)\Big\|_G \\
&\quad\le \frac{1}{\sqrt{\lambda_{\min}}}\left\|x^k-Nx^k-\alpha^{-1}G^{-1}F\left(x^k,\omega\right)-\left[x^*-Nx^*-\alpha^{-1}G^{-1}F(x^*,\omega)\right]\right\|_G \\
&\quad\le \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\left\|x^k-Nx^k-\alpha^{-1}G^{-1}F\left(x^k,\omega\right)-\left[x^*-Nx^*-\alpha^{-1}G^{-1}F(x^*,\omega)\right]\right\| \\
&\quad\le \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\left(\|I-N\|+\alpha^{-1}\left\|G^{-1}\right\|\|M(\omega)\|\right)\left\|x^k-x^*\right\|.
\end{aligned} \tag{5.14}$$

Thus,

$$\begin{aligned}
&\left\|\nabla_x\Theta_k\left(x^k\right)-\nabla_x\Theta_k(x^*)\right\| \\
&\quad\le \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left\|\nabla_x f_\alpha\left(x^k,\omega_i\right)-\nabla_x f_\alpha(x^*,\omega_i)\right\| \\
&\quad= \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\Big\|(I-N)\left[M(\omega_i)x^k+Q(\omega_i)\right]-[M(\omega_i)-\alpha(I-N)G] \\
&\qquad\qquad \times\left[y_\alpha\left(x^k,\omega_i\right)-(I-N)x^k\right] \\
&\qquad\qquad -\{(I-N)[M(\omega_i)x^*+Q(\omega_i)]-[M(\omega_i)-\alpha(I-N)G] \\
&\qquad\qquad\quad \times[y_\alpha(x^*,\omega_i)-(I-N)x^*]\}\Big\| \\
&\quad\le 2\left\|x^k-x^*\right\|\|I-N\|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)\|+\alpha\|I-N\|^2\|G\|\left\|x^k-x^*\right\| \\
&\qquad+\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)-\alpha(I-N)G\|\left\|y_\alpha\left(x^k,\omega_i\right)-y_\alpha(x^*,\omega_i)\right\| \\
&\quad\le 2\left\|x^k-x^*\right\|\|I-N\|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)\|+\alpha\|I-N\|^2\|G\|\left\|x^k-x^*\right\| \\
&\qquad+\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)-\alpha(I-N)G\|\left[\|I-N\|+\alpha^{-1}\left\|G^{-1}\right\|\|M(\omega_i)\|\right]\left\|x^k-x^*\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[2+\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\left(\|G\|\left\|G^{-1}\right\|+1\right)\right]\|I-N\|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)\|\left\|x^k-x^*\right\| \\
&\quad+\left(1+\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)\alpha\|I-N\|^2\|G\|\left\|x^k-x^*\right\| \\
&\quad+\alpha^{-1}\left\|G^{-1}\right\|\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)\|^2\left\|x^k-x^*\right\| \\
&\longrightarrow 0,
\end{aligned}
\tag{5.15}
$$

which means that (5.13) is true.

Next, we show that

$$\lim_{k\to\infty}\nabla_x\Theta_k\left(x^k\right)=\nabla_x\Theta(x^*). \tag{5.16}$$

It follows from Lemma 2.2 and Theorem 4.1 that

$$\lim_{k\to\infty}\nabla_x\Theta_k(x^*)=\lim_{k\to\infty}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\nabla_x f_\alpha(x^*,\omega_i)=\mathbb{E}\nabla_x f_\alpha(x^*,\omega)=\nabla_x\Theta(x^*). \tag{5.17}$$

By (5.13), we have

$$
\begin{aligned}
\left\|\nabla_x\Theta_k\left(x^k\right)-\nabla_x\Theta(x^*)\right\| &\leq \left\|\nabla_x\Theta_k\left(x^k\right)-\nabla_x\Theta_k(x^*)\right\|+\|\nabla_x\Theta_k(x^*)-\nabla_x\Theta(x^*)\| \\
&\longrightarrow 0,
\end{aligned}
\tag{5.18}
$$

which implies that (5.16) is true.

Now we show that x^* is a stationary point of problem (2.5). Since x^k is stationary to problem (2.8), that is, for any $y \in X$,

$$\left\langle\nabla_x\Theta_k\left(x^k\right),y-x^k\right\rangle\geq 0. \tag{5.19}$$

Letting $k \to \infty$ above, we get from (5.16) that

$$\left\langle\nabla_x\Theta(x^*),y-x^*\right\rangle\geq 0. \tag{5.20}$$

Thus, x^* is a stationary point of problem (2.5). This completes the proof. □

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References

[1] P. T. Harker, "Generalized Nash games and quasi-variational inequalities," *European Journal of Operational Research*, vol. 54, pp. 81–94, 1991.
[2] K. Kubota and M. Fukushima, "Gap function approach to the generalized Nash equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 144, no. 3, pp. 511–531, 2010.
[3] J. S. Pang and M. Fukushima, "Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games," *Computational Management Science*, vol. 2, no. 1, pp. 21–56, 2005.
[4] R. P. Agdeppa, N. Yamashita, and M. Fukushima, "Convex expected residual models for stochastic affine variational inequality problems and its application to the traffic equilibrium problem," *Pacific Journal of Optimization*, vol. 6, no. 1, pp. 3–19, 2010.
[5] X. Chen and M. Fukushima, "Expected residual minimization method for stochastic linear complementarity problems," *Mathematics of Operations Research*, vol. 30, no. 4, pp. 1022–1038, 2005.
[6] X. Chen, C. Zhang, and M. Fukushima, "Robust solution of monotone stochastic linear complementarity problems," *Mathematical Programming*, vol. 117, no. 1-2, pp. 51–80, 2009.
[7] H. Fang, X. Chen, and M. Fukushima, "Stochastic R_0 matrix linear complementarity problems," *SIAM Journal on Optimization*, vol. 18, no. 2, pp. 482–506, 2007.
[8] H. Jiang and H. Xu, "Stochastic approximation approaches to the stochastic variational inequality problem," *IEEE Transactions on Automatic Control*, vol. 53, no. 6, pp. 1462–1475, 2008.
[9] G. H. Lin, X. Chen, and M. Fukushima, "New restricted NCP functions and their applications to stochastic NCP and stochastic MPEC," *Optimization*, vol. 56, no. 5-6, pp. 641–953, 2007.
[10] G. H. Lin and M. Fukushima, "New reformulations for stochastic nonlinear complementarity problems," *Optimization Methods & Software*, vol. 21, no. 4, pp. 551–564, 2006.
[11] C. Ling, L. Qi, G. Zhou, and L. Caccetta, "The SC^1 property of an expected residual function arising from stochastic complementarity problems," *Operations Research Letters*, vol. 36, no. 4, pp. 456–460, 2008.
[12] M. J. Luo and G. H. Lin, "Expected residual minimization method for stochastic variational inequality problems," *Journal of Optimization Theory and Applications*, vol. 140, no. 1, pp. 103–116, 2009.
[13] M. J. Luo and G. H. Lin, "Convergence results of the ERM method for nonlinear stochastic variational inequality problems," *Journal of Optimization Theory and Applications*, vol. 142, no. 3, pp. 569–581, 2009.
[14] M. Z. Wang, M. M. Ali, and G. H. Lin, "Sample average approximation method for stochastic complementarity problems with applications to supply chain supernetworks," *Journal of Industrial and Management Optimization*, vol. 7, no. 2, pp. 317–345, 2011.
[15] H. Xu, "Sample average approximation methods for a class of stochastic variational inequality problems," *Asia-Pacific Journal of Operational Research*, vol. 27, no. 1, pp. 103–119, 2010.
[16] C. Zhang and X. Chen, "Stochastic nonlinear complementarity problem and applications to traffic equilibrium under uncertainty," *Journal of Optimization Theory and Applications*, vol. 137, no. 2, pp. 277–295, 2008.
[17] M. Fukushima, "A class of gap functions for quasi-variational inequality problems," *Journal of Industrial and Management Optimization*, vol. 3, no. 2, pp. 165–171, 2007.
[18] K. Taji, "On gap functions for quasi-variational inequalities," *Abstract and Applied Analysis*, vol. 2008, Article ID 531361, 7 pages, 2008.
[19] A. Auslender, *Optimisation: Méthodes Numériques*, Masson, Paris, France, 1976.
[20] M. Fukushima, "Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems," *Mathematical Programming*, vol. 53, no. 1, pp. 99–110, 1992.
[21] P. Billingsley, *Probability and Measure*, John Wiley & Sons, New York, NY, USA, 1995.

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Research Article

Strong Convergence Theorems for a Finite Family of λ_i-Strict Pseudocontractions in 2-Uniformly Smooth Banach Spaces by Metric Projections

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A new hybrid projection algorithm is considered for a finite family of λ_i-strict pseudocontractions. Using the metric projection, some strong convergence theorems of common elements are obtained in a uniformly convex and 2-uniformly smooth Banach space. The results presented in this paper improve and extend the corresponding results of Matsushita and Takahshi, 2008, Kang and Wang, 2011, and many others.

1. Introduction

Let E be a real Banach space and let E^* be the dual spaces of E. Assume that J is the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the generalized duality pairing between E and E^*.

Let C be a closed convex subset of a real Banach space E. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{1.2}$$

for all $x, y \in C$. Also a mapping $T : C \to C$ is called a λ-strict pseudocontraction if there exists a constant $\lambda \in (0,1)$ such that for every $x, y \in C$ and for some $j(x-y) \in J(x-y)$, the following holds:

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \lambda \|(I-T)x - (I-T)y\|^2. \tag{1.3}$$

From (1.3) we can prove that if T is λ-strict pseudo-contractive, then T is Lipschitz continuous with the Lipschitz constant $L = (1+\lambda)/\lambda$.

It is well-known that the classes of nonexpansive mappings and pseudocontractions are two kinds important nonlinear mappings, which have been studied extensively by many authors (see [1–8]).

In [9] Reich considered the Mann iterative scheme $\{x_n\}$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad x_1 \in C \tag{1.4}$$

for nonexpansive mappings, where $\{\alpha_n\}$ is a sequence in $(0,1)$. Under suitable conditions, the author proved that $\{x_n\}$ converges weakly to a fixed point of T. In 2005, Kim and Xu [10] proved a strong convergence theorem for nonexpansive mappings by modified Mann iteration. In 2008, Zhou [11] extended and improved the main results of Kim and Xu to the more broad 2-uniformly smooth Banach spaces for λ-strict pseudocontractive mappings.

On the other hand, by using metric projection, Nakajo and Takahashi [12] introduced the following iterative algorithms for the nonexpansive mapping T in the framework of Hilbert spaces:

$$\begin{aligned}
x_0 &= x \in C, \\
y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \\
C_n &= \{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{1.5}$$

where $\{\alpha_n\} \subset [0, \alpha]$, $\alpha \in [0,1)$, and $P_{C_n \cap Q_n}$ is the metric projection from a Hilbert space H onto $C_n \cap Q_n$. They proved that $\{x_n\}$ generated by (1.5) converges strongly to a fixed point of T.

In 2006, Xu [13] extended Nakajo and Takahashi's theorem to Banach spaces by using the generalized projection.

In 2008, Matsushita and Takahashi [14] presented the following iterative algorithms for the nonexpansive mapping T in the framework of Banach spaces:

$$\begin{aligned}
x_0 &= x \in C, \\
C_n &= \overline{\mathrm{co}}\{z \in C : \|z - Tz\| \leq t_n \|x_n - T x_n\|\}, \\
D_n &= \{z \in C : \langle x_n - z, J(x - x_n) \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap D_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{1.6}$$

where $\overline{\text{co}}C$ denotes the convex closure of the set C, J is normalized duality mapping, $\{t_n\}$ is a sequence in $(0, 1)$ with $t_n \to 0$, and $P_{C_n \cap D_n}$ is the metric projection from E onto $C_n \cap D_n$. Then, they proved that $\{x_n\}$ generated by (1.6) converges strongly to a fixed point of nonexpansive mapping T.

Recently, Kang and Wang [15] introduced the following hybrid projection algorithm for a pair of nonexpansive mapping T in the framework of Banach spaces:

$$
\begin{aligned}
x_0 &= x \in C, \\
y_n &= \alpha_n T_1 x_n + (1-\alpha_n) T_2 x_n, \\
C_n &= \overline{\text{co}}\{z \in C : \|z - T_1 z\| + \|z - T_2 z\| \le t_n \|x_n - y_n\|\}, \\
x_{n+1} &= P_{C_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{1.7}
$$

where $\overline{\text{co}}C$ denotes the convex closure of the set C, $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$, and P_{C_n} is the metric projection from E onto C_n. Then, they proved that $\{x_n\}$ generated by (1.7) converges strongly to a fixed point of two nonexpansive mappings T_1 and T_2.

In this paper, motivated by the research work going on in this direction, we introduce the following iterative for finding fixed points of a finite family of λ_i-strict pseudocontractions in a uniformly convex and 2-uniformly smooth Banach space:

$$
\begin{aligned}
x_0 &= x \in C, \\
y_n &= \sum_{i=1}^{N} \alpha_{n,i} T_i x_n, \\
C_n &= \overline{\text{co}}\left\{z \in C : \sum_{i=1}^{N} \|z - T_i z\| \le t_n \|x_n - y_n\|\right\}, \\
x_{n+1} &= P_{C_n} x, \quad n = 1, 2, \ldots,
\end{aligned}
\tag{1.8}
$$

where $\overline{\text{co}}C$ denotes the convex closure of the set C, $\{\alpha_{n,i}\}$ is N sequences in [0,1] and $\sum_{i=1}^{N} \alpha_{n,i} = 1$ for each $n \ge 0$, $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$, and P_{C_n} is the metric projection from E onto C_n. we prove defined by (1.8) converges strongly to a common fixed point of a finite family of λ_i-strictly pseudocontractions, the main results of Kang and Wang is extended and improved to strictly pseudocontractions.

2. Preliminaries

In this section, we recall the well-known concepts and results which will be needed to prove our main results. Throughout this paper, we assume that E is a real Banach space and C is a nonempty subset of E. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. We also assume that E^* is the dual space of E, and $J : E \to 2^{E^*}$ is the normalized duality mapping. Some properties of duality mapping have been given in [16].

A Banach space E is said to be *strictly convex* if $\|x+y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, $\|x+y\| \leq 2(1-\delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\|, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}. \tag{2.1}$$

E is said to be *smooth* if the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in U$. The modulus of smoothness of E is defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}. \tag{2.3}$$

A Banach space E is said to be *uniformly smooth* if $\rho_E(t)/t \to 0$ as $t \to 0$. A Banach space E is said to be *q-uniformly smooth*, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$.

If E is a reflexive, strictly convex, and smooth Banach space, then for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| = \min_{y\in C}\|y-x\|. \tag{2.4}$$

The mapping $P_C : E \to C$ defined by $P_C x = x_0$ is called the *metric projection* from E onto C. Let $x \in E$ and $u \in C$. Then it is known that $u = P_C x$ if and only if

$$\langle u-y, J(x-u)\rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

For the details on the metric projection, refer to [17–20].

In the sequel, we make use the following lemmas for our main results.

Lemma 2.1 (see [21]). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2 \tag{2.6}$$

for any $x, y \in E$.

Lemma 2.2 (see [11]). *Let C be a nonempty subset of a real 2-uniformly smooth Banach space E with the best smooth constant $K > 0$ and let $T : C \to C$ be a λ-strict pseudocontraction. For $\alpha \in (0,1) \cap (0, \lambda/K^2]$, we define $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then $T_\alpha : C \to E$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

Lemma 2.3 (demiclosed principle, see [22]). *Let E be a real uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a continuous pseudocontractive mapping. Then, $I - T$ is demiclosed at zero.*

Lemma 2.4 (see [23]). *Let C be a closed convex subset of a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing convex continuous function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\left\|T\left(\sum_{j=0}^{m}\mu_j z_j\right) - \sum_{j=0}^{m}\mu_j T z_j\right\|\right) \le \max_{0\le j<k\le m}\left(\|z_j - z_k\| - \|Tz_j - Tz_k\|\right), \tag{2.7}$$

for all $m \ge 1$, $\{\mu_j\}_{j=0}^{m} \in \Delta^m$, $\{z_j\}_{j=0}^{m} \subset C \cap B_r$, and $T \in Lip(C,1)$, where $\Delta^m = \{\{\mu_0,\mu_1,\ldots,\mu_m\} : 0 \le \mu_j \ (0 \le j \le m)$ and $\sum_{j=0}^{m}\mu_j = 1\}$, $B_r = \{x \in E : \|x\| \le r\}$, and $Lip(C,1)$ is the set of all nonexpansive mappings from C into E.

3. Main Results

Now we are ready to give our main results in this paper.

Lemma 3.1. *Let C be a closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E with the best smooth constant $K > 0$, and $T : C \to C$ be a λ-strict pseudocontraction. Then for each $r > 0$, there exists a strictly increasing convex continuous function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\alpha\left\|T\left(\sum_{j=0}^{m}\mu_j z_j\right) - \sum_{j=0}^{m}\mu_j T z_j\right\|\right) \le \alpha\max_{0\le j<k\le m}\left(\|z_j - Tz_j\| + \|z_k - Tz_k\|\right), \tag{3.1}$$

for all $m \ge 1$, $\{\mu_j\}_{j=0}^{m} \in \Delta^m$, $\{z_j\}_{j=0}^{m} \subset C \cap B_r$, where $\alpha \in (0,1) \cap (0,\lambda/K^2]$, $\Delta^m = \{\{\mu_0,\mu_1,\ldots,\mu_m\} : 0 \le \mu_j \ (0 \le j \le m)$ and $\sum_{j=0}^{m}\mu_j = 1\}$, $B_r = \{x \in E : \|x\| \le r\}$.

Proof. Define the mapping $T_\alpha : C \to C$ as $T_\alpha x = (1-\alpha)x + \alpha Tx$, for all $x \in C$. Then T_α is nonexpansive. From Lemma 2.4, there exists a strictly increasing convex continuous function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ and such that

$$\gamma\left(\left\|T_\alpha\left(\sum_{j=0}^{m}\mu_j z_j\right) - \sum_{j=0}^{m}\mu_j T_\alpha z_j\right\|\right) \le \max_{0\le j<k\le m}\left(\|z_j - z_k\| - \|T_\alpha z_j - T_\alpha z_k\|\right). \tag{3.2}$$

Hence

$$\begin{aligned}
\gamma\left(\alpha\left\|T\left(\sum_{j=0}^{m}\mu_j z_j\right)-\sum_{j=0}^{m}\mu_j T z_j\right\|\right) &= \gamma\left(\left\|T_\alpha\left(\sum_{j=0}^{m}\mu_j z_j\right)-\sum_{j=0}^{m}\mu_j T_\alpha z_j\right\|\right)\\
&\le \max_{0\le j<k\le m}\left(\left\|z_j-z_k\right\|-\left\|T_\alpha z_j-T_\alpha z_k\right\|\right)\\
&\le \max_{0\le j<k\le m}\left(\left\|z_j-T_\alpha z_j\right\|+\left\|z_k-T_\alpha z_k\right\|\right)\\
&= \alpha\max_{0\le j<k\le m}\left(\left\|z_j-T z_j\right\|+\left\|z_k-T z_k\right\|\right).
\end{aligned} \tag{3.3}$$

This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed subset of a uniformly convex and 2-uniformly smooth Banach space E with the best smooth constant $K>0$, assume that for each i $(i=1,2,\ldots,N)$, $T_i: C\to C$ is a λ_i-strict pseudocontraction for some $0<\lambda_i<1$ such that $\mathcal{F}=\cap_{i=1}^{N}\mathcal{F}(T_i)\neq\emptyset$. Let $\{\alpha_{n,i}\}$ be N sequences in [0,1] with $\sum_{i=1}^{N}\alpha_{n,i}=1$ for each $n\ge 0$ and $\{t_n\}$ be a sequence in (0,1) with $t_n\to 0$. Let $\{x_n\}$ be a sequence generated by (1.8), where $\overline{\mathrm{co}}\{z\in C:\sum_{i=1}^{N}\|z-T_i z\|\le t_n\|x_n-y_n\|\}$ denotes the convex closure of the set $\{z\in C:\sum_{i=1}^{N}\|z-T_i z\|\le t_n\|x_n-y_n\|\}$ and P_{C_n} is the metric projection from E onto C_n. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}}x$.*

Proof. (I) First we prove that $\{x_n\}$ is well defined and bounded.

It is easy to check that C_n is closed and convex and $\mathcal{F}\subset C_n$ for all $n\ge 0$. Therefore $\{x_n\}$ is well defined.

Put $p=P_{\mathcal{F}}x$. Since $\mathcal{F}\subset C_n$ and $x_{n+1}=P_{C_n}x$, we have that

$$\|x_{n+1}-x\|\le\|p-x\| \tag{3.4}$$

for all $n\ge 0$. Hence $\{x_n\}$ is bounded.

(II) Now we prove that $\|x_n-T_i x_n\|\to 0$ as $n\to\infty$ for all $i\in\{1,2,\ldots,N\}$.

Since $x_{n+1}\in C_n$, there exist some positive integer $m\in\mathbb{N}$ ($\mathbb{N}$ denotes the set of all positive integers), $\{\mu_i\}\in\Delta^m$ and $\{z_i\}_{i=0}^{m}\subset C$ such that

$$\left\|x_{n+1}-\sum_{j=0}^{m}\mu_j z_j\right\|<t_n, \tag{3.5}$$

$$\sum_{i=1}^{N}\left\|z_j-T_i z_j\right\|\le t_n\left\|x_n-y_n\right\| \tag{3.6}$$

for all $j \in \{0,1,\dots,m\}$. Put $r_0 = \sup_{n\ge1}\|x_n - p\|$ and $\lambda = \min_{1\le i\le N}\{\lambda_i\}$. Take $\alpha \in (0,1)\cap(0,\lambda/K^2]$. It follows from Lemma 2.2 and (3.5) that

$$\|x_n - T_i x_n\| = \frac{1}{\alpha}\left\|(T_{i\alpha}x_n - p) + (p - x_n)\right\| \le \frac{2r_0}{\alpha}, \tag{3.7}$$

$$\begin{aligned}\left\|T_i\left(\sum_{j=0}^{m}\mu_j z_j\right) - T_i x_{n+1}\right\| &\le \frac{1}{\alpha}\left(\left\|T_{i\alpha}\left(\sum_{j=0}^{m}\mu_j z_j\right) - T_{i\alpha}x_{n+1}\right\| + (1-\alpha)\left\|\sum_{j=0}^{m}\mu_j z_j - x_{n+1}\right\|\right)\\ &\le \left(\frac{2}{\alpha}-1\right)\left\|\sum_{j=0}^{m}\mu_j z_j - x_{n+1}\right\|\\ &\le \left(\frac{2}{\alpha}-1\right)t_n\end{aligned} \tag{3.8}$$

for all $i \in \{1,2,\dots,N\}$. Moreover, (3.7) implies

$$\|x_n - y_n\| \le \frac{2r_0}{\alpha}. \tag{3.9}$$

It follows from Lemma 3.1, (3.5)–(3.9) that

$$\begin{aligned}\sum_{i=1}^{N}\|x_{n+1} - T_i x_{n+1}\| &\le \sum_{i=1}^{N}\left(\left\|x_{n+1} - \sum_{j=0}^{m}\mu_j z_j\right\| + \left\|\sum_{j=0}^{m}\mu_j (z_j - T_i z_j)\right\|\right.\\ &\qquad\left. + \left\|\sum_{j=0}^{m}\mu_j T_i z_j - T_i\left(\sum_{j=0}^{m}\mu_j z_j\right)\right\| + \left\|T_i\left(\sum_{j=0}^{m}\mu_j z_j\right) - T_i x_{n+1}\right\|\right)\\ &\le \frac{2N}{\alpha}\left\|x_{n+1} - \sum_{j=0}^{m}\mu_j z_j\right\| + \sum_{j=0}^{m}\mu_j\left(\sum_{i=1}^{N}\|z_j - T_i z_j\|\right)\\ &\quad + \sum_{i=1}^{N}\left\|\sum_{j=0}^{m}\mu_j T_i z_j - T_i\left(\sum_{j=0}^{m}\mu_j z_j\right)\right\|\\ &\le \frac{2N}{\alpha}t_n + t_n\|y_n - x_n\| + \sum_{i=1}^{N}\frac{1}{\alpha}\gamma^{-1}\left(\alpha\max_{0\le k<j\le m}\left(\|z_k - T_i z_k\| + \|z_j - T_i z_j\|\right)\right)\\ &\le \frac{2N + 2r_0}{\alpha}t_n + \frac{N}{\alpha}\gamma^{-1}(4r_0 t_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.10}$$

This shows that

$$\|x_n - T_i x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{3.11}$$

for all $i \in \{1,2,\dots,N\}$.

(III) Finally, we prove that $x_n \to p = P_{\mathcal{F}}x$.

It follows from the boundedness of $\{x_n\}$ that there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup v$ as $i \to \infty$. Since for each $i \in \{0,1,\ldots,N\}$, T_i is a λ_i-strict pseudocontraction, then T_i is demiclosed. one has $v \in \mathcal{F}$.

From the weakly lower semicontinuity of the norm and (3.4), we have

$$\begin{aligned} \|p - x\| &\le \|v - x\| \le \liminf_{i\to\infty}\|x_{n_i}\| - x \\ &\le \limsup_{i\to\infty}\|x_{n_i} - x\| \le \|p - x\|. \end{aligned} \tag{3.12}$$

This shows $p = v$ and hence $x_{n_i} \rightharpoonup p$ as $i \to \infty$. Therefore, we obtain $x_n \rightharpoonup p$. Further, we have that

$$\lim_{n\to\infty}\|x_n - x\| = \|p - x\|. \tag{3.13}$$

Since E is uniformly convex, we have $x_n - x \to p - x$. This shows that $x_n \to p$. This completes the proof. □

Corollary 3.3. *Let C be a nonempty closed subset of a uniformly convex and 2-uniformly smooth Banach space E with the best smooth constant $K > 0$, assume that $T : C \to C$ is a λ-strict pseudocontraction for some $0 < \lambda < 1$ such that $\mathcal{F}(T) \ne \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 &= x \in C, \\ C_n &= \overline{\text{co}}\{z \in C : \|z - Tz\| \le t_n\|x_n - Tx_n\|\}, \\ x_{n+1} &= P_{C_n}x, \quad n = 0,1,2,\ldots, \end{aligned} \tag{3.14}$$

where $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$. $\overline{\text{co}}\{z \in C : \|z - Tz\| \le t_n\|x_n - Tx_n\|\}$ denotes the convex closure of the set $\{z \in C : \|z - Tz\| \le t_n\|x_n - Tx_n\|\}$ and P_{C_n} is the metric projection from E onto C_n. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}(T)}x$.

Proof. Set $T_1 = T, T_k = I$ for all $2 \le k \le N$, and $\alpha_{n,1} = 1$, $\alpha_{n,k} = 0$ for all $2 \le k \le N$ in Theorem 3.2. The desired result can be obtained directly from Theorem 3.2. □

Remark 3.4. At the end of the paper, we would like to point out that concerning the convergence problem of iterative sequences for strictly pseudocontractive mappings has been considered and studied by many authors. It can be consulted the references [24–37].

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References

[1] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[2] F. E. Browder, "Fixed-point theorems for noncompact mappings in Hilbert space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 53, pp. 1272–1276, 1965.

[3] F. E. Browder, "Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces," *Archive for Rational Mechanics and Analysis*, vol. 24, pp. 82–90, 1967.

[4] R. E. Bruck,, "Nonexpansive projections on subsets of Banach spaces," *Pacific Journal of Mathematics*, vol. 47, pp. 341–355, 1973.

[5] C. E. Chidume and S. A. Mutangadura, "An example on the Mann iteration method for Lipschitz pseudo-contractions," *Proceedings of the American Mathematical Society*, vol. 129, pp. 2359–2363, 2001.

[6] K. Deimling, "Zeros of accretive operators," *Manuscripta Mathematica*, vol. 13, pp. 365–374, 1974.

[7] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York, NY, USA, 1984.

[8] J. P. Gossez and E. L. Dozo, "Some geometric properties related to the fixed point theory for nonexpansive mappings," *Pacific Journal of Mathematics*, vol. 40, pp. 565–573, 1972.

[9] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[10] T.-H. Kim and H.-K. Xu, "Strong convergence of modified Mann iterations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 61, no. 1-2, pp. 51–60, 2005.

[11] H. Zhou, "Convergence theorems for λ_i-strict pseudo-contractions in 2-uniformly smooth Banach spaces," *Nonlinear Anal*, vol. 69, pp. 3160–3173, 2008.

[12] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[13] H.-K. Xu, "Strong convergence of approximating fixed point sequences for nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 74, no. 1, pp. 143–151, 2006.

[14] S.-y. Matsushita and W. Takahashi, "Approximating fixed points of nonexpansive mappings in a Banach space by metric projections," *Applied Mathematics and Computation*, vol. 196, no. 1, pp. 422–425, 2008.

[15] S. M. Kang and S. Wang, "New hybrid algorithms for nonexpansive mappings in Banach spaces," *International Journal of Mathematical Analysis*, vol. 5, no. 9-12, pp. 433–440, 2011.

[16] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrechtm The Netherlands, 1990.

[17] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Marcel Dekker, New York, NY, USA, 1996.

[18] W. Takahashi, *Nonlinear Functional Analysis*, Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama, Japan, 2000.

[19] F. E. Browder, "Convergence theorems for sequences of nonlinear operators in Banach spaces," *Mathematische Zeitschrift*, vol. 100, pp. 201–225, 1967.

[20] D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Noordhoff, Leyden, The Netherlands, 1978.

[21] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis*, vol. 16, pp. 1127–1138, 1991.

[22] H. Zhou, "Convergence theorems of common fixed points for a finite family of Lipschitz pseudo-contractions in Banach spaces," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 68, no. 10, pp. 2977–2983, 2008.

[23] R. E. Bruck, "On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces," *Israel Journal of Mathematics*, vol. 38, no. 4, pp. 304–314, 1981.

[24] Y. Su, M. Li, and H. Zhang, "New monotone hybrid algorithm for hemi-relatively nonexpansive mappings and maximal monotone operators," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5458–5465, 2011.

[25] S. S. Chang, H. W. J. Lee, C. K. Chan, and J. K. Kim, "Approximating solutions of variational inequalities for asymptotically nonexpansive mappings," *Applied Mathematics and Computation*, vol. 212, no. 1, pp. 51–59, 2009.

[26] H. Zegeye and N. Shahzad, "Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings," *Computers & Mathematics with Applications*, vol. 62, no. 11, pp. 4007–4014, 2011.

[27] X. Qin, S. Huang, and T. Wang, "On the convergence of hybrid projection algorithms for asymptotically quasi-nonexpansive mappings," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 851–859, 2011.

[28] S.-s. Chang, L. Wang, Y.-K. Tang, B. Wang, and L.-J. Qin, "Strong convergence theorems for a countable family of quasi-ϕ-asymptotically nonexpansive nonself mappings," *Applied Mathematics and Computation*, vol. 218, no. 15, pp. 7864–7870, 2012.

[29] W. Nilsrakoo, "Halpern-type iterations for strongly relatively nonexpansive mappings in Banach spaces," *Computers & Mathematics with Applications*, vol. 62, no. 12, pp. 4656–4666, 2011.

[30] S.-S. Chang, H. W. J. Lee, C. K. Chan, and J. A. Liu, "Strong convergence theorems for countable families of asymptotically relatively nonexpansive mappings with applications," *Applied Mathematics and Computation*, vol. 218, no. 7, pp. 3187–3198, 2011.

[31] W. Nilsrakoo and S. Saejung, "Strong convergence theorems by Halpern-Mann iterations for relatively nonexpansive mappings in Banach spaces," *Applied Mathematics and Computation*, vol. 217, no. 14, pp. 6577–6586, 2011.

[32] S. Suantai, W. Cholamjiak, and P. Cholamjiak, "An implicit iteration process for solving a fixed point problem of a finite family of multi-valued mappings in Banach spaces," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1656–1660, 2012.

[33] S. S. Chang, K. K. Tan, H. W. J. Lee, and C. K. Chan, "On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 313, no. 1, pp. 273–283, 2006.

[34] S. S. Chang, H. W. J. Lee, C. K. Chan, and W. B. Zhang, "A modified halpern-type iteration algorithm for totally quasi-ϕ-asymptotically nonexpansive mappings with applications," *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6489–6497, 2012.

[35] X. Qin, S. Y. Cho, and S. M. Kang, "Strong convergence of shrinking projection methods for quasi-ϕ-nonexpansive mappings and equilibrium problems," *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 750–760, 2010.

[36] D. Wu, S.-s. Chang, and G. X. Yuan, "Approximation of common fixed points for a family of finite nonexpansive mappings in Banach space," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 63, no. 5–7, pp. 987–999, 2005.

[37] C. Zhang, J. Li, and B. Liu, "Strong convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces," *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 262–276, 2011.

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Research Article

An Augmented Lagrangian Algorithm for Solving Semiinfinite Programming

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We present a smooth augmented Lagrangian algorithm for semiinfinite programming (SIP). For this algorithm, we establish a perturbation theorem under mild conditions. As a corollary of the perturbation theorem, we obtain the global convergence result, that is, any accumulation point of the sequence generated by the algorithm is the solution of SIP. We get this global convergence result without any boundedness condition or coercive condition. Another corollary of the perturbation theorem shows that the perturbation function at zero point is lower semi-continuous if and only if the algorithm forces the sequence of objective function convergence to the optimal value of SIP. Finally, numerical results are given.

1. Introduction

We consider the semi-infinite programming (SIP):

$$\inf\{f(x) \mid x \in X\}, \tag{1.1}$$

where $X = \{x \in R^n \mid g(x,s) \le 0, \text{for all } s \in \Omega\}$, the functions $f : R^n \to R$ and $g : R^n \times R^m \to R$ are continuously differentiable. $\Omega \subset R^m$ is a nonempty bounded and closed domain. In this paper, we assume that

$$\inf_{x\in R^n} f(x) > -\infty. \tag{1.2}$$

This assumption is very mild, because the objective function $f(x)$ can be replaced by $e^{f(x)}$ if the assumption is not satisfied.

Semi-infinite programming has wide applications such as engineering technology, optimal control, characteristic value calculation, and statistical design. Many methods have been proposed to solve semi-infinite programming (see [1–4]). As we know, the main difficulty for solving SIP is that it has infinite constraints. If transforming the infinite constraints into an integral function, SIP (1.1) is equivalent to a nonlinear programming with finite constraints.

For any given $x \in R^n$ and $s \in \Omega$, let

$$[g(x,s)]_+ = \max\{g(x,s), 0\}. \tag{1.3}$$

Define $\varphi : R^n \to R$ by

$$\varphi(x) = \int_\Omega [g(x,s)]_+ d\mu(s), \tag{1.4}$$

where $\mu \geq 0$ is a given probability measure on Ω, that is, $\int_\Omega d\mu(s) = \mu(\Omega) = 1$. Thus SIP (1.1) can be reformulated as the following nonlinear programming (NP) with one equality constraint:

$$\inf_{x \in R^n} \{f(x) \mid \varphi(x) = 0\}. \tag{1.5}$$

Then nonlinear programming (1.5) has the same optimal solution and optimal value with SIP (1.1).

For nonlinear programming with finite equality constraints, Hestenes [5] and Powell [6] independently proposed an augmented Lagrangian function by incorporating a quadratic penalty term in the conventional Lagrangian function. This augmented Lagrangian function avoids the shortcoming that the conventional Lagrangian function is only suitable for convex function. So the augmented Lagrangian function can be applied to nonconvex optimization problem. Later, the augmented Lagrangian function was extended to inequality constrained optimization problems and thoroughly investigated by Rockafellar [7]. Recently, Yang and Teo [8] and Rückmann and Shapiro [9] introduced the augmented Lagrangian function for SIP (1.1). In [9], necessary and sufficient conditions for the existence of corresponding augmented Lagrange multipliers were presented. [8] proposed a nonlinear Lagrangian method and established that the sequence of optimal values of nonlinear penalty problems converges to that of SIP (1.1), under the assumption that the level set of objective function is bounded. In this paper, using the equivalent relation of semi-infinite programming (1.1) and nonlinear programming (1.5), without any boundness condition, we present an augmented Lagrangian algorithm for SIP (1.1).

We notice that although the constraints of NP (1.5) are finite, but the constraint function is nonsmooth. Therefore, existing gradient-based optimization methods cannot be used to solve NP (1.5) directly. To overcome this inconvenience, we have to smooth the constraint function. For SIP (1.1), [10–13] presented semismooth Newton methods and smoothing Newton methods. They proved that each accumulation point is a generalized stationary point of SIP (1.1). However, at each iteration of these methods, a Hessian matrix needs to be computed. When the size of the problem is large, computing a Hessian matrix is very expensive. Based on exact l_1 penalty function that is approximated by a family of smoothing functions, a smoothed-penalty algorithm for solving NP (1.5) was proposed by [14]. They proved that if the constrained set is bounded or the objective function is coercive, the algorithm generates a sequence whose accumulation points are solutions of SIP (1.1).

In this paper, for SIP (1.1), we present a smooth augmented Lagrangian algorithm by smoothing the classical augmented Lagrangian function [7]. In this algorithm, we need not have to get an exact global optimal solution of unconstraint subproblem at each iteration. It is sufficient to search an inexact solution. It is not difficult to obtain an inexact solution, whenever the evaluation of the integral function is not very expensive. For this algorithm, we establish a perturbation theorem under mild conditions. As a corollary of the perturbation theorem, we obtain the global convergence result, that is, any accumulation point of the sequence generated by the algorithm is the solution of SIP (1.1). We get this global convergence result without any boundedness condition or coercive condition. It is noteworthy that the boundedness of the multiplier sequence is a sufficient condition in many literatures about Lagrangian method (see [15–17]). However, in our algorithm, the multiplier sequence can be unbounded. Another corollary of the perturbation theorem shows that the perturbation function at zero point is lower semi-continuous if and only if the algorithm forces the sequence of objective function convergence to the optimal value of SIP (1.1).

The paper is organized as follows. In the next section, we present a smooth augmented Lagrangian algorithm. In Section 3, we establish the perturbation theorem of the algorithm. By this theorem, we obtain a global convergence property and a sufficient and necessary condition in which the algorithm forces the sequence of objective functions convergence to the optimal value of SIP (1.1). Finally, we give some numerical results in Section 4.

2. Smooth Augmented Lagrangian Algorithm

Before we introduce the algorithm, some definitions and symbols need to be given. For $\varepsilon \geq 0$, we define the relaxed feasible set of SIP (1.1) as follows:

$$R_\varepsilon = \left\{ x \in R^n \mid \int_\Omega \left[g(x,s)\right]_+ d\mu(s) \leq \varepsilon \right\}. \tag{2.1}$$

Then R_0 is the feasible set of SIP (1.1). Let R_0^* be the set of optimal solutions of SIP (1.1). We assume that $R_0^* \neq \emptyset$ in this paper.

The perturbation function is defined as follows:

$$\theta_f(\varepsilon) = \inf_{x \in R_\varepsilon} f(x). \tag{2.2}$$

Thus the optimal value of SIP (1.1) is

$$\theta_f(0) = \inf_{x \in R_0} f(x). \tag{2.3}$$

It is easy to show that $\theta_f(\cdot)$ is upper semi-continuous at the point $\varepsilon = 0$.

For problem (1.5), the corresponding classical augmented Lagrangian function [7] is

$$L(x,\lambda,\rho) = f(x) + \frac{\rho}{2}\left[\varphi(x) + \frac{\lambda}{\rho}\right]^2 - \frac{\lambda^2}{2\rho}, \tag{2.4}$$

where λ is the Lagrangian multiplier and ρ is the penalty parameter. On base of it, we introduce a class of smooth augmented Lagrangian function:

$$f_r(x,\lambda,\rho) = f(x) + \frac{\rho}{2}\left[r\int_\Omega \phi\left(\frac{g(x,s)}{r}\right)d\mu(s) + \frac{\lambda}{\rho}\right]^2 - \frac{\lambda^2}{2\rho}. \tag{2.5}$$

Here r is the approximate parameter.

In the following, we suppose that the continuously differentiable function $\phi : R \to R$ satisfies

(a) $\phi(\cdot)$ is nonnegative and monotone increasing;

(b) for any $t > 0$, $\phi(t) \geq t$;

(c) $\lim_{t\to+\infty}(\phi(t)/t) = 1$.

It is easy to check that there are many continuously differentiable functions satisfying conditions (a), (b), and (c). For example,

$$\begin{aligned}
\phi_1(t) &= \log(1+e^t). \\
\phi_2(t) &= \begin{cases} 2e^t, & t < 0, \\ t + \log(1+t) + 2, & t \geq 0. \end{cases} \\
\phi_3(t) &= \begin{cases} e^t, & t < 0, \\ t + 1, & t \geq 0. \end{cases} \\
\phi_4(t) &= \frac{1}{2}\left(t + \sqrt{t^2+4}\right).
\end{aligned} \tag{2.6}$$

Using conditions (a) and (c), for any $t \in R$, we have

$$\lim_{r\to 0^+} r\phi\left(\frac{t}{r}\right) = \max\{0,t\}. \tag{2.7}$$

From the above equation, under conditions (a)–(c), the smooth function $f_r(x,\lambda,\rho)$ approximates to the classical augmented Lagrangian function $L(x,\lambda,\rho)$ as r approaches to zero, that is,

$$\lim_{r\to 0^+} f_r(x,\lambda,\rho) = L(x,\lambda,\rho) = f(x) + \frac{\rho}{2}\left[\varphi(x) + \frac{\lambda}{\rho}\right]^2 - \frac{\lambda^2}{2\rho}. \tag{2.8}$$

Based on the smooth augmented Lagrangian function $f_\rho(x,\beta,r)$, we present the following smooth augmented Lagrangian algorithm.

Algorithm 2.1. Set $x_0 \in R^n$, $r_0 > 0$, $\varepsilon_0 > 0$, $\rho_0 > 0$, $\lambda_0 \in (0, \min\{1/\sqrt{r_0}, 1/\sqrt{\varepsilon_0}\}]$, $k := 0$.
Step 1. Compute

$$x_k \in \arg\min_{x \in R^n} f_{r_k}(x, \lambda_k, \rho_k). \tag{2.9}$$

Otherwise, seek on inexact global optimal solution x_k satisfying

$$f_{r_k}(x_k, \lambda_k, \rho_k) \leq \inf_{x \in R^n} f_{r_k}(x, \lambda_k, \rho_k) + \varepsilon_k. \tag{2.10}$$

Step 2. Set $r_{k+1} = (1/2)r_k$, $\varepsilon_{k+1} = (1/2)\varepsilon_k$,

$$\begin{gathered}\rho_{k+1} = \begin{cases} \rho_k, & x_k \in R_{\varepsilon_k}; \\ 2\rho_k, & x_k \notin R_{\varepsilon_k}, \end{cases} \\ \lambda_{k+1} \in \left(0, \min\left\{\frac{1}{\sqrt{r_{k+1}}}, \frac{1}{\sqrt{\varepsilon_{k+1}}}\right\}\right]. \end{gathered} \tag{2.11}$$

Step 3. Set $k := k + 1$, go back step 1.

Since $f(x)$ is bounded below and $\phi(\cdot)$ is nonnegative, an inexact solution satisfying (2.10) always exists. Thus Algorithm 2.1 is feasible.

3. Convergence Properties

In this section, by using a perturbation theorem of Algorithm 2.1, we will obtain a global convergence property, a sufficient and necessary condition that Algorithm 2.1 forces the sequence of objective functions convergence to the optimal value of SIP (1.1). To prove the perturbation theorem, we first give the following two lemmas.

Let $\Omega_+(x) = \{s \in \Omega \mid g(x,s) > 0\}$, $\Omega_0(x) = \{s \in \Omega \mid g(x,s) = 0\}$, $\Omega_-(x) = \{s \in \Omega \mid g(x,s) < 0\}$.

Lemma 3.1. *Suppose that the point sequence $\{x_k\}$ is generated by Algorithm 2.1. Then for any $\varepsilon > 0$, there exists a positive integer k_0 such that $x_k \in R_\varepsilon$, for all $k > k_0$.*

Proof.

Case 1. When $k \to +\infty$, ρ_k tends to a finite number. From Algorithm 2.1, there exists a positive integer N_1 such that $x_k \in R_{\varepsilon_k}$ for all $k > N_1$. Notice that $\varepsilon_k \to 0$, so for any $\varepsilon > 0$, there exists a positive integer N_2 such that $\varepsilon_k < \varepsilon$ for all $k > N_2$. Therefore, when $k > \max\{N_1, N_2\}$, we have $x_k \in R_{\varepsilon_k} \subseteq R_\varepsilon$.

Case 2. When $k \to +\infty$, $\rho_k \to +\infty$. We suppose that the conclusion does not hold. Then for $\overline{\varepsilon} > 0$, there exists an infinite subsequence $K \subseteq N = \{1, 2, 3, \ldots\}$ such that $x_k \notin R_{\overline{\varepsilon}}$ for all $k \in K$, that is,

$$\int_\Omega [g(x_k, s)]_+ d\mu(s) > \overline{\varepsilon}. \tag{3.1}$$

Since $\varepsilon_k \to 0$, for the above $\overline{\varepsilon} > 0$, there exists a positive integer N_3 such that $\varepsilon_k < \overline{\varepsilon}$ for all $k > N_3$. Then using (2.10) in Algorithm 2.1, we have

$$f_{r_k}(x_k, \lambda_k, \rho_k) \leq \inf_{x \in R^n} f_{r_k}(x, \lambda_k, \rho_k) + \overline{\varepsilon}. \tag{3.2}$$

Therefore by (3.1), (3.2), and $\phi(t)$ satisfying (a)-(b), for any $k \in K$, $k > N_3$, we derive that

$$\begin{aligned}
\inf_{x \in R^n} f_{r_k}(x, \lambda_k, \rho_k) + \overline{\varepsilon} &\geq f_{r_k}(x_k, \lambda_k, \rho_k) \\
&= f(x_k) + \frac{\rho_k}{2}\left[r_k \int_\Omega \phi\left(\frac{g(x_k, s)}{r_k}\right) d\mu(s) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} \\
&\geq f(x_k) + \frac{\rho_k}{2}\left[r_k \int_{\Omega_+(x_k)} \phi\left(\frac{g(x_k, s)}{r_k}\right) d\mu(s) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} \\
&\geq f(x_k) + \frac{\rho_k}{2}\left[\int_{\Omega_+(x_k)} g(x_k, s) d\mu(s) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} \\
&\geq f(x_k) + \frac{\rho_k}{2}\left[\overline{\varepsilon} + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} \\
&\geq f(x_k) + \frac{\rho_k}{2}\overline{\varepsilon}^2.
\end{aligned} \tag{3.3}$$

Note that $\{f(x_k)\}$ is bounded below and $\rho_k \to +\infty (k \to +\infty)$, then we can obtain that

$$f(x_k) + \frac{\rho_k}{2}\overline{\varepsilon}^2 \longrightarrow +\infty, \quad (k \in K, k \longrightarrow +\infty), \tag{3.4}$$

that is, $\inf_{x \in R^n} f_{r_k}(x, \lambda_k, \rho_k) \to +\infty (k \in K, k \to +\infty)$. However, on the other hand, since $R_0 \neq \emptyset$, we can choose $\overline{x} \in R_0$; by the choice of r_k, ρ_k, λ_k in Algorithm 2.1 and the properties of ϕ, we obtain that

$$\begin{aligned}
\inf_{x \in R^n} f_{r_k}(x, \lambda_k, \rho_k) &\leq f_{r_k}(\overline{x}, \lambda_k, \rho_k) \\
&= f(\overline{x}) + \frac{\rho_k}{2}\left[r_k \int_\Omega \phi\left(\frac{g(\overline{x}, s)}{r_k}\right) d\mu(s) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} \\
&\leq f(\overline{x}) + \frac{\rho_k}{2}\left[r_k \int_\Omega \phi(0) d\mu(s) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} \\
&= f(\overline{x}) + \frac{\rho_k}{2} r_k^2 (\phi(0)\mu(\Omega))^2 + \lambda_k r_k \phi(0)\mu(\Omega) \\
&\leq f(\overline{x}) + \rho_0 r_0^2 (\phi(0)\mu(\Omega))^2 + \sqrt{r_0}\phi(0)\mu(\Omega).
\end{aligned} \tag{3.5}$$

This indicates that $\inf_{x \in R^n} f_{r_k}(x, \lambda_k, \rho_k)$ has an upper bound. It is in contradiction with $\inf_{x \in R^n} f_{r_k}(x, \lambda_k, \rho_k) \to +\infty (k \in K, k \to +\infty)$. □

By using Lemma 3.1, we have the following Lemma 3.2.

Lemma 3.2. *Suppose that the point sequence $\{x_k\}$ is generated by Algorithm 2.1. Then for every accumulation point x^* of $\{x_k\}$, one has $x^* \in R_0$.*

Theorem 3.3. *Suppose that the sequence $\{x_k\}$ is generated by Algorithm 2.1, then*

(i) $\lim_{k\to\infty} f(x_k) = \lim_{\varepsilon\to 0^+}\theta_f(\varepsilon)$;

(ii) $\lim_{k\to\infty} f_{r_k}(x_k, \lambda_k, \rho_k) = \lim_{\varepsilon\to 0^+}\theta_f(\varepsilon)$;

(iii) $\lim_{k\to\infty}(\rho_k/2)[r_k \int_\Omega \phi(g(x_k, s)/r_k)d\mu(s) + \lambda_k/\rho_k]^2 - \lambda_k^2/2\rho_k = 0$.

Proof. Since $\theta_f(\varepsilon)$ is monotonically decreasing with respect to $\varepsilon > 0$ and has below bound, we know $\lim_{\varepsilon\to 0^+}\theta_f(\varepsilon)$ exists and is finite. By Algorithm 2.1, we have $\varepsilon_k \downarrow 0$. Then

$$\lim_{k\to +\infty}\theta_f(\varepsilon_k) = \lim_{\varepsilon\to 0^+}\theta_f(\varepsilon). \tag{3.6}$$

Taking $\delta_k > 0$ and $\delta_k \to 0 (k \to +\infty)$, by the definition of infimum, there exists $z_k \in R_{\varepsilon_k}$ such that

$$f(z_k) \le \theta_f(\varepsilon_k) + \delta_k. \tag{3.7}$$

Since $z_k \in R_{\varepsilon_k}$, that is, $\int_\Omega [g(z_k, s)]_+ d\mu(s) \le \varepsilon_k$.

On the other hand, by Lemma 3.1, for any $\varepsilon > 0$, when k is sufficiently large, we have

$$x_k \in R_\varepsilon. \tag{3.8}$$

Since $\phi(t)$ satisfies conditions (a) and (c), we obtain

$$\lim_{t\to +\infty}\frac{\phi(t)-\phi(0)}{t} = \lim_{t\to +\infty}\frac{\phi(t)}{t} = 1, \qquad \lim_{t\to 0^+}\frac{\phi(t)-\phi(0)}{t} = \lim_{t\to 0^+}\phi'(t) = \phi'(0). \tag{3.9}$$

Therefore, there exists $M > 0$ such that

$$\phi(t) \le Mt + \phi(0), \tag{3.10}$$

for any $t \geq 0$. As stated previously, by the choice of r_k, ε_k, ρ_k, and λ_k in Algorithm 2.1, (3.7), (3.8), and (3.10) derive that for any $\varepsilon > 0$,

$$\begin{aligned}
\theta_f(\varepsilon) &\leq f(x_k) \\
&\leq f(x_k) + \frac{\rho_k}{2}\left[r_k \int_\Omega \phi\left(\frac{g(x_k,s)}{r_k}\right) d\mu(s) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} \\
&\leq \inf_{x\in R^n} f_{r_k}(x,\lambda_k,\rho_k) + \varepsilon_k \\
&\leq f(z_k) + \frac{\rho_k}{2}\left[r_k \int_\Omega \phi\left(\frac{g(z_k,s)}{r_k}\right) d\mu(s) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} + \varepsilon_k \\
&\leq \theta_f(\varepsilon_k) + \delta_k + \frac{\rho_k}{2}\left[r_k \int_\Omega \phi\left(\frac{g(z_k,s)}{r_k}\right) d\mu(s) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} + \varepsilon_k \\
&\leq \theta_f(\varepsilon_k) + \delta_k + \frac{\rho_k}{2}\left[M\varepsilon_k + r_k\phi(0)\mu(\Omega) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k} + \varepsilon_k \\
&\leq \theta_f(\varepsilon_k) + \delta_k + \frac{\rho_k}{2}\left(\max\{r_k,\varepsilon_k\}\left(M + \phi(0)\mu(\Omega)\right)\right)^2 \\
&\quad + \lambda_k \max\{r_k,\varepsilon_k\}\left(M + \phi(0)\mu(\Omega)\right) + \varepsilon_k \\
&\leq \theta_f(\varepsilon_k) + \delta_k + \frac{\rho_0}{2}\max\{r_0,\varepsilon_0\}\left(M + \phi(0)\mu(\Omega)\right)^2 \max\{r_k,\varepsilon_k\} \\
&\quad + \left(M + \phi(0)\mu(\Omega)\right)\max\left\{\sqrt{r_k},\sqrt{\varepsilon_k}\right\} + \varepsilon_k.
\end{aligned} \tag{3.11}$$

From the above inequalities and (3.6), noticing that $r_k\varepsilon_k \to 0(k \to \infty)$, for any $\varepsilon > 0$, we have

$$\begin{aligned}
\theta_f(\varepsilon) &\leq \liminf_{k\to+\infty} f(x_k) \\
&\leq \limsup_{k\to+\infty} f(x_k) \\
&\leq \limsup_{k\to+\infty}\left\{f(x_k) + \frac{\rho_k}{2}\left[r_k \int_\Omega \phi\left(\frac{g(x_k,s)}{r_k}\right) d\mu(s) + \frac{\lambda_k}{\rho_k}\right]^2 - \frac{\lambda_k^2}{2\rho_k}\right\} \\
&\leq \lim_{k\to+\infty} \theta_f(\varepsilon_k) \\
&= \lim_{\varepsilon\to 0^+} \theta_f(\varepsilon).
\end{aligned} \tag{3.12}$$

Then $\lim_{\varepsilon\to 0^+}\theta_f(\varepsilon) = \lim_{k\to+\infty} f(x_k) = \limsup_{k\to+\infty}\{f(x_k) + \frac{\rho_k}{2}[r_k \int_\Omega \phi(g(x_k,s)/r_k)d\mu(s) + \lambda_k/\rho_k]^2 - \lambda_k^2/2\rho_k\}$. So the conclusions (i)–(iii) hold. □

Now, we prove the global convergence of Algorithm 2.1.

Corollary 3.4. *Suppose that the point sequence* $\{x_k\}$ *is generated by Algorithm 2.1. Then every accumulation point of* $\{x_k\}$ *is the optimal solution of the problem* (1.1).

Proof. Let x^* be an accumulation point of $\{x_k\}$; from Lemma 3.2, we have

$$x^* \in R_0. \tag{3.13}$$

By the conclusion (i) of Theorem 3.3 and (3.13), we obtain

$$\lim_{\varepsilon \to 0^+} \theta_f(\varepsilon) = \lim_{k \to \infty} f(x_k) = f(x^*) \geq \theta_f(0). \tag{3.14}$$

Then we get $f(x^*) = \theta_f(0)$, because (3.14) and $\theta_f(\varepsilon)$ are upper semi-continuous at the point $\varepsilon = 0$. □

By using Theorem 3.3, we have the following Corollary 3.5.

Corollary 3.5. $\lim_{k\to\infty} f(x_k) = \theta_f(0)$ *if and only if* $\theta_f(\varepsilon)$ *is lower semi-continuous at the point* $\varepsilon = 0$.

4. Numerical Results

To give some insight into the behavior of the algorithm presented in this paper. It is implemented in Matlab 7.0.4 and runs are made on AMD Athlon(tm) 64 × 2 Dual Core Processor 4800+ with CPU 2.50 GHz and 1.87 GB memory. Tables 1 and 2 show the computational results of the corresponding problems with the following items:

k : number of iterations;

$\mathbf{x}_0$: starting point;

$\phi(t)$: smoothing function;

$\mathbf{x}_k$: the final iteration point;

λ_k: the final Lagrangian multiplier;

$f(x_k)$: the function value of $f(x)$ at the final x_k.

The parameters used in the Algorithm 2.1 are specified as follows:

$$\begin{gathered} r_0 = 1, \quad \varepsilon_0 = 1, \quad \rho_0 = 1, \quad \lambda_0 = 1, \\ \lambda_{k+1} = \min\left\{ \lambda_k + \rho_k r_k \int_\Omega \phi\left(\frac{g(x_k, s)}{r_k}\right) d\mu(s), 10^3 \right\}. \end{gathered} \tag{4.1}$$

Example 4.1 (see [18]). Consider the following:

$$\begin{aligned} \min \quad & 1.21 \exp(x_1) + \exp(x_2) \\ \text{s.t.} \quad & s - \exp(x_1 + x_2) \leq 0, \quad \forall s \in [0, 1]. \end{aligned} \tag{4.2}$$

Table 1: Numerical results of Example 4.1.

k	$\phi(t)$	$\mathbf{x}_k$	λ_k	$f(x_k)$
26	$\log(1+e^t)$	(−0.0968, 0.0938)	367.2453	2.1982
30	$(1/2)(t+\sqrt{t^2+4})$	(−0.0959, 0.0947)	923.9.40	2.1993
19	e^t	(−0.0953, 0.0953)	6.1129	2.1999

Table 2: Numerical results of Example 4.2.

n	k	$\phi(t)$	$\mathbf{x}_k$	λ_k	$f(x_k)$
3	17	$\log(1+e^t)$	(0.0839, 0.4494, 1.0105)	18.5016	0.6472
3	16	$(1/2)(t+\sqrt{t^2+4})$	(0.0839, 0.4494, 1.0108)	18.5869	0.6472
3	16	e^t	(0.0873, 0.4248, 1.0460)	12.3942	0.6484
6	16	$\log(1+e^t)$	(0.00, 1.03, −0.25, 1.24, −1.39, 0.94)	5.3234	0.6161
6	17	$(1/2)(t+\sqrt{t^2+4})$	(−0.00, 1.03, −0.25, 1.23, −1.39, 0.94)	7.5705	0.6161
6	16	e^t	(−0.00, 1.02, −0.25, 1.24, −1.41, 0.95)	4.1742	0.6161
8	17	$\log(1+e^t)$	(−0.00, 1.00, −0.06, 0.77, −1.49, 2.77, −2.41, 0.96)	4.3033	0.6157
8	18	$(1/2)(t+\sqrt{t^2+4})$	(−0.00, 1.00, −0.06, 0.77, −1.47, 2.78, −2.40, 0.96)	7.0902	0.6157
8	16	e^t	(0.00, 1.00, −0.06, 0.78, −1.49, 2.78, −2.42, 0.97)	3.3116	0.6158

We choose the starting point $\mathbf{x}_0 = (0,0)$. This example has the optimal solution $x^* = (-\ln 1.1, \ln 1.1)$.

Example 4.2 (see [18]). Consider the following:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i/i \\ \text{s.t.} \quad & \tan s - \sum_{i=1}^{n} x_i s^{i-1} \le 0, \quad \forall s \in [0,1], \end{aligned} \tag{4.3}$$

for $n = 3$, 6, and 8. We choose zero vectors as the starting points.

Throughout the computational experiments, we use trust region method for solving an unconstrained optimization subproblem at each step. For the corresponding trust region subproblem, we directly use the trust function in Matlab toolbox. The test results of Example 4.1 are summarized in Table 1. We test the three cases for $\phi(t) = \log(1+e^t)$, $\phi(t) = (1/2)(t+\sqrt{t^2+4})$ and $\phi(t) = e^t$, which are, respectively, used as the smoothing approximation functions. k denotes the number of the iteration, λ_k denotes the approximate Lagrangian multiplier at the final iteration, and x_k and $f(x_k)$ are the approximate solution and the objective function at the final iteration. For Example 4.2, we test the results when $n = 3$, $n = 6$, and $n = 8$ in Table 2. Numerical results demonstrate that augmented Lagrangian algorithm established in this paper is a practical and effective method for solving semi-infinite programming problem.

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References

[1] B. Bhattacharjee, P. Lemonidis, W. H. Green Jr., and P. I. Barton, "Global solution of semi-infinite programs," *Mathematical Programming*, vol. 103, no. 2, pp. 283–307, 2005.
[2] R. Hettich and K. O. Kortanek, "Semi-infinite programming: theory, methods, and applications," *SIAM Review*, vol. 35, no. 3, pp. 380–429, 1993.
[3] G. Still, "Discretization in semi-infinite programming: the rate of convergence," *Mathematical Programming*, vol. 91, no. 1, pp. 53–69, 2001.
[4] Y. Tanaka, M. Fukushima, and T. Ibaraki, "A globally convergent SQP method for semi-infinite nonlinear optimization," *Journal of Computational and Applied Mathematics*, vol. 23, no. 2, pp. 141–153, 1988.
[5] M. R. Hestenes, "Multiplier and gradient methods," *Journal of Optimization Theory and Applications*, vol. 4, pp. 303–320, 1969.
[6] M. J. D. Powell, "A method for nonlinear constraints in minimization problems," in *Optimization*, R. Fletcher, Ed., pp. 283–298, Academic Press, London, UK, 1969.
[7] R. T. Rockafellar, "Augmented Lagrange multiplier functions and duality in nonconvex programming," *SIAM Journal on Control and Optimization*, vol. 12, pp. 268–285, 1974.
[8] X. Q. Yang and K. L. Teo, "Nonlinear Lagrangian functions and applications to semi-infinite programs," *Annals of Operations Research*, vol. 103, pp. 235–250, 2001.
[9] J.-J. Rückmann and A. Shapiro, "Augmented Lagrangians in semi-infinite programming," *Mathematical Programming*, vol. 116, no. 1-2, pp. 499–512, 2009.
[10] D.-H. Li, L. Qi, J. Tam, and S.-Y. Wu, "A smoothing Newton method for semi-infinite programming," *Journal of Global Optimization*, vol. 30, no. 2-3, pp. 169–194, 2004.
[11] L. Qi, C. Ling, X. Tong, and G. Zhou, "A smoothing projected Newton-type algorithm for semi-infinite programming," *Computational Optimization and Applications*, vol. 42, no. 1, pp. 1–30, 2009.
[12] L. Qi, A. Shapiro, and C. Ling, "Differentiability and semismoothness properties of integral functions and their applications," *Mathematical Programming*, vol. 102, no. 2, pp. 223–248, 2005.
[13] L. Qi, S.-Y. Wu, and G. Zhou, "Semismooth Newton methods for solving semi-infinite programming problems," *Journal of Global Optimization*, vol. 27, no. 2-3, pp. 215–232, 2003.
[14] M. Gugat and M. Herty, "The smoothed-penalty algorithm for state constrained optimal control problems for partial differential equations," *Optimization Methods & Software*, vol. 25, no. 4–6, pp. 573–599, 2010.
[15] E. G. Birgin, R. A. Castillo, and J. M. Martínez, "Numerical comparison of augmented Lagrangian algorithms for nonconvex problems," *Computational Optimization and Applications*, vol. 31, no. 1, pp. 31–55, 2005.
[16] A. R. Conn, N. I. M. Gould, and P. L. Toint, "A globally convergent augmented Lagrangian algorithm for optimization with general constraints and simple bounds," *SIAM Journal on Numerical Analysis*, vol. 28, no. 2, pp. 545–572, 1991.
[17] P. Tseng and D. P. Bertsekas, "On the convergence of the exponential multiplier method for convex programming," *Mathematical Programming*, vol. 60, no. 1, pp. 1–19, 1993.
[18] A. R. Conn and N. I. M. Gould, "An exact penalty function for semi-infinite programming," *Mathematical Programming*, vol. 37, no. 1, pp. 19–40, 1987.

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Research Article

A Generalized Alternative Theorem of Partial and Generalized Cone Subconvexlike Set-Valued Maps and Its Applications in Linear Spaces

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We first introduce a new notion of the partial and generalized cone subconvexlike set-valued map and give an equivalent characterization of the partial and generalized cone subconvexlike set-valued map in linear spaces. Secondly, a generalized alternative theorem of the partial and generalized cone subconvexlike set-valued map was presented. Finally, Kuhn-Tucker conditions of set-valued optimization problems were established in the sense of globally proper efficiency.

1. Introduction

Generalized convexity plays an important role in set-valued optimization. The generalization of convexity from vector-valued maps to set-valued maps happened in the 1970s. Borwein [1] and Giannessi [2] introduced and studied the cone convexity of set-valued maps. Based on Borwein and Giannessi's work, some authors [3–7] established a series of optimality conditions of set-valued optimization problems under different types of generalized convexity of set-valued maps in topological spaces. Since linear spaces are wider than topological spaces, generalizing some results of the above mentioned references from topological spaces to linear spaces is an interesting topic. Li [8] introduced a cone subconvexlike set-valued map involving the algebraic interior and established Kuhn-Tucker conditions. Huang and Li [9] studied Lagrangian multiplier rules of set-valued optimization problems with generalized cone subconvexlike set-valued maps in linear spaces. When the algebraic interior of the convex cone is empty, Hernández et al. [10] used the relative algebraic interior of the convex cone to introduce cone subconvexlikeness of set-valued maps and investigated Benson proper efficiency of set-valued optimization problems in linear spaces.

The aim of this paper is to study globally proper efficiency of set-valued optimization problems in linear spaces. This paper is organized as follows. In Section 2, we recalled some basic notions and gave some lemmas. In Section 3, we presented a generalized alternative theorem of the partial and generalized cone subconvexlike set-valued map and established Kuhn-Tucker conditions of set-valued optimization problems in the sense of globally proper efficiency.

2. Preliminaries

In this paper, let Y and Z be two real-ordered linear spaces, and let 0 denote the zero element of every space. Let K be a nonempty subset in Y. The cone hull of K is defined as cone $K := \{\lambda k \mid k \in K, \lambda \geq 0\}$. K is called a convex cone if and only if

$$\lambda_1 k_1 + \lambda_2 k_2 \in K, \quad \forall \lambda_1, \lambda_2 \geq 0,\ \forall k_1, k_2 \in K. \tag{2.1}$$

A cone K is said to be pointed if and only if $K \cap (-K) = \{0\}$. A cone K is said to be nontrivial if and only if $K \neq \{0\}$ and $K \neq Y$.

Let Y^* and Z^* stand for the algebraic dual spaces of Y and Z, respectively. Let C and D be nontrivial, pointed, and convex cones in Y and Z, respectively. The algebraic dual cone C^+ of C is defined as $C^+ := \{y^* \in Y^* \mid \langle y, y^* \rangle \geqslant 0, \forall y \in C\}$, and the strictly algebraic dual cone C^{+i} of C is defined as $C^{+i} := \{y^* \in Y^* \mid \langle y, y^* \rangle > 0, \forall y \in C \setminus \{0\}\}$, where $\langle y, y^* \rangle$ denotes the value of the linear functional y^* at the point y. The meaning of D^+ is similar to that of C^+.

Let K be a nonempty subset of Y. The linear hull span K of K is defined as span $K := \{k \mid k = \sum_{i=1}^n \lambda_i k_i, \lambda_i \in \mathbb{R}, k_i \in K,\ i = 1, \ldots, n\}$, and the affine hull aff K of K is defined as aff $K := \{k \mid k = \sum_{i=1}^n \lambda_i k_i, \sum_{i=1}^n \lambda_i = 1, \lambda_i \in \mathbb{R}, k_i \in K,\ i = 1, \ldots, n\}$. The generated linear subspace $L(K)$ of K is defined as $L(K) := \mathrm{span}(K - K)$.

Definition 2.1 (see [11]). Let K be a nonempty subset of Y. The algebraic interior of K is the set

$$\mathrm{cor}K := \left\{k \in K \mid \forall k' \in Y, \exists \lambda' > 0, \forall \lambda \in \left[0, \lambda'\right], k + \lambda k' \in K\right\}. \tag{2.2}$$

Definition 2.2 (see [12]). Let K be a nonempty subset of Y. The relative algebraic interior of K is the set

$$\mathrm{icr}K = \{k \in K \mid \forall v \in \mathrm{aff}\, K - k, \exists \lambda_0 > 0, \forall \lambda \in [0, \lambda_0], k + \lambda v \in K\}. \tag{2.3}$$

Clearly, aff $K - k = L(K)$, for all $k \in K$. Therefore, Definition 2.2 is consistent with the definition of the relative algebraic interior of K in [13, 14]. However, Definition 2.2 seems to be more convenient than the ones in [13, 14].

It is worth noting that if K is a nontrivial and pointed cone in Y, then $0 \notin \mathrm{icr}K$, and if K is a convex cone, then $\mathrm{icr}K$ is a convex set, and $\mathrm{icr}K \cup \{0\}$ is a convex cone.

Lemma 2.3 (see [13]). *If K is a convex cone in Y, then $K + \operatorname{icr} K = \operatorname{icr} K$.*

Lemma 2.4 (see [10, 12, 14]). *If K is a nonempty subset in Y, then*

(a) $\operatorname{aff} K - k = \operatorname{aff} K - K$, *for all* $k \in K$;

if K is convex in Y and $\operatorname{icr} K \neq \emptyset$, *then*

(b) $\operatorname{icr}(\operatorname{icr} K) = \operatorname{icr} K$;

(c) $\operatorname{aff}(\operatorname{icr} K) = \operatorname{aff} K$.

Lemma 2.5 (see [12]). *Let K be a convex set with* $\operatorname{icr}(K) \neq \emptyset$ *in Y. If* $0 \notin \operatorname{icr} K$, *then there exists* $y^* \in Y^* \setminus \{0\}$ *such that*

$$\langle k, y^* \rangle \geq 0, \quad \forall k \in K. \tag{2.4}$$

3. Main Results

Let A be a nonempty set, and let $F : A \rightrightarrows Y$ and $G : A \rightrightarrows Z$ be two set-valued maps on A. Write $F(A) := \bigcup_{x \in A} F(x)$ and $\langle F(x), y^* \rangle := \{\langle y, y^* \rangle \mid y \in F(x)\}$. The meanings of $G(A)$ and $\langle G(x), z^* \rangle$ are similar to those of $F(A)$ and $\langle F(x), y^* \rangle$.

Now, we introduce a new notion of the partial and generalized cone subconvexlike set-valued map.

Definition 3.1. A set-valued map $J = (F, G) : A \rightrightarrows Y \times Z$ is called partial and generalized $C \times D$-subconvexlike on A if and only if $\operatorname{cone}(J(A)) + \operatorname{icr} C \times D$ is a convex set in $Y \times Z$.

The following theorem will give some equivalent characterizations of the partial and generalized $C \times D$-subconvexlike set-valued map in linear spaces.

Theorem 3.2. *Let* $\operatorname{icr} C \neq \emptyset$. *Then the following statements are equivalent:*

(a) *the set-valued map $J : A \rightrightarrows Y \times Z$ is partial and generalized $C \times D$-subconvexlike on A,*

(b) *For all* $(c, d) \in \operatorname{icr} C \times D, \forall x_1, x_2 \in A, \forall \lambda \in]0, 1[$,

$$(c, d) + \lambda J(x_1) + (1 - \lambda) J(x_2) \subseteq \operatorname{cone}(J(A)) + \operatorname{icr} C \times D, \tag{3.1}$$

(c) $\exists c' \in \operatorname{icr} C, \forall x_1, x_2 \in A, \forall \lambda \in]0, 1[, \forall \varepsilon > 0$,

$$\varepsilon(c', 0) + \lambda J(x_1) + (1 - \lambda) J(x_2) \subseteq \operatorname{cone}(J(A)) + C \times D, \tag{3.2}$$

(d) $\exists c'' \in C, \forall x_1, x_2 \in A, \forall \lambda \in]0, 1[, \forall \varepsilon > 0$,

$$\varepsilon(c'', 0) + \lambda J(x_1) + (1 - \lambda) J(x_2) \subseteq \operatorname{cone}(J(A)) + C \times D. \tag{3.3}$$

Proof. (a) $\Rightarrow$ (b). Let $(c, d) \in \operatorname{icr} C \times D, x_1, x_2 \in A, \lambda \in]0,1[, (y_1, z_1) \in J(x_1)$, and $(y_2, z_2) \in J(x_2)$. Clearly,

$$\begin{aligned} (y_1, z_1) + (c, d) &\in \operatorname{cone}(J(A)) + \operatorname{icr} C \times D, \\ (y_2, z_2) + (c, d) &\in \operatorname{cone}(J(A)) + \operatorname{icr} C \times D. \end{aligned} \tag{3.4}$$

Since J is partial and generalized $C \times D$-subconvexlike on A, it follows from (3.4) that

$$\begin{aligned} &(c, d) + \lambda(y_1, z_1) + (1 - \lambda)(y_2, z_2) \\ &\quad = \lambda((y_1, z_1) + (c, d)) + (1 - \lambda)((y_2, z_2) + (c, d)) \in \operatorname{cone}(J(A)) + \operatorname{icr} C \times D, \end{aligned} \tag{3.5}$$

which implies that (3.1) holds.

The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are clear.

(d) $\Rightarrow$ (a). Let $(m_i, n_i) \in \operatorname{cone}(J(A)) + \operatorname{icr} C \times D$ $(i = 1, 2), \lambda \in]0, 1[$. Then there exist $\rho_i \geq 0, x_i \in A, (y_i, z_i) \in J(x_i)$, and $(c_i, d_i) \in \operatorname{icr} C \times D$ $(i = 1, 2)$ such that $(m_i, n_i) = \rho_i(y_i, z_i) + (c_i, d_i)$.

Case one: if $\rho_1 = 0$ or $\rho_2 = 0$, we have $\lambda(m_1, n_1) + (1 - \lambda)(m_2, n_2) \in \operatorname{cone}(J(A)) + \operatorname{icr} C \times D$.

Case two: if $\rho_1 > 0$ and $\rho_2 > 0$, we have

$$\begin{aligned} &\lambda(m_1, n_1) + (1 - \lambda)(m_2, n_2) \\ &\quad = \lambda(\rho_1(y_1, z_1) + (c_1, d_1)) + (1 - \lambda)(\rho_2(y_2, z_2) + (c_2, d_2)) \\ &\quad = [\lambda(c_1, d_1) + (1 - \lambda)(c_2, d_2)] + [\lambda\rho_1(y_1, z_1) + (1 - \lambda)\rho_2(y_2, z_2)] \\ &\quad = \beta\left\{\frac{1}{\beta}[\lambda(c_1, d_1) + (1 - \lambda)(c_2, d_2)] + \left[\frac{\lambda\rho_1}{\beta}(y_1, z_1) + \frac{(1 - \lambda)\rho_2}{\beta}(y_2, z_2)\right]\right\}, \end{aligned} \tag{3.6}$$

where $\beta = \lambda\rho_1 + (1 - \lambda)\rho_2$.

By Lemma 2.4, we obtain

$$\begin{aligned} -c'' \in C - C \subseteq \operatorname{aff} C - C &= \operatorname{aff} C - \frac{1}{\beta}[\lambda c_1 + (1 - \lambda)c_2] \\ &= \operatorname{aff}(\operatorname{icr} C) - \frac{1}{\beta}[\lambda c_1 + (1 - \lambda)c_2]. \end{aligned} \tag{3.7}$$

Since $(1/\beta)[\lambda c_1 + (1 - \lambda)c_2] \in \operatorname{icr} C = \operatorname{icr}(\operatorname{icr} C)$, there exists $\lambda_0 > 0$ such that

$$\frac{1}{\beta}[\lambda c_1 + (1 - \lambda)c_2] + \lambda_0(-c'') \in \operatorname{icr} C. \tag{3.8}$$

By (3.3), (3.6), (3.8), and Lemma 2.3, we have

$$\begin{aligned}
\lambda(m_1,n_1)+(1-\lambda)(m_2,n_2) &= \beta\left\{\frac{1}{\beta}[\lambda(c_1,d_1)+(1-\lambda)(c_2,d_2)]\right. \\
&\quad \left.+\lambda_0(-c'',0)+\left[\lambda_0(c'',0)+\frac{\lambda\rho_1}{\beta}(y_1,z_1)+\frac{(1-\lambda)\rho_2}{\beta}(y_2,z_2)\right]\right\} \\
&= \beta\left\{\left(\frac{1}{\beta}[\lambda c_1+(1-\lambda)c_2]+\lambda_0(-c''),\frac{1}{\beta}[\lambda d_1+(1-\lambda)d_2]\right)\right. \\
&\quad \left.+\left[\lambda_0(c'',0)+\frac{\lambda\rho_1}{\beta}(y_1,z_1)+\frac{(1-\lambda)\rho_2}{\beta}(y_2,z_2)\right]\right\} \\
&\in \beta(\operatorname{icr} C\times D)+\operatorname{cone}(J(A))+C\times D\subseteq \operatorname{cone}(J(A))+\operatorname{icr} C\times D.
\end{aligned} \tag{3.9}$$

Cases one and two imply that $\operatorname{cone}(J(A))+\operatorname{icr} C\times D$ is a convex set in $Y\times Z$. Therefore, (a) holds. □

Remark 3.3. Theorem 3.2 generalizes the sixth item of Proposition 2.4 in [14], Lemma 2.1 in [15], and Lemma 2 in [16].

Now, we will give a generalized alternative theorem of the partial and generalized $C\times D$-subconvexlike map. We consider the following two systems.

System 1. There exists $x_0\in A$ such that $-J(x_0)\cap(\operatorname{icr} C\times D)\neq\emptyset$.
System 2. There exists $(y^*,z^*)\in(C^+\times D^+)\setminus\{(0,0)\}$ such that

$$\langle y,y^*\rangle+\langle z,z^*\rangle\geq 0,\quad \forall(y,z)\in J(A). \tag{3.10}$$

Theorem 3.4 (generalized alternative theorem). *Let* $\operatorname{icr}(\operatorname{cone}(J(A))+\operatorname{icr} C\times D)\neq\emptyset$, *and let the set-valued map* $J: A\rightrightarrows Y\times Z$ *be partial and generalized* $C\times D$*-subconvexlike on* A*. Then,*

(i) *if System 1 has no solutions, then System 2 has a solution;*

(ii) *if* $(y^*,z^*)\in C^{+i}\times D^+$ *is a solution of System 2, then System 1 has no solutions.*

Proof. (i) Firstly, we assert that $(0,0)\notin\operatorname{cone}(J(A))+\operatorname{icr} C\times D$. Otherwise, there exist $x_0\in A$ and $\alpha\geq 0$ such that $(0,0)\in\alpha J(x_0)+\operatorname{icr} C\times D$.

Case one: if $\alpha=0$, then $0\in\operatorname{icr} C$. Since C is a nontrivial, pointed, and convex cone, $0\notin\operatorname{icr} C$. Thus, we obtain a contradiction.

Case two: if $\alpha>0$, then there exists $(y_0,z_0)\in J(x_0)$ such that

$$-(y_0,z_0)\in\frac{1}{\alpha}(\operatorname{icr} C\times D)\subseteq\operatorname{icr} C\times D, \tag{3.11}$$

which contradicts that System 1 has no solutions.

Cases one and two show that our assertion is true. Since the set-valued map J is partial and generalized $C\times D$-subconvexlike on A, $\operatorname{cone}(J(A))+\operatorname{icr} C\times D$ is a convex set in $Y\times Z$. Note that $\operatorname{icr}(\operatorname{cone}(J(A)) + \operatorname{icr} C \times D) \neq \emptyset$. Thus, all conditions of Lemma 2.5 are satisfied. Therefore, there exists $(y^*, z^*) \in (Y^* \times Z^*) \setminus \{(0,0)\}$ such that

$$\langle ry + c, y^* \rangle + \langle rz + d, z^* \rangle \geq 0, \quad \forall r \geq 0, x \in A, y \in F(x), z \in G(x), c \in \operatorname{icr} C, d \in D. \tag{3.12}$$

Letting $r = 1$ in (3.12), we have

$$\langle y + c, y^* \rangle + \langle z + d, z^* \rangle \geq 0, \quad \forall x \in A, y \in F(x), z \in G(x), c \in \operatorname{icr} C, d \in D. \tag{3.13}$$

We again assert that $y^* \in C^+$. Otherwise, there exists $y' \in C$ such that $\langle y', y^* \rangle < 0$. Let $\overline{x} \in A, \overline{y} \in F(\overline{x}), \overline{z} \in G(\overline{x}), \overline{c} \in \operatorname{icr} C$, and $\overline{d} \in D$ be fixed. Then there exists sufficiently large positive number λ such that $\lambda \langle y', y^* \rangle + \langle \overline{y} + \overline{c}, y^* \rangle + \langle \overline{z} + \overline{d}, z^* \rangle < 0$, that is,

$$\langle \overline{y} + (\overline{c} + \lambda y'), y^* \rangle + \left\langle \overline{z} + \overline{d}, z^* \right\rangle < 0. \tag{3.14}$$

By Lemma 2.3, $\overline{c} + \lambda y' \in \operatorname{icr} C$. Thus, (3.14) contradicts (3.13). Therefore, $y^* \in C^+$. Similarly, we can prove that $z^* \in D^+$.

Let $c \in \operatorname{icr} C$ be fixed in (3.13). Then, $\beta c \in \operatorname{icr} C, \forall \beta > 0$. Letting $d = 0$ in (3.13), we have

$$\langle y, y^* \rangle + \beta \langle c, y^* \rangle + \langle z, z^* \rangle \geq 0, \quad \forall x \in A,\ y \in F(x),\ z \in G(x). \tag{3.15}$$

Letting $\beta \to 0$ in (3.15), we obtain

$$\langle y, y^* \rangle + \langle z, z^* \rangle \geq 0, \quad \forall x \in A, y \in F(x), z \in G(x), \tag{3.16}$$

which implies that System 2 has a solution.

(ii) If $(y^*, z^*) \in C^{+i} \times D^+$ is a solution of System 2, then

$$\langle y, y^* \rangle + \langle z, z^* \rangle \geq 0, \quad \forall x \in A,\ y \in F(x),\ z \in G(x). \tag{3.17}$$

We assert that System 1 has no solutions. Otherwise, there exist $p \in F(x_0)$ and $q \in G(x_0)$ such that $-p \in \operatorname{icr} C \subseteq C \setminus \{0\}$ and $-q \in D$. Therefore, we have $\langle p, y^* \rangle + \langle q, z^* \rangle < 0$, which contradicts (3.17). Therefore, our assertion is true. □

Remark 3.5. If $Y \times Z$ is a finite-dimensional space, then the partial and generalized $C \times D$-subconvexlikeness of $J : A \rightrightarrows Y \times Z$ implies that $\operatorname{cone}(J(A)) + \operatorname{icr} C \times D$ is a nonempty convex in $Y \times Z$, which in turn implies that the condition $\operatorname{icr}(\operatorname{cone}(J(A)) + \operatorname{icr} C \times D) \neq \emptyset$ holds trivially.

Remark 3.6. Theorem 3.4 generalizes Theorem 3.7 in [14], Theorem 2.1 in [15], and Theorem 1 in [16].

From now on, we suppose that $\operatorname{icr} C \neq \emptyset$.

Definition 3.7 (see [17]). Let $B \subseteq Y$. $\overline{y} \in B$ be called a global properly efficient point with respect to C (denoted by $\overline{y} \in \mathrm{GPE}(B, C)$) if and only if there exists a nontrivial, pointed, and convex cone C' with $C \setminus \{0\} \subseteq \operatorname{icr} C'$ such that $(B - \overline{y}) \cap (-C' \setminus \{0\}) = \emptyset$.

Now, we consider the following set-valued optimization problem:

$$\begin{aligned} &\text{Min} && F(x) \\ &\text{subject to} && -G(x) \cap D \neq \emptyset. \end{aligned} \tag{3.18}$$

The feasible set of (3.18) is defined by $S := \{x \in A \mid -G(x) \cap D \neq \emptyset\}$.

Definition 3.8. Let $\overline{x} \in S$ be called a global properly efficient solution of (3.18) if and only if there exists $\overline{y} \in F(\overline{x})$ such that $\overline{y} \in \mathrm{GPE}(F(S), C)$. The pair $(\overline{x}, \overline{y})$ is called a global properly efficient element of (3.18).

Now, we will establish Kuhn-Tucker conditions of set-valued optimization problem (3.18) in the sense of globally proper efficiency.

Theorem 3.9. *Suppose that the following conditions hold:*

(i) (x_0, y_0) *is a global properly efficient element of* (3.18)*;*

(ii) *the set-valued map* $I : A \rightrightarrows Y \times Z$ *is partial and generalized* $C \times D$*-subconvexlike on* A*, where* $I(x) = (F(x) - y_0, G(x))$*, for all* $x \in A$*.*

Then, there exists $(y^*, z^*) \in (C^+ \times D^+) \setminus \{(0,0)\}$ such that

$$\inf_{x \in A} \left(\langle F(x), y^* \rangle + \langle G(x), z^* \rangle \right) = \langle y_0, y^* \rangle, \quad \inf \langle G(x_0), z^* \rangle = 0. \tag{3.19}$$

Proof. Since (x_0, y_0) is a global properly efficient element of (3.18), there exists a nontrivial, pointed, and convex cone C' with $C \setminus \{0\} \subseteq \operatorname{icr} C'$ such that

$$-\left(F(x) - y_0\right) \cap \left(C' \setminus \{0\}\right) = \emptyset, \quad \forall x \in A. \tag{3.20}$$

It follows from (3.20) that

$$-\left(F(x) - y_0\right) \cap \operatorname{icr} C = \emptyset, \quad \forall x \in A. \tag{3.21}$$

By (3.21), we obtain

$$-I(x) \cap (\operatorname{icr} C \times D) = \emptyset, \quad \forall x \in A. \tag{3.22}$$

Since I is partial and generalized $C \times D$-subconvexlike on A, it follows from (3.22) and Theorem 3.4 that there exists $(y^*, z^*) \in (C^+ \times D^+) \setminus \{(0,0)\}$ such that

$$\langle F(x) - y_0, y^* \rangle + \langle G(x), z^* \rangle \geq 0, \quad \forall x \in A, \tag{3.23}$$

that is

$$\langle F(x), y^* \rangle + \langle G(x), z^* \rangle \geq \langle y_0, y^* \rangle, \quad \forall x \in A. \tag{3.24}$$

Because $x_0 \in S$, there exists $p \in G(x_0)$ such that $-p \in D$. Since $z^* \in D^+$, we have

$$\langle p, z^* \rangle \leq 0. \tag{3.25}$$

Letting $x = x_0$ in (3.24), we obtain

$$\langle p, z^* \rangle \geq 0. \tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\langle p, z^* \rangle = 0. \tag{3.27}$$

Therefore, we have

$$\langle y_0, y^* \rangle \in \langle F(x_0), y^* \rangle + \langle G(x_0), z^* \rangle. \tag{3.28}$$

By (3.24) and (3.28), we have $\inf_{x \in A}(\langle F(x), y^* \rangle + \langle G(x), z^* \rangle) = \langle y_0, y^* \rangle$. Letting $x = x_0$ in (3.24), we have

$$\langle G(x_0), z^* \rangle \geq 0. \tag{3.29}$$

It follows from (3.27) and (3.29) that $\inf\langle G(x_0), z^* \rangle = 0$. □

The following theorem, which can be found in [17], is a sufficient condition of global properly efficient elements of (3.18).

Theorem 3.10. *Suppose that the following conditions hold:*

(i) $x_0 \in S$,

(ii) *there exist* $y_0 \in F(x_0)$ *and* $(y^*, z^*) \in C^{+i} \times D^+$ *such that*

$$\inf_{x \in A}(\langle F(x), y^* \rangle + \langle G(x), z^* \rangle) \geq \langle y_0, y^* \rangle. \tag{3.30}$$

Then, (x_0, y_0) *is a global properly efficient element of* (3.18).

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References

[1] J. Borwein, "Multivalued convexity and optimization: a unified approach to inequality and equality constraints," *Mathematical Programming*, vol. 13, no. 1, pp. 183–199, 1977.

[2] F. Giannessi, "Theorems of the alternative and optimality conditions," *Journal of Optimization Theory and Applications*, vol. 42, no. 3, pp. 331–365, 1984.

[3] W. D. Rong and Y. N. Wu, "Characterizations of super efficiency in cone-convexlike vector optimization with set-valued maps," *Mathematical Methods of Operations Research*, vol. 48, no. 2, pp. 247–258, 1998.

[4] Z. F. Li, "Benson proper efficiency in the vector optimization of set-valued maps," *Journal of Optimization Theory and Applications*, vol. 98, no. 3, pp. 623–649, 1998.

[5] X. M. Yang, X. Q. Yang, and G. Y. Chen, "Theorems of the alternative and optimization with set-valued maps," *Journal of Optimization Theory and Applications*, vol. 107, no. 3, pp. 627–640, 2000.

[6] X. M. Yang, D. Li, and S. Y. Wang, "Near-subconvexlikeness in vector optimization with set-valued functions," *Journal of Optimization Theory and Applications*, vol. 110, no. 2, pp. 413–427, 2001.

[7] P. H. Sach, "New generalized convexity notion for set-valued maps and application to vector optimization," *Journal of Optimization Theory and Applications*, vol. 125, no. 1, pp. 157–179, 2005.

[8] Z. Li, "The optimality conditions for vector optimization of set-valued maps," *Journal of Mathematical Analysis and Applications*, vol. 237, no. 2, pp. 413–424, 1999.

[9] Y. W. Huang and Z. M. Li, "Optimality condition and Lagrangian multipliers of vector optimization with set-valued maps in linear spaces," *Operation Research Transactions*, vol. 5, no. 1, pp. 63–69, 2001.

[10] E. Hernández, B. Jiménez, and V. Novo, "Weak and proper efficiency in set-valued optimization on real linear spaces," *Journal of Convex Analysis*, vol. 14, no. 2, pp. 275–296, 2007.

[11] J. van Tiel, *Convex Analysis*, John Wiley and Sons, New York, NY, USA, 1984.

[12] S. Z. Shi, *Convex Analysis*, Shanghai Science and Technology Press, Shanghai, China, 1990.

[13] M. Adán and V. Novo, "Efficient and weak efficient points in vector optimization with generalized cone convexity," *Applied Mathematics Letters*, vol. 16, no. 2, pp. 221–225, 2003.

[14] M. Adán and V. Novo, "Partial and generalized subconvexity in vector optimization problems," *Journal of Convex Analysis*, vol. 8, no. 2, pp. 583–594, 2001.

[15] Q. L. Wang and Z. M. Li, "Alternative theorem for (u, O_2)-generalized subconvexlike maps and its application to vector optimization," *Applied Mathematics A*, vol. 20, no. 4, pp. 459–464, 2005.

[16] Z. A. Zhou, "Weak efficient solutions of generalized convex set-valued vector optimization problems," *Journal of Southwest China Normal University*, vol. 30, no. 2, pp. 221–225, 2005.

[17] Z. A. Zhou, X. M. Yang, and J. W. Peng, "Optimality conditions of set-valued optimization problem involving relative algebraic interior in ordered linear spaces," *Optimization*. In press.

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Research Article

Necessary and Sufficient Condition for Mann Iteration Converges to a Fixed Point of Lipschitzian Mappings

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Suppose that E is a real normed linear space, C is a nonempty convex subset of E, $T : C \to C$ is a Lipschitzian mapping, and $x^* \in C$ is a fixed point of T. For given $x_0 \in C$, suppose that the sequence $\{x_n\} \subset C$ is the Mann iterative sequence defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n$, $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $[0,1]$, $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=0}^{\infty} \alpha_n = \infty$. We prove that the sequence $\{x_n\}$ strongly converges to x^* if and only if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that $\limsup_{n\to\infty} \inf_{j(x_n - x^*)\in J(x_n-x^*)} \{\langle T x_n - x^*, j(x_n - x^*)\rangle - \|x_n - x^*\|^2 + \Phi(\|x_n - x^*\|)\} \leq 0$.

1. Introduction

Let E be an arbitrary real normed linear space with dual space E^*, and let C be a nonempty subset of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall\, x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A mapping $T : C \to E$ is called *strongly pseudocontractive* if there exists a constant $k \in (0,1)$ such that, for all $x, y \in C$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq (1-k)\|x-y\|^2. \tag{1.2}$$

T is called *ϕ-strongly pseudocontractive* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that, for all $x, y \in C$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|. \tag{1.3}$$

T is called *generalized Φ-pseudocontractive* (see, e.g., [1]) if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \Phi(\|x-y\|) \tag{1.4}$$

holds for all $x, y \in C$ and for some $j(x-y) \in J(x-y)$.

Let $F(T) = \{x \in C : Tx = x\}$ denote the fixed point set of T. If $F(T) \neq \emptyset$, and (1.3) and (1.4) hold for all $x \in C$ and $y \in F(T)$, then the corresponding mapping T is called *ϕ-hemicontractive* and generalized *Φ-hemicontractive*, respectively. It is well known that these kinds of mappings play important roles in nonlinear analysis.

ϕ-hemicontractive (resp., generalized *Φ-hemicontractive*) mapping is also called uniformly pseudocontractive (resp., uniformly hemicontractive) mapping in [2, 3]. It is easy to see that if T is generalized *Φ-hemicontractive* mapping, then $F(T)$ is singleton.

It is known (see, e.g., [4, 5]) that the class of strongly pseudocontractive mappings is a proper subset of the class of ϕ-strongly pseudocontractive mappings. By taking $\Phi(s) = s\phi(s)$, where $\phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function with $\phi(0) = 0$, we know that the class of ϕ-strongly pseudocontractive mappings is a subset of the class of generalized Φ-pseudocontractive mappings. Similarly, the class of ϕ-hemicontractive mappings is a subset of the class of generalized Φ-hemicontractive mappings. The example in [6] demonstrates that the class of Lipschitzian ϕ-hemicontractive mappings is a proper subset of the class of Lipschitzian generalized Φ-hemicontractive mappings.

It is well known (see, e.g., [7]) that if C is a nonempty closed convex subset of a real Banach space E and $T : C \to C$ is a continuous strongly pseudocontractive mapping, then T has a unique fixed point $p \in C$. In 2009, it has been proved in [8] that if C is a nonempty closed convex subset of a real Banach space E and $T : C \to C$ is a continuous generalized Φ-pseudocontractive mappings, then T has a unique fixed point $p \in C$.

Many results have been proved on convergence or stability of Ishikawa iterative sequences (with errors) or Mann iterative sequences (with errors) for Lipschitzian ϕ-hemicontractive mappings or Lipschitzian generalized Φ-hemicontractive mapping (see, e.g., [4–6, 9–12] and the references therein). In 2010, Xiang et al. [6] proved the following result.

Theorem XCZ (see [6, Theorem 3.2]). *Let E be a real normed linear space, let C be a nonempty convex subset of E, and let $T : C \to C$ be a Lipschitzian generalized Φ-hemicontractive mapping. For given $x_0 \in C$, suppose that the sequence $\{x_n\} \subset C$ is the Mann iterative sequence defined by*

$$x_{n+1} = (1-\beta_n)x_n + \beta_n T x_n, \quad n \ge 0, \tag{1.5}$$

where $\{\beta_n\}$ *is a sequence in* $[0,1]$ *satisfying the following conditions:*

(1) $\sum_{n=0}^{\infty}\beta_n=\infty$,

(2) $\sum_{n=0}^{\infty}\beta_n^2<\infty$.

Then $\{x_n\}$ *converges strongly to the unique fixed point of* T *in* C.

The main purpose of this paper is to give necessary and sufficient condition for the Mann iterative sequence which converges to a fixed point of general Lipschitzian mappings in an arbitrary real normed linear space. As an immediate consequence, we will obtain necessary and sufficient condition for the Mann iterative sequence which converges to a solution of a general Lipschitzian operator equation $Tx=f$.

2. Preliminaries

The following lemmas will be used in the proof of our main results.

Lemma 2.1 (see, e.g., [12]). *Let* E *be a real normed linear space. Then for all* $x,y\in E$, *we have*

$$\|x+y\|^2\leq\|x\|^2+2\langle y,j(x+y)\rangle,\quad\forall j(x+y)\in J(x+y).\tag{2.1}$$

Lemma 2.2 (see, e.g., [13]). *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *be three nonnegative sequences satisfying the following condition:*

$$a_{n+1}\leq(1+b_n)a_n+c_n,\quad\forall\ n\geq n_0,\tag{2.2}$$

where n_0 *is some nonnegative integer,* $\sum_{n=n_0}^{\infty}b_n<\infty$, *and* $\sum_{n=n_0}^{\infty}c_n<\infty$. *Then the limit* $\lim_{n\to\infty}a_n$ *exists.*

Lemma 2.3. *Suppose that* $\varphi:[0,\infty)\to[0,\infty)$ *is a strictly increasing function with* $\varphi(0)=0$ *and there exists a natural number* n_0 *such that* a_n, b_n, ε_n, *and* α_n *are nonnegative real numbers for all* $n\geq n_0$ *satisfying the following conditions:*

(i) $a_{n+1}\leq(1+b_n)a_n-\alpha_n\varphi(a_{n+1})+\alpha_n\varepsilon_n$, for all $n\geq n_0$,

(ii) $\sum_{n=n_0}^{\infty}b_n<\infty$, $\lim_{n\to\infty}\varepsilon_n=0$,

(iii) $\sum_{n=n_0}^{\infty}\alpha_n=\infty$.

Then $\lim_{n\to\infty}a_n=0$.

Proof. Without loss of generality, let $\lim_{n\to\infty}\inf a_n=a$. Now, we will show that $a=0$. Consider its contrary: $a>0$ or $a=\infty$. For any given $r\in(0,a)$, there exists a nonnegative integer $n_1\geq n_0$ such that $a_n\geq r>0$ and $\varepsilon_n<1/2\varphi(r)\leq1/2\varphi(a_{n+1})$ for all $n\geq n_1$. By condition (i), we have

$$\begin{aligned}a_{n+1}&\leq(1+b_n)a_n-\alpha_n\varphi(a_{n+1})+\alpha_n\cdot\frac{1}{2}\varphi(a_{n+1})\\&=(1+b_n)a_n-\frac{1}{2}\alpha_n\varphi(a_{n+1})\\&\leq(1+b_n)a_n,\quad\forall n\geq n_1.\end{aligned}\tag{2.3}$$

Using Lemma 2.2 and condition (ii), we obtain that $\lim_{n\to\infty} a_n$ exists and $\{a_n\}$ is bounded. Suppose that $a_n \le M$ (for all $n \ge n_1$), where M is a nonnegative constant. It follows that

$$a_{n+1} \le (1+b_n)a_n - \frac{1}{2}\alpha_n\varphi(a_{n+1}) \le a_n - \frac{1}{2}\alpha_n\varphi(r) + Mb_n(\forall n \ge n_1). \tag{2.4}$$

Thus,

$$\infty = \frac{1}{2}\varphi(r)\sum_{n=n_1}^{\infty}\alpha_n \le a_{n_1} + M\sum_{n=n_1}^{\infty} b_n < \infty, \tag{2.5}$$

which is a contradiction. Therefore,

$$\liminf_{n\to\infty} a_n = 0. \tag{2.6}$$

By condition (ii), for all $\varepsilon > 0$, there exists a nonnegative integer $n_2 \ge n_0$ such that

$$\varepsilon_n < \varphi(\varepsilon) \quad (\forall n \ge n_2), \qquad \sum_{n=n_2}^{\infty} b_n < \ln 2. \tag{2.7}$$

By (2.6), there exists a natural number $N \ge n_2$ such that $a_N < \varepsilon$. Now, we prove the following inequality (2.8) holds for all $k \ge N$:

$$a_k \le \varepsilon \cdot \exp\left(\sum_{n=N}^{k-1} b_n\right). \tag{2.8}$$

It is obvious that (2.8) holds for $k = N$. Assuming (2.8) holds for some $k \ge N$, we prove that (2.8) holds for $k+1$. Suppose this is not true, that is, $a_{k+1} > \varepsilon \cdot \exp(\sum_{n=N}^{k} b_n)$. Then $a_{k+1} \ge \varepsilon$ and so $\varphi(a_{k+1}) \ge \varphi(\varepsilon)$. Noting that $1 + b_k \le \exp(b_k)$, it follows from condition (i), (2.7), and (2.8) that

$$\begin{aligned} a_{k+1} &\le (1+b_k)a_k - \alpha_k\varphi(a_{k+1}) + \alpha_k\varepsilon_k \\ &\le (1+b_k)a_k - \alpha_k\varphi(\varepsilon) + \alpha_k\varphi(\varepsilon) \\ &\le \varepsilon \cdot (1+b_k)\exp\left(\sum_{n=N}^{k-1} b_n\right) \\ &\le \varepsilon \cdot \exp\left(\sum_{n=N}^{k} b_n\right), \end{aligned} \tag{2.9}$$

which is a contradiction. This implies that (2.8) holds for $k+1$. By induction, (2.8) holds for all $k \geq N$. From (2.7), and (2.8), we have

$$\limsup_{k\to\infty} a_k \leq \varepsilon \cdot \exp\left(\sum_{n=N}^{\infty} b_n\right) < 2\varepsilon. \tag{2.10}$$

Taking $\varepsilon \to 0$, we obtain $\lim_{n\to\infty} \sup a_k = 0$. By (2.6), we have $\lim_{n\to\infty} a_n = 0$. This completes the proof. □

Remark 2.4. Lemma 2.3 is different from Lemma 3 in [14], which requires that $b_n = 0$ for all $n \geq 0$. It is also different from Lemma 2.3 in [6], which requires that $\sum_{n=n_0}^{\infty} \alpha_n \varepsilon_n < \infty$.

3. Main Results

Theorem 3.1. *Let E be a real normed linear space, C be a nonempty convex subset of E, let $T : C \to C$ be a Lipschitzian mapping, and let $x^* \in C$ be a fixed point of T. For given $x_0 \in C$, suppose that the sequence $\{x_n\} \subset C$ is the Mann iterative sequence defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to x^* if and only if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\limsup_{n\to\infty} \inf_{j(x_n-x^*)\in J(x_n-x^*)} \left\{ \langle Tx_n - x^*, j(x_n - x^*)\rangle - \|x_n - x^*\|^2 + \Phi(\|x_n - x^*\|) \right\} \leq 0. \tag{3.2}$$

Proof. First, we prove the sufficiency of Theorem 3.1.

Suppose there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that (3.2) holds. Let

$$\gamma_n = \inf_{j(x_n-x^*)\in J(x_n-x^*)} \left\{ \langle Tx_n - x^*, j(x_n - x^*)\rangle - \|x_n - x^*\|^2 + \Phi(\|x_n - x^*\|) \right\}. \tag{3.3}$$

Then there exists $j(x_n - x^*) \in J(x_n - x^*)$ such that

$$\langle Tx_n - x^*, j(x_n - x^*)\rangle - \|x_n - x^*\|^2 + \Phi(\|x_n - x^*\|) < \gamma_n + \frac{1}{n}, \quad \forall n \geq 1. \tag{3.4}$$

By (3.2), we obtain $\lim_{n\to\infty} \sup \gamma_n \leq 0$. Taking $\varepsilon_n = 1/(n+1) + \max\{\gamma_{n+1}, 0\}$ (for all $n \geq 0$), then

$$\lim_{n\to\infty} \varepsilon_n = 0. \tag{3.5}$$

From (3.1) and (3.4), by using Lemma 2.1, we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Tx_n - x^*)\|^2 \\
&\quad \le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle Tx_n - x^*, j(x_{n+1} - x^*)\rangle \\
&\quad \le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle Tx_{n+1} - x^*, j(x_{n+1} - x^*)\rangle \\
&\qquad + 2\alpha_n\langle Tx_n - Tx_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad \le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\left[\|x_{n+1} - x^*\|^2 - \Phi(\|x_{n+1} - x^*\|) + \gamma_{n+1} + \frac{1}{n+1}\right] \\
&\qquad + 2L\alpha_n\|x_n - x_{n+1}\| \cdot \|x_{n+1} - x^*\| \\
&\quad \le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\left[\|x_{n+1} - x^*\|^2 - \Phi(\|x_{n+1} - x^*\|) + \varepsilon_n\right] \\
&\qquad + 2L\alpha_n\|x_n - x_{n+1}\| \cdot \|x_{n+1} - x^*\|,
\end{aligned} \tag{3.6}$$

where L is the Lipschitzian constant of T. It follows from (3.1) that

$$\begin{aligned}
\|x_n - x_{n+1}\| &= \|\alpha_n(x_n - Tx_n)\| \\
&\le \alpha_n(\|x_n - x^*\| + \|Tx^* - Tx_n\|) \\
&\le \alpha_n(1+L)\|x_n - x^*\|.
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\|x_{n+1} - x^*\|^2 - 2\alpha_n\Phi(\|x_{n+1} - x^*\|) \\
&\quad + 2\alpha_n\varepsilon_n + 2L(1+L)\alpha_n^2\|x_n - x^*\| \cdot \|x_{n+1} - x^*\| \\
&\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\|x_{n+1} - x^*\|^2 - 2\alpha_n\Phi(\|x_{n+1} - x^*\|) \\
&\quad + 2\alpha_n\varepsilon_n + L(1+L)\alpha_n^2\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right).
\end{aligned} \tag{3.8}$$

Setting $a_n = \|x_n - x^*\|^2$(for all $n \ge 0$), $\varphi(s) = 2\Phi(\sqrt{s})$, it follows from (3.8) that

$$\begin{aligned}
a_{n+1} &\le (1-\alpha_n)^2 a_n + 2\alpha_n a_{n+1} - \alpha_n\varphi(a_{n+1}) + 2\alpha_n\varepsilon_n \\
&\quad + L(1+L)\alpha_n^2(a_n + a_{n+1}) \\
&= \left[1 - 2\alpha_n + \alpha_n^2 + L(1+L)\alpha_n^2\right]a_n + \left[2\alpha_n + L(1+L)\alpha_n^2\right]a_{n+1} \\
&\quad - \alpha_n\varphi(a_{n+1}) + 2\alpha_n\varepsilon_n.
\end{aligned} \tag{3.9}$$

It follows from condition (i) that $\lim_{n\to\infty}[2\alpha_n + L(1+L)\alpha_n^2] = 0$. Thus, there exists a natural number n_0 such that $2\alpha_n + L(1+L)\alpha_n^2 \le 1/2$ for all $n \ge n_0$. Let

$$b_n = \frac{1-2\alpha_n+\alpha_n^2+L(1+L)\alpha_n^2}{1-2\alpha_n-L(1+L)\alpha_n^2} - 1 = \frac{\alpha_n^2+2L(1+L)\alpha_n^2}{1-2\alpha_n-L(1+L)\alpha_n^2}, \quad \forall n \ge n_0. \tag{3.10}$$

Since $1/2 \le 1-2\alpha_n - L(1+L)\alpha_n^2 \le 1$ for all $n \ge n_0$, by (3.9) and (3.10), we have

$$\begin{gathered} a_{n+1} \le (1+b_n)a_n - \alpha_n\varphi(a_{n+1}) + 4\alpha_n\varepsilon_n, \quad \forall n \ge n_0, \\ 0 \le b_n \le 2[1+2L(1+L)]\alpha_n^2, \quad \forall n \ge n_0. \end{gathered} \tag{3.11}$$

It follows from condition (i) that $\sum_{n=n_0}^{\infty} b_n < \infty$. Therefore, by (3.5), condition (ii), and Lemma 2.3, we obtain that $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\|x_n - x^*\|^2 = 0$. That is, $\{x_n\}$ converges strongly to x^*.

Finally, we prove the necessity of Theorem 3.1.

Assume that $\{x_n\}$ converges strongly to x^*. Let L be the Lipschitzian constant of T. For all $j(x_n - x^*) \in J(x_n - x^*)$, we have

$$\left|\langle Tx_n - x^*, j(x_n - x^*)\rangle\right| \le L\|x_n - x^*\|^2. \tag{3.12}$$

Taking $\Phi(s) = \sqrt{s}$, then $\Phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function with $\Phi(0) = 0$, and $\lim_{n\to\infty}\Phi(\|x_n - x^*\|) = 0$. From (3.12), we obtain

$$\lim_{n\to\infty} \inf_{j(x_n-x^*)\in J(x_n-x^*)} \left\{\langle Tx_n - x^*, j(x_n-x^*)\rangle - \|x_n - x^*\|^2 + \Phi(\|x_n - x^*\|)\right\} = 0, \tag{3.13}$$

which implies (3.2) holds. This completes the proof of Theorem 3.1. □

Remark 3.2. If $T : C \to C$ is a generalized Φ-hemicontractive mapping, then (3.2) holds. By Theorem 3.1, we obtain Theorem XCZ.

Theorem 3.3. *Let E be a real Banach space, let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a Lipschitzian generalized Φ-pseudocontractive mapping. For given $x_0 \in C$, suppose that the sequence $\{x_n\} \subset C$ is the Mann iterative sequence defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad n \ge 0, \tag{3.14}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:

(1) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(2) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Then $\{x_n\}$ converges strongly to the unique fixed point of T in C.

Proof. By Theorem 2.1 in [8], T has a unique fixed point x^* in C. By Theorem 3.1, $\{x_n\}$ converges strongly to x^*. This completes the proof of Theorem 3.3. □

Theorem 3.4. *Let E be a real normed linear space, let $S : E \to E$ be a Lipschitzian operator, and let $f \in E$ and x^* be a solution of the equation $Sx = f$. For given $x_0 \in E$, suppose that the sequence $\{x_n\}$ is the Mann iterative sequence defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\big(f + x_n - Sx_n\big), \quad n \geq 0, \tag{3.15}$$

where $\{\alpha_n\}$ is a sequences in $[0,1]$ satisfying the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to x^* if and only if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\liminf_{n\to\infty} \sup_{j(x_n - x^*)\in J(x_n - x^*)} \big\{\big\langle Sx_n - Sx^*, j(x_n - x^*)\big\rangle - \Phi(\|x_n - x^*\|)\big\} \geq 0. \tag{3.16}$$

Proof. Define $T : E \to E$ by $Tx = f + x - Sx$. Since $Sx^* = f$, we have $Tx^* = x^*$. From (3.15), we obtain $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n,\ n \geq 0$. Since

$$\begin{aligned}&\big\langle Sx_n - Sx^*, j(x_n - x^*)\big\rangle - \Phi(\|x_n - x^*\|)\\ &\quad = -\Big\{\big\langle Tx_n - x^*, j(x_n - x^*)\big\rangle - \|x_n - x^*\|^2 + \Phi(\|x_n - x^*\|)\Big\}.\end{aligned} \tag{3.17}$$

Therefore,

$$\begin{aligned}&\liminf_{n\to\infty} \sup_{j(x_n - x^*)\in J(x_n - x^*)} \big\{\big\langle Sx_n - Sx^*, j(x_n - x^*)\big\rangle - \Phi(\|x_n - x^*\|)\big\}\\ &\quad = -\limsup_{n\to\infty} \inf_{j(x_n - x^*)\in J(x_n - x^*)} \big\{\big\langle Tx_n - x^*, j(x_n - x^*)\big\rangle - \|x_n - x^*\| + \Phi(\|x_n - x^*\|)\big\}.\end{aligned} \tag{3.18}$$

The condition (3.16) is equivalent to condition (3.2). Since S is a Lipschitzian operator, T is a Lipschitzian mapping. By Theorem 3.1, Theorem 3.4 holds. This completes the proof of Theorem 3.4. □

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References

[1] C. E. Chidume and C. O. Chidume, “Convergence theorems for fixed points of uniformly continuous generalized Φ-hemi-contractive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 303, no. 2, pp. 545–554, 2005.

[2] C. Moore and B. V. C. Nnoli, “Iterative solution of nonlinear equations involving set-valued uniformly accretive operators,” *Computers & Mathematics with Applications*, vol. 42, no. 1-2, pp. 131–140, 2001.

[3] C. E. Chidume and H. Zegeye, “Approximation methods for nonlinear operator equations,” *Proceedings of the American Mathematical Society*, vol. 131, no. 8, pp. 2467–2478, 2003.

[4] M. O. Osilike, “Iterative solution of nonlinear equations of the φ-strongly accretive type,” *Journal of Mathematical Analysis and Applications*, vol. 200, no. 2, pp. 259–271, 1996.

[5] M. O. Osilike, “Iterative solutions of nonlinear φ-strongly accretive operator equations in arbitrary Banach spaces,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 36, pp. 1–9, 1999.

[6] C. H. Xiang, Z. Chen, and K. Q. Zhao, “Some convergence theorems for a class of generalized Φ-hemicontractive mappings,” *Journal of Concrete and Applicable Mathematics*, vol. 8, no. 4, pp. 638–644, 2010.

[7] K. Deimling, “Zeros of accretive operators,” *Manuscripta Mathematica*, vol. 13, pp. 365–374, 1974.

[8] C. H. Xiang, “Fixed point theorem for generalized Φ-pseudocontractive mappings,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 6, pp. 2277–2279, 2009.

[9] Y. Xu, “Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations,” *Journal of Mathematical Analysis and Applications*, vol. 224, no. 1, pp. 91–101, 1998.

[10] X. P. Ding, “Iterative process with errors to nonlinear φ-strongly accretive operator equations in arbitrary Banach spaces,” *Computers & Mathematics with Applications*, vol. 33, no. 8, pp. 75–82, 1997.

[11] R. P. Agarwal, N. J. Huang, and Y. J. Cho, “Stability of iterative processes with errors for nonlinear equations of φ-strongly accretive type operators,” *Numerical Functional Analysis and Optimization*, vol. 22, no. 5-6, pp. 471–485, 2001.

[12] Z. Huang, “Weak stability of Mann and Ishikawa iterations with errors for φ-hemicontractive operators,” *Applied Mathematics Letters*, vol. 20, no. 4, pp. 470–475, 2007.

[13] S. S. Chang, K. K. Tan, H. W. J. Lee, and C. K. Chan, “On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 313, no. 1, pp. 273–283, 2006.

[14] C. E. Chidume and E. U. Ofoedu, “A new iteration process for generalized Lipschitz pseudo-contractive and generalized Lipschitz accretive mappings,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 67, no. 1, pp. 307–315, 2007.

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Research Article

Sufficient Optimality and Sensitivity Analysis of a Parameterized Min-Max Programming

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Sufficient optimality and sensitivity of a parameterized min-max programming with fixed feasible set are analyzed. Based on Clarke's subdifferential and Chaney's second-order directional derivative, sufficient optimality of the parameterized min-max programming is discussed first. Moreover, under a convex assumption on the objective function, a subdifferential computation formula of the marginal function is obtained. The assumptions are satisfied naturally for some application problems. Moreover, the formulae based on these assumptions are concise and convenient for algorithmic purpose to solve the applications.

1. Introduction

In this paper, sufficient optimality and sensitivity analysis of a parameterized min-max programming are given. The paper is triggered by a local reduction algorithmic strategy for solving following nonsmooth semi-infinite min-max-min programming (SIM^{3}P, see [1, 2], etc. for related applications reference):

$$\begin{aligned} &\min_{x} \quad f(x) \\ &\text{s.t.} \quad g(x) = \max_{y\in Y} \min_{1\le i\le q} \{ g_i(x, y) \} \le 0. \end{aligned} \tag{1.1}$$

With the local reduction technique, the SIM^3P can be rewritten as a bilevel programming first, where the lower problem is the following parameterized min-max programming P_x (see [3–5] for related reference of local reduction strategy):

$$\begin{aligned} \min_{y} \quad & g(x,y) = \max_{1\le i\le q}\{-g_i(x,y)\} \\ \text{s.t.} \quad & y \in Y. \end{aligned} \tag{1.2}$$

To make the bilevel strategy applicable to SIM^3P, it is essential to discuss the second-order sufficient optimality of P_x and give sensitivity analysis of the parameterized minimum $y(x)$ and corresponding marginal function $g(x, y(x))$.

Sensitivity analysis of optimization problems is an important aspect in the field of operation and optimization research. Based on different assumptions, many results on kinds of parametric programming have been obtained ([6–9], etc.). Among these, some conclusions on parameterized min-max programming like (1.2) have also been given. For example, based on variation analysis, parameterized continuous programming with fixed constraint was discussed in [7]. Problem like (1.2) can be seen as a special case. Under the inf-compactness condition and the condition objective function is concave with respect to the parameter, directional derivative computational formula of marginal function for (1.2) can be obtained directly. However, concave condition cannot be satisfied for many problems. Recently, Fréchet subgradients computation formula of marginal functions for nondifferentiable programming in Asplund spaces was given ([9]). By using Fréchet subgradients computation formula in [9], subgradient formula of marginal function for (1.2) is direct. But the formula is tedious, if utilizing the formula to construct optimality system of (1.1), the system is so complex that it is difficult to solve the obtained optimality system.

For more convenient computational purpose, the focus of this paper is to establish sufficient optimality and simple computation formula of marginal function for (1.2). Based on Clarke's subdifferential and Chaney's second-order directional derivative, sufficient optimality of the parameterized programming P_x is given first. And then Lipschitzian continuousness of the parameterized isolated minimizer $y(x)$ and the marginal function $g(y(x), x)$ is discussed; moreover, subdifferential computation formula of the marginal function is obtained.

2. Main Results

Let Y in (1.2) be defined as $Y = \{y \in R^m : h_i(y) \le 0,\ i = 1,\ldots,l\}$, where $h_i(\cdot)$ and $i = 1,\ldots,l$, are twice continuously differentiable functions on R^m, and $g_i(\cdot,\cdot)$ in (1.2) are twice continuously differentiable functions on $R^{n\times m}$. In the following, we first give the sufficient optimality condition of (1.2) based on Clarke's subdifferential and Chaney's second-order directional derivative, and then make sensitivity analysis of the parameterized problem P_x.

2.1. Sufficient Optimality Conditions of P_x

Definition 2.1 (see [10]). For a given parameter x, a point $y^* \in Y$ is said to be an local minimum of problem P_x if there exists a neighborhood U of y^* such that

$$g(x,y) \geq g(x,y^*), \quad \forall y \in U \cap Y,\ y \neq y^*. \tag{2.1}$$

Assumption 2.2. For a given parameter x, suppose that P_x satisfying the following constraint qualification:

$$\left\{d \in R^m : \nabla h_i(y)^T d < 0,\ \forall i \in I_h(y),\ y \in Y\right\} \neq \emptyset, \tag{2.2}$$

where $I_h(y) = \{i = \{1,\ldots,l\} : h_i(y) = 0\}$.

For a given parameter x, denote the Lagrange function of P_x as $L(x,y,\lambda) = g(x,y) + \sum_{i=1}^{l} \lambda_i h_i(y)$, then the following holds.

Theorem 2.3. *For a given parameter x, if y^* is a minimum of P_x, Assumption 2.2 holds, then there exists a $\lambda^* \in R_+^l$ such that $0 \in \partial_y L(x,y^*,\lambda)$, where $\partial_y L(x,y^*,\lambda^*)$ denotes the Clarke's subdifferential of $L(x,y^*,\lambda^*)$. Specifically, the following system holds:*

$$0 \in \partial_y g(x,y^*) + \sum_{i=1}^{l} \lambda_i \nabla h_i(y^*), \tag{2.3}$$

where $\partial_y g(x,y)$ denotes Clarke's subdifferential of $g(x,y)$ with respect to y, it can be computed as $\operatorname{co}\{\nabla_y g_i(x,y^*) : i \in I(x,y^*)\}$, $\operatorname{co}\{\cdot\}$ is an operation of making convex hull of the elements, $I(x,y^*) = \{i \in \{1,\ldots,q\} : g(x,y^*) = g(x,y^*)\}$.

Proof. The conclusion is direct from Theorem 3.2.6 and Corollary 5.1.8 in [11]. □

Since $g(x,y) = \max_{1\leq i\leq p}\{g_i(x,y)\}$ is a directional differentiable function (Theorem 3.2.13 in [11]), the directional derivative of $g(x,y)$ with respect to y in direction d can be computed as follows:

$$g_y'(x,y;d) = \max\left\{\xi^T d : \forall \xi \in \partial_y g(x,y),\ \forall d \in R^m\right\}. \tag{2.4}$$

Definition 2.4 (see [10]). Let $f(x)$ is a locally Lipschitzian function on R^n, u be a nonzero vector in R^n. Suppose that

$$d \in \partial_u f(x) = \left\{v \in R^n : \exists\{x_k\},\{v_k\}, \text{s. t. } x_k \xrightarrow{x}, v_k \longrightarrow v, v_k \in \partial f(x_k) \text{ for each } k\right\} \tag{2.5}$$

define Chaney's lower second-order directional derivative as follows:

$$f_-''(x,v,u) = \liminf \frac{f(x_k) - f(x) - v^T(x_k - x)}{t_k^2}, \tag{2.6}$$

taking over all triples of sequences $\{x_k\}$, $\{v_k\}$, and $\{t_k\}$ for which

(a) $t_k > 0$ for each k and $\{x_k\} \to x$;

(b) $t_k \to 0$ and $(x_k - x^*)/t_k$ converges to u;

(c) $\{v_k\} \to v$ with $v_k \in \partial f(x_k)$ for each k.

Similarly, Chaney's upper second-order directional derivative can be defined as

$$f''_+ = \limsup \frac{f(x_k) - f(x) - v^T(x_k - x)}{t_k^2}, \tag{2.7}$$

taking over all triples of sequences $\{x_k\}$, $\{v_k\}$, and $\{t_k\}$ for which (a), (b), and (c) above hold.

For parameterized max-type function $g(x,y) = \max_{1\leq i\leq p}\{-g_i(x,y)\}$, where x is a given parameter, its Chaney's lower and upper second-order directional derivatives can be computed as follows.

Proposition 2.5 (see [12]). *For any given parameter x, Chaney's lower and upper second-order directional derivatives of $g(x,y)$ with respect to y exist; moreover, for any given $0 \neq u \in R^q$, $v \in \partial_u g(x,y)$, it has*

$$\begin{gathered} g''_-(x,y;d) = \min\left\{\frac{1}{2}\sum_{i=1}^{q} a_j u^T \nabla_y^2 g_i(x,y) u : a \in T_u(g,y,v)\right\}, \\ g''_+(x,y;d) = \max\left\{\frac{1}{2}\sum_{i=1}^{q} a_j u^T \nabla_y^2 g_i(x,y) u : a \in T_u(g,y,v)\right\}, \end{gathered} \tag{2.8}$$

where

$$T_u(g,y,v) = \left\{ a \in R_+^q : \begin{array}{l} \exists\{y^{(k)}\}, \{a^{(k)}\}, \{v^{(k)}\},\ \text{such that} \\ (1)\ y^{(k)} \longrightarrow y \quad \text{in direction} \quad u, \\ (2)\ v^{(k)} \longrightarrow v,\ \text{and}\ v^{(k)} \in \partial_y g(x, y^{(k)}),\ k = 1,2,\ldots, \\ (3)\ a^{(k)} \longrightarrow a, a^{(k)} \in E_q, v^{(k)} = \sum_{i=1}^{p} a_i^{(k)} \nabla_y g_i(x, y^{(k)}), \\ (4)\ a_j^{(k)} = 0, \quad \text{for} \quad j \notin K_g(y^{(k)}) \end{array} \right\}, \tag{2.9}$$

where $K_g(y^{(k)}) = \{i \in Q : g_i(x, y^{(n)}) = g(x, y^{(n)}), \exists y^{(n)} \in B(y, 1/n), \forall n \in N\}$, $E_q = \{a \in R_+^q : \sum_{i=1}^{p} a_i = 1\}$, $Q = \{1,\ldots,q\}$, and $B(y,1/n)$ denotes the ball centered in y with radius $1/n$.

Theorem 2.6 (sufficiency theorem). *For a given parameter $x \in R^n$, Assumption 2.2 holds, then there exists $y^* \in R^m$ such that* (2.3) *holds. Moreover, for any feasible direction $d \in R^m$ of Y, that is, $\max\{\nabla h_i(y^*)^T d : 1 \leq i \leq l\} \leq 0$, if d satisfying one of the following conditions:*

(1) $g'_y(x, y^*; d) \neq 0$;

(2) $g'_y(x, y^*; d) = 0$, $\sum_{i=1}^{l} \lambda_i \nabla h_i(y)^T d = 0$, *that is*, $L'_y(x, y; d) = 0$, *and*

$$\min\left\{ \frac{1}{2} \sum_{i=1}^{q} a_i d^T \nabla_y^2 g_i(x, y^*) d : a \in E_q \right\} + \sum_{i=1}^{l} \lambda_i d^T \nabla^2 h_i(y^*) d > 0, \tag{2.10}$$

then y^* *is a local minimum of* P_x.

Proof. (1) If not, then there exists sequences $t_k \downarrow 0$, $d_k \to d$, $y_k = y^* + t_k d_k \in Y$ such that

$$g(x, y_k) < g(x, y^*). \tag{2.11}$$

As a result, $g'_y(x, y^*; d) = \lim_{t\downarrow 0}(g(x, y^* + td) - g(x, y^*))/t = \lim_{k\to+\infty}(g(x, y^* + t_k d_k) - g(x, y^*))/t_k \leq 0$. If $g'_y(x, y^*; d) \neq 0$, then $g'_y(x, y^*; d) < 0$. From (2.4), we know that $\xi^T d < 0$ for all $\xi \in \partial_y g(x, y^*)$. Hence, for the direction $d \in R^m$, we have

$$\xi^T d + \sum_{i=1}^{l} \nabla h_i(y^*)^T d < 0, \quad \xi \in \partial_y g(x, y^*). \tag{2.12}$$

On the other hand, from y^* satisfying (2.3), we know that there exists a $\xi \in \partial_y g(x, y^*)$ such that

$$\xi^T d + \sum_{i=1}^{l} \nabla h_i(y^*)^T d = 0, \tag{2.13}$$

which leads to a contradiction to (2.12).

(2) From Theorem 4 in [10] and Proposition 2.5, the conclusion is direct. □

2.2. Sensitivity Analysis of Parameterized P_x

In the following, we make sensitivity analysis of parameterized min-max programming P_x, that is, study variation of isolated local minimizers and corresponding marginal function under small perturbation of x.

For convenience of discussion, for any given parameter x, denote $y^*(x)$ as a minimizer of P_x, $v(x) = \min\{g(x, y) : y \in Y\}$ as the corresponding marginal function value and make the following assumptions first.

Assumption 2.7. For given $x \in R^n$, the parametric problem P_x is a convex problem, specifically, $g_i(x, y)$ and $i = 1, \ldots, q$ are concave functions with respect to that variables y and $h_j(y)$, $j = 1, \ldots, l$ are convex functions.

Assumption 2.8. Let $I_h(y) = \{i \in L : h_i(y) = 0\}$, $\{\nabla h_i(y) : i \in I_h(y)\}$ are linearly independent.

Definition 2.9 (see Definition 2.1, [13]). For a given $\overline{x}$, $\overline{y} \in Y$ is said to be an isolated local minimum with order i ($i = 1$ or 2) of $P_{\overline{x}}$ if there exists a real $m > 0$ and a neighborhood V of $\overline{y}$ such that

$$g(\overline{x}, y) > g(\overline{x}, \overline{y}) + \frac{1}{2} m \|y - \overline{y}\|^{i}, \quad \forall y \in V \cap Y, \ y \neq \overline{y}. \tag{2.14}$$

Theorem 2.10. *For a given $x \in R^n$, Assumptions 2.2–2.8 hold, then the following conclusions hold:*

(1) *if $y^*(x)$ with corresponding multiplier λ^* is the solution of* (2.3), *then $y^*(x)$ is a unique first-order isolated minimizer of P_x;*

(2) *for any minimum $y^*(x)$, it is a locally Lipschitzian function with respect to x, that is, there exists a $L_1 > 0$, $\delta > 0$ such that*

$$\left\|y^*\left(x^k\right) - y^*(x)\right\| \leq L_1 \left\|x^k - x\right\|, \quad \forall x^k \in U(x, \delta), \ y^*\left(x^k\right) \in Y\left(x^k\right), \tag{2.15}$$

where $Y(x^k)$ denotes minima set of P_{x^k};

(3) *for any minimum $y^*(x)$, marginal function $v(x) = g(x, y^*(x))$ is also a locally Lipschitz function with respect to x, and $\partial v(x) \subseteq S(x)$, where*

$$S(x) = \mathrm{co}\{\nabla_x g_i(x, y^*(x)), \ i \in I(x, y^*(x))\}, \tag{2.16}$$

and $I(x, y^*(x)) = \{i \in \{1, \ldots, q\} : g_i(x, y^*(x)) = g(x, y^*(x))\}$. As a result,

$$\partial v(x) = \left\{ \sum_{i \in I(x, y^*(x))} \lambda_i \nabla_x g_i(x, y^*(x)) : \lambda_i \geq 0, \sum_{i \in I(x, y^*(x))} \lambda_i = 1 \right\}. \tag{2.17}$$

Proof. (1) From Assumption 2.7, it is direct that $y^*(x)$ is a global minimizer of P_x. We only prove $y^*(x)$ is a first-order isolated minimizer.

If the conclusion does not hold, then there exists a sequence $\{y^k\} \in Y(x)$ converging to $y^*(x)$, $y^k \neq y^*(x)$, and a sequence m_k, $m_k > 0$, and m_k converges to 0 such that

$$g\left(x, y^k\right) \leq g(x, y^*(x)) + \frac{1}{2} m_k \left\|y^k - y^*\right\|, \quad y^k \in Y. \tag{2.18}$$

Take $d_k = (y^k - y^*(x))/\|y^k - y^*(x)\|$, for simplicity, we suppose $d_k \to d$, with $\|d\| = 1$. Let $t_k = \|y^k - y^*(x)\|$, then from $y^k \in Y$, $d_k \to d$ and Y is compact, we have

$$y^*(x) + t_k d \in Y, \quad t_k \longrightarrow 0, \tag{2.19}$$

that is,

$$\nabla h_i(y^*(x))^T d \leq 0, \quad \forall i \in I(x, y^*(x)). \tag{2.20}$$

From Assumption 2.8, we know that $\sum_{i\in I(x,y^*(x))} \nabla h_i(y^*(x))^T d \neq 0$. As a result, we have $\sum_{i\in I(x,y^*(x))} \nabla h_i(y^*(x))^T d < 0$.

From the first equation of (2.3), we know that there exists a $z \in \partial_y g(x, y^*(x))$ such that for any feasible direction d, $z^T d = -\sum_{i\in I(x,y^*(x))} \lambda_i \nabla h_i(y^*(x))^T d > 0$. Hence,

$$g'_y(x, y^*(x); d) = \max\left\{\xi^T d : \xi \in \partial_y g(x, y^*(x))\right\} \geq z^T d > 0. \tag{2.21}$$

On the other hand, from $y^*(x)$ is a minimizer, we know that $g'_y(x, y^*(x); d) \geq 0$, this leads to a contradiction;

(2) from Assumption 2.8 and Theorem 3.1 in [13], the conclusion is direct;

(3) since $g(x, y)$ is a locally Lipschitzian function with respect to x and y, then there exists $\delta > 0$, $\delta' > 0$, and $L_2 > 0$ such that for any $x_1 \in U(x, \delta)$, $y \in U(y^*(x), \delta')$, it has

$$\begin{gathered}\left|g(x_1, y^*(x)) - g(x, y^*(x))\right| \leq L_2\|x_1 - x\|, \\ \left|g(x, y) - g(x, y^*(x))\right| \leq L_2\left\|y - y^*(x)\right\|.\end{gathered} \tag{2.22}$$

As to $x_1 \in U(x, \delta)$, from the conclusion in (1.2), there exists a a $L_1 > 0$ such that $\|y^*(x_1) - y^*(x)\| \leq L_1\|x_1 - x\|$. As a result,

$$\begin{aligned}&|v(x_1) - v(x)| \\ &\quad = \left|g(x_1, y^*(x_1) - g(x, y^*(x)))\right| \\ &\quad = \left|g(x_1, y^*(x_1)) - g(x_1, y^*(x)) + g(x_1, y^*(x)) - g(x, y^*(x))\right| \\ &\quad \leq \left|g(x_1, y^*(x_1)) - g(x_1, y^*(x))\right| + \left|g(x_1, y^*(x)) - g(x, y^*(x))\right| \\ &\quad \leq L_2\left\|y^*(x_1) - y^*(x)\right\| + L_2\|x_1 - x\| \leq L_2(1 + L_1)\|x_1 - x\|.\end{aligned} \tag{2.23}$$

Hence, the marginal function $v(x)$ is a local Lipschitzian function with respect to x.

Let $\widehat{S}(x) = \{-\nabla_x g_i(x, y(x)),\ i \in I(x, y(x))\}$, then $S(x) = \mathrm{co}\{\xi, \xi \in \widehat{S}(x)\}$. We prove that $\widehat{S}(x)$ is closed first, that is, prove for any sequence $\{x^k\} \subset R^n$, $x^k \to \overline{x}$, $z^k \in \widehat{S}(x^k)$, $z^k \to z$, it has $z \in \widehat{S}(\overline{x})$.

From $z^k \in \widehat{S}(x^k)$, there exist $y^k \in Y(x^k)$; $i_k \in I(x^k, y^k)$ such that $z^k = -\nabla_x g_{i_k}(x^k, y^k)$. Without loss of generality, suppose that $\{y^k\}$ converges to $\overline{y}$; $\{i_k\}$ converges to $\overline{i}$. From Proposition 3.3 in [14], it has $\overline{y} \in Y(\overline{x})$, and $\overline{i} \in I(\overline{x}, \overline{y})$ and from $\nabla_x g_{\overline{i}}(x, y)$ is a continuous function, it has $z = \lim_{k\to+\infty} z_k = \lim_{k\to+\infty} \nabla_x g_{i_k}(x^k, y^k) = \nabla_x g_{\overline{i}}(\overline{x}, \overline{y}) \in \widehat{S}(x)$. As a result, $\widehat{S}(x)$ is a closed set.

From Theorem 3.2.16 in [11], for any $\xi \in \partial \upsilon(x)$, there exists $x^k \in R^n$, $x^k \to x$ such that $\nabla \upsilon(x^k)$ exists and $\xi = \lim_{k\to+\infty} \nabla \upsilon(x^k)$. In addition, for arbitrary $d \in R^n$, it has

$$\begin{aligned}
\nabla \upsilon\left(x^k\right)^T d = \upsilon'\left(x^k; d\right) &= \lim_{t\downarrow 0} \frac{\upsilon(x^k + td) - \upsilon(x^k)}{t} \\
&= \lim_{t\downarrow 0} \frac{g(x^k + td, y^*(x^k + td)) - g(x^k, y^*(x^k))}{t} \\
&\leq \lim_{t\downarrow 0} \frac{g(x^k + td, y^*(x^k)) - g(x^k, y^*(x^k))}{t} \\
&= \max_{i\in I\left(x^k, y\right)} \left\{ -\nabla_x g_i\left(x^k, y\right)^T d \right\}.
\end{aligned} \tag{2.24}$$

From the definition of $S(x^k)$, $\exists z^k \in S(x^k)$ such that $z^{k^T} d = \max_{i\in I(x^k,y)} \{-\nabla_x g_i(x^k, y)^T d\}$. Hence, it has $\nabla \upsilon(x^k)^T d \leq z^{k^T} d$.

From $z^k \to z \in \widehat{S}(x) \subset S(x)$, $\nabla \upsilon(x^k) \to \xi$ and $\nabla \upsilon(x^k)^T d \leq z^{k^T} d$, it has $\xi^T d \leq z^T d$, that is, for arbitrary $d \in R^n$ and $\xi \in \partial \upsilon(x)$, there exists $z \in S(x)$ such that $\xi^T d \leq z^T d$.

If $\partial \upsilon(x) \subset S(x)$ does not hold, then there exists a $\xi \in \partial \upsilon(x)$ and $\xi \notin S(x)$. From $S(x)$ is a compact convex set and separation theorem ([15]), there exists a $d \in R^n$ such that $\xi^T d < 0$ and for arbitrary $z \in S(x)$, $z^T d \geq 0$, which leads to a contradiction. As a result, $\partial \upsilon(x) \subset S(x)$ holds. From $\partial \upsilon(x) \subset S(x)$ and $S(x) = \mathrm{co}\{\nabla_x g_i(x, y^*(x)),\ i \in I(x, y^*(x))\}$, computation formula (2.17) is direct. □

3. Discussion

In this paper, sufficient optimality and sensitivity analysis of a parameterized min-max programming are given. A rule for computation the subdifferential of $\upsilon(x)$ is established. Though the assumptions in this paper are some restrictive compared to some existing work, the assumptions hold naturally for some applications. Moreover, the obtained computation formula is simple, it is beneficial for establishing a concise first-order necessary optimality system of (1.1), and then constructing effective algorithms to solve the applications.

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References

[1] C. Kirjner-Neto and E. Polak, "On the conversion of optimization problems with max-min constraints to standard optimization problems," *SIAM Journal on Optimization*, vol. 8, no. 4, pp. 887–915, 1998.

[2] E. Polak and J. O. Royset, "Algorithms for finite and semi-infinite min-max-min problems using adaptive smoothing techniques," *Journal of Optimization Theory and Applications*, vol. 119, no. 3, pp. 421–457, 2003.

[3] G.-X. Liu, "A homotopy interior point method for semi-infinite programming problems," *Journal of Global Optimization*, vol. 37, no. 4, pp. 631–646, 2007.
[4] O. Stein and G. Still, "Solving semi-infinite optimization problems with interior point techniques," *SIAM Journal on Control and Optimization*, vol. 42, no. 3, pp. 769–788, 2003.
[5] O. Stein and A. Tezel, "The semismooth approach for semi-infinite programming under the reduction ansatz," *Journal of Global Optimization*, vol. 41, no. 2, pp. 245–266, 2008.
[6] A. Auslender, "Stability in mathematical programming with nondifferentiable data," *SIAM Journal on Control and Optimization*, vol. 22, no. 2, pp. 239–254, 1984.
[7] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, NY, USA, 2000.
[8] J. F. Bonnans and A. Shapiro, "Optimization problems with perturbations: a guided tour," *SIAM Review*, vol. 40, no. 2, pp. 228–264, 1998.
[9] B. S. Mordukhovich, N. M. Nam, and N. D. Yen, "Subgradients of marginal functions in parametric mathematical programming," *Mathematical Programming B*, vol. 116, no. 1-2, pp. 369–396, 2009.
[10] R. W. Chaney, "Second-order sufficient conditions in nonsmooth optimization," *Mathematics of Operations Research*, vol. 13, no. 4, pp. 660–673, 1988.
[11] M. M. Mäkelä and P. Neittaanmäki, *Nonsmooth Optimizatin: Analysis and Algorithms with Application to Optimal Control*, Utopia Press, Singapore, 1992.
[12] L. Huang and K. F. Ng, "Second-order optimality conditions for minimizing a max-function," *Science in China A*, vol. 43, no. 7, pp. 722–733, 2000.
[13] B. S. Mordukhovich, "Sensitivity analysis in nonsmooth optimization," in *Theoretical Aspects of Industrial Design*, D. A. Field and V. Komkov, Eds., pp. 32–46, SIAM, Philadelphia, Pa, USA, 1992.
[14] V. F. Demyanov and A. M. Rubinov, *Constructive Nonsmooth Analysis*, vol. 7, Peter Lang, Frankfurt am Main, Germany, 1995.
[15] R. T. Rochafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, USA, 1970.

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Research Article

Hybrid Method with Perturbation for Lipschitzian Pseudocontractions

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Assume that F is a nonlinear operator which is Lipschitzian and strongly monotone on a nonempty closed convex subset C of a real Hilbert space H. Assume also that Ω is the intersection of the fixed point sets of a finite number of Lipschitzian pseudocontractive self-mappings on C. By combining hybrid steepest-descent method, Mann's iteration method and projection method, we devise a hybrid iterative algorithm with perturbation F, which generates two sequences from an arbitrary initial point $x_0 \in H$. These two sequences are shown to converge in norm to the same point $P_\Omega x_0$ under very mild assumptions.

1. Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and C a nonempty closed convex subset of H. Let $T : C \to C$ be a self-mapping of C. Recall that T is said to be a pseudocontractive mapping if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C, \tag{1.1}$$

and T is said to be a strictly pseudo-contractive mapping if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

For such cases, we also say that T is a k-strict pseudo-contractive mapping. We use $F(T)$ to denote the set of fixed points of T.

It is well known that the class of strictly pseudo-contractive mappings strictly includes the class of nonexpansive mappings which are the mappings T on C such that

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

Iterative methods for nonexpansive mappings have been extensively investigated; see [1–16] and the references therein.

However, iterative methods for strictly pseudo-contractive mappings are far less developed than those for nonexpansive mappings though Browder and Petryshyn initiated their work in 1967; the reason is probably that the second term appearing on the right-hand side of (1.2) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strictly pseudo-contractive mapping T. However, on the other hand, strictly pseudo-contractive mappings have more powerful applications than nonexpansive mappings do in solving inverse problems; see Scherzer [17]. Therefore, it is interesting to develop iterative methods for strictly pseudo-contractive mappings. As a matter of fact, Browder and Petryshyn [18] showed that if a k-strict pseudo-contractive mapping T has a fixed point in C, then starting with an initial $x_0 \in C$, the sequence $\{x_n\}$ generated by the recursive formula:

$$x_{n+1} = \alpha x_n + (1-\alpha)Tx_n, \quad \forall n \ge 0, \tag{1.4}$$

where α is a constant such that $k < \alpha < 1$ converges weakly to a fixed point of T.

Recently, Marino and Xu [19] have extended Browder and Petryshyn's result by proving that the sequence $\{x_n\}$ generated by the following Mann's algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \ge 0 \tag{1.5}$$

converges weakly to a fixed point of T, provided that the control sequence $\{\alpha_n\}$ satisfies the condition that $k < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty$. However, this convergence is in general not strong. It is well known that if C is a bounded and closed convex subset of H, and $T : C \to C$ is a demicontinuous pseudocontraction, then T has a fixed point in C (Theorem 2.3 in [20]). However, all efforts to approximate such a fixed point by virtue of the normal Mann's iteration algorithm failed.

In 1974, Ishikawa [21] introduced a new iteration algorithm and proved the following convergence theorem.

Theorem I (see [21]). *If C is a compact convex subset of a Hilbert space $H, T : C \to C$ is a Lipschitzian pseudocontraction and $x_0 \in C$ is chosen arbitrarily, then the sequence $\{x_n\}_{n\ge 0}$ converges strongly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each positive integer $n \ge 0$ by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the conditions (i) $0 \le \alpha_n \le \beta_n < 1$; (ii) $\beta_n \to 0$ *as* $n \to \infty$; (iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

Since its publication in 1974, it remains an open question whether or not Mann's iteration algorithm converges under the setting of Theorem I to a fixed point of T if the mapping T is Lipschitzian pseudo-contractive. In [22], Chidume and Mutangadura gave an example of a Lipschitzian pseudocontraction with a unique fixed point for which Mann's iteration algorithm fails to converge.

In an infinite-dimensional Hilbert space, Mann and Ishikawa's iteration algorithms have only weak convergence, in general, even for nonexpansive mapping. So, in order to get strong convergence for strictly pseudo-contractive mappings, several attempts have been made based on the CQ method (see, e.g., [19, 23, 24]). The last scheme, in such a direction, seems for us to be the following due to Zhou [25]:

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily,} \\
y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\
z_n = (1-\beta_n)x_n + \beta_n T y_n, \\
C_n = \left\{ z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n\beta_n\left(1 - 2\alpha_n - L^2\alpha_n^2\right)\|x_n - Tx_n\|^2 \right\}, \\
Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0.
\end{gathered} \tag{1.7}$$

He proved, under suitable choice of the parameters α_n and β_n, that the sequence $\{x_n\}$ generated by (1.7) strongly converges to $P_{F(T)}x_0$.

Among classes of nonlinear mappings, the class of pseudocontractions is one of the most important. This is due to the relation between the class of pseudocontractions and the class of monotone mappings (we recall that a mapping A is monotone if $\langle Ax - Ay, x - y\rangle \ge 0$ for all $x, y \in H$). A mapping A is monotone if and only if $(I - A)$ is pseudo-contractive. It is well known (see, e.g., [26]) that if S is monotone, then the solutions of the equation $Sx = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts, especially within the past 30 years or so, have been devoted to iterative methods for approximating fixed points of a pseudo-contractive mapping T (see e.g., [27–32] and the references therein).

Very recently, motivated by the work in [19, 25, 33] and the related work in the literature, Yao et al. [34] suggested and analyzed a hybrid algorithm for pseudo-contractive mappings in Hilbert spaces. Further, they proved the strong convergence of the proposed iterative algorithm for Lipschitzian pseudo-contractive mappings.

Theorem YLM (see [34]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitzian pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that the sequence $\alpha_n \in [a, b]$ for some $a, b \in (0,\ 1/(L+1))$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, let $\{x_n\}$ be the sequence in C generated iteratively by*

$$\begin{gathered}
y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\
C_{n+1} = \left\{ z \in C_n : \left\|\alpha_n (I - T) y_n\right\|^2 \le 2\alpha_n \left\langle x_n - z, (I - T) y_n \right\rangle \right\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \ge 1.
\end{gathered} \tag{1.8}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Inspired by the above research work of Yao et al. [34], in this paper we will continue this direction of research. Let C be a nonempty closed convex subset of a real Hilbert space H. We will propose a new hybrid iterative scheme with perturbed mapping for approximating fixed points of a Lipschitzian pseudo-contractive self-mapping on C. We will establish a strong convergence theorem for this hybrid iterative scheme. To be more specific, let $T : C \to C$ be a L-Lipschitzian pseudo-contractive mapping and $F : C \to H$ a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strong monotone. Let $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset [0,1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. We introduce the following hybrid iterative process with perturbed mapping F. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, two sequences $\{x_n\}, \{y_n\}$ are generated as follows:

$$\begin{gathered} y_n = (1-\alpha_n)x_n + \alpha_n P_C\left[Tx_n - \lambda_n\mu F(Tx_n)\right], \\ C_{n+1} = \Big\{z \in C_n : \left\|\alpha_n(I - P_C(I-\lambda_n\mu F)T)y_n\right\|^2 \le 2\alpha_n\big[\langle x_n - z, (I - P_C(I-\lambda_n\mu F)T)y_n\rangle \\ -\langle Ty_n - P_C(I-\lambda_n\mu F)Ty_n, y_n - z\rangle\big]\Big\} \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 1. \end{gathered} \tag{1.9}$$

It is clear that if $\lambda_n = 0$, for all $n \ge 1$, then the hybrid iterative scheme (1.9) reduces to the hybrid iterative process (1.8). Under very mild assumptions, we obtain a strong convergence theorem for the sequences $\{x_n\}$ and $\{y_n\}$ generated by the introduced method. Our proposed hybrid method with perturbation is quite general and flexible and includes the hybrid method considered in [34] and several other iterative methods as special cases. Our results represent the modification, supplement, extension, and improvement of [34, Algorithm 3.1 and Theorem 3.1]. Further, we consider the more general case, where $\{T_i\}_{i=1}^N$ are N L-Lipschitzian pseudo-contractive self-mappings on C with $N \ge 1$ an integer. In this case, we propose another hybrid iterative process with perturbed mapping F for approximating a common fixed point of $\{T_i\}_{i=1}^N$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, two sequences $\{x_n\}$ and $\{y_n\}$ are generated as follows:

$$\begin{gathered} y_n = (1-\alpha_n)x_n + \alpha_n P_C\left[T_nx_n - \lambda_n\mu F(T_nx_n)\right], \\ C_{n+1} = \Big\{z \in C_n : \left\|\alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2 \le 2\alpha_n\big[\langle x_n - z, (I - P_C(I-\lambda_n\mu F)T_n)y_n\rangle \\ -\langle T_ny_n - P_C(I-\lambda_n\mu F)T_ny_n, y_n - z\rangle\big]\Big\} \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 1, \end{gathered} \tag{1.10}$$

where $T_n := T_{n \bmod N}$, for integer $n \ge 1$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$ (i.e., if $n = jN + q$ for some integers $j \ge 0$ and $0 \le q < N$, then $T_n = T_N$ if $q = 0$ and $T_n = T_q$ if $1 < q < N$). It is clear that if $N = 1$, then the hybrid iterative scheme (1.10) reduces to the hybrid iterative process (1.9). Under quite appropriate conditions, we derive a strong convergence theorem for the sequences $\{x_n\}$ and $\{y_n\}$ generated by the proposed method.

We now give some preliminaries and results which will be used in the rest of this paper. A Banach space X is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in X which converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \neq x. \tag{1.11}$$

It is well known that every Hilbert space H satisfies Opial's condition (see, e.g., [35]). Throughout this paper, we shall use the notations: "$\rightharpoonup$" and "$\to$" standing for the weak convergence and strong convergence, respectively. Moreover, we shall use the following notation: for a given sequence $\{x_n\} \subset X, \omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) := \left\{x \in X : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\right\}. \tag{1.12}$$

In addition, for each point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C, \tag{1.13}$$

where P_C is called the metric projection of H onto C. It is known that P_C is a nonexpansive mapping.

Now we collect some lemmas which will be used in the proof of the main result in the next section. We note that Lemmas 1.1 and 1.2 are well known.

Lemma 1.1. *Let H be a real Hilbert space. There holds the following identity:*

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \quad \forall x, y \in H. \tag{1.14}$$

Lemma 1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{1.15}$$

Lemma 1.3 (see [23]). *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition:*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n \geq 0. \tag{1.16}$$

Then $x_n \to q$.

Lemma 1.4 (see [27]). *Let X be a real reflexive Banach space which satisfies Opial's condition. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a continuous pseudo-contractive mapping. Then, $I - T$ is demiclosed at zero.*

Let $T : C \to C$ be a nonexpansive mapping and $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone, that is, F satisfies the following conditions:

$$\begin{gathered}\|Fx - Fy\| \leq \kappa\|x - y\|, \quad \forall x, y \in C,\\ \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C,\end{gathered} \tag{1.17}$$

respectively. For any given numbers $\lambda \in [0,1)$ and $\mu \in (0,\ 2\eta/\kappa^2)$, we define the mapping $T^{\lambda} : C \to H$:

$$T^{\lambda}x := Tx - \lambda\mu F(Tx), \quad \forall x \in C. \tag{1.18}$$

Lemma 1.5 (see [36]). *If $0 \leq \lambda < 1$ and $0 < \mu < 2\eta/\kappa^2$, then there holds for $T^{\lambda} : C \to H$:*

$$\left\|T^{\lambda}x - T^{\lambda}y\right\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C, \tag{1.19}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1)$.

In particular, whenever $T = I$ the identity operator of H, we have

$$\|(I - \lambda\mu F)x - (I - \lambda\mu F)y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C. \tag{1.20}$$

2. Main Result

In this section, we introduce a hybrid iterative algorithm with perturbed mapping for pseudo-contractive mappings in a real Hilbert space H.

Algorithm 2.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping and $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0, F$ is κ-Lipschitzian and η-strong monotone. Let $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset [0,1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define two sequences: $\{x_n\}$ and $\{y_n\}$ of C as follows:

$$\begin{gathered}y_n = (1 - \alpha_n)x_n + \alpha_n P_C\left[Tx_n - \lambda_n\mu F(Tx_n)\right],\\ C_{n+1} = \Big\{z \in C_n : \|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \leq 2\alpha_n\left[\langle x_n - z, (I - P_C(I - \lambda_n\mu F)T)y_n\rangle\right.\\ \left. - \langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - z\rangle\right]\Big\}\\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1.\end{gathered} \tag{2.1}$$

Now we prove the strong convergence of the above iterative algorithm for Lipschitzian pseudo-contractive mappings.

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitzian pseudo-contractive mapping such that $F(T) \neq \emptyset$, and let $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strong monotone. Assume that $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0,\ 1/(L+1))$ and $\{\lambda_n\} \subset [0, 1)$ such that $\lim_{n\to\infty}\lambda_n = 0$. Take a fixed number $\mu \in (0,\ 2\eta/\kappa^2)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (2.1) converge strongly to the same point $P_{F(T)}x_0$.*

Proof. Firstly, we observe that $P_{F(T)}$ and $\{x_n\}$ are well defined. From [19, 27], we note that $F(T)$ is closed and convex. Indeed, by [27], we can define a mapping $g : C \to C$ by $g(x) = (2I - T)^{-1}$ for every $x \in C$. It is clear that g is a nonexpansive self-mapping such that $F(T) = F(g)$. Hence, by [23, Proposition 2.1 (iii)], we conclude that $F(g) = F(T)$ is a closed convex set. This implies that the projection $P_{F(T)}$ is well defined. It is obvious that $\{C_n\}$ is closed and convex. Thus, $\{x_n\}$ is also well defined.

Now, we show that $F(T) \subset C_n$ for all $n \geq 0$. Indeed, taking $p \in F(T)$, we note that $(I - T)p = 0$, and (1.1) is equivalent to

$$\langle (I-T)x - (I-T)y, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{2.2}$$

Using Lemma 1.1 and (2.2), we obtain

$$\begin{aligned}
&\|x_n - p - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \\
&\quad = \|x_n - p\|^2 - \|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \\
&\qquad - 2\alpha_n\langle (I - P_C(I - \lambda_n\mu F)T)y_n, x_n - p - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle \\
&\quad = \|x_n - p\|^2 - \|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 - 2\alpha_n\langle (I-T)y_n - (I-T)p, y_n - p\rangle \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\qquad - 2\alpha_n\langle (I - P_C(I - \lambda_n\mu F)T)y_n, x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle \\
&\quad \leq \|x_n - p\|^2 - \|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 - 2\alpha_n\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\qquad - 2\alpha_n\langle (I - P_C(I - \lambda_n\mu F)T)y_n, x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle \\
&\quad = \|x_n - p\|^2 - \|x_n - y_n + y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\qquad - 2\alpha_n\langle (I - P_C(I - \lambda_n\mu F)T)y_n, x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle \\
&\quad = \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \\
&\qquad - 2\langle x_n - y_n, y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle \\
&\qquad + 2\alpha_n\langle (I - P_C(I - \lambda_n\mu F)T)y_n, y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\quad = \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \\
&\qquad - 2\langle x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n, y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle
\end{aligned}$$

$$\begin{aligned}
&\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \\
&\quad + 2\left|\langle x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n, y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle\right| \\
&\quad - 2\alpha_n\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle.
\end{aligned} \tag{2.3}$$

Since T is L-Lipschitzian, utilizing Lemma 1.5 we derive

$$\begin{aligned}
&\|(I - P_C(I - \lambda_n\mu F)T)x_n - (I - P_C(I - \lambda_n\mu F)T)y_n\| \\
&\quad \le \|x_n - y_n\| + \|P_C(I - \lambda_n\mu F)Tx_n - P_C(I - \lambda_n\mu F)Ty_n\| \\
&\quad \le \|x_n - y_n\| + \|(I - \lambda_n\mu F)Tx_n - (I - \lambda_n\mu F)Ty_n\| \\
&\quad \le \|x_n - y_n\| + (1 - \lambda_n\tau)\|Tx_n - Ty_n\| \\
&\quad \le \|x_n - y_n\| + \|Tx_n - Ty_n\| \\
&\quad \le (L+1)\|x_n - y_n\|.
\end{aligned} \tag{2.4}$$

From (2.1), we observe that $x_n - y_n = \alpha_n(I - P_C(I - \lambda_n\mu F)T)x_n$. Hence, utilizing Lemma 1.5 and (2.4) we obtain

$$\begin{aligned}
&\left|\langle x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n, y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle\right| \\
&\quad = \alpha_n\big|\langle (I - P_C(I - \lambda_n\mu F)T)x_n - (I - P_C(I - \lambda_n\mu F)T)y_n, \\
&\qquad\qquad y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\rangle\big| \\
&\quad \le \alpha_n\|(I - P_C(I - \lambda_n\mu F)T)x_n - (I - P_C(I - \lambda_n\mu F)T)y_n\| \\
&\qquad \times \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\| \\
&\quad \le \alpha_n(L+1)\|x_n - y_n\|\|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\| \\
&\quad \le \frac{\alpha_n(L+1)}{2}\left(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2\right).
\end{aligned} \tag{2.5}$$

Combining (2.3) and (2.5), we get

$$\begin{aligned}
&\|x_n - p - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2 \\
&\qquad + \alpha_n(L+1)\left(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2\right) \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\quad = \|x_n - p\|^2 + [\alpha_n(L+1) - 1]\left(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\|^2\right) \\
&\qquad - 2\alpha_n\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle \\
&\quad \le \|x_n - p\|^2 - 2\alpha_n\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle.
\end{aligned} \tag{2.6}$$

At the same time, we observe that

$$\begin{aligned}\left\|x_n - p - \alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\right\|^2 &= \left\|x_n - p\right\|^2 - 2\alpha_n\langle x_n - p, (I - P_C(I - \lambda_n\mu F)T)y_n\rangle \\ &\quad + \left\|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\right\|^2.\end{aligned} \tag{2.7}$$

Therefore, from (2.6) and (2.7) we have

$$\begin{aligned}\left\|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\right\|^2 \leq 2\alpha_n[&\langle x_n - p, (I - P_C(I - \lambda_n\mu F)T)y_n\rangle \\ &-\langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - p\rangle],\end{aligned} \tag{2.8}$$

which implies that

$$p \in C_n, \tag{2.9}$$

that is,

$$F(T) \subset C_n, \quad \forall n \geq 0. \tag{2.10}$$

From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \quad \forall y \in C_n. \tag{2.11}$$

Utilizing $F(T) \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0, \quad \forall u \in F(T). \tag{2.12}$$

So, for all $u \in F(T)$ we have

$$\begin{aligned}0 &\leq \langle x_0 - x_n, x_n - u\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|,\end{aligned} \tag{2.13}$$

which hence implies that

$$\|x_0 - x_n\| \leq \|x_0 - u\|, \quad \forall u \in F(T). \tag{2.14}$$

Thus, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{Ty_n\}$.

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \tag{2.15}$$

Hence,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|, \end{aligned} \tag{2.16}$$

and therefore

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|. \tag{2.17}$$

This implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

From Lemma 1.1 and (2.15), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \\ &\longrightarrow 0. \end{aligned} \tag{2.18}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, from $\|x_n - x_{n+1}\| \to 0$ and $\lambda_n \to 0$ it follows that

$$\begin{aligned} &\left\|\alpha_n(I - P_C(I - \lambda_n\mu F)T)y_n\right\|^2 \\ &\quad\leq 2\alpha_n\left[\langle x_n - x_{n+1}, (I - P_C(I - \lambda_n\mu F)T)y_n\rangle - \langle Ty_n - P_C(I - \lambda_n\mu F)Ty_n, y_n - x_{n+1}\rangle\right] \\ &\quad\leq 2\alpha_n\left[\|x_n - x_{n+1}\|\left\|y_n - P_C(I - \lambda_n\mu F)Ty_n\right\| + \left\|Ty_n - P_C(I - \lambda_n\mu F)Ty_n\right\|\left\|y_n - x_{n+1}\right\|\right] \\ &\quad\leq 2\alpha_n\left[\|x_n - x_{n+1}\|\left\|y_n - P_C(I - \lambda_n\mu F)Ty_n\right\| + \left\|Ty_n - (I - \lambda_n\mu F)Ty_n\right\|\left\|y_n - x_{n+1}\right\|\right] \\ &\quad= 2\alpha_n\left[\|x_n - x_{n+1}\|\left\|y_n - P_C(I - \lambda_n\mu F)Ty_n\right\| + \lambda_n\mu\left\|F(Ty_n)\right\|\left\|y_n - x_{n+1}\right\|\right] \\ &\quad\longrightarrow 0. \end{aligned} \tag{2.19}$$

Noticing that $\alpha_n \in [a, b]$ for some $a, b \in (0,\ 1/(L+1))$, thus, we obtain

$$\left\|y_n - P_C(I - \lambda_n\mu F)Ty_n\right\| \longrightarrow 0. \tag{2.20}$$

Also, we note that $\|Ty_n - P_C(I-\lambda_n\mu F)Ty_n\| \le \lambda_n\mu\|F(Ty_n)\| \to 0$. Therefore, we get

$$\|y_n - Ty_n\| \le \|y_n - P_C(I-\lambda_n\mu F)Ty_n\| + \|Ty_n - P_C(I-\lambda_n\mu F)Ty_n\| \longrightarrow 0. \tag{2.21}$$

On the other hand, utilizing Lemma 1.5 we deduce that

$$\begin{aligned}
&\|x_n - P_C(I-\lambda_n\mu F)Tx_n\| \\
&\quad \le \|x_n - y_n\| + \|y_n - P_C(I-\lambda_n\mu F)Ty_n\| + \|P_C(I-\lambda_n\mu F)Ty_n - P_C(I-\lambda_n\mu F)Tx_n\| \\
&\quad \le \|x_n - y_n\| + \|y_n - P_C(I-\lambda_n\mu F)Ty_n\| + \|(I-\lambda_n\mu F)Ty_n - (I-\lambda_n\mu F)Tx_n\| \\
&\quad \le \|x_n - y_n\| + \|y_n - P_C(I-\lambda_n\mu F)Ty_n\| + (1-\lambda_n\tau)\|Ty_n - Tx_n\| \\
&\quad \le \|x_n - y_n\| + \|y_n - P_C(I-\lambda_n\mu F)Ty_n\| + L\|y_n - x_n\| \\
&\quad = (L+1)\|x_n - y_n\| + \|y_n - P_C(I-\lambda_n\mu F)Ty_n\| \\
&\quad = \alpha_n(L+1)\|x_n - P_C(I-\lambda_n\mu F)Tx_n\| + \|y_n - P_C(I-\lambda_n\mu F)Ty_n\|,
\end{aligned} \tag{2.22}$$

that is,

$$\|x_n - P_C(I-\lambda_n\mu F)Tx_n\| \le \frac{1}{1-\alpha_n(L+1)}\|y_n - P_C(I-\lambda_n\mu F)Ty_n\| \longrightarrow 0. \tag{2.23}$$

Meantime, it is clear that

$$\|Tx_n - P_C(I-\lambda_n\mu F)Tx_n\| \le \lambda_n\mu\|F(Tx_n)\| \longrightarrow 0. \tag{2.24}$$

Consequently,

$$\|x_n - Tx_n\| \le \|x_n - P_C(I-\lambda_n\mu F)Tx_n\| + \|Tx_n - P_C(I-\lambda_n\mu F)Tx_n\| \longrightarrow 0. \tag{2.25}$$

Now (2.25) and Lemma 1.4 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T, that is, $\omega_w(x_n) \subset F(T)$. In fact, the inequality (2.14) and Lemma 1.3 ensure the strong convergence of $\{x_n\}$ to $P_{F(T)}x_0$. Since $\|x_n - y_n\| = \|\alpha_n(I - P_C(I-\lambda_n\mu F)T)x_n\| \to 0$, it is immediately known that $\{y_n\}$ converges strongly to $P_{F(T)}x_0$. This completes the proof. □

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$, and let $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strong monotone. Assume that $\{\alpha_n\} \subset [a,b]$ for some $a,b \in (0, 1/2)$ and $\{\lambda_n\} \subset [0,1)$ such that $\lim_{n\to\infty}\lambda_n = 0$. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by* (2.1) *converge strongly to the same point $P_{F(T)}x_0$.*

Corollary 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitzian pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\} \subset [a,b]$ for some*

$a, b \in (0,\ 1/(L+1))$ and $\{\lambda_n\} \subset [0,1)$ such that $\lim_{n\to\infty}\lambda_n = 0$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by the scheme

$$\begin{aligned}
y_n &= (1-\alpha_n)x_n + \alpha_n P_C((1-\lambda_n)Tx_n),\\
C_{n+1} &= \Big\{z \in C_n : \big\|\alpha_n\big(y_n - P_C\big((1-\lambda_n)Ty_n\big)\big)\big\|^2 \le 2\alpha_n\big[\big\langle x_n - z, y_n - P_C\big((1-\lambda_n)Ty_n\big)\big\rangle \\
&\qquad - \big\langle Ty_n - P_C\big((1-\lambda_n)Ty_n\big), y_n - z\big\rangle\big]\Big\}\\
x_{n+1} &= P_{C_{n+1}}x_0
\end{aligned} \tag{2.26}$$

converge strongly to the same point $P_{F(T)}x_0$.

Proof. Put $\mu = 2$ and $F = (1/2)I$ in Theorem 2.2. Then, in this case we have $\kappa = \eta = 1/2$, and hence

$$\left(0, \frac{2\eta}{\kappa^2}\right) = (0,4). \tag{2.27}$$

This implies that $\mu = 2 \in (0,\ 2\eta/\kappa^2) = (0,4)$. Meantime, it is easy to see that the scheme (2.1) reduces to (2.26). Therefore, by Theorem 2.2, we obtain the desired result. □

Corollary 2.5 ([34, Corollary 3.2]). *Let $A : H \to H$ be a L-Lipschitzian monotone mapping for which $A^{-1}(0) \neq \emptyset$. Assume that the sequence $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,\ 1/(L+2))$. Then the sequence $\{x_n\}$ generated by the scheme*

$$\begin{aligned}
y_n &= x_n - \alpha_n Ax_n,\\
C_{n+1} &= \Big\{z \in C_n : \|\alpha_n Ay_n\|^2 \le 2\alpha_n\langle x_n - z, Ay_n\rangle\Big\},\\
x_{n+1} &= P_{C_{n+1}}x_0
\end{aligned} \tag{2.28}$$

strongly converges to $P_{A^{-1}(0)}x_0$.

Proof. Put $\lambda_n = 0$ and $T = I - A$ in Corollary 2.4. Then, it is easy to see that the scheme (2.26) reduces to (2.28). Therefore, by Corollary 2.4, we derive the desired result. □

Next, consider the more general case where Ω is expressed as the intersection of the fixed-point sets of N pseudo-contractive mappings $T_i : C \to C$ with $N \ge 1$ an integer, that is,

$$\Omega = \bigcap_{i=1}^{N} F(T_i). \tag{2.29}$$

In this section, we propose another hybrid iterative algorithm with perturbed mapping for a finite family of pseudo-contractive mappings in a real Hilbert space H.

Algorithm 2.6. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be N pseudo-contractive self-mappings on C with $N \geq 1$ an integer, and let $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strong monotone. Let $\{\alpha_n\} \subset (0,1), \{\lambda_n\} \subset [0,1)$, and take a fixed number $\mu \in (0\ ,2\eta/\kappa^2)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define two sequences $\{x_n\},\ \{y_n\}$ of C as follows:

$$\begin{aligned}
y_n &= (1-\alpha_n)x_n + \alpha_n P_C\left[T_n x_n - \lambda_n \mu F(T_n x_n)\right], \\
C_{n+1} &= \Big\{ z \in C_n : \left\| \alpha_n \left(I - P_C(I - \lambda_n \mu F)T_n\right) y_n \right\|^2 \\
&\qquad \leq 2\alpha_n \left[\left\langle x_n - z, \left(I - P_C(I - \lambda_n \mu F)T_n\right) y_n \right\rangle \right. \\
&\qquad\quad \left. - \left\langle T_n y_n - P_C(I - \lambda_n \mu F)T_n y_n, y_n - z \right\rangle \right] \Big\}, \\
x_{n+1} &= P_{C_{n+1}} x_0, \quad n \geq 1,
\end{aligned} \tag{2.30}$$

where

$$T_n := T_{n \bmod N}, \tag{2.31}$$

for integer $n \geq 1$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$ (i.e., if $n = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_n = T_N$ if $q = 0$ and $T_n = T_q$ if $1 < q < N$).

Theorem 2.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be N L-Lipschitzian pseudo-contractive self-mappings on C such that $\Omega = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $F : C \to H$ be a mapping such that for some constants $\kappa, \eta > 0, F$ is κ-Lipschitzian and η-strong monotone. Assume that $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0,\ 1/(L+1))$ and $\{\lambda_n\} \subset [0,1)$ such that $\lim_{n\to\infty}\lambda_n = 0$. Take a fixed number $\mu \in (0,\ 2\eta/\kappa^2)$. Then the sequences $\{x_n\},\ \{y_n\}$ generated by* (2.30) *converge strongly to the same point $P_\Omega x_0$.*

Proof. Firstly, as stated in the proof of Theorem 2.2, we can readily see that each $F(T_i)$ is closed and convex for $i = 1, 2, \ldots, N$. Hence, Ω is closed and convex. This implies that the projection P_Ω is well defined. It is clear that the sequence $\{C_n\}$ is closed and convex. Thus, $\{x_n\}$ is also well defined.

Now let us show that $\Omega \subset C_n$ for all $n \geq 0$. Indeed, taking $p \in \Omega$, we note that $(I - T_n)p = 0$ and

$$\left\langle (I - T_n)x - (I - T_n)y, x - y \right\rangle \geq 0, \quad \forall x, y \in C. \tag{2.32}$$

Using Lemma 1.1 and (2.32), we obtain

$$\begin{aligned}
&\left\| x_n - p - \alpha_n \left(I - P_C(I - \lambda_n \mu F)T_n\right) y_n \right\|^2 \\
&\quad = \left\| x_n - p \right\|^2 - \left\| \alpha_n \left(I - P_C(I - \lambda_n \mu F)T_n\right) y_n \right\|^2 \\
&\qquad - 2\alpha_n \left\langle \left(I - P_C(I - \lambda_n \mu F)T_n\right) y_n, x_n - p - \alpha_n \left(I - P_C(I - \lambda_n \mu F)T_n\right) y_n \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - p\|^2 - \|\alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\|^2 - 2\alpha_n\langle (I - T_n)y_n - (I - T_n)p, y_n - p\rangle \\
&\quad - 2\alpha_n\langle T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n, y_n - p\rangle \\
&\quad - 2\alpha_n\langle (I - P_C(I - \lambda_n\mu F)T_n)y_n, x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\rangle \\
&\leq \|x_n - p\|^2 - \|\alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\|^2 - 2\alpha_n\langle T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n, y_n - p\rangle \\
&\quad - 2\alpha_n\langle (I - P_C(I - \lambda_n\mu F)T_n)y_n, x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n + y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\|^2 \\
&\quad - 2\alpha_n\langle T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n, y_n - p\rangle \\
&\quad - 2\alpha_n\langle (I - P_C(I - \lambda_n\mu F)T_n)y_n, x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\|^2 \\
&\quad - 2\langle x_n - y_n, y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\rangle \\
&\quad + 2\alpha_n\langle (I - P_C(I - \lambda_n\mu F)T_n)y_n, y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\rangle \\
&\quad - 2\alpha_n\langle T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n, y_n - p\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\|^2 \\
&\quad - 2\langle x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n, y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\rangle \\
&\quad - 2\alpha_n\langle T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n, y_n - p\rangle \\
&\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\|^2 \\
&\quad + 2\left|\langle x_n - y_n - \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n, y_n - x_n + \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\rangle\right| \\
&\quad - 2\alpha_n\langle T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n, y_n - p\rangle.
\end{aligned}
\tag{2.33}
$$

Since each T_i is L-Lipschitzian for $i = 1, 2, \ldots, N$, utilizing Lemma 1.5 we derive

$$
\begin{aligned}
&\|(I - P_C(I - \lambda_n\mu F)T_n)x_n - (I - P_C(I - \lambda_n\mu F)T_n)y_n\| \\
&\quad \leq \|x_n - y_n\| + \|P_C(I - \lambda_n\mu F)T_n x_n - P_C(I - \lambda_n\mu F)T_n y_n\| \\
&\quad \leq \|x_n - y_n\| + \|(I - \lambda_n\mu F)T_n x_n - (I - \lambda_n\mu F)T_n y_n\| \\
&\quad \leq \|x_n - y_n\| + (1 - \lambda_n\tau)\|T_n x_n - T_n y_n\| \\
&\quad \leq \|x_n - y_n\| + \|T_n x_n - T_n y_n\| \\
&\quad \leq (L + 1)\|x_n - y_n\|.
\end{aligned}
\tag{2.34}
$$

From (2.30), we observe that $x_n - y_n = \alpha_n(I - P_C(I - \lambda_n\mu F)T_n)x_n$. Hence, utilizing Lemma 1.5

and (2.34) we obtain

$$\begin{aligned}
&\left|\langle x_n - y_n - \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n, y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\rangle\right| \\
&\quad= \alpha_n\left|\langle (I - P_C(I-\lambda_n\mu F)T_n)x_n - (I - P_C(I-\lambda_n\mu F)T_n)y_n,\right. \\
&\qquad\quad \left. y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\rangle\right| \\
&\quad\le \alpha_n\left\|(I - P_C(I-\lambda_n\mu F)T_n)x_n - (I - P_C(I-\lambda_n\mu F)T_n)y_n\right\| \\
&\qquad\quad \times\left\|y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\| \\
&\quad\le \alpha_n(L+1)\|x_n - y_n\|\left\|y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\| \\
&\quad\le \frac{\alpha_n(L+1)}{2}\left(\|x_n - y_n\|^2 + \left\|y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2\right).
\end{aligned} \tag{2.35}$$

Combining (2.33) and (2.35), we get

$$\begin{aligned}
&\left\|x_n - p - \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2 \\
&\quad\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \left\|y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2 \\
&\qquad + \alpha_n(L+1)\left(\|x_n - y_n\|^2 + \left\|y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2\right) \\
&\qquad - 2\alpha_n\langle T_n y_n - P_C(I-\lambda_n\mu F)T_n y_n, y_n - p\rangle \\
&\quad= \|x_n - p\|^2 + [\alpha_n(L+1) - 1]\left(\|x_n - y_n\|^2 + \left\|y_n - x_n + \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2\right) \\
&\qquad - 2\alpha_n\langle T_n y_n - P_C(I-\lambda_n\mu F)T_n y_n, y_n - p\rangle \\
&\quad\le \|x_n - p\|^2 - 2\alpha_n\langle T_n y_n - P_C(I-\lambda_n\mu F)T_n y_n, y_n - p\rangle.
\end{aligned} \tag{2.36}$$

Meantime, we observe that

$$\begin{aligned}
&\left\|x_n - p - \alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2 \\
&\quad= \|x_n - p\|^2 - 2\alpha_n\langle x_n - p, (I - P_C(I-\lambda_n\mu F)T_n)y_n\rangle \\
&\qquad + \left\|\alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2.
\end{aligned} \tag{2.37}$$

Therefore, from (2.36) and (2.37) we have

$$\begin{aligned}
\left\|\alpha_n(I - P_C(I-\lambda_n\mu F)T_n)y_n\right\|^2 \le 2\alpha_n\big[&\langle x_n - p, (I - P_C(I-\lambda_n\mu F)T_n)y_n\rangle \\
&-\langle T_n y_n - P_C(I-\lambda_n\mu F)T_n y_n, y_n - p\rangle\big],
\end{aligned} \tag{2.38}$$

which implies that

$$p \in C_n, \tag{2.39}$$

that is,

$$\Omega \subset C_n, \quad \forall n \geq 0. \tag{2.40}$$

From $x_n = P_{C_n} x_0$, we have

$$\langle x_0 - x_n, x_n - y \rangle \geq 0, \quad \forall y \in C_n. \tag{2.41}$$

Utilizing $\Omega \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - u \rangle \geq 0, \quad \forall u \in \Omega. \tag{2.42}$$

So, for all $u \in \Omega$ we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - u \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - u \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - u\|, \end{aligned} \tag{2.43}$$

which hence implies that

$$\|x_0 - x_n\| \leq \|x_0 - u\|, \quad \forall u \in \Omega. \tag{2.44}$$

Thus $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{T_n y_n\}$.

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \tag{2.45}$$

Hence,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|, \end{aligned} \tag{2.46}$$

and therefore

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|. \tag{2.47}$$

This implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

From Lemma 1.1 and (2.45), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{2.48}$$

Thus,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.49}$$

Obviously, it is easy to see that $\lim_{n\to\infty}\|x_n - x_{n+i}\| = 0$ for each $i = 1, 2, \ldots, N$. Since $x_{n+1} \in C_{n+1} \subset C_n$, from $\|x_n - x_{n+1}\| \to 0$ and $\lambda_n \to 0$ it follows that

$$\begin{aligned}
&\|\alpha_n(I - P_C(I - \lambda_n\mu F)T_n)y_n\|^2 \\
&\quad\le 2\alpha_n\left[\langle x_n - x_{n+1}, (I - P_C(I - \lambda_n\mu F)T_n)y_n\rangle - \langle T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n, y_n - x_{n+1}\rangle\right] \\
&\quad\le 2\alpha_n\left[\|x_n - x_{n+1}\|\|y_n - P_C(I - \lambda_n\mu F)T_n y_n\| + \|T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n\|\|y_n - x_{n+1}\|\right] \\
&\quad\le 2\alpha_n\left[\|x_n - x_{n+1}\|\|y_n - P_C(I - \lambda_n\mu F)T_n y_n\| + \|T_n y_n - (I - \lambda_n\mu F)T_n y_n\|\|y_n - x_{n+1}\|\right] \\
&\quad= 2\alpha_n\left[\|x_n - x_{n+1}\|\|y_n - P_C(I - \lambda_n\mu F)T_n y_n\| + \lambda_n\mu\|F(T_n y_n)\|\|y_n - x_{n+1}\|\right] \longrightarrow 0.
\end{aligned} \tag{2.50}$$

Noticing that $\alpha_n \in [a, b]$ for some $a, b \in (0,\ 1/(L+1))$, thus, we obtain

$$\|y_n - P_C(I - \lambda_n\mu F)T_n y_n\| \longrightarrow 0. \tag{2.51}$$

Also, we note that $\|T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n\| \le \lambda_n\mu\|F(T_n y_n)\| \to 0$. Therefore, we get

$$\|y_n - T_n y_n\| \le \|y_n - P_C(I - \lambda_n\mu F)T_n y_n\| + \|T_n y_n - P_C(I - \lambda_n\mu F)T_n y_n\| \longrightarrow 0. \tag{2.52}$$

On the other hand, utilizing Lemma 1.5 we deduce that

$$\begin{aligned}
&\left\|x_n - P_C(I-\lambda_n\mu F)T_n x_n\right\| \\
&\quad \le \left\|x_n - y_n\right\| + \left\|y_n - P_C(I-\lambda_n\mu F)T_n y_n\right\| + \left\|P_C(I-\lambda_n\mu F)T_n y_n - P_C(I-\lambda_n\mu F)T_n x_n\right\| \\
&\quad \le \left\|x_n - y_n\right\| + \left\|y_n - P_C(I-\lambda_n\mu F)T_n y_n\right\| + \left\|(I-\lambda_n\mu F)T_n y_n - (I-\lambda_n\mu F)T_n x_n\right\| \\
&\quad \le \left\|x_n - y_n\right\| + \left\|y_n - P_C(I-\lambda_n\mu F)T_n y_n\right\| + (1-\lambda_n\tau)\left\|T_n y_n - T_n x_n\right\| \\
&\quad \le \left\|x_n - y_n\right\| + \left\|y_n - P_C(I-\lambda_n\mu F)T_n y_n\right\| + L\left\|y_n - x_n\right\| \\
&\quad = (L+1)\left\|x_n - y_n\right\| + \left\|y_n - P_C(I-\lambda_n\mu F)T_n y_n\right\| \\
&\quad = \alpha_n(L+1)\left\|x_n - P_C(I-\lambda_n\mu F)T_n x_n\right\| + \left\|y_n - P_C(I-\lambda_n\mu F)T_n y_n\right\|,
\end{aligned} \tag{2.53}$$

that is,

$$\left\|x_n - P_C(I-\lambda_n\mu F)T_n x_n\right\| \le \frac{1}{1-\alpha_n(L+1)}\left\|y_n - P_C(I-\lambda_n\mu F)T_n y_n\right\| \longrightarrow 0. \tag{2.54}$$

Furthermore, it is clear that

$$\left\|T_n x_n - P_C(I-\lambda_n\mu F)T_n x_n\right\| \le \lambda_n\mu\|F(T_n x_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.55}$$

Consequently,

$$\|x_n - T_n x_n\| \le \left\|x_n - P_C(I-\lambda_n\mu F)T_n x_n\right\| + \left\|T_n x_n - P_C(I-\lambda_n\mu F)T_n x_n\right\| \longrightarrow 0, \tag{2.56}$$

and hence for each $i = 1, 2, \dots, N$:

$$\begin{aligned}
\|x_n - T_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\
&\le (L+1)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{2.57}$$

So, we obtain $\lim_{n\to\infty}\|x_n - T_{n+i}x_n\| = 0$ for each $i = 1, 2, \dots, N$. This implies that

$$\lim_{n\to\infty}\|x_n - T_l x_n\| = 0 \quad \text{for each } l = 1, 2, \dots, N. \tag{2.58}$$

Now (2.58) and Lemma 1.4 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T_l. Since l is an arbitrary element in the finite set $\{1, 2, \dots, N\}$, it is known that every weak limit point of $\{x_n\}$ lies in Ω, that is, $\omega_w(x_n) \subset \Omega$. This fact, the inequality (2.44) and Lemma 1.3 ensure the strong convergence of $\{x_n\}$ to $P_\Omega x_0$. Since $\|x_n - y_n\| = \|\alpha_n(I - P_C(I-\lambda_n\mu F)T_n)x_n\| \to 0$, it follows immediately that $\{y_n\}$ converges strongly to $P_\Omega x_0$. This completes the proof. □

Remark 2.8. Algorithm 3.1 in [34] for a Lipschitzian pseudocontraction is extended to develop our hybrid iterative algorithm with perturbation for N-Lipschitzian pseudocontractions; that is, Algorithm 2.6. Theorem 2.7 is more general and more flexible than Theorem 3.1 in [34]. Also, the proof of Theorem 2.7 is very different from that of Theorem 3.1 in [34] because our technique of argument depends on Lemma 1.5. Finally, we observe that several recent results for pseudocontractive and related mappings can be found in [37–42].

References

[1] H. H. Bauschke, "The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 1, pp. 150–159, 1996.

[2] D. Wu, S.-s. Chang, and G. X. Yuan, "Approximation of common fixed points for a family of finite nonexpansive mappings in Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 5–7, pp. 987–999, 2005.

[3] T.-H. Kim and H.-K. Xu, "Strong convergence of modified Mann iterations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 1-2, pp. 51–60, 2005.

[4] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[5] H.-K. Xu, "Strong convergence of an iterative method for nonexpansive and accretive operators," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 631–643, 2006.

[6] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[7] J. G. O'Hara, P. Pillay, and H.-K. Xu, "Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 54, no. 8, pp. 1417–1426, 2003.

[8] N. Shioji and W. Takahashi, "Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 125, no. 12, pp. 3641–3645, 1997.

[9] Y. Yao and J.-C. Yao, "On modified iterative method for nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1551–1558, 2007.

[10] S.-y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.

[11] L.-C. Zeng and J.-C. Yao, "Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 11, pp. 2507–2515, 2006.

[12] L.-C. Zeng, "A note on approximating fixed points of nonexpansive mappings by the Ishikawa iteration process," *Journal of Mathematical Analysis and Applications*, vol. 226, no. 1, pp. 245–250, 1998.

[13] S.-S. Chang, "Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1402–1416, 2006.

[14] H. Zegeye and N. Shahzad, "Viscosity methods of approximation for a common fixed point of a family of quasi-nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 7, pp. 2005–2012, 2008.

[15] C. E. Chidume and C. O. Chidume, "Iterative approximation of fixed points of nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 288–295, 2006.

[16] J. S. Jung, "Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 509–520, 2005.

[17] O. Scherzer, "Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 3, pp. 911–933, 1995.

[18] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[19] G. Marino and H.-K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.

[20] K. Q. Lan and J. H. Wu, "Convergence of approximants for demicontinuous pseudo-contractive maps in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 49, no. 6, pp. 737–746, 2002.

[21] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.

[22] C. E. Chidume and S. A. Mutangadura, "An example of the Mann iteration method for Lipschitz pseudocontractions," *Proceedings of the American Mathematical Society*, vol. 129, no. 8, pp. 2359–2363, 2001.

[23] C. Martinez-Yanes and H.-K. Xu, "Strong convergence of the CQ method for fixed point iteration processes," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 11, pp. 2400–2411, 2006.

[24] T.-H. Kim and H.-K. Xu, "Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 5, pp. 1140–1152, 2006.

[25] H. Zhou, "Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 546–556, 2008.

[26] K. Deimling, "Zeros of accretive operators," *Manuscripta Mathematica*, vol. 13, pp. 365–374, 1974.

[27] H. Zhou, "Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 10, pp. 2977–2983, 2008.

[28] M. O. Osilike and A. Udomene, "Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 2, pp. 431–445, 2001.

[29] C. E. Chidume and H. Zegeye, "Approximate fixed point sequences and convergence theorems for Lipschitz pseudocontractive maps," *Proceedings of the American Mathematical Society*, vol. 132, no. 3, pp. 831–840, 2004.

[30] Y. Yao, Y.-C. Liou, and R. Chen, "Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 12, pp. 3311–3317, 2007.

[31] L.-C. Zeng, N.-C. Wong, and J.-C. Yao, "Strong convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type," *Taiwanese Journal of Mathematics*, vol. 10, no. 4, pp. 837–849, 2006.

[32] A. Udomene, "Path convergence, approximation of fixed points and variational solutions of Lipschitz pseudocontractions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2403–2414, 2007.

[33] W. Takahashi, Y. Takeuchi, and R. Kubota, "Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 276–286, 2008.

[34] Y. Yao, Y.-C. Liou, and G. Marino, "A hybrid algorithm for pseudo-contractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4997–5002, 2009.

[35] K. Geobel and W. A. Kirk, *Topics on Metric Fixed-Point Theory*, Cambridge University, Cambridge, UK, 1990.

[36] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.

[37] L. C. Ceng, D. S. Shyu, and J. C. Yao, "Relaxed composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive mappings," *Fixed Point Theory and Applications*, vol. 2009, Article ID 402602, 16 pages, 2009.

[38] L. C. Ceng, A. Petruşel, and J. C. Yao, "A hybrid method for lipschitz continuous monotone mappings and asymptotically strict pseudocontractive mappings in the intermediate sense," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 1, 2010.

[39] L.-C. Ceng, A. Petruşel, and J.-C. Yao, "Iterative approximation of fixed points for asymptotically strict pseudocontractive type mappings in the intermediate sense," *Taiwanese Journal of Mathematics*, vol. 15, no. 2, pp. 587–606, 2011.

[40] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Strong and weak convergence theorems for asymptotically strict pseudocontractive mappings in intermediate sense," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 2, pp. 283–308, 2010.

[41] L.-C. Ceng, A. Petruşel, S. Szentesi, and J.-C. Yao, "Approximation of common fixed points and variational solutions for one-parameter family of Lipschitz pseudocontractions," *Fixed Point Theory*, vol. 11, no. 2, pp. 203–224, 2010.

[42] D. R. Sahu, N. C. Wong, and J. C. Yao, "A unified hybrid iterative method for solving variational inequalities involving generalized pseudo-contractive mappings," *SIAM Journal on Control and Optimization*. In press.